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FOURIER SERIES OF FUNCTIONS INVOLVING EULER POLYNOMIALS

TAEKYUN KIM, DAE SAN KIM, GWAN-WOO JANG, AND JONGKYUM KWON

Abstract. Recently, T. Kim introduced Fourier series expansions of certain special polynomials and investigated some interesting identities and properties of these polynomials by using those Fourier series. In this paper, we consider three types of functions involving Euler polynomials and derive their Fourier series expansions. Moreover, we express each of them in terms of Benoulli functions.

1. Introduction

Let $E_m(x)$ be the Euler polynomials given by the generating function

$$\frac{2}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!}, \quad \text{(see [1,2,5,7-11,16])}. \tag{1.1}$$

From this equation, we can derive the following relation.

$$E_0 = 1, (E+1)^n + E_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

The Bernoulli polynomials $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}, \quad \text{(see [1,2,5,9])}. \tag{1.2}$$

For any real number x, we let

$$< x >= x - [x] \in [0,1) \tag{1.3}$$

denote the fractional part of x.

Here we will consider the following three types of functions involving Euler polynomials and derive their Fourier series expansions. Further, we will express each of them in terms of Bernoulli functions $B_m(< x >)$.

(1) $\alpha_m(< x >) = \sum_{k=0}^{m} E_k(x)x^{m-k}, (m \geq 1)$;
(2) $\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}E_k(x)x^{m-k}, (m \geq 1)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}E_k(x)x^{m-k}, (m \geq 2)$.

The reader may refer to any book (for example, see [13-15,17]), for elementary facts about Fourier analysis.

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Key words and phrases. Fourier series, Euler polynomials.

As to $\gamma_m(< x >)$, we note that the polynomial identity (1.4) follows immediately from Theorems 4.2 and 4.3, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$.

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(x)x^{m-k}\\&=-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\&+\frac{1}{m}\sum_{s=1}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(x),\end{aligned} \tag{1.4}$$

where $H_m=\sum_{j=1}^{m}\frac{1}{j}$ are the harmonic numbers. The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2 , respectively.

2. The function $\alpha_m(< x >)$

Let $\alpha_m(x)=\sum_{k=0}^{m}E_k(x)x^{m-k},(m\geq 1)$. Then we consider the function $\alpha_m(< x >)=\sum_{k=0}^{m}E_k(< x >)< x >^{m-k}$,
defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is $\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi inx}$,
where

$$\begin{aligned}A_n^{(m)}&=\int_0^1\alpha_m(< x >)e^{-2\pi inx}dx\\&=\int_0^1\alpha_m(x)e^{-2\pi inx}dx.\end{aligned} \tag{2.1}$$

To proceed further, we note the following.

$$\begin{aligned}\alpha_m'(x)&=\sum_{k=0}^{m}\big(kE_{k-1}(x)x^{m-k}+(m-k)E_k(x)x^{m-k-1}\big)\\&=\sum_{k=1}^{m}kE_{k-1}(x)x^{m-k}+\sum_{k=0}^{m-1}(m-k)E_k(x)x^{m-k-1}\\&=\sum_{k=0}^{m-1}(k+1)E_k(x)x^{m-k-1}+\sum_{k=0}^{m-1}(m-k)E_k(x)x^{m-k-1}\\&=(m+1)\sum_{k=0}^{m-1}E_k(x)x^{m-1-k}\\&=(m+1)\alpha_{m-1}(x).\end{aligned} \tag{2.2}$$

So, $\alpha'_m(x) = (m+1)\alpha_{m-1}(x)$. From this, $\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x)$.

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.3}$$

$$\begin{aligned}\alpha_m(1) - \alpha_m(0) &= \sum_{k=0}^{m}(E_k(1) - E_k\delta_{m,k}) \\ &= \sum_{k=0}^{m}((-E_k + 2\delta_{k,0})) - \sum_{k=0}^{m} E_k\delta_{m,k} \\ &= -\sum_{k=0}^{m} E_k + 2 - E_m\end{aligned} \tag{2.4}$$

Thus

$$\alpha_m(1) = \alpha_m(0) \Longleftrightarrow \sum_{k=0}^{m} E_k = 2 - E_m. \tag{2.5}$$

Also,

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(-\sum_{k=0}^{m+1} E_k + 2 - E_{m+1}\right). \tag{2.6}$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.
$Case 1 : n \neq 0$.

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha'_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} + \frac{1}{2\pi in}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right) \\ &= \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} + \frac{1}{2\pi in}\left(\sum_{k=0}^{m-1} E_k - 2 + E_{m-1}\right)\right) + \frac{1}{2\pi in}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right) \\ &= \frac{(m+1)m}{(2\pi in)^2}A_n^{(m-2)} + \frac{m+1}{(2\pi in)^2}\left(\sum_{k=0}^{m-1} E_k - 2 + E_{m-1}\right) + \frac{1}{2\pi in}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right) \\ &= \cdots\end{aligned}$$

$$\begin{aligned}
&= \frac{(m+1)_{m-1}}{(2\pi in)^{m-1}} A_n^{(1)} + \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi in)^j} \left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right) \\
&= -\frac{(m+1)!}{(2\pi in)^m} + \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi in)^j} \left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right) \\
&= \sum_{j=1}^{m} \frac{(m+1)_{j-1}}{(2\pi in)^j} \left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right) \\
&= \frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi in)^j} \left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right),
\end{aligned} \tag{2.7}$$

where $A_n^{(1)} = \int_0^1 \alpha_1(x) e^{-2\pi inx} dx = \int_0^1 (2x - \frac{1}{2}) e^{-2\pi inx} dx = -\frac{2}{2\pi in}$.
$Case2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \left(-\sum_{k=0}^{m+1} E_k + 2 - E_{m+1} \right). \tag{2.8}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers m with $\sum_{k=0}^m E_k = 2 - E_m$ and discontinuous with jump discontinuities at integers for those positive integers m with $\sum_{k=0}^m E_k \neq 2 - E_m$.

We recall the following facts about Bernoulli functions $B_n(< x >)$:
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}. \tag{2.9}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.10}$$

Assume first that m is a positive integer with $\sum_{k=0}^m E_k = 2 - E_m$. Then $\alpha_m(1) = \alpha_m(0)$.
$\alpha_m(< x >)$ is piecewise C^∞, and continuous. So the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}\alpha_m(<x>) &= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &+ \frac{1}{m+2}\sum_{n=-\infty,n\neq 0}^{\infty}\left(\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\right)e^{2\pi inx}\\ &= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &- \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\\ &\times\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\end{aligned} \tag{2.11}$$

$$\begin{aligned}&= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &- \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(<x>) \qquad (2.12)\\ &- \frac{1}{m+2}\binom{m+2}{1}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right)\cdot\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases},\end{aligned}$$

for all $x \in (-\infty, \infty)$.

Hence we obtain the following theorem.

Theorem 2.1. *Let m be a positive integer with $\sum_{k=0}^{m} E_k = 2 - E_m$. Then we have the following.*

(a) $\sum_{k=0}^{m} E_k(<x>)<x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} E_k(<x>)<x>^{m-k}\\ &= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &+ \frac{1}{m+2}\sum_{n=-\infty,n\neq 0}^{\infty}\left(\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\right)e^{2\pi inx},\end{aligned}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b) $$\sum_{k=0}^{m} E_k(<x>)<x>^{m-k} = -\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(<x>),$$

for all $x \in (-\infty, \infty)$, where $B_k(<x>)$ is the Bernoulli function.

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Assume next that m is a positive integer with $\sum_{k=0}^{m} E_k \neq 2 - E_m$. Then $\alpha_m(1) \neq \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise C^∞ , and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right) &= \alpha_m(0) - \frac{1}{2}\sum_{k=0}^{m} E_k + 1 - \frac{1}{2}E_m,\\ &= 1 - \frac{1}{2}\sum_{k=0}^{m-1} E_k,\end{aligned}$$

for $x \in \mathbb{Z}$.

Thus, we get the following theorem.

Theorem 2.2. *Let m be a positive integer with $\sum_{k=0}^{m} E_k \neq 2 - E_m$. Then we have the following.*

$$\begin{aligned}&(a) - \frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &+ \frac{1}{m+2}\sum_{n=-\infty, n\neq 0}^{\infty}\left(\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\right)e^{2\pi i n x}\\ &= \begin{cases}\sum_{k=0}^{m} E_k(< x >) < x >^{m-k}, & for\ x \notin \mathbb{Z},\\ 1 - \frac{1}{2}\sum_{k=0}^{m-1} E_k, & for\ x \in \mathbb{Z}.\end{cases}\end{aligned}$$

$$\begin{aligned}&(b) - \frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(< x >)\\ &= \sum_{k=0}^{m} E_k(< x >) < x >^{m-k},\ for\ x \notin \mathbb{Z},\end{aligned}$$

$$\begin{aligned}&- \frac{1}{m+2}\sum_{j=0, j\neq 1}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(< x >)\\ &= 1 - \frac{1}{2}\sum_{k=0}^{m-1} E_k,\ for\ x \in \mathbb{Z}.\end{aligned}$$

Question: For what values of $m \geq 1$, does $\sum_{k=0}^{m} E_k = 2 - E_m$ hold ?

Remark 2.3. Another expression for $A_0^{(m)} = \int_0^1 \alpha_m(x)dx$ was obtained previously (see [3,4,6,12]) and is

$$\sum_{l=0}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m-l+1}{j}E_{l+j}}{(m-l+1)\binom{l+j}{l}} + \frac{4(-1)^{m+1}}{m+2}E_{m+1}. \tag{2.13}$$

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So, we obtain the following identity.

$$\begin{aligned}&\frac{1}{m+2}\left(-\sum_{k=0}^{m+1}E_k+2-E_{m+1}\right)\\&=\sum_{l=0}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m-l+1}{j}E_{l+j}}{(m-l+1)\binom{l+j}{l}}+\frac{4(-1)^{m+1}}{m+2}E_{m+1}.\end{aligned}$$

3. The fuction $\beta_m(<x>)$

Let $\beta_m(x)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(x)x^{m-k}$, $(m\geq 1)$. Then we will consider the function

$$\beta_m(<x>)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)<x>^{m-k},$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(<x>)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

Before proceeding further, we observe the following:

$$\begin{aligned}\beta_m'(x)&=\sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}E_{k-1}(x)x^{m-k}\right.\\&\left.+\frac{m-k}{k!(m-k)!}E_k(x)x^{m-k-1}\right\}\\&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}E_{k-1}(x)x^{m-k}\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k(x)x^{m-1-k}\\&=\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k(x)x^{m-1-k}\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k(x)x^{m-1-k}\\&=2\beta_{m-1}(x).\end{aligned}\tag{3.1}$$

So, $\beta_m'(x)=2\beta_{m-1}(x)$. This implies that $\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x)$.

$$\int_0^1\beta_m(x)dx=\frac{1}{2}\Big(\beta_{m+1}(1)-\beta_{m+1}(0)\Big).\tag{3.2}$$

$$\begin{aligned}
&\beta_m(1)-\beta_m(0)\\
&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\big(E_k(1)-E_k(0)\delta_{m,k}\big)\\
&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left\{(-E_k+2\delta_{k,0})\right\}.\\
&-\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k\delta_{m,k}\\
&=-\sum_{k=0}^{m}\frac{E_k}{k!(m-k)!}+\frac{2}{m!}-\frac{E_m}{m!}.
\end{aligned}\tag{3.3}$$

So, $\int_0^1\beta_m(x)dx=\frac{1}{2}\left(-\sum_{k=0}^{m+1}\frac{E_k}{k!(m+1-k)!}+\frac{2}{(m+1)!}-\frac{E_{m+1}}{(m+1)!}\right)$.
Also, $\beta_m(1)=\beta_m(0)\Leftrightarrow\sum_{k=0}^{m}\frac{E_k}{k!(m-k)!}=\frac{2}{m!}-\frac{E_m}{m!}$.

Now, we are going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1:$n\neq 0$.

$$\begin{aligned}
B_n^{(m)}&=\int_0^1\beta_m(x)e^{-2\pi inx}dx\\
&=-\frac{1}{2\pi in}\Big[\beta_m(x)e^{-2\pi inx}\Big]_0^1+\frac{1}{2\pi in}\int_0^1\beta_m'(x)e^{-2\pi inx}dx\\
&=-\frac{1}{2\pi in}\Big(\beta_m(1)-\beta_m(0)\Big)+\frac{1}{\pi in}\int_0^1\beta_{m-1}(x)e^{-2\pi inx}dx\\
&=\frac{1}{\pi in}B_n^{(m-1)}-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)\\
&=\frac{1}{\pi in}\Big(\frac{1}{\pi in}B_n^{(m-2)}-\frac{1}{2\pi in}\left(\beta_{m-1}(1)-\beta_{m-1}(0)\right)\Big)-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)\\
&=\frac{1}{(\pi in)^2}B_n^{(m-2)}-\frac{2}{(2\pi in)^2}\left(\beta_{m-1}(1)-\beta_{m-1}(0)\right)-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)\\
&=\cdots\\
&=\frac{1}{(\pi in)^{m-1}}B_n^{(1)}-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\left(\beta_{m-j+1}(1)-\beta_{m-j+1}(0)\right)\\
&=-\frac{1}{(\pi in)^m}+\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1}\frac{E_k}{k!(m-j-k+1)!}-\frac{2}{(m-j+1)!}+\frac{E_{m-j+1}}{(m-j+1)!}\right)\\
&=\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1}\frac{E_k}{k!(m-j-k+1)!}-\frac{2}{(m-j+1)!}+\frac{E_{m-j+1}}{(m-j+1)!}\right),
\end{aligned}\tag{3.4}$$

where $B_n^{(1)}=\int_0^1\beta_1(x)e^{-2\pi inx}dx=\int_0^1(2x-\frac{1}{2})e^{-2\pi inx}dx=-\frac{1}{\pi in}$.

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Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\left(-\sum_{k=0}^{m+1}\frac{E_k}{k!(m+1-k)!} + \frac{2}{(m+1)!} - \frac{E_{m+1}}{(m+1)!}\right). \quad (3.5)$$

Let

$$\Omega_m = \beta_m(1) - \beta_m(0) = -\sum_{k=0}^{m}\frac{E_k}{k!(m-k)!} + \frac{2}{m!} - \frac{E_m}{m!},$$

for $m \geq 1$.

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those positive integers m with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. $\beta_m(< x >)$ is piecewise C^∞, and continuous. So the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >)\\
&= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >) < x >^{m-k}\\
&= \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty}\Big(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx}\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1} \times \Big(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\Big)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_k(< x >)\\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z},\end{cases}
\end{aligned}$$

for all $x \in (-\infty, \infty)$. (3.6)

Now, we obtain the following theorem.

Theorem 3.1. *For each positive integer l, let*

$$\Omega_l = -\sum_{k=0}^{l}\frac{E_k}{k!(l-k)!} + \frac{2}{l!} - \frac{E_l}{l!}.$$

Assume that $\Omega_m = 0$, for a positive integer m. Then we have the following.
(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >) < x >^{m-k} = \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty}\Big(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx},$$

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for all $x \in (-\infty, \infty)$. Here the convergence is uniform.

$$\text{(b)} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k(<x>) <x>^{m-k} = \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_k(<x>),$$

for all $x \in (-\infty, \infty)$. Here $B_k(<x>)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Omega_m \neq 0$. Then, $\beta_m(1) \neq \beta_m(0)$. $\beta_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned} &\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m \\ &= \frac{E_m}{m!} + \frac{1}{2}\left(-\sum_{k=0}^{m} \frac{E_k}{k!(m-k)!} + \frac{2}{m!} - \frac{E_m}{m!} \right) \\ &= \frac{1}{2}\left(\frac{2}{m!} - \sum_{k=0}^{m-1} \frac{E_k}{k!(m-k)!} \right). \end{aligned} \tag{3.7}$$

for $x \in \mathbb{Z}$.

So, we obtain the following theorem.

Theorem 3.2. *For each positive integer l, let*

$$\Omega_l = -\sum_{k=0}^{l} \frac{E_k}{k!(l-k)!} + \frac{2}{l!} - \frac{E_l}{l!}.$$

Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned} &\text{(a)} \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \Big(\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k(<x>) <x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{E_m}{m!} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned}$$

Here the convergence is pointwise.

(b)

$$\begin{aligned} &\sum_{j=0}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k(<x>) <x>^{m-k}, \quad \text{for } x \notin \mathbb{Z}, \\ &\sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \frac{E_m}{m!} + \frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}. \end{aligned}$$

Here $B_k(<x>)$ is the Bernoulli function.

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Question: For what values of $m \geq 1$, does $\sum_{k=0}^{m} \frac{E_k}{k!(m-k)!} = \frac{2}{m!} - \frac{E_m}{m!}$ hold ?

Remark 3.3. In a previous paper (see [3,4,6,12]), it was shown that

$$\int_0^1 \beta_m(x)dx = \sum_{l=0}^{m-1}\sum_{j=1}^{m-l} \frac{(-1)^j \binom{m+1}{l+j} E_{l+j}}{(m+1)!} + \frac{2(-1)^{m+1}E_{m+1}}{(m+1)!}. \tag{3.8}$$

Hence, we have the following identity.

$$\begin{aligned} &\frac{1}{2}\left(-\sum_{k=0}^{m+1} \frac{E_k}{k!(m+1-k)!} + \frac{2}{(m+1)!} - \frac{E_{m+1}}{(m+1)!}\right) \\ &= \sum_{l=0}^{m-1}\sum_{j=1}^{m-l} \frac{(-1)^j \binom{m+1}{l+j} E_{l+j}}{(m+1)!} + \frac{2(-1)^{m+1}E_{m+1}}{(m+1)!}. \end{aligned}$$

4. The fuction $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(x) x^{m-k}, (m \geq 2)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) < x >^{m-k}, \tag{4.1}$$

defined on $(-\infty, \infty)$, which is periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \tag{4.3}$$

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To proceed further, we observe the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\Big(kE_{k-1}(x)x^{m-k} + (m-k)E_k(x)x^{m-k-1}\Big) \\
&= \sum_{k=1}^{m-1} \frac{1}{m-k}E_{k-1}(x)x^{m-k} + \sum_{k=1}^{m-1} \frac{1}{k}E_k(x)x^{m-k-1} \\
&= \sum_{k=0}^{m-2} \frac{1}{m-k-1}E_k(x)x^{m-k-1} + \sum_{k=0}^{m-1} \frac{1}{k}E_k(x)x^{m-k-1} \\
&= \frac{1}{m-1}x^{m-1} + \sum_{k=1}^{m-2} \frac{1}{m-k-1}E_k(x)x^{m-k-1} + \frac{1}{m-1}E_{m-1}(x) \\
&\quad + \sum_{k=1}^{m-2} \frac{1}{k}E_k(x)x^{m-k-1} \\
&= (m-1)\sum_{k=1}^{m-2} \frac{1}{k(m-1-k)}E_k(x)x^{m-1-k} + \frac{1}{m-1}\Big(x^{m-1} + E_{m-1}(x)\Big) \\
&= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}\Big(x^{m-1} + E_{m-1}(x)\Big).
\end{aligned} \tag{4.4}$$

Thus,

$$\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}\Big(x^{m-1} + E_{m-1}(x)\Big).$$

From this, we have

$$\left(\frac{1}{m}\Big(\gamma_{m+1}(x) - \frac{1}{m(m+1)}x^{m+1} - \frac{1}{m(m+1)}E_{m+1}(x)\Big)\right)' = \gamma_m(x).$$

$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}\Big[\gamma_{m+1}(x) - \frac{1}{m(m+1)}x^{m+1} - \frac{1}{m(m+1)}E_{m+1}(x)\Big]_0^1 \\
&= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} - \frac{1}{m(m+1)}\Big(E_{m+1}(1) - E_{m+1}(0)\Big)\Big) \\
&= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} + \frac{2}{m(m+1)}E_{m+1}\Big).
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\big(E_k(1) - E_k(0)\delta_{m,k}\big) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\Big(-E_k(0) + 2\delta_{k,0}\Big) - \sum_{k=1}^{m-1} \frac{1}{k(m-k)}E_k(0)\delta_{m,k} \\
&= -\sum_{k=1}^{m-1} \frac{E_k}{k(m-k)}.
\end{aligned} \tag{4.6}$$

Thus,

$$\gamma_m(1) = \gamma_m(0) \ \Leftrightarrow \ \sum_{k=1}^{m-1} \frac{E_k}{k(m-k)} = 0. \tag{4.7}$$

In addition,

$$\int_0^1 \gamma_m(x)dx = -\frac{1}{m}\left(\sum_{k=1}^{m} \frac{E_k}{k(m-k+1)} + \frac{1}{m(m+1)} - \frac{2}{m(m+1)}E_{m+1}\right). \tag{4.8}$$

Now, we would like to determine the Fourier coefficients $C_n^{(m)}$.
Case 1:$n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big(\gamma_m(1) - \gamma_m(0)\Big) + \frac{m-1}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx \\
&+ \frac{1}{2\pi in(m-1)}\int_0^1 x^{m-1}e^{-2\pi inx}dx + \frac{1}{2\pi in(m-1)}\int_0^1 E_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m,
\end{aligned} \tag{4.9}$$

where , for $l \geq 1$,

$$\int_0^1 E_l(x)e^{-2\pi inx}dx = \begin{cases} 2\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}E_{l-k+1}, & \text{for } n \neq 0, \\ -\frac{2}{l+1}E_{l+1}, & \text{for } n = 0. \end{cases}$$

$$\int_0^1 x^l e^{-2\pi inx}dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}, & \text{for } n \neq 0, \\ \frac{1}{l+1}, & \text{for } n = 0. \end{cases}$$

Here, for $m \geq 2$,

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) = -\sum_{k=1}^{m-1} \frac{E_k}{k(m-k)}, \\
\Theta_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k}, \\
\Phi_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k}.
\end{aligned} \tag{4.10}$$

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$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\Big(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Theta_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_{m-1}\Big) \\
&- \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} - \frac{m-1}{(2\pi in)^2}\Lambda_{m-1} - \frac{1}{2\pi in}\Lambda_m - \frac{m-1}{(2\pi in)^2(m-2)}\Theta_{m-1} \\
&- \frac{1}{(2\pi in)(m-1)}\Theta_m + \frac{2(m-1)}{(2\pi in)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\frac{1}{2}\frac{(m-1)!}{(2\pi in)^{m-1}} - \frac{2(m-1)!}{(2\pi in)^m} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} \\
&- \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1},
\end{aligned} \tag{4.11}
$$

where

$$
\begin{aligned}
&C_n^{(2)} = \int_0^1 \gamma_2(x)e^{-2\pi inx}dx = \int_0^1 (x^2 - \frac{1}{2}x)e^{-2\pi inx}dx = -\frac{1}{2}\frac{1}{2\pi in} - \frac{2}{(2\pi in)^2} \\
&\Lambda_2 = \frac{1}{2},\ \Theta_2 = \frac{1}{2\pi in},\ \Phi_2 = \frac{1}{2\pi in}\times(-\frac{1}{2}).
\end{aligned} \tag{4.12}
$$

Before proceeding further, we note the following.

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$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1}\\
&=-\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\sum_{k=1}^{m-j}\frac{E_k}{k(m-j-k+1)}\\
&=-\frac{1}{m}\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m)_j}{(2\pi in)^j k(m-j-k+1)}E_k\\
&=-\frac{1}{m}\sum_{s=1}^{m-1}\sum_{k=1}^{m-s}\frac{(m)_s}{(2\pi in)^s k(m-s-k+1)}E_k\\
&=-\frac{1}{m}\sum_{s=1}^{m}\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}.
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1}\\
=&\sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^k}E_{m-j-k+1}\\
=&\frac{2}{m}\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m)_{j+k-1}}{(2\pi in)^{j+k}(m-j)}E_{m-j-k+1}\\
=&\frac{2}{m}\sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\\
=&\frac{2}{m}\sum_{s=2}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\sum_{j=1}^{s-1}\frac{1}{m-j}\\
=&\frac{2}{m}\sum_{s=2}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\left(H_{m-1}-H_{m-s}\right)\\
=&\frac{2}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m-s+1}\left(H_{m-1}-H_{m-s}\right).
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}\\
=&\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}.
\end{aligned} \tag{4.15}$$

Putting everything together, we have

$$\begin{aligned} C_n^{(m)} &= \frac{1}{m}\sum_{s=1}^{m}\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)} \\ &-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}+\frac{2}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) \\ &= -\frac{1}{m}\sum_{s=1}^{m}\left(\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\right). \end{aligned} \tag{4.16}$$

Case 2: $n=0$.

$$C_0^{(m)}=\int_0^1\gamma_m(x)dx=-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right). \tag{4.17}$$

Question: For what values of $m\geq 1$, does $\sum_{k=0}^{m}E_k=2-E_m$ hold ?

Remark 4.1. In a previous paper (see [3,4,6,12]), it was shown that

$$\int_0^1\gamma_m(x)dx=\frac{1}{m(m^2-1)}\sum_{l=1}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m+1}{l+j}E_{l+j}}{\binom{m-2}{l-1}}+\frac{2(-1)^{m+1}E_{m+1}}{m(m^2-1)}\sum_{l=1}^{m-1}\frac{(-1)^l}{\binom{m-2}{l-1}}. \tag{4.18}$$

So, we obtain the following identity.

$$\begin{aligned} &-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right) \\ &=\frac{1}{m(m^2-1)}\sum_{l=1}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m+1}{l+j}E_{l+j}}{\binom{m-2}{l-1}}+\frac{2(-1)^{m+1}E_{m+1}}{m(m^2-1)}\sum_{l=1}^{m-1}\frac{(-1)^l}{\binom{m-2}{l-1}}, \end{aligned} \tag{4.19}$$

for $m\geq 2$.

$\gamma_m(<x>)$, $(m\geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(<x>)$ is continuous for those integers $m\geq 2$ with and $\Lambda_m=0$, and discontinuous with jump discontinuities at integers for those integers ≥ 2 with $\Lambda_m\neq 0$.

Assume first that $\Lambda_m=0$. Then $\gamma_m(1)=\gamma_m(0)$. $\gamma_m(<x>)$ is piecewise C^∞ and continuous. So the Fourier series of $\gamma_m(<x>)$ converges uniformly to $\gamma_m(<x>)$, and

$$
\begin{aligned}
&\gamma_m(<x>)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right)\\
&-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\Big(\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})\\
&-\sum_{l=s}^{m-1}\frac{(m)_sE_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\Big)\Big)e^{2\pi inx}\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right)\\
&+\frac{1}{m}\sum_{s=1}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)\\
&\times\Big(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\Big)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right)\\
&+\frac{1}{m}\sum_{s=2}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(<x>)\\
&+\Big(-\sum_{l=1}^{m-1}\frac{E_l}{l(m-l)}\Big)\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z},\end{cases}
\end{aligned}
\tag{4.20}
$$

where $H_m=\sum_{k=1}^{m}\frac{1}{k}$.

Now, we get the following theorem.

Theorem 4.2. *Let m be an integer ≥ 2, with*

$$\Lambda_m=-\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}=0.$$

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Then we have the following.
(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}\\&=-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\&-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\Big(\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})\\&-\sum_{l=s}^{m-1}\frac{(m)_sE_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\Big)\Big)e^{2\pi inx},\end{aligned}$$

for all $x\in(-\infty,\infty)$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}\\&=-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\&+\frac{1}{m}\sum_{s=2}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(<x>),\end{aligned}$$

for all $x\in(-\infty,\infty)$. Here $B_k(<x>)$ is the Bernoulli function.

Assume next that m is an integer ≥ 2 with $\Lambda_m\neq 0$. Then, $\gamma_m(1)\neq\gamma_m(0)$. Hence $\gamma_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0)+\gamma_m(1))=\gamma_m(0)+\frac{1}{2}\Lambda_m=-\frac{1}{2}\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)},$$

for $x\in\mathbb{Z}$.

Hence we obtain the following theorem.

Theorem 4.3. *Let m be an integer ≥ 2, with*

$$\Lambda_m=-\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}\neq 0.$$

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Then, we have the following.
(a)

$$\begin{aligned}
&-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\Big(\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})\\
&-\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\Big)\Big)e^{2\pi inx}\\
&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}, & \textit{for } x\notin\mathbb{Z},\\ -\frac{1}{2}\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}, & \textit{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}$$

Here the convergence is pointwise.
(b)

$$\begin{aligned}
&-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&+\frac{1}{m}\sum_{s=1}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(<x>)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k},\ \textit{for } x\notin\mathbb{Z},
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&+\frac{1}{m}\sum_{s=2}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(<x>)\\
&=-\frac{1}{2}\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)},\ \textit{for } x\in\mathbb{Z}.
\end{aligned}$$

Question: For what values of $m\geq 2$, does $\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}=0$ hold ?

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Higher order generalization of Bernstein type operators defined by (p, q)-integers

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Abstract

In this paper, we introduce the higher order generalization of Bernstein type operators defined by (p, q)-integers. We establish some approximation results for these new operators by using the modulus of continuity.

Keywords and phrases: (p, q)-integers; (p, q)-Bernstein operators; modulus of continuity; approximation theorems.

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1. Introduction and preliminaries

In 1912, S.N Bernstein [4] introduced the following sequence of operators $B_n : C[0,1] \to C[0,1]$ defined for any $n \in \mathbb{N}$ and for any $f \in C[0,1]$ such as

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \; x \in [0,1]. \tag{1.1}$$

In approximation theory, q-type generalization of Bernstein polynomials was introduced by Lupaş [7].

For $f \in C[0,1]$, the generalized Bernstein polynomial based on the q-integers is defined by Phillips [15] as follows

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) \; f\left(\frac{[k]_q}{[n]_q}\right), \; x \in [0,1]. \tag{1.2}$$

Recently, Mursaleen et al. [10] applied (p, q)-calculus in approximation theory and introduced first (p, q)-analogue of Bernstein operators and defined as:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} f\left(\frac{[k]}{p^{k-n}[n]}\right) P_{n,k}(p,q;x), \; 0 < q < p \leq 1, \; x \in [0,1]$$

where

$$P_{n,k}(p,q;x) = p^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x). \tag{1.3}$$

They have also introduced and studied approximation properties based on (p,q)-integers given as: (p,q)-Bernstein-Stancu operators [11], (p,q)-Bernstein-Shurer operators [14] and (p,q)-Bleimann-Butzer-Hahn operators [13]. In the sequel, some more articles on (p,q)-approximation have also been appeared, e.g. [1], [2], [3], [6], [9], [12] and [13].

We recall some basic properties of (p,q)-integers.

The (p,q)-integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q},\ n = 0,1,2,\cdots,\ 0 < q < p \leq 1.$$

The (p,q)-Binomial expansion is

$$(x+y)^n_{p,q} := (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y)$$

and the (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

For $p=1$, all the notions of (p,q)-calculus are reduced to q-calculus. For details on (p,q)-calculus and q-calculus, one can refer [5, 7]. In this paper we use the notation $[n]$ in place of $[n]_{p,q}$.

In [5], (p,q)-derivative of a function $f(x)$ is defined by

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x},\ x \neq 0, \tag{1.4}$$

and the formulae for the (p,q)-derivative for the product of two functions is given as

$$D_{p,q}(fg)(x) = f(px).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(qx), \tag{1.5}$$

also

$$D_{p,q}(fg)(x) = f(qx).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(px). \tag{1.6}$$

Let $r \in \mathbb{N} \cup \{0\}$ be a fixed number. For $f \in C^r[0,1]$ and $m \in \mathbb{N}$, we define r^{th} order (p,q)-Bernstein type operators as follows:

$$B^{[r]}_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{1}{i!} f^{(i)}\left(\frac{[k]}{p^{k-n}[n]}\right)\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i \tag{1.7}$$

In this paper, using the moment estimates from [8], we give the estimates of the central moments for these operators. We also study some approximation properties of an r^{th} order generalization of the operators defined by (1.7) using the techniques of the work on the higher order generalization of q-analogue [16]. Further, we study approximation properties and prove Voronovskaja type theorem for these operators.

If we put $p = 1$, then we get the moments for q-Bernstein operators [8] and the usual generalization higher order q-Bernstein operators [16], respectively.

2. Main results

We have the following elementary result.

Proposition 2.1. *For $n \geq 1$, $0 < q < p \leq 1$*

$$D_{p,q}(1+x)^n_{p,q} = [n](1+qx)^{n-1}_{p,q}. \tag{2.1}$$

Proof. By applying simple calculation on (p,q)-analogue, we have

$$(1+px)^n_{p,q} = p^{n-1}(1+px)(1+qx)^{n-1}_{p,q}, (1+qx)^n_{p,q} = (p^{n-1}+q^n x)(1+qx)^{n-1}_{p,q}. \tag{2.2}$$

Applying (p,q)-derivative and result (2.2) we get the desired result. □

Lemma 2.2. *Let $B_{n,p,q}(f;x)$ be given by (1.7). Then for any $m \in \mathbb{N}$, $x \in [0,1]$ and $0 < q < p \leq 1$ we have*

$$\begin{aligned} B_{n,p,q}\left((t-x)^{m+1}_{p,q};x\right) &= \frac{p^{m+n}x(1-x)}{[n]} D_{p,q}\left\{ B_{n,p,q}\left((t-\frac{x}{p})^m_{p,q};\frac{x}{p}\right)\right\} \\ &+ \frac{p^{m+n-1}[m]x(1-x)}{[n]} B_{n,p,q}\left((t-\frac{qx}{p})^{m-1}_{p,q};\frac{qx}{p}\right) \\ &+ \frac{[m](p^n-q^n)x}{[n]} B_{n,p,q}\left((t-x)^m_{p,q};x\right). \end{aligned}$$

Proof. First of all by using (1.5) and Proposition 2.1, we have

$$D_{p,q}\left(\frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^n \left(t-\frac{x}{p}\right)^m_{p,q} P_{n,k}(p,q;\frac{x}{p})\right)$$

$$= \frac{1}{p^{\frac{n(n-1)}{2}}}\left(\sum_{k=0}^n (t-x)^m_{p,q} D_{p,q}\{P_{n,k}(p,q;\frac{x}{p})\} - \frac{[m]}{p}\sum_{k=0}^n \left(t-\frac{qx}{p}\right)^{m-1}_{p,q} P_{n,k}(p,q;\frac{qx}{p})\right). \tag{2.3}$$

Now in the same way by using (1.5) and Proposition 2.1, we have

$$D_{p,q}\left\{P_{n,k}\left(p,q;\frac{x}{p}\right)\right\} = D_{p,q}\left\{p^{\frac{k(k-1)}{2}}[k]\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left(\frac{x}{p}\right)^k\left(1-\frac{x}{p}\right)^k\right\}$$

$$= p^{\frac{k(k-1)}{2}}\left([k]\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{p^k}x^{k-1}\left(1-\frac{qx}{p}\right)^{n-k}_{p,q} - [n-k]\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{p}x^k\left(1-\frac{qx}{p}\right)^{n-k-1}_{p,q}\right). \tag{2.4}$$

Now by a simple calculation, we have

$$\left(1-\frac{qx}{p}\right)^{n-k}_{p,q} = \frac{1}{p^{n-k}}(p-qx)^{n-k+1}_{p,q} = \frac{1}{p^{n-k}}\frac{1}{(1-x)}(p^{n-k}-q^{n-k}x)(1-x)^{n-k}_{p,q} \tag{2.5}$$

$$\left(1-\frac{qx}{p}\right)_{p,q}^{n-k-1} = \frac{1}{p^{n-k-1}}\frac{1}{(1-x)}(1-x)_{p,q}^{n-k}. \tag{2.6}$$

From (2.4),(2.5) and (2.6), we get

$$D_{p,q}\left\{P_{n,k}\left(p,q;\frac{x}{p}\right)\right\} = \frac{P_{n,k}(p,q;x)}{p^n x(1-x)}\left([k](p^{n-k}-q^{n-k}x)-p^k[n-k]x\right),$$

which implies that

$$D_{p,q}\left\{P_{n,k}\left(p,q;\frac{x}{p}\right)\right\} = \frac{P_{n,k}(p,q;x)}{p^n x(1-x)}\left(p^{n-k}[k]-[n]x\right). \tag{2.7}$$

From (2.3), (2.7), we have

$$\begin{aligned}
&D_{p,q}\left(\sum_{k=0}^n\left(t-\tfrac{x}{p}\right)_{p,q}^m P_{n,k}(p,q;\tfrac{x}{p})\right)\\
&= -\frac{1}{p^{\frac{n(n-1)}{2}}}\frac{[m]}{p}\sum_{k=0}^n\left(t-\frac{qx}{p}\right)_{p,q}^{m-1}P_{n,k}(p,q;\frac{qx}{p})\\
&+ \frac{1}{p^{\frac{n(n-1)}{2}}}\frac{1}{p^n x(1-x)}\sum_{k=0}^n(t-x)_{p,q}^m P_{n,k}(p,q;x)(p^{n-k}[k]-[n]x)\\
&= -\frac{1}{p^{\frac{n(n-1)}{2}}}\frac{[m]}{p}\sum_{k=0}^n\left(t-\frac{qx}{p}\right)_{p,q}^{m-1}P_{n,k}(p,q;\frac{qx}{p})\\
&+ \frac{1}{p^{\frac{n(n-1)}{2}}}\frac{1}{p^n x(1-x)}\sum_{k=0}^n(t-x)_{p,q}^m P_{n,k}(p,q;x)\\
&\times\left(\frac{[n]}{p^m}(p^m t-q^m x)-\frac{[n]}{p^m}(p^m-q^m)x\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&D_{p,q}\left\{B_{n,p,q}\left((t-\tfrac{x}{p})_{p,q}^m;\tfrac{x}{p}\right)\right\}\\
&= -\frac{[m]}{p}B_{n,p,q}\left((t-\frac{qx}{p})_{p,q}^{m-1};\frac{qx}{p}\right)+\frac{[n]}{p^{m+n}x(1-x)}B_{n,p,q}\left((t-x)_{p,q}^{m+1};x\right)\\
&- \frac{[m](p^n-q^n)}{p^{m+n}(1-x)}B_{n,p,q}\left((t-x)_{p,q}^m;x\right).
\end{aligned}$$

This complete the proof of Lemma 2.2. □

Lemma 2.3. *Let $B_{n,p,q}\left((t-x)_{p,q}^m;x\right)$ be a polynomial in x of degree less than or equal to m and the minimum degree of $\frac{1}{[n]}$ is $\lfloor\frac{m+1}{2}\rfloor$. Then for any fixed $m\in\mathbb{N}$ and $x\in[0,1]$, $0<q<p\le 1$ we have*

$$B_{n,p,q}\left((t-x)_{p,q}^m;x\right) = \frac{x(1-x)}{[n]^{\lfloor\frac{m+1}{2}\rfloor}}\sum_{k=0}^{m-2}b_{k,m,n}(p,q)x^k, \tag{2.8}$$

such that the coefficients $b_{k,m,n}(p,q)$ satisfy $\mid b_{k,m,n}(p,q)\mid\le b_m$, $k=1,2,\cdots,m-2$ and b_m does not depend on x,t,p,q; where $\lfloor a\rfloor$ is an integer part of $a\ge 0$.

Proof. Clearly by Lemma 2.2 it is true for $m=2$. Assuming it is true for m, then from the recurrence of Lemma 2.2 and equation (2.8) we easily get

$$B_{n,p,q}\left((t-x)_{p,q}^{m+1};x\right) = \frac{x(1-x)}{[n]^{\lfloor\frac{m+2}{2}\rfloor}}\sum_{k=0}^{m-1} b_{k,m+1,n}(p,q)x^k,$$

where

$$\begin{aligned} b_{k,m+1,n}(p,q) &= \frac{1}{[n]^{\alpha}}\left(p^{m+n-k}[k]+p^{m+n-k-1}q^k\right)b_{k,m,n}(p,q) \\ &- \frac{1}{[n]^{\alpha}}\left(p^{m+n+1-k}[k-1]+[2]p^{m+n-k-1}q^{k-1}\right)b_{k-1,m,n}(p,q) \\ &+ \frac{1}{[n]^{\alpha}}[m](p^n-q^n)b_{k-1,m,n}(p,q)+[m]p^{m+n-k-1}q^k b_{k-1,m-1,n}(p,q) \\ &- [m]p^{m+n-k-1}q^k b_{k-2,m-1,n}(p,q). \end{aligned}$$

Clearly

$$\alpha = 1+\lfloor\frac{m+1}{2}\rfloor-\lfloor\frac{m+2}{2}\rfloor,\ 0\le k\le m-1,$$

which lead us that either $\alpha=0$ or $\alpha=1$.
Since $\mid b_{k,m,n}(p,q)\mid\le b_m$, for $k=m-1$, clearly we have

$$\begin{aligned} \mid b_{k,m+1,n}(p,q)\mid &\le \frac{1}{[n]^{\alpha}}\left(p^{n+1}[m-1]+p^nq^{m-1}\right)b_m+\frac{1}{[n]^{\alpha}}\left(p^{n+2}[m-2]+[2]p^nq^{m-2}\right)b_m \\ &+ \frac{1}{[n]^{\alpha}}[m](p^n-q^n)b_m+[m]p^nq^{m-1}b_{m-1} \\ &+ [m]p^nq^{m-1}b_{m-1} \\ &= \frac{1}{[n]^{\alpha}}\left(p[m-1]+q^{m-1}\right)b_m+\frac{1}{[n]^{\alpha}}\left(p^2[m-2]+[2]q^{m-2}\right)b_m \\ &+ \frac{1}{[n]^{\alpha}}[m]b_m+[m]q^{m-1}b_{m-1}+[m]q^{m-1}b_{m-1} \\ &= b_{m+1},\ k=1,2,\cdots m-1, \end{aligned}$$

and b_m does not depend on x,t,p,q. This complete the proof. □

Remark 2.4. From the Lemma 2.3 we have

$$B_{n,p,q}\left((t-x)_{p,q}^{m};x\right)=x(1-x)Q_{m-2},\ B_{n,p,q}\left((t-x)_{p,q}^{m};x\right)\Big|_{x=0,1}=0, \tag{2.9}$$

where Q_{m-2} is a polynomial of highest degree $m-2$.

From the Lemma 2.2 and Lemma 2.3 we have the following theorem.

Theorem 2.5. *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exits a constant $C_m > 0$ such that for any $x \in [0,1]$, we have*

$$| B_{n,p,q}\left((t-x)^m_{p,q};x\right) |\leq C_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}}.$$

Lemma 2.6. *For any fixed $m \in \mathbb{N}$ and $x \in [0,1]$, $0 < q < p \leq 1$ we have*

$$(t-x)^m = \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^k_{p,q} = \sum_{k=1}^{m} \gamma_{m,k}\left(\frac{p^n-q^n}{[n]}\right)^{m-k} x^{m-k}(t-x)^k_{p,q} \tag{2.10}$$

where

$$\gamma_{m,k} = \begin{cases} \frac{\gamma_{m-1,k-1}}{p^{k-1}} - \frac{[k]\gamma_{m-1,k}}{p^k}, \ k=1,\cdots,m-1, & \gamma_{m,0}=0 \\ 1, & k=m, \end{cases}$$

the coefficients $\gamma_{m,k}$ satisfy $| \gamma_{m,k} | \leq \gamma_m$, $k = 1,\cdots,m$ and γ_m does not depend on x, t, p, q.

Proof. Inductively, for $m = 1$, it is obvious. For $m \geq 1$ the relation (2.10) holds. For $k = 1,\cdots,m$, we have

$$(t-x)^{m+1} = \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^k_{p,q}(t-x)^m, \tag{2.11}$$

We can write

$$t-x = \frac{1}{p^k}\left(p^k t - q^k x - (p-q)[k]_{p,q}x\right) \tag{2.12}$$

(2.11),(2.12) imply that,

$$\begin{aligned}
(t-x)^{m+1} &= \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^{k+1}_{p,q}\frac{1}{p^k} \\
&- \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^k_{p,q}\frac{1}{p^k}(p-q)[k]_{p,q}x \\
&= \frac{\gamma_{m,m}(t-x)^{m+1}_{p,q}}{p^m} + \sum_{k=2}^{m} \frac{1}{p^{k-1}}\gamma_{m,k-1}(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q} \\
&- \frac{\gamma_{m,1}(p-q)^m x^m (t-x)}{p} - \sum_{k=2}^{m} \frac{1}{p^k}[k]\gamma_{m,k}(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q} \\
&= \frac{\gamma_{m,m}(t-x)^{m+1}_{p,q}}{p^m} - \frac{\gamma_{m,1}(p-q)^m x^m (t-x)}{p} \\
&+ \sum_{k=2}^{m} \left(\frac{\gamma_{m,k-1}}{p^{k-1}} - \frac{[k]\gamma_{m,k}}{p^k}\right)(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q} \\
&= \sum_{k=1}^{m+1} \gamma_{m+1,k}(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q},
\end{aligned}$$

where

$$\gamma_{m+1,k} = \begin{cases} \frac{\gamma_{m,k-1}}{p^{k-1}} - \frac{[k]\gamma_{m,k}}{p^k}, \ k = 1, \cdots, m, \quad \gamma_{m,0} = 0 \\ \gamma_{m,m} = 1, \qquad\qquad\qquad\qquad k = m+1. \end{cases}$$

□

Theorem 2.7. *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exits a constant $E_m > 0$ such that for any $x \in [0,1]$, we have*

$$\mid B_{n,p,q}\left((t-x)^m; x\right) \mid \leq E_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}}.$$

Proof. From Lemma 2.6 we have

$$\begin{aligned} \mid B_{n,p,q}\left((t-x)^m, x\right) \mid &\leq \sum_{k=1}^{m} \mid \gamma_{m,k} \mid \left(\frac{p^n - q^n}{[n]}\right)^{m-k} \mid B_{n,p,q}\left((t-x)^k_{p,q}, x\right) \mid \\ &\leq \gamma_m \left(\mid B_{n,p,q}\left((t-x)^m_{p,q}, x\right) \mid + \sum_{k=1}^{m-1} \frac{1}{[n]^{m-k}} \mid B_{n,p,q}\left((t-x)^k_{p,q}, x\right) \mid \right) \end{aligned}$$

By using Theorem 2.5 we have

$$\begin{aligned} \mid B_{n,p,q}\left((t-x)^m, x\right) \mid &\leq \gamma_m \left(\mid B_{n,p,q}\left((t-x)^m_{p,q}, x\right) \mid + \sum_{k=1}^{m-1} \frac{1}{[n]^{m-k}} C_k \frac{x(1-x)}{[n]^{\lfloor \frac{k+1}{2} \rfloor}} \right) \\ &\leq \gamma_m \left(\mid B_{n,p,q}\left((t-x)^m_{p,q}, x\right) \mid + \frac{x(1-x)}{[n]^{1+\lfloor \frac{m}{2} \rfloor}} \sum_{k=1}^{m-1} C_k \right) \\ &\leq \gamma_m \left(C_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} + \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=1}^{m-1} C_k \right) \\ &\leq \left(\gamma_m C_m + \sum_{k=1}^{m-1} C_k \right) \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \\ &= E_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \end{aligned}$$

□

Corollary 2.8. *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exits a constant $K_m > 0$ such that for any $x \in [0,1]$, we have*

$$B_{n,p,q}\left((\mid t-x \mid)^m; x\right) \leq K_m \frac{x(1-x)}{[n]^{\frac{m}{2}}}. \tag{2.13}$$

Proof. For an even m, clearly we have

$$\begin{aligned} B_{n,p,q}\left((\mid t-x\mid)^m;x\right) &= B_{n,p,q}\left((t-x)^m;x\right) \\ &\leq E_m\frac{x(1-x)}{[n]^{\lfloor\frac{m+1}{2}\rfloor}} \\ &= K_m\frac{x(1-x)}{[n]^{\frac{m}{2}}} \end{aligned}$$

In case if m is odd, say $m=2u+1$, we have
$B_{n,p,q}\left((\mid t-x\mid)^{2u+1};x\right)$

$$\begin{aligned} &\leq \sqrt{B_{n,p,q}\left((\mid t-x\mid)^{4u}_{p,q};x\right)}\sqrt{B_{n,p,q}\left((\mid t-x\mid)^{2};x\right)} \\ &\leq \sqrt{E_{4u}\frac{x(1-x)}{[n]^{\lfloor\frac{4u+1}{2}\rfloor}}}\sqrt{E_2\frac{x(1-x)}{[n]^{\lfloor\frac{3}{2}\rfloor}}} \\ &= \sqrt{E_{4u}\frac{x(1-x)}{[n]^{\frac{2u}{2}}}}\sqrt{E_2\frac{x(1-x)}{[n]}} \\ &= K_{2u+1}\frac{x(1-x)}{[n]^{\frac{2u+1}{2}}}. \end{aligned}$$

This complete the proof. □

Theorem 2.9. *Let $B^{[r]}_{n,p,q}(f;x)$ be an operator from $C^r[0,1]\to C^r[0,1]$. Then for $0<q<p\leq 1$ there exits a constant $M(r)$ such that for every $f\in C^r[0,1]$, we have*

$$\| B^{[r]}_{n,p,q}(f;x)\|_{C[0,1]}\leq M(r)\sum_{i=0}^{r}\| f^{(i)}\| = M(r)\| f\|_{C^r[0,1]}\ . \tag{2.14}$$

Proof. Clearly $B^{[r]}_{n,p,q}(f;x)$ is continuous on $[0,1]$. From (1.7) we have

$$B^{[r]}_{n,p,q}(f;x) = \sum_{i=0}^{r}\frac{(-1)^i}{i!}B_{n,p,q}\left((t-x)^i f^{(i)}(t);x\right).$$

From the Corollary 2.8, we have

$$\begin{aligned} \mid B_{n,p,q}\left((t-x)^i f^{(i)}(t);x\right)\mid &\leq \| f^{(i)}\| B_{n,p,q}\left(\mid (t-x)\mid^i;x\right) \\ &\leq K_i\| f^{(i)}\|\, [n]^{-\frac{i}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} \| B^{[r]}_{n,p,q}(f;x)\| &\leq \sum_{i=0}^{r}\frac{(-1)^i}{i!}\| B_{n,p,q}\left((t-x)^i f^{(i)}(t);x\right)\| \\ &\leq M(r)\sum_{i=0}^{r}\| f^{(i)}\|\ . \end{aligned}$$

This complete the proof. □

3. Convergence properties of $B^{[r]}_{n,p,q}(f;x)$

The modulus of continuity of the derivative $f^{(r)}$ is given by

$$\omega\left(f^{(r)};t\right)=\sup\left\{\mid f^{(r)}(x)-f^{(r)}(y)\mid:\mid x-y\mid\leq t,\ x,y\in[0,1]\right\}. \tag{3.1}$$

Theorem 3.1. *Let $0<q<p\leq 1$ and $r\in\mathbb{N}\cup\{0\}$ be a fixed number. Then for $x\in[0,1]$, $n\in\mathbb{N}$ there exits $D_r>0$ such that for every $f\in C^r[0,1]$ the following inequality holds*

$$\mid B^{[r]}_{n,p,q}(f;x)-f(x)\mid\leq D_r\frac{1}{[n]^{\frac{r}{2}}}\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right). \tag{3.2}$$

Proof. Let $r\in\mathbb{N}$. Then for $f\in C^r[0,1]$ at a given point $t\in[0,1]$, we have from the Taylor formula that

$$\begin{aligned} f(x) &= \sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i+\frac{(x-t)^r}{((r-1)!)} \\ &\times \int_0^1(1-u)^{r-1}\left(f^{(r)}(t+u(x-t))-f^{(r)}(t)\right)\mathrm{d}u. \end{aligned}$$

On applying $B^{[r]}_{n,p,q}(f;x)$, we get

$$f(x)-B^{[r]}_{n,p,q}(f;x)=\sum_{k=0}^{n}\frac{(x-\frac{[k]}{p^{k-n}[n]})^r}{(r-1)!}\int_0^1(1-u)^{r-1}P_{n,k}(p,q;x)$$

$$\times\left[f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}+u\left(x-\frac{[k]}{p^{k-n}[n]}\right)\right)-f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}\right)\right]\mathrm{d}u. \tag{3.3}$$

