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A note on non-instantaneous impulsive fractional neutral integro-differential systems with state-dependent delay in Banach spaces

Selvaraj Suganya,* Dumitru Baleanu,† Palaniyappan Kalamani‡ and Mani Mallika Arjunan§

Abstract

In this research, we establish the existence results for non-instantaneous impulsive fractional neutral integro-differential systems with state-dependent delay in Banach space. By utilizing the Banach contraction principle and condensing fixed point theorem coupled with semigroup theory, we build up the desired results. To acquire the main results, our working concepts are that the functions deciding the equation fulfill certain Lipschitz conditions of local type which is similar to the hypotheses [5]. In the end, an example is given to show the abstract theory.

Keywords: Fractional order differential systems, Caputo fractional integral operator, non-instantaneous impulses , state-dependent delay, fixed point theorem, semigroup theory.
MSC 2010: Primary 34K30, 26A33; Secondary 35R10, 47D06.

1 Introduction

Fractional calculus may be considered an old and yet novel topic. In fact, the concepts are almost as old as their more familiar integer-order counterparts. In 1965 Leibniz and L'Hopital had correspondence where they discussed the meaning of the derivative of order one half. Since then, many famous mathematicians have worked on this and related questions, creating the field which is known today as fractional calculus.

The fractional calculus is also considered a novel topic, since it is only during the last three decades that it has been the subject of specialised conferences and treatises. This was stimulated by the fact that many important applications of fractional calculus have been found in numerous diverse and widespread

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fields in science, engineering and finance. Many authors have pointed out that fractional derivatives and integrals are very suitable for modelling the memory and hereditary properties of various materials and processes that are governed by anomalous diffusion. This represents the main advantage of using the fractional derivatives in comparison with classical integer-order models, in which such effects are not taken into account. For more details, we suggest the reader to refer the monographs [6, 24, 47], and the papers [1, 3, 10, 23, 28, 34, 44], and the references cited therein.

Due to the diverse applications in science and technology, functional differential equations turn out to be the most essential branch of research in mathematical sciences. The work on non-integer order functional differential equations with state-dependent delay (abbreviate, SDD), is going on last few years. Furthermore, the study for such kind of the differential equations with SDD, we refer the papers [2, 4, 9, 11–13, 15, 16, 21, 41, 42, 45, 46].

An important feature of real-world dynamic processes that has attracted considerable interest by scientists is the effect of abrupt changes. Hereby, "abrupt" is meant in the sense of a multi-scale problem, i.e. the state of a system changes only slowly for a long time interval, and then undergoes a drastic change within a very short time interval. For example, a football may be flying through the air for several seconds before it changes its flight direction within milliseconds during a collision with a goal post. For the mathematical description of this system, the specification of two sets of equations is appropriate: one for the flight phase, and one for the collision phase.

Several mathematical models can be developed for the football example. In a simplified setting, the motion of the football could be described by the position and velocity of its center of mass, and the encounter with the goal post could be treated as an inelastic collision (i.e. by an immediate change of the football's velocity).

For the description of the collision of the ball with the goal post leads to differential equations in which the velocity experiences, at the time of the collision, a so-called impulse. For additional information on this concept and pertinent advancements of impulsive differential equations (abbreviated, IDEs), for instance [7, 8, 26, 39].

However, in [22, 32], the authors suggested a new class of abstract IDEs for which the impulses are not instantaneous. In particular, in [22], the authors investigate the new type of IDEs with NII of the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}], \, i = 0, 1, 2, \dots, N, \tag{1.1}$$

$$u(t) = g_i(t, u(t)), \quad t \in (t_i, s_i], \, i = 1, 2, \cdots, N, \tag{1.2}$$

$$u(0) = x_0, \tag{1.3}$$

and set up the existence and uniqueness solutions of mild and classical solutions by applying well-known fixed point theorems. From the above system (1.1)-(1.3), we observe that the impulses start instantly at the points t_i and their action continue on a finite time interval $[t_i, s_i]$. As indicated in [22], there are actually several distinct aspirations for the research of this kind of model. For more details on this theory and on its applications, we suggest the reader to refer [14, 22, 30, 32].

Moreover, Pierri et al. [32] have generalized the results of [22], by employing the theory of analytic semigroup and fractional power of closed operators and proven the existence results of solutions for a class of semilinear IDEs with NII in Banach space. Furthermore, in [17, 19, 20, 25], the authors analyzed the different types of IFDEs with NII in Banach spaces under appropriate fixed point theorem. Recently, Suganya et al.[40] researched the existence results for fractional neutral integro-differential system with SDD and NII in Banach space through the utilization of the Hausdorff's measures of non-compactness and Darbo-Sadovskii fixed point theorem.

On the other hand, the existence results for impulsive fractional neutral integro-differential systems(abbreviated, IFNIDS) with SDD and NII in $\mathscr{B}_h$ phase space axioms have not yet been completely examined. This persuade us to explore the existence results of these types of structures with NII in Banach spaces.

Motivated by the effort of the aforementioned papers [5, 17, 21, 22], the principle motivation behind this manuscript is to research the existence of mild solutions for an IFNIDS with SDD of the model

$$\begin{aligned} {}^{C}D_t^{\alpha}\left[z(t) - \mathscr{Q}_1\left(t, z_{\zeta(t,z_t)}, \mathcal{C}z_{\zeta(t,z_t)}\right)\right] &= \mathscr{A}z(t) + \mathscr{Q}_2\left(t, z_{\zeta(t,z_t)}, \mathcal{C}z_{\zeta(t,z_t)}\right) \\ &\quad + \mathscr{Q}_3\left(t, z_{\zeta(t,z_t)}, \mathcal{C}z_{\zeta(t,z_t)}\right),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, N, \qquad (1.4) \end{aligned}$$

$$z(t) = g_i(t, z_{\zeta(t,z_t)}), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, N, \qquad (1.5)$$

$$z_0 = \varphi(t) \in \mathscr{B}_h, \quad t \in (-\infty, 0], \qquad (1.6)$$

where $\mathscr{A}$ denotes the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ in a Banach space $\mathbb{X}$; ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative operator of order α with $0 < \alpha \leq 1$; $\mathscr{I} = [0, T]$ is an operational interval; $\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3 : \mathscr{I} \times \mathscr{B}_h \times \mathscr{B}_h \to \mathbb{X}$, $\zeta : \mathscr{I} \times \mathscr{B}_h \to \mathbb{R}$ are appropriate functions, and $\mathscr{B}_h$ is a phase space outlined in next section. The term $\mathcal{C}z_{\zeta(t,z_t)}$ is given by $\mathcal{C}z_{\zeta(t,z_t)} = \int_0^t K(t,s)(z_{\zeta(s,z_s)})ds$, where $K \in \mathcal{C}(\mathscr{D}, \mathbb{R}^+)$ is the set of all positive functions which are continuous on $\mathscr{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I} : 0 \leq s \leq t \leq T\}$ and $\mathcal{C}^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds$. Here $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_N \leq s_N < t_{N+1} = T$, are prefixed numbers, and $g_i \in C((t_i, s_i] \times \mathscr{B}_h, \mathbb{X})$ for all $i = 1, 2, \dots, N$, is stand for impulsive conditions.

For almost any continuous function z characterized on $(-\infty, T]$ and for almost any $t \geq 0$, we designate by z_t the part of $\mathscr{B}_h$ characterized by $z_t(\theta) = z(t + \theta)$ for $\theta \leq 0$. Now $z_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta \in (-\infty, 0]$ likely the current time t.

Contrary to the recent results, this paper has some useful features including the integral term in the involved functions $\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3$ and define a suitable mild solution of the model (1.4)-(1.6) with the help of probability density function. Then, based on local Lipschitz conditions of the involved functions, we establish the existence results for IFNIDS with SDD and NII of the problem (1.4)-(1.6) under appropriate fixed point theorem, and the outcomes in [17, 21] might be viewed as the particular situations.

We organize the paper as follows. We provide some basis definitions, lemmas and theorems in Section 2 as these are useful for establish our results. Section 3 focuses on the existence of mild solutions for the model (1.4)-(1.6) with the help of the fixed point theorem. Section 4 provides an example to illustrate the acquired abstract concept.

2 Preliminaries

From now on, $\mathbb{X}$ represents Banach space with norm $\|\cdot\|$, $C(\mathscr{I},\mathbb{X})$ denotes the space of all $\mathbb{X}$-valued continuous functions on $\mathscr{I}$ and $\mathscr{L}(\mathbb{X})$ is the Banach space of all linear and bounded operators on $\mathbb{X}$. Furthermore, the notation $B_r(z,\mathbb{X})$ stands for the closed ball with center at z and the radius $r>0$ in $\mathbb{X}$.

Let $\mathscr{A}: D(\mathscr{A})\subset\mathbb{X}\to\mathbb{X}$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operator on $\mathbb{X}$. Let $0\in\varrho(\mathscr{A})$, then it is possible to describe the fractional power $\mathscr{A}^{\mu}, 0<\mu\le 1$, as a closed linear operator on its domain $D(\mathscr{A}^{\mu})$. Moreover, the subspace $D(\mathscr{A}^{\mu})$ is dense in $\mathbb{X}$ and the expression $\|z\|_{\mu}=\|\mathscr{A}^{\mu}z\|$, $z\in D(\mathscr{A}^{\mu})$, defines a norm on $D(\mathscr{A}^{\mu})$. For $0<\nu\le\mu\le 1$, $\mathbb{X}_{\mu}\to\mathbb{X}_{\nu}$ and the imbedding is compact whenever the resolvent operator of $\mathscr{A}$ is compact. Also for every $0<\mu\le 1$, there exists a positive constant $\mathcal{M}_{\mu}$ such that

$$\|\mathscr{A}^{\mu}\mathbb{T}(t)\|\le\frac{\mathcal{M}_{\mu}}{t^{\mu}},\quad 0<t\le T.$$

For additional information about the above preliminaries, we refer to [31, 35].

To portray properly our system, we claim that a function $z:[\sigma,\tau]\to\mathbb{X}$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if z is piecewise continuous and left continuous on $(\sigma,\tau]$. By the symbol $\mathcal{PC}([\sigma,\tau];\mathbb{X})$, we mean the space of normalized piecewise continuous functions from $[\sigma,\tau]$ into $\mathbb{X}$. Specifically, we signify the space $\mathcal{PC}$ established by all functions $z:[0,T]\to\mathbb{X}$ in ways that z is continuous at $t\neq t_i$, $z(t_i^-)=z(t_i)$ and $z(t_i^+)$ exists, for all $i=1,2,\cdots,N$. It is not difficult to find out that $\mathcal{PC}$ is a Banach space having the norm $\|z\|_{\mathcal{PC}}=\sup_{s\in[0,T]}\|z(s)\|$.

Once the delay is infinite, then we should talk about the theoretical phase space $\mathscr{B}_h$ in a beneficial way. Thus, in this manuscript, we deliberate phase spaces $\mathscr{B}_h$ which are same as it was described in [21]. As a result, we bypass the details.

We assume that the phase space $(\mathscr{B}_h,\|\cdot\|_{\mathscr{B}_h})$ is a semi-normed linear space of functions mapping $(-\infty,0]$ into $\mathbb{X}$, and fulfilling the subsequent elementary adages as a result of Fu et al. [18] and Ganga Ram Gautam et al. [21].

If $z:(-\infty,T]\to\mathbb{X}, T>0$, is continuous on $\mathscr{I}$ and $z_0\in\mathscr{B}_h$, then for every $t\in\mathscr{I}$ the accompanying conditions hold:

(P_1) z_t is in $\mathscr{B}_h$;

(P_2) $\|z(t)\|_{\mathbb{X}}\le H\|z_t\|_{\mathscr{B}_h}$;

(P_3) $\|z_t\|_{\mathscr{B}_h}\le\mathscr{E}_1(t)\sup\{\|z(s)\|_{\mathbb{X}}: 0\le s\le t\}+\mathscr{E}_2(t)\|z_0\|_{\mathscr{B}_h}$, where $H>0$ is a constant and $\mathscr{E}_1(\cdot):[0,+\infty)\to[0,+\infty)$ is continuous, $\mathscr{E}_2(\cdot):[0,+\infty)\to[0,+\infty)$ is locally bounded, and $\mathscr{E}_1,\mathscr{E}_2$ are independent of $z(\cdot)$.

For our convenience, denote $\mathscr{E}_1^*=\sup_{s\in\mathscr{I}}\mathscr{E}_1(s)$, $\quad\mathscr{E}_2^*=\sup_{s\in\mathscr{I}}\mathscr{E}_2(s)$.

Define the space

$$\mathscr{B}_T=\{z:(-\infty,T]\to\mathbb{X}\quad\text{such that}\quad z_0\in\mathscr{B}_h\quad\text{and the constraint}\quad z|_{\mathscr{I}}\in\mathcal{PC}\},$$

where $z|_{\mathscr{I}}$ is the constraint of z to the real compact interval on $\mathscr{I}$. The function $\|\cdot\|_{\mathscr{B}_T}$ to be a seminorm in $\mathscr{B}_T$, it is described by

$$\|z\|_{\mathscr{B}_T} = \|\varphi\|_{\mathscr{B}_h} + \sup\{\|z(s)\|_{\mathbb{X}} : s \in [0,T]\},\ z \in \mathscr{B}_T.$$

To stay away from the reiterations of a few definitions utilized as a part of this paper we refer the readers: such as for the definition of the fractional integral, Riemann-Liouville fractional integral operator, the generalized Mittag-Leffler special function, Wright-type function and the Caputo's derivative one can see the papers [17, 35, 40] and the monographs [24, 33, 47].

Currently, we are have the ability to define the mild solution for the problem (1.4)-(1.6). For this, initially we treat the following model

$$^{C}D_t^{\alpha} z(t) = \mathscr{A} z(t) + \mathscr{Q}_2(t), \tag{2.1}$$

$$z(0) = z_0, \tag{2.2}$$

where $^{C}D_t^{\alpha}$ and $\mathscr{A}$ are just like described in (1.4)-(1.6).

By thinking the proofs as in [35, Lemma 6 and Lemma 9], we directly define the mild solution for the model (2.1)-(2.2).

Definition 2.1. *A function $z : \mathscr{I} \to \mathbb{X}$ is considered to be a mild solution of problem (2.1)-(2.2) if $z \in C(\mathscr{I}, \mathbb{X})$ fulfills the accompanying integral equation:*

$$z(t) \ = \mathbb{T}_\alpha(t) z_0 + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2(s)ds, \quad t \in \mathscr{I}.$$

For additional reference about this concept, we suggest the reader to refer[35, 38, 40].

Before we characterize the mild solution for the structure (1.4)-(1.6), finally, we treat the following system

$$^{C}D_t^{\alpha}\big[z(t) - \mathscr{Q}_1(t, z(t))\big] = \mathscr{A} z(t) + \mathscr{Q}_2(t, z(t)) + \mathscr{Q}_3(t, z(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, N, \tag{2.7}$$

$$z(t) = g_i(t, z(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \tag{2.8}$$

$$z(0) = z_0, \tag{2.9}$$

where $^{C}D_t^{\alpha}, g_i(t, z(t))$ and $\mathscr{A}$ are same as defined in (1.4)-(1.6) and $z_0 \in \mathbb{X}, \mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3$ are appropriate functions.

We remark that, the impulses in problem (2.7)-(2.9) start abruptly at the points t_i and their action continues on the interval $[t_i, s_i]$. In addition, the function z takes an abrupt impulse at t_i and follows different rules in the two subintervals $(t_i, s_i]$ and $(s_i, t_{i+1}]$ of the interval $(t_i, t_{i+1}]$. At the point s_i, the function z is continuous.

On the results received in the papers [35–37, 43, 48], first we define the mild solution for the system

(2.7)-(2.9) is given by

$$x(t)=\begin{cases}\mathbb{T}_\alpha(t)[z_0-\mathscr{Q}_1(0,z_0)]+\mathscr{Q}_1(t,z(t))+\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z(s))ds\\ \quad+\int_0^t\mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds,\quad t\in[0,t_1],\\ g_1(t,z(t)),\quad t\in(t_1,s_1],\\ \mathbb{T}_\alpha(t-s_1)d_1+\mathscr{Q}_1(t,z(t))+\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z(s))ds\\ \quad+\int_0^t\mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds,\quad t\in(s_1,t_2],\\ \cdots,\\ g_i(t,z(t)),\quad t\in(t_i,s_i],\ i=1,2,\cdots,N,\\ \mathbb{T}_\alpha(t-s_i)d_i+\mathscr{Q}_1(t,z(t))+\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z(s))ds\\ \quad+\int_0^t\mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds,\quad t\in(s_i,t_{i+1}],\end{cases}$$

where

$$\begin{aligned}d_i&=g_i(s,z(s_i))-\mathscr{Q}_1(s_i,z(s_i))-\int_0^{s_i}\mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s,z(s))ds\\ &\quad-\int_0^{s_i}\mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds,\ i=1,2,\cdots,N.\end{aligned}$$

Remark 2.1. *From the discussion in [40], we clearly see that our definition of mild solution fulfills the given model (2.7)-(2.9).*

In accordance with the above discussion, we determine the mild solution of the model (1.4)-(1.6).

Definition 2.2. *[40, Definition 2.8] A function $z:(-\infty,T]\to\mathbb{X}$ is called a mild solution of the model (1.1)-(1.3) if $z_0=\varphi\in\mathscr{B}_h$, $z(\cdot)|_{\mathscr{I}}\in\mathcal{PC}$ and for each $s\in[0,t)$ the function $\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})$ is integrable and*

$$\begin{aligned}z(t)&=\mathbb{T}_\alpha(t)[\varphi(0)-\mathscr{Q}_1(0,\varphi(0),0)]+\mathscr{Q}_1(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)})\\ &\quad+\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})ds\\ &\quad+\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds\\ &\quad+\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds,\ t\in[0,t_1],\\ &g_i(t,z_{\zeta(t,z_t)}),\quad t\in(t_i,s_i],\ i=1,2,\cdots,N,\\ &\mathbb{T}_\alpha(t-s_i)d_i+\mathscr{Q}_1(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)})\\ &\quad+\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})ds\end{aligned}\tag{2.16}$$

$$+\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds$$
$$+\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds,\ t\in(s_i, t_{i+1}],$$

where

$$d_i = g_i(s_i, z_{\zeta(s_i,z_{s_i})}) - \mathscr{Q}_1(s_i, z_{\zeta(s_i,z_{s_i})}, \mathcal{C}z_{\zeta(s_i,z_{s_i})}) - \int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})ds$$
$$-\int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds - \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_3\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds,$$
$$i = 1, 2, \cdots, N. \tag{2.17}$$

Now, we turn to the statement of Condensing fixed point theorem [21, Theorem 2.9].

Theorem 2.1. *Let B be a convex, bounded and closed subset of Banach space $\mathbb{X}$ and let $P : B \to B$ be a condensing map. Then P has a fixed point.*

3 Existence results

In this section, we show and demonstrate the existence of solutions for the model (1.4)-(1.6) under different fixed point theorems and we consider $\varphi \in \mathscr{B}_h$ a fixed function, $\mathscr{I} = [0, T]$. To simplify writing of the text, in what follows, we assume that $0 \le \zeta(t, \psi) \le t$ for all $\psi \in \mathscr{B}_h$.

Presently, we itemizing the subsequent suppositions:

(H1) The function $\mathscr{Q}_1 : \mathscr{I} \times \mathscr{B}_h \times \mathscr{B}_h \to \mathbb{X}$ is continuous and we can find constants $\beta \in (0, 1), L_{\mathscr{Q}_1} > 0$, $\widetilde{L}_{\mathscr{Q}_1} > 0$ and $L^*_{\mathscr{Q}_1} > 0$ in ways that $\mathscr{Q}_1$ is $\mathbb{X}_\beta$ -valued and fulfills the subsequent assumptions:

$$\| (\mathscr{A})^\beta \mathscr{Q}_1(t, \varphi_1, \psi_1) - (\mathscr{A})^\beta \mathscr{Q}_1(t, \varphi_2, \psi_2) \|_{\mathbb{X}} \le L_{\mathscr{Q}_1} \|\varphi_1 - \varphi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{Q}_1} \| \psi_1 - \psi_2 \|_{\mathscr{B}_h},$$
$$\| (\mathscr{A})^\beta \mathscr{Q}_1(t, \varphi, \psi) \|_{\mathbb{X}} \le L_{\mathscr{Q}_1} \| \psi \|_{\mathscr{B}_h} + L^*_g,$$

where

$$L^*_{\mathscr{Q}_1} = \max_{t\in\mathscr{I}} \|\mathscr{Q}_1(t, 0, 0)\|_{\mathbb{X}}, \quad \text{for all} \quad t \in \mathscr{I} \quad \text{and} \quad \psi, \varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathscr{B}_h.$$

(H2) The function $\mathscr{Q}_2 : \mathscr{I} \times \mathscr{B}_h \times \mathscr{B}_h \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathscr{Q}_2}, \widetilde{L}_{\mathscr{Q}_2} > 0$ and $L^*_{\mathscr{Q}_2} > 0$ in ways that

$$\|\mathscr{Q}_2(t, \varphi_1, \psi_1) - \mathscr{Q}_2(t, \varphi_2, \psi_2)\|_{\mathbb{X}} \le L_{\mathscr{Q}_2}\|\varphi_1 - \varphi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{Q}_2}\|\psi_1 - \psi_2\|_{\mathscr{B}_h},$$

and

$$L^*_{\mathscr{Q}_2} = \max_{t\in\mathscr{I}} \|\mathscr{Q}_2(t, 0, 0)\|_{\mathbb{X}}, \quad \text{for all} \quad t \in \mathscr{I} \quad \text{and} \quad \varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathscr{B}_h.$$

(H3) The function $\mathscr{Q}_3 : \mathscr{I} \times \mathscr{B}_h \times \mathscr{B}_h \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathscr{Q}_3}, \widetilde{L}_{\mathscr{Q}_3} > 0$ and $L^*_{\mathscr{Q}_3} > 0$ in ways that

$$\|\mathscr{Q}_3(t,\varphi_1,\psi_1) - \mathscr{Q}_3(t,\varphi_2,\psi_2)\|_{\mathbb{X}} \le L_{\mathscr{Q}_3}\|\varphi_1 - \varphi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{Q}_3}\|\psi_1 - \psi_2\|_{\mathscr{B}_h},$$

and

$$L^*_{\mathscr{Q}_3} = \max_{t\in\mathscr{I}} \|\mathscr{Q}_3(t,0,0)\|_{\mathbb{X}}, \quad \text{for all} \quad t \in \mathscr{I} \quad \text{and} \quad \varphi_1,\varphi_2,\psi_1,\psi_2 \in \mathscr{B}_h.$$

(H4) The function $g_i : (t_i, s_i] \times \mathscr{B}_h \to \mathbb{X}, i = 1,2,\cdots,N$ are continuous and there exist positive constants $L_{g_i} > 0, L^*_{g_i} > 0$ such that

$$\begin{aligned} \|g_i(t,\varphi_1) - g_i(t,\varphi_2)\|_{\mathbb{X}} &\le L_{g_i}\|\varphi_1 - \varphi_2\|_{\mathscr{B}_h}, \\ \|g_i(t,\varphi)\|_{\mathbb{X}} &\le L_{g_i}\|\varphi\|_{\mathscr{B}_h} + L^*_{g_i}, \end{aligned}$$

where

$$L^*_{g_i} = \max_{t\in(t_i,s_i]} \|g_i(t,0)\|_{\mathbb{X}}, \quad \text{for all} \quad t \in (t_i,s_i] \quad \text{and} \quad \varphi,\varphi_1,\varphi_2 \in \mathscr{B}_h.$$

(H5) For every $r > 0$, there exist constants $L_{\mathscr{Q}_1}(r) > 0, L_{\mathscr{Q}_2}(r) > 0, L_{\mathscr{Q}_3}(r) > 0$ and $L_{g_i}(r) > 0$ such that

$$\begin{aligned} \| (\mathscr{A})^\beta \mathscr{Q}_1(t,\varphi_{t_2},\mathcal{C}\psi_{t_2}) - (\mathscr{A})^\beta \mathscr{Q}_1(t,\varphi_{t_1},\mathcal{C}\psi_{t_1}) \| &\le L_{\mathscr{Q}_1}(r)(1+\mathcal{C}^*)|t_2 - t_1|, \\ \| \mathscr{Q}_2(t,\varphi_{t_2},\mathcal{C}\psi_{t_2}) - \mathscr{Q}_2(t,\varphi_{t_1},\mathcal{C}\psi_{t_1}) \| &\le L_{\mathscr{Q}_2}(r)(1+\mathcal{C}^*)|t_2 - t_1|, \\ \| \mathscr{Q}_3(t,\varphi_{t_2},\mathcal{C}\psi_{t_2}) - \mathscr{Q}_3(t,\varphi_{t_1},\mathcal{C}\psi_{t_1}) \| &\le L_{\mathscr{Q}_3}(r)(1+\mathcal{C}^*)|t_2 - t_1|, \\ \text{and } \|g_i(t,\varphi_{t_2}) - g_i(t,\varphi_{t_1})\| &\le L_{g_i}(r)|t_2 - t_1|, \quad t,t_1,t_2 \in \mathscr{I}, \end{aligned}$$

for all function $z : (-\infty, T] \to \mathbb{X}$ such that $z_0 = \psi \in \mathscr{B}_h, z : \mathscr{I} \to \mathbb{X}$ is continuous and $\max_{0\le s\le T} \|z(s)\| \le r$.

(H6) The function $\zeta : \mathscr{I} \times \mathscr{B}_h \to [0,\infty)$ satisfies:

 (i) For every $\psi \in \mathscr{B}_h$, the function $t \mapsto \zeta(t,\psi)$ is continuous.

 (ii) There exists a constant $L_\zeta > 0$ such that
 $|\zeta(t,\varphi_2) - \zeta(t,\varphi_1)| \le L_\zeta\|\varphi_2 - \varphi_1\|_{\mathscr{B}_h}$, $\varphi_1,\varphi_2 \in \mathscr{B}_h$ for all $t \in \mathscr{I}$.

(H7) The following inequalities holds:

 (i) Let

$$\begin{aligned} \max_{1\le i\le N} \Bigg\{ &\mathcal{M}\mathcal{M}_0 L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + \mathcal{M}L^*_{g_i} + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{T^{\alpha\beta}}{\beta}\right) L^*_{\mathscr{Q}_1} \\ &+ \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{L^*_{\mathscr{Q}_2} + L^*_{\mathscr{Q}_3}\} + (\mathscr{E}^*_1 r + c_n)\Bigg[\mathcal{M}L_{g_i} \\ &+ (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{T^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}) \\ &+ \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\Bigg]\Bigg\} \le r, \quad \text{for some}\, r > 0. \end{aligned}$$

(ii) Let

$$\Lambda = \mathscr{E}_1^* \max_{1\le i\le N} \Bigg[\mathcal{M}(L_{g_i} + 2L_{g_i}(r)L_\zeta) + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}$$
$$+ L_{\mathscr{Q}_1}(r)L^*) + \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3}) + (L_{\mathscr{Q}_2}(r) + L_{\mathscr{Q}_3}(r))L^*\}\Bigg] < 1$$

be such that $0 \le \Lambda < 1$, where $2L_\zeta(1+\mathcal{C}^*) = L^*$.

(H8) The functions $\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3$ and $g_i, i = 1, 2, \cdots, N$ are continuous and there exist $\mu_{\mathscr{Q}_1}(t), \widetilde{\mu}_{\mathscr{Q}_1}(t), \mu_{\mathscr{Q}_2}(t), \widetilde{\mu}_{\mathscr{Q}_2}(t), \mu_{\mathscr{Q}_3}(t), \widetilde{\mu}_{\mathscr{Q}_3}(t), \mu_{g_i} \in C(\mathscr{I}, [0,\infty))$ in a way that

$$\begin{aligned}
\|(\mathscr{A})^\beta \mathscr{Q}_1(t,\varphi_1,\varphi_2)\|_{\mathbb{X}} &\le \mu_{\mathscr{Q}_1}(t)\|\varphi_1\|_{\mathscr{B}_h} + \widetilde{\mu}_{\mathscr{Q}_1}(t)\|\varphi_2\|_{\mathscr{B}_h},\\
\|\mathscr{Q}_2(t,\varphi_1,\varphi_2)\|_{\mathbb{X}} &\le \mu_{\mathscr{Q}_2}(t)\|\varphi_1\|_{\mathscr{B}_h} + \widetilde{\mu}_{\mathscr{Q}_2}(t)\|\varphi_2\|_{\mathscr{B}_h},\\
\|\mathscr{Q}_3(t,\varphi_1,\varphi_2)\|_{\mathbb{X}} &\le \mu_{\mathscr{Q}_3}(t)\|\varphi_1\|_{\mathscr{B}_h} + \widetilde{\mu}_{\mathscr{Q}_3}(t)\|\varphi_2\|_{\mathscr{B}_h},\\
\|g_i(t,\varphi)\|_{\mathbb{X}} &\le \mu_{g_i}(t)\|\varphi\|_{\mathscr{B}_h},\ i = 1,2,\cdots,N,
\end{aligned}$$

for all $t \in \mathscr{I}$ and $\varphi, \varphi_j \in \mathscr{B}_h$, $j = 1, 2$.

Theorem 3.1. *Assume that the conditions (H1)-(H7) hold. Then the structure (1.4)-(1.6) has a unique mild solution on* $(-\infty, T]$.

Proof. We will transform the model (1.4)-(1.6) into a fixed-point problem. Recognize the operator $\Upsilon : \mathscr{B}_T \to \mathscr{B}_T$ specified by

$$(\Upsilon x)(t) = \begin{cases}
\mathbb{T}_\alpha(t)[\varphi(0) - \mathscr{Q}_1(0,\varphi(0),0)] + \mathscr{Q}_1(t, z_{\zeta(t,z_t)}, \mathcal{C}z_{\zeta(t,z_t)}) \\
+ \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds,\ t \in [0, t_1], \\
g_i(t, z_{\zeta(t,z_t)}), \quad t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \\
\mathbb{T}_\alpha(t-s_i)d_i + \mathscr{Q}_1(t, z_{\zeta(t,z_t)}, \mathcal{C}z_{\zeta(t,z_t)}) \\
+ \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds,\ t \in (s_i, t_{i+1}],
\end{cases} \tag{3.1}$$

with d_i, $i = 1, 2, 3, \cdots, N$, defined by (2.17).

In perspective of [40, Theorem 2.1] and for any $z \in \mathbb{X}$ and $\beta \in (0,1)$, we obtain

$$\begin{aligned}
&\|\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}} \\
&= \|\mathscr{A}^{1-\beta}\mathbb{S}_\alpha(t-s)\mathscr{A}^\beta\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}
\end{aligned}$$

$$\leq \left\| \Big[\alpha \int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathscr{A}^{1-\beta}\mathbb{T}((t-s)^\alpha r)dr \Big] \mathscr{A}^\beta \mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}) \right\|_{\mathbb{X}}$$
$$\leq \alpha \mathcal{M}_{1-\beta}(t-s)^{\alpha\beta-1} \left[\int_0^\infty r^\beta \phi_\alpha(r)dr \right] \|\mathscr{A}^\beta \mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}. \tag{3.2}$$

On the other hand, from $\int_0^\infty r^{-q}\psi_\alpha(r)dr = \dfrac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(1+q)}$, for all $q \in [0,1]$ (see [48, Lemma 3.2]), we have

$$\int_0^\infty r^\beta \phi_\alpha(r)dr = \int_0^\infty \frac{1}{r^{\beta\alpha}}\psi_\alpha(r)dr = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}. \tag{3.3}$$

From (3.2) and (3.3), we conclude that

$$\|\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}$$
$$\leq \frac{\alpha\mathcal{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)(t-s)^{1-\alpha\beta}} \|\mathscr{A}^\beta \mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}. \tag{3.4}$$

It is obvious that the function $s \to \mathscr{A}\mathbb{T}_\alpha(t-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})$ is integrable on $[0,t)$ for every $t > 0$.

It is clear that the fixed points of the operator Υ are mild solutions of the model (1.4)-(1.6). We express the function $\widetilde{y}(\cdot) : (-\infty, T] \to \mathbb{X}$ as

$$\widetilde{y}(t) = \begin{cases} \varphi(t), & t \leq 0; \\ \mathbb{T}_\alpha(t)\varphi(0), & t \in \mathscr{I}, \end{cases}$$

then $\widetilde{y}_0 = \varphi$. For every function $x \in C(\mathscr{I}, \mathbb{R})$ with $x(0) = 0$, we allocate as $\widetilde{x}$ is characterized by

$$\widetilde{x}(t) = \begin{cases} 0, & t \leq 0; \\ x(t), & t \in \mathscr{I}. \end{cases}$$

If $z(\cdot)$ obeys (2.16), we are able to split it as $z(t) = \widetilde{y}(t) + x(t)$, $t \in \mathscr{I}$, which suggests $z_t = \widetilde{y}_t + x_t$, for each $t \in \mathscr{I}$ and also the function $x(\cdot)$ obeys

$$x(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0) + \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds, \ t \in [0,t_1], \\ g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}), \quad t \in (t_i, s_i], \ i = 1,2,\cdots,N, \\ \mathbb{T}_\alpha(t-s_i)\widetilde{d}_i + \mathscr{Q}_1(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds, \ t \in (s_i, t_{i+1}], \end{cases}$$

where

$$\begin{aligned}\widetilde{d}_i &= g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\\ &- \mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\\ &- \int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\\ &- \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\\ &- \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds,\ i=1,2,\cdots,N. \quad (3.5)\end{aligned}$$

Let $\mathscr{B}_T^0 = \{x \in \mathscr{B}_T\colon x_0 = 0 \in \mathscr{B}_h\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|x\|_{\mathscr{B}_T^0} = \sup_{t\in\mathscr{I}}\|x(t)\|_{\mathbb{X}} + \|x_0\|_{\mathscr{B}_h} = \sup_{t\in\mathscr{I}}\|x(t)\|_{\mathbb{X}}, \quad x \in \mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. Set $B_r = \{x \in \mathscr{B}_T^0 : \|x\|_{\mathbb{X}} \le r\}$ for some $r \ge 0$; then for each $r, B_r \subset \mathscr{B}_T^0$ is clearly a bounded closed convex set. For $x \in B_r(0, \mathscr{B}_T^0)$, from phase space axioms $(P_1) - (P_3)$ and along with the above discussion, we receive

$$\begin{aligned}&\|x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h}\\ &\le \|x_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h} + \|\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h}\\ &\le \mathscr{E}_1^* \sup_{0\le\tau\le\zeta(s,x_s+\widetilde{y}_s)}\|x(\tau)\|_{\mathbb{X}} + \mathscr{E}_2^*\|x_0\|_{\mathscr{B}_h} + \mathscr{E}_1^* \sup_{0\le\tau\le\zeta(s,x_s+\widetilde{y}_s)}\|\widetilde{y}(\tau)\| + \mathscr{E}_2^*\|\widetilde{y}_0\|_{\mathscr{B}_h}\\ &\le \mathscr{E}_1^* \sup_{0\le\tau\le s}\|x(\tau)\|_{\mathbb{X}} + \mathscr{E}_1^*\|\mathbb{T}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})}\|\varphi(0)\| + \mathscr{E}_2^*\|\varphi\|_{\mathscr{B}_h}\\ &\le \mathscr{E}_1^* \sup_{0\le\tau\le s}\|x(\tau)\|_{\mathbb{X}} + (\mathscr{E}_1^*\mathcal{M}H + \mathscr{E}_2^*)\|\varphi\|_{\mathscr{B}_h}.\end{aligned}$$

In the event that $\|x\|_{\mathbb{X}} < r$, $r > 0$, then

$$\|x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h} \le \mathscr{E}_1^* r + c_n, \quad s \in \mathscr{I}, \tag{3.6}$$

where $c_n = (\mathscr{E}_1^*\mathcal{M}H + \mathscr{E}_2^*)\|\varphi\|_{\mathscr{B}_h}$. We delimit the operator $\overline{\Upsilon} : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$(\overline{\Upsilon}z)(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0) + \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)\\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds,\ t\in[0,t_1],\\ g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}), \quad t\in(t_i,s_i],\ i=1,2,\cdots,N,\\ \mathbb{T}_\alpha(t-s_i)\widetilde{d}_i + \mathscr{Q}_1(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)})\\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds,\ t\in(s_i,t_{i+1}], \end{cases}$$

with $\widetilde{d}_i, i = 1, 2, \cdots, N$, defined by (3.5).

It is vindicated that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ admits a fixed point.

Remark 3.1. *As a result, we have the following estimations:*

$$\begin{aligned}
I_1 &= \|\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0)\|_{\mathbb{X}} \\
&\le \left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\mathscr{Q}_1(0,\varphi,0)\right\|_{\mathbb{X}} \\
&= \mathcal{M}\|(\mathscr{A})^{-\beta}\|\|(\mathscr{A})^{\beta}\mathscr{Q}_1(0,\varphi,0)\|_{\mathbb{X}} \\
&\le \mathcal{M}\mathcal{M}_0\Big[L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + L^*_{\mathscr{Q}_1}\Big], \quad \textit{where} \quad \mathcal{M}_0 = \|(\mathscr{A})^{-\beta}\|. \\
I_2 &= \left\|\mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)\right\|_{\mathbb{X}} \\
&\le \|(\mathscr{A})^{-\beta}\|\Bigg[\|(\mathscr{A})^{\beta}\mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) - (\mathscr{A})^{\beta}\mathscr{Q}_1(t,0,0)\|_{\mathbb{X}} \\
&\quad + \|(\mathscr{A})^{\beta}\mathscr{Q}_1(t,0,0)\|_{\mathbb{X}}\Bigg] \\
&\le \mathcal{M}_0\Bigg[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_1}\Bigg]. \\
I_3 &= \left\|\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}\Bigg[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_1}\Bigg],\ t \in [0,t_1]. \\
I_4 &= \left\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}} \\
&\le \Bigg\|\int_0^t \alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr \\
&\quad \times \mathscr{Q}_2\Bigg(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Bigg)ds\Bigg\|_{\mathbb{X}} \\
&\le \alpha\left[\int_0^\infty r\phi_\alpha(r)dr\right]\int_0^t (t-s)^{\alpha-1}\|\mathbb{T}((t-s)^\alpha r)\|_{\mathscr{L}(\mathbb{X})} \\
&\quad \times \Bigg\|\mathscr{Q}_2\Bigg(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Bigg)ds\Bigg\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\Bigg[(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_2}\Bigg],\ t \in [0,t_1]. \\
I_5 &= \left\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\Bigg[(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_3}\Bigg],\ t \in [0,t_1]. \\
I_6 &= \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)})\|_{\mathbb{X}} \\
&\le \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) - g_i(t,0)\|_{\mathbb{X}} + \|g_i(t,0)\|_{\mathbb{X}} \\
&\le L_{g_i}(\mathscr{E}_1^* r + c_n) + L^*_{g_i},\ t \in (t_i, s_i]. \\
I_7 &= \|g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}
\end{aligned}$$

$$
\begin{aligned}
&\leq L_{g_i}(\mathscr{E}_1^* r + c_n) + L_{g_i}^*,\ t \in (s_i, t_{i+1}]. \\
I_8 &= \|\mathscr{Q}_1(s_i, x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}} \\
&\leq \mathcal{M}_0\Big[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_1}^*\Big],\ t \in (s_i, t_{i+1}]. \\
I_9 &= \Big\|\int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i - s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(s_i)^{\alpha\beta}}{\beta}\Big[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_1}^*\Big],\ t \in (s_i, t_{i+1}]. \\
I_{10} &= \Big\|\int_0^{s_i} \mathbb{S}_\alpha(s_i - s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\Big[(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_2}^*\Big],\ t \in (s_i, t_{i+1}]. \\
I_{11} &= \Big\|\int_0^{s_i} \mathbb{S}_\alpha(s_i - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\Big[(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_3}^*\Big],\ t \in (s_i, t_{i+1}]. \\
I_{12} &= \Big\|\int_0^{t} \mathscr{A}\mathbb{S}_\alpha(t - s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(t_{i+1})^{\alpha\beta}}{\beta}\Big[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_1}^*\Big],\ t \in (s_i, t_{i+1}]. \\
I_{13} &= \Big\|\int_0^{t} \mathbb{S}_\alpha(t - s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)}\Big[(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_2}^*\Big],\ t \in (s_i, t_{i+1}]. \\
I_{14} &= \Big\|\int_0^{t} \mathbb{S}_\alpha(t - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)}\Big[(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_3}^*\Big],\ t \in (s_i, t_{i+1}]. \\
I_{15} &= \Big\|\mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\
&\quad - \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}\right)\Big\|_{\mathbb{X}} \\
&\leq \Big\|\mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\
&\qquad - \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\
&\qquad + \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\
&\qquad - \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}\right)\Big\|_{\mathbb{X}} \\
&\leq \mathcal{M}_0\mathscr{E}_1^*(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*)\|x - \overline{x}\|_{\mathscr{B}_T^0}, \\
I_{16} &= \Big\|\int_0^{t} \mathscr{A}\mathbb{S}_\alpha(t - s)\Big[\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}) \\
&\quad - \mathscr{Q}_1(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)})\Big]ds\Big\|_{\mathbb{X}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{t_1^{\alpha\beta}}{\beta}\mathscr{E}_1^*(L_{\mathscr{Q}_1}+\widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*+L_{\mathscr{Q}_1}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in[0,t_1].\\
I_{17} &= \bigg\|\int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\\
&\quad -\mathscr{Q}_2\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\bigg\|_{\mathbb{X}}\\
&\leq \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_2}+\widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*+L_{\mathscr{Q}_2}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in[0,t_1].\\
I_{18} &= \bigg\|\int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\\
&\quad -\mathscr{Q}_3\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\bigg\|_{\mathbb{X}}\\
&\leq \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_3}+\widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*+L_{\mathscr{Q}_3}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in[0,t_1].\\
I_{19} &= \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)}+\widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)})-g_i(t,\overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}+\widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)})\|_{\mathbb{X}}\\
&\leq \mathscr{E}_1^*[L_{g_i}+2L_{g_i}(r)L_\zeta]\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(t_i,s_i].\\
I_{20} &= \|g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}+\widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})-g_i(s_i,\overline{x}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})}+\widetilde{y}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}\\
&\leq \mathscr{E}_1^*[L_{g_i}+2L_{g_i}(r)L_\zeta]\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].\\
I_{21} &= \|\mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}+\widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}+\widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\\
&\quad -\mathscr{Q}_1(s_i, \overline{x}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})}+\widetilde{y}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}\overline{x}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})}+\widetilde{y}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}\\
&\leq \mathcal{M}_0\mathscr{E}_1^*(L_{\mathscr{Q}_1}+\widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*+L_{\mathscr{Q}_1}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].\\
I_{22} &= \bigg\|\int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})\\
&\quad -\mathscr{Q}_1(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)})\Big]ds\bigg\|_{\mathbb{X}}\\
&\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{(s_i)^{\alpha\beta}}{\beta}\mathscr{E}_1^*(L_{\mathscr{Q}_1}+\widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*+L_{\mathscr{Q}_1}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].\\
I_{23} &= \bigg\|\int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\\
&\quad -\mathscr{Q}_2\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\bigg\|_{\mathbb{X}}\\
&\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_2}+\widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*+L_{\mathscr{Q}_2}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].\\
I_{24} &= \bigg\|\int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\\
&\quad -\mathscr{Q}_3\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\bigg\|_{\mathbb{X}}\\
&\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_3}+\widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*+L_{\mathscr{Q}_3}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].\\
I_{25} &= \bigg\|\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)}+\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \mathscr{Q}_1(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)})\Big] ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(t_{i+1})^{\alpha\beta}}{\beta} \mathscr{E}_1^*(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*)\|x - \overline{x}\|_{\mathscr{B}_T^0},\ t \in (s_i, t_{i+1}]. \\
I_{26} &= \Big\| \int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) \\
&\quad - \mathscr{Q}_2\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big] ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)} \mathscr{E}_1^*(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^* + L_{\mathscr{Q}_2}(r)L^*)\|x - \overline{x}\|_{\mathscr{B}_T^0},\ t \in (s_i, t_{i+1}]. \\
I_{27} &= \Big\| \int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) \\
&\quad - \mathscr{Q}_3\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big] ds\Big\|_{\mathbb{X}} \\
&\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)} \mathscr{E}_1^*(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^* + L_{\mathscr{Q}_3}(r)L^*)\|x - \overline{x}\|_{\mathscr{B}_T^0},\ t \in (s_i, t_{i+1}].
\end{aligned}
$$

Now, we start proving the main proof of this Theorem. We demonstrate that $\overline{\Upsilon}$ maps $B_r(0, \mathscr{B}_T^0)$ into $B_r(0, \mathscr{B}_T^0)$. For any $x(\cdot) \in \mathscr{B}_T^0$, by employing Remark 3.1, we sustain

$$
\begin{aligned}
\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} &= \sum_{i=1}^{5} I_i \\
&\leq \mathcal{M}\mathcal{M}_0 L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}\right) L_{\mathscr{Q}_1}^* \\
&\quad + \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\{L_{\mathscr{Q}_2}^* + L_{\mathscr{Q}_3}^*\} + (\mathscr{E}_1^* r + c_n)\Bigg[\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}) \\
&\quad + \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\Bigg] \\
&\leq r, \quad t \in [0, t_1]. \\
\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} &= I_6 \leq L_{g_i}(\mathscr{E}_1^* r + c_n) + L_{g_i}^*,\ t \in (t_i, s_i], \quad i = 1, 2, \cdots, N. \\
\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} &= \sum_{i=7}^{14} I_i \\
&\leq \max_{1\leq i\leq N}\Bigg\{\mathcal{M}L_{g_i}^* + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)} \cdot \left[\mathcal{M}(s_i)^{\alpha\beta} + (t_{i+1})^{\alpha\beta}\right]\right) L_{\mathscr{Q}_1}^* \\
&\quad + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{L_{\mathscr{Q}_2}^* + L_{\mathscr{Q}_3}^*\}\left[\mathcal{M}(s_i)^\alpha + (t_{i+1})^\alpha\right] + (\mathscr{E}_1^* r + c_n)\Bigg[\mathcal{M}L_{g_i} \\
&\quad + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)} \cdot \{\mathcal{M}(s_i)^{\alpha\beta} + (t_{i+1})^{\alpha\beta}\}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}) \\
&\quad + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\{\mathcal{M}(s_i)^\alpha + (t_{i+1})^\alpha\}\Bigg]\Bigg\} \leq r, \quad t \in (s_i, t_{i+1}],
\end{aligned}
$$

Then, for all $t \in \mathscr{I}$, we conclude that

$$
\begin{aligned}
&\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} \\
&\leq \max_{1\leq i\leq N}\Bigg\{\mathcal{M}\mathcal{M}_0 L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + \mathcal{M}L^*_{g_i} + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{T^{\alpha\beta}}{\beta}\right)L^*_{\mathscr{Q}_1} \\
&\quad + \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{L^*_{\mathscr{Q}_2} + L^*_{\mathscr{Q}_3}\} + (\mathscr{E}^*_1 r + c_n)\Bigg[\mathcal{M}L_{g_i} \\
&\quad + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{T^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}) \\
&\quad + \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\Bigg]\Bigg\} \\
&\leq r.
\end{aligned}
$$

Thus, $\overline{\Upsilon}$ maps the ball $B_r(0, \mathscr{B}^0_T)$ into itself. Now, we prove that $\overline{\Upsilon}$ is a contraction on $B_r(0, \mathscr{B}^0_T)$. Let us consider $x, \overline{x} \in B_r(0, \mathscr{B}^0_T)$, then from estimations $I_j, j = 15, \cdots, 27$, we sustain

$$
\begin{aligned}
\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} &\leq \mathscr{E}^*_1\Bigg[\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{t_1^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*) + \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)} \\
&\quad \{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3}) + L^*(L_{\mathscr{Q}_2}(r) + L_{\mathscr{Q}_3}(r))\}\Bigg]\|x - \overline{x}\|_{\mathscr{B}^0_T},\ t \in [0, t_1]. \\
\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} &\leq \mathscr{E}^*_1[L_{g_i} + 2L_{g_i}(r)L_\zeta]\|x - \overline{x}\|_{\mathscr{B}^0_T},\ t \in (t_i, s_i]. \\
\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} &\leq \mathscr{E}^*_1\max_{1\leq i\leq N}\Bigg[\mathcal{M}L_{g_i} + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)}\cdot\{\mathcal{M}(s_i)^{\alpha\beta} + (t_{i+1})^{\alpha\beta}\}\right) \\
&\quad (L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1} + L_{\mathscr{Q}_1}(r)L^*) + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3}) \\
&\quad + L^*(L_{\mathscr{Q}_2}(r) + L_{\mathscr{Q}_3}(r))\}\{\mathcal{M}(s_i)^\alpha + (t_{i+1})^\alpha\}\Bigg]\|x - \overline{x}\|_{\mathscr{B}^0_T},\ t \in (s_i, t_{i+1}].
\end{aligned}
$$

As a result, for all $t \in \mathscr{I}$, we conclude that

$$
\begin{aligned}
&\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} \\
&\leq \mathscr{E}^*_1\max_{1\leq i\leq N}\Bigg[\mathcal{M}L_{g_i} + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1} + L_{\mathscr{Q}_1}(r)L^*) \\
&\quad + \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3}) + L^*(L_{\mathscr{Q}_2}(r) + L_{\mathscr{Q}_3}(r))\}\Bigg]\|x - \overline{x}\|_{\mathscr{B}^0_T} \\
&\leq \Lambda\|x - \overline{x}\|_{\mathscr{B}^0_T}.
\end{aligned}
$$

From the assumption (H7) and in the perspective of the contraction mapping principle, we understand that $\overline{\Upsilon}$ includes a unique fixed point $x \in \mathscr{B}^0_T$ which is a mild solution of the model (1.4)-(1.6) on $(-\infty, T]$. The proof is now completed. □

Theorem 3.2. *Let the assumption (H8) hold and*

$$\Lambda_1 = \mathscr{E}_1^* \max_{1\le i\le N} \left\{ \mathcal{M}\|\mu_{g_i}\|_\infty + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty + \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\right\} < 1. \tag{3.7}$$

Then the system (1.4)-(1.6) has a mild solution on $\mathscr{I}$.

Proof. Let $\overline{\Upsilon} : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ be the operator same as defined in Theorem 3.1. Now, we demonstrate that $\overline{\Upsilon}$ has a fixed point.

Remark 3.2. *From the hypothesis (H8) along with the above discussion, we sustain*

$$\begin{aligned}
I_{28} &= \|\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0)\|_{\mathbb{X}} \\
&\le \mathcal{M}\mathcal{M}_0\|\mu_{\mathscr{Q}_1}\|_\infty\|\varphi\|_{\mathscr{B}_h}. \\
I_{29} &= \left\|\mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)\right\|_{\mathbb{X}} \\
&\le \mathcal{M}_0(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\}. \\
I_{30} &= \left\|\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\}, \; t\in[0,t_1]. \\
I_{31} &= \left\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty\mathcal{C}^*\right\}, \; t\in[0,t_1]. \\
I_{32} &= \left\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_3}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty\mathcal{C}^*\right\}, \; t\in[0,t_1]. \\
I_{33} &= \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)})\|_{\mathbb{X}} \\
&\le \|\mu_{g_i}\|_\infty(\mathscr{E}_1^* r + c_n), \; t\in(t_i,s_i]. \\
I_{34} &= \|g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}} \\
&\le \|\mu_{g_i}\|_\infty(\mathscr{E}_1^* r + c_n), \; t\in(s_i,t_{i+1}]. \\
I_{35} &= \|\mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}} \\
&\le \mathcal{M}_0(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\}, \; t\in(s_i,t_{i+1}]. \\
I_{36} &= \left\|\int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(s_i)^{\alpha\beta}}{\beta}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\}, \; t\in(s_i,t_{i+1}]. \\
I_{37} &= \left\|\int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}}
\end{aligned}$$

$$\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty \mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{38} = \left\|\int_0^{s_i} \mathbb{S}_\alpha(s_i - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds\right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_3}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty \mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{39} = \left\|\int_0^{t} \mathscr{A}\mathbb{S}_\alpha(t - s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}) ds\right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(t_{i+1})^{\alpha\beta}}{\beta}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty \mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{40} = \left\|\int_0^{t} \mathbb{S}_\alpha(t - s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds\right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty \mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{41} = \left\|\int_0^{t} \mathbb{S}_\alpha(t - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds\right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_3}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty \mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

Now, we will give the main proof of this theorem. We remark that $(\overline{\Upsilon}x)(t) \subset \mathscr{B}_T^0$. Let B_r be the set same as defined in Theorem 3.1, where

$$r \geq \max_{1\leq i\leq N}\Bigg\{\mathcal{M}\mathcal{M}_0\|\mu_{\mathscr{Q}_1}\|_\infty\|\varphi\|_{\mathscr{B}_h} + (\mathscr{E}_1^* r + c_n)\Bigg[\mathcal{M}\|\mu_{g_i}\|_\infty + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{T^{\alpha\beta}}{\beta}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty + \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty) + \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\}\Bigg]\Bigg\}.$$

It is obvious that B_r is closed bounded and convex subset of $\mathscr{B}_T^0$. Let $x \in B_r(0, \mathscr{B}_T^0)$ then for $t \in [0, t_1]$, we receive

$$\|(\overline{\Upsilon}x)\|_{\mathscr{B}_T^0} \leq \mathcal{M}\mathcal{M}_0\|\mu_{\mathscr{Q}_1}\|_\infty\|\varphi\|_{\mathscr{B}_h} + (\mathscr{E}_1^* r + c_n)\Bigg[\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{(t_1)^{\alpha\beta}}{\beta}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty + \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty) + \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\}\Bigg].$$

In the similar manner for $t \in (t_i, s_i]$, we sustain

$$\|(\overline{\Upsilon}x)\|_{\mathscr{B}_T^0} \leq \|\mu_{\mathscr{Q}_1}\|_\infty\|\varphi\|_{\mathscr{B}_h}(\mathscr{E}_1^* r + c_n),\ i = 1, 2, \cdots, N.$$

Similarly, for $t \in (s_i, t_{i+1}]$, we find that

$$\begin{aligned}\|(\overline{\Upsilon}x)\|_{\mathscr{B}_T^0} \leq & \max_{1\leq i\leq N}(\mathscr{E}_1^* r + c_n)\Big\{\mathcal{M}\|\mu_{g_i}\|_\infty \\ & + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)}\cdot\{\mathcal{M}(s_i)^{\alpha\beta}+(t_{i+1})^{\alpha\beta}\}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty + \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty) \\ & + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\}\{\mathcal{M}(s_i)^\alpha + (t_{i+1})^\alpha\}\Big\}.\end{aligned}$$

From this, we notice that $\|\overline{\Upsilon}x\|_{\mathscr{B}_T^0} \leq r$ for every $t \in \mathscr{I}$. Therefore, $\overline{\Upsilon}(B_r) \subseteq B_r$. In order to utilizing the Theorem 2.2, we have to prove that the operator $\overline{\Upsilon}$ is a condensing operator. For this, we split $\overline{\Upsilon}$ by $\overline{\Upsilon} = \overline{\Upsilon}_1 + \overline{\Upsilon}_2$, where

$$(\overline{\Upsilon}_1 x)(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0) + \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds,\ t\in[0,t_1], \\ g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}), \quad t\in(t_i,s_i],\ i=1,2,\cdots,N, \\ \mathbb{T}_\alpha(t-s_i)\Big[g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}) \\ -\mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}) \\ -\int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\Big] \\ +\mathscr{Q}_1(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds,\ t\in(s_i,t_{i+1}], \end{cases}$$

and

$$(\overline{\Upsilon}_2 x)(t) = \begin{cases} \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds,\ t\in[0,t_1], \\ 0, \quad t\in(t_i,s_i],\ i=1,2,\cdots,N, \\ -\mathbb{T}_\alpha(t-s_i)\Big[\int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds \\ \quad - \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\Big] \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds,\ t\in(s_i,t_{i+1}]. \end{cases}$$

Firstly, we show that $\overline{\Upsilon}_1$ is continuous, so we consider a sequence $x^n \to x \in B_r$. In perspective of (3.1), we notice that

$$\|x^n_{\zeta(t,x_t^n+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t^n+\widetilde{y}_t)}\|_{\mathscr{B}_h} \leq \mathscr{E}_1^* r + c_n.$$

Remark 3.3. *By utilizing the hypothesis (H8) and Definition 2.2, we receive:*

(i) For every $t \in [0, t_1]$, *we obtain*

$$\mathscr{Q}_1\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big) \rightarrow \mathscr{Q}_1\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)$$

and since

$$\Big\|\mathscr{Q}_1\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big) - \mathscr{Q}_1\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)\Big\|_{\mathbb{X}} \leq 2I_{29}.$$

(ii) For every $t \in [0, t_1]$, *we obtain*

$$\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big) \rightarrow \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)$$

and since

$$\Big\|\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big) - \mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)\Big]ds\Big\|_{\mathbb{X}} \leq 2I_{30}.$$

(iii) For each $t \in (t_i, s_i]$, *we sustain*

$$g_i\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big) \rightarrow g_i\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)$$

and since

$$\Big\|g_i\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big) - g_i\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)\Big\|_{\mathbb{X}} \leq 2I_{33}.$$

(iv) For every $t \in (s_i, t_{i+1}]$, *we receive*

$$g_i(s_i, x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})}) \rightarrow g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})$$

and since

$$\Big\|g_i(s_i, x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})}) - g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\Big\|_{\mathbb{X}} \leq 2I_{34}.$$

(v) For all $t \in (s_i, t_{i+1}]$, we get

$$\mathscr{Q}_1(s_i, x^n_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x^n_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})})$$
$$\to \mathscr{Q}_1(s_i, x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})})$$

and since

$$\Big\|\mathscr{Q}_1(s_i, x^n_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x^n_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x^n_{s_i}+\widetilde{y}_{s_i})})$$
$$-\mathscr{Q}_1(s_i, x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})})\Big\|_{\mathbb{X}} \leq 2I_{35}.$$

(vi) For every $t \in (s_i, t_{i+1}]$, we obtain

$$\mathscr{A}\mathbb{S}_\alpha(s_i - s)\mathscr{Q}_1\bigg(s, x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}\bigg)$$
$$\to \mathscr{A}\mathbb{S}_\alpha(s_i - s)\mathscr{Q}_1\bigg(s, x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}\bigg)$$

and since

$$\bigg\| \int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i - s)\bigg[\mathscr{Q}_1\bigg(s, x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}\bigg)$$
$$-\mathscr{Q}_1\bigg(s, x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}\bigg)\bigg]ds\bigg\|_{\mathbb{X}} \leq 2I_{36}.$$

(vii) For every $t \in (s_i, t_{i+1}]$, we find that

$$\mathscr{A}\mathbb{S}_\alpha(t - s)\mathscr{Q}_1\bigg(s, x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}\bigg)$$
$$\to \mathscr{A}\mathbb{S}_\alpha(t - s)\mathscr{Q}_1\bigg(s, x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}\bigg)$$

and since

$$\bigg\| \int_0^{t} \mathscr{A}\mathbb{S}_\alpha(t - s)\bigg[\mathscr{Q}_1\bigg(s, x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s, x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x^n_s+\widetilde{y}_s)}\bigg)$$
$$-\mathscr{Q}_1\bigg(s, x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s, x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s, x_s+\widetilde{y}_s)}\bigg)\bigg]ds\bigg\|_{\mathbb{X}} \leq 2I_{39}.$$

From the Remark 3.3, for all $[0, t_1]$, we have

$$\|(\overline{\Upsilon}_1 x^n) - (\overline{\Upsilon}_1 x)\|_{\mathscr{B}^0_T} \leq 2(I_{29} + I_{30}),$$

and for every $t \in (s_i, t_{i+1}]$, we receive

$$\|(\overline{\Upsilon}_1 x^n) - (\overline{\Upsilon}_1 x)\|_{\mathscr{B}^0_T} \leq 2\bigg[\mathcal{M}(I_{34} + I_{35} + I_{36}) + I_{29} + I_{39}\bigg].$$

Since the functions $\mathscr{Q}_1$ and $g_i, i = 1, 2, \cdots, N$ are continuous, so we conclude that

$$\|(\overline{\Upsilon}_1 x^n) - (\overline{\Upsilon}_1 x)\|_{\mathscr{B}^0_T} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\overline{\Upsilon}_1$ is continuous.

