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SOME HERMITE–HADAMARD AND SIMPSON TYPE INEQUALITIES FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS WITH APPLICATIONS

MUHAMMAD IQBAL, MUSTAFA HABIB, NASIR SIDDIQUI,
AND MUHAMMAD MUDDASSAR

Abstract. In this paper, a new general identity for Riemann−Liouville fractional integrals is established. Then by making use of the established identity, we establish some new inequalities of the Simpson and the Hermite−Hadamard type for functions whose absolute values of derivatives are convex. Our results have some relationships with the results, proved in [3, 6, 10], and the analysis used in the proofs is simple.

1. Introduction

Definition 1. *Let $I \subset \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ is said to be convex on I, if for all $a, b \in I$ with $a \leq b$ and $\lambda \in [0, 1]$, satisfies the inequality*

$$f(\lambda a + (1-\lambda) b) \leq \lambda f(a) + (1-\lambda) f(b)$$

The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature (see, e.g.,[[12], p. 137],).These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction for f to be concave.

Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example, ([3],[5]−[6], [9]−[10], [12]) and the references cited therein.

In [10], a variant of Hermite−Hadamard type inequalities was obtained, which follows as:

Theorem 1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \tag{2}$$

In [6] authors proved the following version of Hermite−Hadamard type inequalities:

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Theorem 2. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)| + |f'(b)|}{2} \right) \tag{3}$$

The Simpson's inequality is very important and well known in the literature. For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([7],[13], [16], [17]).

In [16], Sarikaya et al. obtained inequality for differentiable convex mappings which is connected with Simpson's inequality, is as follow:

Theorem 3. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left| \frac{1}{3} \left\{ 2f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right\} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{5(b-a)}{36} \left(\frac{|f'(a)| + |f'(b)|}{2} \right) \tag{4}$$

In [3], the authors generalize some inequalities related to Hermite−Hadamard and Simpsons inequality for functions whose derivatives in absolute value are convex functions as:

Theorem 4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^\circ$ with $a < b$. If $0 \leq \lambda \leq 1$ and $|f'|^{\frac{1}{q}}$ for $q \geq 1$ is a convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned} &\left| (1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ &\leq \frac{b-a}{8} \left(\frac{1}{6}\right)^{\frac{1}{q}} (1 - 2\lambda + 2\lambda^2)^{1-\frac{1}{q}} \\ &\times \Big[\left(\left\{2 - 3\lambda + 2\lambda^3\right\} |f'(a)|^q + \left\{4 - 9\lambda + 12\lambda^2 - 2\lambda^3\right\} |f'(b)|^q \right)^{\frac{1}{q}} \\ &\quad + \left(\left\{4 - 9\lambda + 12\lambda^2 - 2\lambda^3\right\} |f'(a)|^q + \left\{2 - 3\lambda + 2\lambda^3\right\} |f'(b)|^q \right)^{\frac{1}{q}} \Big] \end{aligned} \tag{5}$$

Remark 1. *On letting $\lambda =, 0, 1, \frac{1}{3}$ with $q = 1$, inequality* (5) *reduces to inequalities* (2)*,* (3) *and* (4)*, respectively..*

It is well known that the integral inequalities play an important role in nonlinear analysis. In the recent years, these inequalities have been improved and generalized in a number of ways and a large number of research papers have been written on these inequalities, (see, [1]−[2], [4], [10], [15]) and the references therein.

In recent paper, [10] Sarikaya et. al. proved a variant of Hermite−Hadamard's inequalities in fractional integral forms as follows:

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2} \tag{6}$$

Remark 2. *For $\alpha = 1$, inequality* (6) *reduces to inequality* (1)*.*

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

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Definition 2. *Let $f \in L[a,b]$, the Reimann−Liouville integrals $J_{a^+}^{\alpha}$ and $J_{b^-}^{\alpha}$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > \alpha$$

and

$$J_{b^-}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < \alpha$$

respectively. Here, $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found in ([8] , [11] , [14]).

The aim of this paper is to establish some new Hermite−Hadamard and Simpson type inequalities in the form of fractional integrals for functions whose absolute values of derivatives are convex. we derive a general integral identity via Riemann−Liouville fractional integrals.

2. Main Results

In order to prove our main results we need the following integral identity:

Lemma 1. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a,b) with $a < b$,then the following identity for Riemann−Liouville fractional integrals holds:*

$$\left(1 - \frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] = \frac{b-a}{2^{\alpha+2}} \sum_{n=1}^{4} I_n$$

where

$$\begin{aligned}
I_1 &= \textstyle\int_0^1 \left[(1-t)^\alpha - \lambda\right] f'\left(ta + (1-t)\frac{a+b}{2}\right) dt, \\
I_2 &= \textstyle\int_0^1 \left[\lambda - (1-t)^\alpha\right] f'\left(tb + (1-t)\frac{a+b}{2}\right) dt, \\
I_3 &= \textstyle\int_0^1 \left[2^\alpha - \lambda - (2-t)^\alpha\right] f'\left(t\frac{a+b}{2} + (1-t)a\right) dt, \\
I_4 &= \textstyle\int_0^1 \left[\lambda - 2^\alpha + (2-t)^\alpha\right] f'\left(t\frac{a+b}{2} + (1-t)b\right) dt.
\end{aligned}$$

Proof. Integrating by parts and substituting $u = ta + (1-t)\frac{a+b}{2}$

$$\begin{aligned}
I_1 &= \int_0^1 \left[(1-t)^\alpha - \lambda\right] f'\left(ta + (1-t)\frac{a+b}{2}\right) dt \\
&= \left.\frac{2\left[(1-t)^\alpha - \lambda\right] f'\left(ta + (1-t)\frac{a+b}{2}\right) dt}{a-b}\right|_0^1 \\
&\qquad + \frac{2\alpha}{a-b} \int_0^1 (1-t)^\alpha f'\left(ta + (1-t)\frac{a+b}{2}\right) dt \\
&= \frac{2\lambda}{b-a} f(a) + \frac{2(1-\lambda)}{b-a} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}} \int_a^{\frac{a+b}{2}} (u-a)^{\alpha-1} f(u)du
\end{aligned}$$

Analogously

$$I_2 = \frac{2\lambda}{b-a}f(b) + \frac{2(1-\lambda)}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{a+b}{2}}^{b}(b-u)^{\alpha-1}f(u)du$$

$$I_3 = \frac{2\lambda}{b-a}f(a) + \frac{2(2^\alpha-1-\lambda)}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{a}^{\frac{a+b}{2}}(b-u)^{\alpha-1}f(u)du$$

$$I_4 = \frac{2\lambda}{b-a}f(b) + \frac{2(2^\alpha-1-\lambda)}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{a+b}{2}}^{b}(b-u)^{\alpha-1}f(u)du$$

Multiplying above equalities by $\frac{b-a}{2^{\alpha+2}}$, then adding, to get required identity. ∎

Theorem 6. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \le 1$ on (a,b) with $a < b$ and $0 \le \lambda \le 1$. If $|f'|$ is a convex on $[a,b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$\left|\left(1-\frac{2}{2^\alpha}\lambda\right)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right|$$
$$\le \frac{b-a}{2^{\alpha+2}}\left[\frac{2}{\alpha+1}\left\{1+2^\alpha-(1-\zeta)^{\alpha+1}-(2-\xi)^{\alpha+1}\right\}\right.$$
$$\left. +(1-2\zeta)\lambda + (2^\alpha-\lambda)(1-2\xi)\right](|f'(a)|+|f'(b)|) \quad (7)$$

where $\zeta = 1-\lambda^{\frac{1}{\alpha}}$, and $\xi = 2-(2^\alpha-\lambda)^{\frac{1}{\alpha}}$.

Proof. Let $\zeta = 1-\lambda^{\frac{1}{\alpha}}$, and $\xi = 2-(2^\alpha-\lambda)^{\frac{1}{\alpha}}$ then

$$A = \int_0^1 |(1-t)^\alpha - \lambda|\,dt = \frac{1-2(1-\zeta)^{\alpha+1}}{\alpha+1} + (1-2\zeta)\lambda$$

$$B = \int_0^1 |2^\alpha-(2-t)^\alpha-\lambda|\ dt = \frac{1+2^{\alpha+1}-2(2-\xi)^{\alpha+1}}{\alpha+1} + (2^\alpha-\lambda)(1-2\xi)$$

$$C = \int_0^1 |(1-t)^\alpha-\lambda|\,t\ dt = \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1} + \frac{\lambda}{2}(1-2\zeta^2)$$

$$D = \int_0^1 |2^\alpha-(2-t)^\alpha-\lambda|\,t\ dt$$
$$= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1} + (2^\alpha-\lambda)(\frac{1}{2}-\xi^2)$$

By using the properties of modulus on Lemma 1 and convexity of $|f'|$, we have

$$\left|\left(1-\frac{2}{2^\alpha}\lambda\right)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right|$$
$$\le \frac{b-a}{2^{\alpha+2}}\sum_{n=1}^{4}|I_n| \quad (8)$$

$$\begin{aligned}
|I_1| &\leq \int_0^1 |(1-t)^\alpha - \lambda| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right)\right| dt \\
&= \int_0^1 |(1-t)^\alpha - \lambda| \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| dt \\
&\leq \int_0^1 |(1-t)^\alpha - \lambda| \left\{ \frac{1+t}{2}|f'(a)| + \frac{1-t}{2}|f'(b)| \right\} dt \\
&= \frac{|f'(a)| + |f'(b)|}{2} \int_0^1 |(1-t)^\alpha - \lambda| \ dt + \frac{|f'(a)| - |f'(b)|}{2} \int_0^1 |(1-t)^\alpha - \lambda| t \ dt \\
&= \frac{A}{2}\{|f'(a)| + |f'(b)|\} + \frac{C}{2}\{|f'(a)| - |f'(b)|\}
\end{aligned}$$

Analogously

$$\begin{aligned}
|I_2| &\leq \frac{A}{2}\{|f'(a)| + |f'(b)|\} - \frac{C}{2}\{|f'(a)| - |f'(b)|\} \\
|I_3| &\leq B|f'(a)| - \frac{D}{2}\{|f'(a)| - |f'(b)|\} \\
|I_4| &\leq B|f'(b)| + \frac{D}{2}\{|f'(a)| - |f'(b)|\}
\end{aligned}$$

To get desired result, substituting the above inequalities into the inequality (8). ◻

Corollary 1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[2\lambda^2 - 2\lambda + 1\right](|f'(a)| + |f'(b)|) \quad (9)$$

Proof. Setting $\alpha = 1$ in Theorem 5, we get the required result. ◻

Remark 3. *For setting $\lambda = 0$, inequality (9) reduces to inequality (2).*
For setting $\lambda = 1$, inequality (9) reduces to inequality (3).

Theorem 7. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a, b) with $a < b$ and $0 \leq \lambda \leq 1$. If $|f'|$ is a convex on $[a, b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$\left|\left(1 - \frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+1}}\left[\mu\left|f'\left(\frac{a+b}{2}\right)\right| + \nu\left(\frac{|f'(a)| + |f'(b)|}{2}\right)\right]$$

where

$$\begin{aligned}
\mu = &\tfrac{2^{\alpha+2}+2(1-\zeta)^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \tfrac{2-2(1-\zeta)^{\alpha+2}-2\xi(2-\xi)^{\alpha+1}}{\alpha+1} \\
&+(2^\alpha - \lambda)(\tfrac{1}{2} - \xi^2) - \tfrac{\lambda}{2}(1 - 2\zeta^2) + (1 - 2\zeta)\lambda \\
\nu = &\tfrac{2(2-\xi)^{\alpha+2}-2(1-\zeta)^{\alpha+2}-2^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \tfrac{2^{\alpha+1}-2\zeta(1-\zeta)^{\alpha+1}-2(1-\xi)(2-\xi)^{\alpha+1}}{\alpha+1} \\
&-(2^\alpha - \lambda)(\tfrac{1}{2} - \xi^2) + \tfrac{\lambda}{2}(1 - 2\zeta^2) + (2^\alpha - \lambda)(1 - 2\xi).
\end{aligned}$$

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Proof. By using the properties of modulus on I_1 and convexity of $|f'|$, we have

$$\begin{aligned}
|I_1| &\leq \int_0^1 |(1-t)^\alpha - \lambda| \left| f'\left(ta + (1-t)\frac{a+b}{2} \right) \right| dt \\
&\leq \int_0^1 |(1-t)^\alpha - \lambda| \left\{ t\,|f'(a)| + (1-t)\left| f'\left(\frac{a+b}{2}\right)\right| \right\} dt \\
&\leq (A-C)\left| f'\left(\frac{a+b}{2}\right)\right| + C|f'(a)|
\end{aligned}$$

Analogously

$$\begin{aligned}
|I_2| &\leq (A-C)\left| f'\left(\frac{a+b}{2}\right)\right| + C|f'(b)| \\
|I_3| &\leq D\left| f'\left(\frac{a+b}{2}\right)\right| + (B-D)|f'(a)| \\
|I_4| &\leq D\left| f'\left(\frac{a+b}{2}\right)\right| + (B-D)|f'(b)|
\end{aligned}$$

Substituting the above inequalities into the following inequality

$$\left| \left(1 - \frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \right| \leq \frac{b-a}{2^{\alpha+2}} \sum_{n=1}^{4} |I_n|$$

which completes the proof. ∎

Corollary 2. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left| (1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{b-a}{12}\left[(2-3\lambda+2\lambda^3)\left| f'\left(\frac{a+b}{2}\right)\right| + (1-3\lambda+6\lambda^2-2\lambda^3)\left(\frac{|f'(a)|+|f'(b)|}{2}\right)\right] \tag{10}$$

Proof. Setting $\alpha = 1$ in Theorem 6, we get the required result. ∎

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 8. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a, b) with $a < b$ and $0 \leq \lambda \leq 1$. If $|f'|^{\frac{1}{q}}$ for $q \geq 1$ is a convex on $[a, b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$\left| \left(1 - \frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \right| \leq \frac{b-a}{2^{\alpha+2}} \times$$
$$\left[A\left\{ \left(\frac{A-C}{2A}|f'(a)|^q + \frac{A+C}{2A}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\frac{A+C}{2A}|f'(a)|^q + \frac{A-C}{2A}|f'(b)|^q\right)^{\frac{1}{q}} \right\} \right. \tag{11}$$
$$\left. + B\left\{ \left(\frac{2B-D}{2B}|f'(a)|^q + \frac{D}{2B}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\frac{D}{2B}|f'(a)|^q + \frac{2B-D}{2B}|f'(b)|^q\right)^{\frac{1}{q}} \right\} \right].$$

where

$$\begin{aligned} A &= \frac{1-2(1-\zeta)^{\alpha+1}}{\alpha+1}+(1-2\zeta)\lambda \\ B &= \frac{1+2^{\alpha+1}-2(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(1-2\xi) \\ C &= \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)}-\frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1}+\frac{\lambda}{2}(1-2\zeta^2) \\ D &= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(\frac{1}{2}-\xi^2) \end{aligned}$$

Proof. Using the well-known power-mean integral inequality , we have

$$\begin{aligned} &\left|\left(1-\frac{2}{2^{\alpha}}\lambda\right)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \\ &\leq \frac{b-a}{2^{\alpha+2}}\left[\left(\int_0^1 |(1-t)^{\alpha}-\lambda|\,dt\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1 |(1-t)^{\alpha}-\lambda|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\right. \\ &\qquad \left.+\left(\int_0^1 |(1-t)^{\alpha}-\lambda|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}+ \\ &\left(\int_0^1 |2^{\alpha}-(2-t)^{\alpha}-\lambda|\;dt\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1 |2^{\alpha}-(2-t)^{\alpha}-\lambda|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\qquad \left.\left.+\left(\int_0^1 |2^{\alpha}-(2-t)^{\alpha}-\lambda|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\right]. \end{aligned}$$

Which completes the proof. □

Remark 4. *For $\alpha=1$, inequality (11) reduces to inequality (5).*

Theorem 9. *Let $I\subset\mathbb{R}$ be an open interval, $a,b\in I$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function such that f' is integrable and $0<\alpha\leq 1$ on (a,b) with $a<b$ and $0\leq\lambda\leq 1$. If $|f'|^{\frac{1}{q}}$ for $q\geq 1$ is a convex on $[a,b]$, then the following inequality for Riemann–Liouville fractional integrals holds:*

$$\begin{aligned} &\left|\left(1-\frac{2}{2^{\alpha}}\lambda\right)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\leq\frac{b-a}{2^{\alpha+1}}\times \\ &\left[A\left\{\left(\frac{C}{A}|f'(a)|^q+\left(1-\frac{C}{A}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{C}{A}|f'(b)|^q+\left(1-\frac{C}{A}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}\right. \\ &\left.+B\left\{\left(\left(1-\frac{D}{B}\right)|f'(a)|^q+\frac{D}{B}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left(1-\frac{D}{B}\right)|f'(b)|^q+\frac{D}{B}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}\right]. \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1-2(1-\zeta)^{\alpha+1}}{\alpha+1}+(1-2\zeta)\lambda \\ B &= \frac{1+2^{\alpha+1}-2(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(1-2\xi) \\ C &= \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)}-\frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1}+\frac{\lambda}{2}(1-2\zeta^2) \\ D &= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(\frac{1}{2}-\xi^2) \end{aligned}$$

Corollary 3. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^{\circ}$ with $a < b$. If $0 \leq \lambda \leq 1$ and $|f'|^{\frac{1}{q}}$ for $q \geq 1$ is a convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned} &\left|(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\quad \leq \frac{b-a}{3}\left(\frac{1}{3}\right)^{\frac{1}{q}}(1-2\lambda+2\lambda^2)^{1-\frac{1}{q}} \\ &\quad \times\left[\left(\left\{1-3\lambda+6\lambda^2-2\lambda^3\right\}|f'(a)|^q+\left\{2-3\lambda+2\lambda^3\right\}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\left\{1-3\lambda+6\lambda^2-2\lambda^3\right\}|f'(b)|^q+\left\{2-3\lambda+2\lambda^3\right\}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \end{aligned} \tag{12}$$

Proof. Setting $\alpha = 1$ in Theorem 9, we get the required result. □

3. Applications To Quadrature Formulae

In this section, some particular inequalities which generalize some classical results such as: trapezoid inequality, Simpsons inequality, midpoint inequality and others, are pointed out.

Proposition 1. *(Midpoint Inequality). Under the assumptions Corollary 2 with $\lambda = 0$ in inequality (10), then the following inequality holds,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{24}\left(|f'(a)|+4\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right)$$

Proposition 2. *(Midpoint Inequality). Under the assumptions Corollary 3 with $\lambda = 0$ in inequality (12), then the following inequality holds,*

$$\begin{aligned} \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}&\left[\left(\frac{1}{3}|f'(a)|^q+\frac{2}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{1}{3}|f'(b)|^q+\frac{2}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \end{aligned}$$

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Proposition 3. *(Trapezoid Inequality). Under the assumptions Corollary* 2 *with* $\lambda = 1$ *in inequality* (10)*, then the following inequality holds,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left(|f'(a)| + \left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right)$$

Proposition 4. *(Trapezoid Inequality). Under the assumptions Corollary* 3 *with* $\lambda = 1$ *in inequality* (12)*, then the following inequality holds,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left(\frac{2}{3}|f'(a)|^q + \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{2}{3}|f'(b)|^q + \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

Proposition 5. *Under the assumptions Corollary* 2 *with* $\lambda = \frac{1}{2}$ *in inequality* (10)*, then the following inequality holds,*

$$\left|\frac{1}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{32}\left(|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right)$$

Proposition 6. *Under the assumptions Corollary* 3 *with* $\lambda = \frac{1}{2}$ *in inequality* (12)*, then the following inequality holds,*

$$\left|\frac{1}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{16}\times \left[\left(\frac{1}{2}|f'(a)|^q + \frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{1}{2}|f'(b)|^q + \frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

Proposition 7. *(Simpson Inequality). Under the assumptions Corollary* 2 *with* $\lambda = \frac{1}{3}$ *in inequality* (10)*, then the following inequality holds,*

$$\left|\frac{1}{3}\left\{2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{324}\left(8|f'(a)| + 29\left|f'\left(\frac{a+b}{2}\right)\right| + 8|f'(b)|\right)$$

Proposition 8. *(Simpson Inequality). Under the assumptions Corollary* 3 *with* $\lambda = \frac{1}{3}$ *in inequality* (12)*, then the following inequality holds,*

$$\left|\frac{1}{3}\left\{2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{72}\times \left[\left(\frac{16}{45}|f'(a)|^q + \frac{29}{45}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{16}{45}|f'(b)|^q + \frac{29}{45}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

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The differentiability for fuzzy n-cell mappings and the KKT optimality conditions for a class of fuzzy constrained minimization problem *

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Abstract In this paper, the concept of generalized difference for fuzzy n-cell numbers is presented, and we use the generalized difference to introduce and study the differentiability for fuzzy n-cell mappings. Next, the convexity for fuzzy n-cell mappings is studied, which is based on the concept of the partial ordering of fuzzy n-cell numbers proposed in this study. Finally, using the differentiability and the convexity for fuzzy n-cell mappings, we obtain the KKT optimality conditions for a class of fuzzy constrained minimization problem.

Keywords: Fuzzy n-cell numbers; fuzzy n-cell mappings; differentiability; fuzzy optimization.

1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh [1], enormous researchers have been dedicated on development of various aspects of the theory and applications of fuzzy sets. The occurrence of randomness and imprecision in the real world is inevitable owing to some unexpected situations. Therefore, imposing the uncertainty upon the conventional optimization problems is an interesting research topic.

The theory and methods of mathematical programming are important components of optimization. The importance of the derivative of a function in the study of mathematical programming is well-known. Given that our interest is in fuzzy objective mappings, it is necessary to introduce a concept of derivative for fuzzy mappings. Toward this end, in the fuzzy analysis, there are a variety of notions of derivative for fuzzy mappings. The concept of fuzzy derivative first introduced by Chang and Zadeh [2] in 1972. Since then, numerous definitions of the differentiability for fuzzy mappings have been presented. In 1983, Puri and Ralescu [3] defined the derivative and G-derivative for fuzzy mappings from an open subset of a normal space into n-dimension fuzzy number space E^n by using embedding theorem (which shows how to isometrically embed E^n into a Banach space as a closed convex cone of vertex zero) and Hukuhara difference. In 1987, Kaleva [4] discussed the G-derivative, obtained a sufficient condition of the H-differentiability for fuzzy mappings from $[a, b]$ into E^n and a necessary condition for the H-differentiability of fuzzy mapping from $[a, b]$ into E^1. In 2003, Wang and Wu [5] put forward the concepts of directional derivative, differential and sub-differential for fuzzy mappings from R^n into E^1 by using Hukuhara difference. However, the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [4] and the H-difference of two fuzzy numbers does not always exist [6, 7]. The g-difference between two fuzzy numbers proposed in [7] overcomes these shortcomings of the above discussed concepts and the g-difference of two fuzzy numbers always exists. Based on the g-difference for two fuzzy numbers, Bede [8] introduced and studied new generalized differentiability concepts for fuzzy valued functions in 2013, in particular, a new very general fuzzy differentiability concept was defined and studied, the so-called g-derivative, and it was

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shown that the g-derivative is the most general among all similar definitions.

Motivated both by [8] and the importance of the concept of differentiability for fuzzy optimization, the paper focuses on the concept of differentiability for fuzzy mappings, which is based on the generalized difference of fuzzy n-cell numbers presented in this paper. The Karush-Kuhn-Tucker optimality conditions play an important role in the area of optimization theory and have been studied for over a century. We extend the concept of convexity for real-valued functions to fuzzy n-cell mappings based on the partial ordering $\preceq_c$ introduced in this paper, and then establish the KKT optimality conditions for an optimization problem with a fuzzy n-cell objective mapping.

The remainder of the paper is organised as follows: First of all, we give the preliminary terminology used in the present paper. And then, the generalized difference of fuzzy n-cell numbers is introduced. We use the generalized difference of fuzzy n-cell numbers to study differentiability for fuzzy n-cell mappings, and convexity for fuzzy n-cell mappings based on the partial ordering $\preceq_c$ is discussed in Section 4. At last, using the convexity and differentiability for fuzzy n-cell mappings, section 5 deals with the Karush-Kuhn-Tucker optimality conditions for a class of constrained fuzzy minimization problem.

2. Preliminaries

Throughout this paper, $F(R^n)$ denotes the set of all fuzzy subsets on n-dimensional Euclidean space R^n. A fuzzy set $\widetilde{u}$ on R^n is a mapping $\widetilde{u}: R^n \to [0,1]$. For each fuzzy set $\widetilde{u}$, we denote its r-level set as $[\widetilde{u}]^r = \{x \in R^n : \widetilde{u}(x) \geq r\}$ for any $r \in (0,1]$. The support of $\widetilde{u}$ we denote by supp$\widetilde{u}$ where supp$\widetilde{u} = \{x \in R^n : \widetilde{u}(x) > 0\}$. The closure of supp$\widetilde{u}$ defines the 0-level of $\widetilde{u}$, i.e. $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u})$. Here $cl(M)$ denotes the closure of set M. Fuzzy set $\widetilde{u} \in F(R^n)$ is called a fuzzy number if

(1) $\widetilde{u}$ is a normal fuzzy set, i.e. there exists an $x_0 \in R^n$ such that $\widetilde{u}(x_0) = 1$,

(2) $\widetilde{u}$ is a convex fuzzy set, i.e. $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$,

(3) $\widetilde{u}$ is upper semi-continuous,

(4) $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u}) = cl(\bigcup_{r\in(0,1]}[\widetilde{u}]^r)$ is compact.

We will denote E^n the set of fuzzy numbers [9, 10, 11, 12].

It is clear that any $u \in R^n$ can be regarded as a fuzzy number $\widetilde{u}$ defined by

$$\widetilde{u}(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the fuzzy number $\widetilde{0}$ is defined as $\widetilde{0}(x) = 1$ if $x = 0$, and $\widetilde{0}(x) = 0$ otherwise.

Definition 2.1. [13] If $\widetilde{u} \in E^n$, and $[\widetilde{u}]^r$ is a cell, i.e., for any $r \in [0,1]$,

$$[\widetilde{u}]^r = \prod_{i=1}^{n}[u_i^-(r), u_i^+(r)] = [u_1^-(r), u_1^+(r)] \times [u_2^-(r), u_2^+(r)] \times \cdots \times [u_n^-(r), u_n^+(r)],$$

where $u_i^-(r), u_i^+(r) \in R$ with $u_i^-(r) \leq u_i^+(r)$ $(i = 1, 2, \cdots, n)$, then we call $\widetilde{u}$ a fuzzy n-cell number. Denote the collection of all fuzzy n-cell numbers by $L(E^n)$.

For any $r \in [0,1]$, $l_i[\widetilde{u}]^r = u_i^+(r) - u_i^-(r)$ $(i = 1, 2, \cdots, n)$ is called the r-level length of a fuzzy n-cell number $\widetilde{u}$ with respect to the ith component.

Theorem 2.1. [13] (Representation theorem). If $\widetilde{u} \in L(E^n)$, then for $i = 1, 2, \cdots, n$, $u_i^-(r), u_i^+(r)$ are real-valued functions on $[0,1]$, and satisfy

(1) $u_i^-(r)$ are non-decreasing, left continuous at $r \in (0,1]$ and right continuous at $r = 0$,

(2) $u_i^+(r)$ are non-increasing, left continuous at $r \in (0,1]$ and right continuous at $r = 0$,

(3) $u_i^-(r) \leq u_i^+(r)$ (it is equivalent to $u_i^-(1) \leq u_i^+(1)$).

Conversely if $a_i(r), b_i(r)$ $(i = 1, 2, \cdots, n)$ are real-valued functions on $[0,1]$ which satisfy conditions (1)-(3), then there exists a unique $\widetilde{u} \in L(E^n)$ such that $[\widetilde{u}]^r = \prod_{i=1}^n [a_i(r), b_i(r)]$ for any $r \in [0,1]$.

Theorem 2.2. [13] Let $\widetilde{u}, \widetilde{v} \in L(E^n)$ and $k \in R$. Then for any $r \in [0,1]$,

(1) $[\widetilde{u} + \widetilde{v}]^r = [\widetilde{u}]^r + [\widetilde{v}]^r = \prod_{i=1}^n [u_i^-(r) + v_i^-(r), u_i^+(r) + v_i^+(r)]$,

(2) $[k\widetilde{u}]^r = k[\widetilde{u}]^r = \begin{cases} \prod_{i=1}^n [ku_i^-(r), ku_i^+(r)], & k \geq 0, \\ \prod_{i=1}^n [ku_i^+(r), ku_i^-(r)], & k < 0, \end{cases}$

(3) $[\widetilde{u}\widetilde{v}]^r = \prod_{i=1}^n [\min\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\},$

$\max\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\}].$

Given $\widetilde{u}, \widetilde{v} \in L(E^n)$, the distance $D : L(E^n) \times L(E^n) \to [0, +\infty)$ between $\widetilde{u}$ and $\widetilde{v}$ is defined by the equation

$$D(\widetilde{u}, \widetilde{v}) = \sup_{r \in [0,1]} d([\widetilde{u}]^r, [\widetilde{v}]^r) = \sup_{r \in [0,1]} \max_{1 \leq i \leq n} \{| u_i^-(r) - v_i^-(r) |, | u_i^+(r) - v_i^+(r) |\}.$$

Then $(L(E^n), D)$ is a complete metric space, and satisfies $D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v})$, $D(k\widetilde{u}, k\widetilde{v}) = |k| D(\widetilde{u}, \widetilde{v})$ for any $\widetilde{u}, \widetilde{v}, \widetilde{w} \in L(E^n)$, $k \in R$.

In recent years, several authors have discussed different ordering relation of fuzzy numbers [14]. To the best of our knowledge, very few investigations have been appeared to study ordering relation of fuzzy n-cell numbers. For this reason, an ordering $\preceq_c$ of fuzzy n-cell numbers will be introduced.

Definition 2.2. Let $\tau : L(E^n) \to R^n$ be a vector-valued function defined by

$$\begin{aligned} \tau(\widetilde{u}) &= \left(2\int_0^1 r \frac{\int \cdots \int_{[\widetilde{u}]^r} x_1 dx_1 dx_2 \cdots dx_n}{\int \cdots \int_{[\widetilde{u}]^r} 1 dx_1 dx_2 \cdots dx_n} dr, 2\int_0^1 r \frac{\int \cdots \int_{[\widetilde{u}]^r} x_2 dx_1 dx_2 \cdots dx_n}{\int \cdots \int_{[\widetilde{u}]^r} 1 dx_1 dx_2 \cdots dx_n} dr, \cdots, 2\int_0^1 r \frac{\int \cdots \int_{[\widetilde{u}]^r} x_n dx_1 dx_2 \cdots dx_n}{\int \cdots \int_{[\widetilde{u}]^r} 1 dx_1 dx_2 \cdots dx_n} dr\right) \\ &= \left(\int_0^1 r(\widetilde{u}_1^+(r) + u_1^-(r))dr, \int_0^1 r(\widetilde{u}_2^+(r) + u_2^-(r))dr, \cdots, \int_0^1 r(\widetilde{u}_n^+(r) + u_n^-(r))dr\right), \end{aligned}$$

where $\int_0^1 r \frac{\int \cdots \int_{[\widetilde{u}]^r} x_i dx_1 dx_2 \cdots dx_n}{\int \cdots \int_{[\widetilde{u}]^r} 1 dx_1 dx_2 \cdots dx_n} dr$ $(i = 1, 2, \cdots, n)$ are the Lebesgue integral of $r \frac{\int \cdots \int_{[\widetilde{u}]^r} x_i dx_1 dx_2 \cdots dx_n}{\int \cdots \int_{[\widetilde{u}]^r} 1 dx_1 dx_2 \cdots dx_n}$ $(i = 1, 2, \cdots, n)$ on $[0, 1]$. The vector-valued function τ is called a ranking value function defined on $L(E^n)$.

In this case $\tau(\widetilde{u})$ represents a centroid of the fuzzy n-cell number $\widetilde{u}$. From the ranking value function $\tau(\widetilde{u})$, we consider the following ordering relation $\preceq_c$ on $L(E^n)$.

Definition 2.3. Let $\widetilde{u}, \widetilde{v} \in L(E^n)$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. We say that $\widetilde{u} \preceq_c \widetilde{v}$ ($\widetilde{u}$ precedes $\widetilde{v}$) if $\tau(\widetilde{v}) \in \tau(\widetilde{u}) + C$ $(\tau(\widetilde{v}) - \tau(\widetilde{u}) \in C)$.

Obviously the order relation $\preceq_c$ is reflexive and transitive, and $\preceq_c$ is a partially ordered relation on $L(E^n)$. For $\widetilde{u}, \widetilde{v} \in L(E^n)$, if either $\widetilde{u} \preceq_c \widetilde{v}$ or $\widetilde{v} \preceq_c \widetilde{u}$, then we say that $\widetilde{u}$ and $\widetilde{v}$ are comparable, otherwise non-comparable. If $\widetilde{u}, \widetilde{v} \in E^1$, $C = [0, +\infty) \subseteq R$, then Definition 2.3 coincides with Definition 2.5 from [14].

We say that $\widetilde{u} \prec_c \widetilde{v}$ if $\widetilde{u} \preceq_c \widetilde{v}$ and $\tau(\widetilde{u}) \neq \tau(\widetilde{v})$. Sometimes we may write $\widetilde{v} \succeq_c \widetilde{u}$ (resp. $\widetilde{v} \succ_c \widetilde{u}$) instead of $\widetilde{u} \preceq_c \widetilde{v}$ (resp. $\widetilde{u} \prec_c \widetilde{v}$).

Remark 2.1. Let $\widetilde{u}, \widetilde{v} \in L(E^n)$, $k_1, k_2 \in R$. According to Theorem 2.2 and Definition 2.2, it is easy to verify that $\tau(k_1\widetilde{u} + k_2\widetilde{v}) = k_1\tau(\widetilde{u}) + k_2\tau(\widetilde{v})$.

Theorem 2.3. Let $\widetilde{u}_1, \widetilde{u}_2, \widetilde{v}_1, \widetilde{v}_2, \in L(E^n)$, $k_1, k_2 \in [0, +\infty]$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. If $\widetilde{u}_1 \preceq_c \widetilde{v}_1$ and $\widetilde{u}_2 \preceq_c \widetilde{v}_2$, then $k_1\widetilde{u}_1 + k_2\widetilde{u}_2 \preceq_c k_1\widetilde{v}_1 + k_2\widetilde{v}_2$.

The proof is similar to the proof of Theorem 2.3 in [15]

3. Generalized difference for fuzzy n-cell numbers

Definition 3.1. [16] Let $\widetilde{u}, \widetilde{v} \in L(E^n)$. The generalized difference (g-difference for short) of $\widetilde{u}$ and $\widetilde{v}$ is given by its level sets as

$$[\widetilde{u} \ominus_g \widetilde{v}]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\},\ \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}],$$

where $\beta \in [r, 1]$.

Remark 3.1. If $\widetilde{u}, \widetilde{v} \in E^1$, we have

$$[\widetilde{u} \ominus_g \widetilde{v}]^r = [\inf_{\beta \geq r} \min\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\},\ \sup_{\beta \geq r} \max\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\}],$$

which coincides with Definition 7 of reference [8].

According to Proposition 13 in [7], we have the following conclusion.

Theorem 3.1. Let $\widetilde{u},\widetilde{v}\in L(E^n)$. If $l_i[\widetilde{u}]^r\leq l_i[\widetilde{v}]^r$ or $l_i[\widetilde{u}]^r\geq l_i[\widetilde{v}]^r$ for any $r\in[0,1]$ and $i=1,2,\cdots,n$, then the g-difference $\widetilde{u}\ominus_g\widetilde{v}$ exists and $\widetilde{u}\ominus_g\widetilde{v}\in L(E^n)$.

From now on, throughout this paper, we will assume that the g-difference $\widetilde{u}\ominus_g\widetilde{v}$ for any fuzzy n-cell numbers $\widetilde{u}$ and $\widetilde{v}$ exists.

Theorem 3.2. For any $\widetilde{u},\widetilde{v},\widetilde{w},\in L(E^n)$, we have

(1) $\widetilde{u}\ominus_g\widetilde{u}=\widetilde{0}$, $\widetilde{u}\ominus_g\widetilde{0}=\widetilde{u}$, $\widetilde{0}\ominus_g\widetilde{u}=-\widetilde{u}$,

(2) $\widetilde{u}\ominus_g\widetilde{v}=-(\widetilde{v}\ominus_g\widetilde{u})$,

(3) $k(\widetilde{u}\ominus_g\widetilde{v})=k\widetilde{u}\ominus_g k\widetilde{v}$, for any $k\in R$,

(4) $k_1\widetilde{u}\ominus_g k_2\widetilde{u}=(k_1-k_2)\widetilde{u}$, for any $k_1,k_2\in R$ and $k_1\cdot k_2\geq 0$,

(5) $\widetilde{u}\ominus_g(-\widetilde{v})=\widetilde{v}\ominus_g(-\widetilde{u})$, $(-\widetilde{u})\ominus_g\widetilde{v}=(-\widetilde{v})\ominus_g\widetilde{u}$,

(6) $(\widetilde{u}+\widetilde{v})\ominus_g\widetilde{v}=\widetilde{u}$,

(7) $\widetilde{0}\ominus_g(\widetilde{u}\ominus_g\widetilde{v})=\widetilde{v}\ominus_g\widetilde{u}=(-\widetilde{u})\ominus_g(-\widetilde{v})$,

(8) $\widetilde{u}\ominus_g\widetilde{v}=\widetilde{v}\ominus_g\widetilde{u}=\widetilde{w}$ if and only if $\widetilde{w}=-\widetilde{w}$.

Proof. The proof of (1), (3) are immediate.

(2) According to Definition 3.1, for any $r\in[0,1]$, we have

$$\begin{aligned}
&-[\widetilde{v}\ominus_g\widetilde{u}]^r\\
=\ &-\textstyle\prod_{i=1}^n[\inf_{\beta\geq r}\min\{v_i^-(\beta)-u_i^-(\beta),v_i^+(\beta)-u_i^+(\beta)\},\ \ \sup_{\beta\geq r}\max\{v_i^-(\beta)-u_i^-(\beta),v_i^+(\beta)-u_i^+(\beta)\}]\\
=\ &\textstyle\prod_{i=1}^n[-\sup_{\beta\geq r}\max\{v_i^-(\beta)-u_i^-(\beta),v_i^+(\beta)-u_i^+(\beta)\},\\
&\quad -\inf_{\beta\geq r}\min\{v_i^-(\beta)-u_i^-(\beta),v_i^+(\beta)-u_i^+(\beta)\}]\\
=\ &\textstyle\prod_{i=1}^n[-\sup_{\beta\geq r}(-\min\{u_i^-(\beta)-v_i^-(\beta),u_i^+(\beta)-v_i^+(\beta)\}),\\
&\quad -\inf_{\beta\geq r}(-\max\{u_i^-(\beta)-v_i^-(\beta),u_i^+(\beta)-v_i^+(\beta)\})]\\
=\ &\textstyle\prod_{i=1}^n[\inf_{\beta\geq r}\min\{u_i^-(\beta)-v_i^-(\beta),u_i^+(\beta)-v_i^+(\beta)\},\ \ \sup_{\beta\geq r}\max\{u_i^-(\beta)-v_i^-(\beta),u_i^+(\beta)-v_i^+(\beta)\}]\\
=\ &[\widetilde{u}\ominus_g\widetilde{v}]^r.
\end{aligned}$$

It follows from Theorem 2.2 that $\widetilde{u}\ominus_g\widetilde{v}=-(\widetilde{v}\ominus_g\widetilde{u})$.

(4) For any $r\in[0,1]$, it follows from Definition 3.1 that

$$\begin{aligned}
&[k_1\widetilde{u}\ominus_g k_2\widetilde{u}]^r\\
=\ &\textstyle\prod_{i=1}^n[\inf_{\beta\geq r}\min\{(k_1-k_2)u_i^-(\beta),(k_1-k_2)u_i^+(\beta)\},\ \ \sup_{\beta\geq r}\max\{(k_1-k_2)u_i^-(\beta),(k_1-k_2)u_i^+(\beta)\}].
\end{aligned}$$

If $k_1-k_2\geq 0$, for any $r\in[0,1]$, it is obvious that

$$[k_1\widetilde{u}\ominus_g k_2\widetilde{u}]^r=\prod_{i=1}^n[(k_1-k_2)u_i^-(r),(k_1-k_2)u_i^+(r)]=[(k_1-k_2)\widetilde{u}]^r.$$

On the other hand, if $k_1-k_2<0$, for any $r\in[0,1]$, we have from Theorem 2.2 that

$$\begin{aligned}
&[k_1\widetilde{u}\ominus_g k_2\widetilde{u}]^r\\
=\ &\textstyle\prod_{i=1}^n[\inf_{\beta\geq r}\min\{(k_1-k_2)u_i^-(\beta),(k_1-k_2)u_i^+(\beta)\},\ \ \sup_{\beta\geq r}\max\{(k_1-k_2)u_i^-(\beta),(k_1-k_2)u_i^+(\beta)\}]\\
=\ &\textstyle\prod_{i=1}^n[(k_1-k_2)\sup_{\beta\geq r}\max\{u_i^-(\beta),u_i^+(\beta)\},\ \ (k_1-k_2)\inf_{\beta\geq r}\min\{u_i^-(\beta),u_i^+(\beta)\}]\\
=\ &\textstyle\prod_{i=1}^n[(k_1-k_2)u_i^+(r),(k_1-k_2)u_i^-(r)]\\
=\ &[(k_1-k_2)\widetilde{u}]^r.
\end{aligned}$$

Then $k_1\widetilde{u}\ominus_g k_2\widetilde{u}=(k_1-k_2)\widetilde{u}$.

(5) According to Definition 3.1 and Theorem 2.2, for any $r \in [0,1]$, we have

$$\begin{aligned}
& [\widetilde{u} \ominus_g (-\widetilde{v})]^r \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) + v_i^+(\beta), u_i^+(\beta) + v_i^-(\beta)\},\ \ \sup_{\beta \geq r} \max\{u_i^-(\beta) + v_i^+(\beta), u_i^+(\beta) + v_i^-(\beta)\}] \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{v_i^-(\beta) + u_i^+(\beta), v_i^+(\beta) + u_i^-(\beta)\},\ \ \sup_{\beta \geq r} \max\{v_i^-(\beta) + u_i^+(\beta), v_i^+(\beta) + u_i^-(\beta)\}] \\
= & [\widetilde{v} \ominus_g (-\widetilde{u})]^r.
\end{aligned}$$

Then $\widetilde{u} \ominus_g (-\widetilde{v}) = \widetilde{v} \ominus_g (-\widetilde{u})$. It follows from (3) that $(-\widetilde{u}) \ominus_g \widetilde{v} = (-\widetilde{v}) \ominus_g \widetilde{u}$.

(6) For any $r \in [0,1]$, we have from Theorem 2.2 that

$$\begin{aligned}
[(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{v}]^r &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}, \\
&\qquad \sup_{\beta \geq r} \max\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}] \\
&= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta), u_i^+(\beta)\},\ \ \sup_{\beta \geq r} \max\{u_i^-(\beta), u_i^+(\beta)\}] \\
&= \prod_{i=1}^n [\inf_{\beta \geq r} u_i^-(\beta),\ \ \sup_{\beta \geq r} u_i^+(\beta)] \\
&= \prod_{i=1}^n [u_i^-(r),\ \ u_i^+(r)] \\
&= [\widetilde{u}]^r.
\end{aligned}$$

Then $(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{v} = \widetilde{u}$.

(7) It follows from (1), (2) and (3) that the proof of (7) is immediate.

(8) We have from (2) that the proof of (8) is immediate.

For any $\widetilde{u}, \widetilde{v} \in L(E^n)$, using the method with that Bede proved Proposition 15 in [8], we can show that $D(\widetilde{u}, \widetilde{v}) = D(\widetilde{u} \ominus_g \widetilde{v}, \widetilde{0})$.

4. The differentiability and convexity for fuzzy n-cell mappings

In this work, let M be a convex set of m-dimensional Euclidean space R^m. We consider mapping $\widetilde{F}$ from M into $L(E^n)$, such a mapping is called a fuzzy n-cell mapping. For the sake of brevity, $\widetilde{F}$ is called a fuzzy mapping. For any $r \in [0,1]$, we denote $[\widetilde{F}(t)]^r$ by $F_r(t) = \prod_{i=1}^n [F_i^-(r,t), F_i^+(r,t)]$.

Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping and $\widetilde{u} \in L(E^n)$. For $t_0 \in \text{int}M$, we write $\lim_{t \to t_0} \widetilde{F}(t) = \widetilde{u}$, if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $0 < \|t - t_0\| < \delta$, we have $D(\widetilde{F}(t), \widetilde{u}) < \varepsilon$.

We say that $\widetilde{F}$ is continuous at $t_0 \in \text{int}M$ if $\lim_{t \to t_0} \widetilde{F}(t) = \widetilde{F}(t_0)$.

Theorem 4.1. Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t) : M \to R$ be a real-valued function on M, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \neq 0$. If f is continuous at t_0, then $\widetilde{F}$ is continuous at t_0 and

$$\lim_{t \to t_0} \widetilde{F}(t) = \widetilde{u} \cdot \lim_{t \to t_0} f(t).$$

Proof. Assume that f is continuous at t_0. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $0 < \|t - t_0\| < \delta$, we have $|f(t) - f(t_0)| < \frac{\varepsilon}{D(\widetilde{u}, \widetilde{0})}$. According to the sign-preserving theorem of limit, we have

$$\begin{aligned}
D(\widetilde{F}(t), \widetilde{F}(t_0)) &= D(f(t) \cdot \widetilde{u}, f(t_0) \cdot \widetilde{u}) \\
&= \sup_{r \in [0,1]} \max_{1 \leq i \leq n} \{|f(t) - f(t_0)| \cdot |u_i^-(r)|, |f(t) - f(t_0)| \cdot |u_i^+(r)|\} \\
&= |f(t) - f(t_0)| \sup_{r \in [0,1]} \max_{1 \leq i \leq n} \{|u_i^-(r)|, |u_i^+(r)|\} \\
&= |f(t) - f(t_0)| \cdot D(\widetilde{u}, \widetilde{0}) \\
&< \varepsilon,
\end{aligned}$$

which implies that $\widetilde{F}$ is continuous at t_0.

Definition 4.1. [15] Let $\widetilde{F}: M \to L(E^n)$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If g-difference $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there exist $\widetilde{u}_j \in L(E^n)$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{D(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0), \sum_{j=1}^m \widetilde{u}_j (t_j - t_j^0))}{d(t, t_0)} = 0,$$

then we say that $\widetilde{F}$ is differentiable at t_0 and the fuzzy vector $(\widetilde{u}_1, \ \widetilde{u}_2, \cdots, \widetilde{u}_m)$ is the gradient of $\widetilde{F}$ at t_0, denoted by $\nabla\widetilde{F}(t_0)$, i.e., $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$.

Remark 4.1. Let $\widetilde{F}: M \to L(E^n)$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. Then the gradient $\nabla\widetilde{F}(t_0)$ exists at t_0 if and only if and $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there are $\widetilde{u}_j \in L(E^n)$ $(j = 1, 2, \cdots, m)$, such that

$$\widetilde{u}_j = \lim_{h \to 0} \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}.$$

Here the limit is taken in the metric space $(L(E^n), D)$.

Theorem 4.2. Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t): M \to R$ be a continuous function on M, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \neq 0$. If $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists, then the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 if and only if the gradient $\nabla f = (\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \cdots, \frac{\partial f}{\partial t_m})$ of f exists at t_0 and

$$\widetilde{u} \frac{\partial f}{\partial t_j}|_{t=t_0} = \widetilde{u}_j \ (j = 1, 2, \cdots, m).$$

Proof. It follows from the sign-preserving theorem of limit and Theorem 3.2 that

$$\widetilde{F}(t) \ominus_g \widetilde{F}(t_0) = (f(t) - f(t_0)) \cdot \widetilde{u}.$$

Only if: Taking $h \in R$, such that $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$, then

$$\begin{aligned} & \widetilde{u} \lim_{h \to 0} \tfrac{f(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - f(t_1^0, \cdots, \ t_j^0, \cdots, \ t_m^0)}{h} \\ = \ & \lim_{h \to 0} \tfrac{(f(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - f(t_1^0, \cdots, \ t_j^0, \cdots, \ t_m^0)) \cdot \widetilde{u}}{h} \\ = \ & \lim_{h \to 0} \tfrac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, \ t_j^0, \cdots, \ t_m^0)}{h} \\ = \ & \widetilde{u}_j, \end{aligned}$$

which implies that the gradient $\nabla f = (\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \cdots, \frac{\partial f}{\partial t_m})$ of f exists at t_0 and $\widetilde{u}\frac{\partial f}{\partial t_j}|_{t=t_0} = \widetilde{u}_j$ $(j = 1, 2, \cdots, m)$.

If: Taking $h \in R$, such that $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$, then

$$\begin{aligned} & \lim_{h \to 0} \tfrac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, \ t_j^0, \cdots, \ t_m^0)}{h} \\ = \ & \lim_{h \to 0} \tfrac{(f(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) - f(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0)) \cdot \widetilde{u}}{h} \\ = \ & \tfrac{\partial f}{\partial t_j}|_{t=t_0} \cdot \widetilde{u}, \end{aligned}$$

which implies that the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and $\widetilde{u}_j = \widetilde{u}\frac{\partial f}{\partial t_j}|_{t=t_0}$ $(j = 1, 2, \cdots, m)$.

Theorem 4.3. Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t): M \to R$ be a continuous function on M, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \neq 0$. If $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and f is differentiable at t_0, then $\widetilde{F}$ is differentiable at t_0.

Proof. It follows from the sign-preserving theorem of limit and Theorem 3.2 that $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0) = (f(t) - f(t_0)) \cdot \widetilde{u}$. According to Theorem 4.2, we have

$$\begin{aligned} & D(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0), \textstyle\sum_{j=1}^m \widetilde{u}_j (t_j - t_j^0)) \\ = \ & D(\widetilde{u}(f(t) - f(t_0)), \widetilde{u} \textstyle\sum_{j=1}^m \frac{\partial f}{\partial t_j}|_{t=t_0} (t_j - t_j^0)) \\ = \ & D(\widetilde{u}, \widetilde{0}) \cdot |f(t) - f(t_0) - \textstyle\sum_{j=1}^m \frac{\partial f}{\partial t_j}|_{t=t_0} (t_j - t_j^0)|, \end{aligned}$$

where, $\widetilde{u}_j = \lim_{h\to 0} \frac{\widetilde{F}(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\ominus_g \widetilde{F}(t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}$. This completes the proof.

Convexity plays a key role in mathematical economics, engineering, management science, and optimization theory. Therefore, research into convexity is one of the most important aspects of mathematical programming. Next, using the partial ordering relation $\preceq_c$ we will extend the concept of convexity for real-valued functions to fuzzy mappings.

Definition 4.2. Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy mapping. $\widetilde{F}$ is said to be convex (c.) on M if

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2)$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$.

Remark 4.2. If $\widetilde{F}, \widetilde{G}: M \to L(E^n)$ are convex fuzzy mappings and $\alpha, \beta \geq 0$, then $\alpha\widetilde{F} + \beta\widetilde{G}$ is a convex fuzzy mapping on M.

Theorem 4.4. Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t)\cdot\widetilde{u}$, where $f(t): M \to R$ be a real-valued function on M, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \succeq_c \widetilde{0}$. $\widetilde{F}$ is convex on M if and only if f is convex on M.

Proof. If: Let $C = R^{n+} \subseteq R^n$, where $R^{n+} = \{(x_1, x_2, \cdots, x_n) \in R^n : x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0\}$. Assume that the real-valued function f is convex on M, then for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$, we have

$$f(\lambda t_1 + (1-\lambda)t_2) \leq \lambda f(t_1) + (1-\lambda)f(t_2).$$

According to $\widetilde{u} \succeq_c \widetilde{0}$, we obtain

$$\tau(\widetilde{u})(\lambda f(t_1) + (1-\lambda)f(t_2)) \in \tau(\widetilde{u})f(\lambda t_1 + (1-\lambda)t_2) + C.$$

For any $t_1, t_2 \in M$ and $\lambda \in [0,1]$, it follows from Remark 2.1 that

$$\tau(\lambda\widetilde{u}f(t_1) + (1-\lambda)\widetilde{u}f(t_2)) \in \tau(\widetilde{u}f(\lambda t_1 + (1-\lambda)t_2)) + C,$$

i.e., $\tau(\lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2)) \in \tau(\widetilde{F}(\lambda t_1 + (1-\lambda)t_2)) + C$, which implies that

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2).$$

Therefore, $\widetilde{F}$ is convex on M.

Only if: Let $\widetilde{F}$ be convex on M. For any $t_1, t_2 \in M$ and $\lambda \in [0,1]$,

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2),$$

i.e., $\tau(\lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2)) \in \tau(\widetilde{F}(\lambda t_1 + (1-\lambda)t_2)) + C$, which implies that

$$\tau(\lambda\widetilde{u}f(t_1) + (1-\lambda)\widetilde{u}f(t_2)) \in \tau(\widetilde{u}f(\lambda t_1 + (1-\lambda)t_2)) + C.$$

According to Remark 2.1, we have

$$\tau(\widetilde{u})(\lambda f(t_1) + (1-\lambda)f(t_2)) \in \tau(\widetilde{u})f(\lambda t_1 + (1-\lambda)t_2) + C.$$

For any $t_1, t_2 \in M$ and $\lambda \in [0,1]$, we have from $\widetilde{u} \succeq_c \widetilde{0}$ that

$$f(\lambda t_1 + (1-\lambda)t_2) \leq \lambda f(t_1) + (1-\lambda)f(t_2).$$

Therefore, f is convex on M.

Theorem 4.5. Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = \widetilde{u}_0 + \widetilde{u}_1 f_1(t) + \widetilde{u}_2 f_2(t) + \cdots + \widetilde{u}_l f_l(t)$, where $f_k: M \to R$ be real-valued functions on M, $\widetilde{u}_k \in L(E^n)$ and $\widetilde{u}_k \succeq_C \widetilde{0}$ $(k = 0, 1, 2, \cdots, l)$. If f_k $(k = 1, 2, \cdots, l)$ are convex on M, then $\widetilde{F}$ is convex on M.

Proof. Let f_k $(k = 1, 2, \cdots, l)$ be convex fuzzy-number-valued functions. Then we have

$$f_k(\lambda t_1 + (1-\lambda)t_2) \leq \lambda f_k(t_1) + (1-\lambda)f_k(t_2)$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$. It follows from $\widetilde{u}_k \succeq_C \widetilde{0}$ and Remark 2.1 that

$$\widetilde{u}_k \cdot f_k(\lambda t_1 + (1-\lambda)t_2) \preceq_C \lambda \widetilde{u}_k \cdot f_k(t_1) + (1-\lambda)\widetilde{u}_k \cdot f_k(t_2) \ \ (k = 0, 1, 2, \cdots, l),$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$. According to Theorem 2.3, we have

$$\widetilde{u}_0 + \sum_{k=1}^{l} \widetilde{u}_k \cdot f_k(\lambda t_1 + (1-\lambda)t_2) \preceq_C \lambda(\widetilde{u}_0 + \sum_{k=1}^{l} \widetilde{u}_k \cdot f_k(t_1)) + (1-\lambda)(\widetilde{u}_0 + \sum_{k=1}^{l} \widetilde{u}_k \cdot f_k(t_2)),$$

which implies that

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda F(t_1) + (1-\lambda)F(t_2),$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$. This completes the proof.

5. Convex fuzzy programming

It is well known that convexity plays an important role in the aspect of optimality conditions for mathematical programming. Based on the new concept of convexity for fuzzy mappings, the KKT optimality conditions for a class of fuzzy optimization problems are established.

Let $\widetilde{F}(t), \widetilde{g}_1(t), \widetilde{g}_2(t), \cdots, \widetilde{g}_m(t)$ be fuzzy mappings on M. We consider the following fuzzy constrained minimization problem (FCMP1):

Minimize $\quad \widetilde{F}(t)$

Subject to $\quad \widetilde{g}_k(t) \preceq_c \widetilde{u}_k \ (k = 1, 2, \cdots, l)$,

where $\widetilde{u}_k \in L(E^n)$. In the following arguments the feasible set $T = \{t \in \text{int}M : \widetilde{g}_k(t) \preceq_c \widetilde{u}_k, k = 1, 2, \cdots, l\}$ of fuzzy constrained minimization problem (FCMP1) is assumed to be a compact convex set. If $t_0 \in \text{int}M$ and for no $t \in \text{int}M$ such that $\widetilde{F}(t) \prec_c \widetilde{F}(t_0)$, then t_0 is called an optimal solution or a global optimal solution to the problem FCMP1, $\widetilde{F}(t_0)$ is called the optimal objective value of $\widetilde{F}$.

Let $C = R^{n+} \subseteq R^n$. We have from Definition 2.2 and Definition 2.3 that the problem FCMP1 can be written as following constrained minimization problem (FCMP2):

Minimize $\quad \widetilde{F}(t)$

Subject to $\quad \int_0^1 r[(\widetilde{g}_k(t))_1^+(r) + (\widetilde{g}_k(t))_1^-(r)]dr \leq \int_0^1 r[(\widetilde{u}_k)_1^+(r) + (\widetilde{u}_k)_1^-(r)]dr,\ k = 1, 2, \cdots, l,$
$\int_0^1 r[(\widetilde{g}_k(t))_2^+(r) + (\widetilde{g}_k(t))_2^-(r)]dr \leq \int_0^1 r[(\widetilde{u}_k)_2^+(r) + (\widetilde{u}_k)_2^-(r)]dr,\ k = 1, 2, \cdots, l,$
$\cdots$
$\int_0^1 r[(\widetilde{g}_k(t))_n^+(r) + (\widetilde{g}_k(t))_n^-(r)]dr \leq \int_0^1 r[(\widetilde{u}_k)_n^+(r) + (\widetilde{u}_k)_n^-(r)]dr,\ k = 1, 2, \cdots, l.$

Let $G_s(t) = \int_0^1 r[(\widetilde{g}_k(t))_i^+(r) + (\widetilde{g}_k(t))_i^-(r)]dr - \int_0^1 r[(\widetilde{u}_k)_i^+(r) + (\widetilde{u}_k)_i^-(r)]dr$ $(i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, l,$ $s = i \times k = 1, 2, \cdots, n \times l)$, denoted $p = n \times l$. The problem FCMP2 can be written as follows (FCMP3):

Minimize $\quad \widetilde{F}(t)$

Subject to $\quad G_s(t) \leq 0,\ s = 1, 2, \cdots, p,$

where $G_s : M \to R$ are real-valued functions. It is obvious that the feasible sets of problems (FCMP1) and (FCMP3) are the same.

In the rest of this article, G_s $(s = 1, 2, \cdots, p)$ are assumed to be convex functions on M, and continuously differentiable at $t_0 \in T$.

Theorem 5.1. Suppose that the fuzzy objective function $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t) : M \to R$, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \succeq_c \widetilde{0}$. let f be convex on M and continuously differentiable at $t_0 \in \text{int}M$. If there exist Lagrange multipliers $\mu_s \geq 0$ $(s = 1, 2, \cdots, p)$, such that

(1) $\nabla f(t_0) + \sum_{s=1}^{p} \mu_s \cdot \nabla G_s(t_0) = \mathbf{0}$,

(2) $\mu_s \cdot G_s(t_0) = 0$ $(s = 1, 2, \cdots, p)$,

then t_0 is an optimal solution of FCMP3.

Proof. Since f is convex on M and continuously differentiable at $t_0 \in \text{int}M$. We consider the following constrained optimization problem

Minimize $\quad f(t)$

Subject to $G_s(t) \leq 0$ $(s = 1, 2, \cdots, p)$.
It is obvious that conditions (1) and (2) are the KKT optimality conditions for this optimization problem. Therefore, we conclude that t_0 is an optimal solution of the real-valued objective function f, i.e.,

$$f(t_0) \leq f(t), \tag{5.1}$$

for any $t \in \text{int}M$.

Suppose that t_0 is not a solution of problem FCMP3. Then, there exists a $\bar{t} \in \text{int}M$ such that

$$\widetilde{F}(\bar{t}) \prec_c \widetilde{F}(t_0).$$

Let $C = R^{n+} \subseteq R^n$, we have

$$\tau(f(t_0) \cdot \widetilde{u}) \in \tau(f(\bar{t}) \cdot \widetilde{u}) + C.$$

Thus

$$f(t_0) \cdot \tau(\widetilde{u}) \in f(\bar{t}) \cdot \tau(\widetilde{u}) + C,$$

and

$$f(t_0) \cdot \tau(\widetilde{u}) \neq f(\bar{t}) \cdot \tau(\widetilde{u}).$$

It follows from $\widetilde{u} \succeq_c \widetilde{0}$ that $f(\bar{t}) < f(t_0)$. This is a contradiction with (5.1), then t_0 is an optimal solution of FCMP3.

Example 5.1. Let

$$\widetilde{u}(x_1, x_2) = \begin{cases} 20x_2 - 18, & 0.9 \leq x_2 \leq 0.95,\ 200x_2 - 90 \leq x_1 \leq 290 - 200x_2, \\ -0.1x_1 + 11, & 100 \leq x_1 \leq 110,\ 1.45 - 0.005x_1 \leq x_2 \leq 0.45 + 0.005x_1, \\ -20x_2 + 20, & 0.95 \leq x_2 \leq 1,\ 290 - 200x_2 \leq x_1 \leq 200x_2 - 90, \\ 0.1x_1 - 9, & 90 \leq x_1 \leq 100,\ 0.45 + 0.005x_1 \leq x_2 \leq 1.45 - 0.005x_1, \\ 0, & \text{otherwise}, \end{cases}$$

$C = R^{2+} \subseteq R^2$. Then for any $r \in [0, 1]$,

$$[\widetilde{u}]^r = [90 + 10r, 100 - 10r] \times [0.9 + 0.05r, 1 - 0.05r].$$

According to Definition 2.3, $\tau(\widetilde{u}) = (95, 0.95)$, thus $\widetilde{u} \succeq_c \widetilde{0}$. Suppose that the fuzzy objective function $\widetilde{F} : [1, +\infty) \to L(E^2)$ and $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t) = e^t - t^2$. We consider the following fuzzy constrained minimization problem:

Minimize $\widetilde{F}(t)$,

Subject to $G_1(t) = t^2 - 25 \leq 0, \quad G_2(t) = -t + 1 \leq 0.$

It is obvious that $G_s(t)$ $(s = 1, 2)$ are convex functions on $[1, +\infty)$, and continuously differentiable at $t_0 = 1$.