Now from the definition and properties of modulus of continuity, we have

$$\left|f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}+u\left(x-\frac{[k]}{p^{k-n}[n]}\right)\right)-f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}\right)\right|\leq\omega\left(f^{(r)};u\left|x-\frac{[k]}{p^{k-n}[n]}\right|\right)$$

$$\omega\left(f^{(r)};u\left|x-\frac{[k]}{p^{k-n}[n]}\right|\right)\leq\left(\sqrt{[n]}\left|x-\frac{[k]}{p^{k-n}[n]}\right|+1\right)\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right). \tag{3.4}$$

□

Now for every $0\leq x\leq 1,\ 0<q<p\leq 1,\ k\in\mathbb{N}\cup\{0\},\ n\in\mathbb{N}$ and from (3.3) and (3.4), we get
$\mid B^{[r]}_{n,p,q}(f;x)-f(x)\mid$

$$\leq\frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right)\sum_{k=0}^{n}\left|x-\frac{[k]}{p^{k-n}[n]}\right|^r\left(\sqrt{[n]}\left|x-\frac{[k]}{p^{k-n}[n]}\right|+1\right)P_{n,k}(p,q;x)$$

$$= \frac{1}{r!}\omega\left(f^{(r)}; \frac{1}{\sqrt{[n]}}\right)\left(\sqrt{[n]}B_{n,p,q}\left(\mid x-t\mid^{r+1}; x\right) + B_{n,p,q}\left(\mid x-t\mid^{r}; x\right)\right). \quad (3.5)$$

Using (3.9) and (3.5) for $x \in [0,1]$, we have

$$\begin{aligned} \mid B^{[r]}_{n,p,q}(f;x) - f(x) \mid &\leq \frac{1}{r!}(K_{r+1}+K_r)\left(\frac{1}{\sqrt{[n]}}\right)^r \omega\left(f^{(r)}; \frac{1}{\sqrt{[n]}}\right) \\ &= D_r\left(\frac{1}{\sqrt{[n]}}\right)^r \omega\left(f^{(r)}; \frac{1}{\sqrt{[n]}}\right). \end{aligned}$$

In order to obtain the uniform convergence of $B^{[r]}_{n,p_n,q_n}(f;x)$ to a continuous function f, we take $q = q_n,\ p = p_n$ where $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ satisfying,

$$\lim_n p_n = 1,\ \lim_n q_n = 1. \quad (3.6)$$

Corollary 3.2. *Let $p = p_n,\ q = q_n,\ 0 < q_n < p_n \leq 1$ satisfy (3.6) and $f \in C^r[0,1]$ for a fixed number $r \in \mathbb{N}\cup\{0\}$. Then*

$$\lim_{n\to\infty} [n]^{\frac{r}{2}} \parallel B^{[r]}_{n,k}(f) - f \parallel = 0. \quad (3.7)$$

We say that (cf. [16]) a function $f \in C[0,1]$ belongs to $Lip_M(\alpha)$, $0 < \alpha \leq 1$, provided

$$\mid f(x) - f(y) \mid \leq M \mid x-y\mid^{\alpha},\ (x, y \in [0,1] \text{ and } M > 0). \quad (3.8)$$

Corollary 3.3. *Let $p = p_n,\ q = q_n,\ 0 < q_n < p_n \leq 1$ satisfy (3.6) and $f \in C^r[0,1]$ for a fixed number $r \in \mathbb{N}\cup\{0\}$. If $f^{(r)} \in Lip_M(\alpha)$ then*

$$\parallel B^{[r]}_{n,p,q}(f) - f \parallel = O\left([n]^{-\frac{r+\alpha}{2}}\right). \quad (3.9)$$

Proof. From (3.2) and (3.8), we have

$$\parallel B^{[r]}_{n,p,q}(f) - f \parallel \leq D_r M \frac{1}{[n]^{\frac{r}{2}}}\frac{1}{[n]^{\frac{\alpha}{2}}}.$$

□

Theorem 3.4. *Let $0 < q < p \leq 1$. Suppose that $f \in C^{r+2}[0,1]$, where $r \in \mathbb{N}\cup\{0\}$ is fixed then we have*

$$\begin{aligned} &\Bigg| B^{[r]}_{n,p,q}(f;x) - f(x) - \frac{(-1)^r f^{(r+1)}(x) B_{n,p,q}\left((t-x)^{r+1};x\right)}{(r+1)!} \\ &- \frac{(-1)^r f^{(r+2)}(x) B_{n,p,q}\left((t-x)^{r+2};x\right)}{(r+2)!}\Bigg| \\ &\leq (K_{r+2}+K_{r+4})\frac{x(1-x)}{[n]^{\frac{r}{2}+1}}\sum_{i=0}^{r}\frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right). \end{aligned}$$

Proof. Let $f \in C^{r+2}[0,1]$ and $x \in [0,1]$ for a fixed number $r \in \mathbb{N} \cup \{0\}$ we have $f^{(i)} \in C^{r+2-i}[0,1],\ 0 \leq i \leq r$. Then by Taylor formula we can write

$$f^{(i)}(t) = \sum_{i=0}^{r+2-i} \frac{f^{(i+j)}(x)}{j!}(t-x)^j + R_{r+2-j}(f;t;x), \tag{3.10}$$

where

$$R_{r+2-i}(f;t;x) = \frac{f^{(r+2-i)}(\zeta_{p^{n-k-1}t}) - f^{(r+2-i)}(x)}{(r+2-i)!}(t-x)^{r+2-i},$$

and

$$\mid \zeta_t - x \mid < \mid t - x \mid .$$

Therefore from (1.7) and (3.10) we have

$$\begin{aligned} B^{[r]}_{n,p,q}(f;x) &= \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i}{i!} \sum_{j=0}^{r+2-i} \frac{f^{(i+j)}(x)}{j!} \left(\frac{[k]}{p^{k-n}[n]} - x\right)^j \\ &+ \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i}{i!} R_{r+2-i}(f;t;x) \\ &= I_1 + I_2,\ \{\text{where } t = \frac{[k]}{p^{k-n}[n]}\} \end{aligned}$$

Which implies that

$$\begin{aligned} &\mid B^{[r]}_{n,p,q}(f;x) - I_1 \mid \\ &\qquad = \mid I_2 \mid \\ &\qquad = \left| \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!}(t-x)^{r+2} \right| \\ &\qquad = \left| B_{n,p,q}\left(\sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!}(t-x)^{r+2}; x \right) \right|. \end{aligned}$$

We use the well-known inequality

$$\omega(f, \lambda\delta) \leq (1+\lambda^2)\omega(f,\delta),$$

$$\begin{aligned} \mid f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x) \mid &\leq \omega\left(f^{(r+2-i)}, \mid \zeta_t - x \mid\right) \\ &\leq \omega\left(f^{(r+2-i)}, \mid t - x \mid\right) \\ &\leq \omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)\left(1 + [n](t-x)^2\right). \end{aligned}$$

Hence

$$\mid I_2 \mid \leq \left| B_{n,p,q}\left(\sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} \right| \ \mid t - x \mid^{r+2}; x \right)$$
$$\leq B_{n,p,q}\left(\sum_{i=0}^{r} \frac{1}{i!(r+2-i)!} \omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)(1 + [n](t-x)^2) \mid t - x \mid^{r+2}; x \right)$$

$= \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)$
$\times \left(B_{n,p,q}(\mid t-x\mid^{r+2}; x) + [n]B_{n,p,q}(\mid t-x\mid^{r+4}; x)\right)$
$\leq \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)\left(K_{r+2}\frac{x(1-x)}{[n]^{\frac{r}{2}+1}} + K_{r+4}\frac{x(1-x)}{[n]^{\frac{r}{2}+1}}\right)$
$= (K_{r+2}+K_{r+4})\frac{x(1-x)}{[n]^{\frac{r}{2}+1}}\sum_{i=0}^{r} \frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right).$
Therefore

$$\mid B^{[r]}_{n,p,q}(f;x) - I_1 \mid \leq (K_{r+2}+K_{r+4})\frac{x(1-x)}{[n]^{\frac{r}{2}+1}}\sum_{i=0}^{r}\frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right).$$

Now we simplify for I_1

$$\begin{aligned}
I_1 &= \sum_{k=0}^{n} P_{n,k}(p,q;x)\sum_{i=0}^{r}\frac{(x-\frac{[k]}{p^{k-n}[n]})^i}{i!}\sum_{l=i}^{r+2}\frac{f^{(l)}(x)}{(l-i)!}\left(\frac{[k]}{p^{k-n}[n]}-x\right)^{l-i} \\
&= \sum_{k=0}^{n} P_{n,k}(p,q;x)\sum_{i=0}^{r}\frac{(-1)^i}{i!}\sum_{l=i}^{r}\frac{f^{(l)}(x)}{(l-i)!}\left(\frac{[k]}{p^{k-n}[n]}-x\right)^{l} \\
&+ \sum_{k=0}^{n} P_{n,k}(p,q;x)\sum_{i=0}^{r}\frac{(-1)^i}{i!}\frac{f^{(r+1)}(x)}{(r+1-i)!}\left(\frac{[k]}{p^{k-n}[n]}-x\right)^{r+1} \\
&+ \sum_{k=0}^{n} P_{n,k}(p,q;x)\sum_{i=0}^{r}\frac{(-1)^i}{i!}\frac{f^{(r+2)}(x)}{(r+2-i)!}\left(\frac{[k]}{p^{k-n}[n]}-x\right)^{r+2} \\
&= \sum_{k=0}^{n} P_{n,k}(p,q;x)\sum_{l=0}^{r}\frac{f^{(l)}(x)}{(l)!}\left(\frac{[k]}{p^{k-n}[n]}-x\right)^{l}\sum_{i=0}^{l}\binom{l}{i}(-1)^i \\
&+ \frac{f^{(r+1)}(x)}{(r+1)!}\sum_{k=0}^{n} P_{n,k}(p,q;x)\left(\frac{[k]}{p^{k-n}[n]}-x\right)^{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^i \\
&+ \frac{f^{(r+2)}(x)}{(r+2)!}\sum_{k=0}^{n} P_{n,k}(p,q;x)\left(\frac{[k]}{p^{k-n}[n]}-x\right)^{r+2}\sum_{i=0}^{r}\binom{r+2}{i}(-1)^i.
\end{aligned}$$

For $n \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\}$ we have

$$\sum_{i=0}^{r}\binom{r+1}{i}(-1)^i = (-1)^r,\ \sum_{i=0}^{r}\binom{r+2}{i}(-1)^i = (r+1)(-1)^r$$

Therefore

$$\begin{aligned}
I_1 &= f(x) + \frac{(-1)^r f^{(r+1)}(x) B_{n,p,q}\left((t-x)^{r+1};x\right)}{(r+1)!} \\
&+ \frac{(-1)^r f^{(r+2)}(x) B_{n,p,q}\left((t-x)^{r+2};x\right)}{(r+2)!}.
\end{aligned}$$

This complete the proof. □

Corollary 3.5. *Let $p = p_n$, $q = q_n$, $0 < q_n < p_n \leq 1$ satisfy (3.6) and $f \in C^2[0,1]$ for a fixed number $r \in \mathbb{N} \cup \{0\}$. Then for every $x \in [0,1]$ we have*

$$\left| B^{[r]}_{n,p_n,q_n}(f;x) - f(x) - \frac{f''(x)}{2}\frac{x(1-x)}{[n]} \right| \leq K\frac{x(1-x)}{[n]}\omega\left(f'', [n]^{-\frac{1}{2}}\right),$$

where $K = \frac{K_2+K_4}{2}$. Moreover,

$$\lim_{n\to\infty}[n]\left(B_{n,p_n,q_n}(f;x) - f(x)\right) = \frac{x(1-x)}{2}f''(x)$$

uniformly on $[0,1]$.

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FOURIER SERIES OF FUNCTIONS INVOLVING GENOCCHI POLYNOMIALS

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Abstract. We consider three types of functions involving Genocchi polynomials and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

1. Introduction

Let $G_m(x)$ be the Genocchi polynomials given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!},\quad \text{(see [1,2,12-17,21])}. \tag{1.1}$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned}&G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,\\&G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2+1,\\&G_5(x)=5x^4-10x^3+5x,\ G_6(x)=6x^5-15x^4+15x^2-3,\\&G_7(x)=7x^6-21x^5+35x^3-21x.\end{aligned} \tag{1.2}$$

From the relation $G_m(x)=mE_{m-1}(x)(m\geq 1)$, we have

$$\begin{aligned}&\deg G_m(x)=m-1\ (m\geq 1),\ G_m=mE_{m-1}\ (m\geq 1),\\&G_0=0,\ G_1=1,\ G_{2m+1}=0\ (m\geq 1),\ \text{and}\ G_{2m}\neq 0\ (m\geq 1).\end{aligned} \tag{1.3}$$

Moreover, we have

$$\begin{aligned}&\frac{d}{dx}G_m(x)=mG_{m-1}(x)\ (m\geq 1),\\&G_m(x+1)+G_m(x)=2mx^{m-1}\ (m\geq 0).\end{aligned} \tag{1.4}$$

From these, we have

$$G_m(1)+G_m(0)=2\delta_{m,1},\quad (m\geq 0). \tag{1.5}$$

$$\begin{aligned}\int_0^1 G_m(x)dx&=\frac{1}{m+1}(G_{m+1}(1)-G_{m+1}(0))\\&=\frac{2}{m+1}(-G_{m+1}(0)+\delta_{m,0})\\&=\begin{cases}0, & \text{if } m \text{ is even},\\ -\frac{2}{m+1}G_{m+1}, & \text{if } m \text{ is odd}.\end{cases}\end{aligned} \tag{1.6}$$

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For any real number x, let $< x >= x - [x] \in [0, 1)$ denote the fractional part of x. In this paper, we will study the Fourier series of the following three types of functions involving Genocchi polynomials $G_m(< x >)$.

(1) $\alpha_m(< x >) = \sum_{k=1}^{m} G_k(< x >) < x >^{m-k}, (m \geq 2)$;
(2) $\beta_m(< x >) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k}, (m \geq 2)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(< x >) < x >^{m-k}, (m \geq 2)$.

The reader may refer to any book (for example, see [6,18,22]) for elementary facts about Fourier analysis.

As to $\gamma_m(< x >)$, we note that the polynomial identity (1.7) follows immediately from Theorems 4.1 and 4.2, which can be derived in turn from the Fourier series expansion of $\gamma_m(< x >)$.

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(x) x^{m-k} \\
&= -\frac{1}{m} \left(\sum_{k=1}^{m} \frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)} G_{m+1} \right) \qquad (1.7) \\
&\quad + \frac{1}{m} \sum_{s=1}^{m-1} \binom{m}{s} \left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) - \sum_{l=s}^{m-1} \frac{G_{l-s+1}}{(l-s+1)(m-l)} \right) B_s(x).
\end{aligned}$$

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2 , respectively. It is noteworthy that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(< x >) B_{m-k}(< x >)$ we can derive a slightly different version of the well-known Miki's identity (see [3,5,19,20])

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{2k\,(2m-2k)} B_{2k} B_{2m-2k} \qquad (1.8) \\
&= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{2k} \binom{2m}{2k} B_{2k} B_{2m-2k} + \frac{1}{m} H_{2m-1} B_{2m}, \quad (m \geq 2)\,.
\end{aligned}$$

In addition, we can derive the Faber-Pandharipande-Zagier identity (see [4])

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{2k\,(2m-2k)} \overline{B}_{2k} \overline{B}_{2m-2k} \qquad (1.9) \\
&= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{2k} \binom{2m}{2k} B_{2k} \overline{B}_{2m-2k} + \frac{1}{m} H_{2m-1} \overline{B}_{2m}, \quad (m \geq 2)\,,
\end{aligned}$$

where $\overline{B}_m = \left(\frac{1-2^{m-1}}{2^{m-1}} \right) B_m = \left(2^{1-m} - 1 \right) B_m = B_m \left(\frac{1}{2} \right)$, Some related works can be found in [1,7-11].

2. Fourier series of the first type of functions

In this section, we will study the Fourier series of first type of functions involving Genocchi polynomials. Let $\alpha_m(x) = \sum_{k=1}^{m} G_k(x) x^{m-k}$, $(m \geq 2)$. Note here that $\deg \alpha_m(x) = m-1$. Then we will consider the

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function

$$\alpha_m(<x>) = \sum_{k=1}^{m} G_k(<x>) <x>^{m-k}, \quad (m \geq 2). \tag{2.1}$$

defined on $(-\infty, -\infty)$ which is periodic of period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(<x>) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{2.3}$$

Before proceeding further, we first observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=1}^{m} (kG_{k-1}(x)x^{m-k} + (m-k)G_k(x)x^{m-k-1}) \\ &= \sum_{k=2}^{m} (kG_{k-1}(x)x^{m-k} + \sum_{k=1}^{m-1}(m-k)G_k(x)x^{m-k-1} \\ &= \sum_{k=1}^{m-1}(k+1)G_k(x)x^{m-k-1} + \sum_{k=1}^{m-1}(m-k)G_k(x)x^{m-k-1} \\ &= (m+1)\sum_{k=1}^{m-1} G_k(x)x^{m-1-k} \\ &= (m+1)\alpha_{m-1}(x). \end{aligned} \tag{2.4}$$

From this, we have $\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x)$. Then we have

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)), \tag{2.5}$$

$$\begin{aligned} \alpha_m(1) - \alpha_m(0) &= \sum_{k=1}^{m} (G_k(1) - G_k(0)\delta_{m,k}) \\ = \sum_{k=1}^{m} (-G_k(0) + 2\delta_{k,1} - G_k(0)\delta_{m,k}) &= -\sum_{k=1}^{m} G_k + 2 - G_m, \end{aligned} \tag{2.6}$$

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \sum_{k=1}^{m} G_k = 2 - G_m, \tag{2.7}$$

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2}\left(-\sum_{k=1}^{m+1} G_k + 2 - G_{m+1}\right). \tag{2.8}$$

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We are now ready to determine the Fourier coefficients $A_n^{(m)}$.
Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} + \frac{1}{2\pi i n}\left(\sum_{k=1}^{m} G_k - 2 + G_m\right) \\
&= \frac{m+1}{2\pi i n}\left(\frac{m}{2\pi i n} A_n^{(m-2)} + \frac{1}{2\pi i n}(\sum_{k=1}^{m-1} G_k - 2 + G_{m-1})\right) \\
&\quad + \frac{1}{2\pi i n}\left(\sum_{k=1}^{m} G_k - 2 + G_m\right) \\
&= \frac{(m+1)m}{(2\pi i n)^2} A_n^{(m-2)} + \frac{m+1}{(2\pi i n)^2}\left(\sum_{k=1}^{m-1} G_k - 2 + G_{m-1}\right) \\
&\quad + \frac{1}{2\pi i n}\left(\sum_{k=1}^{m} G_k - 2 + G_m\right) \\
&= \cdots \\
&= \frac{(m+1)_{m-2}}{(2\pi i n)^{m-2}} A_n^{(2)} + \sum_{j=1}^{m-2} \frac{(m+1)_{j-1}}{(2\pi i n)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) \\
&= -\frac{3(m+1)_{m-2}}{(2\pi i n)^{m-1}} + \sum_{j=1}^{m-2} \frac{(m+1)_{j-1}}{(2\pi i n)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) \\
&= \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi i n)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) \\
&= \frac{1}{m+2}\sum_{j=1}^{m-1} \frac{(m+1)_{j}}{(2\pi i n)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right),
\end{aligned} \tag{2.9}$$

where

$$A_n^{(2)} = \int_0^1 (3x-1) e^{-2\pi i n x} dx = -\frac{3}{2\pi i n}. \tag{2.10}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = -\frac{1}{m+2}(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}). \tag{2.11}$$

$\alpha_m(< x >), (m \geq 2)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those integers $m \geq 2$ with $\sum_{k=1}^{m} G_k = 2 - G_m$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\sum_{k=1}^{m} G_k \neq 2 - G_m$.

We need the following facts about Bernoulli functions $B_m(< x >)$:

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(a) for $m \geq 2$,

$$B_m(<x>) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}. \tag{2.12}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(<x>), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}, \end{cases} \tag{2.13}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. Assume first that $m \geq 2$ is an integer with $\sum_{k=1}^{m} G_k = 2 - G_m$. Then $\alpha_m(1) = \alpha_m(0)$. Thus $\alpha_m(<x>)$ is piecewise C^∞, and continuous. So the Fourier series of $\alpha_m(<x>)$ converges uniformly to $\alpha_m(<x>)$, and

$$\begin{aligned}
&\alpha_m(<x>) \\
&= -\frac{1}{m+2}(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}) + \frac{1}{m+2} \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} (\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}) \right) e^{2\pi i n x} \\
&= -\frac{1}{m+2}(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}) \\
&\quad - \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1} \right) \left(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n}}{(2\pi i n)^j} \right) \\
&= -\frac{1}{m+2} \left(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1} \right) - \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1} \right) B_j(<x>) \\
&\quad - \left(\sum_{k=1}^{m} G_k - 2 + G_m) \right) \times \begin{cases} B_1(<x>), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}, \end{cases}
\end{aligned} \tag{2.14}$$

for all $x \in (-\infty, \infty)$. Hence we obtain the following theorem.

Theorem 2.1. *Let $m \geq 2$ be an integer with $\sum_{k=1}^{m} G_k = 2 - G_m$. Then we have the following.*
(a) $\sum_{k=1}^{m} G_k(<x>) <x>^{m-k}$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=1}^{m} G_k(<x>) <x>^{m-k} \\
&= -\frac{1}{m+2} \left(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1} \right) \\
&\quad + \frac{1}{m+2} \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} (\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}) \right) e^{2\pi i n x},
\end{aligned} \tag{2.15}$$

for all $x \in (-\infty, \infty)$. Here the convergence is uniform.

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(b)

$$\begin{aligned}&\sum_{k=1}^{m} G_k(< x >) < x >^{m-k}\\&= -\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m-1}\binom{m+2}{j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) B_j(< x >),\end{aligned} \tag{2.16}$$

for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.

Next, we assume that $m \geq 2$ is an integer with $\sum_{k=1}^{m} G_k \neq 2 - G_m$. Then $\alpha_m(1) \neq \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned}\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) &= \alpha_m(0) - \frac{1}{2}\sum_{k=1}^{m} G_k + 1 - \frac{1}{2}G_m\\&= 1 - \frac{1}{2}\sum_{k=1}^{m-1} G_k.\end{aligned} \tag{2.17}$$

Thus we get the following theorem.

Theorem 2.2. *Let $m \geq 2$ be an integer with $\sum_{k=1}^{m} G_k \neq 2 - G_m$. Then we have the following.*

(a)

$$\begin{aligned}&-\frac{1}{m+2}\left(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}\right) + \frac{1}{m+2}\sum_{n=-\infty,n\neq 0}^{\infty}\left(\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi i n)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right)\right) e^{2\pi i n x}\\&= \begin{cases} \sum_{k=1}^{m} G_k(< x >) < x >^{m-k}, & \text{for } x \in \mathbb{Z}^c,\\ 1 - \frac{1}{2}\sum_{k=1}^{m-1} G_k, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{2.18}$$

Here the convergence is pointwise.

(b)

$$\begin{aligned}&-\frac{1}{m+2}\sum_{j=0}^{m-1}\binom{m+2}{j}(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1})B_j(< x >)\\&= \sum_{k=1}^{m} G_k(< x >) < x >^{m-k}, \text{ for } x \in \mathbb{Z}^c \quad ;\\&-\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m-1}\binom{m+2}{j}(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1})B_j(< x >)\\&= 1 - \frac{1}{2}\sum_{k=1}^{m-1} G_k,\ x \in \mathbb{Z}.\end{aligned} \tag{2.19}$$

Question: For what values of $m \geq 2$, does $\sum_{k=1}^{m} G_k = 2 - G_m$ hold?

3. Fourier series of the second type of functions

Let $\beta_m(x) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(x) x^{m-k}$, $(m \geq 2)$. Then, we consider the function

$$\beta_m(< x >) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k}, \tag{3.1}$$

defined on $(-\infty, -\infty)$ which is periodic with period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{3.3}$$

Before proceeding further, we need the following.

$$\begin{aligned} \beta_m'(x) &= \sum_{k=1}^{m} \left\{ \frac{k}{k!(m-k)!} G_{k-1}(x) x^{m-k} + \frac{m-k}{k!(m-k)!} G_k(x) x^{m-k-1} \right\} \\ &= \sum_{k=2}^{m} \frac{1}{(k-1)!(m-k)!} G_{k-1}(x) x^{m-k} + \sum_{k=1}^{m-1} \frac{1}{k!(m-k-1)!} G_k(x) x^{m-k-1} \\ &= \sum_{k=1}^{m-1} \frac{1}{k!(m-1-k)!} G_k(x) x^{m-1-k} + \sum_{k=1}^{m-1} \frac{1}{k!(m-1-k)!} G_k(x) x^{m-1-k} \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.4}$$

So, $\beta_m'(x) = 2\beta_{m-1}(x)$. From this, we see that

$$\left(\frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x) \tag{3.5}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} (\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.6}$$

We also observe that

$$\begin{aligned} \beta_m(1) - \beta_m(0) &= \sum_{k=1}^{m} \frac{1}{k!(m-k)!} \left(G_k(1) - G_k(0) \delta_{m,k} \right) \\ &= \sum_{k=1}^{m} \frac{1}{k!(m-k)!} (-G_k(0) + 2\delta_{k,1}) - \sum_{k=1}^{m} \frac{G_k(0) \delta_{m,k}}{k!(m-k)!} \\ &= -\sum_{k=1}^{m} \frac{G_k}{k!(m-k)!} + \frac{2}{(m-1)!} - \frac{G_m}{m!}. \end{aligned} \tag{3.7}$$

We put

$$\Omega_m = \beta_m(1) - \beta_m(0) = -\sum_{k=1}^{m} \frac{G_k}{k!(m-k)!} + \frac{2}{(m-1)!} - \frac{G_m}{m!}, \tag{3.8}$$

for $m \geq 2$. Then

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0. \tag{3.9}$$

Moreover,

$$\begin{aligned} \int_0^1 \beta_m(x)dx &= \frac{1}{2}\Omega_{m+1} \\ &= -\frac{1}{2}\left\{\sum_{k=1}^{m+1} \frac{G_k}{k!(m-k+1)!} - \frac{2}{m!} + \frac{G_{m+1}}{(m+1)!}\right\}. \end{aligned} \tag{3.10}$$

Now, we are going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\beta_m(1) - \beta_m(0)) + \frac{1}{\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{1}{\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m \\ &= \frac{1}{\pi in}\left(\frac{1}{\pi in}B_n^{(m-2)} - \frac{1}{2\pi in}\Omega_{m-1}\right) - \frac{1}{2\pi in}\Omega_m \\ &= \frac{1}{(\pi in)^2}B_n^{(m-2)} - \frac{2}{(2\pi in)^2}\Omega_{m-1} - \frac{1}{2\pi in}\Omega_m \\ &= \cdots \\ &= \frac{1}{(\pi in)^{m-2}}B_n^{(2)} - \sum_{j=1}^{m-2} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}, \end{aligned} \tag{3.11}$$

where

$$B_n^{(2)} = \int_0^1 \left(2x - \frac{1}{2}\right)e^{-2\pi inx}dx = -\frac{1}{\pi in}. \tag{3.12}$$

By (3.11) and (3.12), we get

$$\begin{aligned} B_n^{(m)} &= -\frac{1}{(\pi in)^{m-1}} - \sum_{j=1}^{m-2} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\ &= -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}. \end{aligned} \tag{3.13}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.14}$$

Here $\beta_m(<x>), (m \geq 2)$ is piecewise C^∞. Moreover, $\beta_m(<x>)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that $m \geq 2$ is an integer with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. So $\beta_m(<x>)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\beta_m(<x>)$ converges uniformly to $\beta_m(<x>)$, and

$$\begin{aligned}\beta_m(<x>) &= \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \left(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &\quad + \Omega_m \times \begin{cases} B_1(<x>), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}, \end{cases}\end{aligned} \tag{3.15}$$

for all $x \in (-\infty, \infty)$.

Thus we have the following theorem.

Theorem 3.1. *For each integer $l \geq 2$, let*

$$\Omega_l = -\sum_{k=1}^{l} \frac{G_k}{k!(l-k)!} + \frac{2}{(l-1)!} - \frac{G_l}{l!}. \tag{3.16}$$

Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(<x>) <x>^{m-k}$ has the Fourier series expansion

$$\begin{aligned}&\sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x},\end{aligned} \tag{3.17}$$

for all $x \in (-\infty, \infty)$. Here the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(<x>) <x>^{m-k} \\ &= \sum_{j=0, j\neq 1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>),\end{aligned} \tag{3.18}$$

for all $x \in (-\infty, \infty)$. Here $B_k(<x>)$ is the Bernoulli function.

Assume next that $m \geq 2$ is an integer with $\Omega_m \neq 0$. Then $\beta_m(1) \neq \beta_m(0)$, and hence $\beta_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of

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$\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned}
&\frac{1}{2}(\beta_m(0)+\beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m \\
&= \frac{G_m}{m!} + \frac{1}{2}\left(-\sum_{k=1}^{m}\frac{G_k}{k!(m-k)!} + \frac{2}{(m-1)!} - \frac{G_m}{m!}\right) \\
&= \frac{1}{2}\left(\frac{2}{(m-1)!} - \sum_{k=1}^{m-1}\frac{G_k}{k!(m-k)!}\right),
\end{aligned} \tag{3.19}$$

for $x \in \mathbb{Z}$. Hence we obtain the following theorem.

Theorem 3.2. *For each integer $l \geq 2$, let*

$$\Omega_l = -\sum_{k=1}^{l}\frac{G_k}{k!(l-k)!} + \frac{2}{(l-1)!} - \frac{G_l}{l!}. \tag{3.20}$$

Assume that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then we have the following.
(a)

$$\begin{aligned}
&\frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty}\left(\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right)e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m}\frac{1}{k!(m-k)!}G_k(<x>)<x>^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{G_m}{m!} + \frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.21}$$

Here the convergence is pointwise.
(b)

$$\begin{aligned}
&\sum_{j=0}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>) \\
&= \sum_{k=1}^{m}\frac{1}{k!(m-k)!}G_k(<x>)<x>^{m-k},
\end{aligned} \tag{3.22}$$

for $x \in \mathbb{Z}^c$;

$$\begin{aligned}
&\sum_{j=0, j\neq 1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_k(<x>) \\
&= \frac{G_m}{m!} + \frac{1}{2}\Omega_m,
\end{aligned} \tag{3.23}$$

for $x \in \mathbb{Z}$. Here $B_j(<x>)$ is the Bernoulli function.

Remark: For what values of $m \geq 2$, does $\sum_{k=1}^{m}\frac{G_m}{k!(m-k)!} = \frac{2}{(m-1)!} - \frac{G_m}{m!}$ hold?

4. Fourier series of the third type of functions

Let $\gamma_m(x) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(x)x^{m-k}$, $(m \geq 2)$. Then we will consider the function

$$\gamma_m(<x>) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(<x>)<x>^{m-k}, \tag{4.1}$$

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defined on $(-\infty, -\infty)$ which is periodic of period 1. The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi inx}, \tag{4.2}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi inx} dx = \int_0^1 \gamma_m(x) e^{-2\pi inx} dx. \tag{4.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned} \gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left\{ kG_{k-1}(x)x^{m-k} + (m-k)G_k(x)x^{m-k-1} \right\} \\ &= \sum_{k=1}^{m-2} \frac{1}{m-k-1} G_k(x)x^{m-k-1} + \sum_{k=1}^{m-1} \frac{1}{k} G_k(x)x^{m-k-1} \\ &= (m-1)\sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} G_k(x)x^{m-1-k} + \frac{1}{m-1}G_{m-1}(x) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}G_{m-1}(x). \end{aligned} \tag{4.4}$$

Thus,

$$\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}G_{m-1}(x). \tag{4.5}$$

From this, we have

$$\left(\frac{1}{m}(\gamma_{m+1}(x) - \frac{1}{m(m+1)}G_{m+1}(x)) \right)' = \gamma_m(x) \tag{4.6}$$

and

$$\begin{aligned} &\int_0^1 \gamma_m(x) dx \\ =& \left[\frac{1}{m}(\gamma_{m+1}(x) - \frac{1}{m(m+1)}G_{m+1}(x)) \right]_0^1 \\ =& \frac{1}{m}\left((\gamma_{m+1}(1) - \gamma_{m+1}(0)) - \frac{1}{m(m+1)}(G_{m+1}(1) - G_{m+1}(0)) \right) \\ =& \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)}G_{m+1}(0) \right). \end{aligned} \tag{4.7}$$

Observe that

$$\begin{aligned} \gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (G_k(1) - G_k(0)\delta_{m,k}) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (-G_k(0) + 2\delta_{k,1}) - \sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(0)\delta_{m,k} \\ &= -\sum_{k=1}^{m-1} \frac{G_k}{k(m-k)} + \frac{2}{m-1}. \end{aligned} \tag{4.8}$$

So, we have

$$\gamma_m(1) = \gamma_m(0) \Longleftrightarrow \sum_{k=1}^{m-1} \frac{G_k}{k(m-k)} = \frac{2}{m-1}. \tag{4.9}$$

Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(-\sum_{k=1}^{m} \frac{G_k}{k(m-k+1)} + \frac{2}{m} + \frac{2}{m(m+1)}G_{m+1}\right). \tag{4.10}$$

Now, we will determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}[\gamma_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma'_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in}\int_0^1 \left((m-1)\gamma_{m-1}(x) + \frac{1}{m-1}G_{m-1}(x)\right)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{m-1}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx \\
&\quad + \frac{1}{2\pi in(m-1)}\int_0^1 G_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Phi_m,
\end{aligned} \tag{4.11}$$

where

$$\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi in)^k}G_{m-k}, \tag{4.12}$$

and one can show

$$\int_0^1 G_m(x)e^{-2\pi inx}dx = \begin{cases} 2\Omega_{m+1}, & \text{for} \quad n \neq 0, \\ -2\frac{G_{m+1}}{m+1}, & \text{for} \quad n = 0. \end{cases} \tag{4.13}$$

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} + \frac{2}{2\pi in(m-2)}\Phi_{m-1}\right) \\
&\quad - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} - \frac{m-1}{(2\pi in)^2}\Lambda_{m-1} - \frac{1}{2\pi in}\Lambda_m \\
&\quad + \frac{2(m-1)}{(2\pi in)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\frac{1}{m}\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1} + \frac{1}{m}\sum_{j=1}^{m-2}\frac{2(m)_j}{(2\pi in)^j(m-j)}\Phi_{m-j+1},
\end{aligned} \tag{4.14}
$$

where

$$
C_n^{(2)} = \int_0^1 xe^{-2\pi inx}dx = -\frac{1}{2\pi in}. \tag{4.15}
$$

In order to get a final expression for $C_n^{(m)}$, we need to observe the following.

$$
\begin{aligned}
&\sum_{j=1}^{m-2}\frac{2(m)_j}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= \sum_{j=1}^{m-2}\frac{2(m)_j}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j-1}\frac{(m-j)_{k-1}}{(2\pi in)^k}G_{m-j-k+1} \\
&= \sum_{j=1}^{m-2}\sum_{k=1}^{m-j-1}\frac{2(m)_{j+k-1}}{(2\pi in)^{j+k}(m-j)}G_{m-j-k+1} \\
&= 2\sum_{j=1}^{m-2}\frac{1}{m-j}\sum_{s=j+1}^{m-1}\frac{(m)_{s-1}}{(2\pi in)^s}G_{m-s+1} \\
&= 2\sum_{s=2}^{m-1}\frac{(m)_{s-1}}{(2\pi in)^s}G_{m-s+1}\sum_{j=1}^{s-1}\frac{1}{m-j} \\
&= 2\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}),
\end{aligned} \tag{4.16}
$$

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and

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\\
&=\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi in)^j}\left\{-\sum_{k=1}^{m-j}\frac{G_k}{k(m-j-k+1)}+\frac{2}{m-j}\right\}\\
&=-\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m)_jG_k}{(2\pi in)^jk(m-j-k+1)}+2\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi in)^j(m-j)}\\
&=-\sum_{s=1}^{m-1}\sum_{l=s}^{m-1}\frac{(m)_sG_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}+2\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s(m-s)}.
\end{aligned}\tag{4.17}$$

Putting everything altogether,

$$\begin{aligned}
C_n^{(m)}=&\frac{1}{m}\sum_{s=1}^{m-1}\sum_{l=s}^{m-1}\frac{(m)_sG_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\\
&-\frac{2}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s(m-s)}\\
&+\frac{2}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\\
=&-\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\\
&\times\left\{\frac{2}{m-s}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})-\sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right\}.
\end{aligned}\tag{4.18}$$

Case 2: $n=0$.

$$\begin{aligned}
C_0^{(m)}&=\int_0^1\gamma_m(x)dx\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\
&=\frac{1}{m}\left(-\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)}+\frac{2}{m}+\frac{2}{m(m+1)}G_{m+1}\right).
\end{aligned}\tag{4.19}$$

$\gamma_m(<x>),(m\geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(<x>)$ is continuous for those integers $m\geq 2$ with $\Lambda_m=0$ and discontinuous with jump discontinuities at integers for those integers $\Lambda_m\neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. So $\gamma_m(<x>)$ is piecewise C^∞, and continuous. So the Fourier series of $\gamma_m(<x>)$ converges uniformly to $\gamma_m(<x>)$, and

$$\begin{aligned}
&\gamma_m(<x>)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad -\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)\right\}e^{2\pi inx}\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad +\frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)\\
&\qquad \times\left(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad +\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)B_s(<x>)\\
&\quad +\left(\frac{2}{m-1} - \sum_{l=1}^{m-1}\frac{G_l}{l(m-l)}\right)\times\begin{cases} B_1(<x>), & \text{for } x\in\mathbb{Z}^c,\\ 0, & \text{for } x\in\mathbb{Z},\end{cases}
\end{aligned}\tag{4.20}$$

for all $x \in (-\infty, \infty)$. Now, we obtain the following theorem.

Theorem 4.1. *Let $m \geq 2$ be an integer with $\Lambda_m = -\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)} + \frac{2}{m-1} = 0$. Then we have the following.*

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(<x>)<x>^{m-k}$ has the Fourier expansion

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(<x>)<x>^{m-k}\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad -\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)\right\}e^{2\pi inx}
\end{aligned}\tag{4.21}$$

,

for all $x \in (-\infty, \infty)$. Here the convergence is uniform.

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(<x>) <x>^{m-k} \\
&= -\frac{1}{m}\left(\sum_{k=1}^{m} \frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)} G_{m+1}\right) \\
&\quad + \frac{1}{m}\sum_{s=2}^{m-1} \binom{m}{s} \left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) - \sum_{l=s}^{m-1} \frac{G_{l-s+1}}{(l-s+1)(m-l)}\right) B_s(<x>)
\end{aligned} \tag{4.22}$$

for all $x \in (-\infty, \infty)$, where $B_s(<x>)$ is the Bernoulli function.

Assume next that $m \geq 2$ is an integer with $\Lambda_m \neq 0$.Then $\gamma_m(0) \neq \gamma_m(1)$. $\gamma_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers.Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned}
\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) &= \gamma_m(0) + \frac{1}{2}\Lambda_m \\
&= \frac{1}{2}\left(-\sum_{k=1}^{m-1} \frac{G_k}{k(m-k)} + \frac{2}{m-1}\right),
\end{aligned} \tag{4.23}$$

for $x \in \mathbb{Z}$. Hence we have the following theorem.

Theorem 4.2. *Let $m \geq 2$ be an integer with $\Lambda_m = -\sum_{k=1}^{m-1} \frac{G_k}{k(m-k)} + \frac{2}{m-1} \neq 0$. Then we have the following.*
(a)

$$\begin{aligned}
&-\frac{1}{m}\left(\sum_{k=1}^{m} \frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)} G_{m+1}\right) \\
&-\frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty} \left\{\sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}\right)(H_{m-1} - H_{m-s}) - \sum_{l=s}^{m-1} \frac{G_{l-s+1}}{(l-s+1)(m-1)}\right\} e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(<x>) <x>^{m-k}, & \text{for } x \in \mathbb{Z}^c, \\ \frac{1}{2}\left(-\sum_{k=1}^{m-1} \frac{G_k}{k(m-k)} + \frac{2}{m-1}\right), & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.24}$$

Here the convergence is uniform.
(b)

$$\begin{aligned}
&-\frac{1}{m}\left(\sum_{k=1}^{m} \frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)} G_{m+1}\right) \\
&+ \frac{1}{m}\sum_{s=1}^{m-1} \binom{m}{s} \left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) - \sum_{l=s}^{m-1} \frac{G_{l-s+1}}{(l-s+1)(m-l)}\right) B_s(<x>) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(<x>) <x>^{m-k},
\end{aligned} \tag{4.25}$$

for $x \in \mathbb{Z}^c$ *and*

$$\begin{aligned}
&-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)}-\frac{2}{m}-\frac{2}{m(m+1)}G_{m+1}\right)\\
&+\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2}{m-s}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})-\sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)B_s(< x >) \quad (4.26)\\
&=\frac{1}{2}\left(-\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)}+\frac{2}{m-1}\right),
\end{aligned}$$

for $x \in \mathbb{Z}$.

Question For what values of $m \geq 2$, does $\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)} = \frac{2}{m-1}$ hold?

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Lyapunov inequalities of quasi-Hamiltonian systems on time scales

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Abstract In this paper, we obtain several new Lyapunov-type inequalities for the following quasi-Hamiltonian systems

$$x^{\Delta}(t) = -W(t)x(\sigma(t)) - U(t)|y(t)|^{p-2}y(t), \ \ y^{\Delta}(t) = V(t)|x(\sigma(t))|^{q-2}x(\sigma(t)) + W^T(t)y(t)$$

on the time scale interval $[a, b]_{\mathbb{T}} \equiv [a, b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$ ($\sigma(a) < b$), where U and V are real $n \times n$ symmetric matrix-valued functions on $[a, b]_{\mathbb{T}}$ with U being positive definite, W is real $n \times n$ matrix-valued function on $[a, b]_{\mathbb{T}}$ with $I + \mu(t)W$ being invertible, and x, y are real vector-valued functions on $[a, b]_{\mathbb{T}}$.

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1. Introduction

In 1990, Hilger [1] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied some Lyapunov inequalities for dynamic equations on time scales (see [2-4]) during the last few years. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real axis $\mathbb{R}$. On a time scale $\mathbb{T}$, the forward jump operator and the graininess function are defined

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \qquad \mu(t) = \sigma(t) - t,$$

respectively. For the notions used below we refer to [5,6] that provide some basic facts on time scales.

In this paper, we continue this line of investigation and study Lyapunov-type inequalities for the following quasi-Hamiltonian systems

$$x^{\Delta}(t) = -W(t)x(\sigma(t)) - U(t)|y(t)|^{p-2}y(t), \ \ y^{\Delta}(t) = V(t)|x(\sigma(t))|^{q-2}x(\sigma(t)) + W^T(t)y(t), \quad (1.1)$$

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on the time scale interval $[a,b]_{\mathbb{T}} \equiv [a,b]\cap\mathbb{T}$ for some $a,b\in\mathbb{T}$ $(\sigma(a)<b)$, where $p,q\in(0,+\infty)$ and $1/p+1/q=1$, U and V are real $n\times n$ symmetric matrix-valued functions on $[a,b]_{\mathbb{T}}$ with U being positive definite, W is real $n\times n$ matrix-valued function on $[a,b]_{\mathbb{T}}$ with $I+\mu(t)W$ being invertible, and x,y are real vector-valued functions on $[a,b]_{\mathbb{T}}$.

When $n=1$, (1.1) reduces to

$$x^{\Delta}(t)=\alpha(t)x(\sigma(t))+\beta(t)|y(t)|^{p-2}y(t),\quad y^{\Delta}(t)=-\gamma(t)|x(\sigma(t))|^{q-2}x(\sigma(t))-\alpha(t)y(t). \quad (1.2)$$

In 2011, Zhang et al. [7] obtained the following theorem.

Theorem 1.1[7] Suppose that $1-\mu(t)\alpha(t)>0$ and $\beta(t)\geq 0$ for any $t\in\mathbb{T}$ and $a,\ b\in\mathbb{T}^k$ with $\sigma(a)\leq b$. If (1.2) has a real solution $(x(t),y(t))$ satisfying

$$x(a)=0 \text{ or } x(a)x(\sigma(a))<0,\ x(b)=0 \text{ or } x(b)x(\sigma(b))<0,\ \max_{t\in[a,b]_{\mathbb{T}}}|x(t)|>0,$$

then the following inequality holds:

$$\int_a^b|\alpha(t)|\triangle(t)+\Big(\int_a^{\sigma(b)}\beta(t)\triangle(t)\Big)^{\frac{1}{p}}\Big(\int_a^b\max\{\gamma(t),0\}\triangle(t)\Big)^{\frac{1}{q}}\geq 2. \quad (1.3)$$

When $n=1$ and $\mathbb{T}=\mathbb{R}$, Tiryaki et al. [8] obtained the following theorem.

Theorem 1.2[8] Suppose that $\beta(t)>0$ for any $t\in\mathbb{R}$ and $a,\ b\in\mathbb{R}$ with $a<b$. If (1.2) has a real solution $(x(t),y(t))$ satisfying $x(a)=x(b)=0$ and $\max_{t\in[a,b]}|x(t)|>0$, then the following inequalities hold:

$$\int_a^b\frac{\max\{\gamma(t),0\}}{h_a^{1-q}(t)+h_b^{1-q}(t)}dt\geq 1 \quad (1.4)$$

and

$$\int_a^b\max\{\gamma(t),0\}2^{q-2}\Big(\frac{1}{h_a(t)}+\frac{1}{h_b(t)}\Big)^{1-q}dt\geq 1, \quad (1.5)$$

where $h_a(t)=\int_a^t\beta(s)e^{-p\int_t^s\alpha(\tau)d\tau}ds$ and $h_b(t)=\int_t^b\beta(s)e^{-p\int_t^s\alpha(\tau)d\tau}ds$.

For some other related results on Lyapunov-type inequalities, see, e.g. [9-16] and the related references therein.

2. Preliminaries and some lemmas

For any $u\in\mathbb{R}^n$ and any $U\in\mathbb{R}^{n\times n}$ (the space of real $n\times n$ matrices), write

$$|u|=\sqrt{u^Tu}\quad\text{and}\quad|U|=\max_{y\in\mathbb{R}^n-\{O\}}\frac{|Uy|}{|y|},$$

which are called the Euclidean norm of u and the matrix norm of U respectively, where Q^T is the transpose of a $n\times m$ matrix Q. It follows from the definition that for any $y\in\mathbb{R}^n$ and any $U,V\in\mathbb{R}^{n\times n}$,

$$|Uy|\leq|U||y|,\qquad |UV|\leq|U||V|.$$

Write $\mathbb{R}_s^{n\times n} = \{U \in \mathbb{R}^{n\times n} : U^T = U\}$. It is easy to show that for any $U \in \mathbb{R}_s^{n\times n}$,

$$|U| = \max_{\det|\lambda I - U|=0} |\lambda| \quad \text{and} \quad |U^2| = |U|^2,$$

where $\det|\lambda I - U|$ denotes determinant of the matrix $\lambda I - U$. An $U \in \mathbb{R}_s^{n\times n}$ is said to be positive definite (resp. semi-positive definite), written as $U > 0$ (resp. $U \geq 0$), if $y^T U y > 0$ (resp. $y^T U y \geq 0$) for all $y \in \mathbb{R}^n$ with $y \neq 0$. If U is positive definite (resp. semi-positive definite), then there exists a unique positive definite matrix (resp. semi-positive definite matrix), written as $\sqrt{U}$, such that $[\sqrt{U}]^2 = U$.

In this paper, we study Lyapunov-type inequalities of (1.1) which has some solution $(x(t), y(t))$ satisfying

$$x(a) = x(b) = 0 \text{ and } \max_{t\in[a,b]_{\mathbb{T}}} |x(t)| > 0. \tag{2.1}$$

We first introduce the following notions and lemmas.

The point $t \in \mathbb{T}$ is said to be left-dense (resp. left-scattered) if $\rho(t) = t$ (resp. $\rho(t) < t$). The point $t \in \mathbb{T}$ is said to be right-dense (resp. right-scattered) if $\sigma(t) = t$ (resp. $\sigma(t) > t$). If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous provided that f is continuous at right-dense points and has finite left-sided limits at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

For a function $f : \mathbb{T} \longrightarrow \mathbb{R}$, the (delta) derivative $f^{\Delta}(t)$ at $t \in \mathbb{T}$ is defined to be the number (if it exists), such that for given any $\varepsilon > 0$, there is a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. If the (delta) derivative $f^{\Delta}(t)$ exists for every $t \in \mathbb{T}^k$, then we say that f is $\Delta-$ differentiable on $\mathbb{T}$.

Definition 2.1[5] Let $F, f \in C_{rd}(\mathbb{T}, \mathbb{R})$. If $F^{\triangle}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then we definite the Cauchy integral of f by

$$\int_a^b f(t) \triangle t = F(b) - F(a) \text{ for any } a, b \in \mathbb{T}.$$

Lemma 2.2[5] (Holder's inequality) Let $a, b \in \mathbb{T}$ with $a \leq b$ and $f_1, f_2 \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then

$$\int_a^b |f_1(t)f_2(t)| \triangle t \leq \Big(\int_a^b |f_1(t)|^p \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |f_2(t)|^q \triangle t\Big)^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p-1)$.

Lemma 2.3[5] Suppose that $W \in C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ with $I + \mu(t)W(t)$ being invertible and $g \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the initial value problem

$$x^{\Delta}(t) = -W(t)x(\sigma(t)) + g(t),\ x(t_0) = x_0$$

has a unique solution

$$x(t) = e_{\ominus W}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus W}(t, \tau)g(\tau)\Delta\tau,$$

where $(\Theta A)(t) = -[I+\mu(t)A(t)]^{-1}A(t)$ for any $t \in \mathbb{T}^k$ and $e_{\Theta A}(t,t_0)$ is the unique matrix-valued solution of the initial value problem

$$\begin{cases} Y^{\Delta}(t) = (\Theta A)(t)Y(t), \\ Y(t_0) = I. \end{cases}$$

Lemma 2.4[5] Suppose that $A(t)$ and $B(t)$ are differentiable $n \times n$ matrix-valued functions. Then

$$(A(t)B(t))^{\triangle} = A^{\triangle}(t)B(\sigma(t)) + A(t)B^{\triangle}(t) = A(\sigma(t))B^{\triangle}(t) + A^{\triangle}(t)B(t).$$

Lemma 2.5[12] Let $a, b \in \mathbb{T}$ with $a \geq b$ and $x_1(t), x_2(t), \cdots, x_n(t)$ be Δ-integrable on $[a,b]_{\mathbb{T}}$. Write $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$. Then

$$\Big|\int_a^b x(t)\Delta t\Big| = \sqrt{\sum_{i=1}^{n}\Big(\int_a^b x_i(t)\Delta t\Big)^2} \leq \int_a^b \sqrt{\sum_{i=1}^{n} x_i^2(t)}\Delta t = \int_a^b |x(t)|\Delta t.$$

Lemma 2.6[12] Let $V, V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1 \geq V$ (i.e., $V_1 - V \geq 0$) and $x \in \mathbb{R}^n$. Then $x^T V x \leq |V_1||x|^2$.