Next, we prove that the operator $\overline{\Upsilon}_1$ is contraction on $B_r(0,\mathscr{B}_T^0)$. Indeed, let $x,\overline{x}\in B_r(0,\mathscr{B}_T^0)$, for $[0,t_1]$, we sustain

$$\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\le \mathscr{E}_1^*(\|\mu_{\mathscr{Q}_1}\|_\infty+\mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{(t_1)^{\alpha\beta}}{\beta}\right)\|x-\overline{x}\|_{\mathscr{B}_T^0},$$

and for $t\in(t_i,s_i]$, we get

$$\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\le \mathscr{E}_1^*\|\mu_{g_i}\|_\infty\|x-\overline{x}\|_{\mathscr{B}_T^0},$$

and for every $t\in(s_i,t_{i+1}]$, we sustain

$$\begin{aligned}&\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\\&\le \mathscr{E}_1^*\max_{1\le i\le N}\Bigg\{\mathcal{M}\|\mu_{g_i}\|_\infty\\&\quad+\left(\mathcal{M}_0(\mathcal{M}+1)+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)}\cdot\{\mathcal{M}(s_i)^{\alpha\beta}+(t_{i+1})^{\alpha\beta}\}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty+\mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\Bigg\}\|x-\overline{x}\|_{\mathscr{B}_T^0}.\end{aligned}$$

Then for all $t\in\mathscr{I}$, we find that

$$\begin{aligned}&\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\\&\le \mathscr{E}_1^*\max_{1\le i\le N}\left\{\mathcal{M}\|\mu_{g_i}\|_\infty+(\mathcal{M}+1)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty+\mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\right\}\|x-\overline{x}\|_{\mathscr{B}_T^0}\\&\le\Lambda_1\|x-\overline{x}\|_{\mathscr{B}_T^0}.\end{aligned}$$

Since $\Lambda_1<1$, which implies that $\overline{\Upsilon}_1$ is a contraction.

Next, we prove that the operator $\overline{\Upsilon}_2$ is completely continuous on $B_r(0,\mathscr{B}_T^0)$. First, we prove $\overline{\Upsilon}_2$ is continuous, so consider a sequence $x^n\to x\in B_r$. By applying the condition (H8), $I_{31},I_{32},I_{37},I_{38},I_{40},I_{41}$ and in perspective of Remark 3.3, for all $t\in[0,t_1]$, we get

$$\|(\overline{\Upsilon}_2x^n)-(\overline{\Upsilon}_2x)\|_{\mathscr{B}_T^0}\le 2(I_{31}+I_{32}),$$

and for all $t\in(s_i,t_{i+1}]$, we receive

$$\|(\overline{\Upsilon}_2x^n)-(\overline{\Upsilon}_2x)\|_{\mathscr{B}_T^0}\le 2\Big[\mathcal{M}(I_{37}+I_{38})+I_{40}+I_{41}\Big].$$

Since the functions $\mathscr{Q}_2$ and $\mathscr{Q}_3$ are continuous, so we conclude that

$$\|(\overline{\Upsilon}_2x^n)-(\overline{\Upsilon}_2x)\|_{\mathscr{B}_T^0}\to 0\quad\text{as}\quad n\to\infty.$$

Hence $\overline{\Upsilon}_2$ is continuous.

Next, we show that the operator $\overline{\Upsilon}_2$ maps bounded sets into bounded set in B_r. It is enough to show that there exists a positive constant Λ_2 such that for each $x\in B_r$ one has $\|\overline{\Upsilon}_2x\|_{\mathscr{B}_T^0}\le\Lambda_2$. For all $t\in\mathscr{I}$, we obtain

$$\|\overline{\Upsilon}_2x\|_{\mathscr{B}_T^0}\le\frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^*r+c_n)\Big\{(\|\mu_{\mathscr{Q}_2}\|_\infty+\|\mu_{\mathscr{Q}_3}\|_\infty)+\mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty+\|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\Big\}\le\Lambda_2.$$

Finally, we show that $\overline{\Upsilon}_2$ is a family of equi-continuous functions. Let $\tau_1, \tau_2 \in [0, t_1]$ be such that $0 \le \tau_1 < \tau_2 \le t_1$. Then

$$\begin{aligned}
&\|(\overline{\Upsilon}_2 x)(\tau_2) - (\overline{\Upsilon}_2 x)(\tau_1)\|_{\mathbb{X}} \\
&\le \int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_2\big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\big)\|_{\mathbb{X}} ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|\mathbb{S}_\alpha(\tau_2 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_2\big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\big)\|_{\mathbb{X}} ds \\
&\quad + \int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_3\big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\big)\|_{\mathbb{X}} ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|\mathbb{S}_\alpha(\tau_2 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_3\big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\big)\|_{\mathbb{X}} ds \\
&\le (\mathscr{E}_1^* r + c_n)\Big\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\Big\}\Bigg(\int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} ds \\
&\quad + \frac{\mathcal{M}(\tau_2 - \tau_1)^\alpha}{\Gamma(\alpha + 1)}\Bigg),
\end{aligned}$$

and for all $\tau_1, \tau_2 \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}
&\|(\overline{\Upsilon}_2 x)(\tau_2) - (\overline{\Upsilon}_2 x)(\tau_1)\|_{\mathbb{X}} \\
&\le (\mathscr{E}_1^* r + c_n)\Big\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\Big\}\Bigg(\int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} ds \\
&\quad + \frac{\mathcal{M}(\tau_2 - \tau_1)^\alpha}{\Gamma(\alpha + 1)} + \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha + 1)} \|\mathbb{T}_\alpha(\tau_2 - s_i) - \mathbb{T}_\alpha(\tau_1 - s_i)\|_{\mathscr{L}(\mathbb{X})}\Bigg).
\end{aligned}$$

Since T_α and $\mathbb{S}_\alpha$ are strongly continuous, so $\lim_{\tau_2 \to \tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} = 0$, $\lim_{\tau_2 \to \tau_1} \|\mathbb{T}_\alpha(\tau_2 - s_i) - \mathbb{T}_\alpha(\tau_1 - s_i)\|_{\mathscr{L}(\mathbb{X})} = 0$. From this, we conclude that $\|(\overline{\Upsilon}_2 x)(\tau_2) - (\overline{\Upsilon}_2 x)(\tau_1)\|_{\mathbb{X}} \to 0$ as $\tau_2 \to \tau_1$. This proves that $\overline{\Upsilon}_2$ is a family of equi-continuous functions. Hence, the operator $\overline{\Upsilon}_2$ is completely continuous. Therefore the operator $\overline{\Upsilon} = \overline{\Upsilon}_1 + \overline{\Upsilon}_2$ is a condensing operator from $\mathscr{B}_T^0$ into $\mathscr{B}_T^0$, where $\overline{\Upsilon}_1$ is contraction and $\overline{\Upsilon}_2$ is completely continuous. Finally, from Theorem 2.2, we infer that there exists a mild solution of the structure (1.4)-(1.6). This completes the proof. □

4 Example

To prove our theoretical results, we treat the IFNIDS with SDD of the model

$$\begin{aligned}
D_t^\alpha\Bigg[&u(t,z) + \int_{-\infty}^{t} e^{2(s-t)} \frac{u(s - \zeta_1(s)\zeta_2(\|u(s)\|), z)}{49} ds \\
&+ \int_0^t \sin(t-s) \int_{-\infty}^{s} e^{2(\tau - s)} \frac{u(\tau - \zeta_1(\tau)\zeta_2(\|u(\tau)\|), z)}{36} d\tau ds\Bigg] = \frac{\partial^2}{\partial z^2} u(t,z) \\
&+ \int_{-\infty}^{t} e^{2(s-t)} \frac{u(s - \zeta_1(s)\zeta_2(\|u(s)\|), z)}{9} ds
\end{aligned}$$

$$+\int_0^t \sin(t-s)\int_{-\infty}^s e^{2(\tau-s)}\frac{u(\tau-\zeta_1(\tau)\zeta_2(\|u(\tau)\|),z)}{25}d\tau ds$$
$$+\int_{-\infty}^t e^{2(s-t)}\frac{u(s-\zeta_1(s)\zeta_2(\|u(s)\|),z)}{64}ds$$
$$+\int_0^t \sin(t-s)\int_{-\infty}^s e^{2(\tau-s)}\frac{u(\tau-\zeta_1(\tau)\zeta_2(\|u(\tau)\|),z)}{16}d\tau ds,\ (t,z)\in\bigcup_{i=1}^N (s_i,t_{i+1}]\times[0,\pi], \quad (4.1)$$

$$u(t,0)=0=u(t,\pi),\quad t\in[0,T], \quad (4.2)$$

$$u(t,z)=\varphi(t,z),\quad t\le 0,\quad z\in[0,\pi], \quad (4.3)$$

$$u(t,z)=\int_{-\infty}^t e^{2(s-t)}\frac{u(s-\zeta_1(s)\zeta_2(\|u(s)\|),z)}{81}ds,\ (t,z)\in(t_i,s_i]\times[0,\pi],\ i=1,2,\dots,N, \quad (4.4)$$

where ${}^C D_t^q$ is Caputo's fractional derivative of order $0<q<1$, $0=t_0=s_0<t_1<t_2<\cdots<t_{N-1}\le S_N\le t_N\le t_{N+1}=T$ are pre-fixed real numbers and $\varphi\in\mathscr{B}_h$. We consider $\mathbb{X}=L^2[0,\pi]$ having the norm $\|\cdot\|_{L^2}$ and determine the operator $\mathscr{A}:D(\mathscr{A})\subset\mathbb{X}\to\mathbb{X}$ by $\mathscr{A}w=w''$ having the domain

$$D(\mathscr{A})=\{w\in\mathbb{X}: w,w' \quad \text{are absolutely continuous,} \quad w''\in\mathbb{X},\ w(0)=w(\pi)=0\}.$$

Then

$$\mathscr{A}w=\sum_{n=1}^{\infty}n^2\langle w,w_n\rangle w_n,\quad w\in D(\mathscr{A}),$$

where $w_n(s)=\sqrt{\frac{2}{\pi}}\sin(ns)$, $n=1,2,\dots,$. denotes the orthogonal set of eigenvectors of $\mathscr{A}$. It is long familiar that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\ge0}$ in $\mathbb{X}$ and is provided by

$$\mathbb{T}(t)w=\sum_{n=1}^{\infty}e^{-n^2t}\langle w,w_n\rangle w_n,\quad \text{for all}\quad w\in\mathbb{X},\quad \text{and every}\quad t>0.$$

We can find a constant $\mathcal{M}>0$ in a way that $\|\mathbb{T}(t)\|\le\mathcal{M}$. If we fix $\beta=\frac{1}{2}$, then the operator $(\mathscr{A})^{\frac{1}{2}}$ is given by

$$(\mathscr{A})^{\frac{1}{2}}w=\sum_{n=1}^{\infty}-n^2\langle w,w_n\rangle w_n,\quad w\in(D(\mathscr{A})^{\frac{1}{2}}),$$

in which $(D(\mathscr{A})^{\frac{1}{2}})=\left\{\omega(\cdot)\in\mathbb{X}:\sum_{n=1}^{\infty}n^2\langle\omega,w_n\rangle w_n\in\mathbb{X}\right\}$. Then

$$\mathbb{S}_\alpha(t)w=\alpha\int_0^\infty r\phi_\alpha(r)t^{\alpha-1}\mathbb{T}(t^\alpha r)dr,$$
$$=\sum_{n=1}^{\infty}E_{\alpha,\alpha}(-n^2t^\alpha)\langle w,w_n\rangle w_n,\ w\in\mathbb{X}.$$

For the phase space, we choose $h=e^{2s}$, $s<0$, then $l=\int_{-\infty}^0 h(s)ds=\frac{1}{2}<\infty$, for $t\le0$ and determine

$$\|\varphi\|_{\mathscr{B}_h}=\int_{-\infty}^0 h(s)\sup_{\theta\in[s,0]}\|\varphi(\theta)\|_{L^2}ds.$$

Hence, for $(t,\varphi)\in[0,T]\times\mathscr{B}_h$, where $\varphi(\theta)(z)=\varphi(\theta,z)$, $(\theta,z)\in(-\infty,0]\times[0,\pi]$. Set

$$u(t)(z)=u(t,z),\quad \zeta(t,\varphi)=\zeta_1(t)\zeta_2(\|\varphi(0)\|),$$

we have

$$\begin{aligned}
\mathscr{Q}_1(t,\varphi,\overline{\mathscr{H}}\varphi)(z)&=\int_{-\infty}^0 e^{2(s)}\frac{\varphi}{49}ds+(\overline{\mathscr{H}}\varphi)(z),\\
\mathscr{Q}_2(t,\varphi,\widetilde{\mathscr{H}}\varphi)(z)&=\int_{-\infty}^0 e^{2(s)}\frac{\varphi}{9}ds+(\widetilde{\mathscr{H}}\varphi)(z),\\
\mathscr{Q}_3(t,\varphi,\widehat{\mathscr{H}}\varphi)(z)&=\int_{-\infty}^0 e^{2(s)}\frac{\varphi}{64}ds+(\widehat{\mathscr{H}}\varphi)(z),\\
g_i(t,\varphi)(z)&=\int_{-\infty}^0 e^{2(s)}\frac{\varphi}{81}ds,\ i=1,2,\cdots,N,
\end{aligned}$$

where

$$\begin{aligned}
(\overline{\mathscr{H}}\varphi)(z)&=\int_0^t \sin(t-s)\int_{-\infty}^0 e^{2(\tau)}\frac{\varphi}{36}d\tau ds,\\
(\widetilde{\mathscr{H}}\varphi)(z)&=\int_0^t \sin(t-s)\int_{-\infty}^0 e^{2(\tau)}\frac{\varphi}{25}d\tau ds,\\
(\widehat{\mathscr{H}}\varphi)(z)&=\int_0^t \sin(t-s)\int_{-\infty}^0 e^{2(\tau)}\frac{\varphi}{16}d\tau ds,
\end{aligned}$$

then using these configurations, the system (4.1)-(4.4) is usually written in the theoretical form of design (1.4)-(1.6).

To treat this system we assume that $\zeta_i:[0,\infty)\to[0,\infty)$, $i=1,2$ are continuous. Now, we can see that for $t\in[0,1],\varphi,\overline{\varphi}\in\mathscr{B}_h$, we have

$$\begin{aligned}
\|(\mathscr{A})^{\frac12}\mathscr{Q}_1(t,\varphi,\overline{\mathscr{H}}\varphi)\|_{\mathbb{X}}&\le\left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{49}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{36}\right\|d\tau ds\right)^2dz\right)^{\frac12}\\
&\le\left(\int_0^\pi\left(\frac{1}{49}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds+\frac{1}{36}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds\right)^2dz\right)^{\frac12}\\
&\le\frac{\sqrt{\pi}}{49}\|\varphi\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{36}\|\varphi\|_{\mathscr{B}_h}\\
&\le L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h},
\end{aligned}$$

where $L_{\mathscr{Q}_1}+\widetilde{L}_{\mathscr{Q}_1}=\frac{85\sqrt{\pi}}{1764}$, and

$$\begin{aligned}
&\|(\mathscr{A})^{\frac12}\mathscr{Q}_1(t,\varphi,\overline{\mathscr{H}}\varphi)-(\mathscr{A})^{\frac12}\mathscr{Q}_1(t,\overline{\varphi},\overline{\mathscr{H}}\overline{\varphi})\|_{\mathbb{X}}\\
&\le\left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{49}-\frac{\overline{\varphi}}{49}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{36}-\frac{\overline{\varphi}}{36}\right\|d\tau ds\right)^2dz\right)^{\frac12}\\
&\le\left(\int_0^\pi\left(\frac{1}{49}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds+\frac{1}{36}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2dz\right)^{\frac12}
\end{aligned}$$

$$\leq \frac{\sqrt{\pi}}{49}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{36}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}$$
$$\leq L_{\mathscr{Q}_1}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_1}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.$$

Similarly, we conclude

$$\begin{aligned}\|\mathscr{Q}_2(t,\varphi,\widetilde{\mathscr{H}\varphi})\|_{L^2} &\leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{9}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{25}\right\|d\tau ds\right)^2dz\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^\pi\left(\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds+\frac{1}{25}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds\right)^2dz\right)^{\frac{1}{2}}\\ &\leq \frac{\sqrt{\pi}}{9}\|\varphi\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{25}\|\varphi\|_{\mathscr{B}_h}\\ &\leq L_{\mathscr{Q}_2}\|\varphi\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_2}\|\varphi\|_{\mathscr{B}_h},\end{aligned}$$

where $L_{\mathscr{Q}_2}+\widetilde{L}_{\mathscr{Q}_2}=\frac{34\sqrt{\pi}}{225}$, and

$$\begin{aligned}&\|\mathscr{Q}_2(t,\varphi,\widetilde{\mathscr{H}\varphi})-\mathscr{Q}_2(t,\overline{\varphi},\widetilde{\mathscr{H}\overline{\varphi}})\|_{L^2}\\ &\leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{9}-\frac{\overline{\varphi}}{9}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{25}-\frac{\overline{\varphi}}{25}\right\|d\tau ds\right)^2dz\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^\pi\left(\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds+\frac{1}{25}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2dz\right)^{\frac{1}{2}}\\ &\leq \frac{\sqrt{\pi}}{9}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{25}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}\\ &\leq L_{\mathscr{Q}_2}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_2}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.\end{aligned}$$

Correspondingly, we have

$$\begin{aligned}\|\mathscr{Q}_3(t,\varphi,\widehat{\mathscr{H}\varphi})\|_{L^2} &\leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{64}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{16}\right\|d\tau ds\right)^2dz\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^\pi\left(\frac{1}{64}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds+\frac{1}{16}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds\right)^2dz\right)^{\frac{1}{2}}\\ &\leq \frac{\sqrt{\pi}}{64}\|\varphi\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{16}\|\varphi\|_{\mathscr{B}_h}\\ &\leq L_{\mathscr{Q}_3}\|\varphi\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_3}\|\varphi\|_{\mathscr{B}_h},\end{aligned}$$

where $L_{\mathscr{Q}_3}+\widetilde{L}_{\mathscr{Q}_3}=\frac{80\sqrt{\pi}}{1024}$, and

$$\begin{aligned}&\|\mathscr{Q}_3(t,\varphi,\widehat{\mathscr{H}\varphi})-\mathscr{Q}_3(t,\overline{\varphi},\widehat{\mathscr{H}\overline{\varphi}})\|_{L^2}\\ &\leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{64}-\frac{\overline{\varphi}}{64}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{16}-\frac{\overline{\varphi}}{16}\right\|d\tau ds\right)^2dz\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^\pi\left(\frac{1}{64}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds+\frac{1}{16}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2dz\right)^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}&\le \frac{\sqrt{\pi}}{64}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{16}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}\\&\le L_{\mathscr{Q}_3}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_3}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.\end{aligned}$$

Finally,

$$\begin{aligned}\|g_i(t,\varphi)\|_{\mathbb{X}}&=\left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{81}\right\|ds\right)^2dz\right)^{\frac{1}{2}}\\&\le\left(\int_0^\pi\left(\frac{1}{81}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds\right)^2dz\right)^{\frac{1}{2}}\\&\le L_{g_i}\|\varphi\|_{\mathscr{B}_h},\ i=1,2,\cdots,N,\end{aligned}$$

where $L_{g_i}=\frac{\sqrt{\pi}}{81}$, and

$$\begin{aligned}&\|g_i(t,\varphi)-g_i(t,\overline{\varphi})\|_{\mathbb{X}}\\&=\left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{81}-\frac{\overline{\varphi}}{81}\right\|ds\right)^2dz\right)^{\frac{1}{2}}\\&\le\left(\int_0^\pi\left(\frac{1}{81}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2dz\right)^{\frac{1}{2}}\\&\le L_{g_i}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.\end{aligned}$$

Therefore the conditions (H1)-(H6) are all fulfilled. Furthermore, we assume that $\mathscr{E}_1* = 1, \mathcal{M} = 1, \mathcal{M}_0 = 1, \mathcal{M}_{\frac{1}{2}} = 1, \alpha = \frac{1}{2}, T = 1, \mathcal{C}^* = 1$ and $L_\zeta = 1$. Moreover, the appropriate values of the constants $L_{g_i}(r), L_{\mathscr{Q}_1}(r)$, $L_{\mathscr{Q}_2}(r)$ and $L_{\mathscr{Q}_3}(r)$, obtain

$$\begin{aligned}\Lambda=\mathscr{E}_1^*\max_{1\le i\le N}\Bigg[&\mathcal{M}(L_{g_i}+2L_{g_i}(r)L_\zeta)+(\mathcal{M}+1)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(L_{\mathscr{Q}_1}+\mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}\\&+L_{\mathscr{Q}_1}(r)L^*)+\frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2}+L_{\mathscr{Q}_3})+\mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2}+\widetilde{L}_{\mathscr{Q}_3})+(L_{\mathscr{Q}_2}(r)+L_{\mathscr{Q}_3}(r))L^*\}\Bigg]<1\end{aligned}$$

be such that $0 \le \Lambda < 1$, where $2L_\zeta(1+\mathcal{C}^*) = L^*$. Thus the condition (H7) holds. Hence by Theorem 3.1, we realize that the system (4.1)-(4.4) has a unique mild solution on [0,1].

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Characterizations of positive implicative superior ideals induced by superior mappings

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Abstract. The notion of positive implicative superior ideals of BCK-algebras is introduced, and their properties are investigated. Relations between a superior ideal and a positive implicative superior ideal in BCK-algebras are studied, and conditions for a superior ideal to be a positive implicative superior ideal are provided. Characterizations of positive implicative superior ideals induced by superior mappings are discussed.

1. Introduction

Algebras have played an important role in pure and applied mathematics and have its comprehensive applications in many aspects including dynamical systems and genetic code of biology (see [1], [2], [7], and [12]). Starting from the four DNA bases order in the Boolean lattice, Sáanchez et al. [11] proposed a novel Lie Algebra of the genetic code which shows strong connections among algebraic relationship, codon assignments and physicochemical properties of amino acids. A BCK/BCI-algebra (see [3, 4, 10]) is an important class of logical algebras introduced by Iséki and was extensively investigated by several researchers. Jun and Song [5] introduced the notion of BCK-valued functions and investigated several properties. They established block-codes by using the notion of BCK-valued functions, and shown that every finite BCK-algebra determines a block-code. In [6], Jun and Song introduced the notion of superior mapping by using partially ordered sets. Using the superior mapping, they introduced the concept of superior subalgebras and (commutative) superior ideals in BCK/BCI-algebras, and investigated related properties. They also discussed relations among a superior subalgebra, a superior ideal and a commutative superior ideal.

In this paper, we introduce the notion of positive implicative superior ideals of BCK-algebras, and investigate properties. We investigate relations between a superior ideal and a positive implicative superior ideal in BCK-algebras. We provide conditions for a superior ideal to be a positive implicative superior ideal, and discuss characterizations of positive implicative superior ideals.

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2. **Preliminaries**

We display basic definitions and properties of BCK/BCI-algebras that will be used in this paper. For more details of BCK/BCI-algebras, we refer the reader to [3], [8], [9] and [10].

An algebra $\mathcal{L} := (L; *, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in L)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in L)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in L)\ (x * x = 0)$,
(IV) $(\forall x, y \in L)\ (x * y = 0, y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra $\mathcal{L}$ satisfies the following identity:

(V) $(\forall x \in L)\ (0 * x = 0)$,

then $\mathcal{L}$ is called a BCK-algebra.

A BCK-algebra $\mathcal{L}$ is said to be positive implicative if it satisfies:

$$(\forall x, y, z \in L)\,((x * y) * z = (x * z) * (y * z))\,. \tag{2.1}$$

Any BCK/BCI-algebra $\mathcal{L}$ satisfies the following conditions:

$$(\forall x \in L)\,(x * 0 = x)\,, \tag{2.2}$$

$$(\forall x, y, z \in L)\,(x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.3}$$

$$(\forall x, y, z \in L)\,((x * y) * z = (x * z) * y)\,, \tag{2.4}$$

$$(\forall x, y, z \in L)\,((x * z) * (y * z) \leq x * y) \tag{2.5}$$

where $x \leq y$ if and only if $x * y = 0$.

A subset A of a BCK/BCI-algebra $\mathcal{L}$ is called an ideal of $\mathcal{L}$ if it satisfies:

$$0 \in A, \tag{2.6}$$

$$(\forall x, y \in L)\,(x * y \in A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.7}$$

A subset A of a BCK-algebra $\mathcal{L}$ is called a positive implicative ideal of $\mathcal{L}$ if it satisfies (2.6) and

$$(\forall x, y, z \in L)\,((x * y) * z \in A,\ y * z \in A \ \Rightarrow\ x * z \in A)\,. \tag{2.8}$$

Let L be a set of parameters and let U be a partially ordered set with the partial ordering $\preceq$ and the first element e. For a mapping $\tilde{f} : L \to \mathcal{P}(U)$, we consider the mapping

$$||\tilde{f}|| : L \to U,\ x \mapsto \begin{cases} \sup \tilde{f}(x) & \text{if } \exists \sup \tilde{f}(x), \\ e & \text{otherwise,} \end{cases} \tag{2.9}$$

which is called the superior mapping of L with respect to $\left(\tilde{f}, L\right)$. In this case, we say that $\left(\tilde{f}, L\right)$ is a pair on $(U, \preceq)$ (see [6]).

Definition 2.1 ([6])**.** Let $\mathcal{L} := (L, *, 0)$ be a BCK/BCI-algebra. By a superior ideal on $(\mathcal{L}, \tilde{f})$, we mean the superior mapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ which satisfies the following conditions:

$$(\forall x \in L) \left(||\tilde{f}||(0) \preceq ||\tilde{f}||(x) \right), \tag{2.10}$$

$$(\forall x, y \in L) \left(||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\} \right). \tag{2.11}$$

Proposition 2.2 ([6])**.** *If $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$, then $||\tilde{f}||(x) \preceq ||\tilde{f}||(y)$ for all $x, y \in L$ with $x \leq y$.*

3. **Positive implicative superior ideals**

In what follows, let $\mathcal{L} := (L, *, 0)$ be a BCK-algebra unless otherwise specified, where L is a set of parameters.

Definition 3.1. By a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, we mean the superior mapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ which satisfies the condition (2.10) and

$$(\forall x, y, z \in L) \left(||\tilde{f}||(x * z) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\} \right). \tag{3.1}$$

Example 3.2. Let $L = \{0, 1, 2, 3\}$ be a set with a binary operation '$*$' shown in Table 1.

Table 1. Cayley table for the binary operation '$*$'

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [10]). Let $U = \{a, b, c, d, e, f\}$ be ordered as pictured in Figure A3.

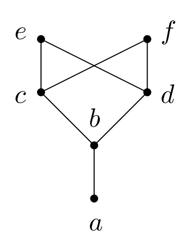

Figure A3

(1) Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{a, b\} & \text{if } x = 0, \\ \{a, d, f\} & \text{if } x = 1, \\ \{b, c, d, f\} & \text{if } x = 2, \\ \{a, b, c\} & \text{if } x = 3. \end{cases}$$

Then the superior mapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ is described as follows: $||\tilde{f}||(0) = b$, $||\tilde{f}||(1) = ||\tilde{f}||(2) = f$ and $||\tilde{f}||(3) = c$. By routine calculations, we know that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$.

(2) Let $(\tilde{g}, L)$ be a pair on $(U, \preceq)$ where $\tilde{g}$ is given as follows:

$$\tilde{g} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{a, b\} & \text{if } x = 0, \\ \{b, c, d, e\} & \text{if } x = 3, \\ \{b, c, d, f\} & \text{if } x \in \{1, 2\}. \end{cases}$$

Then the superior mapping $||\tilde{g}||$ of $\mathcal{L}$ with respect to $(\tilde{g}, L)$ is described as follows: $||\tilde{g}||(0) = b$, $||\tilde{f}||(1) = ||\tilde{f}||(2) = f$ and $||\tilde{f}||(3) = e$. It is not a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ since there does not exist $\sup\{||\tilde{f}||((3 * 2) * 1), ||\tilde{f}||(2 * 1)\}$ because $||\tilde{f}||((3 * 2) * 1) = e$ and $||\tilde{f}||(2 * 1) = f$ are noncomparable.

Example 3.3. Let $U = \{a, b, c, d, e, f\}$ be ordered as pictured in Figure B3.

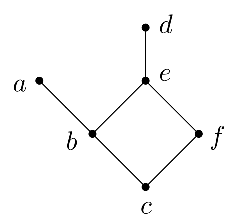

Figure B3

Let $L = \{0, 1, 2, 3, 4\}$ be a set with a binary operation '$*$' shown in Table 2.

TABLE 2. Cayley table for the binary operation '$*$'

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [10]). Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is defined by

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{b, c\} & \text{if } x = 0, \\ \{a, b, e\} & \text{if } x = 1, \\ \{b, e, f\} & \text{if } x = 2, \\ \{a, c, e, f\} & \text{if } x = 3, \\ \{d, e\} & \text{if } x = 4. \end{cases}$$

Then the superior mapping of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ is described as follows: $||\tilde{f}||(0) = b$, $||\tilde{f}||(1) = c$, $||\tilde{f}||(2) = e$, $||\tilde{f}||(3) = c$ and $||\tilde{f}||(4) = d$, and it is neither a superior ideal nor a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$.

Theorem 3.4. *Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$. If $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, then the nonempty set*

$$||\tilde{f}||_\alpha := \{x \in L \mid ||\tilde{f}||(x) \preceq \alpha\}$$

is a positive implicative ideal of $\mathcal{L}$ for all $\alpha \in U$.

Proof. Let $\alpha \in U$ be such that $||\tilde{f}||_\alpha \neq \emptyset$. Clearly $0 \in ||\tilde{f}||_\alpha$. Let $x, y, z \in L$ be such that $(x*y)*z \in ||\tilde{f}||_\alpha$ and $y * z \in ||\tilde{f}||_\alpha$. Then $||\tilde{f}||((x * y) * z) \preceq \alpha$ and $||\tilde{f}||(y * z) \preceq \alpha$. It follows from (3.1) that

$$||\tilde{f}||(x * z) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\} \preceq \alpha.$$

Thus $x * z \in ||\tilde{f}||_\alpha$, and therefore $||\tilde{f}||_\alpha$ is a positive implicative ideal of $\mathcal{L}$. □

Corollary 3.5. *Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$. If $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, then the set*

$$A := \{x \in L \mid ||\tilde{f}||(x) = ||\tilde{f}||(0)\}$$

is a positive implicative ideal of $\mathcal{L}$.

Theorem 3.6. *Every positive implicative superior ideal is a superior ideal.*

Proof. Let $||\tilde{f}||$ be a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. If we take $z = 0$ in (3.1) and use (2.2), then

$$||\tilde{f}||(x) = ||\tilde{f}||(x * 0) \preceq \sup\{||\tilde{f}||((x * y) * 0), ||\tilde{f}||(y * 0)\} = \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\}$$

for all $x, y \in L$. Hence $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. □

The converse of Theorem 3.6 is not true in general as seen in the following example.

Example 3.7. Let $L = \{0, a, b, c\}$ be a set with a binary operation '$*$' shown in Table 3.
Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [10]). Let $U = \{1, 2, 3, \cdots, 8\}$ be ordered as pictured in Figure 1.

TABLE 3. Cayley table for the binary operation '$*$'

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

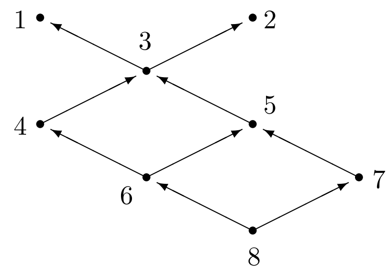

Figure 1

Consider a pair $\left(\tilde{f}, L\right)$ in which $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{6, 8\} & \text{if } x \in \{0\}, \\ \{4, 6, 7\} & \text{if } x \in \{a, b\}, \\ \{2, 3, 5, 6, 7\} & \text{if } x = c. \end{cases}$$

Then the superior mapping $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$ is described as follows: $||\tilde{f}||(0) = 6$, $||\tilde{f}||(a) = ||\tilde{f}||(b) = 3$ and $||\tilde{f}||(c) = 2$. Routine calculations show that $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. But it is not a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ since

$$||\tilde{f}||(b * a) = 3 \not\preceq 6 = \sup\{||\tilde{f}||((b * a) * a), ||\tilde{f}||(a * a)\}.$$

We provide conditions for a superior ideal to be a positive implicative superior ideal.

Theorem 3.8. *For a superior ideal $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$, the following are equivalent.*

(i) $||\tilde{f}||$ *is a positive implicative superior ideal on* $(\mathcal{L}, \tilde{f})$.

(ii) $(\forall x, y \in L) \left(||\tilde{f}||(x * y) \preceq ||\tilde{f}||((x * y) * y)\right)$.

Proof. Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. If we put $z = y$ in (3.1), then

$$\begin{aligned} ||\tilde{f}||(x * y) &\preceq \sup\{||\tilde{f}||((x * y) * y), ||\tilde{f}||(y * y)\} \\ &= \sup\{||\tilde{f}||((x * y) * y), ||\tilde{f}||(0)\} \\ &= ||\tilde{f}||((x * y) * y) \end{aligned}$$

for all $x, y \in L$.

Conversely, let $||\tilde{f}||$ be a superior ideal on $(\mathcal{L}, \tilde{f})$ which satisfies the condition (ii). Note that

$$((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$$

for all $x, y, z \in L$. It follows from (ii), (2.11) and Proposition 2.2 that

$$\begin{aligned}||\tilde{f}||(x * z) &\preceq ||\tilde{f}||((x * z) * z)\\ &\preceq \sup\{||\tilde{f}||(((x * z) * z) * (y * z)), ||\tilde{f}||(y * z)\}\\ &\preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\}\end{aligned}$$

for all $x, y, z \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. □

Theorem 3.9. *For a superior ideal $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$, the following are equivalent.*

(i) *$||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$.*

(ii) $(\forall x, y, z \in L) \left(||\tilde{f}||((x * z) * (y * z)) \preceq ||\tilde{f}||((x * y) * z)\right)$.

Proof. Suppose that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6. Note that

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in L$. It follows from (2.4), Theorem 3.8 and Proposition 2.2 that

$$\begin{aligned}||\tilde{f}||((x * z) * (y * z)) &= ||\tilde{f}||((x * (y * z)) * z)\\ &\preceq ||\tilde{f}||(((x * (y * z)) * z) * z)\\ &\preceq ||\tilde{f}||((x * y) * z)\end{aligned}$$

for all $x, y, z \in L$.

Conversely, let $||\tilde{f}||$ be a superior ideal on $(\mathcal{L}, \tilde{f})$ which satisfies the second condition. Using (2.11) and the second condition, we have

$$\begin{aligned}||\tilde{f}||(x * z) &= \sup\{||\tilde{f}||((x * z) * (y * z)), ||\tilde{f}||(y * z)\}\\ &\preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\}\end{aligned}$$

for all $x, y, z \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. □

Theorem 3.10. *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the condition* (2.10) *and*

$$(\forall x, y, z \in L) \left(||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(((x * y) * y) * z), ||\tilde{f}||(z)\}\right). \tag{3.2}$$

Proof. Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6, and so $||\tilde{f}||$ satisfies the condition (2.10). Using (2.11), (III), (2.2), (2.4) and Theorem 3.9, we have

$$\begin{aligned}||\tilde{f}||(x * y) &\preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(z)\}\\ &= \sup\{||\tilde{f}||(((x * z) * y) * (y * y)), ||\tilde{f}||(z)\}\\ &\preceq \sup\{||\tilde{f}||(((x * z) * y) * y), ||\tilde{f}||(z)\}\\ &= \sup\{||\tilde{f}||(((x * y) * y) * z), ||\tilde{f}||(z)\}\end{aligned}$$

for all $x, y, z \in L$.

Conversely, suppose that $||\tilde{f}||$ satisfies two conditions (2.10) and (3.2). Then

$$\begin{aligned}||\tilde{f}||(x) &= ||\tilde{f}||(x*0)\\ &\preceq \sup\{||\tilde{f}||(((x*0)*0)*z), ||\tilde{f}||(z)\}\\ &= \sup\{||\tilde{f}||(x*z), ||\tilde{f}||(z)\}\end{aligned}$$

for all $x, z \in L$, and so $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. If we take $z = 0$ in (3.2) and use (2.2) and (2.10), then

$$\begin{aligned}||\tilde{f}||(x*y) &\preceq \sup\{||\tilde{f}||(((x*y)*y)*0), ||\tilde{f}||(0)\}\\ &= \sup\{||\tilde{f}||((x*y)*y), ||\tilde{f}||(0)\}\\ &= ||\tilde{f}||((x*y)*y)\end{aligned}$$

for all $x, y \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.8. □

Lemma 3.11. *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$(\forall x, y, z \in L)\left((x*y)*z = 0 \;\Rightarrow\; ||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(y), ||\tilde{f}||(z)\}\right). \tag{3.3}$$

Proof. Assume that $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. Let $x, y, z \in L$ be such that $x*y \leq z$. Then $(x*y)*z = 0$, and so

$$||\tilde{f}||(x*y) \preceq \sup\{||\tilde{f}||((x*y)*z), ||\tilde{f}||(z)\} = \sup\{||\tilde{f}||(0), ||\tilde{f}||(z)\} = ||\tilde{f}||(z)$$

by (2.11) and (2.10). It follows that

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x*y), ||\tilde{f}||(y)\} \preceq \sup\{||\tilde{f}||(z), ||\tilde{f}||(y)\}.$$

Conversely, suppose that the assertion (3.3) is valid. Since

$$(0*x)*x = 0 \text{ and } (x*(x*y))*y = 0$$

for all $x, y \in L$, it follows from (3.3) that

$$||\tilde{f}||(0) \preceq \sup\{||\tilde{f}||(x), ||\tilde{f}||(x)\} = ||\tilde{f}||(x)$$

and

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x*y), ||\tilde{f}||(y)\}$$

for all $x, y \in L$. Therefore $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. □

Corollary 3.12. *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(a_1), ||\tilde{f}||(a_2), \cdots, ||\tilde{f}||(a_n)\} \tag{3.4}$$

for all $x, a_1, a_2, \cdots, a_n \in L$ with $(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$.

Theorem 3.13. *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\} \tag{3.5}$$

for all $x, y, a, b \in L$ with $(((x * y) * y) * a) * b = 0$.

Proof. Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6. Let $x, y, a, b \in L$ be such that $(((x * y) * y) * a) * b = 0$. Using Theorem 3.8(ii) and (3.3), we have

$$||\tilde{f}||(x * y) \preceq ||\tilde{f}||((x * y) * y) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\}.$$

Conversely, suppose that $||\tilde{f}||$ satisfies the condition (3.5) for all $x, y, a, b \in L$ with $(((x*y)*y)*a)*b = 0$. Assume that $(x*u)*v = 0$ for all $x, u, v \in L$. Then $(((x*0)*0)*u)*v = 0$, and so $||\tilde{f}||(x) = ||\tilde{f}||(x*0) \preceq \sup\{||\tilde{f}||(u), ||\tilde{f}||(v)\}$ by (3.5). It follows from Lemma 3.11 that $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. Note that

$$(((x * y) * y) * ((x * y) * y)) * 0 = 0$$

for all $x, y \in L$. Using (3.5) and (2.10), we have

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||((x * y) * y), ||\tilde{f}||(0)\} = ||\tilde{f}||((x * y) * y)$$

for all $x, y \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.8. □

Corollary 3.14. *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(a_1), ||\tilde{f}||(a_2), \cdots, ||\tilde{f}||(a_n)\} \tag{3.6}$$

for all $x, y, a_1, a_2, \cdots, a_n \in L$ with $(\cdots((((x * y) * y) * a_1) * a_2) * \cdots) * a_n = 0$.

Theorem 3.15. *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||((x * z) * (y * z)) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\} \tag{3.7}$$

for all $x, y, z, a, b \in L$ with $(((x * y) * z) * a) * b = 0$.

Proof. Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6. Suppose that $(((x * y) * z) * a) * b = 0$ for all $x, y, z, a, b \in L$. Then

$$||\tilde{f}||((x * z) * (y * z)) \preceq ||\tilde{f}||((x * y) * z) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\}$$

by Theorem 3.9 and Lemma 3.11.

Conversely, suppose that $||\tilde{f}||$ satisfies the condition (3.7) for all $x, y, z, a, b \in L$ with $(((x*y)*z)*a)*b = 0$. Let $x, y, a, b \in L$ be such that $(((x * y) * y) * a) * b = 0$. Then

$$||\tilde{f}||(x * y) = ||\tilde{f}||((x * y) * (y * y)) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\}$$

by (3.7). It follows from Theorem 3.13 that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. □

Corollary 3.16. *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||((x * z) * (y * z)) \preceq \sup\{||\tilde{f}||(a_1), ||\tilde{f}||(a_2), \cdots, ||\tilde{f}||(a_n)\} \tag{3.8}$$

for all $x, y, z, a_1, a_2, \cdots, a_n \in L$ with $(\cdots((((x * y) * z) * a_1) * a_2) * \cdots) * a_n = 0$.

Theorem 3.17. *Let $||\tilde{f}||$ and $||\tilde{g}||$ be superior ideals on $(\mathcal{L}, \tilde{f})$ and $(\mathcal{L}, \tilde{g})$, respectively, such that $||\tilde{f}||(0) = ||\tilde{g}||(0)$ and $||\tilde{g}||(x) \preceq ||\tilde{f}||(x)$ for all $x (\neq 0) \in L$. If $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, then $||\tilde{g}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{g})$.*

Proof. For any $x, y, z \in L$, let $u := (x * y) * z$. Then

$$\begin{aligned}
&||\tilde{g}||(((x * z) * (y * z)) * ((x * y) * z)) = ||\tilde{g}||(((x * z) * (y * z)) * u) \\
&= ||\tilde{g}||(((x * u) * z) * (y * z)) \preceq ||\tilde{f}||(((x * u) * z) * (y * z)) \\
&\preceq ||\tilde{f}||(((x * u) * y) * z) = ||\tilde{f}||(((x * y) * z) * u) \\
&= ||\tilde{f}||(0) = ||\tilde{g}||(0),
\end{aligned}$$

and so $||\tilde{g}||(((x * z) * (y * z)) * ((x * y) * z)) = ||\tilde{g}||(0)$. It follows from (2.11) that

$$\begin{aligned}
&||\tilde{g}||((x * z) * (y * z)) \preceq \sup\{||\tilde{g}||(((x * z) * (y * z)) * ((x * y) * z)), ||\tilde{g}||((x * y) * z)\} \\
&= \sup\{||\tilde{g}||(0), ||\tilde{g}||((x * y) * z)\} = ||\tilde{g}||((x * y) * z).
\end{aligned}$$

Therefore $||\tilde{g}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{g})$ by Theorem 3.9. □

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The uniqueness of meromorphic functions sharing sets in an angular domain *

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Abstract

By using the Tsuji's characteristic, we deal with the uniqueness problem of meromorphic functions sharing sets in an angular domain and obtain some theorems which improve and extend the results given by Zheng, Xuan.

Key words: Meromorphic function; Angular domain; Uniqueness; Tsuji's characteristic.

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1 Introduction and main results

The purpose of this paper is to investigate the uniqueness of meromorphic functions sharing sets in an angular domain by using the Tsuji's characteristic functions of angular domain. It is assumed that the readers are familiar with the notations of the Nevanlinna theory such as $T(r,f), m(r,f), N(r,f)$ and so on, that can be found, for instance, in [5, 17].

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}(=\mathbb{C}\bigcup\{\infty\})$ to denote the extended complex plane, and $\Omega(\subset\mathbb{C})$ to denote an angular domain. Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\Omega\subseteq\mathbb{C}$. Define

$$E(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\quad counting\quad multiplicities\},$$

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$$\overline{E}(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\quad ignoring\quad multiplicities\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

Let f and g be two non-constant meromorphic functions in $\mathbb{C}$. If $E(S,\Omega,f)=E(S,\Omega,g)$, we say that f and g share the set S CM (counting multiplicities) in Ω. If $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$, we say f and g share the set S IM(ignoring multiplicities) in Ω. In particular, when $S=\{a\}$, where $a\in\widehat{\mathbb{C}}$, we say f and g share the value a CM in Ω if $E(S,\Omega,f))=E(S,\Omega,g)$, and we say f and g share the value a IM in Ω if $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$. When $\Omega=\mathbb{C}$, we give the simple notation as before, $E(S,f),\overline{E}(S,f)$ and so on(see [13]).

Let l be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$, we denote by $\overline{E}_l(a,\Omega,f)$ the set of all a-points of f in Ω, where an a-point of multiplicity k is counted one times if $k\le l$ and zero times if $k>l$.

R.Nevanlinna(see [9]) proved the following well-known theorem.

Theorem 1.1 (see [9].) *If f and g are two non-constant meromorphic functions that share five distinct values a_1,a_2,a_3,a_4,a_5 IM in $\Omega=\mathbb{C}$, then $f(z)\equiv g(z)$.*

After his theorems, the uniqueness problems of meromorphic functions sharing values in the whole complex plane attracted many investigations (see [15]). In 2004, Zheng [19] studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$. It is an interesting topic to investigate the uniqueness with shared values in the remaining part of the complex plane removing an unbounded closed set, see [3, 4, 7, 8, 10, 13, 18, 19, 20]. Zheng [20], Cao and Yi [2], Xu and Yi [13] continued to investigate the uniqueness of meromorphic functions sharing five values and four values, Lin, Mori and Tohge [7] and Lin, Mori and Yi [8] investigated the uniqueness of meromorphic and entire functions sharing sets in an angular domain. To state theirs results, we need the following basic notations and definitions of meromorphic functions in an angular domain(see [5, 19, 20]).

In 2009, the present author [14] investigated the uniqueness of meromorphic functions with finite order sharing some values in an angular domain and obtained the following theorem

Theorem 1.2 *(see [14]). Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions, and let $f(z)$ be of finite order λ (lower order μ) and such that for some $a\in\widehat{\mathbb{C}}$ and an integer $p\ge 0,\delta=\delta(a,f^{(p)})>0$. For m pair of real numbers $\{\alpha_j,\beta_j\}$ satisfying*

$$-\pi\le\alpha_1\le\beta_1\le\alpha_2\le\beta_2\le\cdots\le\alpha_m\le\beta_m\le 2\pi$$

and

$$\sum_{j=1}^{m}(\alpha_{j+1}-\beta_j)<\frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}},$$

where $\sigma=\max\{\omega,\mu\}$, $\omega=\max\{\frac{\pi}{\beta_1-\alpha_1},\ldots,\frac{\pi}{\beta_m-\alpha_m}\}$, assume that $a_j(j=1,2,\ldots,q)$ be q distinct complex numbers, and let $k_j(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying

$$k_1\ge k_2\ge\cdots\ge k_p,\tag{1}$$

$$\overline{E}_{k_j}(a_j,\Omega,f)=\overline{E}_{k_j}(a_j,\Omega,g),$$
$$\sum_{j=3}^{q}\frac{k_j}{k_j+1}>2, \tag{2}$$

where $\Omega=\bigcup_{j=1}^{m}\{z:\ \alpha_j\le \arg z\le\beta_j\}$. *If* $\omega<\lambda(f)$, *then* $f(z)\equiv g(z)$.

In 2009, Cao and Yi [2] investigated the uniqueness problem of two transcendental meromorphic functions f,g sharing five values IM in an angular domain and obtained the following result which extended Theorem 1.1 to an angular domain.

Theorem 1.3 (see [2, Theorem 1.3].) *Let f and g be two transcendental meromorphic functions. Given one angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0<\beta-\alpha\le 2\pi$, we assume that f and g share five distinct values $a_j(j=1,2,3,4,5)$ IM in Ω. Then $f(z)\equiv g(z)$, provided that*

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))}=\infty,\quad (r\notin E),$$

where $S_{\alpha,\beta}(r,f)$ is called the Nevanlinna's angular characteristic.

Moreover, Cao and Yi [2] also investigated the two uniqueness problems of two transcendental meromorphic functions f,g sharing four distinct values CM in an angular domain X and f,g sharng two distinct values CM in an angular domain X and the other two distinct values IM in an angular domain X, and they obtained two interesting results which extended the analogous results as in the whole complex plane to an angular domain. In 2011, Xu and Cao [11, 12] improve the results given by Cao and Yi[1, 2] to some extent.

Most recently, Zheng [21] prove the following theorem by using the Tsuji's characteristic to extend the five IM theorem of Nevanlinna's to an angular domain. *The Tsuji's characteristic will be introduced in Section 2.*

Theorem 1.4 *(see [21]). Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Assume that $a_j(j=1,2,\ldots,5)$ be 5 distinct complex numbers. If $\overline{E}(a_j,\Omega,f)=\overline{E}(a_j,\Omega,g)$, then $f(z)\equiv g(z)$.*

In this paper, we will deal with the uniqueness of meromorphic functions sharing sets in an angular domain by using the Tsuji's characteristic and obtain the following results which are improvement of Theorem 1.4.

Theorem 1.5 *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\ne 0$, $S_i\cap S_j=\emptyset$, $(i\ne j)$. Let k_j $(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying (1) and

$$\overline{E}_{k_j)}(S_j,\Omega,f)=\overline{E}_{k_j)}(S_j,\Omega,g),\quad (j=1,2,\ldots,q). \tag{3}$$

Furthermore, let

$$\Theta_T(f) = \sum_a \Theta_T(0, f-a) - \sum_{j=1}^{q}\sum_{s=0}^{l-1} \Theta_T(0, f-(a_j+sb)),$$

$$\begin{aligned} A_1 &= \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta_T(0, f-(a_j+sb))}{k_m+1} + \sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0, f-(a_j+sb))}{k_j+1} \\ &\quad + \frac{(lm-3l+1)k_m}{k_m+1} - \frac{(2l-1)k_n}{k_n+1} + \Theta_0(f) - 2 \end{aligned}$$

and

$$\begin{aligned} A_2 &= \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0, g-(a_j+sb))}{k_n+1} + \sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0, g-(a_j+sb))}{k_j+1} \\ &\quad + \frac{(ln-3l+1)k_n}{k_n+1} - \frac{(2l-1)k_m}{k_m+1} + \Theta_0(g) - 2, \end{aligned}$$

where m and n are positive integers in $\{1, 2, \ldots, q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_1, A_2\} \geq 0, \quad \text{and} \quad \max\{A_1, A_2\} > 0. \tag{4}$$

Then $f_1(z) \equiv f_2(z)$.

From Theorem 1.5, we can get the following corollaries.

Corollary 1.1 *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j = \{a_j, a_j+b, \ldots, a_j+(l-1)b\}, \quad j = 1, 2, \ldots, q,$$

with $b \neq 0$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. Let k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1) and

$$\overline{E}_{k_j)}(S_j, \Omega, f) = \overline{E}_{k_j)}(S_j, \Omega, g), \quad (j = 1, 2, \ldots, q).$$

If

$$\sum_{j=3}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1} + \frac{(2-2l)k_3}{k_3+1} > 2.$$

Then $f(z) \equiv g(z)$.

Proof: Let $m = n = 3$. Since $\Theta_T(f) \geq 0, \Theta_T(g) \geq 0$, $\delta_T(0, f-(a_j+sb)) \geq 0$ and $\delta_T(0, g-(a_j+sb)) \geq 0$ for $j = 1, 2, \ldots, q$, one can deduce from Theorem 1.5 that Corollary 1.1 follows. □

The following corollary is an analog of a result due to Yi (Theorem 10.7 in [15], see also [16]) on $\mathbb{C}$.

Corollary 1.2 *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta \le 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j = \{a_j, a_j + b, \dots, a_j + (l-1)b\}, \quad j = 1, 2, \dots, q,$$

with $b \ne 0$, $q > 4$, $S_i \cap S_j = \emptyset$, $(i \ne j)$. If $\overline{E}(S_j, \Omega, f) = \overline{E}(S_j, \Omega, g), (j = 1, 2, \dots, q)$. Then $f(z) \equiv g(z)$.

Proof: Let $k_1 = k_2 = \ldots = k_q = \infty$. One can deduce from Corollary 1.1 that Corollary 1.2 follows immediately. □

Let $l = 1$. Then it is easily derived the following corollary from Corollary 1.1, which is an analog of the Corollary of Theorem 3.15 in [15].

Corollary 1.3 *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta \le 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Let a_j $(j = 1, 2, \dots, q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and $k_j (j = 1, 2, \dots, q)$ be positive integers or ∞ satisfying (1) and $\overline{E}_{k_j)}(a_j, \Omega, f) = \overline{E}_{k_j)}(a_j, \Omega, g), (j = 1, 2, \dots, q)$. Then*

(i) if $q = 7$, then $f(z) \equiv g(z)$.
(ii) if $q = 6$ and $k_3 \ge 2$, then $f(z) \equiv g(z)$.
(iii) if $q = 5$, $k_3 \ge 3$ and $k_5 \ge 2$, then $f(z) \equiv g(z)$.
(iv) if $q = 5$ and $k_4 \ge 4$, then $f(z) \equiv g(z)$.
(v) if $q = 5$, $k_3 \ge 5$ and $k_4 \ge 3$, then $f(z) \equiv g(z)$.
(vi) if $q = 5$, $k_3 \ge 6$ and $k_4 \ge 2$, then $f(z) \equiv g(z)$.

Another main theorem of this paper is listed as follows.

Theorem 1.6 *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta \le 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \dots, c + a_j w^{l-1}\}, \quad j = 1, 2, \dots, q,$$

with $a_j \ne 0$, $(j = 1, 2, \dots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \ne j)$. Let k_j $(j = 1, 2, \dots, q)$ be positive integers or ∞ satisfying (1) and

$$\overline{E}_{k_j)}(S_j, \Omega, f) = \overline{E}_{k_j)}(S_j, \Omega, g), \quad (j = 1, 2, \dots, q). \tag{5}$$

Furthermore, let

$$\Theta_T(f) = \sum_a \Theta_T(0, f - a) - \sum_{j=1}^{q} \sum_{s=0}^{l-1} \Theta_T(0, f - (c + a_j w^s)),$$

$$\begin{aligned} A_1 \quad = \quad & \frac{\sum_{j=1}^{m-1} \sum_{s=0}^{l-1} \delta_T(0, f - (c + a_j w^s))}{k_m + 1} + \sum_{j=m}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta_T(0, f - (c + a_j w^s))}{k_j + 1} \\ & + \frac{l(m-2)k_m}{k_m + 1} - \frac{l k_n}{k_n + 1} + \Theta_T(f) - 2 \end{aligned}$$

and

$$A_2 = \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0,g-(c+a_jw^s))}{k_n+1} + \sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,g-(c+a_jw^s))}{k_j+1} + \frac{l(n-2)k_n}{k_n+1} - \frac{lk_m}{k_m+1} + \Theta_T(g) - 2,$$

where m and n are positive integers in $\{1,2,\dots,q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_1,A_2\} \geq 0, \quad and \quad \max\{A_1,A_2\} > 0. \tag{6}$$

Then $(f(z)-c)^l \equiv (g(z)-c)^l$.