Since f is a convex function on $[1, +\infty)$, and continuously differentiable at $t_0 = 1$. On the other hand, the condition (1) and (2) from Theorem 5.1 are satisfied for $\mu_1 = 0$, and $\mu_2 = e - 2$. Therefore, $t_0 = 1$ is an optimal solution of FCMP3.

Theorem 5.2. Suppose that the fuzzy objective function $\widetilde{F}(t) = \widetilde{u}_1 f_1(t) + \widetilde{u}_2 f_2(t) + \cdots + \widetilde{u}_l f_l(t)$, where $f_k(t) : M \to R$, $\widetilde{u}_k \in L(E^n)$ and $\widetilde{u}_k \succeq_c \widetilde{0}$ $(k = 1, 2, \cdots, l)$. Let real-valued functions f_k are convex on M and continuously differentiable at $t_0 \in \text{int}M$. If there exist Lagrange multipliers $\mu_s^k \geq 0$ $(s = 1, 2, \cdots, p,\ k = 1, 2, \cdots, l)$, such that

(1) $\nabla f_k(t_0) + \sum_{s=1}^{p} \mu_s^k \cdot \nabla G_s(t_0) = \mathbf{0}$ $(k = 1, 2, \cdots, l)$,

(2) $\mu_s^k \cdot G_s(t_0) = 0$ $(s = 1, 2, \cdots, p,\ k = 1, 2, \cdots, l)$,

then t_0 is an optimal solution of FCMP3.

Proof. Since the real-valued functions $f_k(t)$ $(k = 1, 2, \cdots, l)$ are convex on M and continuously differentiable at $t_0 \in \text{int}M$. We consider the following constrained optimization problem

Minimize $f_k(t)$ $(k = 1, 2, \cdots, l)$,

Subject to $G_s(t) \leq 0$ $(s = 1, 2, \cdots, p)$.
It is obvious that conditions (1) and (2) are the KKT optimality conditions for this optimization problem. Therefore, we conclude that t_0 is an optimal solution of the real-valued objective functions f_k, i.e.,

$$f_k(t_0) \leq f_k(t) \ (k = 1, 2, \cdots, l).$$

for any $t \in \text{int}M$. Let $C = R^{n+} \subseteq R^n$. We have from $\widetilde{u}_k \succeq_c \widetilde{0}$ that

$$\tau(\widetilde{u}_k) f_k(t) \in \tau(\widetilde{u}_k) f_k(t_0) + C \ (k = 1, 2, \cdots, l).$$

Thus,

$$\sum_{k=1}^{l} \tau(\widetilde{u}_k) f_k(t) \in \sum_{k=1}^{l} \tau(\widetilde{u}_k) f_k(t_0) + C.$$

According to Remark 2.1, we obtain

$$\tau(\sum_{k=1}^{l} u_k f_k(t)) \in \tau(\sum_{k=1}^{l} u_k f_k(t_0)) + C,$$

i.e., $\widetilde{F}(t) \in \widetilde{F}(t_0) + C$. Therefore, $\widetilde{F}(t_0) \preceq_c \widetilde{F}(t)$, which implies that t_0 is an optimal solution of FCMP3.

6. Conclusion

The objective of this paper is to introduce the differentiability concept for fuzzy mappings and an ordering relation on the fuzzy n-cell number space is considered. We have used the differentiability for fuzzy mappings to obtain the KKT optimality conditions for fuzzy constrained minimization problem based on the ordering relation $\preceq_c$.

Future research includes studying other types of optimality conditions for fuzzy constrained minimization problem. One alternative is to define the concept of invex function using differentiability and the ordering relation $\preceq_c$ for fuzzy mappings.

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Monotone iterative technique for fractional partial differential equations with impulses *

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Abstract

In this article, we use a monotone iterative technique based on the presence of lower and upper solutions to discuss the existence of mild solutions for the initial value problem of the impulsive time fractional order partial differential equation of Volterra type in an ordered Banach space E

$$\begin{cases} {D_0}^q u(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = x_0, \end{cases}$$

where ${D_0}^q$ is the Caputo fractional derivative of order q, $0 < q < 1$, $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ is a generator of equicontinuous C_0-semigroup, $f \in C(J \times E \times E, E)$, $J = [0, a]$, $a > 0$ is a constant, T is a Volterra integral operator, $0 < t_1 < t_2 < \cdots < t_m < a$, $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$ and $x_0 \in E$. Under wide monotone conditions and the noncompactness measure condition of nonlinearity f, we obtain the existence of extremal mild solutions and unique mild solution between lower and upper solutions. The results obtained generalize the recent conclusions on this topic. An example is also given to illustrate that our results are valuable.

Key Words: Impulsive fractional order integro-differential evolution equation; lower and upper solution; equicontinuous semigroup; measure of noncompactness; monotone iterative technique

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1 Introduction

Fractional order models are found to be more adequate than integer order models in some real world problems. Fractional order derivatives describe the property of memory and heredity of materials, and this is the major advantage of fractional order derivatives compared with integer order derivatives. In recent years, fractional order differential calculus has attracted many physicists, mathematicians and engineers, and notable contributions have been made to both theory and applications of fractional differential equations. It has been found that the differential equations involving fractional order derivatives in time are more realistic to describe many phenomena in practical cases than those of integer order in time. For instance, fractional calculus concepts have been used in the modelling of neurons [1], viscoelastic materials [2]. Other examples from fractional order differential equations can be found in [3-7] and the references therein.

One of the branches of fractional differential equations and dynamics is the theory of time fractional order evolution equations. Since time fractional order semilinear evolution equations are abstract formulations for many problems arising in engineering and physics, time fractional evolution equations have attracted increasing attention in recent years, see [8-16] and the references therein.

In this article, we use a monotone iterative technique based on the presence of lower and upper solutions to discuss the existence of mild solutions for the initial value problem (IVP) of impulsive time fractional order partial differential equation of Volterra type in an ordered Banach space E

$$\begin{cases} {D_0}^q u(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = x_0, \end{cases} \tag{1.1}$$

where ${D_0}^q$ is the Caputo fractional derivative of order q, $0 < q < 1$, $A : D(A) \subset E \to E$ is a closed linear operator, $-A$ generates a C_0-semigroup $S(t)(t \geq 0)$ in E, $J = [0, a]$, $a > 0$ is a constant, $f \in C(J \times E \times E, E)$, $x_0 \in E$, $0 < t_1 < t_2 < \cdots < t_m < a$, $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$, and

$$Tu\,(t) := \int_0^t K(t, s)u(s)ds \tag{1.2}$$

is a Volterra integral operator with integral kernel $K \in C(\triangle, \mathbb{R}^+)$, $\triangle = \{(t, s) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq a\}$, $\Delta u|_{t=t_k}$ stands the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

The monotone iterative technique based on lower and upper solutions is an effective and flexible method, which yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. In 1982, Du and Lakshmikantham [17] established a monotone iterative method for an initial value problem of ordinary differential equation in an ordered banach space. Later, Li [18,19], Chen and Li [20,21] developed the monotone iterative method for the abstract evolution equations in abstract space.

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical background and realistic mathematical model, and hence has been emerging as an important area of investigation in recent years, see [22]. Correspondingly, the existence of solutions to impulsive integro-differential equations in Banach spaces has also been studied by several authors, see for example [23,24] and the references therein. But all of the results mentioned above are for the differential equations of integer order. To the best of the author's knowledge, no results yet exist for the initial value problem of the impulsive time fractional order integro-differential evolution equation (1.1) by using the monotone iterative technique. The purpose of this paper is to establish the monotone iterative method for IVP (1.1) in an ordered Banach space E. Under the positivity assumption for the C_0 semigroup $S(t)$ and some monotone conditions combined with the noncompactness measure condition of nonlinearity f, we obtain the results on the existence and uniqueness of mild solutions for IVP(1.1).

2 Preliminaries

Let $(E, \|\cdot\|)$ be an ordered Banach space with the partial order " $\leq$ ". Then the positive cone $P = \{x \in E \mid x \geq \theta\}$ is normal with normal constant N. Denote $C(J, E)$ the Banach space of all E-value continuous functions with the supremum norm $\|u\|_C = \sup\limits_{t\in J} \|u(t)\|$. Clearly, $C(J, E)$ is also an ordered Banach space, which partial order " $\leq$ " is reduced by the positive cone $P_C = \{u \in C(J, E) \mid u(t) \geq \theta, t \in J\}$. Set

$$PC(J, E) = \{\ u : J \to E \mid u(t) \text{ is continuous at } t \neq t_k,$$
$$\text{left continuous at } t = t_k, \text{ and } u(t_k^+) \text{ exists}, \ k = 1, 2, \cdots, m\ \}.$$

Evidently, $PC(J, E)$ is also an ordered Banach space with the supremum norm $\|u\|_{PC} = \sup_{t\in J} \|u(t)\|$, its partial order " $\leq$ " is reduced by the positive function cone $P_{PC} = \{u \in PC(J, E) \mid u(t) \geq \theta, t \in J\}$. P_{PC} is also a normal with the same normal constant N. For $v, w \in PC(J, E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in PC(J, E) \mid v \leq u \leq w\}$, and for every $t \in J$, we use $[v(t), w(t)]$ to represent the order interval $\{x \in E \mid v(t) \leq x \leq w(t)\}$ in E. Let $J' := J\backslash\{t_1, t_2, \cdots, t_m\}$. We denote by $PC^1(J, E) = \{u \in PC(J, E) \cap C^1(J', E) \mid u'(t_k^+)$ and $u'(t_k^-)$ exist$\}$. Set $L(J, E)$ be the Banach space of all E-valued Bochner integrable functions defined on J with the norm $\|u\|_1 = \int_0^t \|u(t)\|dt$.

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Let $0 < q < 1$. The Caputo fractional order derivative of order q with the lower limit 0 for a function $g \in C^1(J)$ is defined as

$$D_0{}^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{g'(s)}{(t-s)^q}\, ds, \qquad t > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function. See [7].

If g is an abstract function with values in E, the definition of its Caputo fractional order derivative is same. In that case then the integrals appeared in (2.1) is taken in Bochner's sense.

Let $A : D(A) \subset E \to E$ be a closed linear operator and E_1 denote the Banach space $D(A)$ with the graphic norm $\|x\|_1 = \|x\| + \|Ax\|$. We assume that $-A$ generates a C_0-semigroup $S(t)$ $(t \geq 0)$ of linear bounded operators in E. Denote by $\mathscr{L}(E)$ the Banach space of all linear bounded operators in E. By the exponential boundedness of C_0-semigroup, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathscr{L}(E)} \leq M\, e^{\omega t}, \qquad t \geq 0. \tag{2.2}$$

If $\omega = 0$, we call $S(t)$ a uniformly bounded semigroup. Let $h \in L(J, E)$ and consider the initial value problem of the linear time fractional order evolution equation (LIVP)

$$\begin{cases} D_0{}^q u(t) + Au(t) = h(t), & t \in J, \\ u(0) = x_0. \end{cases} \tag{2.3}$$

By [11], we have the following existence and uniqueness result

Lemma 2.1 *Assume that the C_0-semigroup $S(t)$ $(t \geq 0)$ generated by $-A$ is a uniformly bounded and analytic semigroup. If $h \in C(J, E)$ is uniformly Hölder continuous on J, then the linear initial value problem (2.3) has a unique solution expressed by*

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{q-1} V(t-s) h(s) ds \tag{2.4}$$

where $U(t),\ V(t) : [0, \infty) \to \mathscr{L}(E)$ are strongly continuous functions of linear bounded operator value given by

$$\begin{aligned} U(t)x &= \int_0^\infty \zeta_q(\vartheta) S(t^q \vartheta) x d\vartheta, & x \in E,\ t \geq 0, \\ V(t)x &= q \int_0^\infty \vartheta \zeta_q(\vartheta) S(t^q \vartheta) x d\vartheta, & x \in E,\ t \geq 0, \end{aligned} \tag{2.5}$$

where

$$\zeta_q(\vartheta) = \frac{1}{q}\, \vartheta^{-1-(1/q)} \rho_q(\vartheta^{-1/q}) \tag{2.6}$$

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is a probability density function on $(0, +\infty)$, *in which*

$$\rho_q(\vartheta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\vartheta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \vartheta \in (0, +\infty).$$

By [11], the probability density function $\zeta_q(\vartheta)$ given by (2.6) satisfies the following condition

$$\int_0^\infty \zeta_q(\vartheta)d\vartheta = 1, \qquad \int_0^\infty \vartheta\zeta_q(\vartheta)d\vartheta = \frac{1}{\Gamma(1+q)}. \tag{2.7}$$

When $S(t)(t \geq 0)$ is a uniformly bounded C_0-semigroup, we can also define two operator value functions $U(t)$ and $V(t)$ by (2.5). From [11, 12], we have the following result

Lemma 2.2 *Assume that* $S(t)(t \geq 0)$ *is a uniformly bounded* C_0*-semigroup. Then operators* $U(t)$ *and* $V(t)$ *defined by (2.5) have the following properties:*

(1). For evrey $t \geq 0$, $U(t)$ and $V(t)$ are linear bounded operators, and

$$\|U(t)x\| \leq M\|x\|, \qquad \|V(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|, \qquad x \in E. \tag{2.8}$$

(2). $U(t)$ and $V(t)$ are strongly continuous on $[0, +\infty)$.

(3). When $S(t)(t \geq 0)$ is equicontinuous, $U(t)$ and $V(t)$ are continuous in $[0, +\infty)$ by the operator norm.

In this case, by means of Lemma 2.2 (2), the function u gived by (2.5) belongs to $C(J, E)$, we call it a mild solution of the linear fractional order evolution equation (2.3). That is:

Definition 2.1 *Let* $S(t)(t \geq 0)$ *be a uniformly bounded* C_0*-semigroup and* $h \in L(J, E)$*. By the mild solution of the LIVP (2.3), we mean that the function* $u \in C(J, E)$ *satisfying the integral equation*

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{q-1}V(t-s)h(s)ds.$$

Let $h \in PC(J, E)$, $y_k \in E$, $k = 1, 2, \cdots, m$. We consider the initial value problem of the linear impulsive time fractional order evolution equation (LIVP)

$$\begin{cases} {D_0}^q u(t) + Au(t) = h(t), & t \in J, \quad t \neq t_k, \\ \Delta u|_{t=t_k} = y_k, & k = 1, 2, \cdots, m, \\ u(0) = x_0 \in E. \end{cases} \tag{2.9}$$

Let $J_1 = [0, t_1]$, $J_k = (t_{k-1}, t_k]$, $k = 2, 3, \cdots, m+1$, where $t_{m+1} = a$. Using Definition 2.1, from J_1 to J_{m+1} interval by interval, we can easily obtain the following result.

Lemma 2.3 *For every $h \in PC(J,E)$ and $y_k \in E$, $k = 1,2,\cdots,m$, the LIVP (2.9) has a unique mild solution $u \in PC(J,E)$ given by*

$$u(t) = \begin{cases} U(t)x_0 + \int_0^t (t-s)^{\alpha-1}V(t)(t-s)h(s)ds, & t \in J_1, \\ U(t-t_1)(u(t_1)+y_1) + \int_{t_1}^t (t-s)^{q-1}V(t-s)h(s)ds, & t \in J_2, \\ \cdots\ \cdots\ \cdots \\ U(t-t_m)(u(t_m)+y_m) + \int_{t_m}^t (t-s)^{q-1}V(t-s)h(s)ds, & t \in J_{m+1}. \end{cases} \tag{2.10}$$

Remark 2.1 *Note that the operator value functions $U(t)$ and $V(t)$ do not possess the properties of semigroup. The mild solution of the LIVP (2.9) can be expressed only by using piecewise function.*

We consider the nonlinear impulsive time fractional order evolution equation (1.1). By Lemma 2.3, a function $u \in PC(J,E)$ is called a mild solution of IVP (1.1) if u satisfies the piecewise integral equation

$$u(t) = \begin{cases} U(t)x_0 + \int_0^t (t-s)^{\alpha-1}V(t)(t-s)f(s,u(s),Tu(s))ds, & t \in J_1, \\ U(t-t_1)(u(t_1)+I_1(u(t_1))) + \int_{t_1}^t (t-s)^{q-1}V(t-s)f(s,u(s),Tu(s))ds, & t \in J_2, \\ \cdots\ \cdots\ \cdots \\ U(t-t_m)(u(t_m)+I_m(u(t_m))) + \int_{t_m}^t (t-s)^{q-1}V(t-s)f(s,u(s),Tu(s))ds, & t \in J_{m+1}. \end{cases}$$

We will use the monotone iterative method based on lower and upper solutions to discuss the existence of extremal mild solutions for IVP (1.1). Next, we introduce the concepts of lower and upper solutions for IVP (1.1).

Definition 2.2 *If a function $v_0 \in PC^1(J,E) \cap PC(J,E_1)$ and satisfies inequalities*

$$\begin{cases} {D_0}^q v_0(t) + Av_0(t) \le f(t,v_0(t),Tv_0(t)), & t \in J,\ t \ne t_k, \\ \Delta v_0|_{t=t_k} \le I_k(v_0(t_k)), & k = 1,2,\cdots,m, \\ v_0(0) \le x_0, \end{cases} \tag{2.11}$$

we called it a lower solution of IVP (1.1). If all the inequalities of (2.11) are inverse, we call it an upper solution of IVP (1.1).

Our discussion needs that the $S(t)(t \geq 0)$ is a positive C_0-semigroup, that is, $S(t)x \geq \theta$ for any $x \geq \theta$ and $t \geq 0$. For more details of the properties of the positive C_0-semigroup, see [19,25]. Clearly, by the (2.5) we obtain that:

Lemma 2.4 *If $S(t)(t \geq 0)$ is a uniformly bounded positive C_0-semigroup in E, then $U(t)$ and $V(t)$ are positive operators in E for every $t \in [0, +\infty)$.*

Let $C \geq 0$ is a constant, I denote the identity operator in E. It is easy to see that $-(A+C\mathrm{I})$ generates a C_0-semigroup $S_1(t) = e^{-Ct}S(t)\,(t \geq 0)$ in E, and $S_1(t)$ is a positive C_0 semigroup if $S(t)$ is a positive C_0-semigroup. If $C \geq \omega$, then by (2.2), $S_1(t)$ is a a uniformly bounded C_0-semigroup, for more details please see [26]. Hence we can define the corresponding operator value functions $U_1(t)$ and $V_1(t)$ as follows

$$\begin{aligned} U_1(t)x &= \int_0^\infty \zeta_q(\vartheta)S_1(t^q\vartheta)x d\vartheta, \qquad x \in E, \ \ t \geq 0, \\ V_1(t)x &= q\int_0^\infty \vartheta\zeta_q(\vartheta)S_1(t^q\vartheta)x d\vartheta, \quad x \in E, \ \ t \geq 0. \end{aligned} \tag{2.12}$$

$U_1(t)$ and $V_1(t)$ have completely same properties with $U(t)$ and $V(t)$. If the semigroup $S(t)$ is not uniformly bounded, we choose $C \geq \omega$ such that $S_1(t)$ uniformly bounded. In this case, the mild solution of IVP (1.1) can be expressed by $U_1(t)$ and $V_1(t)$.

Next, we recall some properties about the measure of noncompactness that will be used in the proof of our main results. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definitions and properties of the measure of noncompactness, see [27]. For $\forall\ B \subset PC(J,\,E)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset E$. If B is bounded in $PC(J,\,E)$, then $B(t)$ is bounded in E and $\alpha(B(t)) \leq \alpha(B)$.

Lemma 2.5 [27] *Let $B \subset C(J,E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha\Big(\Big\{\int_J u(t) \mid u \in B\Big\}\Big) \leq \int_J \alpha(B(t))dt.$$

Lemma 2.6 [28] *Assume that $B = \{u_n\} \subset PC(J,E)$ is a countable set and there exists a function $m \in L^1(J,\mathbb{R}^+)$ such that for every $n \in \mathbb{N}$*

$$\|u_n(t)\| \leq m(t), \qquad \text{a.e. } t \in J.$$

Then $\alpha(B(t))$ is Lebesgue integral on J, and

$$\alpha\Big(\Big\{\int_J u_n(t)dt\Big\}\Big) \leq 2\int_J \alpha(B(t))dt.$$

Our discussion also need the following generalized Gronwall inequality which can be fund in [29].

Lemma 2.7 *Let $C_0 \geq 0$ be a constant and $a \in L(J)$ be a nonnegative function. If $\varphi \in L(J)$ is nonnegative and satisfies*

$$\varphi \leq a(t) + C_0 \int_0^t (t-s)^{q-1} \varphi(s) ds, \quad t \in J,$$

then

$$\varphi(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(C_0 \Gamma(q))^n}{\Gamma(nq)} (t-s)^{nq-1} a(s) \Big] ds, \quad t \in J.$$

3 Main Results

In this section, we use the monotone iterative method based on lower and upper solutions to discuss the existence of mild solution for IVP (1.1). We assume that the operator $A : D(A) \subset E \to E$ satisfies

(H0) $A : D(A) \subset E \to E$ be a closed linear operator, $-A$ generates a positive and equicontinuous C_0-semigroup $S(t)\,(t \geq 0)$.

Our main results as follows:

Theorem 3.1 *Let E be an ordered Banach space, and let the positive cone P be normal. Assume that $A : D(A) \subset E \to E$ satisfies the assumption (H0), $f \in C(J \times E \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$, and IVP (1.1) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$. If the following conditions are satisfied:*

(H1) There exists a constant $C > 0$ such that

$$f(t, x_2, y_2) - f(t, x_1, y_1) \geq -C(x_2 - x_1),$$

for $\forall\, t \in J$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$, and $Tv_0(t) \leq y_1 \leq y_2 \leq Tw_0(t)$.

(H2) $I_k(x)$ is increasing on the order interval $[v_0(t),\, w_0(t)]$ for $t \in J$, $k = 1, 2, \cdots, m$.

(H3) There exists a constant $L > 0$ such that

$$\alpha(\{f(t, x_n, y_n)\}) \leq L(\alpha(\{x_n\}) + \alpha(\{y_n\})),$$

for $\forall\, t \in J$, and increasing or decreasing monotonic sequences $\{x_n\} \subset [v_0(t), w_0(t)]$ and $\{y_n\} \subset [Tv_0(t), Tw_0(t)]$, *then IVP (1.1) has minimal and maximal mild solutions between v_0 and w_0, and they can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. Without losing the generality, in the assumption (H1) we may assume that $C \geq \omega$, where ω is the growth exponent of $S(t)$ given by (2.2). Then the C_0-semigroup

$$S_1(t) = e^{-Ct}S(t), \qquad t \geq 0 \tag{3.1}$$

generated by $-(A + C\mathrm{I})$ is uniformly bounded, positive and equicontinuous. Let $U_1(t)(t \geq 0)$ and $V_1(t)(t \geq 0)$ be the operator value function defined by (2.12), then they have the properties in Lemma 2.2, specially they satisfy

$$\|U_1(t)\, x\| \leq M\|x\|, \qquad \|V_1(t)\, x\| \leq \frac{M}{\Gamma(q)}\, \|x\|, \qquad t \geq 0, \;\; x \in E. \tag{3.2}$$

For every $u \in PC(J,\, E)$, set

$$G(u)(t) = f(t, u(t), Tu(t)) + Cu(t), \qquad t \in J. \tag{3.3}$$

Then $G : PC(J,\, E) \to PC(J,\, E)$ is a continuous mapping. We rewrite the equation (1.1) to the form of

$$\begin{cases} {D_0}^q u(t) + (A + C\mathrm{I})\, u(t) = G(u)\,(t), \quad t \in J,\; t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = x_0, \end{cases} \tag{3.4}$$

then by Lemma 2.3, the mild solution of this equation, equivalently IVP (1.1), which means that $u \in PC(J,\, E)$ satisfies the piecewise integral equation

$$u(t) = \begin{cases} U_1(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1} V_1(t)(t-s)\, G(u)(s)\, ds, \quad t \in J_1, \\ U_1(t-t_1)(u(t_1) + I_1(u(t_1))) + \displaystyle\int_{t_1}^t (t-s)^{q-1} V_1(t-s)\, G(u)(s)\, ds, \quad t \in J_2, \\ \cdots\;\cdots\;\cdots \\ U_1(t-t_m)(u(t_m) + I_m(u(t_m))) + \displaystyle\int_{t_m}^t (t-s)^{q-1} V_1(t-s) G(u)(s) ds, \; t \in J_{m+1}. \end{cases}$$

We define the mapping $Q : [v_0,\, w_0] \to PC(J,\, E)$ by

$$Qu(t)=\begin{cases}U_1(t)x_0+\displaystyle\int_0^t(t-s)^{\alpha-1}V_1(t)(t-s)\,G(u)(s)\,ds,\ \ t\in J_1,\\ U_1(t-t_1)(u(t_1)+I_1(u(t_1)))+\displaystyle\int_{t_1}^t(t-s)^{q-1}V_1(t-s)\,G(u)(s)\,ds,\ \ t\in J_2,\\ \cdots\cdots\cdots\\ U_1(t-t_m)(u(t_m)+I_m(u(t_m)))+\displaystyle\int_{t_m}^t(t-s)^{q-1}V_1(t-s)G(u)(s)ds,\ t\in J_{m+1},\end{cases} \tag{3.5}$$

then the mild solution of IVP (1.1) is equivalent to the fixed point of Q. Clearly, $Q:[v_0,\,w_0]\to PC(J,\,E)$ is continuous. Since operators $U_1(t)$ and $V_1(t)$ are positive, by the assumptions (H1) and (H2), Q is increasing in $[v_0,\,w_0]$. We use monotone iterative method of increasing operator to find the fixed point of Q.

Firstly, we show that $v_0\le Qv_0$ and $Qw_0\le w_0$.

Let $h(t)={D_0}^q v_0(t)+Av_0(t)+Cv_0(t)$. Then $h\in PC(J,E)$ and v_0 is the unique mild solution of the the linear impulsive time fractional evolution equation (LIVP)

$$\begin{cases}{D_0}^q u(t)+(A+CI)\,u(t)=h(t), & t\in J,\ t\neq t_k,\\ \Delta u|_{t=t_k}=\Delta v_0|_{t=t_k}, & k=1,2,\cdots,m,\\ u(0)=v_0(0)\in E.\end{cases} \tag{3.6}$$

By the definition of lower solution

$$v_0(0)\le x_0,\qquad h(t)\le G(v_0)(t),\quad t\in J',$$

$$\Delta v_0|_{t=t_k}\le I_k(v_0(t_k)),\quad k=1,\,2,\,\cdots,\,m.$$

Hence by Lemma 2.3 and the positivity of operators $U_1(t)$ and $V_1(t)$, we have

$$v_0(t)\ =\ \begin{cases}U_1(t)v_0(0)+\displaystyle\int_0^t(t-s)^{\alpha-1}V_1(t)(t-s)h(s)ds,\ t\in J_1,\\ U_1(t-t_1)(v_0(t_1)+\Delta v_0|_{t=t_1})+\displaystyle\int_{t_1}^t(t-s)^{q-1}V_1(t-s)h(s)ds,\quad t\in J_2,\\ \cdots\cdots\cdots\\ U_1(t-t_m)(v_0(t_m)+\Delta v_0|_{t=t_m})+\displaystyle\int_{t_m}^t(t-s)^{q-1}V_1(t-s)h(s)ds,\ t\in J_{m+1}\end{cases}$$

$$
\begin{aligned}
&\leq \begin{cases} U_1(t)v_0(0) + \int_0^t (t-s)^{\alpha-1} V_1(t)(t-s) G(v_0)(s)\, ds, \ t \in J_1, \\ U_1(t-t_1)(v_0(t_1) + I_1(v_0(t_1))) + \int_{t_1}^t (t-s)^{q-1} V_1(t-s) G(v_0)(s)\, ds, \ t \in J_2, \\ \quad \cdots\ \cdots\ \cdots \\ U_1(t-t_m)(v_0(t_m) + I_m(v_0(t_m))) + \int_{t_m}^t (t-s)^{q-1} V_1(t-s) G(v_0)(s)\, ds, \ t \in J_{m+1} \end{cases} \\
&= Q\, v_0(t).
\end{aligned}
$$

This means that $v_0 \leq Qv_0$. Using a similar method, we can prove that $Qw_0 \leq w_0$. Combining these facts and the increasing property of Q in $[v_0, w_0]$, we see that Q maps $[v_0,\, w_0]$ into itself. Hence, $Q : [v_0,\, w_0] \to [v_0,\, w_0]$ is a continuously increasing operator.

Secondly, we prove that the image set $Q([v_0, w_0])$ is equicontinuous in every interval J_k, $k = 1,\, 2, \cdots,\, m+1$.

For $\forall\, u \in [v_0,\, w_0]$, by the assumptions (H1) and (H2), we have

$$G(v_0) \leq G(u)(t) \leq G(w_0), \quad t \in J,$$

and

$$v_0(t_k) + I_k(v_0(t_k)) \leq u(t_k) + I_k(u(t_k)) \leq w_0(t_k) + I_k(w_0(t_k)), \ \ k = 1, 2, \cdots, m.$$

Hence by the normality of the cone P, there exist positive constant M^* and L_k, $k = 1, 2, \cdots, m$, such that

$$
\begin{aligned}
\|G(u)(t)\| &\leq M^*, \quad t \in J, \\
\|u(t_k) + I_k(u(t_k))\| &\leq L_k, \quad k = 1, 2, \cdots, m.
\end{aligned}
\tag{3.7}
$$

Consider the case of J_1. Let $t^{'}, t^{''} \in J_1$ and $0 < t^{'} < t^{''}$. We show that $\|Qu(t^{''}) - Qu(t^{'})\| \to 0$ independently of u as $t^{''} - t^{'} \to 0$. By the definition of Q, we have

$$
\begin{aligned}
Qu\,(t^{''}) - Qu\,(t^{'}) \ &= \ U_1(t^{''})x_0 - U_1(t^{'})x_0 \\
&\quad + \int_{t^{'}}^{t^{''}} (t^{''} - s)^{q-1} V_1(t^{''} - s) G(u)(s) ds \\
&\quad + \int_0^{t^{'}} [(t^{''} - s)^{q-1} - (t^{'} - s)^{q-1}] V_1(t^{''} - s) G(u)(s) ds \\
&\quad + \int_0^{t^{'}} (t^{'} - s)^{q-1} [V_1(t^{''} - s) - V_1(t^{'} - s)] G(u)(s) ds
\end{aligned}
$$

$$= S_{11} + S_{12} + S_{13} + S_{14},$$

where

$$\begin{aligned} S_{11} &= U_1(t^{''})x_0 - U_1(t^{'})x_0, \\ S_{12} &= \int_{t^{'}}^{t^{''}} (t^{''} - s)^{q-1} V_1(t^{''} - s) G(u)(s) ds, \\ S_{13} &= \int_0^{t^{'}} [(t^{''} - s)^{q-1} - (t^{'} - s)^{q-1}] V_1(t^{''} - s) G(u)(s) ds, \\ S_{14} &= \int_0^{t^{'}} (t^{'} - s)^{q-1} [V_1(t^{''} - s) - V_1(t^{'} - s)] G(u)(s) ds. \end{aligned}$$

Since

$$\|Qu\,(t^{''}) - Qu\,(t^{'})\| \le \|S_{11}\| + \|S_{12}\| + \|S_{13}\| + \|S_{14}\|,$$

we only need to check that $\|S_{1i}\| \to 0$ independently of $u \in [v_0,\, w_0]$ as $t^{''} - t^{'} \to 0$, $i = 1,\, 2,\, 3,\, 4$. For S_{11}, by Lemma 2.2 (2), $U_1(t)x_0$ is continuous on J, hence it is uniformly continuous on J and we have

$$\|S_{11}\| = \|U_1(t^{''})\, x_0 - U_1(t^{'})\, x_0\| \to 0 \quad (t^{''} - t^{'} \to 0).$$

For S_{12} and S_{13}, by (3.2) and (3.7) we have

$$\begin{aligned} \|S_{12}\| &\le \int_{t^{'}}^{t^{''}} (t^{''} - s)^{q-1} \|V_1(t^{''} - s) G(u)(s)\| ds \\ &\le \frac{MM^*}{q\,\Gamma(q)} (t^{''} - t^{'})^q \to 0 \quad (t^{''} - t^{'} \to 0). \end{aligned}$$

$$\begin{aligned} \|S_{13}\| &\le \int_0^{t^{'}} [(t^{'} - s)^{q-1} - (t^{''} - s)^{q-1}]\, \|V_1(t^{''} - s) G(u)(s)\| ds \\ &\le \frac{MM^*}{\Gamma(q)} \int_0^{t^{'}} [(t^{'} - s)^{q-1} - (t^{''} - s)^{q-1}] ds \\ &= \frac{MM^*}{q\,\Gamma(q)} [(t^{'})^q - (t^{''})^q + (t^{''} - t^{'})^q] \\ &\le \frac{MM^*}{q\,\Gamma(q)} (t^{''} - t^{'})^q \to 0 \quad (t^{''} - t^{'} \to 0). \end{aligned}$$

For S_{14}, using (3.2), (3.7), Lemma 2.2(3) and the Lebesgue bounded convergence theorem of integration, we have

$$\|S_{14}\| \le \int_0^{t^{'}} (t^{'} - s)^{q-1} \|V_1(t^{''} - s) - V_1(t^{'} - s)\| \cdot \|G(u)(s)\|\, ds.$$

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$$\begin{aligned}
&\le M^* \int_0^{t'} (t' - s)^{q-1} \|V_1(t'' - s) - V_1(t' - s)\| \, ds \\
&= M^* \int_0^{t'} r^{q-1} \|V_1(t'' - t' + r) - V_1(r)\| \, dr \\
&\le M^* \int_0^{a} r^{q-1} \|V_1(t'' - t' + r) - V_1(r)\| \, dr \to 0 \quad (t'' - t' \to 0).
\end{aligned}$$

As a result, $\|Q(u)(t'') - Q(u)(t')\|$ tends to 0 independently of $u \in [v_0, w_0]$ as $t'' - t' \to 0$, which means that $Q([v_0, w_0])$ is equicontinuous in the interval J_1.

Consider the case of J_2. For $t', t'' \in J_2$ with $t' < t''$, we have

$$\begin{aligned}
Qu\,(t'') - Qu\,(t') &= (\,U_1(t'' - t_1) - U_1(t' - t_1)\,)\,(u(t_1) + I_1(u(t_1))\,) \\
&\quad + \int_{t'}^{t''} (t'' - s)^{q-1} V_1(t'' - s) G(u)(s) \, ds \\
&\quad + \int_{t_1}^{t'} [(t'' - s)^{q-1} - (t' - s)^{q-1}] V_1(t'' - s) G(u)(s) \, ds \\
&\quad + \int_{t_1}^{t'} (t' - s)^{q-1} [V_1(t'' - s) - V_1(t' - s)] G(u)(s) \, ds \\
&= S_{21} + S_{22} + S_{23} + S_{24},
\end{aligned}$$

where

$$\begin{aligned}
S_{21} &= (\,U_1(t'' - t_1) - U_1(t' - t_1)\,)\,(u(t_1) + I_1(u(t_1))\,), \\
S_{22} &= \int_{t'}^{t''} (t'' - s)^{q-1} V_1(t'' - s) G(u)(s) \, ds, \\
S_{23} &= \int_{t_1}^{t'} [(t'' - s)^{q-1} - (t' - s)^{q-1}] V_1(t'' - s) G(u)(s) \, ds, \\
S_{24} &= \int_{t_1}^{t'} (t' - s)^{q-1} [V_1(t'' - s) - V_1(t' - s)] G(u)(s) \, ds.
\end{aligned}$$

It is obvious that

$$\|Qu\,(t'') - Qu\,(t')\| \le \|S_{21}\| + \|S_{22}\| + \|S_{23}\| + \|S_{24}\|.$$

Therefore, we only need to check that $\|S_{2i}\| \to 0$ independently of $u \in [v_0,\, w_0]$ as $t'' - t' \to 0$, $i = 1,\, 2,\, 3,\, 4$. For S_{21}, by Lemma 2.2 (3) and (3.7), we have that

$$\|S_{21}\| = \|(\,U_1(t'' - t_1) - U_1(t' - t_1)\,)\,(u(t_1) + I_1(u(t_1))\,)\|$$

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$$\begin{aligned}
&\leq \|U_1(t^{''}-t_1)-U_1(t^{'}-t_1)\|\cdot\|u(t_1)+I_1(u(t_1))\| \\
&\leq L_1\|U_1(t^{''}-t_1)-U_1(t^{'}-t_1)\|\to 0 \quad (t^{''}-t^{'}\to 0).
\end{aligned}$$

For S_{22}, similarly to S_{12}, we have

$$\|S_{22}\|\leq \frac{MM^*}{q\,\Gamma(q)}(t^{''}-t^{'})^q\to 0 \quad (t^{''}-t^{'}\to 0).$$

For S_{23}, by (3.2) and (3.7) we have

$$\begin{aligned}
\|S_{23}\| &\leq \int_{t_1}^{t^{'}}[(t^{'}-s)^{q-1}-(t^{''}-s)^{q-1}]\,\|V_1(t^{''}-s)G(u)(s)\|ds \\
&\leq \frac{MM^*}{\Gamma(q)}\int_{t_1}^{t^{'}}[(t^{'}-s)^{q-1}-(t^{''}-s)^{q-1}]ds \\
&= \frac{MM^*}{q\,\Gamma(q)}[(t^{'}-t_1)^q-(t^{''}-t_1)^q+(t^{''}-t^{'})^q] \\
&\leq \frac{MM^*}{q\,\Gamma(q)}(t^{''}-t^{'})^q\to 0 \quad (t^{''}-t^{'}\to 0).
\end{aligned}$$

For S_{24}, by (3.7) and lemma 2.2(3), we have

$$\begin{aligned}
\|S_{24}\| &\leq \int_{t_1}^{t^{'}}(t^{'}-s)^{q-1}\|V_1(t^{''}-s)-V_1(t^{'}-s)\|\cdot\|G(u)(s)\|\,ds. \\
&\leq M^*\int_{t_1}^{t^{'}}(t^{'}-s)^{q-1}\|V_1(t^{''}-s)-V_1(t^{'}-s)\|\,ds \\
&= M^*\int_0^{t_1-t^{'}}r^{q-1}\|V_1(t^{''}-t^{'}+r)-V_1(r)\|\,dr \\
&\leq M^*\int_0^{a}r^{q-1}\|V_1(t^{''}-t^{'}+r)-V_1(r)\|\,dr\to 0 \quad (t^{''}-t^{'}\to 0).
\end{aligned}$$

Consequently, $\|Qu\,(t^{''})-Qu\,(t^{'})\|$ tends to 0 independently of $u\in[v_0,w_0]$ as $t^{''}-t^{'}\to 0$. This means that $Q([v_0,w_0])$ is equicontinuous in the interval J_2.

Continuing such a process interval by interval up to J_{m+1}, we can prove that $Q([v_0,\,w_0]$ is equicontinuous in every interval J_k, $k=1,\,2,\,\cdots,\,m+1$.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0,w_0]$ by the iterative schemes

$$v_n=Q\,v_{n-1},\quad w_n=Q\,w_{n-1},\qquad n=1,\,2,\,\cdots. \tag{3.8}$$

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Then from the monotonicity of Q, one can easy to prove that

$$v_0 \le v_1 \le \cdots v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{3.9}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are uniformly convergent in J.

For convenience, let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$. Since $B = Q(B_0) \subset Q([v_0, w_0])$, so that B is equicontinuous in every interval J_k, $k = 2, 3, \cdots, m$. From $B_0 = B \cup \{v_0\}$ it is follows that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J$. Denote

$$\varphi(t) = \alpha(B(t)) = \alpha(B_0(t)), \quad t \in J. \tag{3.10}$$

By Lemma 2.5, $\varphi \in PC(J, \mathbb{R}^+)$. We from J_1 to J_{m+1} interval by interval show that $\varphi(t) \equiv 0$ in J.

For every $t \in J$, by Lemma 2.6 we get that

$$\begin{aligned} \alpha(T(B_0)(t)) &= \alpha\Big(\Big\{ \int_0^t K(t,s)v_{n-1}(s)ds \,\Big|\, n \in \mathbb{N}\Big\}\Big) \\ &\le 2\int_0^t \alpha(\{K(t,s)v_{n-1}(s) \mid n \in \mathbb{N}\})ds \\ &= 2\int_0^t K(t,s)\,\alpha(\{v_{n-1}(s) \mid n \in \mathbb{N}\})ds \\ &\le 2K_0 \int_0^t \varphi(s)ds, \end{aligned}$$

where $K_0 = \max_{(t,s)\in\Delta} K(t,s)$. Therefore

$$\begin{aligned} \int_0^t (t-s)^{q-1}\alpha(T(B_0)(s))\,ds &\le 2K_0 \int_0^t (t-s)^{q-1}\Big[\int_0^s \varphi(r)dr\Big]ds \\ &= \frac{2K_0}{q}\int_0^t (t-r)^q \varphi(r)dr, \\ &\le \frac{2aK_0}{q}\int_0^t (t-s)^{q-1}\varphi(s)ds, \quad t \in J. \end{aligned} \tag{3.11}$$

For $\forall\, t \in J_1$, by (3.5), using Lemma 2.6, the assumption (H3), (3.2) and (3.11), we have

$$\begin{aligned} \varphi(t) &= \alpha(B(t)) = \alpha(\,Q(B_0)(t)\,) \\ &= \alpha\Big(\Big\{U_1(t)x_0 + \int_0^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\,\Big\}\Big) \\ &= \alpha\Big(\Big\{\int_0^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\,\Big\}\Big) \end{aligned}$$

$$
\begin{aligned}
&\le 2\int_0^t \alpha\left(\{(t-s)^{q-1}V_1(t-s)\,G(v_{n-1})(s)\,\}\right)ds\\
&\le \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\left(\{\,G(v_{n-1})(s)\ ds\,\}\right)ds\\
&= \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\left(\{\,f(s,v_{n-1},Tv_{n-1}(s))+Cv_{n-1}(s)\,\}\right)ds\\
&\le \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[L(\,\alpha(B_0(s))+\alpha(\,T(B_0)(s))\,)+C\alpha(B_0(s))\,\right]\,ds\\
&= \frac{2M}{\Gamma(q)}(L+C)\int_0^t (t-s)^{q-1}\varphi(s)\,ds+\frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(T(B_0)(s))\,ds\\
&\le \frac{2M}{\Gamma(q)}\Big(L+C+\frac{2aK_0L}{q}\Big)\int_0^t (t-s)^{q-1}\varphi(s)ds.
\end{aligned}
$$

Hence by Lemma 2.7, $\varphi(t)\equiv 0$ in J_1. In particular, $\alpha(B(t_1))=\alpha(B_0(t_1))=\varphi(t_1)=0$, this means that $B(t_1)$ and $B_0(t_1)$ are precompact in E. Hence, from the continuity of I_1 we obtain that $I_1(B_0(t_1))$ is precompact in E, and $\alpha(I_1(B_0(t_1)))=0$.

For $\forall\, t\in J_2$, since

$$
\begin{aligned}
\alpha(\{U_1(t-t_1)[v_{n-1}(t_1)+I_1(v_{n-1}(t_1))]\}) &\le \alpha(U_1(t-t_1)(B_0(t_1)+I_1(B_0(t_1)))\,)\\
&\le M(\alpha(B_0(t_1))+\alpha(I_1(B_0(t_1)))\,)\ =\ 0,
\end{aligned}
$$

using (3.5) and a similar argument above, we have

$$
\begin{aligned}
\varphi(t) &= \alpha(B(t))=\alpha(\,Q(B_0)(t)\,)\\
&= \alpha\Big(\Big\{U_1(t-t_1)[v_{n-1}(t_1)+I_1(v_{n-1}(t_1))]+\int_{t_1}^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\Big\}\Big)\\
&\le \alpha(\{U_1(t-t_1)[v_{n-1}(t_1)+I_1(v_{n-1}(t_1))]\})\\
&\quad +\alpha\Big(\Big\{\int_{t_1}^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\Big\}\Big)\\
&= \alpha\Big(\Big\{\int_0^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\ \Big\}\Big)\\
&\le 2\int_{t_1}^t \alpha\left(\{(t-s)^{q-1}V_1(t-s)\,G(v_{n-1})(s)\,\}\right)ds\\
&\le \frac{2M}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}\alpha\left(\{\,G(v_{n-1})(s)\,\}\right)ds
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{2M}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}\alpha\,(\{\,f(s,v_{n-1},Tv_{n-1}(s))+Cv_{n-1}(s)\,\})\,ds \\
&\leq \frac{2M}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}[\,L(\alpha(B_0(s))+\alpha(T(B_0)(s)))+C\alpha(B_0(s))\,]\,ds \\
&= \frac{2M}{\Gamma(q)}(L+C)\int_{t_1}^{t}(t-s)^{q-1}\varphi(s)\,ds+\frac{2ML}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}\alpha(T(B_0)(s))\,ds \\
&\leq \frac{2M}{\Gamma(q)}(L+C)\int_{0}^{t}(t-s)^{q-1}\varphi(s)\,ds+\frac{2ML}{\Gamma(q)}\int_{0}^{t}(t-s)^{q-1}\alpha(T(B_0)(s))\,ds \\
&\leq \frac{2M}{\Gamma(q)}\Big(L+C+\frac{2aK_0L}{q}\Big)\int_{0}^{t}(t-s)^{q-1}\varphi(s)ds.
\end{aligned}
$$

Again by Lemma 2.7, $\varphi(t)\equiv 0$ on J_2, from which we obtain that $\alpha(B_0(t_2))=0$, and therefore $\alpha(I_2(B_0(t_2)))=0$. Continuing such a process interval by interval up to J_{m+1}, we can prove that $\varphi\equiv 0$ in every J_k.

Therefore, for every J_k, $\{v_n\}$ is equicontinuous on J_k and $\{v_n(t)\}$ is precompact in E for every $t\in J_k$. By the Arzela-Ascoli theorem, $\{v_n\}$ has a subsequence which is uniformly convergent in J_k. Combining this with the monotonicity (3.9), we easily prove that $\{v_n\}$ itself is uniformly convergent in J_k, $k=1,\,2,\,\cdots,\,m+1$. Consequently, $\{v_n(t)\}$ is uniformly convergent over the whole of J.

Using a similar argument to that for $\{v_n(t)\}$, we can prove that $\{w_n(t)\}$ is also uniformly convergent on J. Hence, $\{v_n\}$ and $\{w_n\}$ are convergent in the Banach space $PC(J,E)$. Set

$$\underline{u}=\lim_{n\to\infty}v_n \quad\text{and}\quad \overline{u}=\lim_{n\to\infty}w_n \quad\text{in}\;\; PC(J,\,E). \tag{3.12}$$

Letting $n\to\infty$ in (3.8) and (3.9), we see that $v_0\leq\underline{u}\leq\overline{u}\leq w_0$ and

$$\underline{u}=Q\,\underline{u} \quad\text{and}\quad \overline{u}=Q\,\overline{u}. \tag{3.13}$$

By the monotonicity of Q, it is easy to prove that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of Q in $[v_0,\,w_0]$, and therefore, they are the minimal and maximal mild solutions of IVP (1.1) in $[v_0,\,w_0]$, respectively.

This completes the proof of Theorem 3.1. □

In Theorem 3.1, if E is weakly sequentially complete, the condition (H3) holds automatically. In fact, when E is an ordered and weakly sequentially complete Banach space, by Theorem 2.2 in paper [30], we know that any monotonic and order-bounded sequence is precompact. Let $\{x_n\}$ and $\{y_n\}$ be two increasing or decreasing sequences in condition (H3), then by condition (H1), $\{\,f(t,\,x_n,\,y_n)+Cx_n\}$ is monotonic and order-bounded sequence. By the property of measure of noncompactness, we have

$$\alpha(\{ f(t, x_n, y_n)\}) \leq \alpha(\{ f(t, x_n, y_n) + Cx_n\}) + C\,\alpha(\{x_n\}) = 0.$$

Hence, condition (H3) holds for any $L > 0$. From Theorem 3.1, we obtain that

Corollary 3.1 *Let E be an ordered and weakly sequentially complete Banach space, and let the positive cone P be normal. Assume that $A : D(A) \subset E \to E$ satisfies the assumption (H0), $f \in C(J \times E \times E, E)$, and $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$. If IVP (1.1) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$ and the assumptions (H1) and (H2) are satisfied, then IVP (1.1) has minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Now we discuss the uniqueness of the mild solution for IVP (1.1) in $[v_0, w_0]$. In theorem 3.1, if replacing the assumption (H3) by the condition:

(H4) There exist positive constants C_1 and C_2 such that

$$f(t, x_2, y_2) - f(t, x_1, y_1) \leq C_1(x_2 - x_1) + C_2(y_2 - y_1),$$

for $\forall\, t \in J$, and $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$, $Tv_0(t) \leq y_1 \leq y_2 \leq Tw_0(t)$,

we have the following uniqueness result.

Theorem 3.2 *Let E be an ordered Banach space, and let the positive cone P be normal. Assume that $A : D(A) \subset E \to E$ satisfies the assumption (H0), $f \in C(J \times E \times E, E)$, and $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$. If IVP(1.1) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$ such that conditions (H1), (H2) and (H4) hold, then IVP (1.1) has a unique mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Proof. We firstly prove that (H1) and (H4) can deduce (H3). For $t \in J$, let $\{x_n\} \subset [v_0(t), w_0(t)]$ and $\{y_n\} \subset [Tv_0(t), Tw_0(t)]$ be two increasing sequences. For $m, n \in \mathbb{N}$ with $m > n$, by (H1) and (H4),

$$\begin{aligned}\theta &\leq (\, f(t, x_m, y_m) - f(t, x_n, y_n)\,) + C(x_m - x_n)\\ &\leq (C + C_1)(x_m - x_n) + C_2(y_m - y_n).\end{aligned}$$

By this and the normality of cone P, we have

$$\|f(t, x_m, y_m) - f(t, x_n, y_n)\|$$

$$\leq N\,\|(C+C_1)(x_m-x_n)+C_2(y_m-y_n)\|+C\,\|x_m-x_n\|$$

$$\leq (C+NC+NC_1)\|x_m-x_n\|+NC_2\|y_m-y_n\|.$$

From this and the definition of the measure of noncompactness, it follows that

$$\begin{aligned}\alpha(\{\,f(t,\,x_n,\,y_n)\}) &\leq (C+NC+NC_1)\,\alpha(\{x_n\})+NC_2\,\alpha(\{y_n\})\\ &\leq L\,(\alpha(\{x_n\})+\alpha(\{y_n\})),\end{aligned}$$

where $L=C+NC+NC_1+NC_2$. If $\{x_n\}$ and $\{y_n\}$ are two decreasing sequences, the above inequality is also valid. Hence (H3) holds.

Therefore, by Theorem 3.1, IVP (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ in $[v_0,\,w_0]$. Let $\{v_n\}$ and $\{w_n\}$ be the sequences defined by the iterative scheme (3.8). Then by the proof of Theorem 3.1, we know that (3.9), (3.12) and (3.13) are valid. We show that $\underline{u}(t)\equiv\overline{u}(t)$ on J. Set

$$\psi(t)=\|\overline{u}(t)-\underline{u}(t)\|,\qquad t\in J, \tag{3.14}$$

then $\psi\in PC(J,\,\mathbb{R}^+)$. We need to show that $\psi(t)\equiv 0$ on J. We from J_1 to J_{m+1} interval by interval show that $\psi(t)\equiv 0$ on J.

For every $t\in J_1$, using (3.5), (3.8), (3.9) and assumption (H1) and (H4), we obtain that

$$\begin{aligned}\theta &\leq \overline{u}(t)-\underline{u}(t)=Q\overline{u}\,(t)-Q\underline{u}\,(t)\\ &= \int_0^t (t-s)^{q-1}V_1(t-s)\,(\,G(\overline{u})(s)-G(\underline{u})(s)\,)\,ds\\ &\leq \int_0^t (t-s)^{q-1}V_1(t-s)\big((C+C_1)(\overline{u}(s)-\underline{u}(s))+C_2(T\overline{u}(s)-T\underline{u}(s))\big)\,ds.\end{aligned}$$

Hence, by the the normality of cone P and (3.2), we have

$$\begin{aligned}\psi(t) &= \|\overline{u}(t)-\underline{u}(t)\|\\ &\leq N\Big\|\int_0^t (t-s)^{q-1}V_1(t-s)\big((C+C_1)(\overline{u}(s)-\underline{u}(s))+C_2(T\overline{u}(s)-T\underline{u}(s))\big)\,ds\Big\|\\ &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t (t-s)^{q-1}\psi(s)\,ds+C_2\int_0^t (t-s)^{q-1}\|T\overline{u}(s)-T\underline{u}(s)\|\,ds\Big]\\ &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t (t-s)^{q-1}\psi(s)\,ds+C_2K_0\int_0^t (t-s)^{q-1}\int_0^s \psi(r)drds\Big]\\ &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t (t-s)^{q-1}\psi(s)\,ds+\frac{C_2K_0}{q}\int_0^t (t-r)^{q}\psi(r)dr\Big]\end{aligned}$$

$$\begin{aligned} &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t (t-s)^{q-1}\psi(s)\,ds + \frac{aC_2K_0}{q}\int_0^t (t-s)^{q-1}\psi(s)ds\Big] \\ &= \frac{MN}{\Gamma(q)}\Big(C+C_1+\frac{aC_2K_0}{q}\Big)\int_0^t (t-s)^{q-1}\psi(s)\,ds. \end{aligned}$$

So we obtain that $\psi(t) \equiv 0$ on J_1 by Lemma 2.7.

For $t \in J_2$, noting that $\underline{u}(t_1) = \overline{u}(t_1)$ and $I_1(\overline{u}(t_1)) = I_1(\underline{u}(t_1))$, using (3.5) and the same argument as above for $t \in J_1$, by (3.5) we can prove that

$$\begin{aligned} \psi(t) &\leq \frac{NM}{\Gamma(q)}\Big(C+C_1+\frac{aC_2K_0}{q}\Big)\int_{t_1}^t (t-s)^{q-1}\psi(s)ds \\ &\leq \frac{NM}{\Gamma(q)}\Big(C+C_1+\frac{aC_2K_0}{q}\Big)\int_0^t (t-s)^{q-1}\psi(s)ds, \qquad t \in J_2. \end{aligned}$$

Again by Lemma 2.7, we have $\psi(t) \equiv 0$ on J_2.

Continuing such a process interval by interval up to J_{m+1}, we obtain that $\psi(t) \equiv 0$ over the whole of J. Hence, $\underline{u}(t) \equiv \overline{u}(t)$ on J and $\widetilde{u} := \underline{u} = \overline{u}$ is a unique mild solution of IVP (1.1) in $[v_0,\, w_0]$, which can be obtained by the monotone iterative procedure (3.8) starting from v_0 or w_0.

This completes the proof of Theorem 3.2. □

Remark 3.1 *Since the condition (H4) can be more easily verified than (H3), the applications of Theorem 3.2 are convenient.*

4 Application

In order to illustrate the applicability of our main results, we consider the initial-boundary value problem of time fractional order parabolic partial differential equation with impulses and integral term

$$\begin{cases} \partial_t{}^q u - \nabla^2 u = g(t,\,x,\,u(t,x),\,Tu(t,x)), \quad (t,x) \in J\times\Omega, \quad t \neq t_k, \\ \Delta u|_{t=t_k} = c_k\, u(t_k,x), \quad x \in \Omega; \quad k = 1,\,2,\,\cdots,\,m, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = \varphi_0(x), \quad x \in \Omega, \end{cases} \tag{4.1}$$

where $\partial_t{}^q$ is the Caputo fractional order partial derivative of order q, $0 < q < 1$, ∇^2 is the Laplace operator, $J = [0,a]$, $a > 0$, $0 < t_1 < t_2 < \cdots < t_m < a$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $g \in C(J\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R})$ and satisfies the growth condition

$$|g(t,\,x,\,\xi,\,\eta)| \leq b_0 + b_1|\xi| + b_2|\eta|, \qquad (t,\,x,\,\xi,\,\eta) \in J\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}, \tag{4.2}$$

with positive constants $b_0,\ b_1,\ b_2$, and

$$Tu(t,x) := \int_0^t K(t,s)u(s,x)ds \tag{4.3}$$

is a Volterra integral operator with integral kernel $K \in C(\triangle, \mathbb{R}^+)$, $\triangle := \{(t,s) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq a\}$; $c_1,\ c_2,\ \cdots,\ c_m$ are positive constants, and the initial value function $\varphi_0 \in L^2(\Omega)$.

Let $E = L^2(\Omega)$, $P = \{u \in L^2(\Omega) \mid u(x) \geq 0,\ a.e.\ x \in \Omega\}$. Then E is a Banach space, P is a normal cone of E. We define an operator A in $L^2(\Omega)$ by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad Au = -\nabla^2 u, \tag{4.4}$$

From [26, Chapter 7, Theorem 3.2], we know that $-A$ generates a positive and analytic semigroup $S(t)\,(t \geq 0)$ in E. Let

$$f(t,\, v,\, w) := g(t,\, x,\, v(x),\, w(x)), \qquad t \in J, \quad v,\, w \in E. \tag{4.5}$$

Then by the condition (4.2), $f : J \times E \times E \to E$ is continuous. Let $I_k = c_k\,\mathrm{I}$, $k = 1,\, 2,\, \cdots,\, m$, where I is the identity operator in $L^2(\Omega)$. For the fuction $u : J \times \Omega \to \mathbb{R}$, let $u(t) = u(t,\cdot)$. Then the equation (4.1) be transformed into the following abstract form of IVP (1.1) in $L^2(\Omega)$

$$\begin{cases} {D_0}^q u(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J, \quad t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = \varphi_0. \end{cases} \tag{4.6}$$

Let λ_1 be the first eigenvalue of A. It is well known that $\lambda_1 > 0$ and it has a unique positive eigenfunction $\phi_1 \in C^2(\Omega) \cap C_0(\overline{\Omega})$ satisfied $\max_{x\in\overline{\Omega}} \phi_1(x) = 1$. Let

$$\mu(t) = 1 + \textstyle\sum_{t>t_k} c_k, \qquad t \in J, \tag{4.7}$$

then $\mu \in PC(J)$ and $\Delta\mu|_{t=t_k} = c_k$, $k = 1,\, 2,\, \cdots,\, m$. In order to solve the problem (4.1), we make the following assumptions:

(A1) $\varphi_0 \in L^2(\Omega)$ and $0 \leq \varphi_0(x) \leq \phi_1(x)$ for $a.e.\ x \in \Omega$.

(A2) $g(t,\, x,\, 0,\, 0) \geq 0$ and $g(t,\, x,\, \mu(t)\phi_1(x),\, \phi_1(x)\, T\mu(t)) \leq \lambda_1\, \mu(t)\, \phi_1(x)$ for every $(x,\, t) \in J \times \overline{\Omega}$.

(A3) In $J \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$, $g(t,\, x,\, \xi,\, \eta)$ has continuous partial derivative ${g_\xi}'(t,\, x,\, \xi,\, \eta)$ and ${g_\eta}'(t,\, x,\, \xi,\, \eta)$.

Theorem 4.1 *If the assumptions (A1)-(A3) are satisfied, then the equation (4.1) has a unique* L^2*-mild solution between* 0 *and* $\mu(t)\phi_1(x)$.

Proof. Consider IVP (4.6). From Definition 2.2 and the assumptions (A1) and (A2) we see that $v_0(t) \equiv 0$ is a lower solution of IVP (4.6) and $w_0(t) = \mu(t)\phi_1$ is an upper solution of IVP (4.6). From (A3) it is easy to verify that f satisfies the assumption (H1) and (H4). Clearly, for $I_k = c_k\mathrm{I}$, $k = 1, 2, \cdots, m$, (H2) holds. Therefore, by Theorem 3.2, IVP (4.6) has a unique mild solution between v_0 and w_0, that is, the equation (4.1) has a unique L^2-mild solution between 0 and $\mu(t)\phi_1(x)$. □

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EVP, MINIMAX THEOREMS AND EXISTENCE OF NONCONVEX EQUILIBRIA IN COMPLETE G-METRIC SPACES

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Abstract. We prove generalized EVP (Ekeland's variational principle) and generalized Takahashi's nonconvex minimization theorem by Ω-distance on G-metric spaces. As a result of last theorems, we get generalized flower petal theorem.

1. introduction

EVP, studied first one in 1972, has found a multitude of applications in different fields of analysis. It has also served to provide simple and elegant proofs of known results. The best references for those are by Ekeland himself: his survey article [2], his book with Aubin [1] and [4].

2. EVP

Definition 2.1. [3] Let $X \neq \varnothing$. The function $G : X \times X \times X \to [0, \infty)$ is said to be G-metric when

(i) $G(u, v, w) = 0$ if $u = v = w$ (coincidence),
(ii) $G(u, u, v) > 0$ for all $u, v \in X$, where $u \neq v$,
(iii) $G(u, u, w) \leq G(u, v, w)$ for all $u, v, w \in X$, with $w \neq v$,
(iv) $G(u, v, w) = G(P\{u, v, w\})$, where p is a permutation of u, v, w (symmetry),
(v) $G(u, v, w) \leq G(u, a, a) + G(a, v, w)$ for all $u, v, w, a \in X$ (rectangle inequality).

In this paper, $\varphi : (-\infty, \infty) \to (0, \infty)$ is a nondecreasing function. We say the function $h : X \to (-\infty, \infty]$ is lower semicontinuous from above (shortly lsca) at $w_0 \in X$ when for every sequence $\{w_n\}$ in X with $w_n \to w_0$ and $h(w_1) \geq h(w_2) \geq \cdots \geq h(w_n) \geq \cdots$, we have

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$h(w_0) \leq \lim_{n\to\infty} h(w_n)$. We say h is lsca on X when h is lsca at every point of X. We say h is proper if $h \neq \infty$.

Definition 2.2. [3] Let (X, G) be a G-metric space.

(1) A sequence $\{u_n\}$ in X is a G-Cauchy sequence when, for every $\varepsilon > 0$, there exists a positive integer n_0 such that if $m, n, l \geq n_0$, then $G(u_m, u_n, u_l) < \varepsilon$.

(2) A sequence $\{u_n\}$ in X is G-convergent to a point $u \in X$ when for every $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$ we have $G(u_m, u_n, u) < \varepsilon$.

Definition 2.3. [5] Let (X, G) be a G-metric space. A function $\Omega : X \times X \times X \to [0, \infty)$ is said to be an Ω-distance on X when

(a) $\Omega(u, v, w) \leq \Omega(u, a, a) + \Omega(a, v, w)$ for all $u, v, w \in X$;

(b) For any $u \in X$, $\Omega(u, ., .) : X \to [0, \infty)$ is lower semicontinuous;

(c) For each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(u, a, a) \leq \delta$ and $\Omega(a, v, w) \leq \delta$ imply $G(u, v, w) \leq \varepsilon$.