3. Main results and proofs

Write

$$\xi(t) = \begin{cases} (\int_a^t |e_{\Theta W}(t,s)|^p |U(s)|^{\frac{p(p-2)}{2}+1} |[\sqrt{U(s)}]^{-1}|^{p(p-2)}\Delta s)^{\frac{q}{p}}, & \text{if} \quad 1 < q < 2, \\ (\int_a^t |e_{\Theta W}(t,s)|^p |U(s)|\Delta s)^{\frac{q}{p}}, & \text{if} \quad q \geq 2, \end{cases} \tag{3.1}$$

and

$$\eta(t) = \begin{cases} (\int_t^b |e_{\Theta W}(t,s)|^p |U(s)|^{\frac{p(p-2)}{2}+1} |[\sqrt{U(s)}]^{-1}|^{p(p-2)}\Delta s)^{\frac{q}{p}}, & \text{if} \quad 1 < q < 2, \\ (\int_t^b |e_{\Theta W}(t,s)|^p |U(s)|\Delta s)^{\frac{q}{p}}, & \text{if} \quad q \geq 2. \end{cases} \tag{3.2}$$

Theorem 3.1 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a,b]_{\mathbb{T}}$, then the following inequality holds:

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))}|V_1(t)| \triangle t \geq 1. \tag{3.3}$$

Proof We claim that $y(t) \not\equiv 0$ ($t \in [a,b]_{\mathbb{T}}$). Indeed, if $y(t) \equiv 0$ ($t \in [a,b]_{\mathbb{T}}$), then the first equation of (1.1) reduces to

$$x^{\Delta}(t) = -W(t)x(\sigma(t)),\ x(a) = 0.$$

By Lemma 2.3, it follows $x(t) = e_{\Theta W}(t,a) \cdot x(a) = 0$, which is a contradiction with (2.1). Moreover, we have $y^T(t)U(t)y(t) \geq 0$ ($\not\equiv 0$) for $t \in [a,b]_{\mathbb{T}}$ since $U(t) > 0$.

Since $(x(t), y(t))$ satisfies the following equality

$$(y^T(t)x(t))^{\Delta} = (x^{\sigma}(t))^T V(t)|x^{\sigma}(t)|^{q-2}x^{\sigma}(t) - y^T(t)U(t)|y(t)|^{p-2}y(t), \tag{3.4}$$

where $x^\sigma(t) = x(\sigma(t))$. By integrating (3.4) from a to b and taking into account that $x(a) = x(b) = 0$, we see

$$\int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t = \int_a^b |y(t)|^{p-2}y^T(t)U(t)y(t) \triangle t > 0. \tag{3.5}$$

For $t \in [a,b]_{\mathbb{T}}$, let $t_0 = a$ and $t_0 = b$ respectively, we obtain from Lemma 2.3 that

$$x(t) = -\int_a^t e_{\ominus W}(t,\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau = -\int_b^t e_{\ominus W}(t,\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau.$$

Which follows that for $t \in [a,b)_{\mathbb{T}}$,

$$x^\sigma(t) = -\int_a^{\sigma(t)} e_{\ominus W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau = \int_{\sigma(t)}^b e_{\ominus W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau.$$

Case I: Assume that $q \geq 2$. Then we have that for $a \leq \tau \leq \sigma(t) \leq b$,

$$\begin{aligned}
& |e_{\ominus W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)| \\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)y(\tau)| \\
=\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{y^T(\tau)U^T(\tau)U(\tau)y(\tau)\}^{\frac{1}{2}} \\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{|\sqrt{U(\tau)}y(\tau)||U(\tau)||\sqrt{U(\tau)}y(\tau)|\}^{\frac{1}{2}} \\
=\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{2}} \\
=\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{2}-\frac{1}{q}} \\
=\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}y(\tau)|^{2(\frac{1}{2}-\frac{1}{q})} \\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{1-\frac{2}{q}}|y(\tau)|^{p-1-\frac{2}{q}} \\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||U(\tau)|^{1-\frac{1}{q}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|y(\tau)|^{p-1-\frac{2}{q}}.
\end{aligned}$$

Combining Lemma 2.2 and Lemma 2.5 we obtain

$$\begin{aligned}
|x^\sigma(t)|^q &= \Big|\int_a^{\sigma(t)} e_{\ominus W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau\Big|^q \\
&\leq \Big(\int_a^{\sigma(t)} |e_{\ominus W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)|\Delta\tau\Big)^q \\
&\leq \Big(\int_a^{\sigma(t)} |e_{\ominus W}(\sigma(t),\tau)||U(\tau)|^{1-\frac{1}{q}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|y(\tau)|^{p-1-\frac{2}{q}}\Delta\tau\Big)^q \\
&\leq \Big(\int_a^{\sigma(t)} |e_{\ominus W}(\sigma(t),\tau)|^p|U(\tau)|\triangle\tau\Big)^{\frac{q}{p}}\Big(\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\Big),
\end{aligned}$$

that is

$$|x^\sigma(t)|^q \leq \xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.6}$$

Similarly, by letting $\eta(t)$ be as in (3.2), for $a \leq \sigma(t) \leq \tau \leq b$, we have

$$|x^\sigma(t)|^q \leq \eta(\sigma(t))\int_{\sigma(t)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\eta(\sigma(t))\xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau \geq |x^\sigma(t)|^q\eta(\sigma(t))$$

and

$$\eta(\sigma(t))\xi(\sigma(t))\int_{\sigma(t)}^{b} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau \geq |x^\sigma(t)|^q\xi(\sigma(t)).$$

Thus

$$|x^\sigma(t)|^q \leq \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

By Lemma 2.6 and (3.5) we see

$$\begin{aligned}
\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t &\leq \int_a^b (|V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau)\Delta t\\
&= \int_a^b |V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\\
&= \int_a^b |V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t)\triangle t\\
&\leq \int_a^b |V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t.
\end{aligned}$$

Since

$$\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t \geq \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t)\triangle t = \int_a^b |y(t)|^{p-2}y^T(t)U(t)y(t)\Delta t > 0,$$

we get

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}|V_1(t)|\triangle t \geq 1.$$

This completes the proof of Case I.

Case II: Assume that $1 < q < 2$. Then $p > 2$. Note that for $a \leq \tau \leq \sigma(t) \leq b$,

$$\begin{aligned}
&|e_{\ominus W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)|\\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)y(\tau)|\\
=\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{y^T(\tau)U^T(\tau)U(\tau)y(\tau)\}^{\frac{1}{2}}\\
=\ & |e_{\ominus W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{(\sqrt{U(\tau)}y(\tau))^T U(\tau)\sqrt{U(\tau)}y(\tau)\}^{\frac{1}{2}}\\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}\sqrt{U(\tau)}y(\tau)|^{p-2}\{|\sqrt{U(\tau)}y(\tau)||U(\tau)||\sqrt{U(\tau)}y(\tau)|\}^{\frac{1}{2}}\\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|\sqrt{U(\tau)}y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|\\
=\ & |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|^{p-1}\\
=\ & |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|^{\frac{2}{q}}|\sqrt{U(\tau)}y(\tau)|^{p-1-\frac{2}{q}}\\
\leq\ & |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|^{\frac{2}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}}\\
=\ & |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
|x^\sigma(t)|^q &= \Big|\int_a^{\sigma(t)} e_{\ominus W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau\Big|^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\ominus A}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)|\Delta\tau\Big)^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}} \\
&\quad \times(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}}\Delta\tau\Big)^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\ominus W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}+1} \\
&\quad \times(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|y(\tau)|^{p-1-\frac{2}{q}}\Delta\tau\Big)^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\ominus W}(\sigma(t),\tau)|^p|(\sqrt{U(\tau)})^{-1}|^{p(p-2)}|U(\tau)|^{\frac{p(p-2)}{2}+1}\triangle\tau\Big)^{\frac{q}{p}} \\
&\quad \times\Big(\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\Big).
\end{aligned}$$

That is

$$|x^\sigma(t)|^q \le \xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.8}$$

Similarly, by letting $\eta(t)$ be as in (3.2), for $a \le \sigma(t) \le \tau \le b$, we have

$$|x^\sigma(t)|^q \le \eta(\sigma(t))\int_{\sigma(t)}^{b} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.9}$$

The rest of the proof is similar to the Case I, we have

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}|V_1(t)|\triangle t \ge 1.$$

This completes the proof of Theorem 3.1

Corollary 3.2 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \ge V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then the following inequality holds:

$$\int_a^b (\xi(\sigma(t))+\eta(\sigma(t)))|V_1(t)|\triangle t \ge 4. \tag{3.10}$$

Proof Note

$$\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))} \le \frac{\xi(\sigma(t))+\eta(\sigma(t))}{4}.$$

It follows from (3.3) that

$$\int_a^b \frac{\xi(\sigma(t))+\eta(\sigma(t))}{4}|V_1(t)|\triangle t \ge 1.$$

That is

$$\int_a^b (\xi(\sigma(t))+\eta(\sigma(t)))|V_1(t)|\triangle t \ge 4.$$

This completes the proof of Corollary 3.2.

Corollary 3.3 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then the following inequality holds:

$$\int_a^b (\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}} |V_1(t)| \triangle t \geq 2. \tag{3.11}$$

Proof Note

$$\xi(\sigma(t)) + \eta(\sigma(t)) \geq 2(\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}}.$$

It follows from (3.3) that

$$\int_a^b \frac{(\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}}}{2} |V_1(t)| \triangle t \geq 1.$$

That is

$$\int_a^b (\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}} |V_1(t)| \triangle t \geq 2.$$

This completes the proof of Corollary 3.3.

Theorem 3.4 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then there exists an $c \in (a, b)$ such that

$$\int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \geq 1 \quad \text{and} \quad \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \geq 1. \tag{3.12}$$

Proof Let

$$F(t) = \int_a^t \xi(\sigma(s))|V_1(s)| \triangle s - \int_t^b \eta(\sigma(s))|V_1(s)| \triangle s.$$

Then we have $F(a) < 0$ and $F(b) > 0$. Hence we can choose an $c \in (a, b)$ such that $F(c) \leq 0$ and $F(\sigma(c)) \geq 0$, that is

$$\int_a^c \xi(\sigma(s))|V_1(s)| \triangle s \leq \int_c^b \eta(\sigma(s))|V_1(s)| \triangle s \tag{3.13}$$

and

$$\int_a^{\sigma(c)} \xi(\sigma(s))|V_1(s)| \triangle s \geq \int_{\sigma(c)}^b \eta(\sigma(s))|V_1(s)| \triangle s. \tag{3.14}$$

From (3.6) and (3.8), we have

$$|V_1(t)||x^\sigma(t)|^q \leq \xi(\sigma(t))|V_1(t)| \int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.15}$$

Note that for $a \leq \tau \leq \sigma(t) \leq \sigma(c) \leq b$. Integrating (3.15) from a to $\sigma(c)$, we obtain

$$\int_a^{\sigma(c)} |V_1(t)||x^\sigma(t)|^q \triangle t \leq \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \Big(\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau \Big) \triangle t$$

$$\leq \int_a^c \xi(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

$$+\xi(\sigma(c))|V_1(c)|(\sigma(c) - c) \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

$$= \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

Similarly, for $a \leq \sigma(c) \leq \sigma(t) \leq \tau \leq b$, we can obtain from (3.7),(3.9) and (3.14) that

$$\int_{\sigma(c)}^b |V_1(t)||x^\sigma(t)|^q \triangle t \leq \int_{\sigma(c)}^b \eta(\sigma(t))|V_1(t)| \triangle t \int_{\sigma(c)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau$$

$$\leq \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_{\sigma(c)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

These yield

$$\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t \leq \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau$$

$$= \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t$$

$$\leq \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t.$$

Since

$$\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t \geq \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t = \int_a^b |y(t)|^{p-2}y^T(t)U(t)y(t)\Delta t > 0,$$

we obtain $\int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \geq 1$.

Next, we have from (3.7) and (3.9) that

$$|x^\sigma(t)|^q|V_1(t)| \leq \eta(\sigma(t))|V_1(t)| \int_{\sigma(t)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.16}$$

Integrating (3.16) from c to b, we obtain that for $a \leq c \leq t \leq \sigma(t) \leq \tau \leq b$,

$$\int_c^b |V_1(t)||x^\sigma(t)|^q \triangle t \leq \int_c^b \eta(\sigma(t))|V_1(t)|\Big(\int_{\sigma(t)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\Big) \triangle t$$

$$\leq \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_{\sigma(c)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

Similarly, for $a \leq \tau \leq \sigma(t) \leq \sigma(c) \leq b$, we can obtain

$$\int_a^c |V_1(t)||x^\sigma(t)|^q \triangle t \leq \int_a^c \xi(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau$$

$$\leq \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

These yield

$$\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t \leq \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^b y^T(t)U(t)y(t)|y(t)|^{p-2}\Delta t$$

$$= \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t$$

$$\leq \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t.$$

Thus, we also obtain $\int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \geq 1$. This completes the proof of Theorem 3.4.

Theorem 3.5 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_\mathbb{T}$, then the following inequalities hold:

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \geq 2, \quad \text{if } q \geq 2,$$

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1} |(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \geq 2, \quad \text{if } 1 < q < 2.$$

Proof From the proof of Theorem 3.1, we have

$$\int_a^b |y(t)|^{p-2} y^T(t) U(t) y(t) \triangle t = \int_a^b |x^\sigma(t)|^{q-2} (x^\sigma(t))^T V(t) x^\sigma(t) \triangle t.$$

It follows from the first equation of (1.1) that for all $a \leq t \leq b$,

$$x(t) = \int_a^t (-W(\tau)x^\sigma(\tau) - U(\tau)|y(\tau)|^{p-2} y(\tau)) \triangle \tau,$$

$$x(t) = \int_t^b (W(\tau)x^\sigma(\tau) + U(\tau)|y(\tau)|^{p-2} y(\tau)) \triangle \tau.$$

Case I: Assume that $q \geq 2$. We have

$$\begin{aligned} |x(t)| &= \Big|\int_a^t (-W(\tau)x^\sigma(\tau) - U(\tau)|y(\tau)|^{p-2} y(\tau)) \triangle \tau\Big| \\ &\leq \int_a^t |W(\tau)x^\sigma(\tau) + U(\tau)|y(\tau)|^{p-2} y(\tau)| \triangle \tau \\ &\leq \int_a^t |W(\tau)x^\sigma(\tau)| \triangle \tau + \int_a^t |U(\tau)|y(\tau)|^{p-2} y(\tau)| \triangle \tau \\ &\leq \int_a^t |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_a^t |U(\tau)|^{1-\frac{1}{q}} (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}} |y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau. \end{aligned}$$

Similarly, we have

$$|x(t)| \leq \int_t^b |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_t^b |U(\tau)|^{1-\frac{1}{q}} (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}} |y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau.$$

Then from Lemma 2.2 and Lemma 2.6, we obtain

$$\begin{aligned} |x(t)| &\leq \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \int_a^b |U(t)|^{1-\frac{1}{q}} (y^T(t)U(t)y(t))^{\frac{1}{q}} |y(t)|^{p-1-\frac{2}{q}} \triangle t\Big] \\ &\leq \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b y^T(t)U(t)y(t)|y(t)|^{p-2} \triangle t\Big)^{\frac{1}{q}}\Big] \\ &= \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |x^\sigma(t)|^{q-2} (x^\sigma(t))^T V(t) x^\sigma(t) \triangle t\Big)^{\frac{1}{q}}\Big] \\ &\leq \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t\Big)^{\frac{1}{q}}\Big]. \end{aligned}$$

Denote $M = \max_{a \le t \le b} |x(t)| > 0$, then

$$M \le \frac{1}{2}\Big[\int_a^b |W(t)| M \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)| M^q \triangle t\Big)^{\frac{1}{q}}\Big].$$

Thus

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \ge 2.$$

Case II: Assume that $1 < q < 2$. Then $p \ge 2$ and

$$\begin{aligned} |x(t)| &\le \int_a^t |W(\tau)x^\sigma(\tau) + U(\tau)|y(\tau)|^{p-2}y(\tau)| \triangle \tau \\ &\le \int_a^t |W(\tau)x^\sigma(\tau)| \triangle \tau + \int_a^t |U(\tau)|y(\tau)|^{p-2}y(\tau)| \triangle \tau \\ &\le \int_a^t |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_a^t |(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}} \\ &\quad \times (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}} |\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}} |y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau. \end{aligned}$$

and

$$\begin{aligned} |x(t)| &\le \int_t^b |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_t^b |(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}} \\ &\quad \times (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}} |\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}} |y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau. \end{aligned}$$

Thus we obtain

$$\begin{aligned} |x(t)| &\le \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \int_a^b |(\sqrt{U(t)})^{-1}|^{p-2}|U(t)|^{\frac{1}{2}} \\ &\quad \times (y^T(t)U(t)y(t))^{\frac{1}{q}} |\sqrt{U(t)}|^{\frac{(p-1)(p-2)}{p}} |y(t)|^{p-1-\frac{2}{q}} \triangle t\Big] \\ &\le \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b (|(\sqrt{U(t)})^{-1}|^{p-2} |\sqrt{U(t)}|^{\frac{(p-1)(p-2)}{p}+1})^p \triangle t\Big)^{\frac{1}{p}} \\ &\quad \times \Big(\int_a^b ((y^T(t)U(t)y(t))^{\frac{1}{q}} |y(t)|^{p-1-\frac{2}{q}})^q \triangle t\Big)^{\frac{1}{q}}\Big] \\ &= \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1} |(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}} \\ &\quad \times \Big(\int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t) x^\sigma(t) \triangle t\Big)^{\frac{1}{q}}\Big] \\ &\le \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1} |(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}} \\ &\quad \times \Big(\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t\Big)^{\frac{1}{q}}\Big]. \end{aligned}$$

Similarly, we also have

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1} |(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \ge 2.$$

This completes the proof of Theorem 3.5.

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A new three-step iterative method for a countable family of pseudo-contractive mappings in Hilbert spaces

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Abstract: In this paper, we propose a new three-step iterative method for a countable family of pseudo-contractive mappings in a real Hilbert space. We also prove the strong convergence of the proposed iterative algorithm under appropriate conditions.

Key words: pseudo-contractive mapping; iterative method; fixed point; strong convergence

AMS subject classification (2000): 47H09

1 Introduction

In this paper, we assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, C is a nonempty closed convex subset of H and $T: C \to C$ is a self-mapping of C. $\mathcal{F}(T)$ denotes the fixed point set of the mapping T. Recall that T is called a k-strictly pseudo-contractive mapping if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall\ x, y \in C, \tag{1.1}$$

and T is called a pseudo-contractive mapping if

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \quad \forall\ x, y \in C. \tag{1.2}$$

It is obvious that $k = 0$, then the mapping T is nonexpansive, that is

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall\ x, y \in C. \tag{1.3}$$

Finding the fixed points of nonexpansive mappings is an important topic in the theory of nonexpansive mappings and has wide applications in a number of applied areas, such as the convex feasibility problem [1, 2], the split feasibility problem [3], image recovery and signal

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processing [4]. After that, as an important generalization of nonexpansive mappings, strictly pseudo-contractive mappings become one of the most interesting studied class of nonexpansive mappings. In fact, strictly pseudo-contractive mappings have more powerful application than nonexpansive mappings do such as in solving inverse problem [5].

Iterative methods for nonexpansive mappings have been extensively investigated (see *e.g.*, [6–16, 31–33] and the references contained therein). However, iterative methods for strictly pseudo-contractive mappings are far less developed than those for nonexpansive mappings and the reason is probably that the second term appearing on the right hand side of (1.1) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strictly pseudo-contractive mapping T.

The most general iterative algorithm for nonexpansive mappings studied by many authors is Mann's iteration algorithm [18] which is as following:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.4}$$

where $x_0 \in C$ is chosen arbitrarily and $\{\alpha_n\}$ is a real sequence in $(0, 1)$. Under the following additional assumptions: (i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, the sequence $\{x_n\}$ generated by (1.4) is generally referred to as Mann's iteration algorithm in the light of [18]. The Mann's iteration algorithm dose not generally converge to a fixed point of T even the fixed point exists. For example, C is a nonempty closed convex and bounded subset of a real Hilbert space, $T : C \to C$ is nonexpansive, one can only prove that the sequence generated by Mann's iteration algorithm (1.4) with the assumptions (i) and (ii) is an approximate fixed point sequence, that is, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. In [19], Reich proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a fixed point of T. To get the sequence $\{x_n\}$ to converge strongly to a fixed point of T (when such a fixed point exists), some type of compactness condition must be additionally imposed either on C (*e.g.*, C is compact) or on T (*e.g.*, T is demicompact or T is semicompact, see [20, 21]).

The first convergence result for k-strictly pseudo-contractive mappings was proposed by Browder and Petryshyn [22] in 1967. They proved that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n, \quad n \geq 0, \tag{1.5}$$

for any starting point $x_0 \in C$ and α is a constant such that $k < \alpha < 1$, then the sequence $\{x_n\}$ converges weakly to a fixed point of k-strictly pseudo-contractive mapping T. In [23], Marino and Xu extended the result of Browder and Petryshyn [22] to Mann's iteration algorithm (1.4), they proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to a fixed point of k-strictly pseudo-contractive mapping T for the conditions that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$.

However, the well known strong convergence result for pseudo-contractive mappings is Ishikawa's iteration algorithm which was proved by Ishikawa [24] in 1974 and it is more general than that of Mann's iteration algorithm (1.4) in some sense. More precisely, he got the following theorem.

Theorem 1.1 ([24]) *Let C be a convex compact subset of a Hilbert space H and let $T : C \to C$ be a Lipschitz pseudo-contractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined iteratively for each $n \geq 1$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive number that satisfy the following there conditions: (i) $0 \leq \alpha_n \leq \beta_n \leq 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. *Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

In 2001, Chidume and Mutangadura [25] gave an example to show that the Mann's iteration algorithm (1.4) failed to be convergent to a fixed point of Lipschitz pseudo-contractive mappings. In order to obtain a strong convergence result for pseudo-contractive mappings without the compactness assumption, Zhou [26] established the hybrid Ishikawa algorithm for Lipschitz pseudo-contractive mappings as following:

Theorem 1.2 ([26]) *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a Lipschitz pseudo-contraction such that $\mathcal{F}(T) \neq \phi$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ satisfying the conditions:* (i) $\alpha_n \leq \beta_n, \forall n \geq 0$; (ii) $\liminf_{n\to\infty} \alpha_n > 0$; (iii) $\limsup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$, $n \geq 0$, *where $L \geq 1$ is the Lipschitzian constant of T. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ z_n = (1-\beta_n)x_n + \beta_n T y_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1-2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x_0, \quad n \geq 0. \end{cases} \tag{1.7}$$

Then, $\{x_n\}$ converges strongly to a fixed point v of T, where $v = P_{\mathcal{F}(T)}(x_0)$.

We observe that the iterative algorithm (1.7) generates a sequence $\{x_n\}$ by projecting x_0 on to the intersection of the suitably constructed closed convex sets C_n and Q_n. Recently, Yao *et al.* [27] introduced the hybrid iterative algorithm which just involved one closed convex set for pseudo-contractive mappings in Hilbert spaces as follows:

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)y_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in N. \end{cases} \tag{1.8}$$

Theorem 1.3 ([27]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping such that $\mathcal{F}(T) \neq \phi$. Assume the sequence $\alpha_n \in [a,b]$ for some $a,b \in (0, \dfrac{1}{L+1})$. Then the sequence $\{x_n\}$ generated by (1.8) converges strongly $P_{\mathcal{F}(T)}(x_0)$.*

In [28], Tang *et al.* proposed the hybrid algorithm (1.8) to the Ishikawa's iteration algorithm (1.6) and got the following result.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0,1]$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)y_n\rangle \\ \qquad\qquad + 2\alpha_n\beta_n L\|x_n - Tx_n\| \cdot \|y_n - x_n + \alpha_n(I-T)y_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1. \end{cases} \tag{1.9}$$

Theorem 1.4 ([28]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping with $L \ge 1$ such that $\mathcal{F}(T) \neq \phi$. Assume the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ satisfying:* (i) $b \le \alpha_n < \alpha_n(L+1)(1+\beta_n L) < a < 1$, *for some $a,b \in (0,1)$;* (ii) $\lim_{n\to\infty} \beta_n = 0$. *Then the sequence $\{x_n\}$ generated by (1.9) converges strongly $P_{\mathcal{F}(T)}(x_0)$.*

Recently, Zegeye *et al.* [29] generalized Ishikawa's iteration algorithm (1.6) to a common fixed point of a finite family of Lipschitz pseudo-contractive mappings and obtained the following theorem.

Theorem 1.5 ([29]) *Let C be a nonempty, closed convex subset of a real Hilbert space H. Let $T_i : C \to C$, $i = 1, 2, \cdots, N$, be a finite family of Lipschitz pseudo-contractive mappings with Lipschitzian constants L_i, for $i = 1, 2, \cdots, N$, respectively. Assume that the interior of $\mathcal{F} := \bigcap_{i=1}^N \mathcal{F}(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T_n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \end{cases} \tag{1.10}$$

where $T_n := T_{n(mod\ N)}$ *and* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *satisfying the following conditions:* (i) $\alpha_n \leq \beta_n, \forall\ n \geq 0$; (ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$; (iii) $\sup_{n\geq 1} \beta_n \leq \beta < \dfrac{1}{\sqrt{1+L^2}+1}$ *for* $L := \max\{L_i : i = 1, 2, \cdots, N\}$. *Then,* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, \cdots, T_N\}$.

In [30], Cheng *et al.* extended the algorithm (1.10) to a countable family of pseudo-contractive mappings and gave a three-step iterative method, which is as follows:

Theorem 1.6 ([30]) *Let* C *be a nonempty, closed convex subset of a real Hilbert space* H*, let* $\{T_n\}_{n=1}^{\infty} : C \to C$ *be a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with Lipschitzian constants* L_n*, let* $L := \sup_{n\geq 1} L_n$*. Assume that the interior of* $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0 \in C$ *by the following algorithm:*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \end{cases} \tag{1.11}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ *satisfying the following conditions:* (i) $\alpha_n \leq \beta_n \leq \gamma_n, \forall\ n \geq 0$; (ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$; (iii) $\sup_{n\geq 1} \gamma_n \leq \gamma$ *with* $\gamma^3 L^4 + 2\gamma^2 L^3 + \gamma^2 L^2 + \gamma L^2 + 2\gamma < 1$*. Then,* $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$.

Remark 1.1 *The condition* (iii) *of the Theorem 1.6 is not correct, it is replaced by* $\sup_{n\geq 1} \gamma_n \leq \gamma$ *with* $\gamma^3 L^4 + 2\gamma^2 L^3 + \gamma^2 L^2 + 2\gamma L^2 + 2\gamma < 1$.

Motivated and inspired by the above works, in this paper, we propose a new three-step iterative method for a countable family of pseudo-contractive mappings in Hilbert spaces and prove its strong convergence theorem under appropriate conditions.

2 Preliminaries

In this section, we recall some definitions and useful results which will be used in the next section.

Definition 2.1 *Let* C *be a subset of a real Hilbert space* H.

(1) *A mapping* $T : C \to H$ *is said to be L-Lipschitz if there exists* $L > 0$ *such that*

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall\ x,\ y \in C.$$

When $L = 1$, T *is nonexpansive. If* $L < 1$, T *is called a contraction. It is easy to see that every contractive mapping is nonexpansive and every nonexpansive mapping is Lipschitz.*

(2) *A countable family of mappings* $\{T_n\}_{n=1}^{\infty}: C \to H$ *is said to be uniformly Lipschitz with Lipschitzian constants* $L_n > 0$, $n \geq 1$, *if there exists* $0 < L := \sup_{n\geq 1} L_n$ *such that*

$$\|T_n x - T_n y\| \leq L\|x - y\|, \quad \forall\ x,\ y \in C,\ n \geq 1.$$

(3) *A countable family of mappings* $\{T_n\}_{n=1}^{\infty} : C \to H$ *is said to be uniformly closed if* $x_n \to x^*$ *and* $\|x_n - T_n x_x\| \to 0$ *imply* $x^* \in \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$.

Definition 2.2 *A mapping* T *with domain* $\mathcal{D}(T)$ *and range* $\mathcal{R}(T)$ *in a real Hilbert space* H *is said to be monotone if the inequality*

$$\|x - y\| \leq \|x - y + s(Tx - Ty)\|$$

holds for every $x,\ y \in \mathcal{D}(T)$ *and for all* $s > 0$.

We observe that

$$\begin{aligned} T \text{ is monotone } &\Leftrightarrow \langle Tx - Ty, x - y\rangle \geq 0 \\ &\Leftrightarrow \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2 + \|Tx - Ty\|^2 \\ &\Leftrightarrow \|Ax - Ay\|^2 \leq \|x-y\|^2 + \|(I-A)x - (I-A)y\|^2,\ A := I - T \\ &\Leftrightarrow A \text{ is pseudo-contractive.} \end{aligned}$$

Furthermore, a zero of T is a fixed point of A, that is,

$$x \in \mathcal{N}(T) := \{x \in \mathcal{D}(T) : Tx = 0\} \Leftrightarrow x \in \mathcal{F}(A) := \{x \in \mathcal{D}(A) : Ax = x\}.$$

Lemma 2.1 *Let* H *be a real Hilbert space. Then for* $\alpha \in [0,1]$ *the following equality*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$$

holds for all $x, y \in H$.

Lemma 2.2 *If the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ *satisfying the following conditions:*

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

where $\alpha = \liminf\limits_{n\to\infty} \alpha_n$, $\beta = \liminf\limits_{n\to\infty} \beta_n$, $\gamma \geq \sup_{n\geq 1} \gamma_n$, *and* $L > 0$ *is a constant. Then, we have*

$$\alpha > 0, \beta > 0 \text{ and } (1-\alpha_n)\gamma_n + \alpha_n\beta_n(\gamma_n^2 L^2 + 2\gamma_n - 1) < 0.$$

Proof. On one hand, it is obvious that $\alpha > 0$, $\beta > 0$ and $\gamma^2 L^2 + 2\gamma - 1 < 0$ because of $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$. And we get that $(1-\alpha_n)\gamma \leq (1-\alpha)\gamma$ and $\alpha_n\beta_n(\gamma^2 L^2 + 2\gamma - 1) \leq \alpha\beta(\gamma^2 L^2 + 2\gamma - 1)$. Then

$$(1-\alpha_n)\gamma + \alpha_n\beta_n(\gamma^2 L^2 + 2\gamma - 1) \leq (1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0.$$

On the other hand, it is easy to know that $(1-\alpha_n)\gamma_n \leq (1-\alpha_n)\gamma$ and $\alpha_n\beta_n(\gamma_n^2+2\gamma_n-1) \leq \alpha_n\beta_n(\gamma^2L^2+2\gamma-1)$. We can obtain

$$(1-\alpha_n)\gamma_n + \alpha_n\beta_n(\gamma_n^2+2\gamma_n-1) \leq (1-\alpha_n)\gamma + \alpha_n\beta_n(\gamma^2L^2+2\gamma-1) < 0.$$

Hence $(1-\alpha_n)\gamma_n + \alpha_n\beta_n(\gamma_n^2L^2+2\gamma_n-1) < 0$. □

3 The main result

Theorem 3.1 *Let C be a nonempty, closed convex subset of a real Hilbert space H, let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with Lipschitzian constants L_n, let $L := \sup_{n\geq 1} L_n$. Assume that the interior of $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2L^2+2\gamma-1) < 0$,

where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \geq \sup_{n\geq 1}\gamma_n$. Then, $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.

Proof. Take $p \in \mathcal{F}$ arbitrarily. By (3.1) and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|(1-\alpha_n)z_n + \alpha_n y_n - p\|^2 \\ &= \|(1-\alpha_n)(z_n-p) + \alpha_n(y_n-p)\|^2 \\ &= (1-\alpha_n)\|z_n-p\|^2 + \alpha_n\|y_n-p\|^2 - \alpha_n(1-\alpha_n)\|z_n-y_n\|^2 \\ &\leq (1-\alpha_n)\|z_n-p\|^2 + \alpha_n\|y_n-p\|^2, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|z_n-p\|^2 &= \|(1-\gamma_n)x_n + \gamma_n T_n x_n - p\|^2 \\ &= \|(1-\gamma_n)(x_n-p) + \gamma_n(T_n x_n - p)\|^2 \\ &= (1-\gamma_n)\|x_n-p\|^2 + \gamma_n\|T_n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &\leq (1-\gamma_n)\|x_n-p\|^2 + \gamma_n(\|x_n-p\|^2 + \|x_n - T_n x_n\|^2) \\ &\quad - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &= \|x_n-p\|^2 + \gamma_n^2\|x_n - T_n x_n\|^2, \end{aligned} \tag{3.3}$$

where the inequality is based on that $\{T_n\}_{n=1}^{\infty}$ is a countable family of pseudo-contractive mappings. Similarly, we can get

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n T_n z_n - p\|^2 \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(T_n z_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T_n z_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T_n z_n\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n(\|z_n - p\|^2 + \|z_n - T_n z_n\|^2) \\
&\quad - \beta_n(1-\beta_n)\|x_n - T_n z_n\|^2. \qquad (3.4)
\end{aligned}$$

In addition, using (3.1), we have that

$$\begin{aligned}
\|z_n - T_n z_n\|^2 &= \|(1-\gamma_n)x_n + \gamma_n T_n x_n - T_n z_n\|^2 \\
&= \|(1-\gamma_n)(x_n - T_n z_n) + \gamma_n(T_n x_n - T_n z_n)\|^2 \\
&= (1-\gamma_n)\|x_n - T_n z_n\|^2 + \gamma_n\|T_n x_n - T_n z_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\
&\leq (1-\gamma_n)\|x_n - T_n z_n\|^2 + \gamma_n L^2\|x_n - z_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\
&= (1-\gamma_n)\|x_n - T_n z_n\|^2 + \gamma_n L^2\|\gamma_n(x_n - T_n x_n)\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\
&= (1-\gamma_n)\|x_n - T_n z_n\|^2 + \gamma_n(\gamma_n^2 L^2 + \gamma_n - 1)\|x_n - T_n x_n\|^2, \qquad (3.5)
\end{aligned}$$

where the inequality is based on that $\{T_n\}_{n=1}^{\infty}$ is a countable family of uniformly Lipschitz mappings. Substituting (3.3) and (3.5) into (3.4), we obtain that

$$\begin{aligned}
\|y_n - p\|^2 &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\Big(\|x_n - p\|^2 + \gamma_n^2\|x_n - T_n x_n\|^2\Big) \\
&\quad + \beta_n\Big((1-\gamma_n)\|x_n - T_n z_n\|^2 + \gamma_n(\gamma_n^2 L^2 + \gamma_n - 1)\|x_n - T_n x_n\|^2\Big) \\
&\quad - \beta_n(1-\beta_n)\|x_n - T_n z_n\|^2 \\
&= \|x_n - p\|^2 + \beta_n\gamma_n(\gamma_n^2 L^2 + 2\gamma_n - 1)\|x_n - T_n x_n\|^2 + \beta_n(\beta_n - \gamma_n)\|x_n - T_n z_n\|^2 \\
&\leq \|x_n - p\|^2 + \beta_n\gamma_n(\gamma_n^2 L^2 + 2\gamma_n - 1)\|x_n - T_n x_n\|^2, \qquad (3.6)
\end{aligned}$$

where the last inequality is based on the condition (i). Therefore, substituting (3.3) and (3.6) into (3.2), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\Big(\|x_n - p\|^2 + \gamma_n^2\|x_n - T_n x_n\|^2\Big) \\
&\quad + \alpha_n\Big(\|x_n - p\|^2 + \beta_n\gamma_n(\gamma_n^2 L^2 + 2\gamma_n - 1)\|x_n - T_n x_n\|^2\Big) \\
&= \|x_n - p\|^2 + \Big((1-\alpha_n)\gamma_n^2 + \alpha_n\beta_n\gamma_n(\gamma_n^2 L^2 + 2\gamma_n - 1)\Big)\|x_n - T_n x_n\|^2. \qquad (3.7)
\end{aligned}$$

According to the conditions and Lemma 2.2, inequality (3.7) implies that

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2. \qquad (3.8)$$

It is obvious that $\lim_{n\to\infty}\|x_n - p\|$ exists, then $\{\|x_n - p\|\}$ is bounded. This implies that $\{x_n\}$, $\{T_n x_n\}$, $\{z_n\}$, $\{T_n z_n\}$ and $\{y_n\}$ are also bounded.

Furthermore, we have that

$$\|x_n - p\|^2 = \|x_n - x_{n+1}\|^2 + \|x_{n+1} - p\|^2 + 2\langle x_{n+1} - p, x_n - x_{n+1}\rangle.$$

This implies

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\|x_n - x_{n+1}\|^2 = \frac{1}{2}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2). \tag{3.9}$$

Moreover, since the interior of $\mathcal{F}$ is nonempty, then there exists $p^* \in \mathcal{F}$ and $r > 0$ such that $p^* + rh \in \mathcal{F}$ whenever $\|h\| \le 1$. Thus, from (3.8), we have

$$\begin{aligned} 0 &\le \langle x_{n+1} - (p^* + rh), x_n - x_{n+1}\rangle + \frac{1}{2}\|x_n - x_{n+1}\|^2 \\ &= \frac{1}{2}(\|x_n - (p^* + rh)\|^2 - \|x_{n+1} - (p^* + rh)\|^2). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we obtain that

$$\begin{aligned} r\langle h, x_n - x_{n+1}\rangle &\le \langle x_{n+1} - p^*, x_n - x_{n+1}\rangle + \frac{1}{2}\|x_n - x_{n+1}\|^2 \\ &= \frac{1}{2}(\|x_n - p^*\|^2 - \|x_{n+1} - p^*\|^2). \end{aligned} \tag{3.11}$$

Since h with $\|h\| \le 1$ is arbitrary, we can take $h = \dfrac{x_n - x_{n+1}}{\|x_n - x_{n+1}\|}$ with $\|h\| = 1$, then

$$\|x_n - x_{n+1}\| \le \frac{1}{2r}(\|x_n - p^*\|^2 - \|x_{n+1} - p^*\|^2).$$

So, for $n > m$, we can get

$$\begin{aligned} \|x_m - x_n\| &= \|(x_m - x_{m+1}) + (x_{m+1} - x_{m+2}) + \cdots + (x_{n-1} - x_n)\| \\ &\le \sum_{i=m}^{n-1}\|x_i - x_{i+1}\| \\ &\le \sum_{i=m}^{n-1}\frac{1}{2r}(\|x_i - p^*\|^2 - \|x_{i+1} - p^*\|^2) \\ &= \frac{1}{2r}(\|x_m - p^*\|^2 - \|x_n - p^*\|^2). \end{aligned}$$

From (3.8), we know that $\{\|x_n - p^*\|^2\}$ converges. Therefore, $\{x_n\}$ is a Cauchy sequence. Since C is closed subset of Hilbert space H, then there exists $x^* \in C$ such that

$$x_n \to x^* \in C. \tag{3.12}$$

Furthermore, from the conditions and Lemma 2.2, we have

$$0 \;<\; \beta\Big((\alpha - 1)\gamma + \alpha\beta(1 - 2\gamma - \gamma^2 L^2)\Big)$$

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$$\begin{aligned} &\leq \gamma_n\Big((\alpha_n-1)\gamma_n+\alpha_n\beta_n(1-2\gamma_n-\gamma_n^2L^2)\Big) \\ &= (\alpha_n-1)\gamma_n^2+\alpha_n\beta_n\gamma_n(1-2\gamma_n-\gamma_n^2L^2). \end{aligned} \tag{3.13}$$

Then, by (3.7) and (3.13), we conclude that

$$\begin{aligned} &\Big((\alpha-1)\gamma\beta+\alpha\beta^2(1-2\gamma-\gamma^2L^2)\Big)\sum_{n=1}^{\infty}\|x_n-T_nx_n\|^2 \\ \leq\ &\sum_{n=1}^{\infty}\Big((\alpha_n-1)\gamma_n^2+\alpha_n\beta_n\gamma_n(1-2\gamma_n-\gamma_n^2L^2)\Big)\|x_n-T_nx_n\|^2 \\ \leq\ &\sum_{n=1}^{\infty}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)<\infty, \end{aligned}$$

from which it follows that

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{3.14}$$

Since $\{T_n\}_{n=1}^{\infty}$ are uniformly closed mappings, then from (3.12) and (3.14), we can obtain

$$x^*\in\bigcap_{n=1}^{\infty}\mathcal{F}(T_n)=\mathcal{F}.$$

The proof is complete. □

Remark 3.1 *We now give an example of a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with the interior of the common fixed points nonempty. This example comes from* [30]. *Suppose that $H := R$ and $C := [-1,1]\in H$. Let $\{T_n\}_{n=1}^{\infty}: C\to C$ be defined by*

$$T_nx:=\begin{cases} x, & x\in[-1,0),\\ (\frac{1}{2^n}+\frac{1}{2})x, & x\in[0,1]. \end{cases}$$

Then $\mathcal{F}:=\bigcap_{n=1}^{\infty}\mathcal{F}(T_n)=[-1,0]$, and hence the interior of the common fixed points is nonempty. Moreover, it is easy to show that $\{T_n\}_{n=1}^{\infty}$ is a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with Lipschitz constant $L:=\sup_{n\geq1}L_n=2$.

For this example, we can let $\alpha_n=\frac{3}{4}+\frac{1}{n+4}$, $\beta_n=\frac{1}{10}+\frac{1}{n+40}$ and $\gamma_n=\frac{3}{20}-\frac{1}{n+40}$ for $n\geq1$. Then $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1)$ and $\beta_n\leq\gamma_n,\forall\ n\geq1$. Furthermore, $\alpha=\liminf_{n\to\infty}\alpha_n=\frac{3}{4}$, $\beta=\liminf_{n\to\infty}\beta_n=\frac{1}{10}$, $\sup_{n\geq1}\gamma_n\leq\frac{3}{20}$, and

$$(1-\alpha)\gamma+\alpha\beta(\gamma^2L^2+2\gamma-1)=(1-\frac{3}{4})\times\frac{3}{20}+\frac{3}{4}\times\frac{1}{10}\times\Big((\frac{3}{20})^2\times2^2+2\times\frac{3}{20}-1\Big)=-\frac{33}{4000}<0.$$

It satisfies all conditions in Theorem 3.1. Hence, from Theorem 3.1, we can obtain the sequence $\{x_n\}$ generated by (3.1) and staring with an arbitrary $x_1\in C$ will converges strongly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

4 Applications

If in Theorem 3.1, we consider a finite family of Lipschitz pseudo-contractive mappings, then we have the following result.

Theorem 4.1 *Let C be a nonempty, closed convex subset of a real Hilbert space H, let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of uniformly closed and Lipschitz pseudo-contractive mappings with Lipschitzian constants L_i, for $i = 1, 2, \cdots, N$, respectively. Assume that the interior of $\mathcal{F} := \bigcap_{i=1}^N \mathcal{F}(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n, \end{cases} \tag{4.1}$$

where $T_n := T_{n(mod\ N)}$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:

(i) $\beta_n \le \gamma_n, \forall\ n \ge 1$,

(ii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \ge \sup_{n\ge1} \gamma_n$, for $L := \max\{L_i : i = 1, 2, \cdots, N\}$. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$.

If in Theorem 3.1, we consider a single Lipschitz pseudo-contractive mapping, then we may add a condition that is $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Theorem 4.2 *Let C be a nonempty, closed convex subset of a real Hilbert space H, let $T : C \to C$ be a Lipschitz pseudo-contractive mapping with Lipschitzian constant L. Assume that the interior of $\mathcal{F}(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:

(i) $\beta_n \le \gamma_n, \forall\ n \ge 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(iii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \ge \sup_{n\ge1} \gamma_n$. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Following the method of the proof of Theorem 3.1, we also obtain that

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + \Big((1-\alpha_n)\gamma_n^2 + \alpha_n\beta_n\gamma_n(\gamma_n^2 L^2 + 2\gamma_n - 1)\Big)\|x_n - Tx_n\|^2,$$

and $x_n \to x^* \in C$. Now, from Lemma 2.2, we get

$$\begin{aligned}
& \Big((\alpha-1)\gamma+\alpha\beta(1-2\gamma-\gamma^2L^2)\Big)\sum_{n=1}^{\infty}\gamma_n\|x_n-Tx_n\|^2 \\
\leq & \sum_{n=1}^{\infty}\gamma_n\Big((\alpha_n-1)\gamma_n+\alpha_n\beta_n(1-2\gamma_n-\gamma_n^2L^2)\Big)\|x_n-Tx_n\|^2 \\
= & \sum_{n=1}^{\infty}\Big((\alpha_n-1)\gamma_n^2+\alpha_n\beta_n\gamma_n(1-2\gamma_n-\gamma_n^2L^2)\Big)\|x_n-Tx_n\|^2 \\
\leq & \sum_{n=1}^{\infty}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)<\infty,
\end{aligned}$$

from which it follows that

$$\liminf_{n\to\infty}\|x_n-Tx_n\|=0,$$

and hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\|x_{n_k}-Tx_{n_k}\|=0.$$

Thus, $x_{n_k}\to x^*$ and the continuity of T imply that $x^*=Tx^*$ and hence $x^*\in\mathcal{F}(T)$. □

Now, we prove a convergence theorem for a countable family of monotone mappings.

Theorem 4.3 *Let H be a real Hilbert space, let $\{T_n\}_{n=1}^{\infty}: H\to H$ be a countable family of uniformly Lipschitz monotone mappings with Lipschitzian constants L_n, let $L:=\sup_{n\geq 1}L_n$. And if $x_n\to x^*$ and $\|T_nx_n\|\to 0$, then $x^*\in\bigcap_{n=1}^{\infty}\mathcal{N}(T_n)$. Assume that the interior of $\mathcal{N}:=\bigcap_{n=1}^{\infty}\mathcal{N}(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1\in C$ by*

$$\begin{cases} z_n=x_n-\gamma_nT_nx_n,\\ y_n=x_n-\beta_n(x_n-z_n)-\beta_nT_nz_n,\\ x_{n+1}=(1-\alpha_n)z_n+\alpha_ny_n,\end{cases} \tag{4.3}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1)$ satisfying the following conditions:

(i) $\beta_n\leq\gamma_n,\forall\ n\geq 1$,

(ii) $(1-\alpha)\gamma+\alpha\beta(\gamma^2L^2+2\gamma-1)<0$,

where $\alpha=\liminf_{n\to\infty}\alpha_n$, $\beta=\liminf_{n\to\infty}\beta_n$ and $\gamma\geq\sup_{n\geq1}\gamma_n$. Then, $\{x_n\}$ converges strongly to $x^*\in\mathcal{N}$.

Proof. Since T_n is monotone if and only if $A_n:=I-T_n$ is pseudo-contractive and $\bigcap_{n=1}^{\infty}\mathcal{F}(A_n)=\bigcap_{n=1}^{\infty}\mathcal{N}(T_n)\neq\emptyset$, then the conclusion follows from Theorem 3.1. □

If in Theorem 4.3, we consider a finite family of monotone mappings and a single monotone mapping, respectively, then we get the following corollaries.

Corollary 4.1 *Let H be a real Hilbert space, let $\{T_i\}_{i=1}^N : H \to H$ be a finite family of Lipschitz monotone mappings with Lipschitzian constants L_i, for $i = 1, 2, \cdots, N$, respectively. And if $x_n \to x^*$ and $\|T_n x_n\| \to 0$, then $x^* \in \bigcap_{i=1}^N \mathcal{N}(T_i)$. Assume that the interior of $\mathcal{N} := \bigcap_{i=1}^N \mathcal{N}(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n T_n x_n, \\ y_n = x_n - \beta_n(x_n - z_n) - \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n) z_n + \alpha_n y_n, \end{cases} \tag{4.4}$$

where $T_n := T_{n(mod\ N)}$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \geq \sup_{n\geq 1} \gamma_n$. for $L := \max\{L_i : i = 1, 2, \cdots, N\}$. Then, $\{x_n\}$ converges strongly to a common zero point of $\{T_1, T_2, \cdots, T_N\}$.

Corollary 4.2 *Let H be a real Hilbert space, let $T : H \to H$ be a Lipschitz monotone mapping with Lipschitzian constant L. Assume that the interior of $\mathcal{N}(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n T x_n, \\ y_n = x_n - \beta_n(x_n - z_n) - \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n) z_n + \alpha_n y_n, \end{cases} \tag{4.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $\sum_{n=1}^\infty \gamma_n = \infty$,

(iii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \geq \sup_{n\geq 1} \gamma_n$. Then, $\{x_n\}$ converges strongly to a zero point of T.

Competing interests

The authors declare that they have no competing interests regarding the publication of this article.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Harmonic analysis in the product of commutative hypercomplex systems

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Abstract

The main aim of this paper is to give integral representations for strongly negative definite functions defined on the product hypercomplex systems. Harmonic properties for strongly negative definite functions are investigated. We construct a Lèvy measure on the product hypercomplex systems, then we study the conditions that guarantee the existence of some integrations having an integrand parts as a function of the constructed kernel. Finally, we give a Lèvy - Khinchin type formula for strongly negative definite functions defined on the product hypercomplex systems.

Keywords. Lèvy – Khinchin; Hypercomplex; Negative definite.

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1. Introduction.

The integral representation of negative definite functions is defined as Lèvy-Khinchin formula. This was established by Lèvy-Khinchin in1930's for $G = \mathbb{R}$. Many author's paid attention to generalize this result in different spaces. It had been extended by Hunt [4] to Lie groups, Parthasarathy et al [8] to locally compact commutative groups, Berg et al [3] to comutative semigroups with identical involution and by Lasser [6] for commutative hypergroups. The main aim of this paper is devoted to find the integral representations for strongly negative definite functions defined on the product dual hypercomplex system. Let Q be a commutative separable locally compact metric space of points $p, q, r, \dots; B(Q)$ is the σ –algebra of Borel subsets on Q and $B_0(Q)$ is the subring of $B(Q)$, which consists of sets with compact closure. We denote by $C(Q)$ the space of continuous functions on Q; $C_b(Q), C_\infty(Q)$ and $C_0(Q)$ consists of bounded, tending to zero at infinity and compactly supported functions from $C(Q)$, respectivly. For a fixed $r \in Q,\ B \in B(Q)$, we will denote by $c(A, B; r)$ a commutative Borel structure measure in $A \in B(Q)$. The hypercomplex system $L_1(Q, dm)$ is the Banach algebra of functions on Q with respect to the multiplicative measure m and convolution " $*$ " defined for any $\phi * \psi \in L_1(Q, dm)$ by:

$$(\phi * \psi)(r) = \int_Q \phi(p)dp \int_Q \psi(q)dq\, c\big(E_p, E_q; r\big)$$

$$= \int_Q \int_Q \phi(p)\psi(q)\, c(p,q;r)\, dm(p)dm(q)$$

$$= \int_Q \int_Q \phi(p)\psi(q)\ dm_r(p,q)$$

The space $C_\infty(Q)$ is a Banach space with norm

$$||.||_\infty = \sup_{r\in Q}|(.)(r)|$$

We will denote by $\mathcal{M}(Q)$, the space of Radon measure on Q, i.e. the space of continuous linear functionals defined on $C_0(Q)$. Let $\mathcal{M}_b(Q) = (\,C_\infty(Q))'$denote the space of bounded Radon measures with norm

$$||\mu||_\infty = \sup\{|\mu(f)|; f \in C_\infty, |f| \le 1\}$$

The topology of simple convergence on functions from in the space of Radon measures, is called vague topology.

