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An international publication of Eudoxus Press, LLC, of TN.

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Existence and global attractiveness of pseudo almost periodic solutions to impulsive partial stochastic neutral functional differential equations

Zuomao Yan* and Fangxia Lu

January 5, 2018

Abstract: In this paper, we introduce a new concept of p-mean piecewise pseudo almost periodic for a stochastic process and establish a new composition theorem about pseudo almost periodic functions under non-Lipschitz conditions. Using this composition theorem, the analytic semigroup theory and fixed point strategy with stochastic analysis theory, we also study the existence and the global attractiveness for p-mean piecewise pseudo almost periodic mild solutions for impulsive partial neutral stochastic neutral functional differential equations. Moreover, an example is given to illustrate the general theorems.

2000 MR Subject Classification: 34A37; 60H10; 35B15; 34F05

Keywords: Impulsive partial stochastic functional differential equations; Pseudo almost periodic functions; Composition theorem; Analytic semigroup theory; Fixed point

1   Introduction

The concept of pseudo almost periodic functions introduced initially by Zhang [1] is an important generalization of the classical almost periodic functions. Since then, there has been an intense interest in studying several extensions of this concept such as asymptotic pseudo almost periodic functions and Stepanov-like pseudo almost periodic functions. Some contributions on pseudo almost periodic type solutions to abstract differential equations have recently been made [2-8] and the references therein. On the other hand, it should be pointed out that noise or stochastic perturbation is unavoidable and omnipresent in nature as well as that in man-made systems. Therefore, we must import the stochastic effects into the investigation of differential systems. The concept of almost periodicity is of great importance in probability for investigating stochastic processes. In fact, the existence of almost periodic, asymptotically almost periodic and pseudo almost periodic solutions for stochastic differential systems has been thoroughly investigated; see [9-18] and reference therein. In particular, Bezandry and Diagana [19,20] introduced the concepts of p-mean pseudo pseudo almost periodicity, and studied the existence of p-mean pseudo almost

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periodic mild solutions to partial stochastic differential equations. Diop et al. [21] obtained the existence, uniqueness and global attractiveness of an p-mean pseudo almost periodic solution for stochastic evolution equation driven by a fractional Brownian motion.

The theory of impulsive differential equations is an important branch of differential equations, which has an extensively physical background [22]. Therefore, it seems interesting to study the various types of impulsive differential equations. The asymptotic properties of solutions of impulsive differential equations have been considered by many authors. For example, Henríquez et al. [23], Liu and Zhang [24], Stamov et al. [25-27] discussed the piecewise almost periodic solutions of impulsive differential equations. Liu and Zhang [28], Chérif [29] established the existence and stability of piecewise pseudo almost periodic solutions to abstract impulsive differential equations. Bainov et al. [30] concerned with the asymptotic equivalence of impulsive differential equations. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. In recent years, several interesting results on impulsive partial stochastic systems have been reported in [31-33] and the references therein. Further, Zhang [34] obtained the existence and uniqueness of almost periodic solutions for a class of impulsive stochastic differential equations with delay by mean of the Banach contraction principle. In [35], the authors investigated the existence and stability of square-mean piecewise almost periodic solutions for nonlinear impulsive stochastic differential equations by using Schauder's fixed point theorem. Neutral differential equations arise in many areas of applied mathematics. For this reason, those equations have been of a great interest during the last few decades. The literature relative to partial neutral stochastic differential equations is quite extensive; for more on this topic and related applications we refer the reader to [36]. Similarly, for more on impulsive partial neutral stochastic functional differential equations we refer to [32,33,37,38]. In this paper, we study the existence and global attractiveness of p-mean piecewise pseudo almost periodic mild solutions to the following impulsive partial neutral stochastic neutral functional differential equations:

$$d[x(t) - h(t, x_t)] = [Ax(t) + g(t, x_t)]dt + f(t, x_t)dW(t), \tag{1}$$
$$t \in R, t \neq t_i, i \in Z,$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in Z, \tag{2}$$

where A is the infinitesimal generator of an exponentially stable analytic semigroup $\{T(t)\}_{t \geq 0}$ on a Hilbert space $L^p(P, H)$ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. The history $x_t \in \mathcal{D}$ with $q > 0$, where x_t being defined by $x_t(\theta) = x(t + \theta)$ for each $\theta \in [-q, 0])$ and $\mathcal{D} = \{\psi : [-q, 0] \to L^p(P, H), \psi$ continuous everywhere except for a finite number of points at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$. The functions h, g, f, I_i, t_i satisfy suitable conditions which will be established later. The no-

tations $x(t_i^+), x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at t_i, respectively.

To the best of our knowledge, the existence and global attractiveness of p-mean piecewise pseudo almost periodic mild solutions for for nonlinear impulsive stochastic system (1)-(2) is an untreated original topic, which in fact is the main motivation of the present paper. Although the papers [34,35] studied the piecewise almost periodic mild solution of impulsive stochastic differential equations, besides the fact that [34,35] applies to the results under the Lipschitz conditions, the class of impulsive stochastic systems is also different from the one studied here. Further, many dynamical control systems arising from realistic models can be described as impulsive partial neutral stochastic functional differential systems. So it is natural to extend the concept of pseudo almost periodicity to dynamical systems represented by these impulsive systems. In the paper, we will introduce the notion of p-mean piecewise pseudo almost periodic for stochastic processes, which, in turn generalizes all the above-mentioned concepts, in particular, the notion of piecewise almost periodic. Then we will establish a new composition theorem for p-mean pseudo almost periodic functions under non-Lipschitz conditions. As an application, we study and obtain the existence and exponential stability of p-mean piecewise pseudo almost periodic mild solutions to system (1)-(2) by using the analytic semigroup theory and Krasnoselskii fixed point theorem with stochastic analysis theory. Such a result generalizes most of known results on the existence of almost periodic solutions of type system (1)-(2). It includes some results of almost periodic and pseudo almost periodic solutions to stochastic differential equations without impulse. Moreover, the results are also new for deterministic systems with impulse.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of p-mean piecewise pseudo almost periodic mild solutions for (1)-(2). In Section 4, we establish the global attractiveness of p-mean piecewise pseudo almost periodic mild solutions for (1)-(2). Finally, an example is given to illustrate our results in Section 5.

2   Preliminaries

Throughout the paper, N, Z, R and R^+ stand for the set of natural numbers, integers, real numbers, positive real numbers, respectively. We assume that $(H, \| \cdot \|), (K, \| \cdot \|_K)$ are real separable Hilbert spaces and $(\Omega, \mathcal{F}, P)$ is supposed to be a filtered complete probability space. Define $L^p(P, H)$, for $p \geq 1$ to be the space of all H-valued random variables V such that $E \| V \|^p = \int_\Omega \| V \|^p \, dP < \infty$. Then $L^p(P, H)$ is a Banach space when it is equipped with its natural norm $\| \cdot \|_p$ defined by $\| V \|_p = (\int_\Omega E \| V \|^p \, dP)^{1/p} < \infty$ for each $V \in L^p(P, H)$. Let $C(R, L^p(P, H)), BC(R, L^p(P, H))$ stand for the collection of all continuous functions from R into $L^p(P, H)$, the Banach space of all bounded continuous functions from R into $L^p(P, H)$, equipped with the sup norm, respectively. We let $L(K, H)$ be the space of all linear bounded operators

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from K into H, equipped with the usual operator norm $\| \cdot \|_{L(K,H)}$; in particular, this is simply denoted by $L(H)$ when $K = H$. Furthermore, $L_2^0(K, H)$ denotes the space of all Q-Hilbert-Schmidt operators from K to H with the norm $\| \psi \|_{L_2^0}^2 = \text{Tr}(\psi Q \psi^*) < \infty$ for any $\psi \in L(K, H)$.

Definition 2.1 ([19]). A stochastic process $x : R \to L^p(P, H)$ is said to be continuous provided that for any $s \in R$,

$$\lim_{t \to s} E \| x(t) - x(s) \|^p = 0.$$

Definition 2.2 ([19]). A stochastic process $x : R \to L^p(P, H)$ is said to be stochastically bounded provided that

$$\lim_{N \to \infty} \limsup_{t \in R} \{P \| x(t) \| > N\} = 0.$$

Let T be the set consisting of all real sequences $\{t_i\}_{i \in Z}$ such that $\gamma = \inf_{i \in Z}(t_{i+1} - t_i) > 0$, $\lim_{i \to \infty} t_i = \infty$, and $\lim_{i \to -\infty} t_i = -\infty$. For $\{t_i\}_{i \in Z} \in T$, let $PC(R, L^p(P, H))$ be the space consisting of all stochastically bounded piecewise continuous functions $f : R \to L^p(P, H)$ such that $f(\cdot)$ is stochastically continuous at t for any $t \notin \{t_i\}_{i \in Z}$ and $f(t_i) = f(t_i^-)$ for all $i \in Z$; let $PC(R \times L^p(P, K), L^p(P, H))$ be the space formed by all stochastically piecewise continuous functions $f : R \times L^p(P, K) \to L^p(P, H)$ such that for any $x \in L^p(P, K)$, $f(\cdot, x) \in PC(R, L^p(P, H))$ and for any $t \in R, f(t, \cdot)$ is stochastically continuous at $x \in L^p(P, K)$.

Definition 2.3 ([19]). A function $f \in C(R, L^p(P, H))$ is said to be p-mean almost periodic if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $E \| f(t + \tau) - f(t) \|^p < \varepsilon$ for all $t \in R$. Denote by $AP(R, L^p(P, H))$ the set of such functions.

Definition 2.4 (Compare with [22]). A sequence $\{x_n\}$ is called p-mean almost periodic if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, i.e., there exists a natural number $l = l(\varepsilon)$, such that for $k \in Z$, there is at least one number $\tilde{q}$ in $[k, k + l]$, for which inequality $E \| x_{n+\tilde{q}} - x_n \|^p < \varepsilon$ holds for all $n \in N$. Denote by $AP(Z, L^p(P, H)))$ the set of such sequences.

Define $l^\infty(Z, L^p(P, H)) = \{x : Z \to L^p(P, H) : \| x \| = \sup_{n \in Z}(E \| x(n) \|^p)^{1/p} < \infty\}$, and

$$PAP_0(Z, L^p(P, H))$$
$$= \left\{ x \in l^\infty(Z, L^p(P, H)) : \lim_{n \to \infty} \frac{1}{2n} \sum_{j=-n}^{n} E \| x(n) \|^p \, dt = 0 \right\}.$$

Definition 2.5. A sequence $\{x_n\}_{n \in Z} \in l^\infty(Z, X)$ is called p-mean pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(Z, L^p(P, H)), x_n^2 \in PAP_0(Z, L^p(P, H))$. Denote by $PAP(Z, L^p(P, H))$ the set of such sequences.

Definition 2.6 (Compare with [22]). For $\{t_i\}_{i \in Z} \in T$, the function $f \in PC(R, L^p(P, H))$ is said to be p-mean piecewise almost periodic if the following conditions are fulfilled:

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(i) $\{t_i^j = t_{i+j} - t_i\}, j \in Z$, is equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set Q_ε of R such that for each $\tau \in Q_\varepsilon$ there is an integer $\tilde{q} \in Z$ such that $|t_{i+\tilde{q}} - t_i - \tau| < \varepsilon$ for all $i \in Z$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\tilde{\delta} = \tilde{\delta}(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of φ and $|t'-t''| < \tilde{\delta}$, then $E \parallel f(t') - f(t'') \parallel^p < \varepsilon$.

(iii) For every $\varepsilon > 0$, there exists a relatively dense set $\tilde{\Omega}(\varepsilon)$ in R such that if $\tau \in \tilde{\Omega}(\varepsilon)$, then
$$E \parallel f(t + \tau) - f(t) \parallel^p < \varepsilon$$
for all $t \in R$ satisfying the condition $|t - t_i| > \varepsilon, i \in Z$. The number τ is called ε-translation number of f.

We denote by $AP_T(R, L^p(P, H))$ the collection of all the p-mean piecewise almost periodic functions. Obviously, the space $AP_T(R, L^p(P, H))$ endowed with the sup norm defined by $\parallel f \parallel_\infty = \sup_{t \in R}(E \parallel f(t) \parallel^p)^{1/p}$ for any $f \in AP_T(R, L^p(P, H))$ is a Banach space. Let $UPC(R, L^p(P, H))$ be the space of all stochastic functions $f \in PC(R, L^p(P, H))$ such that f satisfies the condition (ii) in Definition 2.6.

Definition 2.7. The function $f \in PC(R \times L^p(P, K), L^p(P, H))$ is said to be p-mean piecewise almost periodic in $t \in R$ uniform in $x \in L^p(P, K)$ if for every compact subset $K \subseteq L^p(P, K)$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense subset Ω_ε such that
$$E \parallel f(t + \tau, x) - f(t, x) \parallel^p < \varepsilon$$
for all $x \in K, \tau \in \Omega_\varepsilon$, and $t \in R$ satisfying $|t - t_i| > \varepsilon$. Denote by $AP_T(R \times L^p(P, K), L^p(P, H))$ the set of all such functions.

Similarly as the proof of [22, Lemma 35], one has

Lemma 2.1. Assume that $f \in AP_T(R, L^p(P, H))$, the sequence $\{x_i\}_{i \in Z} \in AP(Z, L^p(P, H))$, and $\{t_i^j\}, j \in Z$ are equipotentially almost periodic. Then, for each $\varepsilon > 0$, there exist relatively dense sets Ω_ε of R and Ω_ε of Z such that

(i) $E \parallel f(t + \tau) - f(t) \parallel^p < \varepsilon$ for all $t \in R, |t - t_i| > \varepsilon, \tau \in \Omega_\varepsilon$ and $i \in Z$.

(ii) $E \parallel x_{i+\tilde{q}} - x_i \parallel^p < \varepsilon$ for all $\tilde{q} \in \Omega_\varepsilon$ and $i \in Z$.

(iii) $E \parallel x_i^{\tilde{q}} - \tau \parallel^p < \varepsilon$ for all $\tilde{q}, \tau \in \Omega_\varepsilon$ and $i \in Z$.

We need to introduce the new space of functions defined for each $q > 0$ by

$$PC_T^0(R, L^p(P, H), q)$$
$$= \left\{ f \in PC(R, L^p(P, H)) : \lim_{t \to \infty} \left(\sup_{\theta \in [t-q, t]} E \parallel f(\theta) \parallel^p \right) = 0 \right\},$$

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$$PAP_T^0(R, L^p(P, H), q) = \left\{ f \in PC(R, L^p(P, H)) : \right.$$

$$\left. \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-q,t]} E \parallel f(\theta) \parallel^p \right) dt = 0 \right\},$$

$$PAP_T^0(R \times L^p(P, K), L^p(P, H), q)$$

$$= \left\{ f \in PC(R \times L^p(P, K), L^p(P, H)) : \right.$$

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-q,t]} E \parallel f(\theta, x) \parallel^p \right) dt = 0$$

uniformly with respect to $x \in \bar{K}$,

$$\text{where } \bar{K} \text{ is an arbitrary compact subset of } L^p(P, K) \left. \right\}.$$

Similar to [4], one has

Lemma 2.2. The spaces $PAP_T^0(R, L^p(P, H), q)$ and $PAP_T^0(R \times L^p(P, K), L^p(P, H), q)$ endowed with the uniform convergence topology are Banach spaces.

Definition 2.8. A function $f \in PC(R, L^p(P, H))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = f_1 + f_2$, where $f_1 \in AP_T(R, L^p(P, H))$ and $f_2 \in PAP_T^0(R, L^p(P, H), q)$. Denoted by $PAP_T(R, L^p(P, H), q)$ the set of all such functions.

$PAP_T(R, L^p(P, H), q)$ is a Banach space with the sup norm $\parallel \cdot \parallel_\infty$.

Similar to [1,28], one has

Remark 2.1. (i) $PAP_T^0(R, L^p(P, H), q)$ is a translation invariant set of $PC(R, L^p(P, H))$. (ii) $PC_T^0(R, L^p(P, H), q) \subset PAP_T^0(R, L^p(P, H), q)$.

Lemma 2.3. Let $\{f_n\}_{n \in N} \subset PAP_T^0(R, L^p(P, H), q)$ be a sequence of functions. If f_n converges uniformly to f, then $f \in PAP_T^0(R, L^p(P, H), q)$.

One can refer to Lemma 2.5 in [6] for the proof of Lemma 2.3.

Definition 2.9. A function $f \in PC(R \times L^p(P, K), L^p(P, H))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = f_1 + f_2$, where $f_1 \in AP_T(R \times L^p(P, K), L^p(P, H))$ and $f_2 \in PAP_T^0(R \times L^p(P, K), L^p(P, H), q)$.

Denoted by $PAP_T(R \times L^p(P, K), L^p(P, H), q)$ the set of all such functions.

We need the following composition of p-mean pseudo almost periodic processes.

Lemma 2.4. Assume $f \in PAP_T(R \times L^p(P, K), L^p(P, H), q)$. Suppose that $f(t, x)$ satisfies

$$E \parallel f(t, x) - f(t, y) \parallel^p \leq \Lambda(E \parallel x - y \parallel^p) \tag{3}$$

for all $t \in R, x, y \in L^p(P, K)$, where Λ is a concave and continuous nondecreasing function from R^+ to R^+ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0+} \frac{ds}{\Lambda(s)} = +\infty$. Here, the symbol $\int_{0+}$ stands for $\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{+\infty}$. If $\phi(\cdot) \in PAP_T(R, L^p(P, K), q)$ then $f(\cdot, \phi(\cdot)) \in PAP_T(R, L^p(P, H), q)$.

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Proof. Assume that $f = f_1 + f_2, \phi = \phi_1 + \phi_2$, where $f_1 \in AP_T(R \times L^p(P, K), L^p(P, H))$, $f_2 \in PAP_T^0(R \times L^p(P, K), L^p(P, H), q)$, $\phi_1 \in AP_T(R, L^p(P, H))$, and $\phi_2 \in PAP_T^0(R, L^p(P, H), q)$. Consider the decomposition

$$f(t, \phi(t)) = f_1(t, \phi_1(t)) + [f(t, \phi(t)) - f(t, \phi_1(t))] + f_2(t, \phi_1(t)).$$

Since $f_1(\cdot, \phi_1(\cdot)) \in AP_T(R, L^p(P, H))$, it remains to prove that both $[f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot))]$ and $f_2(\cdot, \phi_1(\cdot))$ belong to $PAP_T^0(R, L^p(P, H), q)$. Indeed, using (3), it follows that

$$\frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-q,t]} E \parallel f(\theta, \phi(\theta)) - f(\theta, \phi_1(\theta)) \parallel^p \right) dt$$

$$\leq \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-q,t]} \Lambda(E \parallel \phi(\theta) - \phi_1(\theta) \parallel^p) \right) dt$$

$$= \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-q,t]} \Lambda(E \parallel \phi_2(\theta) \parallel^p) \right) dt,$$

noting that Λ is a concave and continuous nondecreasing function and $\Lambda(0) = 0$, we deduce that $\Lambda(E \parallel \phi_2(\theta) \parallel^p) \leq \Lambda(\sup_{\theta \in [t-q,t]} E \parallel \phi_2(\theta) \parallel^p)$, and

$$\frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-q,t]} \Lambda(E \parallel \phi_2(\theta) \parallel^p) \right) dt$$

$$\leq \frac{1}{2r} \int_{-r}^{r} \Lambda\left(\sup_{\theta \in [t-q,t]} E \parallel \phi_2(\theta) \parallel^p \right) dt$$

$$\leq \Lambda\left(\frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-q,t]} E \parallel \phi_2(\theta) \parallel^p \right) dt \right) \to 0 \text{ as } r \to \infty,$$

which implies that $[f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot))] \in PAP_T^0(R, L^p(P, H), q)$.

Since $\phi_1(R)$ is relatively compact in $L^p(P, K)$ and f_1 is uniformly continuous on sets of the form $R \times K$ where $K \subset L^p(P, K)$ is compact subset, for $\varepsilon > 0$ there exists $\xi \in (0, \varepsilon)$ such that

$$E \parallel f_1(t, z) - f_1(t, \tilde{z}) \parallel^p \leq \varepsilon, \ z, \tilde{z} \in \phi_1(R)$$

with $|z - \tilde{z}| < \xi$. Now, fix $z_1, ..., z_n \in \phi_1(R)$ such that $\phi_1(R) \subset \bigcup_{j=1}^{n} B_\xi(z_j, L^p(P, K))$. Obviously, the sets $D_j = \phi_1^{-1}(B_\xi(z_j))$ form an open covering of R, and therefore using the sets $B_1 = D_1, B_2 = D_2 \backslash D_1$ and $B_j = D_j \backslash \bigcup_{k=1}^{j-1} D_k$ one obtains a covering of R by disjoint open sets. For $t \in B_j, \phi_1(t) \in B_\xi(z_j)$,

$$E \parallel f_2(t, \phi_1(t)) \parallel^p$$

$$\leq 3^{p-1} E \parallel f(t, \phi_1(t)) - f(t, z_j) \parallel^p$$

$$\quad + 3^{p-1} E \parallel -f_1(t, \phi_1(t)) + f_1(t, z_j) \parallel^p + 3^{p-1} E \parallel f_2(t, z_j) \parallel$$

$$\leq 3^{p-1} \Lambda(E \parallel \phi_1(t) - z_j \parallel^p) + 3^{p-1} \varepsilon + 3^{p-1} E \parallel f_2(t, z_j) \parallel$$

$$\leq 3^{p-1} \Lambda(\varepsilon) + 3^{p-1} \varepsilon + 3^{p-1} E \parallel f_2(t, z_j) \parallel.$$

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Now using the previous inequality it follows that

$$\frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-q,t]}E\parallel f_1(\theta,\phi_1(\theta))\parallel^p\right)dt$$

$$\leq\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left(\sup_{\theta\in[t-q,t]}E\parallel f_1(\theta,\phi_1(\theta))\parallel^p\right)dt$$

$$\leq 3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left[\sup_{j=1,...,n}\left(\sup_{\theta\in[t-q,t]\cap B_j}\right.\right.$$

$$\times E\parallel f(\theta,\phi_1(\theta))-f(\theta,z_j)\parallel^p\Big)\Big]dt$$

$$+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left[\sup_{j=1,...,n}\left(\sup_{\theta\in[t-q,t]\cap B_j}\right.\right.$$

$$\times E\parallel f_1(\theta,\phi_1(\theta))-f_1(\theta,z_j)\parallel^p\Big)\Big]dt$$

$$+3^{p-1}\frac{1}{2r}\sum_{j=1}^{n}\int_{B_j\cap[-r,r]}\left[\sup_{j=1,...,n}\left(\sup_{\theta\in[t-q,t]\cap B_j}E\parallel f_2(\theta,z_j)\parallel^p\right)\right]dt$$

$$\leq 3^{p-1}\frac{1}{2r}\int_{-r}^{r}[\Lambda(\varepsilon)+\varepsilon]dt$$

$$+3^{p-1}\sum_{j=1}^{n}\frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-q,t]}E\parallel f_2(\theta,z_j)\parallel^p\right)dt.$$

In view of the above it is clear that $f_2(\cdot,\phi_1(\cdot))$ belongs to $PAP_T^0(R,L^p(P,H),q)$. This completes the proof.

Lemma 2.5. Assume the sequence of vector-valued functions $\{I_i\}_{i\in Z}$ is pseudo almost periodic, and there is a concave nondecreasing function from R^+ to R^+ such that $\Lambda_i(0)=0,\Lambda_i(s)>0$ for >0 and $\int_{0+}\frac{ds}{\Lambda_i(s)}=+\infty,$

$$E\parallel I_i(x)-I_i(y)\parallel^p\leq\Lambda_i(E\parallel x-y\parallel^p)$$

for all $x,y\in L^p(P,K),i\in Z$. If $\phi\in PAP_T(R,L^p(P,H),q)\cap UPC(R,L^p(P,H))$ such that $R(\phi)\subset L^p(P,K)$, then $I_i(\phi(t_i))$ is pseudo almost periodic.

Proof. Assume that $\phi=\phi_1+\phi_2$, where $\phi_1\in AP_T(R,L^p(P,H))$, $\phi_2\in PAP_T^0(R,L^p(P,H),q)$. Fix $\phi\in PAP_T(R,L^p(P,H),q)\cap UPC(R,L^p(P,H))$, first we show $\phi(t_i)$ is pseudo almost periodic. One can refer to Lemma 37 in [22] that the sequence $\phi(t_i)$ is almost periodic. Next we need to show that $\phi(t_i)\in PAP_0(Z,L^p(P,H))$. By the hypothesis, $\phi,\phi_1\in UPC(R,L^p(P,H))$, so $\phi_2\in UPC(R,L^p(P,H))$. Let $0<\varepsilon<1$, there exists $0<\xi<\min\{1,\gamma\}$ such that for $t\in(t_i-\xi,t_i),i\in Z$, we have

$$E\parallel\phi_2(t)\parallel^p\leq(1-\varepsilon)E\parallel\phi_2(t_i)\parallel^p,\ i\in Z.$$

Since $t_i^j,i\in Z,j=0,1,...$ are equipotentially almost periodic, $\{t_i^1\}$ is an almost periodic sequence. Here we assume a bound of $\{t_i^1\}$ is M_t and $|t_i|\geq|t_{-i}|$;

therefore,

$$\frac{1}{2t_i} \int_{-t_i}^{t_i} \left(\sup_{\theta \in [t-q,t]} E \parallel \phi_2(\theta) \parallel^p \right) dt$$

$$\geq \frac{1}{2t_i} \sum_{j=-i+1}^{i} \int_{t_j-\xi}^{t_j} \left(\sup_{\theta \in [t-q,t]} E \parallel \phi_2(\theta) \parallel^p \right) dt$$

$$\geq \frac{1}{2t_i} \sum_{j=-i+1}^{i} \xi(1-\varepsilon) E \parallel \phi_2(t_j) \parallel^p$$

$$\geq \frac{\xi(1-\varepsilon)}{M_t} \frac{1}{2t_i} \sum_{j=-i+1}^{i} E \parallel \phi_2(t_j) \parallel^p .$$

Since $\phi_2 \in PAP_T^0(R, L^p(P,H), q)$, it follows from the inequality above that $\phi_2(t_i) \in PAP_0(Z, L^p(P,H))$. Hence, $\phi(t_i)$ is pseudo almost periodic.

Now, we show $I_i(\phi(t_i))$ is pseudo almost periodic. Let

$$I(t,x) = (t-n)I_n(x), n \leq t < n+1, n \in Z,$$

$$\vartheta(t) = (t-n)\phi_n(t_n), n \leq t < n+1, n \in Z.$$

Since $I_n, \phi(t_n)$ are two pseudo almost periodic sequences, Refer to Lemma 1.7.12. in [39], we get that $I \in PAP(R \times L^p(P,K), L^p(P,H))$, $\vartheta \in PAP(R, L^p(P,K))$. For every $t \in R$, there exists a number $n \in Z$ such that $|t-n| \leq 1$, we have for $x_1, x_2 \in L^p(P,K)$,

$$E \parallel I(t,x_1) - I(t,x_2) \parallel^p$$
$$\leq E \parallel I_n(x_1) - I_n(x_2) \parallel^p$$
$$\leq \Lambda_n(E \parallel x_1 - x_2 \parallel^p).$$

Similar to the proof of Lemma 2.4, $I(\cdot, \vartheta(\cdot)) \in PAP(R, L^p(P,H))$. Again, similarly as the proof of Lemma 1.7.12 in [39], we have that $I(i, \vartheta(i))$ is a pseudo almost periodic sequence, that is, $I_i(\phi(t_i))$ is pseudo almost periodic. This completes the proof.

Let $0 \in \rho(A)$, then it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in H and the expression $\parallel x \parallel_\alpha = \parallel A^\alpha x \parallel, x \in D(A^\alpha)$, defines a norm on $D(A^\alpha)$. Hereafter we denote by H_α the Banach space $D(A^\alpha)$ with norm $\parallel x \parallel_\alpha$. Throughout the rest of this paper, we denote by $\parallel \cdot \parallel_{\alpha,\infty}$ the sup norm of the space $PAP_T(R, L^p(P,H_\alpha))$.

Lemma 2.6 ([40]). Let $0 < \alpha \leq \beta \leq 1$. Then the following properties hold:

(a) H_β is a Banach space and $H_\beta \hookrightarrow H_\alpha$ is continuous.

(b) The function $s \to A^\beta T(s)$ is continuous in the uniform operator topology on $(0,\infty)$ and there exists $M_\beta > 0$ such that $\parallel A^\beta T(t) \parallel \leq M_\beta e^{-\delta t} t^{-\beta}$ for each $t > 0$.

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(c) For each $x \in D(A^\beta)$ and $t \geq 0$, $A^\beta T(t)x = T(t)A^\beta x$.

(d) $A^{-\beta}$ is a bounded linear operator in H with $D(A^\beta) = \text{Im}(A^{-\beta})$.

Next, we introduce a useful compactness criterion on $PC(R, L^p(P, H), q)$.

Let $\tilde{h} : R \to R^+$ be a continuous function such that $\tilde{h}(t) \geq 1$ for all $t \in R$ and $\tilde{h}(t) \to \infty$ as $|t| \to \infty$. Define

$$PC_{\tilde{h}}^0(R, L^p(P, H), q)$$
$$= \left\{ f \in PC(R, L^p(P, H)) : \lim_{|t| \to \infty} \left(\sup_{\theta \in [t-q,t]} \frac{E \parallel f(\theta) \parallel^p}{h(\theta)} \right) = 0 \right\}$$

endowed with the norm $\parallel f \parallel_{\tilde{h}} = \sup_{t \in R}(\sup_{\theta \in [t-q,t]} \frac{E\|f(\theta)\|^p}{\tilde{h}(\theta)})$, it is a Banach space.

Lemma 2.7. A set $B \subseteq PC_{\tilde{h}}^0(R, L^p(P, H), q)$ is relatively compact if and only if it verifies the following conditions:

(i) $\lim_{|t| \to \infty} (\sup_{\theta \in [t-q,t]} \frac{E\|f(t)\|^p}{\tilde{h}(t)}) = 0$ uniformly for $f \in B$.

(ii) $B(t) = \{f(t) : f \in B\}$ is relatively compact in $L^p(P, H)$ for every $t \in R$.

(iii) The set B is equicontinuous on each interval $(t_i, t_{i+1})(i \in Z)$.

One can refer to Lemma 4.1 in [28] for the proof of Lemma 2.7.

Lemma 2.8 (Krasnoselskii's Fixed Point Theorem [41]). Let D be a closed, bounded, and convex subset of a Banach space X. Let Ψ_1 and Ψ_2 be operators, defined on D satisfying the conditions:

(a) $\Psi_1 x + \Psi_2 y \in D$ when $x, y \in D$.

(b) The operator Ψ_1 is a contraction.

(c) The operator Ψ_2 is continuous and $\Psi_2(D)$ is contained in a compact set.

Then the equation $\Psi_1 x + \Psi_2 x = x$ has a solution on D.

3 Existence

In this section, we investigate the existence of p-mean piecewise pseudo almost periodic mild solution for system (1)-(2). We begin introducing the followings concepts of mild solutions.

Definition 3.1. An $\mathcal{F}_t$-progressively measurable process $x : [\sigma, \sigma+b) \to H, b > 0$ is called a mild solution of system (1)-(2) on $[\sigma, \sigma + b)$, if $x_\sigma = \varphi \in \mathcal{D}$, the function $s \to AT(t - s)h(s, x_s)$ is integrable on $[0, t)$ for every $\sigma < t < \sigma + b$, and $\sigma \neq t_i, i \in Z$,

$$x(t) = T(t - \sigma)[\varphi(\sigma) - h(\sigma, \varphi)] + h(t, x_t) + \int_\sigma^t AT(t - s)h(s, x_s)ds$$

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$$+ \int_\sigma^t T(t-s)g(s,x_s)ds + \int_\sigma^t T(t-s)f(s,x_s)dW(s)$$

$$+ \sum_{\sigma < t_i < t} T(t-t_i)I_i(x(t_i)), \quad t \in [\sigma, \sigma+b). \tag{4}$$

Additionally, we make the following hypotheses:

(H1) A is the infinitesimal generator of a exponentially stable analytic semi-group $(T(t))_{t \geq 0}$ on $L^p(P,H)$ such that for all $t \geq 0, \parallel T(t) \parallel \leq Me^{-\delta t}$ with $M, \delta > 0$. Moreover, $T(t)$ is compact for $t > 0$.

(H2) There exist constants $\beta, L > 0$ such that $0 < \beta < 1$, the function $h \in PAP_T(R \times \mathcal{D}, L^p(P, H_\beta), q)$, and

$$E \parallel A^\beta h(t_1, \psi_1) - A^\beta h(t_2, \psi_2) \parallel^p \leq L[|t_1 - t_2| + \parallel \psi_1 - \psi_2 \parallel_\mathcal{D}^p],$$
$$t_1, t_2 \in R, \psi_1, \psi_2 \in \mathcal{D},$$

$$E \parallel A^\beta h(t, \psi) \parallel^p \leq L(\parallel \psi \parallel_\mathcal{D}^p + 1), \quad t \in R, \psi \in \mathcal{D}.$$

(H3) The functions $g \in PAP_T(R \times \mathcal{D}, L^p(P, H), q), f \in PAP_T(R \times \mathcal{D}, L^p(P, L_2^0), q)$, and for each $t \in R, \psi_1, \psi_2 \in \mathcal{D}$,

$$E \parallel g(t, \psi_1) - g(t, \psi_2) \parallel^p + E \parallel f(t, \psi_1) - f(t, \psi_2) \parallel_{L_2^0}^p$$
$$\leq \Lambda(E \parallel \psi_1 - \psi_2 \parallel_\mathcal{D}^p),$$

where Λ is a concave and continuous nondecreasing function from R^+ to R^+ such that $\Lambda(0) = 0, \Lambda(s) > 0$ for $s > 0$ and $\int_{0+} \frac{ds}{\Lambda(s)} = +\infty$.

(H4) For any $\rho_1 > 0$, there exist a constant $\mu > 0$ and nondecreasing continuous function $\Theta : R^+ \to R^+$ such that, for all $t \in R$, and $\psi \in \mathcal{D}$ with $E \parallel x \parallel_\mathcal{D}^p > \mu$,
$$E \parallel g(t, \psi) \parallel^p + E \parallel f(t, \psi) \parallel_{L_2^0}^p \leq \rho_1 \Theta(E \parallel \psi \parallel_\mathcal{D}^p).$$

(H5) The functions $I_i \in PAP(Z, L^p(P, H))$, and for each $t \in R, x_1, x_2 \in L^p(P, H), i \in Z$,

$$E \parallel I_i(x_1) - I_i(x_2) \parallel^p \leq \tilde{\Lambda}_i(E \parallel x_1 - x_2 \parallel^p),$$

where $\tilde{\Lambda}_i$ are concave and continuous nondecreasing functions from R^+ to R^+ such that $\tilde{\Lambda}_i(0) = 0, \tilde{\Lambda}_i(s) > 0$ for $s > 0$ and $\int_{0+} \frac{ds}{\tilde{\Lambda}_i(s)} = +\infty$.

(H6) For any $\rho_2 > 0$, there exist a constant $\mu > 0$ and nondecreasing continuous function $\tilde{\Theta}_i : R^+ \to R^+, i \in Z$, such that, for all $t \in R$, and $x \in L^p(P, H)$ with $E \parallel x \parallel^p > \mu$,

$$E \parallel I_i(x) \parallel^p \leq \rho_2 \tilde{\Theta}_i(E \parallel x \parallel^p).$$

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To study the system (1)-(2) we need the following results.

Lemma 3.1. Assume that (H1) holds. If $h \in PAP_T(R, L^p(P, H_\beta), q)$ and if H is the function defined by

$$H(t) := \int_{-\infty}^{t} AT(t - s)h(s)ds$$

for each $t \in R$, then $H \in PAP_T(R, L^p(P, H), q)$.

Proof. Since $h \in PAP_T(R, L^p(P, H_\beta), q)$, there exist $h_1 \in AP_T(R, L^p(P, H_\beta))$ and $h_2 \in PAP_T^0(R, L^p(P, H_\beta), q)$, such that $h = h_1 + h_2$. Then $H(t)$ can be decomposed as

$$H(t) = \int_{-\infty}^{t} AT(t - s)h_1(s)ds + \int_{-\infty}^{t} AT(t - s)h_2(s)ds =: H_1(t) + H_2(t).$$

Next we show that $H_1(t) \in AP_T(R, L^p(P, H))$ and $H_2(t) \in PAP_T^0(R, L^p(P, H), q)$. Thus, the following verification procedure is divided into three steps.

$Step$ 1. $H_1 \in UPC(R, L^p(P, H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By (H1), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{2\tilde{h}_1})^{1/p\beta}$ such that $0 < t' - t'' < \xi$, we have

$$\| T(t' - t'') - I \|^p \leq \frac{\tilde{\delta}_1 \varepsilon}{2\tilde{h}_1},$$

where $\tilde{h}_1 = 2^{p-1}M_{1-\beta}^p(1 - \frac{p(1-\beta)}{p-1})^{1-p} \| h_1 \|_{\beta,\infty}^p$, $\tilde{\delta}_1 = (\Gamma(1 - \frac{p(1-\beta)}{p-1}))^{p-1}\delta^{-p\beta}$.
Using Hölder's inequality, we have

$$E \| H_1(t') - H_1(t'') \|^p$$

$$\leq 2^{p-1}E \left\| \int_{-\infty}^{t''} AT(t'' - s)[T(t' - t'') - I]h_1(s)ds \right\|^p$$

$$+ 2^{p-1}E \left\| \int_{t''}^{t'} AT(t' - s)h_1(s)ds \right\|^p$$

$$\leq 2^{p-1}M_{1-\beta}^p \| T(t' - t'') - I \|^p$$

$$\times \left(\int_{-\infty}^{t''} (t'' - s)^{-\frac{p}{p-1}(1-\beta)}e^{-\delta(t''-s)}ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}E \| A^\beta h_1(s) \|^p ds \right)$$

$$+ 2^{p-1}M_{1-\beta}^p \left(\int_{t''}^{t'} (t' - s)^{-\frac{p}{p-1}(1-\beta)}e^{-\delta(t'-s)}ds \right)^{p-1}$$

$$\times \left(\int_{t'}^{t'} e^{-\delta(t'-s)}E \| A^\beta g_1(s) \|^p ds \right)$$

$$\leq 2^{p-1}M_{1-\beta}^p \| T(t' - t'') - I \|^p$$

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$$\times \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \right)^{p-1} \frac{1}{\delta} \sup_{s \in R} E \parallel h_1(s) \parallel_\beta^p$$

$$+ 2^{p-1} M_{1-\beta}^p \left(\int_{t''}^{t'} (t'-s)^{-\frac{p}{p-1}(1-\beta)} \right)^{p-1} (t'-t'') \sup_{s \in R} E \parallel h_1(s) \parallel_\beta^p$$

$$< 2^{p-1} M_{1-\beta}^p \parallel h_1 \parallel_{\beta,\infty}^p \frac{\tilde{\delta}_1 \varepsilon}{2\tilde{h}_1} \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \right)^{p-1} \delta^{-p\beta}$$

$$+ 2^{p-1} M_{1-\beta}^p \left(1 - \frac{p(1-\beta)}{p-1} \right)^{1-p} \parallel h_1 \parallel_{\beta,\infty}^p \left[\left(\frac{\varepsilon}{2\tilde{h}_1} \right)^{\frac{1}{p\beta}} \right]^{p\beta}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, $H_1 \in UPC(R, L^p(P, H))$.

Step 2. $H_1 \in AP_T(R, L^p(P, H))$.

Let $t_i < t \leq t_{i+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of R formed by ε-periods of F. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \gamma/2\}$, we have

$$E \parallel H_1(t+\tau) - H_1(t) \parallel^p$$

$$\leq E \left\| \int_{-\infty}^t A^{1-\beta} T(t-s) [A^\beta h_1(s+\tau) - A^\beta h_1(s)] ds \right\|^p$$

$$\leq M_{1-\beta}^p \left(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t-s)} ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^t e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \right)$$

$$\leq M_{1-\beta}^p \left(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t-s)} ds \right)^{p-1}$$

$$\times \left[\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \right.$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds$$

$$+ \left. \int_{t_i}^t e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \right].$$

Since $h_1 \in AP_T(R, L^p(P, H_\beta))$, one has

$$E \parallel A^\beta h_1(t+\tau) - A^\beta h_1(t) \parallel^p < \varepsilon$$

for all $t \in [t_j + \eta, t_{j+1} - \eta]$, $j \in Z, j \leq i$, and $t - s \geq t - t_i + t_i - (t_{j+1} - \eta) \geq$

$t - t_i + \gamma(i - 1 - j) + \eta$. Then,

$$\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p$$

$$\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} ds$$

$$\leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_{j+1}+\eta)}$$

$$\leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta\gamma(i-j-1)}$$

$$\leq \frac{\varepsilon}{\delta(1 - e^{-\delta\gamma})},$$

$$\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds$$

$$\leq 2^{p-1} \sup_{s \in R} E \parallel A^\beta h_1(s) \parallel^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} ds$$

$$\leq 2^{p-1} \parallel h_1 \parallel_{\beta,\infty}^p \varepsilon e^{\delta\eta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_j)}$$

$$\leq 2^{p-1} \parallel h_1 \parallel_{\beta,\infty}^p \varepsilon e^{\delta\eta} e^{-\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\delta\gamma(i-j)}$$

$$\leq \frac{2^{p-1} \parallel h_1 \parallel_{\beta,\infty}^p e^{\delta\gamma/2} \varepsilon}{1 - e^{-\delta\gamma}}.$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \leq \tilde{M}_1 \varepsilon,$$

$$\int_{t_i}^{t} e^{-\delta(t-s)} E \parallel A^\beta h_1(s+\tau) - A^\beta h_1(s) \parallel^p ds \leq \tilde{M}_2 \varepsilon,$$

where $\tilde{M}_1, \tilde{M}_2$ are some positive constants. Therefore, we get that $E \parallel H_1(t + \tau) - H_1(t) \parallel^p \leq \bar{N}_1 \varepsilon$ for a positive constant $\bar{N}_1$. Hence, $H_1 \in AP_T(R, L^p(P, H))$.

Step 3. $H_2 \in PAP_T^0(R, L^p(P, H), q)$.

In fact, for $r > 0$, one has

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \parallel H_2(\theta) \parallel^p dt$$

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$$= \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{-\infty}^{\theta} AT(\theta - s)h_2(s)ds \right\|^p dt$$

$$= \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{0}^{\infty} A^{1-\beta}T(s)A^{\beta}h_2(\theta - s)ds \right\|^p dt$$

$$\leq M_{1-\beta}^p \frac{1}{2r} \int_{-r}^{r} \left(\int_{0}^{\infty} s^{-(1-\beta)} e^{-\delta s} ds \right)^{p-1}$$

$$\times \int_{0}^{\infty} e^{-\delta s} \sup_{\theta \in [t-q,t]} E \| A^{\beta}h_2(\theta - s) \|^p \, dsdt$$

$$= M_{1-\beta}^p \left(\int_{0}^{\infty} s^{-(1-\beta)} e^{-\delta s} ds \right)^{p-1}$$

$$\times \int_{0}^{\infty} e^{-\delta s} ds \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \| A^{\beta}h_2(\theta - s) \|^p \, dt.$$

Since $h_2 \in PAP_T^0(R, L^p(P, H_\beta), q)$, it follows that $h_2(\cdot - s) \in PAP_T^0(R, L^p(P, H_\beta), q)$ for each $s \in R$ by Remark 2.1, hence

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{-\infty}^{\theta} AT(\theta - s)h_2(s)ds \right\|^p dt \to 0 \text{ as } r \to \infty$$

for all $s \in R$. Using the Lebesgue's dominated convergence theorem, we have $H_2 \in PAP_T^0(R, L^p(P, H), q)$. This completes the proof.

Lemma 3.2. Assume that (H1) holds. If $g \in PAP_T(R, L^p(P, H), q)$ and if G is the function defined by

$$G(t) := \int_{-\infty}^{t} T(t - s)g(s)ds$$

for each $t \in R$, then $G \in PAP_T(R, L^p(P, H), q)$.

Proof. Since $g \in PAP_T(R, L^p(P, H), q)$, there exist $g_1 \in AP_T(R, L^p(P, H))$ and $g_2 \in PAP_T^0(R, L^p(P, H), q)$, such that $g = g_1 + g_2$. Then $G(t)$ can be decomposed as

$$G(t) = \int_{-\infty}^{t} T(t - s)g_1(s)ds + \int_{-\infty}^{t} T(t - s)g_2(s)ds =: G_1(t) + G_2(t).$$

Next we show that $G_1(t) \in AP_T(R, L^p(P, H))$ and $G_2(t) \in PAP_T^0(R, L^p(P, H), q)$. Thus, the following verification procedure is divided into three steps.

Step 1. $G_1 \in UPC(R, L^p(P, H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By (H1), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{2\tilde{g}_1})^{1/p}$ such that $0 < t' - t'' < \xi$, we have

$$\| T(t' - t'') - I \|^p \leq \frac{\tilde{\delta}_1 \varepsilon}{2\tilde{g}_1},$$

where $\tilde{g}_1 = 2^{p-1} M^p \parallel g_1 \parallel_\infty^p$, $\tilde{\delta}_2 = \delta^{-p}$. Using Hölder's inequality, we have

$$E \parallel G_1(t') - G_1(t'') \parallel^p$$

$$\leq 2^{p-1} E \left\| \int_{-\infty}^{t''} T(t'' - s)[T(t' - t'') - I]g_1(s)ds \right\|^p$$

$$+ 2^{p-1} E \left\| \int_{t''}^{t'} T(t' - s)g_1(s)ds \right\|^p$$

$$\leq 2^{p-1} M^p \parallel T(t' - t'') - I \parallel^p \left(\int_{-\infty}^{t''} e^{-\delta(t''-s)}ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^{t''} e^{-\delta(t''-s)} E \parallel g_1(s) \parallel^p ds \right)$$

$$+ 2^{p-1} M^p \left(\int_{t''}^{t'} e^{-\delta(t'-s)}ds \right)^{p-1} \left(\int_{t''}^{t'} e^{-\delta(t'-s)} E \parallel g_1(s) \parallel^p ds \right)$$

$$\leq 2^{p-1} M^p \parallel T(t' - t'') - I \parallel^p \frac{1}{\delta^p} \sup_{s \in R} E \parallel g_1(s) \parallel^p$$

$$+ 2^{p-1} M^p (t' - t'')^p \sup_{s \in R} E \parallel g_1(s) \parallel^p$$

$$< 2^{p-1} M^p \parallel g_1 \parallel_\infty^p \frac{\tilde{\delta}_2 \varepsilon}{2\tilde{g}_1} \frac{1}{\delta^p} + 2^{p-1} M^p \parallel g_1 \parallel_\infty^p \left[\left(\frac{\varepsilon}{2\tilde{g}_1} \right)^{\frac{1}{p}} \right]^p$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, $G_1 \in UPC(R, L^p(P, H))$.

Step 2. $G_1 \in AP_T(R, L^p(P, H))$.

Let $t_i < t \leq t_{i+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of R formed by ε-periods of F. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \gamma/2\}$, we have

$$E \parallel G_1(t + \tau) - G_1(t) \parallel^p$$

$$\leq E \left\| \int_{-\infty}^{t} T(t - s)[g_1(s + \tau) - g_1(s)]ds \right\|^p$$

$$\leq M^p \left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^{t} e^{-\delta(t-s)} E \parallel g_1(s + \tau) - g_1(s) \parallel^p ds \right)$$

$$\leq M^p \left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds \right)^{p-1}$$

$$\times \left[\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} E \parallel g_1(s + \tau) - g_1(s) \parallel^p ds \right.$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} E \parallel g_1(s + \tau) - g_1(s) \parallel^p ds$$

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$$+ \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)} E \parallel g_1(s+\tau) - g_1(s) \parallel^p ds$$

$$+ \int_{t_i}^{t} e^{-\delta(t-s)} E \parallel g_1(s+\tau) - g_1(s) \parallel^p ds \Big].$$

Since $g_1 \in AP_T(R, L^p(P, H))$, one has

$$E \parallel g_1(t+\tau) - g_1(t) \parallel^p < \varepsilon$$

for all $t \in [t_j + \eta, t_{j+1} - \eta]$, $j \in Z, j \leq i$, and $t - s \geq t - t_i + t_i - (t_{j+1} - \eta) \geq t - t_i + \gamma(i-1-j) + \eta$. Then,

$$\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} E \parallel g_1(s+\tau) - g_1(s) \parallel^p$$

$$\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} ds$$

$$\leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_{j+1}+\eta)}$$

$$\leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta\gamma(i-j-1)}$$

$$\leq \frac{\varepsilon}{\delta(1-e^{-\delta\gamma})},$$

$$\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} E \parallel g_1(s+\tau) - g_1(s) \parallel^p ds$$

$$\leq 2^{p-1} \sup_{s\in R} E \parallel g_1(s) \parallel^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} ds$$

$$\leq 2^{p-1} \parallel g_1 \parallel_\infty^p \varepsilon e^{\delta\eta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_j)}$$

$$\leq 2^{p-1} \parallel g_1 \parallel_\infty^p \varepsilon e^{\delta\eta} e^{-\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\delta\gamma(i-j)}$$

$$\leq \frac{2^{p-1} \parallel g_1 \parallel_\infty^p e^{\delta\gamma/2} \varepsilon}{1-e^{-\delta\gamma}}.$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)} E \parallel g_1(s+\tau) - g_1(s) \parallel^p ds \leq \tilde{M}_3 \varepsilon,$$

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$$\int_{t_i}^{t} e^{-\delta(t-s)} E \parallel g_1(s+\tau) - g_1(s) \parallel^p ds \le \tilde{M}_4 \varepsilon,$$

where $\tilde{M}_3, \tilde{M}_4$ are some positive constants. Therefore, we get that $E \parallel G_1(t + \tau) - G_1(t) \parallel^p \le \bar{N}_2 \varepsilon$ for a positive constant $\bar{N}_2$. Hence, $G_1 \in AP_T(R, L^p(P, H))$.

Step 3. $G_2 \in PAP_T^0(R, L^p(P, H), q)$.

In fact, for $r > 0$, one has

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \parallel G_2(\theta) \parallel^p dt$$

$$= \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{-\infty}^{\theta} T(\theta - s) g_2(s) ds \right\|^p dt$$

$$= \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{0}^{\infty} T(s) g_2(\theta - s) ds \right\|^p dt$$

$$\le M^p \frac{1}{2r} \int_{-r}^{r} \left(\int_{0}^{\infty} e^{-\delta s} ds \right)^{p-1}$$

$$\times \int_{0}^{\infty} e^{-\delta s} \sup_{\theta \in [t-q,t]} E \parallel g_2(\theta - s) \parallel^p ds dt$$

$$= M^p \left(\int_{0}^{\infty} e^{-\delta s} ds \right)^{p-1} \int_{0}^{\infty} e^{-\delta s} ds$$

$$\times \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \parallel g_2(\theta - s) \parallel^p dt.$$

Since $g_2 \in PAP_T^0(R, L^p(P, H_\beta), q)$, it follows that $g_2(\cdot - s) \in PAP_T^0(R, L^p(P, H), q)$ for each $s \in R$ by Remark 2.1, hence

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-p,t]} E \left\| \int_{-\infty}^{\theta} T(\theta - s) g_2(s) ds \right\|^p dt \to 0 \text{ as } r \to \infty$$

for all $s \in R$. Using the Lebesgue's dominated convergence theorem, we have $G_2 \in PAP_T^0(R, L^p(P, H), q)$. This completes the proof.

Lemma 3.3. Assume that (H1) holds. If $f \in PAP_T(R, L^p(P, L_2^0), q)$ and if F is the function defined by

$$F(t) := \int_{-\infty}^{t} T(t - s) f(s) ds$$

for each $t \in R$, then $F \in PAP_T(R, L^p(P, H), q)$.

Proof. Since $f \in PAP_T(R, L^p(P, L_2^0))$, there exist $f_1 \in AP_T(R, L^p(P, L_2^0))$ and $f_2 \in PAP_T^0(R, L^p(P, L_2^0), q)$, such that $f = f_1 + f_2$. Hence,

$$F(t) = \int_{-\infty}^{t} T(t - s) f_1(s) dW(s)$$

$$+ \int_{-\infty}^{t} T(t - s) f_2(s) dW(s) =: F_1(t) + F_2(t).$$

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Next we show that $F_1(t) \in AP_T(R, L^p(P, H))$ and $F_2(t) \in PAP_T^0(R, L^p(P, H), q)$. Thus, the following verification procedure is divided into three steps.