Example 2.4. [5] Let (X, d) be a metric space and $G : X^3 \to [0, \infty)$ defined by $G(u, v, w) = \max\{d(u, v), d(u, w), d(v, w)\}$ for all $u, v, w \in X$. Then $\Omega = G$ is an Ω-distance on X.

Lemma 2.5. [5] *Let (X, G) be a G-metric space and Ω an Ω-distance on X. Let also $\{u_n\}$, $\{v_n\}$ be sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ sequences in $[0, \infty)$ converging to zero and let $u, v, w, a \in X$. Then we have*

(1) *if $\Omega(v, u_n, u_n) \leq \alpha_n$ and $\Omega(u_n, v, w) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(v, v, w) < \varepsilon$ and hence $w = v$;*
(2) *if $\Omega(v_n, u_n, u_n) \leq \alpha_n$ and $\Omega(u_n, u_m, w) \leq \beta_n$ for any $m > n \in \mathbb{N}$, Then $G(v_n, v_m, w) \to 0$ and hence $v_n \to w$;*
(3) *if $\Omega(u_n, u_m, u_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then $\{u_n\}$ is a G-Cauchy sequence;*
(4) *if $\Omega(u_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{u_n\}$ is a G-Cauchy sequence.*

Lemma 2.6. *Let Ω be an Ω-distance on $X \times X \times X$. If $\{u_n\}$ is a sequence in X with $\limsup_{n\to\infty}\{\Omega(u_n, u_m, u_l) : n \leq m \leq l\} = 0$, then $\{u_n\}$ is a G-Cauchy sequence in X.*

Proof. Suppose $\alpha_n = \sup\{\Omega(u_n, u_m, u_l)\}$. We have $\lim_{n\to\infty} \alpha_n = 0$. By Lemma 2.5, we obtain that $\{u_n\}$ is a G-Cauchy sequence in X. □

Lemma 2.7. *Let $f : X \to (-\infty, \infty]$ be a function and Ω a Ω-distance on $X \times X \times X$. We define the set $P(u)$ by*

$$P(u) = \{v \in X : v \neq u,\ \Omega(u, v, v) \leq \varphi(f(u))(f(u) - f(v))\}.$$

If $v \in P(u)$, then we have $f(v) \leq f(u)$ and $P(v) \subseteq P(u)$.

Proof. Let $v \in P(u)$. Then $v \neq u$ and $\Omega(u, v, v) \leq \varphi(f(u))(f(u)-f(v))$. Since $\Omega(u, v, v) \geq 0$ for any $u, v \in X$ and φ is nondecreasing in $(0, \infty)$, we have $f(u) \geq f(v)$. If $P(v) = \emptyset$, then $P(v) \subseteq P(u)$. If $P(v) \neq \emptyset$, then let $w \in P(v)$. We have $w \neq v$ and $\Omega(v, w, w) \leq \varphi(f(v))(f(v) - f(w))$. Then, we have $f(v) \geq f(w)$. Also we have

$$\Omega(u, w, w) \leq \Omega(u, v, v) + \Omega(v, w, w) \leq \varphi\big(f(u)\big)\big(f(u) - f(w)\big).$$

We claim that $w \neq u$. Let $w = u$; then $\Omega(u, w, w) = 0$. On the other hand

$$\Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big) \leq \varphi\big(f(u)\big)\big(f(u) - f(w)\big) = 0.$$

Then $\Omega(u, v, v) = 0$; for each $\varepsilon > 0$, we have $\Omega(u, w, w) = 0 < \delta$, $\Omega(w, v, v) = 0 < \delta \Longrightarrow G(w, v, v) < \varepsilon$. Then $G(w, v, v) = 0$ and $w = v$, which is a contradiction. Therefore $w \in P(u)$ and hence $P(v) \subseteq P(u)$. □

Proposition 2.8. *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Let also Ω be an Ω-distance on $X \times X \times X$. For each $u \in X$, let*

$$P(u) = \big\{v \in X : v \neq u,\ \Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big)\big\}.$$

If $\{u_n\}$ is a sequence in X such that $P(u_n)$ is a nonempty set and $u_{n+1} \in P(u_n)$ for all $n \in \mathbb{N}$, then there exists $u_0 \in X$ such that $u_n \to u_0$ and $u_0 \in \bigcap_{n=1}^{\infty} P(u_n)$.

Also, if $f(u_{n+1}) \leq \inf_{w \in P(u_n)} f(w) + 1/n$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} P(u_n)$ has only one point.

Proof. First we show that $\{u_n\}$ is a Cauchy sequence. Whereas $u_{n+1} \in P(u_n)$, by Lemma 2.7, $f(u_n) \geq f(u_{n+1})$ for all $n \in \mathbb{N}$, so $\{f(u_n)\}$ is nonincreasing. On the other hand, f is bounded from below; then $r = \lim_{n\to\infty} f(u_n)$, so $f(u_n) \geq r$ for all $n \in \mathbb{N}$.

We show that $\limsup_{n\to\infty}\{\Omega(u_n, u_m, u_m) : m > n\} = 0$. We have

$$\begin{aligned}\Omega(u_n, u_m, u_m) &\leq \Omega(u_n, u_{n+1}, u_{n+1}) + \Omega(u_{n+1}, u_m, u_m)\\ &\leq \Omega(u_n, u_{n+1}, u_{n+1}) + \Omega(u_{n+1}, u_{n+2}, u_{n+2}) + \Omega(u_{n+2}, u_m, u_m)\end{aligned}$$

Therefore $\Omega(u_n, u_m, u_m) \le \sum_{j=n}^{m-1} \Omega(u_j, u_{j+1}, u_{j+1}) \le \varphi\big(f(u_1)\big)\big(f(u_n) - r\big)$, for all $m, n \in \mathbb{N}$ with $m > n$. Put $\alpha_n = \varphi\big(f(u_1)\big)\big(f(u_n) - r\big)$ for all $n \in \mathbb{N}$. We have $\sup\{\Omega(u_n, u_m, u_m) : m > n\} \le \alpha_n$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} f(u_n) = r$. We have $\lim_{n\to\infty} \alpha_n = 0$ and

$$\lim_{n\to\infty} \sup\{\Omega(u_n, u_m, u_m) : m > n\} = 0.$$

By Lemma 2.6, $\{u_n\}$ is a G-Cauchy sequence, and X is a G-complete metric space, so there exists $u_0 \in X$ such that $u_n \to u_0$. We show that $u_0 \in \bigcap_{n=1}^{\infty} P(u_n)$. Since f is lsca, then $f(u_0) \le \lim_{n\to\infty} f(u_n) = r \le f(u_k)$. Suppose that $n \in \mathbb{N}$ is fixed for all $m \in \mathbb{N}$ with $m > n$. We have $\Omega(u_n, u_m, u_m) \le \sum_{j=n}^{m-1} \Omega(u_j, u_{j+1}, u_{j+1}) \le \varphi\big(f(u_n)\big)\big((f(u_n) - f(u_0)\big)$. Since $\Omega(u, ., .) : X \to (0, \infty)$ is lower semi continuous, then

$$\Omega(u_n, u_0, u_0) \le \varphi\big(f(u_n)\big)\big(f(u_n) - f(u_0)\big). \tag{2.1}$$

Also $u_0 \ne u_n$ for all $n \in \mathbb{N}$. Suppose contrary, that there exists $j \in \mathbb{N}$ such that $u_0 = u_j$. Since $\Omega(u_j, u_{j+1}, u_{j+1}) \le \varphi\big(f(u_j)\big)\big(f(u_j) - f(u_{j+1})\big) \le \varphi\big(f(u_j)\big)\big(f(u_j) - f(u_0)\big) = 0$, so we would have $\Omega(u_j, u_{j+1}, u_{j+1}) = 0$. Similarly, we would have $\Omega(u_{j+1}, u_{j+2}, u_{j+2}) = 0$. Now, for $\varepsilon > 0$, we would have $\Omega(u_j, u_{j+1}, u_{j+1}) = 0 < \delta$ and $\Omega(u_{j+1}, u_{j+2}, u_{j+2}) = 0 < \delta$. Then $G(u_j, u_{j+2}, u_{j+2}) < \varepsilon$, and by Definition 2.2, we would have $u_j = u_{j+2}$, which is contradiction.

Since $u_{j+1} \in P(u_j)$, then $P(u_{j+1}) \subseteq P(u_j)$ and $u_{j+2} \in P(u_{j+1})$, so $u_{j+2} \in P(u_j)$ and therefore $u_{j+2} \ne u_j$. We conclude that $u_0 \ne u_n$ for all $n \in \mathbb{N}$. By (2.1) we have $u_0 \in \bigcap_{n=1}^{\infty} P(u_n)$, thus $\bigcap_{n=1}^{\infty} P(u_n) \ne \emptyset$. Now we assume that $f(u_{n+1}) \le \inf_{w\in P(u_n)} f(w) + 1/n$ for all $n \in \mathbb{N}$. We show that $\bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$. Let $t \in \bigcap_{n=1}^{\infty} P(u_n)$; then

$$\begin{aligned}\Omega(u_n, t, t) &\le \varphi\big(f(u_n)\big)\big(f(u_n) - f(t)\big)\\ &\le \varphi\big(f(u_1)\big)\big(f(u_n) - \inf_{w\in S(u_n)} f(w)\big)\\ &\le \varphi\big(f(u_1)\big)\big(f(u_n) - f(u_{n+1}) + 1/n).\end{aligned}$$

Let $\beta_n = \varphi\big(f(u_1)\big)\big(f(u_n) - f(u_{n+1}) + 1/n)$, for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \beta_n = 0$, thus $\lim_{n\to\infty} \Omega(u_n, t, t) = 0$; also $\{u_m\}$ is G-Cauchy. Then $\lim_{n\to\infty} \Omega(u_m, u_m, u_n) = 0$, and we obtain $u_n \to t$. By uniqueness, we have $t = u_0$. Then $\bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$. □

Theorem 2.9 (Generalized EVP). *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Let also Ω be an Ω-distance on $X \times X \times X$. Then there exists $t \in X$ such that*

$$\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big) \text{ for all } u \in X \text{ with } u \ne t.$$

Proof. Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that $\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u)-f(v)\big)$. That would mean that $P(u) \neq \emptyset$ for each $u \in X$. Since f is proper, there would exist $u \in X$ such that $f(u) \neq \infty$. We define a sequence $\{u_n\}$ as follows: let $u_1 = x$, and choose $u_2 \in P(u_1)$ such that $f(u_2) \leq \inf_{u \in P(u_1)} f(u) + 1$. Suppose $u_n \in X$ is so defined, and choose $u_{n+1} \in P(u_n)$ such that $f(u_{n+1}) \leq \inf_{u \in P(u_n)} f(u) + 1/n$. By Proposition 2.8, there would exist $u_0 \in X$ such that $\bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$. By Lemma 2.7, $P(u_0) \subseteq \bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$, so $P(u_0) = \{u_0\}$, which is a contradiction. Therefore, there exists $t \in X$ such that

$$\Omega(t,u,u) > \varphi\big(f(t)\big)\big(f(t)-f(u)\big) \text{ for all } u \in X \text{ with } u \neq t. \qquad \square$$

Theorem 2.10 (Generalized Caristi's common fixed point theorem for a family of multivalued maps)**.** *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Let also Ω be an Ω-distance on $X \times X \times X$. Let J be any index set and for each $j \in J$, suppose $P_j : X \to 2^X$ is a multivalued map with nonempty values such that for each $u \in X$, there exists $v = v(u,j) \in P_j(u)$ with*

$$\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u)-f(v)\big). \tag{2.2}$$

Then there exists $t \in X$ such that $t \in \bigcap_{j \in J} P_j(t)$, and $\Omega(t,t,t) = 0$.

Proof. By Theorem 2.9, there exists $t \in X$ such that $\Omega(t,u,u) > \varphi\big(f(t)\big)\big(f(t)-f(u)\big)$ for all $u \in X$ with $u \neq t$. Now we show that $t \in \bigcap_{j \in J} P_j(t)$ and $\Omega(t,t,t) = 0$. According to the assumption, there exists $w(t,j) \in P_j(t)$ such that $\Omega(t,w,w) \leq \varphi\big(f(t)\big)\big(f(t)-f(w(t,j))\big)$. We claim that $w(t,j) = t$, for all $j \in J$. If, on the contrary, $w(t,j_0) \neq t$ for some $j_0 \in J$, then

$$\Omega(t,w,w) \leq \varphi\big(f(t)\big)\big(f(t)-f(w)\big) < \Omega(t,w,w).$$

which is a contradiction. Therefore $t = w(t,j) \in P_j(t)$ for all $j \in P$.

Since $\Omega(t,t,t) \leq \varphi\big(f(t)\big)\big(f(t)-f(t)\big) = 0$, we obtain $\Omega(t,t,t) = 0$. $\square$

Corollary 2.11 (Generalized Caristi's common fixed point theorem for a family of single-valued maps)**.** *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Let also Ω be an Ω-distance on $X \times X \times X$. Let J be any index set and for each $j \in J$, let $g_j : X \to X$ be a single-valued map so that*

$$\Omega\big(u, g_j(u), g_j(u)\big) \leq \varphi\big(f(u)\big)\big(f(u)-f(g_j(u))\big) \tag{2.3}$$

is established for each $u \in X$. Then there exists $t \in X$ such that $g_j(t) = t$ for each $j \in J$ and $\Omega(t,t,t) = 0$.

Proof. Let $P_j : X \to X$ and $P_j(x) = \{g_j(u)\}$, for all $u \in X$ and $j \in J$. Then by Theorem 2.10, $g_j(t) = t$ for each $j \in J$ and $\Omega(t,t,t) = 0$. □

Remark 2.12. (a) Corollary 2.11 implies Theorem 2.10.

Suppose that for each $u \in X$, there exists $v(u,j) \in P_j(u)$ such that

$$\Omega\big(u, v(u,j), v(u,j))\big) \le \varphi\big(f(u)\big)\big(f(u) - f(v(u,j))\big)$$

for each $j \in J$, and let $g_j(u) = v(u,j)$. Then g_j is single-valued map and

$$\Omega\big(u, g_j(u), g_j(u))\big) \le \varphi\big(f(u)\big)\big(f(u) - f(g_j(u))\big)$$

for all $u \in X$. By Corollary 2.11, there exists $t \in X$ such that $t = g_j(t) \in P_j(t)$ for each $j \in J$ and $\Omega(t,t,t) = 0$

(b) Theorem 2.10 implies Theorem 2.9.

Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that

$$\Omega(u,v,v) \le \varphi\big(f(u)\big)\big(f(u) - f(v)\big).$$

Define $P : X \to 2^X \setminus \{\emptyset\}$ by

$$P(u) = \{v \in X : v \neq u, \Omega(u,v,v) \le \varphi\big(f(u)\big)\big(f(u) - f(v)\big)\}.$$

By Theorem 2.10, P has a fixed point $t \in X$; this means $t \in P(t)$. This is a contradiction, because $t \notin P(t)$.

Theorem 2.13 (Nonconvex maximal element theorem for a family of multivalued maps)**.** *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Let also Ω be an Ω-distance on $X \times X \times X$ and J be any index set. For each $j \in J$, let $P_j : X \to 2^X$ be a multivalued map. Suppose that for each $(u,j) \in X \times J$ with $P_j(u) \neq \emptyset$, there exists $v = v(u,j) \in X$ with $v \neq u$ such that (2.2) holds. Then there exists $t \in X$ such that $P_j(t) = \emptyset$ for each $j \in J$.*

Proof. By Theorem 2.9, there exists $t \in X$, such that $\Omega(t,u,u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ for all $u \in X$ with $u \neq t$. We prove that $P_j(t) = \emptyset$ for each $j \in J$. Indeed, if $P_{j_0}(t) \neq \emptyset$, for some $j_0 \in J$, according to the assumption, there would exist $w = w(t,j_0) \in X$ with $w \neq t$ such that $\Omega(t,w,w) \le \varphi\big(f(t)\big)\big(f(t) - f(w)\big)$. Also $\Omega(t,w,w) > \varphi\big(f(t)\big)\big(f(t) - f(w)\big)$, which is a contradiction. □

Remark 2.14. Theorem 2.13 implies Theorem 2.9.

Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that

$$\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big).$$

For each $u \in X$, we define

$$P(u) = \{v \in X : v \neq u,\ \Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big)\}.$$

Then $P(u) \neq \emptyset$ for all $u \in X$. But by Theorem 2.13, there would exist $t \in X$ such that $P(t) = \emptyset$, which is a contradiction.

3. Nonconvex optimization and minimax theorems

Theorem 3.1 (Generalized Takahashi's nonconvex minimization theorem)**.** *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Also, let Ω be an Ω-distance on $X \times X \times X$. Suppose that for any $u \in X$ with $f(u) > \inf_{w \in X} f(w)$ there exists $v \in X$ with $v \neq u$ such that* (2.2) *holds. Then there exists $t \in X$ such that $f(t) = \inf_{w \in X} f(w)$.*

Proof. By Theorem 2.9, there exists $t \in X$ such that $\Omega(t,u,u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$, for all $u \in X$, $u \neq t$. Now we prove that $f(t) = \inf_{w \in X} f(w)$. On the contrary, let $f(t) > \inf_{w \in X} f(w)$. According to the assumption, there would exist $v = v(t) \in X$, with $v \neq t$ such that $\Omega(t,v,v) \leq \varphi\big(f(t)\big)\big(f(t) - f(v)\big)$. Then we would have

$$\Omega(t,v,v) \leq \varphi\big(f(t)\big)\big(f(t) - f(v)\big) < \Omega(t,v,v)$$

which is a contradiction. □

Remark 3.2. Using Theorem 3.1, we can infer Theorem 2.9.

If we could not, then for each $u \in X$, there would exist $v \in X$ with $v \neq u$ such that $\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big)$. By Theorem 3.1, there would exist $t \in X$ such that $f(t) = \inf_{w \in X} f(w)$. According to the assumption, there would exist $z \in X$ with $z \neq u$, such that $\Omega(t,z,z) \leq \varphi\big(f(t)\big)\big(f(t) - f(z)\big) \leq 0$. Then $\Omega(t,z,z) = 0$ and $f(t) = f(z) = \inf_{w \in X} f(w)$. There would exist $w \in X$ with $w \neq z$ such that $\Omega(z,w,w) \leq \varphi\big(f(z)\big)\big(f(z) - f(w)\big) \leq 0$. Then we would have $\Omega(z,w,w) = 0$ and $f(t) = f(z) = f(w) = \inf_{u \in X} f(u)$. Since $\Omega(t,w,w) \leq \Omega(t,z,z) + \Omega(z,w,w)$, then $\Omega(t,w,w) = 0$. For $\varepsilon > 0$ we would have $\Omega(t,z,z) = 0 < \delta$, $\Omega(z,w,w) = 0 < \delta$; then $G(t,w,w) < \varepsilon$, that is, $t = w$. Also for $\varepsilon > 0$ we would have $\Omega(z,t,t) = 0 < \delta$, $\Omega(t,w,w) = 0 < \delta$; then $G(z,w,w) < \varepsilon$ that is, $z = w$, which is a contradiction.

Theorem 3.3 (Nonconvex minimax theorem). *Let $G : X \times X \to (-\infty, \infty]$ be a proper lsca and bounded from below function in the first argument. Suppose that for each $u \in X$ with $\{x \in X : G(u,x) > \inf_{a\in X} G(a,x)\} \neq \emptyset$, there exists $v = v(u) \in X$ with $v \neq u$ such that*

$$\Omega(u,v,v) \leq \varphi\big(G(u,t)\big)\big(G(u,t) - G(v,t)\big) \tag{3.1}$$

for all $t \in \{x \in X : G(u,x) > \inf_{a\in X} G(a,x)\}$.

Then $\inf_{u\in X} \sup_{v\in X} G(u,v) = \sup_{v\in X} \inf_{u\in X} G(u,v)$.

Proof. By Theorem 3.1, for every $v \in X$, there exists $u(v) \in X$ such that $G\big(u(v),v\big) = \inf_{u\in X} G(u,v)$. Then, $\sup_{v\in X} G\big(u(v),v\big) = \sup_{v\in X} \inf_{u\in X} G(u,v)$.

Replacing $u(v)$ by an arbitrary $u \in X$, we obtain

$$\inf_{u\in X} \sup_{v\in X} G(u,v) = \sup_{v\in X} \inf_{u\in X} G(u,v). \qquad \square$$

Theorem 3.4 (Nonconvex equilibrium theorem). *Let G and φ be the same as in Theorem 3.3. Let for each $u \in X$ with $\{x \in X : G(u,x) < 0\} \neq \emptyset$, there exist $v = v(u) \in X$ with $v \neq u$ such that (3.1) holds for all $t \in X$. Then there exists $y \in X$ such that $G(y,v) \geqslant 0$ for all $v \in X$.*

Proof. By Theorem 2.9, for each $w \in X$, there exists $y(w) \in X$ such that $\Omega(y(w),u,u) > \varphi\big(G(y(w),w)\big)\big(G(y(w),w) - G(u,w)\big)$ for all $u \in X$ with $u \neq y(w)$. We show that there exists $y \in X$ such that $G(y,v) \geq 0$ for all $v \in X$. Suppose contrary, that for each $u \in X$ there exists $v \in X$ such that $G(u,v) < 0$. Then for each $u \in X$, $\{x \in X : G(u,x) < 0\} \neq \emptyset$. According to the assumption, there would exist $v = v(y(w))$, $y \neq y(w)$ such that $\Omega(y(w),v,v) \leq \varphi\big(G(y(w),w)\big)\big(G(y(w),w) - G(v,w)\big)$, which is a contradiction. $\square$

Example 3.5. Let $X = [0,1]$ and $G(u,v,w) = \max\{|u-v|, |u-w|, |v-w|\}$. Then (X,G) is a complete G-metric space. Suppose that a,b are positive real numbers with $a \geqslant b$. Let $H : X \times X \to R$ with $H(u,v) = au - bv$. Therefore, the function $u \mapsto H(u,v)$ is proper, lower semicontinuous and bounded from below, and $H(1,v) \geq 0$ for every $v \in X$. Also $H(u,v) \geq 0$ for every $u \in [\frac{b}{a}, 1]$ and every $v \in X$. In fact, for every $u \in [0, \frac{b}{a}]$, $H(u,v) = au - bv < 0$ when $v \in [\frac{a}{b}u, 1]$. Then set $\{x \in X : H(u,x) < 0\} \neq \emptyset$ for every $u \in [0, \frac{b}{a}]$. Let $u, v \in X$, $u \geq v$; we have $u - v = \frac{1}{a}\{(au - bx) - (av - bx)\}$, for every $x \in X$. Define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \frac{1}{a}$. Then $G(u,v,v) \leq \varphi\big(H(u,x)\big)\big(H(u,x) - H(v,x)\big)$, for every $u \geqslant v$, and $u,v,x \in X$. By Theorem 3.4, there exists $y \in X$ such that $H(y,v) \geqslant 0$ for every $v \in X$.

Theorem 3.6. *Let Ω, φ be the same as in Theorem 2.9. For each $j \in J$, let $P_j : X \to X$ be multivalued maps with nonempty values, $g_j, h_j : X \times X \to \mathbb{R}$ be functions and $\{a_j\}$ and $\{b_j\}$ families of real numbers. Suppose that:*

(i) *For each $(u, j) \in X \times J$, there exists $v = v(u, j) \in P_j(u)$ such that $g_j(u, v) \geq a_j$ and $\Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big)$;*
(ii) *For each $(x, j) \in X \times J$, there exists $w = w(x, j) \in P_j(x)$ such that $h_j(x, w) \leq bj$ and $\Omega(x, w, w) \leq \varphi\big(f(x)\big)\big(f(x) - f(w)\big)$.*

Then there exists $u_0 \in P_j(u_o)$ such that $g_j(u_0, u_0) \geq a_j$ and $h_j(u_0, u_0) \leq b_j$ for all $j \in J$ and $\Omega(u_0, u_0, u_0) = 0$.

Proof. By Theorem 2.9, there exists $t \in X$ such that $\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$, for all $u \in X$ with $u \neq t$. For each $j \in J$, by (i) there exists $w_1 = w_1(t, j) \in P_j(t)$ such that $g_j(t, w_1) \geq a_j$ and $\Omega(t, , w_1, w_1) \leq \varphi\big(f(t)\big)\big(f(t) - f(w_1)\big)$. Also according to (ii), there exists $w_2 = w_2(t, j) \in P_j(t)$ such that $h_j(t, w_2) \leq bj$ and $\Omega(t, , w_2, w_2) \leq \varphi\big(f(t)\big)\big(f(t) - f(w_2)\big)$. If $w_1 \neq t$, then $\Omega(t, , w_1, w_1) \leq \varphi\big(f(t)\big)\big(f(t) - f(w_1)\big) < \Omega(t, w_1, w_1)$, which is a contradiction. Therefore $w_1 = t$. Similarly, we have $w_2 = t$. Since $\Omega(t, , t, t) \leq \varphi\big(f(t)\big)\big(f(t) - f(t)\big) = 0$, hence $\Omega(t, t, t) = 0$. □

Remark 3.7. (a) In Theorem 3.6, put $g_j = h_j = F_j$ and $a_j = b_j = c_j$; then there exists $u_0 \in P_j(u_0)$ such that $F_j(u_0, u_0) = c_j$ for all $j \in J$ and $\Omega(u_0, u_0, u_0) = 0$.

(b) In (a), put $P_j(u) = X$ for all $u \in X$; then there exists $u_0 \in X$ such that $F_j(u_0, u_0) = cj$ for all $j \in J$ and $\Omega(u_0, u_0, u_0) = 0$.

Remark 3.8. From Theorem 3.5, we can infer Theorem 2.9.

Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that

$$\Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big).$$

Define $P : X \to X \setminus \{\emptyset\}$ by $P(u) = \{v \in X : v \neq u\}$ and a function $F : X \times X \to \mathbb{R}$ by $F(u, v) = \chi_{P(u)}(v)$, where χ_A is the characteristic function for an arbitrary set A. We would have $v \in P(u) \iff F(u, v) = 1$. Then for each $u \in X$, there would exist $v \in X$ such that $F(u, v) = 1$ and $\Omega(t, , u, u) \leq \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$. According to Remark 3.7(a) with $c = 1$, there would exist $u_0 \in X$ such that $F(u_0, u_0) = 1$ and $\Omega(u_0, u_0, u_0) = 0$. Then $u_0 \in P(u_0)$. This is a contradiction.

4. Applications

Let (X, G) be a G-metric space and $a, b \in X$. Suppose that $\kappa : X \to (0, \infty)$ is a function and Ω a Ω-distance on X. Define

$$\Omega_\varepsilon(a, b, \kappa) = \{u \in X : \varepsilon\Omega(a, u, u) \leq \kappa(a)\big(\Omega(b, a, a) - \Omega(b, u, u)\big)\}$$

such that $\varepsilon \in (0, \infty)$ and $a, b \in X$.

Lemma 4.1. *Let $\Omega, f,$and φ be the same as in Theorem 2.9. Let $\varepsilon > 0$ and Ω be an Ω-distance on X. Suppose that there exists $x \in X$ such that $f(x) < \infty$ and $\Omega(x, x, x) = 0$. Then there exists $t \in X$ such that*

(i) $\varepsilon\Omega(x, t, t) \leq \varphi\big(f(x)\big)\big(f(x) - f(t)\big)$;
(ii) $\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ *for all $u \in X$ with $u \neq t$.*

Proof. Let $x \in X$, $f(x) < +\infty$ and $\Omega(x, x, x) = 0$. Put

$$V = \{u \in X : \varepsilon\Omega(x, u, u) \leq \varphi\big(f(x)\big)\big(f(x) - f(u)\big)\}.$$

The space (V, G) is a nonempty complete G-metric space. By Theorem 2.9, there exists $t \in V$ such that $\varepsilon\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ for all $u \in V$ with $u \neq t$. For any $u \in X \setminus V$, since

$$\begin{aligned}\varepsilon[\Omega(x, t, t) + \Omega(t, u, u)] \geq \varepsilon\Omega(x, u, u) &> \varphi\big(f(x)\big)\big(f(x) - f(u)\big)\\ &\geq \varepsilon\Omega(x, t, t) + \varphi\big(f(t)\big)\big(f(t) - f(u)\big),\end{aligned}$$

we have $\varepsilon\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ for all $u \in X \setminus V$. Then $\varepsilon\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(t)\big)$ for all $u \in X$ with $u \neq t$. □

Theorem 4.2 (Generalized flower petal theorem)**.** *Suppose that N is a proper complete subset of a G-metric space X and $a \in N$. Let Ω be an Ω-distance on X with $\Omega(a, a, a) = 0$. Let $b \in X \setminus N$, $\Omega(b, N, N) = \inf_{u\in N} \Omega(b, u, u) \geq r$, $\Omega(b, a, a) = s > 0$, and let there exist a function $\kappa : X \to (0, \infty)$ satisfying $\kappa(u) = \varphi\big(\Omega(b, u, u)\big)$ for some nondecreasing function $\varphi : (-\infty, \infty] \to (0, \infty)$. Then for each $\varepsilon > 0$, there exists $t \in N \cap \Omega_\varepsilon(a, b, \kappa)$ such that $\Omega_\varepsilon(t, b, \kappa) \cap (N \setminus \{t\}) = \emptyset$ and $\Omega(a, t, t) \leq \varepsilon^{-1}\kappa(a)(s - r)$.*

Proof. The space (N, G) is a complete G-metric space. Consider the function $f : N \to (-\infty, \infty]$ defined by $f(u) = \Omega(b, u, u)$. Since $f(a) = \Omega(b, a, a) = s < \infty$ and $\Omega(b, N, N) = \inf_{u\in N} \Omega(b, u, u) \geq r$, f is a proper lower semicontinuous and bounded from below function. By Lemma 4.2, there exists $t \in N$ such that

(i) $\varepsilon\Omega(a,t,t) \le \kappa(a)\big(f(a)-f(t)\big)$;

(ii) $\varepsilon\Omega(t,u,u) > \kappa(t)\big(f(t)-f(t)\big)$ for all $u \in N$ with $u \ne t$.

Applying (i), we get $t \in N \bigcap \Omega_\varepsilon(a,b,\kappa)$. Applying (i) again, we get $\Omega(a,t,t) \le \varepsilon^{-1}\kappa(a)\big(\Omega(b,a,a) - \Omega(b,t,t)\big) \le \varepsilon^{-1}\kappa(a)(s-r)$. By (ii), we obtain $\varepsilon\Omega(t,u,u) > \kappa(t)\big(\Omega(b,t,t) - \Omega(b,u,u)\big)$ for all $u \in N$ with $u \ne t$. Therefore $u \notin \Omega_\varepsilon(t,b,\kappa)$ for all $u \in N \setminus \{t\}$ or $\Omega_\varepsilon(t,b,\kappa) \cap (N \setminus \{t\}) = \emptyset$. □

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GEOMETRIC PROPERTIES OF BESSEL FUNCTIONS FOR THE CLASSES OF JANOWSKI STARLIKE AND CONVEX FUNCTIONS

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Abstract. Applications of Bessel differential equations have attracted the univalent function theorists in recent years. In the present investigation, we establish certain sufficient conditions for Bessel function to be in the class of Janowski starlike and Janowski convex functions. Further, certain sufficient condition for an integral operator defined using Bessel function to be in the class of Janowski starlike and Janowski convex functions are determined.

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1. **Introduction**

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that are starlike in $\mathbb{U}$ with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha).$$

There has been a continuous interest shown on the Geometric and other related properties of Bessel functions (like hypergeometric functions) after many papers have been published by Baricz [2](see also the other works of Baricz) in recent times. One such problem of Baricz [3] was to find conditions on the triplet p, b and c such that the function $u_{p,b,c}$ is starlike and convex of order α. In earlier investigations, finding conditions on the parameters for which the Gaussian hypergeometric function belong to the various classes of functions have been discussed in detail by Shanmugam [20], Sivasubramanian *et al.* [21] and Sivasubramanian and Sokol [22] (See also [6, 7, 10, 11, 12, 13, 16, 17]).

Let us consider the following second-order linear homogenous differential equation (see, for details, [3]):

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = 0 \quad (b, c, p \in \mathbb{C}). \tag{1.2}$$

The function $\omega_{p,b,c}(z)$, which is called the generalized Bessel function of the first kind of order p, is defined as a particular solution of (1.2). Further, the function $\omega_{p,b,c}(z)$ has the familiar representation

$$\omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma\left(p + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}), \tag{1.3}$$

where Γ stands for the Euler gamma function. The series (1.3) permits the study of Bessel, modified Bessel and spherical Bessel functions all together. Solutions of (1.2) are referred as the generalized Bessel function of order p. The particular solution given by (1.3) is called the generalized Bessel function of the first kind order of p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. By ratio test, the radius of convergence for the series in (1.3) is infinity and hence $\omega_{p,b,c}(z)$ converges everywhere for all $b, c, p \in \mathbb{C}$ and for all $z \in \mathbb{U}$.

It is worth mentioning that, in particular, for $b = c = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order p defined by

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.4}$$

Further, for the choices $c = 1$ and $b = 2$ in (1.3), we obtain the familiar spherical Bessel function of the first kind of order p defined by

$$S_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+3/2)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.5}$$

For the choices of $b = 1$ and $c = -1$ in (1.3), we obtain the modified Bessel function of the first kind of order p defined by

$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.6}$$

From (1.3), it is clear that $\omega(0) = 0$. Therefore, it follows from (1.3) that

$$\omega_{p,b,c}(z) = \left[2^p\Gamma\left(p + \frac{b+1}{2}\right)\right]^{-1} z^p \sum_{n=0}^{\infty} \frac{(-c/4)^n \ \Gamma\left(p+(b+1)/2\right)}{n!\Gamma\left(p+n+(b+1)/2\right)} z^{2n} \quad (z \in \mathbb{C}). \tag{1.7}$$

Let us set

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = \frac{(-c/4)^n \ \Gamma\left(p+(b+1)/2\right)}{n!\Gamma\left(p+n+(b+1)/2\right)}.$$

Hence, (1.7) becomes

$$\omega_{p,b,c}(z) = \left[2^p\Gamma\left(p + \frac{b+1}{2}\right)\right]^{-1} z^p u_{p,b,c}(z^2). \tag{1.8}$$

By using the well-known Pochhammer symbol (or the shifted factorial) $(\lambda)_\mu$ defined, for $\lambda, \mu \in \mathbb{C}$ and in terms of the Euler Γ function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

In view of the fact that $(0)_0 := 1$, the series representation for the function $u_{p,b,c}$ is given by

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n z^n}{(\kappa)_n (n)!} \quad \left(\kappa := p + (b+1)/2 \notin \mathbb{Z}_0^-\right) \tag{1.9}$$

and therefore,

$$z u_{p,b,c}(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1} z^n}{(\kappa)_{n-1}(n-1)!} \quad \left(\kappa := p + (b+1)/2 \notin \mathbb{Z}_0^-\right) \tag{1.10}$$

where $\mathbb{N} := \{1, 2, ...\}$ and $\mathbb{Z}_0^- := \{0, -1, -2, ...\}$. The function $u_{p,b,c}$ is called the generalized and normalized Bessel function of the first kind of order p. We note that by the ratio test, the radius of convergence of the series $u_{p,b,c}$ is infinity. Moreover, the function $u_{p,b,c}$ is analytic in $\mathbb{C}$ and satisfies the differential equation $4z^2u''(z)+4\kappa zu'(z)+czu(z)=0$. Also, if $b,p,c\in\mathbb{C}$ and $\kappa\notin\mathbb{Z}_0^-$, then the function $u_{p,b,c}$ satisfies the recursive relation

$$4ku'_{p,b,c}(z)=-cu_{p+1,b,c}(z) \quad (z\in\mathbb{C}). \tag{1.11}$$

Further, for $z=1$, we denote $u_{p,b,c}(z)$ simply by $u_p(1)$. For $f\in\mathcal{A}$, we define the operator $I_{p,b,c}f(z)$ by

$$I_{p,b,c}\ f(z)=zu_{p,b,c}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}}\frac{a_n\ z^n}{(n-1)!}, \tag{1.12}$$

where $\kappa=p+(b+1)/2\notin\mathbb{Z}_0^-$. In fact, the function $I_{p,b,c}f(z)$ given by (1.12) is an elementary transform of the generalized hypergeometric function. Thus, it is easy to see that

$$I_{p,b,c}f(z)=z{}_0F_1(\kappa;-c/4z)*f(z).$$

For the special choices of $b=c=1$ in (1.12), $I_{p,b,c}f$ reduces to $J_p:\mathcal{A}\to\mathcal{A}$ related with Bessel function, defined by

$$J_p\ f(z)=zu_{p,1,1}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(-1/4)^{n-1}a_n}{(p+1)_{n-1}}\frac{z^n}{(n-1)!}. \tag{1.13}$$

For the special choices of $b=1$ and $c=-1$ in (1.12), $I_{p,b,c}f$ reduces to $M_p:\mathcal{A}\to\mathcal{A}$ related with modified Bessel function, defined by

$$M_pf(z)=zu_{p,1,-1}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{a_n}{(4)^{n-1}(p+1)_{n-1}}\frac{z^n}{(n-1)!} \tag{1.14}$$

where $*$ denotes the usual Hadamard product or convolution of power series.

If f and g are analytic in $\mathbb{U}$, then we say that the function f is *subordinate* to g, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ $(z\in\mathbb{U})$, such that $f(z)=g(w(z))$ $(z\in\mathbb{U})$. We denote this subordination by $f\prec g$ or $f(z)\prec g(z)$ $(z\in\mathbb{U})$.

For $-1\le F<E\le 1$, let

$$\mathcal{S}^*[E,F]=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\frac{1+Ez}{1+Fz}\quad(z\in\mathbb{U})\right\}$$

and

$$\mathcal{K}[E,F]=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\frac{1+Ez}{1+Fz}\quad(z\in\mathbb{U})\right\}.$$

It is fairly straightforward to see that $\mathcal{S}^*[1,-1]$ is the familiar class of starlike functions $\mathcal{S}^*$, $\mathcal{S}^*[1-2\gamma,-1]$ $(0\le\gamma<1)$ is the class of starlike functions of order γ and also the class $\mathcal{S}^*[\lambda,0]$ is denoted by $\mathcal{S}^*_\lambda$. Further, $\mathcal{K}[1,-1]$ is the familiar class of convex functions $\mathcal{K}$, $\mathcal{K}[1-2\gamma,-1]$ $(0\le\gamma<1)$ is the class of convex functions of order γ and also the class $\mathcal{K}[\lambda,0]$ is denoted by $\mathcal{K}_\lambda$. These two classes have been investigated in several works, for example, see [18, 19].

The connection between the Janowski starlike, Janowski convex functions and the Bessel functions is not considered so far. In the present paper, we obtain mapping properties between various subclasses of $\mathcal{S}$ motivated by the works of Anbudurai and Parvatham [1] (see also [5, 10, 11, 12, 16, 17, 18, 24]).

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2. Sufficient conditions for Bessel functions to be in $\mathcal{S}^*[E,F]$ and $\mathcal{K}[E,F]$ involving Jack's Lemma

In the present section, we determine certain sufficient conditions involving Jack's Lemma for $u_{p,b,c}$ and $z\,u_{p,b,c}$ to be in the class of Janowski starlike and Janowski convex functions. To prove the main theorems we need the following lemma.

Lemma 2.1. [9] *Let ω be regular in the unit disk $\mathbb{U}$ with $\omega(0)=0$. If $|\omega(z)|$ attains a maximum value on the circle $|z|=r\,(0\leq r<1)$ at a point z, then $z_1\omega'(z_1)=m\omega(z_1)$ where m is real and $m\geq 1$.*

Lemma 2.2. [18] *Let a function f of the form (1.1) satisfy*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\alpha}\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}<\frac{(E-F)(2+E+E^2)^{\alpha}}{(1+|F|)(1+E)^{2\alpha}} \tag{2.1}$$

for fixed constants E, F and α such that $-1\leq F<E\leq 1$, $\alpha\geq 0$ and $z\in\mathbb{U}$. Then $f\in\mathcal{S}^*[E,F]$.

Theorem 2.1. *Let $f\in\mathcal{A}$. If*

$$\left|(I_{p,b,c}f(z))'-1\right|^{1-\beta}\left|\frac{z(I_{p,b,c}f(z))''}{(I_{p,b,c}f(z))'}\right|^{\beta}<\frac{1}{2^{\beta}}\quad(\beta\geq 0), \tag{2.2}$$

then $I_{p,b,c}f$ is univalent in $\mathbb{U}$.

Proof. We know that

$$I_{p,b,c}f(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\,a^n\,z^n$$

in $\mathcal{A}$. Define ω by $\omega(z)=(I_{p,b,c}f(z))'-1$ for $z\in\mathbb{U}$. Then it follows that ω is analytic in $\mathbb{U}$ with $\omega(0)=0$. In view of (2.2), we have

$$|\omega(z)|^{1-\beta}\left|\frac{z\omega'(z)}{1+\omega(z)}\right|^{\beta}=|\omega(z)|\left|\frac{z\omega'(z)}{\omega(z)}\frac{1}{1+\omega(z)}\right|^{\beta}<\frac{1}{2^{\beta}}. \tag{2.3}$$

Suppose that there exists a point $z_1\in\mathbb{U}$ such that $\max\limits_{|z|\leq|z_1|}|\omega(z)|=|\omega(z_1)|=1$. Then, by Lemma 3.1, we can put

$$\frac{z_1\omega'(z_1)}{\omega(z_1)}=m\geq 1.$$

Therefore, we obtain

$$|\omega(z_1)|\left|\frac{z_1\omega'(z_1)}{\omega(z_1)}\frac{1}{1+\omega(z)}\right|^{\beta}\geq\left(\frac{m}{2}\right)^{\beta}\geq\frac{1}{2^{\beta}}$$

which contradicts the condition (2.3). This shows that $|\omega(z)|=|(I_{p,b,c}f(z))'-1|<1$ which implies that $\Re(I_{p,b,c}f(z))'>0$ for $z\in\mathbb{U}$. Therefore, by the Noshiro-Warschawski theorem [8], $I_{p,b,c}f$ is univalent in $\mathbb{U}$. □

Theorem 2.2. *Let $f\in\mathcal{A}$, $c\in\mathbb{C}$ and $\kappa>0$. If $u_{p,b,c}$ defined by (1.9) satisfies the inequality*

$$\left|\frac{zu'_{p,b,c}(z)}{u_{p,b,c}(z)}\right|<\frac{E-F}{1+|F|}, \tag{2.4}$$

where $-1\leq F<E<1$, $-1\leq F\leq 0$ and $z\in\mathbb{U}$, then $zu_{p,b,c}\in\mathcal{S}^*[E,F]$.

Proof. Let us define a function F by

$$F(z) = zu_{p,b,c}(z) \quad (z \in \mathbb{U}).$$

In view of (2.4), we have

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < \frac{E-F}{1+|F|}. \tag{2.5}$$

An application of Lemma 2.2 with $\alpha = 0$ proves Theorem 2.2. □

Theorem 2.3. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$ and $\kappa > 0$. If $u_{p,b,c}$ defined by (1.9) satisfies the inequality*

$$\left|\frac{zu''_{p,b,c}(z)}{u'_{p,b,c}(z)}\right| < \frac{(E-F)(2+E+E^2)}{(1+|F|)(1+E)^2}, \tag{2.6}$$

where $-1 \le F < E < 1$ and $-1 \le F \le 0$, then $u_{p,b,c}$ is starlike of order $(E+F)/2F$ and type $|F|$ with respect to 1.

Proof. Let $h : \mathbb{U} \to \mathbb{C}$ be defined by

$$h(z) = \frac{u_{p,b,c}(z) - b_0}{b_1}.$$

Then $h \in \mathcal{A}$ and h satisfies

$$\begin{aligned} \left|\frac{zh''(z)}{h'(z)}\right| &= \left|\frac{zu''_{p,b,c}(z)}{u'_{p,b,c}(z)}\right| \\ &< \frac{(E-F)(2+E+E^2)}{(1+|F|)(1+E)^2}. \end{aligned}$$

An application of Lemma 2.2 with $\alpha = 1$ implies that $u_{p,b,c}$ is starlike of order $(E+F)/2F$ and type $|F|$ with respect to 1 as the value of $b_0 = 1$. □

Theorem 2.4. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$ and $\kappa > 0$. If $u_{p,b,c}$ defined by (1.9) satisfies the inequality*

$$\left|\frac{zu'_{p+1,b,c}(z)}{u_{p+1,b,c}(z)}\right| < \frac{E-F}{1+|F|}, \tag{2.7}$$

where $-1 \le F < E < 1$, $-1 \le F \le 0$ and $c \ne 0$, then $u_{p,b,c}(z) \in \mathcal{K}[E,F]$.

Proof. By virtue of Theorem 2.2, $zu_{p+1,b,c} \in \mathcal{S}^*[E,F]$. In view of (1.11), $zu'_{p,b,c}(z) = b_1 u_{p+1,b,c}(z)$, where $b_1 = -c/4\kappa \ne 0$. Therefore, we have $zu'_{p,b,c} \in \mathcal{S}^*[E,F]$, which implies $u_{p,b,c} \in \mathcal{K}[E,F]$. □

Remark 2.1. *Note that, the conclusions of Theorem 2.2, Theorem 2.3 and Theorem 2.4 hold in the disk $|z| < 4/|c|$ where $0 < |c| < 4$ which is larger than the unit disk. By applying as in Theorems 2,3, and 5 of Owa and Srivastava [15] to the function $F(z) = {}_0F_1(\kappa, z)$ and using the transformation $F(z) = u_{p,b,c}(-4z/c)$ and replacing z by $-cz/4$, we obtain that Theorem 2.2, Theorem 2.3 and Theorem 2.4 hold in the disk $|z| < 4/|c|$.*

Theorem 2.5. *Let $c \in \mathbb{C}$, $-1 \le F < E < 1$, $-1 \le F \le 0$ and $\kappa > 0$. If the Bessel's inequality*

$$(1-F)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + (E-F)u_{p,b,|c|}(1) \le 2(E-F) \tag{2.8}$$

is satisfied, then $zu_{p,b,c} \in \mathcal{S}^*[E,F]$.

Proof. A special case of Theorem 3 in [1] gives a sufficient condition for a function $f \in \mathcal{S}^*[E,F]$ and is given by

$$\sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]|A_n| \leq E-F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}.$$

To prove the theorem, we need to show that

$$\begin{aligned} T: &= \sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]|A_n| \\ &= \sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right| \\ &= (1-F)\sum_{n=2}^{\infty} n\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right| - (1-E)\sum_{n=2}^{\infty}\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right| \\ &\leq (1-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + (E-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1-F)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + (E-F)\left(u_{p,b,|c|}(1)-1\right), \end{aligned}$$

which is bounded above by $E-F$ if (2.8) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

Corollary 2.1. *Let $c \in \mathbb{C}$, $-1 \leq F < E < 1$, $-1 \leq F \leq 0$ and $\kappa > 0$. If the Bessel's inequality*

$$\frac{|c|}{4\kappa}\,u_{p+1,b,|c|}(1) + \lambda u_{p,b,c}(1) \leq 2\lambda \tag{2.9}$$

is satisfied, then $zu_{p,b,|c|} \in \mathcal{S}^*_\lambda$.

Theorem 2.6. *Let $c \in \mathbb{C}$ and $\kappa > 0$. If the Bessel's inequality*

$$(1-F)\frac{(|c|/4)^2}{\kappa(\kappa+1)}\,u_{p+2,b,|c|}(1) + (2+E-3F)\frac{|c|}{4\kappa}\,u_{p+1,b,|c|}(1) + (E-F)\,u_{p,b,|c|}(1) \leq 2(E-F) \tag{2.10}$$

is satisfied, then the operator $zu_{p,b,c} \in \mathcal{K}[E,F]$.

Proof. By an analogous similar result [1] mentioned as in the earlier theorem, a sufficient condition for $f \in \mathcal{K}[E,F]$ is that

$$\sum_{n=2}^{\infty} n\left[n(1-F)-(1-E)\right]|A_n| \leq E-F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}.$$

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Then we have to show that

$$T_1 := \sum_{n=2}^{\infty} n\,[n(1-F)-(1-E)]\,|A_n| \leq E-F. \tag{2.11}$$

Writing $n = n-1+1$, and proceeding with the calculation as in the previous theorem, we get

$$\begin{aligned} T_1 &= \sum_{n=2}^{\infty} n\,(n(1-F)-(1-E)) \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| \\ &\leq \sum_{n=2}^{\infty} (n(1-F)-(1-E)) \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} (n(1-F)-(1-E)) \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}. \end{aligned}$$

Breaking the above inequality into two parts and simplifying, we observe that the summation is bounded above by $E-F$ if (2.10) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

Corollary 2.2. *Let $c \in \mathbb{C}$ and $\kappa > 0$. If the Bessel's inequality*

$$\frac{(|c|/4)^2}{\kappa(\kappa+1)} u_{p+2,b,|c|}(1) + (2+\lambda)\frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + \lambda u_{p,b,|c|}(1) \leq 2\lambda \tag{2.12}$$

is satisfied, then the operator $zu_{p,b,c} \in \mathcal{K}_\lambda$.

Remark 2.2. *For the choices of $E = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $F=-1$, each of the above theorems reduces to the results obtained by Baricz* [3].

3. Inclusion properties involving the class of Janowski starlike and convex functions

Let a function $f \in \mathcal{A}$ is said to be in the class $R^\tau(A,B)$ if

$$\left| \frac{f'(z)-1}{\tau(A-B)-B(f'(z)-1)} \right| < 1 \quad (-1 \leq B < A \leq 1; \tau \in \mathbb{C}\setminus\{0\}; z \in \mathrm{U}). \tag{3.1}$$

Clearly, a function f belongs to $R^\tau(A,B)$ if and only if there exists a function w regular in $\mathbb{U}$ satisfying $w(0)=0$ and $|w(z)|<1 \quad z \in \mathbb{U}$ such that

$$1 + \frac{1}{\tau}(f'(z)-1) = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U}).$$

The class $R^\tau(A,B)$ was introduced by Dixit and Pal [6]. For $\tau = 1$, $A=\beta$, $B=-\beta$, $(0<\beta\leq 1)$, $R^\tau(A,B)$ reduces to the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z)-1}{f'(z)+1} \right| < \beta \quad (z \in \mathbb{U};\ \ 0 < \beta \leq 1),$$

which was studied by Caplinger and Cauchy [4] and Padmanaban [14].

Now we aim at investigating various mapping and inclusion properties involving the class of Janowski starlike and Janowski convex functions. To prove the main results we need the following lemmas.

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Lemma 3.1. [6] *Let a function f of the form (1.1) be in $R^\tau(A,B)$. Then*

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A-B)|\tau| z^{n-1}}{1+Bz^{n-1}}\right) dz \quad (n \geq 2;\ z \in \mathbb{U})$$

Lemma 3.2. [6] *Let a function f of the form (1.1) satisfy the inequality*

$$\sum_{n=2}^{\infty} (1+|B|) n |a_n| \leq (A-B)|\tau| \ \ (-1 \leq B < A \leq 1;\ \tau \in \mathbb{C}).$$

Then $f \in R^\tau(A,B)$. The result is sharp for the function

$$f(z) = z + \frac{(A-B)\tau}{(1+|B|)n} z^n \quad (n \geq 2;\ z \in \mathbb{U}).$$

Theorem 3.1. *Let $c \in \mathbb{C}$, $\kappa > 0$. Suppose that $f \in R^\tau(A,B)$. If the Bessel's inequality*

$$2u_{p,b,|c|}(1) - 4\frac{(\kappa-1)}{c}\left(u_{p-1,b,|c|}(1) - 1\right) \leq \frac{1}{(1+B)} + 1 \tag{3.2}$$

is satisfied, then $zu_{p,b,c}(z^2) * f(z) \in R^\tau(A,B)$

Proof. Suppose that $f \in R^\tau(A,B)$. We note that

$$zu_{p,b,c}(z^2) = z + \sum_{n=2}^{\infty} \frac{(\frac{-c}{4})^{n-1}}{(\kappa)_{n-1}(n-1)!}\, z^{2n-1}.$$

By Lemma 3.2, it is enough to show that

$$\sum_{n=2}^{\infty} (1+|B|)\ (2n-1)\ \left| \frac{(\frac{-c}{4})^{n-1}}{(\kappa)_{n-1}(n-1)!}\ a_n \right| \leq (A-B)|\tau|.$$

Then by a similar proof as in the earlier theorem, we get

$$(A-B)\ |\tau|\ (1+|B|)\ \left[2u_{p,b,|c|}(1) - 4\frac{(\kappa-1)}{|c|}\left(u_{p-1,b,|c|}(1) - 1\right) - 1\right] \leq (A-B)\ |\tau|,$$

which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^\tau(A,B)$ and satisfy the condition*

$$u_{p,b,|c|}(1) \leq \frac{1}{1+|B|} + 1. \tag{3.3}$$

Then $I_{p,b,c}\ f \in R^\tau(A,B)$.

Proof. Let f be of the form (1.1) belong to the class $R^\tau(A,B)$. By Lemma 3.2, it suffices to show that

$$\sum_{n=2}^{\infty} n(1+|B|)|A_n| \leq (A-B)|\tau|, \tag{3.4}$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\, a_n.$$

By virtue of Lemma 3.1 and making use of the fact that $|-c/4|^n \leq (|c|/4)^n$, we obtain

$$\begin{aligned} \sum_{n=2}^{\infty} n(1+|B|)\, \left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\, a_n\right| &\leq (1+|B|)\,|\tau|\,(A-B) \sum_{n=2}^{\infty} \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1+|B|)\,|\tau|\,(A-B)\,[u_{p,b,|c|}(1)-1], \end{aligned}$$

which is bounded above by $(A-B)|\tau|$ in view of (3.3). This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$. If the Bessel's inequality*

$$(1-F)u_{p,b,|c|}(1) - (1-E)\frac{4(\kappa-1)}{|c|}u_{p-1,b,|c|}(1) \leq \frac{E-F}{(A-B)|\tau|} + E - F - \frac{(1-E)4(\kappa-1)}{|c|} \quad (3.5)$$

is satisfied, then the operator $I_{p,b,c}\ f \in \mathcal{S}^*[E,F]$.

Proof. Let f be of the form (1.1) belong to the class $R^{\tau}(A,B)$. A special case of Theorem 3 [1] gives a sufficient condition that

$$\sum_{n=2}^{\infty} [n(1-F)-(1-E)]\,|A_n| \leq E-F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\, a_n.$$

Then we have to show that

$$T := \sum_{n=2}^{\infty} [n(1-F)-(1-E)]\,|A_n| \leq E-F. \quad (3.6)$$

Since, $f \in R^{\tau}(A,B)$, in virtue of Lemma 3.1,

$$\begin{aligned} T &\leq \sum_{n=2}^{\infty} [n(1-F)-(1-E)] \left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right| \frac{(A-B)|\tau|}{n} \\ &= (A-B)|\tau| \left[(1-F)\sum_{n=2}^{\infty}\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right| - (1-E)\sum_{n=2}^{\infty}\left|\frac{(-c4)^{n-1}}{(\kappa)_{n-1}(n!)}\right|\right] \\ &\leq (A-B)|\tau| \left[(1-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} - (1-E)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n!)}\right] \\ &= (A-B)|\tau| \left[(1-F)(u_{p,b,|c|}(1)-1) - (1-E)\frac{4(\kappa-1)}{|c|}\left(u_{p-1,b,|c|}(1) - 1 - \frac{|c|/4}{(\kappa-1)}\right)\right], \end{aligned}$$

which is bounded above by $E-F$ if (3.5) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

Corollary 3.1. *Let $c \in \mathbb{C}$, $\kappa > 0$ and $\lambda \in [0,1]$. Suppose that $f \in R^{\tau}(A,B)$ and satisfy the condition*

$$u_{p,b,|c|}(1) \; + \; (\lambda - 1) \, \frac{4(\kappa-1)}{c} \, [u_{p-1,b,|c|}(1) - 1] \; \leq \; \left(\frac{1}{(A-B)\,|\tau|} + 1 \right) \lambda.$$

Then the operator $I_{p,b,c} f \in S^*_{\lambda}$.

Theorem 3.4. *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$. If the Bessel's inequality*

$$(1-F)\frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + (E-F) u_{p,b,|c|}(1) \leq (E-F)\left(\frac{1}{(A-B)|\tau|} + 1\right) \tag{3.7}$$

is satisfied, then the operator $I_{p,b,c} f \in \mathcal{K}[E,F]$.

Proof. Let f be of the form (1.1) belong to the class $R^{\tau}(A,B)$. We need to show (see [1]) that

$$\sum_{n=2}^{\infty} n\left[(n(1-F) - (1-E))\right]|A_n| \leq E - F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\, a_n.$$

Since, $f \in R^{\tau}(A,B)$, in virtue of Lemma 3.1,

$$\begin{aligned}
T: \; &= \; \sum_{n=2}^{\infty} n\left[n(1-F) - (1-E)\right]|A_n| \leq E - F \\
T \; &\leq \; \sum_{n=2}^{\infty} (n(1-F) - (1-E)) \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| (A-B)|\tau| \\
&= \; (A-B)|\tau| \left[(1-F)\sum_{n=2}^{\infty} n \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| - (1-E)\sum_{n=2}^{\infty} \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| \right] \\
&\leq \; (A-B)|\tau| \left[(1-F)\sum_{n=2}^{\infty} \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + (E-F)\sum_{n=2}^{\infty} \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right] \\
&= \; (A-B)|\tau| \left[(1-F)\frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + (E-F)\left(u_{p,b,|c|}(1) - 1\right) \right],
\end{aligned}$$

which is bounded above by $E - F$ if (3.7) is satisfied. This completes the proof of Theorem 3.4. □

For the choices of $E = \lambda$ and $F = 0$, we get the following corollary.

Corollary 3.2. *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$. If the Bessel's inequality*

$$\frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + \lambda\, u_{p,b,|c|}(1) \leq \lambda \left(\frac{1}{(A-B)|\tau|} + 1 \right) \tag{3.8}$$

is satisfied, then the operator $I_{p,b,c} f \in \mathcal{K}_{\lambda}$.

4. Sufficient conditions for Bessel's integral operator to be in the class of $\mathcal{S}^*[E,F]$ and $\mathcal{K}[E,F]$

As in the work of Baricz [3], one can look at other linear operators acting on $u_{p,b,c}$ to obtain similar results. In this section, we make use of this idea in the case of a particular integral operator. We continue our earlier work that was done in the earlier section. That is, we determine sufficient conditions for the integral operator g defined by (4.1) to be in the class of Janowski starlike and Janowski convex functions as follows:

$$\begin{aligned} g(z) &= \int_0^z u_p(t)dt \\ &= z+\sum_{n=2}^{\infty}\frac{b_{n-1}}{n}z^n \\ &= z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n!)}z^n. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $c\in\mathbb{C}$ and $\kappa>0$. Further, let $-1\le F<E<1$ and $-1\le F\le 0$. If the Bessel's inequality*

$$(1-F)u_{p,b,|c|}(1)-(1-E)\frac{4(\kappa-1)}{|c|}u_{p-1,b,|c|}(1)\le 2(E-F)-(1-E)\frac{4(\kappa-1)}{|c|} \tag{4.2}$$

is satisfied, then the function $g\in\mathcal{S}^*[E,F]$ where g is defined by (4.1).

Proof. To prove the theorem, we have to show that

$$T:=\sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]|B_n|\le E-F, \tag{4.3}$$

where

$$B_n=\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}.$$

Then

$$\begin{aligned} T &= \sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}\right| \\ &\le (1-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}-(1-E)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n)!} \\ &= (1-F)\left(u_{p,b,|c|}(1)-1\right)-(1-E)\left[\frac{4(\kappa-1)}{|c|}\left(u_{p-1,b,|c|}(1)-1-\frac{|c|/4}{(\kappa-1)}\right)\right], \end{aligned}$$

which is bounded above by $E-F$ if (4.2) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

Corollary 4.1. *Let $c\in\mathbb{C}$ and $\kappa>0$. Further, let $\lambda\ge 0$. If the Bessel's inequality*

$$u_{p,b,|c|}(1)-(1-\lambda)\frac{4(\kappa-1)}{|c|}u_{p-1,b,|c|}(1)\le 2\lambda+(1-\lambda)\frac{4(\kappa-1)}{|c|} \tag{4.4}$$

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is satisfied, then the function $g \in \mathcal{S}^*_\lambda$ where g is defined by (4.1).

Theorem 4.2. *Let $c \in \mathbb{C}$ and $\kappa > 0$. Further, let $-1 \le F < E < 1$ and $-1 \le F \le 0$ and g be defined as in (4.1). If the Bessel's inequality*

$$(1-F)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + (E-F)u_{p,b,|c|}(1) \le 2(E-F) \tag{4.5}$$

is satisfied, then the integral operator $g \in \mathcal{K}[E,F]$.

Proof. We have to show that

$$T_4 := \sum_{n=2}^{\infty} n\left(n(1-F)-(1-E)\right)|B_n| \le E-F, \tag{4.6}$$

where

$$B_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}.$$

Then

$$\begin{aligned} T_4 &= \sum_{n=2}^{\infty} n\left[n(1-F)-(1-E)\right]\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}\right| \\ &\le \sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]\frac{(|-c/4|)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + (E-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1-F)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + (E-F)\left(u_{p,b,|c|}(1)-1\right), \end{aligned}$$

which is bounded above by $E-F$ if (4.5) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

Corollary 4.2. *Let $c \in \mathbb{C}$ and $\kappa > 0$. Further, let $-1 \le F < E < 1$ and $-1 \le F \le 0$ and g be defined as in (4.1). If the Bessel's inequality*

$$\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + \lambda u_{p,b,|c|}(1) \le 2\lambda \tag{4.7}$$

is satisfied, then $g \in \mathcal{K}_\lambda$.

5. Consequences and observations

Since the study generalized Bessel function permits the study of Bessel, modifed Bessel and spherical Bessel functions all together, each of these Theorems can also be stated for the Bessel, modified Bessel and spherical Bessel functions for special choices of the parameters b and c. However, we leave all these results for the interested readers.

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ON THE JENSEN-TYPE INEQUALITY FOR THE $\bar{g}$-INTEGRAL.

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Abstract. We consider the pseudo-integral with respect to a $\sigma-\oplus$-measure of set-valued functions which was defined by Grbíc et al. Román-Flores et al.(2007) proved the Jensen type inequality for fuzzy integral with respect to a fuzzy measure. In this paper, we prove the Jensen type inequality for the $\bar{g}$-integral with respect to a $\sigma-\oplus_g$-measure under some sufficient conditions.