2. Strongly Negative Definite Functions.

A hypercomplex system $L_1(Q, dm)$ may or may not have a unity. In this paper we will concern our efforts on hypercomplex system with unity. A normal hypercomplex system contain a basis unity if there exists $e \in Q$ such that $e^* = e$ and

$$c(A,B;e) = m(A^* \cap B), \quad A,B \in B(Q).$$

A nonzero measurable and bounded almost everywhere function $Q \ni r \to \chi(r) \in \mathbb{C}$ is said to be a character of the hypercomplex system $L_1(Q, dm)$if for all $A,B \in B_0(Q)$ we have

$$\int_Q c(A,B;r)\,\chi(r)dm(r) = \chi(A)\chi(B)$$

and

$$\int_C \chi(r)\,dm(r) = \chi(C), \ \ C \in B_0(Q).$$

We will denote by X_h the set of all bounded Hermitian characters, i.e.

$$X_h := \{\chi \in C_b(Q); \int_Q c(A,B;r)\,\chi(r)dm(r) = \chi(A)\chi(B),\ \overline{\chi(r)} = \chi(r^*)\,\}$$

A continuous bounded function $\psi: Q \to \mathbb{C}$ is called negative definite if for any $r_1, r_2, \dots, r_n \in Q;\ c_1, c_2, \dots, c_n \in \mathbb{C}$ and $n \in \mathbb{N}$ we have:

$$\sum_{i,j=1}^{n}[\psi(r_i) + \overline{\psi(r_j)} - (R_{r^*_j}\psi)(r_i)]\,c_i\overline{c_j} \geq 0,$$

and a continuous bounded function $\varphi: Q \to \mathbb{C}$ is called positive definite if for any $r_1, r_2, \dots, r_n \in Q;\ c_1, c_2, \dots, c_n \in \mathbb{C}$ and $n \in \mathbb{N}$ we have:

$$\sum_{i,j=1}^{n}(R_{r^*_j}\varphi)(r_i)\,c_i\overline{c_j} \geq 0,$$

where $R_r\ (r \in Q)$, denote the generalized translation operators on$L_1(Q, dm)$.

As pointed out of [1], every positive definite function $\varphi \in P(Q)$ admits a unique representation in the integral form

(2.1) $$\varphi(r) = \hat{\mu}(\chi) = \int_{X_h}\chi(r)d\,\mu(\chi),\ \ \chi \in X_h,$$

where μ is a finite nonnegative regular measure on the space X_h. Conversely, each function have the integral form (1.1) belongs to the set of all positive definite function $P(Q)$.

Let Q_1 and Q_2 be two commutative separable locally compact metric spaces, with identities e_1 and e_2 respectively, and suppose A be a non empty subset of $L_1(Q_1) \times L_1(Q_2)$, then the strongly positive definite function will be defined as follows:

Definition 2.1. A locally bounded continuous measurable function $\Phi \in A$ is called strongly positive definite, if there exists two positive definite functions $\varphi_1 \in P(Q_1)$ and $\varphi_2 \in P(Q_2)$ and a Radon measure $\mu \in \mathcal{M}_+(Q_1 \times Q_2)$, such that

(2.2) $$\hat{\mu}(\chi,\tau) = \begin{cases} \varphi_1(\chi)+\varphi_2(\tau), & (\chi,\tau) \in A \\ 0, & (\chi,\tau) \notin A\,. \end{cases}$$

A locally bounded continuous measurable function $\Psi \in A$ is called strongly negative definite, if $\Psi(e_1, e_2) \geq 0$and $\exp(-t\Psi)$ is strongly positive definite in A for each $t > 0$.

Clearly each strongly positive (negative) definite function is positive (negative) definite but the converse implication does not hold. Negative definiteness is an analogue of one half of Schoenberg's duality result, It is not known for which hypercomplex system, negative definiteness implies strong negative definiteness. The following Lemma is in fact, an adaption of whatever done for hypergroups [7], we will not repeat the proof, wherever the proof for hypergroups can be applied to the hypercomplex with necessary modification.

Lemma 2.2. The sum and the point-wise limit of strongly negative definite functions on hypercomplex are also strongly negative definite.

Theorem 2.3. A function $\Psi: Q \to \mathbb{C}$ is strongly negative definite if and only if the following conditions are satisfied:

(i) $\Psi(e_1, e_2) \geq 0,\ \Psi$ is continuous bounded function ;

(ii) $\overline{\Psi(\boldsymbol{r})} = \Psi(\boldsymbol{r}^*)$ for each $\boldsymbol{r} \in Q_1 \times Q_2$;

(iii) if for any $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ and $\boldsymbol{r}_i = (r_1^i, r_1^i) \in Q_1 \times Q_2$, we have

$$\sum_{i,j=1}^n (R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)\, c_i \overline{c_j} \leq 0.$$

Proof. Suppose that the function Ψ is strongly negative definite. From the above definition of strongly negative definite functions, it is clear that Ψ satisfies (i) and (ii). Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$. Since, every strongly negative definite function is negative definite, so

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^n [\Psi(\boldsymbol{r}_i) + \overline{\Psi(\boldsymbol{r}_j)} - (R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \\ &= \overline{\left(\sum_{j=1}^n c_j\right)} \sum_{i=1}^n [\Psi(\boldsymbol{r}_i)]\, c_i + \left(\sum_{i=1}^n c_i\right) \overline{\sum_{j=1}^n [\Psi(\boldsymbol{r}_j)]} \\ &\quad - \sum_{i,j=1}^n [(R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \\ &= -\sum_{i,j=1}^n [(R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \end{aligned}$$

Conversely, suppose that Ψ satisfies the above conditions. Let $e, \boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$. From (iii) we have

$$\begin{aligned} 0 &\geq \sum_{i,j=0}^n [(R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \\ &= \sum_{i,j=1}^n [(R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} + \overline{c_0} \sum_{i=1}^n [\Psi(\boldsymbol{r}_i)]\, c_i + c_0 \overline{\sum_{j=1}^n [\Psi(\boldsymbol{r}_j)]\, c_j} + \Psi(\boldsymbol{e})|c_0|^2 \\ &= \sum_{i,j=1}^n [\Psi(\boldsymbol{r}_i) + \overline{\Psi(\boldsymbol{r}_j)} - (R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} + \Psi(\boldsymbol{e})|c_0|^2 \end{aligned}$$

This implies

$$\sum_{i,j=1}^n [\Psi(\boldsymbol{r}_i) + \overline{\Psi(\boldsymbol{r}_j)} - (R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \geq \Psi(\boldsymbol{e})|c_0|^2 \geq 0$$

Corollary 2.4. For any functions Φ, Ψ on the product $Q_1 \times Q_2$ we have:

(i) If Ψ belongs to the set of strongly negative definite function on $Q_1 \times Q_2$, then the function $\boldsymbol{r} \to \Psi(\boldsymbol{r}) - \Psi(e_1, e_2)$ is also strongly negative definite function.

(ii) If Φ belongs to the set of strongly positive definite function on $Q_1 \times Q_2$, then the function $\boldsymbol{r} \to \Phi(\boldsymbol{r}) - \Phi(e_1, e_2)$ is also strongly positive definite function.

Proof. Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$. Then we have

$$\sum_{i,j=0}^{n} [R_{\boldsymbol{r}^*_j} (\Psi(\boldsymbol{r}_i) - \Psi(e_1, e_2))]\, c_i \overline{c_j} = \sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i) c_i \overline{c_j} - \Psi(e_1, e_2) \,|\sum_{i=1}^{n} c_i\,|^2$$

$$= \sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*_j} \Psi)(\boldsymbol{r}_i) c_i \overline{c_j} \leq 0$$

This proves the strongly negative definiteness of $\Psi(\boldsymbol{r}) - \Psi(e_1, e_2)$. Similarly, let $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$. Then we find

$$\sum_{i,j=0}^{n} [R_{\boldsymbol{r}^*_j} (\Phi(e_1, e_2) - \Phi(\boldsymbol{r}_i))]\, c_i \overline{c_j} = -\sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*_j} \Phi)(\boldsymbol{r}_i) c_i \overline{c_j} - \Phi(e_1, e_2) \,|\sum_{i=1}^{n} c_i\,|^2$$

$$= -\sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*_j} \Phi)(\boldsymbol{r}_i) c_i \overline{c_j} \leq 0$$

Because Φ belongs to the set of strongly positive definite functions, hence (ii).

Theorem 2.5. For every strongly negative definite function Ψ on the product $Q_1 \times Q_2$ with $\Psi(e_1, e_2) \geq 0$, the function $\frac{1}{\Psi}$ is strongly positive definite function on the product $Q_1 \times Q_2$.

Proof. Suppose Ψ strongly negative definite function on the product $Q_1 \times Q_2$, so $\exp(-t\Psi)$ is strongly positive definite on the product $Q_1 \times Q_2$. This implies

$$|\exp(-t\Psi)| \leq |\exp(-t\Psi(e_1, e_2))| \quad \text{for all } t > 0.$$

It follows, for all $(\chi, \tau) \in \widehat{Q_1 \times Q_2}$ we have

$$\frac{1}{\Psi(\chi, \tau)} = \int_0^\infty \exp\big(-t\Psi(\chi, \tau)\big)\, dt = \int_0^\infty \widehat{\mu_t}(\chi, \tau) dt$$

Where μ_t is the corresponding measure for $\exp(-t\Psi)$. Moreover, applying Lèvy continuity Theorem, there exists a measure $\upsilon \in \mathcal{M}_+(Q_1 \times Q_2)$such that

$$\upsilon(\chi,\tau) := \hat{\upsilon}(\chi,\tau) = \int_0^\infty \widehat{\mu_t}(\chi,\tau)dt$$

and

$$\upsilon(e_1,e_2) = \frac{1}{\Psi(e_1,e_2)} < \infty$$

Consequently, $\upsilon \in \mathcal{M}_+^b(Q_1 \times Q_2)$. This implies the required to prove.

3. Construction of Lèvy measure.

Let $L_1(Q_1 \times Q_2)$ denote a commutative normal hypercomplex system with the product basis $Q_1 \times Q_2$ and basis unity $\boldsymbol{e} = (e_1, e_2)$. A family of bounded Radon measures $(\mu_t)_{t>0}$ will be called a convolution semigroup on $Q_1 \times Q_2$ if it satisfies the following items:

(i) $\mu_t(Q_1 \times Q_2) \leq 1$, for each $t > 0$;
(ii) $\mu_{t_1} * \mu_{t_2} = \mu_{t_1+t_2}$ for each $t_1, t_2 > 0$;
(iii) $\lim_{t\to 0} \mu_t = \in_{\boldsymbol{e}}$, with respect to the vague topology on $\mu \in \mathcal{M}^b(Q_1 \times Q_2)$.

Theorem 3.1. For any strongly negative definite function Ψ on $Q_1 \times Q_2$, there exists a unique convolution semigroup on $Q_1 \times Q_2$ such that Ψ is associated to $(\mu_t)_{t>0}$.

Proof. Firstly, we will prove that, for $(\chi,\tau) \in \widehat{Q_1 \times Q_2}$, the function $t \to \hat{\mu}_t(\chi,\tau)$ is continuous. As pointed out of Ursohn's lemma [9], there exists $f \in C_c(Q_1 \times Q_2)$ that satisfies $f(\boldsymbol{e}) = 1$ *and* $0 \leq f < 1$. Applying the above conditions for the convolution semigroup on $Q_1 \times Q_2$, we have:

$$1 = f(\boldsymbol{e}) = \lim_{t\to 0} < \mu_t, f > \leq \liminf_{t\to 0} \mu_t(Q_1 \times Q_2) \leq \limsup_{t\to 0} \mu_t(Q_1 \times Q_2) \leq 1$$

and this shows that

$$\lim_{t\to 0} \mu_t = \in_{\boldsymbol{e}} \qquad \text{(in the Bernolli topology).}$$

As pointed out of [2], for each $t_1, t_2 > 0$, we have

$$|\hat{\mu}_t(\chi,\tau) - \hat{\mu}_{t_0}(\chi,\tau)| \leq |\hat{\mu}_{|t-t_0|}(\chi,\tau) - 1|$$

the right hand side tends to zero uniformally on compact subset of $\widehat{Q_1 \times Q_2}$, so

$$\lim_{t\to 0} \mu_t = \mu_{t_0} \qquad \text{(in the Bernolli topology).}$$

Secondly, from the definition of strongly negative definite function, there exists a unique determined measures $\mu_t \in \mathcal{M}^b(Q_1 \times Q_2), t > 0$, such that $\hat{\mu}_t(\chi) = \exp(-t\Psi)$ It is clear that, the family $(\mu_t)_{t>0}$ satisfies conditions (i) and (ii). The boundedness of the function Ψon compact subsets of $\widehat{Q_1 \times Q_2}$ implies that

$$\lim_{t\to 0} \hat{\mu}_t(\chi) = \lim_{t\to 0} \exp(-t\Psi) = 1.$$

From [5], there exists a multiplicative measure $\widehat{m}$ on the dual $\widehat{Q_1 \times Q_2}$, such that for every $f \in C_0(Q_1 \times Q_2)$ and $\varepsilon > 0$, there exits $g \in C_0(\widehat{Q_1 \times Q_2})$such that $\widehat{Q_1 \times Q_2}\,\|f - \tilde{g}\| < \varepsilon$ and

$$|\mu_t(f) - \varepsilon_{\boldsymbol{e}}(f)| \le 2\varepsilon + \int_{\widehat{Q_1 \times Q_2}} |g(\chi,\tau)|\,|\hat{\mu}_t(\chi,\tau) - 1| d\widehat{m}(\chi,\tau)$$

this implies (iii).

Let S denote the set of probability and symmetric measures on $Q_1 \times Q_2$ with compact support, i.e.

$$S = \{\sigma;\ \sigma \in \mathcal{M}^1(Q_1 \times Q_2) \cap \mathcal{M}^c(Q_1 \times Q_2), \sigma(\chi,\tau) = \tilde{\sigma}(\chi,\tau)\}$$

Let $(\mu_t)_{t>0}$ be a convolution semigroup on $Q_1 \times Q_2$ and $\Psi: \widehat{Q_1 \times Q_2} \to \mathbb{C}$ the strongly negative definite function associated to $(\mu_t)_{t>0}$. Applying the same technique of [2] for the hypercomplex system instead of semigroups, we can see that, the net $(\frac{1}{t}\mu_t|\widehat{Q_1 \times Q_2}\setminus\{\boldsymbol{e}\})_{t>0}$ of positive measures on $\widehat{Q_1 \times Q_2}\setminus\{\boldsymbol{e}\}$converges vaguely as $t \to 0$ to a measure μ on $\widehat{Q_1 \times Q_2}\setminus\{\boldsymbol{e}\}$, and for every $\sigma \in S$, the function $\Psi * \sigma - \Psi$ is continuous strongly positive definite on $\widehat{Q_1 \times Q_2}$ and the positive bounded measure μ_σ on $Q_1 \times Q_2$ whose Fourier transform is $\Psi * \sigma - \Psi$ satisfies

$$(1 - \tilde{\sigma})\mu = \mu_\sigma | Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}. \tag{3.1}$$

The positive measure μ on $Q_1 \times Q_2\setminus\{\boldsymbol{e}\}$ defined by (3.1) is called the strong Lèvy measure for the convolution semigroup $(\mu_t)_{t>0}$ on $Q_1 \times Q_2$.

Theorem 3.2. Let μ denote the Lèvy measure for the convolution semigroup $(\mu_t)_{t>0}$ on $Q_1 \times Q_2$. Then

$$\int_{Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(r)) d\,\mu(\chi,\tau) < \infty,\ \ (\chi,\tau) \in \widehat{Q_1 \times Q_2}. \tag{3.2}$$

Proof. For $(\chi,\tau) \in \widehat{Q_1 \times Q_2}$, let $\sigma = \frac{1}{2}(\epsilon_{(\chi,\tau)} + \epsilon_{\overline{(\chi,\tau)}}) \in S$; then $\tilde{\sigma} = Re(\chi,\tau)(r)$, substituting in (3.2) we get

$$\int_{Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(r)) d\,\mu(\chi,\tau) = \int_{Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}} (1 - \tilde{\sigma}(r)) d\,\mu(\chi,\tau) = \mu_\sigma | Q_1 \times Q_2 \setminus\{\boldsymbol{e}\} < \infty.$$

4. Integral representation theorem.

A continuous function $h: Q_1 \times Q_2 \to \mathbb{R}$ is called homomorphism if it satisfies $h(\boldsymbol{r}^*) = -h(\boldsymbol{r})$ and $R_{\boldsymbol{r}}h(\boldsymbol{s}) = h(\boldsymbol{r}) + h(\boldsymbol{s}),\ \boldsymbol{r},\boldsymbol{s} \in Q_1 \times Q_2$. Clearly, if $h: Q_1 \times Q_2 \to \mathbb{R}$ is a homomorphism, then the function $\Psi = ih$ is strongly negative definite. A continuous function $q: Q_1 \times Q_2 \to \mathbb{R}$ is called a quadratic form, if it satisfies

(4.1) $$R_{\boldsymbol{r}}q(\boldsymbol{s}) + R_{\boldsymbol{r}*}q(\boldsymbol{s}) = 2q(\boldsymbol{r}) + 2q(\boldsymbol{s}),\ \boldsymbol{r},\boldsymbol{s} \in Q_1 \times Q_2 .$$

Theorem 4.1. Let Ψ be a strongly negative definite function associated the convolution semigroup $(\mu_t)_{t>0}$ on $Q_1 \times Q_2$. If the Lèvy measure μ of $(\mu_t)_{t>0}$ is symmetric, then $Im\Psi$ is a homomorphism.

Proof. As remarked in [2], a continuous function $f: Q_1 \widehat{\times Q_2} \to \mathbb{R}$ which satisfies $f(e_1, e_2) = 0$ is a homomorphism if and only if $f * \upsilon - f = 0$ for all $\upsilon \in S$. Since, $\check{\mu} = \mu$ is equivalent to $\check{\mu}_\sigma = \mu_\sigma$ for each $\sigma \in S$. So, $Im\Psi * \upsilon - Im\Psi = 0$ for each $\sigma \in S$, hence, then $Im\Psi$ is a homomorphism. In particular, we have $i\ Im\Psi$ is strongly negative definite.

Lemma 4.2. For every positive definite symmetric measure μ on the product $Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}$ such that

(4.2) $$\int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(\boldsymbol{r}))d\,\mu(\boldsymbol{r}) < \infty,\ \ (\chi,\tau) \in Q_1 \widehat{\times Q_2} .$$

The function $\Psi_\mu: \widehat{Q_1 \times} Q_2 \to \mathbb{C}$ defined by

(4.3) $$\Psi_\mu := \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(\boldsymbol{r}))d\,\mu(\boldsymbol{r}) < \infty,\ \ (\chi,\tau) \in Q_1 \widehat{\times Q_2} ,$$

is strongly negative definite function.

Proof. To prove the function Ψ_μ is strongly negative definite, we will sufficiently prove that the measure μ is strong Lèvy measure for Ψ_μ. For $f \in C_C^+(Q_1 \widehat{\times Q_2})$ such that $f(\bar{\chi}) = f(\chi)$ and $\int f(\chi)\,dx = 1$, Applying Fubini's Theorem we get

(4.4) $$\begin{aligned}(\Psi_\mu * f)(\chi) &= \int_{Q_1 \widehat{\times} Q_2} \big(R_\rho f\big)(\chi)\Psi_\mu(\rho)d\rho \\ &= \int_{Q_1 \widehat{\times} Q_2} f(\rho) \int_{Q_1 \times Q_2 \setminus \{(e_1,e_2)\}} [1 - Re\chi(r)\rho(r)]d\,\mu(r) \\ &= \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} \big[1 - Re\chi(r)\tilde{f}(r)\big]d\,\mu(r)\end{aligned}$$

Specially, for $\chi = 1$, we have

$$\int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} \big[1 - \tilde{f}(r)\big]d\,\mu(r) = \int f(\rho)\,\Psi_\mu(\rho)d\rho$$

Clearly, $dv(r) = \left[1 - \tilde{f}(r)\right] d\mu(r)$ is positive definite measure on $Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}$, so can be considered as positive definite measure on $Q_1 \times Q_2$. This implies

$$\hat{v}(\chi) = Re\hat{v}(\chi) = \int_{Q_1 \times Q_2 \backslash \{e\}} Re\chi(r)\left[1 - \tilde{f}(r)\right] d\,\mu(r) \quad \text{for } \chi \in \widehat{Q_1 \times Q_2}.$$

Putting $f = \sigma$ in (4.4) implies that

$$\Psi_\mu * \sigma(\chi) - \Psi_\mu(\chi) = \int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} Re\chi(r)[1 - \tilde{\sigma}(r)] d\,\mu(r)$$

So, $\Psi_\mu * \sigma - \Psi_\mu$is the Fourier transform of the measure $[1 - \tilde{\sigma}(r)]|\mu$, this implies μ is the Lèvy measure of Ψ_μ.

<u>Theorem 4.3.(*Main Result*)</u> Let $\Psi: \widehat{Q_1 \times Q_2} \to \mathbb{C}$ be a strongly negative definite function associated the convolution semigroup $(\mu_t)_{t>0}$ with a symmetric positive Lèvy measure μ such that

$$\int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} (1 - Re(\chi, \tau)(r)) d\,\mu(r) < \infty, \quad (\chi, \tau) \in \widehat{Q_1 \times Q_2},$$

Then Ψ admits the integral representation

$$\Psi(\chi, \tau) = \Psi(e) + iIm\Psi + q(\chi, \tau)$$

$$+ \int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} (1 - Re(\chi, \tau)(r)) d\,\mu(r) < \infty, \quad (\chi, \tau) \in \widehat{Q_1 \times Q_2},$$

where

$$q(\chi, \tau) = \lim_{n \to \infty} \left[\frac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2} + \frac{(R^n_{\overline{(\chi,\tau)}}\Psi)(\chi,\tau)}{2n}\right].$$

Proof. Regarding Theorem 4.1, the symmetries of the measure μ implies $h = Im\Psi$ is a homomorphism and ih belongs to the space of strongly negative definite functions on $\widehat{Q_1 \times Q_2}$. Hence, the function $\Psi - CI$ belongs to the space of strongly negative definite functions on $\widehat{Q_1 \times Q_2}$ associated Lèvy measure μ, where $C = \Psi(\boldsymbol{e})$. This implies the function $\Psi^\# = \Psi - CI - ih$ belongs to the space of strongly negative definite functions on $\widehat{Q_1 \times Q_2}$ associated Lèvy measure μ. By virtue of the argument of Theorem 3.2, the integral

$$\Psi_\mu := \int_{Q_1 \times Q_2 \backslash \{e\}} (1 - Re(\chi, \tau)(\boldsymbol{r})) d\,\mu(\boldsymbol{r})$$

is finite for all $(\chi,\tau) \in \widehat{Q_1 \times Q_2}$. Observing Lemma 4.2, we get that, the function $q = \Psi^{\#} - \Psi_\mu$ is a real valued symmetric function with $q(\boldsymbol{e}) = 0$. As remarked in [3], for $\sigma \in S$ we have

$$\Psi^{\#} * \sigma - \Psi^{\#} = \Psi * \sigma - \Psi$$

and

(4.5) $$\Psi_\mu * \sigma - \Psi_\mu = \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} Re\chi(r)[1 - \tilde{\sigma}(r)]d\,\mu(r)$$

Applying (3.1) and (4.5), we get

(4.6) $$q * \sigma - q = (\Psi^{\#} - \Psi_\mu) * \sigma - (\Psi^{\#} - \Psi_\mu) = \hat{\mu}_\sigma(\{\boldsymbol{e}\}) \geq 0$$

As pointed in [2], (4.6) implies that the function q is a nonnegative quadratic form on $\widehat{Q_1 \times Q_2}$. Recalling the integral

$$\Psi_\mu := \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(\boldsymbol{r}))d\,\mu(\boldsymbol{r})$$

By Lemma 4.2 the function Ψ_μ is strongly negative definite. Since every quadratic form satisfies the following relation[2]

$$\lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}q)(\chi,\tau)}{4n^2}\right] = q(\chi,\tau) - \frac{1}{2}(R_{\overline{(\chi,\tau)}}q)(\chi,\tau)$$

So

(4.7) $$q(\chi,\tau) - \tfrac{1}{2}(R_{\overline{(\chi,\tau)}}q)(\chi,\tau)$$

$$= \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2}\right] - \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi_\mu)(\chi,\tau)}{4n^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2}\right] - \lim_{n\to\infty}\frac{1}{4n^2}\int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re((\chi,\tau)(\boldsymbol{r}))^{2n})d\,\mu(\boldsymbol{r})$$

Since the product $Q_1 \times Q_2$ is locally compact, then for every compact K of $\widehat{Q_1 \times Q_2}$, there exists a constant $M_K \geq 0$, a nieghbourhood N_K of **e** and a finite subset S_K of K such that for every element $r \in N_K$ we have

$$\sup_{\mathrm{r}}\{1 - Re(\chi,\tau)(\boldsymbol{r});\ (\chi,\tau) \in K\} \leq M_K \sup_{\mathrm{r}}\{1 - Re(\chi,\tau)(\boldsymbol{r});\ (\chi,\tau) \in S_K\}.$$

If $(\chi,\tau)(\boldsymbol{r}) \neq 0$, let $(\chi,\tau)(\boldsymbol{r}) = \rho\exp(i\vartheta)$ for some $0 < \rho \leq 1$ and $-\pi \leq \vartheta \leq \pi$. Then for

$n \in \mathbb{N}$ the ratio $\frac{\sin(n\vartheta)}{n\vartheta}$ is bounded a way from $Q_1 \times Q_2$ on $[\frac{\pi}{2}, \pi]$, this implies the existence of a positive constant $C \geq 0$ such that

$$\frac{1}{4n^2}(1-\cos(2n\vartheta)) = \frac{1}{2}\left[\frac{\sin(n\vartheta)}{n\vartheta}\right]^2 \left[\frac{\vartheta}{\sin(n\vartheta)}\right]^2 \left[\frac{1-\cos(2\vartheta)}{2}\right]$$

$$\leq C(1-\cos(2\vartheta))$$

Also, we have

$$\frac{1-\rho^{2n}}{4n^2} \leq \frac{1-\rho}{2n} \leq \frac{1-\rho^2}{2}$$

These gives

$$\frac{1}{4n^2}\left(1-Re\left((\chi,\tau)(\boldsymbol{r})\right)^{2n}\right) = \frac{1}{4n^2}(1-\rho^{2n}) + \frac{\rho^{2n}}{4n^2}(1-\cos(2n\vartheta))$$

$$\leq \frac{1-\rho^2}{2} + C\rho^{2n}(1-\cos(2\vartheta))$$

$$\leq \frac{1-\rho^2}{2} + C\rho^{2}(1-\cos(2\vartheta))$$

$$\leq \frac{1-\rho^2}{2} + C(1-Re((\chi,\tau)(\boldsymbol{r}))^2)$$

Applying the dominated convergence theorem gives

$$\frac{1}{4n^2}\int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1-Re((\chi,\tau)(\boldsymbol{r}))^{2n})d\,\mu(\boldsymbol{r}) = 0$$

Substituting in (4.7) gives

(4.8) $$q(\chi,\tau) = \frac{1}{2}(R_{\overline{(\chi,\tau)}}q)(\chi,\tau) + \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2}\right]$$

Observing that

$$(R_{\overline{(\chi,\tau)}}q)(\chi,\tau) = \lim_{n\to\infty}\left[\frac{(R^{n}_{\overline{(\chi,\tau)}}q)(\chi,\tau)}{2n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(R^{n}_{\overline{(\chi,\tau)}}\Psi)(\chi,\tau)}{2n}\right] - \lim_{n\to\infty}\frac{1}{2n}\int\limits_{Q_1\times Q_2\setminus\{\boldsymbol{e}\}}(1-|(\chi,\tau)(\boldsymbol{r})|^{2n})d\,\mu(\boldsymbol{r})$$

But

$$\frac{1}{2n}(1-|(\chi,\tau)(\boldsymbol{r})|^{2n}) \le 1-|(\chi,\tau)(\boldsymbol{r})|^{2}$$

Applying the dominated convergence theorem again gives

$$\lim_{n\to\infty}\frac{1}{2n}\int\limits_{Q_1\times Q_2\setminus\{\boldsymbol{e}\}}(1-|(\chi,\tau)(\boldsymbol{r})|^{2n})d\,\mu(\boldsymbol{r}) = 0$$

and so

$$(R_{\overline{(\chi,\tau)}}q)(\chi,\tau) = \lim_{n\to\infty}\left[\frac{(R^{n}_{\overline{(\chi,\tau)}}\Psi)(\chi,\tau)}{2n}\right]$$

This complete the proof of the Theorem.

5. Conclusion

In this paper integral representations for strongly negative definite functions defined on the product hypercomplex systems is given. Harmonic properties for strongly negative definite functions are investigated. We construct a Lèvy measure on the product hypercomplex systems, then we study the conditions that guarantee the existence of some integrations having an integrand parts as a function of the constructed kernel. Finally, we give a Lèvy - Khinchin type formula for strongly negative definite functions defined on the product hypercomplex systems.

6. Competing Interests

"The authors declare that they have no competing interests."

7. Acknowledgements

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Nonlinear delay fractional difference equations with applications on discrete fractional Lotka–Volterra competition model

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Abstract. The existence and uniqueness of solutions for nonlinear delay fractional difference equations are investigated in this paper. We prove the main results by employing the theorems of Krasnoselskii's Fixed Point and Arzela–Ascoli. As an application of the main theorem, we provide an existence result on the discrete fractional Lotka–Volterra model.

Keywords. Existence and uniqueness; Fractional difference equations; Krasnoselskii Fixed Point Theorem; Arzela-Ascoli's Theorem; Discrete fractional Lotka–Volterra model.

AMS subject classification: 34A08, 34A12, 39A12.

1 Introduction

Fractional differential equations have received a special attention during the last decades since it has been found that these type of equations provide an excellent instruments for the description of memory and hereditary properties of various materials and processes [1, 2, 3]. The problem of the existence of solutions for fractional differential equations, in particular, has been considered in several recent papers; (see Refs. [4, 5, 6, 7, 8] and the references therein).

For the development of the theory of fractional difference equations, which is the discrete counterpart of fractional differential equations, still there exists less interest among researchers. In fact the progress of the theory of fractional difference equations is still in its early stages. Indeed, some mathematicians have recently taken the lead to develop the qualitative properties of fractional difference equations. We name here for instance Atici et. al. [9, 10, 11, 12, 13] who developed the transform methods, properties of initial value problems and studied applications of these equations on the tumor growth, Abdeljawad et. al. [14, 15, 16, 17, 18] who investigated the properties of Riemann and Caputo's fractional sum and difference operators, Anastassiou [19, 20] who defined a Caputo like discrete fractional difference and studied some discrete fractional inequalities, Goodrich [21, 22, 23] who established sufficient conditions for the existence of solutions for initial and boundary value problems of discrete fractional equations and Chen et. al. [24, 25, 26] who studied the stability of certain fractional difference equations. In [27, 28], Wu and Baleanu provided some applied results concerning with certain real life problems described by discrete fractional equations. For further details on these achievements, we recommend the reader to consult the new publications [29, 30].

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Obviously, the existence and uniqueness of solutions are essentially significant concept for differential equations. To the best of authors' knowledge, there are no results concerning with the existence and uniqueness of solutions for nonlinear delay fractional difference equations. The objective of this paper is to cover this gap and study the existence and uniqueness problem for equations of the form

$$\begin{cases} {}^c\nabla_0^\alpha x(t) = f(t, x(t), x(t-\tau)),\ t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},\ \tau \geq 0, \\ x(t) = \phi(t),\ t \in [-\tau, -\tau+1, \ldots, 0], \end{cases} \tag{1}$$

where $f : \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and ${}^c\nabla_0^\alpha$ denotes the Caputo's fractional difference of order $\alpha \in (0, 1)$. To prove our main results, we employ the Krasnoselskii Fixed Point Theorem and the Arzela-Ascoli's Theorem. As an application of the main theorem, we provide an existence result on the discrete fractional Lotka–Volterra model.

2 Preliminaries

Throughout this paper, we will make use of the following notations, definitions and known results of discrete fractional calculus [29]. For any $\alpha, t \in \mathbb{R}$, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},\ t \in \mathbb{R}\backslash\{\ldots, -2, -1, 0\},\ 0^{\overline{\alpha}} = 0, \tag{2}$$

where Γ is the well known Gamma function satisfying $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Definition 1. *Let $x : \mathbb{N}_0 \to \mathbb{R}$, $\rho(s) = s - 1$, $\alpha \in \mathbb{R}^+$ and $\mu > -1$. Then*

1. *The nabla difference of x is defined by*

$$\nabla x(t) = x(t) - x(t-1),\ \ t \in \mathbb{N}_1 = \{1, 2, \ldots\}.$$

2. *The Riemann–Liouville's sum operator of x of order $\alpha > 0$ is defined by*

$$\nabla_0^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} x(s),\ \ t \in \mathbb{N}_1. \tag{3}$$

3. *The Riemann–Liouville's difference operator of x of order $0 < \alpha < 1$ is defined by*

$${}^c\nabla_0^\alpha x(t) = \nabla_0^{-(1-\alpha)} \nabla x(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{t} \big(t - \rho(s)\big)^{\overline{-\alpha}} \nabla x(s),\ \ t \in \mathbb{N}_1. \tag{4}$$

4. *The power rule is defined by*

$$\nabla_0^{-\alpha} t^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t)^{\overline{\alpha+\mu}},\ \ t \in \mathbb{N}_0. \tag{5}$$

Lemma 1. [40] *$x(t)$ denotes a solution of equation (1) if and only if it admits the following representation*

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s-\tau)),\ \ t \in \mathbb{N}_0, \tag{6}$$

and $x(t) = \phi(t),\ t \in [-\tau, -\tau+1, \ldots, 0]$.

The space l_∞ denotes the set of real bounded sequences with respect to the usual supremum norm. We recall that l_∞ is a Banach space.

Definition 2. *A set D of sequences in l_∞ is uniformly Cauchy if for every $\varepsilon > 0$, there exists an integer N such that $|x(t) - x(s)| < \varepsilon$ whenever $t, s > N$ for any $x = \{x(n)\}$ in D.*

The following discrete version of Arzela–Ascoli's Theorem has a crucial role in the proof of our main theorem.

Theorem 1. *(Arzela–Ascoli's Theorem) A bounded, uniformly Cauchy subset D of l_∞ is relatively compact.*

The proof of the main theorem is achieved by employing the following fixed point theorem.

Theorem 2. [31] *(Krasnoselskii Fixed Point Theorem) Let D be a nonempty, closed, convex and bounded subset of a Banach space $(X, \|x\|)$. Suppose that $A : X \to X$ and $B : D \to X$ are two operators such that*

(i) *A is a contraction.*

(ii) *B is continuous and $B(D)$ resides in a compact subset of X,*

(iii) *for any $x, y \in D$, $Ax + By \in D$.*

Then the operator equation $Ax + Bx = x$ has a solution $x \in D$.

3 Main results

We prove our main results under the following assumptions:

(I) $f(t, x(t), y(t)) = f_1(t, x(t)) + f_2(t, x(t), y(t))$, where f_i are Lipschitz functions with Lipschitz constants L_{f_i}, $i = 1, 2$.

(II) $|f_1(t, x(t))| \leq M_1|x(t)|$, $|f_2(t, x(t), y(t))| \leq M_2|x(t)| \times |y(t)|$ for any positive numbers M_1 and M_2.

Let $B(\mathbb{N}_{-\tau}, \mathbb{R})$ denote the set of all bounded functions (sequences). Define the set

$$D = \{x : \ x \in B(\mathbb{N}_{-\tau}, \mathbb{R}), \ |x| \leq r, \ \ t \in \mathbb{N}_0\},$$

where r satisfies

$$|\phi(0)| + \frac{M_1 r + M_2 r^2}{\Gamma(\alpha)} \leq r.$$

Define the operators F_1 and F_2 by

$$F_1 x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f_1(s, x(s)),$$

and

$$F_2 x(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f_2(s, x(s), x(s - \tau)).$$

It is clear that $x(t)$ is a solution of (1) it it is a fixed point of the operator $Fx = F_1 x + F_2 x$.

Theorem 3. *Let conditions* (I)–(II) *hold. Then, equation (1) has a solution in the set* D *provided that* $\frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)} < 1$ *and* $|\phi(0)| + \frac{\left(M_1 r + M_2 r^2\right)C(\alpha)}{\Gamma(\alpha)} \leq r$.

Proof. From the assumptions on the set D, one can easily see that D is a nonempty, closed, convex and bounded set.

Step.1: We prove that F_1 is contractive. We can easily see that

$$\begin{aligned} \left|F_1x(t) - F_1y(t)\right| &= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_1(s,x(s)) - f_1(s,y(s))\right| \\ &\leq \frac{L_{f_1}}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}|x(s)-y(s)| \\ &\leq \frac{L_{f_1}}{\Gamma(\alpha)}\|x-y\|\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}. \end{aligned} \tag{7}$$

By virtue of (2), (3), (5) and since $(t-0)^{\overline{0}} = 1$, one can see that

$$\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(t-0)^{\overline{0}} = \Gamma(\alpha)\nabla_0^{-\alpha}(t-0)^{\overline{0}} = \frac{\Gamma(t+\alpha)}{\alpha\Gamma(t)}.$$

Therefore, (7) becomes

$$\left|F_1x(t) - F_1y(t)\right| \leq \frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)}\|x-y\|, \quad t < T_1,$$

where $C(\alpha) = \frac{\Gamma(T_1+\alpha)}{\alpha\Gamma(T_1)}$ is a positive constant depending on the order α. By the assumption $\frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)} < 1$, we conclude that F_1 is contractive. Furthermore, we obtain for $x \in D$

$$\begin{aligned} \left|F_1x(t) + F_2x(t)\right| &\leq |\phi(0)| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_1(s,x(s)) + f_2(s,x(s),x(s-\tau))\right| \\ &\leq |\phi(0)| + \frac{M_1\|x\| + M_2\|x\|^2}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} \\ &\leq |\phi(0)| + \frac{\left(M_1r + M_2r^2\right)C(\alpha)}{\Gamma(\alpha)}, \end{aligned}$$

which implies that $F_1x + F_2x \in D$. For $x \in D$, we also get

$$|F_2x(t)| \leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_2(s,x(s),x(s-\tau))\right| \leq \frac{\left(M_2r^2\right)C(\alpha)}{\Gamma(\alpha)} \leq r,$$

which implies that $F_2(D) \subset D$.

Step.2: We prove that F_2 is continuous. Let a sequence x_n converge to x. Taking the norm of $F_2x_n(t) - F_2x(t)$, we have

$$\begin{aligned}\big|F_2x_n(t) - F_2x(t)\big| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big|f_2(s,x_n(s),x_n(s-\tau)) - f_2(s,x(s),x(s-\tau))\big| \\ &\leq \frac{L_{f_2}}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\Big(|x_n(s)-x(s)| - \big|x_n(s-\tau)) - x(s-\tau)\big|\Big) \\ &\leq \frac{2L_{f_2}}{\Gamma(\alpha)}\|x_n - x\|\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} = \frac{(2L_{f_2})C(\alpha)}{\Gamma(\alpha)}\|x_n - x\|.\end{aligned}$$

From the above discussion, we conclude that whenever $x_n \to x$, $Fx_n \to Fx$. This proves the continuity of F_2. To prove that $F_2(D)$ resides in a relatively compact subset of l_∞, we let $t_1 \leq t_2 \leq H$ to get

$$\begin{aligned}\big|F_2x(t_2) - F_2x(t_1)\big| &\leq \frac{1}{\Gamma(\alpha)}\Big|\sum_{s=1}^{t_2}(t_2-\rho(s))^{\overline{\alpha-1}}f_2(s,x(s),x(s-\tau)) \\ &- \sum_{s=1}^{t_1}(t_1-\rho(s))^{\overline{\alpha-1}}f_2(s,x(s),x(s-\tau))\Big| \\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}\Big|(t_2-\rho(s))^{\overline{\alpha-1}} - (t_1-\rho(s))^{\overline{\alpha-1}}\Big||f_2(s,x(s),x(s-\tau))| \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{s=t_1+1}^{t_2}\big|(t_2-\rho(s))^{\overline{\alpha-1}}\big||f_2(s,x(s),x(s-\tau))|.\end{aligned}$$

Upon employing condition (II), we obtain

$$\begin{aligned}\big|F_2x(t_2) - F_2x(t_1)\big| &\leq M_2r^2\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}(t_2-\rho(s))^{\overline{\alpha-1}} - \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}(t_1-\rho(s))^{\overline{\alpha-1}} \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{s=t_1+1}^{t_2}(t_2-\rho(s))^{\overline{\alpha-1}}\Big].\end{aligned}$$

By using (3), we get

$$\big|F_2x(t_2) - F_2x(t_1)\big| \leq M_2r^2\Big[\nabla_0^{-\alpha}(t_2-0)^{\overline{0}} - \nabla_0^{-\alpha}(t_1-0)^{\overline{0}} + \nabla_{t_1}^{-\alpha}(t_2-t_1)^{\overline{0}}\Big].$$

From (5), it follows that

$$\big|F_2x(t_2) - F_2x(t_1)\big| \leq \frac{M_2r^2}{\Gamma(\alpha+1)}\Big[t_2^{\overline{0}} - t_1^{\overline{0}} + (t_2-t_1)^{\overline{0}}\Big].$$

This implies that F_2 is bounded and uniformly subset of l^∞. Thus, by virtue of the Discrete Arzela Ascoli's Theorem 1, we conclude that F_2 is relatively compact.

Step.3: It remains to show that for any $x, y \in D$, we have $F_1x(t) + F_2y(t) \in D$. If $x = F_1x(t) + F_2y(t)$, then we have

$$\begin{aligned}|x(t)| &\leq \left|F_1x(t) + F_2y(t)\right| \leq |\phi(0)| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_1(s,x(s)) + f_2(s,y(s),y(s-\tau))\right| \\ &\leq |\phi(0)| + \frac{M_1\|x\| + M_2\|x\|^2}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} \\ &\leq |\phi(0)| + \frac{\big(M_1r + M_2r^2\big)C(\alpha)}{\Gamma(\alpha)},\end{aligned}$$

which implies that $x(t) \in D$.

By employing the Krasnoselskii Fixed Point Theorem, we conclude that there exists $x \in D$ such that $x = Fx = F_1x + F_2x$ which is a fixed point of F. Hence, equation (1) has at least one solution in D.

□

4 Applications

The Lotka–Volterra model has been extensively investigated through different approaches [32, 33, 34, 35, 36, 37]. However, all the above mentioned papers studied the integer order Lotka–Volterra model. In spite of the fact that the study of population and medical models of fractional order has been initiated in [12, 38, 39], there is no literature achieved in the direction of discrete fractional Lotka–Volterra model. Therefore, in this section, we employ Theorem 3 to prove an existence and uniqueness result for the solutions of this model.

For a bounded sequence g on $\mathbb{N}$, we define g^+ and g^- as follows

$$g^+ = \sup_{t\in\mathbb{N}} g(t) \quad \text{and} \quad g^- = \inf_{t\in\mathbb{N}} g(t).$$

Let $f(t, x(t), x(t-\tau)) = x(t)\big(\gamma(t) - \beta(t)x(t-\tau)\big)$ in equation (1), then we have the following discrete fractional Lotka–Volterra model:

$$\begin{cases} {}^c\nabla_0^\alpha x(t) = x(t)\big(\gamma(t) - \beta(t)x(t-\tau)\big),\ t \in \mathbb{N}_0 \\ x(t) = \phi(t),\ t \in [-\tau, -\tau+1, \ldots, 0],\ 0 < \alpha < 1, \end{cases} \tag{8}$$

where the coefficients γ and β satisfy the boundedness relations

$$\gamma^- \leq \gamma(t) \leq \gamma^+, \ \ \beta^- \leq \beta(t) \leq \beta^+,$$

which are medically and biologically feasible. Model (8) represents the interspecific competition in single species with τ denotes the maturity time period.

Denote

$$\overline{f}_1(t, x(t)) = x(t)\gamma(t), \quad \overline{f}_2(t, x(t), x(t-\tau)) = -\beta(t)x(t)x(t-\tau).$$

It follows that the functions $\overline{f}_1$ and $\overline{f}_2$ satisfy the conditions

(III) $|\overline{f}_1(t,x(t))| \le \gamma^+|x(t)|, \;\; |\overline{f}_2(t,x(t),x(t-\tau))| \le \beta^+|x(t)| \times |x(t-\tau)|$.

(IV) $\overline{f}_i$ are Lipschitz functions with Lipschitz constants $\overline{L}_{f_i}, \; i=1,2$.

The solution of model (8) has the form

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s)\Big(\gamma(s) - \beta(s)x(s-\tau)\Big), \;\; t \in \mathbb{N}_0, \tag{9}$$

and $x(t) = \phi(t), \;\; t \in [-\tau, -\tau+1, \ldots, 0]$. Define a function G by

$$Gx(t) = G_1x(t) + G_2x(t),$$

where

$$G_1x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s)\gamma(s),$$

and

$$G_2x(t) = -\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s)\beta(s)x(s-\tau).$$

One can easily employ the same arguments used in the proof of Theorem 3 to complete the proof of the following theorem for equation (8).

Theorem 4. *Let conditions* (III)–(IV) *hold. Then, the model (8) has a solution in the set D provided that* $\frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)} < 1$ *and* $|\phi(0)| + \frac{\left(\gamma^+ r + \beta^+ r^2\right)C(\alpha)}{\Gamma(\alpha)} \le r$.

Remark 1. *The above result can be extended to n species competitive Lotka–Volterra system of the form*

$$\begin{cases} \nabla_0^\alpha x_i(t) = x_i(t)\big(\gamma_i(t) - \sum_{j=1}^n \beta_{ij}(t)x_j(t-\tau_{ij})\big), \; t \in \mathbb{N}_0, \; i = 1,2,\ldots,n. \\ x_i(t) = \phi_i(t), \; t \in [-\tau_i, -\tau_i+1, \ldots, 0], \; 0 < \alpha < 1, \; \tau_i = \max_{1 \le j \le n} \tau_{ij}, \end{cases} \tag{10}$$

where $\gamma^- \le \gamma_i(t) \le \gamma^+, \;\; \beta^- \le \beta_{ij}(t) \le \beta^+$.

Remark 2. *Results of this paper can be carried out for the equation*

$$\begin{cases} \nabla_0^\alpha x(t) = f(t, x(t), x(t-\tau)), \; t \in \mathbb{N}_2 = \{2,3,\ldots\}, \; \tau \ge 0, \\ x(t) = \phi(t), \; t \in [-\tau, -\tau+1, \ldots, 1], \end{cases} \tag{11}$$

where $f : \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* ∇_0^α *denotes the Riemann–Liouville's fractional difference of order* $\alpha \in (0,1)$. *The solution of equation (11) has the form*

$$x(t) = \frac{t^{\overline{\alpha-1}}}{\Gamma(\alpha)}\phi(1) + \frac{1}{\Gamma(\alpha)} \sum_{s=2}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s-\tau)). \tag{12}$$

5 Conclusion

A comprehensive literature survey on the predator–prey type Lotka–Volterra model reveals that a considerable amount of work has already been done by many esteemed researchers during the last century. However the concept of the model related to fractional time derivatives is an original one.

The fractional Lotka–Volterra equation is obtained from the classical equations by replacing the first order time derivative by fractional derivative of order $\alpha \in (0,1)$. One of the most significant outcomes of this evolution equation is the generation of fractional Brownian motions.

It has been discernible that the discrete analogue of ordinary differential equations has tremendous applications in computational analysis and computer simulations. Motivated by this reality, the study of the discrete analogue of fractional differential equations has become pressing and compulsory.

In this paper, we studied the existence and uniqueness of solutions for nonlinear delay fractional difference equations. The main theorem is proved with the help the Krasnoselskii fixed point theorem and the Arzela–Ascoli's Theorem. Prior to the main result, we set forth some notations and definitions which enriched the knowledge of discrete fractional calculus. To demonstrate the applicability of the main theorem, we provide an existence result for the discrete fractional Lotka–Volterra model.

It is to be noted that the analysis carried out in this paper is based on the use of nabla rather than delta operators. Indeed, unlike the delta operator the range of nabla fractional sum and difference operators depends only of the starting point and independent of the order α. This provides exceptional ability to treat skilfully different circumstances throughout the proofs. The delta approach can be obtained from nabla operator through the implementation of the dual identities discussed in [41].

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Some sharp results on NLC-operators in G_p-metric spaces

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Abstract: In this paper we generalize, complement and improve some recent results on NLC-operators established in G_p-metric spaces. Several examples are given to support our theoretical approach.

Keywords: G_p-metric space, NLC-operator, supporting sequence, G_p-complete, fixed point

1 Introduction and preliminaries

Partial metric space and G-metric space are two different generalized metric spaces. In 1994 Matthews [13] introduced partial metric space as follows:

Definition 1.1. Let X be a nonempty set. A partial metric is a mapping $p : X^2 \to [0, +\infty)$ which satisfies that

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$, for all $x, y \in X$;

(p2) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;

(p3) $p(x, y) = p(y, x)$, for all $x, y \in X$;

(p4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$, for all $x, y, z \in X$.

Then the pair (X, p) is called a partial metric space.

It is clear that each (standard) metric space is a partial metric space, while on the contrary it does not hold, in general. In recent years, many authors have obtained lots of fixed point results in partial metric spaces, for example, see [12], [13], [15], [17], [21] and the references therein.

On the other hand, in 2006 Mustafa and Sims [14] introduced another kind of generalized metric space, so-called G-metric space as follows:

Definition 1.2. Let X be a nonempty set. A mapping $G : X^3 \to [0, +\infty)$ is called G-metric if it satisfies the following conditions:

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(G1) $x = y = z \Leftrightarrow G(x, y, z) = 0$ for all $x, y, z \in X$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(P\{x, y, z\})$, where P is a permutation of $x, y, z \in X$ (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the pair (X, G) is called a G-metric space.

Based on this notion, many fixed point results under different contractive conditions have been obtained (see [1], [7]-[10], [14], and the references therein).

In 2011 Zand and Nezhad [23] introduced a concept as a generalization of both partial metric space and G-metric space as follows:

Definition 1.3. Let X be a nonempty set. A mapping $G_p : X^3 \to [0, +\infty)$ is called a G_p-metric if the following conditions are satisfied:

(G_p1) $x = y = z$ if $G_p(x, y, z) = G_p(x, x, x) = G_p(y, y, y) = G_p(z, z, z)$ for all $x, y, z \in X$;

(G_p2) $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$;

(G_p3) $G_p(x, y, z) = G_p(P\{x, y, z\})$, where P is a permutation of $x, y, z \in X$ (symmetry in all three variables);

(G_p4) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the pair (X, G_p) is called a G_p-metric space.

Remark 1.4. It is worth mentioning that authors in [2], [3], [5], [19] and [23] used (G_p2) while in [6], [18] and [20] authors used the following condition:

(G_p2') $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$.

In the former case (X, G_p) is a symmetric G_p-metric space, that is., $G_p(x, x, y) = G_p(x, y, y)$ for all $x, y \in X$. However, in the latter case this does not hold.

Otherwise, each symmetric G-metric space is symmetric G_p-metric space, but the converse is not true (see Example 1 from [23]) as well as each G-metric space is G_p-metric space in the sense of [18]. However, the claim from [23] (page 87, lines 6-,7-) that each G-metric space is also G_p-metric space is false (see [18], page 79). In addition, It is noteworthy that Example 3 in [23] is symmetric G-metric space, and hence it is G_p-metric space. It is also clear that Definition 6 (because (G_p2)) in [23] is superfluous.