From Theorem 1.6, we can get the following corollary immediately.

Corollary 1.4 *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j = \{c+a_j, c+a_jw, \dots, c+a_jw^{l-1}\}, \quad j=1,2,\dots,q,$$

with $a_j \neq 0$, $(j=1,2,\dots,q)$, $q > 2+\frac{2}{l}$, $w=\exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If $\overline{E}(S_j,\Omega,f) = \overline{E}(S_j,\Omega,g)$ for $j=1,2,\dots,q$, then $(f(z)-c)^l \equiv (g(z)-c)^l$.

Proof: Set $m=n=1$ and $k_1=k_2=\dots=\infty$. Since $\Theta_T(f)\geq 0, \Theta_T(g)\geq 0$, $\delta_T(0,f-(a_j+sb))\geq 0$ and $\delta_T(0,g-(a_j+sb))\geq 0$ for $j=1,2,\dots,q$. Then Corollary 1.4 follows immediately from Theorem 1.6. □

2 Preliminaries

In this section, we will introduce some notations of Tsuji's characteristic in an angular domain (see [6, 21]). For meromorphic function f in an angular domain Ω and $\omega = \frac{\pi}{\beta-\alpha}$, we define

$$\mathfrak{M}_{\alpha,\beta}(r,f) = \frac{1}{2\pi}\int_{\arcsin(r^{-\omega})}^{\pi-\arcsin(r^{-\omega})} \log^+\left|f(re^{i(\alpha+\omega^{-1}\theta)}\sin^{\omega^{-1}}\theta)\right| \frac{1}{r^\omega \sin^2\theta}d\theta,$$

$$\mathfrak{N}_{\alpha,\beta}(r,f) = \sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}} \left(\frac{\sin\omega(\beta_n-\alpha)}{|b_n|^\omega} - \frac{1}{r^\omega}\right),$$

where b_n are the poles of $f(z)$ in $\Xi(\alpha,\beta;r) = \{z = re^{i\theta}\theta : \alpha < \theta < \beta, 1 < t \leq r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}$ appearing often according to their multiplicities and then Tsuji characteristic of f is

$$\mathfrak{T}_{\alpha,\beta}(r,f) = \mathfrak{M}_{\alpha,\beta}(r,f) + \mathfrak{N}_{\alpha,\beta}(r,f).$$

We denote by $\mathfrak{n}_{\alpha,\beta}(r,f)$ the number of poles of $f(z)$ in $\Xi(\alpha,\beta;r)$, and then

$$\mathfrak{N}_{\alpha,\beta}(r,f) = \int_1^r \left(\frac{1}{t^\omega} - \frac{1}{r^\omega}\right) d\mathfrak{n}_{\alpha,\beta}(r,f) = \omega\int_1^r \frac{\mathfrak{n}_{\alpha,\beta}(t,f)}{t^{\omega+1}}dt,$$

when pole b_n occurs in the sum $\sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}}$ only once, we denote it by $\overline{\mathfrak{N}}_{\alpha,\beta}(r,f)$. For meromorphic function f in Ω and for all complex numbers a, if

$$\limsup_{r\to\infty}\frac{\mathfrak{T}_{\alpha,\beta}(r,f)}{\log r}=\infty,$$

then f is called transcendental with respect to the Tsuji characteristic[21], and we have the Tsuji deficiency of $f(z)$ as follows

$$\delta_T(a,f;\alpha,\beta)=\liminf_{r\to\infty}\frac{\mathfrak{M}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{\mathfrak{T}_{\alpha,\beta}(r,f)}=1-\limsup_{r\to\infty}\frac{\mathfrak{N}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{\mathfrak{T}_{\alpha,\beta}(r,f)},$$

and

$$\Theta_T(a,f;\alpha,\beta)=1-\limsup_{r\to\infty}\frac{\overline{\mathfrak{N}}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{\mathfrak{T}_{\alpha,\beta}(r,f)},$$

for $a\not\equiv\infty$ and $\delta_T(\infty,f;\alpha,\beta)$ is defined by the above formula with $\mathfrak{M}_{\alpha,\beta}(r,f)$ and $\mathfrak{N}_{\alpha,\beta}(r,f)$ in place of $\mathfrak{M}_{\alpha,\beta}(r,\frac{1}{f-a})$ and $\mathfrak{N}_{\alpha,\beta}(r,\frac{1}{f-a})$, $\Theta_T(\infty,f;\alpha,\beta)$ is defined by the above formula with $\overline{\mathfrak{N}}_{\alpha,\beta}(r,f)$ in place of $\overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a})$. If no confusion occur in the context, then we simply write $\delta_T(a,f)$ for $\delta_T(a,f;\alpha,\beta)$ and $\Theta_T(a,f)$ for $\Theta_T(a,f;\alpha,\beta)$. $\delta_T(a,f)$ is called the Tsuji deficiency of f at a and if $\delta_T(a,f)>0$, then a is said to be a Tsuji deficient value of f. In addition, from ref.[21], we have the following properties of this Tsuji's characteristic

$$\mathfrak{T}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=\mathfrak{T}_{\alpha,\beta}(r,f)+O(1),\tag{7}$$

and the fundamental inequalities

$$(q-2)\mathfrak{T}_{\alpha,\beta}(r,f)\le\sum_{j=1}^{q}\overline{\mathfrak{N}}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+Q_{\alpha,\beta}(r,f),\tag{8}$$

hold for q distinct points $a_j\in\widehat{\mathbb{C}}$,

$$Q_{\alpha,\beta}(r,f)=O(\log^+\mathfrak{T}_{\alpha,\beta}(r,f)+\log r),\quad r\notin E$$

where E denotes a set of r with finite linear measure. It is not necessarily the same for every occurrence in the context. For sake of simplicity, we omit the subscript in all notations and use $\mathfrak{M}(r,f),\mathfrak{N}(r,f),Q(r,f)$ and $\mathfrak{T}(r,f)$ instead of $\mathfrak{M}_{\alpha,\beta}(r,f),\mathfrak{N}_{\alpha,\beta}(r,f),Q_{\alpha,\beta}(r,f)$ and $\mathfrak{T}_{\alpha,\beta}(r,f)$, respectively.

By using Lo Yang's method in dealing with the multiple values problem, we can get the following lemma

Lemma 2.1 *For meromorphic function f in an angular domain Ω and $\omega=\frac{\pi}{\beta-\alpha}$, Let a be an arbitrary complex number, and k be a positive integer. Then*

$$(i)\quad \overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a})\le\frac{k}{k+1}\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a})+\frac{1}{k+1}\mathfrak{N}_{\alpha,\beta}(r,\frac{1}{f-a}),$$

$$(ii)\quad \overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a})\le\frac{k}{k+1}\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a})+\frac{1}{k+1}\mathfrak{T}_{\alpha,\beta}(r,f)+O(1),$$

where $\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a})$ *to denote the zeros of* $f(z)-a$ *in* Ω*, whose multiplicities are no greater than* k *and are counted only once. Likewise, we use* $\overline{\mathfrak{N}}^{(k}_{\alpha,\beta}(r,\frac{1}{f-a})$ *to denote the zeros of* $f(z)-a$ *in* Ω*, whose multiplicities are greater than* k *and are counted only once.*

By using Lemma 2.1 and (8), we can obtain the following lemma

Lemma 2.2 *For meromorphic function* f *in an angular domain* Ω *and* $\omega=\frac{\pi}{\beta-\alpha}$*. Let* $a_1, a_2, \ldots, a_q$ *be* q *distinct complex numbers in the extended complex plane* $\widehat{\mathbb{C}}$*, let* $k_1, k_2, \ldots, k_q$ *be* q *positive integers. Then*

$$(i)\quad (q-2)\mathfrak{T}(r,f)<\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-a_j})+\sum_{j=1}^{q}\frac{1}{k_j+1}\mathfrak{N}(r,\frac{1}{f-a_j})+Q(r,f),$$

$$(ii)\quad (q-2-\sum_{j=1}^{q}\frac{1}{k_j+1})\mathfrak{T}(r,f)<\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-a_j})+Q(r,f),$$

where

$$\mathfrak{N}^{(1)}(r,f)=\mathfrak{N}(r,\frac{1}{f'})+2\mathfrak{N}(r,f)-\mathfrak{N}(r,f'),$$

and $Q(r,f)$ *is stated as in (8).*

From (8) and the definition of transcendental in Tsuji sense, we can get the following lemma.

Lemma 2.3 *(Picard theorem for angular domain) Let* f *be an transcendental meromorphic function in* Ω *in the Tsuji sense. Then* f *has at most two Picard exceptional values in* Ω*.*

3 Proof of Theorem 1.5

Suppose that $f(z)\not\equiv g(z)$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta_T(0,f-d)>0$ and $d\notin\{a_j+sb: j=1,2,\ldots,q \text{ and } s=0,1,\ldots,l-1\}$. We denote them by d_k $(k=1,2,\ldots,\infty)$. Obviously, $\Theta_T(f)=\sum_{k=1}^{\infty}\Theta_T(0,f-d_k)$. Thus there exits a p such that $\sum_{k=1}^{p}\Theta_T(0,f-d_k)>\Theta_T(f)-\varepsilon$ holds for any given $\varepsilon(>0)$. From (8) we have

$$(ql+p-2)\mathfrak{T}(r,f)<\sum_{j=1}^{q}\sum_{s=0}^{n-1}\overline{\mathfrak{N}}(r,\frac{1}{f-(a_j+sb)})+\sum_{k=1}^{p}\overline{\mathfrak{N}}(r,\frac{1}{f-d_k})+Q(r,f).$$

From the definition of deficiency in Tsuji sense, we have

$$\overline{\mathfrak{N}}(r,\frac{1}{f-d_k})<(1-\Theta_T(0,f-d_k))\,\mathfrak{T}(r,f)+Q(r,f).$$

From Lemma 2.1 and the definition of deficiency in Tsuji sense, it follows that for $s \in \{0, 1, \ldots, l-1\}$

$$\begin{aligned}
& \overline{\mathfrak{N}}(r, \frac{1}{f-(a_j+sb)}) \\
\leq\ & \frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r, \frac{1}{f-(a_j+sb)}) + \frac{1}{k_j+1}\mathfrak{N}(r, \frac{1}{f-(a_j+sb)}) \\
<\ & \frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r, \frac{1}{f-(a_j+sb)}) + \frac{1}{k_j+1}\left(1-\delta_T(0, f-(a_j+sb))\right)\mathfrak{T}(r,f) \\
& +Q(r,f).
\end{aligned}$$

Thus, from Lemma 2.2, we have

$$\begin{aligned}
& (ql+p-2)\mathfrak{T}(r,f) \\
<\ & \left\{\sum_{k=1}^{p}(1-\Theta_T(0, f-d_k))\right\}\mathfrak{T}(r,f) + \sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r, \frac{1}{f-(a_j+sb)}) \\
& + \left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1}{k_j+1}\left(1-\delta_T(0, f-(a_j+sb))\right)\right\}\mathfrak{T}(r,f) + Q(r,f).
\end{aligned}$$

Since

$$1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2},$$

we can deduce that

$$\begin{aligned}
& (ql+p-2)\mathfrak{T}(r,f) \\
<\ & \left(p-\Theta_T(f)+\varepsilon\right)\mathfrak{T}(r,f) + \frac{k_m}{k_m+1}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r, \frac{1}{f-(a_j+sb)}) \\
& + \left\{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\left(\frac{k_j}{k_j+1}-\frac{k_m}{k_m+1}\right)\left(1-\delta_T(0, f-(a_j+sb))\right)\right\}\mathfrak{T}(r,f) \\
& + \left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1-\delta_T(0, f-(a_j+sb))}{k_j+1}\right\}\mathfrak{T}(r,f) + Q(r,f),
\end{aligned}$$

that is,

$$\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f) < \sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{\mathfrak{N}}^{k_j)}(r, \frac{1}{f-(a_j+sb)}) + Q(r,f),$$

where

$$B_1 = \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta_T(0, f-(a_j+sb))}{k_m+1} + \sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0, f-(a_j+sb))}{k_j+1} + \Theta_T(f) - 2.$$

Similar to the above discussion, we also have

$$\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_0^{k_j)}(r,\frac{1}{g-(a_j+sb)})+Q(r,g),$$

where

$$B_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0,g-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,g-(a_j+sb))}{k_j+1}+\Theta_T(g)-2.$$

Thus,

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\<\ &\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(a_j+sb)})\\&+Q(r,f)+Q(r,g).\end{aligned}$$

We will prove that $f(z)-g(z)\not\equiv sb$, $s=1,2,\ldots,l-1$. Suppose that $f(z)-g(z)\equiv sb$, $s=1,2,\ldots,l-1.$, we get that a_j $(j=1,2,\ldots,q)$ are the Picard exceptional values of f, and that $a_j+(l-1)b$ $(j=1,2,\ldots,q)$ are the Picard exceptional values of g in Ω. By Lemma 2.3, we can get a contradiction. Similarly, we have $g(z)-f(z)\not\equiv sb$, $s=1,2,\ldots,l-1$.

By using (7) and condition (4), we have

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)})\\\le\ &\overline{\mathfrak{N}}(r,\frac{1}{f-g})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{f-g-sb})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{g-f-sb})\\\le\ &(2l-1)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1).\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(a_j+sb)})\\\le\ &\overline{\mathfrak{N}}(r,\frac{1}{f-g})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{f-g-sb})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{g-f-sb})\\\le\ &(2l-1)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1).\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\<\ &(2l-1)\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+Q(r,f)+Q(r,g),\end{aligned}$$

that is,

$$(A_1-\varepsilon)\,\mathfrak{T}(r,f)+(A_2-\varepsilon)\,\mathfrak{T}(r,g)<Q(r,f)+Q(r,g).$$

Since f and g are transcendental in Tsuji sense and ε is arbitrary, the above inequality contradicts the conditions (4).

Therefore, the proof of Theorem 1.5 is completed

4 The proof of Theorem 1.6

Suppose that $(f(z)-c)^l \not\equiv (g(z)-c)^l$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta_T(0,f-d)>0$ and $d\notin\{c+a_jw^s : j=1,2,\ldots,q\,\text{and}\,s=0,1,\ldots,l-1\}$. We denote them by d_k $(k=1,2,\ldots,\infty)$. Obviously, $\Theta_T(f)=\sum_{k=1}^{\infty}\Theta_T(0,f-d_k)$. Thus there exits a p such that $\sum_{k=1}^{p}\Theta_T(0,f-d_k)>\Theta_T(f)-\varepsilon$ holds for any given $\varepsilon(>0)$.

Using a similar discussion as in the proof of Theorem 1.5, we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\<\ &\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(c+a_jw^s)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(c+a_jw^s)})\\&+Q(r,f)+Q(r,g),\end{aligned}$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta_T(0,f-(c+a_jw^s))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,f-(c+a_jw^s))}{k_j+1}+\Theta_T(f)-2,$$

$$B_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0,g-(c+a_jw^s))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,g-(c+a_jw^s))}{k_j+1}+\Theta_T(g)-2.$$

Furthermore, from the condition (5), (7) and Lemma 2.1, we have

$$\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(c+a_jw^s)})<\overline{\mathfrak{N}}(r,\frac{1}{(f-c)^l-(g-c)^l})\le l(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1),$$

and

$$\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(c+a_jw^s)})<\overline{\mathfrak{N}}(r,\frac{1}{(f-c)^l-(g-c)^l})\le l(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1).$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\<\ &l\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+Q(r,f)+Q(r,g),\end{aligned}$$

that is,

$$(A_1-\varepsilon)\,\mathfrak{T}(r,f)+(A_2-\varepsilon)\,\mathfrak{T}(r,g)<Q(r,f)+Q(r,g).$$

Since f and g are transcendental and ε is arbitrary, the above inequality contradicts (6). Therefore, the proof of Theorem 1.6 is completed.

5 Remarks

Zheng [21] had proved the results related to Tsuji's characteristic and Nevanlinna's characteristic as follows

Lemma 5.1 *(see [21, lemma 2.3.3]) Let $f(z)$ be a meromorphic function in $\Omega(\alpha,\beta)$, for any real number $\varepsilon>0$, $\Omega_\varepsilon=\Omega(\alpha+\varepsilon,\beta-\varepsilon)$. Then for $\varepsilon>0$, we have*

$$\mathfrak{N}(r,f)\le\omega\frac{N(r,\Omega,f)}{r^\omega}+\omega^2\int_1^r\frac{N(t,\Omega,f)}{t^{\omega+1}}dt,$$

and

$$\mathfrak{N}(r,f)\ge\omega c^\omega\frac{N(cr,\Omega_\varepsilon,f)}{r^\omega}+\omega^2c^\omega\int_1^{cr}\frac{N(t,\Omega_\varepsilon,f)}{t^{\omega+1}}dt$$

where $0<c<1$ is a constant depending on ε, $\omega=\frac{\pi}{\beta-\alpha}$ and $N(t,\Omega,f)=\int_1^r\frac{n(t,\Omega,f)}{t}dt$, $n(t,\Omega,f)$ is the number of poles of $f(z)$ in $\Omega\cap\{z:1<|z|\le t\}$.

From Lemma 5.1, we can get that f is transcendental in Tsuji sense if f satisfies condition (9). Thus, we can get the following results

Theorem 5.2 *Let the assumptions of Theorems 1.5-1.6 and Corollaries 1.1-1.4 be given with the exception of that $f(z)$ is transcendental in Tsuji sense. Assume that for some $a\in\widehat{\mathbb{C}}$ and $\varepsilon>0$,*

$$\limsup_{r\to\infty}\frac{N(r,\Omega_\varepsilon,f=a)}{r^\omega\log r}=\infty, \tag{9}$$

where $\omega, N(t,\Omega,f)$ are stated as in Lemma 5.1. Then $f(z)\equiv g(z)$.

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A new generalization of Fibonacci and Lucas $p-$numbers

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Abstract

In this paper, we define a new generalization of the Fibonacci and Lucas $p-$numbers. Further, we build up the tree diagrams for generalized Fibonacci and Lucas $p-$sequence and derive the recurrence relations of these sequences by using these diagrams. Also, we show that the generalized Fibonacci and Lucas $p-$sequences can be reduced into the various number sequences. Finally, we develop Binet formulas for the generalized Fibonacci and Lucas $p-$numbers and present the numerical and graphical results, which obtained by means of the Binet formulas, for specific values of a, b and p.

Keywords: The generalized Fibonacci $p-$numbers, The generalized Lucas $p-$numbers, Binet formula.
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1. Introduction

Fibonacci and Lucas sequences are one of the most popular and fascinating sequences that arise in various situations, especially in mathematics, physics and related fields. The classical Fibonacci and Lucas sequences are defined by $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$, for $n \in \mathbb{N}$, with initial conditions $F_0 = 0, \quad F_1 = 1$ and $L_0 = 2, \quad L_1 = 1$, respectively. One of the most important sources of this area is [1], which was written by Thomas Koshy, and contains numerous applications, generalizations and recurrence relations of Fibonacci and Lucas numbers. In recent years, many authors have studied generalizations of the Fibonacci and Lucas sequences [2–12]. For instance, in [8, 10] the authors defined the generalized Fibonacci $\{q_n\}_{n\in N_0}$ sequence as

$$q_0 = 0, \quad q_1 = 1, \quad q_{n+2} = \begin{cases} aq_{n+1} + q_n, \text{if } n \equiv 0 \pmod 2 \\ bq_{n+1} + q_n, \text{if } n \equiv 1 \pmod 2, \end{cases} \tag{1}$$

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and the generalized Lucas $\{l_n\}_{n\in N_0}$ sequence as in the form

$$l_0 = 2, \quad l_1 = a, \quad l_{n+2} = \begin{cases} bl_{n+1} + l_n, \text{if } n \equiv 0 \pmod 2 \\ al_{n+1} + l_n, \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{2}$$

Stakhov and Rozin introduced Fibonacci and Lucas $p-$numbers, one of the most significant mathematical discoveries of the modern Fibonacci numbers theory, and they presented some properties of this sequence,

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{3}$$

and

$$L_p(n) = L_p(n-1) + L_p(n-p-1), \tag{4}$$

in [13], with the initial conditions $F_p(0) = 0, F_p(1) = 1, F_p(2) = 1, \ldots, F_p(p) = 1$ and $L_p(0) = p+1, L_p(1) = 1, L_p(2) = 1, \ldots, L_p(p) = 1$, respectively. After that, Kocer et al. defined the $m-$extension of the Fibonacci and Lucas $p-$numbers,

$$F_{p,m}(n+p+1) = mF_{p,m}(n+p) + F_{p,m}(n) \tag{5}$$

and

$$L_{p,m}(n+p+1) = mL_{p,m}(n+p) + L_{p,m}(n), \tag{6}$$

with initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, F_{p,m}(3) = m^2, \ldots, F_{p,m}(p+1) = m^p$ and $L_{p,m}(0) = p+1, L_{p,m}(1) = m, L_{p,m}(2) = m^2, L_{p,m}(3) = m^3, \ldots, L_{p,m}(p+1) = m^{p+1}$, where p and n are nonnegative integers and m is a positive real number [14]. The main purpose of the present article is to give a wider generalization of the generalized Fibonacci and Lucas sequence given by (1) and (2), the Fibonacci and Lucas p-sequences given by (3) and (4) and the $m-$extension of the Fibonacci and Lucas p-sequences given by (5) and (6) to introduce a new class of the recurrence numerical sequences called the *generalization of Fibonacci and Lucas $p-$numbers.*

2. Generalized Fibonacci and Lucas $p-$numbers

Definition 2.1. For any positive real numbers a, b and positive integer p, the generalized Fibonacci $p-$sequence $\{f_n\}_{n=0}^{\infty}$ and Lucas $p-$sequence $\{\ell_n\}_{n=0}^{\infty}$ are defined recursively by

$$f_n = \begin{cases} af_{n-1} + f_{n-p-1}, \text{if } n \equiv 0 \pmod 2 \\ bf_{n-1} + f_{n-p-1}, \text{if } n \equiv 1 \pmod 2, \end{cases} \quad \text{and} \quad \ell_n = \begin{cases} b\ell_{n-1} + \ell_{n-p-1}, \text{if } n \equiv 0 \pmod 2 \\ a\ell_{n-1} + \ell_{n-p-1}, \text{if } n \equiv 1 \pmod 2, \end{cases}$$

where $n \geq p+1$ and the initial conditions of f_n and ℓ_n are

$$f_0 = 0, f_1 = 1, f_2 = a, \ldots, f_p = a^{\lfloor \frac{p}{2} \rfloor} b^{\lfloor \frac{p-1}{2} \rfloor} \tag{7}$$

and

$$\ell_0 = p+1, \ell_1 = a, \ell_2 = ab, \ldots, \ell_p = a^{\lfloor \frac{p+1}{2} \rfloor} b^{\lfloor \frac{p}{2} \rfloor}, \tag{8}$$

respectively.

Note that, these sequences can be reduced to different sequences for specific values of p, a and b. It is not difficult to see from the following table that Fibonacci, Lucas, Pell, Pell−Lucas, $k-$Fibonacci, $k-$Lucas, Fibonacci p, Lucas p, Pell p, Pell−Lucas p, $m-$extension of Fibonacci p and $m-$extension of Lucas p-sequences are special cases of generalized Fibonacci and Lucas $p-$sequence.

p	a	b	f_n	ℓ_n
1	1	1	Classical Fibonacci sequence F_n	Classical Lucas sequence L_n
1	2	2	Classical Pell sequence P_n	Classical Pell-Lucas sequence Q_n
1	k	k	$k-$Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$	$k-$Lucas numbers $\{L_{k,n}\}_{n=0}^{\infty}$
p	1	1	Fibonacci $p-$sequence $F_{p,n}$	Lucas $p-$sequence $L_{p,n}$
p	2	2	Pell $p-$sequence $F_{p,n}$	Pell-Lucas $p-$sequence $L_{p,n}$
p	m	m	$m-$extension of Fibonacci $p-$numbers $F_{p,m,n}$	$m-$extension of Lucas $p-$numbers $L_{p,m,n}$

Let a and b be positive real numbers, p be a positive integer and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$. We can construct the tree diagrams for the generalized Fibonacci and Lucas $p-$numbers as:

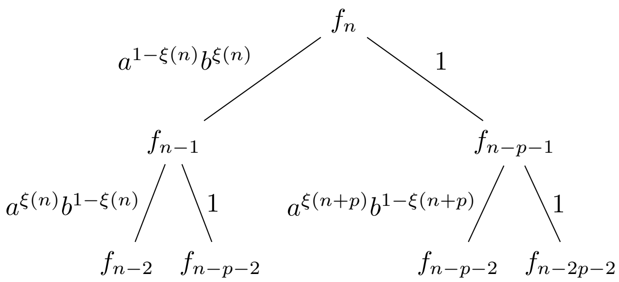

Figure 1: Tree diagram for generalized Fibonacci $p-$numbers

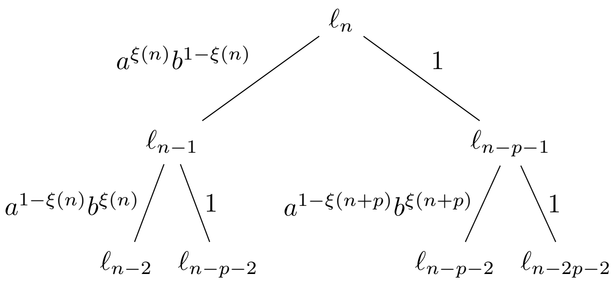

Figure 2: Tree diagram for generalized Lucas $p-$numbers.

By considering Figures 1 and 2, we will derive the recurrence relations for f_n and ℓ_n. First we suppose that p is even. Then, $\{f_n\}$ satisfies the recurrence relation

$$\begin{aligned} f_n &= a^{1-\xi(n)}b^{\xi(n)}f_{n-1} + f_{n-p-1} \\ &= a^{1-\xi(n)}b^{\xi(n)}\left(a^{\xi(n)}b^{1-\xi(n)}f_{n-2} + f_{n-p-2}\right) + a^{\xi(n+p)}b^{1-\xi(n+p)}f_{n-p-2} + f_{n-2p-2} \\ &= abf_{n-2} + a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2} + a^{\xi(n)}b^{1-\xi(n)}f_{n-p-2} + f_{n-2p-2} \\ &= abf_{n-2} + \left(a^{1-\xi(n)}b^{\xi(n)} + a^{\xi(n)}b^{1-\xi(n)}\right)f_{n-p-2} + f_{n-2p-2} \\ &= abf_{n-2} + (a+b)\,f_{n-p-2} + f_{n-2p-2}. \end{aligned}$$

Next, we suppose that p is odd. Then, f_n also satisfies the recurrence relation

$$\begin{aligned}
f_n &= a^{1-\xi(n)}b^{\xi(n)}f_{n-1}+f_{n-p-1}\\
&= a^{1-\xi(n)}b^{\xi(n)}\left(a^{\xi(n)}b^{1-\xi(n)}f_{n-2}+f_{n-p-2}\right)+a^{\xi(n+p)}b^{1-\xi(n+p)}f_{n-p-2}+f_{n-2p-2}\\
&= abf_{n-2}+a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2}+a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2}+f_{n-2p-2}\\
&= abf_{n-2}+2a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2}+f_{n-2p-2}\\
&= abf_{n-2}+2\left(f_{n-p-1}-f_{n-2p-2}\right)+f_{n-2p-2}\\
&= abf_{n-2}+2f_{n-p-1}-f_{n-2p-2}.
\end{aligned}$$

In a similar way, we can easily obtain the same recurrence relation for ℓ_n. Let

$$\alpha_n=\begin{cases} f_n, & \text{if } \alpha_0=0,\alpha_1=1,\alpha_2=a,\ldots,\alpha_p=a^{\lfloor \frac{p}{2}\rfloor}b^{\lfloor \frac{p-1}{2}\rfloor}\\ \ell_n, & \text{if } \alpha_0=p+1,\alpha_1=a,\alpha_2=ab,\ldots,\alpha_p=a^{\lfloor \frac{p+1}{2}\rfloor}b^{\lfloor \frac{p}{2}\rfloor}\end{cases}$$

be a sequence that satisfies both f_n and ℓ_n. Thereby, α_n satisfies the recurrence relation

$$\alpha_n=\begin{cases} ab\alpha_{n-2}+(a+b)\,\alpha_{n-p-2}+\alpha_{n-2p-2}, & \text{if } p \text{ is even,}\\ ab\alpha_{n-2}+2\alpha_{n-p-1}-\alpha_{n-2p-2}, & \text{if } p \text{ is odd.}\end{cases} \tag{9}$$

By considering eq. (9), the characteristic polynomial of α_n is

$$\alpha_p(x)=\begin{cases} x^{2p+2}-abx^{2p}-(a+b)\,x^p-1, & \text{if } p \text{ is even,}\\ x^{2p+2}-abx^{2p}-2x^{p+1}+1, & \text{if } p \text{ is odd.}\end{cases} \tag{10}$$

By taking $r=x^2$, we can express the characteristic equation (10) as

$$\beta_p(r)=\begin{cases} r^{p+1}-abr^p-(a+b)\,r^{\frac{p}{2}}-1, & \text{if } p \text{ is even,}\\ r^{p+1}-abr^p-2r^{\frac{p+1}{2}}+1, & \text{if } p \text{ is odd.}\end{cases} \tag{11}$$

Lemma 2.1. Assume that p is odd. Then the characteristic equation of the generalized Fibonacci and Lucas $p-$numbers $\alpha_p(x)$ does not have multiple roots.

For the other case, it can easily seen that there are no multiple real roots. However, whether there exists complex multiple roots or not is an open problem, and we suggest that interested readers study it with us.

Proof of Lemma. The characteristic equation of the generalized Fibonacci and Lucas $p-$numbers for odd p can be written in the form

$$\alpha_p(x)=(x^{p+1}-1)^2-abx^{2p}$$

and its derivative is

$$\alpha_p'(x)=2(p+1)x^p(x^{p+1}-1)-2pabx^{2p-1}.$$

Then, $\alpha_p(x) = 0$ if and only if

$$ab = \frac{(x^{p+1} - 1)^2}{x^{2p}}$$

and $\alpha_p'(x) = 0$ if and only if

$$ab = \frac{(p+1)x^p(x^{p+1} - 1)}{px^{2p-1}} = \frac{(p+1)x}{p}\frac{x^{p+1} - 1}{x^p}.$$

Upon simplifying, we obtain $ab = \left(\frac{p+1}{p}\right) x\sqrt{ab}$. Therefore, $\alpha_p(x)$ and $\alpha_p'(x)$ vanish for the same x if and only if for some root x of $\alpha_p(x)$,

$$ab = \left(\frac{p+1}{p}\right) x\sqrt{ab} \quad \text{or equivalently, } x = \frac{p\sqrt{ab}}{p+1}.$$

So, for every p and ab, if such an x is a root, it is a multiple root. Let t be a multiple root. Then, $ab = \left(\frac{p+1}{p}\right)^2 t^2$. Since $\alpha_p(t) = 0$, we have

$$\begin{aligned} t^{2p+2} - \left(\frac{p+1}{p}\right)^2 t^{2p+2} - 2t^{p+1} + 1 = 0 \\ -\frac{(2p+1)}{p^2} t^{2p+2} - 2t^{p+1} + 1 = 0 \\ (2p+1)t^{2(p+1)} + 2p^2 t^{p+1} - p^2 = 0. \end{aligned}$$

When treated as a quadratic equation in t^{p+1}, the discriminant is

$$4p^4 + 4p^2(2p+1) = 4p^2(p+1)^2,$$

and therefore, the solutions are

$$t^{p+1} = -p, \frac{p}{2p+1}.$$

But substituting the same ab in $\alpha_p'(t) = 0$, we get

$$\begin{aligned} &2(p+1)t^p(t^{p+1} - 1) - 2p\left(\frac{p+1}{p}\right)^2 t^{2p+1} = 0 \\ &p(t^{p+1} - 1) - (p+1)t^{p+1} = 0 \\ &t^{p+1} = -p \end{aligned}$$

Then, by $ab = \left(\frac{p+1}{p}\right)^2 t^2$, we have

$$ab = \frac{(p+1)^2}{(-p)^{\frac{2p}{p+1}}}.$$

The equation has multiple roots exactly when ab and p are related as above. Note that for odd values of p, ab will be a real number, and then, $ab < 0$. This is a contradiction. Therefore the characteristic equation $\alpha_p(x)$ has distinct roots. The proof is complete. □

We will describe the terms of the sequence $\{\alpha_n\}$ clearly by using the Binet formula. So, we can give the generalized Binet formula for the generalized Fibonacci and Lucas $p-$numbers with the following theorem.

Theorem 2.1. *Suppose that the characteristic equation* (11) *has* $(p+1)$ *distinct roots,* $r_1, r_2, \ldots, r_{p+1}$. *Then* α_n *satisfies the relation*

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\displaystyle\sum_{\substack{1\leq k_1<k_2<\ldots<k_{p+1-j}\leq p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\displaystyle\prod_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-2+\xi(n)} + \frac{1}{\displaystyle\prod_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2p+\xi(n)}\right) r_i^{\lfloor \frac{n}{2}\rfloor}.$$

Proof. Let $r_1, r_2, \ldots, r_{p+1}$ be $(p+1)$ distinct roots of the characteristic equation (11). There are $(2p+2)$ coefficients $k_1, k_2, \ldots, k_{2p+1}, k_{2p+2}$ such that

$$\alpha_n = k_1(\sqrt{r_1})^n + k_2(-\sqrt{r_1})^n + k_3(\sqrt{r_2})^n + k_4(-\sqrt{r_2})^n + \cdots + k_{2p+1}(\sqrt{r_{p+1}})^n + k_{2p+2}(-\sqrt{r_{p+1}})^n.$$

First, we suppose that n is even. Then we obtain

$$\begin{aligned}\alpha_n &= (k_1+k_2)(\sqrt{r_1})^n + (k_3+k_4)(\sqrt{r_2})^n + \cdots + (k_{2p+1}+k_{2p+2})(\sqrt{r_{p+1}})^n \\ &= \sum_{i=1}^{p+1}(k_{2i-1}+k_{2i})(\sqrt{r_i})^n. \end{aligned} \tag{12}$$

In order to determine the coefficients $k_1, k_2, \ldots, k_{2p+1}, k_{2p+2}$, we must solve the linear equation system $\mathbf{V}\boldsymbol{\gamma} = \boldsymbol{\alpha}$, where $\mathbf{V}_{i,j} = r_j^{i-1}$ is a Vandermonde matrix, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{p+1})^T$ and $\boldsymbol{\alpha} = (\alpha_0, \alpha_2, \ldots, \alpha_{2p})^T$ are the column vectors. Considering the cases $n = 0, 2, 4, \ldots, 2p$ in (12), we have the linear equation system

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ r_1 & r_2 & r_3 & \ldots & r_{p+1} \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_{p+1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^p & r_2^p & r_3^p & \ldots & r_{p+1}^p \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p+1} \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_2 \\ \alpha_4 \\ \vdots \\ \alpha_{2p} \end{pmatrix}, \tag{13}$$

where $\gamma_p = k_{2p-1} + k_{2p}$. The $(p+1)\times(p+1)$ Vandermonde matrix can be factorized as $\mathbf{V} = \mathbf{LU}$, (see [15]), where $\mathbf{L}$ is a lower triangular matrix with units on its main diagonal, (i,j)-th element of $\mathbf{L}$ is

$$\mathbf{L}_{i,j} = \begin{cases} 1 & , \text{ if } i = j\ , \\ \displaystyle\sum_{1\leq k_1\leq k_2\leq\ldots\leq k_{i-j}\leq j} r_{k_1}r_{k_2}\ldots r_{k_j} & , \text{ if } i > j \geq 1, \\ 0 & , \text{ if } i < j\ , \end{cases}$$

and $\mathbf{U}$ is an upper triangular matrix, (i,j)-th element of $\mathbf{U}$ is

$$\mathbf{U}_{i,j}=\begin{cases}1 & , \text{ if } i=1\ ,\\ \prod\limits_{\substack{k=1\\k\neq j}}^{i-1}(r_j-r_k) & , \text{ if } i\leq j,\\ 0 & , \text{ if } i>j\ .\end{cases}$$

Now we suppose that $\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\cdots+\lambda_p\mathbf{v}_p+\lambda_{p+1}\mathbf{v}_{p+1}=(0,0,\ldots,0)$, where $\mathbf{v}_k=(1,r_k^1,r_k^2,\ldots,r_k^p)$ is the k-th column vector and $\lambda_1,\lambda_2,\ldots,\lambda_{p+1}$ are real numbers. Then the k-th coordinate

$$\lambda_1+\lambda_2 r_k+\lambda_3 r_k^2+\ldots+\lambda_{p+1}r_k^p=0,$$

which means that r_k is a zero of the polynomial $\xi(r)=\lambda_1+\lambda_2 r+\lambda_3 r^2+\ldots+\lambda_{p+1}r^p$. If the polynomial $\xi(r)$ of degree at most p has $(p+1)$ distinct zeros $r_1,r_2,\ldots,r_{p+1}$ then it must be zero polynomial and we get $\lambda_1=\lambda_2=\ldots=\lambda_{p+1}=0$. So it is easily seen that the vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{p+1}$ are linearly independent. This proves that $\mathbf{V}$ is invertible. So, we can factorize the inverse of the Vandermonde matrix as $\mathbf{V}^{-1}=\mathbf{U}^{-1}\mathbf{L}^{-1}$, (see [16]), where $\mathbf{L}^{-1}$ is a lower triangular matrix with units on its main diagonal, (i,j)-th element of $\mathbf{L}^{-1}$ is

$$\mathbf{L}^{-1}_{i,j}=\begin{cases}1 & , \text{ if } i=j\ ,\\ \mathbf{L}^{-1}_{i-1,j-1}-\mathbf{L}^{-1}_{i-1,j}r_{i-1} & , \ i=2,3,\ldots,p+1; j=2,3,\ldots,i-1,\\ 0 & , \text{ if } i<j\ ,\end{cases}$$

and $\mathbf{U}^{-1}$ is an upper triangular matrix, (i,j)-th element of $\mathbf{U}^{-1}$ is

$$\mathbf{U}^{-1}_{i,j}=\begin{cases}\prod\limits_{\substack{k=1\\k\neq i}}^{j}\dfrac{1}{r_i-r_k} & , \text{ if } i\leq j,\\ 0 & , \text{ if } i>j\ .\end{cases}$$

Therefore (i,j)-th element of $\mathbf{V}^{-1}$ can be written as

$$\mathbf{V}^{-1}_{i,j}=\begin{cases}(-1)^j\dfrac{\sum\limits_{\substack{1\leq k_1<k_2<\ldots<k_{p+1-j}\leq p+1\\k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\leq k\leq p+1\\k\neq i}}(r_i-r_k)} & , \text{ if } 1\leq j<p+1,\\ \dfrac{1}{\prod\limits_{\substack{1\leq k\leq p+1\\k\neq i}}(r_i-r_k)} & , \text{ if } j=p+1\ .\end{cases}$$

Since $\boldsymbol{\gamma} = \mathbf{V}^{-1}\boldsymbol{\alpha}$, the i-th element of $\boldsymbol{\gamma}$ is $\boldsymbol{\gamma_i} = \sum_{j=1}^{p+1} \mathbf{V}_{i,j}^{-1}\alpha_{2j-2}$. So, we have

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\displaystyle\sum_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\displaystyle\prod_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-2} + \frac{\alpha_{2p}}{\displaystyle\prod_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\frac{n}{2}}. \quad (14)$$

Next, we suppose that n is odd. Then we obtain

$$\begin{aligned}\alpha_n &= (k_1-k_2)(\sqrt{r_1})^n + (k_3-k_4)(\sqrt{r_2})^n + \cdots + (k_{2p+1}-k_{2p+2})(\sqrt{r_{p+1}})^n \\ &= \sum_{i=1}^{p+1}(k_{2i-1}-k_{2i})(\sqrt{r_i})^n. \end{aligned} \quad (15)$$

Considering the cases $n = 1, 3, 5, \ldots, 2p+1$ in (15), we have

$$\begin{pmatrix} \sqrt{r_1} & \sqrt{r_2} & \sqrt{r_3} & \cdots & \sqrt{r_{p+1}} \\ \sqrt{r_1}^3 & \sqrt{r_2}^3 & \sqrt{r_3}^3 & \cdots & \sqrt{r_{p+1}}^3 \\ \sqrt{r_1}^5 & \sqrt{r_2}^5 & \sqrt{r_3}^5 & \cdots & \sqrt{r_{p+1}}^5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{r_1}^{2p+1} & \sqrt{r_2}^{2p+1} & \sqrt{r_3}^{2p+1} & \cdots & \sqrt{r_{p+1}}^{2p+1} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_5 \\ \vdots \\ \alpha_{2p+1} \end{pmatrix}, \quad (16)$$

where $\gamma_p = k_{2p-1} - k_{2p}$. It can be easily seen that $\mathbf{L}_{i,j}$ and $\mathbf{L}_{i,j}^{-1}$ are the same as in the previous case. In a similar way, we can find the $(i,j)-$th element of the matrices $\mathbf{U}_{i,j}$ and $\mathbf{U}_{i,j}^{-1}$, respectively, as

$$\mathbf{U}(i,j) = \begin{cases} \sqrt{r_j} & , \text{ if } i = 1 \ , \\ \sqrt{r_j}\displaystyle\prod_{\substack{k=1\\ k\neq j}}^{i-1}(r_j-r_k) & , \text{ if } i \le j, \\ 0 & , \text{ if } i > j \end{cases} \quad \text{and} \quad \mathbf{U}_{i,j}^{-1} = \begin{cases} \displaystyle\prod_{\substack{k=1\\ k\neq i}}^{j} \frac{1}{\sqrt{r_i}(r_i-r_k)} & , \text{ if } i \le j, \\ 0 & , \text{ if } i > j \ . \end{cases}$$

Using these identities, we find the $(i,j)-$th element of $\mathbf{V}^{-1}$ as

$$\mathbf{V}_{i,j}^{-1} = \begin{cases} (-1)^j \dfrac{\displaystyle\sum_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\sqrt{r_i}\displaystyle\prod_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)} & , \text{ if } 1 \le j < p+1, \\ \dfrac{1}{\sqrt{r_i}\displaystyle\prod_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)} & , \text{ if } j = p+1 \ . \end{cases}$$

Thus, α_n satisfies the recurrence relation

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\leq k_1<k_2<\ldots<k_{p+1-j}\leq p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-1} + \frac{\alpha_{2p+1}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\frac{n-1}{2}}. \quad (17)$$

Finally, by combining (14) and (17), we have the generalized Binet formula

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\leq k_1<k_2<\ldots<k_{p+1-j}\leq p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-2+\xi(n)} + \frac{\alpha_{2p+\xi(n)}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\lfloor\frac{n}{2}\rfloor}, \quad (18)$$

which proves the theorem. □

Corollary 2.1. *If we take the initial conditions* $\left\{\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = a, \ldots, \alpha_p = a^{\lfloor\frac{p}{2}\rfloor}b^{\lfloor\frac{p-1}{2}\rfloor}\right\}$ *in* (18)*, we obtain the Binet formula of the generalized Fibonacci* $p-$*numbers as*

$$f_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\leq k_1<k_2<\ldots<k_{p+1-j}\leq p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}f_{2j-2+\xi(n)} + \frac{f_{2p+\xi(n)}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\lfloor\frac{n}{2}\rfloor}. \quad (19)$$

If we take the initial conditions $\left\{\alpha_0 = p+1, \alpha_1 = a, \alpha_2 = ab, \ldots, \alpha_p = a^{\lfloor\frac{p+1}{2}\rfloor}b^{\lfloor\frac{p}{2}\rfloor}\right\}$ *in* (18)*, we obtain the Binet formula of the generalized Lucas* $p-$*numbers as*

$$\ell_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\leq k_1<k_2<\ldots<k_{p+1-j}\leq p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\ell_{2j-2+\xi(n)} + \frac{\ell_{2p+\xi(n)}}{\prod\limits_{\substack{1\leq k\leq p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\lfloor\frac{n}{2}\rfloor}. \quad (20)$$

3. Examples

In this section, we present the numerical results of generalized Fibonacci and Lucas $p-$sequence by specifying p, a and b.

3.1. Case $p = 1$

By considering (18), we obtain the Binet formulas of the generalized Fibonacci and Lucas $1-$numbers as

$$f_n = \left(\frac{f_{2+\xi(n)} - f_{\xi(n)}r_2}{r_1 - r_2}\right) r_1^{\lfloor\frac{n}{2}\rfloor} - \left(\frac{f_{2+\xi(n)} - f_{\xi(n)}r_1}{r_1 - r_2}\right) r_2^{\lfloor\frac{n}{2}\rfloor} \quad (21)$$

and

$$\ell_n = \left(\frac{\ell_{2+\xi(n)} - \ell_{\xi(n)} r_2}{r_1 - r_2}\right) r_1^{\lfloor \frac{n}{2} \rfloor} - \left(\frac{\ell_{2+\xi(n)} - \ell_{\xi(n)} r_1}{r_1 - r_2}\right) r_2^{\lfloor \frac{n}{2} \rfloor}, \tag{22}$$

where $r_1 = \left(\frac{ab+2-\sqrt{a^2b^2+4ab}}{2}\right)$ and $r_2 = \left(\frac{ab+2+\sqrt{a^2b^2+4ab}}{2}\right)$. We give the first few terms of the generalized Fibonacci and Lucas 1−numbers with the following table as

Table 1: Generalized Fibonacci and Lucas 1−numbers for different a and b

(a,b)	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
f_n	$\{0,1,1,2,3,5,8\ldots\}$	$\{0,1,1,3,4,11,15,\ldots\}$	$\{0,1,2,3,8,11,30,\ldots\}$	$\{0,1,2,5,12,29,70,\ldots\}$
ℓ_n	$\{2,1,3,4,7,11,18,\ldots\}$	$\{2,1,4,5,14,19,52,\ldots\}$	$\{2,2,4,10,14,38,52,\ldots\}$	$\{2,2,6,14,34,82,198,\ldots\}$

Moreover, we give the graphical illustration of the first 30 terms of the generalized Fibonacci and Lucas 1−numbers for different values of a and b, (see Figure 3 and Figure 4).

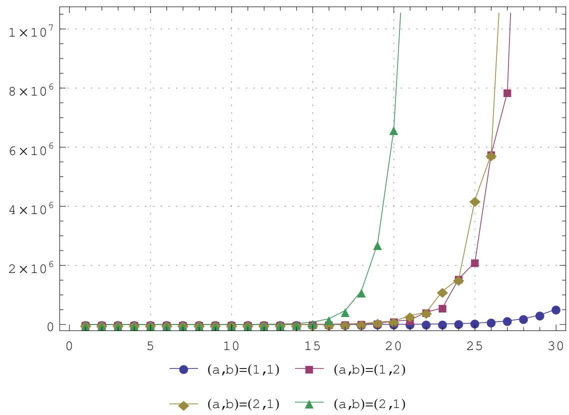

Figure 3: generalized Fibonacci 1−numbers

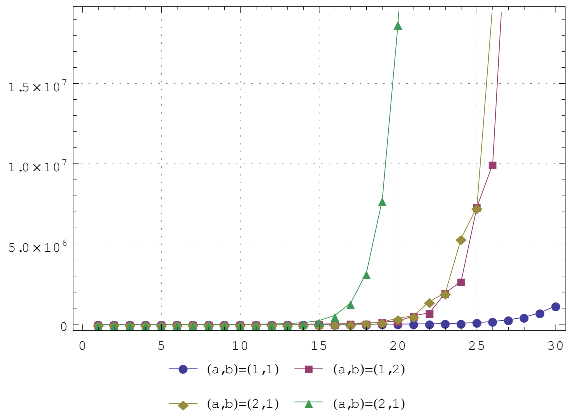

Figure 4: generalized Lucas 1−numbers

3.2. *Case* $p = 2$

Recall the equation (18). The Binet formulas for the generalized Fibonacci and Lucas 2−numbers are

$$\begin{aligned} f_n &= \left(\frac{f_{4+\xi(n)} - (r_2+r_3) f_{2+\xi(n)} + r_2 r_3 f_{\xi(n)}}{(r_1-r_3)(r_1-r_2)}\right) r_1^{\lfloor \frac{n}{2} \rfloor} + \left(\frac{f_{4+\xi(n)} - (r_1+r_3) f_{2+\xi(n)} + r_1 r_3 f_{\xi(n)}}{(r_2-r_3)(r_2-r_1)}\right) r_2^{\lfloor \frac{n}{2} \rfloor} \\ &+ \left(\frac{f_{4+\xi(n)} - (r_1+r_2) f_{2+\xi(n)} + r_1 r_2 f_{\xi(n)}}{(r_3-r_2)(r_3-r_1)}\right) r_3^{\lfloor \frac{n}{2} \rfloor} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \ell_n &= \left(\frac{\ell_{4+\xi(n)} - (r_2+r_3) \ell_{2+\xi(n)} + r_2 r_3 \ell_{\xi(n)}}{(r_1-r_3)(r_1-r_2)}\right) r_1^{\lfloor \frac{n}{2} \rfloor} + \left(\frac{\ell_{4+\xi(n)} - (r_1+r_3) \ell_{2+\xi(n)} + r_1 r_3 \ell_{\xi(n)}}{(r_2-r_3)(r_2-r_1)}\right) r_2^{\lfloor \frac{n}{2} \rfloor} \\ &+ \left(\frac{\ell_{4+\xi(n)} - (r_1+r_2) \ell_{2+\xi(n)} + r_1 r_2 \ell_{\xi(n)}}{(r_3-r_2)(r_3-r_1)}\right) r_3^{\lfloor \frac{n}{2} \rfloor}, \end{aligned} \tag{24}$$

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where

$$r_1 = \frac{\sqrt[3]{2a^3b^3+9a^2b+\sqrt{(2a^3b^3+9a^2b+9ab^2+27)^2-4(a^2b^2+3a+3b)^3}+9ab^2+27}}{3\sqrt[3]{2}} + \frac{\sqrt[3]{2}\left(a^2b^2+3a+3b\right)}{3\sqrt[3]{2a^3b^3+9a^2b+\sqrt{(2a^3b^3+9a^2b+9ab^2+27)^2-4(a^2b^2+3a+3b)^3}+9ab^2+27}} + \frac{ab}{3},$$

$$r_2 = \frac{\left(-1+i\sqrt{3}\right)\sqrt[3]{2a^3b^3+9a^2b+\sqrt{(2a^3b^3+9a^2b+9ab^2+27)^2-4(a^2b^2+3a+3b)^3}+9ab^2+27}}{6\sqrt[3]{2}} + \frac{\left(1+i\sqrt{3}\right)\left(-a^2b^2-3a-3b\right)}{3\ 2^{2/3}\sqrt[3]{2a^3b^3+9a^2b+\sqrt{(2a^3b^3+9a^2b+9ab^2+27)^2-4(a^2b^2+3a+3b)^3}+9ab^2+27}} + \frac{ab}{3}$$

and

$$r_3 = -\frac{\left(1+i\sqrt{3}\right)\sqrt[3]{2a^3b^3+9a^2b+\sqrt{(2a^3b^3+9a^2b+9ab^2+27)^2-4(a^2b^2+3a+3b)^3}+9ab^2+27}}{6\sqrt[3]{2}} + \frac{\left(1-i\sqrt{3}\right)\left(-a^2b^2-3a-3b\right)}{3\ 2^{2/3}\sqrt[3]{2a^3b^3+9a^2b+\sqrt{(2a^3b^3+9a^2b+9ab^2+27)^2-4(a^2b^2+3a+3b)^3}+9ab^2+27}} + \frac{ab}{3}.$$

We give the the first few terms of the generalized Fibonacci and Lucas $2-$numbers with the following table as

Table 2: Generalized Fibonacci and Lucas $2-$numbers for different a and b

(a,b)	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
f_n	$\{0,1,1,1,2,3,4\ldots\}$	$\{0,1,1,2,3,7,9,\ldots\}$	$\{0,1,2,2,5,7,16,\ldots\}$	$\{0,1,2,4,9,20,44,\ldots\}$
ℓ_n	$\{3,1,1,4,5,6,10,\ldots\}$	$\{3,1,2,5,11,13,31,\ldots\}$	$\{3,2,2,7,9,20,27,\ldots\}$	$\{3,2,4,11,24,52,115,\ldots\}$

Moreover, we give the graphical illustration of the first 30 terms of the generalized Fibonacci and Lucas $2-$numbers for different values of a and b, (see Figure 5 and Figure 6).

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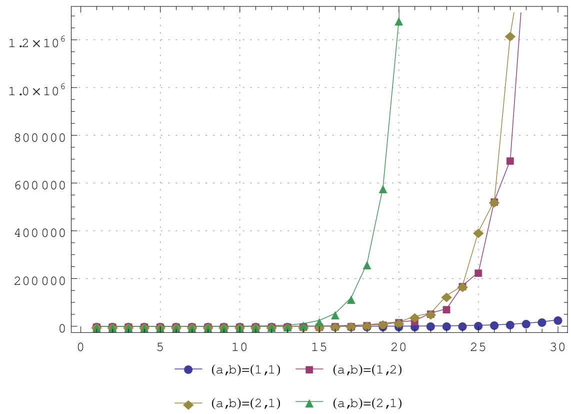

Figure 5: generalized Fibonacci 2−numbers

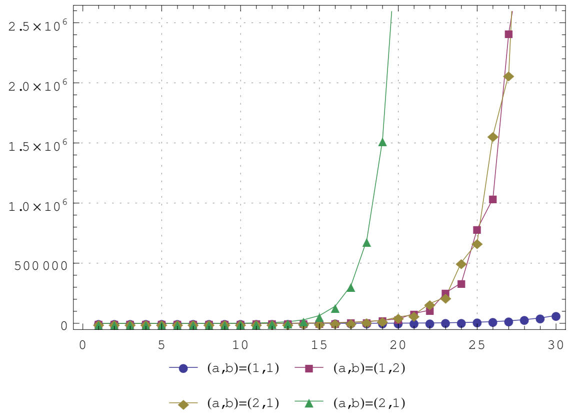

Figure 6: generalized Lucas 2−numbers

4. Conclusion

In this study, for the first time in literature we define a new generalization of Fibonacci and Lucas $p-$numbers. Also, we produce a Binet formula for these sequences by taking different initial conditions. The generalized Binet formula of the Fibonacci and Lucas $p-$numbers can be reduced to different Binet formulas in the literature. For example, if we take $p = 1$, we reduce the Binet formula of $\{f_n\}$ to the bi-periodic Fibonacci numbers $q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \frac{\alpha^n - \beta^n}{\alpha - \beta}$, in [8]. If we take $p = 2$, we reduce the Binet formulas of $\{f_n\}$ and $\{\ell_n\}$ to the γ_n, in [12], Theorem 2.1. As a result, this study contributes to the literature by providing essential information for the generalization of Fibonacci and Lucas $p-$numbers.

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SOME FAMILIES OF GENERATING FUNCTIONS FOR THE GENERALIZED CESÀRO POLYNOMIALS

NEJLA ÖZMEN AND ESRA ERKUŞ-DUMAN

ABSTRACT. The present study deals with some new properties for the generalized Cesàro polynomials. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella functions. Finally, we get several interesting results of this theorem.

1. INTRODUCTION

The Cesàro polynomials $g_n^{(s)}(x)$ are defined by the generating relation (see, for example,[2], p. 449, Problem 20)

$$\sum_{n=0}^{\infty} g_n^{(s)}(x)t^n = (1-t)^{-s-1}(1-xt)^{-1}. \tag{1.1}$$

It is from (1.1) that

$$g_n^{(s)}(x) = \binom{s+n}{n} {}_2F_1\left[-n, 1; -s-n; x\right], \tag{1.2}$$

where ${}_2F_1$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters ($p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned} {}_pF_q\left[\begin{array}{ll} \alpha_1, ..., \alpha_p; & \\ \beta_1, ..., \beta q; & z \end{array}\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ ... \ (\alpha_p)_n}{(\beta_1)_n \ ... \ (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q\left(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta q; z\right). \end{aligned}$$

Here $(\lambda)_\nu$ denotes the Pochhammer symbol defined (in terms of gamma function) by

$$\begin{aligned} (\lambda)_\nu &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ &= \begin{cases} 1, & \text{if } \nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\} \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathbb{N};\ \lambda \in \mathbb{C} \end{cases}. \end{aligned}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function.

Key words and phrases. Generalized Cesàro polynomials, generating function, multilinear and multilateral generating functions, recurrence relation, hypergeometric function.