1. Introduction

Benvenuti-Mesiar [2], Deschrijver [3], J. Fang [4], Mesiar-Pap [9], Ralescu-Adams [10], and Wu-Wang-Ma [12] provided the properties and applications of the generalized fuzzy integral which is a generalization of fuzzy integrals.

The integrals of set-valued functions was introduced by Aumanm [1], and Jang [6,7] and Zhang-Guo [13] investigated some properties of the generalized fuzzy integral of set-valued functions. Not long ago, authors in [8,11] proved the Jensen type inequality for the fuzzy integral and for the generalized Sugeno integral. We consider the pseudo-integral with respect to a $\sigma-\oplus$-measure of set-valued functions which was defined by Grbíc et al [5]. Román-Flores et al. [11] proved the Jensen type inequality for fuzzy integral with respect to a fuzzy measure. In this paper, we prove the Jensen type inequality for $\bar{g}$-integral with respect to a $\sigma-\oplus_g$-measure under some sufficient conditions.

2. Jensen type inequality for the g-integral

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $\overline{\mathbb{R}} = [-\infty,\infty]$ and let $\preccurlyeq$ be a total order on $[a,b]$. We introduce a semiring which is a structure $([a,b],\oplus,\odot)$ as follows.

Key words and phrases. Sugeno integral, $\sigma-\oplus$-measure, $\bar{g}$-integral, Jensen inequality.

Definition 2.1. ([2,7,9]) (1) A function $\oplus : [a,b] \times [a,b] \longrightarrow [a,b]$ is called a pseudo-addition if it is commutative, non-decreasing with respect to $\preccurlyeq$, associative, and with a zero (natural) element denoted by $\mathbf{0}$, that is, for each $x \in [a,b]$, $\mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

(2) A function $\odot : [a,b] \times [a,b] \longrightarrow [a,b]$ is called a pseudo-multiplication if it is commutative, positively non-decreasing, that is, $x \preccurlyeq y$ implies $x \odot z \preccurlyeq y \odot z$ for all $z \in [a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preccurlyeq x\}$, associative and there exists a unit element $\mathbf{1} \in [a,b]$, that is, for each $x \in [a,b], \mathbf{1} \odot x = x$.

(3) The structure $([a,b], \oplus, \odot)$ is called a semiring if $\mathbf{0} \odot x = \mathbf{0}$ and $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, that is, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

(4) A set function $\mu : \Sigma \longrightarrow [a,b]$ is a $\sigma - \oplus$-measure if it satisfies the following two conditions:

(i) $\mu(\emptyset) = \mathbf{0}$ (if $\oplus$ is not idempotent);

(ii) $\mu(\cup_{i=1}^{\infty} A_i) = \oplus_{i=1}^{\infty} \mu(A_i)$ holds for any sequence $(A_i)_{i \in \mathbb{N}}$ of disjoint sets from Σ.

We note that for a real interval $[a,b] = [0,\infty]$, a pseudo-addition $\oplus$ and a pseudo-multiplication $\odot$ are generated by a strictly monotone bijective function $g : [0,\infty] \longrightarrow [0,\infty]$, that is, pseudo-operations are given by

$$x \oplus_g y = g^{-1}(g(x) + g(y)) \text{ and } x \odot_g y = g^{-1}(g(x)g(y)).$$

Now, the pseudo-integral, known as the g-integral, of some measurable function $f : X \longrightarrow [0,\infty]$ is

$$(g) \int_X f d\mu = \int_X^{\oplus_g} f \odot_g d\mu = g^{-1} \left(\int_X (g \circ f) \; d(g \circ \mu) \right) \tag{1}$$

where $g \circ \mu$ is the Lebesgue measure and the integral on the right-hand side of (A) is the Lebesgue integral (see [7,9]). Let (X, Σ, μ) be a $\sigma - \oplus$-measure space. Grbíc et al. [5] defined the pseudo-integral of an interval-valued function F on $A \in \sum$ as follows;

$$\int_A^{\oplus} F \odot d\mu = \{ \int_A^{\oplus} f \odot d\mu | f \in S(F) \} \tag{2}$$

where μ is a $\sigma - \oplus$-measure and $S(F)$ is the set of all selections of F. Let $L^1(\eta)$ be the set of all Lebesgue integrable functions on the Lebesgue space $([0,\infty), \sum, \eta)$ and $f \in L^1_{\oplus}(\mu)$ if and only if $g \circ f \in L^1(g \circ \mu)$. We introduce the definition of g-integrable boundedness of a set-valued function F as follows:

Definition 2.2. ([5]) Let g be a strictly monotone bijective function. A set-valued function F is g -integrable bounded if there is a function $h \in L^1_{\oplus}(\mu)$ such that

(i) $\oplus_{\alpha \in F(x)} \alpha \preccurlyeq h(x)$, for the idempotent pseudo-addition,

(ii) $\sup_{\alpha \in F(x)} \alpha \preccurlyeq h(x)$, for the pseudo-addition given by an increasing generator g,

(iii) $\inf_{\alpha \in F(x)} \alpha \preccurlyeq h(x)$, for the pseudo-addition given by a decreasing generator g.

From Proposition 11 in [5], we note that if F is a pseudo-integrable bounded set-valued function, then F is pseudo-integrable, that is, $\int_X^{\oplus} F \odot d\mu \neq \emptyset$.

Theorem 2.1. *(Theorem 2.4 [5]) Let F be a pseudo-integrable bounded interval-valued function with border functions f_l and f_r. Then we have*

$$\int_X^{\oplus} F \odot d\mu = \left[\int_X^{\oplus} f_l \odot d\mu, \int_X^{\oplus} f_r \odot d\mu\right], \tag{3}$$

Now, we obtain the following Jensen type inequality for the g-integral with respect to a $\sigma - \oplus_g$-measure.

Theorem 2.2. *Let g be a decreasing function and $(X, \sum, g\circ\mu)$ be the Lebesgue measure space and $f \in L^1_{\oplus}\mu$ with $(g)\int_X f d\mu = m$. If $\Phi : [o,\infty) \longrightarrow [0,\infty)$ is strictly increasing function such that $\Phi(x) \leq x$, for every $x \in [0,m]$ and $\Phi(f) \in L^1_{\oplus}(\mu)$, then we have*

$$\Phi\left((g)\int_X f d\mu\right) \leq (g)\int_X \Phi(f) d\mu. \tag{4}$$

Proof. Since $\Phi(f) \leq f$ and g is decrasing,

$$g \circ \Phi(f) \geq g \circ f. \tag{5}$$

By (5) and monotonicity of the Lebesgue integral with respect to $g \circ \mu$, we have

$$\int_X g \circ \Phi(f)\ dg \circ \mu \geq \int_X g \circ f\ dg \circ \mu. \tag{6}$$

Since g^{-1} is decreasing, by (6), we have

$$g^{-1} \int_X g \circ \Phi(f)\ dg \circ \mu \leq g^{-1} \int_X g \circ f\ dg \circ \mu. \tag{7}$$

By (7),

$$\begin{aligned} \Phi\left((g)\int_X f d\mu\right) &= \Phi\left(g^{-1}\int_X g \circ f\ dg \circ \mu\right) \\ &\leq g^{-1}\int_X g \circ f\ dg \circ \mu \\ &\leq g^{-1}\int_X g \circ \Phi(f)\ dg \circ \mu \\ &= (g)\int_X \Phi(f) d\mu. \end{aligned}$$

□

Theorem 2.3. *Let g be an increasing function and $(X, \sum, g \circ \mu)$ be the Lebesgue measure space and $f \in L^1_{\oplus}(\mu)$ with $(g)\int_X f d\mu = m$. If $\Phi : [0,\infty) \longrightarrow [0,\infty)$ is strictly increasing function such that $\Phi(x) \geq x$, for every $x \in [0,m]$, and $\Phi(f) \in L^1_{\oplus}(\mu)$, then*

$$\Phi\left((g)\int_X f d\mu\right) \geq (g)\int_X \Phi(f) d\mu. \tag{8}$$

Proof. Since $\Phi(f) \leq f$ and g is increasing,

$$g \circ \Phi(f) \leq g \circ f. \tag{9}$$

By (9) and monotonicity of the Lebesgue integral with respect to $g \circ \mu$, we have

$$\int_X g \circ \Phi(f)\ dg \circ \mu \leq \int_X g \circ f\ dg \circ \mu. \tag{10}$$

Since g^{-1} is increasing, by (10), we have

$$g^{-1} \int_X g \circ \Phi(f)\ dg \circ \mu \leq g^{-1} \int_X g \circ f\ dg \circ \mu. \tag{11}$$

By (11),

$$\begin{aligned}
\Phi\left((g) \int_X f d\mu\right) &= \Phi\left(g^{-1} \int_X g \circ f\ dg \circ \mu\right) \\
&\geq g^{-1} \int_X (g \circ f)\ dg \circ \mu \\
&\geq g^{-1} \int_X g \circ \Phi(f)\ dg \circ \mu \\
&= (g) \int_X \Phi(f) d\mu.
\end{aligned}$$

□

3. Jensen type inequality for the $\bar{g}$-integral

Let $I([0,\infty])$ be the set of all bounded closed intervals in $[0,\infty]$ as follows :

$$I([0,\infty]) = \{\bar{a} = [a_l, a_r] | a_l, a_r \in [0,\infty] \text{ and } a_l \leq a_r\}$$

For these intervals, we define the order, the strictly order, and strong strictly order of intervals as follows:

Definition 3.1. ([5]) If $\bar{a} = [a_l, a_r], \bar{b} = [b_l, b_r] \in I([0,\infty])$, then we define order ($\leq$), strictly order ($<$), and strong strictly order ($\prec_s$) as follows :
(a) $\bar{a} \leq \bar{b}$ if and only if $a_l \leq b_l$ and $a_r \leq b_r$,
(b) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$,
(c) $\bar{a} \prec_s \bar{b}$ if and only if $a_l < b_l$ and $a_r < b_r$.

Definition 3.2. A mapping $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \times [0,\infty)$ by $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ is called a strictly increasing function if for all $\bar{x} = [x_l, x_r], \bar{y} = [y_l, y_r] \in [0,\infty) \times [0,\infty)$,

$$\bar{x} \prec_s \bar{y} \Rightarrow \overline{\Phi}(\bar{x}) \prec_s \ \overline{\Phi}(y).$$

From Definition 2.2, we directly obtain the following theorem.

Theorem 3.1. *Let* $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty)\times[0,\infty) \longrightarrow [0,\infty)\times[0,\infty)$ *by* $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ *be a mapping. Then* $\overline{\Phi} = [\Phi_l, \Phi_r]$ *is an strictly increasing function if and only if* Φ_l *and* Φ_r *are strictly increasing functions.*

We remark that if $f \in L^1_{\oplus}(\nu) = L^1(g\circ\mu)$, then f is a (g)-integrable function and $g^{-1}\int_X (g\circ g)\ dg\circ\mu$ is finite. By Theorem 2.1, and (1), we obtain the following theorem.

Theorem 3.2. *Let* F *be a pseudo-inequality bounded interval-valued function with border functions* f_l *and* f_r*. If* $\bar{g} = [g_l, g_r]$ *is a monotone function and* $g_l \circ f_l \in L^1(g_l\circ\mu)$ *and* $g_r \circ f_r \in L^1(g_r\circ\mu)$*. Then we have*

$$(\bar{g})\int_X \bar{f}d\mu = \left[(g_l)\int_X f_l d\mu, (g_r)\int_X f_r d\mu\right]. \tag{12}$$

Proof. By (1), we have

$$\begin{aligned}(g_l)\int_X f_l d\mu &= g_l^{-1}\left(\int_X (g_l\circ f)\ dg_l\circ\mu\right)\\ &= \int_X^{\oplus_{g_l}} f\odot_{g_l} d\mu.\end{aligned} \tag{13}$$

and

$$\begin{aligned}(g_r)\int_X f_r d\mu &= g_r^{-1}\left(\int_X (g_r\circ f)\ dg_r\circ\mu\right)\\ &= \int_X^{\oplus_{g_r}} f\odot_{g_r} d\mu.\end{aligned} \tag{14}$$

By (13) and (14), and Theorem 2.1, we have

$$\begin{aligned}(\bar{g})\int_X \bar{f}d\mu &= \int_X^{\oplus_{\bar{g}}} F\odot_{\bar{g}} d\mu\\ &= \left[\int_X^{\oplus_{g_l}} f_l\odot_{g_l} d\mu, \int_X^{\oplus_{g_r}} f_r\odot_{g_r} d\mu\right]\\ &= \left[(g_l)\int_X f_l d\mu, (g_r)\int_X f_r d\mu\right].\end{aligned}$$

□

Finally, we obtain the following Jensen inequality for the $\bar{g}$-integral with respect to a .

Theorem 3.3. *Let* $\bar{g}$ *be a decreasing function and* $(X, \sum, g_s\circ\mu)$ *be the Lebesgue measure space for* $s = l, r$ *and* $g_l\circ f_l \in L^1(g_l\circ\mu)$ *and* $g_r\circ f_r \in L^1(g_r\circ\mu)$ *with* $(g_l)\int_X f_l d\mu = m_l$ *and* $(g_r)\int_X f_r d\mu = m_r$*. If* $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty)\times[0,\infty) \longrightarrow [0,\infty)\times[0,\infty)$ *by* $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ *is strictly increasing function such that* $\overline{\Phi}(x,y) \le (x,y)$ *for every* $(x,y) \in [0,m_l]\times[0,m_r]$ *and* $\Phi_l(f_l) \in L^1(g_l\circ\mu)$ *and* $\Phi_r(f_r) \in L^1(g_r\circ\mu)$*, then we have*

$$\overline{\Phi}\left((\bar{g})\int_X F d\mu\right) \le (\bar{g})\int_X \overline{\Phi}(F)d\mu. \tag{15}$$

Proof. By Theorem 3.1 we have the following two inequalities :

$$(\bar{g})\int_X F d\mu = \left[(g_l)\int_X f_l d\mu, (g_r)\int_X f_r d\mu\right] \tag{16}$$

and

$$(\bar{g})\int_X \overline{\Phi}(F)d\mu = \left[(g_l)\int_X \Phi_l(f_l)d\mu, (g_r)\int_X \Phi_r(f_r)d\mu\right]. \tag{17}$$

By Theorem 2.2, we have

$$\Phi_s\left((g_s)\int_X f_s d\mu\right) \le (g_s)\int_X \Phi_s(f_x)d\mu, \tag{18}$$

for $s = l, r$. By (16), (17) and (18), we obtain the following result :

$$\begin{aligned}
\Phi\left((\bar{g}\int_X F d\mu\right) &= \left[\Phi_l\left((g_l)\int_X f_l d\mu\right), \Phi_r\left((g_r)\int_X f_r d\mu\right)\right] \\
&\le \left[(g_l)\int_X \Phi_l(f_l)d\mu, (g_r)\int_X \Phi_r(f_r)d\mu\right] \\
&= (\bar{g})\int_X \left[\Phi_l(f_l), \Phi_r(f_r)\right] d\mu \\
&= (\bar{g})\int_X \overline{\Phi}(F)d\mu.
\end{aligned}$$

□

Theorem 3.4. *Let $\bar{g}$ be an increasing function and $(X, \sum, g_s \circ \mu)$ be the Legesgue measure space and $g_l \circ f_l \in L^1(g_l \circ \mu)$ and $g_r \circ f_r \in L^1(g_r \circ \mu)$ with $(g_l)\int_X f_l d\mu = m_l$ and $(g_r)\int_X f_r d\mu = m_r$.*

If $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \times [0,\infty)$ by $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ for all $(x,y) \in [0,\infty) \times [0,\infty)$ is strictly increasing such that $\overline{\Phi}(x,y) \ge (x,y)$ for every $(x,y) \in [0,m_l] \times [0,m_r]$ and $\Phi_l(f_l) \in L^1(g_l \circ \mu)$ and $\Phi_r(f_r) \in L^1(g_r \circ \mu)$,

$$\Phi_l\left((\bar{g})\int_X F d\mu\right) \ge (\bar{g})\int_X \overline{\Phi}(F)d\mu. \tag{19}$$

Proof. By using Theorem 2.3,we have

$$\Phi_s\left((g_s)\int_X f_s d\mu\right) \ge (g_s)\int_X \Phi_s(f_s)d\mu \tag{20}$$

for $s = l, r$. By (16), (17) and (20), we obtain the following result:

$$\begin{aligned}
\Phi\left((\bar{g})\int_X F d\mu\right) &= \left[\Phi_l\left((g_l)\int_X f_l d\mu\right), \Phi_r\left((g_r)\int_X f_r d\mu\right)\right] \\
&\ge \left[(g_l)\int_X \Phi_l(f_l)d\mu, (g_r)\int_X \Phi_r(f_r)d\mu\right] \\
&= (\bar{g})\int_X \left[\Phi_l(f_l), \Phi_r(f_r)\right] d\mu \\
&= (\bar{g})\int_X \overline{\Phi}(F)d\mu.
\end{aligned}$$

□

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SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS ON THE CO-ORDINATES

SHU-HONG WANG AND SHAN-HE WU

Abstract. In this paper, operator m-convex and (α, m)-convex function on the co-ordinates are defined, and some new integral inequalities of Hermite-Hadamard type for operator m-convex and (α, m)-convex on the co-ordinates are established.

1. Introduction

Throughout this paper, we adopt the notations: $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$.

The following inequality holds for any convex function f defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave on $[a, b]$. The inequality (1.1) is well known in the literature as Hermite-Hadamard's inequality. We note that the Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The concept of m-convexity was first introduced by G. Toader in [19] (see also [2]) and it is defined as follows:

Definition 1.1 ([19])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y). \tag{1.2}$$

The class of (α, m)-convex functions was also first introduced in [16] and it is defined as follows:

Definition 1.2 ([16])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{1.3}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Also, the m-convex and (α, m)-convex functions on the co-ordinates defined in a rectangle from the plane were introduced as follows.

Definition 1.3 ([17])**.** Let $\triangle := [0, b] \times [0, d]$ be the bidimensional interval in $\mathbb{R}_0^2$ with $b > 0$ and $d > 0$. For some $m \in [0, 1]$, the function $f : \triangle \to \mathbb{R}$ is said to be m-convex if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + m(1-\lambda)w) \le \lambda f(x, y) + m(1-\lambda)f(z, w) \tag{1.4}$$

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holds for all $(x,y),(z,w) \in \triangle$ and $\lambda \in (0,1)$.

Definition 1.4 ([17])**.** For some $m \in [0,1]$, a function $f : \triangle := [0,b] \times [0,d] \subseteq \mathbb{R}_0^2 \to \mathbb{R}$ which is m-convex on $\triangle$ will be called m-convex on the co-ordinates with $b > 0$ and $d > 0$ if the partial mappings

$$f_y : [a,b] \to \mathbb{R}, f_y(u) := f(u,y)$$

and

$$f_x : [c,d] \to \mathbb{R}, f_x(v) := f(x,v)$$

are m-convex for all $y \in [c,d]$ and $x \in [a,b]$.

Definition 1.5 ([17])**.** Let $\triangle := [0,b] \times [0,d]$ be the bidimensional interval in $\mathbb{R}_0^2$ with $b > 0$ and $d > 0$. For some $(\alpha,m) \in [0,1]^2$, the function $f : \triangle \to \mathbb{R}$ is said to be (α,m)-convex if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + m(1-\lambda)w) \le \lambda^\alpha f(x,y) + m(1-\lambda^\alpha)f(z,w) \tag{1.5}$$

holds for all $(x,y),(z,w) \in \triangle$ and $\lambda \in (0,1)$.

Definition 1.6 ([17])**.** For some $(\alpha,m) \in [0,1^2]$, a function $f : \triangle := [0,b] \times [0,d] \subseteq \mathbb{R}_0^2 \to \mathbb{R}$ which is (α,m)-convex on $\triangle$ will be called (α,m)-convex on the co-ordinates with $b > 0$ and $d > 0$ if the partial mappings

$$f_y : [a,b] \to \mathbb{R}, f_y(u) := f(u,y)$$

and

$$f_x : [c,d] \to \mathbb{R}, f_x(v) := f(x,v)$$

are (α,m)-convex for all $y \in [c,d]$ and $x \in [a,b]$.

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of operator functions introduced by S. S. Dragomir in [6].

We review the operator order in $B(H)$ and the continuous functional calculus for a bounded self-adjoint operator. For self-adjoint operators $A, B \in B(H)$, we write $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for every vector $x \in H$, we call it the operator order.

Let A be a bounded self-adjoint linear operator on a complex Hilbert space $(H; \langle .,.\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous complex-valued functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [8], p.3). For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A)$.

With this notation, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)) \tag{1.6}$$

and we call it the continuous functional calculus for a bounded self-adjoint operator A.

A real valued continuous function f on an interval $I \subseteq \mathbb{R}$ is said to be operator convex (operator concave) if the operator inequality

$$f((1-\lambda)A + \lambda B) \le (\ge)(1-\lambda)f(A) + \lambda f(B) \tag{1.7}$$

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holds in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every bounded self-adjoint operators A and B in $B(H)$ whose spectra are contained in I.

In [20], Wang defined operator m-convex and (α, m)-convex functions in the following way:.

Definition 1.7. Let $[0,b] \subseteq \mathbb{R}_0$ with $b > 0$ and K be a convex set of $B(H)^+$. A continuous function $f : [0,b] \to \mathbb{R}$ is said to be operator m-convex on $[0,b]$ for operators in K, if

$$f(tA + m(1-t)B) \le tf(A) + m(1-t)f(B) \tag{1.8}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators A and B in K whose spectra are contained in $[0,b]$ and for some fixed $m \in [0,1]$.

Definition 1.8. Let $[0,b] \subseteq \mathbb{R}_0$ with $b > 0$ and K be a convex set of $B(H)^+$. A continuous function $f : [0,b] \to \mathbb{R}$ is said to be operator (α, m)-convex on $[0,b]$ for operators in K, if

$$f(tA + m(1-t)B) \le t^\alpha f(A) + m(1-t^\alpha)f(B) \tag{1.9}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators A and B in K whose spectra are contained in $[0,b]$ and for some fixed $(\alpha, m) \in [0,1]^2$.

Also, author proved the following inequalities in [20]:

Theorem 1.1 ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le min\left\{\frac{f(A) + \alpha m f\left(\frac{B}{m}\right)}{\alpha+1}, \frac{f(B) + \alpha m f\left(\frac{A}{m}\right)}{\alpha+1}\right\}. \tag{1.10}$$

Theorem 1.2 ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) \le& \frac{1}{2^\alpha}\int_0^1 \left[f(tA + (1-t)B) + m(2^\alpha - 1)f\left(\frac{(1-t)A + tB}{m}\right)\right]\mathrm{d}t \\ \le& \frac{1}{2^{\alpha+1}(\alpha+1)}\left\{f(A) + f(B) + m(\alpha + 2^\alpha - 1)\left[f\left(\frac{A}{m}\right) + f\left(\frac{B}{m}\right)\right]\right. \\ &\left. + \alpha m^2(2^\alpha - 1)\left[f\left(\frac{A}{m^2}\right) + f\left(\frac{B}{m^2}\right)\right]\right\}. \end{aligned} \tag{1.11}$$

Theorem 1.3 ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A) + f(B) + \alpha m\left[f\left(\frac{A}{m}\right) + f\left(\frac{B}{m}\right)\right]}{2(\alpha+1)}. \tag{1.12}$$

Theorem 1.4 ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 \left[f(tA + m(1-t)B) + f(tB + m(1-t)A)\right]\mathrm{d}t \le \frac{(1+m\alpha)[f(A) + f(B)]}{\alpha+1}. \tag{1.13}$$

Theorem 1.5 ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0, 1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$
\begin{aligned}
&f\left(\frac{2-m}{2}B + \frac{m}{2}(mA)\right) \\
\le&\frac{1}{2^\alpha}\int_0^1 \left[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1)f\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right)\right] \mathrm{d}t \\
\le&\frac{1}{2^\alpha(\alpha+1)}\Bigg[f((2-m)B) + m\big(\alpha + 2^\alpha - 1\big)f(mA) \\
&+ m^2\alpha(2^\alpha - 1)f\left(\frac{(2-m)B}{m^2}\right)\Bigg]. \qquad (1.14)
\end{aligned}
$$

For recent results related to Hermite-Hadamard type inequalities are given in [1], [4], [5], [7], [8], [9], [10], [13], [14], and plenty of references therein.

The main purpose of this paper is to establish some new Hadamard type inequalities for operator m)-convex and (α, m)-convex functions on the co-ordinates.

2. OPERATOR CO-ORDINATED m-CONVEX AND (α, m)-CONVEX FUNCTIONS

Let I_1, I_2 be real intervals and let $f : I_1 \times I_2 \to \mathbb{R}$ be a Borel measurable and essentially bounded function. Let $X = (X_1, X_2)$ be a 2-tuple of bounded self-adjoint operators on Hilbert spaces H_1, H_2 such that the spectrum of X_i is contained in I_i for $i = 1, 2$. We say that such a 2-tuple is in the domain of f. If

$$X_i = \int_{I_i} \lambda_i E_i(d\lambda_i), i = 1, 2$$

is the spectral decomposition of X_i where E_i is a bounded positive measure on I_i, we define

$$f(X) = \int_{I_1 \times I_2} f(\lambda_1, \lambda_2)E_1(d\lambda_1) \otimes E_2(d\lambda_2)$$

as a bounded self-adjoint operator on the tensor product $H_1 \otimes H_2$. If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction have the property that

$$f(X_1, X_2) = f_1(X_1) \otimes f_2(X_2),$$

whenever f can be separated as a product $f(t_1, t_2) = f_1(t_1)f_2(t_2)$ of 2 functions each depending on only one variable.

With above functional calculus, we say that a function $f : I_1 \times I_2 \to \mathbb{R}$ is operator convex if f is continuous and the operator inequality

$$f(tX + (1-t)Y) \le tf(X) + (1-t)f(Y) \qquad (2.1)$$

holds for all 2-tuples of self-adjoint operators $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ in the domain of f acting on any Hilbert spaces H_1, H_2 and for all $t \in [0, 1]$.

In [21], Hermite-Hadamard type inequality for the co-ordinated operator convex functions is given.

Theorem 2.1. *Suppose that a continuous function $f : I_1 \times I_2 \subseteq \mathbb{R}^2 \to \mathbb{R}$ is operator convex on the co-ordinates for all 2-tuples of self-adjoint operators in the domain of f acting on any Hilbert spaces H_1, H_2. Then we have the inequalities*

$$f\left(\frac{A+C}{2}, \frac{B+D}{2}\right)$$

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$$
\begin{aligned}
&\le\frac{1}{2}\left[\int_0^1 f\left(tA+(1-t)C,\frac{B+D}{2}\right)\mathrm{d}t+\int_0^1 f\left(\frac{A+C}{2},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda\right]\\
&\le\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda\\
&\le\frac{1}{4}\left[\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t\right.\\
&\quad\left.+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\right]\\
&\le\frac{f(A,B)+f(A,D)+f(C,B)+f(C,D)}{4},
\end{aligned}
\tag{2.2}
$$

where $(A,B),(C,D)\in B(H_1)\otimes B(H_2)$ *with spectra in* $I_1\times I_2$.

For some fundamental results on operator convex and operator monotone functions of several variables, see [11], [12], [15], and the references therein

Now we give the concepts of operator m-convex and (α,m)-convex functions on the co-ordinates.

Definition 2.1. A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ is said to be operator m-convex with $b>0$ and $d>0$ for some fixed $m\in[0,1]$ if the operator inequality

$$f(tX_1+(1-t)Y_1,tX_2+m(1-t)Y_2)\le tf(X_1,X_2)+m(1-t)f(Y_1,Y_2) \tag{2.3}$$

holds for all 2-tuples of self-adjoint operators $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ in the domain of f acting on any Hilbert spaces H_1,H_2 and for all $t\in(0,1)$.

Definition 2.2. A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ which is operator m-convex on $[0,b]\times[0,d]$ with $b>0$ and $d>0$ is said to be operator m-convex on the co-ordinates for some fixed $m\in[0,1]$ if the partial mapping

$$f_{X_2}:I_1\to\mathbb{R},f_{X_2}(u):=f(u,X_2)$$

and

$$f_{X_1}:I_2\to\mathbb{R},f_{X_1}(v):=f(X_1,v)$$

are operator m-convex for all operators $X_2\in B(H_2)$ and $X_1\in B(H_1)$ whose spectra are contained in $[0,d]$ and $[0,b]$, respectively.

Definition 2.3. A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ is said to be operator (α,m)-convex with $b>0$ and $d>0$ for some fixed $(\alpha,m)\in[0,1]^2$ if the operator inequality

$$f(tX_1+(1-t)Y_1,tX_2+m(1-t)Y_2)\le t^\alpha f(X_1,X_2)+m(1-t^\alpha)f(Y_1,Y_2) \tag{2.4}$$

holds for all 2-tuples of self-adjoint operators $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ in the domain of f acting on any Hilbert spaces H_1,H_2 and for all $t\in(0,1)$.

Definition 2.4. A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ which is operator (α,m)-convex on $[0,b]\times[0,d]$ with $b>0$ and $d>0$ is said to be operator (α,m)-convex on the co-ordinates for some fixed $(\alpha,m)\in[0,1]^2$ if the partial mapping

$$f_{X_2}:I_1\to\mathbb{R},f_{X_2}(u):=f(u,X_2)$$

and

$$f_{X_1}:I_2\to\mathbb{R},f_{X_1}(v):=f(X_1,v)$$

are operator (α,m)-convex for all operators $X_2\in B(H_2)$ and $X_1\in B(H_1)$ whose spectra are contained in $[0,d]$ and $[0,b]$, respectively.

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Remark 2.1. It can be easily seen that for $(\alpha, m) \in \{(1,1),(1,m)\}$ one obtains the classes of operator convex and operator m-convex functions of two variables, respectively.

The following lemmas hold:

Lemma 2.1. *For $b > 0$, $d > 0$, and some fixed $m \in [0,1]$, every operator m-convex mapping $f : [0,b] \times [0,d] \subseteq \mathbb{R}_0^2 \to \mathbb{R}$ is operator m-convex on the co-ordinates, but the converse is not generally true.*

Proof. Suppose that f is operator m-convex mapping on $[0,b] \times [0,d]$. Consider $f_{X_1} : [0,d] \to \mathbb{R}, f_{X_1}(v) := f(X_1, v)$. Then for all $t \in (0,1)$ and operators $A, C \in B(H_2)$ with spectra in $[0,d]$, one has

$$\begin{aligned} f_{X_1}(tA + m(1-t)C) &= f(tX_1 + (1-t)X_1, tA + m(1-t)C) \\ &\le tf(X_1, A) + m(1-t)f(X_1, C) = tf_{X_1}(A) + m(1-t)f_{X_1}(C), \end{aligned}$$

where $X_1 \in B(H_1)$ with spectra in $[0,b]$. It shows the operator m-convexity of f_{X_1}.

The fact that $f_{X_2} : [0,b] \to \mathbb{R}, f_{X_2}(u) := f(u, X_2)$ is also operator m-convex on $[0,b]$ for all operators $X_2 \in B(H_2)$ with spectra in $[0,d]$ goes likewise and we shall omit the details.

In [21], authors gave a mapping $f : [0,1]^2 \to \mathbb{R}_0$ defined by $f(r_1, r_2) = r_1 \times r_2$ which is operator convex on the co-ordinates but is not operator convex. We consider the same function with $m = 1$ to prove that the operator m-convexity on the co-ordinates does not imply the operator m-convexity.

The Lemma 2.1 is thus proved. □

Similarly, we state the following elementary results without proof.

Lemma 2.2. *For $b > 0$, $d > 0$, and some fixed $(\alpha, m) \in [0,1]^2$, every operator (α, m)-convex mapping $f : [0,b] \times [0,d] \subseteq \mathbb{R}_0^2 \to \mathbb{R}$ is operator (α, m)-convex on the co-ordinates, but the converse is not generally true.*

3. Hermite-Hadamard type inequalities for operator m-convex and (α, m)-convex functions on the co-ordinates

We will now point out some new inequalities of the Hermite-Hadamard type.

Theorem 3.1. *Let some fixed $(\alpha, m) \in (0,1]^2$ and a continuous function $f : \mathbb{R}_0^2 \to \mathbb{R}$ be operator (α, m)-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of f acting on any Hilbert spaces H_1, H_2. Then one has*

$$\int_0^1 \int_0^1 f(tA + (1-t)C, \lambda B + (1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t \le \frac{min\{v_1, v_2\} + min\{v_3, v_4\}}{2(\alpha+1)}, \tag{3.1}$$

where $(A, B), (C, D) \in B(H_1) \times B(H_2)$ with spectra in $\mathbb{R}_0^2$, and

$$\begin{aligned} v_1 &= \int_0^1 f(tA + (1-t)C, B)\,\mathrm{d}t + \alpha m \int_0^1 f\left(tA + (1-t)C, \frac{D}{m}\right)\mathrm{d}t, \\ v_2 &= \int_0^1 f(tA + (1-t)C, D)\,\mathrm{d}t + \alpha m \int_0^1 f\left(tA + (1-t)C, \frac{B}{m}\right)\mathrm{d}t, \\ v_3 &= \int_0^1 f(A, \lambda B + (1-\lambda)D)\,\mathrm{d}\lambda + \alpha m \int_0^1 f\left(\frac{C}{m}, \lambda B + (1-\lambda)D\right)\mathrm{d}\lambda, \\ v_4 &= \int_0^1 f(C, \lambda B + (1-\lambda)D)\,\mathrm{d}\lambda + \alpha m \int_0^1 f\left(\frac{A}{m}, \lambda B + (1-\lambda)D\right)\mathrm{d}\lambda. \end{aligned} \tag{3.2}$$

Proof. Since the spectrum of $tA + (1-t)C$ and $\lambda B + (1-\lambda)D$ are contained in $\mathbb{R}_0$, and f is continuous, the operator valued integrals $\int_0^1 f(tA+(1-t)C)\,\mathrm{d}t$, $\int_0^1 f(\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda$ and $\int_0^1\int_0^1 f(tA+(1-t)C, \lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda$ exist.

From the operator co-ordinated (α, m)-convexity of f and the inequality (1.10) it is easy to see that

$$\begin{aligned}&\int_0^1 f(tA+(1-t)C, \lambda B+(1-\lambda)D)\,\mathrm{d}t\\ \le & min\Bigg\{\frac{f(A, \lambda B+(1-\lambda)D)) + \alpha m f\big(\frac{C}{m}, \lambda B+(1-\lambda)D)\big)}{\alpha+1},\\ &\frac{f(C, \lambda B+(1-\lambda)D)) + \alpha m f\big(\frac{A}{m}, \lambda B+(1-\lambda)D)\big)}{\alpha+1}\Bigg\}.\end{aligned}$$

Integrating this inequality on $[0,1]$ over λ, we deduce

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C, \lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda\\ \le&\frac{1}{\alpha+1} min\Bigg\{\int_0^1 f(A, \lambda B+(1-\lambda)D)\,\mathrm{d}\lambda + \alpha m\int_0^1 f\left(\frac{C}{m}, \lambda B+(1-\lambda)D\right)\mathrm{d}\lambda,\\ &\int_0^1 f(C, \lambda B+(1-\lambda)D)\,\mathrm{d}\lambda + \alpha m\int_0^1 f\left(\frac{A}{m}, \lambda B+(1-\lambda)D\right)\mathrm{d}\lambda\Bigg\}.\end{aligned} \tag{3.3}$$

By a similar argument we get

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C, \lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\ \le&\frac{1}{\alpha+1} min\Bigg\{\int_0^1 f(tA+(1-t)C, B)\,\mathrm{d}t + \alpha m\int_0^1 f\left(tA+(1-t)C, \frac{D}{m}\right)\mathrm{d}t,\\ &\int_0^1 f(tA+(1-t)C, D)\,\mathrm{d}t + \alpha m\int_0^1 f\left(tA+(1-t)C, \frac{B}{m}\right)\mathrm{d}t\Bigg\}.\end{aligned} \tag{3.4}$$

Summing the inequalities (3.3) and (3.4) and dividing by 2, we get the inequality (3.1).

The proof thus is complete. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1, choosing $\alpha = 1$, we get the inequality for operator m-convex:*

$$\int_0^1\int_0^1 f(tA+(1-t)C, \lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t \le \frac{min\{u_1, u_2\} + min\{u_3, u_4\}}{4}, \tag{3.5}$$

where

$$\begin{aligned}u_1 &= \int_0^1 f(tA+(1-t)C, B)\,\mathrm{d}t + m\int_0^1 f\left(tA+(1-t)C, \frac{D}{m}\right)\mathrm{d}t,\\ u_2 &= \int_0^1 f(tA+(1-t)C, D)\,\mathrm{d}t + m\int_0^1 f\left(tA+(1-t)C, \frac{B}{m}\right)\mathrm{d}t,\\ u_3 &= \int_0^1 f(A, \lambda B+(1-\lambda)D)\,\mathrm{d}\lambda + m\int_0^1 f\left(\frac{C}{m}, \lambda B+(1-\lambda)D\right)\mathrm{d}\lambda,\end{aligned}$$

$$u_4=\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+m\int_0^1 f\left(\frac{A}{m},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda. \tag{3.6}$$

Furthermore, for $\alpha,m=1$ we have

$$\begin{aligned}
&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\
\le&\frac{1}{4}\bigg[\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t\\
&+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\bigg].
\end{aligned}\tag{3.7}$$

Theorem 3.2. *Let some fixed $(\alpha,m)\in(0,1]^2$ and a continuous function $f:\mathbb{R}_0^2\to\mathbb{R}$ be operator (α,m)-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of f acting on any Hilbert spaces H_1,H_2. Then one has*

$$\begin{aligned}
&f\left(\frac{A+C}{2},\frac{B+D}{2}\right)\\
\le&\frac{1}{2^{\alpha+1}}\bigg\{\int_0^1\left[f\left(\frac{A+C}{2},\lambda B+(1-\lambda)D\right)+m(2^\alpha-1)f\left(\frac{A+C}{2},\frac{(1-\lambda)B+\lambda D}{m}\right)\right]\mathrm{d}\lambda\\
&+\int_0^1\left[f\left(tA+(1-t)C,\frac{B+D}{2}\right)+m(2^\alpha-1)f\left(\frac{(1-t)A+tC}{m},\frac{B+D}{2}\right)\right]\mathrm{d}t\bigg\}\\
\le&\frac{1}{2^{\alpha+2}(\alpha+1)}\bigg\{f\left(\frac{A+C}{2},B\right)+f\left(\frac{A+C}{2},D\right)+f\left(A,\frac{B+D}{2}\right)+f\left(C,\frac{B+D}{2}\right)\\
&+m(\alpha+2^\alpha-1)\left[f\left(\frac{A+C}{2},\frac{B}{m}\right)+f\left(\frac{A+C}{2},\frac{D}{m}\right)+f\left(\frac{A}{m},\frac{B+D}{2}\right)+f\left(\frac{C}{m},\frac{B+D}{2}\right)\right]\\
&+\alpha m^2(2^\alpha-1)\bigg[f\left(\frac{A+C}{2},\frac{B}{m^2}\right)+f\left(\frac{A+C}{2},\frac{D}{m^2}\right)\\
&+f\left(\frac{A}{m^2},\frac{B+D}{2}\right)+f\left(\frac{C}{m^2},\frac{B+D}{2}\right)\bigg]\bigg\},
\end{aligned}\tag{3.8}$$

where $(A,B),(C,D)\in B(H_1)\times B(H_2)$ with spectra in $\mathbb{R}_0^2$.

Proof. By operator co-ordinated (α,m)-convexity of f and and the inequality (1.11), we can give

$$\begin{aligned}
&f\left(\frac{A+C}{2},\frac{B+D}{2}\right)\\
\le&\frac{1}{2^\alpha}\int_0^1\left[f\left(tA+(1-t)C,\frac{B+D}{2}\right)+m(2^\alpha-1)f\left(\frac{(1-t)A+tC}{m},\frac{B+D}{2}\right)\right]\mathrm{d}t\\
\le&\frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f\left(A,\frac{B+D}{2}\right)+f\left(C,\frac{B+D}{2}\right)\\
&+m(\alpha+2^\alpha-1)\left[f\left(\frac{A}{m},\frac{B+D}{2}\right)+f\left(\frac{C}{m},\frac{B+D}{2}\right)\right]\\
&+\alpha m^2(2^\alpha-1)\left[f\left(\frac{A}{m^2},\frac{B+D}{2}\right)+f\left(\frac{C}{m^2},\frac{B+D}{2}\right)\right]\bigg\}
\end{aligned}\tag{3.9}$$

and

$$f\left(\frac{A+C}{2},\frac{B+D}{2}\right)$$

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$$\le \frac{1}{2^\alpha}\int_0^1\left[f\left(\frac{A+C}{2},\lambda B+(1-\lambda)D\right)+m(2^\alpha-1)f\left(\frac{A+C}{2},\frac{(1-\lambda)B+\lambda D}{m}\right)\right]\mathrm{d}\lambda$$
$$\le \frac{1}{2^{\alpha+1}(\alpha+1)}\left\{f\left(\frac{A+C}{2},B\right)+f\left(\frac{A+C}{2},D\right)\right.$$
$$+m(\alpha+2^\alpha-1)\left[f\left(\frac{A+C}{2},\frac{B}{m}\right)+f\left(\frac{A+C}{2},\frac{D}{m}\right)\right]$$
$$\left.+\alpha m^2(2^\alpha-1)\left[f\left(\frac{A+C}{2},\frac{B}{m^2}\right)+f\left(\frac{A+C}{2},\frac{D}{m^2}\right)\right]\right\} \tag{3.10}$$

Summing the inequalities (3.9) and (3.10) and dividing by 2, we get the inequality (3.8). The proof is completed. □

Corollary 3.2.1. *Under the assumptions of Theorem 3.2, choosing $\alpha=1$, we get the inequality for operator m-convex:*

$$f\left(\frac{A+C}{2},\frac{B+D}{2}\right)$$
$$\le\frac{1}{4}\left\{\int_0^1\left[f\left(\frac{A+C}{2},\lambda B+(1-\lambda)D\right)+mf\left(\frac{A+C}{2},\frac{(1-\lambda)B+\lambda D}{m}\right)\right]\mathrm{d}\lambda\right.$$
$$\left.+\int_0^1\left[f\left(tA+(1-t)C,\frac{B+D}{2}\right)+mf\left(\frac{(1-t)A+tC}{m},\frac{B+D}{2}\right)\right]\mathrm{d}t\right\}$$
$$\le\frac{1}{16}\left\{f\left(\frac{A+C}{2},B\right)+f\left(\frac{A+C}{2},D\right)+f\left(A,\frac{B+D}{2}\right)+f\left(C,\frac{B+D}{2}\right)\right.$$
$$+2m\left[f\left(\frac{A+C}{2},\frac{B}{m}\right)+f\left(\frac{A+C}{2},\frac{D}{m}\right)+f\left(\frac{A}{m},\frac{B+D}{2}\right)+f\left(\frac{C}{m},\frac{B+D}{2}\right)\right]$$
$$\left.+m^2\left[f\left(\frac{A+C}{2},\frac{B}{m^2}\right)+f\left(\frac{A+C}{2},\frac{D}{m^2}\right)+f\left(\frac{A}{m^2},\frac{B+D}{2}\right)+f\left(\frac{C}{m^2},\frac{B+D}{2}\right)\right]\right\}. \tag{3.11}$$

Furthermore, for $\alpha,m=1$ we have

$$f\left(\frac{A+C}{2},\frac{B+D}{2}\right)$$
$$\le\frac{1}{2}\left[\int_0^1 f\left(\frac{A+C}{2},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda+\int_0^1 f\left(tA+(1-t)C,\frac{B+D}{2}\right)\mathrm{d}t\right]$$
$$\le\frac{1}{4}\left[f\left(\frac{A+C}{2},B\right)+f\left(\frac{A+C}{2},D\right)+f\left(A,\frac{B+D}{2}\right)+f\left(C,\frac{B+D}{2}\right)\right]. \tag{3.12}$$

Theorem 3.3. *Let some fixed $(\alpha,m)\in(0,1]^2$ and a continuous function $f:\mathbb{R}_0^2\to\mathbb{R}$ be operator (α,m)-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of f acting on any Hilbert spaces H_1,H_2. Then one has*

$$\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t$$
$$\le\frac{1}{4(\alpha+1)}\left\{\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t\right.$$
$$+\alpha m\left[\int_0^1 f\left(tA+(1-t)C,\frac{B}{m}\right)\mathrm{d}t+\int_0^1 f\left(tA+(1-t)C,\frac{D}{m}\right)\mathrm{d}t\right]$$
$$+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda$$

$$+\alpha m\left[\int_0^1 f\left(\frac{A}{m},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda+\int_0^1 f\left(\frac{C}{m},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda\right]\Bigg\}, \tag{3.13}$$

where $(A,B),(C,D)\in B(H_1)\times B(H_2)$ *with spectra in* $\mathbb{R}_0^2$*.*

Proof. Using operator co-ordinated (α,m)-convexity of f and the inequality (1.12), we can write

$$\begin{aligned}&\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\\ \le&\frac{1}{2(\alpha+1)}\Bigg\{f(A,\lambda B+(1-\lambda)D)+f(C,\lambda B+(1-\lambda)D)\\ &+\alpha m\left[f\left(\frac{A}{m},\lambda B+(1-\lambda)D\right)+f\left(\frac{C}{m},\lambda B+(1-\lambda)D\right)\right]\Bigg\}.\end{aligned}$$

Integrating this inequality on $[0,1]$ over λ, we deduce

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda\\ \le&\frac{1}{2(\alpha+1)}\Bigg\{\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\\ &+\alpha m\left[\int_0^1 f\left(\frac{A}{m},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda+\int_0^1 f\left(\frac{C}{m},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda\right]\Bigg\}.\end{aligned} \tag{3.14}$$

By a similar argument we get

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\ \le&\frac{1}{2(\alpha+1)}\Bigg\{\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t\\ &+\alpha m\left[\int_0^1 f\left(tA+(1-t)C,\frac{B}{m}\right)\mathrm{d}t+\int_0^1 f\left(tA+(1-t)C,\frac{D}{m}\right)\mathrm{d}t\right]\Bigg\}.\end{aligned} \tag{3.15}$$

Summing the inequalities (3.14) and (3.15) and dividing by 2, we get the inequality (3.13).

The proof thus is complete. □

Corollary 3.3.1. *Under the assumptions of Theorem 3.3, choosing* $\alpha=1$*, we get the inequality for operator* m*-convex:*

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\ \le&\frac{1}{8}\Bigg\{\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t+m\left[\int_0^1 f\left(tA+(1-t)C,\frac{B}{m}\right)\mathrm{d}t\right.\\ &\left.+\int_0^1 f\left(tA+(1-t)C,\frac{D}{m}\right)\mathrm{d}t\right]+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\\ &+m\left[\int_0^1 f\left(\frac{A}{m},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda+\int_0^1 f\left(\frac{C}{m},\lambda B+(1-\lambda)D\right)\mathrm{d}\lambda\right]\Bigg\}.\end{aligned} \tag{3.16}$$

Furthermore, for $\alpha,m=1$ *we have*

$$\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t$$

$$\leq \frac{1}{4}\bigg[\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t$$
$$+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\bigg]. \tag{3.17}$$

Theorem 3.4. *Let some fixed $(\alpha, m) \in (0,1]^2$ and a continuous function $f : \mathbb{R}_0^2 \to \mathbb{R}$ be operator (α, m)-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of f acting on any Hilbert spaces H_1, H_2. Then one has*

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m\alpha}{2(\alpha+1)}\bigg[\int_0^1 f(tA+m(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+m(1-t)C,D)\,\mathrm{d}t$$
$$+\int_0^1 f(A,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda\bigg], \tag{3.18}$$

where $(A, B), (C, D) \in B(H_1) \times B(H_2)$ with spectra in $\mathbb{R}_0^2$.

Proof. Using operator co-ordinated (α, m)-convexity of f and the inequality (1.13), we can write

$$\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t$$
$$\leq \frac{(1+m\alpha)[f(A,\lambda B+m(1-\lambda)D)+f(C,\lambda B+m(1-\lambda)D]}{\alpha+1}.$$

Integrating this inequality on $[0, 1]$ over λ, we deduce

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m\alpha}{\alpha+1}\bigg[\int_0^1 f(A,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda\bigg]. \tag{3.19}$$

By a similar argument we get

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m\alpha}{\alpha+1}\bigg[\int_0^1 f(tA+m(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+m(1-t)C,D)\,\mathrm{d}t\bigg]. \tag{3.20}$$

Summing the inequalities (3.19) and (3.20) and dividing by 2, we get the inequality (3.18).
The proof thus is complete. □

Corollary 3.4.1. *Under the assumptions of Theorem 3.3, choosing $\alpha = 1$, we get the inequality for operator m-convex:*

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m}{4}\bigg[\int_0^1 f(tA+m(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+m(1-t)C,D)\,\mathrm{d}t$$
$$+\int_0^1 f(A,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda\bigg]. \tag{3.21}$$

Furthermore, for $\alpha, m = 1$ we have

$$\begin{aligned}
&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\
\le&\frac{1}{4}\bigg[\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t\\
&+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\bigg].
\end{aligned}\tag{3.22}$$

Theorem 3.5. *Let some fixed $(\alpha, m)\in(0,1]^2$ and a continuous function $f:\mathbb{R}_0^2\to\mathbb{R}$ be operator (α, m)-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of f acting on any Hilbert spaces H_1, H_2. Then one has*

$$\begin{aligned}
&f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
\le&\frac{1}{2^{\alpha+1}}\bigg[\int_0^1 f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\lambda(2-m)D+(1-\lambda)m^2B\bigg)\,\mathrm{d}\lambda\\
&+m(2^\alpha-1)\int_0^1 f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(1-\lambda)(2-m)D+\lambda m^2B}{m}\bigg)\,\mathrm{d}\lambda\\
&+\int_0^1 f\bigg(t(2-m)C+(1-t)m^2A,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\,\mathrm{d}t\\
&+m(2^\alpha-1)\int_0^1 f\bigg(\frac{(1-t)(2-m)C+tm^2A}{m},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\,\mathrm{d}t\bigg]\\
\le&\frac{1}{2^{\alpha+2}(\alpha+1)}\bigg[f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),(2-m)D\bigg)+m\big(\alpha+2^\alpha-1\big)f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),mB\bigg)\\
&+m^2\alpha(2^\alpha-1)f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(2-m)D}{m^2}\bigg)+f\bigg((2-m)C,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m\big(\alpha+2^\alpha-1\big)f\bigg(mA,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m^2\alpha(2^\alpha-1)f\bigg(\frac{(2-m)C}{m^2},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\bigg],
\end{aligned}\tag{3.23}$$

where $(A,B),(C,D)\in B(H_1)\times B(H_2)$ with spectra in $\mathbb{R}_0^2$.

Proof. From operator co-ordinated (α, m)-convexity of f and the inequality (1.14), we can deduce

$$\begin{aligned}
&f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
\le&\frac{1}{2^\alpha}\int_0^1\bigg[f\bigg(t(2-m)C+(1-t)m^2A,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m(2^\alpha-1)f\bigg(\frac{(1-t)(2-m)C+tm^2A}{m},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\bigg]\,\mathrm{d}t\\
\le&\frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f\bigg((2-m)C,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)+m\big(\alpha+2^\alpha-1\big)f\bigg(mA,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m^2\alpha(2^\alpha-1)f\bigg(\frac{(2-m)C}{m^2},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\bigg\}
\end{aligned}\tag{3.24}$$

and

$$
\begin{aligned}
& f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\Big)\\
\le & \frac{1}{2^\alpha}\int_0^1\Big[f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\lambda(2-m)D+(1-\lambda)m^2B\Big)\\
& +m(2^\alpha-1)f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(1-\lambda)(2-m)D+\lambda m^2B}{m}\Big)\Big]\,\mathrm{d}\lambda\\
\le & \frac{1}{2^{\alpha+1}(\alpha+1)}\Big\{f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),(2-m)D\Big)+m\big(\alpha+2^\alpha-1\big)f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),mB\Big)\\
& +m^2\alpha(2^\alpha-1)f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(2-m)D}{m^2}\Big)\Big\}. \qquad (3.25)
\end{aligned}
$$

Summing the inequalities (3.24) and (3.25) and dividing by 2, we get the inequality (3.23).
The proof is completed. □

Corollary 3.5.1. *Under the assumptions of Theorem 3.5, choosing $\alpha=1$, we get the inequality for operator m-convex:*

$$
\begin{aligned}
& f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\Big)\\
\le & \frac{1}{4}\Big[\int_0^1 f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\lambda(2-m)D+(1-\lambda)m^2B\Big)\,\mathrm{d}\lambda\\
& +m\int_0^1 f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(1-\lambda)(2-m)D+\lambda m^2B}{m}\Big)\,\mathrm{d}\lambda\\
& +\int_0^1 f\Big(t(2-m)C+(1-t)m^2A,\frac{2-m}{2}D+\frac{m}{2}(mB)\Big)\,\mathrm{d}t\\
& +m\int_0^1 f\Big(\frac{(1-t)(2-m)C+tm^2A}{m},\frac{2-m}{2}D+\frac{m}{2}(mB)\Big)\,\mathrm{d}t\Big]\\
\le & \frac{1}{16}\Big[f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),(2-m)D\Big)+2mf\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),mB\Big)\\
& +m^2f\Big(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(2-m)D}{m^2}\Big)+f\Big((2-m)C,\frac{2-m}{2}D+\frac{m}{2}(mB)\Big)\\
& +2mf\Big(mA,\frac{2-m}{2}D+\frac{m}{2}(mB)\Big)\\
& +m^2f\Big(\frac{(2-m)C}{m^2},\frac{2-m}{2}D+\frac{m}{2}(mB)\Big)\Big], \qquad (3.26)
\end{aligned}
$$

Furthermore, for $\alpha, m=1$ we have

$$
\begin{aligned}
& f\Big(\frac{A+C}{2},\frac{B+D}{2}\Big)\\
\le & \frac{1}{2}\Big[\int_0^1 f\Big(\frac{A+C}{2},\lambda B+(1-\lambda)D\Big)\,\mathrm{d}\lambda+\int_0^1 f\Big(tA+(1-t)C,\frac{B+D}{2}\Big)\,\mathrm{d}t\Big]\\
\le & \frac{1}{4}\Big[f\Big(\frac{A+C}{2},B\Big)+f\Big(\frac{A+C}{2},D\Big)+f\Big(A,\frac{B+D}{2}\Big)+f\Big(C,\frac{B+D}{2}\Big)\Big]. \qquad (3.27)
\end{aligned}
$$

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ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES: A FIXED POINT APPROACH

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Abstract. Let

$$\begin{aligned} M_1f(x,y): &= \frac{3}{4}f(x+y)-\frac{1}{4}f(-x-y)+\frac{1}{4}f(x-y)+\frac{1}{4}f(y-x)-f(x)-f(y), \\ M_2f(x,y): &= 2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)-f(x)-f(y). \end{aligned}$$

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities

$$N\left(M_1f(x,y)-\rho M_2f(x,y),t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{0.1}$$

and

$$N\left(M_2f(x,y)-\rho M_1f(x,y),t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{0.2}$$

in fuzzy Banach spaces, where ρ is a fixed real number with $\rho \neq 1$.

1. Introduction and preliminaries

Katsaras [19] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [15, 21, 48]. In particular, Bag and Samanta [3], following Cheng and Mordeson [11], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [20]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 25, 26] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [3, 25, 26, 27] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

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The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [25, 28].

Definition 1.2. [3, 28, 26, 27] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [3, 28, 26, 27] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

The stability problem of functional equations originated from a question of Ulam [47] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [39] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [46] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 5, 9, 10, 14, 22, 24, 29, 34, 35, 36, 40, 41, 42, 43, 44, 45, 49, 50]).

We recall a fundamental result in fixed point theory.

Theorem 1.4. [6, 13] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \le \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [7, 8, 30, 31, 38]).

Park [32, 33] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space. Let ρ be a real number with $\rho \ne 1$.

2. Additive-quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces.

Theorem 2.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{4}\varphi(2x, 2y) \le \frac{L}{2}\varphi(2x, 2y) \tag{2.1}$$

for all $x, y \in X$.
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N(M_1 f(x, y) - \rho M_2 f(x, y), t) \ge \frac{t}{t + \varphi(x, y)} \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x, x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \ge \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x, x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. (i) Letting $y = x$ in (2.2), we get

$$N(f(2x) - 2f(x), t) \ge \frac{t}{t + \varphi(x, x)} \tag{2.5}$$

and so

$$N\left(f(x)-2f\left(\frac{x}{2}\right),t\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2},\frac{x}{2}\right)} \tag{2.6}$$

for all $x\in X$.

Consider the set

$$S:=\{g:X\to Y\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+:N(g(x)-h(x),\mu t)\geq\frac{t}{t+\varphi(x,x)},\ \forall x\in X,\forall t>0\right\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that (S,d) is complete (see [23, Lemma 2.1]).

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=2g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x\in X$ and all $t>0$. Hence

$$\begin{aligned}N(Jg(x)-Jh(x),L\varepsilon t)&=N\left(2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right),L\varepsilon t\right)=N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{2}\varepsilon t\right)\\&\geq\frac{\frac{Lt}{2}}{\frac{Lt}{2}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)}\geq\frac{\frac{Lt}{2}}{\frac{Lt}{2}+\frac{L}{2}\varphi(x,x)}=\frac{t}{t+\varphi(x,x)}\end{aligned}$$

for all $x\in X$ and all $t>0$. So $d(g,h)=\varepsilon$ implies that $d(Jg,Jh)\leq L\varepsilon$. This means that

$$d(Jg,Jh)\leq Ld(g,h)$$

for all $g,h\in S$.

It follows from (2.6) that $N\left(f(x)-2f\left(\frac{x}{2}\right),\frac{L}{2}t\right)\geq\frac{t}{t+\varphi(x,x)}$ for all $x\in X$ and all $t>0$. So $d(f,Jf)\leq\frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A:X\to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right)=\frac{1}{2}A(x) \tag{2.7}$$

for all $x\in X$. Since $f:X\to Y$ is odd, $A:X\to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M=\{g\in S:d(f,g)<\infty\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu\in(0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x\in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{L}{2-2L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \ge \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

and so

$$N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), t\right) \ge \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(A(x+y) - A(x) - A(y), t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. So the mapping $A : X \to Y$ is additive.

(ii) Letting $y = x$ in (2.2), we get

$$N\left(\frac{1}{2} f(2x) - 2f(x), t\right) \ge \frac{t}{t + \varphi(x, x)} \tag{2.8}$$

and so

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \ge \frac{\frac{t}{2}}{\frac{t}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} = \frac{t}{t + 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \tag{2.9}$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \ge \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x, x)} = \frac{t}{t + \varphi(x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le L d(g, h)$$

for all $g, h \in S$.

It follows from (2.9) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.10}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

By (2.2),

$$N\left(4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

and so

$$N\left(4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(\frac{1}{2}Q(x+y) + \frac{1}{2}Q(x-y) - Q(x) - Q(y), t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. So the mapping $Q : X \to Y$ is quadratic. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_1 f(x, y) - \rho M_2 f(x, y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.11}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.11). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.12}$$

for all $x, y \in X$..
(i) *Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.
(i) It follows from (2.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.
(ii) It follows from (2.8) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.
The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (2.11). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.11). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(4-2^p)t}{(4-2^p)t + 4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

3. Additive-quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (2.1).*
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N(M_2 f(x, y) - \rho M_1 f(x, y), t) \geq \frac{t}{t + \varphi(x, y)} \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. (i) Letting $y = 0$ in (3.1), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.2}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x,0)}, \ \forall x \in X, \forall t > 0 \right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [23, Lemma 2.1]).

The rest of the proof is similar to the proof of Theorem 2.1 (i).

(ii) Letting $y = 0$ in (3.1), we get

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) = N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x,0)} \tag{3.3}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1 (ii). □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x,y) - \rho M_1 f(x,y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.4}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

Theorem 3.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying (2.12).*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.1). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(1-L)t}{(1-L)t + L\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1-L)t}{(1-L)t + L\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1.

(i) It follows from (3.2) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{2Lt}{2Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

(ii) It follows from (3.3) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{t}{4}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{4}f(2x), Lt\right) \geq \frac{4Lt}{4Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.4). Then $A(x) := N$-$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

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Optimal control of a special predator-prey system with functional response and toxicant

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Abstract

This paper is devoted to the optimal harvesting problem for a diffusive population dynamics with functional response in a polluted environment . C_0-semigroup theory is used to obtain the existence and uniqueness of the positive strong solution for the controlled system. The first order necessary optimality condition is derived by means the technique of tangent-normal cones and adjoint system of the state. The second-order necessary and sufficient optimality conditions are established by making use of the second order Fréchet derivative of the associated Lagrange function.

Keywords: Optimal harvesting; optimal conditions; functional response; toxicant

1 Introduction

The optimal control problems of population dynamics have been widely studied, such as N.C. Apreutesei [1] studied for a Lotka-Volterra system of three differential equations, some necessary conditions of optimality were founded in order to maximize the total number of individuals. W.Ko [2-3] considered a diffusive two-competing-prey and one-predator system with functional response (Beddington-DeAngelis and ratio-dependent), showed the properties for the positive steady-state solutions of the corresponding elliptic system with Robin boundary. Then N.C. Apreutesei [4] studied for a reaction-diffusion system as follows

$$\begin{cases} \dfrac{\partial y_1}{\partial t} = \alpha_1 \Delta y_1 + y_1 g_1(y_1) + u_1 y_1 - y_1 y_2 f(y_1), \\ \dfrac{\partial y_2}{\partial t} = \alpha_2 \Delta y_2 - a y_2 + b y_1 y_2 f(y_1) + c y_2 y_3 h(y_3), \\ \dfrac{\partial y_3}{\partial t} = \alpha_3 \Delta y_3 + y_3 g_3(y_3) + u_3 y_3 - y_3 y_2 h(y_3), \\ \dfrac{\partial y_i}{\partial v}(t,x) = 0,\ on\ \Sigma = [0,T] \times \partial\Omega,\ i = 1,2,3, \\ y_i(0,x) = y_i^0(x),\ x \in\ \Omega,\ i = 1,2,3. \end{cases} \tag{1.1}$$

the author considered the general functional response $y_i f(y_i)$, which contains the classical various Holling type, the existence of an optimal solution and first and second order optimality conditions

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were proved. E.Casas [5] investigated an abstract formulation for optimization problems in some L^p spaces, devoted to reduce the classical gap between the necessary and sufficient conditions for optimization problems in Banach spaces. Other models from population dynamics and optimal control problems can be found in [6-9]. However, those papers were not take into account toxicant factor. Among the practical problems, it determines the real rate of the biological individual and the behavior of individual. To this end, Luo [10-12] first formulated a new age-dependent toxicant population model in an environment with small toxicant capacity, effectively bridge the research between age-structure and polluted environment. Inspired his works, this paper propose a more realistic models with toxicant-population in a small content of the environment.