First our important result in this section is the following:

Proposition 1.5. Every G_p-metric space (X, G_p) in the sense of [18] defines a metric space $\left(X, d_{G_p}\right)$ as follows:

$$d_{G_p}(x, y) = G_p(x, y, y) + G_p(x, x, y) - G_p(x, x, x) - G_p(y, y, y), \text{ for all } x, y \in X.$$

Proof. Using (G_p2), we have $d_{G_p}(x, y) \geq 0$ for all $x, y \in X$. Also, if $x = y$, then $d_{G_p}(x, y) = 0$. Conversely, let $d_{G_p}(x, y) = 0$, then

$$G_p(x, y, y) + G_p(x, x, y) - G_p(x, x, x) - G_p(y, y, y) = 0,$$

that is.,

$$[G_p(x,x,y) - G_p(x,x,x)] + [G_p(x,y,y) - G_p(y,y,y)] = 0,$$

or equivalently, $G_p(x,x,y) = G_p(x,x,x)$ and $G_p(x,y,y) = G_p(y,y,y)$. Further, on account of (G_p4) it implies that $G_p(x,y,y) \leq 2G_p(x,x,y) - G_p(x,x,x) = G_p(x,x,y)$. Similarly it follows that $G_p(x,x,y) \leq G_p(x,y,y)$ for all $x, y \in X$. Then

$$G_p(x,y,x) = G_p(x,x,x) = G_p(y,y,y),$$

thus by (G_p1) it gives $x = y$.

It is obvious that $d_{G_p}(x,y) = d_{G_p}(y,x)$ for all $x, y \in X$.

Finally, we shall prove that

$$d_{G_p}(x,z) \leq d_{G_p}(x,y) + d_{G_p}(y,z),$$

for all $x, y, z \in X$, or equivalently,

$$\begin{aligned} &G_p(x,x,z) + G_p(x,z,z) - G_p(x,x,x) - G_p(z,z,z) \\ \leq\ &G_p(x,x,y) + G_p(x,y,y) - G_p(x,x,x) - G_p(y,y,y) \\ &+ G_p(y,y,z) + G_p(y,z,z) - G_p(y,y,y) - G_p(z,z,z), \end{aligned}$$

that is.,

$$\begin{aligned} &G_p(x,x,z) + G_p(x,z,z) \\ \leq\ &G_p(x,x,y) + G_p(x,y,y) - G_p(y,y,y) + G_p(y,y,z) + G_p(y,z,z) - G_p(y,y,y). \end{aligned}$$

Notice that

$$G_p(x,x,z) = G_p(z,x,x) \leq G_p(z,y,y) + G_p(y,x,x) - G_p(y,y,y)$$

and

$$G_p(x,z,z) \leq G_p(x,y,y) + G_p(y,z,z) - G_p(y,y,y),$$

so the proof is completed. □

Remark 1.6. Our proof of this proposition is more detailed than one of [23].

Further, we announce the following definition with valid approaches which complements Definition 1.9 from [18].

Definition 1.7. Let (X, G_p) be a G_p-metric space and $\{x_n\}$ a sequence in X. Then

(1) $\{x_n\}_{n\in\mathbb{N}}$ is called G_p-convergent to a point $x \in X$ if $\lim\limits_{n,m\to\infty} G_p(x, x_n, x_m) = G_p(x,x,x)$. In this case, we write $x_n \to x$ as $n \to \infty$;

(2) $\{x_n\}$ is called a G_p-Cauchy sequence if $\lim\limits_{n,m\to\infty} G_p(x_n, x_m, x_m) = r \in \mathbb{R}$. Particularly, $\{x_n\}$ is called 0-Cauchy sequence if $r = 0$;

(3) (X, G_p) is called G_p-complete if for every G_p-Cauchy sequence $\{x_n\}$ in X is G_p-convergent to $x \in X$.

Now, we give the following conclusion which corrects Proposition 4 of [23]:

Proposition 1.8. Let (X, G_p) be a symmetric G_p-metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$ the following are equivalent:

(1) $\{x_n\}$ is G_p-convergent to x;

(2) $G_p(x_n, x_n, x) \to G_p(x, x, x)$ as $n \to \infty$;

(3) $G_p(x_n, x, x) \to G_p(x, x, x)$ as $n \to \infty$.

Proof. Since (X, G_p) is symmetric G_p-metric space, then (2) is equivalent to (3). Taking $m = n$ in (1), we speculate that (1) implies (2), thus, (1) implies (3). For the converse we have that

$$\begin{aligned} & G_p(x, x_n, x_m) - G_p(x, x, x) \\ = & G_p(x_n, x_m, x) - G_p(x, x, x) \\ \leq & G_p(x_n, x, x) + G_p(x, x_m, x) - G_p(x, x, x) - G_p(x, x, x) \\ = & [G_p(x_n, x, x) - G_p(x, x, x)] + [G_p(x_m, x, x) - G_p(x, x, x)] \\ \to & 0 + 0 = 0, \text{ as } n, m \to \infty, \end{aligned}$$

then (3) implies (1). We complete the proof. □

Next we generalize Lemma 1.10 from [2] (see also [3], [5], [6], [18], [20]), that is., we announce the following assertion:

Proposition 1.9. Let (X, G_p) be a G_p-metric space in the sense of [18]. Then

(A) if $G_p(x, y, z) = 0$, then $x = y = z$;

(B) if $x \neq y$, then $G_p(x, y, y) > 0$.

Proof. (A) If $x \neq y \neq z \neq x$, then (A) is an immediate consequence of (G_p2') and (G_p1). If for instance, $x \neq y = z$, then $G_p(x, y, z) = G_p(x, y, y) = 0$. In this case, we get $G_p(x, x, x) = G_p(x, x, y) = G_p(y, y, y) = 0$. Indeed, by ($G_p4$) it follows that

$$G_p(x, x, y) \leq G_p(x, y, y) + G_p(y, x, y) - G_p(y, y, y) \leq 2G_p(x, y, y) = 0.$$

Since $G_p(x, x, x) \leq G_p(x, x, y)$ and $G_p(y, y, y) \leq G_p(x, y, y)$ hold for all $x, y \in X$, then we arrive at

$$G_p(x, y, y) = G_p(x, x, y) = G_p(x, x, x) = G_p(y, y, y) = 0,$$

so by (G_p1), we obtain the desired result. □

(B) Let $G_p(x, y, y) = 0$. Now, based on the proof of (A) when $x \neq y = z$, we claim that $x = y$. A contradiction.

2 Auxiliary results

In the sequel, let (X, G_p) be a G_p-metric space in the sense of [18]. First of all, we introduce the following notion:

Definition 2.1. Let (X, G_p) be a G_p-metric space, $\alpha \in (0, 1)$ a constant and $T : X \to X$ a mapping. We say that T is an NLC-operator on X if for each $x \in X$ there is some

$n(x) \in \mathbb{N}$ such that for each $y \in X$ it holds

$$G_p\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \le \max\left\{\alpha G_p(x,x,y), G_p(x,x,x)\right\}. \tag{2.1}$$

For an NLC-operator T and $x \in X$ we define supporting sequence at x as a sequence $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ where $s_0 = 0$ and $s_{k+1} = s_k + n(T^{s_k}x), k \in \mathbb{N}\cup\{0\}$. Also set $J_T(X) = \{x \in X : T^m x = T^{m+1}x \text{ for some } m \in \mathbb{N}\}$.

Remark 2.2. (i) Condition (2.1) implies that for any $i \ge s_k$, it is valid that

$$G_p\left(T^{s_k}x, T^{s_k}x, T^{i}x\right) \le \max\left\{\alpha G_p\left(T^{s_{k-1}}x, T^{s_{k-1}}x, T^{j}x\right), G_p\left(T^{s_{k-1}}x, T^{s_{k-1}}x, T^{s_{k-1}}x\right)\right\}, \tag{2.2}$$

where $j = i - s_k + s_{k-1} \ge s_{k-1}$, and specially that

$$G_p\left(T^{s_k}x, T^{s_k}x, T^{s_k}x\right) \le G_p\left(T^{s_{k-1}}x, T^{s_{k-1}}x, T^{s_{k-1}}x\right). \tag{2.3}$$

Now, fix $x \in X\backslash J_T(X)$. For $k \in \mathbb{N}$ and $i \ge s_k$ use (2.2), repeatedly fix integers $l_j \ge s_j, 0 \le j < k$ and $t_1, t_2, ..., t_k \in \{0,1\}$ such that $l_k := i$, then

$$G_p\left(T^{s_j}x, T^{s_j}x, T^{l_j}x\right) \le \alpha^{t_j} \cdot G_p\left(T^{s_{j-1}}x, T^{s_{j-1}}x, T^{l_{j-1}}x\right)$$

for all $0 \le j \le k$, where

$$t_j = \begin{cases} 1, & \text{if } s_{j-1} < l_{j-1}, \\ 0, & \text{if } s_{j-1} = l_{j-1}. \end{cases}$$

Let us recall $(l_0, l_1, ..., l_{k-1})$ and $(t_1, t_2, ..., t_k)$ as the (k,l)-descent and (k,i)-signature at x, respectively.

Further put $r_{k,i} =: k - h_{k,i}$, where $h_{k,i}$ is a number of zeroes in (k,i)-signature at x.

We shall say that x is Type 1 if there are sequences of positive integers $\{k_m\}_{m\in\mathbb{N}\cup\{0\}}$ and $\{i_m\}_{m\in\mathbb{N}\cup\{0\}}$, one of them is strictly increasing such that for all $m \in \mathbb{N}\cup\{0\}$ we have that $i_m \ge s_m$ and $r_{k_m,i_m} < r_{k_{m+1},i_{m+1}}$.

We shall say that x is Type 2 if x is not Type 1, i.e., there are $k_0, B \in \mathbb{N}$ such that for all $k \ge k_0$ and $i \ge s_k$ it holds $r_{k,i} < B$.

(ii) In the framework of G-metric spaces, condition (2.1) becomes

$$G_p\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \le \alpha G_p(x,x,y), \tag{2.3'}$$

hence, it is iterate contractive condition of Sehgal-Guseman type in this framework (see [11], [16]).

Lemma 2.3. Let T be an NLC-operator on G_p-metric space (X, G_p), $x \notin J_T(X)$, and let $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ be a supporting sequence at x. Then

(a) if $(l_0, l_1, ..., l_{k-1})$ is (k, i_0)-descent at x, then

$$G_p\left(T^{s_k}x, T^{s_k}x, T^{i_0}x\right) \le \alpha^{r_{k,i_0}} \cdot G_p\left(x, x, T^{l_0}x\right),$$
$$G_p\left(T^{s_k}x, T^{s_k}x, T^{i_0}x\right) \le G_p\left(T^{s_j}x, T^{s_j}x, T^{l_j}x\right)$$

for all $0 \le j \le k$, where $l_k := i_0$;

(b) if $P \subseteq \{0,1,...,k-1\}$ and $r_{k,i_0} < \text{card}P$ ($\text{card}P$ is the number of elements of P), then for some $j_0 \in P$ it holds

$$G_p\left(T^{s_k}x, T^{s_k}x, T^{i_0}x\right) \leq G_p\left(T^{s_{j_0}}x, T^{s_{j_0}}x, T^{s_{j_0}}x\right).$$

Proof. Using the definition of $r_{k,i}$, (a) is obvious. To prove (b), under the hypothesis, the set $\{j+1 : j \in P\}$ is subset of $\{1,2,...,k\}$ with $\text{card}(P) > r_{k,i_0}$, so there is some $j_0 \in P$ with $t_{j_0+1} = 0$. Then

$$\begin{aligned} G_p\left(T^{s_{j_0+1}}x, T^{s_{j_0+1}}x, T^{i_0}x\right) &\leq G_p\left(T^{l_{j_0+1}}x, T^{l_{j_0+1}}x, T^{i_0+1}x\right) \\ &\leq \alpha^{t_{j_0}+1} G_p\left(T^{s_{j_0}}x, T^{s_{j_0}}x, T^{l_{j_0}}x\right) \\ &= G_p\left(T^{s_{j_0}}x, T^{s_{j_0}}x, T^{s_{j_0}}x\right), \end{aligned}$$

whereof (a) and $s_{j_0} = l_{j_0}$ have been used. □

Lemma 2.4. Let T be an NLC-operator on G_p-metric space (X, G_p) and $x \in X$, then there is some $M_x > 0$ such that for all $i \geq 0$ it satisfies that

$$G_p\left(x, x, T^i x\right) \leq M_x, \tag{2.4}$$

and so $G_p\left(T^j x, T^j x, T^i x\right) \leq 3M_x$, for each $i, j \in \mathbb{N} \cup \{0\}$.

Proof. If $x \in J_T(X)$, then this is obvious. Thus, let $x \notin J_T(X)$ and set

$$b(x) = G_p(x,x,x) + G_p(x,x,Tx) + \cdots + G_p\left(x,x,T^{n(x)}x\right).$$

Let us prove by induction that

$$G_p\left(x,x,T^i x\right) \leq \frac{1}{1-\alpha} b(x), \text{ for all } i \in \mathbb{N}.$$

Obviously (2.4) is true for $0 \leq k \leq n(x)$. Now assume that the same is valid for some $k \geq n(x)$. Then

$$\begin{aligned} &G_p\left(x,x,T^{k+1}x\right) \leq G_p\left(x,x,T^{n(x)}x\right) + G_p\left(T^{n(x)}x, T^{n(x)}x, T^{k+1}x\right) \\ &\leq G_p\left(x,x,T^{n(x)}x\right) + \max\left\{\alpha G_p\left(x,x,T^{k+1-n(x)}x\right), G_p(x,x,x)\right\} \\ &\leq G_p\left(x,x,T^{n(x)}x\right) + G_p(x,x,x) + \frac{\alpha}{1-\alpha} b(x) \\ &\leq b(x) + \frac{\alpha}{1-\alpha} b(x) \\ &= \frac{1}{1-\alpha} b(x), \end{aligned}$$

so (2.4) is proved with $M_x = \frac{1}{1-\alpha} b(x)$.

Further, we have

$$G_p\left(T^i x, T^j x, T^j x\right) \leq G_p\left(T^i x, x, x\right) + 2G_p\left(T^j x, x, x\right) \leq 3M_x,$$

for all $i, j \in \mathbb{N} \cup \{0\}$. □

Lemma 2.5. Let T be an NLC-operator on G_p-metric space (X, G_p) and $x \in X \backslash J_T(X)$. If x is Type 1, then $\lim_{i,j\to\infty} G_p(T^i x, T^i x, T^j x) = 0$.

Proof. Fix $m \in \mathbb{N} \cup \{0\}$. If $(l_0, ..., l_{k_m - 1})$ is (s_{k_m}, i_m)-descent, then by (a) of Lemma 2.3 we have

$$G_p\left(T^{s_{k_m}}x, T^{s_{k_m}}x, T^{i_m}x\right) \le \alpha^{r_{k_m, i_m}} \cdot G_p\left(x, x, T^{l_0}x\right) \le \alpha^{r_{k_m, i_m}} M_x.$$

In view of $\lim_{m\to\infty} r_{k_m, i_m} = \infty$, it follows that

$$\lim_{m\to\infty} G_p\left(T^{s_{k_m}}x, T^{s_{k_m}}x, T^{i_m}x\right) = \lim_{m\to\infty} G_p\left(T^{s_{k_m}}x, T^{s_{k_m}}x, T^{s_{k_m}}x\right) = 0.$$

For given $\varepsilon > 0$, choose $m_0 \in \mathbb{N}$ such that $\alpha^{m_0} M_x < \varepsilon$ and $G_p\left(T^{s_{k_m}}x, T^{s_{k_m}}x, T^{s_{k_m}}x\right) < \varepsilon$ for all $m \ge m_0$. Let $r_{k_{2m_0}, i} \ge m_0$. Then

$$G_p\left(T^{s_{k_{2m_0}}}x, T^{s_{k_{2m_0}}}x, T^i x\right) \le \alpha^{r_{k_{2m_0}, i}} \cdot M_x \le \alpha^{m_0} M_x < \varepsilon.$$

Now suppose that $r_{k_{2m_0}, i} < m_0$. For $P_i = \{k_{m_0}, ..., k_{2m_0 - 1}\} \subseteq \{0, 1, ..., k_{2m_0} - 1\}$, we have $\operatorname{card}(P) > r_{k_{2m_0}, i}$, so by Lemma 2.3, there exists some $m_0 \le j \le 2m_0 - 1$ such that

$$G_p\left(T^{s_{k_{2m_0}}}x, T^{s_{k_{2m_0}}}x, T^i x\right) \le G_p\left(T^{s_{k_j}}x, T^{s_{k_j}}x, T^{s_{k_j}}x, \right) < \varepsilon$$

for each $i \ge s_{k_{2m_0}}$.

Accordingly, if $i, j \ge s_{k_{2m_0}}$, then

$$\begin{aligned} G_p\left(T^i x, T^i x, T^j x\right) &\le G_p\left(T^{s_{k_{2m_0}}}x, T^{s_{k_{2m_0}}}x, T^j x\right) + G_p\left(T^i x, T^i x, T^{s_{k_{2m_0}}}\right) \\ &\le G_p\left(T^{s_{k_{2m_0}}}x, T^{s_{k_{2m_0}}}x, T^j x\right) + 2G_p\left(T^i x, T^{s_{k_{2m_0}}}x, T^{s_{k_{2m_0}}}x\right) \\ &< 3\varepsilon. \end{aligned}$$

Therefore, we prove that $\lim_{i,j\to\infty} G_p(T^i x, T^i x, T^j x) = 0$. □

Lemma 2.6. Let T be an NLC-operator and $x \in X \backslash J_T(X)$. If x is Type 2, then the sequence $\{T^i x\}_{i\in\mathbb{N}\cup\{0\}}$ is G_p-Cauchy.

Proof. By (2.3), it is easy to see that $\{G_p(T^{s_k}x, T^{s_k}x, T^{s_k}x)\}_{k\in\mathbb{N}\cup\{0\}}$ is a nonincreasing sequence, where $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ is a supporting sequence at x. Then there exists

$$r_x := \lim_k G_p(T^{s_k}x, T^{s_k}x, T^{s_k}x) = \inf_k \{G_p(T^{s_k}x, T^{s_k}x, T^{s_k}x)\}$$

such that it is finite.

At first let us prove that for any $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m \ge m_0$ and $i \ge s_m$, one has

$$G_p\left(T^{s_m}x, T^{s_m}x, T^i x\right) \in (r_x - \varepsilon, r_x + \varepsilon). \tag{2.5}$$

Since x is Type 2 there are $k_0, B \in \mathbb{N}$ such that for all $k \ge k_0$ and all $i \ge s_k$, there holds $r_{k,i} < B$. Let $\varepsilon > 0$, take $m_1 \ge k_0$ such that for all $m \ge m_1$,

$$G_p(T^{s_m}x, T^{s_m}x, T^{s_m}x) \in (r_x - \varepsilon, r_x + \varepsilon). \tag{2.6}$$

Let $m \geq m_1 + B$ and $i \geq s_m$ be arbitrary. For $P = \{m_1, ..., m_1 + B - 1\} \subseteq \{0, 1, ..., m-1\}$, we have card$P \geq B > r_{m,i}$, then there exists $m_1 \leq j \leq m_1 + B - 1$ such that

$$r_x - \varepsilon < G_p\left(T^{s_m}x, T^{s_m}x, T^{s_m}x\right) \leq G_p\left(T^{s_m}x, T^{s_m}x, T^{i}x\right) \leq G_p\left(T^{s_j}x, T^{s_j}x, T^{s_j}x\right) < r_x + \varepsilon.$$

So we get (2.5).

Now let us prove that for any $\varepsilon > 0$, there is $k^* \in \mathbb{N}$ such that for all $i, j \geq k^*$,

$$G_p\left(T^{i}x, T^{i}x, T^{j}x\right) < r_x + \varepsilon. \tag{2.7}$$

Indeed, for any $\varepsilon > 0$, consider m_0 as in (2.5) and let $i, j \geq s_{m_0}$ be arbitrary. Then

$$\begin{aligned} G_p\left(T^{i}x, T^{i}x, T^{j}x\right) &\leq G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{j}x\right) + G_p\left(T^{i}x, T^{i}x, T^{s_{m_0}}x\right) \\ &\quad - G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{s_{m_0}}x\right) \\ &< r_x + \varepsilon + 2G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{i}x\right) - 2G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{s_{m_0}}x\right) \\ &< r_x + \varepsilon + 2\left(r_x + \varepsilon\right) - 2\left(r_x - \varepsilon\right) \\ &= r_x + 5\varepsilon. \end{aligned}$$

To prove $\lim_{i,j\to\infty} G_p\left(T^{i}x, T^{i}x, T^{j}x\right) = r_x$, we only need to show that for any $\varepsilon > 0$, there exists $\widetilde{k} \in \mathbb{N}$ such that for all $i \geq \widetilde{k}$, ones always have

$$r_x - \varepsilon < G_p\left(T^{i}x, T^{i}x, T^{j}x\right). \tag{2.8}$$

Suppose on the contrary, that for any k there is some $i_0 \geq k$ satisfying

$$G_p\left(T^{i_0}x, T^{i_0}x, T^{i_0}x\right) \leq r_x - \varepsilon.$$

Put $z := T^{i_0}x$. Obviously, $x \notin J_T(X)$ implies that $z \notin J_T(X)$. If z is Type 1, then by Lemma 2.5 it follows that

$$0 = \lim_{i,j} G_p\left(T^{i}z, T^{i}z, T^{j}z\right) = \lim_{i,j} G_p\left(T^{i}x, T^{i}x, T^{j}x\right) = r_x,$$

so $\{T^{i}x\}_{i\in\mathbb{N}\cup\{0\}}$ is 0-Cauchy sequence.

Now suppose that z is Type 2, and let $\{q_m\}_{m\in\mathbb{N}\cup\{0\}}$ be a supporting sequence at z. Then, for each $m \in \mathbb{N}\cup\{0\}$,

$$G_p\left(T^{q_m}z, T^{q_m}z, T^{q_m}z\right) \leq G_p\left(z, z, z\right) \leq r_x - \varepsilon,$$

so

$$r_z = \lim_{m} G_p\left(T^{q_m}z, T^{q_m}z, T^{q_m}z\right) \leq r_x - \varepsilon.$$

Note that $r_z < \frac{r_x + r_z}{2}$, then for $j_0 \in \mathbb{N}$, one obtain that

$$G_p\left(T^{j}z, T^{j}z, T^{j}z\right) < \frac{r_x + r_z}{2}, \text{ for all } j \geq j_0.$$

As $\lim_{m\to\infty} G_p(T^{s_m}x, T^{s_m}x, T^{s_m}x) = r_x$, then there is some $m \geq i_0 + j_0$ such that

$$G_p\left(T^j x, T^j x, T^j x\right) > \frac{r_x + r_z}{2},$$

which is impossible, so (2.8) is satisfied. Now, for $s_m - i_0 \geq m - i_0 \geq j_0$, we claim that

$$\frac{r_x + r_z}{2} < G_p(T^{s_m}x, T^{s_m}x, T^{s_m}x) = G_p\left(T^{s_m - i_0}x, T^{s_m - i_0}x, T^{s_m - i_0}x\right) < \frac{r_x + r_z}{2}.$$

In the end, from

$$r_x - \varepsilon < G_p\left(T^i x, T^i x, T^i x\right) \leq G_p\left(T^i x, T^i x, T^j x\right),$$

it follows that $\{T^i x\}_{i\in\mathbb{N}\cup\{0\}}$ is a G_p-Cauchy sequence. □

Lemma 2.7. Let $T : X \to X$ be an operator on G_p-metric space (X, G_p). Suppose that $x \in X$ is a point such that $T^k x = x$ holds for some positive integer k, and there is $y \in X$ such that

$$G_p(y, y, y) = \lim_i G_p\left(y, T^i x, T^i x\right) = \lim_{i,j} G_p\left(T^i x, T^i x, T^j x\right), \tag{2.9}$$

then $Tx = x$.

Proof. Since $T^{k\cdot i}x = x$, then for any $i \in \mathbb{N}\cup\{0\}$, we have that

$$G_p(y, y, y) = \lim_i G_p\left(y, T^{ki}x, T^{ki}x\right) = G_p(y, x, x)$$

and

$$G_p(y, y, y) = \lim_i G_p\left(T^{ki}x, T^{ki}x, T^{ki}x\right) = G_p(x, x, x),$$

so $y = x$. Now (2.9) implies that

$$G_p(x, x, x) = \lim_i G_p\left(x, T^{ki+1}x, T^{ki+1}x\right) = G_p(x, Tx, Tx)$$

and

$$G_p(x, x, x) = \lim_i G_p\left(T^{ki+1}x, T^{ki+1}x, T^{ki+1}x\right) = G_p(Tx, Tx, Tx).$$

Thus, $Tx = x$. □

3 Main results

Both results in this section generalize many existing results in the literature (see [12, Theorem 3.1] and [4, Lemmas 3.-5, Theorems 1 and 2]). Firstly, we announce our first result for NLC-operator in G_p-complete G_p-metric space as follows.

Proposition 3.1. Let T be an NLC-operator on G_p-complete G_p-metric space (X, G_p), then

(1) for each $x \in X$, the sequence $\{T^i x\}_{i\in\mathbb{N}\cup\{0\}}$ G_p-converges to some $v_x \in X$;

(2) for all $x, y \in X$, one has

$$G_p(v_y, v_y, v_x) = \max\{G_p(v_x, v_x, v_x), G_p(v_y, v_y, v_y)\}.$$

Proof. Since (X, G_p) is G_p-complete, then for each $x \in X$, the existence of v_x is assured by Lemma 2.5 and Lemma 2.6. Let us prove (2). Let $x, y \in X$ and $G_p(v_y, v_y, v_y) \geq G_p(v_x, v_x, v_x)$. If $G_p(v_y, v_y, v_y) = 0$, then $v_x = v_y$ and the claim is clear. Thus, assume that $G_p(v_y, v_y, v_y) > 0$.

For any $0 < \varepsilon < \frac{1-\alpha}{2(1+\alpha)} G_p(v_y, v_y, v_y)$, there is some $m_0 \in \mathbb{N}$ such that for all $i, j \geq m_0$, we have

$$\max\{G_p(T^i y, v_y, v_y) - G_p(T^i y, T^i y, T^i y), |G_p(T^i y, T^i y, T^i y) - G_p(v_y, v_y, v_y)|,\\ G_p(T^i y, T^i y, v_y) - G_p(v_y, v_y, v_y)\} < \varepsilon$$

and

$$\max\left\{G_p\left(v_x, v_x, T^j x\right) - G_p\left(v_x, v_x, v_x\right), G_p\left(v_x, T^j x, T^j x\right) - G_p\left(T^j x, T^j x, T^j x\right)\right\} < \varepsilon.$$

For $i, j \geq m_0$, we have

$$\begin{aligned} G_p\left(T^i y, T^i y, T^i x\right) &\leq G_p\left(T^i y, T^i y, v_y\right) - G_p\left(v_y, v_y, v_y\right) \\ &+ G_p\left(v_y, v_y, v_x\right) + G_p\left(v_x, v_x, T^i x\right) - G_p\left(v_x, v_x, v_x\right) \\ &< 2\varepsilon + G_p\left(v_y, v_y, v_x\right) \end{aligned}$$

and

$$\begin{aligned} G_p\left(v_y, v_y, v_x\right) &\leq G_p\left(v_y, v_y, T^i y\right) - G_p\left(T^i y, T^i y, T^i y\right) + G_p\left(v_x, T^j x, T^j x\right) \\ &- G_p\left(T^j y, T^j y, T^j y\right) + G_p\left(T^i y, T^i y, T^j y\right) \\ &< 2\varepsilon + G_p\left(T^i y, T^i y, T^j x\right). \end{aligned}$$

For any $i_0 \geq m_0$ and $i_1 := n\left(T^{i_0} y\right)$, we get

$$\begin{aligned} G_p\left(v_y, v_y, v_x\right) - 2\varepsilon &\leq G_p\left(T^{i_0+i_1} y, T^{i_0+i_1} y, T^{i_0+i_1} x\right) \\ &\leq \max\left\{\alpha G_p\left(T^{i_0} y, T^{i_0} y, T^{i_0} x\right), G_p\left(T^{i_0} y, T^{i_0} y, T^{i_0} y\right)\right\} \\ &< \max\left\{\alpha\left(2\varepsilon + G_p\left(v_y, v_y, v_x\right), \varepsilon + G_p\left(v_y, v_y, v_y\right)\right)\right\}. \end{aligned}$$

If

$$G_p\left(v_y, v_y, v_x\right) - 2\varepsilon < 2\alpha\varepsilon + \alpha G_p\left(v_y, v_y, v_x\right),$$

then

$$G_p\left(v_y, v_y, v_x\right) < 2\varepsilon\left(1+\alpha\right) < G_p\left(v_y, v_y, v_x\right).$$

This is a contradiction. As a consequence, we deduce that

$$G_p\left(v_y, v_y, v_x\right) < 3\varepsilon + G_p\left(v_y, v_y, v_y\right),$$

so

$$G_p\left(v_y, v_y, v_x\right) \leq G_p\left(v_y, v_y, v_y\right).$$

Finally, by (G$_p$2), we speculate that

$$G_p(v_y, v_y, v_x) = G_p(v_y, v_y, v_x) = \max\{G_p(v_x, v_x, v_x), G_p(v_y, v_y, v_y)\}.$$

□

Now, we announce our second result in the framework of G_p-complete G_p-metric spaces.

Theorem 3.2. Let T be an NLC-opertor on G_p-complete G_p-metric space (X, G_p), then there is a fixed point $z \in X$ of T such that $G_p(z, z, z) = \inf\{G_p(v_x, v_x, v_x) : x \in X\}$.

Proof. For $x \in X$, put $r_x := G_p(v_x, v_x, v_x) = \lim_{k\to\infty} G_p(T^{s_k}x, T^{s_k}x, T^{s_k}x)$ for $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ which is the supporting sequence at x. Let $I := \inf\{r_x : x \in X\}$. For $m \geq 1$, take $x_m \in X$ such that for all $i, j \in \mathbb{N}\cup\{0\}$, it holds

$$G_p\left(Tx_m^i, Tx_m^i, Tx_m^j\right) \in \left(I - \frac{1}{m}, I + \frac{1}{m}\right). \tag{3.1}$$

At first we shall prove that $\lim_{m,k\to\infty} G_p(x_m, x_m, x_k) = I$. For $m, k \geq 2$, let $C_{m,k} > 0$ and

$$G_p\left(T^j x_m, T^j x_m, T^i x_k\right) < C_{m,k}, \quad i, j \in \mathbb{N}\cup\{0\}.$$

Fix $m, k \geq 2$ and let $\{s_q\}_{q\in\mathbb{N}\cup\{0\}}$ be the supporting sequence at x_m and let $l \geq 1$ be an integer such that $\alpha^l \cdot C_{m,k} < \frac{1}{k+m}$. Then

$$\begin{aligned}
&G_p(x_m, x_m, x_k)\\
&\leq G_p(x_m, x_m, T^{s_l}x_m) - G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_m) + G_p(T^{s_l}x_m, T^{s_l}x_m, x_k)\\
&\leq G_p(x_m, x_m, T^{s_l}x_m) - G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_m)\\
&\quad + G_p(x_k, T^{s_l}x_k, T^{s_l}x_k) - G_p(T^{s_l}x_k, T^{s_l}x_k, T^{s_l}x_k) + G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k).
\end{aligned}$$

Denote

$$\begin{aligned}
A_{m,k} &:= G_p(x_m, x_m, T^{s_l}x_m) - G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_m) < \frac{2}{m},\\
D_{m,k} &:= G_p(x_k, T^{s_l}x_k, T^{s_l}x_k) - G_p(T^{s_l}x_k, T^{s_l}x_k, T^{s_l}x_k) < \frac{2}{k}.
\end{aligned}$$

At first, assume that

$$G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k) > G_p\left(T^i x_m, T^i x_m, T^i x_m\right) \text{ for all } i \in \{0, 1, ..., s_l\}.$$

Then by (1.2), it is clear that

$$\begin{aligned}
&G_p(T^{s_{j-1}}x_m, T^{s_{l-1}}x_m, T^{s_{l-1}}x_k) \leq \alpha G_p(T^{s_{l_1}}x_m, T^{s_{l_1}}x_m, T^{s_{l_1}}x_k)\\
&\leq \alpha^2 G_p(T^{s_{l-2}}x_m, T^{s_{l-2}}x_m, T^{s_{l-2}}x_k) \leq ...\\
&\leq \alpha^l G_p(x_m, x_m, x_k) \leq \alpha^l \cdot C_{m,k} < \frac{1}{k+n}.
\end{aligned}$$

If $G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k) \leq G_p(T^i x_m, T^i x_m, T^i x_k)$ for some $i \in \{0, ..., s_l\}$, then by (3.1), $G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k) < I + \frac{1}{m}$, so

$$\begin{aligned}
G_p(x_m, x_m, x_k) &\leq A_{m,k} + D_{m,k} + G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k)\\
&< \frac{2}{m} + \frac{2}{k} + I + \frac{1}{m}.
\end{aligned}$$

From the above consideration and

$$I - \frac{1}{m} < G_p(x_m, x_m, x_m) \le G_p(x_m, x_m, x_k),$$

it follows that

$$\lim_{m,k\to\infty} G_p(x_m, x_m, x_m) = I.$$

Now that (X, G_p) is G_p-complete, there is some $u \in X$ such that

$$I = \lim_{m,k\to\infty} G_p(x_m, x_m, x_k) = \lim_{m\to\infty} G_p(x_m, x_m, u) = \lim_{m\to\infty} G_p(x_m, u, u) = G_p(u, u, u).$$

It is easy to see that

$$G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}u\right) = G_p(u, u, u) = I.$$

Now we shall prove that $T^{n(u)}u = u$.

From

$$\begin{aligned} I &= G_p(u, u, u) \le G_p\left(u, u, T^j x_m\right) \\ &\le G_p(u, u, x_m) + G_p\left(x_m, x_m, T^j x_m\right) - G_p(x_m, x_m, x_m) \\ &\le G_p(u, u, x_m) + I + \frac{1}{m} - \left(I - \frac{1}{m}\right) \\ &= G_p(u, u, x_m) + \frac{2}{m}, \end{aligned}$$

it means that

$$\lim_{m\to\infty} G_p\left(u, u, T^j x_m\right) = I, \quad j \in \mathbb{N}.$$

On the other hand,

$$\begin{aligned} I &= G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}u\right) \le G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) \\ &\le \max\left\{\alpha G_p(u, u, x_m), G_p(u, u, u)\right\}, \end{aligned}$$

which implies that

$$\lim_{m\to\infty} G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) = I.$$

Now that

$$\begin{aligned} I &\le G_p\left(u, T^{n(u)}u, T^{n(u)}u\right) \\ &\le G_p\left(u, T^{n(u)}x_m, T^{n(u)}x_m\right) + G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) \\ &\quad - G_p\left(T^{n(u)}x_m, T^{n(u)}x_m, T^{n(u)}x_m\right) \\ &\le 2G_p\left(u, u, T^{n(u)}, x_m\right) - G_p(u, u, u) + G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) - I + \frac{1}{m} \\ &\to I, \quad \text{as } m \to \infty, \end{aligned}$$

so

$$G_p\left(u, T^{n(u)}u, T^{n(u)}u\right) = I = G_p(u, u, u) = G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}u\right),$$

and $T^{n(u)}u = u$. Finally, by utilizing Lemma 2.7, the remaining proof is valid. □

4 Some examples

Now, we give four examples to support our theoretical approach.

Example 4.1. Let $X = \{0, 1, 2\}$ be a set and $G_p : X^3 \to [0, +\infty)$ a mapping satisfying

$$\begin{aligned}
&G_p(x, x, x) = \frac{1}{2} \text{ for all } x \in X, \\
&G_p(0, 0, 1) = G_p(0, 1, 0) = G_p(1, 0, 0) = 1, \\
&G_p(0, 1, 1) = G_p(1, 0, 1) = G_p(1, 1, 0) = 1, \\
&G_p(1, 2, 2) = G_p(2, 1, 2) = G_p(2, 2, 1) = 3, \\
&G_p(0, 0, 2) = G_p(0, 2, 0) = G_p(2, 0, 0) = 3, \\
&G_p(0, 2, 2) = G_p(2, 0, 2) = G_p(2, 2, 0) = 3, \\
&G_p(1, 1, 2) = G_p(1, 2, 1) = G_p(2, 1, 1) = 3.1, \\
&G_p(0, 1, 2) = G_p(0, 2, 1) = G_p(1, 0, 2) = 3.2.
\end{aligned}$$

Then G_p is an asymmetric G_p-metric as $G_p(1, 2, 2) \neq G_p(1, 1, 2)$. Further, (X, G_p) is a G_p-metric space in the sense of [18]. Let $T : X \to X$ be defined by $T0 = T1 = 0, T2 = 1$. We shall prove that T is an NLC-operator where $\alpha = \frac{1}{2}$, while $n(x) = 1$ for all $x \in X$. Indeed, we need to check

$$G_p(Tx, Tx, Ty) \leq \max\left\{\frac{1}{2}G_p(x, x, y), G_p(x, x, x)\right\} \tag{4.1}$$

for all $x, y \in X$ into nine cases as follows:

$$\begin{aligned}
&(1)\quad x = 0,\ y = 0 \Longrightarrow \frac{1}{2} = G_p(T0, T0, T0) \leq \max\left\{\frac{1}{2}G_p(0, 0, 0), G_p(0, 0, 0)\right\} = \frac{1}{2}, \\
&(2)\quad x = 0,\ y = 1 \Longrightarrow \frac{1}{2} = G_p(T0, T0, T1) \leq \max\left\{\frac{1}{2}G_p(0, 0, 1), G_p(0, 0, 0)\right\} = \frac{1}{2}, \\
&(3)\quad x = 0,\ y = 2 \Longrightarrow 1 = G_p(T0, T0, T2) \leq \max\left\{\frac{1}{2}G_p(0, 0, 2), G_p(0, 0, 0)\right\} = \frac{3}{2}, \\
&(4)\quad x = 1,\ y = 0 \Longrightarrow \frac{1}{2} = G_p(T1, T1, T0) \leq \max\left\{\frac{1}{2}G_p(1, 1, 0), G_p(1, 1, 1)\right\} = \frac{1}{2}, \\
&(5)\quad x = 1,\ y = 1 \Longrightarrow \frac{1}{2} = G_p(T1, T1, T1) \leq \max\left\{\frac{1}{2}G_p(1, 1, 1), G_p(1, 1, 1)\right\} = \frac{1}{2}, \\
&(6)\quad x = 1,\ y = 2 \Longrightarrow 1 = G_p(T1, T1, T2) \leq \max\left\{\frac{1}{2}G_p(1, 1, 2), G_p(1, 1, 1)\right\} = \frac{3.1}{2}, \\
&(7)\quad x = 2,\ y = 0 \Longrightarrow 1 = G_p(T2, T2, T0) \leq \max\left\{\frac{1}{2}G_p(2, 2, 0), G_p(2, 2, 2)\right\} = \frac{3}{2}, \\
&(8)\quad x = 2,\ y = 1 \Longrightarrow 1 = G_p(T2, T2, T1) \leq \max\left\{\frac{1}{2}G_p(2, 2, 1), G_p(2, 2, 2)\right\} = \frac{3}{2}, \\
&(9)\quad x = 2,\ y = 2 \Longrightarrow \frac{1}{2} = G_p(T2, T2, T2) \leq \max\left\{\frac{1}{2}G_p(2, 2, 2), G_p(2, 2, 2)\right\} = \frac{1}{2}.
\end{aligned}$$

Hence all cases show that (4.1) is satisfied and then both Proposition 3.1 and Theorem 3.2 are true. □

Example 4.2. Let (X, G_p) be a G_p-metric space where $X = [0, +\infty)$ and $G_p(x, y, z) = \max\{x, y, x\}$. Define $T : X \to X$ by $Tx = \frac{x^2}{3(1+x)}$. We shall prove that T is an NLC-operator, that is, for each $x \in X$, there is $n(x)$ such that for each $y \in X$,

$$G_p\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \leq \max\{\alpha G_p(x, x, y), G_p(x, x, x)\},$$

where $\alpha \in (0, 1)$. Let $\alpha = \frac{1}{3}$. It is easy to see that $n(x) = 1$. Indeed, we shall check that

$$\max\left\{\frac{x^2}{3(1+x)}, \frac{x^2}{3(1+x)}, \frac{y^2}{3(1+y)}\right\} \leq \max\left\{\frac{1}{3}\max\{x, x, y\}, \max\{x, x, x\}\right\}, \quad (4.2)$$

for all $x, y \in [0, +\infty)$.

Consider the following three possible cases.

(i) $y \leq x$. In this case (4.2) becomes:

$$\frac{x^2}{3(1+x)} \leq \max\left\{\frac{1}{3}x, x\right\} = x, \quad (4.3)$$

which is true for any $x \in [0, +\infty)$.

(ii) $\frac{y}{3} \leq x \leq y$. In this case (4.2) becomes:

$$\frac{y^2}{3(1+y)} \leq \max\left\{\frac{1}{3}y, x\right\} = x. \quad (4.4)$$

By virtue of $\frac{y^2}{3(1+y)} = \frac{y}{1+y} \cdot \frac{y}{3} \leq \frac{y}{3} \leq x$, it follows that (4.4) holds.

(iii) $0 \leq x \leq \frac{y}{3}$. Because of $x \leq y$, (4.2) becomes:

$$\frac{y^2}{3(1+y)} \leq \max\left\{\frac{1}{3}y, x\right\} = \frac{1}{3}y. \quad (4.5)$$

Obviously, (4.5) holds for each $y \in [0, +\infty)$.

Hence, (4.2) holds for all $x, y \in [0, +\infty)$, that is., all the conditions of Proposition 3.1 and Theorem 3.2 are satisfied and T has a fixed point (which is $x = 0$). □

Example 4.3. Let $X = \{a, b\}$ be a set with G_p-metric defined by

$$G_p(a, a, a) = 0, \quad G_p(a, a, b) = G_p(a, b, a) = G_p(b, a, a) = 1,$$
$$G_p(b, b, b) = G_p(a, b, b) = G_p(b, a, b) = G_p(b, b, a) = 2.$$

Since $G_p(a, a, b) \neq G_p(a, b, b)$, we get that (X, G_p) is an asymmetric G_p-metric space. Also, we have that for all $x, y \in X$,

$$d_{G_p}(x, y) = G_p(x, x, y) + G_p(x, y, y) - G_p(x, x, x) - G_p(y, y, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y, \end{cases}$$

is a (standard) metric on X. □

Example 4.4. Let $X = \{a, b\}$ be a set with G_p-metric defined by

$$G_p(a, a, a) = 0, \quad G_p(a, a, b) = G_p(a, b, a) = G_p(b, a, a) = 1 = G_p(b, b, b),$$
$$G_p(a, b, b) = G_p(b, a, b) = G_p(b, b, a) = 2.$$

It ensures us that the sequence $\{x_n = a\}$ converges to a. However, conditions (2) and (3) of Proposition 1.8 are not equivalent. Indeed,

$$G_p(x_n, x_n, b) = G_p(a, a, b) \to G_p(b, b, b) \quad (n \to \infty),$$

while

$$G_p(x_n, b, b) = G_p(a, b, b) \nrightarrow G_p(b, b, b) \quad (n \to \infty).$$

Thus, $G_p(\cdot, \cdot,)$ may not be continuous in the sense that $x_n \to x,\ y_n \to y$ and $z_n \to z$ implies $G_p(x_n, y_n, z_n) \to G_p(x, y, z)$. In fact, we take $x_n = y_n = a$ and $z_n = b$ for all $n \in \mathbb{N}$. Further, it is easy to check that $x_n \to b,\ y_n \to a$ and $z_n \to b$ but $G_p(x_n, y_n, z_n) \nrightarrow G_p(b, a, b)$, this is because $G_p(x_n, y_n, z_n) = G_p(a, a, b) = 1 \neq 2 = G_p(b, a, b)$. □

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Most general Self Adjoint Operator Chebyshev-Grüss Inequalities

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Abstract

We demonstrate here most general self adjoint operator Chebyshev-Grüss type inequalities to all cases. We finish with applications.

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1 Motivation

Here we mention the following interesting and motivating results.

Theorem 1 *(Čebyšev, 1882, [2]). Let $f, g : [a,b] \to \mathbb{R}$ absolutely continuous functions. If $f', g' \in L_\infty([a,b])$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \tag{1}$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty .$$

Also we mention

Theorem 2 *(Grüss, 1935, [6]). Let f, g integrable functions from $[a,b]$ into $\mathbb{R}$, such that $m \leq f(x) \leq M$, $\rho \leq g(x) \leq \sigma$, for all $x \in [a,b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \tag{2}$$

$$\leq \frac{1}{4} (M-m)(\sigma-\rho) .$$

2 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defind on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [5, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$.

Equivalently, we use (see [4], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle), \tag{3}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \quad \text{and} \quad g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

In this article we will be using a lot the formula

$$\langle f(U)x, x\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x\rangle), \quad \forall\, x \in H. \tag{4}$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda. \tag{5}$$

Above, $m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \ \ \forall\, \lambda \in \mathbb{R}, \tag{6}$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [7], pp. 256-266, and for more detalis see there pp. 157-266. See also [3].

Some more basics are given (we follow [4], pp. 1-5):

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$, $\forall\, x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x\in H: \|x\|=1} |\langle Ax, x\rangle|. \tag{7}$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$, $\forall\, x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0,\ \alpha_k \in \mathbb{C},\ 0 \leq k \leq n \right\}. \tag{8}$$

If $A \in \mathcal{B}(H)$ (the Banach algebra of all bounded linear operators defined on H, i.e. from H into itself) is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{9}$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{10}$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [4], p. 4, Theorem 7).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\}$$
$$= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|,$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|. \tag{11}$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}. \tag{12}$$

Let $A, B \in \mathcal{B}(H)$, then

$$\|AB\| \leq \|A\| \|B\|, \tag{13}$$

by Banach algebra property.

3 Main Results

Next we present most general Chebyshev-Grüss type operator inequalities based on Theorem 26.9 of [1], p. 404.

Then we specialize them for $n = 1$.

We give

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Theorem 3 *Let $n \in \mathbb{N}$ and $f_1, f_2 \in C^n([a,b])$ with $[m,M] \subset (a,b)$, $m < M$; $g \in C^1([a,b])$ and $g^{-1} \in C^n([a,b])$. Here A is a selfadjoint linear operator on the Hilbert space H with spectrum $Sp(A) \subseteq [m,M]$. We consider any $x \in H$: $\|x\| = 1$.*

Then

$$\langle (\Delta(f_1, f_2; g))(A)x, x\rangle :=$$

$$\Bigg| \langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A)x,x\rangle \cdot \langle f_2(A)x,x\rangle - \frac{1}{2(M-m)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \cdot \right.$$

$$\left\{ \left[\left\langle \left(f_2(A) \int_{m-0}^{M} \left(\int_m^M \left(f_1 \circ g^{-1}\right)^{(k)}(g(t)) (g(\lambda) - g(t))^k dt \right) dE_\lambda \right) x, x \right\rangle - \right.\right.$$

$$\left. \langle f_2(A)x,x\rangle \int_{m-0}^{M} \left(\int_m^M \left(f_1 \circ g^{-1}\right)^{(k)}(g(t)) (g(\lambda) - g(t))^k dt \right) d\langle E_\lambda x, x\rangle \right] +$$

$$\left[\left\langle \left(f_1(A) \int_{m-0}^{M} \left(\int_m^M \left(f_2 \circ g^{-1}\right)^{(k)}(g(t)) (g(\lambda) - g(t))^k dt \right) dE_\lambda \right) x, x \right\rangle - \right.$$

$$\left.\left.\left. \langle f_1(A)x,x\rangle \int_{m-0}^{M} \left(\int_m^M \left(f_2 \circ g^{-1}\right)^{(k)}(g(t)) (g(\lambda) - g(t))^k dt \right) d\langle E_\lambda x, x\rangle \right] \right\} \right\} \Bigg|$$

$$\leq \frac{\|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]}}{(n+1)!(M-m)} \left[\|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_{\infty,[m,M]} + \right.$$

$$\left. \|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_{\infty,[m,M]} \right] \left[\left\| (M1_H - A)^{n+1} \right\| + \left\| (A - m1_H)^{n+1} \right\| \right]. \tag{14}$$

Proof. Call $l_i = f_i \circ g^{-1}$, $i = 1,2$. Then $l_i, l_i', ..., l_i^{(n)}$ are continuous from $g([a,b])$ into $f_i([a,b])$, $i = 1,2$. Hence $\left(f_i \circ g^{-1}\right)^{(n)} \circ g \in C([a,b])$, $i = 1,2$. Here $\{E_\lambda\}_\lambda$ is the spectral family of A.

Next we use Theorem 26.9 of [1], p. 404. We have that $(i = 1,2)$

$$f_i(\lambda) = \frac{1}{M-m} \int_m^M f_i(t) dt +$$

$$\frac{1}{(M-m)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_m^M \left(f_i \circ g^{-1}\right)^{(k)}(g(t)) (g(\lambda) - g(t))^k dt \right\} +$$

$$+ \frac{1}{(n-1)!(M-m)} \int_m^M (g(\lambda) - g(t))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(g(t)) g'(t) K(t,\lambda) dt, \tag{15}$$

$\forall \lambda \in [m,M]$,

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where

$$K(t,\lambda) := \begin{cases} t-m, & m \leq t \leq \lambda \leq M, \\ t-M, & m \leq \lambda < t \leq M. \end{cases} \quad (16)$$

By applying the spectral representation theorem on (15), i.e. integrating against E_λ over $[m,M]$, see (4), we obtain:

$$f_i(A) = \left(\frac{1}{M-m}\int_m^M f_i(t)\,dt\right)1_H +$$

$$\frac{1}{(M-m)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{m-0}^M\left(\int_m^M \left(f_i\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\int_{m-0}^M\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda, \quad (17)$$

$i=1,2$.

We notice that

$$f_1(A)\,f_2(A) = f_2(A)\,f_1(A), \quad (18)$$

to be used next.