$Step$ 1. $F_1 \in UPC(R, L^p(P, H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By (H4), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{2\tilde{f_1}})^{p/2(p-1)}$ such that $0 < t' - t'' < \xi$, we have for $p > 2$,

$$\| T(t' - t'') - I \|^p \leq \frac{\tilde{\delta_3}\varepsilon}{2\tilde{f_1}},$$

where $\tilde{f_1} = 2^{p-1}M^p C_p \| f_1 \|_\infty^p$, $\tilde{\delta_3} = (\frac{p\delta}{p-2})^{(p-2)/2}\frac{p\delta}{2}$. Using Hölder's inequality and the Ito integral [42], we have

$$E \| F_1(t') - F_1(t'') \|^p$$
$$\leq 2^{p-1}E \left\| \int_{-\infty}^{t''} T(t'' - s)[T(t' - t'') - I]f_1(s)dW(s) \right\|^p$$
$$+ 2^{p-1}E \left\| \int_{t''}^{t'} T(t' - s)f_1(s)dW(s) \right\|^p$$
$$\leq 2^{p-1}M^p C_p E \left[\int_{-\infty}^{t''} e^{-2\delta(t''-s)} \| T(t' - t'') - I \|^2 \right.$$
$$\left. \times \| f_1(s) \|_{L_2^0}^2 ds \right]^{p/2}$$
$$+ 2^{p-1}M^p C_p E \left[\int_{t''}^{t'} e^{-2\delta(t'-s)} \| f_1(s) \|_{L_2^0}^2 ds \right]^{p/2}$$
$$\leq 2^{p-1}M^p C_p \| T(t' - t'') - I \|^p \left(\int_{-\infty}^{t''} e^{-\frac{p}{p-2}\delta(t''-s)}ds \right)^{\frac{p-2}{p}}$$
$$\times \left(\int_{-\infty}^{t''} e^{-\frac{p}{2}\delta(t''-s)}ds \right) \sup_{s \in R} \| f_1(s) \|_{L_2^0}^p$$
$$+ 2^{p-1}M^p C_p \left(\int_{t''}^{t'} e^{-\frac{p}{p-2}\delta(t'-s)}ds \right)^{\frac{p-2}{p}}$$
$$\times \left(\int_{t''}^{t'} e^{-\frac{p}{2}\delta(t'-s)}ds \right) \sup_{s \in R} \| f_1(s) \|_{L_2^0}^p$$
$$\leq 2^{p-1}M^p C_p \| f_1 \|_\infty^p \frac{\tilde{\delta_3}\varepsilon}{2\tilde{f_1}} \left(\frac{p\delta}{p-2} \right)^{\frac{p-2}{p}} \frac{p\delta}{2}$$
$$+ 2^{p-1}M^p C_p \| f_1 \|_\infty^p \left[\left(\frac{\varepsilon}{2\tilde{f_1}} \right)^{p/2(p-1)} \right]^{2(p-2)/p}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $p = 2$. Let $\varepsilon > 0$, there exists $0 < \xi < \frac{\varepsilon}{2\tilde{f}_1}$ such that $0 < t' - t'' < \xi$, we have

$$\| T(t' - t'') - I \|^2 \leq \frac{2\delta\varepsilon}{\tilde{f}_1},$$

where $\tilde{f}_1 = 2M^2 \| f_1 \|^2_\infty$. Similar to the above discussion, one has

$E \| F_1(t') - F_1(t'') \|^2$

$$\leq 2M^2 \| T(t' - t'') - I \|^2 \left(\int_{-\infty}^{t''} e^{-2\delta(t''-s)} ds \right) \sup_{s \in R} \| f_1(s) \|^2_{L^0_2}$$

$$+ 2M^2 \left(\int_{t''}^{t'} e^{-2\delta(t'-s)} ds \right) \sup_{s \in R} \| f_1(s) \|^2_{L^0_2}$$

$$\leq 2M^2 \| f_1 \|^2_\infty \frac{2\delta\varepsilon}{2\tilde{f}_1} \left(\int_{-\infty}^{t''} e^{-2\delta(t''-s)} ds \right) + 2M^p \| f_1 \|^2_\infty \left(\frac{\varepsilon}{2\tilde{f}_1} \right)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, $F_1 \in UPC(R, L^p(P, H))$.

Step 2. $F_1 \in AP_T(R, L^p(P, H))$.

Let $t_i < t \leq t_{i+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of R formed by ε-periods of F. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \gamma/2\}$, we have

$E \| F_1(t + \tau) - F_1(t) \|^p$

$$= E \left\| \int_{-\infty}^{t} T(t - s)[f_1(s + \tau) - f_1(s)] dW(s) \right\|^p$$

$$\leq C_p E \left[\int_{-\infty}^{t} \| T(t - s) \|^2 \| f_1(s + \tau) - f_1(s) \|^2_{L^0_2} ds \right]^{p/2}$$

$$\leq C_p M^p E \left[\int_{-\infty}^{t} e^{-2\delta(t-s)} \| f_1(s + \tau) - f_1(s) \|^2_{L^0_2} ds \right]^{p/2}$$

$$\leq C_p M^p \left(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-s)} ds \right)^{\frac{p-2}{p}}$$

$$\times \left[\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|^p_{L^0_2} ds \right.$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|^p_{L^0_2} ds$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|^p_{L^0_2} ds$$

$$+ \left. \int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)} E \| f_1(s + \tau) - f_1(s) \|^p_{L^0_2} ds \right].$$

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Since $f_1 \in AP_T(R, L^p(P, L_2^0))$, one has

$$E \parallel f_1(t+\tau) - f_1(t) \parallel_{L_2^0}^p < \varepsilon$$

for all $t \in [t_j + \eta, t_{j+1} - \eta]$ and $j \in Z, j \leq i$. Then,

$$\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds$$

$$\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-s)} ds$$

$$\leq \frac{2}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta(t-t_{j+1}+\eta)}$$

$$\leq \frac{2\varepsilon}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\gamma(i-j-1)}$$

$$\leq \frac{2\varepsilon}{\delta p(1-e^{-\delta\gamma})},$$

$$\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds$$

$$\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}+\eta} e^{-\frac{p}{2}\delta(t-s)} ds$$

$$\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)}$$

$$\leq 2^{p-1} \sup_{s\in R} E \parallel f_1(s) \parallel_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\gamma(i-j)}$$

$$\leq \frac{2^{p-1} \parallel f_1 \parallel_\infty^p e^{\delta\gamma/4}\varepsilon}{1-e^{-\frac{p}{2}\delta\gamma}}.$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds \leq \tilde{M}_5\varepsilon,$$

$$\int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^p ds \leq \tilde{M}_6\varepsilon,$$

where $\tilde{M}_5, \tilde{M}_6$ are some positive constants. Therefore, we get that $E \parallel F_1(t+\tau) - F_1(t) \parallel^p \leq \bar{N}_3\varepsilon$ for a positive constant $\bar{N}_3$. For $p = 2$, we have

$$E \parallel F_1(t+\tau) - F_1(t) \parallel^2$$

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$$\leq M^2 E \int_{-\infty}^{t} e^{-2\delta(t-s)} \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds$$

$$\leq M^2 \Big[\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-2\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-2\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds$$

$$+ \int_{t_i}^{t} e^{-2\delta(t-s)} E \parallel f_1(s+\tau) - f_1(s) \parallel_{L_2^0}^2 ds \Big].$$

Similarly, we get that $E \parallel F_1(t+\tau) - F_1(t) \parallel^2 \leq \bar{N}_4 \varepsilon$ for a positive constant $\bar{N}_4$.
Hence, $F_1 \in AP_T(R, L^p(P, H))$.

$Step\ 3.$ $F_2 \in PAP_T^0(R, L^p(P, H), q)$.

In fact, for $r > 0$, one has for $p > 2$,

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \parallel F_2(\theta) \parallel^p dt$$

$$= \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{-\infty}^{\theta} T(\theta - s) f_2(s) dW(s) \right\|^p dt$$

$$= \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{0}^{\infty} T(s) f_2(\theta - s) dW(s) \right\|^p dt$$

$$\leq C_p \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left[\int_{0}^{\infty} e^{-2s} \parallel f_2(\theta - s) \parallel_{L_2^0}^2 ds \right]^{p/2} dt$$

$$\leq M^p C_p \frac{1}{2r} \int_{-r}^{r} \left(\int_{0}^{\infty} e^{-\frac{p}{p-2}\delta s} ds \right)^{\frac{p-2}{p}}$$

$$\times \int_{0}^{\infty} e^{-\frac{p}{2}\delta s} \sup_{\theta \in [t-q,t]} E \parallel f_2(\theta - s) \parallel_{L_2^0}^p ds dt$$

$$= M^p C_p \left(\int_{0}^{\infty} e^{-\frac{p-2}{p}\delta s} ds \right)^{\frac{p-2}{p}} \int_{0}^{\infty} e^{-\frac{p}{2}\delta s} ds$$

$$\times \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \parallel f_2(\theta - s) \parallel_{L_2^0}^p dt.$$

For $p = 2$, we have

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \int_{-\infty}^{\theta} T(\theta - s) f_2(s) dW(s) \right\|^2 dt$$

$$\leq M^2 \frac{1}{2r} \int_{-r}^{r} \int_{0}^{\infty} e^{-2s} \sup_{\theta \in [t-q,t]} E \parallel f_2(\theta - s) \parallel_{L_2^0}^2 ds dt$$

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$$= M^2 \left(\int_0^\infty e^{-2\delta s} ds \right) \frac{1}{2r} \int_{-r}^r \sup_{\theta \in [t-q,t]} E \parallel f_2(\theta - s) \parallel_{L_2^0}^p dt.$$

Since $f_2 \in PAP_T^0(R, L^p(P, L_2^0), q)$, it follows that $f_2(\cdot -s) \in PAP_T^0(R, L^p(P, L_2^0), q)$ for each $s \in R$ by Remark 2.1, hence

$$\frac{1}{2r} \int_{-r}^r \sup_{\theta \in [t-q,t]} E \left\| \int_{-\infty}^\theta T(\theta - s) f_2(s) dW(s) \right\|^p dt \to 0 \quad \text{as} \quad r \to \infty$$

for all $s \in R$. Using the Lebesgue's dominated convergence theorem, we have $F_2 \in PAP_T^0(R, L^p(P, H), q)$. This completes the proof.

Lemma 3.4. Assume that (H1) holds. If $\gamma_i \in PAP(Z, L^p(P, H)), i \in Z$ and if $\tilde{\gamma}_i$ is the function defined by

$$R_i(t) := \sum_{t_i < t} T(t - t_i) \gamma_i$$

for each $t \in R$, then $R_i \in PAP_T(R, L^p(P, H), q)$.

Proof. Since $\gamma_i \in PAP(Z, L^p(P, H))$, there exist $\gamma_{1,i} \in AP(Z, L^p(P, H))$ and $\gamma_{2,i} \in PAP_0(Z, L^p(P, H))$, such that $\gamma_i = \gamma_{1,i} + \gamma_{2,i}$. Hence,

$$R_i(t) = \sum_{t_i < t} T(t - t_i) \gamma_{1,i} + \sum_{t_i < t} T(t - t_i) \gamma_{2,i} =: \Pi_{1,i}(t) + \Pi_{2,i}(t).$$

Next we show that $\Pi_{1,i}(t) \in AP_T(R, L^p(P, H))$ and $\Pi_{2,i}(t) \in PAP_T^0(R, L^p(P, H), q)$. Thus, the following verification procedure is divided into three steps.

Step 1. $\Pi_{1,i} \in UPC(R, L^p(P, H))$.

Let $t', t'' \in (t_i, t_{i+1}), i \in Z, t'' < t'$. By (H4), for any $\varepsilon > 0$, we have

$$\parallel T(t' - t'') - I \parallel^p \leq \frac{(1 - e^{-\delta\gamma})^p \varepsilon}{\tilde{\gamma}_1},$$

where $\tilde{\gamma}_1 = M^p \parallel \gamma_{1,i} \parallel_\infty^p$. Using Hölder's inequality, we have

$$E \parallel \Pi_{1,i}(t') - \Pi_{1,i}(t'') \parallel^p$$

$$= E \left\| \sum_{t_i < t'} T(t' - t_i) \gamma_{1,i} - \sum_{t_i < t''} T(t'' - t_i) \gamma_{1,i} \right\|^p$$

$$= E \left\| \sum_{t_i < t''} T(t'' - t_i) [T(t' - t'') - I] \gamma_{1,i} \right\|^p$$

$$\leq M^p \parallel T(t' - t'') - I \parallel^p \left(\sum_{t_i < t''} e^{-\delta(t'' - t_i)} \right)^{p-1}$$

$$\times \left(\sum_{t_i < t''} e^{-\delta(t'' - t_i)} E \parallel \gamma_{1,i} \parallel^p \right)$$

$$\leq M^p \parallel T(t' - t'') - I \parallel^p \left(\sum_{t_i < t''} e^{-\delta(t'' - t_i)} \right)^p \sup_{i \in Z} E \parallel \gamma_{1,i} \parallel^p$$

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$$\leq M^p \frac{(1-e^{-\delta\gamma})^p \varepsilon}{\tilde{\gamma}_1} \left(\sum_{t_i < t''} e^{-\delta(t''-t_i)} \right)^p \parallel \gamma_{1,i} \parallel_\infty^p$$

$$< \varepsilon.$$

Consequently, $\Pi_{1,i} \in UPC(R, L^p(P, H))$.

$\quad$ *Step* 2. $\Pi_{1,i} \in AP_T(R, L^p(P, H))$.

$\quad$ Let $t_i < t \leq t_{i+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of R formed by ε-periods of F. For $\tau \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \gamma/2\}$. By Lemma 2.1, there exists relative dense sets of real numbers Ω_ε and integers Q_ε, for every $\tau \in \Omega_\varepsilon$, there exists at least one number $\tilde{q} \in Q_\varepsilon$ such that $|t^{\tilde{q}} - \tau| < \varepsilon, i \in Z$ and $E \parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel^p < \varepsilon, \tilde{q} \in Q_\varepsilon, i \in Z$. Then,

$$E \parallel \Pi_{1,i}(t+\tau) - \Pi_{1,i}(t) \parallel^p$$

$$= E \left\| \sum_{t_i < t+\tau} T(t+\tau - t_i)\gamma_{1,i} - \sum_{t_i < t} T(t-t_i)\gamma_{1,i} \right\|^p$$

$$\leq E \left[\sum_{t_i < t} \parallel T(t-t_i) \parallel \parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel \right]^p$$

$$\leq M^p E \left[\left(\sum_{t_i < t} e^{-\delta(t-t_i)} \right)^{p-1} \left(\sum_{t_i < t} e^{-\delta(t-t_i)} \parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel^p \right) \right]$$

$$\leq M^p \left(\sum_{t_i < t} e^{-\delta(t-t_i)} \right)^p E \parallel \gamma_{1,i+\tilde{q}} - \gamma_{1,i} \parallel^p$$

$$\leq \frac{M^p \varepsilon}{(1-e^{-\delta\gamma})^p}.$$

Hence, $\Pi_{1,i} \in AP_T(R, L^p(P, H))$.

$\quad$ *Step* 3. $\Pi_{2,i} \in PAP_T^0(R, L^p(P, H), q)$.

$\quad$ For a given $i \in Z$, define the function $v(t)$ by $v(t) = T(t-t_i)\gamma_{2,i}, t_i < t \leq t_{i+1}$, then

$$\lim_{t\to\infty} \sup_{\theta\in[t-q,t]} E \parallel v(\theta) \parallel^p$$

$$= \lim_{t\to\infty} \sup_{\theta\in[t-q,t]} E \parallel T(\theta - t_i)\gamma_{2,i} \parallel^p$$

$$\leq \lim_{t\to\infty} M^p e^{-p\delta(t-t_i)} \sup_{i\in Z} E \parallel \gamma_{2,i} \parallel^p = 0.$$

Thus $v \in PC_T^0(R, L^p(P, H), q) \subset PAP_T^0(R, L^p(P, H), q)$. Define $v_j : R \to L^p(P, H)$ by

$$v_j(t) = T(t - t_{i-j})\gamma_{2,i-j}, t_i < t \leq t_{i+1}, j \in N.$$

So $v_j \in PAP_T^0(R, L^p(P, H), q)$. Moreover,

$$\sup_{\theta\in[t-q,t]} E \parallel v_j(\theta) \parallel^p$$

$$= \sup_{\theta\in[t-q,t]} E \parallel T(\theta - t_{i-j})\gamma_{2,i-j} \parallel^p$$

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$$\leq M^p e^{-p\delta(t-t_{i-j})} \sup_{i \in Z} E \parallel \gamma_{2,i} \parallel^p$$

$$\leq M^p e^{-p\delta(t-t_i)} e^{-p\delta\gamma j} \sup_{i \in Z} E \parallel \gamma_{2,i} \parallel^p .$$

Therefore, the series $\sum_{j=0}^{\infty} v_j$ is uniformly convergent on R. By Lemma 2.3, one has

$$\sum_{t_i < t} T(t - t_i)\gamma_{2,i} = \sum_{j=0}^{\infty} v_j(t) \in PAP_T^0(R, L^p(P, H), q),$$

that is

$$\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [t-q,t]} E \left\| \sum_{t_i < t} T(\theta - t_i)\gamma_{2,i} \right\|^p dt \to 0 \text{ as } r \to \infty.$$

Using the Lebesgue's dominated convergence theorem, we have $\Pi_{2,i} \in PAP_T^0(R, L^p(P, H), q)$ This completes the proof.

Lemma 3.5. If $x \in PAP_T(R, L^p(P, H), q)$, then $t \to x_t$ belongs to $PAP_T(R, \mathcal{D}, q)$.

One can refer to Lemma 3.3 in [4] for the proof of Lemma 3.5.

Now, we establish the existence theorem of p-mean piecewise pseudo almost periodic mild solutions to partial impulsive stochastic differential equation (1)-(2).

Theorem 3.1. Assume that assumptions (H1)-(H6) are satisfied. Then system (1)-(2) has a mild solution $x \in PAP_T(R, L^p(P, H), q)$.

Proof. Let $Y = PAP_T(R, L^p(P, H), q) \cap UPC(R, L^p(P, H))$. Consider the operator $\Psi : Y \to PC(R, L^p(P, H))$ defined by

$$(\Psi x)(t) = \left[h(t, x_t) + \int_{-\infty}^{t} AT(t - s)h(s, x_s)ds \right]$$

$$+ \left[\int_{-\infty}^{t} T(t - s)g(s, x_s)ds + \int_{-\infty}^{t} T(t - s)f(s, x_s)dW(s) \right.$$

$$+ \left. \sum_{t_i < t} T(t - t_i)I_i(x(t_i)) \right] =: (\Psi_1 x)(t) + (\Psi_2 x)(t), \quad t \in R.$$

Obviously, the operator $\Psi_1 + \Psi_2$ has a fixed point if and only if operator Ψ has a fixed point in Y. To prove which we shall employ Lemma 2.8, we divide the proof into several steps.

Step 1. For every $x \in Y$, $\Psi x \in Y$.

Let $x(\cdot) \in Y$, by (H2), (H3), (H5) and Lemmas 2.4, 2.5, we deduce that $h(\cdot, x.), g(\cdot, x.), f(\cdot, x.) \in PAP_T(R, L^p(P, H), q)$ and $I_i(x(t_i)) \in PAP(Z, L^p(P, H))$. Similarly as the proof of Lemmas 3.1-3.5, one has $\Psi x \in Y$.

Step 2. For a closed bounded convex subset B_{r^*} of Y, $\Psi_1 x + \Psi_2 y \in B_{r^*}$, when $x, y \in B_{r^*}$.

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Let $\rho_1, \rho_2 > 0$ be fixed. By (H4) and (H6) it follows that there exist a positive constant μ such that, for all $t \in R$ and $\psi, x \in Y$ with $E \parallel \psi \parallel^p_{\mathcal{D}} > \mu, E \parallel x \parallel^p > \mu$,

$$E \parallel g(t, \psi) \parallel^p + E \parallel f(t, \psi) \parallel^p_{L^0_2} \leq \rho_1 \Theta(E \parallel \psi \parallel^p_{\mathcal{D}}),$$

$$E \parallel I_i(x) \parallel^p \leq \rho_2 \tilde{\Theta}_i(E \parallel x \parallel^p), i \in Z.$$

Let

$$\nu = \sup_{t \in R} \{ E \parallel g(t, \psi) \parallel^p, E \parallel f(t, \psi) \parallel^p_{L^0_2} : E \parallel \psi \parallel^p_{\mathcal{D}} \leq \mu \},$$

$$\nu_1 = \sup_{t \in R, i \in Z} \{ E \parallel I_i(x) \parallel^p : E \parallel x \parallel^p \leq \mu \}.$$

Thus, we have for all $t \in R$,

$$E \parallel g(t, \psi) \parallel^p + E \parallel f(t, \psi) \parallel^p_{L^0_2} \leq \rho_1 \Theta(E \parallel \psi \parallel^p) + \nu, \quad \psi \in \mathcal{D}, \qquad (5)$$

$$E \parallel I_i(x) \parallel^p \leq \rho_2 \tilde{\Theta}_i(E \parallel x \parallel^p) + \nu_1, \quad x \in L^p(P, H), i \in Z. \qquad (6)$$

By (H2), (5), (6), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$E \parallel (\Psi_1 x)(t) + (\Psi_2 y)(t) \parallel^p$$

$$\leq 5^{p-1} E \parallel h(t, x_t) \parallel^p + 5^{p-1} E \left\| \int_{-\infty}^t AT(t-s) h(s, x_s) ds \right\|^p$$

$$+ 5^{p-1} E \left\| \int_{-\infty}^t T(t-s) g(s, y_s) ds \right\|^p$$

$$+ 5^{p-1} E \left\| \int_{-\infty}^t T(t-s) f(s, y_s) dW(s) \right\|^p$$

$$+ 5^{p-1} E \left\| \sum_{t_i < t} T(t-t_i) I_i(y(t_i)) \right\|^p$$

$$\leq 5^{p-1} \parallel A^{-\beta} \parallel E \parallel A^\beta h(s, x_s) \parallel^p$$

$$+ 5^{p-1} M^p_{1-\beta} \left(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t-s)} ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^t e^{-\delta(t-s)} E \parallel A^\beta h(s, x_s) \parallel^p ds \right)$$

$$+ 5^{p-1} M^p \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^t e^{-\delta(t-s)} E \parallel g(s, y_s) \parallel^p ds \right)$$

$$+ 5^{p-1} C_p M^p E \left(\int_{-\infty}^t e^{-2\delta(t-s)} \parallel f(s, y_s) \parallel^2_{L^0_2} ds \right)^{p/2}$$

$$+ 5^{p-1} M^p E \left[\left(\sum_{t_i < t} e^{-\delta(t-t_i)} \right)^{p-1} \left(\sum_{t_i < t} e^{-\delta(t-t_i)} \parallel I_i(y(t_i)) \parallel^p \right) \right]$$

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$$\leq 5^{p-1} \parallel A^{-\beta} \parallel L(\parallel x_t \parallel_{\mathcal{D}}^p + 1)$$

$$+ 5^{p-1} M_{1-\beta}^p \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \right)^{p-1}$$

$$\times \left(\int_{-\infty}^t e^{-\delta(t-s)} L(\parallel x_s \parallel_{\mathcal{D}}^p + 1) ds \right)$$

$$+ 5^{p-1} M^p \frac{1}{\delta^{p-1}} \left(\int_{-\infty}^t e^{-\delta(t-s)} [\rho_1 \Theta(E \parallel y_s \parallel_{\mathcal{D}}^p) + \nu] ds \right)$$

$$+ 5^{p-1} M^p C_p \left(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)} ds \right)^{\frac{p-2}{p}}$$

$$\times \left(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)} [\rho_1 \Theta(E \parallel y_s \parallel_{\mathcal{D}}^p) + \nu] ds \right)$$

$$+ 5^{p-1} M^p \frac{1}{(1 - e^{-\delta\gamma})^{p-1}} \left(\sum_{t_i < t} e^{-\delta(t-t_i)} [\rho_2 \tilde{\Theta}_i(E \parallel y(t_i) \parallel^p) + \nu_1] \right)$$

$$\leq 5^{p-1} \parallel A^{-\beta} \parallel L(\parallel x \parallel_\infty^p + 1)$$

$$+ 5^{p-1} M_{1-\beta}^p \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \right)^{p-1} \delta^{p\beta} L(\parallel x \parallel_\infty^p + 1)$$

$$+ 5^{p-1} M^p \frac{1}{\delta^p} [\rho_1 \Theta(\parallel y \parallel_\infty^p) + \nu]$$

$$+ 5^{p-1} M^p C_p \left(\frac{p-2}{p\delta} \right)^{\frac{p-2}{p}} \frac{2}{p\delta} [\rho_1 \Theta(\parallel y \parallel_\infty^p) + \nu]$$

$$+ 5^{p-1} M^p \frac{1}{(1 - e^{-\delta\gamma})^p} [\rho_2 \sup_{i \in Z} \tilde{\Theta}_i(\parallel y \parallel_\infty^p) + \nu_1].$$

For $p = 2$, we have

$$E \parallel (\Psi_1 x)(t) + (\Psi_2 y)(t) \parallel^2$$
$$\leq 5 \parallel A^{-\beta} \parallel L(\parallel x \parallel_\infty^p + 1)$$
$$+ 5 M_{1-\beta}^2 (\Gamma(1 - 2(1-\beta)) \delta^{2\beta} L(\parallel x \parallel_\infty^p + 1)$$
$$+ 5 M^2 \frac{1}{\delta^2} [\beta \Theta(\parallel y \parallel_\infty^p) + \nu] + 5 M^2 \frac{1}{2\delta} [\beta \Theta(\parallel y \parallel_\infty^p) + \nu]$$
$$+ 5 M^2 \frac{1}{(1 - e^{-\delta\alpha})^2} [\beta \sup_{i \in Z} \tilde{\Theta}_i(\parallel y \parallel_\infty^p) + \nu_1].$$

Note that, for ρ_1, ρ_2 sufficiently small, we can choose $r^* > 0$ such that for $p > 2$,

$$5^{p-1} \parallel A^{-\beta} \parallel L(r^* + 1)$$

$$+ 5^{p-1} M_{1-\beta}^p \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \right)^{p-1} \delta^{p\beta} L(r^* + 1)$$

$$+ 5^{p-1} M^p \frac{1}{\delta^p} [\rho_1 \Theta(r^*) + \nu]$$

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$$+5^{p-1}M^pC_p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\rho_1\Theta(r^*)+\nu]$$

$$+5^{p-1}M^p\frac{1}{(1-e^{-\delta\gamma})^p}[\rho_2\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]\leq r^*, \tag{7}$$

and for $p=2$, we have

$$5\parallel A^{-\beta}\parallel L(r^*+1)$$

$$+5M_{1-\beta}^2(\Gamma(1-2(1-\beta))\delta^{2\beta}L(r^*+1)$$

$$+5M^2\frac{1}{\delta^2}[\beta\Theta(r^*)+\nu]+5M^2\frac{1}{2\delta}[\beta\Theta(r^*)+\nu]$$

$$+5M^2\frac{1}{(1-e^{-\delta\alpha})^2}[\rho_2\sup_{i\in Z}\tilde{\Theta}_i(r^*)+\nu_1]\leq r^*. \tag{8}$$

Let $B_{r^*}=\{x\in Y:\parallel x\parallel_\infty^p\leq r^*\}$ for $r^*>0$. It is easy to see that B_{r^*} is a closed bounded convex subset of Y. Moreover, for all $x,y\in B_{r^*}$,

$$E\parallel(\Psi_1 x)(t)+(\Psi_2 y)(t)\parallel^p\leq r^*.$$

Therefore, $\Psi_1 x+\Psi_2 y\in B_{r^*}$, when $x,y\in B_{r^*}$.

Step 3. Ψ_1 is a contraction.

For $t\in R$, and $x^*,x^{**}\in B_r$. From (H2) and Lemma 2.6, we have

$$E\parallel(\Psi_1 x^*)(t)-(\Psi_1 x^{**})(t)\parallel^p$$

$$\leq 2^{p-1}E\parallel h(t,x_t^*)-h(t,x_t^{**})\parallel^p$$

$$+2^{p-1}E\left\|\int_{-\infty}^t AT(t-s)[h(s,x_s^*)-h(s,x_s^{**})]ds\right\|^p$$

$$\leq 2^{p-1}\parallel A^{-\beta}\parallel E\parallel A^\beta h(t,x_t^*)-A^\beta h(t,x_t^{**})\parallel^p$$

$$+2^{p-1}M_{1-\beta}^p\left(\int_{-\infty}^t(t-s)^{-\frac{p}{p-1}(1-\beta)}e^{-\delta(t-s)}ds\right)^{p-1}$$

$$\times\left(\int_{-\infty}^t e^{-\delta(t-s)}E\parallel A^\beta h(s,x_s^*)-A^\beta h(s,x_s^{**})\parallel^p ds\right)$$

$$\leq 2^{p-1}\parallel A^{-\beta}\parallel L\parallel x_t^*-x_t^{**}\parallel_{\mathcal{D}}^p$$

$$+2^{p-1}M_{1-\beta}^p\left(\Gamma(1-\frac{p(1-\beta)}{p-1})\delta^{\frac{p(1-\beta)}{p-1}-1}\right)^{p-1}$$

$$\times\left(\int_{-\infty}^t e^{-\delta(t-s)}L\parallel x_s^*-x_s^{**}\parallel_{\mathcal{D}}^p ds\right)$$

$$\leq L_0\parallel x^*-x^{**}\parallel_\infty^p.$$

Taking supremum over t,

$$\parallel\Psi_1 x^*-\Psi_1 x^{**}\parallel_\infty^p\leq L_0\parallel x^*-x^{**}\parallel_\infty^p.$$

where $L_0=2^{p-1}[\parallel A^{-\beta}\parallel+M_{1-\beta}^p(\Gamma(1-\frac{p(1-\beta)}{p-1}))^{p-1}\delta^{p\beta}]L$. By (7), we see that $L_0<1$. Hence, Ψ_1 is a contractive operator with constant L_0.

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Step 4. Ψ_2 maps B_{r^*} into an equicontinuous family.

Let $\tau_1, \tau_2 \in (t_i, t_{i+1}), i \in Z, \tau_1 < \tau_2$, and $x \in B_{r^*}$. Then, by (H1), (5), (6), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$E \parallel (\Psi_2 x)(\tau_2) - (\Psi_2 x)(\tau_1) \parallel^p$$

$$\leq 6^{p-1} E \left\| \int_{-\infty}^{\tau_1} T(\tau_1 - s)[T(\tau_2 - \tau_1) - I]g(s, x_s)ds \right\|^p$$

$$+ 6^{p-1} E \left\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s)g(s, x_s)ds \right\|^p$$

$$+ 6^{p-1} E \left\| \int_{-\infty}^{\tau_1} T(\tau_1 - s)[T(\tau_2 - \tau_1) - I]f(s, x_s)dW(s) \right\|^p$$

$$+ 6^{p-1} E \left\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s)f(s, x_s)dW(s) \right\|^p$$

$$+ 3^{p-1} E \left\| \sum_{t_i < \tau_1} T(\tau_1 - t_i)[T(\tau_2 - \tau_1) - I]I_i(x(t_i)) \right\|^p$$

$$\leq 6^{p-1} M^p \parallel T(\tau_2 - \tau_1) - I \parallel^p \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)}ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} E \parallel g(s, x_s) \parallel^p ds \right)$$

$$+ 6^{p-1} M^p \left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)}ds \right)^{p-1}$$

$$\times \left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} E \parallel g(s, x_s) \parallel^p ds \right)$$

$$+ 6^{p-1} M^p C_p E \left[\int_{-\infty}^{\tau_1} e^{-2\delta(\tau_1 - s)} \parallel T(\tau_2 - \tau_1) - I \parallel^2 \right.$$

$$\times \parallel f(s, x_s) \parallel_{L_2^0}^2 ds \Big]^{p/2}$$

$$+ 6^{p-1} M^p C_p E \left[\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2 - s)} \parallel f(s, x_s) \parallel_{L_2^0}^2 ds \right]^{p/2}$$

$$+ 3^{p-1} M^p \parallel T(\tau_2 - \tau_1) - I \parallel^p \left(\sum_{t_i < \tau_1} e^{-\delta(\tau_1 - t_i)} \right)^{p-1}$$

$$\times \left(\sum_{t_i < \tau_1} e^{-\delta(\tau_1 - t_i)} E \parallel I_i(x(t_i)) \parallel^p \right)$$

$$\leq 6^{p-1} M^p \parallel T(\tau_2 - \tau_1) - I \parallel^p \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)}ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} [\rho_1 \Theta(E \parallel x_s \parallel_{\mathcal{D}}^p) + \nu]ds \right)$$

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$$+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-s)}ds\right)^{p-1}$$

$$\times\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\right)$$

$$+6^{p-1}M^pC_p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\int_{-\infty}^{\tau_1}e^{-\frac{p}{p-2}\delta(\tau_1-s)}ds\right)^{\frac{p-2}{p}}$$

$$\times\left(\int_{-\infty}^{\tau_1}e^{-\frac{p}{2}\delta(\tau_1-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\right)$$

$$+6^{p-1}M^pC_p\left(\int_{\tau_1}^{\tau_2}e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\right)^{\frac{p-2}{p}}$$

$$\times\left(\int_{\tau_1}^{\tau_2}e^{-\frac{p}{2}\delta(\tau_2-s)}[\rho_1\Theta(E\parallel x_s\parallel_{\mathcal{D}}^p)+\nu]ds\right)$$

$$+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\sum_{t_i<\tau_1}e^{-\delta(\tau_1-t_i)}\right)^{p-1}$$

$$\times\left(\sum_{t_i<\tau_1}e^{-\delta(\tau_1-t_i)}[\rho_2\tilde{\Theta}_i(E\parallel x(t_i)\parallel^p)+\nu_1]\right)$$

$$\leq 6^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{\delta^p}[\rho_1\Theta(r^*)+\nu]$$

$$+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-s)}ds\right)^p[\rho_1\Theta(r^*)+\nu]$$

$$+6^{p-1}M^pC_p\parallel T(\tau_2-\tau_1)-I\parallel^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}[\rho_1\Theta(r^*)+\nu]$$

$$+6^{p-1}M^pC_p\left(\int_{\tau_1}^{\tau_2}e^{-\frac{p}{p-2}\delta(\tau_2-s)}ds\right)^{\frac{p-2}{p}}$$

$$\times\left(\int_{\tau_1}^{\tau_2}e^{-\frac{p}{2}\delta(\tau_2-s)}ds\right)[\rho_1\Theta(r^*)+\nu]$$

$$+3^{p-1}M^p\parallel T(\tau_2-\tau_1)-I\parallel^p\frac{1}{(1-e^{-\delta\gamma})^p}[\rho_2\tilde{\Theta}_i(r^*)+\nu_1].$$

For $p=2$, we have

$$E\parallel(\Psi x)(\tau_2)-(\Psi x)(\tau_1))\parallel^2$$

$$\leq 6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{1}{\delta^2}[\rho_1\Theta(r^*)+\nu]$$

$$+6M^2\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-s)}ds\right)^2[\rho_1\Theta(r^*)+\nu]$$

$$+6M^2\parallel T(\tau_2-\tau_1)-I\parallel^2\frac{2}{\delta}[\rho_1\Theta(r^*)+\nu]$$

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$$+6M^2 \left(\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-s)} ds \right) [\rho_1 \Theta(r^*) + \nu]$$

$$+3M^2 \parallel T(\tau_2 - \tau_1) - I \parallel^2 \frac{1}{(1-e^{-\delta\gamma})^2} [\rho_2 \tilde{\Theta}_i(r^*) + \nu_1].$$

The right-hand side of the above inequality is independent of $x \in B_{r^*}$ and tends to zero as $\tau_2 \to \tau_1$, since the compactness of $T(t)$ for $t > 0$ implies imply the continuity in the uniform operator topology. Thus, Ψ maps B_{r^*} into an equicontinuous family of functions.

Step 5. $\Psi_2 B_{r^*}$ is precompact.

For each $t \in R$, and let ε be a real number satisfying $0 < \varepsilon < 1$. For $x \in B_{r^*}$, we define

$$(\Psi_{2,\varepsilon} x)(t)$$
$$= T(\varepsilon) \Bigg[\int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)g(s,x_s)ds$$
$$+ \int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)f(s,x_s)dW(s)$$
$$+ \sum_{t_i < t-\varepsilon} T(t-\varepsilon-t_i)I_i(x(t_i)) \Bigg]$$
$$= T(\varepsilon)[(\Psi_2 x)(t-\varepsilon)].$$

Since $T(t)(t > 0)$ is compact, then the set $V_\varepsilon(t) = \{(\Psi_{2,\varepsilon}x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^p(P,H)$ for each $t \in R$. Moreover, for every $x \in B_{r^*}$, we have for $p > 2$,

$$E \parallel (\Psi_2 x)(t) - (\Psi_{2,\varepsilon}x)(t) \parallel^p$$
$$\leq 3^{p-1} E \left\| \int_{t-\varepsilon}^{t} T(t-s)g(s,x_s)ds \right\|^p$$
$$+ 3^{p-1} E \left\| \int_{t-\varepsilon}^{t} T(t-s)f(s,x_s)dW(s) \right\|^p$$
$$+ 3^{p-1} E \left\| \sum_{t-\varepsilon < t_i < t} T(t-t_i)I_i(x(t_i)) \right\|^p$$
$$\leq 3^{p-1} M^p \left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)} ds \right)^{p-1} \left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)} E \parallel g(s,x_s) \parallel^p ds \right)$$
$$+ 3^{p-1} C_p M^p E \left(\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)} \parallel f(s,x_s) \parallel_{L_2^0}^2 ds \right)^{p/2}$$
$$+ 3^{p-1} M^p E \Bigg[\left(\sum_{t-\varepsilon < t_i < t} e^{-\delta(t-t_i)} \right)^{p-1}$$
$$\times \left(\sum_{t-\epsilon < t_i < t} e^{-\delta(t-t_i)} \parallel I_i(x(t_i)) \parallel^p \right) \Bigg]$$

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$$\leq 3^{p-1} M^p \left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)} ds \right)^{p-1}$$

$$\times \left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)} [\rho_1 \Theta(E \parallel x_s \parallel_{\mathcal{D}}^p) + \nu] ds \right)$$

$$+ 3^{p-1} C_p M^p \left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-s)} ds \right)^{\frac{p-2}{p}}$$

$$\times \left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-s)} [\rho_1 \Theta(E \parallel x_s \parallel_{\mathcal{D}}^p) + \nu] ds \right)$$

$$+ 3^{p-1} M^p \left(\sum_{t-\varepsilon < t_i < t} e^{-\delta(t-t_i)} \right)^{p-1}$$

$$\times \left(\sum_{t-\varepsilon < t_i < t} e^{-\delta(t-t_i)} [\rho_2 \tilde{\Theta}_i(E \parallel x_i(t_i) \parallel^p) + \nu_1] \right)$$

$$\leq 3^{p-1} M^p \left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)} ds \right)^{p} [\rho_1 \Theta(r^*) + \nu]$$

$$+ 3^{p-1} C_p M^p \left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-s)} ds \right)^{\frac{p-2}{p}}$$

$$\times \left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-s)} ds \right) [\rho_1 \Theta(r^*) + \nu]$$

$$+ 3^{p-1} M^p \left(\sum_{t-\varepsilon < t_i < t} e^{-\delta(t-t_i)} \right)^{p} [\rho_2 \sup_{i \in Z} \tilde{\Theta}_i(r^*) + \nu_1].$$

For $p = 2$, we have

$$E \parallel (\Psi_2 x)(t) - (\Psi_{2,\varepsilon} x)(t) \parallel^2$$

$$\leq 3M^2 \left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)} ds \right)^2 [\rho_1 \Theta(r^*) + \nu]$$

$$+ 3M^2 \left(\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)} ds \right) [\rho_1 \Theta(r^*) + \nu]$$

$$+ 3M^2 \left(\sum_{t-\varepsilon < t_i < t} e^{-\delta(t-t_i)} \right)^2 [\rho_2 \sup_{i \in Z} \tilde{\Theta}_i(r^*) + \nu_1].$$

Therefore, letting $\varepsilon \to 0$, it follows that there are relatively compact sets $V_\varepsilon(t)$ arbitrarily close to $V(t) = \{(\Psi_2 x)(t) : x \in B_{r^*}\}$, and hence $V(t)$ is also relatively compact in $L^p(P, H)$ for each $t \in R$. Since $\{\Psi_2 x : x \in B_{r^*}\} \subset PC_h^0(R, L^p(P, H), q)$, then $\{\Psi_2 x : x \in B_{r^*}\}$ is a relatively compact set by Lemma 2.7, then Ψ_2 is a compact operator.

Step 6. Ψ_2 is continuous.

Let $\{x^{(n)}\} \subseteq B_{r^*}$ with $x^{(n)} \to x(n \to \infty)$ in Y, then there exists a bounded subset $\bar{K} \subseteq L^p(P, K)$ such that $R(x) \subseteq \bar{K}, R(x^n) \subseteq \bar{K}, n \in N$. By the assumption (H2) and (H4), for any $\varepsilon > 0$, there exists $\xi > 0$ such that $x, y \in K$ and

$\| x - y \|_\infty^p < \xi$ implies that

$$E \| g(s, x_s) - g(s, y_s) \|^p < \varepsilon \quad \text{for all } t \in R,$$

$$E \| f(s, x_s) - f(s, y_s) \|_{L_2^0}^p < \varepsilon \quad \text{for all } t \in R,$$

and

$$E \| I_i(x) - I_i(y) \|^p < \varepsilon \quad \text{for all } i \in Z.$$

For the above ξ there exists n_0 such that $\| x^{(n)} - x \|_\infty^p < \varepsilon$ for $n > n_0$, then for $n > n_0$, we have

$$E \| g(s, x_s^{(n)}) - g(s, x_s) \|^p < \varepsilon \quad \text{for all } t \in R,$$

$$E \| f(s, x_s^{(n)}) - f(s, x_s) \|_{L_2^0}^p < \varepsilon \quad \text{for all } t \in R,$$

and

$$E \| I_i(x^{(n)}) - I_i(x) \|^p < \varepsilon \quad \text{for all } i \in Z.$$

Then, by Hölder's inequality, we have that for $p > 2$,

$$E \| (\Psi_2 x^{(n)})(t) - (\Psi_2 x)(t) \|^p$$

$$\leq 3^{p-1} E \left\| \int_{-\infty}^t T(t-s)[g(s, x_s^{(n)}) - g(s, x_s)] ds \right\|^p$$

$$+ 3^{p-1} E \left\| \int_{-\infty}^t T(t-s)[f(s, x_s^{(n)}) - f(s, x_s)] dW(s) \right\|^p$$

$$+ 3^{p-1} E \left\| \sum_{t_i < t} T(t-t_i)[I_i(x^{(n)}(t_i)) - I_i(x(t_i))] \right\|^p$$

$$\leq 3^{p-1} M^p \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right)^{p-1}$$

$$\times \left(\int_{-\infty}^t e^{-\delta(t-s)} E \| g(s, x_s^{(n)}) - g(s, x_s) \|^p \, ds \right)$$

$$+ 3^{p-1} C_p M^p \left(\int_{-\infty}^t e^{-2\delta(t-s)} E \| f(s, x_s^{(n)}) - f(s, x_s) \|_{L_2^0}^2 \, ds \right)^{p/2}$$

$$+ 3^{p-1} M^p E \left[\left(\sum_{t_i < t} e^{-\delta(t-t_i)} \right)^{p-1} \right.$$

$$\left. \times \left(\sum_{t_i < t} e^{-\delta(t-t_i)} \| I_i(x^{(n)}(t_i)) - I_i(x(t_i)) \|^p \right) \right]$$

$$\leq 3^{p-1} M^p \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right)^p \varepsilon$$

$$+ 3^{p-1} C_p M^p \left(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)} ds \right)^{\frac{p-2}{p}} \left(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)} ds \right) \varepsilon$$

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$$+3^{p-1}M^p\frac{1}{(1-e^{-\delta\gamma})^{p-1}}\left(\sum_{t_i<t}e^{-\delta(t-t_i)}\right)\varepsilon$$

$$\leq 3^{p-1}M^p\left[\frac{1}{\delta^p}+C_p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{2}}\frac{2}{p\delta}+\frac{1}{(1-e^{-\delta\gamma})^p}\right]\varepsilon.$$

For $p=2$, we have

$$E\parallel(\Psi_2 x^{(n)})(t)-(\Psi_2 x)(t)\parallel^2$$

$$\leq 3M^2\left[\frac{1}{\delta^2}+\frac{1}{2\delta}+\frac{1}{(1-e^{-\delta\gamma})^2}\right]\varepsilon.$$

Thus Ψ_2 is continuous on B_{r^*} and Ψ_2 is completely continuous.

Therefore, all the conditions of Lemma 2.8 are satisfied and thus operator Ψ has a fixed point x in B_{r^*}, which is in turn a mild solution of the system (1)-(2), that is

$$x(t)=h(t,x_t)+\int_{-\infty}^t AT(t-s)h(s,x_s)ds$$

$$+\int_{-\infty}^t T(t-s)g(s,x_s)ds+\int_{-\infty}^t T(t-s)f(s,x_s)dW(s)$$

$$+\sum_{t_i<t}T(t-t_i)I_i(x(t_i)),\quad t\in R.$$

Finally, to prove that x satisfies (4) for all $t\geq s$, all $s\in R$. Fix $\sigma,\sigma\neq t_i, i\in Z$, we have for $t\in[\sigma,\sigma+b), b>0$,

$$x(\sigma)=h(\sigma,x_\sigma)+\int_{-\infty}^\sigma AT(\sigma-s)h(s,x_s)ds$$

$$+\int_{-\infty}^\sigma T(\sigma-s)g(s,x_s)ds+\int_{-\infty}^\sigma T(\sigma-s)f(s,x_s)dW(s)$$

$$+\sum_{t_i<\sigma}T(\sigma-t_i)I_i(x(t_i))=\varphi(\sigma).$$

Since $\{T(t):t\geq 0\}$ is an analytic semigroup, we have for all $t\in[\sigma,\sigma+b)$,

$$x(t)$$

$$=h(t,x_t)+\int_{-\infty}^\sigma AT(t-s)h(s,x_s)ds+\int_{-\infty}^\sigma T(t-s)g(s,x_s)ds$$

$$+\int_{-\infty}^\sigma T(t-s)f(s,x_s)dW(s)+\sum_{t_i<\sigma}T(t-t_i)I_i(x(t_i))$$

$$+\int_\sigma^t AT(t-s)h(s,x_s)ds+\int_\sigma^t T(t-s)g(s,x_s)ds$$

$$+\int_\sigma^t T(t-s)f(s,x_s)dW(s)+\sum_{\sigma<t_i<t}T(t-t_i)I_i(x(t_i))$$

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$$= T(t-\sigma)[\varphi(\sigma) - h(\sigma,\varphi)] + h(t,x_t) + \int_\sigma^t T(t-s)g(s,x_s)ds$$

$$+ \int_\sigma^t T(t-s)f(s,x_s)dW(s) + \sum_{\sigma<t_i<t} T(t-t_i)I_i(x(t_i)).$$

Hence $x \in PAP_T(R, L^p(P,H), q)$ is an p-mean piecewise pseudo almost periodic mild solution to system (1)-(2). This completes the proof.

4   Global attractiveness

In this section, we present the global attractiveness of a piecewise pseudo almost periodic solution of (1)- (2). To do this, we also need the following assumptions:

(B1) There exist constants $0 < \beta < 1, l_j > 0, j = 1,2,$ such that

$$E \parallel A^\beta h(t,\psi) \parallel^p \leq l_1 \parallel \psi \parallel_{\mathcal{D}}^p, \quad t \in R, \psi \in \mathcal{D},$$

$$E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel_{L_2^0}^p \leq l_2 \parallel \psi \parallel_{\mathcal{D}}^p, \quad t \in R, \psi \in \mathcal{D}.$$

(B2) There exist constant $c_i > 0, i \in Z$, such that

$$E \parallel I_i(x) \parallel^p \leq c_i E \parallel x \parallel^p, \quad x \in L^p(P,K).$$

Theorem 4.1. Assume that assumptions of Theorem 3.1 hold and, in addition, hypotheses (B1), (B2) are satisfied. Then the piecewise pseudo almost periodic mild solution of (1)-(2) is globally exponentially stable.

Proof. Let $x(\cdot)$ be a fixed point of Ψ in Y. By Theorem 3.1, any fixed point of Ψ is a mild solution of the system (1)-(2). We now can choose a positive constant $\tilde{\beta}$ such that $0 < \tilde{\beta} < \frac{p\delta}{2}$,

$$6^{p-1}M^p \frac{1}{(1-e^{-\delta\gamma})^{p-1}(1-e^{-(\delta-\tilde{\beta})\gamma})} \sup_{i\in Z} c_i < 1,$$

and

$$e^{\tilde{\beta}t}E \parallel x(t) \parallel^p$$

$$\leq 5^{p-1}e^{\tilde{\beta}t}E \parallel h(t,x_t) \parallel^p + 5^{p-1}e^{\tilde{\beta}t}E \left\| \int_{-\infty}^t AT(t-s)h(s,x_s)ds \right\|^p$$

$$+ 5^{p-1}e^{\tilde{\beta}t}E \left\| \int_{-\infty}^t T(t-s)g(s,x_s)ds \right\|^p$$

$$+ 5^{p-1}e^{\tilde{\beta}t}E \left\| \int_{-\infty}^t T(t-s)f(s,x_s)dW(s) \right\|^p$$

$$+ 5^{p-1}e^{\tilde{\beta}t}E \left\| \sum_{t_i<t} T(t-t_i)I_i(x(t_i)) \right\|^p$$

$$= \sum_{j=1}^5 \nu_j.$$

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Now, we estimate the terms on the right-hand side of the above inequality. By (B1) and ν_1, we have

$$\nu_1 \le 5^{p-1} \parallel A^{-\beta} \parallel l_1 e^{\tilde{\beta}t} \parallel x_t \parallel_{\mathcal{D}}^p .$$

For any $x(t) \in L^p(P, H)$ and any $\varepsilon > 0$, there exists a $\tilde{t}_1 > 0$ such that $e^{\tilde{\beta}t} E \parallel x(t-q) \parallel^p < \varepsilon$ for $t > \tilde{t}_1$. Thus, we obtain

$$\nu_1 \le 5^{p-1} \parallel A^{-\beta} \parallel l_1 \varepsilon,$$

which implies $\nu_1 \to 0$ as $t \to \infty$. As to ν_2, for any $x(t) \in L^p(P, H)$, $t \in [-q, \infty)$, we have

$$\nu_2$$
$$\le 5^{p-1} e^{\tilde{\beta}t} M_{1-\beta}^p \left(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}(1-\beta)} e^{-\delta(t-s)} ds \right)^{p-1}$$
$$\times \left(\int_{-\infty}^t e^{-\delta(t-s)} E \parallel A^\beta h(s, x_s) \parallel^p ds \right)$$
$$\le 5^{p-1} M_{1-\beta}^p \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \right)^{p-1}$$
$$\times \left(\int_{-\infty}^t e^{-(\delta-\tilde{\beta})(t-s)} l_1 e^{\tilde{\beta}s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \right).$$

For any $x(t) \in L^p(P, H)$ and any $\varepsilon > 0$, there exists a $\tilde{t}_2 > 0$ such that $e^{\tilde{\beta}t} E \parallel x(s-\theta) \parallel^p < \varepsilon$ for $s > \tilde{t}_2$. Thus, we obtain

$$\nu_2$$
$$\le 5^{p-1} M_{1-\beta}^p \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \right)^{p-1}$$
$$\times \left[\int_{\tilde{t}_2}^t e^{-(\delta-\tilde{\beta})(t-s)} l_1 e^{\tilde{\beta}s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \right)$$
$$+ e^{-(\delta-\tilde{\beta})t} \int_{-\infty}^{\tilde{t}_2} e^{(\delta-\tilde{\beta})s} l_1 e^{\tilde{\beta}s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \right]$$
$$\le 5^{p-1} M_{1-\beta}^p \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \right)^{p-1}$$
$$\times \left[\frac{1}{\delta - \tilde{\beta}} l_1 \varepsilon + e^{-(\delta-\tilde{\beta})t} \int_{-\infty}^{\tilde{t}_2} e^{(\delta-\tilde{\beta})s} l_1 e^{\tilde{\beta}s} E \parallel x_s \parallel_{\mathcal{D}}^p ds \right]$$

Since $e^{-(\delta-\tilde{\beta})t} \to 0$ as $t \to \infty$, then there exists $\tilde{t}_3 \ge \tilde{t}_2$ such that for any $t \ge \tilde{t}_3$,

$$5^{p-1} M_{1-\beta}^p \left(\Gamma(1 - \frac{p(1-\beta)}{p-1}) \delta^{\frac{p(1-\beta)}{p-1} - 1} \right)^{p-1}$$

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$$\times e^{-(\delta-\tilde{\beta})t}\int_{-\infty}^{\tilde{t}_2}e^{(\delta-\tilde{\beta})s}l_1 e^{\tilde{\beta}s}E\parallel x_s\parallel_{\mathcal{D}}^p ds$$

$$\leq \varepsilon - 5^{p-1}M_{1-\beta}^p\left(\Gamma(1-\frac{p(1-\beta)}{p-1})\delta^{\frac{p(1-\beta)}{p-1}-1}\right)^{p-1}\frac{1}{\delta-\tilde{\beta}}l_1\varepsilon.$$

Thus, for any $t\geq \tilde{t}_3$, we obtain $\nu_2\leq\varepsilon$, which implies $\nu_2\to 0$ as $t\to\infty$. As to ν_3, for any $x(t)\in L^p(P,H)$, $t\in[-q,\infty)$, we have

ν_3

$$\leq 5^{p-1}M^p e^{\tilde{\beta}t}\left(\int_{-\infty}^t e^{-\delta(t-s)}ds\right)^{p-1}$$

$$\times\left(\int_{-\infty}^t e^{-\delta(t-s)}E\parallel g(s,x_s)\parallel^p ds\right)$$

$$\leq 5^{p-1}M^p\frac{1}{\delta^{p-1}}\left(\int_{-\infty}^t e^{-(\delta-\tilde{\beta})(t-s)}e^{\tilde{\beta}s}l_2\parallel x_s\parallel^p ds\right).$$

Similar to the discussion of ν_2, we obtain $\nu_2\to 0$ as $t\to\infty$. As to ν_4, for any $x(t)\in L^p(P,H)$, $t\in[-q,\infty)$, we have for $p>2$,

ν_4

$$\leq 5^{p-1}C_p M^p e^{\tilde{\beta}t}E\left(\int_{-\infty}^t e^{-2\delta(t-s)}\parallel f(s,x_s)\parallel_{L_2^0}^2 ds\right)^{p/2}$$

$$\leq 5^{p-1}C_p M^p e^{\tilde{\beta}t}\left(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)}ds\right)^{\frac{p-2}{p}}$$

$$\times\left(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)}l_2 E\parallel x(s)\parallel^p ds\right)$$

$$\leq 5^{p-1}C_p M^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}$$

$$\times\left(\int_{-\infty}^t e^{-(\frac{p\delta}{2}-\tilde{\beta})(t-s)}l_2 e^{\tilde{\beta}s}E\parallel x_s\parallel^p ds\right).$$

Similar to the discussion of ν_2, we obtain $\nu_4\to 0$ as $t\to\infty$. For $p=2$, we have

$$\nu_4\leq 5M^2\left(\int_{-\infty}^t e^{-(2\delta-\tilde{\beta})(t-s)}l_2 e^{\tilde{\beta}s}E\parallel x_s\parallel^2 ds\right).$$

Similarly, we obtain $\nu_4\leq\varepsilon$. By (B2) and Hölder's inequality, we have

ν_5

$$\leq 6^{p-1}M^p e^{\tilde{\beta}t}E\left[\left(\sum_{t_i<t}e^{-\delta(t-t_i)}\right)^{p-1}\left(\sum_{t_i<t}e^{-\delta(t-t_i)}\parallel I_i(x(t_i))\parallel^p\right)\right]$$

$$\leq 6^{p-1}M^p\frac{1}{(1-e^{-\delta\gamma})^{p-1}}e^{\tilde{\beta}t}\left(\sum_{t_i<t}e^{-\delta(t-t_i)}[c_i E\parallel x(t_i)\parallel^p]\right)$$

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$$\leq 6^{p-1} M^p \frac{1}{(1-e^{-\delta\gamma})^p} \left(\sum_{t_i < t} e^{-(\delta-\tilde{\beta})(t-t_i)} [c_i e^{\tilde{\beta} t_i} E \parallel x(t_i) \parallel^p] \right)$$

$$\leq 6^{p-1} M^p \frac{1}{(1-e^{-\delta\gamma})^{p-1}(1-e^{-(\delta-\tilde{\beta})\gamma})} [\sup_{i \in Z} c_i e^{\tilde{\beta} t_i} E \parallel x(t_i) \parallel^p].$$

Then there exists a $\tilde{\theta} \in [-q, \infty)$, such that for any $t \geq \tilde{\theta}$, we have

$$e^{\tilde{\beta} t} E \parallel x(t) \parallel^p$$
$$\leq L^* \sup_{t \in R} e^{\tilde{\beta} t} E \parallel x(t) \parallel^p + (5^{p-1} \parallel A^{-\beta} \parallel^p l_1 + 3)\varepsilon,$$

where $L^* = 6^{p-1} M^p \frac{1}{(1-e^{-\delta\gamma})^{p-1}(1-e^{-(\delta-\tilde{\beta})\gamma})} \sup_{i \in Z} c_i < 1$. Thus we get that

$$\sup_{t \in R} e^{\tilde{\beta} t} E \parallel x(t) \parallel^p \leq \frac{(5^{p-1} \parallel A^{-\beta} \parallel^p l_1 + 3)}{1 - L^*} \varepsilon.$$

It follows that $e^{\tilde{\beta} t} E \parallel x(t) \parallel^p \leq \frac{(5^{p-1} \parallel A^{-\beta} \parallel^p l_1 + 3)}{1-L^*} \varepsilon$, which is implies that $e^{\tilde{\beta} t} E \parallel x(t) \parallel^p \to 0$ as $t \to \infty$. So we conclude that the piecewise pseudo almost periodic mild solution of (1)-(2) is globally exponentially stable. The proof is completed.