The aim of this paper is to seek the maximum of the following functional, which gives the profit from harvesting less the cost of harvesting:

$$J(u,\nu)=\sum_{i=1}^{3}\int_0^T\int_\Omega\big[K_iu_i(t,x)y_i(t,x)-\frac{1}{2}C_iu_i^2(t,x)\big]dxdt-\frac{1}{2}\int_0^T C_4[\nu(t)]^2dt. \tag{OH}$$

where K_i are selling price factors, positive constants C_i and C_4 represents the cost factors of harvesting and the cost factor of administering pollution of environment, respectively; $u=(u_1,u_2,u_3)$ are the proportions of the populations to be harvested, $\nu(t)$ is the exogenous toxicant input rate the moment t, and the state $y=(y_1,y_2,y_3)$ is the solution of the following system corresponding to (u_1,u_2,u_3):

$$\begin{cases}\dfrac{\partial y_1}{\partial t}=\alpha_1\Delta y_1+y_1[g_1(y_1)-r_1c_{10}]-y_1y_2f(y_1)-u_1y_1,\\ \dfrac{\partial y_2}{\partial t}=\alpha_2\Delta y_2-(a-r_2c_{20})y_2+by_1y_2f(y_1)+cy_2y_3h(y_3)-u_2y_2,\\ \dfrac{\partial y_3}{\partial t}=\alpha_3\Delta y_3+y_3[g_3(y_3)-r_3c_{30}]-y_3y_2h(y_3)-u_3y_3,\\ \dfrac{dc_{i0}}{dt}=kc_e(t)-gc_{i0}(t)-mc_{i0}(t), i=1,2,3,\\ \dfrac{dc_e}{dt}=-k_1c_e(t)[y_1(t)+y_2(t)+y_3(t)]+g_1\displaystyle\sum_{i=1}^{3}c_{i0}(t)y_i(t)-hc_e(t)+\nu(t)\end{cases} \tag{1.2}$$

for $(t,x)\in Q$, subject to some Neumann boundary conditions

$$\frac{\partial y_i}{\partial\nu}(t,x)=0,\ on\ \Sigma=[0,T]\times\partial\Omega,\ i=1,2,3$$

and to the initial conditions

$$y_i(0,x)=y_i^0(x),\ x\in\ \Omega,\ i=1,2,3.$$

which descried a diffusive one-predator and two-competing-prey system in a spatially inhomogeneous environment, where $Q=(0,T)\times\Omega$, Ω is a bounded domain in $\mathbb{R}^d(d\geq 1)$ with the boundary $\partial\Omega$ of class $C^{2+\sigma}(\sigma>0)$, we denote by $y_i(t,x)$ the density of individuals of ith population at the moment t and in the location $x\in\overline{\Omega}$. $c_0(t)$ is the concentration of the toxicant in an organism at the moment t, $c_e(t)$ is the concentration of the toxicant in the environment at the moment t. The function u_i is the harvesting rate of population y_i, and the coefficients $\alpha_1,\alpha_2,\alpha_3,a,b,c$ are all positive constants. For the simplicity, we have assumed that f and h depend only on y_1 and on y_3 respectively, but the reasoning and the main results remain true also in the case when f and h depend on y_2 too, parameter a is the per capita death rate of the predator.

The admissible control set is defined as

$$\mathcal{U}_{ad} = \big\{(u,\nu) \in [L^2(Q)]^3 \times L^\infty(0,T)|\ 0 \le u_i(t,x) \le 1,\ a.e\ in\ Q,\ 0 \le \nu(t) \le h\ a.e\ in\ (0,T)\big\}.$$

Throughout this paper, we always assume that:
(H_1) g_1, g_3 are continuous and bounded on $(0,\infty)$;
(H_2) f, h are continuous and positive on $(0,\infty)$ and bounded on bounded sets;
(H_3) $y_i^0 \in H^2(\Omega), y_i^0 > 0$ on Ω and $\partial y_i^0/\partial\nu = 0$ $a.e.\ on\ \partial\Omega,\ i = 1,2,3$;
(H_4) $\nu(\cdot) \in L^2[0,T], 0 \le \nu(t) \le \nu_1 < +\infty$;
(H_5) $0 \le c_{i0}(0) \le 1, 0 \le c_e(0) \le 1$;
(H_6) $g \le k \le g+m, \nu \le h$.

The paper is organized as follows: In section 2, we use results from the semigroup theory and some well-known existence theorems from [13-14] to derive the global existence and uniqueness of a positive strong solution of the controlled system (1.2), Section 3 is devoted to first order necessary optimality conditions for (OH). Necessary and sufficient second order optimality conditions are given in Section 4.

2 Basic properties of the solution

This section concerns the most important properties of the dynamics system with diffusion. Existence, uniqueness and positivity of the solution will be proved. Thus formally, system (1.1) can be written as an infinite dimensional Cauchy problem of the form

$$\begin{cases} \dfrac{dy}{dt}(t) = Ay(t) + F(t,y(t)), \quad t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\}_{t\ge 0}$ on the Banach space $\mathbb{X}$, if $\mathbb{X}$ is a Hilbert space, A is called dissipative if $(Ax,x) \le 0, \forall x \in D(A)$, and $F : [0,T] \times \mathbb{X} \to \mathbb{X}$ is measurable in t and Lipschtiz in $x \in \mathbb{X}$ uniformly with respect to t.

We shall employ a general existence result which we use in the sequel (Proposition 1.2, p.175,[14]).
Theorem 2.1 For each $y_0 \in \mathbb{X}$, the initial value problem (2.1) has a unique mild solution $y \in C([0,T];\mathbb{X})$, and

$$y(t) = S(t)y_0 + \int_0^t S(t-s)F(s,y(s))ds, \quad t \in [0,T].$$

In addition, if $\mathbb{X}$ is a Hilbert space, A is self-adjoint and dissipative on $\mathbb{X}$, and $y_0 \in D(A)$, then the mild solution is in fact a strong solution and $y \in W^{1,2}([\delta,T];X), \forall\delta \in [0,T]$.

Thus, we work in the Hilbert space $H = (L^2(\Omega))^3$, where the operator $A : D(A) \subset H \to H$,

$$A = \begin{pmatrix} \alpha_1\Delta & 0 & 0 \\ 0 & \alpha_2\Delta & 0 \\ 0 & 0 & \alpha_3\Delta \end{pmatrix}, \qquad F(t,y(t)) = \begin{pmatrix} F_1(t,y(t)) \\ F_2(t,y(t)) \\ F_3(t,y(t)) \end{pmatrix},$$

for $y = (y_1,y_2,y_3) \in D(A), D(A) = \Big\{y = (y_1,y_2,y_3) \in (H^2(\Omega))^3, \dfrac{\partial y_i}{\partial\nu} = 0\ on\ \partial\Omega, i = 1,2,3\Big\}$, $y^0 = (y_1^0,y_2^0,y_3^0)$ is the initial value of y, and $F = (F_1,F_2,F_3)$ is the nonlinear term in (2.1), that is

$$\begin{cases} F_1(t,y(t)) = y_1g_1(y_1) - y_1y_2f(y_1) - u_1y_1, \\ F_2(t,y(t)) = -ay_2 + by_1y_2f(y_1) + cy_2y_3h(y_3) - u_2y_2, \\ F_3(t,y(t)) = y_3g_3(y_3) - y_3y_2h(y_3) - u_3y_3, \end{cases} \tag{2.2}$$

Theorem 2.2 Suppose that $y^0 = (y_1^0, y_2^0, y_3^0) \in D(A)$, and $y_i^0 > 0, i = 1,2,3$. Then for each $u \in \mathcal{U}_{ad}$, the system (2.1) has a unique nonnegative solution $(y(t,x), c_0(t), c_e(t))$, such that

(i) $(y_i(t,x), c_{i0}(t), c_e(t)) \in L^\infty(Q) \cap L^2(0,T;H^2(\Omega) \cap L^\infty(0,T;H^1(\Omega))) \times L^\infty(0,T) \times L^\infty(0,T)$,

(ii) $0 \le c_{i0}(t) \le 1, 0 \le c_e(t) \le 1, \forall\, t \in (0,T)$.

Proof Since F is not satisfy Lipschtiz conditions, we cannot apply the theorem 2.1 directly for our problem, usually we use a truncation procedure for F, consider the truncated initial value problem

$$\begin{cases} \dfrac{\partial y^N}{\partial t}(t,x) = Ay^N(t) + F^N(t,y(t)), \quad t \in [0,T], \\ y^N(0) = y_0, \end{cases} \tag{2.3}$$

where $F^N = (F_1^N, F_2^N, F_3^N)$ is obtained from $F = (F_1, F_2, F_3)$, a fixed large number $N > 0$. If $|y_i| \le N$, then y_i in $F(t, y_1, y_2, y_3)$ remains unchanged, if $y_i > N$, then y_i from (2.2) is replaced by N, if $y_i < -N$, then y_i from (2.2) is replaced by $-N$. Thus function F^N becomes Lipschitz continuous with respect to t, according theorem 2.1, the problem (2.3) admits a unique strong solution $y^N = (y_1^N, y_2^N, y_3^N) \in W^{1,2}([\delta,T];H) \cap L^2(0,T;D(A)), \forall \delta \in [0,T]$.

To begin with, we shall that $y \in L^2(0,T;H^2(\Omega) \cap L^\infty(0,T;H^1(\Omega))$. On one hand, from theorem 2.1 we know $y \in L^2(0,T;H^2(\Omega)$, on the other hand, from (2.3) we derive that

$$\int_Q \Big|\frac{\partial y_1^N}{\partial t}\Big|^2 dsdx - 2\alpha_1 \int_Q \frac{\partial y_1^N}{\partial t} \Delta y_1^N dsdx + \alpha^2 \int_Q |\Delta y_1^N|^2 dsdx = \int_Q |F_1(t,y(t))|^2 dsdx,$$

Using the regularity of y_1^N and the Green's formula, we have

$$\int_Q \Big|\frac{\partial y_1^N}{\partial t}\Big|^2 dsdx + 2\alpha_1 \int_\Omega |\nabla y_1^N|^2 dx + \alpha^2 \int_Q |\Delta y_1^N|^2 dsdx = \int_Q |F_1(t,y(t))|^2 dsdx + 2\alpha_1 \int_\Omega |\nabla y_1^0|^2 dx.$$

Since $y_1^N \in W^{1,2}(0,T;H)$ and $y_1^0 \in H^2(\Omega)$, by the Lipschitz property of F_1^N we deduce that

$$2\alpha_1 \int_\Omega |\nabla y_1^N|^2 dx \le \int_Q |y^N| dxds + 2\alpha_1 \int_\Omega |\nabla y_1^0|^2 dx < +\infty.$$

Thus, we have $y_1 \in L^\infty(0,T;H^1(\Omega))$, analogously y_2, y_3 are proved.

Further more, it remains to prove that $y^N \in L^\infty(Q)$, $(c_{i0}, c_e) \in L^\infty(0,T)$. Indeed, consider the following auxiliary initial value problems

$$\begin{cases} \dfrac{\partial \rho_1^N}{\partial t}(t,x) = \Delta\rho_1^N(t) + F_1^N(t,y(t)) - M_N, \quad t \in [0,T], \\ \rho_1^N(0) = y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \end{cases} \tag{2.4}$$

and

$$\begin{cases} \dfrac{\partial \omega_1^N}{\partial t}(t,x) = \Delta\omega_1^N(t) + F_1^N(t,y(t)) + M_N, \quad t \in [0,T], \\ \omega_1^N(0) = y_1^0 + \|y_1^0\|_{L^\infty(\Omega)}, \end{cases} \tag{2.5}$$

where $M_N = \max\Big\{\|F_i^N(\cdot, y(t))\|_{L^\infty(Q)}, \|y_i^0\|_{L^\infty(Q)}, i = 1,2,3\Big\}$.

By theorem 2.1 the function ρ_1^N and ω_1^N in $C([0,T];\mathbb{X})$ is a mild solution to problem (2.4) and (2.5), the solution of these can be written as

$$\rho_1(t) = S(t)(y_1^0 - \|y_1^0\|_{L^\infty(\Omega)}) + \int_0^t S(t-s)(F_1^N(s, y_1, y_2, y_3) - M_N)ds,$$

$$\omega(t) = S(t)(y_1^0 + \|y_1^0\|_{L^\infty(\Omega)}) + \int_0^t S(t-s)(F_1^N(s, y_1, y_2, y_3) + M_N)ds.$$

Remark that their solutions are

$$\rho_1(t,x) = y_1^N(t,x) - M_N t - \|y_1^0\|_{L^\infty(\Omega)}, \quad \omega_1(t,x) = y_1^N(t,x) + M_N t + \|y_1^0\|_{L^\infty(\Omega)},$$

since $|F_1^N(t, y^N)| \leq M_N$, from the comparison principle of linear parabolic equation, we deduce that $\rho_1^N(0) \leq 0,\ \omega_1^N(0) \geq 0$, that is

$$|y_1^N(t,x)| \leq M_N t + \|y_1^0\|_{L^\infty(\Omega)},$$

and in the same manner to prove that y_2^N, y_3^N hold for $(t,x) \in Q$. Therefor, $y_i^N \in L^\infty(Q)$. To prove $(c_{i0}, c_e) \in L^\infty(0,T)$, we define $G : \mathbb{X} \to \mathbb{X}$, from (1.2) we can deduce that

$$G_i(t) = c_{i0}(t) = c_{i0}(0)\exp\{-(g+m)t\} + k\int_0^t c_e(s)\exp\big\{(s-t)(g+m)\big\}ds, i = 1,2,3, \tag{2.6}$$

$$\begin{aligned} G_4(t) = c_e(t) = c_e(0)\exp\Big\{-\int_0^t \big[\sum_{i=1}^3 k_1 y_i(\tau) + h\big]d\tau\Big\} \\ + \int_0^t \Big[g_1\big(\sum_{i=1}^3 c_{i0}(s)y_i(s)\big) + \nu(s)\Big]\exp\Big\{\int_t^s \big[\sum_{i=1}^3 k_1 y_i(\tau) + h\big]d\tau\Big\}ds, \end{aligned} \tag{2.7}$$

if the hypothesis (H_6) hold, it is clear that $0 \leq c_{i0}(t) \leq 1, 0 \leq c_e(t) \leq 1$ and $(c_{i0}, c_e) \in L^\infty(0,T),\ i = 1,2,3.$ the specific process can refer to [15].

Moreover, we shall that y_i^N are positive on Q, to this end, let $y_i^N = (y_i^N)^+ - (y_i^N)^-$, where

$$(y_i^N)^+(t,x) = \sup\{y_i^N(t,x), 0\}, \quad (y_i^N)^-(t,x) = -\inf\{y_i^N(t,x), 0\},\ i = 1,2,3. \tag{2.8}$$

Multiplying the first equation from (2.1) by y_1^N we have

$$\frac{1}{2}\frac{\partial}{\partial t}|(y_1^N)^-|^2 = \alpha_1 (y_1^N)^-\Delta(y_1^N)^- + |(y_1^N)^-|^2\big[g_1(y_1^N) - y_2^N f(y_1^N) - u_1\big]. \tag{2.9}$$

Integrating (2.9) on Ω and using Greens formula we get

$$\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}|(y_1^N)^-|^2 dx = -\alpha_1\int_\Omega |\nabla(y_1^N)^-|^2 dx + \int_\Omega |(y_1^N)^-|^2\big[g_1(y_1^N) - y_2^N f(y_1^N) - u_1\big]dx.$$

By integrating over $[0,t]$, for $t \in [0,T]$, and take into consideration of the uniformly boundedness of y_i^N, it is not difficult to see that there exists a constant $C^N > 0$ depending on N such that

$$\frac{1}{2}\int_\Omega |(y_1^N)^-|^2 dx + \alpha_1\int_0^t\int_\Omega |\nabla(y_1^N)^-|^2 dxds \leq C^N\int_0^t\int_\Omega |y_1^N(s)|^2 dxds.$$

Gronwalls inequality lead to

$$\int_\Omega |(y_1^N)^-|^2 dx \leq 0, \quad \forall t \in [0,T],$$

that is $(y_1^N)^- = 0$, by the definition of (2.8) we conclude that $y_1^N(t,x) > 0$, analogously we get $y_2^N(t,x) > 0$ and $y_3^N(t,x) > 0$.

In addition, we prove the uniqueness of the solution. For any (y^1, c_0^1, c_e^1) and (y^2, c_0^2, c_e^2) are two solutions of problem (1.2), where $y^1 = (y_1^1, y_2^1, y_3^1)$, $c_0^1 = (c_{10}^1, c_{20}^1, c_{30}^1)$, $y^2 = (y_1^2, y_2^2, y_3^2)$, $c_0^2 = (c_{10}^2, c_{20}^2, c_{30}^2)$, we denote by $\varphi = y^1 - y^2$, then φ is the solution of

$$\begin{cases} \dfrac{\partial \varphi_1}{\partial t} = \alpha_1 \Delta \varphi_1 + F_1(t, y_1^1, y_2^1, y_3^1) - F_1(t, y_1^2, y_2^2, y_3^2), \\ \dfrac{\partial \varphi_2}{\partial t} = \alpha_2 \Delta \varphi_2 + F_2(t, y_1^1, y_2^1, y_3^1) - F_2(t, y_1^2, y_2^2, y_3^2), \\ \dfrac{\partial \varphi_3}{\partial t} = \alpha_3 \Delta \varphi_3 + F_3(t, y_1^1, y_2^1, y_3^1) - F_3(t, y_1^2, y_2^2, y_3^2), \\ \dfrac{\partial \varphi_1}{\partial \nu} = \dfrac{\partial \varphi_2}{\partial \nu} = \dfrac{\partial \varphi_3}{\partial \nu} = 0, \ on \ \Sigma, \\ \varphi_1(0,x) = \varphi_2(0,x) = \varphi_3(0,x) = 0. \end{cases} \tag{2.10}$$

Suppose $g_1, g_3, f, h \in C^1[0, \infty)$, g_1, g_3 are bounded and f, h are positive and have at most polynomial growth, then from (2.2) we obtain

$$|F_i(t, y_1^1, y_2^1, y_3^1) - F_i(t, y_1^2, y_2^2, y_3^2)| \le c(|\varphi_1| + |\varphi_2| + |\varphi_3|),$$

where c is a positive constant. Multiplying (2.10) by $\varphi_1, \varphi_2, \varphi_3$ respectively, and integrating on $\Omega_T = \Omega \times (0, t)$ we get

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{3}\int_\Omega |\varphi_i(t)|^2 dx + \sum_{i=1}^{3}\int_{\Omega_T} \alpha_i |\nabla \varphi_i|^2 dsdx &= \sum_{i=1}^{3}\int_{\Omega_T} \varphi_i (F_i(t, y_1^1, y_2^1, y_3^1) - F_i(t, y_1^2, y_2^2, y_3^2))dsdx \\ &\le C\int_0^t \int_\Omega (|\varphi_1(s)|^2 + |\varphi_2(s)|^2 + |\varphi_3(s)|^2)dsdx. \end{aligned} \tag{2.11}$$

From (2.11) and Gronwall's lemma we have

$$\int_\Omega (|\varphi_1(s)|^2 + |\varphi_2(s)|^2 + |\varphi_3(s)|^2) \le 0,$$

which yields that $\varphi_1 = \varphi_2 = \varphi_3 = 0$, thus we have proved the uniqueness of the y_i. However, we can follow by (2.6) and (2.7)

$$|c_0^1(t) - c_0^2(t)| = \sum_{i=1}^{3} |c_{i0}^1(t) - c_{i0}^2(t)| \le 3k \int_0^t |c_e^1(s) - c_e^2(s)|ds, \ i = 1, 2, 3. \tag{2.12}$$

$$|c_e^1(t) - c_e^2(t)| \le M_1 \sum_{i=1}^{3} \int_0^t |c_{i0}^1(s) - c_{i0}^2(s)|ds, \tag{2.13}$$

where M_1 is constant. We define an equivalent norm in $\mathbb{X}$ as follows:

$$\|(c_{i0}, c_e)\|_* = Ess \sup_{t \in (0,T)} e^{-\lambda t} \Big\{ \sum_{i=1}^{3} |c_{i0}(t)| + |c_e(t)| \Big\},$$

by (2.11) and (2.12) we obtain

$$\begin{aligned}
\|G(x^1)-G(x)^2\|_* &= \|G_i(x^1)-G_i(x^2), G_4(x^1)-G_4(x^2)\|_* \\
&\leq M_2 Ess \sup_{t\in(0,T)} e^{-\lambda t}\int_0^t \Big\{\sum_{i=1}^{3}|c_{i0}^1(s)-c_{i0}^2(s)|+|c_e^1(s)-c_e^2(s)|\Big\}ds \\
&\leq M_2 Ess \sup_{t\in(0,T)} e^{-\lambda t}\int_0^t e^{\lambda s}\Big\{e^{-\lambda s}[\sum_{i=1}^{3}|c_{i0}^1(s)-c_{i0}^2(s)|+|c_e^1(s)-c_e^2(s)|]\Big\}ds \\
&\leq M_2\|x^1-x^2\|_* Ess \sup_{t\in(0,T)}\{e^{-\lambda t}\int_0^t e^{\lambda s}ds\} \\
&\leq \frac{M_2}{\lambda}\|x^1-x^2\|_*,
\end{aligned}$$

where M_2 is constant, choose $\lambda > M_2$ yields that G is a strict contraction on $(\mathbb{X}, \|\cdot\|_*)$ and consequently has a unique fixed point.

Thus, the system (1.2) has a unique solution (y_i, c_{i0}, c_e). the proof is completed.

3 Necessary optimality conditions

In this section, we find some necessary optimality conditions in order to maximize the profit from harvesting less the cost of harvesting.

Theorem 3.1 If (u^*,ν^*) is an optimal control and (y^*, c_{i0}^*, c_e^*) is the corresponding optimal state, then

$$\begin{aligned}
u_i^*(t,x) &= \mathcal{L}_i\Big(\frac{(K_i-q_i)y_i^*}{C_i}\Big),\ i=1,2,3,\ a.e.\ in\ Q, \\
\nu^*(t) &= \mathcal{L}_4\Big(\frac{q_7(t)}{C_4}\Big),\ a.e.\ in\ (0,T),
\end{aligned} \tag{3.1}$$

where

$$\mathcal{L}_j(x) = \begin{cases} 0 & x<0 \\ x & 0\leq x\leq H_j, \quad j=1,2,3,4. \\ H_j & x>H_j \end{cases}$$

and $q=(q_1,q_2,\ldots,q_7)$ is the solution of following adjoint system corresponding to (u^*,ν^*).

$$\begin{cases}
\dfrac{\partial q_1}{\partial t} = -\alpha_1\Delta q_1 + [g_1(y_1^*) - r_1 c_{10}^* + y_1^* g_1'(y_1^*) - u_1^* - y_2^* f(y_1^*) - y_1^* y_2^* f'(y_1^*)]q_1 \\
\qquad - [by_2^* f(y_1^*) + by_1^* y_2^* f'(y_1^*)]q_2 + [k_1 c_e^* - g_1 c_{10}^*]q_7 + K_1 u_1^*, \\
\dfrac{\partial q_2}{\partial t} = -\alpha_2\Delta q_2 + y_1^* f(y_1^*)q_1 - [-a - r_2 c_{20}^* + by_1^* f(y_1^*) + cy_3^* h(y_3^*)]q_2 \\
\qquad + y_3^* h(y_3^*)q_3 + [k_1 c_e^* - g_1 c_{20}^*]q_7 + K_2 u_2^*, \\
\dfrac{\partial q_3}{\partial t} = -\alpha_3\Delta q_3 - [g_3(y_3^*) - r_3 c_{30}^* + y_3^* g_3'(y_1^*) - u_3^* - y_2^* h(y_3^*) - y_3^* y_2^* h'(y_3^*)]q_3 \\
\qquad - cy_2^*[h(y_3^*) + y_3^* h'(y_3^*)]q_2 + [k_1 c_e^* - g_1 c_{30}^*]q_7 + K_3 u_3^*, \\
\dfrac{\partial q_j}{\partial t} = (g+m)q_j - g_1 y_i^* q_7,\ j=i+3,\ i=1,2,3, \\
\dfrac{\partial q_7}{\partial t} = -k\sum_{j=4}^{6} q_j + k_1\sum_{i=1}^{3} y_i q_7 + hq_7, \\
q_i(T,x) = 0, \quad x\in\Omega,
\end{cases} \tag{3.2}$$

$$\frac{\partial q_i}{\partial \nu} = 0 \ \ a.e. \ on \ \Sigma, \ i = 1, 2, 3.$$

Proof Existence and uniqueness of the solution q to system (3.2) follows by theorem 2.2. Denote by $\mathcal{N}_{\mathcal{U}_{ad}}(u^*, \nu^*)$ the normal cone at $\mathcal{U}_{ad}$ in (u^*, ν^*),

$$\mathcal{N}_{\mathcal{U}_{ad}}(u^*, \nu^*) = \{v_1 \in L^2(Q), \ v_2 \in L^2(0, T) \text{ satisfying the following formula}\},$$

$$\begin{cases} v_1(t,x) \le 0, & \text{when } u(t,x) = 0, \\ v_1(t,x) = 0, & \text{when } 0 \le u(t,x) \le 1, \\ v_1(t,x) \ge 0, & \text{when } u(t,x) = 1, \end{cases} \quad \begin{cases} v_2(t) \le 0, & \text{when } \nu(t) = 0, \\ v_2(t) = 0, & \text{when } 0 \le \nu(t) \le h, \\ v_2(t) \ge 0, & \text{when } \nu(t) = h, \end{cases}$$

for any given $(\vartheta_1, \vartheta_2) \in \mathcal{T}_{\mathcal{U}_{ad}}(u^*, \nu^*)$ $\vartheta_1 = (\vartheta_{11}, \vartheta_{21}, \vartheta_{31})$, as $\varepsilon > 0$ small enough, $(u^* + \varepsilon\vartheta_1, \nu^* + \varepsilon\vartheta_2) \in \mathcal{U}_{ad}$, we get

$$J(u^* + \varepsilon\vartheta_1, \nu^* + \varepsilon\vartheta_2) \le J(u^*, \nu^*). \tag{3.3}$$

Substituting (2.1) into (3.3) gives that

$$\sum_{i=1}^{3}\int_0^T\int_\Omega K_i(u_i^* + \varepsilon\vartheta_{i1})y_i^\varepsilon dxdt - \frac{1}{2}\sum_{i=1}^{3}\int_0^T\int_\Omega C_i(u_i^* + \varepsilon\vartheta_{i1})^2 dxdt - \frac{1}{2}\int_0^T C_4(\nu^* + \varepsilon\vartheta_2)^2 dt$$
$$\le \sum_{i=1}^{3}\int_0^T\int_\Omega K_i u_i^* y_i^* dxdt - \frac{1}{2}\sum_{i=1}^{3}\int_0^T\int_\Omega C_i[u_i^*]^2 dxdt - \frac{1}{2}\int_0^T C_4[\nu^*]^2 dt,$$

that is

$$\sum_{i=1}^{3}\int_0^T\int_\Omega K_i u_i^* z_i^* dxdt + \sum_{i=1}^{3}\int_0^T\int_\Omega (K_i y_i^* - C_i u_i^*)\vartheta_{i1} dxdt - \int_0^T \nu^*\vartheta_2 dt \le 0, \tag{3.4}$$

where

$$z_i(t,x) = \lim_{\varepsilon\to 0^+}\frac{y_i^\varepsilon(t,x) - y_i^*(t,x)}{\varepsilon}, z_{i+3}(t) = \lim_{\varepsilon\to 0^+}\frac{c_{i0}^\varepsilon(t) - c_{i0}^*(t)}{\varepsilon}, z_7(t) = \lim_{\varepsilon\to 0^+}\frac{c_e^\varepsilon(t) - c_e^*(t)}{\varepsilon}, i = 1, 2, 3,$$

$(y^\varepsilon, c_0^\varepsilon, c_e^\varepsilon)$ is the state corresponding to $(u^* + \varepsilon\vartheta_1, \nu^* + \varepsilon\vartheta_2)$, it follows from the state system (1.2) that $z = (z_1, z_2, \ldots, z_7)$ is the solution of

$$\begin{cases} \frac{\partial z_1}{\partial t} = \alpha_1\Delta z_1 + z_1[g_1(y_1^*) + y_1^* g_1'(y_1^*) - u_1^* - y_2^* f(y_1^*) - y_1^* y_2^* f'(y_1^*)] \\ \qquad - y_1^* z_2 f(y_1^*) - \vartheta_{11} y_1^*, \\ \frac{\partial z_2}{\partial t} = \alpha_2\Delta z_2 + +bz_1[y_2^* f(y_1^*) + y_1^* y_2^* f'(y_1^*)] + z_2[-a - u_2^* + by_1^* f(y_1^*) + cy_3^* h(y_3^*)] \\ \qquad + cz_3[y_2^* h(y_3^*) + y_3^* y_2^* h'(y_3^*)] - \vartheta_{21} y_2^*, \\ \frac{\partial z_3}{\partial t} = \alpha_3\Delta z_3 + z_3[g_3(y_3^*) + y_3^* g_3'(y_1^*) - u_3^* - y_2^* h(y_3^*) - y_3^* y_2^* h'(y_3^*)] \\ \qquad - z_2 y_3^* h(y_3^*) - \vartheta_{31} y_3^*, \\ \frac{\partial z_j}{\partial t} = kz_7(t) - gz_j(t) - mz_j(t), \ j = i + 3, \ i = 1, 2, 3, \\ \frac{\partial z_7}{\partial t} = -k_1 c_e^*(t)\sum_{i=1}^{3} z_i + g_1\sum_{i=1}^{3} c_{i0}(t) z_i(t) + g_1\sum_{i=1}^{3} y_i^* z_j(t) - [k_1\sum_{i=1}^{3} y_i^*(t) + h] z_7(t) + \vartheta_2(t), \end{cases} \tag{3.5}$$

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for all $(t,x) \in Q$, subject to the boundary and initial conditions

$$\frac{\partial q_i}{\partial \nu} = 0 \ \ a.e. \ on \ \Sigma, \ i = 1,2,3.$$

$$z_i(0,x) = z_j(0) = 0, \ x \in \Omega, j = i+3, \ i = 1,2,3.$$

Multiplying the (3.5) by $q_1, q_2, \ldots, q_7$ respectively, integrating on Q and $(0,T)$ and using (3.2) yield

$$\sum_{i=1}^{3} \int_0^T \int_\Omega K_i u_i^*(t,x) z_i(t,x) dxdt = -\sum_{i=1}^{3} \int_0^T \int_\Omega y_i^*(t,x) q_i(t,x) \vartheta_{1i} dxdt + \int_0^T \vartheta_2(t) q_7(t) dt. \quad (3.6)$$

Substituting (3.6) into (3.4) we obtain that

$$\sum_{i=1}^{3} \int_0^T \int_\Omega [(K_i - q_i) y_i^* - C_i u_i^*] \vartheta_{1i} dxdt + \int_0^T (-C_4 \nu^* + q_7) \vartheta_2 dt \leq 0. \quad (3.7)$$

By using the concept of normal cone $\mathcal{U}_{ad}$ at (u^*, ν^*) [16], we get

$$\Big((K_i - q_i) y_i^* - C_i u_i^*, -C_4 \nu^* + q_7 \Big) \in \mathcal{N}_{\mathcal{U}_{ad}}(u^*, \nu^*),$$

the proof is completed by the characteristics properties of the normal vector [14].

4 Second order optimality conditions

In this section, we discuss the second order sufficient conditions for the controlled system, since the second order optimality conditions can be solved by using the second order Fréchet derivative of the associated Lagrange function, so we introduce the Lagrange function firstly,

$$\mathcal{L}(y,u,\nu,q) = J(u,\nu) - \int_Q q(y_t - Ay - F)^{\mathbf{T}} dtdx - \int_\Sigma q\Big(\frac{\partial y}{\partial \nu}\Big)^{\mathbf{T}} dtdx, \quad (4.1)$$

here the upper index $\mathbf{T}$ is the transposed of any matrix and $\frac{\partial y}{\partial \nu} = \Big(\frac{\partial y_1}{\partial \nu}, \frac{\partial y_2}{\partial \nu}, \frac{\partial y_3}{\partial \nu}\Big)$, we employ the method from [17-18], let $X = (y, c_0, c_e)$, $U = (u, v)$, $Q = (q_1, q_2, \cdots, q_7)$, then (4.1) can be written in detail as

$$\begin{aligned}
\mathcal{L}(X,U,Q) = & \int_0^T \int_\Omega \big[K_1 u_1 y_1 + K_2 u_2 y_2 + K_3 u_3 y_3 - \frac{1}{2}(C_1 u_1^2 + C_2 u_2^2 + C_3 u_3^2)\big] dxdt - \frac{1}{2}\int_0^T C_4 \nu^2 dt \\
& + \int_Q \Big(\frac{\partial q_1}{\partial t} y_1 + \frac{\partial q_2}{\partial t} y_2 + \frac{\partial q_3}{\partial t} y_3\Big) dtdx + \int_Q (\alpha_1 y_1 \Delta q_1 + \alpha_2 y_2 \Delta q_2 + \alpha_3 y_3 \Delta q_3) dtdx \\
& + \int_Q [y_1 g_1(y_1) q_1 + y_3 g_3(y_3) q_3 - u_1 y_1 q_1 - u_2 y_2 q_2 - u_3 y_3 q_3 + y_1 y_2 f(y_1)(b q_2 - q_1) \\
& - a y_2 q_2 + y_2 y_3 h(y_3)(c q_2 - q_3)] dtdx + \int_0^T \sum_{j=4}^{6} q_j [k c_e - (g+m) c_{i0}] dt \\
& - \int_0^T q_7 [k_1 c_e (y_1 + y_2 + y_3) + g_1 (c_{10} y_1 + c_{20} y_2 + c_{30} y_3) - h c_e + \nu] dt,
\end{aligned} \quad (4.2)$$

we introduce the Hamiltonian function secondly

$$\begin{aligned}\mathcal{H}(X,U,Q)=&K_1u_1y_1+K_2u_2y_2+K_3u_3y_3+y_1g_1(y_1)q_1+y_3g_3(y_3)q_3-u_1y_1q_1-u_2y_2q_2-u_3y_3q_3\\&+y_1y_2f(y_1)(bq_2-q_1)-ay_2q_2+y_2y_3h(y_3)(cq_2-q_3)+\sum_{j=4}^{6}q_j[kc_e-(g+m)c_{i0}]\\&-q_7[k_1c_e(y_1+y_2+y_3)+g_1(c_{10}y_1+c_{20}y_2+c_{30}y_3)-hc_e+\nu],\end{aligned}$$

Assume that g_1, g_3, f and h are functions of class C^2. If $\bar{u},\bar{\nu}$ is an admissible control and y, q are the corresponding state and adjoint state, then the associated Hessian matrix at (y,u,q) is

$$D^2\mathcal{H}(y,u,\nu,q)=\begin{pmatrix}H_{11}&H_{12}&0&H_{14}&0&0&0\\H_{21}&0&H_{23}&0&H_{25}&0&0\\0&H_{32}&H_{33}&0&0&H_{36}&0\\H_{41}&0&0&-C_1&0&0&0\\0&H_{52}&0&0&-C_2&0&0\\0&0&H_{63}&0&0&-C_3&0\\0&0&0&0&0&0&-C_4\end{pmatrix},$$

and

$$\begin{cases}H_{11}=\bar{q}_1[2g_1'(\bar{y}_1)+\bar{y}_1g_1''(\bar{y}_1)]-\bar{y}_2(\bar{q}_1-b\bar{q}_2)[2f'(\bar{y}_1)+\bar{y}_1f''(\bar{y}_1)],\\H_{12}=-(\bar{q}_1-b\bar{q}_2)[f(\bar{y}_1)+\bar{y}_1f'(\bar{y}_1)],\\H_{23}=-(\bar{q}_3-c\bar{q}_2)[h(\bar{y}_3)+\bar{y}_3f'(\bar{y}_3)],\\H_{33}=\bar{q}_3[2g_3'(\bar{y}_3)+\bar{y}_3g_3''(\bar{y}_3)]-\bar{y}_2(\bar{q}_3-c\bar{q}_2)[2h'(\bar{y}_3)+\bar{y}_3h''(\bar{y}_3)]\\H_{14}=H_{41}=K_1-\bar{q}_1, H_{25}=H_{52}=K_2-\bar{q}_2, H_{36}=H_{63}=K_3-\bar{q}_3.\end{cases}$$

Then, we have

$$\mathcal{L}''(\bar{y},\bar{u},\bar{\nu},\bar{q})[(y,u,\nu),(y,u,\nu)]=\int_Q(y,u,\nu)D^2\mathcal{H}(y,u,q)(y,u,\nu)^{\mathbf{T}}dtdx,$$

that is

$$\begin{aligned}&\mathcal{L}''(\bar{y},\bar{u},\bar{\nu},\bar{q})[(y,u,\nu),(y,u,\nu)]\\&=\int_Q(y_1)^2[\bar{q}_1(2g_1'(\bar{y}_1)+\bar{y}_1g_1''(\bar{y}_1))-\bar{y}_2(\bar{q}_1-b\bar{q}_2)(2f'(\bar{y}_1)+\bar{y}_1f''(\bar{y}_1))]dtdx\\&+\int_Q(y_3)^2[\bar{q}_3(2g_3'(\bar{y}_3)+\bar{y}_3g_3''(\bar{y}_3))-\bar{y}_2(\bar{q}_3-c\bar{q}_2)(2h'(\bar{y}_3)+\bar{y}_3h''(\bar{y}_3))]dtdx\\&+\int_Q2[-y_1y_2(\bar{q}_1-b\bar{q}_2)(f(\bar{y}_1)+\bar{y}_1f'(\bar{y}_1))-y_2y_3(\bar{q}_3-c\bar{q}_2)(h(\bar{y}_3)+\bar{y}_3f'(\bar{y}_3))]dtdx\\&+\int_Q[2u_1y_1(K_1-\bar{q}_1)+2u_2y_2(K_2-\bar{q}_2)+2u_3y_3(K_3-\bar{q}_3)]dtdx\\&-\int_Q(C_1u_1^2+C_2u_2^2+C_3u_3^2)dtdx-\int_0^TC_4\nu^2dt.\end{aligned}$$

Now we can formulate the second order optimality conditions for our problem.

Theorem 4.1 (i) (Second order necessary optimality conditions.) Under the hypotheses of Theorem 3.1, if (u^*,ν^*) is an optimal pair and q is the corresponding adjoint variable, then

$$\mathcal{L}''(y^*,u^*,\nu^*,q)[(y,u,\nu),(y,u,\nu)]\le 0,\quad\forall\ (u,\nu)\in\mathcal{N}_{\mathcal{U}_{ad}}(u^*,\nu^*).$$

(ii) (Second order sufficient optimality conditions.) $\forall\ (u^*, \nu^*) \in \mathcal{U}_{ad}$, together with its corresponding state y^* and adjoint state q, if (y^*, u^*, ν^*, q) satisfies the first order necessary condition (3.1) and the condition

$$\mathcal{L}''(y^*, u^*, \nu^*, q)[(y, u, \nu), (y, u, \nu)] < \kappa(\|v_1\|^2_{L^2(Q)} + \|v_2\|^2_{L^2(0,T)}), \quad \forall\ (v_1, v_2) \in \mathcal{N}_{\mathcal{U}_{ad}},$$

for some $\kappa > 0$, then (y^*, u^*, ν^*) is an optimal local solution of the controlled system (1.2).

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Existence results for new extended vector variational-like inequality

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Abstract

In this paper, we establish and study some new existence theorems for a new extended vector variational-like inequality in a Banach space. The results are proved by using the new definition of $g - f - \eta - \phi - \mu-$quasimonotone of Stampacchia and of Minty type mappings. The obtained results in this article can be viewed as some new and generalized forms which can be applied to several problems.

Keywords: New extended vector variational-like inequality; Existence result; C-convex; KKM-mapping; $g - f - \eta - \phi - \mu-$quasimonotonicity; $g - f - \eta - \phi - \mu-$pseudomonotoncity.

1. Introduction

In 1980, Giannessi introduced a generalization of variational inequality is the vector variational inequality (for short, VVI) in a finite-dimensional Euclidean space, see [8]. For the past years, vector variational inequalities and their generalizations have been studied and applied in various directions. The vector variational-like inequalities is a generalized form of a vector variational inequalities related to the class of η-connected sets which is much more general than the class of convex sets. It well Known that monotonicity plays an important role to proving existence of solutions of vector variational inequalities and vector variational-like inequalities. Some important generalizations of monotonicity, such as quasimonotonicity, proper quasimonotonicity, pseu-domonotonicity, dense pseudomonotonicity, semimonotonicity, have been introduced and considered to study various variational inequalities and other related problems. In [9] Ahmad and Irfan obtained existence results for extended vector variational-like inequality and equilibrium problems by using g-h-η-quasimonotone of Stampacchia and Minty types.

In this paper, we introduce a new definition for a new extended vector variational-like inequality and we define a new and general form of definitions for quasimonotone of Stampacchia and Minty type mappings. We have some ideas to establish some sufficient conditions to guarantee the existence of solutions. The new problems can be viewed as some unified forms of the previous problem, that is, extended vector variational-like inequalities considered and studied by Ahmad and Irfan [9].

Let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset, $C \subset Y$ be a pointed, closed and convex cone in Y such that $intC \neq \emptyset$ where $intC$ denote the

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interior of C. Then for $x, y \in Y$, a partial order $\geq_C$ in Y is defined as

$$x \geq_C y \Leftrightarrow x - y \in C.$$

Let $L(X,Y)$ be the space of all continuous linear mappings from X to Y. Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), g, f : K \to K, \eta : K \times K \to X$ and $\phi, \mu : K \times K \to Y$ are mappings.
We consider the following *new extended vector variational-like inequalities*:

$$(NEVVLI-I) \begin{cases} Find \ \ x \in K such \ that, \\ \left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right) \geq_C 0, \\ \forall y \in K. \end{cases}$$

and

$$(NEVVLI-II) \begin{cases} Find \ \ x \in K such \ that, \\ \left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0, \\ \forall y \in K. \end{cases}$$

Special cases:

(i) If $T_3, T_4, \cdots, T_N \equiv 0$, $T_1 = S, T_2 = T$, $\phi = h$, $\mu \equiv 0$ and $f = g$ then (NEVVLI-I) and (NEVVLI-II) reduces to the following *extended vector variational-like inequalities* considered and studied by Ahmad and Irfan [9]

$$(EVVLI-I) \begin{cases} Find \ \ x \in K such \ that, \\ \langle S(x) + T(x), \eta(g(y), g(x)) \rangle + h(g(y), g(x)) \geq_C 0, \\ \forall y \in K, \end{cases}$$

and

$$(EVVLI-II) \begin{cases} Find \ \ x \in K such \ that, \\ \langle S(y) + T(y), \eta(g(x), g(y)) \rangle + h(g(x), g(y)) \leq_C 0, \\ \forall y \in K, \end{cases}$$

(ii) If $T_2, T_3, \cdots, T_N \equiv 0, T_1 = T$, $\phi = h$, $\mu \equiv 0$ and $f = g = I$ then (NEVVLI-I) and (NEVVLI-II) reduces to the following *vector variational-like inequalities* considered and studied by Ahmad [1]

$$(VVLI-I) \begin{cases} Find \ \ x \in K such \ that, \\ \langle T(x), \eta(y, x) \rangle + h(y, x) \geq_C 0, \quad \forall y \in K, \end{cases}$$

and

$$(VVLI-I) \begin{cases} Find \ \ x \in K such \ that, \\ \langle T(y), \eta(x, y) \rangle + h(x, y) \leq_C 0, \quad \forall y \in K, \end{cases}$$

(iii) If $T_2, T_3, \cdots, T_N \equiv 0, T_1 = T$, $\phi \equiv 0$, $\mu \equiv 0$ and $g = I$ then (NEVVLI-I) and (NEVVLI-II) reduces to the following vector variational-like inequalities considered and studied by Zhao and Xia [12]

$$(VVLI-I) \begin{cases} Find \ \ x \in K such \ that, \\ \langle T(x), \eta(y, x) \rangle \geq_C 0, \quad \forall y \in K, \end{cases}$$

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and

$$(VVLI-I)\begin{cases}Find \ \ x \in K such \ that,\\ \langle T(y), \eta(x,y)\rangle \leq_C 0, \quad \forall y \in K,\end{cases}$$

The following concepts and results are needed for the results.

Definition 1.1. A mapping $f : K \to Y$ is said to be *hemicontinuous* if, for any fixed $x, y \in K$, the mapping $t \mapsto f(x + t(y - x))$ is continuous at 0^+.

Definition 1.2. Let $C : K \to 2^Y$ be a set-valued mapping, $h : K \times K \to Y$ and $g : K \times K \to X$ are the single-valued mappings. Then

(i) $h(\cdot, v)$ is said to be *C-convex* in the first argument if

$$h(tu_1 + (1-t)u_2, v) \in th(u_1, v) + (1-t)h(u_2, v) - C, \forall u_1, u_2 \in K, \quad t \in [0,1],$$

(ii) If K is an affine set, the $\eta(x, y)$ is said to be *affine* with respect to u if for any given $v \in K$

$$\eta(tu_1 + (1-t)u_2, v) = t\eta(u_1, v) + (1-t)\eta(u_2, v), \forall u_1, u_2 \in K, \quad t \in \mathbb{R},$$

with $u = (tu_1 + (1-t)u_2) \in K$.

Definition 1.3. Let $T_1, T_2, \cdots, T_N : K \to L(X, Y), g, f : K \to K, \eta : K \times K \to X$ and $\phi, \mu : K \times K \to Y$ are mappings. Then $T_1, T_2, \cdots, T_N$ are said to be g-f-η-ϕ-μ-pseudomonotone if for any $x, y \in K$,

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right) \geq_C 0,$$
$$\Rightarrow \left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0.$$

Example 1.4. Let $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and

$$T_1(x) = \begin{pmatrix}1\\1\end{pmatrix}, T_2(x) = \begin{pmatrix}2^x\\2^{-x}\end{pmatrix}, T_3(x) = \begin{pmatrix}3^x\\3^{-x}\end{pmatrix}, \cdots, T_N(x) = \begin{pmatrix}N^x\\N^{-x}\end{pmatrix}$$
$$g(x) = 3x, f(x) = 2x, \eta(y, x) = 4y - 5x, \phi(y, x) = \begin{pmatrix}8y - 12x\\6y^2 - 5xy - 11x^2\end{pmatrix},$$
$$\mu(x, y) = \begin{pmatrix}3x - 4y\\4x^2 - 2xy - 6y^2\end{pmatrix}, \forall x, y \in K.$$

Thus

$$\begin{aligned}\eta(g(y), g(x)) &= \eta(3y, 3x)\\ &= 4(3y) - 5(3x)\\ &= 12y - 15x,\end{aligned}$$

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$$\begin{aligned}\phi(f(y), f(x)) &= \phi(2y, 2x)\\ &= \begin{pmatrix} 8(2y) - 12(2x) \\ 6(2y)^2 - 5(2x)(2y) - 11(2x)^2 \end{pmatrix}\\ &= \begin{pmatrix} 16y - 24x \\ 24y^2 - 20xy - 44x^2 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\mu(f(x), f(y)) &= \mu(2y, 2x)\\ &= \begin{pmatrix} 3(2x) - 4(2y) \\ 4(2x)^2 - 2(2x)(2y) - 2(2y)^2 \end{pmatrix}\\ &= \begin{pmatrix} 6x - 8y \\ 16x^2 - 8xy - 24y^2 \end{pmatrix}.\end{aligned}$$

Then $\forall x, y \in K$

$$\begin{aligned}&\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right)\\ &= \begin{pmatrix} 1 + 2^x + 3^x + \cdots + N^x \\ 1 + 2^{-x} + 3^{-x} + \cdots + N^{-x} \end{pmatrix}(12y - 15x) + \begin{pmatrix} 16y - 24x \\ 24x^2 - 20xy - 44y^2 \end{pmatrix} - \begin{pmatrix} 6x - 8y \\ 16y^2 - 8xy - 24x^2 \end{pmatrix}\\ &= \begin{pmatrix} 1 + 2^x + 3^x + \cdots + N^x \\ 1 + 2^{-x} + 3^{-x} + \cdots + N^{-x} \end{pmatrix}(12y - 15x) + \begin{pmatrix} 16y - 24x - 6x + 8y \\ 24y^2 - 20xy - 44x^2 - 16x^2 + 8xy + 24y^2 \end{pmatrix}\\ &= \begin{pmatrix} 1 + 2^x + 3^x + \cdots + N^x \\ 1 + 2^{-x} + 3^{-x} + \cdots + N^{-x} \end{pmatrix}(12y - 15x) + \begin{pmatrix} 24y - 30x \\ 48y^2 - 12xy - 60x^2 \end{pmatrix}\\ &= \begin{pmatrix} 1 + 2^x + 3^x + \cdots + N^x \\ 1 + 2^{-x} + 3^{-x} + \cdots + N^{-x} \end{pmatrix}(12y - 15x) + \begin{pmatrix} 2(12y - 15x) \\ 2(12y - 15x)(y + x) \end{pmatrix}\\ &= \begin{pmatrix} 1 + 2^x + 3^x + \cdots + N^x \\ 1 + 2^{-x} + 3^{-x} + \cdots + N^{-x} \end{pmatrix}(12y - 15x) + \begin{pmatrix} 2 \\ 4(y + x) \end{pmatrix}(12y - 15x)\\ &= (12y - 15x)\left[\begin{pmatrix} 1 + 2^x + 3^x + \cdots + N^x \\ 1 + 2^{-x} + 3^{-x} + \cdots + N^{-x} \end{pmatrix} + \begin{pmatrix} 2 \\ 4(y + x) \end{pmatrix}\right]\\ &= (12y - 15x)\begin{pmatrix} 1 + 2^x + 3^x + \cdots + N^x + 2 \\ 1 + 2^{-x} + 3^{-x} + \cdots + N^{-x} + 4y + 4x \end{pmatrix} \geq_C 0,\end{aligned}$$

implies that $12y \geq 15x$. Thus, $12x \leq 15x \leq 12y \leq 15y$. Therefore, $12x - 15y \leq 0$.

So it follows that

$$
\begin{aligned}
&\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 16x-24y \\ 24y^2-20xy-44x^2 \end{pmatrix} - \begin{pmatrix} 6y-8x \\ 16x^2-8xy-24y^2 \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 16x-24y-6y+8x \\ 24x^2-20xy-44y^2-16y^2+8xy+24x^2 \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 24x-30y \\ 48x^2-12xy-60y^2 \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 2(12x-15y) \\ 2(12x-15y)(x+y) \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 2 \\ 4(x+y) \end{pmatrix} (12x-15y) \\
&= (12x-15y)\left[\begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} + \begin{pmatrix} 2 \\ 4(x+y) \end{pmatrix} \right] \\
&= (12x-15y) \begin{pmatrix} 1+2^y+3^y+\cdots+N^y+2 \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y}+4x+4y \end{pmatrix} \leq_C 0.
\end{aligned}
$$

$\Rightarrow T_1, T_2, \cdots, T_N$ are g-f-η-ϕ-μ-pseudomonotone.

Definition 1.5. A multi-valued operator $S : X \to 2^{X^*}$ is called quasimonotone if for all $x, y \in X$ the following implications hold:

$$\exists x^* \in S(x) : \langle x^*, y-x \rangle > 0 \Rightarrow \exists y^* \in S(y) : \langle y^*, y-x \rangle \geq 0.$$

Definition 1.6. A multi-valued operator $S : X \to 2^{X^*}$ is called *properly quasimonotone* if for all $x_1, x_2, ..., x_n \in X$ and every $y \in Conv\{x_1, x_2, ..., x_n\}$ there exist i such that

$$\forall x_i^* \in S(x_i) : \langle x_i^*, y - x_i \rangle \geq 0.$$

Definition 1.7. A mapping $T : K \to L(X, Y)$ is said to be *properly quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tu, v_i - u \rangle \geq_C 0$ holds for some i. T is said to be *properly quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tv_i, v_i - u \rangle \leq_C 0$ holds for some i.

Definition 1.8. A mapping $T : K \to L(X, Y)$ is said to be *properly g-η-quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tu, \eta(g(v_i), g(u)) \rangle \geq_C 0$ holds for some i. T is said to be *properly g-η-quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$, and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tv_i, \eta(g(v_i), g(u)) \rangle \leq_C 0$ holds for some i.

Definition 1.9. A mapping $T : K \to L(X, Y)$ is said to be *properly g-h-η-quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tu, \eta(g(v_i), g(u)) \rangle + h(g(v_i), g(u)) \geq_C 0$ holds for some i. T is said to be *properly g-h-η-quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$, and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tv_i, \eta(g(u), g(v_i)) \rangle + h(g(u), g(v_i)) \leq_C 0$ holds for some i.

Definition 1.10. A mapping $T : K \to L(X,Y)$ is said to be *properly g-f-η-φ-μ-quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1,2,...,n$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $u := \sum_{i=1}^{n} \lambda_i v_i$, $\langle T(u), \eta(g(v_i), g(u))\rangle + \phi(f(v_i), f(u)) - \mu(f(u), f(v_i)) \geq_C 0$ holds for some i. T is said to be *properly g-f-η-φ-μ-quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1,2,...,n$ with $\sum_{i=1}^{n} \lambda_i = 1$, and $u := \sum_{i=1}^{n} \lambda_i v_i$, $\langle T(v_i), \eta(g(u), g(v_i))\rangle + \phi(f(u), f(v_i)) - \mu(f(v_i), f(u)) \leq_C 0$ holds for some i.

Example 1.11. Let $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and

$$T_1(x) = \begin{pmatrix} 1 \\ x \end{pmatrix}, T_2(x) = \begin{pmatrix} 2 \\ x^2 \end{pmatrix}, T_3(x) = \begin{pmatrix} 3 \\ x^3 \end{pmatrix}, \cdots, T_N(x) = \begin{pmatrix} N \\ x^N \end{pmatrix}$$

$$g(x) = 2x, f(x) = 3x, \eta(y,x) = 7y - 5x, \phi(y,x) = \begin{pmatrix} 5y + 3x \\ 5y^2 + 3x^2 \end{pmatrix},$$

$$\mu(x,y) = \begin{pmatrix} 2x + 3y \\ 2x^2 + 3y^2 \end{pmatrix}, \forall x, y \in K.$$

Thus

$$\begin{aligned} &\eta(g(y), g(x)) = \eta(2y, 2x) \\ &= 7(2y) - 5(2x) \\ &= 14y - 10x, \end{aligned}$$

$$\begin{aligned} &\phi(f(y), f(x)) = \phi(3y, 3x) \\ &= \begin{pmatrix} 5(3y) + 3(3x) \\ 5(3y)^2 + 3(3x)^2 \end{pmatrix} \\ &= \begin{pmatrix} 15y + 9x \\ 45y^2 + 27x^2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\mu(f(x), f(y)) = \mu(3x, 3y) \\ &= \begin{pmatrix} 2(3x) + 3(3y) \\ 2(3x)^2 + 3(3y)^2 \end{pmatrix} \\ &= \begin{pmatrix} 6x + 9y \\ 18x^2 + 27y^2 \end{pmatrix}. \end{aligned}$$

We claim that $T_1, T_2, \cdots, T_N$ are properly g-f-η-ϕ-μ-quasimonotone of Stampacchia type. Suppose to the contrary that there exists $x_i \in K, t_i \geq 0, i = 1,2,...,n$ with $\sum_{i=1}^{n} t_i = 1$ such that

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_i), g(x)) \right\rangle + \phi\left(f(x_i), f(x)\right) - \mu\left(f(x), f(x_i)\right) \ngeq_C 0, i = 1,2,...,n$$

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where $x_i = \sum_{i=1}^{n} \lambda_i x_i$, it follows that

$$\begin{aligned}
&\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_i), g(x)) \right\rangle + \phi\left(f(x_i), f(x)\right) - \mu\left(f(x), f(x_i)\right) \\
&= \begin{pmatrix} 1+2+3+\cdots+N \\ x+x^2+x^3+\cdots+x^N \end{pmatrix} (14x_i - 10x) + \begin{pmatrix} 15x_i + 9x \\ 45{x_i}^2 + 27x^2 \end{pmatrix} - \begin{pmatrix} 6x + 9x_i \\ 18x^2 + 27{x_i}^2 \end{pmatrix} \\
&= \begin{pmatrix} (1+2+3+\cdots+N)(14x_i - 10x) \\ (x+x^2+x^3+\cdots+x^N)(14x_i - 10x) \end{pmatrix} + \begin{pmatrix} 6x_i + 3x \\ 18x_i + 9x \\ \end{pmatrix} \ngeq_C 0 \\
&= \begin{pmatrix} (1+2+3+\cdots+N)(14x_i - 10x) + 6x_i + 3x \\ (x+x^2+x^3+\cdots+x^N)(14x_i - 10x) + 18x_i + 9x \end{pmatrix} \ngeq_C 0
\end{aligned}$$

$i = 1, 2, ..., n,$

which is a contradiction, since

$$(1+2+3+\cdots+N)(14x_i - 10x) + 6x_i + 3x \geq_C 0,$$

and

$$(x+x^2+x^3+\cdots+x^N)(14x_i - 10x) + 18x_i + 9x \geq_C 0,$$

for atleast one i. Thus $T_1, T_2, \cdots, T_N$ are properly g-f-η-ϕ-μ-quasimonotone of Stampacchia type.

Lemma 1.12. *Let $T_1, T_2, \cdots, T_N : K \to L(X, Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ and $g : K \to K$ be mappings. If $T_1, T_2, \cdots, T_N$ are g-f-η-ϕ-μ-pseudomonotone and properly g-f-η-ϕ-μ-quasimonotone of Stampacchia type, then $T_1, T_2, \cdots, T_N$ are properly g-f-η-ϕ-μ-quasimonotone of Minty type.*

Proof. The fact directly follows from Definitions 1.3 and 1.9. □

Definition 1.13. Let D be a nonempty subset of a topological Hausdorff space E. A mapping $G : D \to 2^E$ (the family of nonempty subset of E) is said to be a *KKM mapping* if for any finite subset $\{x_1, x_2, ..., x_n\} \subset D$,
we have $Co\{x_1, ..., x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$. where Co denotes the convex hull operator.

Lemma 1.14 ([6])**.** *Let D be a nonempty subset of a topological Hausdorff vector space E and $G : D \to 2^E$ be a KKM mapping. If $G(x)$ is closed for any $x \in D$, and compact for some $x \in D$, then $\bigcap_{x \in D} G(x) \neq \emptyset$.*

Lemma 1.15. *Let Y be a topological vector space with a pointed, closed and convex cone C such that $intC \neq \emptyset$. If $u, v \in Y$ and $u \notin C$ and $v \in -C$, then $tv + (1-t)u \notin C, \quad \forall t \in (0, 1)$.*

Proof. Assume that $u, v \in Y$ and $u \notin C$ and $v \in -C$. We must to show that $tv + (1-t)u \notin C$ $\forall t \in (0, 1)$. Suppose to the contrary that there exists some $t \in (0, 1)$ such that $tv + (1-t)u \in C$. Since C is cone and $v \in -C$, we have $-tv \in C$. Thus $tv + (1-t)u + (-tv) \in C + C \subset C$ and hence $(1-t)u \in C$. By $(1-t) > 0$ and C is cone, it follows that $\frac{1}{(1-t)}(1-t)u \in C$. So $u \in C$. Which is a contradiction. Hence $tv + (1-t)u \notin C$, $\forall t \in (0, 1)$. This completes the proof. □

2. Existence results

In this section, we establish some existence results for (NEVVLI-I) and (NEVVLI-II) by using Lemma 1.14.