Hence it holds

$$f_2(A)\,f_1(A) = \left(\frac{1}{M-m}\int_m^M f_1(t)\,dt\right)f_2(A) + \quad (19)$$

$$\frac{1}{(M-m)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}f_2(A)\int_{m-0}^M\left(\int_m^M \left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}f_2(A)\cdot$$

$$\int_{m-0}^M\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda,$$

and

$$f_1(A)\,f_2(A) = \left(\frac{1}{M-m}\int_m^M f_2(t)\,dt\right)f_1(A) +$$

$$\frac{1}{(M-m)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}f_1(A)\int_{m-0}^M\left(\int_m^M \left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}f_1(A)\cdot$$

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$$\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda. \quad (20)$$

Here from now on we consider $x\in H:\|x\|=1$; immediately we get

$$\int_{m-0}^{M} d\left\langle E_\lambda x,x\right\rangle = 1.$$

Then it holds $(i=1,2)$

$$\left\langle f_i(A)\,x,x\right\rangle = \left(\frac{1}{M-m}\int_{m}^{M} f_i(t)\,dt\right)+ \quad (21)$$

$$\frac{1}{(M-m)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_i\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle.$$

It follows that

$$\left\langle f_2(A)\,x,x\right\rangle\left\langle f_1(A)\,x,x\right\rangle = \left(\frac{1}{M-m}\int_{m}^{M} f_1(t)\,dt\right)\left\langle f_2(A)\,x,x\right\rangle+\frac{1}{(M-m)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle f_2(A)\,x,x\right\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\left\langle f_2(A)\,x,x\right\rangle\cdot \quad (22)$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle,$$

and

$$\left\langle f_1(A)\,x,x\right\rangle\left\langle f_2(A)\,x,x\right\rangle = \left(\frac{1}{M-m}\int_{m}^{M} f_2(t)\,dt\right)\left\langle f_1(A)\,x,x\right\rangle+\frac{1}{(M-m)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle f_1(A)\,x,x\right\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\left\langle f_1(A)\,x,x\right\rangle\cdot \quad (23)$$

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$$\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle .$$

Furthermore we obtain

$$\langle f_1(A)\,f_2(A)\,x,x\rangle=\left(\frac{1}{M-m}\int_m^M f_1(t)\,dt\right)\langle f_2(A)\,x,x\rangle+\frac{1}{(M-m)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right)x,x\right\rangle\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot \tag{24}$$

$$\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M(g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle,$$

and

$$\langle f_1(A)\,f_2(A)\,x,x\rangle=\left(\frac{1}{M-m}\int_m^M f_2(t)\,dt\right)\langle f_1(A)\,x,x\rangle+\frac{1}{(M-m)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right)x,x\right\rangle\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot \tag{25}$$

$$\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M(g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle.$$

By (24)-(22) we obtain

$$E:=\langle f_1(A)\,f_2(A)\,x,x\rangle-\langle f_1(A)\,x,x\rangle\langle f_2(A)\,x,x\rangle=\frac{1}{(M-m)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right)x,x\right\rangle\right.\right.$$

$$\left.\left.-\langle f_2(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_m^M\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\langle E_\lambda x,x\rangle\right]\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\left[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M(g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle\right. \tag{26}$$

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$$-\left\langle f_2(A)x,x\right\rangle\cdot$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle\Bigg],$$

and by (25)-(23) we have

$$E=\frac{1}{(M-m)}\cdot$$

$$\Bigg\{\sum_{k=1}^{n-1}\frac{1}{k!}\Bigg[\Bigg\langle\Bigg(f_1(A)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\Bigg)x,x\Bigg\rangle$$

$$-\left\langle f_1(A)x,x\right\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\Bigg]\Bigg\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\Bigg[\Bigg\langle\Bigg(f_1(A)\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\Bigg)x,x\Bigg\rangle$$

$$-\left\langle f_1(A)x,x\right\rangle\cdot$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle\Bigg]. \tag{27}$$

Consequently, by adding (26) and (27), we get that

$$2E=\frac{1}{(M-m)}\cdot \tag{28}$$

$$\Bigg\{\sum_{k=1}^{n-1}\frac{1}{k!}\Bigg\{\Bigg[\Bigg\langle\Bigg(f_2(A)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\Bigg)x,x\Bigg\rangle$$

$$-\left\langle f_2(A)x,x\right\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\Bigg]+$$

$$\Bigg[\Bigg\langle\Bigg(f_1(A)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\Bigg)x,x\Bigg\rangle$$

$$-\left\langle f_1(A)x,x\right\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\Bigg]\Bigg\}\Bigg\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\Bigg\{\Bigg[\Bigg\langle\Bigg(f_2(A)\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\Bigg)x,x\Bigg\rangle$$

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$$-\left\langle f_2(A)x,x\right\rangle \cdot$$

$$\left.\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right]+$$

$$\left[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle\right.$$

$$-\left\langle f_1(A)x,x\right\rangle \cdot$$

$$\left.\left.\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right]\right\}.$$

We find that

$$\left\langle f_1(A)f_2(A)x,x\right\rangle-\left\langle f_1(A)x,x\right\rangle\left\langle f_2(A)x,x\right\rangle-\frac{1}{2(M-m)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\{\left[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right)x,x\right\rangle\right.\right.\right.$$

$$\left.-\left\langle f_2(A)x,x\right\rangle\int_{m-0}^{M}\left(\int_m^M\left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right]+$$

$$\left[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right)x,x\right\rangle\right.$$

$$\left.\left.\left.-\left\langle f_1(A)x,x\right\rangle\int_{m-0}^{M}\left(\int_m^M\left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right]\right\}\right\}$$

$$=\frac{1}{2(n-1)!(M-m)}\cdot$$

$$\left\{\left[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle\right.\right.$$

$$-\left\langle f_2(A)x,x\right\rangle \cdot$$

$$\left.\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right]+$$

$$\left[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle\right.$$

$$-\left\langle f_1(A)x,x\right\rangle \cdot$$

$$\left.\left.\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\left\langle E_\lambda x,x\right\rangle\right]\right\}=:R. \tag{29}$$

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Hence we have

$$|R| \leq \frac{1}{2(n-1)!(M-m)} \cdot$$

$$\left\{ \left[\left| \left\langle \left(f_2(A) \int_{m-0}^{M} \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) dE_\lambda \right) x, x \right\rangle \right| \right. \right.$$

$$+ |\langle f_2(A) x, x \rangle|$$

$$\left. \left| \int_{m-0}^{M} \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) d\langle E_\lambda x, x \rangle \right| \right] +$$

$$\left[\left| \left\langle \left(f_1(A) \int_{m-0}^{M} \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_2 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) dE_\lambda \right) x, x \right\rangle \right| \right.$$

$$+ |\langle f_1(A) x, x \rangle| \cdot$$

$$\left. \left. \left| \int_{m-0}^{M} \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_2 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) d\langle E_\lambda x, x \rangle \right| \right] \right\} \tag{30}$$

(here notice that

$$\left| \int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right| \leq$$

$$\int_m^M |g(\lambda) - g(t)|^{n-1} \left| \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) \right| |g'(t)| |K(t,\lambda)| dt \leq$$

$$\left(\int_m^M |\lambda - t|^{n-1} |K(t,\lambda)| dt \right) \|g\|_\infty^{n-1} \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty \|g'\|_\infty = \tag{31}$$

$$\frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} \left[(M-\lambda)^{n+1} + (\lambda - m)^{n+1} \right])$$

$$\leq \frac{1}{2(n-1)!(M-m)} \cdot$$

$$\left\{ \left[\left\| f_2(A) \int_{m-0}^{M} \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) dE_\lambda \right\| \right. \right.$$

$$\left. + \left\| f_1(A) \int_{m-0}^{M} \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_2 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) dE_\lambda \right\| \right]$$

$$+ \left[\|f_2(A)\| \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} \cdot \right.$$

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$$\left[\left\langle \left(M1_H - A\right)^{n+1} x, x\right\rangle + \left\langle \left(A - m1_H\right)^{n+1} x, x\right\rangle\right] +$$

$$\left\| f_1(A) \right\| \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} \cdot$$

$$\left[\left\langle \left(M1_H - A\right)^{n+1} x, x\right\rangle + \left\langle \left(A - m1_H\right)^{n+1} x, x\right\rangle\right]\right\} \tag{32}$$

$$\leq \frac{1}{2(n-1)!(M-m)} \cdot$$

$$\left\{ \|f_2(A)\| \left\| \int_{m-0}^M \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) dE_\lambda \right\| + \right.$$

$$\left. \|f_1(A)\| \left\| \int_{m-0}^M \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_2 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) dE_\lambda \right\| \right]$$

$$+ \left[\frac{\|f_2(A)\| \|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} + \right.$$

$$\left. \frac{\|f_1(A)\| \|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} \right] \cdot$$

$$\left[\left\langle \left(M1_H - A\right)^{n+1} x, x\right\rangle + \left\langle \left(A - m1_H\right)^{n+1} x, x\right\rangle\right]\right\} =: (\xi). \tag{33}$$

Notice here that

$$\left\| \int_{m-0}^M \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) dE_\lambda \right\| =$$

$$\sup_{\|x\|=1} \left| \int_{m-0}^M \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda) dt \right) d\langle E_\lambda x, x\rangle \right|$$

$$\leq \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} \cdot$$

$$\left[\left\| \left(M1_H - A\right)^{n+1} \right\| + \left\| \left(A - m1_H\right)^{n+1} \right\| \right]. \tag{34}$$

A similar estimate to (34) holds for f_2.

Hence we obtain by (33), (34) that

$$(\xi) \leq \frac{1}{(n-1)!(M-m)} \left[\|f_2(A)\| \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} \right.$$

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$$+ \|f_1(A)\| \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty}{n(n+1)} \Bigg] \cdot$$

$$\left[\left\| (M1_H - A)^{n+1} \right\| + \left\| (A - m1_H)^{n+1} \right\| \right] = \tag{35}$$

$$\frac{\|g\|_\infty^{n-1} \|g'\|_\infty}{(n+1)!(M-m)} \cdot$$

$$\left[\|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty + \|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty \right] \cdot$$

$$\left[\left\| (M1_H - A)^{n+1} \right\| + \left\| (A - m1_H)^{n+1} \right\| \right].$$

We have proved that

$$|R| \leq \frac{\|g\|_\infty^{n-1} \|g'\|_\infty}{(n+1)!(M-m)} \cdot$$

$$\left[\|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty + \|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_\infty \right] \cdot \tag{36}$$

$$\left[\left\| (M1_H - A)^{n+1} \right\| + \left\| (A - m1_H)^{n+1} \right\| \right],$$

that is proving the claim.

Above it is $\|\cdot\|_\infty = \|\cdot\|_{\infty,[m,M]}$. ■

We give

Corollary 4 *($n = 1$ case of Theorem 3) For every $x \in H : \|x\| = 1$, we obtain that*

$$|\langle f_1(A) f_2(A) x, x \rangle - \langle f_1(A) x, x \rangle \langle f_2(A) x, x \rangle| \leq \frac{\|g'\|_{\infty,[m,M]}}{2(M-m)} \cdot$$

$$\left[\|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)' \circ g \right\|_{\infty,[m,M]} + \|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)' \circ g \right\|_{\infty,[m,M]} \right] \cdot$$

$$\left[\left\| (M1_H - A)^2 \right\| + \left\| (A - m1_H)^2 \right\| \right]. \tag{37}$$

We present

Theorem 5 *Here all as in Theorem 3. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\langle (\Delta(f_1, f_2; g))(A) x, x \rangle \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]}}{(n-1)!(M-m)} \left(\frac{\Gamma(p(n-1)+1)\Gamma(p+1)}{\Gamma(pn+2)} \right)^{\frac{1}{p}} \cdot$$

$$\left[\|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} + \|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} \right] \cdot$$

$$\left[\left\| (M1_H - A)^{n+\frac{1}{p}} \right\| + \left\| (A - m1_H)^{n+\frac{1}{p}} \right\| \right], \tag{38}$$

where Γ is the gamma function.

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Proof. We observe that

$$\left|\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right| \leq$$

$$\int_m^M |g(\lambda)-g(t)|^{n-1}\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right| |g'(t)|\,|K(t,\lambda)|\,dt \leq$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\int_m^M |\lambda-t|^{n-1}\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right| |K(t,\lambda)|\,dt = \tag{39}$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left[\int_m^\lambda (\lambda-t)^{n-1}(t-m)\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right| dt + \right.$$

$$\left.\int_\lambda^M (M-t)(t-\lambda)^{n-1}\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right| dt\right] \leq$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left[\left(\int_m^\lambda (\lambda-t)^{(p(n-1)+1)-1}(t-m)^{(p+1)-1}\,dt\right)^{\frac{1}{p}} + \right.$$

$$\left.\left(\int_\lambda^M (M-t)^{(p+1)-1}(t-\lambda)^{(p(n-1)+1)-1}\,dt\right)^{\frac{1}{p}}\right]\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]} =$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]} \cdot$$

$$\left(\frac{\Gamma(p(n-1)+1)\,\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[(\lambda-m)^{n+\frac{1}{p}}+(M-\lambda)^{n+\frac{1}{p}}\right], \tag{40}$$

$\forall\ \lambda\in[m,M]$.

So we got so far

$$\left|\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right| \leq$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]} \cdot$$

$$\left(\frac{\Gamma(p(n-1)+1)\,\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[(M-\lambda)^{n+\frac{1}{p}}+(\lambda-m)^{n+\frac{1}{p}}\right], \tag{41}$$

$\forall\ \lambda\in[m,M]$.

Hence it holds

$$\left|\int_{m-0}^M\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right) d\langle E_\lambda x,x\rangle\right| \leq$$

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$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_i\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\cdot$$
$$\left(\frac{\Gamma(p(n-1)+1)\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[\left\|(M1_H-A)^{n+\frac{1}{p}}\right\|+\left\|(A-m1_H)^{n+\frac{1}{p}}\right\|\right], \tag{42}$$

for $i=1,2$.

Thus we derive

$$\left\|\int_{m-0}^{M}\left(\int_m^M(g(\lambda)-g(t))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right\|\le$$
$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_i\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\cdot$$
$$\left(\frac{\Gamma(p(n-1)+1)\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[\left\|(M1_H-A)^{n+\frac{1}{p}}\right\|+\left\|(A-m1_H)^{n+\frac{1}{p}}\right\|\right], \tag{43}$$

for $i=1,2$.

Next we use (42) and (43).

Acting as in the proof of Theorem 3 we find that

$$|R|\overset{(30)}{\le}\frac{1}{2(n-1)!(M-m)}\cdot$$
$$\Bigg\{2\left[\|f_2(A)\|\,\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\cdot\right.$$
$$\left.\left(\frac{\Gamma(p(n-1)+1)\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[\left\|(M1_H-A)^{n+\frac{1}{p}}\right\|+\left\|(A-m1_H)^{n+\frac{1}{p}}\right\|\right]\right]+ \tag{44}$$
$$2\left[\|f_1(A)\|\,\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_2\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\cdot\right.$$
$$\left.\left(\frac{\Gamma(p(n-1)+1)\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[\left\|(M1_H-A)^{n+\frac{1}{p}}\right\|+\left\|(A-m1_H)^{n+\frac{1}{p}}\right\|\right]\right]\Bigg\}=$$
$$\frac{1}{(n-1)!(M-m)}\Bigg\{\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left(\frac{\Gamma(p(n-1)+1)\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\cdot$$
$$\left[\|f_2(A)\|\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}+\|f_1(A)\|\left\|\left(f_2\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\right]\cdot$$
$$\left[\left\|(M1_H-A)^{n+\frac{1}{p}}\right\|+\left\|(A-m1_H)^{n+\frac{1}{p}}\right\|\right]\Bigg\}, \tag{45}$$

proving the claim. ■

We give for $n=1$:

Corollary 6 *(to Theorem 5) It holds*

$$|\langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A) x, x\rangle \langle f_2(A) x, x\rangle| \leq \frac{\|g'\|_{\infty,[m,M]}}{(M-m)(p+1)^{\frac{1}{p}}} \cdot$$

$$\left[\|f_2(A)\| \left\|\left(f_1 \circ g^{-1}\right)' \circ g\right\|_{q,[m,M]} + \|f_1(A)\| \left\|\left(f_2 \circ g^{-1}\right)' \circ g\right\|_{q,[m,M]}\right] \cdot$$

$$\left[\left\|(M1_H - A)^{1+\frac{1}{p}}\right\| + \left\|(A - m1_H)^{1+\frac{1}{p}}\right\|\right]. \tag{46}$$

We continue with

Theorem 7 *All as in Theorem 3. Then*

$$\langle(\Delta(f_1, f_{2;}g))(A) x, x\rangle \leq \frac{(M-m)^{n-1}}{(n-1)!} \|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} \cdot$$

$$\left[\|f_1(A)\| \left\|\left(f_2 \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]} + \|f_2(A)\| \left\|\left(f_1 \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]}\right]. \tag{47}$$

Proof. We observe that

$$\left|\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t, \lambda) dt\right| \leq$$

$$\int_m^M |g(\lambda) - g(t)|^{n-1} |g'(t)| |K(t, \lambda)| \left|\left(f_i \circ g^{-1}\right)^{(n)} (g(t))\right| dt \leq$$

$$\|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} (M-m)^n \left(\int_m^M \left|\left(f_i \circ g^{-1}\right)^{(n)} (g(t))\right| dt\right) =$$

$$\|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} (M-m)^n \left\|\left(f_i \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]}, \quad i = 1, 2. \tag{48}$$

Hence it holds $(i = 1, 2)$

$$\left|\int_{m-0}^M \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t, \lambda) dt\right) d\langle E_\lambda x, x\rangle\right| \leq$$

$$\|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} (M-m)^n \left\|\left(f_i \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]}, \tag{49}$$

the last is valid since

$$\int_{m-0}^M d\langle E_\lambda x, x\rangle = 1, \text{ for } x \in H : \|x\| = 1. \tag{50}$$

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Therefore it holds

$$\left\| \int_{m-0}^{M} \left(\int_{m}^{M} (g(\lambda) - g(t))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (g(t))\, g'(t)\, K(t,\lambda)\, dt \right) dE_\lambda \right\| \leq$$

$$\|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} (M-m)^n \left\| \left(f_i \circ g^{-1}\right)^{(n)} \circ g \right\|_{1,[m,M]}, \tag{51}$$

for $i = 1, 2$.

Acting as in the proof of Theorem 3 we find that

$$|R| \overset{\text{(by (30), (49), (51))}}{\leq} \frac{1}{2(n-1)!(M-m)} \cdot$$

$$\left\{ 2\|f_2(A)\| \|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} (M-m)^n \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_{1,[m,M]} + \right.$$

$$\left. 2\|f_1(A)\| \|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} (M-m)^n \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_{1,[m,M]} \right\} = \tag{52}$$

$$\frac{(M-m)^{n-1}}{(n-1)!} \|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} \cdot$$

$$\left[\|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_{1,[m,M]} + \|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_{1,[m,M]} \right],$$

proving the claim. ■

We finish this section with

Corollary 8 *(to Theorem 7, $n = 1$) It holds*

$$|\langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A) x, x\rangle \langle f_2(A) x, x\rangle| \leq \|g'\|_{\infty,[m,M]} \cdot$$

$$\left[\|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)' \circ g \right\|_{1,[m,M]} + \|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)' \circ g \right\|_{1,[m,M]} \right]. \tag{53}$$

4 Applications

We give

Theorem 9 *Let $f_1, f_2 \in C'([a,b])$ with $[m,M] \subset (a,b)$, $m < M$. Here A is a selfadjoint linear operator on the Hilbert space H with spectrum $Sp(A) \subseteq [m,M]$. We consider any $x \in H : \|x\| = 1$, and $\rho > 0 : M < \ln \rho$.*

Then

$$|\langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A) x, x\rangle \langle f_2(A) x, x\rangle| \leq \frac{e^M}{2(M-m)\rho}$$

$$\left[\|f_2(A)\| \left\|(f_1 \circ \ln \rho t)' \circ \frac{e^t}{\rho}\right\|_{\infty,[m,M]} + \|f_1(A)\| \left\|(f_2 \circ \ln \rho t)' \circ \frac{e^t}{\rho}\right\|_{\infty,[m,M]}\right]$$
$$\left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right]. \quad (54)$$

Proof. Apply Corollary 4 for $g(t) = \frac{e^t}{\rho}$. ∎

We continue with

Theorem 10 *All as in Theorem 9. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$|\langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A) x, x\rangle \langle f_2(A) x, x\rangle| \leq \frac{e^M}{(M-m)(p+1)^{\frac{1}{p}} \rho}$$
$$\left[\|f_2(A)\| \left\|(f_1 \circ \ln \rho t)' \circ \frac{e^t}{\rho}\right\|_{q,[m,M]} + \|f_1(A)\| \left\|(f_2 \circ \ln \rho t)' \circ \frac{e^t}{\rho}\right\|_{q,[m,M]}\right]$$
$$\left[\left\|(M1_H - A)^{1+\frac{1}{p}}\right\| + \left\|(A - m1_H)^{1+\frac{1}{p}}\right\|\right]. \quad (55)$$

Proof. Use of Corollary 6 and $g(t) = \frac{e^t}{\rho}$; $\rho > 0$, $M < \ln \rho$. ∎

We finish article with

Theorem 11 *Here all as in Theorem 9. Then*

$$|\langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A) x, x\rangle \langle f_2(A) x, x\rangle| \leq \frac{e^M}{\rho} \quad (56)$$
$$\left[\|f_1(A)\| \left\|(f_2 \circ \ln \rho t)' \circ \frac{e^t}{\rho}\right\|_{1,[m,M]} + \|f_2(A)\| \left\|(f_1 \circ \ln \rho t)' \circ \frac{e^t}{\rho}\right\|_{1,[m,M]}\right].$$

Proof. Use of Corollary 8. ∎

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FOURIER SERIES OF SUMS OF PRODUCTS OF POLY-BERNOULLI AND GENOCCHI FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We derive Fourier series expansions of three types of sums of products of poly-Bernoulli and Genocchi functions. In addition, we express each of them in terms of Bernoulli functions.

1. Introduction

For any integer r, the poly-Bernoulli polynomials $\mathbb{B}_m^{(r)}(x)$ of index r are given by the generating function

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}(x)\frac{t^m}{m!},\quad (\text{see } [1-4,7-10,12,13,17,18]), \tag{1.1}$$

where $Li_r(x)=\sum_{m=0}^{\infty}\frac{x^m}{m^r}$ is the r-th polylogarithmic function for $r\geq 1$ and a rational function for $r\leq 0$. We observe here that

$$\frac{d}{dx}\left(Li_{r+1}(x)\right)=\frac{1}{x}Li_r(x). \tag{1.2}$$

As to poly-Bernoulli polynomials, we note the following:

$$\begin{aligned}&\frac{d}{dx}\left(\mathbb{B}_m^{(r)}(x)\right)=m\mathbb{B}_{m-1}^{(r)}(x),\ \ (m\geq 1).\\&\mathbb{B}_m^{(1)}(x)=B_m(x),\ \mathbb{B}_0^{(r)}(x)=1,\ \mathbb{B}_m^{(0)}(x)=x^m,\\&\mathbb{B}_m^{(0)}=\delta_{m,0},\ \mathbb{B}_m^{(r+1)}(1)-\mathbb{B}_m^{(r+1)}(0)=\mathbb{B}_{m-1}^{(r)},\ (m\geq 1).\end{aligned} \tag{1.3}$$

The Genocchi polynomials $G_m(x)$ are given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!},\quad (\text{see } [14-16]). \tag{1.4}$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned}&G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,\\&G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2+1,\\&G_5(x)=5x^4-10x^3+5x,\ G_6(x)=6x^5-15x^4+15x^2-3,\\&G_7(x)=7x^6-21x^5+35x^3-21x.\end{aligned} \tag{1.5}$$

From the relation $G_m(x)=mE_{m-1}(x)(m\geq 1)$, we have

$$\begin{aligned}&\deg G_m(x)=m-1\ (m\geq 1),\ G_m=mE_{m-1}\ (m\geq 1),\\&G_0=0,\ G_1=1,\ G_{2m+1}=0\ (m\geq 1),\ \text{and } G_{2m}\neq 0\ (m\geq 1).\end{aligned} \tag{1.6}$$

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In addition,

$$\begin{aligned}&\frac{d}{dx}G_m(x) = mG_{m-1}(x)\ (m \geq 1),\\ &G_m(x+1) + G_m(x) = 2mx^{m-1}\ (m \geq 0).\end{aligned} \tag{1.7}$$

From these, we also have

$$G_m(1) + G_m(0) = 2\delta_{m,1}, \quad (m \geq 0). \tag{1.8}$$

$$\begin{aligned}\int_0^1 G_m(x)dx &= \frac{1}{m+1}(G_{m+1}(1) - G_{m+1}(0))\\ &= \frac{2}{m+1}(-G_{m+1}(0) + \delta_{m,0})\\ &= \begin{cases} 0, & \text{if } \ m \text{ is even,}\\ -\frac{2}{m+1}G_{m+1}, & \text{if } \ m \text{ is odd.}\end{cases}\end{aligned} \tag{1.9}$$

For any real number x, let $< x >= x - [x] \in [0, 1)$ denote the fractional part of x. In this paper, we will study the Fourier series of the following three types of sums of products of poly-Bernoulli and Genocchi functions:

(1) $\alpha_m(< x >) = \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), (m \geq 2)$;
(2) $\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), (m \geq 2)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), (m \geq 2)$.

For some elementary facts about Fourier analysis, the reader may refer to [20,22]. As to $\gamma_m(< x >)$, we note that the polynomial identity (1.10) follows immediately from (4.21) and (4.25), which is derived in turn from the Fourier series expansion of $\gamma_m(< x >)$.

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(x)G_{m-k}(x)\\ &= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right)\\ &\quad + \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right)B_s(x).\end{aligned} \tag{1.10}$$

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from (2.19) and (2.23), and (3.16) and (3.20), respectively. It is worth noting that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_k(< x >)B_{m-k}(< x >)$ we can derive the following polynomial identity:

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{k\,(m-k)}B_k\,(x)\,B_{m-k}\,(x) \\ =&\frac{2}{m^2}\left(B_m + \frac{1}{2}\right) + \frac{2}{m}\sum_{k=1}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k\,(x) + \frac{2}{m}H_{m-1}B_m\,(x)\,, \quad (m \geq 2).\end{aligned} \tag{1.11}$$

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From (1.11), we can derive the following slightly different version of the well-known Miki's identity (see [5,21])

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k\left(2m-2k\right)}B_{2k}B_{2m-2k}\\=&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}+\frac{1}{m}H_{2m-1}B_{2m},\quad(m\geq2)\,.\end{aligned}\tag{1.12}$$

Also, from (1.11) and with $\overline{B}_m=\left(\frac{1-2^{m-1}}{2^{m-1}}\right)B_m=\left(2^{1-m}-1\right)B_m=B_m\left(\frac{1}{2}\right)$, we have

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k\left(2m-2k\right)}\overline{B}_{2k}\overline{B}_{2m-2k}\\=&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}\overline{B}_{2m-2k}+\frac{1}{m}H_{2m-1}\overline{B}_{2m},\quad(m\geq2)\,,\end{aligned}\tag{1.13}$$

which is the Faber-Pandharipande-Zagier identity (see [6]). Some related works can be found in [11,19].

2. Fourier series of functions of the first type

In this section, we will study the Fourier series of first type of sums of products of poly-Bernoulli and Genocchi functions.

$$\alpha_m(x)=\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(x)G_{m-k}(x),\ \ (m\geq2).\tag{2.1}$$

Note here that $\deg\alpha_m(x)=m-1$. We now consider the function

$$\alpha_m(<x>)=\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>),\ \ (m\geq2),\tag{2.2}$$

defined on $(-\infty,-\infty)$, which is periodic of period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi inx},\tag{2.3}$$

where

$$\begin{aligned}A_n^{(m)}&=\int_0^1\alpha_m(<x>)e^{-2\pi inx}dx\\&=\int_0^1\alpha_m(x)e^{-2\pi inx}dx.\end{aligned}\tag{2.4}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}
\alpha_m'(x) &= \sum_{k=0}^{m-1}(k\mathbb{B}_{k-1}^{(r+1)}(x)G_{m-k}(x) + (m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x)) \\
&= \sum_{k=1}^{m-1} k\mathbb{B}_{k-1}^{(r+1)}(x)G_{m-k}(x) + \sum_{k=0}^{m-2}(m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x) \\
&= \sum_{k=0}^{m-2}(k+1)\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x) + \sum_{k=0}^{m-2}(m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x) \\
&= (m+1)\sum_{k=0}^{m-2}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x) \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.5}$$

So, $\alpha_m'(x) = (m+1)\alpha_{m-1}(x)$, and hence

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.6}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{2.7}$$

For $m \geq 2$,

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m-1}\left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \mathbb{B}_0^{(r+1)}(1)G_m(1) - \mathbb{B}_0^{(r+1)}G_m + \sum_{k=1}^{m-1}\left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= -2G_m + 2\delta_{m,1} + \sum_{k=1}^{m-1}\left(\left(\mathbb{B}_k^{(r+1)} + \mathbb{B}_{k-1}^{(r)}\right)(-G_{m-k} + 2\delta_{m-1,k}) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= -2G_m + \sum_{k=1}^{m-1}\left(-2\mathbb{B}_k^{(r+1)}G_{m-k} + 2\mathbb{B}_k^{(r+1)}\delta_{m-1,k} - \mathbb{B}_{k-1}^{(r)}G_{m-k} + 2\mathbb{B}_{k-1}^{(r)}\delta_{m-1,k}\right) \\
&= -2G_m - 2\sum_{k=1}^{m-1}\mathbb{B}_k^{(r+1)}G_{m-k} + 2\mathbb{B}_{m-1}^{(r+1)} - \sum_{k=1}^{m-1}\mathbb{B}_{k-1}^{(r)}G_{m-k} + 2\mathbb{B}_{m-2}^{(r)} \\
&= -2\sum_{k=0}^{m-2}\mathbb{B}_k^{(r+1)}G_{m-k} - \sum_{k=0}^{m-2}\mathbb{B}_k^{(r)}G_{m-k-1} + 2\mathbb{B}_{m-2}^{(r)}.
\end{aligned} \tag{2.8}$$

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0. \tag{2.9}$$

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.10}$$

Now, we are going to determine the Fourier coefficients $A_n^{(m)}$.
Case 1 : $n \neq 0$.

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\alpha_m(1)-\alpha_m(0)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m \\ &= \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} - \frac{1}{2\pi in}\Delta_{m-1}\right) - \frac{1}{2\pi in}\Delta_m \\ &= \frac{(m+1)m}{(2\pi in)^2}A_n^{(m-2)} - \frac{m+1}{(2\pi in)^2}\Delta_{m-1} - \frac{1}{2\pi in}\Delta_m \\ &= \cdots \\ &= \frac{(m+1)_{m-1}}{(2\pi in)^{m-1}}A_n^{(1)} - \sum_{j=1}^{m-1}\frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} \\ &= -\sum_{j=1}^{m-1}\frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} \\ &= -\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}, \end{aligned} \tag{2.11}$$

where

$$A_n^{(1)} = \int_0^1 \alpha_1(x)e^{-2\pi inx}dx = \int_0^1 e^{-2\pi inx}dx = 0. \tag{2.12}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.13}$$

Here we recall the following facts about Bernoulli functions $B_m(< x >)$:
(a) for $m \geq 2$,

$$B_m(< x >) = -m!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m}. \tag{2.14}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}, \end{cases} \tag{2.15}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. $\alpha_m(< x >), (m \geq 2)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Delta_m \neq 0$. Assume first that m is an integer ≥ 2 with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Thus, the Fourier series of $\alpha_m(< x >)$ converges uniformly

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to $\alpha_m(<x>)$, and

$$\begin{aligned}
&\alpha_m(<x>)\\
&=\frac{1}{m+2}\Delta_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\
&=\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=1}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi in}}{(2\pi in)^j}\right)\\
&=\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)\\
&\quad+\frac{1}{m+2}\binom{m+2}{1}\Delta_m\times\begin{cases}B_1(<x>), & \text{for } x\in\mathbb{Z}^c,\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}\tag{2.16}$$

Now, we can state our first theorem.

Theorem 2.1. *For each integer $l\geq 2$, let*

$$\Delta_l=-2\sum_{k=0}^{l-2}\mathbb{B}_k^{(r+1)}G_{l-k}-\sum_{k=0}^{l-2}\mathbb{B}_k^{(r)}G_{l-k-1}+2\mathbb{B}_{l-2}^{(r)}.\tag{2.17}$$

Assume that $\Delta_m=0$, for an integer $m\geq 2$. Then we have the following.

(a) $\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>)$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>)\\
&=\frac{1}{m+2}\Delta_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},
\end{aligned}\tag{2.18}$$

for all $x\in(-\infty,\infty)$, where the convergence is uniform.

(b)

$$\begin{aligned}
&\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>)\\
&=\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>),
\end{aligned}\tag{2.19}$$

for all $x\in(-\infty,\infty)$. Here $B_j(<x>)$ is the Bernoulli function.

Assume next that $m\geq 2$ is an integer with $\Delta_m\neq 0$. Then $\alpha_m(0)\neq\alpha_m(1)$. So $\alpha_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x\in\mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0)+\alpha_m(1))=\alpha_m(0)+\frac{1}{2}\Delta_m,\tag{2.20}$$

for $x\in\mathbb{Z}$. Next, we can state our second theorem.

Theorem 2.2. *For each integer $l \geq 2$, let*

$$\Delta_l = -2\sum_{k=0}^{l-2} \mathbb{B}_k^{(r+1)} G_{l-k} - \sum_{k=0}^{l-2} \mathbb{B}_k^{(r)} G_{l-k-1} + 2\mathbb{B}_{l-2}^{(r)}. \tag{2.21}$$

Assume that $\Delta_m \neq 0$, for an integer ≥ 2. Then we have the following.

(a)

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Delta_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.22}$$

(b)

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &= \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >), \ \text{for } x \in \mathbb{Z}^c; \end{aligned} \tag{2.23}$$

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &\sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Delta_m, \ x \in \mathbb{Z}. \end{aligned} \tag{2.24}$$

3. Fourier series of functions of the second type

Let $\beta_m(x) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(x) G_{m-k}(x)$, $(m \geq 2)$. Observe that

$$\begin{aligned} \beta_m'(x) &= \sum_{k=0}^{m-1} \left\{ \frac{k}{k!(m-k)!}\mathbb{B}_{k-1}^{(r+1)}(x) G_{m-k}(x) + \frac{m-k}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(x) G_{m-k-1}(x) \right\} \\ &= \sum_{k=1}^{m-1} \frac{1}{(k-1)!(m-k)!}\mathbb{B}_{k-1}^{(r+1)}(x) G_{m-k}(x) + \sum_{k=0}^{m-2} \frac{1}{k!(m-k-1)!}\mathbb{B}_k^{(r+1)}(x) G_{m-k-1}(x) \\ &= 2\sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!}\mathbb{B}_k^{(r+1)}(x) G_{m-1-k}(x) \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.1}$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x), \tag{3.2}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}(\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.3}$$

For $m \geq 2$, we have

$$\begin{aligned}
\Omega_m &= \Omega_m(r) = \beta_m(1) - \beta_m(0) \\
&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \frac{1}{m!}\left(\mathbb{B}_0^{(r+1)}(1)G_m(1) - \mathbb{B}_0^{(r+1)}G_m\right) + \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \frac{1}{m!}\left(-2G_m + 2\delta_{m,1}\right) + \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\left(\mathbb{B}_k^{(r+1)} + \mathbb{B}_{k-1}^{(r)}\right)\left(-G_{m-k} + 2\delta_{m-1,k}\right) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= -\frac{2}{m!}G_m + \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(-2\mathbb{B}_k^{(r+1)}G_{m-k} + 2\mathbb{B}_k^{(r+1)}\delta_{m-1,k} - \mathbb{B}_{k-1}^{(r)}G_{m-k} + 2\mathbb{B}_{k-1}^{(r)}\delta_{m-1,k}\right) \\
&= -\frac{2}{m!}G_m - 2\sum_{k=1}^{m-1} \frac{\mathbb{B}_k^{(r+1)}G_{m-k}}{k!(m-k)!} + 2\frac{\mathbb{B}_{m-1}^{(r+1)}}{(m-1)!} - \sum_{k=1}^{m-1} \frac{\mathbb{B}_{k-1}^{(r)}G_{m-k}}{k!(m-k)!} + 2\frac{\mathbb{B}_{m-2}^{(r)}}{(m-1)!} \\
&= -2\sum_{k=0}^{m-2} \frac{\mathbb{B}_k^{(r+1)}G_{m-k}}{k!(m-k)!} - \sum_{k=1}^{m-1} \frac{\mathbb{B}_{k-1}^{(r)}G_{m-k}}{k!(m-k)!} + 2\frac{\mathbb{B}_{m-2}^{(r)}}{(m-1)!}.
\end{aligned} \tag{3.4}$$

Then

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0. \tag{3.5}$$

Also,

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.6}$$

Now, we are going to consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >),\ (m \geq 2), \tag{3.7}$$

defined on $(-\infty, \infty)$, which is periodic with period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{k=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.8}$$

where

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(< x >)e^{-2\pi i n x}dx \\
&= \int_0^1 \beta_m(x)e^{-2\pi i n x}dx.
\end{aligned} \tag{3.9}$$

We are now going to determine the Fourier coefficients $B_n^{(m)}$.

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Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left[\beta_m(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}(\beta_m(1)-\beta_m(0)) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\ &= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m \\ &= \frac{2}{2\pi i n}\left(\frac{2}{2\pi i n}B_n^{(m-2)} - \frac{1}{2\pi i n}\Omega_{m-1}\right) - \frac{1}{2\pi i n}\Omega_m \\ &= \left(\frac{2}{2\pi i n}\right)^2 B_n^{(m-2)} - \frac{2}{(2\pi i n)^2}\Omega_{m-1} - \frac{1}{2\pi i n}\Omega_m \\ &= \cdots \\ &= \left(\frac{2}{2\pi i n}\right)^{m-1} B_n^{(1)} - \sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\ &= -\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}, \end{aligned} \tag{3.10}$$

where

$$B_n^{(1)} = \int_0^1 \beta_1(x) e^{-2\pi i n x} dx = \int_0^1 e^{-2\pi i n x} dx = 0. \tag{3.11}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}. \tag{3.12}$$

$\beta_m(< x >), (m \geq 2)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for an integer $m \geq 2$. Then $\beta_m(0) = \beta_m(1)$. $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} \beta_m(< x >) &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right)e^{2\pi i n x} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.13}$$

Now, we are ready to state our first theorem.

Theorem 3.1. *For each integer $l \geq 2$, let*

$$\Omega_l = -2\sum_{k=0}^{l-2} \frac{\mathbb{B}_k^{(r+1)} G_{l-k}}{k!(l-k)!} - \sum_{k=1}^{l-1} \frac{\mathbb{B}_{k-1}^{(r)} G_{l-k}}{k!(l-k)!} + 2\frac{\mathbb{B}_{l-2}^{(r)}}{(l-1)!}. \tag{3.14}$$

Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x}, \end{aligned} \tag{3.15}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b)

$$\begin{aligned} &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >), \end{aligned} \tag{3.16}$$

for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.

Assume next that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then $\beta_m(0) \neq \beta_m(1)$. Thus $\beta_m(< x >)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned} &\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m \\ &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Omega_m, \end{aligned} \tag{3.17}$$

for $x \in \mathbb{Z}$. We can now state our second theorem.

Theorem 3.2. *For each integer $l \geq 2$, let*

$$\Omega_l = -2\sum_{k=0}^{l-2} \frac{\mathbb{B}_k^{(r+1)} G_{l-k}}{k!(l-k)!} - \sum_{k=1}^{l-1} \frac{\mathbb{B}_{k-1}^{(r)} G_{l-k}}{k!(l-k)!} + 2\frac{\mathbb{B}_{l-2}^{(r)}}{(l-1)!}. \tag{3.18}$$

Assume that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.19}$$

Here the convergence is pointwise.

(b)

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1}+\sum_{j=1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>) \\ &=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>), \end{aligned} \tag{3.20}$$

for $x \in \mathbb{Z}^c$*;*

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>) \\ &=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}G_{m-k}+\frac{1}{2}\Omega_m, \end{aligned} \tag{3.21}$$

for $x \in \mathbb{Z}$*. Here* $B_j(<x>)$ *is the Bernoulli function.*

4. Fourier series of functions of the third type

Let $\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(x)G_{m-k}(x)$, $(m \geq 2)$. We observe the following.

$$\begin{aligned} \gamma_m'(x) &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left\{k\mathbb{B}_{k-1}^{(r+1)}(x)G_{m-k}(x)+(m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x)\right\} \\ &= \sum_{k=0}^{m-2}\frac{1}{m-k-1}\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x)+\sum_{k=1}^{m-1}\frac{1}{k}\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x) \\ &= \frac{1}{m-1}G_{m-1}(x)+\sum_{k=1}^{m-2}\frac{1}{m-1-k}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x)+\sum_{k=1}^{m-2}\frac{1}{k}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x) \\ &= \frac{1}{m-1}G_{m-1}(x)+(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x) \\ &= \frac{1}{m-1}G_{m-1}(x)+(m-1)\gamma_{m-1}(x). \end{aligned} \tag{4.1}$$

From this, we see that

$$\left(\frac{1}{m}(\gamma_{m+1}(x)-\frac{1}{m(m+1)}G_{m+1}(x))\right)'=\gamma_m(x) \tag{4.2}$$

and

$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}\left[\gamma_{m+1}(x) - \frac{1}{m(m+1)}G_{m+1}(x)\right]_0^1 \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(G_{m+1}(1) - G_{m+1}(0))\right) \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(-2G_{m+1}(0) + 2\delta_{m,0})\right) \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2G_{m+1}}{m(m+1)}\right).
\end{aligned} \tag{4.3}$$

For $m \geq 2$, we let

$$\begin{aligned}
&\Lambda_m = \Lambda_m(r) = \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\left(\mathbb{B}_k^{(r+1)} + \mathbb{B}_{k-1}^{(r)}\right)(-G_{m-k} + 2\delta_{m-1,k}) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(-2\mathbb{B}_k^{(r+1)}G_{m-k} + 2\mathbb{B}_k^{(r+1)}\delta_{m-1,k} - \mathbb{B}_{k-1}^{(r)}G_{m-k} + 2\mathbb{B}_{k-1}^{(r)}\delta_{m-1,k}\right) \\
&= -2\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}G_{m-k} + \frac{2}{m-1}\mathbb{B}_{m-1}^{(r+1)} - \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_{k-1}^{(r)}G_{m-k} + \frac{2}{m-1}\mathbb{B}_{m-2}^{(r)}.
\end{aligned} \tag{4.4}$$

So,

$$\gamma_m(1) = \gamma_m(0) \Longleftrightarrow \Lambda_m = 0. \tag{4.5}$$

Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}G_{m+1}\right). \tag{4.6}$$

We are now going to consider

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), \tag{4.7}$$

defined on $(-\infty, \infty)$, which is periodic with period 1. The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.8}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx. \tag{4.9}$$

Now, we want to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$
\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}[\gamma_m(x) e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}(\gamma_m(1) - \gamma_m(0)) + \frac{1}{2\pi i n}\int_0^1 \left(\frac{1}{m-1} G_{m-1}(x) + (m-1)\gamma_{m-1}(x)\right) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} + \frac{1}{2\pi i n(m-1)}\int_0^1 G_{m-1}(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} + \frac{1}{2\pi i n(m-1)}\Phi_m,
\end{aligned}
\tag{4.10}
$$

where

$$
\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi i n)^k} G_{m-k}, \tag{4.11}
$$

and one can show

$$
\int_0^1 G_l(x) e^{-2\pi i n x} dx \quad = \begin{cases} \sum_{k=1}^{l-1} \frac{2(l)_{k-1}}{(2\pi i n)^k} G_{l-k+1}, & \text{for} \quad n \neq 0, \\ -\frac{2G_{l+1}}{l+1}, & \text{for} \quad n = 0. \end{cases} \tag{4.12}
$$

We observe that

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m + \frac{2}{2\pi i n(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi i n}\left(\frac{m-2}{2\pi i n} C_n^{(m-2)} - \frac{1}{2\pi i n}\Lambda_{m-1} + \frac{2}{2\pi i n(m-2)}\Phi_{m-1}\right) \\
&\quad - \frac{1}{2\pi i n}\Lambda_m + \frac{2}{2\pi i n(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi i n)^2} C_n^{(m-2)} - \frac{m-1}{(2\pi i n)^2}\Lambda_{m-1} - \frac{1}{2\pi i n}\Lambda_m \\
&\quad + \frac{2(m-1)}{(2\pi i n)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi i n(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)_{m-2}}{(2\pi i n)^{m-2}} C_n^{(2)} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi i n)^j (m-j)}\Phi_{m-j+1} \\
&= -\frac{(m-1)!}{(2\pi i n)^{m-1}}\Lambda_2 - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi i n)^j (m-j)}\Phi_{m-j+1} \\
&= -\frac{1}{m}\sum_{j=1}^{m-1} \frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1} + \frac{1}{m}\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)}\Phi_{m-j+1},
\end{aligned}
\tag{4.13}
$$

where

$$\begin{aligned} C_n^{(2)} &= \int_0^1 \gamma_2(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} \left[\gamma_2(x) e^{-2\pi i n x} \right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \gamma_2'(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} (\gamma_2(1) - \gamma_2(0)) = -\frac{1}{2\pi i n} \Lambda_2. \end{aligned} \tag{4.14}$$

In order to get a final expression for $C_n^{(m)}$, we observe the following.

$$\begin{aligned} &\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \Phi_{m-j+1} \\ &= \sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1}}{(2\pi i n)^k} G_{m-j-k+1} \\ &= \sum_{j=1}^{m-2} \sum_{k=1}^{m-j-1} \frac{2(m)_{j+k-1}}{(2\pi i n)^{j+k} (m-j)} G_{m-j-k+1} \\ &= 2 \sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{s=j+1}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} G_{m-s+1} \\ &= 2 \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} G_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} \\ &= 2 \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}). \end{aligned} \tag{4.15}$$

Putting everything altogether, we have

$$\begin{aligned} C_n^{(m)} &= -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \Lambda_{m-s+1} + \frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \\ &= -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left\{ \Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \right\}. \end{aligned} \tag{4.16}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x) dx = \frac{1}{m} \left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)} \right). \tag{4.17}$$

$\gamma_m(< x >), (m \geq 2)$ is piecewise C^∞. In addition, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $\Lambda_m \neq 0$.

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Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. $\gamma_m(< x >)$ is piecewise C^∞, and continuous. So the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >)\\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right)\\
&\quad - \frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\right\}e^{2\pi inx}\\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right)\\
&\quad + \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\\
&\times\left(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right)\\
&\quad + \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(< x >)\\
&\quad + \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c,\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned} \tag{4.18}$$

Now, we can state our first result.

Theorem 4.1. *For each integer $l \geq 2$, let*

$$\begin{aligned}
\Lambda_l = &-2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\mathbb{B}_k^{(r+1)}G_{l-k} + \frac{2}{l-1}\mathbb{B}_{l-1}^{(r+1)}\\
&-\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\mathbb{B}_{k-1}^{(r)}G_{l-k} + \frac{2}{l-1}\mathbb{B}_{l-2}^{(r)}.
\end{aligned} \tag{4.19}$$

Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >)$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >)\\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right)\\
&\quad - \frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\right\}e^{2\pi inx},
\end{aligned} \tag{4.20}$$

for all $x \in (-\infty, \infty)$. Here the convergence is uniform.

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\right) \\
&\quad + \frac{1}{m} \sum_{s=2}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right) B_s(< x >)
\end{aligned} \tag{4.21}$$

for all $x \in (-\infty, \infty)$. Here $B_s(< x >)$ is the Bernoulli function.

Assume next that $m \geq 2$ is an integer with $\Lambda_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$. $\gamma_m(< x >)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned}
\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) &= \gamma_m(0) + \frac{1}{2}\Lambda_m \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Lambda_m,
\end{aligned} \tag{4.22}$$

for $x \in \mathbb{Z}$. Next, we can state our second result.

Theorem 4.2. *For each integer $l \geq 2$, let*

$$\Lambda_l = -2\sum_{k=1}^{l-1} \frac{1}{k(l-k)} \mathbb{B}_k^{(r+1)} G_{l-k} + \frac{2}{l-1}\mathbb{B}_{l-1}^{(r+1)} - \sum_{k=1}^{l-1} \frac{1}{k(l-k)} \mathbb{B}_{k-1}^{(r)} G_{l-k} + \frac{2}{l-1}\mathbb{B}_{l-2}^{(r)}. \tag{4.23}$$

Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\begin{aligned}
&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\
&\quad - \frac{1}{m} \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right)\right) e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Lambda_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.24}$$

Here the convergence is pointwise.

(b)

$$\begin{aligned}
&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\
&\quad + \frac{1}{m} \sum_{s=1}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right) B_s(< x >) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >),
\end{aligned} \tag{4.25}$$

for $x \in \mathbb{Z}^c$ *and*

$$\begin{aligned} &\frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\ &\quad + \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right)B_s(< x >) \\ &\sum_{s=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}G_{m-k} + \frac{1}{2}\Lambda_m, \end{aligned} \tag{4.26}$$

for $x \in \mathbb{Z}$.

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Convergence of the Newton-HSS Method under the Lipschitz Condition with the L-average

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Abstract: Under the hypothesis that the Jacobian matrix satisfies the center Lipschitz condition with the L-average, we prove the local convergence of the Newton-HSS method, which is used to solve large sparse systems of nonlinear equations with positive definite Jacobian matrices at the solution points. Numerical results are given to examine its feasibility and effectiveness.

Keywords: Large sparse systems; Nonlinear equations; Newton-HSS method; Center Lipschtiz condition with the L-average.

AMS classifications: 65F10, 65F50, 65W05,

1 Introduction

In this paper, we consider the following system of nonlinear equations

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is nonlinear and continuously differentiable, $\mathbb{D}$ is an open convex subset of the n-dimensional complex linear space $\mathbb{C}^n$. The Jacobian matrix $F^{'}(x) \in \mathbb{C}^{n\times n}$ is sparse, nonsymmetric and positive definite. This kind of nonlinear equations can be derived in many areas of scientific computing and engineering applications [1, 2, 3].

The most classic and important iterative method for the system of nonlinear equations (1.1) is Newton's method [14, 15], which can be formulated as

$$x_{k+1} = x_k - F^{'}(x_k)^{-1}F(x_k), \quad k = 0, 1, \cdots,$$

where $x_0 \in \mathbb{D}$ is a given initial vector. Obviously, at the k-th iteration step, it is necessary to solve the so-called Newton equation

$$F^{'}(x_k)s_k = -F(x_k), \tag{1.2}$$

which is the dominant task in implementations of the Newton method, then get the $k+1$-th iterative vector $x_{k+1} = x_k + s_k$. Bai and Guo [5], Guo and Duff [10], first used the HSS iteration [4] to solve approximately the Newton equation (1.2), and used inexact Newton method [8] as the outer solver, presented the Newton-HSS method for solving

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the system of nonlinear equations (1.1), and gave the convergence theorems under the Lipschtiz continuous conditions. The following is HSS iterative method, which is used to solve non-Hermitian positive-definite linear system $Ax = b$ [4].

Algorithm 1.1. HSS
1. *Given an initial guess* $x_0 \in \mathbb{C}^n$*, and positive constant* α*.*
2. *Split the linear matrix* A *into its Hermitian part* H *and skew-Hermitian part* S

$$H = \frac{1}{2}(A + A^*) \text{ and } S = \frac{1}{2}(A - A^*).$$

3. *For* $k = 0, 1, 2, \cdots$*, compute* x_{k+1} *using the following iteration scheme until* $\{x_k\}$ *satisfies the stopping criterion:*

$$\begin{cases} (\alpha I + H)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\ (\alpha I + S)x_{k+1} = (\alpha I - H)x_{k+\frac{1}{2}} + b, \end{cases} \tag{1.3}$$

where I *denotes the identity matrix.*

Recently, using HSS method as the inner solver for solving Newton equations (1.2), and the modified Newton method as the outer solver, Chen et al. have proposed the modified Newton-HSS method for the system of nonlinear equations (1.1). They have proved the convergence theorems under Hölder continuous condition, which is weaker than the usual Lipschtiz condition. When using Newton's method to solve equations (1.1), Guo [11] studied its semi-local convergence property, which is as brief as Newton-Kantorovich theorem [9, 12], under the hypothese that the derivative satisfies center Lipschtiz condition with the L-average, which is weaker than Hölder condition and Lipschtiz condition, and has gotten a lot of attention and been extensively studied [13, 17, 18].

The following conditions were introduced by Wang in [13], named the center Lipschitz condition with the L-average.

Definition 1.1. *Let* Y *be a Banach space and let* $x_* \in \mathbb{C}^n$*. Let* G *be a mapping from* $\mathbb{C}^n$ *to* Y*. Then* G *is said to satisfy the center Lipschitz condition with the L-average on* $B(x_*, r)$ *if*

$$\|G(x) - G(x_*)\| \le \int_0^{\|x-x_*\|} L(u)du \quad for \quad each \quad x \in B(x_*, r);$$

In this paper, motivated by the idea of [11], the main work is to study the local convergence theorem of the Newton-HSS method under the hypothese that the derivative satisfies the center Lipschitz condition with the L-average. The organization of the paper is as follows. In Section 2, we introduce the Newton-HSS iterative method. In Section 3, we first give some lemmas which are useful for our main result, then present the new local convergence theorems under the hypothese that the derivative satisfies the center Lipschitz condition with the L-average. An numerical example is given to illustrate the applications of the results in our paper in Section 4. Finally, in Section 5, some conclusions are given.

2 Newton-HSS iteration

For Jacobian matrix $F^{'}(x)$, let

$$H(x) = \frac{1}{2}(F^{'}(x) + F^{'}(x)^*)$$

be its Hermitian part,

$$S(x) = \frac{1}{2}(F^{'}(x) - F^{'}(x)^*)$$

be the skew-Hermitian part, the following is the Newton-HSS method [5, 10].