5 An example

Consider following partial stochastic differential equations of the form

$$d[z(t,x) - \varpi_1(t, z(t-q, x))] = \frac{\partial^2}{\partial x^2} z(t,x)dt + \varpi_2(t, z(t-q, x))dt$$
$$+ \varpi_3(t, z(t-q, x))dW(t), t \in R, t \neq t_i, i \in Z, x \in [0, \pi], \quad (9)$$

$$\Delta z(t_i, x) = \beta_i z(t_i, x), i \in Z, x \in [0, \pi], \quad (10)$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \in R, \quad (11)$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. In this system, $\beta_i \in PAP(Z, R)$, $t_i = i + \frac{1}{4}|\sin i + \sin \sqrt{2}i|$, $\{t_i^j\}, i \in Z, j \in Z$ are equipotentially almost periodic and $\gamma = \inf_{i \in Z}(t_{i+1} - t_i) > 0$, one can see [22] for more details.

Let $H = L^2([0, 1])$ with the norm $\parallel \cdot \parallel$ and define the operators $A : H \to H$ by $A\omega = \omega''$ with the domain

$$D(A) := \{\omega \in H : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in H, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = -\sum_{n=1}^{\infty} n^2 \langle \omega, z_n \rangle z_n, \ \omega \in D(A),$$

where $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), n = 1, 2, 3, ...$, is an orthogonal set of eigenvector of A.

The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}}\omega = \sum_{n=1}^{\infty} n^{\frac{3}{2}} \langle \omega, z_n \rangle z_n$$

on the space

$$D((-A)^{\frac{3}{4}}) = \left\{ \omega(\cdot) \in H : \sum_{n=1}^{\infty} n^{\frac{3}{2}} \langle \omega, z_n \rangle z_n \in H \right\},$$

and $(-A)^{-\frac{3}{4}}\omega = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \langle \omega, z_n \rangle z_n$ for every $\omega \in H$ and $\| (-A)^{-\frac{3}{4}} \|$ is bounded. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t)(t \geq 0)$ in H, and is given (See [40]) by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)\langle \omega, \omega_n \rangle \omega_n, \quad \omega \in H,$$

that satisfies $\| T(t) \| \leq \exp(-\pi^2 t), t \geq 0$ and satisfies (H1).

Let Let $y(t) = z(t, x), t \in [-q, \infty), x \in [0, \pi]$. Taking

$$A^{\frac{3}{4}} h(t, y_t)(x) = \varpi_1(t, z(t-q, x)),$$

$$g(t, y_t)(x) = \varpi_2(t, z(t-q, x)),$$

$$f(t, y_t)(x) = \varpi_3(t, z(t-q, x)),$$

and

$$I_i(y)(x) = \beta_i z(t_i, x), \quad i \in Z.$$

Then, the above equation (9)-(11) can be written in the abstract form as the system (1)-(2).

From Theorem 3.1, it follows that the following proposition holds.

Proposition 5.1. Let $\varpi_1, \varpi_2, \varpi_3$ satisfy (H2)-(H6), then system (9)-(11) has an p-mean piecewise pseudo almost periodic mild solution on R.

In the above example, we can take

$$\varpi_1(t, z(t-q, \cdot)) = \tilde{k}_1[\sin t + \sin \sqrt{2}t + l(t)] \sin z(t-q, \cdot),$$

$$\varpi_2(t, z(t-q, \cdot)) = \tilde{k}_2[\sin t + \sin \sqrt{2}t + l(t)] \sin z(t-q, \cdot),$$

$$\varpi_3(t, z(t-q, \cdot)) = \tilde{k}_3[\sin t + \sin \sqrt{2}t + l(t)] \sin z(t-q, \cdot),$$

and

$$\beta_i z(t_i, \cdot) = \tilde{c}_i[\sin i + \sin \sqrt{2}i + l(i)] \sin z(t_i, \cdot), i \in Z,$$

where $\tilde{k}_j > 0, j = 1, 2, 3$ and $\tilde{c}_i > 0, i \in Z, l \in UPC(R, R)$ defined by

$$l(t) = \begin{cases} 0, & t \leq 0, \\ e^{-t}, & t \geq 0. \end{cases}$$

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From [3], $\sin t + \sin \sqrt{2} t$ is almost periodic. On the other hand,

$$\frac{1}{2r} \int_{-r}^{r} |l(t)|^p dt$$

$$= \frac{1}{2r} \int_{0}^{r} |l(t)|^p dt = \frac{1}{2r} \int_{0}^{r} e^{-pt} dt = \frac{1}{2r} \frac{1 - e^{-pr}}{p}.$$

Consequently

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |l(t)|^p dt = 0.$$

Taking

$$A^{\frac{3}{4}} h(t, y_t)(x) = \tilde{k}_1 [\sin t + \sin \sqrt{2} t + l(t)] \sin z(t - q, x),$$

$$g(t, y_t)(x) = \tilde{k}_2 [\sin t + \sin \sqrt{2} t + l(t)] \sin z(t - q, x),$$

$$f(t, y_t)(x) = \tilde{k}_3 [\sin t + \sin \sqrt{2} t + l(t)] \sin z(t - q, x),$$

and

$$I_i(y)(x) = \tilde{c}_i [\sin i + \sin \sqrt{2} i + l(i)] \sin z(t_i, x), i \in Z.$$

Thus, one has

$$E \parallel A^{\frac{3}{4}} h(t, \psi) - A^{\frac{3}{4}} h(t_1, \psi_1) \parallel^p$$

$$\leq ((2 + \sqrt{2}) \tilde{k}_1)^p [|t - t_1| + \parallel \psi - \psi_1 \parallel_{\mathcal{D}}^p],$$

$$E \parallel g(t, \psi) - g(t, \psi_1) \parallel^p \leq (3 \tilde{k}_2)^p \parallel \psi - \psi_1 \parallel_{\mathcal{D}}^p,$$

$$E \parallel f(t, \psi) - f(t, \psi_1) \parallel^p \leq (3 \tilde{k}_3)^p \parallel \psi - \psi_1 \parallel_{\mathcal{D}}^p,$$

and $E \parallel A^{\frac{3}{4}} h(t, \psi) \parallel^p \leq (3 \tilde{k}_1)^p \parallel \psi \parallel_{\mathcal{D}}^p$, $E \parallel g(t, \psi) \parallel^p \leq (3 \tilde{k}_2)^p \parallel \psi \parallel_{\mathcal{D}}^p$, $E \parallel f(t, \psi) \parallel^p \leq (3 \tilde{k}_3)^p \parallel \psi \parallel_{\mathcal{D}}^p$, for all $(t, \psi), (t_1, \psi_1) \in R \times \mathcal{D}$. Further,

$$E \parallel I(y) - I(y_1) \parallel^p \leq (3 \tilde{c}_i)^p \parallel y - y_1 \parallel^p,$$

and $E \parallel I(y) \parallel^p \leq (3 \tilde{c}_i)^p \parallel y \parallel^p$, for all $(t, y), (t, y_1) \in L^p(P, H))$. Then, all conditions in Theorem 4.1 are satisfied. Hence, the system (9)-(11) has an p-mean piecewise globally exponentially stable pseudo almost periodic mild solution.

Acknowledgments

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FOURIER SERIES OF FUNCTIONS ASSOCIATED WITH POLY-GENOCCHI POLYNOMIALS

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ABSTRACT. In this paper, we will consider three types functions associated with poly-Genocchi polynomials and find their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

1. INTRODUCTION

Let r be any integer. Then we recall that

$$Li_r(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^r}, \ \text{(see [1,6])},$$

is the *rth polylogarithm function* for $r \geq 1$, and a rational function for $r \leq 0$. Here we note

$$\frac{d}{dx} Li_{r+1}(x) = \frac{1}{x} Li_r(x).$$

The *poly-Bernoulli polynomials* $\mathbb{B}_m^{(r)}(x)$ of index r are given by

$$\frac{Li_r(1 - e^{-t})}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} \mathbb{B}_m^{(r)}(x) \frac{t^m}{m!}. \tag{1.1}$$

For $x = 0$, $\mathbb{B}_m^{(r)} = \mathbb{B}_m^{(r)}(0)$ are called *poly-Bernoulli numbers* of index r. Note here that $\mathbb{B}_m^{(1)}(x) = B_m(x)$ are the *Bernoulli polynomials* given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

Here we mention in passing that our definition in (1.1) of poly-Bernoulli polynomials was introduced in [4]. Analogously to the construction of poly-Bernoulli polynomials, the poly-Genocchi polynomials $\mathbb{G}_m^{(r)}(x)$ of index r are defined by

$$\frac{2Li_r(1 - e^{-t})}{e^t + 1} e^{xt} = \sum_{m=0}^{\infty} \mathbb{G}_m^{(r)}(x) \frac{t^m}{m!}. \tag{1.2}$$

When $x = 0$, $\mathbb{G}_m^{(r)} = \mathbb{G}_m^{(r)}(0)$ are called *poly-Genocchi numbers of index r*. Observe here that $\mathbb{G}_m^{(1)}(x) = G_m(x)$ are the Genocchi polynomials given by

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{m=0}^{\infty} G_m(x) \frac{t^m}{m!}.$$

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The poly-Genocchi polynomials $\mathbb{G}_m^{(r)}(x)$ were first introduced in [3], where they were called poly-Euler polynomials and denoted by $\mathbb{E}_m^{(r)}(x)$. However, for the obvious reason it seems more appropriate to call them poly-Genocchi polynomials rather than poly-Euler polynomials. For the definitions of Genocch numbers and polynomials, the reader refers to [2, 7, 8] and [3], respectively.

For poly-Bernoulli polynomials and poly-Genocchi polynomials, we will need the following facts.

$$\frac{d}{dx}\mathbb{B}_m^{(r)}(x) = m\mathbb{B}_{m-1}^{(r)}(x), \ (m \geq 1), \ \mathbb{B}_0^{(r)}(x) = 1$$

$$\mathbb{B}_m^{(0)}(x) = x^m, \ \mathbb{B}_m^{(0)} = \delta_{m,0},$$

$$\frac{d}{dx}\mathbb{G}_m^{(r)}(x) = m\mathbb{G}_{m-1}^{(r)}(x), \ (m \geq 1),$$

$$\mathbb{G}_m^{(r+1)}(1) + \mathbb{G}_m^{(r+1)} = 2\mathbb{B}_{m-1}^{(r)}, \ (m \geq 1), \tag{1.3}$$

$$\mathbb{G}_0^{(r)}(x) = 0, \ \mathbb{G}_1^{(r)}(x) = 1, \ \deg\mathbb{G}_m^{(r)}(x) = m - 1, \ (m \geq 1). \tag{1.4}$$

Here we obtain (1.3) by differentiating with respect to t both sides of the identity

$$\sum_{m=0}^{\infty}\left(\mathbb{G}_m^{(r+1)}(1) + \mathbb{G}_m^{(r+1)}\right)\frac{t^m}{m!} = 2Li_{r+1}(1 - e^{-t}),$$

which follows immediately from (1.2).

Let $E_m(x)$ be the *Euler polynomials* defined by

$$\frac{2}{e^t + 1}e^{xt} = \sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},$$

and let

$$Li_r(1 - e^{-t}) = \sum_{n=1}^{\infty}a_n\frac{t^n}{n!} = t + \sum_{n=2}^{\infty}a_n\frac{t^n}{n!}.$$

Then in view of (1.2) we have

$$\sum_{m=0}^{\infty}\mathbb{G}_m^{(r)}(x)\frac{t^m}{m!} = \sum_{m=1}^{\infty}\left(\sum_{l=0}^{m-1}\binom{m}{l}a_{m-l}E_l(x)\right)\frac{t^m}{m!},$$

from which (1.4) follows.

In [5], $\mathbb{G}_m^{(r+1)}(x)$ are expressed as linear combinations of Euler polynomials and of Genocchi polynomials as follows.

$$\mathbb{G}_m^{(r+1)}(x) = \sum_{j=0}^{m-1}\binom{m}{j}\mathbb{B}_{m-j-1}^{(r)}E_j(x)$$

$$= \frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)}G_j(x), \ (m \geq 1).$$

We will need the following facts about Bernoulli functions $B_m(\langle x\rangle)$, where for any real number x, $\langle x\rangle = x - \lfloor x\rfloor \in [0, 1)$ denotes the fractional part of x :

(a) for $m \geq 2$,

$$B_m(\langle x\rangle) = -m!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

In this paper, we will consider the following three types of functions $\alpha_m(\langle x \rangle)$, $\beta_m(\langle x \rangle)$, and $\gamma_m(\langle x \rangle)$ associated with poly-Genocchi polynomials and find their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

(a) $\alpha_m(\langle x \rangle) = \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(x) x^{m-k}, \ (m \geq 2);$

(b) $\beta_m(\langle x \rangle) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(x) x^{m-k}, \ (m \geq 2);$

(c) $\gamma_m(\langle x \rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{G}_k^{(r+1)}(x) x^{m-k}, \ (m \geq 2).$

2. FOURIER SERIES OF FUNCTIONS OF THE FIRST TYPE

Let

$$\alpha_m(x) = \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(x) x^{m-k}, \ (m \geq 2).$$

Then we will consider the function

$$\alpha_m(\langle x \rangle) = \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k}, \ (m \geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\alpha_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx},$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(\langle x \rangle) e^{-2\pi inx} dx = \int_0^1 \alpha_{,}(x) e^{-2\pi inx} dx.$$

Before proceeding further, we need to observe the following.

$$\alpha'_m(x) = \sum_{k=1}^{m} \left(k \mathbb{G}_{k-1}^{(r+1)}(x) x^{m-k} + (m-k) \mathbb{G}_k^{(r+1)}(x) x^{m-k-1} \right)$$

$$= \sum_{k=2}^{m} k \mathbb{G}_{k-1}^{(r+1)}(x) x^{m-k} + \sum_{k=1}^{m-1} (m-k) \mathbb{G}_k^{(r+1)}(x) x^{m-k-1}$$

$$= \sum_{k=1}^{m-1} (k+1) \mathbb{G}_k^{(r+1)}(x) x^{m-k-1} + \sum_{k=1}^{m-1} (m-k) \mathbb{G}_k^{(r+1)}(x) x^{m-k-1}$$

$$= (m+1) \alpha_{m-1}(x).$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2} \right)' = \alpha_m(x),$$

and

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \left(\alpha_{m+1}(1) - \alpha_{m+1}(0) \right).$$

For each integer $m \geq 2$, we let

$$
\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=1}^{m} \left(\mathbb{G}_k^{(r+1)}(1) - \mathbb{G}_k^{(r+1)} \delta_{m,k} \right) \\
&= \sum_{k=1}^{m} \left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)} - \mathbb{G}_k^{(r+1)} \delta_{m,k} \right) \\
&= 2\sum_{k=1}^{m} \mathbb{B}_{k-1}^{(r)} - \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)} - \mathbb{G}_m^{(r)}.
\end{aligned}
$$

Then

$$
\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0
$$

$$
\Longleftrightarrow 2\sum_{k=1}^{m} \mathbb{B}_{k-1}^{(r)} = \sum_{k=1}^{m} \mathbb{G}_k^{(r+1)} + \mathbb{G}_m^{(r+1)},
$$

and

$$
\int_0^1 \alpha_m(x)\,dx = \frac{1}{m+2} \Delta_{m+1}.
$$

We are now going to determine the Fourier coefficients $A_n^{(m)}$.

Case $1 : n \neq 0$.

$$
\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n} \left[\alpha_m(x) e^{-2\pi i n x} \right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \alpha_m'(x) e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n} (\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n} \int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x}\,dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n} \Delta_m,
\end{aligned}
$$

from which by induction on m we can deduce tat

$$
A_n^{(m)} = -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1}.
$$

Case $2 : n = 0$.

$$
A_0^{(m)} = \int_0^1 \alpha_m(x)\,dx = \frac{1}{m+2} \Delta_{m+1}.
$$

$\alpha_m(\langle x \rangle)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\alpha_m(\langle x \rangle)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$, and discontinuous with jump discontinuities for those integers $m \geq 2$ with $\Delta_m \neq 0$.

Assume first that m is an integer ≥ 2 with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\alpha_m(\langle x \rangle)$

converges uniformly to $\alpha_m(\langle x \rangle)$, and

$$\alpha_m(\langle x \rangle) = \frac{1}{m+2} \Delta_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx}$$

$$= \frac{1}{m+2} \Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} \left(-j! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^j} \right)$$

$$= \frac{1}{m+2} \Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x \rangle)$$

$$+ \Delta_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Now, we can state our first theorem.

Theorem 2.1. *For each integer $l \geq 2$, we let*

$$\Delta_l = 2 \sum_{k=1}^{l} \mathbb{B}_{k-1}^{(r)} - \sum_{k=1}^{l} \mathbb{G}_k^{(r+1)} - \mathbb{G}_l^{(r+1)}.$$

Assume that $\Delta_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k}$ *has the Fourier series expansion*

$$\sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k}$$

$$= \frac{1}{m+2} \Delta_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx},$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=1}^{m} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k} = \frac{1}{m+2} \Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x \rangle),$$

for all $x \in \mathbb{R}$.

Assume next that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\alpha_m(\langle x \rangle)$ converges pointwise to $\alpha_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) = \alpha_m(0) + \frac{1}{2} \Delta_m.$$

We can now state our second theorem.

Theorem 2.2. *For each integer $l \geq 2$, we let*

$$\Delta_l = 2 \sum_{k=1}^{l} \mathbb{B}_{k-1}^{(r)} - \sum_{k=1}^{l} \mathbb{G}_k^{(r+1)} - \mathbb{G}_l^{(r+1)}.$$

Assume that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}$$

$$=\begin{cases} \sum_{k=1}^{m}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, & for\ x\notin\mathbb{Z}, \\ \mathbb{G}_m^{(r+1)} + \frac{1}{2}\Delta_m, & for\ x\in\mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m+2}\sum_{j=0}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle) = \sum_{k=1}^{m}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, \ for\ x\notin\mathbb{Z};$$

$$\frac{1}{m+2}\sum_{\substack{j=0 \\ j\neq 1}}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle) = \mathbb{G}_m^{(r+1)} + \frac{1}{2}\Delta_m, \ for\ x\in\mathbb{Z}.$$

3. Fourier series of functions of the second type

Let

$$\beta_m(x) = \sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-k}, \ (m\geq 2).$$

Then we will consider the function

$$\beta_m(\langle x\rangle) = \sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, \ (m\geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)} = \int_0^1\beta_m(\langle x\rangle)e^{-2\pi inx}dx = \int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

To continue further, we need to observe the following.

$$\beta_m'(x) = \sum_{k=1}^{m}\left(\frac{k}{k!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k} + \frac{m-k}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}\right)$$

$$=\sum_{k=2}^{m}\frac{1}{(k-1)!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k} + \sum_{k=1}^{m-1}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}$$

$$=\sum_{k=1}^{m-1}\frac{1}{k!(m-1-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k} + \sum_{k=1}^{m-1}\frac{1}{k!(m-1-k)!}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k}$$

$$=2\beta_{m-1}(x).$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x),$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\beta_{m+1}(1) - \beta_{m+1}(0)\right).$$

For each integer $m \geq 2$, we put

$$\Omega_m = \beta_m(1) - \beta_m(0)$$

$$= \sum_{k=1}^m \frac{1}{k!(m-k)!}\left(\mathbb{G}_k^{(r+1)}(1) - \mathbb{G}_k^{(r+1)}\delta_{m,k}\right)$$

$$= \sum_{k=1}^m \frac{1}{k!(m-k)!}\left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)} - \mathbb{G}_k^{(r+1)}\delta_{m,k}\right)$$

$$= 2\sum_{k=1}^m \frac{\mathbb{B}_{k-1}^{(r)}}{k!(m-k)!} - \sum_{k=1}^m \frac{\mathbb{G}_k^{(r+1)}}{k!(m-k)!} - \frac{\mathbb{G}_m^{(r+1)}}{m!}.$$

Then

$$\beta_m(0) = \beta_m(1) \iff \Omega_m = 0$$

$$\iff 2\sum_{k=1}^m \frac{\mathbb{B}_{k-1}^{(r)}}{k!(m-k)!} = \sum_{k=1}^m \frac{\mathbb{G}_k^{(r+1)}}{k!(m-k)!} + \frac{\mathbb{G}_m^{(r+1)}}{m!},$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

Now, we would like to determine the Fourier coefficients $B_n^{(m)}$.
Case $1: n \neq 0$.

$$B_n^{(m)} = \int_0^1 \beta_m(x)e^{-2\pi inx}dx$$

$$= -\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx$$

$$= -\frac{1}{2\pi in}(\beta_m(1) - \beta_m(0)) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx$$

$$= \frac{2}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m,$$

from which by induction on m we can deduce that

$$B_n^{(m)} = -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}.$$

Case $2: n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(\langle x \rangle)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\beta_m(\langle x \rangle)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that m is an integer ≥ 2 with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. $\beta_m(\langle x \rangle)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\beta_m(\langle x \rangle)$

converges uniformly to $\beta_m(\langle x \rangle)$, and

$$\beta_m(\langle x \rangle) = \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x}$$

$$= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \left(-j! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right)$$

$$= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x \rangle) + \Omega_m \times \left\{ \begin{array}{ll} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{array} \right.$$

Now, we are ready to state our first theorem.

Theorem 3.1. *For each integer $l \geq 2$, we let*

$$\Omega_l = 2 \sum_{k=1}^{l} \frac{\mathbb{B}_{k-1}^{(r)}}{k!(l-k)!} - \sum_{k=1}^{l} \frac{\mathbb{G}_k^{(r+1)}}{k!(l-k)!} - \frac{\mathbb{G}_l^{(r+1)}}{l!}.$$

Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following

(a) $\sum_{k=1}^{m} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k}$ *has the Fourier series expansion*

$$\sum_{k=1}^{m} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k} = \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=1}^{m} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k} = \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x \rangle),$$

for all $x \in \mathbb{R}$.

Next, we assume that m is an integer ≥ 2 with $\Omega_m \neq 0$. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(\langle x \rangle)$ converges pointwise to $\beta_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

We are now ready to state our second theorem.

Theorem 3.2. *For each integer $l \geq 2$, we let*

$$\Omega_l = 2 \sum_{k=1}^{l} \frac{\mathbb{B}_{k-1}^{(r)}}{k!(l-k)!} - \sum_{k=1}^{l} \frac{\mathbb{G}_k^{(r+1)}}{k!(l-k)!} - \frac{\mathbb{G}_l^{(r+1)}}{l!}.$$

Assume that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \right) e^{2\pi inx}$$

$$= \begin{cases} \sum_{k=1}^m \frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{m!}\mathbb{G}_m^{(r+1)} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\sum_{j=0}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle) = \sum_{k=1}^m \frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, \text{ for } x \notin \mathbb{Z};$$

$$\sum_{\substack{j=0 \\ j\neq 1}}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle) = \frac{1}{m!}\mathbb{G}_m^{(r+1)} + \frac{1}{2}\Omega_m, \text{ for } x \in \mathbb{Z}.$$

4. FOURIER SERIES OF FUNCTIONS OF THE THIRD TYPE

Let

$$\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(x)x^{m-k}, \ (m \geq 2).$$

Then we will consider the function

$$\gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\langle x\rangle^{m-k}, \ (m \geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)}e^{2\pi inx},$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(\langle x\rangle)e^{-2\pi inx}dx = \int_0^1 \gamma_m(x)e^{-2\pi inx}dx.$$

To proceed further, we need to observe the following.

$$\gamma_m'(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(k\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k} + (m-k)\mathbb{G}_k^{(r+1)}(x)x^{m-k-1} \right)$$

$$= \sum_{k=2}^{m-1} \frac{1}{m-k}\mathbb{G}_{k-1}^{(r+1)}(x)x^{m-k} + \sum_{k=1}^{m-1} \frac{1}{k}\mathbb{G}_k^{(r+1)}(x)x^{m-k-1}$$

$$= \sum_{k=1}^{m-2} \frac{1}{m-1-k}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k} + \sum_{k=1}^{m-1} \frac{1}{k}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k}$$

$$= (m-1)\sum_{k=1}^{m-2} \frac{1}{k(m-1-k)}\mathbb{G}_k^{(r+1)}(x)x^{m-1-k} + \frac{1}{m-1}\mathbb{G}_{m-1}^{r+1}(x)$$

$$= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}\mathbb{G}_{m-1}^{(r+1)}(x).$$

From this, we have

$$\left(\frac{1}{m} \left(\gamma_{m+1}(x) - \frac{1}{m(m+1)} \mathbb{G}_{m+1}^{(r+1)}(x) \right) \right)' = \gamma_m(x),$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m} \left[\gamma_{m+1}(x) - \frac{1}{m(m+1)} \mathbb{G}_{m+1}^{(r+1)}(x) \right]_0^1$$

$$= \frac{1}{m} \left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} \left(\mathbb{G}_{m+1}^{(r+1)}(1) - \mathbb{G}_{m+1}^{(r+1)}(0) \right) \right)$$

$$= \frac{1}{m} \left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)} \left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)} \right) \right).$$

For each integer $m \geq 2$, we let

$$\Lambda_m = \gamma_m(1) - \gamma_m(0)$$

$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left(\mathbb{G}_k^{(r+1)}(1) - \mathbb{G}_k^{(r+1)} \delta_{m,k} \right)$$

$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)} - \mathbb{G}_k^{(r+1)} \delta_{m,k} \right)$$

$$= 2 \sum_{k=1}^{m-1} \frac{\mathbb{B}_{k-1}^{(r)}}{k(m-k)} - \sum_{k=1}^{m-1} \frac{\mathbb{G}_k^{(r+1)}}{k(m-k)}.$$

Then

$$\gamma_m(0) = \gamma_m(1) \Longleftrightarrow \Lambda_m = 0$$

$$\Longleftrightarrow 2 \sum_{k=1}^{m-1} \frac{\mathbb{B}_{k-1}^{(r)}}{k(m-k)} = \sum_{k=1}^{m-1} \frac{\mathbb{G}_k^{(r+1)}}{k(m-k)},$$

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m} \left(\Lambda_{m+1} + \frac{2}{m(m+1)} \left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)} \right) \right).$$

Now, we are going to determine the Fourier coefficients $C_n^{(m)}$.

Case $1 : n \neq 0$.

Observe first that

$$\int_0^1 \mathbb{G}_m^{(r+1)}(x)e^{-2\pi inx}dx = \begin{cases} 2 \sum_{k=1}^{m-1} \frac{(m)_{k-1}}{(2\pi in)^k} \left(\mathbb{G}_{m-k+1}^{(r+1)} - \mathbb{B}_{m-k}^{(r)} \right) & \text{, for } n \neq 0, \\ \frac{2}{m+1} \left(\mathbb{B}_m^{(r)} - \mathbb{G}_{m+1}^{(r+1)} \right), & \text{for } n = 0. \end{cases}$$

Then we have

$$C_n^{(m)} = -\frac{1}{2\pi in} \left[\gamma_m(x)e^{-2\pi inx} \right]_0^1 + \frac{1}{2\pi in} \int_0^1 \gamma_m'(x)e^{-2\pi inx}dx$$

$$= -\frac{1}{2\pi in}(\gamma_m(1) - \gamma_m(0)) + \frac{1}{2\pi in} \int_0^1 \left((m-1)\gamma_{m-1}(x) + \frac{1}{m-1}\mathbb{G}_{m-1}^{(r+1)}(x) \right) e^{-2\pi inx}dx$$

$$= -\frac{1}{2\pi in}\Lambda_m + \frac{m-1}{2\pi in}C_n^{(m-1)} + \frac{1}{2\pi in(m-1)} \int_0^1 \mathbb{G}_{m-1}^{(r+1)}(x)e^{-2\pi inx}dx$$

$$= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Phi_m,$$

where

$$\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \left(\mathbb{G}_{m-k}^{(r+1)} - \mathbb{B}_{m-k-1}^{(r)} \right), \qquad (4.1)$$

From (4.1), by induction on m we can show that

$$C_n^{(m)} = -\frac{1}{m} \sum_{j=1}^{m-1} \frac{(m)_j}{(2\pi i n)^j} \Lambda_{m-j+1} + \frac{1}{m} \sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \Phi_{m-j+1}.$$

We now observe that

$$\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \Phi_{m-j+1}$$

$$= \sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1}}{(2\pi i n)^k} \left(\mathbb{G}_{m-j-k+1}^{(r+1)} - \mathbb{B}_{m-j-k}^{(r)} \right)$$

$$= \sum_{j=1}^{m-2} \sum_{k=1}^{m-j-1} \frac{2(m)_{j+k-1}}{(2\pi i n)^{j+k} (m-j)} \left(\mathbb{G}_{m-j-k+1}^{(r+1)} - \mathbb{B}_{m-j-k}^{(r)} \right)$$

$$= 2 \sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{s=j+1}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right)$$

$$= 2 \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right) \sum_{j=1}^{s-1} \frac{1}{m-j}$$

$$= 2 \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right) (H_{m-1} - H_{m-s})$$

$$= 2 \sum_{s=1}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right) (H_{m-1} - H_{m-s})$$

$$= 2 \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \frac{\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}}{m-s+1} (H_{m-1} - H_{m-s}).$$

Putting everything altogether,

$$C_n^{(m)} = -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left(\Lambda_{m-s+1} - \frac{2(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)})}{m-s+1} (H_{m-1} - H_{m-s}) \right).$$

Case $2 : n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x) dx$$

$$= \frac{1}{m} \left(\Lambda_{m+1} + \frac{2}{m(m+1)} \left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)} \right) \right).$$

$\gamma_m(\langle x \rangle)$, $(m \geq 2)$ is piecewise C^∞. In addition, $\gamma_m(\langle x \rangle)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(\langle x \rangle)$ is piecewise C^∞, and continuous. Thus Fourier series of $\gamma_m(\langle x \rangle)$ converges uniformly to $\gamma_m(\langle x \rangle)$, and

$$\gamma_m(\langle x \rangle) = \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right)$$

$$- \frac{1}{m}\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right)\right\}e^{2\pi i n x}$$

$$= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right)$$

$$+ \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right)$$

$$\times \left(-s!\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right)$$

$$= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right)$$

$$+ \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right)B_s(\langle x \rangle)$$

$$+ \Lambda_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Now, we can state our first theorem.

Theorem 4.1. *For each integer $l \geq 2$, we let*

$$\Lambda_l = 2\sum_{k=1}^{l-1}\frac{\mathbb{B}_{k-1}^{(r)}}{k(l-k)} - \sum_{k=1}^{l-1}\frac{\mathbb{G}_k^{(r+1)}}{k(l-k)}.$$

Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\langle x \rangle^{m-k}$ *has the Fourier series expansion*

$$\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\langle x \rangle^{m-k}$$

$$= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}\left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)}\right)\right)$$

$$- \frac{1}{m}\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} - \frac{2\left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)}\right)}{m-s+1}(H_{m-1} - H_{m-s})\right)\right\}e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k}$$

$$= \frac{1}{m} \sum_{\substack{s=0 \\ s \neq 1}}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2 \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right)}{m-s+1} (H_{m-1} - H_{m-s}) \right) B_s(\langle x \rangle),$$

for all $x \in \mathbb{R}$.

Assume next that m is an integer ≥ 2 with $\Lambda_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(\langle x \rangle)$ converges pointwise to $\gamma_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2} \left(\gamma_m(0) + \gamma_m(1) \right) = \frac{1}{2} \Lambda_m,$$

for $x \in \mathbb{Z}$.

We can now state our second theorem.

Theorem 4.2. *For each integer $l \geq 2$, we let*

$$\Lambda_l = 2 \sum_{k=1}^{l-1} \frac{\mathbb{B}_{k-1}^{(r)}}{k(l-k)} - \sum_{k=1}^{l-1} \frac{\mathbb{G}_k^{(r+1)}}{k(l-k)}.$$

Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\frac{1}{m} \left(\Lambda_{m+1} + \frac{2}{m(m+1)} \left(\mathbb{G}_{m+1}^{(r+1)} - \mathbb{B}_m^{(r)} \right) \right)$$

$$- \frac{1}{m} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left\{ \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left(\Lambda_{m-s+1} - \frac{2 \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right)}{m-s+1} (H_{m-1} - H_{m-s}) \right) \right\} e^{2\pi i n x}$$

$$= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{2} \Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m} \sum_{s=0}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2 \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right)}{m-s+1} (H_{m-1} - H_{m-s}) \right) B_s(\langle x \rangle)$$

$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbb{G}_k^{(r+1)}(\langle x \rangle) \langle x \rangle^{m-k}, \quad \text{for } x \notin \mathbb{Z};$$

$$\frac{1}{m} \sum_{\substack{s=0 \\ s \neq 1}}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2 \left(\mathbb{G}_{m-s+1}^{(r+1)} - \mathbb{B}_{m-s}^{(r)} \right)}{m-s+1} (H_{m-1} - H_{m-s}) \right) B_s(\langle x \rangle)$$

$$= \frac{1}{2} \Lambda_m, \quad \text{for } x \in \mathbb{Z}.$$

5. Acknowledgements

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Tracy-Singh Products and Classes of Operators

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Abstract

We investigate relationship between Tracy-Singh products and certain classes of Hilbert space operators. We show that the normality, hyponormality, paranormality of operators are preserved by Tracy-Singh products. Operators of class-$\mathcal{A}$ type are also preserved under Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary.

Keywords: Tracy-Singh product, tensor product, normality, class $\mathcal{A}$ operator
Mathematics Subject Classifications 2010: 47A05, 47A80, 47B20, 47B47.

1 Introduction

Tensor product of bounded linear operators plays a crucial role in functional analysis and operator theory. Many algebraic-order-analytic properties of operators are preserved under taking tensor products, but by no means all of them. Importance results on tensor product involving certain classes of operators (e.g. positive, unitary, normal, compact) have been noticed by many mathematicians from the beginning of the theory to nowadays (e.g. [22]). In the last two decades, the concepts of normality, hyponormality, and paranormality have been introduced and investigated by many authors, see e.g., [5, 13, 21]. Relations between tensor products and class-$\mathcal{A}$ type operators also have received much attention, e.g., [10, 11, 12, 19, 20]. See more information about classes of operators in the monograph [7].

Recently, the notion of tensor product was extended to the Tracy-Singh product for Hilbert space operators in [15]. It was shown that compactness,

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positivity and strict-positivity of operators are preserved under Tracy-Singh products [15, 16].

In this paper, we investigate relationship between Tracy-Singh products and certain classes of operators. We divide such classes into three categories. The first category consists of nilpotent, (skew)-Hermitian, (co)isometry, and unitary operators. The second one contains operator normality, hyponormality, and paranormality. The last one is the class-$\mathcal{A}$ type operators, which includes class $\mathcal{A}(k)$, class $\mathcal{A}$, quasi-class $(\mathcal{A}, k)$, quasi-class $\mathcal{A}$, $*$-class $\mathcal{A}$, quasi-$*$-class $\mathcal{A}$, and quasi-$*$-class $(\mathcal{A}, k)$ operators. We will show that the mentioned properties of operators are preserved under taking Tracy-Singh products. Moreover, we obtain necessary and sufficient conditions for the Tracy-Singh product of two operators to be normal, quasinormal, (co)isometry, and unitary operators.

The paper is structured as follows. The next section supplies some prerequisites about the tensor product and the Tracy-Singh product of operators. Next, we discuss relationship between Tracy-Singh products and the normality, hyponormality, and paranormality of operators. Then we consider Tracy-Singh products and certain properties of operators–being nilpotent, (skew)-Hermitian, (co)isometry, and unitary. The last section deals with class $\mathcal{A}$ type operators.

2 Preliminaries

In what follows, $\mathbb{H}$ and $\mathbb{K}$ denote complex separable Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathcal{B}(X, Y)$ the Banach space of bounded linear operators from X into Y, equipped with the operator norm $\|\cdot\|$ and abbreviate $\mathcal{B}(X, X)$ to $\mathcal{B}(X)$. For Hermitian operators A and B on the same Hilbert space, we use the notation $A \geqslant B$ to mean that $A - B$ is a positive operator.

In order to define the Tracy-Singh product, we have to fix the orthogonal decompositions of Hilbert spaces, namely,

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{l=1}^{n} \mathbb{K}_l$$

where all $\mathbb{H}_i$'s and $\mathbb{K}_k$'s are Hilbert spaces. Any operator $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathcal{B}(\mathbb{H}_j, \mathbb{H}_i)$ and $B_{kl} \in \mathcal{B}(\mathbb{K}_l, \mathbb{K}_k)$ for each i, j, k, l. Then the *Tracy-Singh product* of A and B is defined to be

$$A \boxtimes B = \left[\left[A_{ij} \otimes B_{kl} \right]_{kl} \right]_{ij}, \tag{1}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_k$ into itself. Note that when $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$.

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Lemma 1 ([15])**.** *Algebraic and order properties of the Tracy-Singh product for operators are listed here (provided that every operation is well-defined):*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*

2. *Compatibility with adjoints: $(A \boxtimes B)^* = A^* \boxtimes B^*$.*

3. *Compatibility with ordinary products: $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.*

4. *Compatibility with powers: $(A \boxtimes B)^r = A^r \boxtimes B^r$ for any $r \in \mathbb{N}$.*

5. *Compatibility with inverses: if A and B are invertible, then $A \boxtimes B$ is invertible with $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*

6. *Positivity: if $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*

7. *Monotonicity: if $A_1 \geqslant B_1$ and $A_2 \geqslant B_2$, then $A_1 \boxtimes A_2 \geqslant B_1 \boxtimes B_2$.*

Lemma 2 ([15])**.** *Let $A = [A_{ij}] \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be operator matrices. Then each (i, j)-block of $A \boxtimes B$ is $A_{ij} \boxtimes B$.*

Analytic properties of the Tracy-Singh product for operators are listed below.

Lemma 3 ([16])**.** *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$. Then we have*

(i) $\dfrac{1}{mn}\|A\|\|B\| \leqslant \|A \boxtimes B\| \leqslant mn\|A\|\|B\|.$

(ii) *$|A \boxtimes B| = |A| \boxtimes |B|$, here the absolute value of A is defined by $|A| = (A^*A)^{\frac{1}{2}}$.*

(iii) *If A and B are positive operators, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any nonnegative real α.*

Lemma 4. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$.*

(i) *The condition $A \boxtimes B = 0$ holds if and only if $A = 0$ or $B = 0$.*

(ii) *If $A \boxtimes B = A \boxtimes C$ and $A \neq 0$, then $B = C$.*

(iii) *If $B \boxtimes A = C \boxtimes A$ and $A \neq 0$, then $B = C$.*

Proof. From the norm estimation in Lemma 3(i), one can deduce property (i). Properties (ii) and (iii) follow from (i) and the bilinearity of Tracy-Singh product in Lemma 1. $\qquad\square$

Lemma 5 ([21])**.** *Let $A, C \in \mathcal{B}(\mathbb{H})$ and $B, D \in \mathcal{B}(\mathbb{K})$ be nonzero operators. Then $A \otimes B = C \otimes D$ if and only if there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $C = \alpha A$ and $D = \alpha^{-1} B$.*

Proposition 6. *Let $A = [A_{ij}]_{i,j=1}^{m,m}, C = [C_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n}, D = [D_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij}, B_{kl}, C_{ij} and D_{kl} are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B = C \boxtimes D$ if and only if there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $C = \alpha A$ and $D = \alpha^{-1} B$.*

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Proof. If $C = \alpha A$ and $D = \alpha^{-1} B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$, then by Lemma 1,

$$C \boxtimes D = (\alpha A) \boxtimes (\alpha^{-1} B) = \alpha \alpha^{-1} (A \boxtimes B) = A \boxtimes B.$$

Assume that $A \boxtimes B = C \boxtimes D$. By using Lemma 2, we get $A_{ij} \boxtimes B = C_{ij} \boxtimes D$ for all $i, j = 1, \ldots, m$. For any fixed $i, j \in \{1, \ldots, m\}$, we have $A_{ij} \otimes B_{kl} = C_{ij} \otimes D_{kl}$ for all $k, l = 1, \ldots, n$. For each $i, j \in \{1, \ldots, m\}$ and $k, l \in \{1, \ldots, n\}$, by applying Lemma 5, there exists $\alpha_{ij,kl} \in \mathbb{C} \setminus \{0\}$ such that $C_{ij} = \alpha_{ij,kl} A_{ij}$ and $D_{kl} = \alpha_{ij,kl}^{-1} B_{kl}$. For any fixed $i, j \in \{1, \ldots, m\}$, we have $C_{ij} = \alpha_{ij,kl} A_{ij}$ for all $k, l = 1, \ldots, n$. This implies that $\alpha_{ij,11} = \cdots = \alpha_{ij,nn} = \alpha_{ij}$. For any fixed $k, l \in \{1, \ldots, n\}$, we have $D_{kl} = \alpha_{ij}^{-1} B_{kl}$ for all $i, j = 1, \ldots, m$. It follows that $\alpha_{11} = \cdots = \alpha_{mm} = \alpha$. Thus $C_{ij} = \alpha A_{ij}$ and $D_{kl} = \alpha^{-1} B_{kl}$ for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Therefore $C = \alpha A$ and $D = \alpha^{-1} B$. □

Recall that the commutator of A and B in $\mathcal{B}(\mathbb{H})$ is defined by

$$[A, B] = AB - BA.$$

Proposition 7. *Let $A, C \in \mathcal{B}(\mathbb{H})$ and $B, D \in \mathcal{B}(\mathbb{K})$.*

(i) *If $[A, C] \geqslant 0$ and $[B, D] \geqslant 0$, then $[A \boxtimes B, C \boxtimes D] \geqslant 0$.*

(ii) *If $[A, C] \leqslant 0$ and $[B, D] \leqslant 0$, then $[A \boxtimes B, C \boxtimes D] \leqslant 0$.*

(iii) *If $[A, C] = 0$ and $[B, D] = 0$, then $[A \boxtimes B, C \boxtimes D] = 0$.*

Proof. (i) Since $AC \geqslant CA$ and $BD \geqslant DB$, we have $AC \boxtimes BD \geqslant CA \boxtimes DB$ by Lemma 1. Then

$$[A \boxtimes B, C \boxtimes D] = AC \boxtimes BD - CA \boxtimes DB \geqslant 0.$$

The assertion (ii) follows from (i) and the fact that $-[X, Y] = [Y, X]$ for any operators X and Y. The assertion (iii) follows from (i) and (ii). □

3   Tracy-Singh products and operator normality

In this section, we discuss normality of Tracy-Singh products of operators. The contents can be divided into three parts. The first part deals with general properties of normality, the second one concerns hyponormality, and the last one consists of paranormality.

3.1   Normality

Recall the following types of operator normality; see e.g. [7, Chapter 2] and [17] for more details.

Definition 8. *An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- normal *if $[T^*, T] = 0$;*

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- binormal *if* $[T^*T, TT^*] = 0$ *;*

- quasinormal *if* $[T, T^*T] = 0$ *;*

- posinormal *if* $TT^* = T^*PT$ *for some positive operator P.*

Stochel [21] showed that for non-zero $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$, the tensor product $A \otimes B$ is normal (resp. quasinormal) if and only if A and B are normal (resp. quasinormal). Now, we will extend this result to the case of Tracy-Singh products.

Theorem 9. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \dots, m$ and $k, l = 1, \dots, n$. Then $A \boxtimes B$ is normal if and only if so are A and B.*

Proof. If A and B are normal, then by Lemma 1 and Proposition 7 we have

$$[(A \boxtimes B)^*, A \boxtimes B] = [A^* \boxtimes B^*, A \boxtimes B] = [A^*, A] \boxtimes [B^*, B] = 0,$$

i.e., $A \boxtimes B$ is also normal. Conversely, suppose that $A \boxtimes B$ is normal. Note that

$$A^*A \boxtimes B^*B = (A \boxtimes B)^*(A \boxtimes B) = (A \boxtimes B)(A \boxtimes B)^* = AA^* \boxtimes BB^*.$$

By Proposition 6, there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $AA^* = \alpha A^*A$ and $BB^* = \alpha^{-1}B^*B$. Since AA^* and A^*A are positive, we have $\alpha > 0$. Then

$$\|A\|^2 = \|AA^*\| = \|\alpha A^*A\| = \alpha\|A\|^2,$$
$$\|B\|^2 = \|BB^*\| = \|\alpha^{-1}B^*B\| = \alpha^{-1}\|B\|^2.$$

We arrive at $\alpha = 1$, meaning that both A and B are normal. $\square$

Theorem 10. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \dots, m$ and $k, l = 1, \dots, n$. Then $A \boxtimes B$ is quasinormal if and only if so are A and B.*

Proof. Assume that A and B are quasinormal. Since $[A, A^*A] = 0$ and $[B, B^*B] = 0$, we have

$$[A \boxtimes B, (A \boxtimes B)^*(A \boxtimes B)] = [A \boxtimes B, A^*A \boxtimes B^*B] = 0.$$

Hence, $A \boxtimes B$ is quasinormal. Suppose that $A \boxtimes B$ is quasinormal. Note that

$$\begin{aligned} AA^*A \boxtimes BB^*B &= (A \boxtimes B)(A \boxtimes B)^*(A \boxtimes B) \\ &= (A \boxtimes B)^*(A \boxtimes B)^2 \\ &= A^*A^2 \boxtimes B^*B^2. \end{aligned}$$

Then there exists $\alpha \in \mathbb{C}\setminus\{0\}$ such that $A^*A^2 = \alpha AA^*A$ and $B^*B^2 = \alpha^{-1}BB^*B$. This in turn implies that

$$(A^2)^*A^2 = A^*(A^*A^2) = \alpha(A^*A)^2,$$
$$(B^2)^*B^2 = B^*(B^*B^2) = \alpha^{-1}(B^*B)^2.$$

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Since $(A^2)^*A^2$ and $\alpha(A^*A)^2$ are positive, we conclude $\alpha > 0$. We have

$$\alpha\|A\|^4 \;=\; \alpha\|(A^*A)^2\|^2 \;=\; \|(A^2)^*A^2\| \;=\; \|A^2\|^2 \;\leqslant\; \|A\|^4$$

and, similarly, $\alpha^{-1}\|B\|^4 \leqslant \|B\|^4$. This forces $\alpha = 1$ and, thus, both A and B are quasinormal. $\qquad\square$

Proposition 11. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: binormal, posinormal.*

Proof. The assertion for binormality follows from Lemma 1 and Proposition 7. Now, suppose that $AA^* = A^*PA$ and $BB^* = B^*QB$ for some positive operators P and Q. By Lemma 1, we get

$$\begin{aligned}
(A \boxtimes B)(A \boxtimes B)^* \;&=\; AA^* \boxtimes BB^* \;=\; A^*PA \boxtimes B^*QB \\
&=\; (A \boxtimes B)^*(P \boxtimes Q)(A \boxtimes B).
\end{aligned}$$

According to Lemma 1, $P \boxtimes Q$ is positive. Therefore $A \boxtimes B$ is posinormal. $\quad\square$

3.2 Hyponormality

Recall the following hyponormal structures of operators; see e.g. [1, 4, 13] and [7, Chapter 2] for more information.