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In addition, we have the following relationship between the Cesàro polynomials $g_n^{(s)}(x)$ and the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [2] :

$$g_n^{(s)}(x) = P_n^{(s+1,-s-n-1)}(2x-1).$$

In 2011, Lin et. al. [3] introduced the generalized Cesàro polynomials as follows:

$$g_n^{(s)}(\lambda, x) = \binom{s+n}{n} \, {}_2F_1\left[-n, \lambda; -s-n; x\right]. \tag{1.3}$$

It is noted that the special case $\lambda = 1$ of (1.3) reduces immediately to the Cesàro polynomials defined by (1.2).

The four Appell functions of two variables, denoted by F_1, F_2, F_3 and F_4, were generalized by Lauricella functions of n variables which are denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ [2] and

$$F_A^{(2)} = F_2,\ F_B^{(2)} = F_3,\ F_C^{(2)} = F_4,\ F_D^{(2)} = F_1.$$

A further generalization of the familiar Kampé de Fériet hypergeometric function in two variables is due to Srivastava and Daoust [4] who defined the generalized Lauricella (or the Srivastava-Daoust) function as follows:

$$F_{C:D^{(1)};...;D^{(n)}}^{A:B^{(1)};...;B^{(n)}}\left(\begin{array}{c} \left[(a):\theta^{(1)},...,\theta^{(n)}\right]: \quad \left[(b^{(1)}):\phi^{(1)}\right]; \quad ...; \quad \left[(b^{(n)}):\phi^{(n)}\right]; \\ \left[(c):\psi^{(1)},...,\psi^{(n)}\right]: \quad \left[(d^{(1)}):\delta^{(1)}\right]; \quad ...; \quad \left[(d^{(n)}):\delta^{(n)}\right]; \end{array} z_1,...,z_n\right)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \Omega(m_1,...,m_n) \frac{z_1^{m_1}}{m_1!} ... \frac{z_n^{m_n}}{m_n!},$$

where, for convenience,

$$\Omega(m_1,...,m_n) := \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta_j^{(1)}+...+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C} (c_j)_{m_1\psi_j^{(1)}+...+m_n\psi_j^{(n)}}} \frac{\prod_{j=1}^{B^{(1)}} (b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}} (d_j^{(1)})_{m_1\delta_j^{(1)}}} ... \frac{\prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$$\theta_j^{(k)} \ (j = 1, ..., A;\ k = 1, ..., n), \text{ and } \phi_j^{(k)} \ (j = 1, ..., B^{(k)};\ k = 1, ..., n),$$

$$\psi_j^{(k)} \ (j = 1, ..., C;\ k = 1, ..., n), \quad \text{and } \delta_j^{(k)} \ (j = 1, ..., D^{(k)};\ k = 1, ..., n)$$

are real constants and $\left(b_{B^{(k)}}^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)} \ (j = 1, ..., B^{(k)};\ k = 1, ..., n)$$

with similar interpretations for other sets of parameters [1].

For a suitably bounded non-vanishing multiple sequence $\{\Omega(m_1; m_2, ..., m_s)\}_{m_1,m_2,...,m_s \in \mathbb{N}_0}$ of real or complex parameters, let $\phi_n(u_1; u_2, ..., u_s)$ of s (real or complex) variables

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$u_1; u_2, ..., u_s$ defined by

$$\begin{aligned}\phi_n(u_1; u_2, ..., u_s) \quad : \quad &= \sum_{m_1=0}^{n} \sum_{m_2,...,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\ &\times \Omega\left(f(m_1, m_2, ..., m_s); m_2, ..., m_s\right) \frac{u_1^{m_1}}{m_1!} \cdots \frac{u_s^{m_s}}{m_s!} \qquad (1.4)\end{aligned}$$

where, for convenience,

$$((b))_{m_1\phi} = \prod_{j=1}^{B} (b_j)_{m_1\phi_j} \quad \text{and} \quad ((d))_{m_1\delta} = \prod_{j=1}^{D} (d_j)_{m_1\delta_j}.$$

The main object of this paper to study different properties of the generalized Cesàro polynomials. We give various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella functions.

2. Generating Functions

Theorem 2.1. *The generalized Cesàro polynomials* $g_n^{(s)}(\lambda, x)$ *have the following generating function:*

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) t^n = (1-t)^{-s-1}(1-xt)^{-\lambda}. \qquad (2.1)$$

Proof. If we denote the right-hand side of (2.1) by G, then we obtain

$$\begin{aligned} G &= \sum_{n=0}^{\infty} (s+1)_n \frac{t^n}{n!} \sum_{m=0}^{\infty} (\lambda)_m \frac{(xt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(s+1)_n (\lambda)_m}{n! m!} x^m t^{n+m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(s+1)_{n-m} (\lambda)_m}{(n-m)! m!} x^m t^n. \end{aligned}$$

By using the formula

$$(s+1)_{n-m} = \frac{(s+n-m)!}{s!},$$

we get

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(s+1)_{n-m} (\lambda)_m}{(n-m)! m!} x^m t^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{s+n}{n} (-n)_m \frac{1}{(-s-n)_m} \frac{(\lambda)_m}{m!} x^m t^n \\ &= \sum_{n=0}^{\infty} \binom{s+n}{n} {}_2F_1[-n, \lambda; -s-n; x] t^n \\ &= \sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) t^n. \end{aligned}$$

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□

Lemma 2.2. *The following generating function holds true:*

$$\sum_{n=0}^{\infty}\binom{n+m}{n}g_{n+m}^{(s)}(\lambda,x)t^{n}=(1-t)^{-s-m-1}(1-xt)^{-\lambda}g_{m}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right). \quad (2.2)$$

Proof. If we write $t+u$ instead of t in (2.1) , we get

$$\begin{aligned}\sum_{n=0}^{\infty}g_{n}^{(s)}(\lambda,x)(t+u)^{n} &= (1-t-u)^{-s-1}(1-xt-xu)^{-\lambda}\\ \sum_{n=0}^{\infty}g_{n}^{(s)}(\lambda,x)\sum_{m=0}^{n}\binom{n}{m}t^{n-m}u^{m} &= (1-t)^{-s-1}\left(1-\frac{u}{1-t}\right)^{-s-1}(1-xt)^{-\lambda}\left(1-\frac{xu}{1-xt}\right)^{-\lambda}.\end{aligned}$$

Replacing n by $n+m$ in last relation, we may write that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}g_{n+m}^{(s)}(\lambda,x)t^{n}u^{m}=(1-t)^{-s-1}(1-xt)^{-\lambda}\sum_{m=0}^{\infty}(1-t)^{-m}g_{m}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right)u^{m}$$

From the coefficients of u^m on the both sides of the equality, one can get the desired result. □

Lemma 2.3. *The following addition formula holds for the generalized Cesàro polynomials $g_n^{(s)}(\lambda,x)$:*

$$g_{n}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)=\sum_{k=0}^{n}g_{n-k}^{(s_1)}(\lambda_1,x)g_{k}^{(s_2)}(\lambda_2,x). \quad (2.3)$$

Proof. Replacing s by s_1+s_2+1 and λ by $\lambda_1+\lambda_2$ in (2.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}g_{n}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)t^{n} &= (1-t)^{-s_1-s_2-2}(1-xt)^{-(\lambda_1+\lambda_2)}\\ &= (1-t)^{-s_1-1}(1-xt)^{-\lambda_1}(1-t)^{s_2-1}(1-xt)^{-\lambda_2}\\ &= \sum_{n=0}^{\infty}g_{n}^{(s_1)}(\lambda_1,x)t^{n}\sum_{k=0}^{\infty}g_{k}^{(s_2)}(\lambda_2,x)t^{k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}g_{n}^{(s_1)}(\lambda_1,x)g_{k}^{(s_2)}(\lambda_2,x)t^{n+k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}g_{n-k}^{(s_1)}(\lambda_1,x)g_{k}^{(s_2)}(\lambda_2,x)t^{n}.\end{aligned}$$

From the coefficients of t^n on the both sides of the last equality, one can get the desired result. □

3. Bilinear and Bilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating fuctions for the generalized Cesàro polynomials $g_n^{(s)}(\lambda,x)$ given by (1.3) .

We begin by stating the following theorem.

Theorem 3.1. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* μ,ψ*, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k \qquad (a_k\neq 0) \tag{3.1}$$

and

$$\Theta_{n,p}^{\mu,\psi}(\lambda,x;y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}(\lambda,x)\,\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $p\in\mathbb{N}$*, we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(\lambda,x;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n=(1-t)^{-s-1}(1-xt)^{-\lambda}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.2}$$

provided that each member of (3.2) exists.

Proof. For convenience, let S denote the first member of the assertion (3.2) of Theorem 3.1. Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}(\lambda,x)\,\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n-pk}.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\, g_n^{(s)}(\lambda,x)\,\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n \\ &= \sum_{n=0}^{\infty}g_n^{(s)}(\lambda,x)\,t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k \\ &= (1-t)^{-s-1}(1-xt)^{-\lambda}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

Theorem 3.2. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* μ,ψ*, let*

$$\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;y_1,...,y_r;z):=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)\Omega_{\mu+\psi k}(y_1,...,y_r)z^k,$$

where $a_k\neq 0\,,n,p\in\mathbb{N}$ *and the notation* $[n/p]$ *means the greatest integer less than or equal* n/p.

Then, for $p\in\mathbb{N}$ *, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l g_{n-k}^{(s_1)}(\lambda_1,x)g_{k-pl}^{(s_2)}(\lambda_2,x)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l=\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;y_1,...,y_r;z) \tag{3.3}$$

provided that each member of (3.3) exists.

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Proof. For conveninace, let T denote the first member of the assertion (3.3). Then, upon substituting for the polynomials $g_n^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)$ from the (2.3) into the left-hand side of (3.3), we obtain

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l g_{n-k-pl}^{(s_1)}(\lambda_1,x)g_k^{(s_2)}(\lambda_2,x)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \sum_{l=0}^{[n/p]} a_l\left(\sum_{k=0}^{n-pl} g_{n-k-pl}^{(s_1)}(\lambda_1,x)g_k^{(s_2)}(\lambda_2,x)\right)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \sum_{l=0}^{[n/p]} a_l g_{n-pl}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;y_1,...,y_r;z). \end{aligned}$$

□

Theorem 3.3. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,p,q}\left[\lambda,x;y_1,...,y_r;t\right]:=\sum_{n=0}^{\infty} a_n g_{m+qn}^{(s)}(\lambda,x)\Omega_{\mu+pn}(y_1,...,y_r)t^n$$

where $a_n\neq 0$ and

$$\theta_{n,p,q}(y_1,...,y_r;z):=\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r)z^k.$$

Then, for $p\in\mathbb{N}$; we have

$$\begin{aligned} &\sum_{n=0}^{\infty} g_{m+n}^{(s)}(\lambda,x)\theta_{n,p,q}(y_1,...,y_r;z)t^n \\ &= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\Lambda_{\mu,p,q}\left(\lambda,\frac{x(1-t)}{1-xt};y_1,...,y_r;z(\frac{t}{1-t})^q\right) \quad (3.4) \end{aligned}$$

provided that each member of (3.4) exists.

Proof. For convenience, let T denote the first member of the assertion (3.4) of Theorem 3.3. Then,

$$T=\sum_{n=0}^{\infty} g_{m+n}^{(s)}(\lambda,x)\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r)z^k t^n.$$

Replacing n by $n+qk$ and then using (2.2), we may write that

$$\begin{aligned} T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+n+qk}{n} g_{m+n+qk}^{(s)}(\lambda,x)a_k\,\Omega_{\mu+pk}(y_1,...,y_r)z^k t^{n+qk} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{m+n+qk}{n} g_{m+n+qk}^{(s)}(\lambda,x)t^n\right)a_k\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k \\ &= \sum_{k=0}^{\infty}(1-t)^{-s-m-qk-1}(1-xt)^{-\lambda}g_{m+qk}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right)a_k\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k \\ &= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\sum_{k=0}^{\infty}(1-t)^{-qk}g_{m+qk}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right)a_k\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k \\ &= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\Lambda_{\mu,p,q}\left(\lambda,\frac{x(1-t)}{1-xt};y_1,...,y_r;z(\frac{t}{1-t})^q\right) \end{aligned}$$

which completes the proof. □

4. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r) = \Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)$$

in Theorem 3.1 , where the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ [5], generated by

$$\begin{gathered}(1-x_1t)^{-\alpha}e^{(x_2+...+x_r)\,t} = \sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)\,t^n \\ \left(\alpha\in\mathbb{C}\,;\ |t|<\left\{|x_1|^{-1}\right\}\right).\end{gathered} \tag{4.1}$$

Thus, we have the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ and the generalized Cesàro polynomials.

Corollary 4.1. *If*

$$\begin{aligned}\Lambda_{\mu,\psi}(y_1,...,y_r;w) \quad : &= \sum_{k=0}^{\infty}a_k\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)w^k \\ &\quad (a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})\,,\end{aligned}$$

then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}(\lambda,x)\,\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)\frac{w^k}{t^{pk}}t^n \\ &= (1-t)^{-s-1}(1-xt)^{-\lambda}\Lambda_{\mu,\psi}(y_1,...,y_r;w)\end{aligned} \tag{4.2}$$

provided that each member of (4.2) exists.

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Remark 4.1. *Using the generating relation (4.1) for the multivariable polynomials* $\Phi_n^{(\alpha)}(x_1,...,x_r)$ *and getting* $a_k=1$, $\mu=0$, $\psi=1$ *in Corollary 4.1, we find that*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} g_{n-pk}^{(s)}(\lambda,x)\,\Phi_k^{(\alpha)}(y_1,...,y_r)w^k t^{n-pk}\\ =\;&(1-t)^{-s-1}(1-xt)^{-\lambda}(1-y_1w)^{-\alpha}e^{(y_2+...+y_r)\;w},\\ &\left(|w|<\left\{|y_1|^{-1}\right\}\right).\end{aligned}$$

If we set $r=1$ and

$$\Omega_{\mu+\psi k}(y_1)=g_{\mu+\psi k}^{(s_3)}(\lambda_3,y)$$

in Theorem 3.2, we have the following bilinear generating functions for the generalized Cesàro polynomials.

Corollary 4.2. *If*

$$\begin{aligned}\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;\lambda_3,y;z)\quad:&=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)g_{\mu+\psi k}^{(s_3)}(\lambda_3,y)z^k\\ (a_k\quad\neq&\quad 0\,,\,\mu,\psi\in\mathbb{C})\end{aligned}$$

then, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l g_{n-k}^{(s_1)}(\lambda_1,x)g_{k-pl}^{(s_2)}(\lambda_2,x)g_{\mu+\psi l}^{(s_3)}(\lambda_3,y)z^l=\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;\lambda_3,y;z)\quad(4.3)$$

provided that each member of (4.3) exists.

Remark 4.2. *Using (4.3) and taking* $a_l=1$, $\mu=0$, $\psi=1$, $x=y$, $p=1$ *in Corollary 4.2, we have*

$$\sum_{n=0}^{\infty}\sum_{l=0}^{n}g_{n-l}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)g_l^{(s_3)}(\lambda_3,x)z^n=(1-z)^{-(s_1+s_2+s_3)-3}(1-xz)^{-(\lambda_1+\lambda_2+\lambda_3)}.$$

Finally, choosing

$$s=r \text{ and } \Omega_{\mu+\psi k}(y_1,...,y_r)=u_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$$

in Theorem 3.3, where the Erkus-Srivastava polynomials $u_n^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$ is generated by [7]

$$\begin{aligned}\prod_{j=1}^{r}\left\{(1-x_jt^{m_j})^{-\alpha_j}\right\}\quad&=\quad\sum_{n=0}^{\infty}u_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)\,t^n,\\ (\alpha_j\quad\in\quad&\mathbb{C}\;\;(j=1,...,r)\;;\;|t|<\min\left\{|x_1|^{-1/m_1},...,|x_r|^{-1/m_r}\right\}\end{aligned}$$

we get a family of the bilateral generating functions for the Erkus-Srivastava polynomials and the generalized Cesàro polynomials as follows:

Corollary 4.3. *If*

$$\begin{aligned}\Lambda_{m,q}\left[\lambda,x;y_1,...,y_r;z\right]\quad:&=\sum_{k=0}^{\infty}a_k g_{m+qk}^{(s)}(\lambda,x)u_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)z^k\\ &(a_k\neq 0,\;m\in\mathbb{N}_0,k\neq 0)\end{aligned}$$

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and

$$\theta^{p,\mu}_{n,m,q}(y_1,...,y_r;z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k u^{(\alpha_1,...,\alpha_r)}_{\mu+\psi k}(y_1,...,y_r) z^k$$

where $n,p \in \mathbb{N}$*, then we have*

$$\sum_{n=0}^{\infty} g^{(s)}_{m+n}(\lambda,x)\theta^{p,\mu}_{n,m,q}(y_1,...,y_r;z)t^n \tag{4.4}$$
$$= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\Lambda_{m,q}\left[\lambda, \frac{x(1-t)}{1-xt}; y_1,...,y_r; z\left(\frac{t}{1-t}\right)^q\right]$$

provided that each member of (4.4) exists.

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 3.1, Theorem 3.2, Theorem 3.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the generalized Cesáro polynomials given explicitly by (1.3).

5. Miscellaneous Properties

In this section we give some properties for the generalized Cesàro polynomials $g^{(s)}_n(\lambda,x)$ given by (1.3).

Theorem 5.1. *The generalized Cesàro polynomials* $g^{(s)}_n(\lambda,x)$ *have the following integral representation:*

$$g^{(s)}_n(\lambda,x) = \frac{1}{\Gamma(s+1)\Gamma(\lambda)}\int_0^\infty\int_0^\infty e^{-(u_1+u_2)}\frac{(u_1+u_2x)^n}{n!}u_1^s u_2^{\lambda-1}du_1du_2. \tag{5.1}$$

Proof. If we use the identity

$$a^{-v} = \frac{1}{\Gamma(v)}\int_0^\infty e^{-at}t^{v-1}dt, \qquad (Re(v)>0)$$

on the right-hand side of the generating function (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} g^{(s)}_n(\lambda,x)t^n &= \frac{1}{\Gamma(s+1)}\int_0^\infty e^{-(1-t)u_1}u_1^s du_1 \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-xt)u_2}u_2^{\lambda-1}du_2 \\
&= \frac{1}{\Gamma(s+1)\Gamma(\lambda)}\int_0^\infty\int_0^\infty e^{-(u_1+u_2)}e^{(u_1+u_2x)t}u_1^s u_2^{\lambda-1}du_1du_2 \\
&= \frac{1}{\Gamma(s+1)\Gamma(\lambda)}\int_0^\infty\int_0^\infty e^{-(u_1+u_2)}\sum_{n=0}^{\infty}\frac{(u_1+u_2x)^n}{n!}t^n u_1^s u_2^{\lambda-1}du_1du_2.
\end{aligned}$$

From the coefficents of t^n on the both sides of the last equality, one can get the desired result. □

We now discuss some miscellaneous recurrence relations of the generalized Cesàro polynomials. By differentiating each member of the generating function relation (2.1) with respect to x and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k), \tag{5.2}$$

we arrive at the following (differential) recurrence relation for the generalized Cesàro polynomials:

$$\frac{\partial}{\partial x} g_n^{(s)}(\lambda,x) - x\frac{\partial}{\partial x} g_{n-1}^{(s)}(\lambda,x) = \lambda g_{n-1}^{(s)}(\lambda,x), \quad n \geq 1. \tag{5.3}$$

On the other hand, by differentiating each member of the generating function relation (2.1) with respect to x, we have

$$\frac{\partial}{\partial x} g_n^{(s)}(\lambda,x) = \lambda \sum_{k=0}^{n-1} x^{n-k-1} g_k^{(s)}(\lambda,x). \tag{5.4}$$

If we consider (5.3) and (5.4), we can easily get the following recurrence relation for the generalized Cesàro polynomials :

$$\sum_{k=0}^{n-1} x^{n-k-1} g_k^{(s)}(\lambda,x) - \sum_{k=0}^{n-2} x^{n-k-1} g_k^{(s)}(\lambda,x) = g_{n-1}^{(s)}(\lambda,x).$$

Besides, by differentiating each member of the generating function relation (2.1) with respect to t, we have the following another recurrence relation for these polynomials:

$$(n+1)g_{n+1}^{(s)}(\lambda,x) = \sum_{m=0}^{n}\left[(s+1)\, g_{n-m}^{(s)}(\lambda,x) + \lambda x^{m+1} g_{n-m}^{(s)}(\lambda,x)\right].$$

6. The Generalized Lauricella Functions

In the present section, we derive various families of bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions.

Theorem 6.1. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x)\phi_n(u_1;u_2,...,u_k)t^n \\ =\;& (1-t)^{-s-1}(1-xt)^{-\lambda} \sum_{m_1,p,m_2,...,m_k=0}^{\infty} \frac{((b))_{(m_1+p)\phi}(s+1)_{m_1}(\lambda)_p}{((d))_{(m_1+p)\delta}} \\ &\times\Omega(f((m_1+p),m_2,...,m_k);m_2,...,m_k)\frac{\left(\frac{u_1 t}{t-1}\right)^{m_1}}{m_1!}\frac{\left(\frac{u_1 xt}{xt-1}\right)^{p}}{p!}\frac{u_2^{m_2}}{m_2!}...\frac{u_k^{m_k}}{m_k!},\end{aligned}$$

where $\phi_n(u_1;u_2,...,u_k)$ is given by (1.4).

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Proof. By using the relationship (2.2), it is easily observed that

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x)\phi_n(u_1;u_2,...,u_k)t^n \\
=&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x)\sum_{m_1=0}^{n}\sum_{m_2,...,m_k=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\
&\qquad\times\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right)\frac{u_1^{m_1}}{m_1!}...\frac{u_k^{m_k}}{m_k!}t^n \\
=&\sum_{m_1,m_2,...,m_k=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) \\
&\qquad\times(-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_s^{m_k}}{m_k!}\,(1-t)^{-s-m_1-1}(1-xt)^{-\lambda}g_{m_1}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right) \\
=&(1-t)^{-s-1}(1-xt)^{-\lambda}\sum_{m_1,m_2,...,m_k=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) \\
&\qquad\times(-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_k^{m_k}}{m_k!}(1-t)^{-m_1}\binom{s+m_1}{m_1}\sum_{p=0}^{m_1}\frac{(-m_1)_p(\lambda)_p}{(-s-m_1)_p p!}\left(\frac{x(1-t)}{1-xt}\right)^p \\
=&(1-t)^{-s-1}(1-xt)^{-\lambda} \\
&\times\sum_{m_1,p,m_2,...,m_k=0}^{\infty}\frac{((b))_{(m_1+p)\phi}}{((d))_{(m_1+p)\delta}}\Omega\left(f((m_1+p),m_2,...,m_k);m_2,...,m_k\right)(s+1)_{m_1}(\lambda)_p \\
&\qquad\frac{\left(\frac{u_1t}{t-1}\right)^{m_1}}{m_1!}\frac{\left(\frac{u_1xt}{xt-1}\right)^p}{p!}\frac{u_2^{m_2}}{m_2!}...\frac{u_k^{m_k}}{m_k!}.
\end{aligned}$$

□

By appropriately choosing the multiple sequence $\Omega(m_1,m_2,...,m_s)$ in Theorem 6.1, we obtain several interesting results as follows which give bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions.

I.By letting

$$\begin{aligned}
&\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) \\
=&\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_k\theta_j^{(k)}}}{\prod_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+...+m_k\psi_j^{(k)}}}\frac{\prod_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}}...\frac{\prod_{j=1}^{B^{(k)}}(b_j^{(k)})_{m_k\phi_j^{(k)}}}{\prod_{j=1}^{D^{(k)}}(d_j^{(k)})_{m_k\delta_j^{(k)}}}
\end{aligned}$$

in Theorem 6.1,we get the following result.

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Corollary 6.2. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) F^{A:B+1;B^{(2)};...;B^{(k)}}_{E:D;D^{(2)};...;D^{(k)}} \left(\begin{array}{l} \left[(a):\theta^{(1)},...,\theta^{(k)}\right]: \quad [-n:1], \quad [(b):\phi]; \quad [(b^{(2)}):\phi^{(2)}]; \quad ...; \quad [(b^{(k)}):\phi^{(k)}]; \\ \left[(c):\psi^{(1)},...,\psi^{(k)}\right]: \qquad [(d):\delta]; \quad [(d^{(2)}):\delta^{(2)}]; \quad ...; \quad [(d^{(k)}):\delta^{(k)}]; \\ \qquad\qquad u_1, u_2, ..., u_k \end{array} \right) t^n$$

$$= (1-t)^{-s-1}(1-xt)^{-\lambda} F^{A+B:1;1;B^{(2)};...;B^{(k)}}_{E+D:0;0;D^{(2)};...;D^{(k)}} \left(\begin{array}{l} \left[(e):\varphi^{(1)},...,\varphi^{(k+1)}\right]: \quad [s+1:1], \quad [\lambda:1]; \quad [(b^{(2)}):\phi^{(2)}]; \quad ...; \quad [(b^{(k)}):\phi^{(k)}]; \\ \left[(f):\xi^{(1)},...,\xi^{(k+1)}\right]: \quad - \quad -; \quad [(d^{(2)}):\delta^{(2)}]; \quad ...; \quad [(d^{(k)}):\delta^{(k)}]; \\ \qquad\qquad (\frac{u_1 t}{t-1}), (\frac{u_1 xt}{xt-1}), u_2, ..., u_k \end{array} \right)$$

where the coefficients $e_j, f_j, \varphi_j^{(k)}$ *and* $\xi_j^{(k)}$ *are given by*

$$e_j = \begin{cases} a_j & (1 \le j \le A) \\ b_{j-A} & (A < j \le A+B), \end{cases}$$

$$f_j = \begin{cases} c_j & (1 \le j \le E) \\ d_{j-E} & (E < j \le E+D), \end{cases}$$

$$\varphi_j^{(r)} = \begin{cases} \theta_j^{(1)} & (1 \le j \le A;\ 1 \le r \le 2) \\ \theta_j^{(r-1)} & (1 \le j \le A;\ 2 < r \le k+1) \\ \phi_{j-A} & (A < j \le A+B;\ 1 \le r \le 2) \\ 0 & (A < j \le A+B;\ 2 < r \le k+1) \end{cases}$$

and

$$\xi_j^{(r)} = \begin{cases} \psi_j^{(1)} & (1 \le j \le E;\ 1 \le r \le 2) \\ \psi_j^{(r-1)} & (1 \le j \le E;\ 2 < r \le k+1) \\ \delta_{j-E} & (E < j \le E+D;\ 1 \le r \le 2) \\ 0 & (E < j \le E+D;\ 2 < r \le k+1) \end{cases}$$

respectively.

II. *Upon setting*

$$\Omega\left(f(m_1, m_2, ..., m_k); m_2, ..., m_k\right) = \frac{(a)_{m_1+...+m_k}(b_2)_{m_2}...(b_k)_{m_k}}{(c_1)_{m_1}...(c_k)_{m_k}}$$

and

$$\phi = \delta = 0 \quad (\text{that is, } \phi_1 = ... = \phi_B = \delta_1 = ... = \delta_D = 0)$$

in Theorem 6.1,we obtain the following result.

Corollary 6.3. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x) F_A^{(k)}\left[a,-n,b_2,...,b_k;c_1,...,c_k;u_1,u_2,...,u_k\right] t^n \\ = \quad & (1-t)^{-s-1}(1-xt)^{-\lambda} F^{1:1;1;1;...;1}_{1:0;0;1;...;1} \\ & \left(\begin{array}{lllllll} [(a):1,...,1]: & [s+1:1]; & [\lambda:1]; & [b_2:1]; & ...; & [b_k:1]; \\ \left[(c_1):\psi^{(1)},...,\psi^{(k+1)}\right]: & -; & -; & [c_2:1]; & ...; & [c_k:1]; \end{array} \right. \\ & \left. \qquad (\frac{u_1 t}{t-1}),(\frac{u_1 xt}{xt-1}),u_2,...,u_k \right),\end{aligned}$$

where the coefficients $\psi^{(\eta)}$ *are given by*

$$\psi^{(\eta)} = \begin{cases} 1, & (1 \le \eta \le 2) \\ 0, & (2 < \eta \le k+1) \end{cases}.$$

III. *If we put*

$$\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) = \frac{(a_1^{(1)})_{m_2}...(a_1^{(k-1)})_{m_k}(a_2^{(1)})_{m_2}...(a_2^{(k-1)})_{m_k}}{(c)_{m_1+\cdots+m_k}}$$

and

$$B=1, b_1=b, \phi_1=1 \text{ and } \delta=0$$

in Theorem 6.1,we obtain Corollary 6.4 below.

Corollary 6.4. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x) F_B^{(k)}\left[-n,a_1^{(1)},...,a_1^{(k-1)},b,a_2^{(1)},...,a_2^{(k-1)};c;u_1,u_2,...,u_k\right] t^n \\ = \quad & (1-t)^{-s-1}(1-xt)^{-\lambda} F^{1:1;1;2;...;2}_{1:0;0;0;...;0} \\ & \left(\begin{array}{lllllll} \left[(b):\theta^{(1)},...,\theta^{(k+1)}\right]: & [s+1:1]; & [\lambda:1]; & \left[a^{(1)}:1\right]; & ...; & \left[a^{(k-1)}:1\right]; \\ [(c):1,...,1]: & -; & -; & -; & ...; & -; \end{array} \right. \\ & \left. \qquad (\frac{u_1 t}{t-1}),(\frac{u_1 xt}{xt-1}),u_2,...,u_k \right),\end{aligned}$$

where the coefficients $\theta^{(\eta)}$ *are given by*

$$\theta^{(\eta)} = \begin{cases} 1, & (1 \le \eta \le 2) \\ 0, & (2 < \eta \le k+1) \end{cases}.$$

IV.*By letting*

$$\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) = \frac{(a)_{m_1+...+m_k}(b_2)_{m_2}...(b_s)_{m_k}}{(c)_{m_1+...+m_k}}$$

and

$$\phi=\delta=0,$$

in Theorem 6.1,we obtain the following result.

Corollary 6.5. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x) F_D^{(k)}\left[a,-n,b_2,...,b_k;c;u_1,u_2,...,u_k\right] t^n$$
$$= (1-t)^{-s-1}(1-xt)^{-\lambda} F_D^{(k+1)}\left[a,s+1,\lambda,b_2,...,b_k;c;(\frac{u_1 t}{t-1}),(\frac{u_1 xt}{xt-1}),u_2,...,u_k\right].$$

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On uniqueness of meromorphic functions sharing one small function

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Abstract: In this paper, we prove the uniqueness theorems of meromorphic functions concerning shared small functions. Let f and g be meromorphic functions with finite order. We prove $f \equiv tg$ under some conditions, when $f^n f^{(k)}$ and $g^n g^{(k)}$ share a small function $a(z)$. We improve the results of Wang and Gao in [7].

Keywords: meromorphic function; shared value; uniqueness.

1. Introduction and results

In this article, a meromorphic function means meromorphic in the whole complex plane. We shall use the standard notations in the Nevanlinna value distribution theorem of meromorphic functions such as $T(r,f)$, $N(r,f)$, $m(r,f)$, $\overline{N}(r,f)$, etc. (see [1],[2]) For any nonconstant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying

$$\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)}=0,$$

possibly outside of a finite linear measure in R^+.

Let f and g be two nonconstant meromorphic functions. A meromorphic function $a(z)$ is called a small function with respect to f provided that $T(r,a)=S(r,f)$. Note that the set of all small functions of f is a field. Let $a(z)$ be a small function with respect to f and g. We say that f and g share $a(z)$ CM(IM) provided that $f-a(z)$ and $g-a(z)$ have same zeros counting multiplicities (ignoring multiplicities).

Throughout this paper, we need the following definitions.

$$\Theta(b,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,b;f)}{T(r,f)},$$

where b is a value in the extended complex plane.

The order of growth ρ of f:

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{\log r}.$$

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The exponent of convergence of $a-$value of f:

$$\begin{aligned}\lambda(a,f) &= \inf\left\{\tau>0:\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\tau}}<\infty\right\}\\ &= \limsup_{r\to\infty}\frac{\log^+ n(r,\frac{1}{f-a})}{\log r}\\ &= \limsup_{r\to\infty}\frac{\log^+ N(r,\frac{1}{f-a})}{\log r}.\end{aligned}$$

where $\{a_n\}$ is the sequence of a-value of $f(z)$. If $a=0$, we denote $\lambda(0,f)$ by $\lambda(f)$. If $a=\infty$, we denote $\lambda(\infty,f)$ by $\lambda(\frac{1}{f})$.

In 1920's, Nevanlinna [3] proved the famous four-valued theorem, which is an important result about uniqueness of meromorphic functions. Then many results about meromorphic functions that share more than or equal to two values have been obtained (see [4]). In 1997, Yang and Hua [5] studied meromorphic functions sharing one value.

Theorem A. (see [5]). Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let $n\geq 11$ be a positive integer. If f^nf' and g^ng' share 1 CM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$, or $f(z)\equiv tg(z)$ for a constant such that $t^{n+1}=1$.

In 2002, Fang and Qiu [6] investigated meromorphic functions sharing fixed point.

Theorem B. (see [6]). Let $f(z)$ and $g(z)$ be two nonconstant meromorphic(entire) functions and let $n\geq 11(n\geq 6)$ be a positive integer. If f^nf' and g^ng' share z CM, then either $f(z)=c_1e^{cz^2}$, $g(z)=c_2e^{-cz^2}$, where c_1, c_2, and c are three constants satisfying $4(c_1c_2)^{n+1}c^2=-1$, or $f(z)\equiv tg(z)$ for a constant such that $t^{n+1}=1$.

In 2007, Wang and Gao [7] extended Theorem B, in which they studied meromorphic functions sharing a small function.

Theorem C. (see [7]). Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let $a(z)(\not\equiv 0)$ be a common small function with respect to them, and let $n\geq 11$ be a positive integer. If f^nf' and g^ng' share $a(z)$ CM, then either $f^nf'g^ng'\equiv a(z)^2$, or $f(z)\equiv tg(z)$ for a constant such that $t^{n+1}=1$.

In this paper we replace f' by $f^{(k)}$ in Theorem C and obtain the following results which improve Theorem C.

Theorem 1. Let f and g be two transcendental meromorphic functions with finite order and $\max\{\lambda(f),\lambda(\frac{1}{f})\}<1$. Let k, n be two positive integers and $a(z)(\not\equiv 0)$ be a small function of f and g. If $f^nf^{(k)}$ and $g^ng^{(k)}$ share $a(z)$ CM, f and g share 0 CM, and $n>k+4+\sqrt{k^2+4k+25}$, then $f\equiv tg$ for a constant such that $t^{n+1}=1$ or $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$.

Remark: When $k=1$, $n\geq 11>5+\sqrt{30}$. Let $a(z)=z$, $f(z), g(z)$ in the Theorem B satisfy the conditions of Theorem 1 and $f^nf'g^ng'\equiv z^2$.

Theorem 2. Let f and g be two transcendental meromorphic functions with finite order and $\max\left\{\lambda(f),\lambda(\frac{1}{f})\right\}<1$. Let k, n be two positive integers and $a(z)(\not\equiv 0)$ be a small function of f and g. If $f^nf^{(k)}$ and $g^ng^{(k)}$ share $a(z)$ IM, f and g share 0 CM and $\Delta=\min\{\Delta_1,\Delta_2\}>6k+20-n$, where

$$\begin{aligned}\Delta_1 &= (3k+4)\Theta(\infty,f)+(2k+3)\Theta(\infty,g)+4\Theta(0,f)+3\Theta(0,g)\\ &+ \delta_{k+2}(0,f)+\delta_{k+2}(0,g)+2\delta_{k+1}(0,f)+\delta_{k+1}(0,g),\end{aligned}$$

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$$\begin{aligned}\Delta_2 &= (3k+4)\Theta(\infty,g)+(2k+3)\Theta(\infty,f)+4\Theta(0,g)+3\Theta(0,f)\\ &+ \delta_{k+2}(0,g)+\delta_{k+2}(0,f)+2\delta_{k+1}(0,g)+\delta_{k+1}(0,f),\end{aligned}$$

then $f\equiv tg$ for a constant such that $t^{n+1}=1$ or $f^n f^{(k)}g^n g^{(k)}\equiv a(z)^2$.

2.Lemmas

In this section, we present some lemmas which will be needed in the sequel. We now explain some definitions and notions which are used in this paper.

Definition 1. Let k be a positive integer. We denote by $N_{k)}\left(r,\frac{1}{f-a}\right)$ the counting function for the zeros of $f-a$ with multiplicity $\le k$ and by $\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}\left(r,\frac{1}{f-a}\right)$ be the counting function for the zeros of $f-a$ with multiplicity $\ge k$, and $\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$ be the corresponding one for which the multiplicity is not counted. Set

$$N_k\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

Definition 2. For $b\in\mathbb{C}\cup\{\infty\}$ we put

$$\delta_k(b,f)=1-\limsup_{r\to\infty}\frac{N_k(r,b;f)}{T(r,f)}.$$

Definition 3. Let f and g be two non-constant meromorphic functions such that f and g share the value 1IM. Let z_0 be a zero of $f-1$ with multiplicity p, and a zero of $g-1$ with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those common zeros of $f-1$ and $g-1$ satisfying $p>q$. Similarly we define $\overline{N}_L(r,1;g)$. In addition, we denote by $N_E^{1)}\left(r,\frac{1}{f-1}\right)$ the counting function of those common simple 1-points of $f(z)$ and $g(z)$.

Lemma 2.1([9], Lemma 3) Let f be a nonconstant meromorphic function and k be a positive integer. Then

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\le N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f) \qquad (2.1).$$

Lemma 2.2([10], Lemma 2) Let f be a nonconstant meromorphic function. If f and g share 1 IM, then

$$\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)<\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}(r,f)+S(r,f). \qquad (2.2)$$

Lemma 2.3([7,11], Lemma 3) Let f and g be two transcendental meromorphic functions defined in the complex plane $\mathbb{C}$. If f and g share 1 CM, then one of the following cases will occur:

(i)$T(r,f)\le N_2(r,\frac{1}{f})+N_2(r,\frac{1}{g})+N_2(r,f)+N_2(r,g)+S(r,f),$ (2.3)

(ii)$f\equiv g$ or $fg\equiv 1$.

Lemma 2.4 Let f and g be two meromorphic functions with finite order defined in the complex plane $\mathbb{C}$ and let k, n be two positive integers. If f and g share 0 CM, $\max\left\{\lambda(f),\lambda(\frac{1}{f})\right\}<1$ and $f^nf^{(k)}\equiv g^ng^{(k)}$, then $f\equiv tg$ for a constant such that $t^{n+1}=1$.

Proof: Let z_0 be a pole of f with multiplicity p, then by $f^nf^{(k)}\equiv g^ng^{(k)}$, z_0 is also a pole of g with multiplicity p, thus f and g share ∞ CM. Since f and g share 0 CM, f and g have same zeros and poles. Suppose f and g have the forms $f=\frac{h_1(z)}{h_2(z)}e^{\alpha(z)}$, $g=\frac{h_1(z)}{h_2(z)}e^{\beta(z)}$, where $h_1(z)$, $h_2(z)$ are entire functions, and $\alpha(z)$, $\beta(z)$ are polynomials.

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Let $h(z) = \frac{h_1(z)}{h_2(z)}$. Thus f and g have the forms $f = h(z)e^{\alpha(z)}$, $g = h(z)e^{\beta(z)}$. Since $\max\left\{\lambda(f), \lambda(\frac{1}{f})\right\} < 1$, $\rho(h) < 1$.

First we assume $k = 1$, then $f^n f' \equiv g^n g'$, by integration we have $f^{n+1} = g^{n+1} + c$, where c is a constant. From the condition that f and g share 0 CM we have $c = 0$, thus $f \equiv tg$ where $t^{n+1} = 1$.

Next we assume $k = 2$, then

$$f^n f'' = e^{(n+1)\alpha} h^n [h(\alpha')^2 + 2h'\alpha' + h\alpha'' + h''],$$

$$g^n g'' = e^{(n+1)\beta} h^n [h(\beta')^2 + 2h'\beta' + h\beta'' + h''],$$

thus $f^n f'' \equiv g^n g''$ gives

$$e^{(n+1)\alpha}[h(\alpha')^2 + 2h'\alpha' + h\alpha'' + h''] \equiv e^{(n+1)\beta}[h(\beta')^2 + 2h'\beta' + h\beta'' + h'']. \tag{2.4}$$

If $\alpha \not\equiv \beta$, we set $\alpha - \beta = C$ where C is constant or $\alpha - \beta = P(z)$ where $P(z)$ is a polynomial.

Let $\alpha - \beta = C$. We have $f = C_1 g$ and replace (2.4) by the following equality

$$C_1^{n+1} g^n g'' = g^n g'',$$

then $f \equiv tg$ where $t^{n+1} = 1$.

Let $\alpha - \beta = P(z)$. We replace (2.4) by the following equality

$$e^{(n+1)P(z)}[h(\alpha')^2 + 2h'\alpha' + h\alpha'' + h''] \equiv [h(\beta')^2 + 2h'\beta' + h\beta'' + h''].$$

$P(z)$ is a nonconstant polynomial, then $\rho(e^{(n+1)P(z)}) \geq 1$. Since $\rho(h) < 1$ and α, β are polynomials,

$$\rho\left(\frac{h(\alpha')^2 + 2h'\alpha' + h\alpha'' + h''}{h(\beta')^2 + 2h'\beta' + h\beta'' + h''}\right) < 1.$$

It is a contradiction.

At last we consider $f^n f^{(k)} \equiv g^n g^{(k)}$. By calculation we get

$$f^n f^{(k)} = e^{(n+1)\alpha} h^n [h(\alpha')^k + D(\alpha', \alpha'', \cdots, \alpha^{(k)}, h, h', h'', \cdots, h^{(k)})],$$

where D is a differential polynomial concerning α and h.
Similarly,

$$g^n g^{(k)} = e^{(n+1)\beta} h^n [h(\beta')^k + D(\beta', \beta'', \cdots, \beta^{(k)}, h, h', h'', \cdots, h^{(k)})].$$

Thus $f^n f^{(k)} \equiv g^n g^{(k)}$ yields

$$\begin{aligned} & e^{(n+1)(\alpha-\beta)}[h(\alpha')^k + D(\alpha', \alpha'', \cdots, \alpha^{(k)}, h, h', h'', \cdots, h^{(k)})] \\ \equiv \; & [h(\beta')^k + D(\beta', \beta'', \cdots, \beta^{(k)}, h, h', h'', \cdots, h^{(k)})] \end{aligned} \tag{2.5}$$

If $\alpha - \beta = C$, as the same proof of the case where $k = 2$, we obtain $f \equiv tg$ where $t^{n+1} = 1$.
If $\alpha - \beta = P(z)$ where $P(z)$ is a polynomial, from (2.5) we get a contraction. The proof of Lemma 2.4 is completed.

3. The Proof of Theorem 1

Let

$$F=\frac{f^nf^{(k)}}{a(z)},\quad G=\frac{g^ng^{(k)}}{a(z)}. \tag{3.1}$$

Then F and G share 1 CM. This and Lemma 2.1 imply that

$$\begin{aligned}
N_2\left(r,\frac{1}{F}\right) &= N_2\left(r,\frac{a(z)}{f^nf^{(k)}}\right)\le N_2\left(r,\frac{1}{f^nf^{(k)}}\right)+N_2(r,a(z))\\
&\le 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\
&= \frac{2}{n}\left[n\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}}\right)\right]\\
&+ \left(1-\frac{2}{n}\right)N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\
&\le \frac{2}{n}N(r,\frac{1}{f^nf^{(k)}})+S(r,f)\\
&+ (1-\frac{2}{n})\frac{1}{n+1}\left[nN_k\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}}\right)+nk\overline{N}(r,f)\right]\\
&\le \frac{2}{n}N\left(r,\frac{1}{f^nf^{(k)}}\right)+\left(1-\frac{2}{n}\right)\frac{1}{n+1}N\left(r,\frac{1}{f^nf^{(k)}}\right)\\
&+ \left(1-\frac{2}{n}\right)\frac{nk}{n+1}\overline{N}(r,f)+S(r,f)\\
&\le \frac{3}{n+1}N\left(r,\frac{1}{f^nf^{(k)}}\right)+\frac{(n-2)k}{n+1}\overline{N}(r,f)+S(r,f). \qquad (3.2)
\end{aligned}$$

Thus we have

$$\begin{aligned}
N_2\left(r,\frac{1}{F}\right)+N_2(r,F) &\le \frac{3}{n+1}N\left(r,\frac{1}{f^nf^{(k)}}\right)+\frac{(n-2)k}{n+1}\overline{N}(r,f)\\
&+ 2\overline{N}(r,f)+S(r,f)\\
&= \frac{3}{n+1}N\left(r,\frac{1}{f^nf^{(k)}}\right)+\frac{(n-2)k+2n+2}{n+1}\overline{N}(r,f)+S(r,f)\\
&\le \frac{3}{n+1}N\left(r,\frac{1}{F}\right)+\frac{(n-2)k+2n+2}{(n+1)^2}N(r,f^nf^{(k)})+S(r,f)\\
&\le \frac{3}{n+1}N(r,\frac{1}{F})+\frac{(n-2)k+2n+2}{(n+1)^2}N(r,F)+S(r,f)\\
&\le \frac{(n-2)k+5n+5}{(n+1)^2}T(r,F)+S(r,f). \qquad (3.3)
\end{aligned}$$

Since

$$\begin{aligned}
nT(r,f) &= T\left(r,\frac{f^nf^{(k)}}{a(z)}\cdot\frac{a(z)}{f^{(k)}}\right)+S(r,f)\\
&\le T(r,F)+T\left(r,\frac{a(z)}{f^{(k)}}\right)+S(r,f)\\
&\le T(r,F)+T(r,f^{(k)})+S(r,f)\\
&\le T(r,F)+T(r,\frac{f^{(k)}}{f})+T(r,f)+S(r,f)\\
&\le T(r,F)+N\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{f^{(k)}}{f}\right)+T(r,f)+S(r,f)\\
&\le T(r,F)+N(r,f)+k\overline{N}(r,f)+T(r,f)+S(r,f)\\
&\le T(r,F)+(k+2)T(r,f)+S(r,f), \qquad (3.4)
\end{aligned}$$

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we have $S(r,f)=S(r,F)$. Combining with (3.3), we obtain

$$N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\le\frac{(n-2)k+5n+5}{(n+1)^2}T(r,F)+S(r,F). \tag{3.5}$$

Similarly, we get

$$N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\le\frac{(n-2)k+5n+5}{(n+1)^2}T(r,G)+S(r,G) \tag{3.6}$$

By Lemma 2.3, if (i) holds, then we have

$$T(r,F)+T(r,G)\le\frac{2(n-2)k+10n+10}{(n+1)^2}[T(r,F)+T(r,G)]+S(r,F)+S(r,G),$$

this contradicts with the assumption $n>k+4+\sqrt{k^2+4k+25}$, therefore, $f^nf^{(k)}\equiv g^ng^{(k)}$ or $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$. If $f^nf^{(k)}\equiv g^ng^{(k)}$, by Lemma 2.4, we obtain $f\equiv tg$. This completes the proof of Theorem 1.

4. The Proof of Theorem 2

Let

$$F=\frac{f^nf^{(k)}}{a(z)},\quad G=\frac{g^ng^{(k)}}{a(z)}, \tag{4.1}$$

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{4.2}$$

Suppose that $H\not\equiv 0$, from the condition that $f^nf^{(k)}$ and $g^ng^{(k)}$ share $a(z)$IM, we have F and G share 1IM, so the common simple 1-point of F and G is the zero of H, and

$$\begin{aligned}\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right) &= \overline{N}_E^{1)}\left(r,\frac{1}{G-1}\right)\\ &\le N\left(r,\frac{1}{H}\right)\le T(r,H)+O(1)\\ &\le N(r,H)+S(r,f)+S(r,g), \end{aligned} \tag{4.3}$$

From (4.2) we have

$$\begin{aligned}N(r,H) \le\ & \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ +\ & \overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g).\end{aligned} \tag{4.4}$$

where $\overline{N}_0\left(r,\frac{1}{F'}\right)$ is the counting function of the zeros of F', which are not the zeros of F and $F-1$ and $\overline{N}_0\left(r,\frac{1}{G'}\right)$ is the counting function of the zeros of G', which are not the zeros of G and $G-1$. Noting

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that F and G share 1 IM, we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right) &+ \overline{N}\left(r,\frac{1}{G-1}\right)\\ &= 2\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)+2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &+ 2\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g)\\ &\le \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)\\ &+ 3\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\ &+ 2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)\\ &+ \overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g). \qquad (4.5)\end{aligned}$$

It is easy to see that

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right) &+ 2\overline{N}_L\left(r,\frac{1}{G-1}\right)+2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)\\ &\le N\left(r,\frac{1}{G-1}\right)\le T(r,G)+O(1). \qquad (4.6)\end{aligned}$$

This and (4.5) give

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) &\le \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &+ \overline{N}(r,F)+\overline{N}(r,G)+2\overline{N}_L(r,\frac{1}{F-1})\\ &+ \overline{N}_L(r,\frac{1}{G-1})+T(r,G)+\overline{N}_0(r,\frac{1}{F'})\\ &+ \overline{N}_0(r,\frac{1}{G'})+S(r,f)+S(r,g). \qquad (4.7)\end{aligned}$$

By the second fundamental theorem and (4.7), we have

$$\begin{aligned}T(r,F)+T(r,G) &\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F-1}\right)\\ &+ \overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g)\\ &\le T(r,G)+2\overline{N}(r,F)+2\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\ &+ 2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g).\end{aligned}$$

Thus

$$\begin{aligned}T(r,F) &\le 2\overline{N}(r,F)+2\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\ &+ 2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g). \qquad (4.8)\end{aligned}$$

From Lemma 2.2, we obtain

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) \leq \overline{N}\left(r, F\right) + \overline{N}\left(r, \frac{1}{F}\right), \tag{4.9}$$

$$\overline{N}_L\left(r, \frac{1}{G-1}\right) \leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right). \tag{4.10}$$

Taking (4.9) and (4.10) into (4.8) and combining with Lemma 2.1 , we get

$$\begin{aligned} T(r,F) &\leq 4\overline{N}(r,F) + 3\overline{N}(r,G) + N_2\left(r, \frac{1}{F}\right) \\ &+ N_2\left(r, \frac{1}{G}\right) + 2\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f) + S(r,g) \\ &\leq 4\overline{N}(r,f) + 3\overline{N}(r,g) + N_2\left(r, \frac{1}{f^n}\right) \\ &+ N_2\left(r, \frac{1}{f^{(k)}}\right) + N_2\left(r, \frac{1}{g^n}\right) + N_2\left(r, \frac{1}{g^{(k)}}\right) \\ &+ 2\overline{N}\left(r, \frac{1}{f^n}\right) + 2\overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{g^n}\right) \\ &+ \overline{N}\left(r, \frac{1}{g^{(k)}}\right) + S(r,f) + S(r,g) \\ &\leq 4\overline{N}(r,f) + 3\overline{N}(r,g) + 4\overline{N}(r, \frac{1}{f}) \\ &+ 3\overline{N}\left(r, \frac{1}{g}\right) + N_2\left(r, \frac{1}{f^{(k)}}\right) + N_2\left(r, \frac{1}{g^{(k)}}\right) \\ &+ 2\overline{N}(r, \frac{1}{f^{(k)}}) + \overline{N}\left(r, \frac{1}{g^{(k)}}\right) + S(r,f) + S(r,g) \\ &\leq (3k+4)\overline{N}(r,f) + (2k+3)\overline{N}(r,g) + 4\overline{N}\left(r, \frac{1}{f}\right) \\ &+ 3\overline{N}\left(r, \frac{1}{g}\right) + N_{2+k}\left(r, \frac{1}{f}\right) + N_{2+k}\left(r, \frac{1}{g}\right) \\ &+ 2N_{1+k}\left(r, \frac{1}{f}\right) + N_{1+k}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g). \end{aligned} \tag{4.11}$$

On the other hand, from (3.4) we have

$$T(r,F) \geq (n-k-1)T(r,f) + S(r,f). \tag{4.12}$$

Thus from (4.11) and (4.12), it follows that

$$\begin{aligned} (n-k-1)T(r,f) &\leq (3k+4)\overline{N}(r,f) + (2k+3)\overline{N}(r,g) \\ &+ 4\overline{N}\left(r, \frac{1}{f}\right) + 3\overline{N}\left(r, \frac{1}{g}\right) + N_{2+k}\left(r, \frac{1}{f}\right) \\ &+ N_{2+k}\left(r, \frac{1}{g}\right) + 2N_{1+k}\left(r, \frac{1}{f}\right) + N_{1+k}\left(r, \frac{1}{g}\right) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{4.13}$$

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In a similar way, we can get

$$\begin{aligned}(n-k-1)T(r,g) &\leq (3k+4)\overline{N}(r,g)+(2k+3)\overline{N}(r,f)\\ &+ 4\overline{N}\left(r,\frac{1}{g}\right)+3\overline{N}\left(r,\frac{1}{f}\right)+N_{2+k}\left(r,\frac{1}{g}\right)\\ &+ N_{2+k}\left(r,\frac{1}{f}\right)+2N_{1+k}\left(r,\frac{1}{g}\right)+N_{1+k}\left(r,\frac{1}{f}\right)\\ &+ S(r,f)+S(r,g). \qquad (4.14)\end{aligned}$$

We suppose that there exists a set I of infinite measure such that $T(r,g) \leq T(r,f)$ for $r \in I$. Hence by (4.13) we have

$$\begin{aligned}nT(r,f) &\leq [(6k+20)-(3k+3)\Theta(\infty,f)-(2k+3)\Theta(\infty,g)-4\Theta(0,f)\\ &- 3\Theta(0,g)-\delta_{k+2}(0,f)-\delta_{k+2}(0,g)-2\delta_{k+1}(0,f)-\delta_{k+1}(0,g)+\epsilon]T(r,f)+S(r,f)\end{aligned}$$

for $r \in I$ and $0 < \epsilon < \Delta_1 - (6k+20-n)$. And we obtain $T(r,f) \leq S(r,f)$ for $r \in I$, which is a contradiction.

Similarly, if $T(r,f) \leq T(r,g)$ for $r \in I$, by (4.14) we have

$$\begin{aligned}nT(r,g) &\leq [(6k+20)-(3k+3)\Theta(\infty,g)-(2k+3)\Theta(\infty,f)-4\Theta(0,g)\\ &- 3\Theta(0,f)-\delta_{k+2}(0,g)-\delta_{k+2}(0,f)-2\delta_{k+1}(0,g)-\delta_{k+1}(0,f)+\epsilon]T(r,g)+S(r,g)\end{aligned}$$

for $r \in I$ and $0 < \epsilon < \Delta_2 - (6k+20-n)$, which also is a contradiction.

Hence $H(z) \equiv 0$. That is

$$\frac{F''}{F'}-\frac{2F'}{F-1} \equiv \frac{G''}{G'}-\frac{2G'}{G-1}.$$

Without loss of generality, next we suppose that there exists a set I of infinite measure such that $T(r,g) \leq T(r,f)$ for $r \in I$. By integrating two sides of the above equality we get

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \qquad (4.15)$$

where $A(\neq 0)$ and B are constants.

We consider the following three cases:

Case 1. Let $B \neq 0$ and $A = B$. If $B = -1$, from (4.15) we obtain $FG \equiv 1$, that is $f^n f^{(k)} g^n g^{(k)} \equiv a(z)^2$. If $B \neq -1$, from (4.15) we get $\frac{1}{G} = \frac{BF}{(1+B)F-1}$, thus $\overline{N}(r, \frac{1}{F-\frac{1}{B+1}}) = \overline{N}(r, \frac{1}{G})$. By the second fundamental theorem, (4.12) and Lemma 2.1, we have

$$\begin{aligned}
(n-k-1)T(r,f) &\leq T(r,F)+S(r,F)\\
&\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{1}{B+1}}\right)+S(r,F)\\
&= \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)\\
&\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\\
&+ \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)\\
&\leq (k+1)\overline{N}(r,f)+k\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\
&+ N_{1+k}\left(r,\frac{1}{f}\right)+N_{1+k}\left(r,\frac{1}{g}\right)+S(r,f)\\
&\leq (3k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)\\
&+ 4\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{g}\right)+N_{2+k}\left(r,\frac{1}{f}\right)\\
&+ N_{2+k}\left(r,\frac{1}{g}\right)+2N_{1+k}\left(r,\frac{1}{f}\right)\\
&+ N_{1+k}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned}$$

That is,

$$\begin{aligned}
(n-k-1)T(r,f) &\leq (3k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)\\
&+ 4\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{g}\right)+N_{2+k}\left(r,\frac{1}{f}\right)\\
&+ N_{2+k}\left(r,\frac{1}{g}\right)+2N_{1+k}\left(r,\frac{1}{f}\right)\\
&+ N_{1+k}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned}$$

Thus we obtain

$$[\Delta_1-(5k+18-n)]T(r,f)\leq S(r,f),$$

which is a contradiction.

Case 2. Let $B\neq 0$ and $A=B$. If $B=-1$, we have from (4.15) that $G=\frac{A}{-F+A+1}$, therefore $\overline{N}(r,\frac{1}{-F+A+1})=\overline{N}(r,G)$. By the second fundamental theorem and the same argument as in case 1, we get a contradiction.

If $B\neq -1$, we have from (4.15) that $G-(1+\frac{1}{B})=\frac{-A}{B^2(F+\frac{A-B}{B})}$, therefore $\overline{N}\left(r,\frac{1}{F+\frac{A-B}{B}}\right)=\overline{N}(r,G)$. Next by the second fundamental theorem and the same argument as in case 1, we get a contradiction.