Lemma 2.1. *Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ be mappings and $g, f : K \to K$ is affine mapping satisfying the following conditions:*

(a) *$T_1, T_2, \cdots, T_N$ are g-f-η-ϕ-μ-pseudomonotone;*

(b) *for any fixed $x \in X$, the mapping $y \mapsto \left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle$ is hemicontinuous and $\phi(f(x), f(y))$ and $\mu(f(y), f(x))$ are continuous with $\{z_t\} \to x_0 \in K, z_t \in K$;*

(c) *$\phi(\cdot, f(y))$ is C-convex in the first variable and $\phi(f(x), f(x)) = 0, \forall x \in K$;*

(d) *$\mu(f(y), \cdot)$ is C-concave in the second variable and $\mu(f(x), f(x)) = 0, \forall x \in K$;*

(e) *$\eta(\cdot, g(y))$ is affine in the first variable and $\eta(g(x), g(x)) = 0, \forall x \in K$.*

Then for any $x_0 \in K$, the following statements are equivalent

(I) $\left\langle \sum_{i=1}^{N} T_i(x_0), \eta(g(x), g(x_0)) \right\rangle + \phi\left(f(x), f(x_0)\right) - \mu\left(f(x_0), f(x)\right) \geq_C 0,$

(II) $\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_0), g(x)) \right\rangle + \phi\left(f(x_0), f(x)\right) - \mu\left(f(x), f(x_0)\right) \leq_C 0.$

Proof. $T_1, T_2, \cdots, T_N$ are g-f-η-ϕ-μ-pseudomonotone, it follows that $(I) \Rightarrow (II)$.
$(II) \Rightarrow (I)$. Suppose that (II) holds for any $x_0 \in K$

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0.$$

For arbitrary $z \in K$, letting $z_t = (1-t)x_0 + tz, \ \ t \in (0,1)$, we have $z_t \in K$ by convexity of K. Hence we have

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0), g(z_t)) \right\rangle + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \leq_C 0.$$

Now we show that

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle + \phi\left(f(z), f(z_t)\right) - \mu\left(f(z_t), f(z)\right) \geq_C 0.$$

Suppose to the contrary that there exists some $t \in (0,1)$ such that

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle + \phi\left(f(z), f(z_t)\right) - \mu\left(f(z_t), f(z)\right) \not\geq_C 0.$$

As C is a convex cone and in veiw of (c), (d) and (e) we get

$$\begin{aligned}
0 &= \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z_t), g(z_t)) \right\rangle + \phi\left(f(z_t), f(z_t)\right) - \mu\left(f(z_t), f(z_t)\right) \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g((1-t)x_0 + tz), g(z_t)) \right\rangle + \phi\left(f((1-t)x_0 + tz), f(z_t)\right) \\
&\quad - \mu\left(f(z_t), f((1-t)x_0 + tz)\right) \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), \eta((1-t)g(x_0) + tg(z), g(z_t)) \right\rangle + \phi\left((1-t)f(x_0) + tf(z), f(z_t)\right) \\
&\quad - \mu\left(f(z_t), (1-t)f(x_0) + tf(z)\right) \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), (1-t)\eta(g(x_0)g(z_t)) + t\eta(g(z), g(z_t)) \right\rangle + \phi\left((1-t)f(x_0) + tf(z), f(z_t)\right) \\
&\quad - \mu\left(f(z_t), (1-t)f(x_0) + tf(z)\right) \\
&\leq_C \left\langle \sum_{i=1}^{N} T_i(z_t), (1-t)\eta(g(x_0)g(z_t)) + t\eta(g(z), g(z_t)) \right\rangle + (1-t)\phi\left(f(x_0), f(z_t)\right) \\
&\quad + t\phi(f(z), f(z_t)) - [(1-t)\mu\left(f(z_t), f(x_0)\right) + t\mu(f(z_t), f(z))] \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), (1-t)\eta(g(x_0)g(z_t)) \right\rangle + \left\langle \sum_{i=1}^{N} T_i(z_t), t\eta(g(z), g(z_t)) \right\rangle \\
&\quad + (1-t)\phi\left(f(x_0), f(z_t)\right) + t\phi(f(z), f(z_t)) - (1-t)\mu\left(f(z_t), f(x_0)\right) - t\mu(f(z_t), f(z)) \\
&= (1-t)\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) \right\rangle + t\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle \\
&\quad + (1-t)\phi\left(f(x_0), f(z_t)\right) + t\phi(f(z), f(z_t)) - (1-t)\mu\left(f(z_t), f(x_0)\right) - t\mu(f(z_t), f(z)) \\
&= t\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) + \phi(f(z), f(z_t)) - t\mu(f(z_t), f(z)) \right\rangle \right\} \\
&\quad + (1-t)\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \right\rangle \right\} \\
&\in t\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) + \phi(f(z), f(z_t)) - t\mu(f(z_t), f(z)) \right\rangle \right\} \\
&\quad + (1-t)\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \right\rangle \right\} - C,
\end{aligned}$$

which implies that

$$\begin{aligned}
&t\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) + \phi(f(z), f(z_t)) - t\mu(f(z_t), f(z)) \right\rangle \right\} \\
&+ (1-t)\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \right\rangle \right\} \\
&\in C.
\end{aligned}$$

Which is a contradiction. Hence

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle + \phi\left(f(z), f(z_t)\right) - \mu\left(f(z_t), f(z)\right) \geq_C 0.$$

Condition (b) implies that

$$\left\langle \sum_{i=1}^{N} T_i(x_0), \eta(g(z), g(x_0)) \right\rangle + \phi\left(f(z), f(x_0)\right) - \mu\left(f(x_0), f(z)\right) \geq_C 0, \forall x \in K.$$

This completes the proof. □

Theorem 2.2. *Let X and Y be two real Banach spaces and $K \subset X$ a nonempty, compact and convex set. Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ and $g, f : K \to K$ are the mappings satisfying the following conditions:*

(i) *for any fixed $y \in X$, the mapping $x \mapsto \left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle$, $\phi(f(y), f(x))$ and $\mu(f(x), f(y))$ are continuous;*
(ii) *$T_1, T_2, \cdots, T_N$ are properly g-f-η-ϕ-μ-quasimonotone of Stampacchia type;*
(iii) *for all $x \in K, \eta(g(x), g(x)) = 0$ and $\phi(f(x), f(x)) = 0 = \mu(f(x), f(x))$.*

Then there exists $x \in K$ such that

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right) \geq_C 0, \quad \forall y \in K.$$

Proof. Define a multivalued mapping $M_1 : K \to 2^K$ by

$$M_1(z) = \left\{ \left\langle \sum_{i=1}^{N} T_i(x), \eta(g(z), g(x)) \right\rangle + \phi\left(f(z), f(x)\right) - \mu\left(f(x), f(z)\right) \geq_C 0 \right\}, \quad \forall z \in K,$$

then $M_1(z)$ is nonempty for each $z \in K$. We claim that M_1 is a KKM mapping. In fact if it is not the case then there exists $\{x_1, x_2, ..., x_n\} \subset K$, $x = \sum_{i=1}^{n} t_i x_i$ with $t_i > 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} t_i = 1$ such that $x \notin \bigcup_{i=1}^{m} M_1(x_i)$.

This implies that

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_i), g(x)) \right\rangle + \phi\left(f(x_i), f(x)\right) - \mu\left(f(x), f(x_i)\right) \geq_C 0.$$

This contradicts condition (ii). Therefore M_1 is a KKM mapping; Now we prove that for any $z \in K, M_1(z)$ is closed.

In veiw of (i), let there exists a net $\{x_n\} \subset M_1(z)$ such that $x_n \to x \in K$. Because

$$\left\langle \sum_{i=1}^{N} T_i(x_n), \eta(g(z), g(x_n)) \right\rangle + \phi\left(f(z), f(x_n)\right) - \mu\left(f(x_n), f(z)\right) \geq_C 0, \quad \forall n,$$

we have

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(z), g(x)) \right\rangle + \phi\left(f(z), f(x)\right) - \mu\left(f(x), f(z)\right) \geq_C 0.$$

Hence $x \in M_1(z)$ and so $M_1(z)$ is closed. It follows from the compactness of K and closedness of $M_1(z) \subset K$, that $M_1(z)$ is compact. Thus by Lemma 1.14, we have $\bigcap_{z\in K} M_1(z) \neq \emptyset$. Hence there exist $x \in K$ such that

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right) \geq_C 0 \quad \forall y \in K.$$

This completes the proof. □

Theorem 2.3. *Let K be a nonempty, bounded, closed and convex subset a real reflexive Banach space X and Y a real Banach space. Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ and $g, f : K \to K$ are the mappings satisfying the following conditions:*

(i) *for any fixed $y \in X$, the mapping $\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(\cdot), g(y)) \right\rangle$, $\phi(f(\cdot), f(y))$ and $\mu(f(y), f(\cdot))$ are lower semicontinuous;*

(ii) *$T_1, T_2, \cdots, T_N$ are properly g-f-η-ϕ-μ-quasimonotone of Minty type;*

(iii) *forall $x \in K, \eta(g(x), g(x)) = 0$ and $\phi(f(x), f(x)) = 0 = \mu(f(x), f(x))$.*

Then there exists $x \in K$ such that

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0, \quad \forall y \in K.$$

Proof. Define a multivalued mapping $M_2 : K \to 2^K$ by

$$M_2(z) = \left\{ x \in K : \left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0 \right\}, \quad \forall z \in K.$$

then $M_2(z)$ is nonempty for each $z \in K$. We claim that M_2 is not KKM mapping, then there exists $\{x_1, x_2, ..., x_n\} \subset K$, $x = \sum_{i=1}^{n} t_i x_i$ with $t_i > 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} t_i = 1$ such that $x \notin \bigcup_{i=1}^{m} M_2(x_i)$.

This implies that

$$\left\langle \sum_{i=1}^{N} T_i(x_i), \eta(g(x), g(x_i)) \right\rangle + \phi\left(f(x), f(x_i)\right) - \mu\left(f(x_i), f(x)\right) \not\leq_C 0, \quad i = 1, 2, ..., n.$$

This contradicts condition (ii). Therefore M_2 is a KKM mapping. Since K is bounded, $M_2(z)$ is bounded. From (ii), we have $M_2(z)$ is convex. Next, we will show that $M_2(z)$ closed.

In veiw of (i), let there exists a net $\{x_n\} \subset M_2(z)$ such that $x_n \to x \in K$. Because

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x_n), g(y)) \right\rangle + \phi\left(f(x_n), f(y)\right) - \mu\left(f(y), f(x_n)\right) \leq_C 0, \quad \forall n,$$

we have

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0.$$

Hence $x \in M_2(z)$ and so $M_2(z)$ is closed.

Since X is reflexive, $M_2(z)$ is weakly compact for all $z \in K$. It follows from Lemma 1.14, that $\bigcap_{z\in K} M_2(z) \neq \emptyset$. Hence there exist $x \in K$ such that

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0, \quad \forall y \in K.$$

This completes the proof. □

It is useful to mention that the result of Theorem 2.2 can be viewed as an improvement of the following corollary.

Corollary 2.4 ([9, Theorem 2.1])**.** *Let X and Y be two real Banach spaces and $K \subset X$ a nonempty, compact and convex set. Let $S, T : K \to L(X, Y), \eta : K \times K \to X, h : K \times K \to Y$ and $g : K \to K$ are the mappings satisfying the following conditions:*

(a) for any fixed $y \in X$, the mapping $x \mapsto \langle S(y) + T(y), \eta(g(x), g(y))\rangle$ and $h(g(x), g(y))$ are continuous;

(b) S and T are properly g-h-η-quasimonotone of Stampacchia type;

(c) for all $x \in K, \eta(g(x), g(x)) = 0 = h(g(x), g(x))$.

Then there exists $x \in K$ such that

$$\langle S(x) + T(x), \eta(g(y), g(x))\rangle + h(g(y), g(x)) \geq_C 0, \quad \forall y \in K.$$

Proof. By taking $T_3, T_4, \cdots, T_N \equiv 0$, $T_1 = S, T_2 = T$, $\phi = h$, $\mu \equiv 0$ and $f = g$ in Theorem 2.2, we obtain the desired results. □

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Existence of solutions for a new semi-linear evolution equations with impulses *

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Abstract

By using monotone iterative technique and operator semigroup theorem, we consider the existence of mild solutions for a class of nonlocal semi-linear evolution equation with not instantaneous impulses in ordered Banach spaces. Finally, an example is given to show the existence results.

Key Words: evolution equation; not instantaneous impulses; operator semigroup; upper and lower solutions; monotone iterative technique; mild solutions
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1 Introduction

The impulsive differential equations are used to describe mathematical models of many real processes and phenomena studied by physical, chemical, biological, population dynamics, industrial robotics, economics, engineering and so on, see [1]. Applied impulsive mathematical models have become an active research subject in nonlinear science and have attracted more attention in many fields , see[2-4] and references therein.

For more details on differential equations with "abrupt and instantaneous" impulses, one can see for instance the monographs[5-7] and the references therein. By means of monotone iterative method coupled with lower and upper solutions, some sufficient

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conditions for the existence of solutions of impulsive integro-differential equations were established in [8]. Recently, the existence results to impulsive differential equations with nonlocal conditions was studied in [9-15]. Moreover, Chen, Li and Yang[16] used the perturbation method and monotone iterative technique in the presence of lower and upper solutions to discuss the existence of mild solutions for the nonlocal impulsive evolution equation in ordered Banach spaces.

However, it seems that the models with instantaneous impulses could not explain the certain dynamics of evolution processes in pharmacotherapy. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. Hernandez and O'Regan[17] and Pierri et al.[18] initially studied on Cauchy problems for a new type first order evolution equations with not instantaneous impulses of the form:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = h_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ u(0) = u_0. \end{cases}$$

Wang and Li[19], Yu et al.[20] considered periodic boundary value problems for nonlinear evolution equations with non instantaneous impulses. Wang et al.[21] discussed a class of new fractional differential equations with not instantaneous impulses.

However, to the best of our knowledge, the existence mild solutions for nonlocal evolution equations with not instantaneous impulses by means of monotone iterative technique has not been investigated yet. Motivated by this consideration, in this paper, we discuss the existence of mild solutions for the nonlocal evolution equation with not instantaneous impulses in an ordered Banach space X

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = h_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m, \\ u(0) = g(u), \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ in X; $0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < t_3 \leq s_3 < \cdots < t_{m-1} \leq s_{m-1} < t_m \leq s_m < t_{m+1} = a$ are pre-fixed numbers, $J = [0, a]$, $a > 0$ is a constant; $f \in C([0, a], X)$. $h_i \in C([t_i, s_i] \times X, X)$ for all $i = 1, 2, \cdots, m$.

2 Preliminaries

Throughout this paper, Let X be a Banach space, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ in X. Denote

$$M \equiv \sup_{t\in J} \|T(t)\|,$$

which is a finite number. For more details of the properties of the operator semigroups and positive C_0-semigroup, we refer to the monographs[22, 23] and [24].

Let X be an ordered Banach space with the norm $\|\cdot\|$ and partial order " $\leq$ ", whose positive cone $K = \{x \in X | x \geq \theta\}$ is normal with normal constant N. Let $C(J, X)$ with the norm $\|u\|_C = \max_{t\in J} \|u(t)\|$, then $C(J, X)$ is an ordered Banach space induced by the convex cone $K_C = \{u \in C(J, X) \mid u(t) \geq 0, t \in J\}$, and K_C is also a normal cone.

Let $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $J'' = J \setminus \{0, t_1, t_2, \cdots, t_m\}$. Evidently, $PC(J, X) = \{u : J \to X \mid u(t)$ is continuous in J', and left continuous at t_k, and $u(t_k^+)$ exists, $k = 1, 2, \cdots, m\}$. $PC(J, X)$ is a Banach space with the norm $\|\cdot\|_{PC} = \sup_{t\in J} \|u(t)\|$. Evidently, $PC(J, X)$ is also an order Banach space with the partial order " $\leq$ " induced by the positive cone $K_{PC} = \{u \in PC(J, X)|u(t) \geq \theta, t \in J\}$. K_{PC} is normal with the same normal constant N. For $v, w \in PC(J, X)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in PC(J, X) \mid v \leq u \leq w\}$ in $PC(J, X)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in X \mid v(t) \leq u(t) \leq w(t), t \in J\}$ in X. We use X_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. For more details and definitions of the partial and cone, we refer to the monographs [25, 26].

Definition 2.1. If functions $v_0 \in PC(J, X) \cap C^1(J'', X) \cap C(J', X_1)$ satisfy

$$\begin{cases} v_0'(t) + Av_0(t) \leq f(t, v_0(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ v_0(t) \leq h_i(t, v_0(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ v_0(0) \leq g(v_0), \end{cases} \tag{2.1}$$

we call v_0 a lower solution of problem (1.1); if all the inequalities of (2.1) are inverse, we call it an upper solution of problem (1.1).

Next, we recall some properties of measure of noncompactness that will be used in the proof of our main results. Let $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness of the bounded set.For the details of the definition and properties of the measure of noncompactness, see [25]. The following lemmas are needed in our arguments.

Lemma 2.3.([27]) *Let $B \subset C(J, X)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha(B) = \max_{t\in J} \alpha(B(t)) = \alpha(B(J)).$$

Lemma 2.4. ([28]) *Let $B = \{u_n\} \subset C(J,X)(n = 1, 2, \cdots)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt \mid n \in \mathbb{N}\Big\}\Big) \leq 2\int_J \alpha(B(t))dt. \tag{2.2}$$

3 Linear nonlocal problem

Let $I = [t_0, t]$, $t_0 \geq 0$. It is well-known ([22])that for any $x_0 \in D(A)$ and $h \in C^1(I,X)$, the initial value problem of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in I, \\ u(t_0) = x_0, \end{cases} \tag{3.1}$$

has a unique classical solution $u \in C^1(I,X) \cap C(I,X_1)$ expressed by

$$u(t) = T(t-t_0)x_0 + \int_{t_0}^t T(t-s)h(s)ds, t \in I \tag{3.2}$$

If $x_0 \in X$ and $h \in C(I,X)$, the function u given by (3.2) belongs to $C(I,X)$, which is known as a mild solution of IVP(3.1).

To prove our main results, for any $h \in PC(J,X)$ and $y_i \in PC(J,X), i = 1, 2, \cdots, m$, we consider the linear nonlocal evolution equation with not instantaneous impulses in X

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = y_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ u(0) = g(u). \end{cases} \tag{3.3}$$

Theorem 3.1. *Let X be a Banach space, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ in X. For any $h \in PC(J,X)$, $y_i \in PC(J,X), i = 1, 2, \cdots, m$, $g : PC(J,X) \to X$, problem(3.3) has a unique mild solution $u \in PC(J,X)$ given by*

$$\begin{cases} u(t) = T(t)g(u) + \int_0^t T(t-\tau)h(\tau)d\tau, \quad t \in [0, t_1]; \\ u(t) = y_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m; \\ u(t) = T(t-s_i)y_i(s_i) + \int_{s_i}^t T(t-\tau)h(\tau)d\tau, \quad t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m. \end{cases} \tag{3.4}$$

Proof Let $t \in [0, t_1]$, problem(3.3) is equivalent to the linear nonlocal evolution equation without impulse

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in [0, t_1], \\ u(0) = g(u). \end{cases} \tag{3.5}$$

Then (3.5) has a unique mild solution $u \in C([0, t_1], X)$ given by

$$u(t) = T(t)g(u) + \int_0^t T(t - \tau)h(\tau)d\tau.$$

Let $t \in (t_i, s_i]$, then $u(t) = y_i(t), \quad i = 1, 2, \cdots, m$.

Let $t \in (s_i, t_{i+1}]$, problem(3.3) is equivalent to the initial value problem of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in (s_i, t_{i+1}], , i = 1, 2, \cdots, m, \\ u(s_i) = y_i(s_i). \end{cases} \tag{3.6}$$

Then (3.6) has a unique mild solution $u \in C([s_i, t_{i+1}], X)$ given by

$$u(t) = T(t - s_i)y_i(s_i) + \int_{s_i}^t T(t - \tau)h(\tau)d\tau.$$

Inversely, we can verify directly that the function $u \in PC(J, X)$ defined by(3.4) is a mild solution of problem(3.3). Hence problem(3.3) has a unique mild solution $u \in PC(J, X)$ given by (3.4). This completes the proof.

Remark 3.2. *In Theorem 3.1, let X be an ordered Banach space,$-A$ generate a positive C_0-semigroup $T(t)(t \geq 0)$ in X. For any $h \geq \theta$,$g \geq \theta$ and $y_i \geq \theta, i = 1, 2, \cdots, m$, then the mild solution of problem(3.3) is a positive solution.*

4 The main results

Now, we are in a position to state and prove our main results of this section.

Theorem 4.1. *Let X be an ordered Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a compact and positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. Assume that problem(1.1) has lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that the following conditions are satisfied:*

(H1) There exists a constant $C \geq 0$ such that

$$f(t, x_2) - f(t, x_1) \geq -C(x_2 - x_1), \quad t \in J,$$

for any $t \in J$, and $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$.

(H2) The impulsive functions $h_i (i = 1, 2, \cdots, m)$ are satisfy the conditions

$$h_i(t, x_2) \geq h_i(t, x_1), \;\; i = 1, 2, \cdots, m,$$

for $\forall t \in J, v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$.

(H3) The nonlocal function $g(u)$ is increasing in u for $u \in [v_0, w_0]$.

(H4) $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$ are compact operators.

(H5) $g : PC(J, X) \to X$ is compact operator.

Then the problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.

Proof It is easy to see that $-(A + CI)$ generates a positive compact semigroup $S(t) = e^{-Ct}T(t)$. Define $D = [v_0, w_0]$. Let $\overline{M} = \sup_{t\in J} \|S(t)\|$, we define an operator $Q : D \to PC(J, X)$ by

$$(Qu)(t) = \begin{cases} S(t)g(u) + \int_0^t S(t-\tau)(f(\tau, u(\tau)) + Cu(\tau))d\tau, \quad t \in [0, t_1]; \\ h_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m; \\ S(t - s_i)h_i(s_i, u(s_i)) + \int_{s_i}^t S(t-\tau)(f(\tau, u(\tau)) + Cu(\tau))d\tau, \\ t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m. \end{cases} \tag{4.1}$$

Since f, h_i and g are continuous, so $Q : D \to PC(J, X)$ is continuous. Clearly, from Theorem 3.1, the mild solutions of problem (1.1) are equivalent to the fixed point of operator Q.

(i) We show $Q : D \to PC(J, X)$ is an increasing operator.

For $\forall x_1, x_2 \in D$ and $x_1 \leq x_2$, from the assumptions (H1) and (H2), we have

$$f(t, x_1(t)) + Cx_1(t) \leq f(t, x_2(t)) + Cx_2(t), t \in J. \tag{4.2}$$

and

$$h_i(t, x_1(t)) \leq h_i(t, x_2(t)), i = 1, 2, \cdots, m. \tag{4.3}$$

Combining the positive of C_0-semigroup $S(t)$ with (4.2), (4.2) and (H3), we have

$$\begin{aligned} & S(t)g(x_1) + \int_0^t S(t-\tau)(f(\tau, x_1(\tau)) + Cx_1(\tau))d\tau \\ \leq \;\; & S(t)g(x_2) + \int_0^t S(t-\tau)(f(\tau, x_2(\tau)) + Cx_2(\tau))d\tau, \quad t \in [0, t_1]; \\ & h_i(t, x_1(t)) \leq h_i(t, x_2(t)), t \in (t_i, s_i], \quad i = 1, 2, \cdots, m; \end{aligned}$$

$$
\begin{aligned}
& S(t-s_i)h_i(s_i,x_1(s_i)) + \int_{s_i}^{t} S(t-\tau)(f(\tau,x_1(\tau)) + Cx_1(\tau))d\tau \\
\leq\ & S(t-s_i)h_i(s_i,x_2(s_i)) + \int_{s_i}^{t} S(t-\tau)(f(\tau,x_2(\tau)) + Cx_2(\tau))d\tau, \\
& t \in (s_i, t_{i+1}], \quad i = 1,2,\cdots,m.
\end{aligned}
$$

Namely, $Q : D \to PC(J,X)$ is an increasing operator.

(ii) We show $v_0 \leq Q(v_0)$, $Q(w_0) \leq w_0$.

Let

$$
\begin{cases}
v_0'(t) + Av_0(t) + Cv_0(t) = \bar{f}(t) \quad t \in (s_i, t_{i+1}], i = 0,1,2,\cdots,m, \\
v_0(t) = \bar{h}_i(t), \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m. \\
v_0(0) = \bar{g}(v_0),
\end{cases}
\tag{4.4}
$$

by the definition of v_0, we have

$$
\begin{cases}
\bar{f}(t) \leq f(t,v_0(t)) + Cv_0(t), \quad t \in (s_i, t_{i+1}], i = 0,1,2,\cdots,m, \\
\bar{h}_i(t) \leq h_i(t,v_0(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m. \\
\bar{g}(v_0) \leq g(v_0),
\end{cases}
\tag{4.5}
$$

By Theorem 3.1, (4.5)and (4.6), we have

$$
v_0(t) = \begin{cases}
S(t)\bar{g}(v_0) + \int_0^t S(t-\tau)\bar{f}(\tau)d\tau, \quad t \in [0,t_1]; \\
\bar{h}_i(t), \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m; \\
S(t-s_i)\bar{h}_i(s_i) + \int_{s_i}^t S(t-\tau)\bar{f}(\tau)d\tau, \\
t \in (s_i, t_{i+1}], \quad i = 1,2,\cdots,m.
\end{cases}
\tag{4.6}
$$

and

$$
\begin{aligned}
& S(t)\bar{g}(v_0) + \int_0^t S(t-\tau)\bar{f}(\tau)d\tau \\
\leq\ & S(t)g(v_0) + \int_0^t S(t-\tau)(f(\tau,v_0(\tau)) + Cv_0(\tau))d\tau, \quad t \in [0,t_1]; \\
& \bar{h}_i(t) \leq h_i(t,v_0(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m; \\
& S(t-s_i)\bar{h}_i(s_i) + \int_{s_i}^t S(t-\tau)\bar{f}(\tau)d\tau \\
\leq\ & S(t-s_i)h_i(s_i,v_0(s_i)) + \int_{s_i}^t S(t-\tau)(f(\tau,v_0(\tau)) + Cv_0(\tau))d\tau, \\
& t \in (s_i, t_{i+1}], \quad i = 1,2,\cdots,m.
\end{aligned}
$$

Namely, $v_0(t) \leq Q(v_0)(t)$. Similarly, it can be shown that $Q(w_0)(t) \leq w_0(t)$. Therefore, $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.

(iii) We prove that the operator Q has fixed points on $[v_0, w_0]$.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ by the iterative scheme

$$v_n = Q(v_{n-1}), \quad w_n = Q(w_{n-1}), \quad n = 1, 2, \cdots \tag{4.7}$$

Then from the monotonicity of operator Q it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{4.8}$$

Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J. Let $B = \{v_n \mid n \in \mathbb{N}\}$, $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$, then $B_0 = \{v_0\} \cup B$ and $B = Q(B_0)$.

For any $v_{n-1} \in B_0$, let

$$\begin{aligned}
(Q_1 v_{n-1})(t) = {} & S(t)g(u) + \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau, \quad t \in [0, t_1]; \\
(Q_2 v_{n-1})(t) = {} & h_i(t, v_{n-1}(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m; \\
(Q_3 v_{n-1})(t) = {} & S(t-s_i)h_i(s_i, v_{n-1}(s_i)) + \int_{s_i}^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau, \\
& t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m.
\end{aligned}$$

For $0 < t \leq a$, by the assumption (H1), we know that

$$f(t, v_0(t)) + Cv_0(t) \leq f(t, v_{n-1}(t)) + Cv_{n-1}(t) \leq f(t, w_0(t)) + Cw_0(t).$$

Since $f(t, v_0(t))$ and $f(t, w_0(t))$ are continuous in compact set $[0, a]$, so their image sets are compact sets in X, namely image sets are bounded. Combining this fact with the normality of cone K in X, we have $\exists M_1 > 0$, $\forall v_{n-1} \in B_0$,

$$\begin{aligned}
& \|f(t, v_{n-1}(t)) + Cv_{n-1}(t)\| \\
\leq {} & \|f(t, v_0(t)) + Cv_0(t)\| + N_0\|f(t, w_0(t)) + Cw_0(t) - f(t, v_0(t)) - Cv_0(t)\| \qquad (4.9) \\
\leq {} & M_1.
\end{aligned}$$

Case 1. For interval $[0, t_1]$ and any $0 < \epsilon < t_1$, let

$$(W_1 v_{n-1})(t) := \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau$$

and

$$(W_1^\epsilon v_{n-1})(t) := \int_0^{t-\epsilon} S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau,$$

then

$$\begin{aligned}
& \|(W_1 v_{n-1})(t)) - (W_1^{\epsilon} v_{n-1})(t)\| \\
= \; & \| \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \\
& - \int_0^{t-\epsilon} S(t-s)\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau\| \\
\leq \; & \int_{t-\epsilon}^t \|S(t-\tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
\leq \; & \overline{M} M_1 \epsilon.
\end{aligned}$$

Therefore, $Y_1(t) \triangleq \{(W_1 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in X by using the total boundedness.

On the other hand, by the assumption (H5), $\{S(t)g(v_{n-1}) \mid v_{n-1} \in B_0\}$ is precompact in X due to the compactness of $S(t)$. Therefore, $\{(Q_1 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in X for $t \in [0, t_1]$.

Case 2. For $t \in (t_i, s_i], i = 1, 2, \cdots, m$, the set $\{(Q_2 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in X by the assumption (H4).

Case 3. For $t \in (s_i, t_{i+1}], i = 1, 2, \cdots, m$, similar to the case 1, $\{(Q_3 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in X by (4.9)and the assumption (H4).

Hence, $\{v_n(t)\} = \{Q(v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in X for $t \in J$, combining this fact with the monotonicity of $\{v_n\}$, we easily prove that $\{v_n(t)\}$ is convergent. Let $\{v_n(t)\} \to \underline{u}(t)$ in $t \in J$. Similarly, we prove that $\{w_n(t)\} \to \overline{u}(t)$ in $t \in J$.

Evidently $\{v_n(t)\}, \{w_n(t)\} \in PC(J, X)$, so $\underline{u}(t)$ and $\overline{u}(t)$ is bounded integrable in J). Since for any $t \in J$, $v_n(t) = Q(v_{n-1})(t)$, $w_n(t) = Q(w_{n-1})(t)$, letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have $\underline{u}(t) = Q(\underline{u})(t)$, $\overline{u}(t) = Q(\overline{u})(t)$ and $\underline{u}(t), \overline{u}(t) \in PC(J, X)$. Combining this with monotonicity (4.8), we have $v_0(t) \leq \underline{u}(t) \leq \overline{u}(t) \leq w_0(t)$.

Next, we prove that $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal fixed points of Q in $[v_0, w_0]$, respectively. In fact, for any $u^* \in [v_0, w_0], Q(u^*) = u^*$, we have $v_0 \leq u^* \leq w_0$ and $v_1 = Q(v_0) \leq Q(u^*) = u^* \leq Q(w_0) = w_1$. Continuing such progress, we get $v_n \leq u^* \leq w_n$. Letting $n \to \infty$, we get $\underline{u}(t) \leq u^* \leq \overline{u}(t)$. Therefor, $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal mild solutions of the problem (1.1) between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively.□

From the Theorem4.1, we obtain the following result.

Theorem4.2. *Let X be an ordered Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator* *and $-A$ generate a compact and positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. Assume that problem(1.1) has lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2), (H3) and the following conditions are satisfied:*

(H6) $\{h_i(\cdot, x_n)\}(i = 1, 2, \cdots, m)$ are precompact in X, for any increasing or decreasing monotonic sequence $\{x_n\} \subset [v_0, w_0]$.

(H7) $\{g(x_n)\}$ is a precompact set in X, for any increasing or decreasing monotonic sequence $\{x_n\} \subset [v_0, w_0]$.

Then problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.

Next, we discuss the existence of the mild solutions for problem (1.1) under the function g is continuous in PC(J,X) and noncompactness measure conditions.

Theorem 4.3. *Let X be an ordered Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate an equicontinuous and positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Assume that problem(1.1) has lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2) and (H3) hold, and satisfy:*

(H8) There exist a constant $L > 0$ such that

$$\alpha(\{f(t, x_n)\}) \leq L\alpha(\{x_n\}),$$

for all $t \in J$, and increasing or decreasing sequence $\{x_n\} \subset [v_0(t), w_0(t)]$.

(H9) There exist constants $0 < L_i < 1(i = 1, 2, \cdots, m)$ such that

$$\alpha(\{h_i(t, x_n)\}) \leq L_i\alpha(\{x_n\}), (i = 1, 2, \cdots, m),$$

for all $t \in J$, and increasing or decreasing sequence $\{x_n\} \subset [v_0(t), w_0(t)]$.

(H10) There exist a constant $L^{'} > 0$ such that

$$\alpha(\{g(x_n)\}) \leq L^{'}\alpha(\{x_n\}),$$

for all $t \in J$, and increasing or decreasing sequence $\{x_n\} \subset [v_0(t), w_0(t)]$.

(H11) $\overline{M}[L_i + L^{'} + 2(L + C)a] < 1(i = 1, 2, \cdots, m)$.

Then the problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.

Proof From Theorem 4.1, we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous. Furthermore, if conditions (H1), (H2) and (H3) are satisfied, the iterative sequences $\{v_n\}$ and

$\{w_n\}$ defined by (4.7) satisfying (4.8). Therefore, for any $t \in J$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are monotone and order-bounded sequences in X.

Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J. Since $T(t)(t \geq 0)$ is an equicontinuous C_0-semigroup, so $S(t)(t \geq 0)$ also is an equicontinuous C_0-semigroup.

Let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$, by (4.8) and the normality of the positive cone K, then B and B_0 are bounded.

(i) We prove that $Q(B_0)$ is equicontinuous in $PC(J, X)$.

Combining (H2) and (H3) with the normality of cone K in X, we have $\exists M_2 > 0, M_3 > 0$, $\forall v_{n-1} \in B_0$,

$$\|g(v_{n-1})\| \leq \|g(v_0)\| + N_0\|g(w_0) - g(v_0)\| \leq M_2. \tag{4.10}$$

$$\|h_i(t, v_{n-1}(t))\| \leq \|h_i(t, v(t))\| + N_0\|h_i(t, w_0(t)) - g(h_i(t, v_0(t))\| \leq M_3. \tag{4.11}$$

Case 1. For $\forall t', t'' \in [0, t_1]$ and $t' < t''$, by (4.9) and (4.10) we have that

$$\begin{aligned}
& \|(Qv_{n-1})(t'') - (Qv_{n-1})(t')\| \\
= \; & \|S(t'')g(v_{n-1}) + \int_0^{t''} S(t'' - \tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \\
& -S(t')g(v_{n-1}) - \int_0^{t'} S(t' - \tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau\| \\
\leq \; & \|S(t'') - S(t')\|\|g(v_{n-1})\| \\
& + \int_0^{t'} \|S(t'' - \tau) - S(t' - \tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
& + \int_{t'}^{t''} \|S(t'' - \tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
\leq \; & M_2\|S(t')\|\|S(t'' - t') - I\| + M_1 \int_0^{t'} \|S(t'' - \tau) - S(t' - \tau)\|d\tau + \overline{M}M_1(t'' - t') \\
\leq \; & M_2\overline{M}\|S(t'' - t') - I\| + M_1 \int_0^{t'} \|S(t'' - t' + \tau) - S(\tau)\|d\tau + \overline{M}M_1(t'' - t') \\
\to \; & 0(t'' - t' \to 0).
\end{aligned}$$

Case 2. For $\forall t', t'' \in (t_i, s_i](i = 1, 2, \cdots, m)$ and $t' < t''$, we have that

$$\|(Qv_{n-1})(t'') - (Qv_{n-1})(t')\| = \|h_i(t'', v_{n-1}(t'')) - h_i(t', v_{n-1}(t'))\| \to 0(t'' - t' \to 0).$$

Case 3. For $\forall t', t'' \in (s_i, t_{i+1}] (i = 1, 2, \cdots, m)$ and $t' < t''$, by (4.9) and (4.11) we have that

$$\begin{aligned}
& \|(Qv_{n-1})(t'') - (Qv_{n-1})(t')\| \\
= \ & \|S(t'')h_i(s_i, v_{n-1}(s_i)) + \int_{s_i}^{t''} S(t''-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \\
& -S(t')h_i(s_i, v_{n-1}(s_i)) - \int_{s_i}^{t'} S(t'-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau\| \\
\leq \ & \|S(t'') - S(t')\|\|h_i(s_i, v_{n-1}(s_i))\| \\
& + \int_{s_i}^{t'} \|S(t''-\tau) - S(t'-\tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
& + \int_{t'}^{t''} \|S(t''-\tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
\leq \ & M_3\|S(t')\|\|S(t''-t') - I\| + M_1 \int_{s_i}^{t'} \|S(t''-\tau) - S(t'-\tau)\|d\tau + \overline{M}M_1(t''-t') \\
\leq \ & M_3\overline{M}\|S(t''-t') - I\| + M_1 \int_0^{t'-s_i} \|S(t''-t'+\tau) - S(\tau)\|d\tau + \overline{M}M_1(t''-t') \\
\to \ & 0 (t''-t' \to 0).
\end{aligned}$$

Therefore, $Q(B_0)$ is equicontinuous in $PC(J, X)$.

(ii) We prove that $\alpha(B(t)) = 0$ for $t \in J$.

It follows from $B_0 = \{v_0\} \cup B$ that $\alpha(B(t)) = \alpha(B_0(t))$ for $t \in J$.

Case 1. For $t \in [0, t_1]$, by Lemma 2.3 and Lemma 2.4, we have that

$$\begin{aligned}
& \alpha(B(t)) = \alpha((QB_0)(t)) \\
= \ & \alpha\Big(\Big\{S(t)g(v_{n-1}) + \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\Big\}\Big) \\
\leq \ & \alpha(\{S(t)g(v_{n-1}) \mid n \in \mathbb{N}\}) \\
& +\alpha\Big(\Big\{\int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\Big\}\Big) \\
\leq \ & \overline{M}\alpha(\{g(v_{n-1}) \mid n \in \mathbb{N}\}) + 2\int_0^t \alpha(\{S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\}) \\
\leq \ & \overline{M}L'\alpha(B_0) + 2\overline{M}\int_0^t (L+C)\alpha(B_0(\tau))d\tau
\end{aligned}$$

$$\leq \overline{M}L^{'}\max_{t\in J}\alpha(B(t))+2\overline{M}(L+C)a\max_{t\in J}\alpha(B(t))$$

$$\leq \overline{M}[L^{'}+2(L+C)a]\max_{t\in J}\alpha(B(t)).$$

Case 2. For $t\in(t_i,s_i], i=1,2,\cdots,m$, by (H9) we have

$$\alpha(B(t))=\alpha((QB_0)(t))=\alpha\Big(\Big\{h_i(t,v_{n-1}(t))\mid n\in\mathbb{N}\Big\}\Big)$$

$$\leq L_i\alpha(B_0(t))\leq L_i\max_{t\in J}\alpha(B(t))<\max_{t\in J}\alpha(B(t)).$$

Case 3. For $t\in(s_i,t_{i+1}](i=1,2,\cdots,m)$, we have

$$\alpha(B(t))=\alpha((QB_0)(t))$$

$$=\alpha\Big(\Big\{S(t-s_i)h_i(s_i,v_{n-1}(s_i))+\int_{s_i}^{t}S(t-\tau)(f(\tau,v_{n-1}(\tau))+Cv_{n-1}(\tau))d\tau\mid n\in\mathbb{N}\Big\}\Big)$$

$$\leq \alpha(\{S(t-s_i)h_i(s_i,v_{n-1}(s_i))\mid n\in\mathbb{N}\})$$

$$+\alpha\Big(\Big\{\int_{s_i}^{t}S(t-\tau)(f(\tau,v_{n-1}(\tau))+Cv_{n-1}(\tau))d\tau\mid n\in\mathbb{N}\Big\}\Big)$$

$$\leq \overline{M}\alpha(\{h_i(s_i,v_{n-1}(s_i))\mid n\in\mathbb{N}\})$$

$$+2\int_{s_i}^{t}\alpha(\{S(t-\tau)(f(\tau,v_{n-1}(\tau))+Cv_{n-1}(\tau))d\tau\mid n\in\mathbb{N}\})$$

$$\leq \overline{M}L_i\alpha(B_0)+2\overline{M}\int_{s_i}^{t}(L+C)\alpha(B_0(\tau))d\tau$$

$$\leq \overline{M}L_i\max_{t\in J}\alpha(B(t))+2\overline{M}(L+C)a\max_{t\in J}\alpha(B(t))$$

$$\leq \overline{M}[L_i+2(L+C)a]\max_{t\in J}\alpha(B(t)).$$

By (H11), we have $\alpha(B(t))<\max_{t\in J}\alpha(B(t))$, then $\alpha(B(t))=0$ in $t\in J$. Therefore, $\{v_n(t)\}$ is precompact in X for $t\in J$, combining this fact with the monotonicity of $\{v_n\}$, we easily prove that $\{v_n(t)\}$ is convergent. Let $\{v_n(t)\}\to\underline{u}(t)$ in $t\in J$. The same idea can be used to prove that $\{w_n(t)\}\to\overline{u}(t)$ in $t\in J$. Similar to the proof of Theorem 4.1, we know that $\underline{u}(t)$ and $\overline{u}(t)$ are the problem(1.1) between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively. This completes the proof of Theorem 4.3. □

Remark 4.4. Analytic semigroup and differentiable semigroup are equicontinuous semigroup ([22]). In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup are analytic semigroup. Therefore, Theorem 4.3. has extensive applicability.

we discuss the existence of the mild solutions for problem (1.1) under the positive cone is regular.

Theorem 4.5. *Let X be an ordered Banach space, whose positive cone K is regular. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Assume that problem(1.1) has lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2) and (H3) are satisfied.*
Then the problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.

Proof From Theorem 4.1 we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator. Similarly, the two sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are defined in $[v_0, w_0]$ by the iterative scheme (4.7). By conditions (H1), (H2) and (H3), then $\{v_n(t)\}$ and $\{w_n(t)\}$ are ordered-monotonic and ordered-bounded sequences in X.

Using the regularity of the cone K, any ordered-monotonic and ordered-bounded sequence in X is convergent. Similar to the proof of Theorem 4.1, we know that $\underline{u}(t)$ and $\overline{u}(t)$ are the problem(1.1) between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively. This completes the proof of Theorem 4.5. □

Corollary 4.6. *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Assume that problem(1.1) has lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2) and (H3) are satisfied. Then the problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.*

Proof In an ordered and weakly sequentially complete Banach space, the normal cone K is regular. Then the proof is complete. □

Next, we discuss the existence of mild solutions of problem (1.1), when we don't assume the lower and upper solutions of problem (1.1) be exist.

Theorem 4.7. *Let X be an ordered Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and*

$-A$ generate a positive and compact C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Suppose that conditions (H1)–(H5) hold and the following condition is satisfied:

(H12) $\exists b \geq 0$, $h \in PC(J, X), h \geq \theta$, $y_i(s_i) \in D(A), y_i \geq \theta, i = 1, 2, \cdots, m$ and $g(u) \in D(A), g(u) \geq \theta$ for any $u \in PC(J, X)$, such that

$$f(t,u) \leq bu + h(t), \quad h_i(t,u) \leq y_i(t), \quad u \geq 0;$$

$$bu - h(t) \leq f(t,u), \quad -y_i(t) \leq h_i(t,u), \quad u \leq 0.$$

Then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.

Proof For $h(t) \geq \theta, y_i(t) \geq \theta$, we consider the linear nonlocal evolution equation with not instantaneous impulses in X

$$\begin{cases} u'(t) + Au(t) - bu(t) = h(t), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = y_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ u(0) = g(u), \end{cases} \tag{4.12}$$

Since $-(A - bI)$ generate a positive C_0-semigroup $S(t) = e^{bt}T(t)(t \geq 0)$ in X. By Theorem 3.1 and assumption (H12), we know that the problem (4.12) has a unique positive solution $u^* \geq \theta$. Let $v_0 = -u^*, w_0 = u^*$, by the conditions (H1)–(H3) and (H12), we get

$$\begin{cases} v_0'(t) + Av_0(t) = bv_0(t) - h(t) \leq f(t, v_0(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ v_0(t) = -y_i(t) \leq h_i(t, v_0(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ v_0(0) = -g(-v_0) \leq g(v_0), \end{cases} \tag{4.13}$$

and

$$\begin{cases} w_0'(t) + Aw_0(t) = bw_0(t) + h(t) \geq f(t, w_0(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ w_0(t) = y_i(t) \geq h_i(t, w_0(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ w_0(0) \geq g(w_0), \end{cases} \tag{4.14}$$

So, we inferred that v_0 and w_0 are a lower solution and an upper solution of the problem (1.1), respectively. Therefore by Theorem 4.1., the conclusion holds. Then the proof is complete.□

Meanwhile, we can obtain the following results from Theorem 4.2, 4.3, 4.5 and Corollary 4.6, respectively.

Corollary 4.8. *Let X be an ordered Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive and compact C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and map a monotonic set into a precompact set and conditions (H1)–(H3) and (H12) hold, then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.*

Corollary 4.9. *Let X be an ordered Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and for any monotonic sequence $\{x_n\}$ satisfy conditions (H8)–(H11) and conditions (H1)–(H3) as well as (H12) hold, then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.*

Corollary 4.10. *Let X be an ordered Banach space, whose positive cone K is regular. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and conditions (H1)–(H3) as well as (H12) hold, then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.*

Corollary 4.11. *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)(t \geq 0)$ in X. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and conditions (H1)–(H3) as well as (H12) hold, then the problem (1.1) has minimal and maximal mild solutions, which can be obtained by monotone iterative procedure.*

5 Application

In this section, we present one example, which indicates how our abstract results can be applied to concrete problems.

Example 5.1. Consider the following nonlocal parabolic partial differential equation with not instantaneous impulses:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + A(x,D)u(x,t) = f(x,t,u(x,t)), \quad x \in \Omega, \\ t \in J, \quad t \in (s_i, t_{i+1}], i = 0,1,2,\cdots,m, \\ u(x,t) = h_i(x,t,u(x,t)), \quad x \in \Omega, \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m, \\ Bu = 0, \quad (x,t) \in \partial\Omega \times J, \\ u(x,0) = g(u), \quad x \in \Omega, \end{cases} \tag{5.1}$$

where $J = [0,a]$, $a > 0$ is a constant, $0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < t_3 \leq s_3 < \cdots < t_{m-1} \leq s_{m-1} < t_m \leq s_m < t_{m+1} = a$ are pre-fixed numbers, integer $n \geq 1, \Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$,

$$A(x,D) = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial y_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} + a_0(x)$$

is a strongly elliptic operator of second order, coefficient functions $a_{ij}(x), a_i(x)$ and $a_0(x)$ are Hölder continuous in Ω, $Bu = b_0(x)u + \delta\frac{\partial u}{\partial n}$ is a regular boundary operator on $\partial\Omega$, $f : \overline{\Omega} \times J \times \mathbb{R} \to \mathbb{R}$ is continuous, $h_i : \overline{\Omega} \times J \times \mathbb{R} \to \mathbb{R}$ are also continuous, $i = 1,2,\cdots,m$, g is a continuous function.

Let $X = L^p(\Omega)$ with $p > n+2$, $K = \{u \in L^p(\Omega) \mid u(x) \geq 0 \ \ a.e. \ \ x \in \Omega\}$, and define the operator A as follows:

$$D(A) = \{u \in W^{2,p}(\Omega) \mid Bu = 0\}, \ Au = A(x,D)u.$$

We know that X is a Banach space, K is a regular cone of X, and $-A$ generates a positive and analytic C_0-semigroup $T(t)(t \geq 0)$ in X (see [22]). Define $u(t) = u(\cdot,t), f(t,u(t)) = f(\cdot,t,u(\cdot,t)), h_i(t,u(t)) = h_i(\cdot,t,u(\cdot,t))$, then system (5.1) can be reformulated as problem (1.1) in X. We assume that the following conditions hold:

(i) Let $f(x,t,0) \geq 0, h_i(x,t,0) \geq 0, g(0) \geq 0, x \in \Omega$.

(ii) There exist $w = w(x,t) \in PC(\Omega \times J) \cap C^{2,1}(\Omega \times J'')$, and $w(x,t) \geq 0, x \in \Omega, t \in J$ such that

$$\begin{cases} \frac{\partial}{\partial t}w(x,t) + A(x,D)w(x,t) \geq f(x,t,w(x,t)), \quad x \in \Omega, \\ t \in J, \quad t \in (s_i, t_{i+1}], i = 0,1,2,\cdots,m, \\ w(x,t) \geq h_i(x,t,w(x,t)), \quad x \in \Omega, \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m, \\ Bw = 0, \quad (x,t) \in \partial\Omega \times J, \\ w(x,0) \geq g(w), \quad x \in \Omega, \end{cases}$$

(iii) The partial derivative $f'_u(x,t,u)$ is continuous on any bounded domain.

(iv) For any $u_1, u_2 \in [0, w(x,t)]$ with $u_1 \leq u_2$, for any $x \in \Omega, i = 1,2,\cdots,m$,we have

$$h_i(x,t,u_1(x,t)) \leq h_i(x,t,u_2(x,t)), g(u_1) \leq g(u_1)$$

Theorem 5.2. If assumptions (i), (ii), (iii) and (iv) are satisfied, then the impulsive parabolic partial differential equation (5.1) has minimal and maximal mild solutions between 0 and $w(x,t)$, which can be obtained by a monotone iterative procedure starting from 0 and $w(x,t)$, respectively.

Proof From assumptions (i) and(ii) we know that 0 and $w(x,t)$ are lower and upper solutions of problem (5.1), respectively. (iii) implies that condition (H1) is satisfied. (iv) implies that conditions (H2) and (H2) are satisfied. So, by Theorem 4.5., we have the result. Then the proof is complete.□

6 Conclusions

In this paper, we consider the existence of mild solutions for the new nonlocal evolution equation with impulses. We initially use the monotone iterative technique to the problem under new impulsive conditions. Hence the results are new.

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Eigenvalue for a system of Caputo fractional differential equations *

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Abstract: In this article, we study the existence of positive solutions for a system of nonlinear differential equations of mixed Caputo fractional orders

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, & 0 < t < 1, \\ {}^cD_{0+}^{\beta}v(t) + \mu g(t, u(t), v(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(0) = 0, \\ v(0) = v'(0) = v''(1) = v'''(0) = 0, \end{cases}$$

where $3 < \alpha, \beta \leq 4$ are real numbers, ${}^cD_{0+}^{\alpha}, {}^cD_{0+}^{\beta}$ are the Caputo fractional derivatives, and $f, g : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are given continuous functions. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are obtained.

Keywords: Fractional order differential equation, Positive solution, Existence, Krasnoselskii's fixed point theorem, Eigenvalue.

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1 Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, economics, control theory, see [4, 8]. Recently, fractional differential equations have been of great interest, there are a large number of papers dealing with the existence of positive solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis (such as upper and lower solution method, Leray-Schauder theory, etc.), see [1, 2, 5, 7, 9, 10]. In this paper, we consider the system of Caputo fractional differential equations

$$\begin{cases} {}^cD_{0+}^{\alpha}u\left(t\right) + \lambda f\left(t, u\left(t\right), v\left(t\right)\right) = 0, & 0 < t < 1, \\ {}^cD_{0+}^{\beta}v\left(t\right) + \mu g\left(t, u\left(t\right), v\left(t\right)\right) = 0, & 0 < t < 1, \\ u\left(0\right) = u'\left(0\right) = u''\left(1\right) = u'''\left(0\right) = 0, \\ v\left(0\right) = v'\left(0\right) = v''\left(1\right) = v'''\left(0\right) = 0, \end{cases} \tag{1.1}$$

where $3 < \alpha, \beta \leq 4$ are real numbers, ${}^cD_{0+}^{\alpha}, {}^cD_{0+}^{\beta}$ are the Caputo fractional derivatives, and f, g : $[0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are given continuous functions. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are obtained.

This paper is organized as follows. In Section 2, we present some basic definitions and properties from the fractional calculus theory. In Section 3, based on the Krasnoselskii's fixed point theorem, we prove two existence theorems of the positive solutions for BVP (1.1). In section 4, an example is presented to illustrate the main results.

2 Preliminaries

Let us start with the necessary definitions which are used throughout this paper.

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Definition 2.1 ([2]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \qquad t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([3, 8]). For a function $f : (0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order is defined as

$${}^{c}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, \quad t > 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α.

Lemma 2.1 ([3, 9]). Let $\alpha > 0$, then fractional differential equation ${}^{c}D_{0+}^{\alpha} u(t) = 0$ has solutions

$$u(t) = C_1 + C_2 t + \cdots + C_n t^{n-1}, C_i \in \mathbb{R}, i = 1, 2, \cdots, n, n = [\alpha] + 1.$$

Lemma 2.2 ([3, 9]). Let $\alpha > 0$, then

$$I_{0+}^{\alpha}\, {}^{c}D_{0+}^{\alpha} u(t) = u(t) + C_1 + C_2 t + \cdots + C_n t^{n-1}, C_i \in \mathbb{R}, i = 1, 2, \cdots, n, n = [\alpha] + 1.$$

In the following, we present Green's function of BVP (1.1).

Lemma 2.3 . Let $h_1 \in C[0, 1]$ and $3 < \alpha \leq 4$, the unique solution of problem

$${}^{c}D_{0+}^{\alpha} u(t) + h_1(t) = 0, 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(0) = u''(1) = u'''(0) = 0, \tag{2.2}$$

is

$$u(t) = \int_0^1 G_1(t, s) h_1(s)\mathrm{d}s,$$

where

$$G_1(t, s) = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} - 2(t-s)^{\alpha-1}}{2\Gamma(\alpha)}, 0 \leq s \leq t \leq 1, \\ \dfrac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}, \qquad 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Here $G_1(t, s)$ is called the Green's function of BVP (2.1) and (2.2).

Proof. We may apply Lemma 2.2 to reduce (2.1) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} h_1(t) + C_1 + C_2 t + C_3 t^2 + C_4 t^3,$$

for some $C_1, C_2, C_3, C_4 \in \mathbb{R}$. Consequently, the general solution of (2.1) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_1(s)\mathrm{d}s + C_1 + C_2 t + C_3 t^2 + C_4 t^3.$$

By (2.2), there are $C_1 = C_2 = C_4 = 0$, and $C_3 = \dfrac{1}{2\Gamma(\alpha-2)} \displaystyle\int_0^1 (1-s)^{\alpha-3} h_1(s)\mathrm{d}s$.
Therefore, the unique solution of problem (2.1) and (2.2) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_1(s)\mathrm{d}s + \frac{1}{2\Gamma(\alpha-2)} \int_0^1 t^2(1-s)^{\alpha-3} h_1(s)\mathrm{d}s \\ &= \int_0^t \left[\frac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right] h_1(s)\mathrm{d}s + \int_t^1 \frac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} h_1(s)\mathrm{d}s \\ &= \int_0^t \frac{(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} - 2(t-s)^{\alpha-1}}{2\Gamma(\alpha)} h_1(s)\mathrm{d}s + \int_t^1 \frac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} h_1(s)\mathrm{d}s \\ &= \int_0^1 G_1(t, s) h_1(s)\mathrm{d}s. \end{aligned}$$

The proof is finished.

Lemma 2.4 . The function $G_1(t,s)$ defined by (2.3) possesses the following properties:

(1) $G_1(t,s) > 0$, for $t,s \in (0,1)$;

(2) $\min\limits_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \frac{1}{16}\max\limits_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s)$, for $s \in (0,1)$.

Proof. Let $g_1(t,s) = \frac{(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} - 2(t-s)^{\alpha-1}}{2\Gamma(\alpha)}$, $\quad g_2(t,s) = \frac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}$.

(1) Since $3 < \alpha \le 4, 0 < s \le t < 1$, so

$$(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} > 2t^2(1-s)^{\alpha-3} > 2t^2(t-s)^{\alpha-3} \ge 2(t-s)^{\alpha-1},$$

therefore, $g_1(t,s) > 0$, obviously, $g_2(t,s) > 0$, thus $G_1(t,s) > 0$, for $t,s \in (0,1)$.

(2) Since

$$\frac{\partial g_1(t,s)}{\partial t} = \frac{(\alpha-1)(\alpha-2)\,t\,(1-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} > 0, \qquad \frac{\partial g_2(t,s)}{\partial t} = \frac{t(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} > 0,$$

so $G_1(t,s)$ is monotone increasing function for t.

Thus,

$$0 \le G_1(t,s) \le \max_{0\le t\le 1} G_1(t,s) = G_1(1,s), \qquad t,s \in [0,1].$$

Noticing that

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) = G_1(\frac{1}{4},s) = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} - 32\left(\frac{1}{4}-s\right)^{\alpha-1}}{32\Gamma(\alpha)}, & s \in (0,\frac{1}{4}], \\ \dfrac{(1-s)^{\alpha-3}}{32\Gamma(\alpha-2)}, & s \in [\frac{1}{4},1). \end{cases}$$

$$\max_{0\le t\le 1} G_1(t,s) = G_1(1,s) = \frac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} - 2(1-s)^{\alpha-1}}{2\Gamma(\alpha)}, \qquad s \in (0,1).$$

Next we proof

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s).$$

When $0 < s \le \frac{1}{4}$, since $3 < \alpha \le 4$, we have

$$(\frac{1}{4}-s)^{\alpha-1} = (\frac{1}{4})^{\alpha-1}(1-4s)^{\alpha-1} \le (\frac{1}{4})^2(1-4s)^{\alpha-1} < \frac{1}{16}(1-s)^{\alpha-1},$$

so

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s).$$

When $\frac{1}{4} \le s < 1$, we obtain

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) = \frac{(1-s)^{\alpha-3}}{32\Gamma(\alpha-2)} = \frac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3}}{32\Gamma(\alpha)},$$

$$\frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s) = \frac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3}}{32\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{16\Gamma(\alpha)}.$$

Obvious that,

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s), \qquad s \in (0,1).$$

The proof is finished.

Lemma 2.5 . If the function $f \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, then the unique solution of BVP (1.1) satisfied

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge \frac{1}{16}\|u\|.$$

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Proof. From lemma 2.3, we known

$$u(t)=\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\le\int_0^1 \max_{0\le t\le 1} G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s,$$

and

$$\|u\|=\max_{0\le t\le 1}|u(t)|=\max_{0\le t\le 1}\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\le\int_0^1 \max_{0\le t\le 1} G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s.$$

From lemma 2.4, we have

$$\begin{aligned}\min_{\frac14\le t\le\frac34} u(t)&=\min_{\frac14\le t\le\frac34}\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\\&\ge\frac{1}{16}\int_0^1 \max_{0\le t\le 1} G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\ge\frac{1}{16}\max_{0\le t\le 1}\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s=\frac{1}{16}\|u\|.\end{aligned}$$

The proof is finished.

Similarly, we can obtain $G_2(t,s)$ if α is replaced by β,

$$G_2(t,s)=\begin{cases}\dfrac{(\beta-1)(\beta-2)t^2(1-s)^{\beta-3}-2(t-s)^{\beta-1}}{2\Gamma(\beta)}, & 0\le s\le t\le 1,\\ \dfrac{t^2(1-s)^{\beta-3}}{2\Gamma(\beta-2)}, & 0\le t\le s\le 1.\end{cases}\tag{2.4}$$

The function $G_2(t,s)$ defined by (2.4) have the same properties with $G_1(t,s)$, so

$$\min_{\frac14\le t\le\frac34} G_2(t,s)\ge\frac{1}{16}\max_{0\le t\le 1} G_2(t,s)=\frac{1}{16}G_2(1,s),\, s\in(0,1).$$

Lemma 2.6 ([6]). Let E be a Banach space, and let $P\subset E$ be a cone in E. Assume Ω_1,Ω_2 be two open subsets of E with $\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T:P\to P$ be a completely continuous operator such that either

(i)$\|Tw\|\le\|w\|, w\in P\cap\partial\Omega_1, \|Tw\|\ge\|w\|, w\in P\cap\partial\Omega_2$, or

(ii)$\|Tw\|\ge\|w\|, w\in P\cap\partial\Omega_1, \|Tw\|\le\|w\|, w\in P\cap\partial\Omega_2$

holds. Then T has a fixed point in $P\cap\overline{\Omega}_2\backslash\Omega_1$.

3 Main results and proof

In this section, we establish the existence of positive solutions for BVP (1.1). For convenience, we introduce the following notations

$$\begin{aligned}
f_0&=\liminf_{u+v\to 0^+}\min_{t\in\left[\frac14,\frac34\right]}\frac{f(t,u,v)}{u+v}, & g_0&=\liminf_{u+v\to 0^+}\min_{t\in\left[\frac14,\frac34\right]}\frac{g(t,u,v)}{u+v},\\
f^0&=\limsup_{u+v\to 0^+}\max_{t\in[0,1]}\frac{f(t,u,v)}{u+v}, & g^0&=\limsup_{u+v\to 0^+}\max_{t\in[0,1]}\frac{g(t,u,v)}{u+v},\\
f_\infty&=\liminf_{u+v\to\infty}\min_{t\in\left[\frac14,\frac34\right]}\frac{f(t,u,v)}{u+v}, & g_\infty&=\liminf_{u+v\to\infty}\min_{t\in\left[\frac14,\frac34\right]}\frac{g(t,u,v)}{u+v},\\
f^\infty&=\limsup_{u+v\to\infty}\max_{t\in[0,1]}\frac{f(t,u,v)}{u+v}, & g^\infty&=\limsup_{u+v\to\infty}\max_{t\in[0,1]}\frac{g(t,u,v)}{u+v}.
\end{aligned}$$

By using the Green's functions $G_i(t,s)\,(i=1,2)$, from Section 2, the problem (1.1) can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases}u(t)=\lambda\displaystyle\int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s, 0\le t\le 1,\\ v(t)=\mu\displaystyle\int_0^1 G_2(t,s)g(s,u(s),v(s))\mathrm{d}s, 0\le t\le 1.\end{cases}$$

We consider the Banach space $X = C[0,1]$ with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$.

We define the cone $P \subset Y$ by

$$P = \left\{ (u,v) \in Y | u(t) \ge 0, v(t) \ge 0, \min_{\frac{1}{4}\le t\le \frac{3}{4}} (u(t)+v(t)) \ge \frac{1}{16} \|(u,v)\|_Y, t \in [0,1] \right\}.$$

For $\lambda, \mu > 0$, we define the operators $T_1, T_2 : Y \to X$ and $T : Y \to Y$ respectively by

$$\begin{cases} T_1(u,v)(t) = \lambda \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s, 0 \le t \le 1, \\ T_2(u,v)(t) = \mu \int_0^1 G_2(t,s) g(s,u(s),v(s)) \mathrm{d}s, 0 \le t \le 1, \end{cases}$$

and $T(u,v) = (T_1(u,v), T_2(u,v)), (u,v) \in Y$. Thus, the solutions of BVP (1.1) are the fixed points of the operator T.

Lemma 3.1. $T : P \to P$ is a completely continuous operator.

Proof. Let $(u,v) \in P$ be an arbitrary element. From the definition $T_1(u,v)$ and Lemma 2.4, we get

$$\|T_1(u,v)\| = \max_{0\le t\le 1} |T_1(u,v)(t)| \le \int_0^1 \max_{0\le t\le 1} G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s,$$

$$\begin{aligned} \min_{\frac{1}{4}\le t\le \frac{3}{4}} T_1(u,v)(t) = \min_{\frac{1}{4}\le t\le \frac{3}{4}} \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s &\ge \frac{1}{16} \int_0^1 \max_{0\le t\le 1} G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s \\ &\ge \frac{1}{16} \max_{0\le t\le 1} \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s = \frac{1}{16} \|T_1(u,v)\|. \end{aligned}$$

In the similar manner, we deduce $\min_{\frac{1}{4}\le t\le \frac{3}{4}} T_2(u,v)(t) \ge \frac{1}{16} \|T_2(u,v)\|$.

Thus we have

$$\min_{\frac{1}{4}\le t\le \frac{3}{4}} (T_1(u,v)(t) + T_2(u,v)(t)) \ge \frac{1}{16} (\|T_1(u,v)\| + \|T_2(u,v)\|) \ge \frac{1}{16} T\|(u,v)\|_Y.$$

Hence $T(u,v) \in P$, that is $T(P) \subset P$.

According to the Arzela-Ascoli theorem, we can easily get that $T : P \to P$ is a completely continuous operator. The proof is completed.