Definition 12. *Let $p > 0$ be a constant. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- hyponormal *if $[T^*, T]$ is positive ;*

- *p-hyponormal if $(T^*T)^p \geqslant (TT^*)^p$;*

- quasihyponormal *if $T^*[T^*, T]T$ is positive ;*

- *p-quasihyponormal if $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$;*

- cohyponormal *if T^* is hyponormal ;*

- log-hyponormal *if T is invertible and $\log(T^*T) \geqslant \log(TT^*)$.*

Definition 13. *Let $T \in \mathcal{B}(\mathbb{H})$ have the polar decomposition $T = U|T|$ where U is a unitary operator. The* Aluthge transformation *of T is defined by*

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Then T is said to be

- *w-hyponormal if $|\tilde{T}| \geqslant |T| \geqslant |\tilde{T}^*|$;*

- *iw-hyponormal if T is invertible and $|T| \geqslant \left(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.*

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Theorem 14. *Let $A \in \mathcal{B}(\mathbb{H})$, $B \in \mathcal{B}(\mathbb{K})$, and let $p > 0$ be a constant. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: hyponormal, p-hyponormal, cohyponormal, quasihyponormal, p-quasihyponormal.*

Proof. The assertions for hyponormality and cohyponormality follow from Lemma 1 and Proposition 7. The assertion for p-hyponormality is done by applying Lemmas 1 and 3. Now, suppose that A and B are quasihyponormal. By Lemma 1, we obtain

$$(A \boxtimes B)^* \left[(A \boxtimes B)^*, A \boxtimes B \right] (A \boxtimes B)$$
$$= (A^* \boxtimes B^*)((A^* \boxtimes B^*)(A \boxtimes B) - (A \boxtimes B)(A^* \boxtimes B^*))(A \boxtimes B)$$
$$= (A^* \boxtimes B^*)(A^* \boxtimes B^*)(A \boxtimes B)(A \boxtimes B) - (A^* \boxtimes B^*)(A \boxtimes B)(A^* \boxtimes B^*)(A \boxtimes B)$$
$$= A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B.$$

Since $A^*A^*AA - A^*AA^*A = A^*[A^*, A]A \geqslant 0$ and $B^*B^*BB - B^*BB^*B = B^*[B^*, B]B \geqslant 0$, we have by Lemma 1 that

$$A^*A^*AA \boxtimes B^*B^*BB - A^*AA^*A \boxtimes B^*BB^*B \geqslant 0.$$

Hence, $(A \boxtimes B)^* \left[(A \boxtimes B)^*, A \boxtimes B \right] (A \boxtimes B) \geqslant 0$. This means that $A \boxtimes B$ is quasihyponormal.

Assume that A and B are p-quasihyponormal. Lemmas 1 and 3 together imply that

$$(A \boxtimes B)^* \left((A \boxtimes B)^*(A \boxtimes B) \right)^p (A \boxtimes B)$$
$$= (A^* \boxtimes B^*) (A^*A \boxtimes B^*B)^p (A \boxtimes B)$$
$$= A^*(A^*A)^pA \boxtimes B^*(B^*B)^pB$$
$$\geqslant A^*(AA^*)^pA \boxtimes B^*(BB^*)^pB$$
$$= (A \boxtimes B)^* (AA^* \boxtimes BB^*)^p (A \boxtimes B)$$
$$= (A \boxtimes B)^* ((A \boxtimes B)(A \boxtimes B)^*)^p (A \boxtimes B).$$

This show that $A \boxtimes B$ is p-quasihyponormal. $\qquad\square$

Kim [13] investigated the tensor product of log-hyponormal (reps. w-hyponormal, iw-hyponormal) operators. Now, we consider the case of Tracy-Singh products.

Lemma 15 ([6])**.** *Let S and T be positive invertible operators. Then $\log T \geqslant \log S$ if and only if $T^p \geqslant \left(T^{\frac{p}{2}} S^p T^{\frac{p}{2}} \right)^{\frac{1}{2}}$ for all $p \geqslant 0$.*

Theorem 16. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be positive invertible operators. If A and B are log-hyponormal, then $A \boxtimes B$ is also log-hyponormal.*

Proof. Assume that A and B are log-hyponormal operators. Since A and B are invertible, Lemma 1 implies that $A \boxtimes B$ is invertible. Using Lemmas 1 and 3,

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we obtain that for any $p \geqslant 0$,

$$
\begin{aligned}
&[(A \boxtimes B)^*(A \boxtimes B)]^p \\
&\quad = (A^*A \boxtimes B^*B)^p \\
&\quad = (A^*A)^p \boxtimes (B^*B)^p \\
&\quad \geqslant [(A^*A)^{\frac{p}{2}}(AA^*)^p(A^*A)^{\frac{p}{2}}]^{\frac{1}{2}} \boxtimes [(B^*B)^{\frac{p}{2}}(BB^*)^p(B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [(A^*A)^{\frac{p}{2}}(AA^*)^p(A^*A)^{\frac{p}{2}} \boxtimes (B^*B)^{\frac{p}{2}}(BB^*)^p(B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [(A^*A \boxtimes B^*B)^{\frac{p}{2}}(AA^* \boxtimes BB^*)^p(A^*A \boxtimes B^*B)^{\frac{p}{2}}]^{\frac{1}{2}} \\
&\quad = [((A \boxtimes B)^*(A \boxtimes B))^{\frac{p}{2}}((A \boxtimes B)(A \boxtimes B)^*)^p((A \boxtimes B)^*(A \boxtimes B))^{\frac{p}{2}}]^{\frac{1}{2}}.
\end{aligned}
$$

By Lemma 15, we have $\log(A \boxtimes B)^*(A \boxtimes B) \geqslant \log(A \boxtimes B)(A \boxtimes B)^*$. This means that $A \boxtimes B$ is log-hyponormal. $\square$

Lemma 17 ([1]). *An operator $T \in \mathcal{B}(\mathbb{H})$ is w-hyponormal if and only if $|T| \geqslant \left(|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$ and $|T^*| \leqslant \left(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.*

Theorem 18. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. If A and B are w-hyponormal, then $A \boxtimes B$ is also w-hyponormal.*

Proof. Assume that A and B are w-hyponormal. By applying Lemmas 1 and 3, we have

$$
\begin{aligned}
|A \boxtimes B| &= |A| \boxtimes |B| \\
&\geqslant \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}}\right)^{\frac{1}{2}} \boxtimes \left(|B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A|^{\frac{1}{2}}|A^*||A|^{\frac{1}{2}} \boxtimes |B|^{\frac{1}{2}}|B^*||B|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A|^{\frac{1}{2}} \boxtimes |B|^{\frac{1}{2}}\right)(|A^*| \boxtimes |B^*|)\left(|A|^{\frac{1}{2}} \boxtimes |B|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|A \boxtimes B|^{\frac{1}{2}}|(A \boxtimes B)^*||A \boxtimes B|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
|(A \boxtimes B)^*| &= |A^*| \boxtimes |B^*| \\
&\leqslant \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \boxtimes \left(|B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}} \boxtimes |B^*|^{\frac{1}{2}}|B||B^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= \left[\left(|A^*|^{\frac{1}{2}} \boxtimes |B^*|^{\frac{1}{2}}\right)(|A| \boxtimes |B|)\left(|A^*|^{\frac{1}{2}} \boxtimes |B^*|^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\
&= \left(|(A \boxtimes B)^*|^{\frac{1}{2}}|A \boxtimes B||(A \boxtimes B)^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

By Lemma 17, the operator $A \boxtimes B$ is w-hyponormal. $\square$

Corollary 19. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be invertible operators. If A and B are iw-hyponormal, then $A \boxtimes B$ is also iw-hyponormal.*

Proof. It follows from Lemma 1, Proposition 18 and the fact that every iw-hyponormal operator is w-hyponormal and every invertible w-hyponormal operator is iw-hyponormal ([13]). $\qquad\square$

3.3 Paranormality

Consider the following paranormality of operators; see [2, 3, 14, 18].

Definition 20. *Let $M \geqslant 1$ be a constant. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- *M-paranormal if $M^2 T^{*2} T^2 - 2\alpha T^* T + \alpha^2 I \geqslant 0$ for all $\alpha > 0$;*

- *paranormal if $T^{*2} T^2 - 2\alpha T^* T + \alpha^2 I \geqslant 0$ for all $\alpha > 0$;*

- *M^*-paranormal if $M^2 T^{*2} T^2 - 2\alpha T T^* + \alpha^2 I \geqslant 0$ for all $\alpha > 0$;*

- *$*$-paranormal if $T^{*2} T^2 - 2\alpha T T^* + \alpha^2 I \geqslant 0$ for all $\alpha > 0$.*

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is an isometry if $T^* T = I$; it is called an involution if $T^2 = I$.

Proposition 21. *Let $A \in \mathcal{B}(\mathbb{H})$, $X \in \mathcal{B}(\mathbb{K})$ and let $M \geqslant 1$ be a constant. If X is an isometry and A is M-paranormal (resp. paranormal), then $A \boxtimes X$ and $X \boxtimes A$ are M-paranormal (resp. paranormal).*

Proof. Assume that A is M-paranormal and X is an isometry. It follows that for any $\alpha > 0$ we have

$$M^2 (A \boxtimes X)^{*2} (A \boxtimes X)^2 - 2\alpha (A \boxtimes X)^* (A \boxtimes X) + \alpha^2 (I \boxtimes I)$$
$$= M^2 A^{*2} A^2 \boxtimes X^{*2} X^2 - 2\alpha A^* A \boxtimes X^* X + \alpha^2 I \boxtimes I$$
$$= M^2 A^{*2} A^2 \boxtimes I - 2\alpha A^* A \boxtimes I + \alpha^2 I \boxtimes I$$
$$= \left(M^2 A^{*2} A^2 - 2\alpha A^* A + \alpha^2 I \right) \boxtimes I$$
$$\geqslant 0.$$

Thus $A \boxtimes X$ is M-paranormal. Similarly, the operator $X \boxtimes A$ is M-paranormal. The case of paranormality is just the case of M-paranormality when $M = 1$. $\quad\square$

Proposition 22. *Let $A \in \mathcal{B}(\mathbb{H})$, $X \in \mathcal{B}(\mathbb{K})$ and let $M \geqslant 1$ be a constant. If X is a self-adjoint involution and A is an M^*-paranormal (resp. $*$-paranormal) operator, then $A \boxtimes X$ and $X \boxtimes A$ are M-paranormal (resp. $*$-paranormal).*

Proof. The proof is similar to that of Proposition 21. $\qquad\square$

Ando [2] showed that for any paranormal operator A, the tensor products $A \otimes I$ and $I \otimes A$ are paranormal. The next result is an extension of this fact to the case of Tracy-Singh products.

Corollary 23. *Let $A \in \mathcal{B}(\mathbb{H})$ and let $M \geqslant 1$ be a constant. If A satisfies one of the following properties, then the same property hold for $A \boxtimes I$ and $I \boxtimes A$: paranormal, M-paranormal, $*$-paranormal, M^*-paranormal.*

4 Tracy-Singh products and operators of type nilpotent, Hermitian, and isometry

In this section, we discuss relationship between Tracy-Singh products and certain classes of operators, namely, nilpotent operators, (skew)-Hermitian operators, (co)isometry operators, and unitary operators. Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is said to be nilpotent if $T^k = 0$ for some natural number k.

Proposition 24. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. Then $A \boxtimes B$ is nilpotent if and only if A or B is nilpotent.*

Proof. It follows directly from Lemmas 1 and 4. $\qquad\square$

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is Hermitian if $T^* = T$, and T is skew-Hermitian if $T^* = -T$. It follows from Lemma 1 that the Tracy-Singh product of Hermitian operators is also Hermitian. The Tracy-Singh product of two skew-Hermitian operators is Hermitian. The Tracy-Singh product between a Hermitian operator and a skew-Hermitian operator is skew-Hermitian.

Proposition 25. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ be nonzero operators.*

1. *Assume $A \boxtimes B$ is Hermitian. Then A is Hermitian (resp. skew-Hermitian) if and only if B is Hermitian (resp. skew-Hermitian).*

2. *Assume $A \boxtimes B$ is skew-Hermitian. Then A is Hermitian (resp. skew-Hermitian) if and only if B is skew-Hermitian (resp. Hermitian).*

Proof. It follows directly from Lemmas 1 and 4. $\qquad\square$

Recall that an operator $T \in \mathcal{B}(\mathbb{H})$ is a coisometry if $TT^* = I$. A unitary operator is an operator which is both an isometry and a coisometry. Stochel [21] gave a necessary and sufficient condition for $A \otimes B$ to be an isometry (resp. a coisometry, unitary). Now, we will extend this result to the case of Tracy-Singh products.

Proposition 26. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B$ is an isometry (resp. a coisometry) if and only if so are αA and $\alpha^{-1} B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

Proof. If αA and $\alpha^{-1} B$ are isometries, then by Lemma 1,

$$
\begin{aligned}
(A \boxtimes B)^* (A \boxtimes B) &= A^* A \boxtimes B^* B \\
&= (\alpha A)^* (\alpha A) \boxtimes (\alpha^{-1} B)^* (\alpha^{-1} B) \\
&= I \boxtimes I.
\end{aligned}
$$

Suppose that $A \boxtimes B$ is an isometry. Then $A^* A \boxtimes B^* B = I \boxtimes I$. Thus, by Proposition 6, there exists $\beta \in \mathbb{C} \setminus \{0\}$ such that $\beta A^* A = I$ and $\beta^{-1} B^* B = I$. Setting $\alpha = \sqrt{\beta}$, we obtain $(\alpha A)^* (\alpha A) = I$ and $(\alpha^{-1} B)^* (\alpha^{-1} B) = I$. Hence αA and $\alpha^{-1} B$ are isometries. The proof for the case of coisometry is similar to that of isometry. $\qquad\square$

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Theorem 27. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices such that A_{ij} and B_{kl} are nonzero operators for all $i, j = 1, \ldots, m$ and $k, l = 1, \ldots, n$. Then $A \boxtimes B$ is unitary if and only if so are αA and $\alpha^{-1}B$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

Proof. If αA and $\alpha^{-1}B$ are unitary, then Lemma 1 implies

$$(A \boxtimes B)^*(A \boxtimes B) = A^*A \boxtimes B^*B = (\alpha A)^*(\alpha A) \boxtimes (\alpha^{-1}B)^*(\alpha^{-1}B) = I.$$

Similarly, we have $(A \boxtimes B)(A \boxtimes B)^* = I$. Conversely, suppose that $A \boxtimes B$ is unitary. We know that $A \boxtimes B$ is both an isometry and a coisometry. By Proposition 26, there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that αA and $\alpha^{-1}B$ are isometries, and βA and $\beta^{-1}B$ are coisometries. We have $(\alpha A)^*(\alpha A) = I = (\beta A)(\beta A)^*$ and

$$(\alpha^{-1}B)^*(\alpha^{-1}B) = I = (\beta^{-1}B)(\beta^{-1}B)^*.$$

Since $A \boxtimes B$ is normal, so are A and B (Theorem 9). Then $\alpha^2 AA^* = \alpha^2 A^*A = \beta^2 AA^*$ and $\alpha^{-2}BB^* = \alpha^{-2}B^*B = \beta^{-2}BB^*$. Since $\alpha, \beta > 0$, it comes to the conclusion that $\alpha = \beta$. Hence αA and $\alpha^{-1}B$ are unitary. $\qquad \square$

5 Tracy-Singh products and class-$\mathcal{A}$ type operators

The following classes of operators bring attention to operator theorists; see more information in [8, 9, 11, 12, 20].

Definition 28. *Let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathbb{H})$ is said to be*

- *class $\mathcal{A}$ if $|T^2| \geqslant |T|^2$;*

- *class $\mathcal{A}(k)$ if $\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geqslant |T|^2$;*

- *quasi-class $\mathcal{A}$ if $T^*|T^2|T \geqslant T^*|T|^2T$;*

- *quasi-class $(\mathcal{A}, k)$ if $T^{*k}|T^2|T^k \geqslant T^{*k}|T|^2T^k$;*

- *$*$-class $\mathcal{A}$ if $|T^2| \geqslant |T^*|^2$;*

- *quasi-$*$-class $\mathcal{A}$ if $T^*|T^2|T \geqslant T^*|T^*|^2T$;*

- *quasi-$*$-class $(\mathcal{A}, k)$ if $T^{*k}|T^2|T^k \geqslant T^{*k}|T^*|^2T^k$.*

The next theorem shows that such classes of operators are preserved under Tracy-Singh products.

Theorem 29. *Let $A \in \mathcal{B}(\mathbb{H})$, $B \in \mathcal{B}(\mathbb{K})$, and let $k \in \mathbb{N}$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: class $\mathcal{A}(k)$, class $\mathcal{A}$, quasi-class $(\mathcal{A}, k)$, quasi-class $\mathcal{A}$, $*$-class $\mathcal{A}$, quasi-$*$-class $\mathcal{A}$, quasi-$*$-class $(\mathcal{A}, k)$.*

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Proof. Assume that A and B are class $\mathcal{A}(k)$. By Lemmas 1 and 3, we get

$$
\begin{aligned}
[(A \boxtimes B)^*|A \boxtimes B|^{2k}(A \boxtimes B)]^{\frac{1}{k+1}} &= \left[(A^* \boxtimes B^*)\left(|A|^{2k} \boxtimes |B|^{2k}\right)(A \boxtimes B)\right]^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A \boxtimes B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&= \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}} \boxtimes \left(B^*|B|^{2k}B\right)^{\frac{1}{k+1}} \\
&\geqslant |A|^2 \boxtimes |B|^2 \\
&= |A \boxtimes B|^2.
\end{aligned}
$$

Hence $A \boxtimes B$ is a class $\mathcal{A}(k)$ operator. Now, assume that A and B are quasi-class $\mathcal{A}(k)$. Applying Lemmas 1 and 3, we get

$$
\begin{aligned}
(A \boxtimes B)^{k*}|(A \boxtimes B)^2|(A \boxtimes B)^k &= (A^{k*} \boxtimes B^{k*})(|A^2| \boxtimes |B^2|)(A^k \boxtimes B^k) \\
&= A^{k*}|A^2|A^k \boxtimes B^{k*}|B^2|B^k \\
&\geqslant A^{k*}|A|^2A^k \boxtimes B^{k*}|B|^2B^k \\
&= (A^{k*} \boxtimes B^{k*})(|A|^2 \boxtimes |B|^2)(A^k \boxtimes B^k) \\
&= (A \boxtimes B)^{k*}|A \boxtimes B|^2(A \boxtimes B)^k.
\end{aligned}
$$

Hence, $A \boxtimes B$ is a quasi-class $\mathcal{A}(k)$ operator. The proof for class $\mathcal{A}$ (resp. quasi-class $\mathcal{A}$) is done by replacing $k = 1$ in the case of class $\mathcal{A}(k)$ (resp. quasi-class $(\mathcal{A}, k)$). The proof for the case of quasi $*$-class $(\mathcal{A}, k)$ is similar to that of quasi-class $(\mathcal{A}, k)$. Similarly, the proof for $*$-class $\mathcal{A}$ (resp. quasi-$*$-class A) is done by replacing $k = 0$ (resp. $k = 1$) in the case of quasi-$*$-class $(\mathcal{A}, k)$. $\qquad\square$

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On unicity theorems of difference of entire or meromorphic functions

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ABSTRACT. In this article, we investigate the uniqueness problems of differences of meromorphic functions and obtain some results which can be viewed as discrete analogues of the results given by Yi. An example is given to show the results in this paper are best possible.

1 INTRODUCTION

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see, e.g., [7, 18]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r with finite linear measure. Then the meromorphic function α is called a small function of $f(z)$, if $T(r, \alpha) = S(r, f)$. If $f(z) - \alpha$ and $g(z) - \alpha$ have same zeros, counting multiplicity (ingoring multiplicity), then we say $f(z)$ and $g(z)$ share the small function α CM (IM). For a small function α related to $f(z)$, we define

$$\delta(\alpha, f) = \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{f - \alpha}\right)}{T(r, f)}.$$

In 1976, Yang [17] proposed the following problem:

Suppose that $f(z)$ and $g(z)$ are two entire functions such that $f(z)$ and $g(z)$ share 0 CM and $f'(z)$ and $g'(z)$ share 1 CM. What can be said about the relationship between $f(z)$ and $g(z)$?

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In [13], Yi answered the question posed by C. C. Y. These results may be stated as follows:

Theorem A. Let $f(z)$ and $g(z)$ be two nonconstant entire functions. Assume that $f(z)$ and $g(z)$ share 0 CM, $f'(z)$ and $g'(z)$ share 1 CM and $\delta(0, f) > \frac{1}{2}$. Then $f'(z)g'(z) \equiv 1$ unless $f(z) \equiv g(z)$.

Currently, there has been an increasing interest in studying difference equations in the complex plane. For example, Halburd and Korhonen [3, 4] established a version of Nevanlinna theory based on difference operators. Ishizaki and Yanagihara [7] deveploped a version of Wiman-Valiron theory for difference equations of entire functions of small growth. Also Chiang and Feng [1] has a difference version of Wiman-Valiron.

The main purpose of this paper is to establish partial difference counterparts of Theorem A. Our results can be stated as follows:

Theorem 1.1. Let $c_j, a_j, b_j (j = 1, 2, \cdots, k)$ be complex constants, and let $f(z)$ and $g(z)$ be two nonconstant entire functions of finite order. Assume that $f(z)$ and $g(z)$ share 0 CM, $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$ and $L(g) = \sum_{i=1}^{k} b_i g(z + c_i)$ share 1 CM and $\delta(0, f) > \frac{2}{3}$. Then $L(f)L(g) \equiv 1$ or $L(f) \equiv L(g)$.

Theorem 1.2. Let $c \in \mathbb{C} \setminus \{0\}$, and let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions of finite order satisfying $f(z + c)$ and $g(z + c)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ CM. If

$$N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + 2\overline{N}(r, f) < (\lambda + o(1))T(r), \qquad (1.1)$$

where $\lambda < 1$ and $T(r) = \max\{T(r, f), T(r, g)\}$, then $f(z)g(z) \equiv 1$ or $f(z) \equiv g(z)$.

The following example shows that Theorem 1.2 is exact.

Example 1.1. Let $f(z) = e^{2z} + e^z, g(z) = e^{-2z} - e^{-z}$. We have that $f(z + c)$ and $g(z + c)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ CM and

$$N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + 2\overline{N}(r, f) = (\lambda + o(1))T(r),$$

but $f(z) \not\equiv g(z)$ and $f(z)g(z) \not\equiv 1$.

2   Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas.
The following lemma is a difference analogue of the logarithmic derivative lemma.
Lemma 2.1 [3] *Let $f(z)$ be a meromorphic function of finite order and let c be a non-zero*

complex number. Then we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f).$$

Lemma 2.2 *Let $f(z)$ be a nonconstant entire function of finite order, and let $c_i, a_i (i = 1, 2, \cdots, k)$ be complex constants. Then*

$$T\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) \leq T(r, f(z)) + S(r, f),$$

$$N\left(r, \frac{1}{\sum_{i=1}^{k} a_i f(z + c_i)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Proof of Lemma 2.2. By Lemma 2.1, we have

$$T\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) = m\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) = m\left(r, f(z) \frac{\sum_{i=1}^{k} a_i f(z + c_i)}{f(z)}\right)$$

$$\leq \sum_{i=1}^{k} m\left(r, \frac{f(z + c_i)}{f(z)}\right) + m(r, f(z)) + O(1) = T(r, f(z)) + S(r, f). \tag{2.1}$$

$$m\left(r, \frac{1}{f(z)}\right) = m\left(r, \frac{1}{\sum_{i=1}^{k} a_i f(z + c_i)} \frac{\sum_{i=1}^{k} a_i f(z + c_i)}{f(z)}\right) \leq m\left(r, \frac{1}{\sum_{i=1}^{k} a_i f(z + c_i)}\right) + S(r, f). \tag{2.2}$$

From the first main theory and (2.2), we obtain

$$T(r, f(z)) - N\left(r, \frac{1}{f(z)}\right) \leq T\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) - N\left(r, \frac{1}{\sum_{i=1}^{k} a_i f(z + c_i)}\right) + S(r, f). \tag{2.3}$$

By (2.1) and (2.3), we deduce

$$N\left(r, \frac{1}{\sum_{i=1}^{k} a_i f(z + c_i)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f). \tag{2.4}$$

Lemma 2.3 *Assume that the conditions of Theorem 1.1 are satisfied. Then*

$$T(r, f(z)) = O\left(T(r, \sum_{i=1}^{k} a_i f(z + c_i))\right) \quad for \quad r \notin E,$$

$$T(r, g(z)) = O\left(T(r, \sum_{i=1}^{k} a_i f(z + c_i))\right) \quad for \quad r \notin E,$$

$$T\left(r, \sum_{i=1}^{k} b_i g(z + c_i)\right) = O\left(T(r, \sum_{i=1}^{k} a_i f(z + c_i))\right) \quad for \quad r \notin E,$$

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where E is a set of finite linear measure.

Proof of Lemma 2.3. By the first main theory and (2.2), we have

$$(\delta(0,f) + o(1))T(r, f(z)) \le m\Big(r, \frac{1}{f(z)}\Big) + S(r, f)$$

$$\le m\Big(r, \frac{1}{\sum_{i=1}^{k} a_i f(z + c_i)}\Big) + S(r, f) \le T\Big(r, \sum_{i=1}^{k} a_i f(z + c_i)\Big) + S(r, f).$$

And so

$$T(r, f(z)) \le \Big(\frac{1}{\delta(0,f)} + o(1)\Big)T\Big(r, \sum_{i=1}^{k} a_i f(z + c_i)\Big) + S(r, f). \tag{2.5}$$

Hence, we have

$$T(r, f(z)) = O\Big(T(r, \sum_{i=1}^{k} a_i f(z + c_i)\Big) \quad r \notin E.$$

By the second main theorem, the first main theory, Lemma 2.2 and (2.5), we have

$$T\Big(r, \sum_{i=1}^{k} b_i g(z + c_i)\Big)$$

$$< N\Big(r, \frac{1}{\sum_{i=1}^{k} b_i g(z + c_i)}\Big) + N\Big(r, \frac{1}{\sum_{i=1}^{k} b_i g(z + c_i) - 1}\Big) + S\Big(r, \sum_{i=1}^{k} b_i g(z + c_i)\Big)$$

$$\le N\Big(r, \frac{1}{g(z)}\Big) + N\Big(r, \frac{1}{\sum_{i=1}^{k} b_i g(z + c_i) - 1}\Big) + S(r, g)$$

$$= N\Big(r, \frac{1}{f(z)}\Big) + N\Big(r, \frac{1}{\sum_{i=1}^{k} a_i f(z + c_i) - 1}\Big) + S(r, g)$$

$$\le (1 - \delta(0,f) + o(1))T(r, f(z)) + T\Big(r, \sum_{i=1}^{k} a_i f(z + c_i)\Big) + S(r, g)$$

$$\le \Big(\frac{1}{\delta(0,f)} + o(1)\Big)T\Big(r, \sum_{i=1}^{k} a_i f(z + c_i)\Big) + S(r, f) + S(r, g). \tag{2.6}$$

Using the method similar to the proof of (2.3), we have

$$T(r, g(z)) - N\Big(r, \frac{1}{g(z)}\Big) \le T\Big(r, \sum_{i=1}^{k} b_i g(z + c_i)\Big) + S(r, g). \tag{2.7}$$

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From (2.5)-(2.7), we obtain

$$T(r, g(z)) \leq N\left(r, \frac{1}{g(z)}\right) + T\left(r, \sum_{i=1}^{k} b_i g(z + c_i)\right) + S(r, g)$$

$$\leq N\left(r, \frac{1}{f(z)}\right) + \left(\frac{1}{\delta(0, f)} + o(1)\right) T\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) + S(r, g)$$

$$\leq (1 - \delta(0, f) + o(1)) T(r, f(z)) + \left(\frac{1}{\delta(0, f)} + o(1)\right) T\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) + S(r, g)$$

$$\leq \left(\frac{2}{\delta(0, f)} - 1 + o(1)\right) T\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) + S(r, g) + S(r, f),$$

that is

$$T(r, g(z)) = O\left(T(r, \sum_{i=1}^{k} a_i f(z + c_i))\right) \quad r \notin E. \tag{2.8}$$

From Lemma 2.2 and (2.8), we get

$$T\left(r, \sum_{i=1}^{k} b_i g(z + c_i)\right) = O\left(T(r, \sum_{i=1}^{k} a_i f(z + c_i))\right) \quad r \notin E.$$

Lemma 2.3 thus is be proved.

Lemma 2.4 [10] *Let f_1, f_2 and f_3 be three entire functions satisfying*

$$\sum_{i=1}^{3} f_i \equiv 1.$$

If $f_1 \not\equiv$ constant, and

$$\sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) \leq (\lambda + o(1)) T(r) \quad (r \notin E),$$

where $T(r) = \max\{T(r, f_i) | i = 1, 2, 3)\}$, and $\lambda < 1$, then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Proof of Theorem 1.1.
Since $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$ and $L(g) = \sum_{i=1}^{k} b_i g(z + c_i)$ share 1 CM, we have

$$\frac{L(f) - 1}{L(g) - 1} = e^{p(z)}, \tag{2.9}$$

where $p(z)$ is polynomial.
Let $f_1 = L(f), f_2 = e^{p(z)}, f_3 = -e^{p(z)} L(g)$, by (2.9) and Lemma 2.2, we have

$$f_1 + f_2 + f_3 \equiv 1,$$

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$$N\left(r, \frac{1}{f_1}\right) \le N\left(r, \frac{1}{f}\right) + S(r, f), \tag{2.10}$$

$$N(r, f_2) = N\left(r, \frac{1}{e^{p(z)}}\right) = 0, \tag{2.11}$$

$$N(r, f_3) = N\left(r, \frac{1}{-e^p L(g)}\right) = N\left(r, \frac{1}{L(g)}\right) \le N\left(r, \frac{1}{g}\right) + S(r, g) = N\left(r, \frac{1}{f}\right) + S(r, g). \tag{2.12}$$

By Lemma 2.3, (2.5), (2.10)-(2.12) and $\delta(0, f) > \frac{2}{3}$, we have

$$\sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) \le 2N\left(r, \frac{1}{f}\right) + S(r, f) + S(r, g)$$

$$\le 2(1 - \delta(0, f) + o(1))T(r, f) + S(r, f) + S(r, g)$$

$$\le 2\left(\frac{1}{\delta(0, f)} - 1 + o(1)\right)T(r, L(f)) + S(r, L(f))$$

$$= (\lambda + o(1))T(r, L(f)) \quad r \notin E,$$

where $\lambda = 2\left(\frac{1}{\delta(0, f)} - 1\right) < 1$. From Lemma 2.4, we have

$$f_2 \equiv 1 \quad \text{or} \quad f_3 \equiv 1.$$

If $f_2 \equiv 1$, by (2.9), we obtain $\frac{L(f)-1}{L(g)-1} = e^{p(z)} \equiv 1$. Hence, we have $L(f) \equiv L(g)$.

If $f_3 \equiv 1$, that is $-e^{p(z)} L(g) \equiv 1$. So $L(g) = -e^{-p(z)}$. By $L(f) + 1 + e^{p(z)} = 1$, we have $L(f) = -e^{p(z)}$. So $L(f)L(g) \equiv 1$. Theorem 1.1 thus is proved.

3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemmas.

Lemma 3.1 [14] *Let f_1 and f_2 be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If $c_1 f_1 + c_2 f_2 = c_3$, then*

$$T(r, f_1) < N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 3.2 [12] *Let $f_1, f_2, \cdots, f_n$ be linearly independent meromorphic functions satisfying $\sum_{i=1}^{n} f_i \equiv 1$. Then for $j = 1, 2, \cdots, n$, we have*

$$T(r, f_j) < \sum_{i=1}^{n} N\left(r, \frac{1}{f_i}\right) + N(r, f_j) + N(r, D) - \sum_{i=1}^{n} N(r, f_i) - N(r, \frac{1}{D})$$

$$+O(\log r + \log T_n(r)) \quad \text{for} \quad r \notin E,$$

where D denotes the Wronskian

$$D = \begin{vmatrix} f_1, & f_2, & \cdots, & f_n \\ f_1', & f_2', & \cdots, & f_n' \\ \cdots, & \cdots, & \cdots, & \cdots \\ f_1^{(n-1)}, & f_2^{(n-1)}, & \cdots, & f_n^{(n-1)} \end{vmatrix}$$

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and $T_n(r)$ denotes the maximum of $T(r, f_i), i = 1, 2, \cdots, n$.

Lemma 3.3 [14] *Let f_1, f_2 and f_3 be three nonconstant meromorphic functions satisfying $\sum_{i=1}^{3} f_i \equiv 1$, and let $g_1 = -\frac{f_3}{f_2}, g_2 = \frac{1}{f_2}, g_3 = -\frac{f_1}{f_2}$. If f_1, f_2 and f_3 are linearly independent, then g_1, g_2 and g_3 are linearly independent.*

Lemma 3.4 [8] *Let $f(z)$ be a meromorphic function of finite order, $c \neq 0$ be fixed. Then*

$$\overline{N}(r, f(z + c)) \leq \overline{N}(r, f(z)) + S(r, f),$$

$$N\left(r, \frac{1}{f(z + c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Remark. 1 Using the same method of Lemma 3.4, we obtain

$$N\left(r, \frac{1}{f(z + c)}\right) \geq N\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

From this and Lemma 3.4, we deduce

$$N\left(r, \frac{1}{f(z + c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Lemma 3.5 [5, 6] *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then*

$$T(r, f(z + |c|)) = T(r, f(z)) + S(r, f).$$

Remark. 2 It is shown in [2, p. 66], that for $c \in \mathbb{C} \setminus \{0\}$, we have

$$(1 + o(1))T(r - |c|, f(z)) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f(z))$$

hold as $r \to \infty$, for a general meromorphic. By this and Lemma 3.5, we obtain

$$T(r, f(z + c)) = T(r, f(z)) + S(r, f)$$

Lemma 3.6 [7] *Let $P(z)$ be a polynomial of degree m, and let $F(z) = \frac{R(z)}{Q(z)}$, where $Q(z)$ is a polynomial of degree $n \geq 1$ and $R(z)$ is a polynomial of degree less than n which has no common factor with $Q(z)$. Then*

$$T(r, P) = N(r, \frac{1}{P}) + O(1) = m \log r + O(1),$$

$$T(r, F) = n \log r + O(1),$$

$$N(r, \frac{1}{F}) \leq (n - 1) \log r + O(1),$$

$$T(r, F + P) = N(r, \frac{1}{F + P}) + O(1) = (n + m) \log r + O(1).$$

Lemma 3.7 *Let $f(z)$ be an nonconstant finite order meromorphic function, and let $g(z)$ be a rational function such that $f(z + c)$ and $g(z + c)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ CM. If*

$$N(r) < (\lambda + o(1))T(r), \tag{3.1}$$

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where $N(r) = \max\{N(r, \frac{1}{f}), N(r, \frac{1}{g})\}$, *and* $\lambda(< 1)$ *is a positive constant, then* $f(z) \equiv g(z)$.
Proof of Lemma 3.7. If $f(z)$ is a transcendental function, then

$$T(r, g) = S(r, f).$$

By this, $f(z + c)$ and $g(z + c)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ CM, we obtain

$$N\left(r, \frac{1}{f(z+c)-1}\right) = N\left(r, \frac{1}{g(z+c)-1}\right) \leq T(r, g(z+c)) + O(1) = S(r, f),$$

$$N(r, f) = N(r, g) \leq T(r, g) = S(r, f).$$

By the second main theorem and Remark 1, we deduce that

$$T(r, f(z)) < N\left(r, \frac{1}{f(z)}\right) + N(r, f(z)) + N\left(r, \frac{1}{f(z)-1}\right) + S(r, f)$$

$$= N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+c)-1}\right) + S(r, f)$$

$$= N\left(r, \frac{1}{f(z)}\right) + S(r, f). \tag{3.2}$$

(3.1) and (3.2) imply that

$$T(r) < (\lambda + o(1))T(r),$$

a contradiction.
If $f(z)$ is a polynomial, by $f(z)$ and $g(z)$ share ∞ CM, we see that $g(z)$ is a polynomial.
By Lemma 3.6, we obtain that

$$T(r, f) = N\left(r, \frac{1}{f}\right) + O(1),$$

$$T(r, g) = N\left(r, \frac{1}{g}\right) + O(1),$$

and hence $T(r) = N(r) + O(1)$. From this and (3.1), we get

$$T(r) < (\lambda + o(1))T(r),$$

a contradiction.
If $f(z)$ is a rational function which is not a polynomial, then $f(z + c)$ is not a constant.
Since $f(z + c)$ and $g(z + c)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ CM, we deduce

$$\frac{f(z+c)-1}{g(z+c)-1} \equiv \alpha,$$

where α is a constant. And so

$$f(z+c) - \alpha g(z+c) \equiv 1 - \alpha. \tag{3.3}$$

Let $g(z) = \frac{m(z)}{n(z)} + P_1(z)$, where $P_1(z)$ is a polynomial, $n(z)$ is a polynomial of degree $n \geq 1$ and $m(z)$ is a polynomial of degree less than n which has no common factor with $n(z)$. By (3.3), we have

$$f(z) = \frac{\alpha m(z)}{n(z)} + P_2(z),$$

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where $P_2(z) = 1 - \alpha + \alpha P_1(z)$. By Lemma 3.6 and (3.1), we have $P_1(z) \equiv 0$ and $P_2(z) \equiv 0$. Hence $\alpha = 1$. So $f(z) \equiv g(z)$. Lemma 3.7 is thus be proved.

Proof of Theorem 1.2.

By Lemma 3.7, it may be assumed that $f(z)$ and $g(z)$ are two transcendental meromorphic functions. By $f(z+c)$ and $g(z+c)$ share $1\,CM$, $f(z)$ and $g(z)$ share $\infty\,CM$, we obtain

$$f(z+c) - 1 = e^{h(z)}(g(z+c) - 1),$$

where $h(z)$ is a polynomial, and so

$$f(z+c) + e^{h(z)} - e^{h(z)}g(z+c) = 1. \tag{3.4}$$

Let $f_1 = f(z+c), f_2 = e^{h(z)}, f_3 = -e^{h(z)}g(z+c)$ and let $T_1(r) = \max\{T(r, f_i)|i = 1, 2, 3\}$. By (3.4), we have

$$\sum_{i=1}^{3} f_i \equiv 1, \tag{3.5}$$

$$\sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) = N\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{g(z+c)}\right), \tag{3.6}$$

$$T_1(r) = O(T(r)). \tag{3.7}$$

We divide the proof in two parts:

Case 1. Suppose that f_1, f_2 and f_3 are linearly independent. From Lemma 3.2 and (3.7), we deduce

$$T(r, f(z+c)) < \sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) + N(r, D) - N(r, f_2) - N(r, f_3) + o(T(r)), \tag{3.8}$$

for $r \notin E$, where

$$D = \begin{vmatrix} f_1, & f_2, & f_3 \\ f_1', & f_2', & f_3' \\ f_1'', & f_2'', & f_3'' \end{vmatrix}$$

By the above and (3.5), we obtain

$$D = \begin{vmatrix} f_2', & f_3' \\ f_2'', & f_3'' \end{vmatrix}$$

Now combining this and Lemma 3.4, we deduce

$$N(r, D) - N(r, f_2) - N(r, f_3) \leq N(r, g''(z+c)) - N(r, g(z+c))$$
$$= 2\overline{N}(r, g(z+c)) \leq 2\overline{N}(r, g(z)) + S(r, g) = 2\overline{N}(r, f(z)) + S(r, g). \tag{3.9}$$

By Remark 1, we have

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f), \quad N\left(r, \frac{1}{g(z+c)}\right) = N\left(r, \frac{1}{g(z)}\right) + S(r, g). \tag{3.10}$$

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By (3.6), (3.10), we have

$$\sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) = N\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{g(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{g(z)}\right) + o(T(r)),$$
(3.11)

for $r \notin E$. By (3.8), (3.9), (3.11) and Remark 2, we have

$$T(r, f(z)) = T(r, f(z+c)) + S(r, f) < N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{g(z)}\right) + 2\overline{N}(r, f(z)) + O(T(r)),$$
(3.12)

for $r \notin E$.

Let $g_1 = -\frac{f_3}{f_2} = g(z+c), g_2 = \frac{1}{f_2} = e^{-h(z)}, g_3 = \frac{-f_1}{f_2} = -e^{-h(z)}f(z+c)$. By (3.4), we have

$$\sum_{i=1}^{3} g_i \equiv 1.$$

From Lemma 3.3, we see that g_1, g_2 and g_3 are linearly independent. Using the similar method as above, we obtain that

$$T(r, g(z)) < N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{g(z)}\right) + 2\overline{N}(r, f(z)) + o(T(r)),$$
(3.13)

for $r \notin E$. From (3.12) and (3.13), we know that

$$T(r) < N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + 2\overline{N}(r, f) + o(T(r)),$$

for $r \notin E$. From (1.1) and (3.13), we have

$$T(r) < (\lambda + o(1))T(r),$$
(3.14)

(3.14) is impossible.

Case 2. Suppose that f_1, f_2, f_3 are linearly dependent. Then there exist three constants $(c_1, c_2, c_3) \neq (0, 0, 0)$ satisfying

$$\sum_{i=1}^{3} c_i f_i = 0.$$
(3.15)

If $c_1 = 0$, (3.15) yields $c_2 \neq 0, c_3 \neq 0$ and $f_3 = -\frac{c_2}{c_3}f_2$. So $g(z+c) = \frac{c_2}{c_3}$, a contradiction. Hence we have, $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1}f_2 - \frac{c_3}{c_1}f_3.$$
(3.16)

(3.5) and (3.16) imply that

$$\left(1 - \frac{c_2}{c_1}\right)f_2 + \left(1 - \frac{c_3}{c_1}\right)f_3 = 1.$$
(3.17)

Next we deal with the following three subcases.

Subcase 2.1. Assume that $c_1 = c_2$. Then by (3.17), we have $c_1 \neq c_3$ and

$$f_3 = \frac{c_1}{c_1 - c_3}.$$

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And so

$$g(z+c) = -\frac{c_1}{c_1 - c_3} e^{-h(z)}. \tag{3.18}$$

Hence $g(z)$ is an entire function, and $N\left(r, \frac{1}{g(z+c)}\right) = 0$. By (3.4) and (3.18), we obtain that

$$f(z+c) - \frac{c_1}{c_1 - c_3} \frac{1}{g(z+c)} = -\frac{c_3}{c_1 - c_3}. \tag{3.19}$$

If $c_3 \neq 0$, by (3.18), (3.19), Lemma 3.1 and the first main theory, we deduce that

$$T(r, f(z+c)) < N\left(r, \frac{1}{f(z+c)}\right) + S(r, f), \tag{3.20}$$

$$T(r, g(z+c)) < N\left(r, \frac{1}{f(z+c)}\right) + S(r, f). \tag{3.21}$$

From (3.20), (3.21), Remark 1 and Remark 2, we have

$$T(r, f(z)) = T(r, f(z+c)) + S(r, f) < N\left(r, \frac{1}{f(z+c)}\right) + S(r, f) = N\left(r, \frac{1}{f(z)}\right) + S(r, f),$$

and

$$T(r, g(z)) = T(r, g(z+c)) + S(r, f) < N\left(r, \frac{1}{f(z+c)}\right) + S(r, f) = N\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

By the above, we deduce that

$$T(r) < N\left(r, \frac{1}{f}\right) + o(T(r)) \quad for \quad r \notin E,$$

which contradicts our assumption (1.1). Thus $c_3 = 0$. Hence from (3.19), we have $f(z + c)g(z + c) \equiv 1$, so, $f(z)g(z) \equiv 1$.

Subcase 2.2. Assume that $c_1 = c_3$. From (3.17), we have $c_1 \neq c_2$ and $f_2 = \frac{c_1}{c_1 - c_2}$.
And so

$$e^{h(z)} = \frac{c_1}{c_1 - c_2}. \tag{3.22}$$

By (3.4) and (3.22), we obtain that

$$f(z+c) - \frac{c_1}{c_1 - c_2} g(z+c) = -\frac{c_2}{c_1 - c_2}. \tag{3.23}$$

If $c_2 \neq 0$, from (3.23), Lemma 3.1, Remark 1 and Remark 2, we obtain that

$$\begin{aligned}
T(r, f(z)) &= T(r, f(z+c)) + S(r, f) \\
&< N\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{g(z+c)}\right) + \overline{N}(r, f(z)) + S(r, f) \\
&= N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{g(z)}\right) + \overline{N}(r, f(z)) + S(r, f).
\end{aligned} \tag{3.24}$$

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Using the similar method as above, we can get

$$T(r,g) < N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + S(r,f). \tag{3.25}$$

By (3.24) and (3.25), we obtain that

$$T(r) < N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + o(T(r)) \quad \text{for} \quad r \notin E,$$

a contradiction. Thus $c_2 = 0$, and by (3.23), we obtain that $f(z+c) = g(z+c)$. Hence $f(z) = g(z)$.

Case 2.3. Assume that $c_1 \neq c_2$ and $c_1 \neq c_3$. From (3.17), we obtain

$$(c_1 - c_3)g(z+c) + c_1 e^{-h(z)} = c_1 - c_2. \tag{3.26}$$

By (3.26), Lemma 3.1, Remark 1 and Remark 2, we obtain that

$$T(r,g(z)) = T(r,g(z+c)) + S(r,g) < N\left(r,\frac{1}{g(z+c)}\right) + S(r,g) = N\left(r,\frac{1}{g(z)}\right) + S(r,g). \tag{3.27}$$

By (3.4) and (3.26), we have

$$(c_3 - c_1)f(z+c) + (c_3 - c_2)e^{h(z)} = c_3. \tag{3.28}$$

If $c_3 \neq 0$, by (3.28), Lemma 3.1, Remark 1 and Remark 2, we obtain that

$$T(r,f(z)) = T(r,f(z+c)) + S(r,f) < N\left(r,\frac{1}{f(z+c)}\right) + S(r,f) = N\left(r,\frac{1}{f(z)}\right) + S(r,f). \tag{3.29}$$

By (3.27) and (3.29), we see

$$T(r) < N(r,\frac{1}{f}) + N\left(r,\frac{1}{g}\right) + o(T(r)) \quad \text{for} \quad r \notin E,$$

which contradicts the assumption (1.1).

If $c_3 = 0$, by (3.28), we obtain

$$f(z+c) = -\frac{c_2}{c_1}e^{h(z)}. \tag{3.30}$$

Hence $f(z)$ is an entire function, and $N(r,\frac{1}{f(z+c)}) = 0$. From (3.26) and (3.30), we have

$$c_2\frac{1}{f(z+c)} - c_1 g(z+c) = c_2 - c_1. \tag{3.31}$$

By (3.31), Lemma 3.1, Remark 1 and Remark 2, we have

$$T(r,f(z)) = T(r,f(z+c)) + S(r,f) < N\left(r,\frac{1}{g(z+c)}\right) + S(r,f) = N\left(r,\frac{1}{g(z)}\right) + S(r,f). \tag{3.32}$$

By (3.27) and (3.32), we have

$$T(r) < N\left(r,\frac{1}{g}\right) + o(T(r)) \quad \text{for} \quad r \notin E,$$

which contradicts the assumption (1.1).

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Acknowledgements

The author would like to thank the referee for his/her helpful suggestions and comments.

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On the existence and behavior of the solutions for some difference equations systems

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ABSTRACT

The main goal of this paper, is to study the existence of solutions for a class of nonlinear systems of difference equations of order four, in four-dimensional with the initial conditions are real numbers. Moreover, we study some behavior such as the periodicity and boundedness of solutions for such systems. Finally, some numerical examples are presented and graphed by Matlab.

Keywords: Difference equations, Recursive sequences, Periodic solutions, System of difference equations.

Mathematics Subject Classification: 39A10, 39A11, 39A99, 34C99.

1. INTRODUCTION

In this paper, we are concerned with the existence of solutions for the rational systems of difference equations of order four in four-dimensional case

$$
\begin{array}{ll}
x_{n+1} & = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}z_{n-2}t_{n-1}x_n}, \\
z_{n+1} & = \frac{z_{n-3}}{\pm 1 \pm z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{\pm 1 \pm t_{n-3}x_{n-2}y_{n-1}z_n},
\end{array}
$$

with the initial conditions are real numbers. Although we study the dynamics of these solutions such as the periodicity and boundedness and give some numerical examples for the systems.

The study of nonlinear difference equations systems of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations systems of order greater than one come from the results for rational difference equations [1-45]. Therefore, the study of rational difference equations systems of order greater than one is worth further consideration.

Recently, Din et al. [3] studied the local asymptotic stability of an equilibrium point, periodicity behavior of positive solutions, and global character of an equilibrium point of a fourth-order system of rational difference equations

$$
x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_{n-3}y_{n-2}y_{n-1}y_n}, \ y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_{n-3}x_{n-2}x_{n-1}x_n}.
$$

El-Dessoky [4] has investigated the solutions of the rational equation systems

$$
x_{n+1} = \frac{y_{n-1}y_{n-2}}{x_n(\pm 1 \pm y_{n-1}y_{n-2})}, \ y_{n+1} = \frac{x_{n-1}x_{n-2}}{y_n(\pm 1 \pm x_{n-1}x_{n-2})}.
$$

Yalçınkaya [5] got the sufficient conditions for the global asymptotic stability of the system of nonlinear difference equations

$$
x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \ y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.
$$

In [6] Yang et al. studied the system of high order rational difference equations

$$x_n = \frac{a}{y_{n-p}}, \ y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}.$$

In [7], Kulenovic et al. obtained the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1} = \frac{x_n+a}{y_n+b}, \ y_{n+1} = \frac{y_n+c}{z_n+d}, \ z_{n+1} = \frac{z_n+e}{x_n+f}.$$

In [8] El-Dessoky obtained the form of the solutions and periodicity of some systems of rational difference equations

$$x_{n+1} = \frac{z_{n-3}}{a_1+b_1 z_n y_{n-1}x_{n-2}z_{n-3}}, \ y_{n+1} = \frac{x_{n-3}}{a_2+b_2 x_n z_{n-1}y_{n-2}x_{n-3}}, z_{n+1} = \frac{y_{n-3}}{a_3+b_3 y_n x_{n-1}z_{n-2}y_{n-3}}.$$

2. SYSTEMS AND THE FORMULATION OF THEIR SOLUTIONS

Here we interest to investigate the following system of some rational difference equations

$$
\begin{aligned}
x_{n+1} &= \frac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n}, \\
z_{n+1} &= \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{1+t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned} \tag{1}
$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

THEOREM 2.1. *Suppose that $\{x_n, \ y_n, \ z_n, \ t_n\}$ are solutions of system (1). Then for $n = 0, \ 1, \ 2, \ ...$, we see that*

$$x_{4n-3} = x_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}, \ x_{4n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})},$$

$$x_{4n-1} = x_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}, \ x_{4n} = x_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})},$$

$$y_{4n-3} = y_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}, \ y_{4n-2} = y_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})},$$

$$y_{4n-1} = y_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}, \ y_{4n} = y_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})},$$

$$z_{4n-3} = z_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}, \ z_{4n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})},$$

$$z_{4n-1} = z_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}, \ z_{4n} = z_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})},$$

$$t_{4n-3} = t_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}, \ t_{4n-2} = t_{-2}\prod_{i=0}^{n-1}\frac{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})},$$

$$t_{4n-1} = t_{-1}\prod_{i=0}^{n-1}\frac{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}, \ t_{4n} = t_0\prod_{i=0}^{n-1}\frac{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})},$$

where $\prod_{i=0}^{-1}B_i = 1$.

Proof: For $n = 0$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$, that is,

$$x_{4n-7} = x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}, \ x_{4n-6} = x_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})},$$

$$x_{4n-5} = x_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}, \ x_{4n-4} = x_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})},$$

$$y_{4n-7} = y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}, \ y_{4n-6} = y_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})},$$

$$y_{4n-5} = y_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}, \ y_{4n-4} = y_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})},$$

$$z_{4n-7} = z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}, \ z_{4n-6} = z_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})},$$

$$z_{4n-5} = z_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}, \ z_{4n-4} = z_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})},$$

$$t_{4n-7} = t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}, \ t_{4n-6} = t_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})},$$

$$t_{4n-5} = t_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}, \ t_{4n-4} = t_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})}.$$

We deduce from system (1) that

$$x_{4n-3} = \frac{x_{4n-7}}{1+t_{4n-4}z_{4n-5}y_{4n-6}x_{4n-7}}$$

$$= \frac{x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}}{1+\left[\begin{array}{c}t_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})}z_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+3)t_0z_{-1}y_{-2}x_{-3})}\\ y_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+2)t_0z_{-1}y_{-2}x_{-3})}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}\end{array}\right]}$$

$$= \frac{x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}}{1+\left[t_0z_{-1}y_{-2}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+4)t_0z_{-1}y_{-2}x_{-3})}\right]} = \frac{x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}}{1+\frac{t_0z_{-1}y_{-2}x_{-3}}{(1+(4i-4)t_0z_{-1}y_{-2}x_{-3})}}$$

$$= x_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})}\frac{(1+(4i-4)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i-3)t_0z_{-1}y_{-2}x_{-3})} = x_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)t_0z_{-1}y_{-2}x_{-3})}{(1+(4i+1)t_0z_{-1}y_{-2}x_{-3})},$$

$$y_{4n-3} = \frac{y_{4n-7}}{1+x_{4n-4}t_{4n-5}z_{4n-6}y_{4n-7}}$$

$$= \frac{y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}}{1+\left[\begin{array}{c}x_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})}t_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+3)x_0t_{-1}z_{-2}y_{-3})}\\ z_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+2)x_0t_{-1}z_{-2}y_{-3})}y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}\end{array}\right]}$$

$$= \frac{y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}}{1+\left[x_0t_{-1}z_{-2}y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+4)x_0t_{-1}z_{-2}y_{-3})}\right]}$$

$$= \frac{y_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}}{1+\frac{x_0t_{-1}z_{-2}y_{-3}}{(1+(4i-4)x_0t_{-1}z_{-2}y_{-3})}} = y_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)x_0t_{-1}z_{-2}y_{-3})}{(1+(4i+1)x_0t_{-1}z_{-2}y_{-3})}.$$

$$z_{4n-3} = \frac{z_{4n-7}}{1+y_{4n-4}x_{4n-5}t_{4n-6}z_{4n-7}}$$

$$= \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}}{1+\left[\begin{array}{l} y_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})}x_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+3)y_0x_{-1}t_{-2}z_{-3})} \\ t_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+2)y_0x_{-1}t_{-2}z_{-3})}z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})} \end{array}\right]}$$

$$= \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}}{1+y_0x_{-1}t_{-2}z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+4)y_0x_{-1}t_{-2}z_{-3})}} = z_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)y_0x_{-1}t_{-2}z_{-3})}{(1+(4i+1)y_0x_{-1}t_{-2}z_{-3})}.$$

Finally from Eq.(1), we see that

$$t_{4n-3} = \frac{t_{4n-7}}{1+z_{4n-4}y_{4n-5}x_{4n-6}t_{4n-7}}$$

$$= \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}}{1+\left[\begin{array}{l} z_0\prod_{i=0}^{n-2}\frac{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})}y_{-1}\prod_{i=0}^{n-2}\frac{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+3)z_0y_{-1}x_{-2}t_{-3})} \\ x_{-2}\prod_{i=0}^{n-2}\frac{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+2)z_0y_{-1}x_{-2}t_{-3})}t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})} \end{array}\right]}$$

$$= \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}}{1+\left[z_0\prod_{i=0}^{n-2}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+4)z_0y_{-1}x_{-2}t_{-3})}\right]} = t_{-3}\prod_{i=0}^{n-1}\frac{(1+(4i)z_0y_{-1}x_{-2}t_{-3})}{(1+(4i+1)z_0y_{-1}x_{-2}t_{-3})}.$$

Similarly we can prove the other relations. This completes the proof.