Algorithm 2.1. Newton-HSS
1. *Given an initial guess x_0, positive constants α and tol, and positive integer sequence $\{l_k\}_{k=0}^{\infty}$;*
2. *for $k = 0, 1, \cdots$ until $\|F(x_k)\| \le tol\|F(x_0)\|$ do:*

2.1. *Set $d_{k,0} = 0$;*
2.2. *for $l = 0, 1, \cdots, l_k - 1$, apply Algorithm HSS to the linear system (1.2):*

$$\begin{cases}(\alpha I + H(x_k))d_{k,l+\frac{1}{2}} = (\alpha I - S(x_k))d_{k,l} - F(x_k), \\ (\alpha I + S(x_k))d_{k,l+1} = (\alpha I - H(x_k))d_{k,l+\frac{1}{2}} - F(x_k),\end{cases}$$

and obtain d_{k,l_k} such that

$$\|F(x_k) + F^{'}(x_k)d_{k,l_k}\| \le \eta_k\|F(x_k)\| \qquad \eta_k \in [0, 1). \tag{2.1}$$

2.3. *Set $x_{k+1} = x_k + d_{k,l_k}$.*

Denote

$$\begin{aligned} B(\alpha; x) &= \frac{1}{2\alpha}(\alpha I + H(x))(\alpha I + S(x)), \\ C(\alpha; x) &= \frac{1}{2\alpha}(\alpha I - H(x))(\alpha I - S(x)), \\ T(\alpha; x) &= (\alpha I + S(x))^{-1}(\alpha I - H(x))(\alpha I + H(x))^{-1}(\alpha I - S(x)). \end{aligned} \tag{2.2}$$

Thus we have the following formulas

$$\begin{aligned} T(\alpha; x) &= B(\alpha; x)^{-1}C(\alpha; x), \\ F^{'}(x) &= B(\alpha; x) - C(\alpha; x), \\ F^{'}(x)^{-1} &= (I - T(\alpha; x))^{-1}B(\alpha; x)^{-1}. \end{aligned} \tag{2.3}$$

From the Newton-HSS method we can get [10]

$$x_{k+1} = x_k - (I - T_k^{l_k})F^{'}(x_k)^{-1}F(x_k), \tag{2.4}$$

here $T_k := T(\alpha; x_k)$.

3 Local convergence theorem under the center Lipschitz condition with the L-average

In this section, we establish a new local convergence theorem for the Newton-HSS method under the assumption that the derivative satisfies the center Lipschitz condition with the L-average, which is weaker than Hölder condition and Lipschtiz condition. Firstly, we give the assumption.

Assumption 3.1. *Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$ of a point $x_* \in \mathbb{D}$ at which $F'(x)$ is continuous, positive definite, and $F(x_*) = 0$. Assume the following conditions hold for all $x \in B(x_*, r) \subset \mathbb{N}_0$, where $B(x_*, r)$ denotes an open ball centered at x_* with radius r:*
(A1) (The Bounded Condition) there exist positive constants β, γ and δ such that

$$max\{\|H(x_*)\|, \|S(x_*)\|\} \leq \beta, \;\; \|F^{'}(x_*)^{-1}\| \leq \gamma.$$

(A2) (The Center Lipschitz Condition with the L-average) there exist positive integrable functions $L_h(u)$ and $L_s(u)$ such that,

$$\|H(x) - H(x_*)\| \leq \int_0^{\rho(x)} L_h(u)du,$$
$$\|S(x) - S(x_*)\| \leq \int_0^{\rho(x)} L_s(u)du,$$

here $\rho(x) = \|x - x_*\|$.

Let $L(u) = L_h(u) + L_s(u)$, thus $L(u)$ is a positive valued integrable function on $[0, +\infty)$. Before giving the main theorem, we list a series of useful lemmas as follows for our purpose. Lemma 3.1 is taken from [8], and we will give a proof of Lemma 3.2.

Lemma 3.1. *Define $\chi(t) = \frac{1}{t}\int_0^t L(u)(t-u)du, \;\; t \geq 0$, Then χ is increasing on $[0, +\infty)$.*

Lemma 3.2. *Under Assumption 3.1, if $\gamma \int_0^r L(u)du < 1$, then for $x \in B(x_*, r) \subset \mathbb{N}_0$, $F'(x)^{-1}$ exists, and*

$$(1) \;\; \|F'(x) - F'(x_*)\| \leq \int_0^{\rho(x)} L(u)du,$$
$$(2) \;\; \|F'(x)^{-1}\| \leq \frac{\gamma}{1 - \gamma \int_0^{\rho(x)} L(u)du},$$
$$(3) \;\; \|F(x)\| \leq \left(\frac{1}{\rho(x)} \int_0^{\rho(x)} L(u)(\rho(x) - u)du + 2\beta\right)\|x - x_*\|,$$
$$(4) \;\; \|x - x_* - F'(x)^{-1}F(x)\|$$
$$\leq \frac{\gamma}{1 - \gamma \int_0^{\rho(x)} L(u)du}\left(\frac{1}{\rho(x)} \int_0^{\rho(x)} L(u)(\rho(x) - u)du + \int_0^{\rho(x)} L(u)du\right)\|x - x_*\|.$$

Proof. By Assumption 3.1, $F'(x) = H(x)+S(x)$, perturbation lemma [14], and condition $\gamma\int_0^r L(u)du < 1$, it is easy to get (1) and (2).

For (3), from integral mean-value theorem and Assumption 3.1, we first have

$$\begin{aligned}
&\|F(x) - F(x_*) - F'(x_*)(x - x_*)\| \\
=&\|\int_0^1 (F'(x + t(x - x_*)) - F'(x_*))dt(x - x_*)\| \\
\leq&\int_0^1\int_0^{t\rho(x)} L(u)dudt\|x - x_*\| \\
=&\frac{1}{\rho(x)}\int_0^{\rho(x)} L(u)(\rho(x) - u)du\|x - x_*\|,
\end{aligned} \tag{3.5}$$

thus, together with $\|F'(x_*)\| = \|H(x_*) + S(x_*)\| \leq 2\beta$, we have

$$\begin{aligned}
\|F(x)\| &\leq \|F(x) - F(x_*) - F'(x_*)(x - x_*)\| + \|F'(x_*)(x - x_*)\| \\
&\leq (\frac{1}{\rho(x)}\int_0^{\rho(x)} L(u)(\rho(x) - u)du + 2\beta)\|x - x_*\|.
\end{aligned}$$

For (4), by integral mean-value theorem, Assumption 3.1, and (3.5), we can get

$$\begin{aligned}
&\|x - x_* - F'(x)^{-1}F(x)\| \\
=&\| - F'(x)^{-1}(F(x) - F(x_*) - F'(x_*)(x - x_*) + F'(x_*)(x - x_*) - F'(x)(x - x_*))\| \\
\leq&\|F'(x)^{-1}\|(\|F(x) - F(x_*) - F'(x_*)(x - x_*)\| + \|F'(x_*) - F'(x)\|\|x - x_*\|) \\
\leq&\frac{\gamma}{1 - \gamma\int_0^{\rho(x)} L(u)du}(\frac{1}{\rho(x)}\int_0^{\rho(x)} L(u)(\rho(x) - u)du + \int_0^{\rho(x)} L(u)du)\|x - x_*\|.
\end{aligned}$$

□

Then we can give the following local convergence theorem of the Newton-HSS method under the center Lipschtiz condition with the L-average .

Theorem 3.1. *Assume that Assumption 3.1 holds with $r \in (0, r_*)$, here r_* is defined by $r_* := \min\{r_1, r_2\}$, where r_1 and r_2 satisfy*

$$\int_0^{r_1} L(u)du = 2(\alpha + \beta)\left(\sqrt{\frac{\alpha\tau\theta}{(2 + \tau\theta)\gamma(\alpha + \beta)^2} + 1} - 1\right), \tag{3.6}$$

$$\frac{1}{r_2}\int_0^{r_2} L(u)(2r_2 - u)du = \frac{1 - 2\beta\gamma((\tau + 1)\theta)^{l_*}}{2\gamma}, \tag{3.7}$$

and with $l_* = \liminf_{k\to\infty} l_k$ satisfying

$$l_* > \lfloor\frac{\ln 2\beta\gamma}{\ln((\tau + 1)\theta)}\rfloor, \tag{3.8}$$

where the symbol $\lfloor \cdot \rfloor$ is used to denote the smallest integer no less than the corresponding real number, $\tau \in (0, (1-\theta)/\theta)$, and

$$\theta \equiv \theta(\alpha; x_*) = \|T(\alpha; x_*)\| \leq \max_{\lambda \in \sigma(H(x_*))} \frac{|\alpha - \lambda|}{\alpha + \lambda} \equiv \sigma(\alpha; x_*) < 1.$$

Then, for any $x_0 \in B(x_*, r)$, and any sequence $\{l_k\}_{k=0}^{\infty}$, the iteration sequence $\{x_k\}_{k=0}^{\infty}$ generated by Algorithm Newton-HSS is well defined and converges to x_*. Moreover, it holds that

$$\limsup_{k \to \infty} \|x_k - x_*\|^{\frac{1}{k}} \leq g(r_*; l_*),$$

here,

$$g(t, l) := \frac{\gamma}{1 - \gamma \int_0^t L(u)du} \left(\frac{2}{t} \int_0^t L(u)(t-u)du + \int_0^t L(u)du + 2\beta\gamma((\tau+1)\theta)^l\right).$$

Proof. First of all, we will show the following estimate about the iterative matrix $T(\alpha; x)$ of the linear solver: if $x \in B(x_*, r)$, then

$$\|T(\alpha; x)\| < (\tau + 1)\theta < 1.$$

In fact, from the definition of $B(\alpha; x)$ in (2.2) and Assumption 3.1, denote $\rho(x) = \|x - x_*\|$, we can get

$$\begin{aligned}
&\|B(\alpha; x) - B(\alpha; x_*)\| \\
\leq &\frac{1}{2}\|H(x) - H(x_*) + S(x) - S(x_*)\| + \frac{1}{2\alpha}\|H(x)S(x) - H(x_*)S(x_*)\| \\
\leq &\frac{1}{2}\int_0^{\rho(x)} L(u)du \\
&+ \frac{1}{2\alpha}\|(H(x) - H(x_*) + H(x_*))(S(x) - S(x_*)) + (H(x) - H(x_*))S(x_*)\| \\
\leq &\frac{1}{2}\int_0^{\rho(x)} L(u)du + \frac{1}{2\alpha}\left[\left(\int_0^{\rho(x)} L_h(u)du + \beta\right)\int_0^{\rho(x)} L_s(u)du + \beta\int_0^{\rho(x)} L_h(u)du\right] \\
\leq &\frac{1}{2}\int_0^{\rho(x)} L(u)du + \frac{1}{2\alpha}\left(\frac{(\int_0^{\rho(x)} L(u)du)^2}{4} + \beta\int_0^{\rho(x)} L(u)du\right) \\
\leq &\frac{1}{8\alpha}\left(\int_0^{\rho(x)} L(u)du\right)^2 + \frac{\alpha+\beta}{2\alpha}\int_0^{\rho(x)} L(u)du.
\end{aligned} \tag{3.9}$$

Similarly, we have

$$\|C(\alpha; x) - C(\alpha; x_*)\| \leq \frac{1}{8\alpha}\left(\int_0^{\rho(x)} L(u)du\right)^2 + \frac{\alpha+\beta}{2\alpha}\int_0^{\rho(x)} L(u)du. \tag{3.10}$$

Then from (2.3), it follows that

$$\|B(\alpha; x_*)^{-1}\| = \|(I - T(\alpha; x_*))F'(x_*)^{-1}\| \leq (1+\theta)\gamma < 2\gamma.$$

Therefore from (3.6), we obtain

$$\begin{aligned}
&\|I - B(\alpha; x_*)^{-1}B(\alpha; x)\| \\
&\leq \|B(\alpha; x_*)^{-1}\| \cdot \|B(\alpha; x) - B(\alpha; x_*)\| \\
&\leq \gamma \frac{(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du}{4\alpha} \\
&< 1.
\end{aligned}$$

Hence using the perturbation lemma, we get $B(\alpha; x)^{-1}$ exists, and

$$\begin{aligned}
&\|B(\alpha; x)^{-1}\| \\
&\leq \frac{\|B(\alpha; x_*)^{-1}\|}{1 - \|I - B(\alpha; x_*)^{-1}B(\alpha; x)\|} \\
&\leq \frac{8\alpha\gamma}{4\alpha - \gamma[(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du]}.
\end{aligned} \tag{3.11}$$

Hence, together with (3.6), (3.9)-(3.11), the estimate about the gap between inner iterative matrix $T(\alpha; x)$ and $T(\alpha; x_0)$ is obtained as follows:

$$\begin{aligned}
&\|T(\alpha; x) - T(\alpha; x_*)\| \\
&= \|B(\alpha; x)^{-1}(C(\alpha; x) - C(\alpha; x_*)) - B(\alpha; x)^{-1}(B(\alpha; x) - B(\alpha; x_*))B(\alpha; x_*)^{-1}C(\alpha; x_*)\| \\
&\leq \frac{2\gamma[(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du]}{4\alpha - \gamma[(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du]} \\
&< \tau\theta.
\end{aligned}$$

Consequently,

$$\|T(\alpha; x)\| \leq \|T(\alpha; x) - T(\alpha; x_*)\| + \|T(\alpha; x_*)\| < (\tau+1)\theta < 1. \tag{3.12}$$

Next, we turn to estimate the error about the Newton-HSS iteration sequence $\{x_k\}$ defined by (2.4). Clearly, from $\int_0^{\rho(x)} L(u)du < \frac{1}{\rho(x)}\int_0^{\rho(x)} L(u)(2\rho(x) - u)du$ and Lemma 3.2, it holds that $\gamma\int_0^{\rho(x)} L(u)du < 1$, hence, using Lemma 3.1, Lemma 3.2, (3.7), (3.8)

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and (3.12), we obtain

$$\begin{aligned}
&\|x_{k+1} - x_*\| \\
=&\|x_k - x_* - F'(x_k)^{-1}F(x_k) + T_k^{l_k}F(x_k)^{-1}F(x_k)\| \\
\leq&\|x_k - x_* - F'(x_k)^{-1}F(x_k)\| + \|T_k^{l_k}\|\|F(x_k)^{-1}\|\|F(x_k)\| \\
\leq&\frac{\gamma}{1-\gamma\int_0^{\rho(x_k)}L(u)du}\Big(\frac{1}{\rho(x_k)}\int_0^{\rho(x_k)}L(u)(\rho(x_k)-u)du + \int_0^{\rho(x_k)}L(u)du\Big)\|x_k - x_*\| \\
&+ \frac{((\tau+1)\theta)^{l_k}\gamma}{1-\gamma\int_0^{\rho(x_k)}L(u)du}\Big(\frac{1}{\rho(x_k)}\int_0^{\rho(x_k)}L(u)(\rho(x_k)-u)du + 2\beta\Big)\|x_k - x_*\| \\
\leq&\frac{\gamma}{1-\gamma\int_0^{\rho(x_k)}L(u)du}\Big(\frac{2}{\rho(x_k)}\int_0^{\rho(x_k)}L(u)(\rho(x_k)-u)du + \int_0^{\rho(x_k)}L(u)du \\
&+ 2\beta\gamma((\tau+1)\theta)^{l_k}\Big)\|x_k - x_*\| \\
:=&g(\rho(x_k); l_k)\|x_k - x_*\| \\
\leq&g(r_*, l_*)\|x_k - x_*\| \\
\leq&g(r_2, l_*)\|x_k - x_*\| \\
<&\|x_k - x_*\|,
\end{aligned}$$

when $x_k \in B(x_*, r_*)$, here, we have used the notation

$$g(t,l) := \frac{\gamma}{1-\gamma\int_0^t L(u)du}\Big(\frac{2}{t}\int_0^t L(u)(t-u)du + \int_0^t L(u)du + 2\beta\gamma((\tau+1)\theta)^l\Big).$$

Thus, we can further prove that $\{x_k\} \subset B(x_*, r)$ with the estimates

$$\|x_{k+1} - x_*\| \leq g(r_*, l_*)\|x_k - x_*\|, \quad k = 0, 1, 2, \cdots. \tag{3.13}$$

In fact, for $k = 0$ we have $\|x_0 - x_*\| < r$, as $x_0 \in B(x_*, r)$. Together with $g(r_*, l_*) < 1$, it follows from (3.13) that

$$\|x_1 - x_*\| \leq g(r_*, l_*)\|x_0 - x_*\| < r,$$

hence, $x_1 \in B(x_*, r)$. Suppose that $x_k \in B(x_*, r)$, then using (3.13) again, we have

$$\|x_{k+1} - x_*\| \leq g(r_*, l_*)\|x_k - x_*\| < r,$$

hence, $x_{k+1} \in B(x_*, r)$. Moreover, we have

$$\|x_k - x_*\| \leq g(r_*, l_*)\|x_k - x_*\| \leq g(r_*, l_*)^{k+1}\|x_0 - x_*\|.$$

Now the proof is complete. □

Remark 1. *If we assume the integrable functions $L_h(u)$ and $L_s(u)$ in (A2) are positive constants L_h and L_s, respectively, then the center Lipschtiz condition with the L-average becomes usual Lipschtiz condition. If both integrable functions are $\frac{L}{p}u^{p-1}$, where $0 < p < 1$, L is a positive constant, then the center Lipschtiz condition with the L-average becomes usual Hölder condition. Therefore, Theorem 3.1 is an extension of Theorem 3.2 in [10]. If the outer solver, Newton method, is changed to the modified Newton method, then 3.1 is an extension of Theorem 3.1 in [7].*

4 Application

In this section, we apply the main result on a two-demensional nonlinear convection-diffusion equation.

Example 1. Consider the following two-dimensional nonlinear convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy}) + q_1 u_x + q_2 u_y = u^c, & (x, y) \in \Omega, \\ u(x, y) = 0, & on\ (x, y) \in \partial\Omega, \end{cases} \tag{4.14}$$

where c is a rational number, $\Omega = (0,1) \times (0,1)$, $\partial\Omega$ is the boundary. q_1 and q_2 are positive constants used to measure magnitudes of the convective terms. By applying the centered finite difference scheme on the equidistant discretization grid with the stepsize $h = 1/(N+1)$, the system of nonlinear equations (1.1) is obtained with the following form

$$F(x) = Mx + h^2\phi(x) = 0,$$

where N is a prescribed positive integer,

$$M = (T_x \otimes I + I \otimes T_y),$$
$$\phi(x) = (x_1^c, x_2^c, \cdots, x_n^c)^T,$$

with $T_x = tridiag(-1 - Re_1, 2, -1 + Re_1)$, $T_y = tridiag(-1 - Re_2, 2, -1 + Re_2)$, here, $Re_j = \frac{1}{2}q_j h$, $j = 1, 2$, $Re = \max\{Re_1, Re_2\}$ is the mesh Reynolds number, $\otimes$ is the Kronecker product, and $n = N \times N$.

Obviously, $x_* = 0$ is a solution of (4.14), and it is easy to get $F^{'}(x) = M + ch^2 diag(x_1^{c-1}, x_2^{c-1}, \cdots, x_n^{c-1})$. Hence $F'(x_*) = M$. Moreover, we have

$$\|F^{'}(x) - F^{'}(x_*)\| \leq ch^2\|x - x_*\|^{c-1} = \int_0^{\rho(x)} L(u)du, \quad x \in B(x_*, r) \subset \Omega,$$

where $\|\cdot\|$ denotes the 2-norm, $L(u) = c(c-1)h^2u^{c-2}$. Hence, the center Lipschtiz condition with the L average is satisfied.

Thus we can obtain the convergence result of nonlinear equation (4.14).

Corollary 4.1. *Consider (4.14), define $r_* := \min\{r_1, r_2\}$, where r_1 and r_2 satisfy*

$$r_1^{c-1} = \frac{2(\alpha+\beta)}{ch^2}\Big(\sqrt{\frac{\alpha\tau\theta}{(2+\tau\theta)\gamma(\alpha+\beta)^2} + 1} - 1\Big),$$

$$r_2^{c-1} = \frac{1 - 2\beta\gamma((\tau+1)\theta)^{l_*}}{2(c+1)\gamma h^2},$$

and $l_* = \liminf_{k\to\infty} l_k$ *satisfies*

$$l_* > \lfloor \frac{\ln 2\beta\gamma}{\ln((\tau+1)\theta)} \rfloor,$$

where the symbol $\lfloor\cdot\rfloor$ *is used to denote the smallest integer no less than the corresponding real number,* $\tau \in (0, (1-\theta)/\theta)$*, and*

$$\theta \equiv \theta(\alpha; x_*) = \|T(\alpha; x_*)\| \leq \max_{\lambda\in\sigma(H(x_*))} \frac{|\alpha - \lambda|}{\alpha + \lambda} \equiv \sigma(\alpha; x_*) < 1.$$

Then, for any $x_0 \in B(x_*, r)$*, and any sequence* $\{l_k\}_{k=0}^{\infty}$*, the iteration sequence* $\{x_k\}_{k=0}^{\infty}$ *generated by Algorithm Newton-HSS is well defined and converges to* x_**. Moreover, it holds that*

$$\limsup_{k\to\infty} \|x_k - x_*\|^{\frac{1}{k}} \leq g(r_*; l_*),$$

here,

$$g(t, l) := \frac{\gamma}{1 - c\gamma h^2 t^{c-1}}((c+2)h^2 t^{c-1} + 2\beta\gamma((\tau+1)\theta)^l).$$

Remark 2. *For equation (4.14), if* $c = 4/3$ *or* $c = 3/2$*, then equation (4.14) becomes the equation studied in [6] and [7], respectively, hence satisfies Hölder condition. If* $c = 2$*, thus* $L(u) = 2h^2$ *is a positive constant, hence equation (4.14) satisfies Lipschtiz condition.*

Now we consider the numerical results of the corollary. We choose $c = 2$. In the following computation, the stopping criterion for the outer Newton method is set to be

$$\frac{\|F(x_k)\|_2}{\|F(x_0)\|_2} \leq 10^{-10},$$

and the prescribed tolerance for controlling the accuracy of the HSS iteration is set to be $\eta_k = \eta$. Let the initial guess $x_0 = 1$, then parameters β, γ, can be estimated from Assumption 3.1. Take positive constants $q_1 = q$, $q_2 = 1/h$, and adopt experimentally optimal parameter α, which yields the smallest value of $\|x_{k+1} - x_*\|$, then θ and τ can be estimated from the definition of $\|T(\alpha; x_0)\|$ and the estimation of $\|T(\alpha; x)\|$, respectively, and the Newton-HSS method is examined for different problem size $n = N \times N$, different quantity $q = q_1$ and different tolerance η, from the values of $\|x_{k+1} - x_*\|$, $\frac{\|x_{k+1}-x_*\|}{\|x_k - x_*\|}$. We list the numerical results in Tables 4.2 and 4.3.

Table 4.1. The optimal value α for the Newton-HSS method

N	q=600			q=800			q=1000		
	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$
30	0.9	3.1	1.8	1.3	3.3	1	0.6	0.6	1.1
40	1	3.3	2.1	1	0.7	1.5	1	3.9	0.6
50	3.7	2.1	4.9	0.7	3.1	3.4	0.7	0.5	2.9

Table 4.2. Values of $\|x_{k+1} - x_*\|$ for different N and q ($\eta = 0.1$)

k	N=30			N=40			N=50		
	q=600	q=800	q=1000	q=600	q=800	q=1000	q=600	q=800	q=1000
1	1.5115	1.8882	3.1749	2.1595	1.1569	3.1537	1.6414	2.1067	2.0363
2	0.1171	0.2152	0.2880	0.1665	0.1154	0.2779	0.0877	0.1133	0.1477
3	0.0087	0.0156	0.0268	0.0116	0.0085	0.0219	0.0075	0.0079	0.0121
4	6.49e-04	0.0011	0.0026	8.30e-04	8.90e-04	0.0021	5.03e-04	7.05e-04	9.03e-04
5	6.05e-05	9.33e-05	2.61e-04	5.97e-05	7.90e-05	1.83e-04	4.16e-05	5.63e-05	7.18e-05
6	5.80e-06	7.70e-06	2.21e-05	4.11e-06	5.98e-06	1.30e-05	3.55e-06	5.03e-06	6.28e-06
7	5.26e-07	5.21e-07	1.53e-06	3.12e-07	4.52e-07	7.06e-07	2.73e-07	3.32e-07	6.01e-07
8	3.23e-08	3.86e-08	1.58e-07	2.35e-08	3.14e-08	4.35e-08	1.96e-08	2.54e-08	5.40e-08
9	2.51e-09	3.36e-09	1.32e-08	1.80e-09	2.64e-09	3.81e-09	1.34e-09	2.10e-09	4.50e-09
10	1.46e-10	2.14e-10	9.05e-10	1.73e-10	2.51e-10	3.10e-10	8.91e-11	1.81e-10	3.59e-10

In Table 4.2, we present the values of $\|x_{k+1} - x_*\|$, corresponding to the problem size $N = 30$, 40 and 50, and parameter $q = 600$, 800 and 1000, respectively, for the inner tolerance $\eta = 0.1$. From the table, we can see that the sequence $\{x_k\}$ generated by the Newton-HSS method converges to the solution x_* in all these situations.

Table 4.3. Values of $\frac{\|x_{k+1}-x_*\|}{\|x_k - x_*\|}$ for different N and q ($\eta = 0.1$)

k	N=30			N=40			N=50		
	q=600	q=800	q=1000	q=600	q=800	q=1000	q=600	q=800	q=1000
1	0.0504	0.0629	0.1058	0.0540	0.0392	0.0788	0.0328	0.0421	0.0407
2	0.0775	0.0114	0.0907	0.0771	0.0736	0.0881	0.0534	0.0538	0.0725
3	0.0741	0.0726	0.0931	0.0699	0.0736	0.0788	0.0858	0.0693	0.0819
4	0.0742	0.0729	0.0957	0.0713	0.1048	0.0966	0.0669	0.0898	0.0747
5	0.0933	0.0820	0.1017	0.0720	0.0888	0.0863	0.0827	0.0799	0.0795
6	0.0959	0.0825	0.0848	0.0688	0.0756	0.0712	0.0854	0.0893	0.0874
7	0.0906	0.0676	0.0689	0.0759	0.0757	0.0542	0.0768	0.0659	0.0958
8	0.0614	0.0742	0.1037	0.0752	0.0694	0.0616	0.0718	0.0767	0.0898
9	0.0778	0.0869	0.0835	0.0766	0.0840	0.0876	0.0686	0.0826	0.0833
10	0.0582	0.0636	0.0686	0.0961	0.0953	0.0815	0.0663	0.0860	0.0799

In Table 4.3, we present the values of $\frac{\|x_{k+1}-x_*\|}{\|x_k - x_*\|}$ corresponding to the problem size $N = 30$, 40 and 50, and parameter $q = 600$, 800 and 1000, respectively, for the inner tolerance $\eta = 0.1$. From the table, we can observe that all the values of $g(r_*, l_*)$ are smaller than 0.11.

5 Conclusion

The Newton-HSS method is a considerable method for solving large sparse nonlinear systems with non-Hermitian positive definite Jacobian matrices. In this paper, Under the hypothesis that the Jacobian matrix satisfies the center Lipschtiz condition with the L-average, which is weaker than Hölder condition and Lipschtiz condition, we establish the local convergence theorem for the Newton-HSS method. Finally, a numerical example is given to confirm the concrete applications of the results of our paper.

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Oscillation for Fractional Neutral Functional Differential Systems*

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Abstract

In this paper, we discuss the linear autonomous system of neutral delay differential equations with Riemann–Liouville fractional derivative

$$D^{\alpha}\left[x(t)+\sum_{j=1}^{l}P_jx(t-\tau_j)\right]+\sum_{i=1}^{n}Q_ix(t-\delta_i)=0$$

where $D^{\alpha}x(t)=[D^{\alpha_1}x_1(t),D^{\alpha_2}x_2(t),...,D^{\alpha_m}x_m(t)]^T$ is Riemann–Liouville fractional derivative, the coefficients $P_j(j=1,2,...,l)$ and $Q_i(i=1,2,...,n)$ are real $m\times m$ matrices and the delays $\tau_j(j=1,2,...,l)$ and $\delta_i(i-1,2,...,n)$ are non-negative real numbers. Sufficient conditions for all solutions of the given equation to be oscillatory are obtained by using fractional calculus and Laplace transform.

Key words and phrases: Fractional neutral differential equations; Riemann–Liouville derivative; Oscillation; Laplace transform.
AMS (MOS) Subject Classifications: 15A60; 26A33; 34A30; 34K11; 44A10.

1 Introduction

Fractional differential equations have gained considerable importance due to their application in various disciplines, such as physics, mechanics, chemistry, engineering, etc. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Podlubny[1], Kilbas et al.[2], Diethelm[3], Zhou[4, 5], the recent papers[6, 7, 8, 9] and the references therein.

On the other hand, the objective of oscillation theory is to acquire as much information as possible about the qualitative properties of solutions of differential equations. Oscillation theory of functional differential equations with integer derivative has been developed in the past thirty years. The several monographs by Ladde et al.[10], Györi and Ladas[11], Gopalsamy[12], Erbe et al.[13], Agarwal et al.[14] summarize a lot of important works in this area.

However, to the best of our knowledge, there are few results on oscillation for fractional differential equations. Recently, Grace, Agarwal and Wong, et al.[15], Bolat[16], Duan, Wang and Fu[17], Harikrishnan, Prakash and Nieto[18] investigated oscillation and forced oscillation of fractional-order delay differential equations.

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In this paper, we discuss the neutral functional differential equations with Riemann–Liouville fractional derivative

$$D^{\alpha}\left[x(t)+\sum_{j=1}^{l}P_j x(t-\tau_j)\right]+\sum_{i=1}^{n}Q_i x(t-\delta_i)=0, \tag{E}$$

where $x(t) = [x_1(t), x_2(t), ..., x_m(t)]^T$, $D^{\alpha}x(t) = [D^{\alpha_1}x_1(t), D^{\alpha_2}x_2(t), ..., D^{\alpha_m}x_m(t)]^T$ is Riemann–Liouville fractional derivative of order $0 < \alpha, \alpha_r < 1$, $\alpha_r = p_r/q_r$, p_r, q_r are co-prime, for $r = 1, 2, ..., m$, and $P_j, Q_i \in \mathbb{R}^{m\times m}$, $j = 1, 2, ..., l$, $i = 1, 2, ..., n$, the delays $\tau_j (j = 1, 2, ..., l)$ and $\delta_i(i - 1, 2, ..., n)$ are non-negative real numbers.

Our aim is to establish sufficient conditions for oscillation of the system (E). In the next section, we introduce some useful preliminaries. In section 3, we obtain various sufficient conditions for oscillation of all solutions to the system (E) by using fractional calculus and Laplace transform.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 2.1 *[2] Let $[a, b](-\infty < a < b < \infty)$ be a finite interval and let $AC[a, b]$ be the space of functions f which are absolutely continuous on $[a, b]$. It is known [see Kolmogorov and Fomin ([16], p.338) that $AC[a, b]$ coincides with the space of primitives of Lebesgue summable functions:*

$$f(x)\in AC[a,b]\ \Rightarrow f(x)=c+\int_a^x \psi(t)dt\ (\psi(t)\in L(a,b)).$$

Definition 2.2 *[2] The fractional integral of order α with the lower limit zero for a function f is defined as*

$$(I^{\alpha}f)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\ t>0,\ 0<\alpha<1,$$

provided the right side is point-wise defined on $[0, b]$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 *[2] Riemann-Liouville derivative of order α with the lower limit zero for a function f can be written as*

$$(D^{\alpha}f)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(s)}{(t-s)^{\alpha}}ds,\ t>0,\ 0<\alpha<1.$$

Firstly, we consider the fractional delay differential systems

$$D^{\alpha}x(t)+\sum_{i=1}^{n}P_i x(t-\tau_i)=0,\quad t\ge 0, \tag{1}$$

where $x(t) = [x_1(t), x_2(t), ..., x_m(t)]^T$, $D^{\alpha}x(t) = [D^{\alpha_1}x_1(t), D^{\alpha_2}x_2(t), ..., D^{\alpha_m}x_m(t)]^T$ is Riemann–Liouville fractional derivative of order $0 < \alpha, \alpha_i < 1$, $\alpha_j = p_j/q_j$, p_j, q_j are odd numbers, for $i = 1, 2, ..., m$, and $P_i \in \mathbb{R}^{m\times m}$, $\tau_i \in [0, \infty)$ for $i = 1, 2, ..., n$.

Without loss of generality we will assume the coefficients of P_i of (1) are all nonzero and that $\tau_1 = \max\{\tau_1, ..., \tau_n\}$.

Definition 2.4 *By a solution of (1) in $[0, \infty)$ with initial function $\varphi \in AC[-\tau_1, 0]$ we mean a function $x \in AC[-\tau_1, \infty)$ such that $x(t) = \varphi(t)$, $t \in [-\tau_1, 0]$, $(D^{\alpha}x)(t)$ exists and $x(t)$ satisfies (1) in $[0, \infty)$. A solution $x(t) = [x_1(t), ..., x_m(t)]^T$ of system (1) is said to oscillate if every component $x_i(t)$ of the solution has arbitrarily large zeros. Otherwise the solution is called non-oscillatory.*

We recall some facts about Laplace transforms. If $X(s)$ is the Laplace transform of $x(t)$,

$$X(s)=(Lx)(s)=\int_0^{\infty}e^{-st}x(t)dt,$$

then the abscissa of convergence of $X(s)$ is defined by

$$b = \inf\{\delta \in \mathbb{R} : X(\delta)\ exists\}.$$

Then $X(s)$ is analytic for $Re(s) > b$.

We call a function $x(t)$ to be eventually positive if there exists a $c \geq 0$ such that $x_c(t) > 0$ for all $t > 0$, where $x_c(t) = x(t + c)$.

For any m-dimensional vector $x = (x_1, x_2, ..., x_m)^T \in \mathbb{R}^m$, $\|x\|$ denotes its norm. For any $m \times m$ real matrix A, the associated matrix norm is then defined by $\|A\| = \max_{\|x\|=1} \|Ax\|$. Denote $\mu(P_i)$ is the logarithmic norm with $\mu(P_i) = \max_{\|u\|=1}(P_i u, u)$.

Lemma 2.5 *[2] Let $(LD^\alpha x)(s)$ is the Laplace transform of the Riemann–Liouville fractional derivative of order α with the lower limit 0 for a function x, and $X(s)$ is the Laplace transform of $x(t) \in AC[0, b]$, for any $b > 0$, and the following estimate*

$$|x(t)| \leq Ae^{p_0 t} \quad (t > b > 0)$$

holds for constants $A > 0$ and $p_0 > 0$. Then the relation

$$(LD^\alpha x)(s) = s^\alpha BX(s) - (I^{1-\alpha}x)(0),\ 0 < \alpha < 1$$

is valid for $Re(s) > p_0$, where

$$X(s) = [X_1(s), X_2(s), ..., X_m(s)]^T, \quad s^\alpha = [s^{\alpha_1}, s^{\alpha_2}, ..., s^{\alpha_m}]$$

$$(I^{1-\alpha}x)(0) = \Big((I^{1-\alpha_1}x_1)(0), (I^{1-\alpha_2}x_2)(0)], ..., (I^{1-\alpha_m}x_m)(0)\Big)^T,$$

$$B = [B_1, B_2, ..., B_m]^T\ B_i = (b_{ij})_{m\times m}$$

$$b_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Lemma 2.6 *[19] If $X(s)$ is the Laplace transform of a non-negative function $x(t)$ and has abscissa of convergence $b > -\infty$, then $X(s)$ has a singularity at the point $s = b$.*

Lemma 2.7 *[20] Let $v, w : [0, \infty) \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $a > 0$ and $0 < \beta < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^\beta} ds,$$

then there exists a constant $k = k(\beta)$ such that

$$v(t) \leq w(t) + ka \int_0^t \frac{w(s)}{(t-s)^\beta} ds$$

for every $t \in [0, \infty)$.

3 Main Results

In this section, we present our main results.

Lemma 3.1 *For any $c \in \mathbb{R}$, the Laplace transform $X_c(s)$ of $x_c(t)$ exists and has the same abscissa of convergence as $X(s)$.*

Proof. Given that

$$\begin{aligned} X_c(s) &= \int_0^\infty e^{-st}x_c(t)dt = \int_0^\infty e^{-st}x(t+c)dt = e^{sc}\int_c^\infty e^{-st}x(t)dt \\ &= e^{sc}[X(s) - \int_0^c e^{-st}x(t)dt]. \end{aligned}$$

Since the last integral defines an entire function of the complex variable s, therefore $X(s)$ and $X_c(s)$ converge or diverge for the same values of s, and have their singularities at the same points. This completes the proof. □

Lemma 3.2 *The solution of equation (1) has an exponent estimate*

$$x(t) = o(e^{q_0 t}) \qquad (t > b > 0)$$

for constant $q_0 > 0$.

Proof. Taking Riemann-Liouville integral of equation (1), we get

$$\begin{aligned} x(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}Bx_0 - \sum_{i=1}^n P_i \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}Bx(s-\tau_i)ds \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}Bx_0 - \sum_{i=1}^n P_i F_i(t), \end{aligned} \tag{2}$$

where

$$x_0 = (I^{1-\alpha}x)(0),$$

$$F_i(t) = \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}Bx(s-\tau_i)ds, \quad \frac{t^{\alpha-1}}{\Gamma(\alpha)} = \left[\frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}, \frac{t^{\alpha_2-1}}{\Gamma(\alpha_2)}, ..., \frac{t^{\alpha_m-1}}{\Gamma(\alpha_m)}\right].$$

As $AC[-\tau_1, 0]$ is the Banach space with the norm $\|\varphi\|_{AC} = [\|\varphi_1\|_{AC}, \|\varphi_2\|_{AC}, ..., \|\varphi_m\|_{AC}]^T$. Then we have

$$\begin{aligned} \|F_i(t)\| &\le \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}B\|x(s-\tau_i)\|ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B \max_{s-\tau_i\le\eta\le s}\|x(\eta)\|ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B \max_{s-\tau_1\le\eta\le s}\|x(\eta)\|ds, \end{aligned}$$

or

$$\|F_i(t)\| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B\Big(\max_{s-\tau_1\le\eta\le s}\|x(\eta)\| + \|\varphi\|_{AC}\Big)ds. \tag{3}$$

From (3), it follows that

$$\begin{aligned} \|x(t)\| &\le \frac{b^{\alpha-1}}{\Gamma(\alpha)}B|x_0| + \sum_{i=1}^n \frac{\|P_i\|}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}B\Big(\max_{s-\tau_1\le\eta\le s}\|x(\eta)\| + \|\varphi\|_{AC}\Big)ds\right] \\ &\le \frac{b^{\alpha-1}}{\Gamma(\alpha)}B|x_0| + \frac{np}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha}B\|\varphi\|_{AC} + \frac{np}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B\max_{s-\tau_1\le\eta\le s}\|x(\eta)\|ds, \end{aligned}$$

where $p = \max\{\|P_i\|\}$, for $i = 1, 2, ..., n$.

Next we introduce a nondecreasing function $m(t)$ as

$$m(t) = \frac{b^{\alpha-1}}{\Gamma(\alpha)}B|x_0| + \frac{np}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha}B\|\|\varphi\|_{AC}\|.$$

By Lemma 2.7, there exists a number α_0 in $\{\alpha_i\}$ such that

$$\|x(t)\| \leq \max_{t-\tau_1\leq s\leq t} \|x(s)\| \leq m(t) + \frac{knp}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} m(s)ds \leq m(t)\Big(1 + \frac{knp}{\alpha_0\Gamma(\alpha_0)} t^{\alpha_0}\Big). \tag{4}$$

Obviously, from (4) we infer that $x(t)$ has an exponent estimate. The proof is complete. □

Theorem 3.3 *If the characteristic equation*

$$\det\left(\lambda^\alpha B + \sum_{i=1}^n P_i e^{-\lambda\tau_i}\right) = 0 \tag{5}$$

has no real roots, then every solution of (1) is oscillatory, where $\lambda^\alpha = [\lambda^{\alpha_1}, \lambda^{\alpha_2}, ..., \lambda^{\alpha_m}]$.

Proof. For the sake of contradiction, let us assume that (5) has no real roots and that (1) has a non-oscillatory solution $x(t) = [x_1(t), ..., x_m(t)]^T$. This means that one of the components of $x(t)$ is non-oscillatory. Without loss of generality we assume that the component $x_1(t)$ is eventually positive, such that for some $c \geq 0$, $x_c(t) > 0$ for $t \geq 0$. As (1) is autonomous, it follows by Lemma 3.1 that $X_1(s)$ and $X_c(s)$ have the same convergence. Then we assume that $x_1(t) > 0$ for $t \geq -\tau_1$. Taking Laplace transform of both sides of (1), we obtain

$$s^\alpha BX(s) - (I^{1-\alpha}x)(0) + \sum_{i=1}^n P_i \int_0^\infty e^{-st}x(t-\tau_i)dt = 0,$$

i.e.

$$s^\alpha BX(s) - (I^{1-\alpha}x)(0) + \sum_{i=1}^n P_i e^{-s\tau_i} X(s) + \sum_{i=1}^n P_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st}x(t)dt = 0.$$

Hence

$$(s^\alpha B + \sum_{i=1}^n P_i e^{-s\tau_i})X(s) = (I^{1-\alpha}x)(0) - \sum_{i=1}^n P_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st}x(t)dt. \tag{6}$$

Let

$$F(s) = s^\alpha B + \sum_{i=1}^n P_i e^{-s\tau_i},\ x_0 = (I^{1-\alpha}x)(0),$$

$$\Phi(s) = x_0 - \sum_{i=1}^n P_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st}x(t)dt.$$

Then, from (6) we get

$$F(s)X(s) = \Phi(s),\ Re(s) > b. \tag{7}$$

Since $\det[F(s)] = 0$ has no real roots, $\det[F(s)] > 0,\ s \in \mathbb{R}$. By Cramer's rule, we have

$$X_1(s) = \frac{\det[D(s)]}{\det[F(s)]},\ Re(s) > b, \tag{8}$$

where

$$D(s) = \begin{pmatrix} \Phi_1(s) & F_{12}(s) & \cdots & F_{1m}(s) \\ \vdots & \vdots & & \vdots \\ \Phi_m(s) & F_{m2}(s) & \cdots & F_{mm}(s) \end{pmatrix},$$

$\Phi_i(s)$ is the ith component of the vector $\Phi(s)$ and $F_{ij}(s)$ is the (i,j)th component of the matrix $F(s)$. Clearly, for all $i, j = 1, 2, ..., m$ the functions $\Phi_i(s)$ and $F_{ij}(s)$ are entire and hence $\det[D(s)]$ and $\det[F(s)]$ are also entire functions.

Since $\det[F(s)] > 0$ for $s \in \mathbb{R}$, so $\det[D(s)]/\det[F(s)]$ holds for $s \in \mathbb{R}$ and thus (8) becomes

$$X_1(s) = \frac{\det[D(s)]}{\det[F(s)]},\ s \in \mathbb{R}. \tag{9}$$

As $x_1(t) > 0$, it follows that $X_1(s) > 0$ for all $s \in \mathbb{R}$ and, by $\det[F(s)] > 0$ $s \in \mathbb{R}$ and (9), $\det[D(s)] > 0$ $s \in \mathbb{R}$. Now one can see from the definitions of $D(s)$, $F(s)$ and $\Phi(s)$ that there exist positive constants M, β, and s_0 such that

$$\det[D(s)] \leq Me^{-\beta s} \quad for\ s \leq -s_0. \tag{10}$$

Since $\det[F(s)]$ is a continuous function in the variables $s, e^{-s\tau_1}, ..., e^{-s\tau_n}$, and $\det[F(s)] > 0,\ s \in \mathbb{R}$, it follows that there exists a positive number m_0 such that

$$\det[F(s)] \geq m_0 \quad for\ s \in \mathbb{R}. \tag{11}$$

From (9), (10) and (11), it follows that

$$X_1(s) = \int_0^\infty e^{-st} x_1(t)dt \geq \int_T^\infty e^{-st} x_1(t)dt \geq e^{-sT} \int_T^\infty x_1(t)dt > 0$$

and so

$$0 < \int_T^\infty x_1(t)dt \leq \frac{M}{m_0} e^{s(T-\beta)} \to 0 \quad as\ s \to -\infty.$$

This implies that $x_1(t) \equiv 0$ for $t \geq T$, which is a contradiction. The proof is complete. □

In Theorem 3.3, the characteristic equation (5) plays an important role in the investigation of the oscillation of equation (1). However, to determine whether (5) has a real root, is quite an issue in itself. In the following we derive some sufficient conditions for the oscillation of equation (1) which can easily be applied.

Before proceeding for it, we need the following lemma which is interesting in its own right.

Lemma 3.4 *Assume that*
$P_i \in \mathbb{R}^{m\times m}$, $\tau_i \geq 0$ *for* $i = 1, 2, ..., n$, *and* $\bar{\alpha} = \min\{\alpha_j\}$, *for* $j = 1, 2, ..., m$ *with*

$$\sum_{i=1}^n \mu(-P_i)e^{-\lambda\tau_i} < 0 \quad for \quad \lambda \in \mathbb{R} \tag{12}$$

and

$$\inf_{\lambda<0} \left[\frac{1}{\lambda^{\bar{\alpha}}} \sum_{i=1}^n \mu(-P_i)e^{-\lambda\tau_i}\right] > 1. \tag{13}$$

Then every solution of (1) oscillates.

Proof. Assume, for the sake of contraction, that (1) has a non-oscillatory solution. Then, by Theorem 3.3, the characteristic equation (5) has a real root λ_0. In consequence, there exists a vector $u \in \mathbb{R}^n$ with $\|u\| = 1$ such that

$$\left(\lambda_0^\alpha B + \sum_{i=1}^n P_i e^{-\lambda_0\tau_i}\right)u = 0,$$

i.e.

$$\lambda_0^\alpha Bu = -\sum_{i=1}^n P_i e^{-\lambda_0\tau_i} u.$$

Hence

$$\begin{aligned} \lambda_0^{\bar{\alpha}} &= (\lambda_0^{\bar{\alpha}} u, u) \leq (\lambda_0^\alpha Bu, u) = (-\sum_{i=1}^n P_i e^{-\lambda_0\tau_i} u, u) \\ &= (-\sum_{i=1}^n P_i u, u)e^{-\lambda_0\tau_i} \leq \sum_{i=1}^n \mu(-P_i)e^{-\lambda_0\tau_i}. \end{aligned}$$

Then by (12), $\lambda_0 < 0$ such that

$$\left[\frac{1}{\lambda_0^{\bar{\alpha}}} \sum_{i=1}^n \mu(-P_i)e^{-\lambda_0\tau_i}\right] \leq 1 \quad \text{or} \quad -\lambda_0 \geq \left[\sum_{i=1}^n -\mu(-P_i)\right]^{\frac{1}{\bar{\alpha}}}. \tag{14}$$

This contradicts (13) and completes the proof. □

Theorem 3.5 *Assume that for each $i = 1, 2, ..., n$,*

$$P_i \in \mathbb{R}^{m\times m}, \quad \tau_i \geq 0 \quad and \quad \mu(-P_i) \leq 0.$$

Then each of the following two conditions is sufficient for the oscillation of all solutions of (1):

(i) $\sum_{i=1}^{n} -\mu(-P_i)\tau_i\Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}} > \frac{1}{e};$

(ii) $\Big[\prod_{i=1}^{n}(-\mu(-P_i))\Big]^{\frac{1}{n}} \sum_{i=1}^{n} \tau_i\Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}} > \frac{1}{e}.$

Proof. We employ Lemma 3.4. As $\mu(-P_i) \leq 0$, (12) is satisfied and so it suffices to establish (13). First, assume that (i) holds. Then, by using the inequality $e^x \geq ex$, we see that for all $\lambda < 0$,

$$\begin{aligned}
\frac{1}{\lambda^{\bar{\alpha}}}\sum_{i=1}^{n}\mu(-P_i)e^{-\lambda\tau_i} &\geq \frac{1}{\lambda^{\bar{\alpha}}}\sum_{i=1}^{n}\mu(-P_i)e(-\lambda\tau_i)\\
&= -e\frac{1}{\lambda^{\bar{\alpha}}}\sum_{i=1}^{n}\mu(-P_i)\tau_i\lambda^{\bar{\alpha}}(-\lambda)^{1-\bar{\alpha}}\\
&\geq e\sum_{i=1}^{n} -\mu(-P_i)\tau_i\Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}},
\end{aligned}$$

which, together with (i), implies that (13) holds. Next, assume that (ii) holds. Then, by using the arithmetic mean–geometric mean inequality we find that for all $\lambda < 0$,

$$\begin{aligned}
\frac{1}{\lambda^{\bar{\alpha}}}\sum_{i=1}^{n}\mu(-P_i)e^{-\lambda\tau_i} &= -\frac{1}{\lambda^{\bar{\alpha}}}\sum_{i=1}^{n} -\mu(-P_i)e^{-\lambda\tau_i}\\
&\geq -\frac{1}{\lambda^{\bar{\alpha}}}n\Big[\prod_{i=1}^{n} -\mu(-P_i)e^{-\lambda\tau_i}\Big]^{\frac{1}{n}}\\
&= -\frac{1}{\lambda^{\bar{\alpha}}}n\Big[\prod_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1}{n}}\exp\Big(-\frac{1}{n}\lambda\sum_{i=1}^{n}\tau_i\Big)\\
&\geq -\frac{1}{\lambda^{\bar{\alpha}}}\Big[\prod_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1}{n}}e\Big(-\lambda\sum_{i=1}^{n}\tau_i\Big)\\
&= \Big[\prod_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1}{n}}e(-\lambda)^{1-\bar{\alpha}}\sum_{i=1}^{n}\tau_i\\
&\geq e\Big[\prod_{i=1}^{n}(-\mu(-P_i))\Big]^{\frac{1}{n}}\sum_{i=1}^{n}\tau_i\Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}}.
\end{aligned}$$

From this and (ii) it follows that (13) holds. The proof is complete. □

As a special case of the delay differential system with one delay,

$$(D^{\alpha}x)(t) + Px(t-\tau) = 0, \tag{15}$$

where

$$P \in \mathbb{R}^{m\times m} \quad and \quad \tau \geq 0,$$

the conditions (i) and (ii) coincide and each reduces to

$$[-\mu(-P)]^{\frac{1}{\bar{\alpha}}}\tau > \frac{1}{e}. \tag{16}$$

Note that (16) is sharp in the sense that the lower bound $1/e$ cannot be improved. Moreover, when P is a scalar, (16) is a sufficient condition for the oscillation of all solutions to equation (15).

For the delay differential system (15), we also have the following explicit sufficient condition for the oscillation of all solutions.

Theorem 3.6 *Assume that*

$$P \in \mathbb{R}^{m\times m} \quad and \quad \tau \geq 0.$$

If P has no real eigenvalues in the interval $(-\infty, 1/(e\tau)^{\bar{\alpha}}]$ (when $\tau = 0$, replace $1/(e\tau)^{\bar{\alpha}}$ by $+\infty$), then every solution of (15) oscillates.