Case 3. Let $B=0$, then we obtain from (4.15) that $G=\frac{F}{A}+1-\frac{1}{A}$. If $A\neq 1$, then $\overline{N}\left(r,\frac{1}{G-\frac{A-1}{A}}\right)=\overline{N}\left(r,\frac{1}{F}\right)$. By the second fundamental theorem and the same argument as in case 1, we get a contradiction.

Thus $A=1$, so we have $F\equiv G$, that is $f^nf^{(k)}\equiv g^ng^{(k)}$. By Lemma 2.4 we have $f\equiv tg$ for a constant such that $t^{n+1}=1$. From the above proof we obtain $f\equiv tg$ or $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$.

The proof of Theorem 2 is completed.

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Stochastic elastic equation in d-dimensional space driven by multiplicative multi-parameter fractional white noise

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Abstract

Stochastic elastic equation in d-dimensional space driven by multiplicative multi-parameter fractional white noise are considered. By using the Wiener chaos expansion and undetermined coefficient methods, we obtain the existence and uniqueness of the solution in a distribution space. The Lyapunov exponents and the Hölder continuity in the distribution space of the solution are also estimated.

Key Words: Fractional Brownian motion; Multi-parameter fractional white noise; Winner Chaos expansion; Asymptotic behavior.

1 Introduction

The subject of stochastic calculus with respect to fractional Brownian motion has gained considerable popularity and importance due to its frequent appearance in a wide variety of physical phenomena, such as hydrology, economic, telecommunications and medicine. Many contributions for stochastic calculus with respect to fractional Brownian motion have emerged in the last decades, see [3, 7, 23, 24]. Since some physical phenomena are naturally modeled by stochastic partial differential equations and the randomness can be described by multi-parameter fractional white noise, it is important to study the problems of solutions of stochastic partial differential equations with multi-parameter fractional white noise. Many studies of the solutions of stochastic equations with fractional white noise have emerged recently, see [8, 15, 19, 25, 28] and reference therein.

In [14], Y. Hu studied the existence and uniqueness of the solution for a stochastic heat equation by chaos expansion, and estimated the Lyapunov exponent in a distribution space of the solution. In [11], the authors considered a class of hyperbolic stochastic

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partial differential equation driven by a space time fractional noise. In [1, 27], some authors considered stochastic wave equation with fractional Gaussian noise. A stochastic generalized Burgers equations driven by fractional noises has been studied in [20]. In [3], the authors investigated stochastic Poisson equation and quasi-linear heat equation driven by multi-parameter fractional white noise. In a recent paper [30], the authors considered a 1-dimensional stochastic Burgers' equation driven by a genuine cylindrical fBm with Hurst parameter $H > \frac{1}{4}$. They proved the regularities of the solution to the linear stochastic problem corresponding to the stochastic Burgers' equation and then obtained the global existence and uniqueness results of the stochastic Burgers' equation. For more contributions on stochastic calculus with fractional noise, we refer the reader to [4, 16–18, 20, 22, 26, 29, 30] and reference therein.

It should be noted that most of the papers and books on stochastic partial differential equations with fractional noise are devoted to the case of additive noise. However, there are few papers that consider the case of multiplicative fractional noise [2, 9, 14, 18]. Enlightened by the above contributions, in this paper we will consider a stochastic elastic equation driven by multiplicative multi-parameter white noise.

Let $H = (h_0, h_1, \dots, h_d)$ with $\frac{1}{2} < h_i < 1, i = 0, 1, \dots, d$. Consider the following stochastic elastic equation

$$\frac{\mathrm{d}u_t(t,x)}{\mathrm{d}t} + \Delta^2 u(t,x) = u(t,x) \diamond W^H(t,x), \tag{1.1}$$

where $t > 0, x \in D = (0,1)^d$ with initial and boundary conditions

$$u(0,x) = u_0(x), \frac{\partial u}{\partial t}(0,x) = \upsilon_0(x), \quad x \in D,$$

$$u(t,x) = \Delta u(t,x) = 0, \quad \text{on } \partial D.$$

We assume that u_0 and υ_0 are deterministic functions defined on $[0,1]^d$, u_0 and Δu_0 both vanish at ∂D, W^H is a $(d+1)$-dimensional fractional white noise and $u \diamond W^H$ is Wick product which will be defined in section 2.

Stochastic elastic equation driven by Brownian motion has been studied by many authors, (see [5, 6, 12, 21, 31]). However, there are few papers consider stochastic elastic equation driven by fractional Brownian motion. Such equation is a fourth order partial differential equation and has very wide applications in structural engineering. As an engineering problem, it has its applications in beams, bridges and other structures, see [5, 26].

Our aim in this paper is to obtain the existence, uniqueness, the asymptotic behavior and the Hölder continuity of the mild solution of problem (1.1) in a distribution space. The keys to the proof are the Wiener chaos expansion of the solution and the undetermined coefficient method.

Throughout the paper, we use the letter C denotes a constant that may not be the same form one occurrence to anther, even in the same line. We express the dependence on some parameters by writing the parameters as arguments, e.g. $C = C(H)$.

The remaining of this paper is organized as follows. In section 2, we give preliminaries of the stochastic integral with respect to multi-parameter fractional white noise. In section 3, we prove the the existence, uniqueness, asymptotic behavior and Hölder continuity in a distribution space of the solutions of problem (1.1).

2 Stochastic integral with respect to multi-parameter fractional white noise

In this section, we introduce the definition of stochastic integral with respect to the $d-$parameter fractional Brownian fields for Hurst index $H = (h_1, \cdots, h_d), (\frac{1}{2} < h_i < 1, i = 1, \cdots, d)$ by using the fractional white noise analysis method. For more contributions about white noise analysis, we refer the reader to [3, 13].

Let $h \in (0, 1)$. A fractional Brownian motion $B_t^h, t \geq 0$, of Hurst index h is a continuous Gaussian stochastic process, such that for all $s, t \in \mathbb{R}_+$,

$$B_0^h = 0, \quad \mathbb{E}(B_t^h) = 0, \quad \mathbb{E}(B_t^h B_s^h) = \frac{1}{2}(|t|^{2h} + |s|^{2h} - |t-s|^{2h}). \tag{2.1}$$

Since we are concerned with the fractional Brownian motions of multi-parameter, some notations must be introduced. Let $u = (u_1, \cdots, u_d) \in \mathbb{R}^d$, denote $\mathrm{d}u = \mathrm{d}u_1 \cdots \mathrm{d}u_d$. Fix h with $\frac{1}{2} < h < 1$. We put

$$\phi_h(s, t) = h(2h-1)|s-t|^{2h-2}, \quad s, t \in \mathbb{R}, \tag{2.2}$$

and

$$\phi_H(u, v) = \prod_{i=1}^{d} \phi_{h_i}(u_i, v_i), \quad u = (u_1, \cdots, u_d), v = (v_1, \cdots, v_d) \in \mathbb{R}^d \tag{2.3}$$

for $H = (h_1, \cdots, h_d), (\frac{1}{2} < h_i < 1, i = 1, \cdots, d)$.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ be the dual of $\mathcal{S}(\mathbb{R}^d)$, i.e., $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$.

The action of $\omega \in \mathcal{S}'(\mathbb{R}^d)$ on $f \in \mathcal{S}(\mathbb{R}^d)$ is given by $\langle \omega, f \rangle = \int_{\mathbb{R}^d} \omega(x) f(x) \mathrm{d}x$. Denote

$$\langle f, g \rangle_H = \int_{\mathbb{R}^{2d}} f(u) g(v) \phi_H(u, v) \mathrm{d}u \mathrm{d}v, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{2.4}$$

If we equip $\mathcal{S}(\mathbb{R}^d)$ with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|f\|_H = \langle f, f \rangle_H$, then the completion of $\mathcal{S}(\mathbb{R}^d)$, denote by $L^2_{\phi_H}(\mathbb{R}^d)$, becomes a separable Hilbert space. Now let $\Omega = \mathcal{S}'(\mathbb{R}^d)$. The map $f \to e^{-\frac{1}{2}\|f\|_H^2}$ is positive definite on $\mathcal{S}(\mathbb{R}^d)$, by the Bochner-Minlos theorem [13], there exists a probability measure $\mathbb{P}^H$ on the Borel subset $\mathcal{B}(\Omega)$ of Ω such that

$$\int_\Omega e^{i\langle \omega, f \rangle} \mathrm{d}\mathbb{P}^H(\omega) = e^{-\frac{1}{2}\|f\|_H^2}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{2.5}$$

Let $\mathbb{E}$ denotes the expectation under the probability measure $\mathbb{P}^H$, then

$$\mathbb{E}[\langle \cdot, f \rangle] = 0, \quad \mathbb{E}[\langle \cdot, f \rangle^2] = \|f\|_H^2. \tag{2.6}$$

Now define a square integrable stochastic field $B^H(x), x \in \mathbb{R}^d$ as

$$B^H(x) = B^H(x, \omega) = \langle \omega, I_{[0,x]}(\cdot) \rangle, \tag{2.7}$$

where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d, I_{[0,x]} = I_{[0,x_1]} \cdots I_{[0,x_d]}$ and for every $i \in \{1, 2, \cdots, d\}$,

$$I_{[0,x_i]}(y_i) = \begin{cases} 1 & 0 \le y_i \le x_i, \\ -1 & x_i \le y_i \le 0, \text{ excetp } x_i = y_i = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

From (2.7), we see that $B^H(x)$ is a Gaussian field and for every $x, y \in \mathbb{R}^d$,

$$\mathbb{E}[B^H(x)] = 0, \quad \mathbb{E}[B^H(x) B^H(y)] = \frac{1}{2^d} \prod_{i=1}^d (|x_i|^{2h_i} + |y_i|^{2h_i} - |x_i - y_i|^{2h_i}), \tag{2.9}$$

where we have used the well known identity

$$\int_0^{x_i} \int_0^{y_i} \phi_h(s, t) \mathrm{d}s \mathrm{d}t = \frac{1}{2}(|x_i|^{2h_i} + |y_i|^{2h_i} - |x_i - y_i|^{2h_i}). \tag{2.10}$$

By Kolmogorov's continuity theorem, $B^H(x)$ has a continuous version. The fractional Brownian field is defined as the continuous version of $B^H(x)$, which we sill denote it by $B^H(x)$.

Similarly to the case of stochastic integral with respect to fractional Brownian motion for deterministic function [3, 7, 10], we have the following lemma.

Lemma 2.1 *If $f \in L^2_H(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d} f(x) \mathrm{d}B^H(x)$ is well-defined Gaussian random variable and*

$$\mathbb{E}\left[\int_{\mathbb{R}^d} f(x) \mathrm{d}B^H(x)\right] = 0, \quad \mathbb{E}\left[\int_{\mathbb{R}^d} f(x) \mathrm{d}B^H(x)\right]^2 = \int_{\mathbb{R}^{2d}} f(u) f(v) \phi_H(u, v) \mathrm{d}u \mathrm{d}v. \tag{2.11}$$

The Hermite polynomials $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-\frac{1}{2}x^2}), \quad n = 0, 1, 2, \ldots. \tag{2.12}$$

For example, the first Hermite polynomials are

$$h_0(x) = 1, \quad h_1(x) = x, \quad h_2(x) = x^2 - 1, \quad h_3(x) = x^3 - 3x, \cdots. \tag{2.13}$$

Let $\xi_n(x)$ be the Hermite functions defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x)e^{-\frac{x^2}{2}}, \quad n = 1, 2, \ldots. \tag{2.14}$$

It is proved in [13] that $\xi_n \in \mathcal{S}(\mathbb{R})$ and the collection $\{\xi_n\}_{n=1}^{\infty}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$. The most important property of ξ_n used in this paper is

$$|\xi_n(x)| \leq \begin{cases} Cn^{-\frac{1}{12}}, & |x| \leq 2\sqrt{n}, \\ Ce^{-\gamma x^2}, & |x| > 2\sqrt{n}, \end{cases} \quad n = 1, 2, \ldots, \tag{2.15}$$

where constants C and γ are independent of n.

Lemma 2.2 *[3] Let $\frac{1}{2} < h < 1$. The fractional integral $I_-^{h-\frac{1}{2}}$ is defined by*

$$I_-^{h-\frac{1}{2}} f(u) = c_h \int_u^{\infty} (t-u)^{h-\frac{3}{2}} f(t)\mathrm{d}t, \tag{2.16}$$

where $c_h = \sqrt{h(2h-1)\Gamma(3/2-h)/(\Gamma(h-1/2)\Gamma(2-2h))}$ and Γ denotes the gamma function. Then $I_-^{h-\frac{1}{2}}$ is an isometry from $L^2_{\phi_h}(\mathbb{R})$ to $L^2(\mathbb{R})$.

Now we define $\eta_n^h(u) = (I_-^{h-\frac{1}{2}})^{-1}(\xi_n)(u)$. Then by Lemma 2.2 and the properties of ξ_n, $\{\eta_n^h\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2_{\phi_h}(\mathbb{R})$.

Let $\delta = (\delta_1, \ldots, \delta_d)$ denote d-dimensional multi-indices with $\delta_i \in \mathbb{N}, i = 1, \ldots, d$, where $\mathbb{N}$ is the set of natural numbers. Then the family of tensor products

$$\eta_\delta(x_1, x_2, \ldots, x_d) := \eta_{\delta_1}^{h_1}(x_1)\eta_{\delta_2}^{h_2}(x_2)\cdots\eta_{\delta_d}^{h_d}(x_d), \quad \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d \tag{2.17}$$

forms an orthonormal basis of $L^2_{\phi_H}(\mathbb{R}^d)$. Let $\{\delta^i = (\delta_1^i, \ldots, \delta_d^i)\}_{i=1}^{\infty}$ be a fixed ordering of $\mathbb{N}^d$. From a detailed proof in [13], we can assume that the ordering has the properties that

$$i < j \Rightarrow \delta_1^i + \delta_2^i + \cdots + \delta_d^i \leq \delta_1^j + \delta_2^j + \cdots + \delta_d^j. \tag{2.18}$$

and

$$j^{\frac{1}{d}} \leq \delta_1^j \cdot \delta_2^j \cdots \delta_d^j \leq j. \tag{2.19}$$

Let $\mathcal{J} = \left(\mathbb{N}_0^{\mathbb{N}}\right)_c$ denote the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ with elements $\alpha_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. Denote $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$. Define

$$e_j(x) = \eta_{\delta^j}(x) = \eta_{\delta_1^j}^{h_1}(x_1)\eta_{\delta_2^j}^{h_2}(x_2)\cdots\eta_{\delta_d^j}^{h_d}(x_d), \quad x = (x_1, \ldots, x_d), j = 1, 2, \ldots. \tag{2.20}$$

Then $\{e_j\}$ forms an orthonormal basis of $L^2_{\phi_H}(\mathbb{R}^d)$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{J}$, define

$$\mathcal{H}_\alpha(\omega) = \prod_{i=1}^{m} h_{\alpha_i}(\langle \omega, e_i \rangle). \tag{2.21}$$

Then we have the following fractional Wiener Ito chaos expansion theorem.

Theorem 2.1 *[3] The family $\{\mathcal{H}_\alpha\}_{\alpha\in\mathcal{J}}$ constitutes an orthogonal basis for $L^2(\mathbb{P}^H)$ and for $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J}$,*

$$\|\mathcal{H}_\alpha\|^2_{L^2(\mathbb{P}^H)} = \mathbb{E}[\mathcal{H}_\alpha^2] = \alpha! = \alpha_1!\alpha_2!\cdots. \tag{2.22}$$

Moreover, if $F \in L^2(\mathbb{P}^H)$, then there exist constants $c_\alpha \in \mathbb{R}, \alpha \in \mathcal{J}$, such that

$$F(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega), \tag{2.23}$$

where the convergence holds in $L^2(\mathbb{P}^H)$ and

$$\|F\|^2_{L^2(\mathbb{P}^H)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2. \tag{2.24}$$

Now we compute the Wiener Ito chaos expansion of the fractional Brownian field $B^H(x)$. By (2.7),

$$\begin{aligned} B^H(x) &= \langle \omega, I_{[0,x]}\rangle = \langle \omega, \sum_{i=1}^{\infty}\langle I_{[0,x]}, e_i\rangle_H e_i\rangle = \sum_{i=1}^{\infty}\langle I_{[0,x]}, e_i\rangle_H \langle \omega, e_i\rangle \\ &= \sum_{i=1}^{\infty}\left[\int_0^x \int_{\mathbb{R}^d} e_i(v)\phi_H(u,v)\mathrm{d}v\mathrm{d}u\right]\mathcal{H}_{\varepsilon^{(i)}}(\omega), \end{aligned} \tag{2.25}$$

where $\varepsilon^{(i)} = (0, \ldots, 0, 1, 0, \ldots)$ denote the ith unit vector.

The fractional Hida test function and distribution spaces are defined as follows.

Definition 2.1 *The fractional Hida test function space $(\mathcal{S})_H = \bigcap_{k=0}^{\infty}(\mathcal{S})_{H,k}$, where $(\mathcal{S})_{H,k}$ is the set of all $\psi(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega) \in L^2(\mathbb{P}^H)$ such that*

$$\|\psi\|^2_{H,k} = \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty, \ k \in \mathbb{N}, \tag{2.26}$$

where $(2\mathbb{N})^{\gamma} = \prod_{j=1}^{\infty}(2j)^{\gamma_j}$ *if* $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{J}$. *The fractional Hida distribution space* $(\mathcal{S})_H^* = \bigcup_{q=0}^{\infty}(\mathcal{S})_{H,-q}^*$, *where* $(\mathcal{S})_{H,-q}^*$ *is the set of all formal expansions* $G(\omega) = \sum_{\alpha\in\mathcal{J}} b_\alpha \mathcal{H}_\alpha(\omega)$ *such that*

$$\|G\|_{H,-q}^2 = \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty, \; q \in \mathbb{N}. \tag{2.27}$$

The family of seminorms $\|\cdot\|_{H,k}, k \in \mathbb{N}$ gives rise to a topology on $(\mathcal{S})_H$ and $(\mathcal{S})_H^*$ can be identified with the dual of $(\mathcal{S})_H$ by the action

$$\ll G, \psi \gg = \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha b_\alpha. \tag{2.28}$$

Definition 2.2 $G : \mathbb{R}^d \to (\mathcal{S})_H^*$ *is* $\mathrm{d}x$*-integrable in* $(\mathcal{S})_H^*$ *if*

$$\ll G(x), \psi \gg \in L^1(\mathbb{R}^d), \quad \textit{for all } \psi \in (\mathcal{S})_H. \tag{2.29}$$

If $G : \mathbb{R}^d \to (\mathcal{S})_H^*$ *is* $\mathrm{d}x$*-integrable in* $(\mathcal{S})_H^*$, *We define* $\int_{\mathbb{R}^d} G(x)\mathrm{d}x$ *to be the unique element of* $(\mathcal{S})_H^*$ *such that*

$$\ll \int_{\mathbb{R}^d} G(x)\mathrm{d}x, \psi \gg = \int_{\mathbb{R}^d} \ll G(x), \psi \gg \mathrm{d}x, \quad \textit{for all } \psi \in (\mathcal{S})_H. \tag{2.30}$$

The fractional noise $W^H(x)$ is defined by the formal derivative of $B^H(x)$ in $(\mathcal{S})_H^*$,

$$W^H(x) = \sum_{i=1}^{\infty} \left[\int_{\mathbb{R}^d} e_i(\upsilon)\phi_H(x,\upsilon)\mathrm{d}\upsilon \right] \mathcal{H}_{\varepsilon(i)}(\omega). \tag{2.31}$$

Then, $W^H(x) \in (\mathcal{S})_H^*$ and $\frac{\mathrm{d}}{\mathrm{d}x} B^H(x) = W^H(x)$ in $(\mathcal{S})_H^*$.

Definition 2.3 *The Wick product* $F \diamond G$ *of* $F(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega) \in (\mathcal{S})_H^*$ *and* $G(\omega) = \sum_{\alpha\in\mathcal{J}} b_\alpha \mathcal{H}_\alpha(\omega) \in (\mathcal{S})_H^*$ *is defined by*

$$F \diamond G(\omega) = \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega) = \sum_{\gamma\in\mathcal{J}} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) \mathcal{H}_\gamma(\omega). \tag{2.32}$$

Based on the preparations above, now we define the fractional Wick Ito Skorohod integral as follows.

Definition 2.4 *Suppose* $G : \mathbb{R}^d \to (\mathcal{S})_H^*$ *is a given function and* $G(x) \diamond W^H(x)$ *is* $\mathrm{d}x$*-integrable in* $(\mathcal{S})_H^*$. *Then the fractional Wick Ito Skorohod integral* $\int_{\mathbb{R}^d} G(x)\mathrm{d}B^H(x)$ *is defined by*

$$\int_{\mathbb{R}^d} G(x)\mathrm{d}B^H(x) = \int_{\mathbb{R}^d} G(x) \diamond W^H(x)\mathrm{d}x. \tag{2.33}$$

For a interval in $\mathbb{R}^d$, *the integral can be defined as*

$$\int_0^x G(y)\mathrm{d}B^H(y) = \int_{\mathbb{R}^d} G(y) I_{[0,x]}(y)\mathrm{d}B^H(y). \tag{2.34}$$

3 Main results

Let $H = (h_0, h_1, \ldots, h_d)$ with $\frac{1}{2} < h_i < 1, i = 0, 1, \ldots, d$. In this section, we consider the following stochastic elastic equation

$$\frac{\mathrm{d}u_t(t,x)}{\mathrm{d}t} + \Delta^2 u(t,x) = u(t,x) \diamond W^H(t,x), \tag{3.1}$$

where $t > 0, x \in D = (0,1)^d$, with initial and boundary conditions

$$u(0,x) = u_0(x), \frac{\partial u}{\partial t}(0,x) = \upsilon_0(x), \quad x \in D,$$

$$u(t,x) = \Delta u(t,x) = 0, \quad \text{on } \partial D.$$

We assume that u_0 and υ_0 are deterministic functions defined on $[0,1]^d$, u_0 and Δu_0 both vanish at $x = 0$ and $x = 1$, and that W^H is a $(d+1)$-dimensional fractional white noise has been defined in section 2.

Let $r = (r_1, \ldots, r_d) \in \mathbb{N}^d, x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Define $\varphi_r(x) = \sqrt{2^d} \prod_{i=1}^d \sin(r_i \pi x_i)$. Then $\{\varphi_r\}_{r \in \mathbb{N}^d}$ satisfy the boundary conditions of (3.1) and compose of a complete orthonormal system on $L^2(D)$ which diagonalize Δ with

$$\lambda_r = \pi^2 |r|^2 = \pi^2 \sum_{i=1}^d r_i^2, \tag{3.2}$$

the corresponding eigenvalues.

For a given function $g : D \to \mathbb{R}$ and $\rho \in \mathbb{R}$, define

$$\|g\|_{\rho,2} := \left(\sum_{r \in \mathbb{N}^d} (1 + |r|^2)^\rho |\langle g, \varphi_r \rangle|^2 \right)^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $L^2(D)$, and denote by $H^{\rho,2}(D)$ the set of functions $g : D \to \mathbb{R}$ such that $\|g\|_{\rho,2} < \infty$. Notice that $H^{\rho,2}(D)$ is a subspace of the fractional Sobolev space of fractional differential order α and integrability order $p = 2$ (see [27]). For a special case $\rho = 0$, it is clear that $H^{\rho,2}(D) = L^2(D)$ and we will denote $\| \cdot \|_{0,2}$ by $\| \cdot \|$. By Parseval's identity, we have

$$\|g\|^2 = \sum_{r \in \mathbb{N}^d} |\langle g, \varphi_r \rangle|^2, \quad \forall g \in L^2(D). \tag{3.3}$$

Since the fundamental solution of

$$\upsilon_{tt} + \Delta^2 \upsilon = 0,$$

$$\upsilon = \Delta u = 0 \text{ on } \partial D, \quad \upsilon|_{t=0} = \phi(x), \ \upsilon_t|_{t=0} = \psi(x) \text{ on } D$$

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is given by

$$\upsilon(t,x)=\sum_{r\in\mathbb{N}^d}\frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D\psi(y)\varphi_r(y)\mathrm{d}y+\sum_{r\in\mathbb{N}^d}\cos(\lambda_r t)\varphi_r(x)\int_D\phi(y)\varphi_r(y)\mathrm{d}y, \tag{3.4}$$

we can define the solution of (3.1) as follows.

Definition 3.1 *A random field $u=u(t,x):\mathbb{R}^+\times D\times\Omega\to\mathbb{R}$ is said to be a solution of (3.1), if*

(i) $u=u(t,x):\mathbb{R}^+\times D\times\Omega\to\mathbb{R}$ is jointly measurable.

(ii) There exists constant $q\in\mathbb{N}$, such that for almost all $x\in D$ and $t\geq 0$,

$$\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\mathrm{d}B^H(s,y)$$

is well defined as an element of $(\mathcal{S})^*_{H,-q}$, and that

$$\int_D\left\|\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\mathrm{d}B^H(s,y)\right\|^2_{H,-q}\mathrm{d}x<\infty,\forall t\geq 0.$$

(iii) It holds in $(\mathcal{S})^*_{H,-q}$ that

$$\begin{aligned}u(t,x) &= \sum_{r\in\mathbb{N}^d}\frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D\upsilon_0(y)\varphi_r(y)\mathrm{d}y\\ &\quad+\sum_{r\in\mathbb{N}^d}\cos(\lambda_r t)\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y\\ &\quad+\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\mathrm{d}B^H(s,y).\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}U_0(t,x) &= \sum_{r\in\mathbb{N}^d}\frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D\upsilon_0(y)\varphi_r(y)\mathrm{d}y\\ &\quad+\sum_{r\in\mathbb{N}^d}\cos(\lambda_r t)\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y.\end{aligned}$$

Then by the definition of the stochastic integral, the solution of (3.1), if it exists, can be written as

$$u(t,x)=U_0(t,x)+\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\diamond W^H(s,y)\mathrm{d}s\mathrm{d}y. \tag{3.6}$$

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If we assume that the solution of (3.1) exists in $(\mathcal{S})_H^*$ and it has a formal expansions

$$u(t,x) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t,x)\mathcal{H}_\alpha, \tag{3.7}$$

where $c_\alpha(t,x), \alpha \in \mathcal{J}$ are coefficients of Wiener chaos expansion of u, which are undetermined. Let

$$c_{\alpha-\varepsilon^{(i)}} = 0 \ \text{ if } \alpha_i = 0. \tag{3.8}$$

Then by the formal expansion of W^H, we obtain that

$$u(t,x) \diamond W^H(t,x) = \sum_{\alpha\in\mathcal{J}} \left[\sum_{i=1}^{\infty} c_{\alpha-\varepsilon^{(i)}}(t,x) \left(\int_{\mathbb{R}^{d+1}} e_i(v)\phi_H(t,x;v)\mathrm{d}v\right)\right] \mathcal{H}_\alpha(\omega). \tag{3.9}$$

Brought (3.9) into (3.6), we derive that

$$\begin{aligned} u(t,x) - U_0(t,x) &= \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r} \varphi_r(x)\varphi_r(y)u(s,y) \diamond W^H(s,y)\mathrm{d}s\mathrm{d}y. \\ &= \sum_{|\alpha|\geq 1} \left[\sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r} \varphi_r(x)\varphi_r(y) \sum_{i=1}^{\infty} c_{\alpha-\varepsilon^{(i)}}(s,y) \right. \\ &\quad \left. \times \left(\int_{\mathbb{R}^{d+1}} e_i(v)\phi_H(s,y;v)\mathrm{d}v\right) \mathrm{d}s\mathrm{d}y\right] \mathcal{H}_\alpha. \end{aligned} \tag{3.10}$$

Therefore, by (3.10) and (3.7), we get $c_\alpha(t,x) = U_0(t,x)$ if $\alpha = 0$ and for $|\alpha| \geq 1$,

$$\begin{aligned} c_\alpha(t,x) &= \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r} \varphi_r(x)\varphi_r(y) \sum_{i=1}^{\infty} c_{\alpha-\varepsilon^{(i)}}(s,y) \\ &\quad \times \left(\int_{\mathbb{R}^{d+1}} e_i(v)\phi_H(s,y;v)\mathrm{d}v\right) \mathrm{d}s\mathrm{d}y. \end{aligned} \tag{3.11}$$

We will need the following preliminaries and lemmas to estimate c_α.

The boundedness and Hölder continuity of U_0 are given by the following lemma.

Lemma 3.1 *Assume that $v_0 \in H^{\varrho,2}(D)$ for some $\varrho \geq -2$ and $u_0 \in H^{\rho,2}(D)$ for some $\rho \geq 0$, then $U_0(t,\cdot) \in L^2(D)$, and*

$$\sup_{t\in\mathbb{R}^+} \|U_0(t,\cdot)\| < +\infty. \tag{3.12}$$

Moreover, if $v_0 \in H^{\varrho,2}(D)$ for some $\varrho \geq 0$ and $u_0 \in H^{\rho,2}(D)$ for some $\rho \geq 2$, then, for any $t,s \in \mathbb{R}^+$ with $|t-s| < 1$,

$$\|U_0(t,\cdot) - U_0(s,\cdot)\|^2 \leq C|t-s|^2. \tag{3.13}$$

Proof. It is clear that $\|U_0(t,\cdot)\| \le \|I(t,\cdot)\| + \|J(t,\cdot)\|$, with

$$\begin{aligned} I(t,x) &= \sum_{r\in\mathbb{N}^d} \frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D \upsilon_0(y)\varphi_r(y)\mathrm{d}y, \\ J(t,x) &= \sum_{r\in\mathbb{N}^d} \cos(\lambda_r t)\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y. \end{aligned}$$

By (3.2), (3.3) and the assumptions of υ_0, we have

$$\|I(t,\cdot)\|^2 = \sum_{r\in\mathbb{N}^d} |< I(t,\cdot),\varphi_r >|^2$$
$$= \sum_{r\in\mathbb{N}^d} \frac{\sin^2(\lambda_r t)}{\lambda_r^2}|<\varphi_r,\upsilon_0>|^2 \le C\sum_{r\in\mathbb{N}^d}\frac{|<\varphi_r,\upsilon_0>|^2}{|r|^4} < +\infty.$$

Similarly, we have

$$\|J(t,\cdot)\|^2 \le C\sum_{r\in\mathbb{N}^d} |<\varphi_r,u_0>|^2 < +\infty.$$

Thus, the first part of the lemma if proved. For the second part, we have

$$\|U_0(t,\cdot) - U_0(s,\cdot)\|^2 \le 2(\|I_1(t,s;\cdot)\|^2 + \|I_2(t,s;\cdot)\|^2),$$

where

$$\begin{aligned} I_1(t,s;x) &= \sum_{r\in\mathbb{N}^d} \frac{\sin(\lambda_r t)-\sin(\lambda_r s)}{\lambda_r}\varphi_r(x)\int_D \upsilon_0(y)\varphi_r(y)\mathrm{d}y, \\ I_2(t,s;x) &= \sum_{r\in\mathbb{N}^d} (\cos(\lambda_r t)-\cos(\lambda_r s))\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y. \end{aligned}$$

By the assumptions on u_0, υ_0, we get

$$\begin{aligned} \|I_1(t,s;\cdot)\|^2 &= \sum_{r\in\mathbb{N}^d} |< I_1(t,s;\cdot),\varphi_r >|^2 \\ &= \sum_{r\in\mathbb{N}^d} \frac{|\sin(\lambda_r t)-\sin(\lambda_r s)|^2}{\lambda_r^2}|<\varphi_r,\upsilon_0>|^2 \\ &\le C(t-s)^2\sum_{r\in\mathbb{N}^d} |<\varphi_r,\upsilon_0>|^2 \le C(t-s)^2, \end{aligned}$$

and

$$\begin{aligned} \|I_2(t,s;\cdot)\|^2 &= \sum_{r\in\mathbb{N}^d} |< I_2(t,s;\cdot),\varphi_r >|^2 \\ &= \sum_{r\in\mathbb{N}^d} |\cos(\lambda_r t)-\cos(\lambda_r s)|^2|<\varphi_r,\upsilon_0>|^2 \\ &\le C(t-s)^2\sum_{r\in\mathbb{N}^d} |\lambda_r|^2|<\varphi_r,\upsilon_0>|^2 \\ &\le C(t-s)^2\sum_{r\in\mathbb{N}^d} (1+|r|^2)^2|<\varphi_r,\upsilon_0>|^2 \le C(t-s)^2. \end{aligned}$$

Bringing together the above estimates, we obtain

$$\|U_0(t,\cdot)-U_0(s,\cdot)\|^2 \le C(t-s)^2,$$

with a constant C independent of t. Thus the lemma is proved. ¶

Let $\Phi_i(t,x)=\int_{\mathbb{R}^{d+1}} e_i(s,y)\phi_H(t,x;s,y)\mathrm{d}s\mathrm{d}y$, then we have the following lemma.

Lemma 3.2 *There exists constant $C=C(h)$, such that*

$$\sup_{x\in D}|\Phi_i(t,x)| \le C(h)i^{\sigma(h)}(1\vee t^{h-\frac{1}{2}}), \quad i\in\mathbb{N}, \tag{3.14}$$

where $h=\max\{h_0,h_1,\ldots,h_d\}, 1\vee t^{h-\frac{1}{2}}=\max\{1,t^{h-\frac{1}{2}}\}$ *and*

$$\sigma(h)=\begin{cases} \frac{1}{4}(h-\frac{2}{3}), & \text{if } h<\frac{2}{3},\\ \frac{1}{2}(h-\frac{2}{3}), & \text{if } h\ge\frac{2}{3}.\end{cases} \tag{3.15}$$

Proof. By the definitions of e_i and ϕ_H, we have

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}} e_i(s,y)\phi_H(t,x;s,y)\mathrm{d}s\mathrm{d}y\\ =\ &\int_{\mathbb{R}}\eta^{h_0}_{\delta^i_0}(s)\phi_{h_0}(t,s)\mathrm{d}s\prod_{k=1}^{d}\int_{\mathbb{R}}\eta^{h_k}_{\delta^i_k}(y)\phi_{h_k}(x,y)\mathrm{d}y.\end{aligned} \tag{3.16}$$

Let $\frac{1}{2}<h<1$ and $n\in\mathbb{N}$. It is proved in [?] that

$$\phi_h(t,s)=c_h^2\int_{-\infty}^{t\wedge s}(s-u)^{h-\frac{3}{2}}(t-u)^{h-\frac{3}{2}}\mathrm{d}u. \tag{3.17}$$

where $c_h=\sqrt{h(2h-1)\Gamma(3/2-h)/(\Gamma(h-1/2)\Gamma(2-2h))}$. Therefore, by (2.15) and Lemma 3.2, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\eta^h_n(s)\phi_h(t,s)\mathrm{d}s\right| = c_h^2\left|\int_{\mathbb{R}}\eta^h_n(s)\int_{-\infty}^{t\wedge s}(s-u)^{h-\frac{3}{2}}(t-u)^{h-\frac{3}{2}}\mathrm{d}u\mathrm{d}s\right|\\ =\ & c_h^2\left|\int_{-\infty}^{t}(t-u)^{h-\frac{3}{2}}\mathrm{d}u\int_u^{+\infty}(s-u)^{h-\frac{3}{2}}\eta^h_n(s)\mathrm{d}s\right|\\ =\ & c_h^2\left|\int_{-\infty}^{t}(t-u)^{h-\frac{3}{2}}\frac{1}{c_h}I_-^{h-\frac{1}{2}}\eta^h_n(u)\mathrm{d}u\right| = c_h\left|\int_{-\infty}^{t}(t-u)^{h-\frac{3}{2}}\xi_n(u)\mathrm{d}u\right|\\ \le\ & c_h\left[\int_{|u|\le 2\sqrt{n}}(t-u)^{h-\frac{3}{2}}n^{-\frac{1}{12}}\mathrm{d}u+\int_{|u|>2\sqrt{n}}(t-u)^{h-\frac{3}{2}}e^{-\gamma u^2}\mathrm{d}u\right].\end{aligned} \tag{3.18}$$

If $0\le t\le 2\sqrt{n}$, it follows from (3.18) that

$$\begin{aligned}&\int_{\mathbb{R}}\eta^h_n(s)\phi_h(t,s)\mathrm{d}s\\ \le\ & c_h\left[\int_{-\infty}^{-2\sqrt{n}}\left[(t-u)^{h-\frac{3}{2}}\frac{1}{u}\right]ue^{-\gamma u^2}\mathrm{d}u+\int_{-2\sqrt{n}}^{t}(t-u)^{h-\frac{3}{2}}n^{-\frac{1}{12}}\mathrm{d}u\right]\\ \le\ & c_h\left[\frac{e^{-4\gamma n}}{4\gamma}n^{-\frac{1}{2}}(t+2\sqrt{n})^{h-\frac{3}{2}}+\frac{1}{h-\frac{1}{2}}n^{-\frac{1}{12}}(t+2\sqrt{n})^{h-\frac{1}{2}}\right]\\ \le\ & C(h)n^{-\frac{1}{12}}(t+2\sqrt{n})^{h-\frac{1}{2}} \le C(h)n^{\frac{1}{2}(h-\frac{2}{3})}(1\vee t^{h-\frac{1}{2}}).\end{aligned} \tag{3.19}$$

For the case of $t > 2\sqrt{n}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \eta_n^h(s)\phi_h(t,s)\mathrm{d}s \le c_h \Bigg[\int_{-2\sqrt{n}}^{2\sqrt{n}} (t-u)^{h-\frac{3}{2}} n^{-\frac{1}{12}} \mathrm{d}u \\
+ & \int_{-\infty}^{-2\sqrt{n}} \left[(t-u)^{h-\frac{3}{2}} \frac{1}{u} \right] u e^{-\gamma u^2} \mathrm{d}u + \int_{2\sqrt{n}}^{t} (t-u)^{h-\frac{3}{2}} e^{-\gamma u^2} \mathrm{d}u \Bigg] \\
\le & \ c_h \left[\frac{e^{-4\gamma n}(t+2\sqrt{n})^{h-\frac{3}{2}}}{4\gamma n^{\frac{1}{2}}} + \frac{n^{-\frac{1}{12}}(t+2\sqrt{n})^{h-\frac{1}{2}} - (n^{-\frac{1}{12}} - e^{-4\gamma n})(t-2\sqrt{n})^{h-\frac{1}{2}}}{h-\frac{1}{2}} \right] \\
\le & \ C(h) n^{-\frac{1}{12}} (t+2\sqrt{n})^{h-\frac{1}{2}} \le C(h) n^{\frac{1}{2}(h-\frac{2}{3})} (1 \vee t^{h-\frac{1}{2}}). \qquad (3.20)
\end{aligned}$$

With the estimates of (3.19), (3.20), we obtain from (3.16) that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{d+1}} e_i(s,y)\phi_H(t,x;s,y) \right| \le C(h_0,h_1,\cdots,h_d) \prod_{k=0}^{d} (\delta_k^i)^{\frac{1}{2}(h_k-\frac{2}{3})} (1 \vee t^{h_1-\frac{1}{2}}) \\
\le & \ C(h_0,h_1,\cdots,h_d) (\prod_{k=0}^{d} \delta_k^i)^{\frac{1}{2}(h-\frac{2}{3})} (1 \vee t^{h-\frac{1}{2}}) \le C(h) i^{\sigma(h)} (1 \vee t^{h-\frac{1}{2}}), \qquad (3.21)
\end{aligned}$$

where $h = \max\{h_0, h_1, \cdots, h_d\}$ and $\sigma(h)$ is given by (3.15). This proves the lemma. ¶

Using the above lemmas, now we can estimate $c_\alpha(t,x)$.

Lemma 3.3 *Assume that all the assumptions of Lemma 3.1 are satisfied, then for every $\alpha \in \mathcal{J}$ and $t \ge 0$, $c_\alpha(t,\cdot) \in L^2(D)$. Moreover, there exists constant $C = C(h)$, such that*

$$\|c_\alpha(t,\cdot)\|^2 \le C(h)^{2|\alpha|} \frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!} (t^2 \vee t^{2h+1})^{|\alpha|}. \qquad (3.22)$$

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $C_\alpha(t) = \|c_\alpha(t,\cdot)\|^2$. By Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
C_\alpha(t) &= \sum_{r\in\mathbb{N}^d} | < c_\alpha(t,\cdot), \varphi_r > |^2 \\
&= \sum_{r\in\mathbb{N}^d} \left| \int_0^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r} \varphi_r(y) \sum_{i=1}^{m} c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \right|^2 \\
&\le m \sum_{r\in\mathbb{N}^d} \sum_{i=1}^{m} \left| \int_0^t \frac{\sin(\lambda_r(t-s))}{\lambda_r} \int_D \varphi_r(y) c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}y\mathrm{d}s \right|^2 \\
&\le m \sum_{r\in\mathbb{N}^d} \sum_{i=1}^{m} \int_0^t \left| \frac{\sin(\lambda_r(t-s))}{\lambda_r} \right|^2 \mathrm{d}s \int_0^t \left| \int_D \varphi_r(y) c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}y \right|^2 \mathrm{d}s \\
&\le Ctm \sum_{i=1}^{m} \int_0^t \sum_{r\in\mathbb{N}^d} | < \varphi_r, c_{\alpha-\varepsilon^{(i)}}(s,\cdot)\Phi_i(s,\cdot) > |^2 \mathrm{d}s \\
&\le Ctm \sum_{i=1}^{m} \int_0^t \|c_{\alpha-\varepsilon^{(i)}}(s,\cdot)\Phi_i(s,\cdot)\|^2 \mathrm{d}s. \qquad (3.23)
\end{aligned}$$

Owing to Lemma 3.2, $\sup_{x\in D}|\Phi_i(t,x)| \le C(h)i^{\sigma(h)}(1\vee t^{h-\frac{1}{2}})$. Hence, by (3.23),

$$\begin{aligned} C_\alpha(t) &\le Ctm\sum_{i=1}^m\int_0^t \|c_{\alpha-\varepsilon^{(i)}}(s,\cdot)\Phi_i(s,\cdot)\|^2\mathrm{d}s \\ &\le C(h)^2tm\sum_{i=1}^m i^{2\sigma(h)}\int_0^t C_{\alpha-\varepsilon^{(i)}}(s)(1\vee(s^{h-\frac{1}{2}}))^2\mathrm{d}s \\ &\le C(h)^2(t\vee t^{2h})m\sum_{i=1}^m i^{2\sigma(h)}\int_0^t C_{\alpha-\varepsilon^{(i)}}(s)\mathrm{d}s. \end{aligned} \tag{3.24}$$

Now let $n=|\alpha|=\alpha_1+\cdots+\alpha_m$. By iterating the above equation, we get

$$\begin{aligned} &C_\alpha(t) \le C(h)^2(t\vee t^{2h})m\sum_{i=1}^m i^{2\sigma(h)}\int_0^t C_{\alpha-\varepsilon^{(i)}}(s)\mathrm{d}s \\ \le\ & C(h)^{2n}(t\vee t^{2h})^n[1^{\alpha_1}2^{\alpha-2}\cdots m^{\alpha_m}]\sum_{i_1,i_2,\ldots,i_n=1}^m (i_1i_2\cdots i_n)^{2\sigma(h)}\times \\ &\int_0^t\int_0^s\int_0^{s_1}\cdots\int_0^{s_{n-2}} C_{\alpha-\varepsilon^{(i_1)}-\varepsilon^{(i_2)}\ldots-\varepsilon^{(i_n)}}(s_{n-1})\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1\mathrm{d}s. \end{aligned} \tag{3.25}$$

Since for some $\beta\in\mathcal{J}$ with $\beta_j=0$ we have $C_{\beta-\varepsilon^{(j)}}=0$, and by Lemma 3.1, $C_0(t)=\|U_0(t,\cdot)\|$ is bounded on $\mathbb{R}^+$, thus we derive from (3.25) that

$$\begin{aligned} C_\alpha(t) &\le C(h)^{2n}(t\vee t^{2h})^n[1^{\alpha_1}2^{\alpha-2}\cdots m^{\alpha_m}]\frac{n!}{\alpha_1!\cdots\alpha_m!}[1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}]^{2\sigma(h)}\times \\ &\quad \int_0^t\int_0^s\int_0^{s_1}\cdots\int_0^{s_{n-2}} C_0(s_{n-1})\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1\mathrm{d}s \\ &\le C(h)^{2n}(t\vee t^{2h})^n\frac{n!}{\alpha_1!\cdots\alpha_m!}[1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}]^{2\sigma(h)+1}\frac{t^n}{n!} \\ &= C(h)^{2|\alpha|}\frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!}(t^2\vee t^{2h+1})^{|\alpha|}. \end{aligned} \tag{3.26}$$

This proves the lemma. ¶

The following lemma will be used in the proof of our main results.

Lemma 3.4 *Let constants $a>0, b\in\mathbb{R}, q\in\mathbb{R}$. Then the sufficient and necessary condition for*

$$\sum_{\alpha\in\mathcal{J}}(a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha}<\infty \tag{3.27}$$

is $q>\max\{b+1, b\log_2 a\}$.

Proof. For $\alpha=(\alpha_1,\alpha_2,\ldots,)\in\mathcal{J}$, let $\alpha_0=\max\{j;\alpha_j\ne 0\}$. Then

$$\sum_{\alpha\in\mathcal{J}}(a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha}=\sum_{n=0}^\infty\sum_{\alpha\in\mathcal{J},\alpha_0=n}(a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha}:=\sum_{n=0}^\infty a_n.$$

We assume $q > b\log_2 a$ and $q > b+1$, then $a_0 = 1$ and for $n \geq 1$,

$$\begin{aligned} a_n &= \sum_{\alpha\in\mathcal{J},\alpha_0=n} (a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha} = \sum_{\alpha_1,\ldots,\alpha_{n-1}\geq 0,\alpha_n\geq 1} (a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha} \\ &= \left(\sum_{\alpha_n=1}^{\infty}(an)^{b\alpha_n}(2n)^{-q\alpha_n}\right)\left[\prod_{j=1}^{n-1}\left(\sum_{\alpha_j=0}^{\infty}(aj)^{b\alpha_j}(2j)^{-q\alpha_j}\right)\right] \\ &= \left(\sum_{\alpha_n=1}^{\infty}\left[\frac{(an)^b}{(2n)^q}\right]^{\alpha_n}\right)\left[\prod_{j=1}^{n-1}\left(\sum_{\alpha_j=0}^{\infty}\left[\frac{(aj)^b}{(2j)^q}\right]^{\alpha_j}\right)\right] \\ &= \frac{(an)^b}{(2n)^q-(an)^b}\prod_{j=1}^{n-1}\frac{(2j)^q}{(2j)^2-(aj)^b} = \frac{(an)^b}{(2n)^q}\prod_{j=1}^{n}\frac{(2j)^q}{(2j)^2-(aj)^b}. \end{aligned} \tag{3.28}$$

By (3.28), we have

$$\begin{aligned} \frac{a_n}{a_{n+1}} &= \frac{(an)^b(2n+2)^q}{(an+a)^b(2n)^q}\times\frac{(2n+2)^q-(an+a)^b}{(2n+2)^q} \\ = &\left(\frac{n}{n+1}\right)^b\left[\left(\frac{n+1}{n}\right)^q-\frac{(an+a)^b}{(2n)^q}\right] = \left(1+\frac{1}{n}\right)^{q-b}-\frac{(an)^b}{(2n)^q}. \end{aligned} \tag{3.29}$$

This gives

$$\frac{a_n}{a_{n+1}}-1 \geq \frac{q-b}{n}-\frac{(an)^b}{(2n)^q}. \tag{3.30}$$

Hence

$$\lim\inf_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) \geq q-b > 1. \tag{3.31}$$

Therefore, by Abel's criterion for convergence, $\sum_{n=0}^{\infty} a_n < \infty$. Conversely, if $q \leq b\log_2 a$, then, by (3.28), $a_n = \infty$. If $q = b+1$, then, by (3.29), $\lim\inf_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = 1-\frac{1}{2}(\frac{a}{2})^b < 1$, by Abel's criterion, $\sum_{n=0}^{\infty} a_n = \infty$. Hence, condition $q > \max\{b+1, b\log_2 a\}$ is sufficient and necessary for the convergence of (3.27). ¶

We now present our main results.

Theorem 3.1 *Assume that all the conditions in Lemma 3.1 are satisfied, then there exists a unique solution of (3.1) in the sense of Definition 3.1.*

Proof. From the above analyses and lemmas, we know that the stochastic field $u(t,x)$ with a formal expansions

$$u(t,x) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t,x)\mathcal{H}_\alpha, \tag{3.32}$$

where $c_\alpha(t,x) = U_0(t,x)$ if $\alpha = 0$ and for $|\alpha| \geq 1$,

$$\begin{aligned} c_\alpha(t,x) &= \sum_{r\in\mathbb{N}^d}\int_0^t\int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)\sum_{i=1}^{\infty} c_{\alpha-\varepsilon^{(i)}}(s,y) \\ &\quad\times\left(\int_{\mathbb{R}^{d+1}} e_i(v)\phi_H(s,y;v)\mathrm{d}v\right)\mathrm{d}s\mathrm{d}y, \end{aligned} \tag{3.33}$$

is a solution of (3.1) in the sense of Definition 3.1, if for almost every $x \in D$ and $t \geq 0$, it belongs to $(\mathcal{S})^*_{H,-q}$ for some $q \in \mathbb{N}$ and $\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x < +\infty, (\forall t \geq 0)$. Now we compute $\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x$. By Lemma 3.3, we have

$$\begin{aligned}\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x &= \sum_{\alpha\in\mathcal{J}} \alpha!\|c_\alpha(t,\cdot)\|^2(2\mathbb{N})^{-q\alpha}\\ \leq \sum_{\alpha\in\mathcal{J}} & \frac{C(h)^{2|\alpha|}(\mathbb{N})^{(2\sigma(h)+1)\alpha}(t^2\vee t^{2h+1})^{|\alpha|}(2\mathbb{N})^{-q\alpha}}{\alpha!}. \end{aligned} \tag{3.34}$$

It is prove in [13] that for $\alpha \in \mathcal{J}$,

$$|\alpha|! \leq \alpha!(2\mathbb{N})^{2\alpha}. \tag{3.35}$$

Therefore,

$$\begin{aligned}\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x &\leq \sum_{\alpha\in\mathcal{J}} \frac{C(h)^{2|\alpha|}(\mathbb{N})^{(2\sigma(h)+1)\alpha}(t^2\vee t^{2h+1})^{|\alpha|}(2\mathbb{N})^{-q\alpha}(2\mathbb{N})^{2\alpha}}{|\alpha|!}\\ = \sum_{n=0}^{\infty}\frac{(t^2\vee t^{2h+1})^n}{n!} & \sum_{|\alpha|=n}\left(C(h)^{\frac{2}{2\sigma(h)+1}}\mathbb{N}\right)^{(2\sigma(h)+1)\alpha}(2\mathbb{N})^{-(q-2)\alpha}. \end{aligned} \tag{3.36}$$

If we choose a natural number $q > \max\{4+2\sigma(h), 2+2\log_2 C(h)\}$, then by Lemma 3.4, $\sum_{|\alpha|=n}\left(C(h)^{\frac{1}{2\sigma(h)+1}}\mathbb{N}\right)^{(2\sigma(h)+1)\alpha}(2\mathbb{N})^{-(q-2)\alpha} < \infty$. Thus

$$\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x \leq C(h)\sum_{n=0}^{\infty}\frac{(t^2\vee t^{2h+1})^n}{n!} = C(h)e^{t^2\vee t^{2h+1}}. \tag{3.37}$$

Now we prove the uniqueness. Assume that stochastic fields $u(t,x), v(t,x)$ with formal expansions

$$u(t,x) = \sum_{\alpha\in\mathcal{J}} a_\alpha(t,x)\mathcal{H}_\alpha,\ v(t,x) = \sum_{\alpha\in\mathcal{J}} b_\alpha(t,x)\mathcal{H}_\alpha \tag{3.38}$$

are two solutions of (3.1) in the sense of Definition 3.1. Then $u(t,x) - v(t,x)$ is a solution of of (3.1) in the sense of Definition 3.1 with zero initial conditions, that is, $a_0(t,x) - b_0(t,x) = 0$. Therefore, similarly to the proof of Lemma 3.3, we can derive that

$$\begin{aligned}&\|a_\alpha(t,\cdot) - b_\alpha(t,\cdot)\|^2\\ \leq\ & C(h)^{2n}(t\vee t^{2h})^n[1^{\alpha_1}2^{\alpha-2}\cdots m^{\alpha_m}]\frac{n!}{\alpha_1!\cdots\alpha_m!}[1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}]^{2\sigma(h)}\times\\ & \int_0^t\int_0^s\int_0^{s_1}\cdots\int_0^{s_{n-2}}\|a_0(s_{n-1},\cdot) - b_0(s_{n-1},\cdot)\|^2\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1\mathrm{d}s = 0. \end{aligned} \tag{3.39}$$

Thus the theorem follows. ¶

The following corollary deals with the asymptotic properties of $u(t,x)$ in $(\mathcal{S})^*_{H,-q}$.

Corollary 3.1 *Let $u(t,x)$ be the solution of (3.1) and $q > \max\{4+2\sigma(h), 2+2\log_2 C(h)\}$, where $C(h)$ is given by (3.37), then $u(t,x) \in (\mathcal{S})^*_{H,-q}$ for almost all $(t,x) \in \mathbb{R}^+ \times [0,1]$, and*

$$\limsup_{t\to+\infty} \frac{\ln(\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x)}{t^{2h+1}} < \infty. \tag{3.40}$$

Proof. By (3.37), we know that for sufficiently large t,

$$\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x \quad \le \quad C(h)e^{t^{2h+1}}. \tag{3.41}$$

Therefore

$$\limsup_{t\to+\infty} \frac{\ln(\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x)}{t^{2h+1}} \le \limsup_{t\to+\infty} \frac{\ln(C(h)) + t^{2h+1}}{t^{2h+1}} = 1. \tag{3.42}$$

This shows the corollary. ¶

We now study the Hölder property of the trajectories of the solution of equation (3.1) in the distribution space $(\mathcal{S})^*_{H,-q}$.

Theorem 3.2 *Fix $T > 0$ and let $u(t,x)$ be the solution of (3.1) in time interval $[0,T]$ with initial conditions $v_0 \in H^{\beta_1,2}(D)$ for some $\beta_1 \ge 0$ and $u_0 \in H^{\beta_2,2}$ for some $\beta_2 \ge 2$, then there exists $q \in \mathbb{N}$, such that for any $t,\tau \in [0,T]$ with $|t-\tau| < 1$,*

$$\int_D \|u(t,x) - u(\tau,x)\|^2_{H,-q}\mathrm{d}x \le C|t-\tau|. \tag{3.43}$$

Proof. Let $0 \le \tau < t \le T$. First we estimate the term $\|c_\alpha(t,\cdot) - c_\alpha(\tau,\cdot)\|$. Since

$$\begin{aligned}
& c_\alpha(t,x) - c_\alpha(\tau,x) \\
= & \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y) \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
& -\sum_{r\in\mathbb{N}^d} \int_0^\tau \int_D \frac{\sin(\lambda_r(\tau-s))}{\lambda_r}\varphi_r(x)\varphi_r(y) \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
= & \sum_{r\in\mathbb{N}^d} \int_\tau^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y) \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
& +\sum_{r\in\mathbb{N}^d} \int_0^\tau \int_D \frac{\sin(\lambda_r(t-s)) - \sin(\lambda_r(\tau-s))}{\lambda_r}\varphi_r(x)\varphi_r(y) \\
& \times \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
\doteq & I(t,\tau;x) + J(t,\tau;x).
\end{aligned}$$

By using the methods used in the proof of Lemma 3.3, we can derive that

$$\begin{aligned}
\|I(t,\tau;\cdot)\|^2 & \le (t-\tau)^2 C(h)^{2|\alpha|} \frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!}(T^2 \vee T^{2h+1})^{|\alpha|}, \\
\|J(t,\tau;\cdot)\|^2 & \le (t-\tau) C(h)^{2|\alpha|} \frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!}(T^2 \vee T^{2h+1})^{|\alpha|}.
\end{aligned}$$

Thus we have $\|c_\alpha(t,\cdot)-c_\alpha(\tau,\cdot)\|^2 \le C(t-\tau)C(h)^{2|\alpha|}\frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!}(T^2\vee T^{2h+1})^{|\alpha|}$. By the definition of the norm $\|\cdot\|^2_{H,-q}$, we can divide the term $\int_D \|u(t,x)-u(\tau,x)\|^2_{H,-q}\mathrm{d}x$ into two terms, that is

$$\begin{aligned}
&\int_D \|u(t,x)-u(\tau,x)\|^2_{H,-q}\mathrm{d}x \\
\le\ & 2\left(\|U_0(t,\cdot)-U_0(\tau,\cdot)\|^2+\sum_{\alpha\in\mathcal{J},|\alpha|\ge 1}\alpha!\|c_\alpha(t,\cdot)-c_\alpha(\tau,\cdot)\|^2(2\mathbb{N})^{-q\alpha}\right) \\
\le\ & C(t-\tau)\sum_{\alpha\in\mathcal{J}}\left[(C(h)(T\vee T^{h+\frac{1}{2}}))^{\frac{2}{2\sigma(h)+1}}\mathbb{N}\right]^{(2\sigma(h)+1)\alpha}(2\mathbb{N})^{-q\alpha}.
\end{aligned}$$

Bringing together the above estimates and by Lemma 3.1, Lemma 3.3, we obtain that

$$\int_D \|u(t,x)-u(\tau,x)\|^2_{H,-q}\mathrm{d}x \quad\le\quad C(h,T)(t-\tau),$$

if $q>\max\{2\sigma(h)+2, 2\log_2 C(h)(T\vee T^{h+\frac{1}{2}})\}$, where $C(h)$ is given by (3.37). The proof is finished. ¶

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FOURIER SERIES OF SUMS OF PRODUCTS OF EULER FUNCTIONS AND THEIR APPLICATIONS

TAEKYUN KIM, DAE SAN KIM, GWAN-WOO JANG, AND LEE CHAE JANG

Abstract. We consider three types of sums of products of Euler functions and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

1. Introduction

Let $E_m(x)$ be the Euler polynomials given by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\quad \text{(see [1, 2, 14])}. \tag{1.1}$$

For any real number x, we let

$$<x>=x-[x]\in[0,1) \tag{1.2}$$

denote the fractional part of x.