Lemma 1. Let $\{x_n,\ y_n,\ z_n,\ t_n\}$ be a positive solution of system (1), then every solution of system (1) is bounded and converges to zero.

Proof: It follows from System (1) that

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n} < x_{n-3}, \quad y_{n+1} = \frac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n} < y_{n-3},$$

$$z_{n+1} = \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n} < z_{n-3}, \quad t_{n+1} = \frac{t_{n-3}}{1+t_{n-3}x_{n-2}y_{n-1}z_n} < t_{n-3},$$

we see that

$$x_{n+1} < x_{n-3}, \quad y_{n+1} < y_{n-3}, \ z_{n+1} < z_{n-3}, \quad t_{n+1} < t_{n-3},$$

Then the subsequences $\{x_{4n-3}\}_{n=0}^{\infty}, \{x_{4n-2}\}_{n=0}^{\infty}, \{x_{4n-1}\}_{n=0}^{\infty}, \{x_{4n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-3},\ x_{-2},\ x_{-1},\ x_0\}$. Also, for the other subsequences of the main sequences $\{y_n\}, \{z_n\}$, and $\{t_n\}_{n=0}^{\infty}$.

Lemma 2. If $x_i,\ y_i,\ z_i,\ t_i,\ i = -3,\ -2,\ -1,\ 0$ arbitrary real numbers and let $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of system (1) then the following statements are true:-

(i) If $x_{-3} = 0$, then we have $x_{4n-3} = 0$ and $y_{4n-2} = y_{-2},\ z_{4n-1} = z_{-1},\ t_{4n} = t_0$.

(ii) If $x_{-2} = 0$, then we have $x_{4n-2} = 0$ and $t_{4n-3} = t_{-3},\ z_{4n} = z_0,\ y_{4n-1} = y_{-1}$.

(iii) If $x_{-1} = 0$, then we have $x_{4n-1} = 0$ and $z_{4n-3} = z_{-3},\ t_{4n-2} = t_{-2},\ y_{4n} = y_0$.

(iv) If $x_0 = 0$, then we have $x_{4n} = 0$ and $y_{4n-3} = y_{-3},\ z_{4n-2} = z_{-2},\ t_{4n-1} = t_{-1}$.

(v) If $y_{-3} = 0$, then we have $y_{4n-3} = 0$ and $z_{4n-2} = z_{-2},\ t_{4n-1} = t_{-1},\ x_{4n} = x_0$.

(vi) If $y_{-2} = 0$, then we have $y_{4n-2} = 0$ and $x_{4n-3} = x_{-3}$, $z_{4n-1} = z_{-1}$, $t_{4n} = t_0$.

(vii) If $y_{-1} = 0$, then we have $y_{4n-1} = 0$ and $t_{4n-3} = t_{-3}$, $z_{4n} = z_0$, $x_{4n-2} = x_{-2}$.

(viii) If $y_0 = 0$, then we have $y_{4n} = 0$ and $z_{4n-3} = z_{-3}$, $t_{4n-2} = t_{-2}$, $x_{4n-1} = x_{-1}$.

(ix) If $z_{-3} = 0$, then we have $z_{4n-3} = 0$ and $t_{4n-2} = t_{-2}$, $y_{4n} = y_0$, $x_{4n-1} = x_{-1}$.

(x) If $z_{-2} = 0$, then we have $z_{4n-2} = 0$ and $y_{4n-3} = y_{-3}$, $t_{4n-1} = t_{-1}$, $x_{4n} = x_0$.

(xi) If $z_{-1} = 0$, then we have $z_{4n-1} = 0$ and $x_{4n-3} = x_{-3}$, $y_{4n-2} = y_{-2}$, $t_{4n} = t_0$.

(xii) If $z_0 = 0$, then we have $z_{4n} = 0$ and $t_{4n-3} = t_{-3}$, $y_{4n-1} = y_{-1}$, $x_{4n-2} = x_{-2}$.

(xiii) If $t_{-3} = 0$, then we have $t_{4n-3} = 0$ and $z_{4n} = z_0$, $y_{4n-1} = y_{-1}$, $x_{4n-2} = x_{-2}$.

(ivx) If $t_{-2} = 0$, then we have $t_{4n-2} = 0$ and $z_{4n-3} = z_{-3}$, $y_{4n} = y_0$, $x_{4n-1} = x_{-1}$.

(vx) If $t_{-1} = 0$, then we have $t_{4n-1} = 0$ and $y_{4n-3} = y_{-3}$, $z_{4n-2} = z_{-2}$, $x_{4n} = x_0$.

(vxi) If $t_0 = 0$, then we have $t_{4n} = 0$ and $x_{4n-3} = x_{-3}$, $y_{4n-2} = y_{-2}$, $z_{4n-1} = z_{-1}$.

Proof: The proof follows directly from the expressions of the solutions of system (1).

THEOREM 2.2. *Let $\{x_n, y_n, z_n, t_n\}$ are solutions of the system*

$$x_{n+1} = \frac{x_{n-3}}{-1+x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{-1+y_{n-3}z_{n-2}t_{n-1}x_n},$$

$$z_{n+1} = \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{1-t_{n-3}x_{n-2}y_{n-1}z_n}, \tag{2}$$

with the initial values are arbitrary real numbers satisfies $t_0z_{-1}y_{-2}x_{-3} = y_0x_{-1}t_{-2}z_{-3} \neq \pm 1$, $z_0y_{-1}x_{-2}t_{-3} = x_0t_{-1}z_{-2}y_{-3} \neq 1$, $z_0y_{-1}x_{-2}t_{-3} = x_0t_{-1}z_{-2}y_{-3} \neq \frac{1}{2}$. Then the solution are given by the following formulae for $n = 0, 1, 2, ...,$

$$x_{4n-3} = \frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n}, \ x_{4n-2} = \frac{(-1)^n x_{-2}(-1+z_0y_{-1}x_{-2}t_{-3})^n}{(-1+2z_0y_{-1}x_{-2}t_{-3})^n},$$

$$x_{4n-1} = \frac{x_{-1}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n}, \ x_{4n} = x_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n,$$

$$y_{4n-3} = \frac{y_{-3}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n}, \ y_{4n-2} = y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n,$$

$$y_{4n-1} = \frac{(-1)^n y_{-1}(-1+2z_0y_{-1}x_{-2}t_{-3})^n}{(-1+z_0y_{-1}x_{-2}t_{-3})^n}, \ y_{4n} = y_0\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^n,$$

$$z_{4n-3} = \frac{z_{-3}}{(1+y_0x_{-1}t_{-2}z_{-3})^n}, \ z_{4n-2} = \frac{z_{-2}(-1+x_0t_{-1}z_{-2}y_{-3})^n}{(-1+2x_0t_{-1}z_{-2}y_{-3})^n},$$

$$z_{4n-1} = \frac{z_{-1}}{(1+t_0z_{-1}y_{-2}x_{-3})^n}, \ z_{4n} = (-1)^n z_0\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n,$$

$$t_{4n-3} = \frac{(-1)^n t_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n}, \ t_{4n-2} = t_{-2}\left(1+y_0x_{-1}t_{-2}z_{-3}\right)^n,$$

$$t_{4n-1} = \frac{t_{-1}(-1+2x_0t_{-1}z_{-2}y_{-3})^n}{(-1+x_0t_{-1}z_{-2}y_{-3})^n}, \ t_{4n} = t_0\left(1+t_0z_{-1}y_{-2}x_{-3}\right)^n.$$

Proof: As the proof of Theorem 1.

THEOREM 2.3. *Suppose that $\{x_n, y_n, z_n, t_n\}$ are solutions of the system*

$$x_{n+1} = \frac{x_{n-3}}{-1+x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{-1+y_{n-3}z_{n-2}t_{n-1}x_n},$$

$$z_{n+1} = \frac{z_{n-3}}{-1+z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{-1+t_{n-3}x_{n-2}y_{n-1}z_n}, \tag{3}$$

with $t_0z_{-1}y_{-2}x_{-3} = z_0y_{-1}x_{-2}t_{-3} = y_0x_{-1}t_{-2}z_{-3} = x_0t_{-1}z_{-2}y_{-3} \neq 1$, then all solutions of the system are unbounded if $t_0z_{-1}y_{-2}x_{-3} = z_0y_{-1}x_{-2}t_{-3} = y_0x_{-1}t_{-2}z_{-3} = x_0t_{-1}z_{-2}y_{-3} \neq 2$, and takes the form

$$x_{4n-3} = \frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n}, \ x_{4n-2} = x_{-2}\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n,$$

$$x_{4n-1} = \frac{x_{-1}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n}, \ x_{4n} = x_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n,$$

$$y_{4n-3} = \frac{y_{-3}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n}, \ y_{4n-2} = y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n,$$

$$y_{4n-1} = \frac{y_{-1}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n}, \ y_{4n} = y_0\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^n,$$

$$z_{4n-3} = \frac{z_{-3}}{(-1+y_0x_{-1}t_{-2}z_{-3})^n}, \ z_{4n-2} = z_{-2}\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^n,$$

$$z_{4n-1} = \frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n}, \ z_{4n} = z_0\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^n,$$

and

$$t_{4n-3} = \frac{t_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^n}, \ t_{4n-2} = t_{-2}\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^n,$$

$$t_{4n-1} = \frac{t_{-1}}{(-1+x_0t_{-1}z_{-2}y_{-3})^n}, \ t_{4n} = t_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n.$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is

$$x_{4n-7} = \frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}, \ x_{4n-6} = x_{-2}\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^{n-1},$$

$$x_{4n-5} = \frac{x_{-1}}{(-1+y_0x_{-1}t_{-2}z_{-3})^{n-1}}, \ x_{4n-4} = x_0\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^{n-1},$$

$$y_{4n-7} = \frac{y_{-3}}{(-1+x_0t_{-1}z_{-2}y_{-3})^{n-1}}, \ y_{4n-6} = y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^{n-1},$$

$$y_{4n-5} = \frac{y_{-1}}{(-1+z_0y_{-1}x_{-2}t_{-3})^{n-1}}, \ y_{4n-4} = y_0\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^{n-1},$$

$$z_{4n-7} = \frac{z_{-3}}{(-1+y_0x_{-1}t_{-2}z_{-3})^{n-1}}, \ z_{4n-6} = z_{-2}\left(-1+x_0t_{-1}z_{-2}y_{-3}\right)^{n-1},$$

$$z_{4n-5} = \frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}, \ z_{4n-4} = z_0\left(-1+z_0y_{-1}x_{-2}t_{-3}\right)^{n-1},$$

$$t_{4n-7} = \frac{t_{-3}}{(-1+z_0y_{-1}x_{-2}t_{-3})^{n-1}}, \ t_{4n-6} = t_{-2}\left(-1+y_0x_{-1}t_{-2}z_{-3}\right)^{n-1},$$

$$t_{4n-5} = \frac{t_{-1}}{(-1+x_0t_{-1}z_{-2}y_{-3})^{n-1}}, \ t_{4n-4} = t_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^{n-1}.$$

It follows from System (3) that

$$x_{4n-3} = \frac{x_{4n-7}}{-1+t_{4n-4}z_{4n-5}y_{4n-6}x_{4n-7}} = \frac{\frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}\right]}$$

$$= \frac{\frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{[-1+t_0z_{-1}y_{-2}x_{-3}]} = \frac{x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},$$

$$y_{4n-2} = \frac{y_{4n-6}}{-1+x_{4n-3}t_{4n-4}z_{4n-5}y_{4n-6}} = \frac{y_{-2}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}\right]}$$

$$= \frac{y_{-2}(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})}\right]} = y_{-2}\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n,$$

$$z_{4n-1} = \frac{z_{4n-5}}{-1+y_{4n-2}x_{4n-3}t_{4n-4}z_{4n-5}} = \frac{\frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^n(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}\right]}$$

$$= \frac{\frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}}{[-1+t_0z_{-1}y_{-2}x_{-3}]} = \frac{z_{-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n},$$

$$t_{4n} = \frac{t_{4n-4}}{-1+z_{4n-1}y_{4n-2}x_{4n-3}t_{4n-4}} = \frac{t_0(1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}(-1+t_0z_{-1}y_{-2}x_{-3})^n(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{(-1+t_0z_{-1}y_{-2}x_{-3})^n(-1+t_0z_{-1}y_{-2}x_{-3})^n}\right]}$$

$$= \frac{t_0(-1+t_0z_{-1}y_{-2}x_{-3})^{n-1}}{\left[-1+\frac{t_0z_{-1}y_{-2}x_{-3}}{(-1+t_0z_{-1}y_{-2}x_{-3})}\right]} = t_0\left(-1+t_0z_{-1}y_{-2}x_{-3}\right)^n.$$

Also, we can prove the other relations similarly. The proof is complete.

THEOREM 2.4. *If the sequences $\{x_n, \ y_n, \ z_n, \ t_n\}$ are solutions of difference equation system (3) such that* $t_0z_{-1}y_{-2}x_{-3} = z_0y_{-1}x_{-2}t_{-3} = y_0x_{-1}t_{-2}z_{-3} = x_0t_{-1}z_{-2}y_{-3} = 2$. *Then all solutions of system (3) are periodic*

with period four and takes the form

$$
\begin{aligned}
x_{4n-3} &= x_{-3}, \ x_{4n-2} = x_{-2}, \ x_{4n-1} = x_{-1}, \ x_{4n} = x_0. \\
y_{4n-3} &= y_{-3}, \ y_{4n-2} = y_{-2}, \ y_{4n-1} = y_{-1}, \ y_{4n} = y_0. \\
z_{4n-3} &= z_{-3}, \ z_{4n-2} = z_{-2}, \ z_{4n-1} = z_{-1}, \ z_{4n} = z_0. \\
t_{4n-3} &= t_{-3}, \ t_{4n-2} = t_{-2}, \ t_{4n-1} = t_{-1}, \ t_{4n} = t_0.
\end{aligned}
$$

Or

$$
\begin{aligned}
\{x_n\} &= \{x_{-3}, \ x_{-2}, \ x_{-1}, \ x_0, \ x_{-3}, \ x_{-2}, ...\}. \\
\{y_n\} &= \{y_{-3}, \ y_{-2}, \ y_{-1}, \ y_0, \ y_{-3}, \ y_{-2}, ...\}. \\
\{z_n\} &= \{z_{-3}, \ z_{-2}, \ z_{-1}, \ z_0, \ z_{-3}, \ z_{-2}, ...\}. \\
\{t_n\} &= \{t_{-3}, \ t_{-2}, \ t_{-1}, \ t_0, \ t_{-3}, \ t_{-2}, ...\}.
\end{aligned}
$$

Proof: The proof follows from the previous Theorem and will be omitted.

The following theorems can be proved similarly.

3. OTHER SYSTEMS:

In this section, we get the solutions of the following systems of the difference equations

$$
\begin{aligned}
x_{n+1} &= \frac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n}, \\
z_{n+1} &= \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{1-t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
x_{n+1} &= \frac{x_{n-3}}{1+x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n} \\
z_{n+1} &= \frac{z_{n-3}}{1-z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{1-t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
x_{n+1} &= \frac{x_{n-3}}{1-x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{1-y_{n-3}z_{n-2}t_{n-1}x_n}, \\
z_{n+1} &= \frac{z_{n-3}}{1+z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{1+t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
x_{n+1} &= \frac{x_{n-3}}{-1+x_{n-3}y_{n-2}z_{n-1}t_n}, \ y_{n+1} = \frac{y_{n-3}}{1+y_{n-3}z_{n-2}t_{n-1}x_n}, \\
z_{n+1} &= \frac{z_{n-3}}{1-z_{n-3}t_{n-2}x_{n-1}y_n}, \ t_{n+1} = \frac{t_{n-3}}{-1+t_{n-3}x_{n-2}y_{n-1}z_n}.
\end{aligned}
\tag{7}
$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

THEOREM 3.1. *If $\{x_n, y_n, z_n, \ t_n\}$ are solutions of difference equation system (4). Then for $n = 0, \ 1, \ 2, \ ...,$*

$$
\begin{aligned}
x_{4n-3} &= x_{-3} \prod_{i=0}^{n-1} \frac{(1+(2i)t_0 z_{-1} y_{-2} x_{-3})}{(1+(2i+1)t_0 z_{-1} y_{-2} x_{-3})}, \ x_{4n-2} = x_{-2} \prod_{i=0}^{n-1} \frac{(1+(2i-1)z_0 y_{-1} x_{-2} t_{-3})}{(1+(2i)z_0 y_{-1} x_{-2} t_{-3})}, \\
x_{4n-1} &= x_{-1} \prod_{i=0}^{n-1} \frac{(1+(2i)y_0 x_{-1} t_{-2} z_{-3})}{(1+(2i+1)y_0 x_{-1} t_{-2} z_{-3})}, \ x_{4n} = x_0 \prod_{i=0}^{n-1} \frac{(1+(2i+1)x_0 t_{-1} z_{-2} y_{-3})}{(1+(2i+2)x_0 t_{-1} z_{-2} y_{-3})}, \\
y_{4n-3} &= y_{-3} \prod_{i=0}^{n-1} \frac{(1+(2i)x_0 t_{-1} z_{-2} y_{-3})}{(1+(2i+1)x_0 t_{-1} z_{-2} y_{-3})}, \ y_{4n-2} = y_{-2} \prod_{i=0}^{n-1} \frac{(1+(2+1)t_0 z_{-1} y_{-2} x_{-3})}{(1+(2i+2)t_0 z_{-1} y_{-2} x_{-3})}, \\
y_{4n-1} &= y_{-1} \prod_{i=0}^{n-1} \frac{(1+(2i)z_0 y_{-1} x_{-2} t_{-3})}{(1+(2i+1)z_0 y_{-1} x_{-2} t_{-3})}, \ y_{4n} = y_0 \prod_{i=0}^{n-1} \frac{(1+(2i+1)y_0 x_{-1} t_{-2} z_{-3})}{(1+(2i+2)y_0 x_{-1} t_{-2} z_{-3})}, \\
z_{4n-3} &= z_{-3} \prod_{i=0}^{n-1} \frac{(1+(2i)y_0 x_{-1} t_{-2} z_{-3})}{(1+(2i+1)y_0 x_{-1} t_{-2} z_{-3})}, \ z_{4n-2} = z_{-2} \prod_{i=0}^{n-1} \frac{(1+(2i+1)x_0 t_{-1} z_{-2} y_{-3})}{(1+(2i+2)x_0 t_{-1} z_{-2} y_{-3})},
\end{aligned}
$$

$$z_{4n-1} = z_{-1} \prod_{i=0}^{n-1} \frac{(1+(2i+2)t_0 z_{-1} y_{-2} x_{-3})}{(1+(2i+3)t_0 z_{-1} y_{-2} x_{-3})}, \quad z_{4n} = z_0 \prod_{i=0}^{n-1} \frac{(1+(2i+1)z_0 y_{-1} x_{-2} t_{-3})}{(1+(2i+4)z_0 y_{-1} x_{-2} t_{-3})},$$

$$t_{4n-3} = t_{-3} \prod_{i=0}^{n-1} \frac{(1+(2i)z_0 y_{-1} x_{-2} t_{-3})}{(1+(2i-1)z_0 y_{-1} x_{-2} t_{-3})}, \quad t_{4n-2} = t_{-2} \prod_{i=0}^{n-1} \frac{(1+(2i+1)y_0 x_{-1} t_{-2} z_{-3})}{(1+(2i)y_0 x_{-1} t_{-2} z_{-3})},$$

$$t_{4n-1} = t_{-1} \prod_{i=0}^{n-1} \frac{(1+(2i+2)x_0 t_{-1} z_{-2} y_{-3})}{(1+(2i+1)x_0 t_{-1} z_{-2} y_{-3})}, \quad t_{4n} = t_0 \prod_{i=0}^{n-1} \frac{(1+(2i+3)t_0 z_{-1} y_{-2} x_{-3})}{(1+(2i+2)t_0 z_{-1} y_{-2} x_{-3})},$$

where $\prod_{i=0}^{-1} B_i = 1$.

THEOREM 3.2. *The form of the solutions of system (5) are given by the following formulae:*

$$x_{4n-3} = \frac{x_{-3}}{(1+t_0 z_{-1} y_{-2} x_{-3})^n}, \quad x_{4n-2} = (-1)^n x_{-2} \left(-1 + z_0 y_{-1} x_{-2} t_{-3}\right)^n,$$

$$x_{4n-1} = \frac{x_{-1}(-1+2y_0 x_{-1} t_{-2} z_{-3})^n}{(-1+y_0 x_{-1} t_{-2} z_{-3})^n}, \quad x_{4n} = (-1)^n x_0 \left(-1 + x_0 t_{-1} z_{-2} y_{-3}\right)^n,$$

$$y_{4n-3} = \frac{y_{-3}}{(1+x_0 t_{-1} z_{-2} y_{-3})^n}, \quad y_{4n-2} = \frac{y_{-2}(1+t_0 z_{-1} y_{-2} x_{-3})^n}{(1+2t_0 z_{-1} y_{-2} x_{-3})^n},$$

$$y_{4n-1} = \frac{y_{-1}}{(1+z_0 y_{-1} x_{-2} t_{-3})^n}, \quad y_{4n} = (-1)^n y_0 \left(-1 + y_0 x_{-1} t_{-2} z_{-3}\right)^n,$$

$$z_{4n-3} = \frac{(-1)^n z_{-3}}{(-1+y_0 x_{-1} t_{-2} z_{-3})^n}, \quad z_{4n-2} = z_{-2} \left(1 + x_0 t_{-1} z_{-2} y_{-3}\right)^n,$$

$$z_{4n-1} = \frac{z_{-1}(1+2t_0 z_{-1} y_{-2} x_{-3})^n}{(1+t_0 z_{-1} y_{-2} x_{-3})^n}, \quad z_{4n} = z_0 \left(1 + z_0 y_{-1} x_{-2} t_{-3}\right)^n,$$

$$t_{4n-3} = \frac{(-1)^n t_{-3}}{(-1+z_0 y_{-1} x_{-2} t_{-3})^n}, \quad t_{4n-2} = \frac{t_{-2}(-1+y_0 x_{-1} t_{-2} z_{-3})^n}{(-1+2y_0 x_{-1} t_{-2} z_{-3})^n},$$

$$t_{4n-1} = \frac{(-1)^n t_{-1}}{(-1+x_0 t_{-1} z_{-2} y_{-3})^n}, \quad t_{4n} = t_0 \left(1 + t_0 z_{-1} y_{-2} x_{-3}\right)^n,$$

where $t_0 z_{-1} y_{-2} x_{-3} \neq -1$, $t_0 z_{-1} y_{-2} x_{-3} \neq -\frac{1}{2}$, $y_0 x_{-1} t_{-2} z_{-3} \neq 1$, $y_0 x_{-1} t_{-2} z_{-3} \neq \frac{1}{2}$, $z_0 y_{-1} x_{-2} t_{-3} = x_0 t_{-1} z_{-2} y_{-3} \neq \pm 1$.

THEOREM 3.3. *Let $\{x_n, y_n, z_n, t_n\}$ are solutions of difference equation system (6) with $t_0 z_{-1} y_{-2} x_{-3} \neq 1$, $t_0 z_{-1} y_{-2} x_{-3} \neq \frac{1}{2}$, $x_0 t_{-1} z_{-2} y_{-3} = z_0 y_{-1} x_{-2} t_{-3} \neq \pm 1$, $y_0 x_{-1} t_{-2} z_{-3} \neq -1$, $y_0 x_{-1} t_{-2} z_{-3} \neq -\frac{1}{2}$,then for $n = 0, 1, 2, ...$,*

$$x_{4n-3} = \frac{(-1)^n x_{-3}}{(-1+t_0 z_{-1} y_{-2} x_{-3})^n}, \quad x_{4n-2} = x_{-2} \left(1 + z_0 y_{-1} x_{-2} t_{-3}\right)^n,$$

$$x_{4n-1} = \frac{x_{-1}(1+2y_0 x_{-1} t_{-2} z_{-3})^n}{(1+y_0 x_{-1} t_{-2} z_{-3})^n}, \quad x_{4n} = x_0 \left(1 + x_0 t_{-1} z_{-2} y_{-3}\right)^n,$$

$$y_{4n-3} = \frac{(-1)^n y_{-3}}{(-1+x_0 t_{-1} z_{-2} y_{-3})^n}, \quad y_{4n-2} = \frac{y_{-2}(-1+t_0 z_{-1} y_{-2} x_{-3})^n}{(-1+2t_0 z_{-1} y_{-2} x_{-3})^n},$$

$$y_{4n-1} = \frac{(-1)^n y_{-1}}{(-1+z_0 y_{-1} x_{-2} t_{-3})^n}, \quad y_{4n} = y_0 \left(1 + y_0 x_{-1} t_{-2} z_{-3}\right)^n,$$

$$z_{4n-3} = \frac{z_{-3}}{(1+y_0 x_{-1} t_{-2} z_{-3})^n}, \quad z_{4n-2} = (-1)^n z_{-2} \left(-1 + x_0 t_{-1} z_{-2} y_{-3}\right)^n,$$

$$z_{4n-1} = \frac{z_{-1}(-1+2t_0 z_{-1} y_{-2} x_{-3})^n}{(-1+t_0 z_{-1} y_{-2} x_{-3})^n}, \quad z_{4n} = (-1)^n z_0 \left(-1 + z_0 y_{-1} x_{-2} t_{-3}\right)^n,$$

$$t_{4n-3} = \frac{t_{-3}}{(1+z_0 y_{-1} x_{-2} t_{-3})^n}, \quad t_{4n-2} = \frac{t_{-2}(1+y_0 x_{-1} t_{-2} z_{-3})^n}{(1+2y_0 x_{-1} t_{-2} z_{-3})^n},$$

$$t_{4n-1} = \frac{t_{-1}}{(1+x_0 t_{-1} z_{-2} y_{-3})^n}, \quad t_{4n} = (-1)^n t_0 \left(-1 + t_0 z_{-1} y_{-2} x_{-3}\right)^n.$$

THEOREM 3.4. *Suppose that the initial conditions of the system (7) are arbitrary real numbers satisfies $t_0 z_{-1} y_{-2} x_{-3} \neq 1$, $t_0 z_{-1} y_{-2} x_{-3} \neq \frac{1}{2}$, $x_0 t_{-1} z_{-2} y_{-3} = z_0 y_{-1} x_{-2} t_{-3} \neq \pm 1$, $y_0 x_{-1} t_{-2} z_{-3} \neq 1$, $y_0 x_{-1} t_{-2} z_{-3} \neq \frac{1}{2}$, and if $\{x_n, y_n, z_n, t_n\}$ are solutions of system (7). Then for $n = 0, 1, 2, ...$,*

$$x_{4n-3} = \frac{x_{-3}}{(-1+t_0 z_{-1} y_{-2} x_{-3})^n}, \quad x_{4n-2} = x_{-2} \left(-1 + z_0 y_{-1} x_{-2} t_{-3}\right)^n,$$

$$x_{4n-1} = \frac{(-1)^n x_{-1}(-1+2y_0 x_{-1} t_{-2} z_{-3})^n}{(-1+y_0 x_{-1} t_{-2} z_{-3})^n}, \quad x_{4n} = x_0 \left(-1 + x_0 t_{-1} z_{-2} y_{-3}\right)^n,$$

$$
\begin{aligned}
y_{4n-3} &= \frac{y_{-3}}{(1+x_0 t_{-1} z_{-2} y_{-3})^n}, \; y_{4n-2} = \frac{y_{-2}(-1+t_0 z_{-1} y_{-2} x_{-3})^n}{(-1+2t_0 z_{-1} y_{-2} x_{-3})^n}, \\
y_{4n-1} &= \frac{y_{-1}}{(1+z_0 y_{-1} x_{-2} t_{-3})^n}, \; y_{4n} = (-1)^n y_{-2} \left(-1 + t_0 z_{-1} y_{-2} x_{-3}\right)^n, \\
z_{4n-3} &= \frac{(-1)^n z_{-3}}{(-1+y_0 x_{-1} t_{-2} z_{-3})^n}, \; z_{4n-2} = z_{-2} \left(1 + x_0 t_{-1} z_{-2} y_{-3}\right)^n, \\
z_{4n-1} &= \frac{z_{-1}(-1+2t_0 z_{-1} y_{-2} x_{-3})^n}{(-1+t_0 z_{-1} y_{-2} x_{-3})^n}, \; z_{4n} = z_0 \left(1 + z_0 y_{-1} x_{-2} t_{-3}\right)^n, \\
t_{4n-3} &= \frac{t_{-3}}{(-1+z_0 y_{-1} x_{-2} t_{-3})^n}, \; t_{4n-2} = \frac{(-1)^n t_{-2}(-1+y_0 x_{-1} t_{-2} z_{-3})^n}{(-1+2y_0 x_{-1} t_{-2} z_{-3})^n}, \\
t_{4n-1} &= \frac{t_{-1}}{(-1+x_0 t_{-1} z_{-2} y_{-3})^n}, \; t_{4n} = t_0 \left(-1 + t_0 z_{-1} y_{-2} x_{-3}\right)^n.
\end{aligned}
$$

4. NUMERICAL EXAMPLES

Here we consider some numerical examples for the previous systems to illustrate the results.

Example 1. We consider the system (1) with the initial conditions $x_{-3} = .16$, $x_{-2} = -.3$, $x_{-1} = 7$, $x_0 = -1.3$, $y_{-3} = .2$, $y_{-2} = -.4$, $y_{-1} = .51$, $y_0 = 1$, $z_{-3} = -.8$, $z_{-2} = .4$, $z_{-1} = 5$, $z_0 = .74$, $t_{-3} =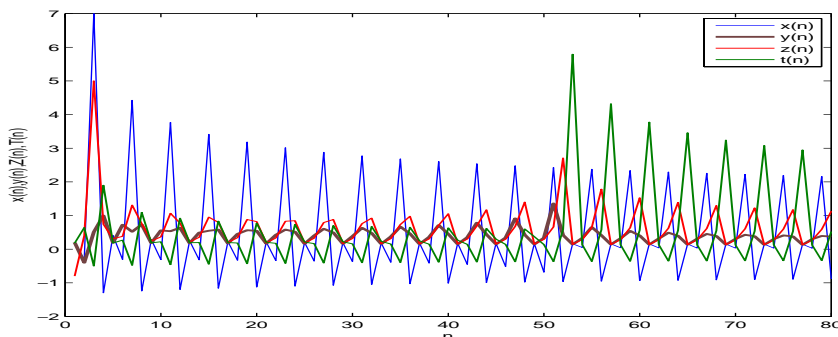 .18$, $t_{-2} = .64$, $t_{-1} = -.5$ and $t_0 = 1.9$. (See Fig. 1). Also, see Figure 2 to see the behavior of the solutions of System (1) with initials conditions $x_{-3} = .16$, $x_{-2} = .3$, $x_{-1} = 0$, $x_0 = 1.3$, $y_{-3} = .2$, $y_{-2} = .4$, $y_{-1} = .51$, $y_0 = 1$, $z_{-3} = .8$, $z_{-2} = .4$, $z_{-1} = .5$, $z_0 = .74$, $t_{-3} = .8$, $t_{-2} = .64$, $t_{-1} = .5$ and $t_0 = 1.9$.
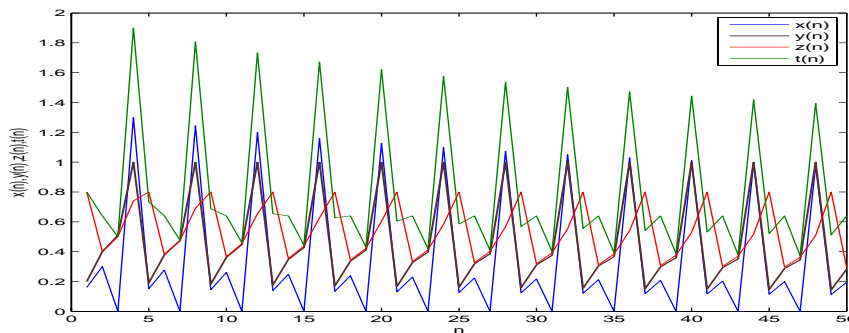

Figure 1. Plot of the system (1).

Figure 2. Draw the behavior of the solution of the system (1).

Example 2. See Figure (3) for an example for the system (2) with the initial values $x_{-3} = .6$, $x_{-2} = .3$, $x_{-1} = .19$, $x_0 = -.3$, $y_{-3} = .2$, $y_{-2} = .4$, $y_{-1} = .56$, $y_0 = .91$, $z_{-3} = .28$, $z_{-2} = .4$, $z_{-1} = .65$, $z_0 = .37$, $t_{-3} = .8$, $t_{-2} = .64$, $t_{-1} = .5$ and $t_0 = .7$.
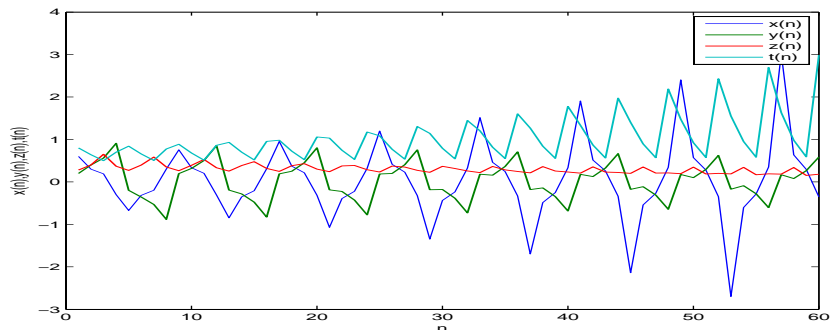

Figure 3. Sketch the behavior of the solution of the system (2).

Example 3. If we take the initial conditions as follows $x_{-3} = .6$, $x_{-2} = .3$, $x_{-1} = -.19$, $x_0 = -.3$, $y_{-3} = .2$, $y_{-2} = .04$, $y_{-1} = .56$, $y_0 = .91$, $z_{-3} = .28$, $z_{-2} = -.4$, $z_{-1} = .49$, $z_0 = .37$, $t_{-3} = -.8$, $t_{-2} = -.64$, $t_{-1} = .5$ and $t_0 = .7$, for the difference system (3), see Fig. 4.
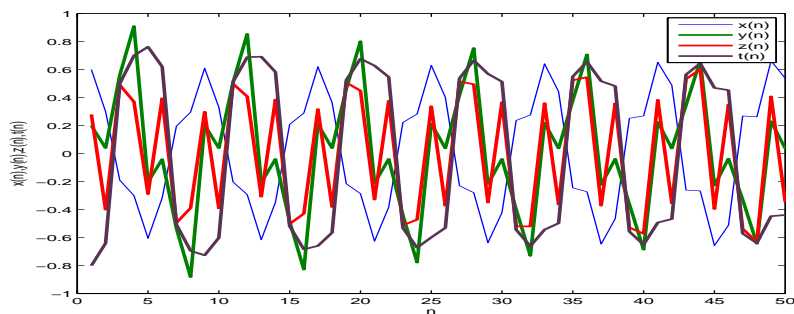

Figure 4. Plot of system (3).

Example 4. Figure (5) shows the periodicity behavior of the solution of the difference system (3) with the initial conditions $x_{-3} = 6$, $x_{-2} = -.3$, $x_{-1} = 9$, $x_0 = -8$, $y_{-3} = 1/9$, $y_{-2} = -9$, $y_{-1} = 5$, $y_0 = .1$, $z_{-3} = 20$, $z_{-2} = 6$, $z_{-1} = -.7$, $z_0 = .2$, $t_{-3} = -20/3$, $t_{-2} = 1/9$, $t_{-1} = -3/8$ and $t_0 = 10/189$.
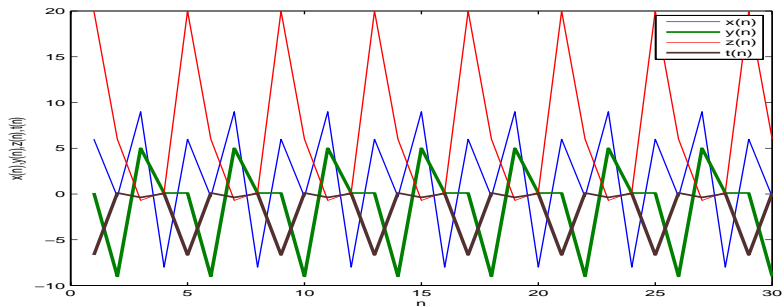

Figure 5. Plot the behavior of the solution of the difference system (3).

Acknowledgements

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Applications of soft sets to BCC-ideals in BCC-algebras

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Abstract. The notions of a union soft ideal and a union soft BCC-ideal of a BCC-algebra are introduced and some related properties of them are investigated. A quotient structure of BCC-algebra using a uni-soft BCC-ideal is constructed and some related properties are studied.

1. Introduction

Y. Kormori [10] introduced a notion of a BCC-algebras, and W. A. Dudek [3] redefined the notion of BCC-algebras by using a dual from of the ordinary definition of Y. Kormori. In [6], J. Hao introduced the notion of ideals in a BCC-algebra and studied some related properties. W. A. Dudek and X. Zhang [4] introduced a BCC-ideals in a BCC-algebra and described connections between such BCC-ideals and congruences. S. S. Ahn and S. H. Kwon [1] defined a topological BCC-algebra and investigated some properties of it.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [15]. In response to this situation Zadeh [16] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [17]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [13]. Maji et al. [12] and Molodtsov [13] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [13] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Maji et al. [12] described the application of soft set theory to a decision making problem. Maji et al. [11] also studied

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several operations on the theory of soft sets. Jun [8] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [2, 5, 7, 9, 14] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we introduce the notions of a union soft ideal and a union soft BCC-ideal of a BCC-algebra and investigated some related properties of them. A quotient structure of BCC-algebra using a uni-soft BCC-ideal is constructed and some related properties are studied.

2. Preliminaries

By a BCC-*algebra* [3] we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: for all $x, y, z \in X$,

(a1) $((x * y) * (z * y)) * (x * z) = 0$,
(a2) $0 * x = 0$,
(a3) $x * 0 = x$,
(a4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call X a BCC-*algebra*. In X, we can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$. Then $\leq$ is a partial order on X.

A BCC-algebra X has the following properties: for any $x, y \in X$,

(b1) $x * x = 0$,
(b2) $(x * y) * x = 0$,
(b3) $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebra (see [3]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(b4) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Let $(X, *, 0_X)$ and $(Y, *, 0_Y)$ be BCC-algebras. A mapping $\varphi : X \rightarrow Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for all $x, y \in X$. A non-empty subset S of a BCC-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of a BCI-algebra X is called an *ideal* [6] of X if it satisfies:

(c1) $0 \in I$,
(c2) $x * y, y \in I \Rightarrow x \in I$ for all $x, y \in X$.

I is called an BCC-*ideal* [4] of X if it satisfies (c1) and

(c3) $(x * y) * z, y \in I \Rightarrow x * z \in I$, for all $x, y, z \in X$.

Theorem 2.1. [6] *In a BCC-algebra, an ideal is a subalgebra.*

Theorem 2.2. [4] *In a BCC-algebra, a BCC-ideal is an ideal.*

Corollary 2.3. [4] *Any BCC-ideal of a BCC-algebra is a subalgebra.*

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Let X be a BCC-algebra and let I be a BCC-ideal of X. Define a relation $\sim^I$ on X by $x \sim y$ if and only if $x * y, y * x \in I$ for any $x, y \in X$. Then it is a congruence relation on X [4]. Denote by $[x]_I$ the equivalence class containing x, i.e., $[x]_I := \{y \in X | x \sim^I y\}$ and let $X/I := \{[x]_I | x \in X\}$.

Theorem 2.4. *If I is a BCC-algebra X, then the quotient algebra X/I is a BCC-algebra.*

A soft set theory is introduced by Molodtsov [13].

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.5. [13] A soft set (f, A) over U is defined to be the set of ordered pairs $(f, A) := \{(x, f(x)) : x \in E, f(x) \in \mathscr{P}(U)\}$, where $f : E \to \mathscr{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

For $\epsilon \in A$, $f(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (f, A). Clearly, a soft set is not a set. For a soft set (f, A) of X and a subset γ of U, the *γ-exclusive set* of (f, A), defined to be the set $e_A(f; \gamma) := \{x \in A | f(x) \subseteq \gamma\}$.

For any soft sets (f, X) and (g, X) of X, we call (f, X) a *soft subset* of (g, X), denoted by $(f, X) \tilde{\subseteq} (g, X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The *soft union* of (f, X) and (g, X), denoted by $(f, X) \tilde{\cup} (g, X)$, is defined to be the soft set $(f \tilde{\cup} g, X)$ of X over U in which $f \tilde{\cup} g$ is defined by $(f \tilde{\cup} g)(x) := f(x) \cup g(x)$ for all $x \in X$. The *soft intersection* of (f, X) and (g, X), denoted by $(f, X) \tilde{\cap} (g, X)$, is defined to be the soft set $(f \tilde{\cap} g, M)$ of X over U in which $f \tilde{\cap} g$ is defined by $(f \tilde{\cap} g)(x) := f(x) \cap g(x)$ for all $x \in M$.

3. Uni-soft BCC-ideals

In what follows, let X be a BCC-algebra unless otherwise specified.

Definition 3.1. A soft set (f, X) over U is called a *union soft subalgebra* (briefly, *uni-soft subalgebra*) of a BCC-algebra X over U if it satisfies:

(3.0) $f(x * y) \subseteq f(x) \cup f(y)$ for all $x, y \in X$.

Proposition 3.2. *Every uni-soft subalgebra (f, X) of a BCC-algebra X over U satisfies the following inclusion:*

(3.1) $f(0) \subseteq f(x)$ for all $x \in X$.

Proof. Using (3.0) and (b1), we have $f(0) = f(x * x) \subseteq f(x) \cup f(x) = f(x)$ for all $x \in X$. $\square$

Example 3.3. Let $(U := \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a BCC-algebra [6] with the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

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Let (f, X) be a soft set over U defined as follows:

$$f : X \to \mathscr{P}(U), \ x \mapsto \begin{cases} 4\mathbb{Z} & \text{if } x = 0, \\ 2\mathbb{Z} & \text{if } x \in \{1, 2\}, \\ \mathbb{Z} & \text{if } x = 3. \end{cases}$$

It is easy to check that (f, X) is a uni-soft subalgebra of X over U.

Theorem 3.4. *A soft set (f, X) of a BCC-algebra X over U is a uni-soft subalgebra of X over U if and only if the γ-exclusive set $e_X(f; \gamma)$ is a subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $e_X(f; \gamma) \neq \emptyset$.*

Proof. Assume that (f, X) is a uni-soft subalgebra of X over U. Let $x, y \in X$ and let $\gamma \in \mathscr{P}(U)$ be such that $x, y \in e_X(f; \gamma)$. Then $f(x) \subseteq \gamma$ and $f(y) \subseteq \gamma$. It follows from (3.0) that $f(x * y) \subseteq f(x) \cup f(y) \subseteq \gamma$ Hence $x * y \in e_X(f; \gamma)$. Thus $e_X(f; \gamma)$ is a subalgebra of X.

Conversely, suppose that $e_X(f; \gamma)$ is a subalgebra X for all $\gamma \in \mathscr{P}(U)$ with $e_X(f; \gamma) \neq \emptyset$. Let $x, y \in X$, be such that $f(x) = \gamma_x$ and $f(y) = \gamma_y$. Take $\gamma = \gamma_x \cup \gamma_y$. Then $x, y \in e_X(f; \gamma)$ and so $x * y \in e_X(f; \gamma)$ by assumption. Hence $f(x * y) \subseteq \gamma = \gamma_x \cup \gamma_y = f(x) \cup f(y)$. Thus (f, X) is a uni-soft subalgebra of X over U. $\square$

Theorem 3.5. *Every subalgebra of a BCC-algebra X can be represented as a γ-exclusive set of a uni-soft subalgebra of X over U.*

Proof. Let A be a subalgebra of a BCC-algebra X. For a subset γ of U, define a soft set (f, X) over U by

$$f : X \to \mathscr{P}(U), \ x \mapsto \begin{cases} \gamma & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $A = e_X(f; \gamma)$. We now prove that (f, X) is a uni-soft subalgebra of X over U. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$ because A is a subalgebra of X. Hence $f(x) = f(y) = f(x * y) = \gamma$, and so $f(x * y) \subseteq f(x) \cup f(y)$. If $x \in A$ and $y \notin A$, then $f(x) = \gamma$ and $f(y) = U$ which imply that $f(x * y) \subseteq f(x) \cup f(y) = \gamma \cup U = U$. Similarly, if $x \notin A$ and $y \in A$, then $f(x * y) \subseteq f(x) \cup f(y)$. Obviously, if $x \notin A$ and $y \notin A$, then $f(x * y) \subseteq f(x) \cup f(y)$. Therefore (f, X) is a uni-soft subalgebra of X over U. $\square$

Any subalgebra of a BCC-algebra X may not be represented as a γ-exclusive set of a uni-soft subalgebra (f, X) of X over U in general (see Example 3.6).

Example 3.6. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a BCC-algebra as in Example 3.3. Consider a soft set (f, X) which is given by

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$$f : X \to \mathscr{P}(U), \ x \mapsto \begin{cases} \{0\} & \text{if } x = 0, \\ \{0,3\} & \text{if } x \in \{1,2,3\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft subalgebra of X over U. The γ-exclusive set of (f, X) are described as follows:

$$e_X(f; \gamma) = \begin{cases} \{0\} & \text{if } \gamma = \{0\}, \\ X & \text{if } \gamma \in \{\{0,3\}, \{0,2,3\}, X\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 2\}$ cannot be a γ-exclusive set $e_X(f; \gamma)$ since there is no $\gamma \subseteq U$ such that $e_X(f; \gamma) = \{0, 2\}$.

Definition 3.7. A soft set (f, X) over U is called a *union soft ideal* (briefly, *uni-soft ideal*) of a BCC-algebra X over U if it satisfies (3.1) and

(3.2) $f(x) \subseteq f(x * y) \cup f(y)$ for all $x, y \in X$.

Proposition 3.8. *Every uni-soft ideal of a BCC-algebra X over U is a uni-soft subalgebra of X over U.*

Proof. Put $x := x * y$ and $y := x$ in (3.2). Then we have $f(x * y) \subseteq f((x * y) * x) \cup f(x)$. Using (b2) and (3.1), we obtain $f(x*y) \subseteq f((x*y)*x) \cup f(x) = f(0) \cup f(x) \subseteq f(y) \cup f(x) = f(x) \cup f(y)$ for all $x, y \in X$. Hence (f, X) is a uni-soft subalgebra of X over U. $\qquad\square$

Theorem 3.9. *A soft set (f, X) of a BCC-algebra X over U is a uni-soft ideal of X over U if and only if the γ-exclusive set $e_X(f; \gamma)$ is a ideal of X for all $\gamma \in \mathscr{P}(U)$ with $e_X(f; \gamma) \neq \emptyset$.*

Proof. Similar to Theorem 3.4. $\qquad\square$

Proposition 3.10 *Every uni-soft ideal (f, X) of a BCC-algebra over U satisfies the following properties:*

(i) $(\forall x \in X)(x \leq y \Rightarrow f(x) \subseteq f(y))$,
(ii) $(\forall x, y, z \in X)(x * y \leq z \Rightarrow f(x) \subseteq f(y) \cup f(z))$.

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. It follows from (3.2) and (3.1) that $f(x) \subseteq f(x * y) \cup f(y) = f(0) \cup f(y) = f(y)$.
(ii) Let $x, y, z \in X$ be such that $x * y \leq z$. By (3.2) and (3.1), we have $f(x * y) \subseteq f((x * y) * z) \cup f(z) = f(0) \cup f(z) = f(z)$. Hence $f(x) \subseteq f(x * y) \cup f(y) \subseteq f(z) \cup f(y) = f(y) \cup f(z)$. $\qquad\square$

The following corollary is easily proved by induction.

Corollary 3.11. *Every uni-soft ideal of a BCC-algebra X over U satisfies the following condition:*

(3.3) $(\cdots (x * a_1) * \cdots) * a_n = 0 \Rightarrow f(x) \subseteq \bigcup_{k=1}^n f(a_k)$ for all $x, a_1, \cdots, a_n \in X$.

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Theorem 3.12. *If (f, X) and (g, X) are uni-soft ideals of a BCC-algebra X over U, then the union $(f, X)\tilde{\cup}(g, X)$ of (f, X) and (g, X) is a uni-soft ideal of X over U.*

Proof. For any $x \in X$, we have $(f\tilde{\cup}g)(0) = f(0) \cup g(0) \subseteq f(x) \cup g(x) = (f\tilde{\cup}g)(x)$. Let $x, y \in X$. Then we have $(f\tilde{\cup}g)(x) = f(x) \cup g(x) \subseteq (f(x * y) \cup f(y)) \cup (g(x * y) \cup g(y)) = (f(x * y) \cup g(x * y)) \cup (f(y) \cup g(y)) = (f\tilde{\cup}g)(x * y) \cup (f\tilde{\cup}g)(y)$. Hence $(f, X)\tilde{\cup}(g, X)$ is a uni-soft ideal of X over U. $\qquad\square$

The soft intersection of uni-soft ideals of a BCC-algebra X may not be a uni-soft ideal of X over U (see Example 3.13).

Example 3.13. Let $E = X$ be the set of parameters, and let $U := \mathbb{Z}$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a BCC-algebra with the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let (f, X) and (g, X) be soft sets over $U = \mathbb{Z}$ defined as follows:

$$f : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} 9\mathbb{Z} & \text{if } x \in \{0, 1, 2\}, \\ 3\mathbb{Z} & \text{if } x \in \{2, 3\}, \end{cases}$$

and

$$g : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} 12\mathbb{Z} & \text{if } x = 0, \\ 3\mathbb{Z} & \text{if } x = 3, \\ \mathbb{Z} & \text{if } x \in \{1, 2\}. \end{cases}$$

Then (f, X) and (g, X) are uni-soft ideals of X over U. But $(f, X)\tilde{\cap}(g, X)$ is not a uni-soft ideal of X over U, since $(f\tilde{\cap}g)(2) = f(2) \cap g(2) = 3\mathbb{Z} \cap \mathbb{Z} = 3\mathbb{Z} \nsubseteq (f\tilde{\cap}g)(2 * 1) \cup (f\tilde{\cap}g)(1) = (f(1) \cap g(1)) \cup (f(1) \cap g(1)) = f(1) \cap g(1) = 9\mathbb{Z} \cap \mathbb{Z} = 9\mathbb{Z}$.

Definition 3.14. A soft set (f, X) over U is called a *union soft BCC-ideal* (briefly, *uni-soft BCC-ideal*) of a BCC-algebra X over U if it satisfies (3.1) and

(3.4) $f(x * z) \subseteq f((x * y) * z) \cup f(y)$ for all $x, y, z \in X$.

Lemma 3.15. *Every uni-soft BCC-ideal of a BCC-algebra X over U is a uni-soft ideal of X over U.*

Proof. Put $z := 0$ in (3.4). Using (a3), we have $f(x*0) = f(x) \subseteq f((x*y)*0)\cup f(y) = f(x*y)\cup f(y)$ for all $x, y \in X$. Hence (f, X) is a uni-soft ideal of X over U. $\qquad\square$

Applications of soft sets to BCC-ideals in BCC-algebras

Corollary 3.16. *Every uni-soft BCC-ideal of a BCC-algebra X over U is a uni-soft subalgebra of X over U.*

The converse of Proposition 3.8 and Lemma 3.15 need not be a true, in general (see Example 3.17).