Proof. For $\tau = 0$, this result follows immediately from Theorem 3.1. So assume $\tau > 0$. Note that the characteristic equation $\det(\lambda^\alpha B + Pe^{-\lambda\tau}) = 0$ has a real root λ_0, that is, $\det(\lambda_0^\alpha e^{\lambda_0\tau}B + P) = 0$ if and only if $\mu_0^\alpha = -\lambda_0^\alpha e^{\lambda_0\tau}$ is a real eigenvalue of P. For convenience, we take one element $\lambda_0^{\alpha_i}$ of λ_0^α:

$$\mu_0^{\alpha_i} = -\lambda_0^{\alpha_i} e^{\lambda_0\tau}. \tag{17}$$

Observe that (17) holds if $\lambda_0^{\alpha_i} + \mu_0^{\alpha_i} e^{-\lambda_0\tau} = 0$, that is, the equation $\lambda^{\alpha_i} + \mu_0^{\alpha_i} e^{-\lambda\tau} = 0$ has a real root. If $\mu_0 \leq 1/e\tau$, then the eigenvalue μ_0^α of P should lie in the interval $(-\infty, 1/(e\tau)^{\bar{\alpha}}]$. The proof is complete. □

Definition 3.7 ([13]) *We say that (1) is oscillatory, globally in the delays, if for all $\tau_i \geq 0$ for $i = 1, 2, ..., n$, every solution of (1) oscillates.*

The following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.8 *Equation (15) is oscillatory globally in the delay τ if P has no real eigenvalues.*

Next, we consider the linear autonomous system of neutral delay differential equations

$$D^\alpha\left[x(t) + \sum_{j=1}^{l} P_j x(t-\tau_j)\right] + \sum_{i=1}^{n} Q_i x(t-\delta_i) = 0, \tag{18}$$

where the coefficients P_j and Q_i are real $m \times m$ matrices and the delays τ_j and δ_i are non-negative real numbers. Associated with (18), the characteristic equation is

$$det\left(\lambda^\alpha B + \lambda^\alpha \sum_{j=1}^{l} P_j e^{-\lambda\tau_j} + \sum_{i=1}^{n} Q_i e^{-\lambda\delta_i}\right) = 0. \tag{19}$$

Lemma 3.9 *If $lp < 1$, then the solution of equation (18) has an exponent estimate*

$$\|x(t)\| \leq A_0 e^{b_0 t} \qquad (t > b > 0)$$

for constants $A_0 > 0$ and $b_0 > 0$.

Proof. Let $\delta = \max\{\delta_i\}$, $j = 1, 2, ..., l$, $\bar{\delta} = \max\{\tau_1, \ \delta\}$, $q = \max\{\|Q_i\|\}$ $i = 1, 2, ..., n$, and take $x_0 = (I^{1-\alpha}x)(0) + \sum_{j=1}^{l} P_j (I^{1-\alpha}x)(-\tau_j)$ with $x(t) \in AC[0, b]$. Then there exists a constant M such that $\|x(t)\| \leq M$. Then, for $t > b$,

$$\begin{aligned}
\|x(t)\| &\leq c\|x_0\| + lp\|x(t-\tau_j)\| + \frac{nq}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0 - 1} \|x(s-\delta_i)\| ds \\
&\leq c\|x_0\| + lp \max_{t-\tau_i \leq s \leq t} \|x(s)\| + \frac{nq}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} \max_{s-\delta_i \leq \eta \leq s} \|x(\eta)\| ds \\
&\leq c\|x_0\| + lp(M + \|\varphi\|_{AC} + \max_{b \leq s \leq t} \|x(s)\|) \\
&\quad + \frac{nq}{\Gamma(\alpha_0)} \int_0^b (t-s)^{\alpha_0-1} \max_{s-\bar{\delta} \leq \eta \leq b} \|x(\eta)\| ds + \frac{nq}{\Gamma(\alpha_0)} \int_b^t (t-s)^{\alpha_0-1} \max_{b \leq \eta \leq s} \|x(\eta)\| ds
\end{aligned}$$

$$\begin{aligned}&\leq c\|x_0\| + (M + \|\varphi\|_{AC})\left(lp + \frac{nq}{\Gamma(\alpha_0)}\frac{t^{\alpha_0}}{\alpha_0}\right) + lp\max_{b\leq s\leq t}\|x(s)\| \\ &\quad + \frac{nq}{\Gamma(\alpha_0)}\int_b^t (t-s)^{\alpha_0-1}\max_{b\leq\eta\leq s}\|x(\eta)\|ds.\end{aligned}$$

Set

$$a = c\|x_0\| + (M + \|\varphi\|_{AC})\left(lp + \frac{nq}{\Gamma(\alpha_0)}\frac{t^{\alpha_0}}{\alpha_0}\right),$$

which yields

$$\max_{b\leq s\leq t}\|x(s)\| \leq \frac{1}{1-lp}\left(a + \frac{nq}{\Gamma(\alpha_0)}\int_b^t (t-s)^{\alpha_0-1}\max_{b\leq\eta\leq s}\|x(\eta)\|ds\right),$$

Consequently, by Lemma 3.2, we obtain

$$\begin{aligned}\|x(t)\| &\leq \max_{b\leq s\leq t}\|x(s)\| \leq \frac{a(t)}{1-lp} + \frac{1}{(1-lp)^2}\frac{knq}{\Gamma(\alpha_0)}\int_0^t (t-s)^{\alpha_0-1}a(s)ds \\ &\leq \frac{a(t)}{1-lp}\left(1 + \frac{1}{1-lp}\frac{knq}{\alpha_0\Gamma(\alpha_0)}t^{\alpha_0}\right).\end{aligned}$$

The proof is complete. □

A slight modification in the proof of Theorem 3.3 shows that the following result is also true.

Theorem 3.10 *Assume that for* $j = 1, 2, ..., l$ *and* $i = 1, 2, ..., n$,

$$P_j, Q_i \in \mathbb{R}^{m\times m}, \quad \tau_j \in (0, \infty) \quad and \quad \delta_i \in [0, \infty).$$

If the characteristic equation (19) has no real roots, then every solution of (18) oscillates.

Proof. If we modify the functions $F(s)$ and $\Phi(s)$, defined in Theorem 3.3, as

$$F(s) = s^\alpha B + s^\alpha B\sum_{j=1}^{l} P_j e^{-s\tau_j} + \sum_{i=1}^{n} Q_i e^{-s\delta_i},$$

$$\Phi(s) = x_0 - \sum_{i=1}^{n} Q_i e^{-s\delta_i}\int_{-\delta_i}^0 e^{-st}x(t)dt - s^\alpha B\sum_{j=1}^{l} P_j e^{-s\tau_j}\int_{\tau_j}^0 e^{-st}x(t)dt + \sum_{j=1}^{l} P_j(I^{1-\alpha}x)(-\tau_j).$$

Then, following the method of proof for Theorem 3.3, one can complete the proof. □

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FIXED POINT RESULTS FOR A PAIR OF MULTI DOMINATED MAPPINGS ON A SMALLEST SUBSET IN K-SEQUENTIALLY DISLOCATED QUASI METRIC SPACE WITH AN APPLICATION

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Abstract. The aim of this paper is to establish fixed point results for semi α_*-dominated multivalued pair of mappings satisfying generalized locally α_*-ψ-type contractive conditions for a pair of multivalued dominated mappings in complete dislocated quasi metric space. Applications have been given and an example has been constructed to demonstrate the novelty of our results.

1. Introduction and preliminaries

Let $H : Z \to Z$ be a mapping. A point $x \in Z$ is said to be a fixed point of Z if $x = Zx$. Fixed point results are a tool to estimate the unique solution of nonlinear functional equations. Many results appeared in literature related to the fixed point of mappings which are contractive on the whole domain. It may happen that $H : Z \to Z$ is not a contraction but is a contraction on a subset of Z. It is possible for one to get fixed point for such mappings if they satisfiy certain conditions. It has been shown the existence of fixed point for such mappings that fulfill certain conditions on a closed ball by Beg et al. [8] (see also [3, 4, 5, 14, 24, 25, 26, 27]).

Many authors established fixed point theorems in complete dislocated metric spaces. The idea of dislocated topologies has useful applications in the context of logic programming semantics (see [11]). Dislocated metric space (metric-like space) (see [17]) is a generalization of partial metric space (see [16]). Furthermore, dislocated quasi metric space (quasi-metric-like space) (see [8, 21, 30, 31]) generalized the idea of dislocated metric space and quasi-partial metric space (see [18, 25]).

Nadler [20] initiated the study of fixed point theorems for the multivalued mappings (see also [7]). Several results on multivalued mappings have been observed [1, 10, 19, 29]. Asl et al. [6] gave the idea of α_*-ψ contractive multifunctions, α_*-admissible mapping and got some fixed point conclusions for these multifunctions (see also [2, 12]).

In this paper, we evaluate some fixed point results forα_*-ψ-contractive type multivalued α_*-dominated mappings in a closed ball in left(right) K-sequentially complete dislocated quasi metric space. Moreover, we give examples of multivalued mappings which are α_*-dominated but not α_*-admissible. We give the following definitions and results which will be needed in the sequel

Definition 1.1. [30] Let X be a nonempty set and let $d_q : X \times X \to [0, \infty)$ be a function, called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x, y, z \in X$:

(i) If $d_q(x, y) = d_q(y, x) = 0$, then $x = y$;

(ii) $d_q(x, y) \le d_q(x, z) + d_q(z, y)$.

The pair (X, d_q) is called a dislocated quasi metric space.

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It is clear that if $d_q(x,y) = d_q(y,x) = 0$, then from (i), $x = y$. But if $x = y$, $d_q(x,y)$ may not be 0. It is observed that if $d_q(x,y) = d_q(y,x)$ for all $x, y \in X$, then (X, d_q) becomes a dislocated metric space (metric-like space) (X, d_l). For $x \in X$ and $\varepsilon > 0$, $B_{d_q}(x,\varepsilon) = \{y \in X : d_q(x,y) < \varepsilon$ and $d_q(y,x) < \varepsilon\}$ and $\overline{B_{d_q}(x,\varepsilon)} = \{y \in X : d_q(x,y) \leq \varepsilon$ and $d_q(y,x) \leq \varepsilon\}$ are an open ball and a closed ball in (X, d_q), respectively. Also $B_{d_l}(x,\varepsilon) = \{y \in X : d_q(x,y) \leq \varepsilon\}$ is a closed ball in (X, d_l).

Example 1.2. [8] Let $X = R^+ \cup \{0\}$ and $d_q(x,y) = x + \max\{x,y\}$ for any $x, y \in X$.

(i) If $d_q(x,y) = d_q(y,x) = 0$, then $x+\max\{x,y\} = y+\max\{y,x\} = 0$, which implies that $x = y = 0$.

(ii) Case 1: If $x \geq y$, then $d_q(x,y) = x + \max\{x,y\} = 2x$. Let $z \in X$. If $z \leq x$, then

$$\begin{aligned} d_q(x,z) + d_q(z,y) &= x + \max\{x,z\} + z + \max\{z,y\} \\ &= x + x + z + \max\{z,y\} \geq 2x = d_q(x,y). \end{aligned}$$

If $z > x$, then $d_q(x,z) + d_q(z,y) = x + z + z + z \geq 2x = d_q(x,y)$.

Case 2: If $x < y$, then $d_q(x,y) = x + y$. If $z \geq y$, then $d_q(x,z) + d_q(z,y) = x + z + z + z \geq x + y = d_q(x,y)$. If $z < y$, then $d_q(x,z) + d_q(z,y) = x + \max\{x,z\} + z + y \geq x + y = d_q(x,y)$.

Hence both the conditions of Definition 1.1 hold and so $d_q(x,y) = x+\max\{x,y\}$ defines a dislocated quasi metric on X.

Definition 1.3. [8] Let (X, d_q) be a dislocated quasi metric space.

(a) A sequence $\{x_n\}$ in (X, d_q) is called left (resp., right) K-Cauchy if for all $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n > m \geq n_0$ (resp., for all $m > n \geq n_0$), $d_q(x_m, x_n) < \varepsilon$.

(b) A sequence $\{x_n\}$ is called dislocated quasi-converges (for short, d_q-converges) to x if $\lim_{n\to\infty} d_q(x_n, x) = \lim_{n\to\infty} d_q(x, x_n) = 0$ or for any $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n > n_0$, $d_q(x, x_n) < \varepsilon$ and $d_q(x_n, x) < \varepsilon$. In this case, x is called a d_q-limit of $\{x_n\}$.

(c) (X, d_q) is called left (resp., right) K-sequentially complete if every left (resp., right) K-Cauchy sequence in X converges to a point $x \in X$ such that $d_q(x,x) = 0$.

Definition 1.4. Let (X, d_q) be a dislocated quasi metric space. Let K be a nonempty subset of X and $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$\begin{aligned} d_q(x,K) &= d_q(x,y_0), \text{ where } d_q(x,K) = \inf_{y\in K} d_q(x,y), \\ d_q(K,x) &= d_q(y_0,x), \text{ where } d_q(K,x) = \inf_{y\in K} d_q(y,x). \end{aligned}$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set.

We denote by $P(X)$ the set of all proximinal subsets of X.

Definition 1.5. [22] Let $(S,T) : X \to P(X)$ and $\beta : X \times X \to [0, +\infty)$ be a function. We say that the pair (S,T) is $\beta_\star$-admissible if for all $x, y \in X$,

$$\beta(x,y) \geq 1 \text{ implies } \beta_\star(Tx, Sy) \geq 1 \text{ and } \beta_\star(Tx, Sy) \geq 1.$$

Again the pair (S,T) is said to be β-admissible if $\beta(x,y) \geq 1$ implies $\beta(a,b) \geq 1$ for all $a \in Sx$ and $b \in Ty$.

Let Ψ denote the family of all nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ. If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

Definition 1.6. [22] Let (X, d) be a complete metric space and $\beta : X \times X \to [0, +\infty)$ be a mapping. Let $(S,T) : X \to P(X)$ be a multifunction and $\psi \in \Psi$. We say that the pair (S,T) is a $\beta_\star$-ψ contractive multifunction whenever

$$\begin{aligned} \beta_\star(Tx, Sy)H(Tx, Sy) &\leq \psi(d(x,y)) \ \forall\, x, y \in X, \\ \beta_\star(Sx, Ty)H(Sx, Ty) &\leq \psi(d(x,y)) \ \forall\, x, y \in X, \end{aligned}$$

where $\beta_\star(Tx, Sy) = \inf\{\beta(a, b) : a \in Tx,\ b \in Sy\}$.

Definition 1.7. [27] Let (X, d_l) be a dislocated metric space, $S : X \to P(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, +\infty)$. Let $A \subseteq X$. Then we say that S is semi α_*-admissible on A, whenever $\alpha(x, y) \geq 1$ implies $\alpha_\star(Sx, Sy) \geq 1$ for all $x \in A$, where $\alpha_*(Sx, Sy) = \inf\{\alpha(a, b) : a \in Sx, b \in Sx\}$. If $A = X$, then we say that S is α_*-admissible on X.

Definition 1.8. Let (X, d_l) be a dislocated metric space, $S, T : X \to P(X)$ be multivalued mappings and $\alpha : X \times X \to [0, +\infty)$. Let $A \subseteq X$. Then we say that S is semi α_*-dominated on A, whenever $\alpha_*(x, Sx) \geq 1$ for all $x \in A$, where $\alpha_*(x, Sx) = \inf\{\alpha(x, b) : b \in Sx\}$. If $A = X$, then we say that S is α_*-dominated on X.

Definition 1.9. [23] The function $H_{d_q} : P(X) \times P(X) \to X$, defined by

$$H_{d_q}(A, B) = \max\{\sup_{a \in A} d_q(a, B),\ \sup_{b \in B} d_q(A, b)\},$$

is called a dislocated quasi Hausdorff metric on $P(X)$. Also $(P(X), H_{d_q})$ is known as a dislocated quasi Hausdorff metric space.

Lemma 1.10. [23] *Let (X, d_q) be a dislocated quasi metric space. Let $(P(X), H_{d_q})$ be a dislocated quasi Hausdorff metric space on $P(X)$. Then, for all $A, B \in P(X)$ and for each $a \in A$, there exists $b_a \in B$ such that $H_{d_q}(A, B) \geq d_q(a, b_a)$ and $H_{d_q}(B, A) \geq d_q(b_a, a)$, where $d_q(a, B) = d_q(a, b_a)$ and $d_q(B, a) = d_q(b_a, a)$.*

Example 1.11. Let $X = \mathbb{R}$. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x > y \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

Define the multivalued mappings $S, T : X \to P(X)$ by

$$Sx = \{[x - 4, x - 3] \text{ if } x \in X\},$$

$$Ty = \{[y - 2, y - 1] \text{ if } y \in X\}.$$

Suppose $x = 3$ and $y = 2$. Since $3 > 2$, $\alpha(3, 2) \geq 1$. Now $\alpha_\star(S3, T2) = \inf\{\alpha(a, b) : a \in S3, b \in T2\} = \frac{1}{2} \ngeq 1$, which means that $\alpha_\star(S3, T2) < 1$, that is, the pair (S, T) is not $\alpha_\star$-admissible. Also $\alpha_\star(S3, S2) \ngeq 1$ and $\alpha_\star(T3, T2) \ngeq 1$. This implies that S and T are not $\alpha_\star$-admissible individually. Since $\alpha_\star(x, Sx) = \inf\{\alpha(x, b) : b \in Sx\} \geq 1$ for all $x \in X$, S is an $\alpha_\star$-dominated mapping. Similarly, $\alpha_\star(y, Ty) = \inf\{\alpha(y, b) : b \in Ty\} \geq 1$. Hence it is clear that S and T are $\alpha_\star$-dominated but not $\alpha_\star$-admissible.

Lemma 1.12. [23] *Every closed ball Y in a left (right) K-sequentially complete dislocated quasi metric space X is left (right) K-sequentially complete.*

2. Main results

Let (X, d_q) be a dislocated quasi metric space, $x_0 \in X$ and $S, T : X \to P(X)$ be multifunctions on X. Let $x_1 \in Sx_0$ be an element such that $d_q(x_0, Sx_0) = d_q(x_0, x_1)$. Let $x_2 \in Tx_1$ be such that $d_q(x_1, Tx_1) = d_q(x_1, x_2)$. Let $x_3 \in Sx_2$ be such that $d_q(x_2, Sx_2) = d_q(x_2, x_3)$.Continuing this process, we construct a sequence $\{x_n\}$ of points in X such that $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n = 0, 1, 2, ...$ Also $d_q(x_{2n}, Sx_{2n}) = d_q(x_{2n}, x_{2n+1})$ and $d_q(x_{2n+1}, Tx_{2n+1}) = d_q(x_{2n+1}, x_{2n+2})$. We denote this iterative sequence by $\{TS(x_n)\}$. We say that $\{TS(x_n)\}$ is a sequence in X generated by x_0.

Theorem 2.1. *Let (X, d_q) be a left (right) K-sequentially complete dislocated quasi metric space. Suppose there exists a function $\alpha : X \times X \to [0, \infty)$. Let $r > 0$, $x_0 \in \overline{B_{d_q}(x_0, r)}$ and $S, T : X \to P(X)$ be semi α_*-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_q(x, y) = \max\{d_q(x, y), d_q(x, Sx), d_q(y, Ty)\}$, the following hold:*

$$\max\{\alpha_*(x, Sx)H_{dq}(Sx, Ty), \alpha_*(y, Ty)H_{dq}(Ty, Sx)\} \leq \min\{\psi(D_q(x, y)), \psi(D_q(y, x))\} \tag{2.1}$$

for all $x, y \in \overline{B_{d_q}(x_0, r)} \cap \{TS(x_n)\}$ with either $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ whenever $x \in Sy$, and

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Then $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0, r)}$. Also if the inequality (2.1) holds for x^* and either $\alpha(x_n, x^*) \geq 1$ or $\alpha(x^*, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have a common fixed point x^* in $\overline{B_{d_q}(x_0, r)}$ and $d_q(x^*, x^*) = 0$.

Proof. Consider a sequence $\{TS(x_n)\}$ generated by x_0. Then, we have $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n = 0, 1, 2, ...$ Also $d_q(x_{2n}, Sx_{2n}) = d_q(x_{2n}, x_{2n+1})$, $d_q(x_{2n+1}, Tx_{2n+1}) = d_q(x_{2n+1}, x_{2n+2})$. By Lemma 1.10, we have

$$d_q(x_{2n}, x_{2n+1}) \leq H_{d_q}(Tx_{2n-1}, Sx_{2n}), \tag{2.3}$$

$$d_q(x_{2n+1}, x_{2n+2}) \leq H_{d_q}(Sx_{2n}, Tx_{2n+1}) \tag{2.4}$$

for all $n = 1, 2, ...$ From (2.2), we have

$$\max\{d_q(x_1, x_0), d_q(x_0, x_1)\} \leq \sum_{i=0}^{j} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r.$$

It follows that, $d_q(x_1, x_0) \leq r$ and $d_q(x_0, x_1) \leq r$. Hence we have

$$x_1 \in \overline{B_{d_q}(x_0, r)}.$$

Let $x_2, \cdots, x_j \in \overline{B_{d_q}(x_0, r)}$ for some $j \in \mathbb{N}$. Since $S, T : X \to P(X)$ are semi α_*-dominated mappings on $\overline{B_{d_q}(x_0, r)}$, $\alpha_*(x_{2i}, Sx_{2i}) \geq 1$ and $\alpha_*(x_{2i+1}, Tx_{2i+1}) \geq 1$. Since $\alpha_*(x_{2i}, Sx_{2i}) \geq 1$, $\inf\{\alpha(x_{2i}, b) : b \in Sx_{2i}\} \geq 1$. Also $x_{2i+1} \in Sx_{2i}$ and so $\alpha(x_{2i}, x_{2i+1}) \geq 1$. Now by (2.3), we obtain

$$\begin{aligned}
d_q(x_{2i+1}, x_{2i+2}) &\leq H_{d_q}(Sx_{2i}, Tx_{2i+1}) \leq \max\{\alpha_*(x_{2i}, Sx_{2i})H_{d_q}(Sx_{2i}, Tx_{2i+1}), \\
&\quad \alpha_*(x_{2i+1}, Tx_{2i+1})H_{d_q}(Tx_{2i+1}, Sx_{2i})\} \\
&\leq \min\{\psi(D_q(x_{2i}, x_{2i+1})), \psi(D_q(x_{2i+1}, x_{2i}))\} \leq \psi(D_q(x_{2i}, x_{2i+1})) \\
&\leq \psi(\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i}, Sx_{2i}), d_q(x_{2i+1}, Tx_{2i+1})\}) \\
&\leq \psi(\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\}) \\
&\leq \psi(\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\}).
\end{aligned}$$

If $\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\} = d_q(x_{2i+1}, x_{2i+2})$, then $d_q(x_{2i+1}, x_{2i+2}) \leq \psi(d_q(x_{2i+1}, x_{2i+2}))$, which contradicts to the fact that $\psi(t) < t$ for all $t > 0$. So $\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\} = d_q(x_{2i}, x_{2i+1})$. Hence we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \psi(d_q(x_{2i}, x_{2i+1})). \tag{2.5}$$

Since $\alpha_*(x_{2i-1}, Tx_{2i-1}) \geq 1$ and $x_{2i} \in Tx_{2i-1}$, $\alpha(x_{2i-1}, x_{2i}) \geq 1$. Now by (2.4), we have

$$\begin{aligned} d_q(x_{2i}, x_{2i+1}) &\leq H_{d_q}(Tx_{2i-1}, Sx_{2i}) \leq \max\{\alpha_*(x_{2i}, Sx_{2i})H_{d_q}(Sx_{2i}, Tx_{2i-1}), \\ &\quad \alpha_*(x_{2i-1}, Tx_{2i-1})H_{d_q}(Tx_{2i-1}, Sx_{2i})\} \\ &\leq \min\{\psi(D_q(x_{2i}, x_{2i-1})), \psi(D_q(x_{2i-1}, x_{2i}))\} \leq \psi(D_q(x_{2i}, x_{2i-1})) \\ &\leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, Sx_{2i}), d_q(x_{2i-1}, Tx_{2i-1})\}) \\ &\leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i-1}, x_{2i})\}) \\ &\leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i-1}, x_{2i})\}). \end{aligned}$$

If $\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i-1}, x_{2i})\} = d_q(x_{2i}, x_{2i+1})$, then $d_q(x_{2i}, x_{2i+1}) \leq \psi(d_q(x_{2i}, x_{2i+1}))$, which contradicts to the fact that $\psi(t) < t$ for all $t > 0$. Hence we obtain

$$d_q(x_{2i}, x_{2i+1}) \leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i-1}, x_{2i})\}).$$

If $\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i-1}, x_{2i})\} = d_q(x_{2i-1}, x_{2i})$, then

$$d_q(x_{2i}, x_{2i+1})) \leq \psi(d_q(x_{2i-1}, x_{2i})).$$

Since ψ is a nondecreasing function,

$$\psi(d_q(x_{2i}, x_{2i+1})) \leq \psi^2(d_q(x_{2i-1}, x_{2i})).$$

By (2.5), we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \psi^2(d_q(x_{2i-1}, x_{2i})). \tag{2.6}$$

If $\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i-1}, x_{2i})\} = d_q(x_{2i}, x_{2i-1})$, then

$$d_q(x_{2i+1}, x_{2i+2}) \leq \psi^2(d_q(x_{2i}, x_{2i-1})) \tag{2.7}$$

By (2.6) and (2.7), we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \max\{\psi^2(d_q(x_{2i}, x_{2i-1})), \psi^2(d_q(x_{2i-1}, x_{2i}))\}.$$

Continuing in this way, we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \max\{\psi^{2i+1}(d_q(x_1, x_0)), \psi^{2i+1}(d_q(x_0, x_1))\} \tag{2.8}$$

Similarly, we have

$$d_q(x_{2i}, x_{2i+1}) \leq \max\{\psi^{2i}(d_q(x_1, x_0)), \psi^{2i}(d_q(x_0, x_1))\}. \tag{2.9}$$

By (2.8) and (2.9), we obtain

$$d_q(x_j, x_{j+1}) \leq \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} \text{ for some } j \in \mathbb{N}. \tag{2.10}$$

By Lemma 1.10 and (2.1), we have

$$\begin{aligned} d_q(x_{2i+2}, x_{2i+1}) &\leq H_{d_q}(Tx_{2i+1}, Sx_{2i}) \leq \max\{\alpha_*(x_{2i}, Sx_{2i})H_{d_q}(Sx_{2i}, Tx_{2i+1}), \\ &\quad \alpha_*(x_{2i+1}, Tx_{2i+1})H_{d_q}(Tx_{2i+1}, Sx_{2i})\} \\ &\leq \min\{\psi(D_q(x_{2i}, x_{2i+1})), \psi(D_q(x_{2i+1}, x_{2i}))\}. \end{aligned}$$

By the same reasoning as in the proof of (**??**), we have

$$d_q(x_{j+1}, x_j) \leq \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} \text{ for some } j \in \mathbb{N}. \tag{2.10}$$

Now

$$\begin{aligned} d_q(x_0, x_{j+1}) &\leq d_q(x_0, x_1) + ... + d_q(x_j, x_{j+1}) \\ &\leq d_q(x_0, x_1) + ... + \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} \\ &\leq \sum_{i=0}^{j} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r. \end{aligned} \tag{2.11}$$

Also

$$\begin{aligned} d_q(x_{j+1}, x_0) &\leq d_q(x_{j+1}, x_j) + ... + d_q(x_1, x_0) \\ &\leq \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} + ... + d_q(x_1, x_0) \\ &\leq \sum_{i=0}^{j} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r. \end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have $x_{j+1} \in \overline{B_{dq}(x_0, r)}$. Hence by mathematical induction $x_n \in \overline{B_{dq}(x_0, r)}$ for all $n \in \mathbb{N}$. Therefore, $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$. Since $S, T : X \to P(X)$ are semi α_*-dominated mappings on $\overline{B_{d_q}(x_0, r)}$, $\alpha_*(x_n, Sx_n) \geq 1$ and $\alpha_*(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N}$. Now (2.8) and (2.9) can be written as

$$d_q(x_n, x_{n+1}) \leq \max\{\psi^n(d_q(x_1, x_0)), \psi^n(d_q(x_0, x_1))\} \text{ for all } n \in \mathbb{N}. \tag{2.13}$$

$$d_q(x_{n+1}, x_n) \leq \max\{\psi^n(d_q(x_1, x_0)), \psi^n(d_q(x_0, x_1))\} \text{ for all } n \in \mathbb{N}. \tag{2.14}$$

Fix $\varepsilon > 0$ and let $k_1(\varepsilon) \in \mathbb{N}$ such that $\sum_{k \geq k_1(\varepsilon)} \max\{\psi^k(d_q(x_1, x_0)), \psi^k(d_q(x_0, x_1))\} < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > k_1(\varepsilon)$. Then we obtain

$$\begin{aligned} d_q(x_n, x_m) &\leq \sum_{k=n}^{m-1} d_q(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \max\{\psi^k(d_q(x_1, x_0)), \psi^k(d_q(x_0, x_1))\} \text{ by (2.13)}, \\ d_q(x_n, x_m) &\leq \sum_{k \geq k_1(\varepsilon)} \max\{\psi^n(d_q(x_1, x_0)), \psi^n(d_q(x_0, x_1))\} < \varepsilon. \end{aligned}$$

Thus we obtain that $\{TS(x_n)\}$ is a left K-Cauchy sequence in $(\overline{B_{d_q}(x_0, r)}, d_q)$.

Similarly, by (2.14), we have

$$d_q(x_m, x_n) \leq \sum_{k=n}^{m-1} d_q(x_{k+1}, x_k) < \varepsilon.$$

Hence $\{TS(x_n)\}$ is a right K-Cauchy sequence in $(\overline{B_{d_q}(x_0, r)}, d_q)$. Since every closed ball in left(right) K-sequentially complete dislocated quasi metric space is left(right) K-sequentially complete, there exists $x^* \in \overline{B_{dq}(x_0, r)}$ such that $\{TS(x_n)\} \to x^*$, that is,

$$\lim_{n \to \infty} d_q(x_n, x^*) = \lim_{n \to \infty} d_q(x^*, x_n) = 0. \tag{2.15}$$

Now

$$\begin{aligned} d_q(x^*, Tx^*) &\leq d_q(x^*, x_{2n+1}) + d_q(x_{2n+1}, Tx^*) \\ &\leq d_q(x^*, x_{2n+1}) + H_{d_q}(Sx_{2n}, Tx^*) \text{ by Lemma 1.10}. \end{aligned} \tag{2.16}$$

Since $\alpha_*(x^*,Tx^*)\geq 1$, $\alpha_*(x_{2n},Sx_{2n})\geq 1$ and $\alpha(x_{2n},x^*)\geq 1$, we obtain

$$\begin{aligned} H_{d_q}(Sx_{2n},Tx^*) &\leq \max\{\alpha_*(x_{2n},Sx_{2n})H_{d_q}(Sx_{2n},Tx^*),\alpha_*(x^*,Tx^*)H_{dq}(Tx^*,Sx_{2n})\} \\ &\leq \min\{\psi(D_q(x_{2n},x^*)),\psi(D_q(x^*,x_{2n}))\} \\ &\leq \psi(\max\{d_q(x_{2n},x^*),d_q(x_{2n},x_{2n+1}),d_q(x^*,Tx^*)\}) \\ &\leq \psi(\max\{d_q(x_{2n},x^*),d_q(x_{2n},x^*)+d_q(x^*,x_{2n+1}),d_q(x^*,Tx^*)\}). \end{aligned} \tag{2.17}$$

By (2.16) and (2.17), we have

$$d_q(x^*,Tx^*)\leq d_q(x^*,x_{2n+1})+\psi(\max\{d_q(x_{2n},x^*),d_q(x_{2n},x^*)+d_q(x^*,x_{2n+1}),d_q(x^*,Tx^*)\}).$$

Letting $n\to\infty$, and by (2.15), we obtain $d_q(x^*,Tx^*)\leq\psi(d_q(x^*,Tx^*))$ and hence $d_q(x^*,Tx^*)=0$. Now

$$\begin{aligned} d_q(Tx^*,x^*) &\leq d_q(Tx^*,x_{2n+1})+d_q(x_{2n+1},x^*) \\ &\leq H_{d_q}(Tx^*,Sx_{2n})+d_q(x_{2n+1},x^*),\ \text{by Lemma 1.10.} \end{aligned}$$

By using a similar argument, we obtain $d_q(Tx^*,x^*)=0$ or $x^*\in Tx^*$.

Similarly, by Lemma 1.10, (2.15) and

$$d_q(x^*,Sx^*)\leq d_q(x^*,x_{2n+2})+d_q(x_{2n+2},Sx^*),$$

we can show that $d_q(x^*,Sx^*)=0$ and $x^*\in Sx^*$.

Similarly, $d_q(Sx^*,x^*)=0$. Hence S and T have a common fixed point x^* in $\overline{B_{d_q}(x_0,r)}$. Now

$$d_q(x^*,x^*)\leq d_q(x^*,Tx^*)+d_q(Tx^*,x^*)\leq 0.$$

This implies that $d_q(x^*,x^*)=0$. □

Corollary 2.2. *Let (X,d_q) be a left (right) K-sequentially complete dislocated quasi metric space. Suppose that there exists a function $\alpha:X\times X\to[0,\infty)$. Let $r>0$, $x_0\in\overline{B_{d_q}(x_0,r)}$ and $S:X\to P(X)$ be a semi α_*-dominated mapping on $\overline{B_{d_q}(x_0,r)}$. Assume that, for some $\psi\in\Psi$ and $D_q(x,y)=\max\{d_q(x,y),d_q(x,Sx),d_q(y,Sy)\}$, the following hold:*

$$\max\{\alpha_*(x,Sx)H_{dq}(Sx,Sy),\ \alpha_*(y,Sy)H_{dq}(Sy,Sx)\}\leq\min\{\psi(D_q(x,y)),\psi(D_q(y,x))\} \tag{2.18}$$

for all $x,y\in\overline{B_{d_q}(x_0,r)}\cap\{S(x_n)\}$ with either $\alpha(x,y)\geq 1$ or $\alpha(y,x)\geq 1$, and

$$\sum_{i=0}^{n}\max\{\psi^i(d_q(x_1,x_0),\psi^i(d_q(x_0,x_1))\}\leq r \text{ for all } n\in\mathbb{N}\cup\{0\}.$$

Then $\{S(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{S(x_n)\}\to x^*\in\overline{B_{d_q}(x_0,r)}$. Also, if (2.18) holds for x^* and either $\alpha(x_n,x^*)\geq 1$ or $\alpha(x^*,x_n)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$, then S has a fixed point x^* in $\overline{B_{d_q}(x_0,r)}$ and $d_q(x^*,x^*)=0$.

Corollary 2.3. *Let (X,d_l) be a complete dislocated metric space. Suppose that there exists a function $\alpha:X\times X\to[0,\infty)$. Let $r>0$, $x_0\in\overline{B_{d_l}(x_0,r)}$ and $S,T:X\to P(X)$ be semi α_*-dominated mappings on $\overline{B_{d_l}(x_0,r)}$. Assume that, for some $\psi\in\Psi$ and $D_l(x,y)=\max\{d_l(x,y),d_l(x,Sx),d_l(y,Ty)\}$, the following hold:*

$$\max\{\alpha_*(x,Sx)H_{d_l}(Sx,Ty),\ \alpha_*(y,Ty)H_{d_l}(Sx,Ty)\}\leq\psi(D_l(x,y)) \tag{2.19}$$

for all $x,y\in\overline{B_{d_l}(x_0,r)}\cap\{TS(x_n)\}$ with either $\alpha(x,y)\geq 1$ or $\alpha(y,x)\geq 1$, and

$$\sum_{i=0}^{n}\psi^i(d_l(x_0,x_1))\leq r \text{ for all } n\in\mathbb{N}\cup\{0\}.$$

Then $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0,r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_l}(x_0,r)}$. Also if (2.19) holds for x^* and either $\alpha(x_n,x^*) \geq 1$ or $\alpha(x^*,x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have a common fixed point x^* in $\overline{B_{d_l}(x_0,r)}$ and $d_q(x^*,x^*)=0$.

Corollary 2.4. *Let (X,d_l) be a complete dislocated metric space. Suppose that there exists a function $\alpha : X \times X \to [0,\infty)$. Let $r>0$, $x_0 \in \overline{B_{d_l}(x_0,r)}$ and $S : X \to P(X)$ be a semi α_*-dominated mapping on $\overline{B_{d_l}(x_0,r)}$. Assume that, for some $\psi \in \Psi$ and $D_l(x,y) = \max\{d_l(x,y), d_l(x,Sx), d_l(y,Sy)\}$, the following hold:*

$$\max\{\alpha_*(x,Sx)H_{d_l}(Sx,Sy),\ \alpha_*(y,Sy)H_{d_l}(Sx,Sy)\} \leq \psi(D_l(x,y)) \tag{2.20}$$

for all $x,y \in \overline{B_{d_l}(x_0,r)} \cap \{S(x_n)\}$ with either $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$, and

$$\sum_{i=0}^{n} \psi^i(d_l(x_0,x_1)) \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Then $\{S(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0,r)}$ and $\{S(x_n)\} \to x^* \in \overline{B_{d_l}(x_0,r)}$. Also, if (2.20) holds for x^* and either $\alpha(x_n,x^*) \geq 1$ or $\alpha(x^*,x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S has a fixed point x^* in $\overline{B_{d_l}(x_0,r)}$ and $d_q(x^*,x^*)=0$.

Let X be a nonempty set , $\preceq$ a partial order on X and $A \subseteq X$. We say that $a \preceq B$ whenever for all $b \in B$, we have $a \preceq b$. A mapping $S : X \to P(X)$ is said to be semi dominated on A if $a \preceq Sa$ for each $a \in A \subseteq X$. If $A = X$, then $S : X \to P(X)$ is said to be dominated.

Corollary 2.5. *Let $(X,\preceq,d_q)$ be a left (right) K-sequentially ordered complete dislocated quasi metric space. Let $r>0$, $x_0 \in \overline{B_{d_q}(x_0,r)}$ and $S,T : X \to P(X)$ be semi dominated mappings on $\overline{B_{d_q}(x_0,r)}$. Assume that, for some $\psi \in \Psi$ and $D_q(x,y) = \max\{d_q(x,y), d_q(x,Sx), d_q(y,Ty)\}$, the following hold:*

$$\max\{H_{dq}(Sx,Ty), H_{dq}(Ty,Sx)\} \leq \min\{\psi(D_q(x,y)), \psi(D_q(y,x))\} \tag{2.21}$$

for all $x,y \in \overline{B_{d_q}(x_0,r)} \cap \{TS(x_n)\}$ with either $x \preceq y$ or $y \preceq x$, and

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(x_1,x_0), \psi^i(d_q(x_0,x_1))\} \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.22}$$

Then $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0,r)}$. Also if (2.21) holds for x^* and either $x_n \preceq x^*$ or $x^* \preceq x_n$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have a common fixed point x^* in $\overline{B_{d_q}(x_0,r)}$ and $d_q(x^*,x^*)=0$.

Proof. Let $\alpha : X \times X \to [0,+\infty)$ be a function defined by $\alpha(x,y)=1$ for all $x \in \overline{B_{d_q}(x_0,r)}$ with either $x \preceq y$ or $y \preceq x$, and $\alpha(x,y)=0$ for all other elements $x,y \in X$. Since S and T are the semi dominated mappings on $\overline{B_{d_q}(x_0,r)}$, $x \preceq Sx$ and $x \preceq Tx$ for all $x \in \overline{B_{d_q}(x_0,r)}$. This implies that $x \preceq b$ for all $b \in Sx$ and $x \preceq c$ for all $c \in Tx$. So $\alpha(x,b)=1$ for all $b \in Sx$ and $\alpha(x,c)=1$ for all $c \in Tx$. This implies that $\inf\{\alpha(x,y) : y \in Sx\} = 1$ and $\inf\{\alpha(x,y) : y \in Tx\} = 1$. Hence $\alpha_*(x,Sx)=1$ and $\alpha_*(x,Tx)=1$ for all $x \in \overline{B_{d_q}(x_0,r)}$. So $S,T : X \to P(X)$ are semi α_*-dominated mappings on $\overline{B_{d_q}(x_0,r)}$. Moreover, (2.21) can be written as

$$\max\{\alpha_*(x,Sx)H_{dq}(Sx,Ty),\ \alpha_*(y,Ty)H_{dq}(Ty,Sx)\} \leq \min\{\psi(D_q(x,y)), \psi(D_q(y,x))\}$$

for all x,y in $\overline{B_{d_q}(x_0,r)} \cap \{TS(x_n)\}$ with either $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$. Also, (2.22) holds. Then by Theorem 2.1, $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0,r)}$. Now, $x_n, x^* \in \overline{B_{d_q}(x_0,r)}$ and either $x_n \preceq x^*$ or $x^* \preceq x_n$ implies that either $\alpha(x_n,x^*) \geq 1$ or $\alpha(x^*,x_n) \geq 1$.

So all the conditions of Theorem 2.1 are satisfied. Hence by Theorem 2.1, S and T have a common fixed point x^* in $\overline{B_{d_q}(x_0, r)}$ and $d_q(x^*, x^*) = 0$. □

Example 2.6. Let $X = Q^+ \cup \{0\}$ and let $d_l : X \times X \to X$ be a complete dislocated quasi metric on X defined by

$$d_q(x, y) = x + y \text{ for all } x, y \in X.$$

Define multivalued mappings $S, T : X \times X \to P(X)$ by,

$$Sx = \begin{cases} [\frac{x}{3}, \frac{2}{3}x] \text{ if } x \in [0, 1] \cap X \\ [x, x+1] \text{ if } x \in (1, \infty) \cap X \end{cases}$$

and

$$Tx = \begin{cases} [\frac{x}{4}, \frac{3}{4}x] \text{ if } x \in [0, 1] \cap X \\ [x+1, x+3] \text{ if } x \in (1, \infty) \cap X. \end{cases}$$

Considering $x_0 = 1$, $r = 8$, we get $\overline{B_{d_q}(x_0, r)} = [0, 7] \cap X$. Now $d_q(x_0, Sx_0) = d_q(1, S1) = d_q(1, \frac{1}{3}) = \frac{4}{3}$. So we obtain a sequence $\{TS(x_n)\} = \{1, \frac{1}{12}, \frac{1}{144}, \frac{1}{1728}, ...\}$ in X generated by x_0. Also $\overline{B_{d_q}(x_0, r)} \cap \{TS(x_n)\} = \{1, \frac{1}{12}, \frac{1}{144}, ...\}$. Let $\psi(t) = \frac{4t}{5}$ and

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in [0, 1] \\ \frac{3}{2} \quad \text{otherwise.} \end{cases}$$

Now if $x, y \notin \overline{B_{d_q}(x_0, r)} \cap \{TS(x_n)\}$, then we have the following cases.

Case 1. If $\max\{\alpha_*(x, Sx)H_{d_q}(Sx, Ty),\ \alpha_*(y, Ty)H_{d_q}(Ty, Sx)\} = \alpha_*(x, Sx)H_{d_q}(Sx, Ty)$, then for $x = 2$ and $y = 3$, we have

$$\alpha_*(2, S2)Hd_q(S2, T3) = \frac{3}{2}(8) > \psi(D_q(x, y)) = \frac{28}{5}.$$

Case 2. If $\max\{\alpha_*(x, Sx)H_{d_q}(Sx, Ty),\ \alpha_*(y, Ty)H_{dq}(Ty, Sx)\} = \alpha_*(y, Ty)H_{dq}(Ty, Sx)$, then for $x = 2$and $y = 3$, we have

$$\alpha_*(3, T3)Hd_q(T3, S2) = \frac{3}{2}(8) > \psi(D_q(y, x)) = \frac{28}{5}.$$

So the contractive condition does not hold on thewhole space X.

Now, for all $x, y \in \overline{B_{d_q}(x_0, r)} \cap \{TS(x_n)\}$, we have the following.

Case 3. If $\max\{\alpha_*(x, Sx)H_{d_q}(Sx, Ty),\ \alpha_*(y, Ty)H_{d_q}(Ty, Sx)\} = \alpha_*(x, Sx)H_{d_q}(Sx, Ty)$, then we have

$$\begin{aligned}
\alpha_*(x, Sx)H_{d_q}(Sx, Ty) &= 1[\max\{\sup_{a \in Sx} d_q(a, Ty), \sup_{b \in Ty} d_q(Sx, b)\}] \\
&= \max\{\sup_{a \in Sx} d_q(a, [\frac{y}{4}, \frac{3y}{4}]), \sup_{b \in Ty} d_q([\frac{x}{3}, \frac{2x}{3}], b)\} \\
&= \max\{d_q(\frac{2x}{3}, [\frac{y}{4}, \frac{3y}{4}]), d_q([\frac{x}{3}, \frac{2x}{3}], \frac{3y}{4})\} \\
&= \max\{d_q(\frac{2x}{3}, \frac{y}{4}), d_q(\frac{x}{3}, \frac{3y}{4})\} \\
&= \max\{\frac{2x}{3} + \frac{y}{4}, \frac{x}{3} + \frac{3y}{4}\} \\
&\leq \psi(\max\{x + y, \frac{4x}{3}, \frac{5y}{4}\}) = \psi(D_q(x, y)).
\end{aligned}$$

Case 4. If $\max\{\alpha_*(x,Sx)H_{d_q}(Sx,Ty),\, \alpha_*(y,Ty)H_{d_q}(Ty,Sx)\} = \alpha_*(y,Ty)H_{d_q}(Ty,Sx)$, then we have

$$\begin{aligned}
\alpha_*(y,Ty)H_{d_q}(Ty,Sx) &= 1[\max\{\sup_{b\in Ty} d_q(Sx,b),\ \sup_{a\in Sx} d_q(a,Ty)\}] \\
&= \max\{\sup_{b\in Ty} d_q([\frac{x}{3},\frac{2x}{3}],b),\ \sup_{a\in Sx} d_q(a,[\frac{y}{4},\frac{3y}{4}])\} \\
&= \max\{d_q([\frac{x}{3},\frac{2x}{3}],\frac{3y}{4}),d_q(\frac{2x}{3},[\frac{y}{4},\frac{3y}{4}])\} \\
&= \max\{d_q(\frac{x}{3},\frac{3y}{4}),d_q(\frac{2x}{3},\frac{y}{4})\} \\
&= \max\{\frac{x}{3}+\frac{3y}{4},\frac{2x}{3}+\frac{y}{4}\} \\
&\le \psi(\max\{y+x,\frac{5y}{4},\frac{4x}{3}\}) = \psi(D_q(y,x)).
\end{aligned}$$

So the contractive condition holds on $\overline{B_{d_q}(x_0,r)} \cap \{TS(x_n)\}$. Also

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(x_1,x_0),\psi^i(d_q(x_0,x_1))\} = \frac{4}{3}\sum_{i=0}^{n}(\frac{4}{5})^i < 8 = r.$$

Hence all the conditions of Theorem 2.1 are satisfied. Now we have $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{TS(x_n)\} \to 0 \in \overline{B_{d_q}(x_0,r)}$. Also $\alpha(x_n,0) \ge 1$ or $\alpha(0,x_n) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$. Moreover, 0 is a common fixed point of S and T.

3. Fixed point results for graphic contractions

In this section, we present an application of Theorem 2.1 in graph theory. Jachymski [15] proved the contraction principle for mappings on a metric space with a graph. Let (X,d) be a metric space and $\triangle$ represent the diagonal of the cartesian product $X \times X$. Assume that G is a directed graph and $V(G)$ is the set of vertices along with X and the set $E(G)$ denotes the edges of X included all loops, i.e., $E(G) \supseteq \triangle$. If G has no parallel edges, then we can unify G with pair $(V(G),E(G))$. Furthermore, we consider G as a weighted graph (see [15]) which showing to each edge the distance between its vertices. If l and m are the vertices in a graph G, then a path in G from l to m of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=o}^{N}$ of $N+1$ vertices such that $l_o = l$, $l_N = m$ and $(l_{n-1},l_n) \in E(G)$ where $i = 1,2,\cdots N$. Hussain et al. [13] established fixed points for ψ-graphic contraction with an application to integral equations. A graph G is connected if there is a path between any two vertices (for more details, see [9, 14, 28]).

Definition 3.1. Let X be a nonempty set and $G = (V(G),E(G))$ be a graph such that $V(G) = X$. Then $S : X \to CB(X)$ is said to be semi graph dominated on $A \subseteq X$ if, for each $x \in A$, $(x,y) \in E(G)$ for all $y \in Sx$. If $A = X$, then we say that S is graph dominated on X.

Theorem 3.2. *Let (X,d_q) be a complete dislocated quasi metric space endowed with a graph G. Let $r > 0$, $x_0 \in \overline{B_{d_q}(x_0,r)}$, $S,T : X \to P(X)$ mappings and $\{TS(x_n)\}$ be a sequence in X generated by x_0. Assume that the following hold:*

(i) S and T are semi graph dominated on $\overline{B_{d_q}(x_0,r)}$;

(ii) there exists $\psi \in \Psi$ and $D_q(x,y) = \max\{d_q(x,y),d_q(x,Sx),d_q(y,Ty)\}$ such that

$$\max\left\{H_{d_q}(Sx,Ty),H_{d_q}(Ty,Sx)\right\} \le \min\{\psi(D_q(x,y)),\psi(D_q(y,x))\} \tag{3.1}$$

for all $x,y \in \overline{B_{d_q}(x_0,r)} \cap \{TS(x_n)\}$ with $(x,y) \in E(G)$ or $(y,x) \in E(G)$;

(iii) $\sum_{i=0}^{n} \max\{\psi^i(d_q(x_1,x_0),\psi^i(d_q(x_0,x_1))\} \le r$ for all $n \in \mathbb{N}\cup\{0\}$.

Then $\{TS(x_n)\}$ *is a sequence in* $\overline{B_{d_q}(x_0,r)}$ *and* $\{TS(x_n)\} \to x^*$. *Also if* $(x_n, x^*) \in E(G)$ *or* $(x^*, x_n) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* (3.1) *holds for* x^*, *then* S *and* T *have a common fixed point* x^* *in* $\overline{B_{d_q}(x_0,r)}$.

Proof. Define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \in \overline{B_{d_q}(x_0,r)},\ (x,y) \in E(G) \text{ or } (y,x) \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Since S and T are semi graph dominated on $\overline{B_{d_q}(x_0,r)}$, for $x \in \overline{B_{d_q}(x_0,r)}$, $(x,y) \in E(G)$ for all $y \in Sx$ and $(x,y) \in E(G)$ for all $y \in Tx$. So $\alpha(x,y) = 1$ for all $y \in Sx$ and $\alpha(x,y) = 1$ for all $y \in Tx$. This implies that $\inf\{\alpha(x,y) : y \in Sx\} = 1$ and $\inf\{\alpha(x,y) : y \in Tx\} = 1$. Hence $\alpha_*(x,Sx) = 1$, $\alpha_*(x,Tx) = 1$ for all $x \in \overline{B_{d_q}(x_0,r)}$. So $S,T : X \to P(X)$ are semi α_*-dominated mappings on $\overline{B_{d_q}(x_0,r)}$. Moreover, (3.1) can be written as

$$\max\{\alpha_*(x,Sx)H_{dq}(Sx,Ty),\ \alpha_*(y,Ty)H_{dq}(Ty,Sx)\} \le \min\{\psi(D_q(x,y)), \psi(D_q(y,x))\}$$

for all x,y in $\overline{B_{d_q}(x_0,r)} \cap \{TS(x_n)\}$ with either $\alpha(x,y) \ge 1$ or $\alpha(y,x) \ge 1$. Also (iii) holds. Then, by Theorem 2.1, we have $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0,r)}$. Now $x_n, x^* \in \overline{B_{d_q}(x_0,r)}$ and either $(x_n,x^*) \in E(G)$ or $(x^*,x_n) \in E(G)$ implies that either $\alpha(x_n,x^*) \ge 1$ or $\alpha(x^*,x_n) \ge 1$. So all the conditions of Theorem 2.1 are satisfied. Hence by Theorem 2.1, S and T have a common fixed point x^* in $\overline{B_{d_q}(x_0,r)}$ and $d_q(x^*,x^*) = 0$. □

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