Here we will consider the following three types of sums of products of Euler functions and derive their Fourier series expansions. Further, we will express each of them in terms of Bernoulli functions $B_m(<x>)$.

(1) $\alpha_m(<x>)=\sum_{k=0}^{m}E_k(<x>)E_{m-k}(<x>),(m\geq 1)$;
(2) $\beta_m(<x>)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>),(m\geq 1)$;
(3) $\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)E_{m-k}(<x>),(m\geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [8,13,15,18]).

As to $\gamma_m(<x>)$, we note that the polynomial identity (1.3) follows immediately from (4.16), which is in turn derived from the Fourier series expansion of $\gamma_m(<x>)$.

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(x)E_{m-k}(x)\\&=-\frac{4}{m}\sum_{s=0,s\neq 1}^{m}\binom{m}{s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})B_s(x),\end{aligned} \tag{1.3}$$

where $H_m=\sum_{j=1}^{m}\frac{1}{j}$ are the harmonic numbers.

The obvious polynomial identities can be derived also for $\alpha_m(<x>)$ and $\beta_m(<x>)$ from (2.21) and (2.24), and (3.15) and (3.18), respectively. It is remarkable that from the Fourier series expansion of the

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function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(<x>) B_{m-k}(<x>)$ we can derive the following corresponding polynomial identity:

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x) B_{m-k}(x) \\ =&\frac{2}{m^2}\left(B_m+\frac{1}{2}\right)+\frac{2}{m}\sum_{k=1}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k(x)+\frac{2}{m}H_{m-1}B_m(x), \quad (m \geq 2).\end{aligned} \tag{1.4}$$

Simple modification of (1.3) yields

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} B_{2k}(x) B_{2m-2k}(x)+\frac{2}{2m-1}B_1(x)B_{2m-1}(x) \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}(x)+\frac{1}{m}H_{2m-1}B_{2m}(x) \\ &+\frac{2}{2m-1}B_1(x)B_{2m-1}, \quad (m \geq 2).\end{aligned} \tag{1.5}$$

Letting $x=0$ in (1.4) gives a slightly different version of the well-known Miki's identity (see [3,6,16,17]):

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} B_{2k} B_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}+\frac{1}{m}H_{2m-1}B_{2m}, \quad (m \geq 2).\end{aligned} \tag{1.6}$$

Setting $x=\frac{1}{2}$ in (1.5) with $\overline{B}_m=\left(\frac{1-2^{m-1}}{2^{m-1}}\right)B_m=\left(2^{1-m}-1\right)B_m=B_m\left(\frac{1}{2}\right)$, we have

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} \overline{B}_{2k} \overline{B}_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}\overline{B}_{2m-2k}+\frac{1}{m}H_{2m-1}\overline{B}_{2m}, \quad (m \geq 2),\end{aligned} \tag{1.7}$$

which is the Faber-Pandharipande-Zagier identity (see [4]). Some related works can be found in [9-12].

2. Fourier series of functions of the first type

In this section, we consider the function

$$\alpha_m(<x>)=\sum_{k=0}^{m} E_k(<x>)E_{m-k}(<x>), \quad (m \geq 1) \tag{2.1}$$

defined on $(-\infty,-\infty)$ which is periodic of period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >)e^{-2\pi i n x}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx. \end{aligned} \tag{2.3}$$

Before proceeding further, we observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m}(kE_{k-1}(x)E_{m-k}(x) + (m-k)E_k(x)E_{m-k-1}(x)) \\ &= \sum_{k=1}^{m}(kE_{k-1}(x)E_{m-k}(x) + \sum_{k=0}^{m-1}(m-k)E_k(x)E_{m-k-1}(x)) \\ &= \sum_{k=0}^{m-1}(k+1)E_k(x)E_{m-k-1}(x) + \sum_{k=0}^{m-1}(m-k)E_k(x)E_{m-k-1}(x)) \\ &= (m+1)\sum_{k=0}^{m-1}E_k(x)E_{m-1-k}(x) \\ &= (m+1)\alpha_{m-1}(x). \end{aligned} \tag{2.4}$$

From this, we have $\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x)$.Then we have

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{2.5}$$

Noting that $E_m(x+1) + E_m(x) = 2x^m$, we see that $E_m(1) + E_m(0) = 2\delta_{m,0}$. So, we have

$$\begin{aligned} &\alpha_m(1) - \alpha_m(0) \\ &= \sum_{k=0}^{m}(E_k(1)E_{m-k}(1) - E_k(0)E_{m-k}(0)) \\ &= \sum_{k=0}^{m}((-E_k(0) + 2\delta_{k,0})(-E_{m-k}(0) + 2\delta_{m-k,0}) - E_k(0)E_{m-k}(0)) \\ &= \sum_{k=0}^{m}(-2\delta_{m-k,0}E_k(0) - 2E_{m-k}(0)\delta_{k,0} + 4\delta_{k,0}\delta_{m-k,0}) \\ &= 4\delta_{m,0} - 4E_m(0), \quad (m \geq 0). \end{aligned} \tag{2.6}$$

Thus, for $m \geq 1$,

$$\alpha_m(1) - \alpha_m(0) = -4E_m. \tag{2.7}$$

Also,

$$\begin{aligned} \int_0^1 \alpha_m(x)dx &= \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)) \\ &= -\frac{4}{m+2}E_{m+1}. \end{aligned} \tag{2.8}$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} (\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n} \int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} + \frac{2}{\pi i n} E_m \\
&= \frac{m+1}{2\pi i n} \left(\frac{m}{2\pi i n} A_n^{(m-2)} + \frac{2}{\pi i n} E_{m-1} \right) + \frac{2}{\pi i n} E_n \\
&= \frac{(m+1)m}{(2\pi i n)^2} A_n^{(m-2)} + \frac{m+1}{2\pi i n} \frac{2}{\pi i n} E_{m-1} + \frac{2}{\pi i n} E_n \\
&= \frac{(m+1)m}{(2\pi i n)^2} \left(\frac{m-1}{2\pi i n} A_n^{(m-3)} + \frac{2}{\pi i n} E_{m-2} \right) + \frac{m+1}{2\pi i n} \frac{2}{\pi i n} E_{m-1} + \frac{2}{\pi i n} E_m \\
&= \frac{(m+1)m(m-1)}{(2\pi i n)^3} A_n^{(m-3)} + \frac{(m+1)m}{(2\pi i n)^2} \frac{2}{\pi i n} E_{m-2} + \frac{m+1}{2\pi i n} \frac{2}{\pi i n} E_{m-1} + \frac{2}{\pi i n} E_m \\
&= \cdots \\
&= \frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} A_n^{(1)} + \sum_{k=1}^{m-1} \frac{(m+1)_{k-1}}{(2\pi i n)^{k-1}} \frac{2}{\pi i n} E_{m-k+1},
\end{aligned} \tag{2.9}$$

where

$$A_n^{(1)} = \int_0^1 \alpha_1(x) e^{-2\pi i n x} dx = \int_0^1 (2x-1) e^{-2\pi i n x} dx = -\frac{1}{\pi i n}. \tag{2.10}$$

Hence

$$\begin{aligned}
A_n^{(m)} &= -\frac{2(m+1)_{m-1}}{(2\pi i n)^m} + 4 \sum_{k=1}^{m-1} \frac{(m+1)_{k-1}}{(2\pi i n)^k} E_{m-k+1} \\
&= \frac{4}{m+2} \sum_{k=1}^{m} \frac{(m+2)_k}{(2\pi i n)^k} E_{m-k+1}.
\end{aligned} \tag{2.11}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = -\frac{4}{m+2} E_{m+1}. \tag{2.12}$$

We recall here that

$$B_1 = -\frac{1}{2}, B_{2n+1} = 0, \text{ for } n \geq 1, (-1)^{n+1} B_{2n} > 0, \tag{2.13}$$

(see [5], Proposition 15.1.1), and

$$E_n = -\frac{1}{n+1}(2^{n+2}-2) B_{n+1} \ (n \geq 0), \quad \text{(see [3])}. \tag{2.14}$$

From these, we see that

$$E_{2n} = 0 \ (n \geq 1), \ E_{2n-1} \neq 0 \ (n \geq 1), \text{ and } E_0 = 1. \tag{2.15}$$

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From these and (2.7), we observe that

$$\begin{aligned} &\alpha_m(1) = \alpha_m(0)(\alpha_m(1) \neq \alpha_m(0)) \Longleftrightarrow E_m = 0(E_m \neq 0) \\ &\Longleftrightarrow m \text{ is an even positive integer } (m \text{ is an odd positive integer}). \end{aligned} \tag{2.16}$$

Here $\alpha_m(< x >)$ is piecewise C^∞. In addition, $\alpha_m(< x >)$ is continuous for all even positive integers m and discontinuous with jump discontinuities at integers for all odd positive integers m.

We now recall the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}. \tag{2.17}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}, \end{cases} \tag{2.18}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. Assume first that m is an even positive integer. Then $\alpha_m(1) = \alpha_m(0)$. Thus $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned} &\alpha_m(< x >) \\ &= -\frac{4}{m+2}E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{4}{m+2} \sum_{k=1}^{m} \frac{(m+2)_k}{(2\pi i n)^k} E_{m-k+1} \right) e^{2\pi i n x} \\ &= -\frac{4}{m+2}E_{m+1} - \frac{4}{m+2} \sum_{k=1}^{m} \binom{m+2}{k} E_{m-k+1} \left(-k! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \right) \\ &= -\frac{4}{m+2}E_{m+1} - \frac{4}{m+2} \sum_{k=2}^{m} \binom{m+2}{k} E_{m-k+1} B_k(< x >) \\ &\quad - \frac{4}{m+2}\binom{m+2}{1} E_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\ &= -\frac{4}{m+2} \sum_{k=0, k\neq 1}^{m} \binom{m+2}{k} E_{m-k+1} B_k(< x >), \end{aligned} \tag{2.19}$$

for all $x \in (-\infty, \infty)$. Hence we get the following theorem.

Theorem 2.1. *Let m be an even positive integer. Then we have the following.*

(a) $\sum_{k=0}^{m} E_k(< x >)E_{m-k}(< x >)$ has the Fourier series expansion

$$\begin{aligned} &\sum_{k=0}^{m} E_k(< x >)E_{m-k}(< x >) \\ &= -\frac{4}{m+2}E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{4}{m+2} \sum_{k=1}^{m} \frac{(m+2)_k}{(2\pi i n)^k} E_{m-k+1} \right) e^{2\pi i n x}, \end{aligned} \tag{2.20}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=0}^{m} E_k(<x>)E_{m-k}(<x>)\\&=-\frac{4}{m+2}\sum_{k=0,k\neq 1}^{m}\binom{m+2}{k}E_{m-k+1}B_k(<x>)\end{aligned} \tag{2.21}$$

for all $x \in (-\infty, \infty)$, where $B_k(<x>)$ is the Bernoulli function.

Assume next that m is an odd positive integer. Then $\alpha_m(1) \neq \alpha_m(0)$, and hence $\alpha_m(<x>)$ is piecewise C^{∞} and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0)+\alpha_m(1)) = \alpha_m(0) - 2E_m, \tag{2.22}$$

for $x \in \mathbb{Z}$., Thus we get the following theorem.

Theorem 2.2. *Let m be an odd positive integer. Then we have the following.*
(a)

$$\begin{aligned}&\sum_{n=-\infty,n\neq 0}^{\infty}\left(\frac{4}{m+2}\sum_{k=1}^{m}\frac{(m+2)_k}{(2\pi in)^k}E_{m-k+1}\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}E_k(<x>)E_{m-k}(<x>), & \text{for } x\in\mathbb{Z}^c,\\ \sum_{k=0}^{m}E_kE_{m-k}-2E_m, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned} \tag{2.23}$$

(b)

$$\begin{aligned}&-\frac{4}{m+2}\sum_{k=1}^{m}\binom{m+2}{k}E_{m-k+1}B_k(<x>)\\&=\sum_{k=0}^{m}E_k(<x>)E_{m-k}(<x>),\end{aligned} \tag{2.24}$$

for $x \in \mathbb{Z}^c$;

$$\begin{aligned}&-\frac{4}{m+2}\sum_{k=2}^{m}\binom{m+2}{k}E_{m-k+1}E_k(<x>)\\&=\sum_{k=0}^{m}E_kE_{m-k}-2E_m,\end{aligned} \tag{2.25}$$

for $x \in \mathbb{Z}$.

3. Fourier series of functions of the second type

In this section, we consider the function

$$\beta_m(<x>) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>), \quad (m \geq 1) \tag{3.1}$$

defined on $(-\infty, -\infty)$ which is periodic of period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(< x >) e^{-2\pi inx} dx \\ &= \int_0^1 \beta_m(x) e^{-2\pi inx} dx. \end{aligned} \tag{3.3}$$

Before proceeding further, we observe the following.

$$\begin{aligned} \beta_m'(x) &= \sum_{k=0}^{m} \left\{ \frac{k}{k!(m-k)!} E_{k-1}(x) E_{m-k}(x) + \frac{m-k}{k!(m-k)!} E_k(x) E_{m-k-1}(x) \right\} \\ &= \sum_{k=1}^{m} \frac{1}{(k-1)!(m-k)!} E_{k-1}(x) E_{m-k}(x) + \sum_{k=0}^{m-1} \frac{1}{k!(m-k-1)!} E_k(x) E_{m-k-1}(x) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k(x) E_{m-1-k}(x) + \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k(x) E_{m-1-k}(x) \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.4}$$

So, $\beta_m'(x) = 2\beta_{m-1}(x)$. From this, we have

$$\left(\frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x) \tag{3.5}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2}(\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.6}$$

Using $E_m(1) + E_m(0) = 2\delta_{m,0}$, we observe that

$$\begin{aligned} \beta_m(1) - \beta_m(0) &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left(E_k(1) E_{m-k}(1) - E_k(0) E_{m-k}(0) \right) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left\{ (-E_k(0) + 2\delta_{k,0})(-E_{m-k}(0) + 2\delta_{m-k,0}) - E_k(0) E_{m-k}(0) \right\} \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left\{ -2\delta_{k,0} E_{m-k}(0) - 2E_k(0)\delta_{m-k,0} + 4\delta_{k,0}\delta_{m-k,0} \right\} \\ &= -\frac{4}{m!}(E_m(0) - \delta_{m,0}). \end{aligned} \tag{3.7}$$

So, for $m \geq 1$,

$$\beta_m(1) - \beta_m(0) = -\frac{4}{m!} E_m. \tag{3.8}$$

Also, we have

$$\begin{aligned}\int_0^1 \beta_m(x)dx &= \frac{1}{2}(\beta_{m+1}(1)-\beta_{m+1}(0))\\ &= \frac{1}{2}\left(-\frac{4}{(m+1)!}\right)(E_{m+1}(0)-\delta_{m+1,0})\\ &= -\frac{2}{(m+1)!}E_{m+1}.\end{aligned} \tag{3.9}$$

Now, we are ready to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}[\beta_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\beta_m(1)-\beta_m(0)) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{1}{\pi in}B_n^{(m-1)} + \frac{2}{m!}\frac{1}{\pi in}E_m\\ &= \frac{1}{\pi in}\left(\frac{1}{\pi in}B_n^{(m-2)} + \frac{2}{(m-1)!}\frac{1}{\pi in}E_{m-1}\right) + \frac{2}{m!}\frac{1}{\pi in}E_m\\ &= \frac{1}{(\pi in)^2}B_n^{(m-2)} + \frac{2}{(m-1)!}\frac{1}{(\pi in)^2}E_{m-1} + \frac{2}{m!}\frac{1}{\pi in}E_m\\ &= \frac{1}{(\pi in)^2}\left(\frac{1}{\pi in}B_n^{(m-3)} + \frac{2}{(m-2)!}\frac{1}{\pi in}E_{m-2}\right) + \frac{2}{(m-1)!}\frac{1}{(\pi in)^2}E_{m-1} + \frac{2}{m!}\frac{1}{\pi in}E_m\\ &= \frac{1}{(\pi in)^3}B_n^{(m-3)} + \frac{2}{(m-2)!}\frac{1}{(\pi in)^3}E_{m-2} + \frac{2}{(m-1)!}\frac{1}{(\pi in)^2}E_{m-1} + \frac{2}{m!}\frac{1}{\pi in}E_m\\ &= \cdots\\ &= \frac{1}{(\pi in)^{m-1}}B_n^{(1)} + \sum_{k=1}^{m-1}\frac{2}{(m-k+1)!}\frac{1}{(\pi in)^k}E_{m-k+1}\\ &= -\frac{1}{(\pi in)^m} + \sum_{k=1}^{m-1}\frac{2}{(m-k+1)!}\frac{1}{(\pi in)^k}E_{m-k+1}\\ &= 2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi in)^k}.\end{aligned} \tag{3.10}$$

Case 2: $n = 0$. By (3.9), we see that

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = -\frac{2}{(m+1)!}E_{m+1}. \tag{3.11}$$

From (3.8), we observe that

$$\begin{aligned}&\beta_m(1) = \beta_m(0)(\beta_m(1) \neq \beta_m(0)) \iff E_m = 0(E_m \neq 0)\\ &\qquad \iff m \text{ is an even positive integer } (m \text{ is an odd positive integer}).\end{aligned} \tag{3.12}$$

Here $\beta_m(<x>)$ is piecewise C^∞. In addition, $\beta_m(<x>)$ is continuous for all even positive integers m and discontinuous with jump discontinuities at integers for all odd positive integers m.

Assume first that m is an even positive integer. Then $\beta_m(1) = \beta_m(0)$. So $\beta_m(<x>)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(<x>)$ converges uniformly to $\beta_m(<x>)$, and

$$\begin{aligned}
&\beta_m(<x>) \\
&= -\frac{2}{(m+1)!}E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi i n)^k}\right)e^{2\pi i n x} \\
&= -\frac{2}{(m+1)!}E_{m+1} - \frac{1}{(m+1)!}\sum_{k=1}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}\left(-k!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^k}\right) \\
&= -\frac{2}{(m+1)!}E_{m+1} - \frac{1}{(m+1)!}\sum_{k=2}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(<x>) \\
&\quad -\frac{4}{(m+1)!}\binom{m+1}{1}E_m \times \begin{cases} B_1(<x>), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z} \end{cases} \\
&= -\frac{1}{(m+1)!}\sum_{k=0,k\neq 1}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(<x>),
\end{aligned} \tag{3.13}$$

for all $x \in (-\infty, \infty)$.

Hence we get the following theorem.

Theorem 3.1. *Let m be an even positive integer. Then we have the following.*

(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>)$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>) \\
&= -\frac{2}{(m+1)!}E_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi i n)^k}\right)e^{2\pi i n x},
\end{aligned} \tag{3.14}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b)

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>) \\
&= -\frac{1}{(m+1)!}\sum_{k=0,k\neq 1}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(<x>),
\end{aligned} \tag{3.15}$$

for all $x \in (-\infty, \infty)$, where $B_k(<x>)$ is the Bernoulli function.

Assume next that m is an odd positive integer. Then $\beta_m(1) \neq \beta_m(0)$, and hence $\beta_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) - \frac{2}{m!}E_m, \tag{3.16}$$

for $x \in \mathbb{Z}$., Thus we get the following theorem.

Theorem 3.2. *Let m be an odd positive integer. Then we have the following.*

(a)

$$\begin{aligned}&\sum_{n=-\infty,n\neq 0}^{\infty}\left(2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi in)^k}\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>), & \text{for} \quad x\in\mathbb{Z}^c,\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_kE_{m-k}-\frac{2}{m!}E_m, & \text{for} \quad x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{3.17}$$

Here the convergence is pointwise.

(b)

$$\begin{aligned}&-\frac{1}{(m+1)!}\sum_{k=1}^{m}2^{k+1}\binom{m+2}{k}E_{m-k+1}B_k(<x>)\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>),\end{aligned}\tag{3.18}$$

for $x\in\mathbb{Z}^c$*;*

$$\begin{aligned}&-\frac{1}{(m+1)!}\sum_{k=2}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(<x>)\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_kE_{m-k}-\frac{2}{m!}E_m,\end{aligned}\tag{3.19}$$

for $x\in\mathbb{Z}$*.*

4. Fourier series of functions of the third type

In this section, we consider the function

$$\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)E_{m-k}(<x>)\tag{4.1}$$

defined on $(-\infty,-\infty)$ which is periodic of period 1. The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},\tag{4.2}$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.\tag{4.3}$$

To proceed further, we note the following.

$$\begin{aligned}\gamma_m'(x) &= \sum_{k=1}^{m-1}\frac{1}{m-k}E_{k-1}(x)E_{m-k}(x) + \sum_{k=1}^{m-1}\frac{1}{k}E_k(x)E_{m-k-1}(x)\\ &= \sum_{k=0}^{m-2}\frac{1}{m-1-k}E_k(x)E_{m-1-k}(x) + \sum_{k=1}^{m-1}\frac{1}{k}E_k(x)E_{m-1-k}(x)\\ &= \frac{2}{m-1}E_{m-1}(x) + (m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}E_k(x)E_{m-1-k}(x)\\ &= (m-1)\gamma_{m-1}(x) + \frac{2}{m-1}E_{m-1}(x).\end{aligned} \tag{4.4}$$

So,

$$\gamma_m'(x) = (m-1)\gamma_m(x) + \frac{2}{m-1}E_{m-1}(x). \tag{4.5}$$

From this, we note that

$$\frac{1}{m}\left(\gamma_{m+1}(x) - \frac{2}{m(m+1)}E_{m+1}(x)\right)' = \gamma_m(x). \tag{4.6}$$

$$\begin{aligned}&\int_0^1 \gamma_m(x)dx\\ &= \left[\frac{1}{m}(\gamma_{m+1}(x) - \frac{2}{m(m+1)}E_{m+1}(x))\right]_0^1\\ &= \frac{1}{m}(\gamma_{m+1}(1) - \gamma_{m+1}(0)) - \frac{2}{m^2(m+1)}(E_{m+1}(1) - E_{m+1}(0))\\ &= \frac{1}{m}(\gamma_{m+1}(1) - \gamma_{m+1}(0)) - \frac{2}{m^2(m+1)}(-2E_{m+1}(0) + 2\delta_{m+1,0})\\ &= \frac{1}{m}(\gamma_{m+1}(1) - \gamma_{m+1}(0)) + \frac{4}{m^2(m+1)}E_{m+1}.\end{aligned} \tag{4.7}$$

Observe that

$$\begin{aligned}\gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}(E_k(1)E_{m-k}(1) - E_k(0)E_{m-k}(0))\\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left((-E_k(0) + 2\delta_{k,0})(-E_{m-k}(0) + 2\delta_{m-k,0}) - E_k(0)E_{m-k}(0)\right)\\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(-2\delta_{k,0}E_{m-k}(0) - 2E_k(0)\delta_{m-k,0} + 4\delta_{k,0}\delta_{m-k,0}\right)\\ &= 0.\end{aligned} \tag{4.8}$$

Thus, for $m \geq 2$, $\gamma_m(1) - \gamma_m(0) = 0$. Also,

$$\int_0^1 \gamma_m(x)dx\frac{1}{m}(\gamma_{m+1}(x) - \gamma_{m+1}(0)) + \frac{4}{m^2(m+1)}E_{m+1}(0) = \frac{4}{m^2(m+1)}E_{m+1}. \tag{4.9}$$

We can show that

$$\int_0^1 E_{m-1}(x)e^{-2\pi inx}dx = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k} E_{m-k}. \tag{4.10}$$

Now, we are ready to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}[\gamma_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma'_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in}\int_0^1 (m-1)\gamma_{m-1}(x) + \frac{2}{m-1}E_{m-1}(x))e^{-2\pi inx}dx \\
&= \frac{1}{2\pi in}\left\{(m-1)\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx + \frac{2}{m-1}\int_0^1 E_{m-1}(x)e^{-2\pi inx}dx\right\} \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} + \frac{1}{2\pi in}\frac{4}{m-1}\sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k} \\
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} + \frac{4}{m-2}\frac{1}{2\pi in}\sum_{k=1}^{m-2}\frac{(m-2)_{k-1}}{(2\pi in)^k}E_{m-k-1}\right) + \frac{1}{2\pi in}\frac{4}{m-1}\sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k} \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} + \frac{m-1}{(2\pi in)^2}\frac{4}{m-2}\sum_{k=1}^{m-2}\frac{(m-2)_{k-1}}{(2\pi in)^k}E_{m-k-1} \\
&\quad + \frac{1}{2\pi in}\frac{4}{m-1}\sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k} \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} + \sum_{l=1}^{m-2}\frac{(m-1)_{l-1}}{(2\pi in)^l}\frac{4}{m-l}\sum_{k=1}^{m-l}\frac{(m-l)_{k-1}}{(2\pi in)^k}E_{m-k-l+1} \\
&= -\frac{2(m-1)!}{(2\pi in)^m} + \sum_{l=1}^{m-2}\frac{(m-1)_{l-1}}{(2\pi in)^l}\frac{4}{m-l}\sum_{k=1}^{m-l}\frac{(m-l)_{k-1}}{(2\pi in)^k}E_{m-k-l+1} \\
&= \sum_{l=1}^{m-1}\frac{(m-1)_{l-1}}{(2\pi in)^l}\frac{4}{m-l}\sum_{k=1}^{m-l}\frac{(m-l)_{k-1}}{(2\pi in)^k}E_{m-k-l+1} \\
&= \frac{4}{m}\sum_{l=1}^{m-1}\frac{1}{m-l}\sum_{k=1}^{m-l}\frac{(m)_{k+l-1}}{(2\pi in)^{k+l}}E_{m-k-l+1} \\
&= \frac{4}{m}\sum_{s=2}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\sum_{l=1}^{s-1}\frac{1}{m-l} \\
&= \frac{4}{m}\sum_{s=2}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}),
\end{aligned} \tag{4.11}$$

where

$$c_n^{(2)} = \int_0^1 \gamma_2(x)e^{-2\pi inx}dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right) e^{-2\pi inx}dx = -\frac{2}{(2\pi in)^2}. \tag{4.12}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{4}{m^2(m+1)}E_{m+1}. \tag{4.13}$$

As $\gamma_m(1) = \gamma_m(0)$, for all $m \geq 2$, $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >)\\
&= \frac{4}{m^2(m+1)}E_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(\frac{4}{m}\sum_{s=2}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)e^{2\pi inx}\\
&= \frac{4}{m^2(m+1)}E_{m+1} - \frac{4}{m}\sum_{s=2}^{m}\binom{m}{s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\left(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&= \frac{4}{m^2(m+1)}E_{m+1} - \frac{4}{m}\sum_{s=2}^{m}\binom{m}{s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})B_s(< x >)\\
&= -\frac{4}{m}\sum_{s=0,s\neq 1}^{m}\binom{m}{s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})B_s(< x >)
\end{aligned} \tag{4.14}$$

Finally, we obtain the following theorem.

Theorem 4.1. *Let m be an integer ≥ 2. Then we have the following.*

(a) $\sum_{k=0}^{m}\frac{1}{k(m-k)}E_k(< x >)E_{m-k}(< x >)$ has the Fourier expansion

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)E_{m-k}(< x >)\\
&= \frac{4}{m^2(m+1)}E_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(\frac{4}{m}\sum_{s=2}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)e^{2\pi inx},
\end{aligned} \tag{4.15}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)E_{m-k}(< x >)\\
&= -\frac{4}{m}\sum_{s=0,s\neq 1}^{m}\binom{m}{s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})B_s(< x >),
\end{aligned} \tag{4.16}$$

for all $x \in (-\infty, \infty)$, where $B_s(< x >)$ is the Bernoulli function.

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Quotient subtraction algebras by an int-soft ideal

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of an intersectional soft subalgebra and an intersectional soft ideal of a subtraction algebra are introduced, and related properties are investigated. A quotient structure of a subtraction algebra using an intersectional soft ideal is constructed.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [16]. In response to this situation Zadeh [17] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [18]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [14]. Maji et al. [13] and Molodtsov [14] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [14] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number

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of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [13] described the application of soft set theory to a decision making problem. Maji et al. [12] also studied several operations on the theory of soft sets. Aktaş and Çağman [4] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [9] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 5, 7, 8, 15] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the an intersectional soft sets in a subalgebra (an ideal) of a subtraction algebra. We introduce the notion of an intersectional soft subalgebra (ideal) of a subtraction algebra, and investigate some related properties. We consider a new construction of a quotient subtraction algebra induced by an int-soft ideal. Also we investigated some related properties.

2. **Preliminaries**

We review some definitions and properties that will be useful in our results (see [10]).

By a *subtraction algebra* we mean an algebra $(X, *, 0)$ with a single binary operation "$-$" that satisfies the following conditions: for any $x, y, z \in S$,

(S1) $x - (y - x) = x$,
(S2) $x - (x - y) = y - (y - x)$,
(S3) $(x - y) - z = (x - z) - y$.

The subtraction determines an order relation on X: $a \leq b$ if and only if $a - b = 0$, where $0 = a - a$ ia an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebras in the sense of [2], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Hence $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true:

(a1) $(x - y) - y = x - y$,
(a2) $x - 0 = x$ and $0 - x = 0$,
(a3) $(x - y) - x = 0$,

(a4) $x - (x - y) \leq y$,
(a5) $(x - y) - (y - x) = x - y$,
(a6) $x - (x - (x - y)) = x - y$,
(a7) $(x - y) - (z - y) \leq x - z$,
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$,
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$,
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$,
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

A non-empty subset A of a subtraction algebra X is called a *subalgebra* ([10]) of X if $x - y \in A$ for any $x, y \in A$. A non-empty subset I of a subtraction algebra X is called an *ideal* ([10]) of X if

(I1) $0 \in I$,
(I2) $x - y, y \in I$ imply $x \in I$ for any $x, y, z \in X$.

A mapping $f : X \to Y$ of subtraction algebras is called a *homomorphism* if $f(x-y) = f(x)-f(y)$ for all $x, y \in X$.

Molodtsov [12] defined the soft set in the following way: Let U be an initial universe set and let E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

A fair $(\tilde{f}, A)$ is called a *soft set* over U, where $\tilde{f}$ is a mapping given by $\tilde{f} : X \to \mathscr{P}(U)$.

In other words, a soft set over U is parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the set $(\tilde{f}, A)$. A soft set over U can bd represented by the set of ordered pairs:

$$(\tilde{f}, A) = \{(x, \tilde{f}(x)) | x \in A, \tilde{f}(x) \in \mathscr{P}(U)\},$$

where $\tilde{f} : X \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$. Clearly, a soft set is not a set.

For a soft set $(\tilde{f}, A)$ of X and a subset γ of U, the γ-*inclusive set* of $(\tilde{f}, A)$, defined to be the set

$$i_A(\tilde{f}; \gamma) := \{x \in A | \gamma \subseteq \tilde{f}(x)\}.$$

3. Intersectional soft subalgebras

In what follows let X denote a subtraction algebra unless otherwise specified.

Definition 3.1. A soft set $(\tilde{f}, X)$ over U is called an *intersectional soft subalgebra* (briefly, *int-soft subalgebra* of X if it satisfies:

(3.1) $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$ for all $x, y \in X$.

Proposition 3.2. *Every int-soft subalgebra $(\tilde{f}, X)$ of a subtraction algebra X satisfies the following inclusion:*

(3.2) $\tilde{f}(x) \subseteq \tilde{f}(0)$ *for all* $x \in X$.

Proof. Using (3.1), we have $\tilde{f}(x) = \tilde{f}(x) \cap \tilde{f}(x) \subseteq \tilde{f}(x - x) = \tilde{f}(0)$ for all $x \in X$. □

Example 3.3. Let $(U = \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a subtraction algebra ([11]) with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Let $(\tilde{f}, X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0 \\ 2\mathbb{Z} & \text{if } x \in \{1, 2\} \\ 4\mathbb{Z} & \text{if } x = 3. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over U.

Theorem 3.4. *A soft set $(\tilde{f}, X)$ of a subtraction algebra X over U is an int-soft subalgebra of X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is a subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

Proof. Assume that $(\tilde{f}, X)$ is an int-soft subalgebra over U. Let $x, y \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x, y \in i_X(\tilde{f}; \gamma)$. Then $\gamma \subseteq \tilde{f}(x)$ and $\gamma \subseteq \tilde{f}(y)$. It follows from (3.1) that $\gamma \subseteq \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$ Hence $x - y \in i_X(\tilde{f}; \gamma)$. Thus $i_X(\tilde{f}, X)$ is a subalgebra of X.

Conversely, suppose that $i_X(\tilde{f}; \gamma)$ is a subalgebra X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$. Let $x, y \in X$, be such that $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in i_X(\tilde{f}; \gamma)$ and so $x - y \in i_X(\tilde{f}; \gamma)$ by assumption. Hence $\tilde{f}(x) \cap \tilde{f}(y) = \gamma_x \cap \gamma_y = \gamma \subseteq \tilde{f}(x - y)$. Thus $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

The subalgebra $i_X(\tilde{f}; \gamma)$ in Theorem 3.4 is called the *inclusive subalgebra* of X.

Theorem 3.5. *Every subalgebra of a subtraction algebra can be represented as a γ-inclusive set of an int-soft subalgebra.*

Proof. Let A be a subalgebra of a subtraction algebra X. For a subset γ of U, define a soft set $(\tilde{f}, X)$ over U by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

Obviously, $A = i_X(\tilde{f}; \gamma)$. We now prove that $(\tilde{f}, X)$ is an int-soft subalgebra over U. Let $x, y \in X$. If $x, y \in A$, then $x - y \in A$ because A is a subalgebra of X. Hence $\tilde{f}(x) = \tilde{f}(y) = \tilde{f}(x - y) = \gamma$, and so $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$. If $x \in A$ and $y \notin A$, then $\tilde{f}(x) = \gamma$ and $\tilde{f}(y) = \emptyset$ which imply that $\tilde{f}(x) \cap \tilde{f}(y) = \gamma \cap \emptyset = \emptyset \subseteq \tilde{f}(x - y)$. Similarly, if $x \notin A$ and $y \in A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$. Obviously, if $x \notin A$ and $y \notin A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$. Therefore $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

Any subalgebra of a subtraction algebra X may not be represented as a γ-inclusive set of an int-soft subalgebra $(\tilde{f}, X)$ over U in general (see the following example).

Example 3.6. Consider a subtraction algebra $X = \{0, 1, 2, 3\}$ which is given Example 3.3. Consider a soft set $(\tilde{f}, X)$ which is given by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{0, 1\} & \text{if } x = 0 \\ \{1\} & \text{if } x \in \{1, 2, 3\} \end{cases}$$

Then $(\tilde{f}, X)$ is an int-soft subalgebra over U. The γ-inclusive set of $(\tilde{f}, X)$ are described as follows:

$$i_X(\tilde{f}; \gamma) = \begin{cases} X & \text{if } \gamma \in \{\emptyset, \{1\}\} \\ \{0\} & \text{if } \gamma \in \{\{0\}, \{0, 1\}\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 2\}$ cannot be a γ-inclusive set $i_X(\tilde{f}; \gamma)$ since there is no $\gamma \subseteq U$ such that $i_X(\tilde{f}; \gamma) = \{0, 2\}$.

We make a new int-soft subalgebra from old one.

Theorem 3.7. *Let $(\tilde{f}, X)$ be a soft set of a subtraction algebra X over U. Define a soft set $(\tilde{f}^*, X)$ of X over U by*

$$\tilde{f}^* : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in i_X(\tilde{f}; \gamma) \\ \emptyset & \text{otherwise} \end{cases}$$

where γ is a non-empty subset subset of U. If $(\tilde{f}, X)$ is an int-soft subalgebra of X, then so is $(\tilde{f}^*, X)$.

Proof. If $(\tilde{f}, X)$ is an int-soft subalgebra over U, then $i_X(\tilde{f};\gamma)$ is a subalgebra of X for all $\gamma \subseteq U$ by Theorem 3.6. Let $x, y \in X$. If $x, y \in i_X(\tilde{f};\gamma)$, then $x - y \in i_X(\tilde{f};\gamma)$. Hence we have

$$\tilde{f}^*(x) \cap \tilde{f}^*(y) = \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x-y) = \tilde{f}^*(x-y).$$

If $x \notin i_X(\tilde{f};\gamma)$ or $y \notin i_X(\tilde{f};\gamma)$, then $\tilde{f}^*(x) = \emptyset$ or $\tilde{f}^*(y) = \emptyset$. Thus

$$\tilde{f}^*(x) \cap \tilde{f}^*(y) = \emptyset \subseteq \tilde{f}^*(x-y).$$

Therefore $(\tilde{f}^*, X)$ is an int-soft subalgebra over U. □

Definition 3.8. A soft set $(\tilde{f}, X)$ over U is called an *intersectional ideal* (briefly, *int-soft ideal*) of X if it satisfies (3.2) and

(3.3) $\tilde{f}(x-y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$ for all $x, y \in X$.

Example 3.9. (1) Let $E = X$ be the set of parameters and let $U = X$ be the universe set where $X = \{0, a, b, c\}$ is a subtraction algebra ([3]) with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Let $(\tilde{f}, X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{0, a\} \\ \gamma_1 & \text{if } x \in \{b, c\} \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$ It is easy to check that $(\tilde{f}, X)$ is an int-soft ideal of X.

(2) In Example 3.3, $(\tilde{f}, X)$ is an int-soft subalgebra of X. But it is not an int-soft ideal of X, since $\tilde{f}(3-1) \cap \tilde{f}(2) = 2\mathbb{Z} \nsubseteq 4\mathbb{Z} = \tilde{f}(3)$.

Proposition 3.10. *Every int-soft ideal $(\tilde{f}, X)$ of a subtraction algebra X satisfies the following inclusion:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow \tilde{f}(y) \subseteq \tilde{f}(x))$,

(ii) $(\forall x, y, z \in X)(\tilde{f}((x-y)-z) \cap \tilde{f}(y) \subseteq \tilde{f}(x-z))$.

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x - y = 0$. Hence $\tilde{f}(y) = \tilde{f}(0) \cap \tilde{f}(y) = \tilde{f}(x-y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$.

(ii) Let $x, y, z \in X$. Using (S3) and (3.3), we have $\tilde{f}((x-y)-z)) \cap \tilde{f}(y) = \tilde{f}((x-z)-y)) \cap \tilde{f}(y) \subseteq \tilde{f}(x-z)$. □

Theorem 3.11. *Let $(\tilde{f}, X)$ be a soft set of X over U. Then $(\tilde{f}, X)$ is an int-soft ideal of X over U if and only if*

(3.4) $(\forall x, y, z \in X)(x - y \leq z \Rightarrow \tilde{f}(z) \cap \tilde{f}(y) \subseteq \tilde{f}(x))$.

Proof. Assume that $(\tilde{f}, X)$ is an int-soft ideal of X over U. Let x, y and $z \in X$ be such that $x - y \leq z$. By Proposition 3.10(i) and (3.3), we have $\tilde{f}(z) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$.

Conversely, suppose that $(\tilde{f}, X)$ satisfies (3.4). By (a2), we get $0 \leq x$ for any $x \in X$. Using Proposition 3.10(i), we have $\tilde{f}(x) \subseteq \tilde{f}(0)$ for any $x \in X$. By (a4), we have $x - (x - y) \leq y$ for any $x, y \in X$. It follows from (3.4) that $\tilde{f}(y) \cap \tilde{f}(x - y) \subseteq \tilde{f}(x)$. Hence (3.3) hold. Therefore $(\tilde{f}, X)$ is an int-soft ideal of X over U. □

Theorem 3.12. *A soft set $(\tilde{f}, X)$ of X over U is an int-soft ideal of a subtraction algebra X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is an ideal of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

Proof. Similar to Theorem 3.4. □

The ideal $i_X(\tilde{f}; \gamma)$ in Theorem 3.12 is called the *inclusive ideal* of X.

Proposition 3.13. *Let $(\tilde{f}, X)$ be a soft set of a subtraction algebra X over U. Then the set $X_{\tilde{f}} := \{x \in X | \tilde{f}(x) = \tilde{f}(0)\}$ is an ideal of X.*

Proof. Obviously, $0 \in X_{\tilde{f}}$. Let $x, y \in X$ be such that $x - y \in X_{\tilde{f}}$ and $y \in X_{\tilde{f}}$. Then $\tilde{f}(x - y) = \tilde{f}(0)$ and $\tilde{f}(y) = \tilde{f}(0)$. By (3.3), we have $\tilde{f}(0) = \tilde{f}(x - y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$. It follows from (3.2) that $\tilde{f}(x) = \tilde{f}(0)$. Hence $x \in X_{\tilde{f}}$. Therefore $X_{\tilde{f}}$ is an ideal of X. □

4. Quotient subtraction algebras induced by an int-soft ideal

Let $(\tilde{f}, X)$ be an int-soft ideal of a subtraction algebra X. For any $x, y \in X$, we define a binary operation " $\sim^{\tilde{f}}$ " on X as follows:

$$x \sim^{\tilde{f}} y \Leftrightarrow \tilde{f}(x - y) = \tilde{f}(y - x) = \tilde{f}(0).$$

Lemma 4.1. *The operation $\sim^{\tilde{f}}$ is an equivalence relation on a subtraction algebra X.*

Proof. Obviously, it is reflexive and symmetric. Let x, y and $z \in X$ be such that $x \sim^{\tilde{f}} y$ and $y \sim^{\tilde{f}} z$. Then $\tilde{f}(x - y) = \tilde{f}(y - x) = \tilde{f}(0)$ and $\tilde{f}(y - z) = \tilde{f}(z - y) = \tilde{f}(0)$. By (a7), we have $(x - z) - (y - z) \leq x - y$ and $(z - x) - (y - x) \leq z - y$. Using (3.4) and (3.2), we have $\tilde{f}(0) = \tilde{f}(x - y) \cap \tilde{f}(y - z) \subseteq \tilde{f}(x - z) \subseteq \tilde{f}(0)$ and $\tilde{f}(0) = \tilde{f}(z - y) \cap \tilde{f}(y - x) \subseteq \tilde{f}(z - x) \subseteq \tilde{f}(0)$. Hence $\tilde{f}(x - z) = \tilde{f}(z - x) = \tilde{f}(0)$. Thus $x \sim^{\tilde{f}} z$, that is, $\sim^{\tilde{f}}$ is transitive. Therefore $\sim^{\tilde{f}}$ is an equivalence relation. □

Lemma 4.2. *For any $x, y, u, v \in X$, if $x \sim^{\tilde{f}} y$ and $u \sim^{\tilde{f}} v$, then $x - u \sim^{\tilde{f}} y - v$.*

Proof. Let $x, y, u, v \in X$ be such that $x \sim^{\tilde{f}} y$ and $u \sim^{\tilde{f}} v$. Then $\tilde{f}(x-y) = \tilde{f}(y-x) = \tilde{f}(0)$ and $\tilde{f}(u-v) = \tilde{f}(v-u) = \tilde{f}(0)$. Since $(x-u)-(y-u) \leq x-y$ and $(y-u)-(x-u) \leq y-x$, it follows from Proposition 3.10(i) that $\tilde{f}(0) = \tilde{f}(x-y) \leq \tilde{f}((x-u)-(y-u))$ and $\tilde{f}(0) = \tilde{f}(y-x) \leq \tilde{f}((y-u)-(x-u))$. By (3.2), we have $\tilde{f}((x-u)-(y-u)) = \tilde{f}(0)$ and $\tilde{f}((y-u)-(x-u)) = \tilde{f}(0)$. Hence $x-u \sim^{\tilde{f}} y-u$.

By (a4), (a9) and (S3), we have $(y-(y-v))-u = (y-u)-(y-v) \leq v-u$. Using Proposition 3.10(i), we obtain $\tilde{f}(0) = \tilde{f}(v-u) \leq \tilde{f}((y-u)-(y-v))$. It follows from (3.2) that $\tilde{f}((y-u)-(y-v)) = \tilde{f}(0)$. By a similar way, we get $\tilde{f}((y-v)-(y-u)) = \tilde{f}(0)$. Hence $y-v \sim^{\tilde{f}} y-u$. Since $\sim^{\tilde{f}}$ is symmetric, we have $y-u \sim^{\tilde{f}} y-v$. Since $\sim^{\tilde{f}}$ is transitive, $x-u \sim^{\tilde{f}} y-v$. Therefore $\sim^{\tilde{f}}$ is a congruence relation on X. □

Denote $\tilde{f}_x$ and $X/\tilde{f}$ the equivalence class containing x and the set of all equivalence classes of X, respectively, i.e.,

$$\tilde{f}_x := \{y \in X | y \sim^{\tilde{f}} x\} \text{ and } X/\tilde{f} := \{\tilde{f}_x | x \in X\}.$$

Define a binary relation $-$ on $X/\tilde{f}$ as follows:

$$\tilde{f}_x - \tilde{f}_y = \tilde{f}_{x-y}$$

for all $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$. Then this operation is well-defined by Lemma 4.2.

Theorem 4.3. *If $(\tilde{f}, X)$ is an int-soft ideal of a subtraction algebra X, then the quotient $X/\tilde{f} := (X/\tilde{f}; -)$ is a subtraction algebra.*

Proof. Straightforward. □

Proposition 4.4. *Let $\mu : X \to Y$ be an epimorphism of subtraction algebras. If $(\tilde{f}, Y)$ is an int-soft ideal of Y, then $(\tilde{f} \circ \mu, X)$ is an int-soft ideal of X.*

Proof. For any $x \in X$, we have $(\tilde{f} \circ \mu)(x) = \tilde{f}(\mu(x)) \subseteq \tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f} \circ \mu)(0_X)$ and $(\tilde{f} \circ \mu)(x) = \tilde{f}(\mu(x)) \supseteq \tilde{f}(\mu(x) -_Y a) \cap \tilde{f}(a)$ for any $a \in Y$. Let y be any preimage of a under μ. Then we have

$$\begin{aligned}(\tilde{f} \circ \mu)(x) &\supseteq \tilde{f}(\mu(x) -_Y a)) \cap \tilde{f}(a) \\ &= \tilde{f}(\mu(x) - \mu(y)) \cap \tilde{f}(\mu(y)) \\ &= \tilde{f}(\mu(x -_X y) \cap \tilde{f}(\mu(y)) \\ &= (\tilde{f} \circ \mu)(x -_X y) \cap (\tilde{f} \circ \mu)(y).\end{aligned}$$

Hence $\tilde{f} \circ \mu$ is an int-soft ideal of X. □

Proposition 4.5. *Let $(\tilde{f}, X)$ be an int-soft ideal of a subtraction algebra X. The mapping $\gamma : X \to X/\tilde{f}$, given by $\gamma(x) := \tilde{f}_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = \tilde{f}_0\} = X_{\tilde{f}}$.*

Proof. Let $\tilde{f}_x \in X/\tilde{f}$. Then there exists an element $x \in X$ such that $\gamma(x) = \tilde{f}_x$. Hence γ is surjective. For any $x, y \in X$, we have

$$\gamma(x - y) = \tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y = \gamma(x) - \gamma(y).$$

Thus γ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X|\gamma(x) = \tilde{f}_0\} = \{x \in X|x \in \tilde{f}_0\} = \{x \in X|x \sim^{\tilde{f}} 0\} = \{x \in X|\tilde{f}(x) = \tilde{f}(0)\} = X_{\tilde{f}}$. It completes the proof. □

Proposition 4.6. *Let a soft set $(\tilde{f}, X)$ over U of a subtraction algebra X be an int-soft ideal of X. If J is an ideal of X, then $J/\tilde{f}$ is an ideal of $X/\tilde{f}$.*

Proof. Let a soft set $(\tilde{f}, X)$ over U of a subtraction algebras X be an int-soft ideal of X and let J be an ideal of X. Then $0 \in J$. Hence $\tilde{f}_0 \in J/\tilde{f}$. Let $\tilde{f}_x, \tilde{f}_y \in J/\tilde{f}$ such that $\tilde{f}_x - \tilde{f}_y \in J/\tilde{f}$ and $\tilde{f}_y \in J/\tilde{f}$. Since $\tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y$, we have $x - y, y \in J$. Since J is an ideal of X, we have $x \in J$. Hence $\tilde{f}_x \in J/\tilde{f}$. □

Theorem 4.7. *If J^* is an ideal of a quotient subtraction algebra $X/\tilde{f}$, then there exists an ideal $J = \{x \in X|\tilde{f}_x \in J^*\}$ in X such that $J/\tilde{f} = J^*$.*

Proof. Since J^* is an ideal of $X/\tilde{f}$, $\tilde{f}_0 \in J^*$. Hence $0 \in J$. Let $\tilde{f}_x, \tilde{f}_y \in J/\tilde{f}$ be such that $\tilde{f}_x - \tilde{f}_y, \tilde{f}_y \in J^*$. Since $\tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y$, we have $x - y, y \in J$. Since J^* is an ideal of $J/\tilde{f}$, $\tilde{f}_x \in J/\tilde{f}$ and so $x \in J$. Therefore J is an ideal of X. By Proposition 4.6, we have

$$\begin{aligned} J/\tilde{f} &= \{\tilde{f}_j|j \in J\} \\ &= \{\tilde{f}_j|\exists \tilde{f}_x \in J^* \text{ such that } j \sim^{\tilde{f}} x\} \\ &= \{\tilde{f}_j|\exists \tilde{f}_x \in J^* \text{ such that } \tilde{f}_x = \tilde{f}_j\} \\ &= \{\tilde{f}_j|\tilde{f}_j \in J^*\} = J^*. \end{aligned}$$

□

Theorem 4.8. *Let a soft set $(\tilde{f}, X)$ over U be an int-soft ideal of a subtraction algebra X. If J is an ideal of X, then $\dfrac{X/\tilde{f}}{J/\tilde{f}} \cong X/J$.*

Proof. Note that $\dfrac{X/\tilde{f}}{J/\tilde{f}} = \{[\tilde{f}_x]_{J/\tilde{f}}|\tilde{f} \in X/\tilde{f}\}$. If we define $\varphi : \dfrac{X/\tilde{f}}{J/\tilde{f}} \to X/J$ by $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = [x]_J = \{y \in X|x \sim^J y\}$, then it is well defined. In fact, suppose that $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. Then $\tilde{f}_x \sim^{J/\tilde{f}} \tilde{f}_y$ and so $\tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y, \tilde{f}_{y-x} = \tilde{f}_y - \tilde{f}_x \in J/\tilde{f}$. Hence $y - x, x - y \in J$. Therefore

$x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[\tilde{f}_x]_{J/\tilde{f}}, [\tilde{f}_y]_{J/\tilde{f}} \in \dfrac{X/\tilde{f}}{J/\tilde{f}}$, we have

$$\begin{aligned}\varphi([\tilde{f}_x]_{J/\tilde{f}} - [\tilde{f}_y]_{J/\tilde{f}}) &= \varphi([\tilde{f}_x - \tilde{f}_y]_{J/\tilde{f}}) = [x-y]_J \\ &= [x]_J - [y]_J = \varphi([\tilde{f}_x]_{J/\tilde{f}}) - \varphi([\tilde{f}_y]_{J/\tilde{f}}).\end{aligned}$$

Hence φ is a homomorphism.

Obviously, φ is onto. Finally, we show that φ is one-to-one. If $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = \varphi([\tilde{f}_y]_{J/\tilde{f}})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $\tilde{f}_a \in [\tilde{f}_x]_{J/\tilde{f}}$, then $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_x$ and hence $\tilde{f}_{a-x}, \tilde{f}_{x-a} \in J/\tilde{f}$. It follows that $a-x, x-a \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a-y, y-a \in J$ and so $\tilde{f}_{a-y}, \tilde{f}_{y-a} \in J/\tilde{f}$. Therefore $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_y$. Hence $\tilde{f}_a \in [\tilde{f}_y]_{J/\tilde{f}}$. Thus $[\tilde{f}_x]_{J/\tilde{f}} \subseteq [\tilde{f}_y]_{J/\tilde{f}}$. Similarly, we obtain $[\tilde{f}_y]_{J/\tilde{f}} \subseteq [\tilde{f}_x]_{J/\tilde{f}}$. Therefore $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. It is completes the proof. □

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Characterization of weak sharp solutions for generalized variational inequalities in Banach spaces

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Abstract

In this paper, we study the solution set of variational like-inequalities (in this sense we are called η-variational inequalities) and introduce the notion of a weak sharp set of solutions to η-variational inequality problem in reflexive, strictly convex and smooth Banach space. We also present sufficient conditions for the relevant mapping to be constant on the solutions. Moreover, we characterize the weak sharpness of the solutions of η-variational inequality by primal gap function.

Keywords: η-variational inequality, Gap function, Weakly sharp solution

1. Introduction

Burke and Ferris [2] introduced the concept of a weak sharp minimum to present sufficient conditions for the finite identification, by iterative algorithm, of local minima associated with mathematical programming in space $\mathbb{R}^n$. Patriksson [7] has generalized the concept of the weak sharpness of the solution set of a variational inequality problem (in short, VIP). Their concepts have been extended by Marcotte and Zhu [6] to introduce another the notion of weak sharp solutions for variational inequalities. They also characterized the weak sharp solutions in terms of a dual gap function for variational inequalities. The relevant results have been obtained by Zhang et al. [12]. It is further study by Wu and Wu [9–11]. Hu and Song [4] have extended the results of weak sharpness for the solutions of VIP under some continuity and monotonicity assumptions in Banach space. They also introduce the notion of weak sharp set of solutions to a variational inequality problem in a reflexive, strictly convex and smooth Banach space and present its several equivalent conditions. Liu and Wu [5] studied weak sharp solutions for the variational inequality in terms of its primal gap function. They also characterized the weak sharpness of the solution set of VIP in terms of primal gap function. Recently, AL-Hamidan et al. [1] give some characterization of weak sharp solutions for the VIP without considering the primal or dual gap function.

In this paper, we provide some general two concepts of Liu and Wu [5] and Hu and Song [4] to study the weak sharpness of solution set of η-variational inequality problem in Banach space. We also give some characterizations of weak sharp solutions for the η-VIP and also present its several equivalent conditions. Our purpose in this paper is to develop the weak sharpness result in space $\mathbb{R}^n$.

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The paper is organized as follows. Section 2 discuss the new concepts of the Gateaux differentiable and Lipschitz continuity of the primal gap function and we also introduce the main definitions. Several equivalent conditions for F to be constant are discuss and present some relationship among C^η, C_η, $\Gamma(x^*)$, and $\Lambda(x^*)$ in Section 3. Finally, section 4 addresses the weak sharpness of C^η in terms of the primal gap function is characterized.

2. Preliminaries and formulations

Let E be a real Banach space with is topological dual space E^* and $\langle \cdot\,, \cdot \rangle$ denote the pairing between E and E^* respectively. For a mapping from $\eta : E \times E$ to E . Let g be a mapping from E into Banach space Y. The mapping g is called directionally differentiable at a point $x \in E$ in a direction $v \in E$ if the limit

$$g'(x, v) := \lim_{t \downarrow 0} \frac{g(x + tv) - g(x)}{t}$$

exists. We say that g is directionally differentiable at x, if g is directionally differentiable at x in every direction $v \in E$.

The mapping g is called Gateaux differentiable at x if g is directionally differentiable at x and the directional derivative $g'(x, v)$ is linear and continuous in v and we denote this operator by $\nabla g(x)$, i.e. $\langle \nabla g(x), v\rangle = g'(x, v)$.

Definition 2.1. Let g be a mapping from E into Banach space Y. The mapping g is called η-Gateaux differentiable at x if g is Gateaux differentiable at x and there exists a unique $\xi \in E^*$ such that $\langle \xi, \eta(v, 0)\rangle = \langle \nabla g(x), v\rangle$, $\forall v \in E$. We denote this operator by $\nabla_\eta g(x)$ i.e. $\langle \nabla_\eta g(x), \eta(v, 0)\rangle = g'(x, v)$.

We defined η-subdifferential of a proper convex function f at $x \in E$ is given by

$$\partial_\eta f(x) := \{x^* \in E^* : \langle x^*, \eta(y, x)\rangle \le f(y) - f(x), \forall y \in E\}.$$

Let C be a closed convex subset of E. The mapping $P_C : E \to 2^C$ defined by

$$P_C(x) := \{y \in C : \|x - y\| = d(x, C)\},$$

is called *the metric projection operator*.

We known that if E is a reflexive and strictly convex Banach space, P_C is a single-valued mapping.

The duality mappings $J : E \to 2^{E^*}$ and $J^* : E^* \to 2^E$ are defined by

$$J(x) = \{x^* \in E^* : \langle x^*, x\rangle = \|x^*\|_*^2 = \|x\|^2\}, \forall x \in E$$

and

$$J^*(x^*) = \{x \in E : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|_*^2\}, \forall x^* \in E^*.$$

We know the following (see [8])

(i) if E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone.