Example 3.17. Let $(U := \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCC-algebra [4] with the following table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Let (f, X) be a soft set over U defined as follows:

$$f : X \to \mathscr{P}(U), \ x \mapsto \begin{cases} 3\mathbb{Z} & \text{if } x \in \{0, 1, 2, 3\}, \\ \mathbb{Z} & \text{if } x = 4. \end{cases}$$

It is easy to check that (f, X) is a uni-soft subalgebra of X over U, but not a uni-soft ideal of X over U, since $f(4) = \mathbb{Z} \nsubseteq f(4 * 3) \cup f(3) = f(3) \cup f(3) = 3\mathbb{Z}$. Consider a uni-soft set (g, X) which is given by

$$g : X \to \mathscr{P}(U), \ x \mapsto \begin{cases} 2\mathbb{Z} & \text{if } x \in \{0, 1\}, \\ \mathbb{Z} & \text{if } x \in \{2, 3, 4\}. \end{cases}$$

It is easy to show that (g, X) is a uni-soft ideal of X over U. But it is not a uni-soft BCC-ideal of X over U, since $g(4 * 3) = g(3) = \mathbb{Z} \nsubseteq g((4 * 1) * 3) \cup g(1) = g(0) \cup g(1) = 2\mathbb{Z}$.

Example 3.18. Let $(U := \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3, 4, 5\}$ is a BCC-algebra [4] with the following table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let (f, X) be a soft set over U defined as follows:

$$f : X \to \mathscr{P}(U), \ x \mapsto \begin{cases} 5\mathbb{Z} & \text{if } x \in \{0, 1, 2, 3, 4\}, \\ \mathbb{Z} & \text{if } x = 5. \end{cases}$$

It is easy to check that (f, X) is a uni-soft BCC-ideal of X over U .

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Theorem 3.19. *A soft set* (f, X) *of a BCC-algebra* X *over* U *is a uni-soft BCC-ideal of* X *over* U *if and only if the* γ-*exclusive set* $e_X(f; \gamma)$ *is a BCC-ideal of* X *for all* $\gamma \in \mathscr{P}(U)$ *with* $e_X(f; \gamma) \neq \emptyset$.

Proof. Suppose that (f, X) is a uni-soft BCC-ideal of X over U. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $(x * y) * z \in e_X(f; \gamma)$ and $y \in e_X(f; \gamma)$. Then $f((x * y) * z) \subseteq \gamma$ and $f(y) \subseteq \gamma$. It follows from (3.1) and (3.4) that $f(0) \subseteq f(x * z) \subseteq f((x * y) * z) \cup f(y) \subseteq \gamma$. Hence $0 \in e_X(f; \gamma)$ and $x * z \in e_X(f; \gamma)$, and therefore $e_X(f; \gamma)$ is a BCC-ideal of X.

Conversely, assume that $e_X(f; \gamma)$ is a BCC-ideal of X for all $\gamma \in \mathscr{P}([0,1])$ with $e_X(f; \gamma) \neq \emptyset$. For any $x \in X$, let $f(x) = \gamma$. Then $x \in e_X(f; \gamma)$. Since $e_X(f; \gamma)$ is a BCC-ideal of X, we have $0 \in e_X(f; \gamma)$ and so $f(0) \subseteq f(x) = \gamma$. For any $x, y, z \in X$, let $f((x * y) * z) = \gamma_{(x*y)*z}$ and $f(y) = \gamma_y$. Let $\gamma := \gamma_{(x*y)*z} \cup \gamma_y$. Then $(x * y) * z \in e_X(f; \gamma)$ and $y \in e_X(f; \gamma)$ which imply that $x * z \in e_X(f; \gamma)$. Hence $f(x * z) \subseteq \gamma = \gamma_{(x*y)*z)} \cup \gamma_y = f((x * y) * z) \cup f(y)$. Thus (f, X) is a uni-soft BCC-ideal of X over U. $\square$

Proposition 3.20. *Let* (f, X) *be a uni-soft BCC-ideal of a BCC-algebra* X *over* U. *Then* $X_f := \{x \in X | f(x) = f(0)\}$ *is a BCC-ideal of* X.

Proof. Clearly, $0 \in X_f$. Let $(x * y) * z, y \in X_f$. Then $f((x * y) * z) = f(0)$ and $f(y) = f(0)$. It follows from (3.4) that $f(x * z) \subseteq f((x * y) * z) \cup f(y) = f(0)$. By (3.1), we get $f(x * z) = f(0)$. Hence $x * z \in X_f$. Therefore X_f is a BCC-ideal of X. $\square$

4. Quotient BCC-ideals induced by uni-soft BCC-ideals

Let (f, X) be a uni-soft BCC-ideal of a BCC-algebra X over U. For any $x, y \in X$, we define a binary operation " $\sim^f$ " on X as follows:

$$x \sim^f y \Leftrightarrow f(x * y) = f(y * x) = 0.$$

Lemma 4.1. *The operation " $\sim^f$ " is an equivalence relation on a BCC-algebra* X.

Proof. Obviously, $\sim^f$ is both reflexive and symmetric. Let $x, y, z \in X$ be such that $x \sim^f y$ and $y \sim^f z$. Then $f(x*y) = f(0) = f(y*x)$ and $f(y*z) = f(0) = f(z*y)$. Since $(x*z)*(y*z) \leq x*y$ and $(z*x)*(y*x) \leq z*y$, it follows from Proposition 3.10(ii) that $f(x*z) \subseteq f(y*z) \cup f(x*y) = f(0)$ and $f(z * x) \subseteq f(y * x) \cup f(z * y) = f(0)$. By (3.1), we have $f(x * z) = f(0) = f(z * x)$ and so $x \sim^f z$. Therefore " $\sim^f$ " is an equivalence relation on X. $\square$

Lemma 4.2. *For any* x, y *in a BCC-algebra* X, *if* $x \sim^f y$, *then* $x * z \sim^f y * z$ *and* $z * x \sim^f z * y$ *for all* $z \in X$.

Proof. Let $x, y, z \in X$ be such that $x \sim^f y$. Then $f(x*y) = f(0) = f(y*x)$. Since $(x*z)*(y*z) \leq x*y$ and $(y*z)*(x*z) \leq y*x$, it follows from Proposition 3.10(i) that $f((x*z)*(y*z)) \subseteq f(x*y) = f(0)$ and $f((y*z)*(x*z)) \subseteq f(y*x) = f(0)$. Thus $f((x*z)*(y*z)) = f(0) = f((y*z)*(x*z))$, and so $x*z \sim^f y*z$. Since $((z*x)*(y*x))*(z*y) = 0$, we have $f((z*x)*(z*y)) \subseteq f(((z*x)*(y*$

$x)) * (z * y)) \cup f(y * x) = f(0) \cup f(y * x) = f(y * x) = f(0)$. Since $((z * y) * (x * y)) * (z * x) = 0$, we have $f((z * y) * (z * x)) \subseteq f(((z * y) * (x * y)) * (z * x)) \cup f(x * y) = f(0) \cup f(x * y) = f(x * y) = f(0)$. By (3.1), we have $f((z * x) * (z * y)) = f(0)$ and $f((z * y) * (z * x)) = f(0)$. Therefore $x * z \sim^f y * z$ and $z * x \sim^f z * y$. $\square$

Using Lemma 4.2 and the transitivity of $\sim^f$, we have the following Lemma.

Lemma 4.3. *For any x, y, u, v in a BCC-algebra X, if $x \sim^f y$ and $u \sim^f v$, then $x * u \sim^f y * v$.*

By Lemmas 4.1, 4.2 and 4.3, the operation " $\sim^f$ " is a congruence relation on a BCC-algebra X. Denote by f_x the equivalence class containing $x \in X$, and by X/f the set of all equivalence classes of X, i.e., $f_x := \{y \in X | y \sim^f x\}$ and $X/f := \{f_x | x \in X\}$. Define a binary operation $\bullet$ on X/f as follows: for all $f_x, f_y \in X/f$, $f_x \bullet f_y := f_{x*y}$. Then this operation is well-defined by Lemma 4.3.

Theorem 4.4. *If (f, X) is a uni-soft BCC-ideal of a BCC-algebra X over U, then the quotient $X/f := (X/f, \bullet, f_0)$ is a BCC-algebra.*

Proof. Straightforward. $\square$

Proposition 4.5. *Let $\mu : (X, *, 0_X) \to (Y, *, 0_Y)$ be an epimorphism of BCC-algebras. If (g, Y) is a uni-soft BCC-ideal of Y over U, then $(g \circ \mu, X)$ is a uni-soft BCC-ideal of X over U.*

Proof. For any $x \in X$, we have $(g \circ \mu)(0) = g(\mu(0_X)) = g(0_Y) \subseteq g(\mu(x)) = (g \circ \mu)(x)$. For any $x, y \in X$, we have $(g \circ u)(x * z) = g(\mu(x * z)) = g(\mu(x) * \mu(z)) \subseteq g((\mu(x) *_Y a) * \mu(z)) \cup g(a)$ for any $a \in Y$. Let y be any preimage of a under μ. Then $(g \circ \mu)(x * z) \subseteq g((\mu(x) *_Y a) * \mu(z)) \cup g(a) = g((\mu(x) *_Y \mu(y)) * \mu(z)) \cup g(\mu(y)) = g(\mu((x *_X y) *_X z)) \cup g(\mu(y)) = (g \circ \mu)((x *_X y) *_X z) \cup (g \circ \mu)(y)$. Hence $g \circ \mu$ is a uni-soft BCC-ideal of X over U. $\square$

Theorem 4.6. *Let $\mu : (X, *, 0_X) \to (Y, *, 0_Y)$ be an epimorphism of BCC-algebras. If (g, Y) is a uni-soft BCC-ideal of Y over U, then the quotient algebra $X/(g \circ \mu) := (X/(g \circ \mu), \bullet_X, (g \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/g := (Y/g, \bullet_Y, g_{0_Y})$.*

Proof. By Theorem 4.4 and Proposition 4.5, $X/(g \circ \mu) := (X/(g \circ \mu), \bullet_X, (g \circ \mu)_{0_X})$ and and $Y/g := (Y/g, \bullet_Y, g_{0_Y})$ are BCC-algebras. Define a map

$$\eta : X/(g \circ \mu) \to Y/g, \ (g \circ \mu)_x \mapsto g_{\mu(x)}$$

for all $x \in X$. Then the function η is well-defined. In fact, assume that $(g \circ \mu)_x = (g \circ \mu)_y$ for all $x, y \in X$. Then we have $g(\mu(x) *_Y \mu(y)) = g(\mu(x *_X y)) = (g \circ \mu)(x *_X y) = (g \circ \mu)(0_X) = g(\mu(0_X)) = g(0_Y)$ and $g(\mu(y) *_Y \mu(x)) = g(\mu(y *_X x)) = (g \circ \mu)(y *_X x) = (g \circ \mu)(0_X) = g(\mu(0_X)) = g(0_Y)$. Hence $g_{\mu(x)} = g_{\mu(y)}$.

For any $(g \circ \mu)_x, (g \circ \mu)_y \in X/(g \circ \mu)$, we have $\eta((g \circ \mu)_x \bullet_X (g \circ \mu)_y) = \eta((g \circ \mu)_{x*y}) = g_{\mu(x *_X y)} = g_{\mu(x) *_Y \mu(y)} = g_{\mu(x)} \bullet_Y g_{\mu(y)} = \eta((g \circ \mu)_x) \bullet_Y \eta((g \circ \mu)_y))$. Therefore η is a homomorphism. Let $g_a \in Y/g$. Then there exists $x_0 \in X$ such that $\mu(x_0) = a$ since μ is surjective. Hence $\eta((g \circ \mu)_{x_0}) = g_{\mu(x_0)} = g_a$ and so η is surjective.

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Let $x, y \in X$ be such that $g_{\mu(x)} = g_{\mu(y)}$. Then we have $(g \circ \mu)(x *_X y) = g(\mu(x *_X y)) = g(\mu(x) *_Y \mu(y)) = g(0_Y) = g(\mu(0_X)) = (g \circ \mu)(0_X)$ It follows that $(g \circ \mu)_x = (g \circ \mu)_y$. Thus η is injective. This completes the proof. $\qquad\square$

The homomorphism $\pi : X \to X/g$, $x \to g_x$, is called the *natural homomorphism* of X onto X/g. In Theorem 4.6, if we define natural homomorphisms $\pi_X : X \to X/g \circ \mu$ and $\pi_Y : Y \to Y/g$ then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ \mu$, i.e., the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \mu\ } & Y \\
\pi_X \downarrow & & \pi_Y \downarrow \\
X/(g \circ \mu) & \xrightarrow{\ \eta\ } & Y/g.
\end{array}
$$

Proposition 4.7. *Let (f, X) be a uni-soft BCC-ideal of a BCC-algebra X over U. The mapping $\gamma : X \to X/f$, given by $\gamma(x) := f_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = f_0\} = X_f$.*

Proof. Let $f_x \in X/f$. Then there exists an element $x \in X$ such that $\gamma(x) = f_x$. Hence γ is surjective. For any $x, y \in X$, we have $\gamma(x * y) = f_{x*y} = f_x \bullet f_y = \gamma(x) \bullet \gamma(y)$. Thus γ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = f_0\} = \{x \in X | x \sim^f 0\} = \{x \in X | f(x) = f(0)\} = X_f$. $\qquad\square$

Proposition 4.8. *Let (f, X) be a uni-soft BCC-ideal of a BCC-algebra X over U. If J is a BCC-ideal of X, then J/f is a BCC-ideal of X/f.*

Proof. Let (f, X) be a uni-soft BCC-ideal of X over U and let J be a BCC-ideal of X. Since $0 \in J$, we have $f_0 \in J/f$. For any $x, y, z \in J$, $(x * y) * z \in J$ and $y \in J$, we get $x * z \in J$. Let $(f_x \bullet f_y) \bullet f_z, f_y \in J/f$. Then $(f_x \bullet f_y) \bullet f_z = f_{(x*y)*z} \in J/f$ and $f_y \in J/f$ imply $f_x \bullet f_z \in J/f$. Thus J/f is a BCC-ideal of X/f. $\qquad\square$

Theorem 4.9. *Let (f, X) be a uni-soft BCC-ideal of a BCC-algebra X over U. If J^* is a BCC-ideal of a BCC-algebra X/f, then there exists a BCC-ideal $J = \{x \in X | f_x \in J^*\}$ in X such that $J/f = J^*$.*

Proof. Since J^* is a BCC-ideal of X/f, $(f_x \bullet f_y) \bullet f_z = f_{(x*y)*z}$, $f_y \in J^*$ imply $f_x \bullet f_z = f_{x*z} \in J^*$ for any $f_x, f_y, f_z \in X/f$. Thus $(x * y) * z, y \in J$ imply $x * z \in J$ for any $x, y, z \in X$. Therefore J is a BCC-ideal of X. By Proposition 4.8, we have $J/f = \{f_j | j \in J\} = \{f_j | \exists f_x \in J^*$ such that $j \sim^f x\} = \{f_j | \exists f_x \in J^*$ such that $f_x = f_j\} = \{f_j | f_j \in J^*\} = J^*$. $\qquad\square$

Theorem 4.10. *Let (f, X) be a uni-soft BCC-ideal of a BCC-algebra X over U. If J is a BCC-ideal of X, then $\dfrac{X/f}{J/f} \cong X/J$.*

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Proof. Note that $\dfrac{X/f}{J/f} = \{[f_x]_{J/f} | f \in X/f\}$. If we define $\varphi : \dfrac{X/f}{J/f} \to X/J$ by $\varphi([f_x]_{J/f}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[f_x]_{J/f} = [f_y]_{J/f}$. Then $f_x \sim^{J/f} f_y$ and so $f_{x*y} = f_x \bullet f_y \in J/f$ and $f_{y*x} = f_y \bullet f_x \in J/f$. Hence $x * y \in J$ and $y * x \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[f_x]_{J/f}, [f_y]_{J/f} \in \dfrac{X/f}{J/f}$, we have $\varphi([f_x]_{J/f} \bullet [f_y]_{J/f}) = \varphi([f_x \bullet f_y]_{J/f}) = [x * y]_J = [x]_J * [y]_J = \varphi([f_x]_{J/f}) * \varphi([f_y]_{J/f})$. Hence φ is a homomorphism.

Obviously, φ is onto. Finally, we show that φ is one-to-one. If $\varphi([f_x]_{J/f}) = \varphi([f_y]_{J/f})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $f_a \in [f_x]_{J/f}$, then $f_a \sim^{J/f} f_x$ and hence $f_{a*x}, f_{x*a} \in J/f$. It follows that $a * x, x * a \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y, y * a \in J$ and so $f_{a*y}, f_{y*a} \in J/f$. Therefore $f_a \sim^{J/f} f_y$. Hence $f_a \in [f_y]_{J/f}$. Thus $[f_x]_{J/f} \subseteq [f_y]_{J/f}$. Similarly, we obtain $[f_y]_{J/f} \subseteq [f_x]_{J/f}$. Therefore $[f_x]_{J/f} = [f_y]_{J/f}$. This completes the proof. $\square$

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FIXED POINT THEOREMS FOR VARIOUS CONTRACTION CONDITIONS IN DIGITAL METRIC SPACES

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Abstract. In this paper, we prove the existence of fixed points for Kannan contraction, Chatterjea contraction and Reich contraction in setting of digital metric spaces. These digital contractions are the applications of metric fixed point theory contractions.

1. Introduction

The basic tool of metric fixed point theory is the Banach contraction principle, which states that "Let T be a mapping from a complete metric space (X, d) into itself satisfying

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then T has a unique fixed point."

This principle gives existence and uniqueness of fixed points and methods for obtaining approximate fixed points. This principle was generalized by several authors by using different types of minimal commutative along with continuity one of the mappings. In finding common fixed point generally we include the following steps:

  (i) A commutative type condition,
  (ii) Completeness of the space or completeness of the range space of one or more mappings,
  (iii) A relation between the ranges of mappings,
  (iv) Continuity of one or more mappings,
  (v) A contractive type condition.

This principle was further generalized by using different types of properties such as E.A. property, Common Limit Range property along with containment of range spaces instead of continuity of mappings.

The topological fixed point theory involves the study of spaces with the fixed point property. Moreover, topology is the study of geometric problems that does not depend only on the exact shape of the objects, but rather it acts on a space. In topology, generally we consider infinitely many points in arbitrary small neighborhood of a point. To consider finite number of points in a neighborhood, the concept of digital topology was introduced by Rosenfeld [13].

In fact, digital topology is the study of geometric and topological properties of digital image using geometric and algebraic topology. The digital images have been used in computer sciences such as image processing, computer graphics. For detail one can refer to [1, 8, 11]. Digital topology also provides a mathematical basis for image processing operations. Further, digital topology is a developing area in 2D and 3D digital images. For a difference in general topology and digital topology, see Figure 1.
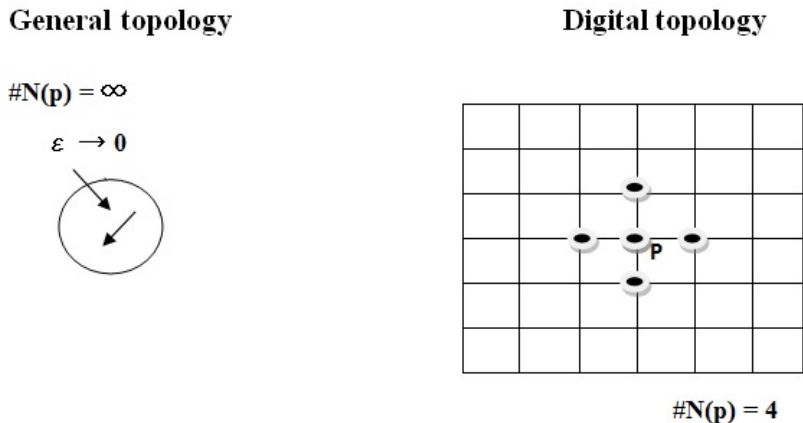
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FIGURE 1. Neighboorhood in general and digital topology

The elements of 2D image array are called pixels and the elements of 3D image array are called voxels. Each pixel or voxel is associated with lattice points (A point with integer coordinate) in the plane or in 3D-space. A lattice point associated with a pixel or voxel has values 0 and 1. The pixel or voxel that has value 0 is called a black point and the pixel or voxel that has value 1 is called a white point.

2. TOPOLOGICAL STRUCTURE OF DIGITAL METRIC SPACES

Let $\mathbb{Z}^n$, $n \in \mathbb{N}$, be the set of points in the Euclidean n-dimensional space with integer coordinates.

Definition 2.1. [4]. Let l, n be positive integers with $1 \leq l \leq n$. Consider two distinct points

$$p = (p_1, p_2, ..., p_n), \quad q = (q_1, q_2, ..., q_n) \in \mathbb{Z}^n.$$

The points p and q are k_l-adjacent if there are at most l indices i such that $|p_i - q_i| = 1$, and for all other indices j, $|p_j - q_j| \neq 1$, $p_j = q_j$.

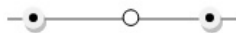(i) Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$ (see Figure 2).

FIGURE 2. 2-adjacency

(ii) Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if the points are distinct and differ by at most 1 in each coordinate.
(iii) Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if the points are 8-adjacent and differ in exactly one coordinate (see Figure 3).

(iv) Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if the points are distinct and differ by at most 1 in each coordinate.
(v) Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if the points are 26-adjacent and differ by at most 2 coordinates.
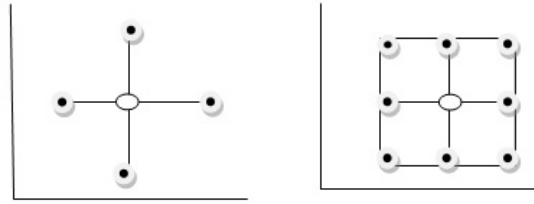
FIGURE 3. 4-adjacency and 8-adjacency

(vi) Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if the points are 18-adjacent and differ in exactly one coordinate (see Figure 4).
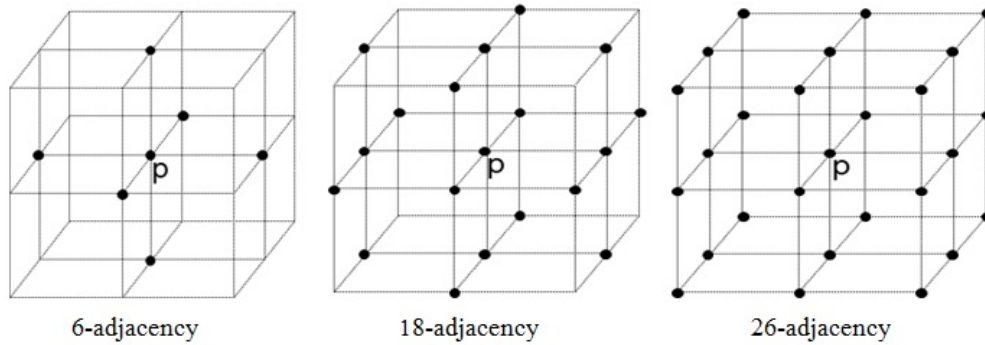

FIGURE 4. Adjacencies in $\mathbb{Z}^3$

One can easily note that the coordination number of Na in the crystal structure of NaCl is 6 which is equal to adjacency relation in digital images of Figure 5.
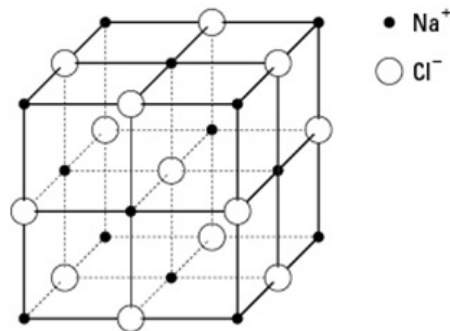

FIGURE 5. Crystal structure of NaCl

Definition 2.2. A digital image is a pair (X, κ), where $\emptyset \neq X \subset \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation on X. Technically, a digital image (X, κ) is an undirected graph whose vertex set is the set of members of X and whose edge set is the set of unordered pairs $\{x_0, x_1\} \subset X$ such that $x_0 \neq x_1$ and x_0 and x_1 are κ-adjacent.

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The notion of digital continuity in digital topology was developed by Rosenfeld [14] for studying 2D and 3D digital images. Boxer [2] gave the digital version of several notions of topology and Ege and Karaca [6] studied various digital continuous functions.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers, respectively. Boxer [3] defined a κ-neighbor of $p \in \mathbb{Z}^n$ which is a point of $\mathbb{Z}^n$ that is κ-adjacent to p where $\kappa \in \{2, 4, 6, 8, 18, 26\}$ and $n \in \{1, 2, 3\}$. The set

$$N_\kappa(p) = \{q \mid q \text{ is } \kappa - \text{adjacent to } p\}$$

is called the κ-neighborhood of p. Boxer [2] defined a digital interval as

$$[a, b]_\mathbb{Z} = \{z \in \mathbb{Z} \mid a \leq z \leq b\},$$

where $a, b \in \mathbb{Z}$ and $a < b$. A digital image $X \subset \mathbb{Z}^n$ is κ-connected [9] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, \ldots, r - 1$.

Definition 2.3. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images and $f : X \to Y$ be a function.

(i) If for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y, then f is said to be (κ_0, κ_1)-continuous [3].

(ii) f is (κ_0, κ_1)-continuous if for every κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent in Y [3].

(iii) If f is (κ_0, κ_1)-continuous, bijective and f^{-1} is (κ_1, κ_0)-continuous, then f is called (κ_0, κ_1)-isomorphism and denoted by $X \cong_{(\kappa_0, \kappa_1)} Y$.

Now we start with digital metric space (X, d, κ) with κ-adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$ as follows.

Definition 2.4. [6] Let (X, κ) be a digital image set. Let d be a function from $(X, \kappa) \times (X, \kappa) \to \mathbb{Z}^n$ satisfying all the properties of metric space. The triplet (X, d, κ) is called a digital metric space.

Proposition 2.5. [8] *A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) is a Cauchy sequence if and only if there is $\alpha \in \mathbb{N}$ such that $d(x_n, x_m) \lneq 1$ for all $n, m \gneq \alpha$.*

Theorem 2.6. [8] *For a digital metric space (X, d, κ), if a sequence $\{x_n\} \subset X \subset \mathbb{Z}^n$ is a Cauchy sequence then there is $\alpha \in \mathbb{N}$ such that we have $x_n = x_m$ for all $n, m \gneq \alpha$.*

Proposition 2.7. [8] *A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) converges to a limit $l \in X$ if for all $\epsilon \gneq 0$, there is $\alpha \in \mathbb{N}$ such that $d(x_n, l) \lneq \epsilon$ for all $n \gneq \alpha$.*

Proposition 2.8. [8] *A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that $x_n = l$ for all $n \gneq \alpha$.*

Theorem 2.9. [8] *A digital metric space (X, d, κ) is complete.*

Definition 2.10. [6] Let (X, d, κ) be any digital metric space. A self map f on a digital metric space is said to be a digital contraction if there exists a $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

Proposition 2.11. [6] *Every digital contraction map $f : (X, d, \kappa) \to (X, d, \kappa)$ is digitally continuous.*

Proposition 2.12. [8] *In a digital metric space (X, d, κ), consider two points x_i, x_j in a sequence $\{x_n\} \subset X$ such that they are κ-adjacent. Then they have the Euclidean distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most $\sqrt{t}$ depending on the position of the two points.*

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3. Main results

In 2015, Ege and Karaca [6] proved Banach contraction principle in the setting of digital metric spaces. With the motivation of Banach contraction principle in digital metric spaces, we prove Kannan, Chatterjea and Reich contraction fixed point theorems in the setting of digital metric spaces.

The following theorem is the digital version of Kannan contraction fixed point theorem [10].

Theorem 3.1. *Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation between the objects of X. Let (X, d, κ) be a digital metric space and S be a self map on X satisfying the following:*

$$d(Sx, Sy) \leq \alpha\{d(x, Sx) + d(y, Sy)\} \tag{3.1}$$

for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$. Then S has a unique fixed point in X.

Proof. Let $x_0 \in X$ and consider the iterate of sequence $x_{n+1} = Sx_n$. Now

$$d(x_1, x_2) = d(Sx_0, Sx_1) \leq \alpha\{d(x_0, Sx_0) + d(x_1, Sx_1)\},$$

i.e.,

$$d(x_1, x_2) \leq \frac{\alpha}{1 - \alpha} d(x_0, x_1).$$

Similarly, we have

$$d(x_2, x_3) \leq \frac{\alpha}{1 - \alpha} d(x_1, x_2)$$

$$\leq (\frac{\alpha}{1 - \alpha})^2 d(x_0, x_1)$$

and so on

$$d(x_n, x_{n+1}) \leq (\frac{\alpha}{1 - \alpha})^n d(x_0, x_1),$$

$$d(x_{n+1}, x_{n+2}) \leq (\frac{\alpha}{1 - \alpha})^{n+1} d(x_0, x_1).$$

Let $\beta = \frac{\alpha}{1-\alpha}$. Then we can rewrite the above statement as follows:

$$d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1),$$

$$d(x_{n+1}, x_{n+2}) \leq \beta^{n+1} d(x_0, x_1).$$

If we use the triangle inequality repeatedly, then we obtain the following:

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k})$$

$$\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{n+k-1}) d(x_0, x_1)$$

$$\leq \frac{\beta^n}{1 - \beta} d(x_0, x_1).$$

Since $0 \leq \beta < 1$, $\frac{\beta^n}{1-\beta} d(x_0, x_1) \to 0$ as $n \to \infty$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence in (X, d, κ). By Theorem 2.9, there exists a limit point v and due to (κ, κ)-continuity of S, we have

$$S(v) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} x_{n+1} = v.$$

Therefore, S has a fixed point.

Now we show that S has a unique fixed point. If a and b are fixed points of S, then

$$d(a, b) = d(Sa, Sb) \leq \alpha\{d(a, Sa) + d(b, Sb)\}$$

$$= \{d(a, a) + d(b, b)\} = 0.$$

As a result, $d(a, b) = 0$ and so $a = b$. $\qquad\square$

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Now we prove the digital version of Chatterjea fixed point theorem [5] as follows:

Theorem 3.2. *Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X. Let (X, d, κ) be a digital metric space and S be a self map on X satisfying the following:*

$$d(Sx, Sy) \le \alpha\{d(x, Sy) + d(y, Sx)\}$$

for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$. Then S has a unique fixed point in X.

Proof. Let $x_0 \in X$ and consider the iterate sequence $x_n = Sx_{n-1}$. Now

$$
\begin{aligned}
d(x_1, x_2) = d(Sx_0, Sx_1) &\le \alpha\{d(x_0, Sx_1) + d(x_1, Sx_0)\} \\
&\le \alpha\{d(x_0, x_2) + d(x_1, x_1)\} \\
&\le \alpha\{d(x_0, x_1) + d(x_1, x_2)\},
\end{aligned}
$$

i.e.,

$$(1 - \alpha)d(x_1, x_2) \le \alpha d(x_0, x_1),$$

$$d(x_1, x_2) \le \frac{\alpha}{1 - \alpha} d(x_0, x_1).$$

In a similar way, we get the following:

$$d(x_2, x_3) \le \frac{\alpha}{1 - \alpha} d(x_1, x_2) \le (\frac{\alpha}{1 - \alpha})^2 d(x_0, x_1),$$

$$d(x_n, x_{n+1}) \le (\frac{\alpha}{1 - \alpha})^n d(x_0, x_1),$$

$$d(x_{n+1}, x_{n+2}) \le (\frac{\alpha}{1 - \alpha})^{n+1} d(x_0, x_1).$$

If we take $\beta = \frac{\alpha}{1-\alpha}$, then we obtain

$$d(x_n, x_{n+1}) \le \beta^n d(x_0, x_1),$$

$$d(x_{n+1}, x_{n+2}) \le \beta^{n+1} d(x_0, x_1).$$

From the triangle inequality, we conclude

$$
\begin{aligned}
d(x_n, x_{n+k}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\
&\le (\beta^n + \beta^{n+1} + \ldots + \beta^{n+k-1})d(x_0, x_1) \\
&\le \frac{\beta^n}{1 - \beta} d(x_0, x_1).
\end{aligned}
$$

Since $0 \le \beta < 1$, $\frac{\beta^n}{1-\beta}d(x_0, x_1) \to 0$ as $n \to \infty$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence in (X, d, κ) and (X, d, κ) is a digital complete metric space. So there is a limit point u and by the (κ, κ)-continuity of S, we have

$$S(u) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} x_{n+1} = u.$$

Therefore, S has a fixed point.

To show the uniqueness, let a and b be fixed points of S. Then from the hypothesis, we get

$$
\begin{aligned}
d(a, b) = d(Sa, Sb) &\le \alpha\{d(a, Sa) + d(b, Sb)\} \\
&= \{d(a, a) + d(b, b)\} = 0.
\end{aligned}
$$

As a result, $d(a, b) = 0$ and so $a = b$. $\qquad\square$

Reich fixed point theorem [12] can be given as follows in digital images.

Theorem 3.3. *If S is a mapping on a digital metric space (X, d, κ) into itself satisfying the following*

$$d(Sx, Sy) \leq ad(x, Sx) + bd(y, Sy) + cd(x, y)$$

for all $x, y \in X$ and all nonnegative real numbers a, b, c with $a + b + c < 1$. Then S has a unique fixed point in X.

Proof. Let $x_0 \in X$. Defining the sequence $x_{n+1} = Sx_n$, we get the following:

$$\begin{aligned} d(x_1, x_2) = d(Sx_0, Sx_1) &\leq ad(x_0, Sx_0) + bd(x_1, Sx_1) + cd(x_0, x_1) \\ &\leq ad(x_0, x_1) + bd(x_1, x_2) + cd(x_0, x_1) \\ &\leq \frac{a+c}{1-b} d(x_0, x_1). \end{aligned}$$

Similarly, we have

$$d(x_n, x_n + 1) \leq \beta^n d(x_0, x_1),$$

where $\beta = \frac{a+c}{1-b}$ and $\beta < 1$. The triangle inequality gives the following:

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{n+k-1}) d(x_0, x_1) \\ &\leq \frac{\beta^n}{1-\beta} d(x_0, x_1). \end{aligned}$$

Since $0 \leq \beta < 1$, $\frac{\beta^n}{1-\beta} d(x_0, x_1)$ as $n \to \infty$. Then we can say that $\{x_n\}$ is a Cauchy sequence in (X, d, κ). There exists a limit point w such that

$$S(w) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} x_{n+1} = w$$

by the completeness of (X, d, κ). Hence S has a fixed point. It can be easily shown that this fixed point is unique. $\square$

We give an example about Theorem 3.1.

Example 3.4. Consider the minimal simple closed 18-surface $MSS'_{18} = \{c_i : i \in [0, 5]_{\mathbb{Z}}\}$ (see Figure 6).
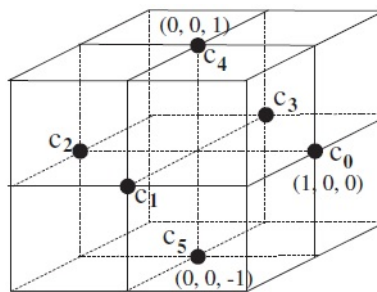

FIGURE 6. MSS'_{18} [7]

Let $S : MSS'_{18} \to MSS'_{18}$ be any digital map satisfying the inequality (3.1). Consider a point such as c_0 in MSS'_{18} and take $S(c_0) = c' \in MSS'_{18}$. For the point $c_i \in N_{18}(c_0, 1)$, $i \in \{1, 3, 4, 5\}$, we have

$$d(S(c_i), S(c_0)) \leq \alpha\{d(c_i, S(c_i)) + d(c_0, S(c_0))\} \leq \alpha\{\sqrt{2} + \sqrt{2}\} = 2\sqrt{2}\alpha$$

since the maximum distance between different 18-adjacent points in MSS'_{18} is $\sqrt{2}$ by Proposition 2.12. Since $0 < \alpha < \frac{1}{2}$, we get $d(S(c_i), S(c_0)) \lneq \sqrt{2}$. As a result, $d(S(c_i), S(c_0)) = 0$ implies that $S(c_i) = S(c_0) = c'$ from the property of MSS'_{18}. This procedure can be applied all points in MSS'_{18} since c_0 is an arbitrary point. Therefore, S is a constant map. By Theorem 3.1, we can say that S has a fixed point.

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Fixed points of Círíc type ordered F-contractions on partial metric spaces

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Abstract. In this paper, by considering both F-contraction and fixed point results on ordered partial metric spaces, we introduce a pair of Círíc type ordered F-contractions on an ordered partial metric space. Then we give a common fixed point theorem for such contractions. We give an example showing that our main theorem is applicable, but both results of Durmaz *et al.* [11] and Wardowski [18] are not. We also discuss that this fixed point result can be applied to show the existence of solution of an integral equation.

1. Introduction

Matthews [12] introduced the concept of partial metric spaces and proved an analogue of Banach fixed point theorem in partial metric spaces. In fact, a partial metric space is a generalization of metric space in which the self distances $p(r_1, r_1)$ of elements of a space may not be zero and follows the inequality $p(r_1, r_1) \leq p(r_1, r_2)$. After this remarkable contribution, many authors took interest in partial metric spaces and its topological properties and presented several well known fixed point results in the framework of partial metric spaces (see [1, 2, 5, 6, 7, 8, 14] and references therein).

Banach presented a landmark fixed point result (Banach Contraction Principle). This result proved a gateway for the fixed point researchers and opened a new door in metric fixed point theory. A number of efforts have been made to enrich and generalize Banach Contraction Principle (see [9, 10] and references therein). Following Banach, in 2012, Wardowski [18] presented a new contraction (known as F-contraction). Since 2012, a number of fixed point results have been established by using F-contraction or ordered F-contraction (see [3, 11, 13, 15, 17]).

Wardowski [18] presented the concept of F-contraction. Then some generalizations of F-contractions including multivalued case are obtained in [4, 3]. In this article, we prove a common fixed point theorem for a pair ordered F-contractions in complete partial metric spaces. An example is constructed to illustrate this result and to show that our result generalizes the result established by Durmaz *et al.* [11]. We apply the mentioned theorem to show the existence of solution of implicit type integral equations.

---

2. Preliminaries

Throughout this paper, we denote $(0, \infty)$ by $\mathbb{R}^+$, $[0, \infty)$ by $\mathbb{R}_0^+$, $(-\infty, +\infty)$ by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Following concepts and results will be required for the proofs of main results.

Definition 1. [18] A mapping $T : M \to M$ is said to be an F-contraction if it satisfies the following condition

$$(d(T(r_1), T(r_2))) > 0 \Rightarrow \tau + F(d(T(r_1), T(r_2)) \leq F(d(r_1, r_2))) \qquad (2.1)$$

for all $r_1, r_2 \in M$ and some $\tau > 0$. Here $F : \mathbb{R}^+ \to \mathbb{R}$ is a function satisfying the following properties.

- (F_1) : F is strictly increasing.
- (F_2) : For each sequence $\{r_n\}$ of positive numbers $\lim_{n \to \infty} r_n = 0$ if and only if $\lim_{n \to \infty} F(r_n) = -\infty$.
- (F_3) : There exists $\theta \in (0, 1)$ such that $\lim_{\alpha \to 0^+} (\alpha)^\theta F(\alpha) = 0$.

Wardowski [18] established the following result using F-contraction.

Theorem 1. [18] *Let (M, d) be a complete metric space and $T : M \to M$ be an F-contraction. Then T has a unique fixed point $\upsilon \in M$ and for every $r_0 \in M$ the sequence $\{T^n(r_0)\}$ for all $n \in \mathbb{N}$ is convergent to υ.*

Recently, Durmaz *et al.* [11] presented an ordered version of Theorem 1.

Theorem 2. *Let $(M, \preceq, d)$ be an ordered complete metric space and $T : M \to M$ be an ordered F-contraction. Let T be a nondecreasing mapping and there exists $r_0 \in M$ such that $r_0 \preceq T(r_0)$. If T is continuous or M is regular, then T has a fixed point.*

We denote by Δ_F the set of all functions satisfying the conditions $(F_1) - (F_3)$.

Example 1. [18] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies $(F_1) - (F_3)$ for any $\kappa \in (0, 1)$. Each mapping $T : M \to M$ satisfying (2.1) is an F-contraction such that

$$d(T(r_1), T(r_2)) \leq e^{-\tau} d(r_1, r_2), \text{ for all } r_1, r_2 \in M, T(r_1) \neq T(r_2).$$

Obviously, for all $r_1, r_2 \in M$ such that $T(r_1) = T(r_2)$, the inequality

$$d(T(r_1), T(r_2)) \leq e^{-\tau} d(r_1, r_2)$$

holds, that is, T is a Banach contraction.

Remark 1. From (F_1) and (2.1) it is easy to conclude that every F-contraction is necessarily continuous.

Definition 2. [12] Let M be a nonempty set and assume that the function $p : M \times M \to \mathbb{R}_0^+$ satisfies the following properties:

- (p_1) $r_1 = r_2 \Leftrightarrow p(r_1, r_1) = p(r_1, r_2) = p(r_2, r_2)$,
- (p_2) $p(r_1, r_1) \leq p(r_1, r_2)$,
- (p_3) $p(r_1, r_2) = p(r_2, r_1)$,

(p_4) $p(r_1, r_3) \leq p(r_1, r_2) + p(r_2, r_3) - p(r_2, r_2)$

for all $r_1, r_2, r_3 \in M$. Then p is called a partial metric on M and the pair (M, p) is known as partial metric space.

In [12], Matthews proved that every partial metric p on M induces a metric $d_p : M \times M \to \mathbb{R}_0^+$ defined by

$$d_p(r_1, r_2) = 2p(r_1, r_2) - p(r_1, r_1) - p(r_2, r_2)$$

for all $r_1, r_2 \in M$.

Notice that a metric on a set M is a partial metric p such that $p(r, r) = 0$ for all $r \in M$ and $p(r_1, r_2) = 0$ implies $r_1 = r_2$ (using (p_1) and (p_2)).

Matthews [12] established that each partial metric p on M generates a T_0 topology $\tau(p)$ on M. The base of topology $\tau(p)$ is the family of open p-balls $\{B_p(r, \epsilon) : r \in M, \ \epsilon > 0\}$, where $B_p(r, \epsilon) = \{r_1 \in M : p(r, r_1) < p(r, r) + \epsilon\}$ for all $r \in M$ and $\epsilon > 0$. A sequence $\{r_n\}_{n \in \mathbb{N}}$ in (M, p) converges to a point $r \in M$ if and only if $p(r, r) = \lim_{n \to \infty} p(r, r_n)$.

Definition 3. [12] Let (M, p) be a partial metric space.

(1) A sequence $\{r_n\}_{n \in \mathbb{N}}$ in (M, p) is called a Cauchy sequence if $\lim_{n,m \to \infty} p(r_n, r_m)$ exists and is finite.
(2) A partial metric space (M, p) is said to be complete if every Cauchy sequence $\{r_n\}_{n \in \mathbb{N}}$ in M converges, with respect to $\tau(p)$, to a point $r \in X$ such that $p(r, r) = \lim_{n,m \to \infty} p(r_n, r_m)$.

The following lemma will be helpful in the sequel.

Lemma 1. [12]

(1) A sequence r_n is a Cauchy sequence in a partial metric space (M, p) if and only if it is a Cauchy sequence in metric space (M, d_p)
(2) A partial metric space (M, p) is complete if and only if the metric space (M, d_p) is complete.
(3) A sequence $\{r_n\}_{n \in \mathbb{N}}$ in M converges to a point $r \in M$, with respect to $\tau(d_p)$ if and only if $\lim_{n \to \infty} p(r, r_n) = p(r, r) = \lim_{n,m \to \infty} p(r_n, r_m)$.
(4) If $\lim_{n \to \infty} r_n = \upsilon$ such that $p(\upsilon, \upsilon) = 0$ then $\lim_{n \to \infty} p(r_n, r) = p(\upsilon, r)$ for every $r \in M$.

In the following example, we shall show that there are mappings which are not F-contractions in metric spaces, nevertheless, such mappings follow the conditions of F-contraction in partial metric spaces.

Example 2. Let $M = [0, 1]$ and define partial metric by $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \in M$. The metric d induced by partial metric p is given by $d(r_1, r_2) = |r_1 - r_2|$ for all $r_1, r_2 \in M$. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ and T by

$$T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0, 1); \\[2mm] 0 & \text{if } r = 1. \end{cases}$$

Then T is not an F-contraction in a metric space (M, d). Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, $d(T(r_1), T(r_2)) > 0$ and we have

$$
\begin{aligned}
\tau + F\left(d(T(r_1), T(r_2))\right) &\leq F\left(d(r_1, r_2)\right), \\
\tau + F\left(d(T(1), T(\frac{5}{6}))\right) &\leq F\left(d(1, \frac{5}{6})\right), \\
\tau + F\left(d(0, \frac{1}{6})\right) &\leq F\left(\frac{1}{6}\right), \\
\frac{1}{6} &< \frac{1}{6},
\end{aligned}
$$

which is a contradiction for all possible values of τ. Now if we work in partial metric space (M, p), we get a positive answer, that is,

$$
\begin{aligned}
\tau + F\left(p(T(r_1), T(r_2))\right) &\leq F\left(p(r_1, r_2)\right) \text{ implies} \\
\tau + F\left(\frac{1}{6}\right) &\leq F(1),
\end{aligned}
$$

which is true.

Similarly, for all other points in M our claim proves true.

Definition 4. Let $(M \preceq)$ be a partially ordered set. Two mappings $S, T \colon M \to M$ are said to be weakly increasing mappings if $S(m) \preceq TS(m)$ and $T(m) \preceq ST(m)$ hold for all $m \in M$.

Example 3. Let $M = \mathbb{R}^+$ be endowed with usual order and usual topology. Let $S, T \colon M \to M$ be given by

$$
S(m) = \begin{cases} m^{\frac{1}{2}} & \text{if } m \in [0, 1] \\ m^2 & \text{if } m \in (1, \infty) \end{cases} \quad \text{and } T(m) = \begin{cases} m & \text{if } m \in [0, 1] \\ 2m & \text{if } m \in (1, \infty). \end{cases}
$$

Then the pair (S, T) is weakly increasing mappings, where T is a discontinuous mapping.

3. Main results

We begin with the following definitions.

Definition 5. Let $(M, \preceq)$ be an ordered set and p be a metric on M. Then the triplet $(M, \preceq, p)$ is known as an ordered partial metric space. If (M, p) is complete, then $(M, \preceq, p)$ is called an ordered complete partial metric space. Moreover, M is regular if the ordered partial metric space $(M, \preceq, p)$ provides the following condition:

$$
\begin{cases}
\text{If } \{r_n\} \subset M \text{ is a nondecreasing (nonincreasing) sequence with } r_n \to r, \\
\text{then } r_n \preceq r \ (r \preceq r_n) \text{ for all } n.
\end{cases}
$$

Definition 6. Let $(M, \preceq, p)$ be an ordered partial metric space and $S, T \colon M \to M$ be two mappings. Let

$$
\gamma = \{(h, k) \in M \times M : h \preceq k, p(S(h), T(k)) > 0\}.
$$

We say the mappings S and T are a pair of Círíc type ordered F-contractions if there exist $F \in \Delta_F$ and $\tau > 0$ such that for all $(h, k) \in \gamma$,

$$
\tau + F(p(S(h), T(k))) \leq F\left(\mathcal{M}(h, k)\right), \tag{3.1}
$$

where

$$\mathcal{M}(h,k) = \max \left\{ p(h,k), p(h,S(h)), p(k,T(k)), \frac{p(k,S(h)) + p(h,T(k))}{2} \right\}.$$

The following lemma will be useful in the sequel.

Lemma 2. *Let $(M, \preceq, p)$ be an ordered complete partial metric space and S, T be a pair of Círić type ordered F-contractions. Then for each $i = 0, 1, 2, 3, \ldots$ $p(r_{2i}, r_{2i+1}) = 0$ implies $p(r_{2i+1}, r_{2i+2}) = 0$.*

Proof. Let $r_0 \in M$ be an initial point and take $r_1 = S(r_0)$ and $r_2 = T(r_1)$. Then by induction we can construct an iterative sequence r_n of points in M such a way that $r_{2i+1} = S(r_{2i})$ and $r_{2i+2} = T(r_{2i+1})$, where $i = 0, 1, 2, \ldots$. We argue by contradiction that $p(r_{2i+1}, r_{2i+2}) > 0$. We note that

$$
\begin{aligned}
\mathcal{M}(r_{2i}, r_{2i+1}) &= \max \left\{ p(r_{2i}, r_{2i+1}), p(r_{2i}, S(r_{2i})), p(r_{2i+1}, T(r_{2i+1})), \frac{p(r_{2i+1}, S(r_{2i})) + p(r_{2i}, T(r_{2i+1}))}{2} \right\} \\
&= \max \left\{ p(r_{2i}, r_{2i+1}), p(r_{2i}, r_{2i+1}), p(r_{2i+1}, r_{2i+2}), \frac{p(r_{2i+1}, r_{2i+1}) + p(r_{2i}, r_{2i+2})}{2} \right\} \\
&= \max \{ 0, p(r_{2i+1}, r_{2i+2}) \} = p(r_{2i+1}, r_{2i+2}).
\end{aligned}
$$

Consider $\tau + F(p(r_{2i+1}, r_{2i+2})) = \tau + F(p(S(r_{2i}), T(r_{2i+1})))$. From (3.1), we have

$$
\begin{aligned}
\tau + F\left(p(r_{2i+1}, r_{2i+2})\right) &= \tau + F\left(p(S(r_{2i}), T(r_{2i+1}))\right) \\
&\leq F\left(\mathcal{M}(r_{2i}, r_{2i+1})\right) \\
&\leq F\left(p(r_{2i+1}, r_{2i+2})\right)
\end{aligned}
$$

for all $i \in \mathbb{N} \cup \{0\}$, which gives a contradiction. Hence $p(r_{2i+1}, r_{2i+2})) = 0$. $\square$

The following theorem is one of the main results.

Theorem 3. *Let $(M, \preceq, p)$ be an ordered complete partial metric space and $S, T : M \to M$ be a pair of Círić type ordered F-contractions. If S, T are two weakly increasing mappings and there exists $r_0 \in M$ such that $r_0 \preceq S(r_0)$, then there exists a point v such that $p(v, v) = 0$. Assume that either one of S, T is continuous or M is regular. Then S, T have a common fixed point.*

Proof. We begin with the following observation:

$$\mathcal{M}(h, k) = 0 \text{ if and only if } h = k \text{ is a common fixed point of } (S, T).$$

Indeed, if $h = k$ is a common fixed point of (S, T), then $T(k) = T(h) = h = k = S(k) = S(h)$ and

$$
\begin{aligned}
\mathcal{M}(h, k) &= \max \left\{ p(h, k), p(h, S(h)), p(k, T(k)), \frac{p(k, S(h)) + p(h, T(k))}{2} \right\} \\
&= p(h, h).
\end{aligned}
$$

If $p(h, h) > 0$, then from the contractive condition (3.1), we get

$$\tau + F(p(h, h)) = \tau + F(p(S(h), T(k))) \leq F(p(h, h)),$$

which is a contradiction. Thus $p(h, h) = 0$ entails $\mathcal{M}(h, h) = 0$.

Conversely, if $\mathcal{M}(h,k) = 0$, then it is easy to check that $h = k$ is a common fixed point of S and T.