Next, for $\alpha_1, \alpha_2, \overline{\alpha}_1, \overline{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1, \overline{\alpha}_1 + \overline{\alpha}_2 = 1$, we define the numbers L_1, L_2, L_3, L_4 by

$$L_1 = \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s) \mathrm{d}s}{256 \overline{\alpha}_1}, \qquad L_2 = \frac{\int_0^1 G_1(1,s) \mathrm{d}s}{\alpha_1}, \qquad L_3 = \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(1,s) \mathrm{d}s}{256 \overline{\alpha}_2}, \qquad L_4 = \frac{\int_0^1 G_2(1,s) \mathrm{d}s}{\alpha_2}.$$

Theorem 3.1. If $f^0, g^0, f_\infty, g_\infty \in (0,\infty)$, $\frac{1}{L_1 f_\infty} < \frac{1}{L_2 f^0}$ and $\frac{1}{L_3 g_\infty} < \frac{1}{L_4 g^0}$ hold, then for any $\lambda \in \left(\frac{1}{L_1 f_\infty}, \frac{1}{L_2 f^0} \right)$ and $\mu \in (\frac{1}{L_3 g_\infty}, \frac{1}{L_4 g^0})$, BVP (1.1) has at least one positive solution $(u(t), v(t)), t \in [0,1]$.

Proof. When $\lambda \in \left(\frac{1}{L_1 f_\infty}, \frac{1}{L_2 f^0} \right)$ and $\mu \in \left(\frac{1}{L_3 g_\infty}, \frac{1}{L_4 g^0} \right)$, choosing $\varepsilon > 0$, such that

$$\frac{1}{L_1(f_\infty - \varepsilon)} \le \lambda \le \frac{1}{L_2(f^0 + \varepsilon)}, \quad \frac{1}{L_3(g_\infty - \varepsilon)} \le \mu \le \frac{1}{L_4(g^0 + \varepsilon)} \tag{3.1}$$

By the definition of f^0, g^0, there exists $R_1 > 0$, such that for all $t \in [0,1], u, v \in \mathbb{R}^+$, with $0 \le u + v \le R_1$, we have

$$f(t,u,v) \le (f^0 + \varepsilon)(u+v), \qquad g(t,u,v) \le (g^0 + \varepsilon)(u+v). \tag{3.2}$$

Now define the set $\Omega_1 = \{(u,v) \in Y, \|(u,v)\|_Y < R_1\}$. Let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_1$, so $u(t) + v(t) \le R_1$ for all $t \in [0,1]$, thus

$$\begin{aligned} T_1(u,v)(t) = &\lambda \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s \le \lambda \int_0^1 G_1(1,s)(f^0 + \varepsilon)(u(s)+v(s)) \mathrm{d}s \\ &\le \lambda (f^0 + \varepsilon) \int_0^1 G_1(1,s) \mathrm{d}s \|(u,v)\|_Y \le \frac{1}{L_2} \int_0^1 G_1(1,s) \mathrm{d}s \|(u,v)\|_Y \\ &\le \alpha_1 \|(u,v)\|_Y. \end{aligned}$$

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Therefore, $\|T_1(u,v)\| \leq \alpha_1 \|(u,v)\|_Y$.

In the similar manner, we deduce

$$T_2(u,v)(t) \leq \mu(g^0+\varepsilon)\int_0^1 G_2(1,s)\mathrm{d}s\,\|(u,v)\|_Y \leq \alpha_2 \|(u,v)\|_Y.$$

So, $\|T_2(u,v)\| \leq \alpha_2 \|(u,v)\|_Y$.

Then for $(u,v) \in P \cap \partial\Omega_1$, we deduce

$$\|T(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \leq (\alpha_1+\alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By the definition of f_∞, g_∞, there exists $R_2 > 0$, such that for all $t \in [0,1], u, v \in \mathbb{R}^+$, with $u+v \geq R_2$, we have

$$f(t,u,v) \geq (f_\infty - \varepsilon)(u+v), \qquad g(t,u,v) \geq (g_\infty - \varepsilon)(u+v). \tag{3.3}$$

Now define the set $\Omega_2 = \{(u,v) \in Y, \|(u,v)\|_Y < R_2\}$. Let $(u,v) \in P \cap \partial\Omega_2$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_2$, so $u(t)+v(t) \geq \frac{1}{16}\|(u,v)\|_Y$ for all $t \in [\frac{1}{4}, \frac{3}{4}]$, thus, it follows from Lemma 2.4 that

$$\begin{aligned} T_1(u,v)(t) =&\lambda \int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \geq \lambda \int_{\frac14}^{\frac34} G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \\ \geq&\lambda \int_{\frac14}^{\frac34} \frac{1}{16}G_1(1,s)(f_\infty-\varepsilon)(u(s)+v(s))\mathrm{d}s = \frac{\lambda(f_\infty-\varepsilon)}{16}\int_{\frac14}^{\frac34} G_1(1,s)(u(s)+v(s))\mathrm{d}s \\ \geq&\frac{\lambda(f_\infty-\varepsilon)}{256}\int_{\frac14}^{\frac34} G_1(1,s)\mathrm{d}s\,\|(u,v)\|_Y \geq \frac{1}{256L_1}\int_{\frac14}^{\frac34} G_1(1,s)\mathrm{d}s\,\|(u,v)\|_Y \geq \overline{\alpha}_1\|(u,v)\|_Y. \end{aligned}$$

Therefore, $\|T_1(u,v)\| \geq \overline{\alpha}_1 \|(u,v)\|_Y$.

Similarly, we have

$$T_2(u,v)(t) \geq \frac{\mu(g_\infty-\varepsilon)}{256}\int_{\frac14}^{\frac34} G_2(1,s)\mathrm{d}s\,\|(u,v)\|_Y \geq \overline{\alpha}_2\|(u,v)\|_Y.$$

So, $\|T_2(u,v)\| \geq \overline{\alpha}_2 \|(u,v)\|_Y$.

Then for $(u,v) \in P \cap \partial\Omega_2$, we deduce

$$\|T(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \geq (\overline{\alpha}_1+\overline{\alpha}_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By using Lemma 2.6, we conclude that T has a fixed point $(u,v) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $R_1 \leq \|u\|+\|v\| \leq R_2$.

Theorem 3.2. If $f_0, g_0, f^\infty, g^\infty \in (0,\infty)$, $\frac{1}{L_1 f_0} < \frac{1}{L_2 f^\infty}$ and $\frac{1}{L_3 g_0} < \frac{1}{L_4 g^\infty}$ hold, then for any $\lambda \in (\frac{1}{L_1 f_0}, \frac{1}{L_2 f^\infty})$ and $\mu \in (\frac{1}{L_3 g_0}, \frac{1}{L_4 g^\infty})$, BVP (1.1) has at least one positive solution $(u(t), v(t)), t \in [0,1]$.

Proof. When $\lambda \in \left(\frac{1}{L_1 f_0}, \frac{1}{L_2 f^\infty}\right)$ and $\mu \in (\frac{1}{L_3 g_0}, \frac{1}{L_4 g^\infty})$, choosing $\varepsilon > 0$, such that

$$\frac{1}{L_1(f_0-\varepsilon)} \leq \lambda \leq \frac{1}{L_2(f^\infty+\varepsilon)}, \quad \frac{1}{L_3(g_0-\varepsilon)} \leq \mu \leq \frac{1}{L_4(g^\infty+\varepsilon)}. \tag{3.4}$$

By the definition of f_0, g_0, there exists $R_3 > 0$, such that for all $t \in [0,1], u, v \in \mathbb{R}^+$, with $0 \leq u+v \leq R_3$, we have

$$f(t,u,v) \geq (f_0 - \varepsilon)(u+v), \qquad g(t,u,v) \geq (g_0 - \varepsilon)(u+v). \tag{3.5}$$

Now define the set $\Omega_3 = \{(u,v) \in Y, \|(u,v)\|_Y < R_3\}$. Let $(u,v) \in P \cap \partial\Omega_3$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_3$, so $u(t)+v(t) \geq \frac{1}{16}\|(u,v)\|_Y$ for all $t \in [\frac{1}{4}, \frac{3}{4}]$, thus, it follows from Lemma 2.4 that

$$\begin{aligned} T_1(u,v)(t) =&\lambda \int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \geq \lambda \int_{\frac14}^{\frac34} G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \\ \geq&\lambda \int_{\frac14}^{\frac34} \frac{1}{16}G_1(1,s)(f_0-\varepsilon)(u(s)+v(s))\mathrm{d}s = \frac{\lambda(f_0-\varepsilon)}{16}\int_{\frac14}^{\frac34} G_1(1,s)(u(s)+v(s))\mathrm{d}s \\ \geq&\frac{\lambda(f_0-\varepsilon)}{256}\int_{\frac14}^{\frac34} G_1(1,s)\mathrm{d}s\|(u,v)\|_Y \geq \frac{1}{256L_1}\int_{\frac14}^{\frac34} G_1(1,s)\mathrm{d}s\|(u,v)\|_Y \geq \overline{\alpha}_1\|(u,v)\|_Y. \end{aligned}$$

Therefore, $\|T_1(u,v)\| \geq \overline{\alpha}_1 \|(u,v)\|_Y$.

In the similar manner, we deduce

$$T_2(u,v)(t) \geq \frac{\mu(g_0-\varepsilon)}{256}\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(1,s)\mathrm{d}s \|(u,v)\|_Y \geq \overline{\alpha}_2 \|(u,v)\|_Y.$$

So, $\|T_2(u,v)\| \geq \overline{\alpha}_2 \|(u,v)\|_Y$.

Then for $(u,v) \in P \cap \partial\Omega_3$, we deduce

$$\|T(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \geq (\overline{\alpha}_1 + \overline{\alpha}_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

Next, we define the functions $f^*, g^* : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+, f^*(t,x) = \max\limits_{0\leq u+v\leq x} f(t,u,v), g^*(t,x) = \max\limits_{0\leq u+v\leq x} g(t,u,v), t \in [0,1], x \in \mathbb{R}^+$. Then $f(t,u,v) \leq f^*(t,x), g(t,u,v) \leq g^*(t,x)$ for all $t \in [0,1]$, $u \geq 0, v \geq 0$ and $u+v \leq x$. The functions $f^*(t,\cdot), g^*(t,\cdot)$ are nondecreasing for every $t \in [0,1]$, and satisfy the conditions

$$\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x} \leq f^\infty, \qquad \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x} \leq g^\infty.$$

Therefore, for $\varepsilon > 0$, there exists $\overline{R}_4 > 0$, such that for all $x \geq \overline{R}_4$ and $t \in [0,1]$, we can get

$$\frac{f^*(t,x)}{x} \leq \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x} + \varepsilon \leq f^\infty + \varepsilon,$$

$$\frac{g^*(t,x)}{x} \leq \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x} + \varepsilon \leq g^\infty + \varepsilon,$$

so $f^*(t,x) \leq (f^\infty+\varepsilon)x, g^*(t,x) \leq (g^\infty+\varepsilon)x$.

We consider $R_4 \geq \overline{R}_4 + R_3$ and define the set $\Omega_4 = \{(u,v) \in Y, \|(u,v)\|_Y < R_4\}$. Let $(u,v) \in P \cap \partial\Omega_4$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_4$ or equivalently $\|u\| + \|v\| = R_4$. By the definition of f^*, g^*, we can get for all $t \in [0,1]$,

$$f(t,u(t),v(t)) \leq f^*(t,\|(u,v)\|_Y), \qquad g(t,u(t),v(t)) \leq g^*(t,\|(u,v)\|_Y). \tag{3.6}$$

Thus

$$\begin{aligned} T_1(u,v)(t) =& \lambda\int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \leq \lambda\int_0^1 G_1(1,s)f^*(t,\|(u,v)\|_Y)\mathrm{d}s \\ \leq& \lambda\int_0^1 G_1(1,s)(f^\infty+\varepsilon)R_4\mathrm{d}s = \lambda(f^\infty+\varepsilon)\int_0^1 G_1(1,s)\mathrm{d}s\|(u,v)\|_Y \\ \leq& \frac{1}{L_2}\int_0^1 G_1(1,s)\mathrm{d}s\|(u,v)\|_Y \leq \alpha_1\|(u,v)\|_Y. \end{aligned}$$

Therefore, $\|T_1(u,v)\| \leq \alpha_1 \|(u,v)\|_Y$.

Similarly, we have

$$T_2(u,v)(t) \leq \mu(g^\infty+\varepsilon)\int_0^1 G_2(1,s)\mathrm{d}s\|(u,v)\|_Y \leq \alpha_2\|(u,v)\|_Y.$$

So, $\|T_2(u,v)\| \leq \alpha_2 \|(u,v)\|_Y$.

Then for $(u,v) \in P \cap \partial\Omega_4$, we deduce

$$\|T(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \leq (\alpha_1+\alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By using Lemma 2.6, we conclude that T has a fixed point $(u,v) \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$ such that $R_3 \leq \|u\| + \|v\| \leq R_4$.

4 Example

Example 4.1. Consider the following system of fractional differential equations

$$\begin{cases} {}^cD_{0+}^{\frac{7}{2}}u(t) + \lambda f(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ {}^cD_{0+}^{\frac{10}{3}}v(t) + \mu g(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(0) = 0, \\ v(0) = v'(0) = v''(1) = v'''(0) = 0, \end{cases} \tag{4.1}$$

In the system (4.1), $\alpha = \frac{7}{2}, \beta = \frac{10}{3}$ and

$$f(t,u,v) = \frac{(t+1)[p_1(u+v) + q_1 e^{-(u+v)}](u+v)}{u+v+1},$$
$$g(t,u,v) = \frac{(t+1)^2[p_2(u+v) + q_2 e^{-(u+v)}](u+v)}{u+v+1},$$

for $t \in [0,1], u, v \geq 0$, where $p_1, p_2, q_1, q_2 > 0$.

We deduce $L_1 \approx \frac{0.0007}{\overline{\alpha}_1}, L_2 \approx \frac{0.2902}{\alpha_1}, L_3 \approx \frac{0.0007}{\overline{\alpha}_2}, L_4 \approx \frac{0.3120}{\alpha_2}$. We have $f^0 = 2q_1, g^0 = 4q_2, f_\infty = p_1, g_\infty = p_2$. For $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$, we consider $\overline{\alpha}_1 = \alpha_1, \overline{\alpha}_2 = \alpha_2$.

Then, the conditions $\frac{1}{L_1 f_\infty} < \frac{1}{L_2 f^0}$ and $\frac{1}{L_3 g_\infty} < \frac{1}{L_4 g^0}$ become

$$L_1 p_1 > 2L_2 q_1, \qquad L_3 p_2 > 4L_4 q_2.$$

For example, if $\dfrac{p_1}{q_1} \geq 830$ and $\dfrac{p_2}{q_2} \geq 1783$, then the above conditions are satisfied. Therefore, by Theorem 3.1, there exists one positive solution $(u(t), v(t)), t \in [0,1]$.

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A COMMON FIXED POINT THEOREM FOR A PAIR OF GENERALIZED CONTRACTION MAPPINGS WITH APPLICATIONS

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Abstract. In this article, we introduce abscissa dominating function $\mathbb{F} : [0,\infty)^2 \to \mathbb{R}$ and define a generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping which retrieves Banach's contraction, Geraghty type contraction and weak contraction as particular cases. We establish a common fixed points theorem for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings in complete partial metric spaces and apply this theorem to show the existence of solution of system of integral equations. This result and its consequences generalize many existing results both in partial metric spaces and metric spaces. We give examples to illustrate our results and to express the usefulness of these results in the literature.

1. Introduction

A partial metric on a nonempty set X is a function $p : X \times X \to [0,\infty)$ such that

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \le p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

Partial metrics were introduced in [12] as a generalization of the notion of metric to allow non-zero self distance for the purpose of modeling partial objects in reasoning about data flow networks. The self distance $p(x,x)$ is to be understood as a quantification of the extent to which x is unknown. A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. Matthews [12] proved an analogue of Banach's fixed point theorem in partial metric spaces. After this remarkable fixed point theorem, many authors took interest in partial metric spaces and its topological properties and established many well known fixed point results successfully (see [1, 2, 3, 4, 5, 6, 7, 8, 11]).

In this paper, continuing the study of fixed point theorems in partial metric spaces, we shall establish a common fixed points theorem for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings and shall discuss its consequences. The result proved in this paper generalizes many existing results in the literature (see [5, 7, 8, 14]). We explain hypotheses of our result through an example. In the last section of this paper, we apply this theorem to show the existence of solution of system of integral equations.

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2. Preliminaries

Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Following concepts and results will be required for the proofs of main results. Matthews [12] proved that every partial metric p on X induces a metric $d_p : X \times X \to \mathbb{R}_0^+$ by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{2.1}$$

for all $x, y \in X$.

Notice that a metric on a set X is a partial metric p such that $p(x,x) = 0$ for all $x \in X$. Following [12], each partial metric p on X generates a T_0 topology $\tau(p)$ on X. The base of the topology $\tau(p)$ is the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \ \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

Definition 1. [12] Let (X,p) be a partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(2) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 2. [15] Let $S : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ be two functions. Then S is said to be α-admissible if

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(S(x),S(y)) \geq 1 \ \forall \ x,y \in X.$$

Definition 3. [10] Let $S : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ be two functions. Then S is said to be a triangular α-admissible mapping if

(1) $\alpha(x,y) \geq 1$ implies $\alpha(S(x),S(y)) \geq 1$,
(2) $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ imply $\alpha(x,y) \geq 1$

for all $x,y,z \in X$.

Definition 4. [1] Let $S,T : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ be two functions. The pair (S,T) is said to be triangular α-admissible if

(1) $\alpha(x,y) \geq 1$ implies $\alpha(S(x),T(y)) \geq 1$ and $\alpha(T(x),S(y)) \geq 1$,
(2) $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ imply $\alpha(x,y) \geq 1$

for all $x,y,z \in X$.

The following lemma will be helpful in the sequel.

Lemma 1. [12]

(1) A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$, with respect to $\tau(d_p)$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(3) If $\lim_{n\to\infty} x_n = \upsilon$ such that $p(\upsilon,\upsilon) = 0$ then $\lim_{n\to\infty} p(x_n,y) = p(\upsilon,y)$ for every $y \in X$.

Lemma 2. [5] Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = S(x_n)$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

Lemma 3. [1] Let $S, T : X \to X$ be triangular α-admissible mappings. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$. Define sequence $x_{2i+1} = S(x_{2i})$, and $x_{2i+2} = T(x_{2i+1})$, where $i = 0, 1, 2, \ldots$.. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

Definition 5. A continuous function $\mathbb{F} : [0, \infty)^2 \to \mathbb{R}$ is called an abscissa dominating function if for any $u, v \in \mathbb{R}_0^+$, the following conditions hold:

(1) $\mathbb{F}(u, v) < u$,
(2) If $\mathbb{F}(u, v) = u$, then either $u = 0$ or $v = 0$.

An extra condition $\mathbb{F}(0, 0) = 0$ could be imposed in some cases if required. Let Δ_c denote the class of all abscissa dominating functions.

Example 1. (1) $\mathbb{F}(u, v) = u - v$.
(2) $\mathbb{F}(u, v) = ru$, for some $r \in (0, 1)$.
(3) $\mathbb{F}(u, v) = \dfrac{u}{(1+v)^r}$ for some $r \in (0, \infty)$.
(4) $\mathbb{F}(u, v) = \dfrac{\log(t + a^u)}{(1+v)}$, for some $a > 1$.
(5) $\mathbb{F}(u, v) = (u + l)^{\frac{1}{(1+v)^r}} - l$, $l > 1$, for $r \in (0, \infty)$.
(6) $\mathbb{F}(u, v) = u\beta(u)$, where $\beta : \mathbb{R}_0^+ \to [0, 1)$. and continuous.
(7) $\mathbb{F}(u, v) = u\pi^{-1/2} \int_0^\infty \dfrac{e^{-j}}{\sqrt{j} + v}\, dj$.

Let Φ denote the class of the functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which satisfy the following conditions:

(a) φ is continuous;
(b) $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$,

and Ψ denote the class of all the functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which satisfy the following conditions:

(1) ψ is increasing;
(2) $\psi(t) > 0, t > 0$ and $\psi(t) = 0$ imply $t = 0$.

3. Main results

This section contains definitions, a common fixed point result for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings in the setting of partial metric spaces and examples to support this result. We begin with following definitions.

Definition 6. Let (X,p) be a partial metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Mappings $S, T : X \to X$ are called a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping if for all $x, y \in X$, the contractive condition

$$\alpha(x, y)\psi\left(p(S(x), T(y))\right) \leq \mathbb{F}(\psi(M(x, y)), \varphi(M(x, y))) \tag{3.1}$$

holds, where $\mathbb{F} \in \Delta_c, \psi \in \Psi, \varphi \in \Phi$ and

$$M(x,y) = \max\left\{p(x,y), \frac{p(x,S(x))p(y,T(y))}{1+p(x,y)}, \frac{p(x,S(x))p(y,T(y))}{1+p(S(x),T(y))}\right\}.$$

If we set $S = T$ in (3.1), then we obtain the following contractive condition

$$\alpha(x,y)\psi\left(p(T(x),T(y))\right) \le \mathbb{F}(\psi(N(x,y)),\varphi(N(x,y))),$$

where

$$N(x,y) = \max\left\{p(x,y), \frac{p(x,T(x))p(y,T(y))}{1+p(x,y)}, \frac{p(x,T(x))p(y,T(y))}{1+p(T(x),T(y))}\right\}.$$

The following theorem is one of the main results.

Theorem 1. *Let (X,p) be a complete partial metric space, $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S,T : X \to X$ are continuous mappings satisfying the following conditions:*

1. *(S,T) is a pair of $(\alpha,\mathbb{F},\psi,\varphi)$-contraction mappings,*
2. *(S,T) is triangular α-admissible,*
3. *there exists $x_0 \in X$ such that $\alpha(x_0,S(x_0)) \ge 1$,*
4. *$\alpha(x,y) \ge 1$ for all $x,y \in Fix(T,S)$.*

Then (S,T) have a unique common fixed point.

Proof. We begin with the following observation. $M(x,y) = 0$ if and only if $x = y$ is a common fixed point of (S,T). Indeed, if $x = y$ is a common fixed point of (S,T), then $T(y) = T(x) = x = y = S(y) = S(x)$ and

$$M(x,y) = \max\left\{p(x,x), \frac{p(x,x)p(x,x)}{1+p(x,x)}, \frac{p(x,x)p(x,x)}{1+p(x,x)}\right\} = p(x,x).$$

From the contractive condition (3.1), we get

$$\psi\left(p(x,x)\right) = \psi\left(p(S(x),T(y))\right) \le \alpha(x,y)\psi\left(p(S(x),T(y))\right) \le \mathbb{F}\left(\psi\left(M(x,y)\right),\varphi\left(M(x,y)\right)\right),$$

which is only possible if $p(x,x) = 0$. So $M(x,y) = 0$.

Conversely, if $M(x,y) = 0$, then by (P_1) and (P_2) it is easy to check that $x = y$ is a fixed point of S and T.

On the other hand, if $M(x,y) > 0$, we construct an iterative sequence x_n of points in X such a way that $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$ where $i = 0,1,2,\ldots$. We observe that if $x_n = x_{n+1}$, then x_n is a common fixed point of S and T. Suppose that $x_n \ne x_{n+1}$ for all $n \ge 0$. Since $\alpha(x_0,x_1) \ge 1$ and the pair (S,T) is α-admissible, by Lemma 3, we have

$$\alpha(x_n,x_{n+1}) \ge 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Thus, for $\mathbb{F} \in \Delta_c$, we have

$$\begin{aligned}\psi\left(p(x_{2i+1},x_{2i+2})\right) &= \psi\left(p(S(x_{2i}),T(x_{2i+1}))\right) \le \alpha(x_{2i},x_{2i+1})\psi\left(p(S(x_{2i}),T(x_{2i+1}))\right)\\ &\le \mathbb{F}\left(\psi\left(M(x_{2i},x_{2i+1})\right),\varphi\left(M(x_{2i},x_{2i+1})\right)\right)\end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$.

Now

$$\begin{aligned} M(x_{2i},x_{2i+1}) &= \max\left\{\begin{array}{l} p(x_{2i},x_{2i+1}), \dfrac{p(x_{2i},S(x_{2i}))p(x_{2i+1},T(x_{2i+1}))}{1+p(x_{2i},x_{2i+1})}, \\ \dfrac{p(x_{2i},S(x_{2i}))p(x_{2i+1},T(x_{2i+1}))}{1+p(S(x_{2i}),T(x_{2i+1}))} \end{array}\right\} \\ &= \max\left\{p(x_{2i},x_{2i+1}), \frac{p(x_{2i},x_{2i+1})p(x_{2i+1},x_{2i+2})}{1+p(x_{2i},x_{2i+1})}, \frac{p(x_{2i},x_{2i+1})p(x_{2i+1},x_{2i+2})}{1+p(x_{2i+1},x_{2i+2})}\right\} \\ &\leq \max\left\{p(x_{2i},x_{2i+1}), p(x_{2i+1},x_{2i+2})\right\}. \end{aligned}$$

From the definition of $\mathbb{F}$, the case $M(x_{2i},x_{2i+1}) = p(x_{2i+1},x_{2i+2})$ is impossible. Indeed, if $x_{2i+1} \neq x_{2i+2}$, then

$$\begin{aligned} \psi\left(p(x_{2i+1},x_{2i+2})\right) &\leq \mathbb{F}\left(\psi\left(M(x_{2i},x_{2i+1})\right), \varphi\left(M(x_{2i},x_{2i+1})\right)\right) \\ &< \psi\left(M(x_{2i},x_{2i+1})\right) = \psi\left(p(x_{2i+1},x_{2i+2})\right), \end{aligned}$$

which is a contradiction. Therefore, $M(x_{2i},x_{2i+1}) = p(x_{2i},x_{2i+1})$. Thus

$$\begin{aligned} \psi\left(p(x_{2i+1},x_{2i+2})\right) &\leq \mathbb{F}\left(\psi\left(M(x_{2i},x_{2i+1})\right), \varphi\left(M(x_{2i},x_{2i+1})\right)\right) \\ &\leq \mathbb{F}\left(\psi\left(p(x_{2i},x_{2i+1})\right), \varphi\left(p((x_{2i},x_{2i+1})\right) < \psi\left(p(x_{2i},x_{2i+1})\right)\right) \end{aligned}$$

and so

$$\psi\left(p(x_{2i+1},x_{2i+2})\right) < \psi\left(p(x_{2i},x_{2i+1})\right).$$

The definition of ψ implies that

$$p(x_{2i+1},x_{2i+2}) < p(x_{2i},x_{2i+1}).$$

Thus

$$p(x_{n+1},x_{n+2}) < p(x_n,x_{n+1}), \text{ for all } n \in \mathbb{N}\cup\{0\}. \tag{3.3}$$

Hence we deduce that the sequence $\{p(x_n,x_{n+1})\}_{n\in\mathbb{N}}$ is nonnegative and nonincreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} p(x_n,x_{n+1}) = r$. We assert that $r = 0$. Suppose, on contrary, that $r > 0$. If $r > 0$, then letting $n \to +\infty$ in the following inequality

$$\psi\left(p(x_{n+1},x_{n+2})\right) \leq \mathbb{F}\left(\psi\left(p(x_n,x_{n+1})\right), \varphi\left(p((x_n,x_{n+1})\right)\right) \leq \psi\left(p(x_n,x_{n+1})\right), \tag{3.4}$$

we get

$$\psi(r) \leq \mathbb{F}(\psi(r),\varphi(r)) < \psi(r),$$

which is a contradiction. Thus $r = 0$. Hence

$$\lim_{n\to+\infty} p(x_{n-1},x_n) = 0. \tag{3.5}$$

Now, we claim that the sequence $\{x_n\}$ is a Cauchy sequence in (X,p). Suppose, on contrary, that $\{x_n\}$ is not a Cauchy sequence. Then $\lim_{n,m\to\infty} p(x_n,x_m) \neq 0$ and there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that m_k is the smallest index for $m_k > n_k > k$ satisfying

$$p(x_{m_k},x_{n_k}) \geq \epsilon. \tag{3.6}$$

This means that

$$p(x_{m_k},x_{n_{k-1}}) < \epsilon. \tag{3.7}$$

By the triangle inequality, we have

$$\begin{aligned}\epsilon &\leq p(x_{m_k}, x_{n_k}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) - p(x_{n_{k-1}}, x_{n_{k-1}}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + p(x_{n_{k-1}}, x_{n_k}).\end{aligned}$$

That is,

$$\epsilon < \epsilon + p(x_{n_{k-1}}, x_{n_k}) \tag{3.8}$$

for all $k \in \mathbb{N}$. In the view of (3.8), (3.5), we have

$$\lim_{k\to\infty} p(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.9}$$

Again using the triangle inequality, we have

$$\begin{aligned}p(x_{m_k}, x_{n_k}) &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) - p(x_{m_{k+1}}, x_{m_{k+1}}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}) - p(x_{n_{k+1}}, x_{n_{k+1}}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k})\end{aligned}$$

and

$$\begin{aligned}p(x_{m_{k+1}}, x_{n_{k+1}}) &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) - p(x_{m_k}, x_{m_k}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}) - p(x_{n_k}, x_{n_k}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}).\end{aligned}$$

Taking the limit as $k \to +\infty$ and using (3.5) and (3.9), we obtain

$$\lim_{k\to+\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \tag{3.10}$$

By Lemma 3 and $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$, we have

$$\begin{aligned}\psi\left(p(x_{n_{k+1}}, x_{m_{k+2}})\right) &= \psi\left(p(S(x_{n_k}), T(x_{m_{k+1}}))\right) \leq \alpha(x_{n_k}, x_{m_{k+1}})\psi\left(p(S(x_{n_k}), T(x_{m_{k+1}}))\right) \\ &\leq \mathbb{F}\left(\psi\left(M(x_{n_k}, x_{m_{k+1}})\right), \varphi\left(M(x_{n_k}, x_{m_{k+1}})\right)\right).\end{aligned}$$

This implies that $\lim_{k\to\infty} p(x_{n_k}, x_{m_{k+1}}) = 0 < \epsilon$, which is a contradiction. So $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence in (X, p). From (2.1), we obtain that $d_p(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore, $\lim_{n,m\to\infty} d_p(x_n, x_m) = 0$ and thus by Lemma 1, $\{x_n\}$ is a Cauchy sequence in both (X, p) and (X, d_p). Since (X, p) is a complete partial metric space, by Lemma 1, (X, d_p) is also a complete metric space. Thus there exists $\upsilon \in X$ such that $x_n \to \upsilon$, that is, $\lim_{n\to\infty} d_p(x_n, \upsilon) = 0$. Then again from Lemma 1, we get

$$\lim_{n\to\infty} p(\upsilon, x_n) = p(\upsilon, \upsilon) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.11}$$

Due to $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, it follows from (3.11) that $p(\upsilon, \upsilon) = 0$ and $\{x_n\}$ converges to υ with respect to $\tau(p)$. Moreover, $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$. Now the continuity of T implies

$$\upsilon = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} T(x_{2n+1}) = T(\lim_{n\to\infty} x_{2n+1}) = T(\upsilon).$$

Analogously, $\upsilon = S(\upsilon)$. Thus we have $S(\upsilon) = T(\upsilon) = \upsilon$. Hence (S, T) have a common fixed point. Now we show that υ is the unique common fixed point of S and T. Assume the contrary, that is, there exists $\omega \in X$ such that $\upsilon \neq \omega$ and $\omega = T(\omega)$. From the contractive condition (3.1), we have

$$\psi\left(p(\upsilon,\omega)\right) \leq \mathbb{F}\left(\psi\left(M(\upsilon,\omega)\right),\varphi\left(M(\upsilon,\omega)\right)\right) < \psi\left(M(\upsilon,\omega)\right),$$

but

$$M(\upsilon,\omega) = \max\left\{p(\upsilon,\omega), \frac{p(\upsilon,S(\upsilon))p(\omega,T(\omega))}{1+p(\upsilon,\omega)}, \frac{p(\upsilon,S(\upsilon))p(\omega,T(\omega))}{1+p(S(\upsilon),T(\omega))}\right\}.$$

This implies that

$$M(\upsilon,\omega) = p(\upsilon,\omega).$$

This means that $p(\upsilon,\omega) < p(\upsilon,\omega)$, which is a contradiction and so $p(\upsilon,\omega) = 0$. Consequently, υ is a unique common fixed point of the pair (S, T). □

It is also possible to remove the continuity of the mappings S and T by replacing a weaker condition: (C) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to \upsilon \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, \upsilon) \geq 1$ for all k.

Theorem 2. *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S, T : X \to X$ are mappings such that*

1. *(S, T) is a pair of $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings,*
2. *(S, T) is triangular α-admissible,*
3. *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$,*
4. *$\alpha(x, y) \geq 1$ for all $x, y \in Fix(T, S)$,*
5. *(C) holds.*

Then (S, T) have a unique common fixed point.

Proof. Following the proof of Theorem 1, we know that $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$ as $n \to +\infty$. We only have to show that υ is a common fixed point of the pair (S, T). Due to the hypothesis (4), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, \upsilon) \geq 1$ for all k. Now by using (3.1) for all k, we have

$$\begin{aligned}\psi\left(p(x_{2n_k+1},T(\upsilon))\right) &= \psi\left(p(S(x_{2n_k}),T(\upsilon))\right) \leq \alpha(x_{2n_k},\upsilon)\psi\left(p(S(x_{2n_k}),T(\upsilon))\right) \\ &\leq \mathbb{F}\left(\psi\left(M(x_{2n_k},\upsilon)\right),\varphi\left(M(x_{2n_k},\upsilon)\right)\right)\end{aligned}$$

and so

$$\psi\left(p(x_{2n_k+1},T(\upsilon))\right) \leq \mathbb{F}\left(\psi\left(M(x_{2n_k},\upsilon)\right),\varphi\left(M(x_{2n_k},\upsilon)\right)\right),$$

which implies that

$$p(x_{2n_k+1},T(\upsilon)) \leq M(x_{2n_k},\upsilon). \tag{3.12}$$

On the other hand, we obtain

$$M(x_{2n_k},\upsilon) = \max\left\{p(x_{2n_k},\upsilon), \frac{p(x_{2n_k},S(x_{2n_k}))p(\upsilon,T(\upsilon))}{1+p(x_{2n_k},\upsilon)}, \frac{p(x_{2n_k},S(x_{2n_k}))p(\upsilon,T(\upsilon))}{1+p(S(x_{2n_k}),T(\upsilon))}\right\}.$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} M(x_{2n_k},\upsilon) \leq \max\left\{p(\upsilon,S(\upsilon)),p(\upsilon,T(\upsilon))\right\}. \tag{3.13}$$

Case I. Assume that $\lim_{k\to\infty} M(x_{2n_k}, \upsilon) = p(\upsilon, T(\upsilon))$. Suppose that $p(\upsilon, T(\upsilon)) > 0$. Otherwise, the result is obvious. Letting $k \to \infty$ in (3.12), we obtain that $p(\upsilon, T(\upsilon)) < p(\upsilon, T(\upsilon))$, which is a contradiction. Thus we obtain that $p(\upsilon, T(\upsilon)) = 0$. Due to $(PM1)$ and $(PM2)$, we have $\upsilon = T(\upsilon)$.
Case II. Assume that $\lim_{k\to\infty} M(x_{2n_k}, \upsilon) = p(\upsilon, S(\upsilon))$. Then arguing as above, we get $\upsilon = S(\upsilon)$. Thus $\upsilon = T(\upsilon) = S(\upsilon)$. □

If we set $T = S$ and $M(x, y) = \max\left\{p(x, y), \frac{p(x, S(x))p(y, S(y))}{1 + p(x, y)}, \frac{p(x, S(x))p(y, S(y))}{1 + p(S(x), S(y))}\right\}$ in Theorems 1 and 2, then we obtain the following results.

Corollary 1. *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S : X \to X$ is a continuous mapping such that*

(1) *S is a $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping,*
(2) *S is triangular α-admissible,*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0) \geq 1$,*
(4) *$\alpha(x, y) \geq 1$ for all $x, y \in Fix(S)$.*

Then S has a unique fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

Corollary 2. *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that S satisfies the following conditions:*

(1) *S is a $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping,*
(2) *S is triangular α-admissible,*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$,*
(4) *$\alpha(x, y) \geq 1$ for all $x, y \in Fix(S)$,*
(5) *(C) holds.*

Then S has a unique fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

Remark 1. *For a partial metric space (X, p), we have the following observations:*

(1) *If we set $p(x, x) = 0$ and $\mathbb{F}(x, y) = \beta(x)x$ for all $x, y \in X$ in Corollaries 1 and 2, then we obtain the results presented by Chandok* [4].

(2) *If we set $M(x, y) = \max\{p(x, y), p(x, S(x)), p(y, S(y))\}$, $p(x, x) = 0$ and $\mathbb{F}(x, y) = \beta(x)x$ for all $x, y \in X$ in Theorems 1 and 2, then the results presented by Cho et al.* [5] *can be viewed as particular cases of Theorems 1 and 2.*

4. Consequences

The following corollaries shall support our claim that Theorem 1 is a generalized version of many corresponding results and shorten the proofs of many results presented in the literature.

The results established in [14] can be viewed as particular cases of Corollary 3.

Corollary 3. ([14]) *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Let $S, T : X \to X$ be a pair of self-mappings such that*

(1) *(S, T) is a pair of Geraghty type contraction mappings,*
(2) *(S, T) is triangular α-admissible,*

(3) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, S(x_0)) \geq 1$,
(4) $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T, S)$,
(5) *either* S, T *are continuous or the condition (C) holds.*

Then (S, T) *have a unique common fixed point* v *in* X *with* $p(v, v) = 0$.

Proof. Setting $\mathbb{F}(x, y) = x\beta(x)$, $\psi(t) = t$, $\varphi(t) = t$ in Theorem 1, we obtain the required result. □

Corollary 4. ([14]) *Let* (X, p) *be a complete partial metric space and* $\alpha : X \times X \to \mathbb{R}_0^+$ *be a function. Let* $S, T : X \to X$ *be a pair of self-mappings such that*

(1) *the pair* (S, T) *satisfies*

$$\alpha(x, y)p(S(x), T(y)) \leq \kappa M(x, y) \text{ where } \kappa \in (0, 1),$$

(2) (S, T) *is triangular* α*-admissible,*
(3) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, S(x_0)) \geq 1$,
(4) $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T, S)$,
(5) *either* S, T *are continuous or the condition (C) holds.*

Then (S, T) *have a unique common fixed point* v *in* X *with* $p(v, v) = 0$.

Proof. Setting $\mathbb{F}(x, y) = \kappa x$, $\psi(t) = t$, $\varphi(t) = t$ in Theorem 1, we obtain the required result. □

Corollary 5 generalizes the results proved in [13].

Corollary 5. ([13]) *Let* (X, p) *be a complete partial metric space and* $\alpha : X \times X \to \mathbb{R}_0^+$ *be a function. Let* $S, T : X \to X$ *be a pair of self-mappings such that*

(1) *the pair* (S, T) *satisfies*

$$\alpha(x, y)\psi(p(S(x), T(y))) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

(2) (S, T) *is triangular* α*-admissible,*
(3) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, S(x_0)) \geq 1$,
(4) $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T, S)$,
(5) *either* S, T *are continuous or thecondition (C) holds.*

Then (S, T) *have a unique common fixed point* v *in* X *with* $p(v, v) = 0$.

Proof. Setting $\mathbb{F}(x, y) = x - y$ in Theorem 1, we obtain the required result. □

To illustrate the results proved in this paper and to show the superiority of a pair of $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings than the contractions used in [4, 5], we present the following example.

Example 2. Let $X = \{1, 2, 3\}$. Define $p : X \times X \to \mathbb{R}_0^+$ by

$$\begin{aligned} p(1,3) &= p(3,1) = \frac{5}{7}, p(1,1) = \frac{1}{10}, p(2,2) = \frac{2}{10}, p(3,3) = \frac{3}{10}, \\ p(1,2) &= p(2,1) = \frac{3}{7}, p(2,3) = p(3,2) = \frac{4}{7}. \end{aligned}$$

It is easy to check that p is a partial metric and define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in X; \\ 0 & \text{if otherwise.} \end{cases}$$

Define the mappings $S, T : X \to X$ as follows:

$$\begin{aligned} S(x) &= 1 \text{ for each } x \in X, \\ T(1) &= T(3) = 1, \ T(2) = 3. \end{aligned}$$

In addition, define $\mathbb{F}(x, y) = \beta(x)x$ for all $x \in X$, where $\beta : \mathbb{R}_0^+ \to [0, 1)$ defined by $\beta(M(x, y)) = \frac{9}{10}$ for all $x, y \in X$. Note that $S(x)$ and $T(x)$ belong to X and are continuous. The pair (S, T) is α-admissible. Indeed, $\alpha(x, y) = 1$ implies $\alpha(S(x), T(y)) = 1$. We shall show that the condition (3.1) in Theorem 2 is satisfied. If $x = 2$, $y = 3$, then $\alpha(2, 3) = 1$ and

$$\begin{aligned} M(2,3) &= \max\left\{p(2,3), \frac{p(2,S(2))p(3,T(3))}{1+p(2,3)}, \frac{p(2,S(2))p(3,T(3))}{1+p(S(2),T(3))}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{9}{20}, \frac{9}{14}\right\} = \frac{9}{14}, \end{aligned}$$

$p(S(2), T(3)) = p(1, 1) = \frac{1}{10}$. Now

$$\frac{1}{10} = \alpha(2,3)p(S(2),T(3)) \leq \beta(M(2,3))M(2,3) = \frac{81}{140}$$

holds.

Similarly, for other cases ($x = 1$, $y = 3$ and $x = 2$, $y = 1$), it is easy to check that the contractive condition (3.1) in Theorem 1 is satisfied. Consequently, all the conditions (1-4) of Theorem 1 are satisfied. Hence (S, T) have a unique common fixed point ($x = 1$). Nevertheless, the contractive condition (3) in [5] does not hold for this particular case. Indeed, for $x = 2$, $y = 3$,

$$\begin{aligned} M(2,3) &= \max\{d(2,3), d(2,T(2)), d(3,T(3))\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}\right\} = \frac{5}{7}, \end{aligned}$$

$$\alpha(2,3)d(T(2),T(3)) = \frac{5}{7} \nleq \frac{9}{14} = \beta(M(2,3))M(2,3).$$

Similarly, the contractive condition (2.1) in [4] does not hold for this particular case. Indeed, for $x = 2$, $y = 3$ and $\psi(t) = t$,

$$\begin{aligned} M(2,3) &= \max\left\{d(2,3), d(2,T(2)), d(3,T(3)), \frac{d(2,T(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,T(2))d(3,T(3))}{1+d(T2,T3)}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{20}{77}, \frac{5}{21}\right\} = \frac{5}{7}, \end{aligned}$$

$$\alpha(2,3)\psi(d(T(2),T(3))) = \frac{5}{7} \nleq \frac{9}{14} = \beta(\psi(M(2,3)))\,\psi(M(2,3)).$$

Here we have assumed that $p(x, y) = d(x, y)$ for all $x, y \in X$ such that $x \neq y$.

5. Application to system of integral equations

In this section, we shall apply Theorem 1 to show the existence of solution of a pair of simultaneous Volterra-Hammerstein integral equations

$$x(t) = f(t) + \lambda \int_0^1 K(t,s)F_n(s,x(s))\,ds, \tag{5.1}$$

$$y(t) = f(t) + \lambda \int_0^1 K(t,s)G_n(s,y(s))\,ds \tag{5.2}$$

for all $t \in [0,1]$, where $f(t)$ is known, $K(t,s)$, $F_n(s,x(s))$ and $G_n(s,y(s))$ are real-valued functions that are measurable both in t and s on $[0,1]$, and λ is a real number.

Let $X = L^1([0,1],\mathbb{R})$ and $p(x,y) = d(x,y) + c_n$ for all $x,y \in X$, where

$$d(x,y) = \|x(s) - y(s)\|_X = \int_0^1 |x(s) - y(s)|\ ds$$

and $\{c_n\}$ is a sequence of positive real numbers satisfying $c_n \to 0$ as $n \to \infty$. It is easy to verify that (X,p) is a complete partial metric space. We define $\mathbb{F} : [0,\infty)^2 \to \mathbb{R}$ by $\mathbb{F}(x,y) = \beta(x)x$ for all $x \in X$ and $\psi(t) = t$.

Let Θ represent the class of functions $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with the following properties

(1) ϕ is increasing,
(2) For each $t > 0$, $\phi(t) < t$,
(3) $\int_0^1 \phi(t)\,dt \leq \phi\left(\int_0^1 t\,dt\right)$,
(4) $\beta(t) = \frac{\phi(t)}{t} \in S$.

For examples, $\phi(t) = \frac{1}{5}t$, $\phi(t) = \frac{t}{1+t}$ are elements of Θ.

Now we present the main result regarding application of Theorem 1.

Theorem 3. *Assume that the following hypotheses are satisfied:*

(C_1)

$$\int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\ dt = R_1 < +\infty.$$

(C_2) $F, G \in L^1[0,1]$ *are such that, for all* $s \in [0,1]$ *and* $x,y \in L^1[0,1]$,

$$|F_n(s,x(s)) - G_n(s,y(s))| \leq \phi(x(s) - y(s)),\ \text{as } n \to \infty.$$

Then the system of integral equations (5.1) *and* (5.2) *has a solution for each* λ *with* $\lambda R_1 < 1$.

Proof. We define the operators, for all $x,y \in X$,

$$Sx(t) = f(t) + \lambda \int_0^1 K(t,s)F_n(s,x(s))\,ds,$$

$$Ty(t) = f(t) + \lambda \int_0^1 K(t,s)G_n(s,y(s))\,ds.$$

Then S and T are operators from X into itself. Indeed, we have

$$\begin{aligned} |Sx| &\leq |f(t)| + |\lambda| \int_0^1 |K(t,s)F_n(s,x(s))|\, ds \\ &\leq |f(t)| + |\lambda| \sup_{0\leq s\leq 1} |K(t,s)| \int_0^1 |F_n(s,x(s))|\, ds. \end{aligned}$$

By the assumptions (C_1) and (C_2), we obtain

$$\int_0^1 |Sx|\, dt \leq |\lambda| \int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\, dt \int_0^1 |F_n(s,x(s))|\, ds + \int_0^1 |f(t)|\, dt < +\infty.$$

This implies that $Sx \in X$.

Similarly $Ty \in X$.

Now consider

$$\begin{aligned} p(Sx,Ty) &= d(Sx,Ty) + c_n \\ &= \|Sx - Ty\| + c_n \\ &= \int_0^1 |Sx(t) - Ty(t)|\ dt + c_n \\ &= \int_0^1 \left| \lambda \int_0^1 K(t,s)F_n(s,x(s))\, ds - \lambda \int_0^1 K(t,s)G_n(s,y(s))\, ds \right| dt + c_n \\ &= \int_0^1 \left| \lambda \int_0^1 K(t,s)\left[F_n(s,x(s)) - G_n(s,y(s))\right]\ ds \right| dt + c_n \\ &\leq |\lambda| \int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\, dt \int_0^1 |F_n(s,x(s)) - G_n(s,y(s))|\ ds + c_n \end{aligned}$$

for all $x, y \in X$.

Letting $n \to \infty$, we get

$$\begin{aligned} p(Sx,Ty) &\leq |\lambda| R_1 \int_0^1 \phi\left(|x(s) - y(s)|\right)\ ds \\ &\leq |\lambda| R_1 \phi\left(d(x,y)\right) < \phi\left(d(x,y)\right) \leq \phi\left(p(x,y)\right). \end{aligned}$$

Thus

$$\begin{aligned} p(Sx,Ty) &\leq \phi\left(p(x,y)\right) \leq \phi\left(M(x,y)\right) = \frac{\phi\left(M(x,y)\right)}{M(x,y)} M(x,y), \\ p(Sx,Ty) &\leq \beta\left(M(x,y)\right) M(x,y) \end{aligned}$$

for all $x, y \in X$.

Finally, we define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in X; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for all $x, y \in X$, we have

$$\alpha(x,y)\psi\left(p(S(x),T(y))\right) \leq \mathbb{F}\left(\psi\left(M(x,y)\right), \varphi\left(M(x,y)\right)\right).$$

Apparently, $\alpha(x,y) = 1$ and $\alpha(y,z) = 1$ imply $\alpha(x,z) = 1$ for all $x, y, z \in X$. Moreover, $\alpha(x,y) = 1$ implies $\alpha(S(x),T(y)) = 1$ and $\alpha(T(x),S(y)) = 1$ and so (S,T) is a triangular α-admissible pair of mappings. Hence all the hypotheses of Theorem 1 are satisfied. Consequently, the mappings S and T have a common fixed point which is the solution of system of integral equations (5.1) and (5.2). □

6. Conclusion

This paper presents a common fixed point theorem for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings. The presented theorem not only generalizes and improvea many new and classical results in fixed point theory but also proves a short method to show the existence of fixed points. The authors believe that the use of abscissa dominating function to find fixed points of various contraction mappings makes significant and useful contribution in the existing literature.

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Inner-outer factorization on the Nevanlinna space in a strip

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Abstract In this paper, we prove a famous harmonic majorant lemma, and by applying this lemma to $\log^+|f|$, we claim that $\log^+|f|$ has indeed a harmonic majorant for every function f in the Nevanlinna space in a strip instead of the usual assumption on $\log^+|f|$ having a harmonic majorant in the same setting. By using the conformal mapping from a strip onto the unit disk and the inner-outer factorization theorem on the Nevanlinna space in the unit disk, we obtain an inner-outer factorization theorem on the Nevanlinna space in such a strip.

Key words Nevanlinna space; strip; inner-outer factorization; harmonic majorant

1 Introduction

Let $\mathbb{C}$ be the complex plane. We denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{U}$ and its boundary by $\partial\mathbb{U}$. Let $H(\mathbb{U})$ be the space of all holomorphic functions in $\mathbb{U}$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{U})$ (see [1–3]) is the set of $f \in H(\mathbb{U})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

The Nevanlinna space $H^0(\mathbb{U})$ (see [1–3]) is the set of $f \in H(\mathbb{U})$ for which

$$\|f\|_{H^0} = \sup_{0<r<1} \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta\right\} < \infty.$$

Let $e^{i\theta_0}$ be a point of $\partial\mathbb{U}$. We write $\lim_{z\to e^{i\theta_0}} f(z) = A$ nontangentially if for every open triangular sector D in $\mathbb{U}$ with vertex at $e^{i\theta_0}$, $f(z) \to A$ as $z \to e^{i\theta_0}$ within D.

For a sequence $\{z_n\}$ in $\mathbb{U}$ satisfying $\sum_n (1 - |z_n|) < \infty$, the following function

$$B(z) = z^k \prod_n \frac{|z_n|(z_n - z)}{z_n(1 - z\overline{z_n})} \tag{1.1}$$

is called a Blaschke product, where k is a nonnegative integer. Note that $\{z_n\}$ may be finite, or even empty. If $\{z_n\}$ is empty, then denote $B(z) = z^k$.

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A function $g \in H(\mathbb{U})$ is said to be an inner function if it is bounded and has nontangential limit whose modulus is 1 almost everywhere on $\partial\mathbb{U}$. The following function

$$S(z) = \exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\} \tag{1.2}$$

is called a singular inner function, where $\mu(t)$ is a bounded nondecreasing singular function. One can show that every inner function has a factorization $e^{i\gamma}B(z)S(z)$, where $B(z)$ is from (1.1), $S(z)$ is from (1.2). For more details on inner function, we refer to [4–8].

A function $h \in H(\mathbb{U})$ is called an outer function if there exists a positive function φ with $\log\varphi \in L^1(\partial\mathbb{U})$ and a complex number c with $|c| = 1$ such that

$$h(z) = c\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\varphi(e^{it})dt\right\}.$$

Let $S_a = \{x+iy : x \in \mathbb{R}, 0 < y < a\}(a > 0)$ be a strip in $\mathbb{C}$. We denote its boundary by $\partial S_a = L_0 \bigcup L_a$, where $L_b = \{t + ib : t \in \mathbb{R}\}$ $(b \in \mathbb{R})$. Let $H(S_a)$ be the space of all analytic functions in S_a. The Nevanlinna space $H^0(S_a)$ is the set of $f \in H(S_a)$ with

$$\|f\|_{H^0} = \sup_{0<y<a}\exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\log^+|f(x+iy)|dx\right\} < \infty.$$

If $\alpha > 0$ and $z_0 = x_0 + iy_0 \in \partial S_a$, then the angular domain in S_a with vertex z_0 and aperture $\alpha > 0$ is the region

$$\Gamma_\alpha(z_0) = \{x + iy \in S_a : |x - x_0| < \alpha|y - y_0|\}.$$

Let $\mathscr{M}(\mathbb{R})$ be the set of finite complex valued Borel measures, then $\mathscr{M}(\mathbb{R})$ is a Banach space with the norm $\|\mu\|_{\mathscr{M}} = \int_{\mathbb{R}} d|\mu|(x)$, where $|\mu|$ is the total variation of $\mu \in \mathscr{M}(\mathbb{R})$. Moreover, by Riesz representation theorem, $\mathscr{M}(\mathbb{R})$ is the dual space of $C_0(\mathbb{R})$ in the sense of isomorphism (see [9]).

It is well known that every $f \in H^p(\mathbb{U})$ $(p > 0)$ has a unique canonical factorization $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function. Motivated by this result, inner-outer factorization of analytic functions in some other spaces were studied (see $\mathcal{Q}_p$ spaces [10], Besov-type spaces [11, 12]). However, there is a sharp structural difference between functions in the Hardy space and that in the Nevanlinna space. In factoring functions in the Nevanlinna space $H^0(\mathbb{U})$, the singular factor is replaced by a quotient of two singular inner functions. That is the following theorem, which can be found in [1, 3].

Theorem A *If $f \in H^0(\mathbb{U})$, $f \not\equiv 0$, then $f^*(e^{i\theta}) = \lim_{z\to e^{i\theta}} f(z)$ exists nontangentially at almost every $\theta \in [-\pi,\pi)$ and $\log|f^*(e^{i\theta})| \in L^1([-\pi,\pi))$. Moreover, $f(z)$ can be written by*

$$f(z) = cB(z)G(z)S(z),$$

where c is a constant with $|c| = 1$. $B(z)$ is the Blaschke product of the form (1.1), where $\{z_n\}$ are the zeros of $f(z)$; and

$$G(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|f^*(e^{it})|dt\right\}$$

is an outer function in $\mathbb{U}$; and

$$S(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu_s(t)\right\}$$

is a quotient of two singular inner functions in $\mathbb{U}$, where μ_s is a singular signed measure on $[-\pi, \pi)$ with finite total variation.

Inner-outer factorization on the Nevanlinna space in other domains were studied (see [3, 13]). However, most of the results on the unit disk or other domains were obtained on the premise that $\log^+|f|$ had a harmonic majorant (see [1, 13, 14]). In this paper, we prove that $\log^+|f|$ has a harmonic majorant indeed for every function f in the Nevanlinna space in a strip. Based on this fact, we obtain the existence of nontangential limits of $f \in H^0(S_a)$ as follows:

Theorem 1.1 *If $f \in H^0(S_a)$ and $f(z) \not\equiv 0$, then*

$$f^*(t) = \lim_{\substack{z\to t \\ z\in\Gamma_\alpha(t)}} f(z),\ \ f^*(t+ia) = \lim_{\substack{z\to t+ia \\ z\in\Gamma_\alpha(t+ia)}} f(z) \tag{1.3}$$

exist nontangentially at almost every $t \in \mathbb{R}$, and

$$\int_{-\infty}^{\infty}\frac{\big|\log|f^*(t)|\big| + \big|\log|f^*(t+ia)|\big|}{\cosh\frac{\pi}{a}t}dt < \infty. \tag{1.4}$$

Further, $f^*(t),\ f^*(t+ia) \neq 0$ at almost every $t \in \mathbb{R}$.

Next, we give a similar factorization on $H^0(S_a)$ as Theorem A, which is also called inner-outer factorization.

Theorem 1.2 *Let $f \in H^0(S_a)$, $f \not\equiv 0$, then the zeros $\{z_n\}$ of f satisfy*

$$\sum_n \frac{e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n}{1+e^{\frac{2\pi}{a}x_n}} < \infty,\ \ z_n = x_n + iy_n, \tag{1.5}$$

and there exist two singular signed measures $\mu_{1,s}$ on L_0 and $\mu_{2,s}$ on L_a such that

$$\int_{\mathbb{R}}\frac{\big|\log|f^*(t)|\big|}{\cosh\frac{\pi}{a}t}dt + \int_{\mathbb{R}}\frac{\big|\log|f^*(t+ia)|\big|}{\cosh\frac{\pi}{a}t}dt + \int_{\mathbb{R}}\frac{d|\mu_{1,s}|(t)}{\cosh\frac{\pi}{a}t} + \int_{\mathbb{R}}\frac{d|\mu_{2,s}|(t)}{\cosh\frac{\pi}{a}t} < \infty. \tag{1.6}$$

Moreover, f can be written by

$$f(z) = cG(z)S(z)B(z),$$

where c is a complex constant with $|c| = 1$,

$$\begin{aligned} G(z) = &\exp\left\{\frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t} - e^{\frac{\pi}{a}z}} - \frac{e^{\frac{2\pi}{a}t}}{1 + e^{\frac{\pi}{a}t}}\right)\log|f^*(t)|dt\right. \\ &\left.-\frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t} + e^{\frac{\pi}{a}z}} - \frac{e^{\frac{2\pi}{a}t}}{1 + e^{\frac{2\pi}{a}t}}\right)\log|f^*(t + ia)|dt\right\}; \end{aligned}$$

and

$$\begin{aligned} S(z) = &\exp\left\{i\tau_1 e^{-\frac{\pi}{a}z} - i\tau_2 e^{\frac{\pi}{a}z} + \frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t} - e^{\frac{\pi}{a}z}} - \frac{e^{\frac{2\pi}{a}t}}{1 + e^{\frac{2\pi}{a}t}}\right)d\mu_{1,s}(t)\right. \\ &\left.-\frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t} + e^{\frac{\pi}{a}z}} - \frac{e^{\frac{2\pi}{a}t}}{1 + e^{\frac{2\pi}{a}t}}\right)d\mu_{2,s}(t)\right\}, \end{aligned}$$

where τ_1, τ_2 are real numbers; and

$$B(z) = \left(\frac{e^{\frac{\pi}{a}z} - i}{e^{\frac{\pi}{a}z} + i}\right)^k \prod_n \frac{e^{\frac{\pi}{a}z} - e^{\frac{\pi}{a}z_n}}{e^{\frac{\pi}{a}z} - e^{\frac{\pi}{a}\bar{z}_n}} e^{i\theta(z_n)},$$

where k is a nonnegative integer and

$$e^{i\theta(z_n)} = \frac{(e^{\frac{\pi}{a}\bar{z}_n} + i)(e^{\frac{\pi}{a}\bar{z}_n} - i)}{|e^{\frac{\pi}{a}z_n} + i||e^{\frac{\pi}{a}z_n} - i|}.$$

2 Proofs of main results

In this section, we will give the proofs of our main results in Section 1. To this end, we need the following lemmas.

Lemma 2.1 *If $v(z)$ is subharmonic in S_a and it satisfies*

$$C = \sup_{0<y<a}\int_{-\infty}^{\infty}|v(x + iy)|dx < \infty, \tag{2.1}$$

then

$$v(x + iy) \leq \frac{2C}{\pi\min\{y, a - y\}}.$$

Proof Let $z = x + iy \in S_a$, $\rho = \min\{y, a - y\}$, $D(z, \rho) = \{w : |w - z| < \rho\}$. By the mean property, it establishes that

$$\begin{aligned} v(z) &\le \int_{D(z,\rho)} v(\zeta)\frac{d\lambda(\zeta)}{\pi\rho^2} \le \int_{D(z,\rho)} |v(\zeta)|\frac{d\lambda(\zeta)}{\pi\rho^2} \\ &\le \frac{1}{\pi\rho^2}\int_{y-\rho}^{y+\rho}\int_{\mathbb{R}} |v(\xi + i\eta)|d\xi d\eta \\ &\le \frac{2C}{\pi\rho} = \frac{2C}{\pi\min\{y, a - y\}}. \end{aligned}$$

□

Lemma 2.2 *Let*

$$P_y^{\pm}(x) = \frac{\sin\frac{\pi}{a}y}{2a(\cosh\frac{\pi}{a}x \mp \cos\frac{\pi}{a}y)}$$

and $P_y(x) = P_y^+(x) + P_y^-(x)$. *Then*

(1) $P(x, y) = P_y(x)$ *is harmonic in* S_a*;*

(2) $P_y(x) > 0, x + iy \in S_a$*;*

(3) $\int_{\mathbb{R}} P_y^{\pm}(x)dx = \frac{1}{2} \pm (\frac{1}{2} - \frac{y}{a}), \int_{\mathbb{R}} P_y(x)dx = 1$*;*

(4) $\int_{|x|>\delta} P_y(x)dx \to 0\ (y \to 0)$ *and* $\int_{|x|>\delta} P_y(x)dx \to 0\ (y \to a)$*, where* $\delta > 0$ *is a constant.*

Proof The proofs of these facts follow from the following relations (see [15]):

A. $P_y^{\pm}(x) = \frac{1}{a}\mathrm{Im}(\pm 1 - e^{\frac{\pi}{a}z})^{-1}$;

B. $-\cosh\frac{\pi}{a}x < \cos\frac{\pi}{a}y < \cosh\frac{\pi}{a}x$;

C. $\frac{d}{dx}(\frac{1}{\pi}\arctan(\tan\frac{\pi}{2a}y \cdot \tanh\frac{\pi}{2a}x)) = P_y^-(x),\ P_y^+(x) = P_{a-y}^-(x)$;

D. $\cos\frac{\pi}{a}y \in (-1, 1),\ e^{-\frac{\pi}{a}|x|}(\cosh\frac{\pi}{a}x - 1)$ is even and increasing for $\delta \le x < +\infty$, which implies that

$$|P_y^{\pm}(x)| \le \frac{e^{-\frac{\pi}{a}|x| + \frac{\pi}{a}\delta}\sin\frac{\pi}{a}y}{2a(\cosh\frac{\pi}{a}\delta - 1)}\quad (|x| \ge \delta).$$

□

Lemma 2.3 *Let* $\tilde{P}_y^{\pm}(x) = \frac{\sin\frac{\pi}{a-2y_0}y}{2(a-2y_0)(\cosh\frac{\pi}{a-2y_0}x \mp \cos\frac{\pi}{a-2y_0}y)}$, $0 < y_0 < \frac{a}{2}$*, then there exist* $A_1, A_2 > 0$ *depending on* y_0 *and* y*, such that*

$$\begin{aligned} |P_y^+(x) - \tilde{P}_{y-y_0}^+(x)| \le& \frac{1}{1 - \cos\frac{\pi}{a}y}\left|\frac{1}{2a} - \frac{1}{2(a - 2y_0)}\right| + A_1\left|\frac{\pi}{a - 2y_0} - \frac{\pi}{a}\right| \\ &+ \frac{|\frac{\pi}{a}y - \frac{\pi}{a-2y_0}(y - y_0)|}{2(a - 2y_0)(1 - \cos\frac{\pi}{a}y)} \\ &+ \frac{|\frac{\pi}{a}y - \frac{\pi}{a-2y_0}(y - y_0)|}{2(a - 2y_0)(1 - \cos\frac{\pi}{a}y)(1 - \cos\frac{\pi}{a-2y_0}(y - y_0))} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}|P_y^-(x)-\tilde{P}_{y-y_0}^-(x)| \le & \frac{1}{1+\cos\frac{\pi}{a}y}\left|\frac{1}{2a}-\frac{1}{2(a-2y_0)}\right| + A_2\left|\frac{\pi}{a-2y_0}-\frac{\pi}{a}\right| \\ &+\frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1+\cos\frac{\pi}{a}y)} \\ &+\frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1+\cos\frac{\pi}{a}y)(1+\cos\frac{\pi}{a-2y_0}(y-y_0))} \end{aligned} \tag{2.3}$$

hold for every $y_0 < y < a - y_0$.