Thought out this paper, we let $\eta : E \times E$ to E be satisfy the following condition;

(i) η is continuous on $E \times E$;
(ii) for any $x, y \in E$, $\eta(x, y) = -\eta(y, x)$;

(iii) for any $x, y \in E$ and α, β are scalars, $\eta(\alpha x + \beta y, 0) = \alpha\eta(x, 0) + \beta\eta(y, 0)$;
(iv) there exists $k > 0$ such that $\|\eta(x, y)\| = k\|x - y\|$ for all $x, y \in E$;
(v) $\eta(E \times \{0\}) = E$.

For a mapping g from a Banach space E into Banach space Y, we say that g is η-locally Lipschitz on E if for any $\overline{x} \in E$ there exist $\delta > 0$ and $L \geq 0$ such that

$$\|g(x) - g(y)\| \leq L\|\eta(x, y)\|, \quad \text{for all} \quad x, y \in B(\overline{x}, \delta).$$

The following results are importance:

Lemma 2.2 ([3])**.** *Let E be a Banach space, $J : E \to 2^{E^*}$ a duality mapping and $\Phi(\|x\|) = \int_0^{\|x\|} ds$, $0 \neq x \in X$. Then $J(x) = \partial\Phi(\|x\|)$.*

Lemma 2.3. *Assume that E is a reflexive, strictly convex and smooth Banach space. Let C be a closed convex subset of E and $\hat{x} \in C$. Then the following are equivalent:*

(i) *$\hat{x}$ is a best approximation to x :$\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|$.*
(ii) *the inequality $\langle J(\eta(x, \hat{x})), \eta(y, \hat{x})\rangle \leq 0, \quad \forall y \in C$ holds.*

Proof. $(ii) \Rightarrow (i)$ For each $x \in E$. Let $\hat{x} \in C$ such that

$$\langle J(\eta(x, \hat{x})), \eta(y, \hat{x})\rangle \leq 0 \quad \forall y \in C.$$

Then

$$\begin{aligned}
\|\eta(x, \hat{x})\|\|J(\eta(x, \hat{x}))\|_* &= \langle J(\eta(x, \hat{x})), \eta(x, \hat{x})\rangle \\
&\leq \langle J(\eta(x, \hat{x})), \eta(x, \hat{x})\rangle + \langle J(\eta(x, \hat{x})), \eta(\hat{x}, y)\rangle, \quad \forall y \in C \\
&= \langle J(\eta(x, \hat{x})), \eta(x, y)\rangle, \quad \forall y \in C \\
&\leq \|J(\eta(x, \hat{x}))\|_* \|\eta(x, y)\|, \quad \forall y \in C.
\end{aligned}$$

Hence, $\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|$.

$(i) \Rightarrow (ii)$ For each $x \in E$. Suppose that $\hat{x} \in C$ such that

$$\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|.$$

Since C is convex, we obtain that

$$\|\eta(x, \hat{x})\| \leq \|\eta(x, (1 - t)\hat{x} + ty)\|, \quad \forall y \in C \quad \text{and} \quad t \in [0, 1],$$

which implies that

$$\Phi(\|\eta(x, \hat{x})\|) - \Phi(\|\eta(x, (1 - t)\hat{x} + ty)\|), \quad \forall y \in C \quad \text{and} \quad t \in [0, 1],$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ give by $\Phi(x) = \int_0^x ds$, for all $x \in \mathbb{R}_+$.

By Lemma 2.2, $J(z) = \partial\Phi(\|z\|)$. It follows that

$$\begin{aligned}
\langle J(\eta(x, (1 - t)\hat{x} + ty)), \eta(x, \hat{x}) - \eta(x, (1 - t)\hat{x} + ty)\rangle &\leq \Phi(\|\eta(x, \hat{x})\|) - \Phi(\|\eta(x, (1 - t)\hat{x} + ty)\|) \\
&\leq 0, \quad \forall y \in C \quad \text{and} \quad t \in [0, 1],
\end{aligned}$$

that is,

$$t\langle J(\eta(x, (1 - t)\hat{x} + ty)), \eta(y, \hat{x})\rangle \leq 0, \quad \forall y \in C \quad \text{and} \quad t \in [0, 1]$$

Therefore,

$$\langle J(\eta(x,(1-t)\hat{x}+ty)),\eta(y,\hat{x})\rangle \le 0, \quad \forall y \in C \text{ and } t \in [0,1].$$

Taking $t \to 0$, we have

$$\langle J(\eta(x,\hat{x})),\eta(y,\hat{x})\rangle \le 0, \quad \forall y \in C.$$

□

Remark 2.4. By definition of η for each $x \in E$ if $\hat{x} = P_C(x)$ then $\|\eta(x,\hat{x})\| = \inf_{y\in C}\|\eta(x,y)\|$.

If C is a closed convex subset of E and $\overline{x} \in C$, then the η-tangent cone to C at $\overline{x}$ has the form

$$\begin{aligned} T_C^\eta(\overline{x}) &= \{d \in E : \text{there exists a bounded sequence } \{d_k\} \subseteq X \text{ with } \eta(d_k,0) \to d, t_k \downarrow 0 \\ &\quad \text{such that } \overline{x}+t_kd_k \in C, \quad \forall k \in \mathbb{N}\}. \end{aligned}$$

In the above, denote $x_k = \overline{x}+t_kd_k \in C$. Taking the limit as $k \to +\infty$, $t_k \to 0$, which implies that $t_kd_k \to 0$, thereby leading to $x_k \to \overline{x}$. Also from construction,

$$\frac{\eta(x_k,\overline{x})}{t_k} = \eta(d_k,0) \to d.$$

Thus, the η-tangent cone can be equivalently expressed as

$$\begin{aligned} T_C^\eta(\overline{x}) &= \{d \in E : \text{there exists sequence } \{x_k\} \subseteq C \text{ with } x_k \to \overline{x}, t_k \downarrow 0 \\ &\quad \text{such that } \frac{\eta(x_k,\overline{x})}{t_k} \to d\}. \end{aligned}$$

Proposition 2.5. *Consider a set $C \subseteq E$ and $\overline{x} \in C$. Then the following hold:*

(i) *$T_C^\eta(\overline{x})$ is closed;*
(ii) *If C is convex, $T_C^\eta(\overline{x})$ is the closure of the cone generated by $\eta(C \times \{\overline{x}\})$, that is, $T_C^\eta(\overline{x}) = \overline{cone(\eta(C \times \{\overline{x}\}))}$*

Proof. (i) Suppose that $\{d_k\} \subseteq T_C^\eta(\overline{x})$ such that $d_k \to d$. Since $d_k \in T_C^\eta(\overline{x})$, there exist $\{x_k^r\} \subseteq C$ with $x_k^r \to \overline{x}$ and $\{t_k^r\} \subseteq \mathbb{R}_+$ with $t_k^r \to 0$ such that

$$\frac{\eta(x_k^r,\overline{x})}{t_k^r} \to d_k, \quad \forall k \in \mathbb{N}.$$

For a fixed k, there exists $\overline{r}$ such that

$$\|\frac{\eta(x_k^r,\overline{x})}{t_k^r} - d_k\| < \frac{1}{k}, \quad \forall r \ge \overline{r}.$$

Taking $k \to +\infty$, one can generate a sequence $\{x_k\} \subseteq C$ with $x_k \to \overline{x}$ and $t_k \downarrow 0$ such that

$$\frac{\eta(x_k,\overline{x})}{t_k} \to d.$$

Thus, $d \in T_C^\eta(\overline{x})$. Hence, $T_C^\eta(\overline{x})$ is closed.

(ii) Suppose that $d \in T_C^\eta(\overline{x})$, which implies that there exist $\{x_k\} \subseteq C$ with $x_k \to \overline{x}$ and $\{t_k\} \subseteq \mathbb{R}_+$ with $t_k \to 0$ such that

$$\frac{\eta(x_k,\overline{x})}{t_k} \to d.$$

Observe that $\eta(x_k,\overline{x}) \in \eta(C \times \{\overline{x}\})$. Since $t_k > 0$, $\frac{1}{t_k} > 0$. Therefore,

$$\frac{\eta(x_k,\overline{x})}{t_k} \in cone\ \eta(C \times \{\overline{x}\}).$$

Thus, $d \in \overline{cone(\eta(C \times \{\overline{x}\}))}$. Hence, $T_C^\eta(\overline{x}) \subseteq \overline{cone(\eta(C \times \{\overline{x}\}))}$.

Conversely, for each $x \in C$. Define a sequence

$$x_k = \overline{x} + \frac{1}{k}(x - \overline{x}), \quad \forall k \in \mathbb{N}.$$

By the convexity of C, it is obvious that $\{x_k\} \subseteq C$. Taking $k \to +\infty$, $x_k \to \overline{x}$, by construction, we obtain that

$$k\eta(x_k, \overline{x}) = \eta(x, \overline{x}).$$

Set $t_k = \frac{1}{k} > 0$, $t_k \to 0$ such that $\frac{\eta(x_k,\overline{x})}{t_k} \to \eta(x, \overline{x})$, which implies that $\eta(x, \overline{x}) \in T_C^\eta(\overline{x})$.

Since $x \in C$ is arbitrary, $\eta(C \times \{\overline{x}\}) \subseteq T_C^\eta(\overline{x})$. Because $T_C^\eta(\overline{x})$ is cone, we have $cone(\eta(C \times \{\overline{x}\})) \subseteq T_C^\eta(\overline{x})$. By (i), $T_C^\eta(\overline{x})$ is closed, which implies that $\overline{cone(\eta(C \times \{\overline{x}\}))} \subseteq T_C^\eta(\overline{x})$. □

The η-normal cone to C at $\overline{x}$ is defined by $N_C^\eta(\overline{x}) := [T_C^\eta(\overline{x})]^\circ$, where

$$A^\circ := \{x^* \in E^* : \langle x^*, x\rangle \leq 0, \forall x \in A\}.$$

If C is convex, then

$$N_C^\eta(\overline{x}) := \begin{cases} \{x^* \in E^* : \langle x^*, \eta(c, \overline{x})\rangle \leq 0 \text{ for all } c \in C\} & \text{if } \overline{x} \in C, \\ \emptyset, & \text{if } \overline{x} \notin C. \end{cases}$$

Let C be a nonempty closed convex subset of reflexive, strictly convex and smooth Banach space E. For a mapping F from E into E^*, the *η-variational inequality problem* [η-VIP] is to find a vector $x^* \in C$ such that

$$\langle F(x^*), \eta(x, x^*)\rangle \geq 0 \text{ for all } x \in C. \tag{2.1}$$

We denote the solution set of the η-VIP by C^η

The *η-dual variational inequality problem* [η-DVIP] is to find a vector $x_* \in C$ such that

$$\langle F(x), \eta(x, x_*)\rangle \geq 0 \text{ for all } x \in C. \tag{2.2}$$

We denote the solution set of the η-DVIP by C_η

Definition 2.6. The mapping $F : E \to E^*$ is said to be:

(i) *η-monotone* on C if $\langle F(x) - F(y), \eta(y, x)\rangle \geq 0$ for all $x, y \in C$;

(ii) *η-pseudomonotone* at $x \in C$ if for each $y \in C$ there holds

$$\langle F(x), \eta(y, x)\rangle \geq 0 \Rightarrow \langle F(y), \eta(y, x)\rangle \geq 0;$$

(iii) *η-pseudomonotone$^+$* on C if it is η-pseudomonotone at each point in C and, for all $x, y \in C$,

$$\left.\begin{array}{l} \langle F(y), \eta(x, y)\rangle \geq 0 \\ \langle F(x), \eta(x, y)\rangle = 0 \end{array}\right\} \Rightarrow F(x) = F(y).$$

Now, we define the *primal gap function* $g(x)$ associated with η-VIP (2.1) as

$$g(x) := \sup_{y \in C}\{\langle F(x), \eta(x, y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Gamma(x) := \{y \in C : \langle F(x, \eta(x, y))\rangle = g(x)\}.$$

Similarly, we define the *dual gap function* $G(x)$ associated with η-DVIP (2.2) as

$$G(x) := \sup_{y \in C}\{\langle F(y), \eta(x, y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Lambda(x) := \{y \in C : \langle F(y, \eta(x, y))\rangle = G(x)\}.$$

3. Sufficient condition for constancy of F on C^η and some properties of the primal gap function

In this section, we discuss about relations among C^η, C_η, $\Gamma(x^*)$, and $\Lambda(x^*)$. We study sufficient condition for F to be constant on C^η and also study the η-Lipschitz continuity and η-subdifferentiability of the primal gap function g in terms of the mapping F.

Proposition 3.1. *Let $\hat{x} \in C$. Then*

(i) $\hat{x} \in C^\eta \Leftrightarrow g(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Gamma(\hat{x})$;
(ii) $\hat{x} \in C_\eta \Leftrightarrow G(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Lambda(\hat{x})$.

Proof. (i) Consider

$$\begin{aligned} \hat{x} \in C^\eta \quad &\Leftrightarrow \quad \langle F(\hat{x}), \eta(y, \hat{x})\rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad \langle F(\hat{x}), \eta(\hat{x}, y)\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad g(\hat{x}) = 0. \end{aligned}$$

And we also consider

$$\begin{aligned} \hat{x} \in \Gamma(\hat{x}) \quad &\Leftrightarrow \quad \langle F(\hat{x}), \eta(\hat{x}, \hat{x})\rangle = g(\hat{x}) \\ &\Leftrightarrow \quad 0 = g(\hat{x}). \end{aligned}$$

Similarly, we can obtain (ii). □

Proposition 3.2. *If F is η-pseudomonotone on C, $C^\eta \subseteq C_\eta$.*

Proof. Immediate from the definitions. □

The following proposition we present a sufficient condition for F to be constant on C^η.

Proposition 3.3. *Let F be η-pseudomonotone$^+$ on C^η. Then F is constant on C^η*

Proof. Let $x_1, x_2 \in C^\eta$. Since F is η-pseudomonotone$^+$ on C^η, we have

$$\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0 \;\text{ and }\; \langle F(x_2), \eta(x_1, x_2)\rangle \geq 0.$$

By pseudomonotonicity of F on C^η, we have

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \;\text{ it follows that }\; \langle F(x_1), \eta(x_1, x_2)\rangle = 0.$$

Since F is η-pseudomonotone$^+$ on C^η and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, implies that $F(x_1) = F(x_2)$. Hence, F is constant on C^η. □

Proposition 3.4. *Let F be η-pseudomonotone$^+$ on C and $x^* \in C^\eta$. Then $C^\eta = \Lambda(x^*)$ and F is constant on $\Lambda(x^*)$.*

Proof. First, we prove that F is constant on $\Lambda(x^*)$. For $x^* \in C^\eta$ and $c \in C$, we have

$$\langle F(x^*), \eta(c, x^*)\rangle \geq 0. \text{ Since } F \text{ is pseudomonotone, we get that}$$

$$\langle F(c), \eta(c, x^*)\rangle \geq 0, \forall c \in C. \text{ It follows that } G(x^*) = 0.$$

For $c \in \Lambda(x^*)$, we have

$$\langle F(c), \eta(c, x^*)\rangle = -G(x^*) = 0 \;\text{ and hence }\; F(c) = F(x^*).$$

It sufficient to show that $C^\eta = \Lambda(x^*)$.
($\subseteq$): Let $y^* \in C^\eta$. Then $\langle F(y^*), \eta(x^*, y^*)\rangle \geq 0$. Since $x^* \in C^\eta \subseteq C_\eta$, we have

$$\langle F(z), \eta(z, x^*)\rangle \geq 0, \forall z \in C.$$

It follows that $G(x^*) = 0$, and $\langle F(y^*), \eta(y^*, x^*)\rangle \geq 0$. Therefore,

$$\langle F(y^*), \eta(x^*, y^*)\rangle = 0 = G(x^*), \quad \text{that is,} \quad y^* \in \Lambda(x^*). \text{ Thus } \ C^\eta \subseteq \Lambda(x^*).$$

($\supseteq$): Let $y^* \in \Lambda(x^*)$. Then $\langle F(y^*), \eta(x^*, y^*)\rangle = G(x^*) \geq 0$. Since $x^* \in C^\eta$, we have

$$\langle F(x^*), \eta(y, x^*)\rangle \geq 0, \forall y \in C.$$

Note that $x^* \in \Lambda(x^*)$, we have $F(x^*) = F(y^*)$. Consider, for all $y \in C$,

$$0 \leq \langle F(y^*), \eta(y, x^*)\rangle = \langle F(y^*), \eta(y, y^*)\rangle + \langle F(y^*), \eta(y^*, x^*)\rangle$$

implies $0 \leq \langle F(y^*), \eta(x^*, y^*)\rangle \leq \langle F(y^*), \eta(y, y^*)\rangle, \forall y \in C$. Therefore, $C^\eta = \Lambda(x^*)$. □

Proposition 3.5. *Suppose that F be η-pseudomonotone on C and $x^* \in C^\eta$. If F is constant on $\Gamma(x^*)$ then F is constant on C^η. And hence*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

Proof. Since F is η-pseudomonotone on C, we have $C^\eta \subseteq C_\eta$. Let $y^* \in C_\eta$. Then

$$\langle F(x^*), \eta(x^*, y^*)\rangle \geq 0.$$

By assumption, we obtain that $g(x^*) = 0$ and hence $\langle F(x^*), \eta(y^*, x^*)\rangle \geq 0$. It follows that

$$\langle F(x^*), \eta(x^*, y^*)\rangle = 0 = g(x^*). \text{ Thus } \ y^* \in \Gamma(x^*).$$

Therefore $C^\eta \subseteq C_\eta \subseteq \Gamma(x^*)$. Let $z^* \in \Gamma(x^*)$. Then $\langle F(x^*), \eta(x^*, z^*)\rangle = g(x^*) = 0$.
From above $x^* \in C^\eta \subseteq \Gamma(x^*)$ and F is constant on $\Gamma(x^*)$, we obtain that $F(x^*) = F(z^*)$. Since $x^* \in C^\eta$, we have $\langle F(x^*), \eta(z, x^*)\rangle \geq 0, \forall z \in C$.

It follows that, for all $z \in C$,

$$\begin{aligned} 0 &\leq \langle F(z^*), \eta(z, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle + \langle F(z^*), \eta(z^*, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle + \langle F(x^*), \eta(z^*, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle. \end{aligned}$$

This implies that $z^* \in C^\eta$. Thus $\Gamma(x^*) \subseteq C^\eta$ and hence

$$C^\eta = C_\eta = \Gamma(x^*).$$

It sufficient to prove that $\Gamma(x^*) = \Lambda(x^*)$. For $c \in \Gamma(x^*)$, we have

$$\langle F(x^*), \eta(x^*, c)\rangle = g(x^*) = 0,$$

so $\langle F(c), \eta(x^*, c)\rangle = 0 = G(x^*)$. Therefore,

$$c \in \Lambda(x^*), \quad \text{which implies that } \ \Gamma(x^*) \subseteq \Lambda(x^*).$$

Now let $c \in \Lambda(x^*)$. Then

$$\langle F(c), \eta(x^*, c)\rangle = G(x^*) = 0.$$

The pseudomonotonicity of F on C implies that $\langle F(x^*), \eta(x^*, c)\rangle \geq 0$. In this case,

$$\langle F(x^*), \eta(x^*, c)\rangle = 0 = g(x^*) \quad \text{since} \quad x^* \in C^\eta.$$

Thus $c \in \Gamma(x^*)$ and hence $\Lambda(x^*) \subseteq \Gamma(x^*)$. Therefore

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

□

Proposition 3.6. *Let F be η-pseudomonotone$^+$ on C. Then, for $x^* \in C^\eta$, F is constant on $\Gamma(x^*)$ if and only if*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

Proof. ($\Rightarrow$)Suppose that F is constant on $\Gamma(x^*)$. By Proposition 3.5, we obtain that

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

($\Leftarrow$)Assume that $C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*)$. Let $x_1, x_2 \in \Gamma(x^*)$. Then

$$\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0 \quad \text{and} \quad \langle F(x_2), \eta(x_1, x_2)\rangle \geq 0 \quad \text{, because } x_1, x_2 \in C^\eta.$$

Since F is η-pseudomonotone and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, we obtain that

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \quad \text{, that is ,} \quad \langle F(x_1), \eta(x_2, x_1)\rangle \leq 0.$$

Thus $\langle F(x_1), \eta(x_2, x_1)\rangle = 0$. Since F is η-pseudomonotone$^+$ on C, we have $F(x_1) = F(x_2)$.

□

Proposition 3.7. *Let F be η-pseudomonotone$^+$ on C. Then the following are equivalent:*

(i) *F is constant on $\Gamma(x^*)$ for each $x^* \in C^\eta$.*
(ii) *$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^\eta$.*
(iii) *$C^\eta = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^\eta$.*
(iv) *$C^\eta = \Gamma(x^*)$ for each $x^* \in C^\eta$.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate. It suffices to show $(iv) \Rightarrow (i)$. Suppose that $C^\eta = \Gamma(x^*)$ for each $x^* \in C^\eta$. Let $x^* \in C^\eta$ and $x_1, x_2 \in \Gamma(x^*)$. Then $x_1, x_2 \in C^\eta$ and $\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0$ and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$.

Since F is η-pseudomonotone and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, we obtain that

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \quad \text{, that is ,} \quad \langle F(x_1), \eta(x_2, x_1)\rangle \leq 0.$$

Thus $\langle F(x_1), \eta(x_2, x_1)\rangle = 0$. Since F is η-pseudomonotone$^+$ on C, we have $F(x_1) = F(x_2)$. □

Next we prove that if F is η-locally Lipschitz on C^η, then so is g.

Lemma 3.8. *Let C be compact. If F is η-locally Lipschitz on C^η, then g is also η-locally Lipschitz on C^η.*

Proof. Suppose that F is η-locally Lipschitz on C^η. Let x^* be any element in C^η. Then there exist $\delta > 0$ and $L_0 \geq 0$ such that

$$\|F(x) - F(y)\| \leq L_0\|\eta(x, y)\| \quad \text{and} \quad \|F(x)\| \leq L_0 \quad \text{for all} \quad x, y \in B(x^*, \delta).$$

Let $c \in \Gamma(x)$ with $x \in B(x^*, \delta)$. Then

$$\begin{aligned} g(x) - g(y) &\leq \langle F(x), \eta(x,c)\rangle - \langle F(y), \eta(y,c)\rangle \\ &= \langle F(x), \eta(x,y)\rangle + \langle F(x), \eta(y,c)\rangle - \langle F(y), \eta(y,c)\rangle \\ &= \langle \eta(x,y)\rangle + \langle F(x) - F(y), \eta(y,c)\rangle \\ &\leq \|F(x)\|\|\eta(x,y)\| + \|F(x) - F(y)\|\|\eta(y,c)\| \\ &\leq L_0\|\eta(x,y)\| + L_0\|\eta(x,y)\|\|\eta(y,c)\|. \end{aligned}$$

By the compactness of C and definition of η implies that there exists a constant $M \geq 0$ such that

$$\|\eta(y,c)\| \leq M \quad \text{for all} \quad y \in B(x^*, \delta) \quad \text{and} \quad c \in C.$$

We set $L = L_0 + L_0 M$, we obtain that

$$g(x) - g(y) \leq L\|\eta(x,y)\|.$$

We can conclude that g is η-locally Lipschitz on C^η. □

The following Proposition 3.2 we present the η-subdifferential of g at $x^* \in C^\eta$ is a singleton under sufficient condition.

Proposition 3.9. *Let F be η-monotone on X and $x^* \in C^\eta$. Suppose that g is finite on X and η-Gateaux differentiable at x^*. Then $\partial_\eta g(x^*) = \{F(x^*)\}$.*

Proof. Since $x^* \in C^\eta$, we have $g(x^*) = 0$. For each $y \in X$ and F is η-monotone, we obtain that

$$g(y) - g(x^*) \geq \langle F(y), \eta(y, x^*)\rangle \geq \langle F(x^*), \eta(y, x^*)\rangle.$$

Hence $F(x^*) \in \partial_\eta g(x^*)$.

Let $z \in \partial_\eta g(x^*)$. Then for each $v \in X$ and $t > 0$, we get that

$$g(x^* + tv) - g(x^*) \geq \langle z, \eta(x^* + tv, x^*)\rangle = t\langle z, \eta(v, 0)\rangle,$$

that is,

$$\frac{g(x^* + tv) - g(x^*)}{t} \geq \langle z, \eta(v, 0)\rangle.$$

By the η-Gateaux differentiability of g at x^* implies that

$$\langle \nabla_\eta g(x^*), \eta(v,0)\rangle = \lim_{t \to 0} \frac{g(x^* + tv) - g(x^*)}{t} \geq \langle z, \eta(v, 0)\rangle.$$

Therefore, $\langle z - \nabla_\eta g(x^*), \eta(v,0)\rangle \leq 0$, for all $v \in X$. By definition of η we can set $\eta(v,0) = z - \nabla_\eta g(x^*)$, we have $\|z - \nabla_\eta g(x^*)\|^2 \leq 0$. This implies that $z = \nabla_\eta g(x^*)$, and hence $\partial_\eta g(x^*) = \{\nabla_\eta g(x^*)\} = \{F(x^*)\}$. □

4. Weak sharpness of C^η

Throughout this paper, we assume that C^η and C_η are nonempty and that E is a reflexive, strictly convex, and smooth Banach space. We introduce the notion of weak sharpness solution for generalized variational inequality(η-VIP).

Definition 4.1. The solution set C^η is said to be weakly sharp, if F satisfies

$$-F(x^*) \in int \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ \quad \text{for all} \quad x^* \in C^\eta.$$

Theorem 4.2. *Let F be η-monotone on E and constant on $\Gamma(x^*)$ for some $x^* \in C^\eta$. Suppose that g is η-Gateaux differentiable, η-locally Lipschitz on C^η, and $g(x) < +\infty$ for all $x \in E$. Then C^η is weakly sharp if and only if there exists a positive number α such that*

$$\alpha d^\eta_{C^\eta}(x) \leq g(x) \quad \text{for all} \quad x \in C, \tag{4.1}$$

where $d^\eta_{C^\eta}(x) := \inf_{y \in C^\eta} \|\eta(x, y)\|$.

Proof. On the given assumption and by Proposition 3.5, we obtain that

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

If C^η is weakly sharp, then for any $x^* \in C^\eta$ there exists $\alpha > 0$ such that

$$\alpha B_{E^*} \subseteq F(x^*) + \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ, \tag{4.2}$$

where B_{E^*} is the open unit ball in E^*.

Since F is constant on $\Gamma(x^*)$, α satisfies (4.2) for all $x^* \in C^\eta$. Therefore, for every $y \in B_{E^*}$, we have

$$\alpha y - F(x^*) \in \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ \subseteq [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ.$$

Thus, for every $z \in [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]$. It follows that

$$\langle \alpha y - F(x^*), z \rangle \leq 0. \tag{4.3}$$

Taking $y = \frac{Jz}{\|Jz\|_*}$ in (4.3), we get that, for each $x^* \in C^\eta$,

$$\alpha \|Jz\|_* = \frac{\alpha}{\|Jz\|_*} \langle Jz, z \rangle \leq \langle F(x^*), z \rangle.$$

This implies that for every $z \in [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]$, we have

$$\alpha \|z\| \leq \langle F(x^*), z \rangle.$$

For any $x \in C$, set $\overline{x} = P_{C^\eta}(x)$, we have $\eta(x, \overline{x}) \in T_C(\overline{x}) \cap J^* N_{C^\eta}(\overline{x})$ by Proposition 2.5 and lemma 2.3. Therefore,

$$\langle F(x^*), \eta(x, \overline{x}) \rangle \geq \alpha \|\eta(x, \overline{x})\| = \alpha d^\eta_{C^\eta}(x).$$

Conversely, suppose that there exists $\alpha > 0$ such that

$$\alpha d^\eta_{C^\eta}(x) \leq g(x) \quad \text{for each} \quad x \in C.$$

We claim that

$$\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ \quad \text{for each} \quad x^* \in C^\eta. \tag{4.4}$$

If $T_C(x^*) \cap J^* N_{C^\eta}(x^*) = \{0\}$ for $x^* \in C^\eta$, then $[T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\eta = E$ and $\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\eta$, trivially. So it suffices to prove (4.4) to hold if $T_C(x^*) \cap J^* N_{C^\eta}(x^*) \neq \{0\}$ for $x^* \in C^\eta$. Now let $0 \neq z \in T_C(x^*) \cap J^* N_{C^\eta}(x^*)$. By definition of η there exists a unique $v \in E$ such that $z = \eta(v, 0)$. Then

$$\langle J(\eta(v, 0)), \eta(v, 0) \rangle > 0 \quad \text{and} \quad \langle J(\eta(v, 0)), \eta(y^*, x^*) \rangle \leq 0 \quad \text{for each} \quad y^* \in C^\eta,$$

which implies that C^η is separated from $x^* + v$ by the hyperplane

$$H_v = \{x \in E : \langle J(\eta(v, 0)), \eta(x, x^*) \rangle = 0\} = \{x \in E : \langle J(\eta(v, 0)), \eta(x, 0) \rangle = \langle J(\eta(v, 0)), \eta(x^*, 0) \rangle\}.$$

Thus we can write

$$H_v = \{x \in E : \langle J(\eta(v,0)), \eta(x,0)\rangle = \beta\}, \quad \text{where} \quad \beta = \langle J(\eta(v,0)), \eta(x^*,0)\rangle.$$

Since $\eta(v,0) \in T_C(x^*)$, for each positive sequence $\{t_i\}$ decreasing to 0, there exists a sequence $\{v_i\}$ such that $\{\eta(v_i,0)\}$ converging to $\eta(v,0)$ and $x^* + t_i v_i \in C$ for sufficiently large i . By definition of η, we obtain that v_i converging to v. Thus $\langle J(\eta(v,0)), \eta(v_i,0)\rangle > 0$ holds for sufficiently large i, and hence we suppose that $x^* + t_i v_i$ lies in the open set $\{x \in E : \langle J(\eta(v,0)), \eta(x,x^*)\rangle > 0\}$. Therefore,

$$d^{\eta}_{C^\eta}(x^* + t_i v_i) \geq d^{\eta}_{H_v}(x^* + t_i v_i). \tag{4.5}$$

For each $x \in E$. We set

$$y := x - \left[\frac{\langle J(\eta(v,0)), \eta(x,0)\rangle - \beta}{\|J(\eta(v,0))\|_*^2}\right] v.$$

A straightforward computation show that $\langle J(\eta(v,0)), \eta(y,0)\rangle = \beta$, i.e., $y \in H_v$.
Furthermore, for any $z \in H_v$, we have

$$\begin{aligned}\langle J(\eta(x,y)), \eta(z,y)\rangle &= \left[\frac{\langle J(\eta(v,0)), \eta(x,0)\rangle - \beta}{\|J(\eta(v,0))\|_*^2}\right] \left(\langle J(\eta(v,0)), \eta(z,0)\rangle - \langle J(\eta(v,0)), \eta(y,0)\rangle\right) \\ &= 0.\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} d^{\eta}_{H_v}(x^* + t_i v_i) &= \|\eta(x^* + t_i v_i, y)\| \\ &= \frac{t_i \langle J(\eta(v,0)), \eta(v_i,0)\rangle}{\|J(\eta(v,0))\|_*^2} \|\eta(v,0)\| \\ &= \frac{t_i \langle J(\eta(v,0)), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}, \end{aligned}$$

and hence, by (4.1),

$$g(x^* + t_i v_i) \geq \alpha d_{C^\eta}(x^* + t_i v_i) \geq \alpha t_i \frac{\langle J\eta(v,0), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}.$$

By Proposition 3.1, $g(x^*) = 0$ for any $x^* \in C^\eta$, so

$$g(x^* + t_i v_i) - g(x^*) = g(x^* + t_i v_i) \geq \alpha t_i \frac{\langle J\eta(v,0), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}.$$

Since g is η-locally Lipschitz and η-Gateaux differentiable on C^η, there hold

$$\|g(x^* + t_i v_i) - g(x^* + t_i v)\| \leq L t_i \|\eta(v_i, v)\|$$

for some $L > 0$ and all sufficiently large i and

$$\begin{aligned} \langle \nabla_\eta g(x^*), \eta(v,0)\rangle &= \lim_{i\to\infty} \frac{g(x^* + t_i v) - g(x^*)}{t_i} \\ &= \lim_{i\to\infty} \frac{g(x^* + t_i v_i) - g(x^*)}{t_i} \geq \alpha\|\eta(v,0)\|. \end{aligned}$$

By Proposition 3.9, $\nabla_\eta g(x^*) = F(x^*)$. Thus

$$\langle F(x^*), \eta(v,0)\rangle \geq \alpha\|\eta(v,0)\|.$$

This implies that for each $w \in B_{E^*}$,

$$\langle \alpha w - F(x^*), \eta(v,0)\rangle = \langle \alpha w, \eta(v,0)\rangle - \langle F(x^*), \eta(v,0)\rangle \leq \alpha\|\eta(v,0)\| - \alpha\|\eta(v,0)\| = 0.$$

Hence $\alpha B_{E^*} - F(x^*) \subseteq [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ$, that is,

$$\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ.$$

This shows that C^η is weakly sharp since F is constant on C^η. □

Corollary 4.3 ([5]). *Let F be monotone on R^n and constant on $\Gamma(x^*)$ for some $x^* \in C^*$. Suppose that g is Gateaux differentiable, locally Lipschitz on C^*, and $g(x) < +\infty$ for all $x \in R^n$. Then C^* is weakly sharp if and only if there exists a positive number α such that*

$$\alpha d_{C^*}(x) \leq g(x) \quad \textit{for all} \quad x \in C.$$

Proof. By applying above Theorem 4.2, if we define $\eta(x,y) = x - y$, for all $x, y \in E$ and space $E = \mathbb{R}^n$, then C^η can be reduce to C^*, where C^* is the solution set of variational inequalities. Moreover, the mapping g is Gateaux differentiable and locally Lipschitz on C^*. □

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR n-TIMES DIFFERENTIABLE (α, m)-LOGARITHMICALLY CONVEX FUNCTIONS

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Abstract. In this paper, some new integral inequalities of Hermite-Hadamard type are presented for functions whose nth derivatives in absolute value are (α, m)-logarithmically convex. From our results, several inequalities of Hermite-Hadamard type can be derived in terms of functions whose first and second derivatives in absolute value are (α, m)-logarithmically convex functions as special cases. Our results may provide refinements of some results for (α, m)-logarithmically convex functions already exist in the most recent concerned literature of inequalities.

1. Introduction

Let us first refresh our knowledge how the following definition of classical convex functions is generalized.

Definition 1. *A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The inequalities in (1.1) are swapped if f is a concave function.

The definition of convex functions plays an important role in the theory of convex analysis and in many other branches of pure and applied mathematics. A number of remarkable and significant results in the theory of inequality hinge on this definition.

One of the momentous results which uses the notion of convexity is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : \emptyset \neq I \subseteq \mathbb{R} \to \mathbb{R}$, is a convex function of single variable, a, $b \in I$ with $a < b$. The inequalities in (1.2) are celebrated as Hermite-Hadamard inequality and are overturned if f is a concave function.

The inequalities (1.2) have been target of extensive research because of its usefulness and usages in the theory of inequalities and in various other branches of mathematics. A vast literature is reported on the Hermite-Hadamard type inequalities during the past few years which generalize, improve and extend the inequalities (1.2), see for example [6, 12, 13, 14, 15, 17, 19, 23, 28] and closely related references therein.

The classical convexity has been generalized in diverse ways such as s-convexity, m-convexity, (α, m)-convexity, h-convexity, logarithmic-convexity, s- logarithmic convexity, (α, m)- logarithmic convexity and h-logarithmic-convexity but we will focus on the following generalizations of the classical convexity to prove our results.

Definition 2. [2, 33, 34] *If a function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies*

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda}[f(y)]^{1-\lambda} \tag{1.3}$$

for all x, $y \in I$, $\lambda \in [0, 1]$, the function f is called logarithmically convex on I. If the inequality (1.3) reverses, the function f is called logarithmically concave on I.

The above stated concept logarithmically convex functions is further generalized as in the definitions below.

Definition 3. [9] *A function $f : [0, b] \to (0, \infty)$ is said to be m-logarithmically convex if*

$$f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all x, $y \in [0, b]$, $t \in [0, 1]$ and $m \in (0, 1]$.

Definition 4. [9] *A function $f : [0, b] \to (0, \infty)$ is said to be (α, m)-logarithmically convex if*

$$f(tx + m(1-t)y) \leq [f(x)]^{t^{\alpha}} [f(y)]^{m(1-t^{\alpha})}$$

holds for all x, $y \in [0, b]$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$.

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It is also obvious that if $m = 1$ in Definition 3 and if $(\alpha, m) = (1, 1)$ in Definition 4, the notion of m-logarithmic convexity and (α, m)-logarithmic convexity recapture the notion of usual logarithmic convexity.

Many papers have been written by a number of mathematicians concerning Hermite-Hadamard type inequalities for different classes of convex functions see for instance the recent papers [2, 3, 4, 7, 8, 9, 16, 18, 24, 25, 27, 29, 31, 32, 33, 35] and the references within these papers.

The main purpose of the present paper is to establish new Hermite-Hadamard type integral inequalities by using the notion of m- and (α, m)-logarithmically convex functions and a new identity for n-times differentiable functions from [19]in Section 2.

2. Main Results

We will use the following Lemmas to establish our main results in this section.

Lemma 1. [19] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a, b])$ for $n \in \mathbb{N}$, where a, $b \in I^\circ$ with $a < b$, we have the identity*

$$\begin{aligned}(2.1)\quad & \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt \\ &\quad + \frac{(-1)^n (b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right)dt,\end{aligned}$$

where an empty sum is understood to be nil.

Lemma 2. [20] *If $\mu > 0$ and $n \in \mathbb{N} \cup \{0\}$, then*

$$(2.2)\quad \int_0^1 t^n \mu^t dt = \begin{cases} \frac{(-1)^{n+1} n!}{(\ln\mu)^{n+1}} + n!\mu \sum_{k=0}^n \frac{(-1)^k}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1 \\ \frac{1}{n+1}, & \mu = 1. \end{cases}$$

Lemma 3. *If $\mu > 0$ and $\mathbb{N} \cup \{0\}$, then*

$$(2.3)\quad E(n;\mu) := \int_0^1 (1-t)^n \mu^t dt = \begin{cases} \frac{n!\mu}{(\ln\mu)^{n+1}} - n!\sum_{k=0}^n \frac{1}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1 \\ \frac{1}{n+1}, & \mu = 1. \end{cases}$$

Proof. By making the substitution $t = 1 - u$ in Lemma 2, we get (2.3). □

Lemma 4. [7] *For $\alpha > 0$ and $\mu > 0$, we have*

$$(2.4)\quad G(\alpha;\mu) := \int_0^1 (1-t)^{\alpha-1}\mu^t dt = \sum_{k=1}^{\infty} \frac{(\ln\mu)^{k-1}}{(\alpha)_k} < \infty,$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1).$$

From Lemma 3 and Lemma 4, by simple computations we get the following results.

Lemma 5. *If $\mu > 0$ and $n \in \mathbb{N}$, then*

$$(2.5)\quad F(n;\mu) := nE(n-1;\mu) - E(n;\mu) = \begin{cases} \frac{n!\mu(\ln\mu-1)}{(\ln\mu)^{n+1}} + \frac{1}{\ln\mu} - n!\sum_{k=1}^n \frac{\ln\mu-1}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1 \\ \frac{n}{n+1}, & \mu = 1. \end{cases}$$

Lemma 6. *For $\alpha > 0$ and $\mu > 0$, we have*

$$(2.6)\quad H(\alpha;\mu) := nG(\alpha;\mu) - G(\alpha+1;\mu) = \sum_{k=1}^{\infty} \frac{(n\alpha+nk-\alpha)(\ln\mu)^{k-1}}{(\alpha)_{k+1}} < \infty,$$

where

$$(\alpha)_{k+1} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k).$$

Lemma 7. [35] *Let $0 < \xi \le 1 \le \eta$, $0 \le \lambda \le 1$ and $0 < s \le 1$. Then*

$$(2.7)\quad \xi^{\lambda^s} \le \xi^{s\lambda} \text{ and } \eta^{\lambda^s} \le \eta^{s\lambda+1-s}.$$

Theorem 1. *Let $I\supset[0,\infty)$ be an open interval and let $f:I\to(0,\infty)$ be a function such that $f^{(n)}$ exists on I. If $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $0\le a<b<\infty$ and $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, $q\in[1,\infty)$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \le\frac{(b-a)^n}{2^{n+1}n!}\left(\frac{n}{n+1}\right)^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}, \tag{2.8}$$

where $F(n;\xi)$ is defined in Lemma 5, $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$ and

$$\theta=\begin{cases}\frac{\alpha}{2}, & 0<\mu\le1\\ 1-\frac{\alpha}{2}, & \mu>1.\end{cases}$$

Proof. From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le\frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\left(\int_0^1(1-t)^{n-1}(n-1+t)\,dt\right)^{1-\frac{1}{q}}\\ &\quad\times\left\{\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\},\end{aligned}\tag{2.9}$$

where $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.
It is obvious that

$$\int_0^1(1-t)^{n-1}(n-1+t)\,dt=\frac{n}{n+1}. \tag{2.10}$$

When $0<\mu\le1$, by using Lemma 5 and Lemma 7, we obtain

$$\begin{aligned}&\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}}+\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}}\\ &\le\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)}dt\right)^{\frac{1}{q}}+\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)}dt\right)^{\frac{1}{q}}\\ &=\mu^{\frac{\alpha}{2}}\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha qt}{2}}dt\right)^{\frac{1}{q}}+\mu^{\frac{\alpha}{2}}\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha qt}{2}}dt\right)^{\frac{1}{q}}\\ &=\mu^{\frac{\alpha}{2}}\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}\tag{2.11}$$

When $\mu>1$, by using Lemma 5 and Lemma 7, we have

$$\begin{aligned}&\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}}+\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}}\\ &\le\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}}+\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}}\\ &=\mu^{1-\frac{\alpha}{2}}\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha qt}{2}}dt\right)^{\frac{1}{q}}+\mu^{1-\frac{\alpha}{2}}\left(\int_0^1(1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha qt}{2}}dt\right)^{\frac{1}{q}}\\ &=\mu^{1-\frac{\alpha}{2}}\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}\tag{2.12}$$

A combination of (2.9)-(2.12) gives the desired result. □

Corollary 1. *Suppose the assumptions of Theorem 1 are satisfied and if $q=1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{F\left(n;\mu^{-\frac{\alpha}{2}}\right)+F\left(n;\mu^{\frac{\alpha}{2}}\right)\right\}, \tag{2.13}$$

where $F(n;\xi)$ is defined in Lemma 5, $\mu=\frac{|f^{(n)}(a)|}{|f^{(n)}(\frac{b}{m})|^m}$ and θ is defined in Theorem 1.

Corollary 2. *Under the assumptions of Theorem 1, if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{\left[F\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}, \tag{2.14}$$

where $\mu=\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}$, θ is defined in Theorem 1 and

$$F(1;\xi)=\begin{cases}\frac{1}{\ln\xi}\left[\xi+\frac{1-\xi}{\ln\xi}\right], & \xi\neq 1\\ \frac{1}{2}, & \xi=1.\end{cases}$$

Corollary 3. *Corollary 2 with $q=1$ gives the following result*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left|f'\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{\left[F\left(1;\mu^{-\frac{\alpha}{2}}\right)\right]+\left[F\left(1;\mu^{\frac{\alpha}{2}}\right)\right]\right\}, \tag{2.15}$$

where $F(1;\xi)$ and μ are defined as in Corollary 2 and θ is as defined in Theorem 1.

Corollary 4. *Suppose the assumptions of Theorem 1 are fulfilled and if $n=2$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left|f''\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{\left[F\left(2;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(2;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}, \tag{2.16}$$

where $\mu=\frac{|f''(a)|}{|f''(\frac{b}{m})|^m}$, θ is as defined in Theorem 1 and

$$F(2;\xi)=\begin{cases}\frac{2\xi\ln\xi-(\ln\xi)^2-2\xi+2}{(\ln\xi)^3}, & \xi\neq 1,\\ \frac{2}{3}, & \xi=1.\end{cases}$$

Corollary 5. *If $q=1$ in Corollary 4, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{16}\left|f''\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{F\left(2;\mu^{-\frac{\alpha}{2}}\right)+F\left(2;\mu^{\frac{\alpha}{2}}\right)\right\}, \tag{2.17}$$

where θ is defined in Theorem 1 and μ, $F(2;\xi)$ are defined in Corollary 4.

Theorem 2. *Let $I\supset[0,\infty)$ be an open interval and let $f:I\to(0,\infty)$ be a function such that $f^{(n)}$ exists on I. If $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $0\le a<b<\infty$ and $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, $q\in(1,\infty)$, we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right]^{1-\frac{1}{q}}}{2^{n+1}n!}\left(\frac{q-1}{2q-1}\right)^{1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta\\ &\quad\times\left\{\left[G\left(nq-q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(nq-q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{2.18}$$

where $\mu=\frac{|f^{(n)}(a)|}{|f^{(n)}(\frac{b}{m})|^m}$, $G(\alpha;\xi)$ is defined in Lemma 4 and θ is defined in Theorem 1.

Proof. Using Lemma 1, the Hölder inequality and the (α, m)-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[0, \frac{b}{m}\right]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.19}$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1 (n-1+t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 (1-t)^{q(n-1)}\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1 (1-t)^{q(n-1)}\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)\right\}^{1/q}.$$

The proof follows by using similar arguments as in proving Theorem 1, using Lemma 4 and Lemma 7. □

Corollary 6. *Under the assumptions of Theorem 2, if $n = 1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.20}$$
$$\leq \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1/q}\left|f'\left(\frac{b}{m}\right)\right|^m\mu^\theta\left\{\left[G\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},$$

where $\mu = \frac{\left|f'(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$,

$$G(1;\xi)=\sum_{k=1}^{\infty}\frac{(\ln\xi)^{k-1}}{k!}<\infty$$

and θ is defined in Theorem 1.

Corollary 7. *Under the assumptions of Theorem 2, if $n = 2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.21}$$
$$\leq \frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-\frac{1}{q}}}{16}\left(\frac{q-1}{2q-1}\right)^{1/q}\left|f''\left(\frac{b}{m}\right)\right|^m\mu^\theta$$
$$\times\left\{\left[G\left(q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},$$

where $\mu = \frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$,

$$G(q+1;\xi)=\sum_{k=1}^{\infty}\frac{(\ln\xi)^{k-1}}{(q+1)_k}<\infty$$

and θ is defined in Theorem 1.

Theorem 3. *Let $I \supset [0,\infty)$ be an open interval and let $f : I \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I. If $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where $0 \leq a < b < \infty$ and $\left|f^{(n)}\right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0,1] \times (0,1]$, $q \in (1,\infty)$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.22}$$
$$\leq \frac{n^{n+1-\frac{1}{q}}(b-a)^n}{2^{n+1}n!}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},$$

where $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq 1\\ 1, & \xi=1\end{cases},$$

$$B(z;\alpha,\beta)=\int_0^z t^{\alpha-1}(1-t)^{\beta-1}\,dt, 0\leq z\leq 1, \alpha>0, \beta>0$$

is the incomplete Beta function and θ is defined in Theorem 1.

Proof. Using Lemma 1, the Hölder inequality and the (α, m)-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[0, \frac{b}{m}\right]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.23}$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\}.$$

By using Lemma 7 and the fact that

$$\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt$$
$$=n^{\frac{nq+q-1}{q-1}}\int_0^{\frac{1}{n}}t^{\frac{(n-1)q}{q-1}}(1-t)^{\frac{q}{q-1}}\,dt=n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right),$$

we get the required inequality (2.22) from (2.23). □

Corollary 8. *Suppose the assumptions of Theorem 3 are satisfied and if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\}, \tag{2.24}$$

where $\mu=\frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq 1\\ 1, & \xi=1\end{cases}$$

and θ are defined as in Theorem 1.

Corollary 9. *Suppose the assumptions of Theorem 3 are satisfied and if $n=2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.25}$$
$$\leq\frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},$$

where $\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq 1\\ 1, & \xi=1\end{cases},$$

$B(z;\alpha,\beta)$ is the incomplete Beta function as defined in Theorem 3 and θ is defined as in Theorem 1.

Theorem 4. *Let $I\supset[0,\infty)$ be an open interval and let $f:I\subset[0,\infty)\to(0,\infty)$ be a function such that $f^{(n)}$ exists on I. If $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $0\leq a<b<\infty$ and $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, $q\in(1,\infty)$ for $0\leq r\leq(n-1)q$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.26}$$
$$\leq\frac{(b-a)^n}{2^{n+1}n!}\left[\frac{(q-1)\left(n^2q-nr-2n+r+1\right)}{(nq-r-1)(nq+q-r-2)}\right]^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta$$
$$\times\left\{\left[H\left(r+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left(r+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.$$

$\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$, *$\theta$ is defined in Theorem 1 and $H(\alpha;\xi)$ is defined in Lemma 6.*

Proof. From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.27}$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1(1-t)^{(nq-q-r)/(q-1)}(n-1+t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1(1-t)^r(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1(1-t)^r(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\},$$

The rest of the proof is similar to that of the proof of Theorem 2 by using Lemma 6 and Lemma 7. □

Corollary 10. *Suppose the assumptions of Theorem 4 are fulfilled and if $r = 0$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.28}$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left[\frac{(q-1)\left(n^2q-2n+1\right)}{(nq-1)(nq+q-2)}\right]^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta\left\{\left[H\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.$$

Corollary 11. *Suppose the assumptions of Theorem 4 are fulfilled and if $r = (n-1)q$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.29}$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left[\frac{2nq-2n-q+1}{2(q-1)}\right]^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta$$
$$\times\left\{\left[H\left((n-1)q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left((n-1)q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.$$

Remark 1. *Several interesting inequalities for m-logarithmically convex functions can be obtained by setting $\alpha = 1$ in the results presented in this section. However, we leave the details to the interested reader.*

Remark 2. *We can get several interesting inequalities for logarithmically convex functions by setting $\alpha = 1$ and $m = 1$ in the results proved above. However, the details are left to the interested reader.*

3. Applications to Special Means

For positive real numbers $a > 0$, $b > 0$, we consider the following means

$$A(a,b)=\frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ H(a,b)=\frac{2ab}{a+b},$$

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\\ a & a=b,\end{cases}$$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\neq 0,-1 \text{ and } a\neq b,\\ \frac{b-a}{\ln b-\ln a}, & p=-1 \text{ and } a\neq b,\\ I(a,b), & p=0 \text{ and } a\neq b,\\ a, & a=b.\end{cases}$$

It is well known that A, G, H, $L = L_{-1}$, $I = L_0$ and L_p are called the arithmetic, geometric, harmonic, identric, exponential and generalized logarithmic means of positive real numbers a and b.

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

Theorem 5. *Let $0 < a < b \leq 1$, $r < 0$, $r \neq -1$ and $q \geq 1$.*

(1) *If $r \neq -2$, then*

$$\left|A\left(a^{r+1}, b^{r+1}\right) - L_{r+1}^{r+1}(a,b)\right|$$
$$\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{2}{q}} |r+1| \left(\frac{1}{qr(\ln b - \ln a)}\right)^{1/q} \left\{ b^{r/2}\left[b^{qr/2} - L\left(a^{qr/2}, b^{qr/2}\right)\right]^{1/q} \right.$$
$$\left. + a^{r/2}\left[L\left(a^{qr/2}, b^{qr/2}\right) - a^{qr/2}\right]^{1/q} \right\}.$$

(2) *If $r = -2$, then*

$$\left|\frac{1}{H(a,b)} - \frac{1}{L(a,b)}\right|$$
$$\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} \left(\frac{1}{q(\ln a - \ln b)}\right)^{1/q} \left\{ b^{-1}\left[b^{-q} - L\left(a^{-q}, b^{-q}\right)\right]^{1/q} \right.$$
$$\left. + a^{-1}\left[L\left(a^{-q}, b^{-q}\right) - a^{-q}\right]^{1/q} \right\}.$$

Proof. Let $f(x) = \frac{x^{r+1}}{r+1}$ for $0 < x \leq 1$. Then $\left|f'(x)\right| = x^r$ and

$$\ln\left|f'(\lambda x + (1-\lambda)y)\right|^q$$
$$\leq \lambda \ln\left|f'(x)\right|^q + (1-\lambda)\ln\left|f'(y)\right|^q$$

for $x, y \in (0,1]$, $\lambda \in [0,1]$ and $q \geq 1$. This shows that $\left|f'(x)\right|^q = x^{rq}$ is logarithmically convex function on $(0,1]$ so that we have $(\alpha, m) = (1,1)$, $\mu = \left|\frac{f'(a)}{f'(b)}\right|$ and $\theta = \frac{1}{2}$.

Since $\left|f'(a)\right| > \left|f'(b)\right| = b^r \geq 1$, hence

$$\mu = \left|\frac{f'(a)}{f'(b)}\right| = \left(\frac{a}{b}\right)^r$$

and

$$\left[f'\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{ \left[F\left(1; \mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(1; \mu^{\frac{\alpha q}{2}}\right)\right]^{1/q} \right\}$$
$$= \sqrt{f'(b) f'(a)} \left\{ \left[F\left(1; \mu^{-\frac{q}{2}}\right)\right]^{1/q} + \left[F\left(1; \mu^{\frac{q}{2}}\right)\right]^{1/q} \right\}$$
$$= \sqrt{a^r b^r} \left\{ \left[F\left(1; \mu^{-\frac{q}{2}}\right)\right]^{1/q} + \left[F\left(1; \mu^{\frac{q}{2}}\right)\right]^{1/q} \right\}$$
$$= \left(\frac{2}{qr(\ln b - \ln a)}\right)^{1/q} \left\{ b^{r/2}\left[b^{qr/2} - L\left(a^{qr/2}, b^{qr/2}\right)\right]^{1/q} \right.$$
$$\left. + a^{r/2}\left[L\left(a^{qr/2}, b^{qr/2}\right) - a^{qr/2}\right]^{1/q} \right\}.$$

Substituting the above quantities in Corollary 2, we get the required results. □

Remark 3. *Many interesting inequalities of means can be obtained from the other results of Section 2, however, the details are left to the readers.*

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Positive solutions for p-Laplacian fractional difference equation with a parameter *

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Abstract: In this paper, we consider a boundary value problem for fractional difference equation with p-Laplacian operator involving a parameter

$$\begin{cases} \Delta[\phi_p(\Delta_C^\nu u)](t) + \lambda^{p-1} f(t+\nu-1, u(t+\nu-1)) = 0, t \in [0, b-1]_{\mathbb{N}_0}, \\ \Delta u(\nu-2) = \Delta_C^\nu u(0) = 0, u(\nu+b) = \gamma u(\eta), \end{cases}$$

where $1 < \nu \le 2$ is a real number, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\gamma \in (0,1)$, $\eta \in (\nu, \nu+b)$, Δ_C^ν denotes the discrete Caputo fractional difference of order ν, $f : [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}} \times [0, +\infty) \to (0, +\infty)$ is a continuous function, $b \ge 3$ is an integer, $\lambda > 0$ is a parameter. We study the existence of positive solutions to this problem by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones.

Keywords: boundary value problem; discrete fractional calculus; existence of solutions; p-Laplacian operator; Guo-Krasnosel'skii theorem.

Mathematics Subject Classification 2010: 39A12; 26A33; 34B15

1 Introduction

In recent years, fractional differential equations have received increasing attention. With the development of computer, it is well known that discrete analogues of differential equations can be very useful, especially for using computer to simulate the behavior of solutions for certain dynamic equations. More recent works to the discrete fractional calculus can be find in [1–7] and references contained therein. For example, Y. Pan and Z. Han et al. [5] studied the the existence and nonexistence of positive solutions to a boundary value problem for fractional difference equation with a parameter

$$-\Delta^\nu y(t) = \lambda f(t+\nu-1, y(t+\nu-1)), t \in [0, b+1]_{\mathbb{N}_0},$$
$$y(\nu-2) = y(\nu+b+1) = 0,$$

where $1 < \nu \le 2$ is a real number, $f : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to (0, +\infty)$ is a continuous function, $b \ge 2$ is an integer, λ is a parameter. The eigenvalue intervals of boundary value problem to a nonlinear fractional difference equation are considered by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones, some sufficient conditions to the nonexistence of positive solutions for the boundary value problem are established.

Differential equations with p-Laplacian operator are applied in real life, especially in physics and engineering [8]. Some theories of fractional difference equations with p-Laplacian operator are just beginning to be investigated. W. Lv [9] investigated the following boundary value problem for fractional difference equation involving a p-Laplacian operator

$$\Delta[\phi_p(\Delta_C^\alpha u)](t) = f(t+\alpha-1, u(t+\alpha-1)), t \in [0, b]_{\mathbb{N}_0},$$
$$u(\alpha-2) = \beta_1 u(\alpha+b+1),$$

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$$\Delta u(\alpha-2) = \Delta u(\alpha-1) = \beta_2 \Delta u(\alpha+b),$$

where $1 < \alpha \le 2$, $b \in \mathbb{N}_1$, $\beta_1 \ne 1$, $\beta_2 \ne 1$, Δ is the forward difference operator with step size 1, Δ_C^α denotes the discrete Caputo fractional difference of order α, $f : [\alpha-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ϕ_p is the p-Laplacian operator. Some existence and uniqueness results are obtained by using the Banach contraction mapping theorem.