If $\mathbb{M}(r_1, r_2) > 0$ for all $r_1, r_2 \in M$, then by the given assumptions there exists $r_0 \in M$ such that $r_0 \preceq S(r_0)$. Take $r_1 = S(r_0)$ and $r_2 = T(r_1)$. Then by induction we can construct an iterative sequence r_n of points in M such a way that $r_{2i+1} = S(r_{2i})$ and $r_{2i+2} = T(r_{2i+1})$, where $i = 0, 1, 2, \ldots$. Since $r_0 \preceq S(r_0)$ and S, T are weakly increasing mappings, we obtain

$$r_1 = S(r_0) \preceq TS(r_0) = T(r_1) = r_2 = T(r_1) \preceq ST(r_1) = S(r_2) = r_3.$$

Iteratively, we obtain

$$r_0 \preceq r_1 \preceq r_2 \preceq \cdots \preceq r_{n-1} \preceq r_n \preceq r_{n+1} \preceq \cdots.$$

Now if $p(S(r_{2i}), T(r_{2i+1})) = 0$, then using Lemma 2, we can conclude that r_{2i} is a common fixed point of S, T. If $p(S(r_{2i}), T(r_{2i+1})) > 0$, then $(r_{2i}, r_{2i+1}) \in \gamma$, since $r_{2i} \preceq r_{2i+1}$. From the contractive condition (3.1), we get

$$
\begin{aligned}
\tau + F\left(p(r_{2i+1}, r_{2i+2})\right) &= \tau + F\left(p(S(r_{2i}), T(r_{2i+1}))\right) \\
&\leq F\left(\mathcal{M}(r_{2i}, r_{2i+1})\right)
\end{aligned}
\tag{3.2}
$$

for all $i \in \mathbb{N} \cup \{0\}$, where

$$
\begin{aligned}
\mathcal{M}(r_{2i}, r_{2i+1}) &= \max\left\{ p(r_{2i}, r_{2i+1}), p(r_{2i}, S(r_{2i})), p(r_{2i+1}, T(r_{2i+1})), \frac{p(r_{2i+1}, S(r_{2i})) + p(r_{2i}, T(r_{2i+1}))}{2} \right\} \\
&= \max\left\{ p(r_{2i}, r_{2i+1}), p(r_{2i}, r_{2i+1}), p(r_{2i+1}, r_{2i+2}), \frac{p(r_{2i+1}, r_{2i+1}) + p(r_{2i}, r_{2i+2})}{2} \right\} \\
&= \max\left\{ p(r_{2i}, r_{2i+1}), p(r_{2i+1}, r_{2i+2}) \right\}.
\end{aligned}
$$

If $\mathcal{M}(r_{2i}, r_{2i+1}) = p(r_{2i+1}, r_{2i+2})$, then due to (F_1) and (3.2), we get a contradiction. Thus, for $\mathcal{M}(r_{2i}, r_{2i+1}) = p(r_{2i}, r_{2i+1})$, we have

$$F\left(p(r_{2i+1}, r_{2i+2})\right) \leq F\left(p(r_{2i}, r_{2i+1})\right) - \tau \tag{3.3}$$

for all $i \in \mathbb{N} \cup \{0\}$. Also since $r_{2i+1} \preceq r_{2i+2}$, $p(S(r_{2i+2}), T(r_{2i+1})) > 0$. Otherwise, by Lemma 2, r_{2i+1} is a common fixed point of S, T. Thus $(r_{2i+1}, r_{2i+2}) \in \gamma$ and note that

$$
\begin{aligned}
\mathcal{M}(r_{2i+2}, r_{2i+1}) &= \max\left\{ \begin{array}{l} p(r_{2i+2}, r_{2i+1}), p(r_{2i+2}, S(r_{2i+2})), p(r_{2i+1}, T(r_{2i+1})), \\ \dfrac{p(r_{2i+1}, S(r_{2i+2})) + p(r_{2i+2}, T(r_{2i+1}))}{2} \end{array} \right\} \\
&= \max\left\{ \begin{array}{l} p(r_{2i+2}, r_{2i+1}), p(r_{2i+2}, r_{2i+3}), p(r_{2i+1}, r_{2i+2}), \\ \dfrac{p(r_{2i+1}, r_{2i+3}) + p(r_{2i+2}, r_{2i+2})}{2} \end{array} \right\} \\
&= \max\left\{ p(r_{2i+2}, r_{2i+1}), p(r_{2i+2}, r_{2i+3}) \right\}.
\end{aligned}
$$

Again the case $\mathcal{M}(r_{2i+2}, r_{2i+1}) \leq p(r_{2i+2}, r_{2i+3})$ is not possible. So, for the other case, the contractive condition (3.1) implies

$$F\left(p(r_{2i+2}, r_{2i+3})\right) \leq F\left(p(r_{2i+1}, r_{2i+2})\right) - \tau \tag{3.4}$$

for all $i \in \mathbb{N} \cup \{0\}$. By (3.3) and (3.4), we have

$$F\left(p(r_{n+1}, r_{n+2})\right) \leq F\left(p(r_n, r_{n+1})\right) - \tau \tag{3.5}$$

for all $n \in \mathbb{N} \cup \{0\}$. By (3.5), we obtain

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_{n-2}, r_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_0, r_1)\right) - n\tau. \tag{3.6}$$

By (3.6), we obtain $\lim_{n \to \infty} F\left(p(r_n, r_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n \to \infty} p(r_n, r_{n+1}) = 0. \tag{3.7}$$

From the property (F_3) of F-contraction, there exists $\kappa \in (0, 1)$ such that

$$\lim_{n \to \infty} \left((p(r_n, r_{n+1}))^\kappa F\left(p(r_n, r_{n+1})\right)\right) = 0. \tag{3.8}$$

By (3.6), for all $n \in \mathbb{N}$, we obtain

$$(p(r_n, r_{n+1}))^\kappa \left(F\left(p(r_n, r_{n+1})\right) - F\left(p(r_0, x_1)\right)\right) \leq -\left(p(r_n, r_{n+1})\right)^\kappa n\tau \leq 0. \tag{3.9}$$

Considering (3.7), (3.8) and letting $n \to \infty$ in (3.9), we have

$$\lim_{n \to \infty} \left(n\left(p(r_n, r_{n+1})\right)^\kappa\right) = 0. \tag{3.10}$$

Since (3.10) holds, there exists $n_1 \in \mathbb{N}$ such that $n\left(p(r_n, r_{n+1})\right)^\kappa \leq 1$ for all $n \geq n_1$ or

$$p(r_n, r_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{3.11}$$

Using (3.11), we get, for $m > n \geq n_1$,

$$
\begin{aligned}
p(r_n, r_m) &\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + \ldots + p(r_{m-1}, r_m) \\
&\quad - \sum_{j=n+1}^{m-1} p(r_j, r_j) \\
&\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + \ldots + p(r_{m-1}, r_m) \\
&= \sum_{i=n}^{m-1} p(r_i, r_{i+1}) \\
&\leq \sum_{i=n}^{\infty} p(r_i, r_{i+1}) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}.
\end{aligned}
$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m \to \infty} p(r_n, r_m) = 0$. Hence $\{r_n\}$ is a Cauchy sequence in (M, p). Due to Lemma 1, $\{r_n\}$ is a Cauchy sequence in (M, d_p). Since (M, p) is a complete partial metric space, (M, d_p) is a complete metric space and as a result there exists $v \in M$ such that $\lim_{n \to \infty} d_p(r_n, v) = 0$. Moreover, by Lemma 1

$$\lim_{n \to \infty} p(v, r_n) = p(v, v) = \lim_{n,m \to \infty} p(r_n, r_m). \tag{3.12}$$

Since $\lim_{n,m\to\infty} p(r_n, r_m) = 0$, from (3.12), we deduce that

$$p(v, v) = 0 = \lim_{n\to\infty} p(v, r_n). \tag{3.13}$$

Now from (3.13) it follows that $r_{2n+1} \to v$ and $r_{2n+2} \to v$ as $n \to \infty$ with respect to $\tau(p)$. Suppose that T is continuous. Then

$$v = \lim_{n\to\infty} r_n = \lim_{n\to\infty} r_{2n+1} = \lim_{n\to\infty} r_{2n+2} = \lim_{n\to\infty} T(r_{2n+1}) = T(\lim_{n\to\infty} r_{2n+1}) = T(v).$$

Now we show that $v = S(v)$. Suppose on contrary that $p(v, S(v)) > 0$. Regarding $v \preceq v$ together with the contractive condition (3.1), we obtain

$$\begin{aligned}
\tau + F(p(v, S(v))) &= \tau + F(p(S(v), T(v))) \\
&\leq F(\mathcal{M}(v, v)), \\
F(p(v, S(v))) &< F(p(v, v)),
\end{aligned}$$

which is a contradiction. Thus $p(v, S(v)) = 0$ and due to (p_1), (p_2) we conclude that $v = S(v)$. Consequently, we have $S(v) = T(v) = v$, that is, (S, T) have a common fixed point v.

In the other case, using the assumption that M is regular, we have that $r_n \preceq v$ for all $n \in \mathbb{N}$. To show that v is a common fixed point of S, T, we split the proof into two cases.

(1) $r_n = v$ for some n. Then there exists $i_0 \in \mathbb{N}$ such that $r_{2i_0} = v$. Consider $S(v) = S(r_{2i_0}) = r_{2i_0+1} \preceq v$ and also $v = r_{2i_0} \preceq r_{2i_0+1} = S(v)$. Thus $v = S(v)$ and from (3.1), we have $v = T(v)$.

(2) $r_n \neq v$ for all n. Suppose that $p(v, S(v)) > 0$. Since $\lim_{n\to\infty} r_{2i} = v$, there exists $\mathcal{N} \in \mathbb{N}$ such that

$$p(r_{2i+1}, S(v)) > 0 \text{ and } p(r_{2i}, v) < \frac{p(v, S(v))}{2} \text{ for all } i \geq \mathcal{N}.$$

Moreover,

$$\mathcal{M}(r_{2i}, v) = \max\left\{ p(r_{2i}, v), p(r_{2i}, S(r_{2i})), p(v, T(v)), \frac{p(v, S(r_{2i})) + p(r_{2i}, T(v))}{2} \right\},$$

$$\mathcal{M}(r_{2i}, v) \leq \frac{p(v, S(v))}{2} \text{ for all } i \geq \mathcal{N}.$$

So $(r_{2i}, v) \in \gamma$ and S and T satisfy the generalized rational type ordered F-contraction. Thus

$$\begin{aligned}
\tau + F(p(r_{2i+1}, S(v))) &= \tau + F(p(S(r_{2i}), T(v))) \\
&\leq F(\mathcal{M}(r_{2i}, v)), \\
F(p(v, S(v))) &< F(\frac{p(v, S(v))}{2}) \text{ as } i \to \infty,
\end{aligned}$$

which is a contradiction. Therefore, $p(v, S(v)) = 0$ and due to (p_1), (p_2) we conclude that $v = S(v)$ and from (3.1) we have $v = T(v)$. Thus (S, T) have a common fixed point v. $\qquad \square$

We denote the set of common fixed points of S, T by $\mathrm{Fix}(S, T)$.

Remark 2. If we assume that $\mathrm{Fix}(S, T)$ in Theorem 3 is a chain along with existing conditions, then it is a singleton set (common fixed point is unique). Indeed, if ω is another common fixed

point of S, T, then $\omega \preceq \upsilon$. Also $p(S(\upsilon), T(\omega)) > 0$ (otherwise $\upsilon = \omega$) and so $(\upsilon, \omega) \in \gamma$. From the contractive condition (3.1), we have

$$
\begin{aligned}
\tau + F(p(\upsilon, \omega)) &= \tau + F(p(S(\upsilon), T(\omega))) \\
&\leq F(\mathcal{M}(\upsilon, \omega)),
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
\mathcal{M}(\upsilon, \omega) &= \max\left\{ p(\upsilon, \omega), p(\upsilon, S(\upsilon)), p(\omega, T(\omega)), \frac{p(\omega, S(\upsilon)) + p(\upsilon, T(\omega))}{2} \right\} \\
&= p(\upsilon, \omega).
\end{aligned}
$$

From (3.14), we have

$$
F(p(\upsilon, \omega)) < F(p(\upsilon, \omega)),
$$

which leads to a contradiction. Hence $\upsilon = \omega$ and υ is a unique common fixed point of a pair (S, T).

Remark 3. If $\mathrm{Fix}(S, T)$ is not a chain and there exists z in M such that every element in the orbit $O_T(z) = \{z, T(z), T^2(z), \ldots\}$ is comparable to υ, ω, then $\upsilon = \omega$ (υ is unique) provided that S and T are Círíc type ordered F-contractions.

Proof. Assume that υ, ω are in $\mathrm{Fix}(S, T)$ and there exists an element $z \in M$ such that every element of $O_T(z) = \{z, T(z), T^2(z), \ldots\}$ is comparable to υ, ω and hence $(T^{n-1}(z), S^{n-1}(\upsilon))$ and $(T^{n-1}(z), S^{n-1}(\omega))$ are elements of γ for each $n \geq 1$. Due to (3.1), we have

$$
\begin{aligned}
\tau + F(p(\upsilon, T^n(z))) &= \tau + F(p(S^n(\upsilon), T^n(z))) \\
&\leq F(\mathcal{M}(S^{n-1}(\upsilon), T^{n-1}(z))),
\end{aligned}
\tag{3.15}
$$

where

$$
\begin{aligned}
\mathcal{M}\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) &= \max\left\{ \begin{array}{c} p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right), p\left(S^{n-1}(\upsilon), S^n(\upsilon)\right), p\left(T^{n-1}(z), T^n(z)\right), \\ \dfrac{p\left(T^{n-1}(z), S^n(\upsilon)\right) + p\left(S^{n-1}(\upsilon), T^n(z)\right)}{2} \end{array} \right\} \\
&= p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) = p\left(\upsilon, T^{n-1}(z)\right).
\end{aligned}
$$

Thus, from (3.15), we deduce that $\{p(\upsilon, T^n(z))\}$ is a nonnegative decreasing sequence which in turn converges to 0.

Similarly, we can show that $\{p(\omega, T^n(z))\}$ is a nonnegative decreasing sequence, which converges to 0. Consequently, $\upsilon = \omega$. $\qquad \square$

The following example illustrates Theorem 3 and shows that the condition (3.1) is more general than contractivity condition given by Durmaz *et al.* ([11]).

Example 4. Let $M = [0, 1]$ and define $p(r_1, r_2) = \max\{r_1, r_2\}$. Let $\prec_1$ be defined by $r_1 \prec_1 r_2$ if and only if $r_2 \leq r_1$ for all $r_1, r_2 \in M$. Then $r_1 \prec_1 r_2$ is a partial order on M and $(M, \prec_1, p)$ is a complete ordered partial metric space. Moreover, define $d(r_1, r_2) = |r_1 - r_2|$. Then $(M, \prec_1, d)$ is a complete ordered metric space. Define the mappings $S, T : M \to M$ as follows:

$$
T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0, 1); \\ 0 & \text{if } r = 1 \end{cases} \qquad \text{and } S(r) = \frac{3r}{7} \text{ for all } r \in M.
$$

Clearly, S, T are weakly increasing self mappings with respect to $\prec_1$. Define the function $F :$ $R^+ \to R$ by $F(r) = \ln(r)$ for all $r \in R^+ > 0$. Let $r_1, r_2 \in M$ such that $p(S(r_1), T(r_2)) > 0$ and suppose that $r_2 \prec_1 r_1$. Then

$$\mathcal{M}(r_1, r_2) = \max\left\{ r_2, \frac{r_1 r_2}{1 + r_1}, \frac{r_1 r_2}{1 + \max\left\{ \frac{3r_1}{7}, \frac{r_2}{5} \right\}} \right\}.$$

Since $\frac{r_1}{1+r_1} < 1$ and $\frac{r_1}{1+\max\left\{ \frac{3r_1}{7}, \frac{r_2}{5} \right\}} < 1$, we have that $\mathcal{M}(r_1, r_2) = r_2$.

In a similar way, if $r_1 \prec_1 r_2$, then we obtain that $\mathcal{M}(r_1, r_2) = r_1$, i.e., $\mathcal{M}(r_1, r_2) = p(r_1, r_2)$. Let $\tau \leq \ln(\frac{7}{3})$. Since $(r_1, r_2) \in \gamma$

$$
\begin{aligned}
\tau + (p(S(r_1), T(r_2))) &= \tau + \ln\left(\max\left\{ \frac{3r_1}{7}, \frac{r_2}{5} \right\} \right) \\
&\leq \ln(\frac{7}{3}) + \ln\left(\max\left\{ \frac{3p(r_1, r_2)}{7}, \frac{p(r_1, r_2)}{5} \right\} \right) \\
&= \ln(\frac{7}{3}) + \ln\left(\frac{3p(r_1, r_2)}{7} \right) = \ln(p(r_1, r_2)) \\
&= F(\mathcal{M}(r_1, r_2)).
\end{aligned}
$$

Thus the contractive condition (3.1) is satisfied for all $r_1, r_2 \in M$ with $L = 0$. Hence all the hypotheses of Theorem 3 are satisfied. Note that (S, T) have a unique common fixed point $r = 0$. As we have seen in Example 2, T is not an F-contraction in $(M, \prec_1, d)$. Thus we cannot apply Theorem 1 and hence Theorem 2. The following corollary generalizes Theorem 2.

The following corollary generalizes the results in [13].

Corollary 1. *Let $(M, \preceq, p)$ be a complete ordered partial metric space and $T : M \to M$ be a mapping such that $r_0 \preceq T(r_0)$. Assume that*

   (1) *either T is a continuous mapping or M is regular,*
   (2) *T is a Ćirić type ordered F-contraction.*

Then T has a unique fixed point v in M such that $p(v, v) = 0$.

Proof. Setting $S = T$ in Theorem 3, we obtain the required result. □

4. APPLICATION OF THEOREM 3

This section contains an existence result which shows the usefulness of Theorem 3 in establishing existence of solution of implicit type integral equation:

$$\mathcal{A}(t, u(r, t)) = \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, u(\theta, \phi)) \, d\theta d\phi, \tag{4.1}$$

where $u \in \mathcal{U} = \mathcal{L}\left[C([0, a]) \times [0, a]\right]$=Lebesgue measurable space, $t, \theta, \phi \in I_a = [0, a]$. For $u \in \mathcal{U}$, define norm as: $\|u\| = \max_{t \in [0,a]} \{|u(t)|\}$. Let $\mathcal{U}$ be endowed with the partial metric $p : \mathcal{U} \times \mathcal{U} \to \mathbb{R}_0^+$ defined by

$$p(u, v) = d(u, v) + c = \max_{t \in [0,a]} |u(r, t) - v(r, t)| + c \text{ for all } u, v \in \mathcal{U}.$$

Also, $\mathcal{U}$ can be equipped with order $\prec_2$ defined by $u \prec_2 v$ if and only if $v(r,t) \leq u(r,t)$. Obviously, $(\mathcal{U}, \|\cdot\|)$ is a Banach space and $(\mathcal{U}, \prec_2, p)$ is a complete ordered partial metric space.

Theorem 4. *Assume that*

(a) *for all $u, v \in \mathcal{U}$ and $\kappa = |u(r_1, t) - v(r_1, t)| + c$*

$$|\mathcal{A}(t, u(r_1, t)) - \mathcal{A}(t, v(r_1, t))| + c \leq (\kappa) e^{-\tau} \text{ for each } t \in I_a,$$

(b) $\mathcal{H}(t, \theta, \phi, u(\theta, \phi)) \leq \frac{1}{a^2} u(r_1, t)$ *for all $t \in I_a$,*

(c) *for all $t, \theta, \phi \in I_a$,*

$$\mathcal{A}\left(t, \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, u(\theta, \phi)) \, d\theta d\phi\right) \leq \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, u(\theta, \phi)) \, d\theta d\phi,$$

(d) $\mathcal{H}(t, \theta, \phi, \mathcal{A}(\theta, u(\theta, \phi))) \geq \dfrac{1}{a^2} \mathcal{A}(t, u(r_1, t)).$

Then implicit integral equation (4.1) has a solution in $\mathcal{U}$.

Proof. Firstly, define $S(u(r_1, t)) = \mathcal{A}(t, u(r_1, t))$ and $T(u(r_1, t)) = \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, u(\theta, \phi)) \, d\theta d\phi$. We show that S,T are weakly increasing mappings. Consider

$$
\begin{aligned}
S(T(u(r_1, t))) &= \mathcal{A}(t, T(u(r_1, t))) \\
&= \mathcal{A}\left(t, \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, u(\theta, \phi)) \, d\theta d\phi\right) \\
&\leq \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, u(\theta, \phi)) \, d\theta d\phi = T(u(r_1, t)) \text{ using (c)}
\end{aligned}
$$

and

$$
\begin{aligned}
T(S(u(r_1, t))) &= \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, S(u(\theta, \phi))) \, d\theta d\phi \\
&= \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, \mathcal{A}(\theta, u(\theta, \phi))) \, d\theta d\phi \\
&\leq \frac{1}{a^2} \int_0^a \int_0^a \mathcal{A}(t, u(r_1, t)) \, d\theta d\phi = \mathcal{A}(t, u(r_1, t)) \text{ due to (b).}
\end{aligned}
$$

Thus $S(T(u(r_1, t))) \leq T(u(r_1, t))$ and $T(S(u(r_1, t))) \leq S(u(r_1, t))$ for all $t \in I_a$ imply that S,T are weakly increasing mappings with respect to $\prec_2$.
Secondly, consider

$$
\begin{aligned}
p\left(S(u), T(v)\right) &= \max_{t \in I_a} |S(u(r_1, t)) - T(v(r_2, t))| + c \\
&= \max_{t \in I_a} \left|\mathcal{A}(t, u(r_1, t)) - \int_0^a \int_0^a \mathcal{H}(t, \theta, \phi, v(\theta, \phi)) \, d\theta d\phi\right| + c \\
&\leq \max_{t \in I_a} \left|\mathcal{A}(t, u(r_1, t)) - \frac{1}{a^2} \int_0^a \int_0^a \mathcal{A}(t, v(r_1, t)) \, d\theta d\phi\right| + c \text{ using (d)} \\
&= \max_{t \in I_a} |\mathcal{A}(t, u(r_1, t)) - \mathcal{A}(t, v(r_1, t))| + c \\
&\leq \max_{t \in I_a} (\kappa) e^{-\tau} \text{ using (a)} \\
&\leq e^{-\tau} p(u, v).
\end{aligned}
$$

So

$$\tau + \ln(p\left(S(u), T(v)\right)) \quad \leq \quad \ln(p(u,v)) \leq \ln(\mathcal{M}(u,v)).$$

Thus by taking $F(r) = \ln(r)$, we have

$$\tau + F(p\left(S(u), T(v)\right)) \leq F(\mathcal{M}(u,v)).$$

Hence by Theorem 3 the integral equation (4.1) has a solution in $\mathcal{L}\left[C\left([0,a]\right) \times [0,a]\right]$. $\qquad \square$

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The Theoretical Analysis of l_1-TV Compressive Sensing Model for MRI Image Reconstruction

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July 27, 2017

Abstract

One of the main tasks in MRI image reconstruction is to catch the picture characteristics such as interfaces and textures from incomplete frequency data, and the iteration methods are the most useful methods to the minimization problem. A new viewpoint of choosing the iteration stopping rules in image reconstruction problem is proposed. The reconstruction model based on compressive sensing theory consists of a data matching term and two penalty terms, wavelet sparse and total variation regularization term. Then the Bregman iteration with lagged diffusivity fixed point iteration is used to solve the corresponding nonlinear Euler-Lagrange equation of image reconstruction model with incomplete frequency data. A real MRI image is used to test the proposed method in numerical experiments with different stopping rules. The theoretical analysis illustrate that although the norm of objective functional decreases with respect to the number of iteration, it cannot ensure the reconstructed image is the desired optimization image.

MSC(2010): 65M32, 65T50, 65T60, 65K10

Keywords: image reconstruction; MRI image; total variation; wavelet transform; regularization

1   Introduction

Image processing can be roughly divided into three kinds of problems, namely, image deblurring, image enhancement and image restoration, and the main purpose is to obtain the clear image with interfaces and textures from its noisy measurement. For a bounded connected domain $\Omega \subset \mathbb{R}^2$ (a rectangle in general [1]), let $u(\mathbf{x}), \mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$ be the grey function of an image defined in Ω. In general, we can get the degradation data $b^\sigma(\mathbf{x})$ with blurring noisy process, such as moving blurry, Gaussian blurry, white Gaussian noise, impulse noise (salt and pepper noise) as well as Poisson noise [2, 3]. The optimization scheme is one of the classical way to reconstruct u from b^σ, i.e., minimizes the Tikhonov cost functional

$$J(u) = \frac{1}{2}\|K \circ u - b^\sigma\|_{L^2(\Omega)}^2 + \alpha L \circ u \tag{1}$$

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with some penalty term $L \circ u$ and the regularization parameter $\alpha > 0$, where the operator L represents the *a-priori* regularity image u. Obviously, all of terms in (1) are continuous version, i.e., $u(x)$ is defined in Ω everywhere, to unify the basic idea of this scheme with our numerical implementations, we describe all these terms by the finite dimensional approximation of $u(\mathbf{x}) \in \mathbb{R}^{N \times N}$ with components $u_{i,j}(i, j = 1, \cdots, N)$.

In many engineering configurations, instead of the spatial noisy data $b_{i,j}^{\sigma}$ for each pixel $\Omega_{i,j}$, the practical measurement data may be the incomplete frequency data, or the finite number of discrete frequencies at the band-limited interval. For example, in the application of magnetic resonance imaging (MRI) image reconstruction, data collected by an MR scanner are, roughly speaking, in the frequency domain (called k-space) rather than the spatial domain. One of the main stage for MRI is the k-space data acquisition. In this stage, energy from a radio frequency pulse is directed to a small section of the targeted anatomy at a time. As a result, the protons within that area are forced to spin in a certain frequency and get aligned to the direction of the magnet. Upon stopping the radio frequency, the physical system gets back to its normal state and releases energy that is then recorded for analysis. This process is repeated until enough data is collected for reconstructing a high quality image in the second stage. This process is based on the compressive sensing (CS), and this kind of MRI image reconstruction problem is called CS-MRI image reconstruction. For more details about these contents see [4, 5, 6] and references therein. In this case, the data-matching term in (1) should be replaced by $\|\mathcal{P} \circ \mathcal{F}[u] - \mathcal{P} \circ \hat{b}^{\delta}\|_2^2$, where $\mathcal{F}$ is the two-dimensional discrete Fourier transform converting the spatial matrix $u \in \mathbb{R}^{N \times N}$ into frequency matrix $\mathcal{F}[u] \in \mathbb{C}^{N \times N}$, while $\mathcal{P}$ is a linear operator specifying the incomplete frequency data from $\mathbb{C}^{N \times N}$, $\hat{b}^{\delta} \in \mathbb{C}^{N \times N}$ is the noisy frequency data, $\|\cdot\|_2$ denotes the Euclidean norm.

In the case of CS-MRI, the recovery of u from $\hat{b}^{\delta}$ is equivalent to solving the l_0 problem:

$$\min_u \{\|\Psi \circ u\|_0 : \|\mathcal{P} \circ \mathcal{F}[u] - \mathcal{P} \circ \hat{b}^{\delta}\|_2^2 \leq \delta^2\}, \tag{2}$$

where $\|\cdot\|_0$ is the number of nonzero components of the objective, and orthogonal wavelet operator $\Psi : \mathbb{R}^{N \times N} \to \mathbb{R}^{N^2 \times 1}$ is based on the orthogonal wavelet basis $\psi_{i,j}(i, j = 1, \cdots, N)$ [7]. However, it is well-known that (2) is a NP-hard problem, and as usually, we replace it by the l_1-minimizing problem:

$$\min_u \{\|\Psi \circ u\|_1, \|\mathcal{P} \circ \mathcal{F}[u] - \mathcal{P} \circ \hat{b}^{\delta}\|_2^2 \leq \delta^2\}, \tag{3}$$

which yields sparse solutions under some conditions [8], $\|\cdot\|_1$ denotes the l_1 norm.

2   A theoretical analysis for l_1-TV optimization model

As usual, the image u has the obvious edges such as the interfaces in MRI images. So it is natural to also cooperate this *a-priori* information into the reconstruction model by considering the total variation (TV) penalty. So it is natural to consider the following unconstraint cost functional:

$$J(u) := \frac{1}{2}\|\mathcal{P} \circ \mathcal{F}[u] - \mathcal{P} \circ \hat{b}^{\delta}\|_2^2 + \alpha_1\|\Psi \circ u\|_1 + \alpha_2|u|_{TV}, \tag{4}$$

where α_1, α_2 are positive regularization parameters that determine the penalty terms. Therefore, the image reconstruction problem is the following l_1-TV optimization model

$$\arg \min_{u \in \mathbb{R}^{N \times N}} J(u) = \frac{1}{2}\|\mathcal{P} \circ \mathcal{F}[u] - \mathcal{P} \circ \hat{b}^{\delta}\|_2^2 + \alpha_1\|\Psi \circ u\|_1 + \alpha_2|u|_{TV}. \tag{5}$$

Suppose $\mathbf{u} \in \mathbb{R}^{N^2 \times 1}$ is a vector formed by stacking the columns of a two-dimensional MRI image array $u := (u_{i,j})$, $i, j = 1, \cdots, N$. Since (5) is not a differential, we added a small positive parameter β [9, 10] and presented the optimization model as the following minimizing convex perturbed form

$$\arg \min_{\mathbf{u} \in \mathbb{R}^{N^2 \times 1}} J(\mathbf{u}) = \frac{1}{2} \|\mathbf{PFu} - \mathbf{P}\hat{\mathbf{b}}^\delta\|_2^2 + \alpha_1 \|\Psi\mathbf{u}\|_{1,\beta} + \alpha_2 |\mathbf{u}|_{TV,\beta}, \tag{6}$$

in which two regularization terms are defined as

$$\|\Psi\mathbf{u}\|_{1,\beta} = \sum_{i=1}^{N^2} \sqrt{((\Psi \circ u)_i)^2 + \beta}, \quad |\mathbf{u}|_{TV,\beta} = \sum_{i,j=1}^{N} \sqrt{|\nabla_{i,j}u|^2 + \beta},$$

where $\nabla_{i,j}u = (\nabla_{i,j}^x u, \nabla_{i,j}^y u)$ is defined under periodic boundary condition

$$\nabla_{i,j}^x u = \begin{cases} u_{i+1,j} - u_{i,j}, & \text{if } i < m, \\ u_{1,j} - u_{m,j}, & \text{if } i = m. \end{cases} \quad \nabla_{i,j}^y u = \begin{cases} u_{i,j+1} - u_{i,j}, & \text{if } j < n, \\ u_{i,1} - u_{i,n}, & \text{if } j = n. \end{cases}$$

for $i, j = 1, \cdots, N$, $|\nabla_{i,j}u| = \sqrt{(\nabla_{i,j}^x u)^2 + (\nabla_{i,j}^y u)^2}$. $\mathbf{P}$ is an $N^2 \times N^2$ matrix consisting of sampling matrix P (an $N \times N$ matrix generating from the identity matrix I by setting its some rows as null vectors), and $\mathbf{F} \in \mathbb{C}^{N^2 \times N^2}$ is the two-dimensional discrete Fourier matrix defined in Fourier matrix $F \in \mathbb{C}^{N \times N}$ with the components $F_{m,n} = e^{-i2\pi mn/N}$.

The objective function in the problem (6) is strictly convex and differentiable with respect to variable $\mathbf{u}$ and its global minimizer is unique [11] when $\alpha_1 = 0$. The solution of (6) with small enough β can better approximate to the solution of the minimizing (4). Because the solution of the minimization problem for (4) can be regarded as the limit of the solution of (6) when $\beta \to 0$.

There are a number of numerical methods for solving the image reconstruction model (6), like fixed-point continuation method [12], split Bregman method [13], gradient project method [14], fast alternating minimization method [15], the variable splitting method [16], the operator-splitting algorithm [17] and fast iterative shrinkage-thresholding algorithm [18]. Meanwhile the conjugate gradient method (CGM) [19] is also very efficient approach to solve (6) in CS-MRI, and the former work [10] is better than CGM. In this paper, a fast scheme with different iteration stopping rules is proposed to solve the objective problem (6), which is based on Bregman method [20] and lagged diffusivity fixed point iteration [21]. The numerical experiments are shown to compare proposed method with the one in former work [10]. The fast iterative scheme for proposed model (6) is as follows Algorithm 1.

Now we give the theoretical analysis based on regularization for the CS-MRI image reconstruction problem. The objective optimization problem (6) can be rewritten as

$$\arg \min_{u \in \mathbb{R}^{N \times N}} J(u) = \frac{1}{2} \|\mathcal{P} \circ \mathcal{F}[u] - \mathcal{P} \circ \hat{b}^\delta\|_2^2 + \alpha_1 \|\Psi \circ u\|_{1,\beta} + \alpha_2 |u|_{TV,\beta}. \tag{7}$$

Assume $L_\alpha \circ u = \alpha_1 \|\Psi \circ u\|_{1,\beta} + \alpha_2 |u|_{TV,\beta}$. In order to get the approximate solution $u^\star$ to the above problem, we change the optimization problem (7) to the equation below

$$((\mathcal{PF})^*\mathcal{PF} + L_\alpha) \circ u^\star = (\mathcal{PF})^*\mathcal{P} \circ \hat{b}^\delta, \tag{8}$$

or

$$u^\star = ((\mathcal{PF})^*\mathcal{PF} + L_\alpha)^{-1}(\mathcal{PF})^*\mathcal{P} \circ \hat{b}^\delta, \tag{9}$$

Algorithm 1 The fast iterative scheme based on Bregman iteration for minimizing $J(\mathbf{u})$

Input: frequency input $\{\hat{b}_{i,j}^{\delta}|i,j = 1,\cdots,N\}$, sampling matrix $\mathbf{P} \in \mathbb{R}^{N^2 \times N^2}$, and parameters $\alpha_1, \alpha_2, \beta$.

Do iteration from $k = 0$ with $\hat{b}^{(0)} = \Theta, u^{(0)} = \Theta$.

While (the stopping rule is not satisfied)

{ Compute:

$$\hat{\mathbf{b}}^{(k+1)} = \hat{\mathbf{b}}^{\delta} + \left(\hat{\mathbf{b}}^{(k)} - \mathbf{PFu}\right),$$

$$\mathbf{u}^{(k+1)} = \arg \min_{\mathbf{u} \in \mathbb{R}^{\mathbf{N^2} \times 1}} \left\{\alpha_1 \|\Psi\mathbf{u}\|_{1,\beta} + \alpha_2 |\mathbf{u}|_{TV,\beta} + \frac{1}{2}\|\mathbf{PFu} - \mathbf{P}\hat{\mathbf{b}}^{(k+1)}\|_2^2\right\},$$

$k \leftarrow k + 1$. }

End do

$\mathbf{u}^{\star} := \mathbf{u}^{(k)}$.

End

---

where $*$ means the conjugate transpose. Let operator $R_\alpha = ((\mathcal{PF})^*\mathcal{PF} + L_\alpha)^{-1}(\mathcal{PF})^*\mathcal{P}$. When R_α is the regularization strategy [22], we have

$$
\begin{aligned}
\|u^{\star} - u\|_2 &\leq \left\|R_\alpha \circ \hat{b}^{\delta} - R_\alpha \circ \hat{b}\right\|_2 + \left\|R_\alpha \circ \hat{b} - u\right\|_2 \\
&\leq \|R_\alpha\|_2 \cdot \left\|\hat{b}^{\delta} - \hat{b}\right\|_2 + \|R_\alpha \circ (K_F \circ u) - u\|_2 \\
&\leq \delta \|R_\alpha\|_2 + \|R_\alpha K_F \circ u - u\|_2,
\end{aligned}
\tag{10}
$$

where K_F is an operator with Fourier transform in the classical way to reconstruct u from frequency data $\hat{b}$ based on (1), i.e., $K_F \circ u = \hat{b}$.

The iteration scheme can be based on Bregman iteration method [20], so the regularization parameter is seen as discrete regularization parameter, like $\alpha_1, \alpha_2, \beta$ and iteration number k. With the inequations above, the regularization error $\|u^{\star} - u\|_2$ can be seen as two parts: the ill-posed of model, and the error when R_α tends to K_F^{-1}. In [23], the error $\|u^{\star} - u\|_2$ is of the optimal value at some iteration step. When iteration number $k \to \infty$ which beyonds that iteration step, objective function $J(u^{\star}) \to 0$ but the error $\|u^{\star} - u\|_2 \nrightarrow 0$.

Now, we provide the similar conclusion in finite dimension. As we all know, the penalty terms in (7) are the important functions to the objective problem. However, the ill-posed problem (6) requires only the incomplete frequency data $\mathbf{P}\hat{\mathbf{b}}^{\delta}$. Therefore, we should optimize the objective function $\|\mathbf{PFu} - \mathbf{P}\hat{\mathbf{b}}^{\delta}\|_2^2$. In the other words, there also exists a $\mathbf{u}^{\dagger}$ (i.e. the exact solution) such that

$$
\left\|\mathbf{PFu}^{\dagger} - \mathbf{P}\hat{\mathbf{b}}^{\delta}\right\|_2 = \inf_{\mathbf{u} \in \mathbb{R}^{N^2 \times 1}} \left\|\mathbf{PFu} - \mathbf{P}\hat{\mathbf{b}}^{\delta}\right\|_2 = 0.
\tag{11}
$$

Noticing that the minimizing sequence $\{\mathbf{u}_{k,\alpha} : k = 1, 2, \cdots\}$ only has the convergence

$$
\lim_{k \to \infty} J(\mathbf{u}_{k,\alpha}) = J(\mathbf{u}^{\star}),
\tag{12}
$$

there is no convergence for the norm $\|\mathbf{u}_{k,\alpha} - \mathbf{u}^{\star}\|$ in general. So we need to identify the behavior of $\mathbf{u}_{k,\alpha}$ as $k \to \infty$ and $\alpha_i \to 0 (i = 1, 2)$.

Theorem 2.1. *There exists a subsequence $\left\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots \right\} \subset \left\{\mathbf{u}_{k,\alpha} : k = 1, 2, \cdots \right\}$ such that*

$$\lim_{j\to\infty} \left\| \mathbf{u}_{k_j,\alpha} - \mathbf{u}_{\alpha} \right\|_2 = 0. \tag{13}$$

Proof. Since (12), we easily know that

$$\lim_{k\to\infty} J(\mathbf{u}_{k,\alpha}) = J(\mathbf{u}^\star) = \inf_{\mathbf{u}\in\mathbb{R}^{N^2\times 1}} J(\mathbf{u}).$$

According to the finite dimension domain, there exists a subsequence of $\mathbf{u}_{k,\alpha}$ denoted by $\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots \}$ such that $\mathbf{u}_{k_j,\alpha} \to \mathbf{u}_{\alpha}$ as $j \to \infty$. Hence the limit of the norm $\left\| \mathbf{u}_{k_j,\alpha} - \mathbf{u}_{\alpha} \right\|_2$ equals to 0. The proof is complete. $\square$

From Theorem 2.1, we have some sequence $\alpha_m \to 0$ as $m \to \infty$, that means

$$\lim_{m\to\infty} \lim_{j\to\infty} \mathbf{u}_{k_j,\alpha_m} = \lim_{m\to\infty} \mathbf{u}_{\alpha_m} := \bar{\mathbf{u}}, \tag{14}$$

$$\lim_{m\to\infty} \lim_{j\to\infty} J\left(\mathbf{u}_{k_j,\alpha_m}\right) = 0. \tag{15}$$

Noticing (11) above, (15) is equal to $\|\mathbf{PFu}^\dagger - \mathbf{P\hat{b}}^{\delta}\|_2$. It reveals the exact meaning of the approximate solution to our problem by $\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots \}$, while $\{\mathbf{u}_{k_j,\alpha} : j = 1, 2, \cdots \}$ can converge to some $\bar{\mathbf{u}}$. But it is worth noting that $\bar{\mathbf{u}}$ cannot be ensured theoretically to be the exact solution $\mathbf{u}^\dagger$. Therefore the iteration number could not be too big. Even though the approximate solution $u^\star$ is the minimization for optimization model, it could not be the best solution for image reconstruction problem, i.e., $u^\star \nrightarrow u^\dagger$.

3   Numerical Experiments

In this section, the proposed fast algorithm with different iteration stopping rules is shown to solve the objective problem (6), which is compared with the method in [10]. All tests are performed in MATLAB 7.10 on a laptop with an Intel Core i5 CPU M460 processor and 2 GB of memory.

The signal to noise ratio (SNR) and relative error (ReErr) are used to measure the quality of the reconstructed images. The definitions of SNR and relative error are given as follows

$$\mathrm{SNR} = 20\lg\left(\frac{\|u\|_2}{\left\|u - u^{(k)}\right\|_2}\right), \tag{16}$$

$$\mathrm{ReErr} = \frac{\left\|u^{(k)} - u\right\|_2^2}{\|u\|_2^2}, \tag{17}$$

where $u^{(k)}$ and u are the reconstructed and original images, respectively. The CPU time is used to evaluate the speed of MRI image reconstruction.

As usual, the iteration stopping rule is one of the following three conditions:

$$J(u^{(k)}) \le \delta, \quad \frac{\left\|u^{(k)} - u^{(k-1)}\right\|_2}{\left\|u^{(k)}\right\|_2} \le \delta, \quad k = K_0, \tag{18}$$

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which mean the norm of objective function in (6), the relative difference between successive iteration for the reconstructed image, and maximum number of iterations K_0. In order to illustrate the efficient of the theoretical analysis in Section 2, we use the maximum iterations as the stopping rule.

Firstly, the performance of Algorithm 1 in solving model (6) for a real MRI brain image is shown in Figure 1, which is compared with the efficient method in [10]. Let additive Gaussian noise level in frequency domain is $\delta = 0.01$, i.e., adding 1% additive noise on the frequency of original image. The parameters $\alpha_1 = 0.01, \alpha_2 = \beta = 0.0001$ which are the same as the comparison algorithm. To the sampling matrix $\mathbf{P}$, we choose the radial sampling method with 22×8 views on frequency data. The tests results are shown in Figure 1(c)(d) which are based on stopping rule $K_0 = 60$ in Algorithm 1 and $K_0 = 100$ in the comparison algorithm, respectively. The SNR in (c) and (d) is 38.2321dB, 37.7443dB respectively, and the relative error is $4.0866 \times 10^{-5}, 1.8770 \times 10^{-4}$ respectively. From these data, the reconstruction is efficient with these parameters in the fast iteration scheme based on the iterations K_0 as stopping rule. However, ever though the iterations number in (d) is bigger than the one in (c), the reconstructed image (d) is not clearer than (c).

Next our numerical experiment is to illustrate the relationship between reconstruction error (Err) and objective function $J\left(u^{(k)}\right)$. The reconstruction error between reconstructed image and original image is used to evaluate the exactitude of ill-posed problem, which defined as follows

$$\text{Err} = \|u^{(k)} - u\|_2. \tag{19}$$

We take different iteration step by step, i.e., $10, 20, 30, \cdots$. The sampling mask in the frequency space separately take $22 \times 8, 22 \times 10, 22 \times 12$ views in radial sampling method. We still assume the above mentioned parameters in the tests. The fitting curve of error $\|u^{(k)} - u\|_2$ and $J\left(u^{(k)}\right)$ in different iteration numbers are shown in Figure 2.

From Figure 2, we find that although the stopping criterion is satisfied, the reconstruction error $\|u^{(k)} - u\|_2$ is of the optimal value at some iteration step. It means that the more iteration is better for reconstruction is not true. Meanwhile, the convergence and error analysis are the same with theoretical analysis in the above section. Therefore, the choice of iteration stopping rule, especially the iteration number K_0, is one of the most important factor of MRI image reconstruction.
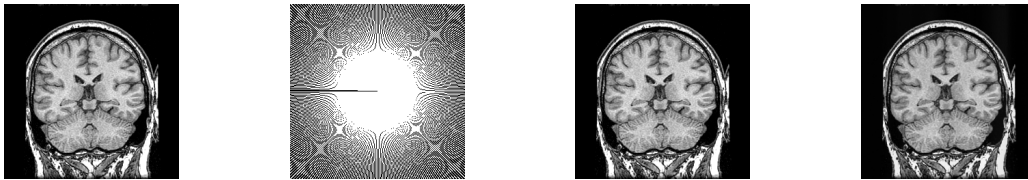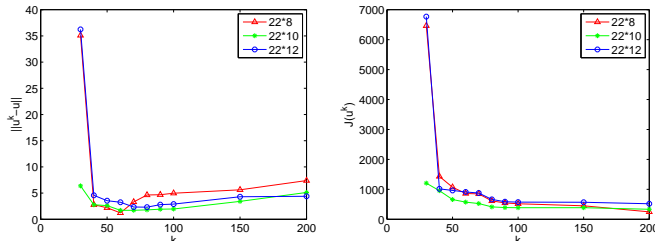

Figure 1: (a) Original image; (b) Sampling mask: 22×8 views; (c) Reconstructed image with $K_0 = 60$; (d) Reconstructed image with $K_0 = 100$.

Figure 2: Fitting curve of error and $J(u^{(k)})$ in different iteration numbers.

4 Conclusion

The l_1-TV optimization model based on compressive sensing was established to reconstruct MRI images. Bregman method and lagged diffusivity fixed point iteration are used to solve the modified reconstruction model, and a fast iteration scheme with error estimate analysis is proposed. Based on Tikhonov regularization theory, a theoretical analysis on iteration stopping rules is proposed. A real MRI brain image is employed to test in the numerical experiments and the results demonstrate that proposed method and theoretical analysis is very efficient in CS-MRI image reconstruction.

Acknowledgements

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On Fibonacci Z-sequences and their logarithm functions

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Abstract. In this paper we discuss the concept of a Z-sequence and use it to define the Fibonacci Z-sequence. Based on Z-sequences one may define analogs of logarithm functions. The special case of the logarithm function associated with the Fibonacci Z-sequence is of interest, since the recursive property of this sequence permits a more detailed study of these functions. They are similar to ordinary logarithm functions which may be based on Z-sequences $\{a^n\}_{n \in Z}$, where $a > 1$.

1. Introduction and Preliminaries

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. One of the amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio (section) rather quickly as they go to infinity probably has a good deal to do with the observation made in the previous sentence. Surveys and connections of the type just mentioned are provided in [1] and [2] for a very minimal set of examples of such texts, while in [3] an application (observation) concerns itself with a theory of a particular class of means which has apparently not been studied in the fashion done there by two of the authors the present paper. Surprisingly novel perspectives are still available.

Kim and Neggers [6] showed that there is a mapping $D : M \to DM$ on means such that if M is a Fibonacci mean so is DM, that if M is the harmonic mean, then DM is the arithmetic mean, and if M is a Fibonacci mean, then $\lim_{n \to \infty} D^n M$ is the golden section mean. Surprisingly novel perspectives are still available and will presumably continue to be so for the future as long as mathematical investigations continue to be made.

Han et al. [4] considered several properties of Fibonacci sequences in arbitrary groupoids. They discussed Fibonacci sequences in both several groupoids and groups. The present authors [7] introduced the notion of generalized Fibonacci sequences over a groupoid and discussed these in particular for the case where the groupoid contains idempotents and pre-idempotents. Using the notion of Smarandache-type P-algebras they obtained several relations on groupoids which are derived from generalized Fibonacci sequences.

In [5] Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x+2) = f(x+1) + f(x)$, and developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. Moreover, they showed that if f is a Fibonacci function then $\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. The present authors [8] discussed Fibonacci functions using the (ultimately) periodicity and we also discuss the exponential

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Fibonacci functions. Especially, given a non-negative real-valued function, we obtain several exponential Fibonacci functions.

The present authors [9] introduced the notions of Fibonacci (co-)derivative of real-valued functions, and found general solutions of the equations $\triangle(f(x)) = g(x)$ and $(\triangle + I)(f(x)) = g(x)$. Moreover, they [10] defined and studied a function $F : [0, \infty) \to \mathbf{R}$ and extensions $F : \mathbf{R} \to \mathbf{C}$, $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ which are continuous and such that if $n \in \mathbf{Z}$, the set of all integers, then $F(n) = F_n$, the nth Fibonacci number based on $F_0 = F_1 = 1$. If x is not an integer and $x < 0$, then $F(x)$ may be a complex number, e.g., $F(-1.5) = \frac{1}{2} + i$. If $z = a + bi$, then $\widetilde{F}(z) = F(a) + iF(b-1)$ defines complex Fibonacci numbers. In connection with this function (and in general) they defined a Fibonacci derivative of $f : \mathbf{R} \to \mathbf{R}$ as $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$ so that if $(\triangle f)(x) \equiv 0$ for all $x \in \mathbf{R}$, then f is a (real) Fibonacci function. A complex Fibonacci derivative $\widetilde{\triangle}$ is given as $\widetilde{\triangle} f(a + bi) = \triangle f(a) + i \triangle f(b-1)$ and its properties are discussed in same detail.

2. **Fibonacci logarithm**

Let $\mathcal{F} = \{F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \cdots, \}$ be the *Fibonacci sequence* where F_k denotes the kth Fibonacci number. Let $\mathcal{F}^* = \{F_0^* = 1, F_1^* = 2, F_2^* = 3, F_3^* = 5, \cdots\}$ denote the *short Fibonacci sequence* and let $E(\mathcal{F}^*) = \{F_0^* = 1, F_1^* = 2, F_{-1}^* = \frac{1}{2}, F_2^* = 3, F_{-2}^* = \frac{1}{3}, \cdots\}$ denote the *extended short Fibonacci sequence* or the *Fibonacci Z-sequence*. In general, by a *Z-sequence* $S = \{a_0, a_1, a_{-1}, a_2, a_{-2}, \cdots\}$ we mean a sequence of positive real numbers satisfying $a_i < a_{i+1}$ for all $i \in Z$ where $\lim_{k \to \infty} a_k = \infty$ and $\lim_{k \to -\infty} a_k = 0$.

In general, for a *Z-sequence* S, we say that a positive real number x has *S-characteristic* k if k is the unique integer such that $a_k \le x < a_{k+1}$. Thus, if $S = E(\mathcal{F}^*)$, then $F_k^* \le x < F_{k+1}^*$ means that x has $E(\mathcal{F}^*)$-characteristic k. Given the context, we shall refer to this number as the *Fibonacci characteristic* of x.

For example, if $x = 1.2$, then $F_0^* = 1 < 1.2 < 2 = F_1^*$, and hence its Fibonacci characteristic is 0. Again, if $x = \frac{1}{10}$, then $F_4^* = F_5 = 8 < \frac{1}{x} = 10 < 13 = F_5^*$ whence $F_{-5}^* < x < F_{-4}^*$, i.e, $x = \frac{1}{10}$ has Fibonacci characteristic -5, while 10 has Fibonacci characteristic 4. In discussing characteristics we keep in mind that for $k \ge 1$, $F_k^* = F_{k+1}$, e.g., $F_1^* = F_2 = 2$.

We have a rule for computing Fibonacci characteristics of numbers $0 < x < 1$. Indeed, compute the Fibonacci characteristic of $\frac{1}{x}$. If it is $n > 0$, then the Fibonacci characteristic of x is $-(n+1)$. Computing or estimating the Fibonacci characteristics of numbers $x \ge 1$ will be a topic of interest to be discussed below.

Suppose now that $F_k^* \le x < F_{k+1}^*$. Then the *Fibonacci mantissa* of x is defined as the number α such that

$$\frac{(F_{k+1}^*)^\alpha}{(F_k^*)^{\alpha-1}} = x$$

We note that if $x = F_k^*$, then $(F_{k+1}^*/F_k^*)^\alpha = 1$, and hence $\alpha = 0$. Also, if $x = F_{k+1}^*$, then $(F_{k+1}^*/F_k^*)^{\alpha-1} = 1$, and hence $\alpha = 1$. Hence, $0 \le \alpha < 1$ for numbers x such that $F_k^* \le x < F_{k+1}^*$.

Finally, we define the *Fibonacci logarithm* $\log_{\mathcal{F}}(x) = k + \alpha$, where k is the Fibonacci characteristic of x and where α is the Fibonacci mantissa of x. F is called the *pseudo base* of the logarithm. We simply denote $\log_{\mathcal{F}}(x)$ by $\log_F(x)$. It is our purpose in this paper to discuss the Fibonacci logarithm function of the positive real variable x and to make several observations as a consequence.

On Fibonacci Z-sequences and their logarithm functions

3. Fibonacci logarithm $\log_F(x)$

We begin by noting that $\log_F(x)$ is continuous everywhere. If $F_k^* \leq x < F_{k+1}^*$, then $\log_F(x) = k + \alpha = log_F(F_k^*) + \alpha$. We will show that $\log_F(x)$ is differentiable at x. As we have seen above, if α for F_{k+1}^* is computed relative to F_k^*, then it equals 1 and hence $\log_F(F_k^*) + 1 = k + 1 = \log_F(F_{k+1}^*)$ as well. Hence $\lim_{x^- \to F_{k+1}^*} \log_F(x) = \lim_{x^+ \to F_{k+1}^*} \log_F(x) = k + 1$, establishing continuity at that point.

Theorem 3.1. *If $\log_F(x)$ is the Fibonacci logarithm function, then its derivative is*

$$(log_F(x))' = \frac{1}{x} \frac{1}{ln(F_{k+1}^*/F_k^*)} \tag{3.1}$$

when $F_k^* < x < F_{k+1}^*$ (where ln means the natural logarithm function.)