Proof Firstly, we have

$$\begin{aligned}P_y^+(x)-\tilde{P}_{y-y_0}^+(x) =& \frac{\sin\frac{\pi}{a}y}{2a(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)} - \frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))} \\ =& I_1+I_2+I_3+I_4,\end{aligned}$$

where

$$\begin{aligned} I_1 &= \left[\frac{1}{2a}-\frac{1}{2(a-2y_0)}\right]\frac{\sin\frac{\pi}{a}y}{\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y}, \\ I_2 &= \frac{\sin\frac{\pi}{a}y-\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)}, \\ I_3 &= \frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)}\frac{\cos\frac{\pi}{a}y-\cos\frac{\pi}{a-2y_0}(y-y_0)}{(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))}, \\ I_4 &= \frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)}\frac{\cosh\frac{\pi}{a-2y_0}x-\cosh\frac{\pi}{a}x}{(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))}. \end{aligned}$$

Obviously,

$$|I_1| \le \frac{1}{1-\cos\frac{\pi}{a}y}\left|\frac{1}{2a}-\frac{1}{2(a-2y_0)}\right|.$$

By mean value theorem of differentials, it is easy to see that

$$\begin{aligned} |I_2| &\le \frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)}, \\ |I_3| &\le \frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)(1-\cos\frac{\pi}{a-2y_0}(y-y_0))}, \\ I_4 &= \frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)}\frac{(\frac{\pi}{a-2y_0}-\frac{\pi}{a})x\cdot\sinh\xi}{(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))}, \end{aligned}$$

where ξ is between $\frac{\pi}{a}x$ and $\frac{\pi}{a-2y_0}x$. Note that there exists $M>0$ such that $|x|e^{-\frac{\pi}{a}|x|}\le \frac{1}{4}$, $\cosh\frac{\pi}{a}x\ge 2$ and $\cosh\frac{\pi}{a-2y_0}x\ge 2$ for all $|x|>M$. Therefore, we have

$$\frac{|x|}{\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y}\le\frac{|x|}{\frac{1}{2}\cosh\frac{\pi}{a}x}\le\frac{4|x|}{e^{\frac{\pi}{a}|x|}}\le 1$$

and

$$\frac{|\sinh\xi|}{\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0)}\le\frac{|\sinh\xi|}{\frac{1}{2}\cosh\frac{\pi}{a-2y_0}x}\le\frac{2e^{|\xi|}}{\cosh\frac{\pi}{a-2y_0}x}\le 4.$$

Hence

$$|I_4|\le\frac{2}{a-2y_0}\left|\frac{\pi}{a-2y_0}-\frac{\pi}{a}\right|.$$

If $|x|\le M$, then

$$|\sinh\xi|\le\sinh\frac{\pi}{a}|x|+\sinh\frac{\pi}{a-2y_0}|x|\le\sinh\frac{\pi}{a}M+\sinh\frac{\pi}{a-2y_0}M,$$

which follows that

$$|I_4|\le\frac{1}{2(a-2y_0)}\frac{M(\sinh\frac{\pi}{a}M+\sinh\frac{\pi}{a-2y_0}M)}{(1-\cos\frac{\pi}{a}y)(1-\cos\frac{\pi}{a-2y_0}(y-y_0))}\left|\frac{\pi}{a-2y_0}-\frac{\pi}{a}\right|.$$

Let

$$A_1=\max\left\{\frac{2}{a-2y_0},\frac{M(\sinh\frac{\pi}{a}M+\sinh\frac{\pi}{a-2y_0}M)}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)(1-\cos\frac{\pi}{a-2y_0}(y-y_0))}\right\},$$

then

$$|I_4|\le A_1\left|\frac{\pi}{a-2y_0}-\frac{\pi}{a}\right|,\quad x\in\mathbb{R},$$

(2.2) is thus proved. Similarly, we can prove (2.3). □

Lemma 2.4 (Harmonic Majorant) *Let $v(z)$ be a nonnegative subharmonic function in S_a satisfying (2.1), then*

$$M_v(y)=\int_{\mathbb{R}}v(x+iy)dx$$

is convex in $(0,a)$ and there exist two positive measures $\mu_1,\ \mu_2\in\mathscr{M}(\mathbb{R})$ with $\|\mu_1\|,\ \|\mu_2\|\le C$ such that

$$u(x+iy)=\int_{-\infty}^{\infty}P_y^+(x-t)d\mu_1(t)+\int_{-\infty}^{\infty}P_y^-(x-t)d\mu_2(t).$$

Moreover, $v(z)\le u(z)$ for all $z\in S_a$.

Proof There exists a sequence $\{y_k\}$ such that $\lim_{k\to\infty}y_k=0$. By (2.1), $\{v_{y_k}\},\{v_{a-y_k}\}$ are bounded linear functionals on $C_0(\mathbb{R})$ and they are uniformly bounded, where $v_y(x)=v(x+iy)$. Based on Banach-Alaoglu theorem, there exist $\mu_1,\mu_2\in\mathscr{M}(\mathbb{R})$ and a subsequence

$\{y_{k_j}\}$ such that $\{v_{y_{k_j}}\}$ converges weakly to μ_1 as $j \to \infty$ and $\{v_{a-y_{k_j}}\}$ converges weakly to μ_2 as $j \to \infty$. That is, for each $\varphi \in C_0(\mathbb{R})$,

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} v(t + iy_{k_j})\varphi(t)dt = \int_{-\infty}^{\infty} \varphi(t)d\mu_1(t),$$

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} v(t + i(a - y_{k_j}))\varphi(t)dt = \int_{-\infty}^{\infty} \varphi(t)d\mu_2(t).$$

Accordingly, we obtain that

$$\begin{aligned}\|\mu_1\| &= \sup\left\{\Big|\int_{\mathbb{R}} \varphi(t)d\mu_1(t)\Big| : \varphi \in C_0(\mathbb{R}), \|\varphi\|_\infty = 1\right\}\\ &\le \liminf_{j\to\infty} \int_{\mathbb{R}} v(t + iy_{k_j})dt \le C,\end{aligned}$$

$$\begin{aligned}\|\mu_2\| &= \sup\left\{\Big|\int_{\mathbb{R}} \varphi(t)d\mu_2(t)\Big| : \varphi \in C_0(\mathbb{R}), \|\varphi\|_\infty = 1\right\}\\ &\le \liminf_{j\to\infty} \int_{\mathbb{R}} v(t + i(a - y_{k_j})dt \le C.\end{aligned}$$

Because of $\varphi(t) = P_y^+(x - t)(\text{or } P_y^-(x - t)) \in C_0(\mathbb{R})$, in particular, we have

$$\begin{aligned}&\lim_{j\to\infty} \int_{-\infty}^{\infty} v(t + iy_{k_j})P_y^+(x - t)dt + \lim_{j\to\infty} \int_{-\infty}^{\infty} v(t + i(a - y_{k_j}))P_y^-(x - t)dt\\ =&\int_{-\infty}^{\infty} P_y^+(x - t)d\mu_1(t) + \int_{-\infty}^{\infty} P_y^-(x - t)d\mu_2(t) \triangleq u(x + iy).\end{aligned}$$

For any fixed $0 < y_0 < y_1 < a$, let $r = \frac{y_1 - y_0}{a}$, then the function

$$\tilde{u}(z) = \int_{\mathbb{R}} v(t + iy_0)P_y^+(x - t)dt + \int_{\mathbb{R}} v(t + iy_1)P_y^-(x - t)dt$$

is harmonic in S_a (see [15, Theorem 1]). Assume that $\varepsilon > 0$ and $A > \exp\{\frac{C}{\pi\varepsilon \min\{y_0, a-y_1\}}\} + 1$. Since $v(rz + iy_0)$ is subharmonic in the set $\{z = x + iy : x \in \mathbb{R}, -\frac{y_0}{r} < y < \frac{a-y_0}{r}\}$, then there exist two sequences of continuous functions $\{u_n^{(1)}(t)\}$ and $\{u_n^{(2)}(t)\}$ decreasing to $v(rt + iy_0)$ and $v(rt + iy_1)$ on $[-A, A]$, respectively. Let

$$U_n(z) = \int_{-A}^{A} P_y^+(x - t)u_n^{(1)}(t)dt + \int_{-A}^{A} P_y^-(x - t)u_n^{(2)}(t)dt,$$

then by Lemma 2.2, $U_n(z)$ is harmonic in S_a and

$$|U_n(z)| \le \max\{\max_{|t|\le A} u_n^{(1)}(t), \max_{|t|\le A} u_n^{(2)}(t)\} = A_n.$$

Therefore, the function

$$V_n(z) = v(rz + iy_0) - 2\varepsilon \log |z + i| - U_n(z)$$

is subharmonic in S_a, and by Lemma 2.1, we speculate that

$$V_n(z) \leq \frac{2C}{\pi \min\{y_0, a - y_1\}} - \varepsilon \log |x^2 + (y+1)^2| + A_n \to -\infty \ (z \to \infty, 0 < y < a).$$

It follows that

$$\limsup_{z\to t, 0<y<a} V_n(z) \leq \frac{2C}{\pi \min\{y_0, a - y_1\}} - 2\varepsilon \log A \leq 0$$

and

$$\limsup_{z\to t+ia, 0<y<a} V_n(z) \leq \frac{2C}{\pi \min\{y_0, a - y_1\}} - 2\varepsilon \log A \leq 0$$

for $t \in \mathbb{R}$, $|t| > A$. If $t \in \mathbb{R}$, $|t| \leq A$, then

$$\limsup_{z\to t, 0<y<a} V_n(z) \leq v(rt + iy_0) - u_n^{(1)}(t) \leq 0,$$

and

$$\limsup_{z\to t+ia, 0<y<a} V_n(z) \leq v(rt + iy_1) - u_n^{(2)}(t) \leq 0.$$

By [3, Theorem 4.3.11], we derive that $V_n(z) \leq 0$ on S_a. Take $n \to \infty$, then $A \to \infty$, and then let $\varepsilon \to 0$, we obtain

$$v(rz + iy_0) \leq \int_{-\infty}^{\infty} P_y^+(x - t) v(rt + iy_0) dt + \int_{-\infty}^{\infty} P_y^-(x - t) v(rt + iy_1) dt.$$

Hence, for every $0 < y_0 < y < y_1 < a$, we have

$$\begin{aligned} v(z) &\leq \int_{-\infty}^{\infty} P^+_{\frac{y-y_0}{r}} \left(\frac{x}{r} - t\right) v(rt + iy_0) dt + \int_{-\infty}^{\infty} P^-_{\frac{y-y_0}{r}} \left(\frac{x}{r} - t\right) v(rt + iy_1) dt \\ &= \int_{-\infty}^{\infty} \frac{1}{r} P^+_{\frac{y-y_0}{r}} \left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_0) dt + \int_{-\infty}^{\infty} \frac{1}{r} P^-_{\frac{y-y_0}{r}} \left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_1) dt. \end{aligned} \tag{2.4}$$

Moreover, by Lemma 2.2,

$$\begin{aligned} \int_{-\infty}^{\infty} v(x + iy) dx \leq &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{r} P^+_{\frac{y-y_0}{r}} \left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_0) dt dx \\ &+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{r} P^-_{\frac{y-y_0}{r}} \left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_1) dt dx \\ &= \left(1 - \frac{y - y_0}{ar}\right) \int_{-\infty}^{\infty} v(t + iy_0) dt + \frac{y - y_0}{ar} \int_{-\infty}^{\infty} v(t + iy_1) dt \\ &= \frac{y_1 - y}{y_1 - y_0} \int_{-\infty}^{\infty} v(t + iy_0) dt + \frac{y - y_0}{y_1 - y_0} \int_{-\infty}^{\infty} v(t + iy_1) dt. \end{aligned}$$

Thus, we conclude that $M_v(y)$ is convex in $(0,a)$. Let $y_1 = a - y_0$, then $\frac{1}{r}P^{\pm}_{\frac{y-y_0}{r}}(\frac{x}{r}-\frac{t}{r}) = \tilde{P}^{\pm}_{y-y_0}(x-t)$ and (2.4) becomes

$$v(z) \le \int_{-\infty}^{\infty} \tilde{P}^{+}_{y-y_0}(x-t)v(t+iy_0)dt + \int_{-\infty}^{\infty} \tilde{P}^{-}_{y-y_0}(x-t)v(t+i(a-y_0))dt,$$

where $0 < y_0 < y < a - y_0 < a$. By Lemma 2.3, one has

$$\lim_{j\to\infty}\int_{-\infty}^{\infty} |\tilde{P}^{+}_{y-y_{k_j}}(x-t) - P^{+}_{y}(x-t)|v(t+iy_{k_j})dt = 0,$$

$$\lim_{j\to\infty}\int_{-\infty}^{\infty} |\tilde{P}^{-}_{y-y_{k_j}}(x-t) - P^{-}_{y}(x-t)|v(t+i(a-y_{k_j}))dt = 0.$$

Therefore, it is not hard to verify that

$$\begin{aligned} v(z) \le& \lim_{j\to\infty}\int_{-\infty}^{\infty}[v(t+iy_{k_j})P^{+}_{y}(x-t) + v(t+i(a-y_{k_j}))P^{-}_{y}(x-t)]dt \\ =& \int_{-\infty}^{\infty} P^{+}_{y}(x-t)d\mu_1(t) + \int_{-\infty}^{\infty} P^{-}_{y}(x-t)d\mu_2(t) \\ =& u(z) \end{aligned}$$

for all $z = x + iy,\ 0 < y < a$. It completes the proof. □

Next, we will apply this lemma to the function $\log^+|f|$ to prove Theorem 1.1 and Theorem 1.2, where $f \in H^0(S_a)$.

Lemma 2.5 ([2, 14]) *If v is subharmonic in $\mathbb{U}$, then the following statements are equivalent:*
(i) v has a harmonic majorant in $\mathbb{U}$;
(ii) $\sup\limits_{0<r<1}\{\frac{1}{2\pi}\int_{-\pi}^{\pi} v(re^{i\varphi})d\varphi\} < \infty$.

Proof of Theorem 1.1 Since $\log^+|f|$ is subharmonic and it satisfies (2.1), it follows from Lemma 2.4 that there exist two positive measures $\mu_1, \mu_2 \in \mathscr{M}(\mathbb{R})$ such that

$$u(x+iy) = \int_{-\infty}^{\infty} P^{+}_{y}(x-t)d\mu_1(t) + \int_{-\infty}^{\infty} P^{-}_{y}(x-t)d\mu_2(t),$$

and $\log^+|f|(z) \le u(z)$ for all $z \in S_a$. The conformal mapping

$$\beta(z) = \frac{e^{\frac{\pi}{a}z} - i}{e^{\frac{\pi}{a}z} + i} \tag{2.5}$$

from S_a onto $\mathbb{U}$ maps ∂S_a onto $\partial\mathbb{U}\backslash\{1,-1\}$ conformally (to be precise, there exists a continuous and strictly increasing function $\theta_1(t)$ from $\mathbb{R}$ onto $(-\pi, 0)$ such that $e^{i\theta_1(t)} = \beta(t)$; and there exists a continuous and strictly decreasing function $\theta_2(t)$ from $\mathbb{R}$ onto $(0,\pi)$ such that $e^{i\theta_2(t)} = \beta(t+ia)$). Its inverse mapping is

$$\alpha(w) = \frac{a}{\pi}\log\frac{i(1+w)}{1-w}$$

(take the analytic branch $\log 1 = 0$). Then the function $\log^+|f|(\alpha(w))$ has a harmonic majorant $u(\alpha(w))$ in $\mathbb{U}$. According to Lemma 2.5, one has $F(w) = f(\alpha(w)) \in H^0(\mathbb{U})$. Therefore, by Theorem A, $F(w) = f(\alpha(w))$ has a nontangential limit $F^*(e^{i\theta})$ at almost every $\theta \in [-\pi, \pi)$ and $\log|F^*(e^{i\theta})| \in L^1([-\pi, \pi))$. Since α and β are conformal, the limits in (1.3) exist nontangentially for almost every $t \in \mathbb{R}$. By virtue of

$$e^{i\theta_1(t)} = \beta(t) = \frac{e^{\frac{\pi}{a}t} - i}{e^{\frac{\pi}{a}t} + i}, \quad e^{i\theta_2(t)} = \beta(t+ia) = \frac{e^{\frac{\pi}{a}(t+ia)} - i}{e^{\frac{\pi}{a}(t+ia)} + i},$$

which follows that $d\theta_1 = \frac{\frac{2\pi}{a}e^{\frac{\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}dt$ and $d\theta_2 = \frac{-\frac{2\pi}{a}e^{\frac{\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}dt$. Thus, (1.4) follows immediately from the following identities:

$$\begin{aligned}\int_{-\pi}^{\pi} \big|\log|F^*(e^{i\theta})|\big| d\theta =& \int_{-\infty}^{\infty} \frac{\frac{2\pi}{a}e^{\frac{\pi}{a}t}\big|\log|f^*(t)|\big|}{1+e^{\frac{2\pi}{a}t}}dt \\ &+ \int_{-\infty}^{\infty} \frac{\frac{2\pi}{a}e^{\frac{\pi}{a}t}\big|\log|f^*(t+ia)|\big|}{1+e^{\frac{2\pi}{a}t}}dt \\ =&\frac{\pi}{a}\int_{-\infty}^{\infty} \frac{\big|\log|f^*(t)|\big| + \big|\log|f^*(t+ia)|\big|}{\cosh\frac{\pi}{a}t}dt.\end{aligned}$$

□

Proof of Theorem 1.2 According to Theorem 1.1, $f(z)$ has nontangential limits $f^*(t)$ and $f^*(t+ia)$ and they satisfy (1.4). Since $\log^+|f|$ is subharmonic and it satisfies (2.1), then by Lemma 2.4, $\log^+|f|$ has a harmonic majorant in S_a. It follows that $F(w) = f(\alpha(w)) \in H(\mathbb{U})$ and $\log^+|F|$ has a harmonic majorant in $\mathbb{U}$. By Lemma 2.5, we have $F \in H^0(\mathbb{U})$. Then, by Theorem A, $F(w)$ has a nontangential limit $F^*(e^{i\theta})$ at almost every $\theta \in [-\pi, \pi)$ and $\log|F^*(e^{i\theta})| \in L^1([-\pi, \pi))$. Furthermore, F can be written by

$$F(w) = c_1 G_1(w) B_1(w) S_1(w),$$

where c_1 is a constant with $|c_1| = 1$, and

$$G_1(w) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+w}{e^{i\theta}-w}\log|F^*(e^{i\theta})|d\theta\right\}$$

is an outer function in $\mathbb{U}$; and

$$S_1(w) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+w}{e^{i\theta}-w}d\nu_s(\theta)\right\}$$

is a quotient of two singular inner functions in $\mathbb{U}$, where ν_s is a singular signed measure on $[-\pi, \pi)$ with finite total variation;

$$B_1(w) = w^k \prod_n \Big(\frac{\beta(z_n) - w}{1 - \overline{\beta(z_n)}w}\Big)\Big(\frac{\overline{\beta(z_n)}}{|\beta(z_n)|}\Big)$$

is a Blaschke product in $\mathbb{U}$, where k is a nonnegative integer and

$$\sum_n (1-|\beta(z_n)|^2) \leq 2\sum_n (1-|\beta(z_n)|) < \infty. \tag{2.6}$$

Therefore,

$$\begin{aligned}
\log|G_1(\beta(z))| =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t)+\beta(z)}{\beta(t)-\beta(z)}\Big)|\beta'(t)|\log|f^*(t)|dt \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t+ia)+\beta(z)}{\beta(t+ia)-\beta(z)}\Big)|\beta'(t+ia)|\log|f^*(t+ia)|dt, \\
\log|S_1(\beta(z))| =& \frac{1}{2\pi}\mathrm{Re}\frac{-1+\beta(z)}{-1-\beta(z)}\nu_s(\{-\pi\})+\frac{1}{2\pi}\mathrm{Re}\frac{1+\beta(z)}{1-\beta(z)}\nu_s(\{0\}) \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t)+\beta(z)}{\beta(t)-\beta(z)}\Big)d\nu_s(\theta_1(t)) \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t+ia)+\beta(z)}{\beta(t+ia)-\beta(z)}\Big)d\nu_s(\theta_2(t)).
\end{aligned}$$

Making use of (2.5), we have

$$1-|\beta(z_n)|^2 = \frac{4e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n}{1+e^{\frac{2\pi}{a}x_n}+2e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n} \geq \frac{e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n}{1+e^{\frac{2\pi}{a}x_n}}, \tag{2.7}$$

and

$$\Big(\frac{\beta(z_n)-w}{1-\overline{\beta(z_n)}w}\Big)\Big(\frac{\overline{\beta(z_n)}}{|\beta(z_n)|}\Big) = \frac{e^{\frac{\pi}{a}z_n}-e^{\frac{\pi}{a}z}}{e^{\frac{\pi}{a}\bar{z}_n}-e^{\frac{\pi}{a}z}}e^{i\theta(z_n)},$$

where

$$e^{i\theta(z_n)} = \frac{(e^{\frac{\pi}{a}\bar{z}_n}+i)(e^{\frac{\pi}{a}\bar{z}_n}-i)}{|e^{\frac{\pi}{a}z_n}+i||e^{\frac{\pi}{a}z_n}-i|}.$$

Moreover,

$$\begin{aligned}
\frac{1}{2\pi}\mathrm{Re}\frac{-1+\beta(z)}{-1-\beta(z)}\nu_s(\{-\pi\}) &= \frac{\nu_s(\{-\pi\})}{2\pi}\mathrm{Re}(ie^{-\frac{\pi}{a}z}), \\
\frac{1}{2\pi}\mathrm{Re}\frac{1+\beta(z)}{1-\beta(z)}\nu_s(\{0\}) &= \frac{\nu_s(\{0\})}{2\pi}\mathrm{Re}(-ie^{\frac{\pi}{a}z}), \\
\frac{1}{2\pi}\mathrm{Re}\frac{\beta(t)+\beta(z)}{\beta(t)-\beta(z)}|\beta'(t)| &= \mathrm{Re}\left\{\frac{1}{ai}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}-e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}\right)\right\}, \\
\frac{1}{2\pi}\mathrm{Re}\frac{\beta(t+ia)+\beta(z)}{\beta(t+ia)-\beta(z)}|\beta'(t+ia)| &= \mathrm{Re}\left\{-\frac{1}{ai}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}+e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}\right)\right\}.
\end{aligned}$$

Let $\tau_1 = \frac{\nu_s(\{-\pi\})}{2\pi}, \tau_2 = \frac{\nu_s(\{0\})}{2\pi}$. Define two singular signed measures $\mu_{1,s}$ on L_0 and $\mu_{2,s}$ on L_a by

$$d\mu_{1,s}(t) = |\beta'(t)|^{-1} d\nu_s(\theta_1(t)) = \frac{a}{\pi}\cosh\frac{\pi}{a}t\, d\nu_s(\theta_1(t)),$$
$$d\mu_{2,s}(t) = |\beta'(t+ia)|^{-1} d\nu_s(\theta_2(t)) = \frac{a}{\pi}\cosh\frac{\pi}{a}t\, d\nu_s(\theta_2(t)),$$

then $\log|G_1(\beta(z))| = \log|G(z)|$ and $\log|S_1(\beta(z))| = \log|S(z)|$. Therefore, there exist two constants c_2, c_3 with $|c_2| = |c_3| = 1$ such that $G_1(\beta(z)) = c_2 G(z), S_1(\beta(z)) = c_3 S(z)$. Let $c = c_1 c_2 c_3, B(z) = B_1(\beta(z))$, then $f(z) = cG(z)S(z)B(z)$. Accordingly, (1.5) follows instantly from (2.6) and (2.7). Since ν_s is finite, then (1.6) follows from (1.4). It completes the proof. □

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Controllability of Stochastic Evolution Differential Equations Driven by Fractional Brownian Motion and Poisson Jumping Processes *

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Abstract

In this article, we investigate stochastic evolution differential equations driven by fractional Brownian motions and Poisson random measure processes. At first, we discuss the existence and unique of the mild solution by using Banach fixed point principle. Secondly, sufficient conditions for the complete controllability of the stochastic evolution systems are formulated and proved by using the C_0-semigroup theory and stochastic analysis techniques. In the end, an example is presented to illustrate our main results.

Key words: *Stochastic evolution equation; Fractional Brownian motion; Poisson noise process; Mild solution; Complete controllability.*

1 Introduction

In this paper, we will study the problem having the following form:

$$\begin{cases} dx(t) = [Ax(t) + Bu(t) + f(t, x(t))]dt + \sigma(t)dB_Q^H(t) + \displaystyle\int_Z h(t, x(t), y)\widetilde{N}(dt, dy), \\ \qquad\qquad t \in J = [0, T] \\ x(0) = x_0. \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ on a separable Hilbert space X, $B_Q^H(t)$ is a fractional Brownian motion (fBm for short) with Hurst index $H \in (\frac{1}{2}, 1)$ on a real and separable Hilbert space K. $f : J \times X \to X, \sigma : J \to L_2^0(K, X), h : J \times X \times Z \to X$ are Borel measurable functions. Here $L_2^0(K, X)$ denotes the space of all Q-Hilbert-Schmidt operators from K into X. The control function $u(t)$ takes value in $V = L_2(J, U)$, and U is a Hilbert space, B is a linear operator from V into $L_2(J, X)$. $\widetilde{N}(dt, dy)$ is the compensated Poisson measure which will be given in the below.

Recently, stochastic differential systems have attracted a great attention since it arises naturally in mathematical modeling of various phenomena in the social and natural sciences, such as pricing an option, forecasting the growth of population and determining optimal portfolio of investments, for example one can see [16, 29, 32] and the references therein. Prato and Giuseppe [33] researched stochastic equations in infinite dimensions. Luo and Taniguchi [26] considered the existence and uniqueness of non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jumps. For the literatures on controllability of stochastic system with impulsive effect, one can see the papers [18, 24, 27, 36] and references therein.

It's well known that the noise or perturbations of a stochastic differential system are typically modeled by a Brownian motion as such a process is Gauss-Markov and has independent increments. However, many researchers have found that empirical data from many physical phenomena with the standard Brownian motion is often shown not to be an effective process to use in a model. A family of processes that seems to have wide physical applicability is fractional Brownian motion (fBm). Since it was first introduced by Kolmogorov in 1940, Mandelbrot and Ness discussed the applications of the fBm process in later. Since then, based on different settings, various forms of equations have been studied. For example, the case of finite-dimensional equations has been studied by Besalú and Rovira [5], Jérémie Unterberger [39], Dung [9], León and Tindel [23], for the case of infinite-dimensional systems in a Hilbert space have been considered by Boufoussi and Hajji [7], Caraballo, Garrido-Atienza and Taniguchi [8], and Ahmed [11]. Furthermore, the stochastic differential equations driven by a Poisson process can be widely found in applications from various fields such as storage systems, queueing systems, economic systems and neurophysiology systems, for example, one can see [1, 20, 35]. SPDEs with Poisson jump process is an important step for the study of SPDEs with Lévy process. In recent years, there is quite a substantial amount of work that has been done in this field. Hausenblas[12] dealt with SPDEs driven by Poisson random measures with non-Lipschitz coefficients in Banach spaces. Laukajtys and Słomiński [22] considered the penalization method for a reflected SDE driven by

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Poisson jumps. Later, Ren and Wu [34] established existence and uniqueness for solutions of multivalued, finite dimensional SDEs driven by Poisson point processes, where the maximal monotone operator has the whole space as domain.

On the other hand, one of the well-known qualitative behaviors of a dynamical system is controllability, which was first introduced by Kalman [17] in 1963. It means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Recently, many researchers take attention to the study of the controllability for a variety of differential dynamical systems. For example, Leiva [13] considered the exact controllability of the suspension bridge model proposed by Lazer and McKenna. Liu and Li [25] studied the controllability of impulsive fractional evolution inclusions in Banach spaces. The approximate controllability for a class of semilinear abstract equations was discussed by Zhou [40]. For more detailed, one can see [4, 14, 37]. For the controllability problem there are different methods for various types of nonlinear stochastic systems. Subalakshmi and Balachandran [37] studied the approximate controllability of nonlinear stochastic impulsive systems in Hilbert spaces by using Nussbaum's fixed point theorem. In [19], using a stochastic Lyapunov-like approach, sufficient conditions for stochastic ϵ-controllability are formulated. Balachandran etal.[3] researched the controllability of semilinear stochastic integrodifferential systems by using the Picard type iteration. By using the contraction mapping principle, Mahmudov and Zorlu studied the controllability [28] for non-linear stochastic systems.

By contrast, there has not been very much research of stochastic differential equations driven both by fractional Brownian motion and by Poisson noise processes. By using the extended form of Krylov-type estimate for the combined noise of fBM and compound Poisson, Bai and Ma [2] studied the existence of the strong solutions for the stochastic differential equation driven by fractional Brownian motion and Poisson point processes. Hajji and Lakhel [10] discussed the existence of the neutral stochastic functional differential equation driven by fractional Brownian motion and Poisson point processes. To the best of our knowledge, there is no paper researched the complete controllability of stochastic differential equations driven by fractional Brownian motions and poisson noise processes. Thus, we shall make the first attempt to discuss such problem in this paper.

The rest of this paper is organized as follows. In the next section, we will introduce some useful preliminaries on the data. In Section 3, some sufficient conditions are established to guarantee the existence and uniqueness of mild solutions of the system (1.1). In Section 4, we will study the completely controllability for stochastic evolution systems. Finally, we present an example to illustrate our main results.

2 Preliminaries

Now, we introduce some basic definitions and preliminaries which are used throughout this paper. Throughout this article, we use the following notations:

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0,T]\}, P)$ be a complete probability space satisfying the standard conditions, which means that the filtration $\mathcal{F}_t, t \in [0,T]$ is right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets. Let $L_2(\Omega, X) = L_2(\Omega, \mathcal{F}_t, X)$ be the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variables with values in X. Moreover, let $L_2^{\mathcal{F}}(J, X)$ be the Hilbert space of all square integrable and $\mathcal{F}_t$-adapted measurable processes with values in X. Further, let $C(J, L_2(\Omega, X)) := C(J, L_2(\Omega, \mathcal{F}_t, X))$ be the Banach space of continuous maps from J into $L_2(\Omega, X)$ satisfying $\sup_{t\in J} E\|x(t)\|^2 < \infty$ with the norm $\|x\| = (\sup_{t\in J} E\|x(t)\|_X^2)^{\frac{1}{2}}$.

Now, we present some basic definitions on fractional Brownian motion (fBm).

Definition 2.1. The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B_t^H = \{B_t^H, \mathcal{F}_t, t \in [0,T]\}$, having the properties $B_0^H = 0, EB_t^H = 0$, and $EB_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$.

Let $T > 0$, for a linear space Λ, there exists a $R-$valued step function $\phi \in \Lambda$ on $[0,T]$, such that

$$\phi(t) = \sum_{i=1}^{n-1} z_i \chi_{(t_i, t_{i+1}]}(t),$$

where $t \in [0,T], z_i \in R$ and $0 = t_1 < t_2 < \cdots < t_n = T$. For any $\phi \in \Lambda$, the Wiener integral with respect to B^H can be defined as

$$\int_0^T \phi(s) dB^H(s) = \sum_{i=1}^{n-1} z_i (B^H(t_{i+1}) - B^H(t_i)).$$

Let $\mathcal{H}$ be a Hilbert space, which is defined as the closure of Λ with respect to the scalar product $\langle \chi_{[0,t]}, \chi_{(0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$. Then the mapping

$$\phi = \sum_{i=1}^{n-1} z_i \chi_{(t_i, t_{i+1}]} \mapsto \int_0^T \phi(s) dB^H(s)$$

is an isometry between Λ and the linear space span $\{B^H(t) : t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{B^H(t) : t \in [0,T]\}$ (see [38]). The image of an element $\phi \in \mathcal{H}$ by this isometry is called the Wiener integral of ϕ with respect to B^H. Our next aim is to give an explicit expression of this integral.

Now, let us consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (z-s)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} dz,$$

where $c_H = \left(\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$ ($B(\cdot,\cdot)$ denote the Beta function), and $t > s$. It is easily shown that

$$\frac{\partial K_H(t,s)}{\partial t} = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Let $\mathcal{K}_H : \Lambda \to L^2([0,T])$ be the linear operator, which is defined as

$$\mathcal{K}_H \phi(s) = \int_s^t \phi(t) \frac{\partial K_H}{\partial t}(t,s) dt.$$

Then $(\mathcal{K}_H \chi_{[0,T]})(s) = K_H(t,s)\chi_{[0,T]}(s)$, and $\mathcal{K}_H$ is an isometry between Λ and $L_2([0,T])$ which can be extended to $\mathcal{H}$.

We denote $L_2^{\mathcal{H}}([0,T]) = \{\phi \in \mathcal{H} : \mathcal{K}_H \phi \in L_2([0,T])\}$, then for $H > \frac{1}{2}$, we get

$$L^{\frac{1}{H}}([0,T]) \subset L_2^{\mathcal{H}}([0,T]).$$

Moreover, the following lemma hold:

Lemma 2.2 ([30]). *For* $\phi \in L^{\frac{1}{H}}([0,T])$,

$$H(2H-1) \int_0^T \int_0^T |\phi(r)||\phi(z)||r-u|^{2H-2} dr dz \le c_H \|\phi\|^2_{L^{\frac{1}{H}}([0,T])}.$$

Let $(X, |\cdot|_X, \langle\cdot,\cdot\rangle_X)$ and $(K, |\cdot|_K, \langle\cdot,\cdot\rangle_K)$ be separable Hilbert spaces. $L(K,X)$ denotes the space of all bounded linear operator from K to X and $Q \in L(K,X)$ is a non-negative self adjoint operator. Denote by $L_2^0(K,X)$ the space of all $\xi \in L(K,X)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, the norm is given by

$$|\xi|^2_{L_2^0(K,X)} = |\xi Q^{\frac{1}{2}}|^2_{HS} = tr(\xi Q \xi^*).$$

Then ξ is a Q-Hilbert-Schmidt operator from K to X.

Let $\{B_n^H(t)\}_{n\in N}$ be a sequence of two-side one-dimensional fBm mutually independent on the complete probability space $(\Omega, \mathcal{F}, P)$, $\{e_n\}_{n\in N}$ be a complete orthonormal basis in K. We define the K-valued stochastic process $B_Q^H(t)$ as

$$B_Q^H(t) = \sum_{n=1}^{\infty} B_n^H(t) Q^{\frac{1}{2}e_n}, t \ge 0.$$

If Q is a non-negative self-adjoint trace class operator, then the series $\sum_{n=1}^{\infty} B_n^H(t) Q^{\frac{1}{2}e_n}, t \ge 0$ converges in the space K, i.e., it holds that $B_Q^H(t) \in L_2(\Omega, K)$. Then, we can say that $B_Q^H(t)$ is a K-valued Q-cylindrical fBm with covariance operator Q.

Definition 2.3. Let $\psi : [0,T] \to L_2^0(K,X)$ such that

$$\sum_{n=1}^{\infty} \|\mathcal{K}_H(\psi Q^{\frac{1}{2}}) e_n\|_{L_2([0,T],X)} < \infty. \tag{2.1}$$

Then for $t \ge 0$, its stochastic integral with respect to the fBm B_Q^H is defined as

$$\int_0^t \psi(s) dB_Q^H(s) = \sum_{n=1}^{\infty} \int_0^t \psi(s) Q^{\frac{1}{2}} e_n dB_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \Big(\mathcal{K}_H(\psi Q^{\frac{1}{2}} e_n)\Big)(s) dw(s),$$

where w is a Wiener process.

Notice that if

$$\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_{L^{\frac{1}{H}}([0,T],X)} < \infty, \tag{2.2}$$

then in particular (2.2) holds, which follows immediately from (2.1).

The following lemma is obtained as a simple application of Lemma 2.2.

Lemma 2.4 ([30]). *For any $\psi : [0,T] \to L_2^0(K,X)$ such that $\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_U$ is uniformly convergent for $t \in [0,T]$, and for any $p, q \in [0,T]$ with $p > q$,*

$$E\left\| \int_q^p \psi(s) dB_Q^H(s) \right\|_X^2 \leq cH(2H-1)(p-q)^{2H-1} \sum_{n=1}^{\infty} \int_q^p \|\psi Q^{\frac{1}{2}} e_n\|_X^2 ds.$$

Then

$$E\left\| \int_q^p \psi(s) dB_Q^H(s) \right\|_X^2 \leq cH(2H-1)(p-q)^{2H-1} \int_q^p \|\psi(s)\|_{L_2^0}^2 ds, \tag{2.3}$$

where $c = c(H)$.

In the follow, we give the definition of the Poission random measure.

Let $\{p(t) : t \in J\}$ be a Poisson point process, and take its value in a measure in a measurable space $(Z, \mathcal{B}(Z))$ with a σ-finite intensity measure $\mu(dy)$. We denote the Poisson counting measure as $N(dt, dy)$, which is induced by $p(\cdot)$, and the compensating martingale measure by

$$\widetilde{N}(dt, dy) = N(dt, dy) - \mu(dy)dt.$$

For investigated our main results, we shall give the following lemma.

Lemma 2.5. *Let the space $M_\mu^\theta(J \times \Omega \times (K - \{0\}), H), (\theta \geq 2)$ be the set of all random process $L(t,y)$ with values in H, predictable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that*

$$E\left(\int_0^T \int_Z \|L(t,y)\|_X^\theta \mu(dy)dt \right) < \infty.$$

Assume $L \in M_\mu^2(J \times \Omega \times (K - \{0\}), H) \cap M_\mu^4(J \times \Omega \times (K - \{0\}), H)$, then for any $t \in J$, we have

$$\begin{aligned} E\left[\sup_{0\leq r\leq t} \left\| \int_0^r \int_Z S(r-s)L(s,y)\widetilde{N}(dy,ds) \right\|_H^2 \right] \quad &\leq \quad l\left\{ E\left(\int_0^t \int_Z \|L(s,y)\|_H^2 \mu(dy)ds \right) \right. \\ &\left. + E\left(\int_0^t \int_Z \|L(s,y)\|_H^4 \mu(dy)ds \right)^{\frac{1}{2}} \right\} \end{aligned}$$

for some number $l > 0$ dependent on $T > 0$.

Now, we define the mild solution of the system(1.1) as follows.

Definition 2.6. A X-valued process $x(t)$ is called a mild solution of (1.1), if $x(0) = x_0, x(t) \in C(J, L_2(\Omega, X))$, for each $0 \leq t \leq T$, the following integral equation satisfies:

$$\begin{aligned} x(t) \quad = \quad & S(t)x_0 + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s,x(s))dt + \int_0^t S(t-s)\sigma(s)dB_Q^H(s) \\ & + \int_0^t \int_Z S(t-s)h(t,x(t),y)N(ds,dy). \end{aligned} \tag{2.4}$$

3 Existence result

The purpose of this section is to study the existence of mild solutions for problem (1.1). Our main method is the Banach contraction fixed point theorem.

At first, we assume that the following hypotheses be held:

(H_1) The C_0-semigroup $\{S(t)\}_{t\geq 0}$ is linear and bounded in X [31], i.e., there exists a constant $M > 0$, such that

$$\|S(t)\| \leq M.$$

(H_2) There exist constants $L_1, L_2 > 0$ such that

$$\|f(t,x_1) - f(t,x_2)\|^2 \leq L_1\|x_1 - x_2\|^2$$
$$\|f(t,x)\|^2 \leq L_2(1 + \|x\|^2)$$

for all $x_1, x_2, x \in X$ and a.e. $t \in J$.

(H_3) There are some constants $L_3, L_4 > 0$ such that

$$\int_Z \|h(t,x_1(t),y) - h(t,x_2(t),y)\|^2 \mu(dy) \leq L_3\|x_1(t) - x_2(t)\|^2,$$

$$\int_Z \|h(t,x(t),y)-h(t,x(t),y)\|^4\mu(dy) \le L_4\|x_1(t)-x_2(t)\|^4,$$

for all $x_1, x_2, x \in X$ and a.e. $t \in J$.

(H_4) There are some constants $L_5, L_6 > 0$ such that

$$\int_Z \|h(t,x(t),y)\|^2\mu(dy) \le L_5(1+\|x(t)\|^2),$$
$$\int_Z \|h(t,x(t),y)\|^4\mu(dy) \le L_6(1+\|x(t)\|^4)$$

for all $x_1, x_2, x \in X$ and a.e. $t \in J$.

(H_5) The function $\sigma : [0,\infty) \to L_2^0(K,X)$ satisfies $\int_0^T \|\sigma(s)\|^2_{L_2^0} ds < \infty$.

Now, we consider the existence result for system (1.1).

Theorem 3.1. *Assume that hypotheses $(H_1)-(H_5)$ hold. Then for any $u \in L_2(J,U)$ the stochastic system (1.1) has a unique mild solution on J, if*

$$2T^2[M^2L_1 + l(L_3+\sqrt{L_4})] < 1.$$

Proof. We define an operator $F : C(J,L_2(\Omega,X)) \to C(J,L_2(\Omega,X))$ by

$$\begin{aligned}(Fx)(t) &= S(t)x_0 + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s,x(s))ds \\ &\quad + \int_0^t S(t-s)\sigma(s)dB_Q^H(s) + \int_0^t\int_Z S(t-s)h(s,x(s),y)N(ds,dy).\end{aligned}$$

Using the contraction mapping principle, we will show that the operator F has a fixed point. To prove this, we subdivide the proof into four steps.

Step 1. For any $x \in C(J,L_2(\Omega,X))$, we show that F maps $C(J,L_2(\Omega,X))$ into itself.

For all $x \in C(J,L_2(\Omega,X))$, we have

$$\begin{aligned}
&E\|(Fx)(t)\|^2 \\
\le\ & 5E\|S(t)x_0\|^2 + 5E\left\|\int_0^t S(t-s)Bu(s)ds\right\|^2 + 5E\left\|\int_0^t S(t-s)f(s,x(s))ds\right\|^2 \\
&+5E\left\|\int_0^t S(t-s)\sigma(s)dB_Q^H(s)\right\|^2 + 5E\left\|\int_0^t\int_Z S(t-s)h(t,x(t),y)N(ds,dy)\right\|^2 \\
\le\ & 5M^2\left[E\|x_0\|^2 + E\|Bu\|^2T^2 + TL_2(1+E\|x\|_C^2) + cH(2H-1)T^{2H-1}\int_0^T \|\sigma(s)\|^2_{L_Q^0(V,U)}ds\right] \\
&+l\left\{E\left(\int_0^t\int_Z \|h(t,x(t),y)\|_H^2\mu(dy)ds\right) + E\left(\int_0^t\int_Z \|h(t,x(t),y)\|_H^4\mu(dy)ds\right)^{\frac{1}{2}}\right\} \\
\le\ & 5M^2\left[E\|x_0\|^2 + E\|Bu\|^2T^2 + TL_2(1+E\|x\|_C^2) + cH(2H-1)T^{2H-1}\int_0^T \|\sigma(s)\|^2_{L_Q^0(V,U)}ds\right] \\
&+l\left[L_5\int_0^t E(1+\|x(s)\|^2)ds + \sqrt{L_6}\left(\int_0^t E(1+\|x(s)\|^4)ds\right)^{\frac{1}{2}}\right] \\
\le\ & 5M^2\Bigg[E\|x_0\|^2 + E\|Bu\|^2T^2 + TL_2(1+E\|x\|_C^2) \\
&+cH(2H-1)T^{2H-1}\int_0^T \|\sigma(s)\|^2_{L_Q^0(V,U)}ds\Bigg] + l(L_5T+\sqrt{L_6T})E(1+\|x(s)\|^2) \qquad (3.1)
\end{aligned}$$

for all $t \in J$.

From the inequality (3.1) and the assumptions, one can see that there exists $M_1 > 0$ such that

$$E\|(Fx)(t)\|^2 \le M_1(1+T\sup_{s\in J} E\|x(s)\|^2)$$

for all $t \in J$. Thus, F maps $C(J,L_2(\Omega,X))$ into itself.

Step 2. We prove that F is a contraction mapping.

Let $x_1, x_2 \in C(J,L_2(\Omega,X))$, for $t \in J$ we have

$$\begin{aligned}
&E\|(Fx_1)(t)-(Fx_2)(t)\|^2 \\
\le\ & E\left\|\int_0^t S(t-s)[f(s,x_1(s))-f(s,x_2(s))]ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_Z S(t-s)[h(t,x_1(t),y)-h(t,x_2(t),y)]N(ds,dy)\Big\|^2\\
\le\;& 2M^2T^2L_1\sup_{t\in[0,T]}E\|x_1(t)-x_2(t)\|_{X^2}+2l\Big\{E\Big(\int_0^t\int_Z\|h(t,x_1(t),y)-h(t,x_2(t),y)\|_C^2\mu(dy)ds\Big)\\
&+E\Big(\int_0^t\int_Z\|h(t,x_1(t),y)-h(t,x_2(t),y)\|_C^4\mu(dy)ds\Big)^{\frac12}\Big\}\\
\le\;& 2M^2T^2L_1\sup_{s\in J}E\|x_1(s)-x_2(s)\|_C^2+2l(L_3+\sqrt{L_4})\int_0^t E\|x_1(s)-x_2(s)\|^2ds\\
\le\;& 2T^2[M^2L_1+l(L_3+\sqrt{L_4})]\sup_{s\in J}E\|x_1(s)-x_2(s)\|_C^2.
\end{aligned}
$$

Since $2T^2[M^2L_1+l(L_3+\sqrt{L_4})]<1$, then F is a contraction mapping and hence there exists a unique fixed point $x(\cdot)$ in $C(J,L_2(\Omega,X))$ which is the mild solution of problem (1.1). □

4 Controllability results

In this section, we discuss the controllability results for System (1.1). Before starting, we consider the following assumption:

(H_5) The linear operator $L_0^T\in L_2(U,X)$ is defined by

$$L_0^Tu=\int_0^T S(T-s)Bu(s)ds.$$

has an inverse operator $(L_0^T)^{-1}$ which takes values in $L_2(J,U)\setminus\ker L_0^T$, where $\ker L_0^T=\{x\in L_2(J,U),L_0^Tx=0\}$, and there are positive constants M_b,M_L such that $\|B\|^2\le M_b$, $\|(L_0^T)^{-1}\|^2\le M_L$.

To the readers' convenience, we give the definitions of controllability as follows.

Definition 4.1. System (1.1) is said to be completely controllable on the interval J if

$$\mathcal{R}_t(x_0)=C(J,L_2(\Omega,X)),$$

that is, all the points in $C(J,L_2(\Omega,X))$ can be exactly reached from arbitrary initial condition $x(0)=x_0$ and x_T at time T.

Theorem 4.2. *Assume that hypotheses $(H_1)-(H_5)$ hold. Then the stochastic system (1.1) is completely controllable on J, if*

$$3\Big\{TM^2\Big(L_1T+2M^2M_b^2M_L[M^2L_1T+(L_3+\sqrt{L_4})]T^2\Big)+l(L_3+\sqrt{L_4})\Big\}<1.$$

Proof. Fix $T>0$ and let $\mathcal{Z}_T=C(J,L_2(\Omega,X))$ be the Banach space of all functions from J into $L_2(\Omega,X)$, endowed with the supremum norm

$$\|\mu\|_{\mathcal{Z}_T}=\Big(\sup_{t\in[0,T]}E\|\mu(t)\|^2\Big)^{\frac12}.$$

Let's consider the set

$$G_T=\{x\in\mathcal{Z}_T:x(0)=x_0\}.$$

We easily know that G_T is a closed subset of $\mathcal{Z}_T$ equipped with norm $\|\cdot\|_{\mathcal{Z}_T}$.

By assumption (H_5), one can choose the feedback control function $u_x(t)$ as

$$
\begin{aligned}
u_x(t)\;=\;& B^*S^*(T-t)E\Big\{(L_0^T)^{-1}(x_T-S(T)x_0-\int_0^T S(T-s)f(s,x(s))ds\\
&-\Big[\int_0^T S(T-s)\sigma(s)dB_Q^H(s)+\int_0^T\int_Z S(T-s)h(s,x(s),y)N(ds,dy)\Big]|\mathcal{F}_t\Big\}.
\end{aligned}
$$

We will prove that if we use this control $u_x(t)$, the operator Φ define on $\|\cdot\|_{\mathcal{Z}_T}$ by

$$
\begin{aligned}
\Phi(x)(t)\;=\;& S(t)x_0+\int_0^t S(t-s)BB^*S^*(T-s)E\Big[(L_0^T)^{-1}\Big(x_b-S(T)x_0\\
&-\int_0^T S(T-\eta)f(s,x(\eta))d\eta-\Big[\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)\\
&+\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big]|\mathcal{F}_t\Big]ds+\int_0^t S(t-s)f(s,x(s))ds
\end{aligned}
$$

$$+\int_0^t S(t-s)\sigma(s)dB_Q^H(s)+\int_0^t\int_Z S(t-s)h(s,x(s),y)N(ds,dy).$$

has a fixed point on J.

To prove that the operator Φ has a fixed point on J, we divide the subsequent proof into the following two steps.

Step 1. For any $x\in G_T$, let's prove that $t\to\Phi(x)(t)$ is continuous on J in the $L_2(\Omega,X)$-sense.

Let $0<t<t+\delta<T$, here $t,t+\delta$ are belong to J, and $\delta>0$ is sufficiently small. Then we have

$$\begin{aligned}
&E\|\Phi(x)(t+\delta)-\Phi(x)(t)\|^2\\
\leq\;& 9E\|S(t+\delta)x_0-S(t)x_0]\|^2+9E\Big\|\int_0^t[S(t+\delta-s)-S(t-s)]f(s,x(s))ds\Big\|^2\\
&+9E\Big\|\int_t^{t+\delta}S(t+\delta-s)f(s,x(s))ds\Big\|^2+9E\Big\|\int_0^t[S(t+\delta-s)-S(t-s)]B\\
&\times B^*S^*(T-s)(L_0^T)^{-1}\Big(x_T-S(T)x_0-\int_0^T S(T-\eta)f(\eta,x(\eta))d\eta\\
&-\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)-\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big)ds\Big\|^2\\
&+9E\Big\|\int_t^{t+\delta}S(t+\delta-s)BB^*S^*(T-s)(L_0^T)^{-1}\Big(x_T-S(T)x_0-\int_0^T S(T-\eta)f(\eta,x(\eta))d\eta\\
&-\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)-\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big)ds\Big\|^2\\
&+9E\Big\|\int_0^t[S(t+\delta-s)-S(t-s)]\sigma(s)dB_Q^H(s)\Big\|^2+9E\Big\|\int_t^{t+\delta}S(t-s)\sigma(s)dB_Q^H(s)\Big\|^2.\\
&+9E\Big\|\int_0^t\int_Z[S(t+\delta-s)-S(t-s)]h(t,x(t),y)N(ds,dy)\Big\|^2\\
&+9E\Big\|\int_t^{t+\delta}\int_Z S(t+\delta-s)h(t,x(t),y)N(ds,dy)\Big\|^2\\
\leq\;& 9\sum_{i=1}^{9}I_9.
\end{aligned}$$

We can easily know that

$$I_1\leq\|S(t+\delta)-S(t)\|^2E\|x_0\|^2\to0\qquad\text{as }\delta\to0.$$

By using the well-known Hölder's inequality, we get

$$I_2\leq t\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\sup_{s\in J}E(1+\|x(s)\|^2)ds\to0\qquad\text{as }\delta\to0.$$

$$I_3\leq M^2t\int_t^{t+\delta}\sup_{s\in J}E(1+\|x(s)\|^2)ds\to0\qquad\text{as }\delta\to0.$$

By Hölder's inequality again, Lemma 2.4 and the condition (H_3), we get

$$\begin{aligned}
I_4\;\leq\;& 5t\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\|B\|^4\|L_0^T\|^2\Big(E\|x_T\|^2+M^2E\|x_0\|^2\\
&+E\Big\|\int_0^T S(T-\eta)f(\eta,x(\eta))d\eta\Big\|^2+E\Big\|\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)\Big\|^2\\
&+E\Big\|\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big\|^2\Big)ds\\
\leq\;& 5M_b^2M_L\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\Big(E\|x_T\|^2+M^2E\|x_0\|^2\\
&+M^2T\int_0^T\sup_{\eta\in J}E(1+\|x(\eta)\|^2)d\eta+M^2cH(2H-1)T^{2H-1}\int_0^T\|\sigma(\eta)\|^2_{L_2^0}d\eta\\
&+l\Big\{E\Big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^2\mu(dy)d\eta\Big)+E\Big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^4\mu(dy)d\eta\Big)^{\frac12}\Big\}\Big)ds\\
\leq\;& 5M_b^2M_L\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\Big(E\|x_T\|^2+M^2E\|x_0\|^2
\end{aligned}$$

$$+M^2T\int_0^T \sup_{\eta\in J} E(1+\|x(\eta)\|^2)d\eta + M^2cH(2H-1)T^{2H-1}\int_0^T \|\sigma(\eta)\|^2_{L_2^0}d\eta$$
$$+l(L_5T+\sqrt{L_6T})E(1+\|x(s)\|^2)\Big)ds.$$

Hence, by Lebesgue's dominated convergence, one can know that $I_4\to 0$ as $\delta\to 0$.

For a similar way, we obtain

$$\begin{aligned}
I_5 &\leq 5M^2M_b^2M_L\int_t^{t+\delta}\Big(E\|x_T\|^2+M^2E\|x_0\|^2+M^2T\int_0^T\sup_{\eta\in J}E(1+\|x(\eta)\|^2)d\eta\\
&\quad +M^2cH(2H-1)T^{2H-1}\int_0^T\|\sigma(\eta)\|^2_{L_2^0}d\eta\\
&\quad +L_4M^2\int_0^T\sup_{\eta\in J}E(1+\|x(\eta)\|^2)d\eta\Big)ds\to 0 \text{ as } \delta\to 0.
\end{aligned}$$

$$\begin{aligned}
I_6 &\leq cH(2H-1)t^{2H-1}\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\|\sigma(s)\|^2_{L_2^0}ds \text{ as } \delta\to 0.\\
I_7 &\leq cH(2H-1)M^2\delta^{2H-1}\int_t^{t+\delta}\|\sigma(s)\|^2_{L_2^0}ds \text{ as } \delta\to 0.\\
I_8 &\leq (L_5T+\sqrt{L_6T})\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\sup_{s\in J}E(1+\|x(s)\|^2)ds \text{ as } \delta\to 0.\\
I_9 &\leq l(L_5T+\sqrt{L_6T})\int_t^{t+\delta}\sup_{s\in J}E(1+\|x(s)\|^2)ds \text{ as } \delta\to 0.
\end{aligned}$$

Then, by the strong continuous of $S(t)$ and the Lebesgue's dominated convergence theorem, we know that the right hand of $I_i(i=1,\cdots,9)$ tends to 0 as $\delta\to 0$. Hence, $\Phi(x)(t)$ is continuous on J in the $L_2(\Omega,X)$-sense.

Next, we prove that Φ is a contraction mapping. Let $x,z\in C(J,L_2(\Omega,X))$ are two mild solution of (1.1), then

$$\begin{aligned}
&E\|\Phi(x)(t)-\Phi(z)(t)\|^2_H\\
\leq\ &3E\Big\|\int_0^t S(t-s)[f(s,x(s))-f(s,z(s))]ds\Big\|^2\\
&+3E\Big\|\int_0^t S(t-s)B(s)[u_x(s)-u_z(s)]ds\Big\|^2\\
&+3E\Big\|\int_0^t\int_Z S(t-s)[h(s,x(s),y)-h(s,z(s),y)]N(ds,dy)\Big\|^2\\
\leq\ &3J_1+3J_2+3J_3.
\end{aligned}$$

We can easily show that

$$J_1 \leq TM^2L_1T\sup_{t\in J}E\|x(t)-z(t)\|^2_H, \tag{4.1}$$
$$J_3 \leq l(L_3+\sqrt{L_4})\sup_{t\in J}E\|x(t)-z(t)\|^2. \tag{4.2}$$

Since

$$\begin{aligned}
&E\|u_x(t)-u_z(t)\|^2\\
\leq\ &E\Big\|B^*S^*(T-t)(L_0^T)^{-1}\Big(\int_0^T S(T-s)(f(s,x(s))-f(s,z(s)))ds\\
&-\int_0^T\int_Z S(T-s)[h(s,x(s),y)-h(s,z(s),y)]N(ds,dy)\Big)\Big\|^2\\
\leq\ &2M^2M_bM_L\Big(M^2T\int_0^T E\|f(s,x(s))-f(s,z(s))\|^2ds\\
&+l\Big\{E\big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^2\mu(dy)d\eta\big)+E\big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^4\mu(dy)d\eta\big)^{\frac12}\Big\}\Big)\\
\leq\ &2M^2M_bM_L[M^2L_1T+(L_3+\sqrt{L_4})]\int_0^T\sup_{s\in[0,t]}E\|x(s)-z(s)\|^2ds,
\end{aligned}$$

we have

$$\sup_{t\in J}E\|u_x(t)-u_z(t)\|^2\leq 2M^2M_bM_L[M^2L_1T+(L_3+\sqrt{L_4})]T\sup_{t\in J}E\|x(t)-z(t)\|^2.$$

Hence

$$J_2 \leq 2M^4 M_b^2 M_L[M^2 L_1 T + (L_3 + \sqrt{L_4})]T^2 \sup_{t\in J} E\|x(t) - z(t)\|^2, \tag{4.3}$$

By inequalities (4.1)-(4.3), we get

$$\begin{aligned} & E\|\Psi(x)(t) - \Psi(z)(t)\|^2 \\ \leq\ & 3\Big\{TM^2\Big(L_1 T + 2M^2 M_b^2 M_L[M^2 L_1 T + (L_3 + \sqrt{L_4})]T^2\Big) + l(L_3 + \sqrt{L_4})\Big\}\|x - z\|_C^2. \end{aligned}$$

Since $3\Big\{TM^2\Big(L_1 T + 2M^2 M_b^2 M_L[M^2 L_1 T + (L_3 + \sqrt{L_4})]T^2\Big) + l(L_3 + \sqrt{L_4})\Big\} < 1$. Therefore, Φ is a contraction mapping and hence there exists a unique fixed point $x(\cdot)$ in $C(J, L_2(\Omega, X))$ which is the mild solution of system (1.1). Thus, system (1.1) is complete controllable on J.

□

5 An example

Let's consider the following stochastic partial differential equations driven by fractional Brownian motion and Poisson noise process:

$$\begin{cases} dx(\theta,t) = \Big[\dfrac{\partial^2 x(\theta,t)}{\partial\theta^2} + F(\theta,t,x(\theta,t)) + g(\theta,t)\Big]dt + \sigma(t)dB^H(t) \\ \qquad\qquad + \displaystyle\int_Z \cos tx(\theta,t)\mu\widetilde{N}(dt,d\mu), \text{ in } \Omega\times[0,\tau], \\ x(\theta,t) = 0, \qquad \text{on } \partial\Omega\times[0,\tau], \\ x(\theta,0) = x_0(\theta), \qquad \theta\in\Omega, \end{cases} \tag{5.1}$$

where B^H is a fractional Brownian motion, Ω is a bounded open set in $R, F : \Omega \times J \times R \to R$ is nonlinear function, measurable with respect to θ and almost everywhere continuous with respect to t. Let $\{q(t), t \in [0,\tau]\}$ be the Poisson jump process taking values in the space $H = [0,\infty)$ with a σ-finite intensity measure $\lambda(d\mu)$ on the completely probability space $(\Sigma, \mathcal{F}, P)$. We denote the Poisson counting measure as $N(dt, d\mu)$, which is induced by $q(\cdot)$, and compensating martingale measure given by

$$\widetilde{N}(dt, d\mu) = N(dt, d\mu) - \lambda(d\mu)dt.$$

Take $X = Y = U = L^2([0,\tau])$ and the operator $A : D(A) \subset X \to X$ is defined by

$$Ax = x'',$$

$$D(A) = \{x \in X : x,\ x' \text{ are absolutely continuous},\ x'' \in X,\ x(0) = x(\pi) = 0\}.$$

Then, A can be written as

$$Ax = -\sum_{n=1}^{\infty} n^2(x, x_n)x_n, \qquad x \in D(A),$$

where $x_n(x) = \sqrt{2/\pi}\sin ny (n = 1, 2, \cdots)$ is an orthonormal basis of X. It is well known that A is the infinitesimal generator of a differentiable semigroup $T(t)(t > 0)$ in X given by

$$T(t)x = \sum_{n=1}^{\infty} \exp^{-n^2 t}(x, x_n)x_n, \quad x \in X, \qquad \text{and} \quad \|T(t)\| \leq e^{-1} < 1 = M.$$

In order to define the operator $Q : Y \to R$, we choose a sequence $\{l_n\}_{n\in N} \subset R^+$, let $Qe_n = l_n e_n$, and assume that

$$\text{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{l_n} < \infty.$$

Thus, we define the fractional Brownian motion in Y as

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{l_n}\gamma_n^H(t)e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\gamma_n^H\}_{n\in n}$ is a sequence of one-dimensional fractional Brownian motion mutually independent.

Let $x(t)(\cdot) = x(\cdot, t), f(t, x)(\cdot) = F(\cdot, t, x(\cdot))$. Define the bounded operator $B : U \to X$ by $Bu(t)(\theta) = g(\theta, t), \theta \in \Omega, u \in U$. Hence, by the above choice, it's easily known that the system (5.1) can be written into (1.1) and all the conditions of Theorem 4.2 are satisfied. Then by the Theorem 4.2, the stochastic partial differential equations driven by fractional Brownian motion and Poisson noise process is completely controllable on $[0, \tau]$.

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