In this paper, we discuss the following boundary value problem for fractional difference equation with p-Laplacian operator

$$\Delta[\phi_p(\Delta_C^\nu u)](t) + \lambda^{p-1} f(t+\nu-1, u(t+\nu-1)) = 0, t \in [0, b-1]_{\mathbb{N}_0}, \quad (1.1)$$

$$\Delta u(\nu-2) = \Delta_C^\nu u(0) = 0, u(\nu+b) = \gamma u(\eta), \quad (1.2)$$

where $1 < \nu \le 2$ is a real number, $\gamma \in (0,1), \eta \in (\nu, \nu+b)$, Δ_C^ν denotes the discrete Caputo fractional difference of order ν, $f : [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}} \times [0, +\infty) \to (0, +\infty)$ is a continuous function, $b \ge 3$ is an integer, $\lambda > 0$ is a parameter. ϕ_p is the p–Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$.

Our work presented in this article has the following features which are worth emphasizing.

(i) As far as we know, there are not many results available concerning with three–point boundary value problem of fractional difference equation which Δ_C^ν is the standard Caputo fractional difference.

(ii) We consider the boundary value problem with p-Laplacian which arises in the modeling of different physical and natural phenomena.

(iii) We investigate the intervals of parameter λ for boundary value problem to a nonlinear fractional difference equation with p-Laplacian.

The plan of the paper is as follows. In Section 2, we shall present some definitions and lemmas in order to prove our main results, the corresponding Green function and some properties of the Green function. In Section 3, we shall deduce the existence of positive solutions to problem (1.1)–(1.2) by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones. In Section 4, we give some examples to illustrate the theorems.

2 Preliminaries

For the convenience of the reader, we give some necessary basic definitions and lemmas that will be important to us in what follows.

Definition 2.1 ([6]) We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any t and ν, for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 ([7]) Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Then the ν-th fractional sum of f(based at a) at the point $t \in \mathbb{N}_{a+\nu}$ is defined by

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s).$$

Note that by our convention on delta sums we can extend the domain of $\Delta_a^{-\nu} f$ to $\mathbb{N}_{a+\nu-N}$, where N is the unique positive integer satisfying $N-1 < \nu \le N$, by noting that

$$\Delta_a^{-\nu} f(t) = 0, t \in \mathbb{N}_{a+\nu-N}^{a+\nu-1}.$$

Definition 2.3 ([10]) The ν-th Caputo fractional difference of a function $f : \mathbb{N}_a \to \mathbb{R}$, for $\nu > 0$, $\nu \notin \mathbb{N}$, is defined by

$$\begin{aligned} \Delta_C^\nu f(t) &= \Delta^{-(n-\nu)} \Delta^n f(t) \\ &= \tfrac{1}{\Gamma(n-\nu)} \Sigma_{s=a}^{t-n+\nu} (t-s-1)^{\underline{n-\nu-1}} \Delta^n f(s), \end{aligned}$$

for $t \in \mathbb{N}_{a+n-\nu}$, where n is the smallest integer greater than or equal to ν and Δ^n is the n-th forward difference operator. If $\nu = n$, then $\Delta_C^\nu f(t) = \Delta^n f(t)$.

Lemma 2.1 *([11]) Assume that $\nu > 0$ and f is defined on $\mathbb{N}_a$. Then*

$$\Delta^{-\nu}\Delta_C^{\nu}f(t) = f(t) + C_0 + C_1 t + \cdots + C_{n-1}t^{\underline{n-1}},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$, and n is the smallest integer greater than or equal to ν.

Lemma 2.2 *([6]) Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then*

$$\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}.$$

Lemma 2.3 *([6]) For t and s, for which both $(t-s-1)^{\underline{\nu}}$ and $(t-s-2)^{\underline{\nu}}$ are defined, we find that*

$$\Delta_s[(t-s-1)^{\underline{\nu}}] = -\nu(t-s-2)^{\underline{\nu-1}}.$$

Lemma 2.4 *Let $f : [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}} \times [0, +\infty) \to (0, +\infty)$ be given. A function u is a solution of the (1.1)–(1.2), if and only if it has the form*

$$\begin{aligned} u(t) = \quad & \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ & + \tfrac{\lambda\gamma}{1-\gamma}\textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big), t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}, \end{aligned} \tag{2.2}$$

where $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\Gamma(\nu)}\begin{cases}(\nu+b-s-1)^{\underline{\nu-1}} - (t-s-1)^{\underline{\nu-1}}, 0 \le s < t-\nu+1 \le b, \\ (\nu+b-s-1)^{\underline{\nu-1}}, 0 \le t-\nu+1 \le s \le b.\end{cases} \tag{2.3}$$

Proof. If $u(t)$ is a solution to (1.1)–(1.2). Then from (1.1), together with condition $\Delta_C^{\nu}u(0) = 0$, we find that

$$\begin{aligned}[\phi_p(\Delta_C^{\nu}u)](t) \quad &= \phi_p(\Delta_C^{\nu}u(0)) - \lambda^{p-1}\textstyle\sum_{s=0}^{t-1} f(s+\nu-1, u(s+\nu-1)) \\ &= -\lambda^{p-1}\textstyle\sum_{s=0}^{t-1} f(s+\nu-1, u(s+\nu-1)), t \in [0,b]_{\mathbb{N}_0},\end{aligned}$$

so

$$\Delta_C^{\nu}u(t) = -\lambda\phi_q\left(\sum_{s=0}^{t-1} f(s+\nu-1, u(s+\nu-1))\right), t \in [0,b]_{\mathbb{N}_0},$$

in view of Lemma 2.1, we have

$$u(t) = -\frac{\lambda}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) + C_0 + C_1 t, t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}. \tag{2.6}$$

Furthermore, (2.3) implies that

$$\Delta u(t) = -\frac{\lambda}{\Gamma(\nu-1)}\sum_{s=0}^{t-(\nu-1)}(t-s-1)^{\underline{\nu-2}}\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) + C_1, t \in [\nu-2, \nu+b-1]_{\mathbb{N}_{\nu-2}}.$$

By condition $\Delta u(\nu-2) = 0$, we can get that $C_1 = 0$. Then we obtain

$$u(t) = -\frac{\lambda}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) + C_0, t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}. \tag{2.8}$$

Now

$$u(\nu+b) = -\frac{\lambda}{\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) + C_0,$$

$$\gamma u(\eta) = -\frac{\lambda\gamma}{\Gamma(\nu)} \sum_{s=0}^{\eta-\nu} (\eta - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) + \gamma C_0,$$

by condition $u(\nu + b) = \gamma u(\eta)$, we obtain that

$$\begin{aligned} C_0 = & \tfrac{\lambda}{(1-\gamma)\Gamma(\nu)} \textstyle\sum_{s=0}^{b} (\nu + b - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) \\ & - \tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)} \textstyle\sum_{s=0}^{\eta-\nu} (\eta - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right). \end{aligned}$$

Now, substitution of C_0 and C_1 into (2.6) gives

$$\begin{aligned} u(t) = & -\tfrac{\lambda}{\Gamma(\nu)} \textstyle\sum_{s=0}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) \\ & + \tfrac{\lambda}{(1-\gamma)\Gamma(\nu)} \textstyle\sum_{s=0}^{b} (\nu + b - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) \\ & - \tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)} \textstyle\sum_{s=0}^{\eta-\nu} (\eta - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right), t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}, \end{aligned}$$

splitting the second sum in two parts on the basis of the following equality

$$\frac{1}{\Gamma(\nu)} + \frac{\gamma}{(1-\gamma)\Gamma(\nu)} = \frac{1}{(1-\gamma)\Gamma(\nu)},$$

therefore,

$$\begin{aligned} u(t) = & -\tfrac{\lambda}{\Gamma(\nu)} \textstyle\sum_{s=0}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) \\ & + \tfrac{\lambda}{\Gamma(\nu)} \textstyle\sum_{s=0}^{b} (\nu + b - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) \\ & + \tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)} \textstyle\sum_{s=0}^{b} (\nu + b - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) \\ & - \tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)} \textstyle\sum_{s=0}^{\eta-\nu} (\eta - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right), t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}, \end{aligned}$$

which is equivalent to (2.1) that

$$\begin{aligned} u(t) = & \lambda \textstyle\sum_{s=0}^{b} G(t, s) \phi_q \big(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \big) \\ & + \tfrac{\lambda\gamma}{1-\gamma} \textstyle\sum_{s=0}^{b} G(\eta, s) \phi_q \big(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \big), t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}. \end{aligned}$$

On the other hand, if the function $u(t)$ satisfies to (2.1), then $u(\nu + b) = \gamma u(\eta)$. What's more, function $u(t)$ defined by (2.2) can transform to (2.8) that

$$u(t) = -\frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right) + C_0, t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}.$$

Then we find that

$$\Delta u(t) = -\frac{\lambda}{\Gamma(\nu - 1)} \sum_{s=0}^{t-(\nu-1)} (t - s - 1)^{\underline{\nu-2}} \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau + \nu - 1, u(\tau + \nu - 1)) \right), t \in [\nu - 2, \nu + b - 1]_{\mathbb{N}_{\nu-2}}, \tag{2.16}$$

and

$$\begin{aligned} \Delta_C^{\nu} u(t) & = -\Delta_C^{\nu} \left\{ \Delta^{-\nu} \left[\lambda \phi_q \left(\textstyle\sum_{s=0}^{t-1} f(s + \nu - 1, u(s + \nu - 1)) \right) \right] \right\} \\ & = -\phi_q \left(\textstyle\sum_{s=0}^{t-1} \lambda^{p-1} f(s + \nu - 1, u(s + \nu - 1)) \right), t \in [0, b]_{\mathbb{N}_0}. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we see that $\Delta u(\nu-2) = \Delta_C^\nu u(0) = 0$.
From the above proofs, it is to say function $u(t)$ meets the boundary condition (1.2). Then we will proof that $u(t)$ satisfies the fractional difference equation (1.1). Taking p-Laplacian operators on sides of (2.17), we find that

$$[\phi_p(\Delta_C^\nu u)](t) = -\sum_{s=0}^{t-1} \lambda^{p-1} f(s+\nu-1, u(s+\nu-1)), t \in [0,b]_{\mathbb{N}_0}. \tag{2.18}$$

By equation (2.18), function $\Delta[\phi_p(\Delta_C^\nu u)](t)$ has the form

$$\Delta[\phi_p(\Delta_C^\nu u)](t) = -\lambda^{p-1} f(t+\nu-1, u(t+\nu-1)), t \in [0, b-1]_{\mathbb{N}_0},$$

which shows that if (1.1)–(1.2) has a solution, then it can be represented by (2.2) and that every function of the form (2.2) is a solution of (1.1)–(1.2), which completes the proof.

Lemma 2.5 *([12]) Let ν be any positive real number and a, b be two real numbers such that $\nu < a \leq b$. Then the following are valid.*
(i) $\frac{1}{x^{\underline{\nu}}}$ is a decreasing function for $x \in (0, +\infty)_{\mathbb{N}}$.
(ii) $\frac{(a-x)^{\underline{\nu}}}{(b-x)^{\underline{\nu}}}$ is a decreasing function for $x \in [0, a-\nu)_{\mathbb{N}}$.

Lemma 2.6 *The function $G(t,s)$ defined by (2.3) has the following properties:*
1. $0 \leq G(t,s) \leq G(s+\nu-1, s)$, for $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$ and $s \in [0,b]_{\mathbb{N}_0}$;
2. there exists a positive number $\kappa \in (0,1)$ such that

$$\min_{t \in [\frac{(b+\nu)}{4}, \frac{3(b+\nu)}{4}]} G(t,s) \geq \kappa \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = \kappa G(s+\nu-1, s),$$

for $s \in [0,b]$.

Proof. 1. For $0 \leq s < t-\nu+1 \leq b$, we get

$$(\nu+b-s-1)^{\underline{\nu-1}} \geq (t-s-1)^{\underline{\nu-1}},$$

which implies $G(t,s) \geq 0$. For $0 \leq t-\nu+1 \leq s \leq b$, clear $G(t,s) \geq 0$.

On the other hand, case (1): $0 \leq s < t-\nu+1 \leq b$,

$$\Delta_t G(t,s) = -\frac{(\nu-1)(t-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)} < 0;$$

case (2): $0 \leq t-\nu+1 \leq s \leq b$,

$$\Delta_t G(t,s) = 0.$$

Combining the above two cases, we have the function $G(t,s)$ is non-increasing of t, thus

$$G(t,s) \leq G(s+\nu-1, s), s \in [0,b]_{\mathbb{N}_0}.$$

2. For $s \geq t-\nu+1$ and $\frac{(b+\nu)}{4} \leq t \leq \frac{3(b+\nu)}{4}$, we have

$$\frac{G(t,s)}{G(s+\nu-1,s)} = 1.$$

For $s < t-\nu+1$ and $\frac{(b+\nu)}{4} \leq t \leq \frac{3(b+\nu)}{4}$, we have that

$$\begin{aligned} \frac{G(t,s)}{G(s+\nu-1,s)} &= \frac{(\nu+b-s-1)^{\underline{\nu-1}} - (t-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}} \\ &= 1 - \frac{(t-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}} \\ &\geq 1 - \frac{(\frac{3(b+\nu)}{4}-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}. \end{aligned}$$

By Lemma 2.5 (ii),

$$\frac{(\frac{3(b+\nu)}{4}-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}$$

is decreasing for $0 \le s < \frac{3(b+\nu)}{4}-\nu+1$. Hence

$$\frac{G(t,s)}{G(s+\nu-1,s)} \ge 1-\frac{(\frac{3(b+\nu)}{4}-1)^{\underline{\nu-1}}}{(\nu+b-1)^{\underline{\nu-1}}},$$

which implies

$$\min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]} G(t,s) \ge \kappa G(s+\nu-1,s),$$

where $\kappa = 1-\frac{(\frac{3(b+\nu)}{4}-1)^{\underline{\nu-1}}}{(\nu+b-1)^{\underline{\nu-1}}}$.

Lemma 2.7 *([12]) Let B be a Banach space and let $P\subseteq B$ be a cone. Assume that Ω_1 and Ω_2 are open subsets contained in B such that $0\in\Omega_1$ and $\overline{\Omega_1}\subseteq\Omega_2$. Assume, further, that $T: P\cap(\overline{\Omega_2}\setminus\Omega_1)\to P$ is a completely continuous operator. If either*
(1)$\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_1$ and $\|Tu\|\ge\|u\|$ for $u\in P\cap\partial\Omega_2$; or
(2)$\|Tu\|\ge\|u\|$ for $u\in P\cap\partial\Omega_1$ and $\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_2$.
Then T has at least one fixed point in $P\cap(\overline{\Omega_2}\setminus\Omega_1)$.

Define the Banach space B by

$$B=\{u:[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\to\mathbb{R}\}$$

with norm $\|u\|=\max\{|u(t)|, t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\}$.

Define the cone

$$P=\left\{u\in B \mid u(t)\ge 0, t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}, \min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]_{\mathbb{N}_{\nu-2}}} u(t)\ge\sigma\|u\|, \sigma=\kappa(1-\gamma)\right\}. \tag{2.21}$$

From Lemma 2.4, we know that u is a solution of (1.1)–(1.2) if and only if u is a fixed point of the operator $T: B\to B$ defined by

$$\begin{aligned} Tu(t)= \;& \lambda\textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\big) \\ & +\tfrac{\lambda\gamma}{1-\gamma}\textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\big), t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}. \end{aligned} \tag{2.22}$$

Lemma 2.8 *Let T be defined as in (2.22) and P as in (2.21). Then $T: P\to P$ is completely continuous.*

Proof. Note that T is a summation operator on a discrete finite set, so T is trivially completely continuous. We have that

$$\|Tu\|\le\frac{\lambda}{1-\gamma}\sum_{s=0}^{b}\frac{(\nu+b-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\right). \tag{2.23}$$

For all $u \in P$, it follows from (2.23) that

$$\begin{aligned}\min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]}(Tu)(t) = & \min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]} \lambda \sum_{s=0}^{b} G(t,s)\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) \\ & + \frac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) \\ & - \frac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\sum_{s=0}^{\eta-\nu}(\eta-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) \\ & \geq \kappa\lambda\sum_{s=0}^{b}\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s)\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) \\ & = \kappa(1-\gamma)\frac{\lambda}{1-\gamma}\sum_{s=0}^{b} G(s+\nu-1,s)\phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) \\ & \geq \sigma\|Tu\|.\end{aligned}$$

It is obvious that $(Tu)(t) \geq 0$ whenever $u \in P$, thus, $T : P \to P$ as desired.

3 Existence of positive solutions

In this section, we will show the existence of positive solutions for boundary value problem (1.1)–(1.2). Let us put

$$\varphi(l) = \min\{f(t,u), (t,u) \in [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}} \times [0,l]\},$$

$$\psi(l) = \max\{f(t,u), (t,u) \in [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}} \times [0,l]\}.$$

(H1) The function $f(t,u)$ satisfies $\lim_{u\to 0^+} \frac{\max_{t\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}} f(t,u)}{u^{p-1}} = 0.$

(H2) The function $f(t,u)$ satisfies $\lim_{u\to +\infty} \frac{\min_{t\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}} f(t,u)}{u^{p-1}} = +\infty.$

Set $l_0 = \lceil \frac{(b+\nu)}{4} - \nu + 1\rceil, l_1 = \lfloor \frac{3(b+\nu)}{4} - \nu + 1\rfloor$, $K = \max G(t,s)$, for $(t,s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0}$. For convenience, we denote

$$A = \sum_{s=0}^{b}\phi_q(s), B = \sum_{s=l_0}^{l_1}\phi_q(s).$$

Theorem 3.1 *If $f \in C([\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}} \times [0,+\infty) \to (0,+\infty))$ and there exist two positive constants $\alpha_2 > \alpha_1$ such that*

$$\phi_p(\delta_1\alpha_1) \leq \varphi(\alpha_1), \psi(\alpha_2) \leq \phi_p(\delta_2\alpha_2)$$

hold, where δ_1, δ_2 are positive constants satisfying

$$\frac{\delta_2 A}{1-\gamma} < \kappa\delta_1 B,$$

then for each

$$\lambda \in \left(\left(\kappa\delta_1 KB\right)^{-1}, \left(\frac{\delta_2 KA}{1-\gamma}\right)^{-1}\right),$$

the problem (1.1)–(1.2) has at least one positive solution.

Proof. Let $\Omega_1 = \{u \in B : \|u\| \leq \alpha_1\}$. For any $u \in P$ with $\|u\| = \alpha_1$, we have $\phi_p(\delta_1\alpha_1) \leq \varphi(\alpha_1) \leq f(t+\nu-1, u(t+\nu-1))$ for $(t+\nu-1, u(t+\nu-1)) \in [0, b-1]_{\mathbb{N}_0} \times [0, \alpha_1]$. We have

$$\begin{aligned}
|Tu(t)| &= \lambda \sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\
&\quad + \tfrac{\lambda\gamma}{1-\gamma} \sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\
&\geq \lambda \sum_{s=l_0}^{l_1} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\
&\geq \kappa\lambda K \sum_{s=l_0}^{l_1} \phi_q\big(\sum_{\tau=0}^{s-1} \varphi(\alpha_1)\big) \\
&\geq \kappa\lambda K \sum_{s=l_0}^{l_1} \phi_q\big(\sum_{\tau=0}^{s-1} \phi_p(\delta_1\alpha_1)\big) \\
&= \alpha_1\delta_1\lambda\kappa K B \\
&\geq \alpha_1.
\end{aligned}$$

So,

$$\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1. \tag{3.2}$$

On the other hand, let $\Omega_2 = \{u \in B : \|u\| \leq \alpha_2\}$. For any $u \in P$ with $\|u\| = \alpha_2$, we have $f(t+\nu-1, u(t+\nu-1)) \leq \psi(\alpha_2) \leq \phi_p(\delta_2\alpha_2)$ for $(t+\nu-1, u(t+\nu-1)) \in [0, b-1]_{\mathbb{N}_0} \times [0, \alpha_2]$. We have

$$\begin{aligned}
|Tu(t)| &= \lambda \sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\
&\quad + \tfrac{\lambda\gamma}{1-\gamma} \sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\
&\leq K\lambda \sum_{s=0}^{b} \phi_q\left(\sum_{\tau=0}^{s-1} \psi(\alpha_2)\right) + \tfrac{\lambda\gamma}{1-\gamma} K \sum_{s=0}^{b} \phi_q\left(\sum_{\tau=0}^{s-1} \psi(\alpha_2)\right) \\
&\leq K \tfrac{\lambda}{1-\gamma} \sum_{s=0}^{b} \phi_q\left(\sum_{\tau=0}^{s-1} \phi_p(\delta_2\alpha_2)\right) \\
&= \delta_2\alpha_2\lambda \tfrac{K}{1-\gamma} A \\
&\leq \alpha_2.
\end{aligned}$$

Hence,

$$\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2. \tag{3.4}$$

Consequently, from (3.2) and (3.4), we may invoke Lemma 2.6 to deduce that T has a fixed point in the set $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. Then the theorem is proved.

Theorem 3.2 *Suppose that conditions (H1)–(H2) hold. If there exist a sufficient small positive constant δ_3 and sufficient large δ_4 such that*

$$\delta_3 \frac{A}{1-\gamma} < \kappa\delta_4 B$$

holds, then for each

$$\lambda \in \left(\left(\delta_4\kappa K B\right)^{-1}, \left(\frac{\delta_3 K A}{1-\gamma}\right)^{-1}\right),$$

then problem (1.1)–(1.2) has at least one positive solution.

Proof. Because of condition (H1), there exist $\beta_1 > 0$ and a sufficient small constant $\delta_3 > 0$ such that

$$f(t,u) \leq (\delta_3 u)^{p-1}, 0 < u \leq \beta_1. \tag{3.5}$$

So for $u \in P$ with $\|u\| = \beta_1$, by (2.22) and (3.5), we have for all $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$,

$$\begin{aligned} \|Tu\| &= \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\quad + \tfrac{\lambda\gamma}{1-\gamma}\textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\le \lambda\|u\|\delta_3 KA + \|u\|\tfrac{\lambda\gamma}{1-\gamma}\delta_3 KA \\ &= \lambda\|u\|\delta_3 K\tfrac{1}{1-\gamma}A \\ &\le \|u\|. \end{aligned} \tag{3.6}$$

Thus, if we choose $\Omega_1 = \{u \in B : \|u\| \le \beta_1\}$, then (3.6) implies that

$$\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_1. \tag{3.7}$$

On the other hand, condition (H2) implies that there exist a number $0 < \beta_1 < \beta_2$ and a sufficient large constant δ_4 such that

$$f(t,u) \ge (\delta_4 u)^{p-1}, u \ge \beta_2. \tag{3.8}$$

And then we set $\beta_2^* = \frac{\beta_2}{\sigma} > \beta_2$. Then, $u \in P$ and $\|u\| = \beta_2^*$ implies $\min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]} u(t) \ge \sigma\|u\| = \beta_2$, thus $u(t) \ge \beta_2$, for all $t \in [\frac{(b+\nu)}{4}, \frac{3(b+\nu)}{4}]$. Therefore, for all $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$, by (2.11) and (3.8), we have that

$$\begin{aligned} Tu(t) &= \lambda\textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\quad + \tfrac{\lambda\gamma}{1-\gamma}\textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\ge \lambda\textstyle\sum_{s=l_0}^{l_1} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\ge \lambda\kappa\textstyle\sum_{s=l_0}^{l_1} K\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\ge \lambda\|u\|\delta_4\kappa KB \\ &\ge \|u\|. \end{aligned} \tag{3.9}$$

Hence, if we choose $\Omega_2 = \{u \in B : \|u\| \le \beta_2^*\}$, from (3.9) we have that

$$\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_2. \tag{3.10}$$

Consequently, from (3.7) and (3.10), we may invoke Lemma 2.6 to deduce that T has a fixed point in the set $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. Then the theorem is proved.

4 Examples

In this section, we will present some examples to illustrate main results.

Example 4.1 Suppose that $\nu = \frac{3}{2}, b = 9, p = \frac{3}{2}$. Take $\gamma = 0.1, \alpha_1 = 2$ and $\alpha_2 = 80000$. Then $f(t,u) = t^2 + \sin u + 6$ and problem (1.1)–(1.2) becomes

$$\Delta[\phi_p(\Delta_C^{\frac{3}{2}}u)](t) + \lambda^{p-1} f(t,u) = 0, t \in [0,8]_{\mathbb{N}_0}, \tag{4.1}$$

$$\Delta u(\nu-2) = \Delta_C^{\frac{3}{2}}u(0) = 0, u(\nu+b) = 0.1u(\eta). \tag{4.2}$$

Make $\delta_1 = 15$ and $\delta_2 = \frac{1}{10}$. By calculation, we have $K = \max_{(t,s)\in[-\frac{1}{2},\frac{21}{2}]_{\mathbb{N}_0}\times[0,9]_{\mathbb{N}_0}} G(t,s) \approx 3.524$, $\sqrt{30} = \phi_p(\delta_2\alpha_1) < \varphi(\alpha_1) = 6$ and $88 = \psi(\alpha_2) < \phi_p(\delta_1\alpha_2) \approx 89$. Then $\delta_1 K\frac{1}{1-\gamma}\sum_{s=0}^{b}\phi_q(s) \approx$

$3.524 \times \frac{1}{9} \times 285 \approx 111.6$ and $\delta_2 \kappa K \sum_{s=\lceil \frac{(b+\nu)}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(b+\nu)}{4} - \nu + 1 \rfloor} \phi_q(s) \approx 15 \times 0.0871 \times 3.524 \times 135 \approx 690$. So, the conditions of Theorem 3.1 are satisfied. Then the boundary value problem (4.1)–(4.2) has at least one positive solution for each $\lambda \in (0.001, 0.009)$.

Example 4.2 Suppose that $\nu = \frac{3}{2}, b = 9, p = \frac{3}{2}$. Take $\gamma = 0.1, \delta_3 = \frac{1}{1000}$ and $\delta_4 = 2$. Then $f(t,u) = (t+1)u^{\frac{3}{2}}$, and problem (1.1)–(1.2) becomes

$$\Delta[\phi_p(\Delta_C^{\frac{3}{2}} u)](t) + \lambda^{p-1} f(t,u) = 0, t \in [0,8]_{\mathbb{N}_0}, \tag{4.3}$$

$$\Delta u(\nu - 2) = \Delta_C^{\frac{3}{2}} u(0) = 0, u(\nu + b) = 0.1u(\eta). \tag{4.4}$$

In addition, we have $K = \max_{(t,s)\in[-\frac{1}{2},\frac{21}{2}]_{\mathbb{N}_0}\times[0,9]_{\mathbb{N}_0}} G(t,s) \approx 3.524$, $\lim_{u\to 0^+} \frac{\max_{t\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}} f(t,u)}{u^{p-1}} = 0$ and $\lim_{u\to+\infty} \frac{\min_{t\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}} f(t,u)}{u^{p-1}} = +\infty$. Then $\delta_3 K \frac{1}{1-\gamma} \sum_{s=0}^{b} \phi_q(s) \approx \frac{1}{1000} \times 3.524 \times \frac{10}{9} \times 285 \approx 1.116$ and $\delta_4 \kappa K \sum_{s=\lceil \frac{(b+\nu)}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(b+\nu)}{4} - \nu + 1 \rfloor} \phi_q(s) \approx 2 \times 0.0871 \times 3.524 \times 135 \times \approx 90.90$. So, the conditions of Theorem 3.2 are satisfied. Then the boundary value problem (4.3)–(4.4) has at least one positive solution for each $\lambda \in (0.012, 0.896)$.

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Weighted Composition Operators from analytic Morrey spaces into Zygmund spaces*

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Abstract

In this paper we characterize the boundedness and compactness of the weighted composition operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

Keywords Analytic Morrey space, Zygmund space; Weighted composition operator; Boundedness; Compactness

2010 MR Subject Classification 47B38, 30D99, 30H05

1 Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane and $H(D)$ denote the set of all analytic functions on D. Let $u, \varphi \in H(D)$, where φ is an analytic self-map of D. Then the well-known *weighted composition operator* uC_φ on $H(D)$ is defined by $uC_\varphi(f)(z) = u(z)\cdot(f\circ\varphi(z))$ for $f \in H(D)$ and $z \in D$. Weighted composition operators can be regarded as a generalization of multiplication operators M_u and composition operators C_φ. In 2001, Ohno and Zhao studied the weighted composition operators on the classical Bloch space β in [18], which has led many researchers to study this operator on other Banach spaces of analytic functions. The boundedness and compactness of it have been studied on various Banach spaces of analytic functions, such as Hardy, Bergman, BMOA, Bloch-type spaces, see, e.g. [4, 6, 11, 29].

For an arc $I \subset \partial\mathbb{D}$, let $|I| = \frac{1}{2\pi}\int_I |d\zeta|$ be the normalized arc length of I,

$$f_I = \frac{1}{|I|}\int_I f(\zeta)\frac{|d\zeta|}{2\pi}, f \in H(D),$$

and $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in D : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I\}.$$

Clearly, if $I = \partial D$, then $S(I) = D$.

Let $\mathcal{L}^{2,\lambda}(D)$ represent the analytic $Morrey$ spaces of all analytic functions $f \in H^2$ on D such that

$$\sup_{I\subset\partial D}\Big(\frac{1}{|I|^\lambda}\int_I |f(\zeta) - f_I|^2\frac{|d\zeta|}{2\pi}\Big)^{1/2} < \infty,$$

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where $0 < \lambda \le 1$ and the Hardy space H^2 consists of analytic functions f in D satisfying

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty.$$

From Theorem 3.1 of [25] or Theorem 3.21 of [26], we can define the norm of function $f \in \mathcal{L}^{2,\lambda}(D)$ and its equivalent formula as follows

$$\begin{aligned} \|f\|_{\mathcal{L}^{2,\lambda}} &= |f(0)| + \sup_{I \subset \partial D} \big(\frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1-|z|^2) dm(z)\big)^{1/2} \\ &\approx |f(0)| + \sup_{a \in D} \big((1-|a|^2)^{1-\lambda} \int_D |f'(z)|^2 (1-|\varphi_a(z)|^2) dm(z)\big)^{1/2}. \end{aligned}$$

Similarly to the relation between $BMOA$ space and $VMOA$ space, we have that $f \in \mathcal{L}_0^{2,\lambda}(D)$, the little analytic $Morrey$ spaces, if $f \in \mathcal{L}^{2,\lambda}(D)$ and

$$\lim_{|I|\to 0} \big(\frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi}\big)^{1/2} = 0.$$

Clearly, $\mathcal{L}_0^{2,1}(D) = VMOA$. The following lemma gives equivalent conditions of $\mathcal{L}_0^{2,\lambda}$. The proof is similar to that of Theorem 6.3 in [10], we omit the details.

Lemma 1.1 *Suppose that* $0 < \lambda < 1$ *and* $f \in H(D)$. *Let* $a \in D$, $\varphi_a(z) = \dfrac{a-z}{1-\overline{a}z}$. *Then the following statements are equivalent.*

(i) $f \in \mathcal{L}_0^{2,\lambda}(D)$;

(ii) $\lim_{|a|\to 1} (1-|a|^2)^{1-\lambda} \int_D |f'(z)|^2 (1-|\varphi_a(z)|^2) dm(z) = 0$;

(iii) $\lim_{|a|\to 1} (1-|a|^2)^{1-\lambda} \int_D |f'(z)|^2 \log \dfrac{1}{|\varphi_a(z)|} dm(z) = 0$.

It is known that $\mathcal{L}^{2,1}(D) = BMOA$ and if $0 < \lambda < 1$, $BMOA \subsetneq \mathcal{L}^{2,\lambda}(D)$. For more information on $BMOA$ and $VMOA$, see [10].

The Zygmund space $\mathcal{Z}$ consists of all analytic functions f defined on D such that

$$z(f) = \sup\{(1-|z|^2)|f''(z)| : z \in D\} < +\infty.$$

From a theorem of Zygmund (see [37, vol. I, p. 263] or [8, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if f is continuous in the close unit disk $\overline{D} = \{z : |z| \le 1\}$ and the boundary function $f(e^{i\theta})$ such that

$$\sup_{h>0,\theta} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

An analytic function $f \in H(D)$ is said to belong to the little Zymund space $\mathcal{Z}_0$ consists of all $f \in \mathcal{Z}$ satisfying $\lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0$. It can easily proved that $\mathcal{Z}$ is a Banach space under the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + z(f)$$

and the polynomials are norm-dense in closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$. For some other information on this space and some operators on it, see, for example, [12, 13, 15].

$Morrey$ space was initially introduced in 1938 by Morrey [17] to show that certain systems of partial differential equations (PDEs) had Hölder continuous solutions. In the past, $Morrey$ space has been studied heavily in different areas. For example, Adams and Xiao studied $Morrey$

spaces which is defined on Euclidean spaces $\mathbb{R}^n$ by potential theory and Hausdorff capacity in [1, 2]. Wang, Xiao [23] studied holomorphic Campanato spaces on the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. Wang and Xiao [24] characterized the first and second preduals of the analytic *Morrey* spaces $\mathcal{L}^{2,\lambda}$ on the unit disk. Xiao and Yuan [28] studied the analytic Campanato spaces (including the analytic *Morrey* spaces) in terms of the Möbius mappings and the Littlewood-Paley forms. Xiao and Xu [27] studied the composition operators of $\mathcal{L}^{2,\lambda}$ spaces. Li, Liu and Lou[14] studied the Volterra-type operators on $\mathcal{L}^{2,\lambda}$ spaces. Zhuo and Ye [36] considered this operators from $\mathcal{L}^{2,\lambda}$ spaces to the classical Bloch space.

In 2006, the boundedness of composition operators on the Zygmund space $\mathcal{Z}$ was first studied by Choe, Koo, and Smith in [3]. Later, many researchers have studied composition operators and weighted composition operators acting on the Zygmund space $\mathcal{Z}$. Li and Stević in [12] studied the boundedness and compactness of the generalized composition operators on Zygmund spaces and Bloch type spaces. Ye and Hu in [34] characterized boundedness and compactness of weighted composition operators on the Zygmund space $\mathcal{Z}$. Esmaeili and Lindström in [9] studied weighted composition operators from Zygmund type spaces to Bloch type spaces and their essential norms. Sanatpour and Hassanlou in [20] gave the essential norms of this operators between Zygmund-type spaces and Bloch-type spaces. See also [7, 19, 22, 29, 30, 31, 32, 33, 35] for corresponding results for weighted composition operators from one Banach space of analytic functions to another. In this paper we consider the weighted composition operators from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

Notations: For two functions F and G, if there is a constant $C > 0$ dependent only on indexes $p, \lambda ...$ such that $F \leq CG$, then we say that $F \lesssim G$. Furthermore, denote that $F \approx G$ (F is comparable with G) whenever $F \lesssim G \lesssim F$.

2 Auxiliary results

In order to prove the main results of this paper. we need some auxiliary results.

Lemma 2.1 *Let* $0 < \lambda < 1$. *If* $f \in \mathcal{L}^{2,\lambda}$, *then*

(i) $|f(z)| \lesssim \dfrac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{2}}}$ *for every* $z \in D$;

(ii) $|f'(z)| \lesssim \dfrac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{3-\lambda}{2}}}$ *for every* $z \in D$;

(iii) $|f''(z)| \lesssim \dfrac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{5-\lambda}{2}}}$ *for every* $z \in D$.

Proof (i) and (ii) are from Lemma 2.5 in [14]. For any $f \in \mathcal{L}^{2,\lambda}$. Fix $z \in D$ and let $\rho = \dfrac{1+|z|}{2}$, by the Cauchy integral formula, we obtain that

$$|f''(z)| = |\frac{1}{2\pi i}\int_{|\xi|=\rho}\frac{f'(\xi)}{(\xi-z)^2}\,d\xi| \leq \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-\rho^2)^{\frac{3-\lambda}{2}}}\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho\,d\theta}{|\rho e^{i\theta}-z|^2} = \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-\rho^2)^{\frac{3-\lambda}{2}}}\frac{\rho}{\rho^2-|z|^2} \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{5-\lambda}{2}}}.$$

Hence (iii) holds.

Lemma 2.2 *Let* $0 < \lambda < 1$. *If* $f \in \mathcal{L}_0^{2,\lambda}$, *then*

(i) $\lim\limits_{|z|\to 1}(1-|z|^2)^{\frac{3-\lambda}{2}}|f'(z)| = 0$;

(ii) $\lim\limits_{|z|\to 1}(1-|z|^2)^{\frac{1-\lambda}{2}}|f(z)| = 0$;

(iii) $\lim\limits_{|z|\to 1}(1-|z|^2)^{\frac{5-\lambda}{2}}|f''(z)| = 0$.

The proof of (i) is similar to that of Lemma 2.5 in [14], and we easily obtain (ii) and (iii) by (i). These details are omitted here.

Lemma 2.3 *Suppose* $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ *is a bounded operator, then* $uC_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{Z}$ *is a bounded operator.*

The proof is similar to that of Lemma 2.3 in [33]. The details are omitted.

3 Boundedness of uC_φ

In this section we characterize the boundedness of the weighted composition operator uC_φ from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

Theorem 3.1 *Let* u *be an analytic function on the unit disc* D, *and* φ *an analytic self-map of* D. *Then* uC_φ *is a bounded operator from the analytic Morrey spaces* $\mathcal{L}^{2,\lambda}$ *to the Zygmund space* $\mathcal{Z}$ *if and only if the following are satisfied:*

$$\sup_{z\in D}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty; \tag{3.1}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty; \tag{3.2}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty. \tag{3.3}$$

Proof Suppose uC_φ is bounded from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Using functions $f(z) = 1$, $f(z) = z$ and and $f(z) = z^2$ in $\mathcal{L}^{2,\lambda}$, we have

$$u \in \mathcal{Z}, \tag{3.4}$$

$$\sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)+\varphi(z)u''(z)| < +\infty, \tag{3.5}$$

and

$$\sup_{z\in D}(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z)+\varphi^2(z)u''(z)+2u(z)(\varphi(z)\varphi''(z)+(\varphi'(z))^2)| < \infty.$$

Since $\varphi(z)$ is a self-map, we get

$$K_1 = \sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| < +\infty \tag{3.6}$$

and

$$K_2 = \sup_{z\in D}(1-|z|^2)|(\varphi'(z))^2u(z)| < +\infty. \tag{3.7}$$

Fix $a \in D$ with $|a| > \frac{1}{2}$, we take the test functions:

$$f_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} - 2\frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\frac{5-\lambda}{2}}} + \frac{(1-|a|^2)^3}{(1-\bar{a}z)^{\frac{7-\lambda}{2}}} \tag{3.8}$$

for $z \in D$. Then, arguing as the proof of Lemma 3.2 in [14] we obtain that $f_a \in \mathcal{L}^{2,\lambda}$ and $\sup_a \|f_a\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$. Since $f_a(a) = 0$, $f_a'(a) = 0$, $f_a''(a) = \dfrac{2\bar{a}}{(1-|a|^2)^{\frac{5-\lambda}{2}}}$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned}
\|f_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim \|uC_\varphi f_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_a)''(z)| \\
&= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_a'(\varphi(z)) \\
&+ f_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f_a(\varphi(z))|.
\end{aligned}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned}
\|f_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)\big)f_{\varphi(\lambda)}'(\varphi(\lambda)) \\
&+ f_{\varphi(\lambda)}''(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda) + u''(\lambda)f_{\varphi(\lambda)}(\varphi(\lambda))| \\
&= (1-|\lambda|^2)|(\varphi'(\lambda))^2u(\lambda)\frac{2\overline{\varphi(\lambda)}}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}}| \\
&\ge \frac{(1-|\lambda|^2)|\varphi'(\lambda))^2u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}}.
\end{aligned}$$

For $\forall\lambda \in D$ with $|\varphi(\lambda)| \le \frac{1}{2}$, by (3.7), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|\varphi'(\lambda))^2u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}} \le (\frac{4}{3})^{\frac{5-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|\varphi'(\lambda))^2u(\lambda)| < +\infty.$$

Hence (3.3) holds.

Next, we will show that (3.2) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take another test functions:

$$g_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} - \frac{12-2\lambda}{7-\lambda}\frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\frac{5-\lambda}{2}}} + \frac{5-\lambda}{7-\lambda}\frac{(1-|a|^2)^3}{(1-\bar{a}z)^{\frac{7-\lambda}{2}}} \tag{3.9}$$

for $z \in D$. Then $g_a \in \mathcal{L}^{2,\lambda}$ and $\sup_a \|g_a\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$(see [14]). Since $g_a(a) = 0$, $g_a''(a) = 0$, $g_a'(a) = \dfrac{-2\bar{a}}{(7-\lambda)(1-|a|^2)^{\frac{3-\lambda}{2}}}$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned}
\|g_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim \|uC_\varphi g_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi g_a)''(z)| \\
&= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)g_a'(\varphi(z)) \\
&+ g_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)g_a(\varphi(z))|.
\end{aligned}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned} \|g_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)g'_{\varphi(\lambda)}(\varphi(\lambda)) \\ &+ g''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda)+u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)\frac{-2\overline{\varphi(\lambda)}}{(7-\lambda)(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}| \\ &\geq \frac{1}{7-\lambda}\frac{(1-|\lambda|^2)|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}. \end{aligned}$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (3.6), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}\leq(\frac{4}{3})^{\frac{3-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)|<+\infty.$$

Hence (3.2) holds.

Finally we will show (3.1) holds. Let

$$h_a(z)=\frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} \tag{3.10}$$

for $z\in D$. It is easily proved that $\sup_{\frac{1}{2}<|a|<1}\|h_a\|_{\mathcal{L}^{2,\lambda}}\lesssim 1$. Then,

$$\begin{aligned} \|h_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim \|uC_\varphi h_a\|_{\mathcal{Z}}\geq(1-|z|^2)|(uC_\varphi h_a)''(z)| \\ &\geq (1-|z|^2)|u''(z)h_a(\varphi(z))|-(1-|z|^2)|(2\varphi'(z)u'(z)+\varphi''(z)u(z))h'_a(\varphi(z))| \\ &- (1-|z|^2)|h''_a(\varphi(z))(\varphi'(z))^2u(z)|. \end{aligned}$$

Therefore, by Lemma 2.1, (3.2) and (3.3), we obtain that

$$\begin{aligned} \sup_{z\in D}(1-|z|^2)|u''(z)h_a(\varphi(z))| &\leq \sup_{z\in D}(1-|z|^2)|(2\varphi'(z)u'(z)+\varphi''(z)u(z))h'_a(\varphi(z))| \\ &+ \sup_{z\in D}(1-|z|^2)|h''_a(\varphi(z))(\varphi'(z))^2u(z)|+C\|h_a\|_{\mathcal{L}^{2,\lambda}} \\ &\lesssim \sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|h_a\|_{\mathcal{L}^{2,\lambda}} \\ &+ \sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|h_a\|_{\mathcal{L}^{2,\lambda}}+\|h_a\|_{\mathcal{L}^{2,\lambda}}<\infty. \end{aligned}$$

Let $a=\varphi(z)$, it follows that

$$\sup_{z\in D}(1-|z|^2)|u''(z)h_a(\varphi(z))|=\sup_{z\in D}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}<\infty.$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (3.4), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|u''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{1-\lambda}{2}}}=(\frac{4}{3})^{\frac{1-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|u''(\lambda)|<\infty.$$

Hence (3.1) holds.

Conversely, suppose that (3.1), (3.2), and (3.2) hold. For $f \in \mathcal{L}^{2,\lambda}$, by Lemma 2.1, we have the following inequality:

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\
&\leq (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))| \\
&\lesssim \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} \\
&+ \frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}+\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} \\
&\lesssim \|f\|_{\mathcal{L}^{2,\lambda}},
\end{aligned}$$

and

$$\begin{aligned}
&|u(0)f(\varphi(0))|+|u'(0)f(\varphi(0))|+|u(0)f'(\varphi(0))\varphi'(0)| \\
&\leq \Big(\frac{|u(0)|+|u'(0)|}{(1-|\varphi(0)|^2)^{\frac{1-\lambda}{2}}}+\frac{|u(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{\frac{3-\lambda}{2}}}\Big)\|f\|_{\mathcal{L}^{2,\lambda}}.
\end{aligned}$$

This shows that uC_φ is bounded. This completes the proof of Theorem 3.1.

Theorem 3.2 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then uC_φ is bounded from the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ if and only if $u \in \mathcal{Z}_0$, (3.1), (3.2) and (3.3) hold, and the following are satisfied:*

$$\lim_{|z|\to 1}(1-|z|^2)|u(z)(\varphi'(z))^2| = 0; \tag{3.11}$$

$$\lim_{|z|\to 1}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| = 0. \tag{3.12}$$

Proof Suppose that uC_φ is bounded from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$. Then $u = uC_\varphi 1 \in \mathcal{Z}_0$. Also $u\varphi = uC_\varphi z \in \mathcal{Z}_0$, thus

$$(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)+\varphi(z)u''(z)| \longrightarrow 0 \quad (|z| \to 1^-).$$

Since $|\varphi| \leq 1$ and $u \in \mathcal{Z}_0$, we have $\lim_{|z|\to}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| = 0$. Hence (3.12) holds.

Similarly, $uC_\varphi z^2 \in \mathcal{Z}_0$, then

$$(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z)+\varphi^2(z)u''(z)+2u(z)(\varphi(z)\varphi''(z)+(\varphi'(z))^2)| \longrightarrow 0 \quad (|z| \to 1^-).$$

By (3.12), $|\varphi| \leq 1$ and $u \in \mathcal{Z}_0$, we get that $\lim_{|z|\to 1}(1-|z|^2)|u(z)(\varphi'(z))^2| = 0$, i. e. that (3.11) holds.

On the other hand, by Lemma 2.3 and Theorem 3.1, we obtain that (3.1), (3.2) and (3.3) hold.

Conversely, let

$$M_1 = \sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty;$$

$$M_2 = \sup_{z\in D} \frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty;$$

$$M_3 = \sup_{z\in D} \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty.$$

For $\forall f \in \mathcal{L}_0^{2,\lambda}$, by Lemma 2.2, given $\epsilon > 0$ there is a $0 < \delta < 1$ such that $(1-|z|^2)^{\frac{3-\lambda}{2}}|f'(z)| < \frac{\epsilon}{3M_1}$, $(1-|z|^2)^{\frac{5-\lambda}{2}}|f''(z)| < \frac{\epsilon}{3M_2}$ and $(1-|z|^2)^{\frac{1-\lambda}{2}}|f(z)| < \frac{\epsilon}{3M_3}$ for all z with $\delta < |z| < 1$.

If $|\varphi(z)| > \delta$, it follows that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)\big|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f(\varphi(z))\big| \\
&\le (1-|z|^2)\big|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f'(\varphi(z))\big| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)| + (1-|z|^2)|u''(z)f(\varphi(z))| \\
&< \frac{(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\frac{\epsilon}{3M_1} \\
&+ \frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\frac{\epsilon}{3M_2} + \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\frac{\epsilon}{3M_3} \\
&< \epsilon,
\end{aligned}$$

We know that there exists a constant K such that $|f(z)| \le K$, $|f'(z)| \le K$ and $|f''(z)| \le K$ for all $|z| \le \delta$.

If $|\varphi(z)| \le \delta$, it follows that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)\big|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f(\varphi(z))\big| \\
&\le K(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| \\
&+ K(1-|z|^2)|(\varphi'(z))^2u(z)| + K(1-|z|^2)|u''(z)|.
\end{aligned}$$

Thus we conclude that $(1-|z|^2)|(uC_\varphi(f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $uC_\varphi f \in \mathcal{Z}_0$ for all $f \in \mathcal{L}_0^{2,\lambda}$. On the other hand, uC_φ is bounded from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$ by Theorem 3.1. Hence uC_φ is a bounded operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$.

Corollary 3.1 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if the following are satisfied:*

$$\sup_{z\in D} \frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty; \tag{3.13}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}<\infty. \tag{3.14}$$

Corollary 3.2 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ if and only if $\varphi \in \mathcal{Z}_0$, (3.13) and (3.14) holds.*

Proof By Theorem 3.2, C_φ is a bounded operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if $\varphi \in \mathcal{Z}_0$, $\lim_{|z|\to 1^-}(1-|z|^2)|(\varphi'(z))^2| = 0$, (3.13) and (3.14) hold. However, by (1.5) in [33], That $\varphi \in \mathcal{Z}_0$ implies that $\lim_{|z|\to 1^-}(1-|z|^2)|(\varphi'(z))^2| = 0$. Then, C_φ is a bounded operator if and only if $\varphi \in \mathcal{Z}_0$, (3.13) and (3.14) hold.

In the formulation of lemma, we use the notation M_u on $H(D)$ defined by $M_u f = uf$ for $f \in H(D)$.

Corollary 3.3 *The pointwise multiplier $M_u : \mathcal{L}^{2,\lambda} \to \mathcal{Z}$ is a bounded operator if and only if $u = 0$.*

4 Compactness of uC_φ

In order to prove the compactness of uC_φ, we require the following lemmas.

Lemma 4.1 *Suppose that uC_φ be a bounded operator from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$, then uC_φ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of D. We have $\|uC_\varphi(f_n)\|_{\mathcal{Z}} \to 0$, as $n \to \infty$.*

The proof is similar to that of Proposition 3.11 in [5] . The details are omitted.

Lemma 4.2 *Let $U \subset \mathcal{Z}_0$. Then U is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in U}(1-|z|^2)|f''(z)| = 0.$$

The proof is similar to that of Lemma 1 in [16], we omit it.

Theorem 4.1 *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Suppose that uC_φ is a bounded operator from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$. Then uC_φ is compact if and only if the following are satisfied:*

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0; \tag{4.1}$$

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0; \tag{4.2}$$

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} = 0. \tag{4.3}$$

Proof Suppose that uC_φ is compact from $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist, then (4.1), (4.2) and (4.3) are automatically satisfied. Without loss of generality we may suppose that $|\varphi(z_n)| > \frac{1}{2}$ for all n. We take the test functions

$$f_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}} - \frac{2(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{5-\lambda}{2}}} + \frac{(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^{\frac{7-\lambda}{2}}}. \tag{4.4}$$

By the proof of Theorem 3.1 we know that that $\sup_n \|f_n\|_{\mathcal{L}^{2,\lambda}} \le C < \infty$. Then $\{f_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 4.1. Note that $f_n(\varphi(z_n)) \equiv 0$, $f_n'(\varphi(z_n)) \equiv 0$ and $f_n''(\varphi(z_n)) = \dfrac{2\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}$.

It follows that

$$\|uC_\varphi f_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))f_n'(\varphi(z_n))$$

$$+u(z_n)f_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)f_n(\varphi(z_n))|$$

$$= 2(1-|z_n|^2)|(\varphi'(z_n))^2 u(z_n)\frac{\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}|$$

$$\ge \frac{(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}.$$

Then $\lim\limits_{n\to\infty} \dfrac{(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}} = 0$. Thus (4.3) holds.

Next, let

$$g_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}} - \frac{12-2\lambda}{7-\lambda}\frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{5-\lambda}{2}}} + \frac{5-\lambda}{7-\lambda}\frac{(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^{\frac{7-\lambda}{2}}}. \tag{4.5}$$

We similarly obtain that $\{g_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(g_n)\|_{\mathcal{Z}} = 0$ by Lemma 4.1. Note that $g_n(\varphi(z_n)) \equiv 0$, $g_n''(\varphi(z_n)) \equiv 0$ and $g_n'(\varphi(z_n)) = \dfrac{-2(\overline{\varphi(z_n)})^2}{(7-\lambda)(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}$. It follows that

$$\|uC_\varphi g_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))g_n'(\varphi(z_n))$$

$$+u(z_n)g_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)g_n(\varphi(z_n))|$$

$$= (1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|\frac{2|\overline{\varphi(z_n)}|^2}{(7-\lambda)(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}$$

$$\ge \frac{(1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|}{(7-\lambda)(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}.$$

Then $\lim\limits_{n\to\infty} \dfrac{(1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}} = 0$. Thus (4.2) holds.

Finally, let

$$h_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}}. \tag{4.6}$$

We know that $\{h_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty}\|uC_\varphi(h_n)\|_{\mathcal{Z}} = 0$ by Lemma 4.1. Note that $h_n(\varphi(z_n)) = \dfrac{1}{(1-|\varphi(z_n)|^2)^{\frac{1-\lambda}{2}}}$, $h_n'(\varphi(z_n)) = \dfrac{3-\lambda}{2(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}$ and $h_n''(\varphi(z_n)) = \dfrac{(3-\lambda)(5-\lambda)}{4(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}$.

It follows that

$$\begin{aligned}\|uC_\varphi h_n\|_{\mathcal{Z}} &\geq (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n))h_n'(\varphi(z_n)) \\ &\quad + u(z_n)h_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)h_n(\varphi(z_n))|.\end{aligned}$$

Then,

$$\begin{aligned}\frac{(1-|z_n|^2)|u''(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{1-\lambda}{2}}} &\leq \|uC_\varphi h_n\|_{\mathcal{Z}} + (1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|\frac{(3-\lambda)(5-\lambda)}{4(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}} \\ &\quad + (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n))|\frac{3-\lambda}{2(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}},\end{aligned}$$

hence (4.1) holds by (4.2) and (4.3). The proof of the necessary is completed.

Conversely, suppose that (4.1), (4.2), and (4.3) hold. Since uC_φ is a bounded operator, by Theorem 3.1, we have

$$M_1 = \sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty;$$

$$M_2 = \sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty;$$

$$M_3 = \sup_{z\in D}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty.$$

Let $\{f_n\}$ be a bounded sequence in $\mathcal{L}^{2,\lambda}$ with $\|f_n\|_{\mathcal{L}^{2,\lambda}} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of D. We only prove $\lim\limits_{n\to\infty}\|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 4.1. By the assumption, for any $\epsilon > 0$, there is a constant δ, $0<\delta<1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \epsilon, \qquad \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \epsilon,$$

and

$$\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \epsilon.$$

Let $K = \{w \in D : |w| \leq \delta\}$. Noting that K is a compact subset of D, we get that

$$\begin{aligned}
z(uC_\varphi f_n) &= \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_n)''(z)| \\
&\le \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{z\in D}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{z\in D}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\lesssim 3\epsilon + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 3\epsilon + M_1\sup_{w\in K}|f_n'(w)| + M_2\sup_{w\in K}|f_n''(w)| + M_3\sup_{w\in K}|f_n(w)|.
\end{aligned}$$

As $n\to\infty$,

$$\|uC_\varphi f_n\|_{\mathcal{Z}} \to 0.$$

Hence uC_φ is compact. This completes the proof of Theorem 4.1.

Theorem 4.2 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then uC_φ is compact from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if the following are satisfied:*

$$\lim_{|z|\to1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}=0; \tag{4.7}$$

$$\lim_{|z|\to1}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}=0; \tag{4.8}$$

$$\lim_{|z|\to1}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}=0. \tag{4.9}$$

Proof Assume (4.7), (4.8), and (4.9) hold. From Theorem 4.2, we know that uC_φ is bounded from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$. Suppose that $f\in\mathcal{L}_0^{2,\lambda}$ with $\|f\|_{\mathcal{L}^{2,\lambda}}\le1$. We obtain that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\
&\le (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)| + (1-|z|^2)|u''(z)f(\varphi(z))| \\
&\lesssim \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} \\
&+ \frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} + \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}},
\end{aligned}$$

thus

$$\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \le 1\}$$

$$\lesssim \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}$$

$$+\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} + \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}},$$

and it follows that

$$\lim_{|z|\to 1} \sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \le 1\} = 0,$$

hence $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is compact by Lemma 4.2.

Conversely, suppose that $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is compact.

First, it is obvious that $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is bounded, then by Theorem 3.2, we have $u \in \mathcal{Z}_0$ and that (3.11) and (3.12) hold. On the other hand, by Lemma 4.2 we have

$$\lim_{|z|\to 1} \sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \le M\} = 0,$$

for some $M > 0$.

Next, noting that the proof of Theorem 3.1 and the fact that the functions given in (3.8) are in $\mathcal{L}_0^{2,\lambda}$ and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} = 0$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by (3.11), we easily have

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}$$

$$\le (\frac{4}{3})^{\frac{5-\lambda}{2}} \lim_{|z|\to 1} (1-|z|^2)|u(z)(\varphi'(z))^2| = 0.$$

Thus (4.9) holds. Also, the second statement, that (4.8), is proved similarly. We omitted it here.

Similarly, noting that the functions given in (3.10) are in $\mathcal{L}_0^{2,\lambda}$ and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} \lesssim \lim_{|z|\to 1} \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}$$

$$+ \lim_{|z|\to 1} \frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} + \lim_{|z|\to 1} (1-|z|^2)|(uC_\varphi h_a)''(z)|,$$

for $|\varphi(z)| > \frac{1}{2}$. So by (4.8) and (4.9), it follows that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by $u \in \mathcal{Z}_0$, we easily have

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = \lim_{|z|\to 1} (\frac{4}{3})^{\frac{1-\lambda}{2}} (1-|z|^2)|u''(z)| = 0.$$

This completes the proof of Theorem 4.2.

Corollary 4.1 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from the analytic Morrey space $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if C_φ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0.$$

Corollary 4.2 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} = 0$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0.$$

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