Proof. We compute $\log_F(x)$ and $\log_F(x + h)$, where x and $x + h$ are both in the open interval (F_k^*, F_{k+1}^*). Accordingly both have the same Fibonacci characteristic k. Assume α and β are Fibonacci mantissas of x and $x + h$ respectively. Then $\log_F(x + h) - \log_F(x) = \beta - \alpha$, the difference of the Fibonacci mantissas. Consider the following.

$$\frac{x + h}{x} = \frac{(F_{k+1}^*)^\beta/(F_k^*)^{\beta-1}}{(F_{k+1}^*)^\alpha/(F_k^*)^{\alpha-1}} \tag{3.2}$$
$$= (F_{k+1}^*/F_k^*)^{\beta-\alpha}$$

It follows that

$$ln(1 + \frac{h}{x}) = (\beta - \alpha)ln(F_{k+1}^*/F_k^*) \tag{3.3}$$

Hence, we obtain

$$log_F(x + h) - \log_F(x) = \beta - \alpha$$
$$= \frac{ln(1 + \frac{h}{x})}{ln(F_{k+1}^*/F_k^*)} \tag{3.4}$$

It follows that

$$\lim_{h \to 0} \frac{\log_F(x + h) - \log_F(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{h}ln(1 + \frac{h}{x})}{ln(F_{K+1}^*/F_k^*)}$$
$$= \frac{1}{x} \frac{1}{ln(F_{k+1}^*/F_k^*)}, \tag{3.5}$$

proving the theorem. □

If $b := F_{k+1}^*/F_k^*$, then the usual logarithm function $\log_b(x)$ with the base b has the derivative $(log_b(x))' = \frac{1}{x} \frac{1}{ln(b)}$, and hence in Theorem 3.1, $(log_F(x))' = (log_b(x))'$ on the open interval (F_k^*, F_{k+1}^*), i.e., the functions $\log_F(x)$ and $\log_b(x)$ differ by a constant. Let $C_k := \log_F(F_k^*) - \log_b(F_k^*)$. We need to find an upper bound for C_k. Given any particular value of k, one can of course immediately determine C_k precisely. For example, if $k = 5$, then $F_5^* = 13, F_6^* = 21$ and $b = 21/13$, so that $C_5 = log_F(13) - log_{\frac{21}{13}}(13) = 5 - log_{\frac{21}{13}}(13) = -0.34840144$. However, in order to obtain an improved sense of the behavior of C_k as a function of k, it may be better to determine a fairly simple bound for C_k which is a function of k itself. If we let $\log_F(x) := \log_b(x) + C_k$ and $t_k := \log_b(F_k^*)$, then $b^{t_k} = F_k^*$ and $t_k = ln(F_k^*)/ln(b) = ln(F_k^*)/ln(F_{k+1}^*/F_k^*)$. Since $1 < F_{k+1}^*/F_k^* \leq 2$, it follows that $ln(\frac{F_{k+1}^*}{F_k^*}) \leq ln2 \leq ln(e) = 1$, so that $t_k > ln(F_k^*)$, and hence

$$C_k = k - t_k < k - ln(F_k^*) \tag{3.6}$$

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If $-k < 0$, then $F^*_{-k} = 1/F_k{}^*$ and $F^*_{-k+1} = 1/F^*_{k-1}$, so that if we let $b^* := F^*_{-k+1}/F^*_{-k} = F^*_k/F^*_{k-1}$, so that we may determine $(\log_F(x))' = \frac{1}{x}\frac{1}{ln(b^*)}$ for $F^*_{-k} \leq x < F^*_{-k+1}$ or $1/F^*_k \leq x < 1/F^*_{-k+1}$. If we let $\log_F(x) := \log_b(x) + C_{-k}$ for some C_{-k} where $F^*_{-k} \leq x < F^*_{-k+1}$ and if we let $t_{-k} := \log_{b^*}(F^*_{-k})$, then $t_{-k} = \frac{ln(F^*_{-k})}{ln(b^*)} < -ln(F^*_k)$. Hence we obtain:

$$C_{-k} = \log_F(F^*_{-k}) - \log_{b^*}(F^*_{-k})$$
$$= -k - t_{-k} \tag{3.7}$$
$$< -k + ln(F^*_k)$$

From this we have $k - ln(F^*_k) < -C_{-k}$. We summarize:

Proposition 3.2. *If $C_k := \log_F(F^*_k) - \log_b(F^*_k)$ then it has a bound $C_k < k - ln(F^*_k) < -C_{-k}$.*

Note that $\log F(x)$ is not differentiable at the F^*_k's. There is both a left-derivative and right-derivative at that point. Indeed, for the left derivative we get

$$\lim_{x^- \to F^*_k} \frac{1}{x}\frac{1}{ln(b^*)} = \frac{1}{F^*_k}\frac{1}{ln(b^*)}, \tag{3.8}$$

while for the right derivative it is:

$$\lim_{x^+ \to F^*_k} \frac{1}{x}\frac{1}{ln(b)} = \frac{1}{F^*_k}\frac{1}{ln(b)}, \tag{3.9}$$

Hence, we define the *saltus (jump)* at F^*_k to be

$$\Delta(F^*_k) = \frac{1}{F^*_k}\left[\frac{1}{ln(b)} - \frac{1}{ln(b^*)}\right]$$
$$= \frac{1}{F^*_k}\left[\frac{ln(\frac{b^*}{b})}{ln(b)ln(b^*)}\right]$$

where $b^*/b = (F^*_k/F^*_{k-1})/(F^*_{k+1}/F^*_k) = (F^*_k)^2/F^*_{k-1}F^*_{k+1}$. We recall that $(F^*_k)^2 = F^*_{k-1}F^*_{k+1} + (-1)^{k+1}$, and thus we have

$$\frac{b^*}{b} = \begin{cases} < 1 & \text{if } k \text{ is even,} \\ > 1 & \text{otherwise} \end{cases} \tag{3.10}$$

It follows that

$$ln(\frac{b^*}{b}) = \begin{cases} < 0 & \text{if } k \text{ is even,} \\ > 0 & \text{otherwise} \end{cases} \tag{3.11}$$

Since $F^*_k ln(b)ln(b^*) > 0$, $\Delta(F^*_k)$'s sign is determined by the sign of $ln(\frac{b^*}{b})$. Hence we obtain:

Proposition 3.3. *If $\Delta(F^*_k)$ is the saltus at F^*_k, then*

$$\Delta(F^*_k) = \begin{cases} < 0 & \text{if } k \text{ is even,} \\ > 0 & \text{otherwise} \end{cases} \tag{3.12}$$

When $k = 0$, $\frac{b^*}{b} = (F^*_0)^2/F^*_{-1}F^*_1 = 1$ and so $ln(b^*/b) = 0$. Thus $\Delta(F^*_0) = 0$, i.e., $\log_F(x)$ is differentiable at $F^*_0 = 1$.

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For negative integers, replacing F_{-k}^* by $\frac{1}{F_k^*}$, the saltus formula $\Delta(F_{-k}^*)$ is transformed to

$$
\begin{aligned}
\Delta(F_{-k}^*) &= \frac{1}{F_{-k}^*} \frac{ln(b^*/b)}{ln(b)ln(b^*)} \\
&= F_k^* \frac{ln[F_{k-1}^* F_{k+1}^*/(F_k^*)^2]}{ln(\frac{F_{k+1}^*}{F_k^*})ln(\frac{F_k^*}{F_{k-1}^*})}
\end{aligned}
\tag{3.13}
$$

For example, for $k = 3$, we obtain $\frac{F_4^* F_2^*}{(F_3^*)^2} = \frac{8 \cdot 3}{25} = \frac{24}{25} < 1$. Of course, $\frac{(F_{-3})^2}{F_{-4} F_{-2}} = (1/5)^2/(\frac{1}{8})(\frac{1}{3}) = (\frac{1}{25})/(\frac{1}{24})$ when computed directly.

The Fibonacci number F_k^* is said to be *elliptical* if $(F_k^*)^2/F_{k+1}^* F_{k-1}^* > 1$; *parabolic* if $(F_k^*)^2/F_{k+1}^* F_{k-1}^* = 1$; *hyperbolic* if $(F_k^*)^2/F_{k+1}^* F_{k-1}^* < 1$.

For $k \geq 1$, F_k^* is elliptical if k is odd; hyperbolic if k is even. The only parabolic Fibonacci number is $F_0^* = 1$. If $-k < 0$, then F_{-k}^* is elliptical if k is even and F_{-k}^* is hyperbolic if k is odd.

Proposition 3.4. $\log_F(x) + \log_F(\frac{1}{x}) = C_k + C_{-k}$.

Proof. If $x \neq F_k^*$ for all $k \in Z$, then $F_k^* \leq x < F_{k+1}^*$ yields $\log_F(x) = k + \alpha = \log_b(x) + C_k$. Now, $F_{-(k+1)}^* < \frac{1}{x} < F_{-k}^*$ and $F_{-k}^*/F_{-(k+1)}^* = b$, so that $\log_F(\frac{1}{x}) = \log_b(\frac{1}{x}) + C_{-k} = -\log_b(x) + C_{-k}$, proving the proposition. □

4. **Determining the Fibonacci Mantissa of** x

Suppose that $F_k^* \leq x < F_{k+1}^*$, i.e., x has Fibonacci characteristic k. If $f(x) := \log_F(x)$, then we may determine $f(x)$ from its Taylor series around F_k^* provided we consider the derivatives at F_k^* to be the right-hand derivatives. Thus, we will write $f(x) = x + \alpha$, with $f(F_k^*) = k$ and hence also

$$
\alpha = \sum_{n=1}^{\infty} f^{(n)}(F_k^*) \frac{(x - F_k^*)^n}{n!}
\tag{4.1}
$$

On the interval (F_k^*, F_{k+1}^*), the derivative $(\log_F(x))' = f'(x) = \frac{1}{x} \frac{1}{ln(b)}$, $b = F_{k+1}^*/F_k^*$. Hence the nth derivative $f^n(x)$ is $\frac{(-1)^{n-1}(n-1)!}{x^n} \frac{1}{ln(b)}$. Thus, $f^{(n)}(F_k^*) \frac{(x-F_k^*)^n}{n!} = \frac{(-1)^{n-1}}{n ln(b)}(\frac{x-F_k^*}{F_k^*})^n$, whence the fact that $x - F_k^* < F_{k+1}^* - F_k^* = F_{k-1}^*$ guarantees that $|(x - F_k^*)/F_k^*| < |F_{k-1}^*/F_k^*| < 1$, so that the series for α converges by the ratio-test. We summarize:

Theorem 4.1. If $f(x) = \log_F(x)$ with $F_k^* \leq x < F_{k+1}^*$, then $f(x) = k + \alpha$, where

$$
\alpha = \sum_{n=1}^{\infty} f^{(n)} = \frac{(-1)^{n-1}}{n \, ln(b)}\left[\frac{x - F_k^*}{F_k^*}\right]^n
$$

In particular, if $x = F_{k+1}^*$, then $\alpha = 1$, and we obtain an expression

$$
1 = \frac{1}{ln(b)}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{F_{k+1}^* - F_k^*}{F_k^*}\right]^n\right],
\tag{4.2}
$$

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so that we obtain an expression for $ln(b)$ as follows:

$$ln(b) = ln\left[\frac{F^*_{k+1}}{F^*_k}\right]$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left[\frac{F^*_{k+1} - F^*_k}{F^*_k}\right]^n \tag{4.3}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left[\frac{F^*_{k-1}}{F^*_k}\right]^n$$

From the expression for α as an infinite series, if we consider only the first term α_1, where

$$\alpha_1 = f^{(1)}(F^*_k)(x - F^*_k) = \frac{1}{F^*_k}(x - F^*_k)\frac{1}{ln\left[\frac{F^*_{k+1}}{F^*_k}\right]},$$

we may rewrite $ln(F^*_{k+1}/F^*_k) = ln(1 + F^*_{k-1}/F^*_k)$. If we approximate this expression by F^*_{k-1}/F^*_k, i.e., $ln(1 + F^*_{k-1}/F^*_k) \sim F^*_{k-1}/F^*_k$, then we obtain a further approximation:

$$\alpha_1 = \frac{1}{F^*_k}\frac{x - F^*_k}{\frac{F^*_{k-1}}{F^*_k}} = \frac{x - F^*_k}{F^*_{k-1}}$$

Now $F^*_{k-1} = F^*_{k+1} - F^*_k$ is the length of the interval (F^*_k, F^*_{k+1}) and thus α^*_1 represents the mantissa corresponding to the straight line connecting (F^*_k, k) with $(F^*_{k+1}, k + 1)$. See the following figure:
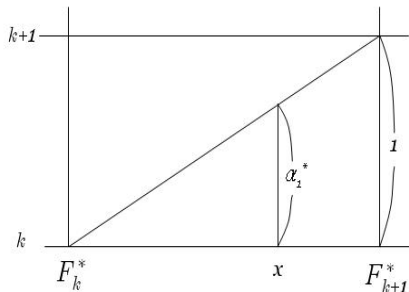
We may thus use the α^*_1 estimate to construct a piecewise-linear average Fibonacci logarithm $\log^*_F(c) = k + \alpha^*_1$ for $F^*_k \leq x < F^*_{k+1}$. We summarize:

Proposition 4.2. If we define $\log^*_F(x) = k + \alpha^*_1$ for $F^*_k \leq x < F^*_{k+1}$, then

(i) $\log^*_F(F^*_k) = \log_F(F^*_k)$ for all integers k,

(ii) $\log^*_F(F^*_k)$ is continuous and differentiable on non-Fibonacci numbers,

(iii) if x has Fibonacci characteristic k, then $(\log^*_F(x))' = \frac{1}{F^*_{k-1}}$.

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Thus, for $\log_F^*(x)$ we have the saltus $\Delta^*(F_k^*)$ as follow:

$$
\begin{aligned}
\Delta^*(F_k^*) &= \frac{1}{F_k^*} - \frac{1}{F_{k-1}^*} \\
&= \frac{F_{k-1}^* - F_k^*}{F_k^* F_{k-1}^*} \\
&= -\frac{F_{k-2}^*}{F_k^* F_{k-1}^*}
\end{aligned}
\tag{4.4}
$$

Hence, e.g., $\Delta^*(F_5^*) = -\frac{F_3^*}{F_5^* F_4^*} = -\frac{5}{104}$.

Returning to Theorem 4.1, a second level approximation for α is $\alpha_1 + \alpha_2$, where $\alpha_2 = f^{(2)}(F_k^*)(x - F_k^*)^2 \frac{1}{2!} = \frac{-1}{2ln(b)} \left[\frac{x - F_k^*}{F_k^*} \right]^2$. Now $ln(b) = ln(1 + F_k^*/F_k^*)$, where in the first order approximation we approximate $ln(b) = ln(1 + A)$ by A itself. In order to be fair to the second order approximation we should be fair to $ln(b)$ as well and use $ln(b) = A - A^2/2$ from its MacLaurin expansion. This affects the α_1 estimate as well. Indeed, we obtain a new estimate for α_1 as:

$$
\alpha_1 = \frac{1}{F_k^*} \frac{x - F_k^*}{\frac{F_{k-1}^*}{F_k^*} - \frac{1}{2}(\frac{F_{k-1}^*}{F_k^*})^2} = \frac{2F_k^*(x - F_k^*)}{F_{k-1}^*(2F_k^* - F_{k-1}^*)}
$$

and for α_2 we find that

$$
\alpha_2 = \frac{-(x - F_k^*)^2}{F_{k-1}^*(2 - F_{k-1}^*)} = \frac{(x - F_k^*)^2}{F_k^*(F_{k-1}^* - 2)}
$$

In general, if we wish to make an nth order approximation for α, we write $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, where $\alpha_n = f^{(n)}(F_k^*)(x - F_k^*)^n/n! = \frac{(-1)^{n-1}}{nln(b)} \left[\frac{x - F_k^*}{F_k^*} \right]^n$.

At the same time we use an nth order approximation for $ln(b) = ln(1 + F_{k-1}^*/F_k^*) = ln(1 + A)$ by setting it equal to $ln(b) = A - A^2/2 + \cdots + (-1)^{n-1}A^n/n$, and recomputing $\alpha_1, \cdots, \alpha_{n-1}$ in terms of the nth order approximation of $ln(b)$ as well.

5. Concluding remark and future works

In this paper we discussed Fibonacci Z-sequences and obtained some interesting results on Fibonacci logarithm functions. Based on Fibonacci logarithm functions, we shall discuss on Fibonacci exponential functions and obtain several properties on it.

Acknowledgement

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1

HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR MT$_{(r;g,m,\varphi)}$-PREINVEX FUNCTIONS

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ABSTRACT. In the present paper, the notion of MT$_{(r;g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving MT$_{(r;g,m,\varphi)}$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for MT$_{(r;g,m,\varphi)}$-preinvex functions that are twice differentiable via Riemann-Liouville fractional integrals are established. Some applications to special means are also given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

1. INTRODUCTION AND PRELIMINARIES

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In (see [12],[22]), Tunç and Yildirim defined the following so-called MT-convex function:

Definition 1.2. A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to the class of MT(I), if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.2}$$

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In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (please see [3],[4],[9]-[17], [24], [25]).

Fractional calculus (see [21]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$ is the classical gamma function (see [26]-[31], [32] [33]-[35]). Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (please see [6]-[21]).

Definition 1.4. (see [2]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0,+\infty)$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x,y \in I, \ t \in [0,1].$$

Definition 1.5. (see [5]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x,y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y-x$, but the converse is not necessarily true. For more details (see [5],[7]).

Definition 1.6. (see [8]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x,y \in K$ and $t \in [0,1]$, we have that

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k}f(\gamma_k) + R_m^{\star}|f|, \qquad (1.3)$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [18]).

Recently, Liu (see [19]) obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.4 of P-function.

Also in (see [20]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of $\text{MT}_{(r;g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving $\text{MT}_{(r;g,m,\varphi)}$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for nonnegative $\text{MT}_{(r;g,m,\varphi)}$-preinvex functions that are twice differentiable via fractional integrals are given. In Section 4, some applications to special means are given. In Section 5, some conclusions and future research are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

2. New integral inequalities for $\text{MT}_{(r;g,m,\varphi)}$-preinvex functions

Definition 2.1. (see [1]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next give new definition, to be referred as $MT_{(r;g,m,\varphi)}$-preinvex function.

Definition 2.3. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$, $g : [0, 1] \longrightarrow (0, 1)$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0, +\infty)$ is said to be $MT_{(r;g,m,\varphi)}$-preinvex function with respect to η, if

$$f(m\varphi(y) + g(t)\eta(\varphi(x), \varphi(y), m)) \leq M_r(f(\varphi(x)), f(\varphi(y)), m; g(t)) \quad (2.1)$$

holds for any fixed $m \in (0, 1]$ and for all $x, y \in I$, $t \in [0, 1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m; g(t))$$

$$= \begin{cases} \left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f^r(\varphi(x)) + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f^r(\varphi(y)) \right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \\ \left[f(\varphi(x)) \right]^{\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}} \left[f(\varphi(y)) \right]^{\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2.4. In Definition 2.3, it is worthwhile to note that the class $MT_{(r;g,m,\varphi)}(I)$ is a generalization of the class $\text{MT}(I)$ given in Definition 1.2 for $r = m = 1$ with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x, y \in I$, $g(t) = t$, $\forall t \in (0, 1)$.

Let give below a nontrivial example for motivation of this new interesting class of $MT_{(r;g,m,\varphi)}$-preinvex functions.

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Example 2.5. $f_1, f_2 : (1, +\infty) \longrightarrow (0, +\infty)$, $f_1(x) = x^p$, $f_2(x) = (1+x)^p$, $p \in \left(0, \frac{1}{1000}\right)$; $h : [1, 3/2] \longrightarrow (0, +\infty)$, $h(x) = (1+x^2)^k$, $k \in \left(0, \frac{1}{100}\right)$, are simple examples of the new class of $MT_{(1;t,m,x)}$-preinvex functions with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $g(t) = t$, $r = 1$, for any fixed $m \in (0, 1]$, but they are not convex.

In this section, in order to prove our main results regarding some new integral inequalities involving $MT_{(r;g,m,\varphi)}$-preinvex functions, we need the following new interesting lemma:

Lemma 2.6. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Then for any fixed $m \in (0, 1]$ and $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1}$$

$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)].$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$

$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q$$

$$\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)]$$

$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1}$$

$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)].$$

$\square$

The following definition will be used in the sequel.

Definition 2.7. The Euler beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}.$$

Theorem 2.8. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous function on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Let $k > 1$ and $0 < r \le 1$. If $f^{\frac{k}{k-1}}$ is $MT_{(r;g,m,\varphi)}$-preinvex function on an open m-invex set K for any fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

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$$\leq \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times \left[A_2^r(g(t);r)f^{\frac{rk}{k-1}}(\varphi(a)) + A_1^r(g(t);r)f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}},$$

where

$$B(g(t);k,p,q) = \int_0^1 g^{kp}(t)(1-g(t))^{kq}d[g(t)];$$

$$A_1(g(t);r) = \int_{1-g(1)}^{1-g(0)} \left(\sqrt{\frac{1-t}{t}}\right)^{\frac{1}{r}} dt;$$

$$A_2(g(t);r) = \int_{g(0)}^{g(1)} \left(\sqrt{\frac{1-t}{t}}\right)^{\frac{1}{r}} dt.$$

Proof. Let $k > 1$ and $0 < r \leq 1$. Since $f^{\frac{k}{k-1}}$ is $MT_{(r;g,m,\varphi)}$-preinvex function on K, combining with Lemma 2.6, Hölder inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \left[\int_0^1 g^{kp}(t)(1-g(t))^{kq}d[g(t)]\right]^{\frac{1}{k}}$$

$$\times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times \left[\int_0^1 \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f^r(\varphi(b))^{\frac{k}{k-1}} + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f^r(\varphi(a))^{\frac{k}{k-1}}\right)^{\frac{1}{r}} d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\leq \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times \left\{ \left(\int_0^1 \left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}\right)^{\frac{1}{r}} f^{\frac{k}{k-1}}(\varphi(b))d[g(t)]\right)^r \right.$$

$$\left. + \left(\int_0^1 \left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}}\right)^{\frac{1}{r}} f^{\frac{k}{k-1}}(\varphi(a))d[g(t)]\right)^r \right\}^{\frac{k-1}{rk}}$$

$$= \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times \left[A_2^r(g(t);r)f^{\frac{rk}{k-1}}(\varphi(a)) + A_1^r(g(t);r)f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}}.$$

$\square$

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Corollary 2.9. *Under the same conditions as in Theorem 2.8 for* $r = 1$ *and* $g(t) = t$, *we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$\leq \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \beta^{\frac{1}{k}}(kp+1, kq+1)$$

$$\times \left[f^{\frac{k}{k-1}}(\varphi(a)) + f^{\frac{k}{k-1}}(\varphi(b))\right]^{\frac{k-1}{k}}.$$

Theorem 2.10. *Let* $\varphi : I \longrightarrow K$ *be a continuous function and* $g : [0, 1] \longrightarrow (0, 1)$ *is a differentiable function. Assume that* $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0, +\infty)$ *is a continuous function on* K° *with respect to* $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, *for* $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. *Let* $l \geq 1$ *and* $0 < r \leq 1$. *If* f^l *is* $MT_{(r;g,m,\varphi)}$-*preinvex function on an open m-invex set* K *for any fixed* $m \in (0, 1]$, *then for any fixed* $p, q > 0$, *we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rl}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times \left[B^r\left(g(t);\frac{1}{2r},2pr-1,2qr+1\right)f^{rl}(\varphi(a))\right.$$

$$\left. + B^r\left(g(t);\frac{1}{2r},2pr+1,2qr-1\right)f^{rl}(\varphi(b))\right]^{\frac{1}{rl}}.$$

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is $MT_{(r;g,m,\varphi)}$-preinvex function on K, combining with Lemma 2.6, the well-known power mean inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $m \in (0, 1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1} \int_0^1 \left[g^p(t)(1-g(t))^q\right]^{\frac{l-1}{l}} \left[g^p(t)(1-g(t))^q\right]^{\frac{1}{l}}$$
$$\times f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)]\right]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 g^p(t)(1-g(t))^q f^l(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\right]^{\frac{1}{l}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times \left[\int_0^1 g^p(t)(1-g(t))^q \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f^r(\varphi(b))^l + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f^r(\varphi(a))^l\right)^{\frac{1}{r}} d[g(t)]\right]^{\frac{1}{l}}$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rl}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times \left\{ \left(\int_0^1 g^{p+\frac{1}{2r}}(t)(1-g(t))^{q-\frac{1}{2r}} f^l(\varphi(b)) d[g(t)] \right)^r \right.$$

$$\left. + \left(\int_0^1 g^{p-\frac{1}{2r}}(t)(1-g(t))^{q+\frac{1}{2r}} f^l(\varphi(a)) d[g(t)] \right)^r \right\}^{\frac{1}{rl}}$$

$$= \left(\frac{m}{2} \right)^{\frac{1}{rl}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times \left[B^r \left(g(t); \frac{1}{2r}, 2pr-1, 2qr+1 \right) f^{rl}(\varphi(a)) \right.$$

$$\left. + B^r \left(g(t); \frac{1}{2r}, 2pr+1, 2qr-1 \right) f^{rl}(\varphi(b)) \right]^{\frac{1}{rl}}.$$

$\square$

Corollary 2.11. *Under the same conditions as in Theorem 2.10 for $r = 1$ and $g(t) = t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x) dx$$

$$\leq \left(\frac{m}{2} \right)^{\frac{1}{l}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times \left[\beta \left(p+\frac{1}{2}, q+\frac{3}{2} \right) f^l(\varphi(a)) + \beta \left(p+\frac{3}{2}, q+\frac{1}{2} \right) f^l(\varphi(b)) \right]^{\frac{1}{l}}.$$

3. HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR $MT_{(r;g,m,\varphi)}$-PREINVEX FUNCTIONS

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for $MT_{(r;g,m,\varphi)}$-preinvex functions via fractional integrals, we need the following new fractional integral identity:

Lemma 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Assume that $f : K \longrightarrow \mathbb{R}$ be a twice differentiable function on K° and f'' is integrable on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for $\alpha > 0$, we have*

$$\frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times \left[(1-g^{\alpha+1}(1)) f'(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)) \right.$$

$$\left. - (1-g^{\alpha+1}(0)) f'(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)) \right]$$

$$+ \frac{\eta^{\alpha+1}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times \left[(1-g^{\alpha+1}(1)) f'(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)) \right.$$

$$\left. - (1-g^{\alpha+1}(0)) f'(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)) \right]$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Big[g^{\alpha}(1)f(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))$$

$$-g^{\alpha}(0)f(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m))\Big]$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Big[g^{\alpha}(1)f(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))$$

$$-g^{\alpha}(0)f(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m))\Big]$$

$$-\frac{\alpha}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg[\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)}(t-m\varphi(a))^{\alpha-1}f(t)dt$$

$$+\int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)}(t-m\varphi(b))^{\alpha-1}f(t)dt\Bigg]$$

$$=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \qquad (3.1)$$

Proof. A simple proof of the equality can be done by performing two integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Let denote

$$I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \qquad (3.2)$$

Using relation (3.2), the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

Theorem 3.2. *Let $\varphi : I \longrightarrow A$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f : A \longrightarrow (0, +\infty)$ be a twice differentiable function on A°. If f''^q is nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function, $q > 1$, $p^{-1} + q^{-1} = 1$, then for $\alpha > 0$ and $0 < r \le 1$, we have*

$$|I_{f,g,\eta,\varphi}(x; \alpha, m, a, b)| \le \left(\frac{m}{2}\right)^{\frac{1}{rq}} \frac{C^{\frac{1}{p}}(g(t); p, \alpha)}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times \left\{ |\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} \left[A_2^r(g(t); r) f''(\varphi(a))^{rq} + A_1^r(g(t); r) f''(\varphi(x))^{rq} \right]^{\frac{1}{rq}} \right.$$

$$\left. + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2} \left[A_2^r(g(t); r) f''(\varphi(b))^{rq} + A_1^r(g(t); r) f''(\varphi(x))^{rq} \right]^{\frac{1}{rq}} \right\}, \quad (3.3)$$

where $C(g(t); p, \alpha) = \int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)]$.

Proof. Suppose that $q > 1$ and $0 < r \le 1$. Using relation (3.2), nonnegative $MT_{(r;g,m,\varphi)}$-preinvexity of f''^q, Hölder inequality, Minkowski inequality and taking the modulus, we have

$$|I_{f,g,\eta,\varphi}(x; \alpha, m, a, b)| \le \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times \int_0^1 |1 - g^{\alpha+1}(t)| \big| f''(m\varphi(a) + g(t)\eta(\varphi(x), \varphi(a), m)) \big| d[g(t)]$$

$$+ \frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times \int_0^1 |1 - g^{\alpha+1}(t)| \big| f''(m\varphi(b) + g(t)\eta(\varphi(x), \varphi(b), m)) \big| d[g(t)]$$

$$\le \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|} \left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)] \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 f''(m\varphi(a) + g(t)\eta(\varphi(x), \varphi(a), m))^q d[g(t)] \right)^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|} \left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)] \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 f''(m\varphi(b) + g(t)\eta(\varphi(x), \varphi(b), m))^q d[g(t)] \right)^{\frac{1}{q}}$$

$$\le \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|} \left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)] \right)^{\frac{1}{p}}$$

$$\times \left[\int_0^1 \left(\frac{m\sqrt{g(t)}}{2\sqrt{1 - g(t)}} f''(\varphi(x))^{rq} + \frac{m\sqrt{1 - g(t)}}{2\sqrt{g(t)}} f''(\varphi(a))^{rq} \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|} \left(\int_0^1 (1 - g^{\alpha+1}(t))^p d[g(t)] \right)^{\frac{1}{p}}$$

$$\times \left[\int_0^1 \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f''(\varphi(x))^{rq} + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f''(\varphi(b))^{rq} \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{1}{q}}$$

$$\leq \left(\frac{m}{2} \right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} C^{\frac{1}{p}}(g(t);p,\alpha)$$

$$\times \left\{ \left(\int_0^1 \left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}} \right)^{\frac{1}{r}} f''(\varphi(x))^q d[g(t)] \right)^r \right.$$

$$\left. + \left(\int_0^1 \left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}} \right)^{\frac{1}{r}} f''(\varphi(a))^q d[g(t)] \right)^r \right\}^{\frac{1}{rq}}$$

$$+ \left(\frac{m}{2} \right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} C^{\frac{1}{p}}(g(t);p,\alpha)$$

$$\times \left\{ \left(\int_0^1 \left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}} \right)^{\frac{1}{r}} f''(\varphi(x))^q d[g(t)] \right)^r \right.$$

$$\left. + \left(\int_0^1 \left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}} \right)^{\frac{1}{r}} f''(\varphi(b))^q d[g(t)] \right)^r \right\}^{\frac{1}{rq}}$$

$$= \left(\frac{m}{2} \right)^{\frac{1}{rq}} \frac{C^{\frac{1}{p}}(g(t);p,\alpha)}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times \left\{ |\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} \left[A_2^r(g(t);r) f''(\varphi(a))^{rq} + A_1^r(g(t);r) f''(\varphi(x))^{rq} \right]^{\frac{1}{rq}} \right.$$

$$\left. + |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2} \left[A_2^r(g(t);r) f''(\varphi(b))^{rq} + A_1^r(g(t);r) f''(\varphi(x))^{rq} \right]^{\frac{1}{rq}} \right\}.$$

$\square$

Corollary 3.3. *Under the same conditions as in Theorem 3.2 for $r = 1$, $g(t) = t$ and $f'' \leq K$, we get*

$$\left| \frac{-\eta(\varphi(x),\varphi(a),m)^{\alpha+1} f'(m\varphi(a)) - \eta(\varphi(x),\varphi(b),m)^{\alpha+1} f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \right.$$

$$+ \frac{\eta(\varphi(x),\varphi(a),m)^{\alpha} f(m\varphi(a) + \eta(\varphi(x),\varphi(a),m)) + \eta(\varphi(x),\varphi(b),m)^{\alpha} f(m\varphi(b) + \eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$- \frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\left. \times \left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) + J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \right] \right|$$

$$\leq \frac{K}{(1+\alpha)^{1+\frac{1}{p}}} \left(\frac{m\pi}{2} \right)^{\frac{1}{q}} \left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)} \right)^{\frac{1}{p}}$$

$$\times \left[\frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{|\eta(\varphi(b), \varphi(a), m)|} \right].$$

Theorem 3.4. *Let* $\varphi : I \longrightarrow A$ *be a continuous function and* $g : [0, 1] \longrightarrow (0, 1)$ *is a differentiable function. Suppose* $A \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}$ *for any fixed* $m \in (0, 1]$ *and let* $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. *Assume that* $f : A \longrightarrow (0, +\infty)$ *be a twice differentiable function on* A°. *If* f''^q *is nonnegative* $MT_{(r;g,m,\varphi)}$*-preinvex function,* $q \geq 1$, *then for* $\alpha > 0$ *and* $0 < r \leq 1$, *we have*

$$|I_{f,g,\eta,\varphi}(x; \alpha, m, a, b)|$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|} \left(g(1) - g(0) - \frac{g^{\alpha+2}(1) - g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times \Big[\left(A_2^r(g(t); r) - A_4^r(g(t); r, \alpha)\right) f''(\varphi(a))^{rq}$$

$$+ \left(A_1^r(g(t); r) - A_3^r(g(t); r, \alpha)\right) f''(\varphi(x))^{rq} \Big]^{\frac{1}{rq}}$$

$$+ \left(\frac{m}{2}\right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|} \left(g(1) - g(0) - \frac{g^{\alpha+2}(1) - g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times \Big[\left(A_2^r(g(t); r) - A_4^r(g(t); r, \alpha)\right) f''(\varphi(b))^{rq}$$

$$+ \left(A_1^r(g(t); r) - A_3^r(g(t); r, \alpha)\right) f''(\varphi(x))^{rq} \Big]^{\frac{1}{rq}}, \tag{3.4}$$

where

$$A_3(g(t); r, \alpha) = \int_{1-g(1)}^{1-g(0)} t^{-\frac{1}{2r}} (1-t)^{\frac{1}{2r}+\alpha+1} dt;$$

$$A_4(g(t); r, \alpha) = \int_{g(0)}^{g(1)} t^{-\frac{1}{2r}+\alpha+1} (1-t)^{\frac{1}{2r}} dt.$$

Proof. Suppose that $q \geq 1$ and $0 < r \leq 1$. Using relation (3.2), nonnegative $MT_{(r;g,m,\varphi)}$-preinvexity of f''^q, the well-known power mean inequality, Minkowski inequality and taking the modulus, we have

$$|I_{f,g,\eta,\varphi}(x; \alpha, m, a, b)| \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times \int_0^1 |1 - g^{\alpha+1}(t)| |f''(m\varphi(a) + g(t)\eta(\varphi(x), \varphi(a), m))| d[g(t)]$$

$$+ \frac{|\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|}$$

$$\times \int_0^1 |1 - g^{\alpha+1}(t)| |f''(m\varphi(b) + g(t)\eta(\varphi(x), \varphi(b), m))| d[g(t)]$$

$$\leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b), \varphi(a), m)|} \left(\int_0^1 (1 - g^{\alpha+1}(t)) d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 (1 - g^{\alpha+1}(t)) f''(m\varphi(a) + g(t)\eta(\varphi(x), \varphi(a), m))^q d[g(t)]\right)^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} \left(\int_0^1 (1-g^{\alpha+1}(t))d[g(t)] \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 (1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))^q d[g(t)] \right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} \left(\int_0^1 (1-g^{\alpha+1}(t))d[g(t)] \right)^{1-\frac{1}{q}}$$

$$\times \left[\int_0^1 (1-g^{\alpha+1}(t)) \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f''(\varphi(x))^{rq} + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f''(\varphi(a))^{rq} \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} \left(\int_0^1 (1-g^{\alpha+1}(t))d[g(t)] \right)^{1-\frac{1}{q}}$$

$$\times \left[\int_0^1 (1-g^{\alpha+1}(t)) \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} |f''(\varphi(x))|^q + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f''(\varphi(b))^{rq} \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{1}{q}}$$

$$\leq \left(\frac{m}{2} \right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} \left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2} \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left(\int_0^1 \left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}} \right)^{\frac{1}{r}} (1-g^{\alpha+1}(t))f''(\varphi(x))^q d[g(t)] \right)^r \right.$$

$$\left. + \left(\int_0^1 \left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}} \right)^{\frac{1}{r}} (1-g^{\alpha+1}(t))f''(\varphi(a))^q d[g(t)] \right)^r \right\}^{\frac{1}{rq}}$$

$$+ \left(\frac{m}{2} \right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} \left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2} \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left(\int_0^1 \left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}} \right)^{\frac{1}{r}} (1-g^{\alpha+1}(t))f''(\varphi(x))^q d[g(t)] \right)^r \right.$$

$$\left. + \left(\int_0^1 \left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}} \right)^{\frac{1}{r}} (1-g^{\alpha+1}(t))f''(\varphi(b))^q d[g(t)] \right)^r \right\}^{\frac{1}{rq}}$$

$$= \left(\frac{m}{2} \right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} \left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2} \right)^{1-\frac{1}{q}}$$

$$\times \left[(A_2^r(g(t);r) - A_4^r(g(t);r,\alpha)) f''(\varphi(a))^{rq} \right.$$

$$\left. + (A_1^r(g(t);r) - A_3^r(g(t);r,\alpha)) f''(\varphi(x))^{rq} \right]^{\frac{1}{rq}}$$

$$+ \left(\frac{m}{2} \right)^{\frac{1}{rq}} \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|} \left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2} \right)^{1-\frac{1}{q}}$$

$$\times \left[(A_2^r(g(t);r) - A_4^r(g(t);r,\alpha)) f''(\varphi(b))^{rq} \right.$$

$$\left. + (A_1^r(g(t);r) - A_3^r(g(t);r,\alpha)) f''(\varphi(x))^{rq} \right]^{\frac{1}{rq}}.$$

$$\square$$

Corollary 3.5. *Under the same conditions as in Theorem 3.4 for $r = 1$, $g(t) = t$ and $f'' \leq K$, we get*

$$\left| \frac{-\eta(\varphi(x), \varphi(a), m)^{\alpha+1} f'(m\varphi(a)) - \eta(\varphi(x), \varphi(b), m)^{\alpha+1} f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b), \varphi(a), m)} \right.$$

$$+ \frac{\eta(\varphi(x), \varphi(a), m)^\alpha f(m\varphi(a) + \eta(\varphi(x), \varphi(a), m)) + \eta(\varphi(x), \varphi(b), m)^\alpha f(m\varphi(b) + \eta(\varphi(x), \varphi(b), m))}{\eta(\varphi(b), \varphi(a), m)}$$

$$- \frac{\Gamma(\alpha+1)}{\eta(\varphi(b), \varphi(a), m)}$$

$$\left. \times \left[J^\alpha_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) + J^\alpha_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \right] \right|$$

$$\leq \frac{K}{\alpha+1} \left(\frac{\alpha+1}{\alpha+2} \right)^{1-\frac{1}{q}} \left(\frac{m}{2} \right)^{\frac{1}{q}} \left(\pi - \frac{\sqrt{\pi}(\alpha+1)\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha+3)} \right)^{\frac{1}{q}}$$

$$\times \left[\frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{|\eta(\varphi(b), \varphi(a), m)|} \right].$$

Remark 3.6. For different choices of positive values $0 < r \leq 1$, for any fixed $m \in (0, 1]$, for a particular choices of a differentiable function $g : [0, 1] \longrightarrow (0, 1)$, for example: $e^{-(t+1)}$, $\sin\left(\frac{\pi(t+1)}{3}\right)$, $\cos\left(\frac{\pi(t+1)}{3}\right)$, etc, and a particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbb{R}$, x^n for all $x > 0$ and for all $n \in \mathbb{N}$, etc, by Theorem 3.2 and Theorem 3.4 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities.

4. Applications to special means

Definition 4.1. (see [23]) A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following internality property:

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}.$$

For a Mean function the following holds, $M(x, x) = x$, $\forall x \in \mathbb{R}_+$.

We consider some means for arbitrary positive real numbers α, β ($\alpha \neq \beta$).

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1.$$

(5) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)}.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

(8) The weighted p-power mean:

$$M_p \begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{pmatrix} = \left(\sum_{i=1}^n \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0 \, (i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Recently, the bivariate means have attracted the attention of many researchers, many remarkable inequalities can be found in the literature [36]-[46]. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(x), \varphi(y)) :$ $[\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \times [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, $\varphi : I \longrightarrow A$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Therefore one can obtain various inequalities using the results of Section 3 for these means as follows: Replace $\eta(\varphi(x), \varphi(y), m)$ with $\eta(\varphi(x), \varphi(y))$ and setting $\eta(\varphi(x), \varphi(y)) = M(\varphi(x), \varphi(y))$, $\forall x, y \in I$, for value $m = 1$ in (3.3) and (3.4), one can obtain the following interesting inequalities involving means:

$$|I_{f,g,M(\cdot,\cdot),\varphi}(x; \alpha, 1, a, b)| \leq \left(\frac{1}{2}\right)^{\frac{1}{rq}} \frac{C^{\frac{1}{p}}(g(t); p, \alpha)}{(\alpha + 1)M(\varphi(a), \varphi(b))}$$

$$\times \left\{ M^{\alpha+2}(\varphi(a), \varphi(x)) \left[A_2^r(g(t); r) f''(\varphi(a))^{rq} + A_1^r(g(t); r) f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}} \right.$$

$$\left. + M^{\alpha+2}(\varphi(b), \varphi(x)) \left[A_2^r(g(t); r) f''(\varphi(b))^{rq} + A_1^r(g(t); r) f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}} \right\}, \quad (4.1)$$

$$|I_{f,g,M(\cdot,\cdot),\varphi}(x; \alpha, 1, a, b)|$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{rq}} \frac{M^{\alpha+2}(\varphi(a), \varphi(x))}{(\alpha + 1)M(\varphi(a), \varphi(b))} \left(g(1) - g(0) - \frac{g^{\alpha+2}(1) - g^{\alpha+2}(0)}{\alpha + 2}\right)^{1 - \frac{1}{q}}$$

$$\times \left[\left(A_2^r(g(t); r) - A_4^r(g(t); r, \alpha)\right) f''(\varphi(a))^{rq}\right.$$

$$\left. + \left(A_1^r(g(t); r) - A_3^r(g(t); r, \alpha)\right) f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}$$

$$+ \left(\frac{1}{2}\right)^{\frac{1}{rq}} \frac{M^{\alpha+2}(\varphi(b), \varphi(x))}{(\alpha + 1)M(\varphi(a), \varphi(b))} \left(g(1) - g(0) - \frac{g^{\alpha+2}(1) - g^{\alpha+2}(0)}{\alpha + 2}\right)^{1 - \frac{1}{q}}$$

$$\times \left[\left(A_2^r(g(t);r) - A_4^r(g(t);r,\alpha) \right) f''(\varphi(b))^{rq} \right.$$

$$\left. + \left(A_1^r(g(t);r) - A_3^r(g(t);r,\alpha) \right) f''(\varphi(x))^{rq} \right]^{\frac{1}{rq}}. \tag{4.2}$$

Letting $M(\varphi(x), \varphi(y)) = A, G, H, P_r, I, L, L_p, M_p, \forall x, y \in I$ in (4.1) and (4.2), we get the inequalities involving means for a particular choices of nonnegative twice differentiable $MT_{(r;g,1,\varphi)}$-preinvex function f. The details are left to the interested reader.

5. Conclusions

In this paper, we proved some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving $MT_{(r;g,m,\varphi)}$-preinvex functions. Also, we established some new Hermite-Hadamard type integral inequalities for nonnegative $MT_{(r;g,m,\varphi)}$-preinvex functions via Riemann-Liouville fractional integrals. These results provide new estimates on these types.

Motivated by this new interesting class of $MT_{(r;g,m,\varphi)}$-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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UMBRAL CALCULUS APPROACH TO DEGENERATE POLY-GENOCCHI POLYNOMIALS

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ABSTRACT. In this paper, we apply umbral calculus techniques in order to derive explicit exprissions, some properties, recurrence relations and identities for degenerate poly-Genocchi polynomials. Furthermore, we derive several explicit expressions of degenerate poly-Genocchi polynomials as linear combinations of some of the well-known families of special polynomials.

1. Review on umbral calculus

The purpose of this paper is to use umbral calculus in order to derive some new and interesting expressions, recurrence relations and identities for degenerate poly-Genocchi polynomials. To do that we first recall the umbral calculus very briefly. For a complete treatment, the reader may refer to [10].

Let $\mathcal{F}$ be the algebra of all formal power series in the single variable t with the coefficients in the field $\mathbb{C}$ of complex numbers:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \;\middle|\; a_k \in \mathbb{C} \right\}. \tag{1.1}$$

Let $\mathbb{P} = \mathbb{C}[x]$ denote the ring of polynomials in x with the coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. For $L \in \mathbb{P}^*$, $p(x) \in \mathbb{P}$, $< L \,|\, p(x) >$ denotes the action of the linear functional L on $p(x)$. For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, the linear functional $< f(t) \,|\, \cdot >$ on $\mathbb{P}$ is defined by

$$< f(t) \,|\, x^n > = a_n, \quad (n \geq 0). \tag{1.2}$$

For $L \in \mathbb{P}^*$, let $f_L(t) = \sum_{k=0}^{\infty} \left\langle L | x^k \right\rangle \frac{t^k}{k!} \in \mathcal{F}$. Then we evidently have $\left\langle f_L(t) | x^n \right\rangle = \left\langle L | x^n \right\rangle$, and the map $L \to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathcal{F}$. Thus $\mathcal{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in t. So an element $f(t) \in \mathcal{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra, the study of which is the umbral calculus.

The order $o(f(t))$ of $0 \neq f(t) \in \mathcal{F}$ is the smallest integer k such that the coefficients of t^k does not vanish. In particular, for $0 \neq f(t) \in \mathcal{F}$, it is called an invertible series if $o(f(t)) = 0$ and a delta series if $o(f(t)) = 1$.

Let $f(t), g(t) \in \mathcal{F}$, with $o(g(t)) = 0$, $o(f(t)) = 1$. Then there exists a unique sequence of polynomials $S_n(x)$ ($\deg S_n(x) = n$) such that $\left\langle g(t)f(t)^k | S_n(x) \right\rangle = n! \delta_{n,k}$, for $n, k \geq 0$. Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, which is concisely denoted by $S_n(x) \sim (g(t), f(t))$.

1

2   Umbral calculus approach to degenerate poly-Genocchi polynomials

It is known that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}, \tag{1.3}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

Let $p_n(x) \sim (1, f(t)), q_n(x) \sim (1, l(t))$. Then the transfer formula says that

$$q_n(x) = x \left(\frac{f(t)}{l(t)} \right)^n x^{-1} p_n(x), \ (n \geq 1). \tag{1.4}$$

Let $S_n(x) \sim (g(t)), f(t))$. Then we have the Sheffer identity:

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} S_k(x) p_{n-k}(y), \tag{1.5}$$

where $p_n(x) = g(t) S_n(x) \sim (1, f(t))$.

For $S_n(x) \sim (g(t), f(t))$,

$$f(t) S_n(x) = n S_{n-1}(x). \tag{1.6}$$

Also, we have the recurrence formula:

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} S_n(x). \tag{1.7}$$

Assume that $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$. Then

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \tag{1.8}$$

where

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k | x^n \right\rangle. \tag{1.9}$$

Finally, we also need the following: for any $h(t) \in \mathcal{F}$, $p(x) \in \mathbb{P}$,

$$\left\langle h(t) | x p(x) \right\rangle = \left\langle \partial_t h(t) | p(x) \right\rangle. \tag{1.10}$$

For $s_n(x) \sim (g(t), f(t))$,

$$\frac{d}{dx} S_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} < \bar{f}(t) | x^{n-l} > S_l(x). \tag{1.11}$$

2. Introduction

Let r be any integer, and let $0 \neq \lambda \in \mathbf{C}$. The series $Li_r(x)$ defined by

$$Li_r(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^r}, \text{ (see [3–9])}, \tag{2.1}$$

is the r-th polylogarithmic function for $r \geq 1$, and a rational function for $r \leq 0$. One immediate property of this is

$$\frac{d}{dx} (Li_{r+1}(x)) = \frac{1}{x} Li_r(x). \tag{2.2}$$

The degenerate poly-Genocchi polynomials $\gamma_n^{(r)}(\lambda, x)$ of index r are given by

$$\frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \gamma_n^{(r)}(\lambda, x)\frac{t^n}{n!}. \tag{2.3}$$

For $x = 0$, $\gamma_n^{(r)}(\lambda) = \gamma_n^{(r)}(\lambda, 0)$ are called degenerate poly-Genocchi numbers of index r. In particular for $r = 1$, $\gamma_n(\lambda, x) = \gamma_n^{(1)}(\lambda, x)$ may be called degenerate Genocchi polynomials and are given by

$$\frac{2t}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \gamma_n(\lambda, x)\frac{t^n}{n!}. \tag{2.4}$$

They are a degenerate version of the poly-Genocchi polynomials $G_n^{(r)}(x)$ of index r given by

$$\frac{2Li_r(1 - e^{-t})}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!}. \tag{2.5}$$

These poly-Genocchi polynomials were first introduced in [3] under the name of poly-Euler polynomials and with the notation $\mathbb{E}_n^{(r)}(x)$. However, it seems more appropriate to call them poly-Genocchi polynomials, as $G_n(x) = G_n^{(1)}(x)$ are the 'classical' Genocchi polynomials defined by

$$\frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{2.6}$$

Clearly, $\lim_{\lambda \to 0} \gamma_n^{(r)}(\lambda, x) = G_n^{(r)}(x)$. Also, we recall that the degenerate Euler polynomials $\mathcal{E}_n(\lambda, x)$ were introduced in [1] by Carlitz and are given by

$$\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda, x)\frac{t^n}{n!}. \tag{2.7}$$

The degenerate poly-Bernoulli polynomials $\beta_n^{(r)}(\lambda, x)$ of index r are defined in $[6, 7]$ as

$$\frac{Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(r)}(\lambda, x)\frac{t^n}{n!}. \tag{2.8}$$

When $x = 0$, $\beta_n^{(r)}(\lambda) = \beta_n^{(r)}(\lambda, 0)$ are called degenerate poly-Bernoulli numbers of index r. For $r = 1$, $\beta_n(\lambda, x) = \beta_n^{(1)}(\lambda, x)$ are called degenerate Bernoulli polynomials and given by

$$\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda, x)\frac{t^n}{n!}, \tag{2.9}$$

which were introduced again by Carlitz in [1]. They are a degenerate version of the poly-Bernoulli polynomials $B_n^{(r)}(x)$ of index r given by

$$\frac{Li_r(1 - e^{-t})}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \tag{2.10}$$

We note here that this definition of poly-Bernoulli polynomials is slightly different from its original definition(cf. $[4, 5]$). Obviously, $\lim_{\lambda \to 0} \beta_n^{(r)}(\lambda, x) = B_n^{(r)}(x)$, and $B_n(x) = B^{(1)}(x)$ are the 'classical' Bernoulli polynomials defined by

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{2.11}$$

Write $Li_r(1 - e^{-t}) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n \frac{t^n}{n!}$. Then from (2.3) and (2.7), we see that

$$\sum_{n=0}^{\infty} \gamma_n^{(r)}(\lambda, x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \Big(\sum_{l=0}^{n-1} \binom{n}{l} a_{n-l} \mathcal{E}_l(\lambda, x) \Big) \frac{t^n}{n!}. \tag{2.12}$$

In turn, (2.12) implies that

$$\gamma_0^{(r)}(\lambda, x) = 0, \gamma_1^{(r)}(\lambda, x) = 1, deg\gamma_n^{(r)}(\lambda, x) = n - 1, (n \geq 1). \tag{2.13}$$

In this paper, we would like to apply umbral calculus techniques([2, 7, 9, 10]) in order to derive explicit exprissions, some properties, recurrence relations and identities for degenerate poly-Genocchi polynomials. However, sometimes we can not apply the umbral calculus techniques directly to $\gamma_n^{(r)}(\lambda, x)$, since $\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$ is not invertible and hence $\gamma_n^{(r)}(\lambda, x)$ is not a Sheffer sequence. Nevertheless, from (2.3) and (2.13), we note that

$$\frac{2Li_r(1 - e^{-t})}{t((1 + \lambda t)^{\frac{1}{\lambda}} + 1)}(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n + 1} \frac{t^n}{n!}. \tag{2.14}$$

Thus we see from (2.14) that $\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1}$ is the Sheffer sequence for the pair $\left(\frac{(e^{\lambda t}-1)(e^t+1)}{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t} - 1) \right)$, namely

$$\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n + 1} \sim \left(\frac{(e^{\lambda t} - 1)(e^t + 1)}{2\lambda Li_r(1 - e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t} - 1) \right). \tag{2.15}$$

Furthermore, we give some explicit expressions of degenerate poly-Genocchi polynomials as linear combinations of some of the well-known families of special polynomials.

3. Main results

The following Theorems 2.1 and 2.2 are mentioned in [9] and obtained in [7, 9], and will be useful in deriving several explicit expressions of degenerate poly-Genocchi polynomials as linear combinations of degenerate Euler or degenerate Genocchi polynomials.

Theorem 3.1. *For all integers $r \geq 2$, and $n \geq 0$, we have the following identities.*

$$\left\langle \frac{Li_r(1 - e^{-t})}{t} \mid x^n \right\rangle = \frac{1}{n + 1} \sum_{m=1}^{n+1} (-1)^{m+n-1} \frac{m!}{m^r} S_2(n + 1, m)$$

$$= \sum_{m=0}^{n} \binom{n}{m} B_m^{(r)} \frac{1}{n - m + 1}$$

$$= \frac{1}{n + 1} B_n^{(r-1)}$$

$$= n! \sum_{j_1, \cdots, j_{r-1} \geq 0, j_1 + \cdots + j_{r-1} = n} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)}.$$

Theorem 3.2. *For all integers $r \geq 2$, and $n \geq 0$, we have the following identities.*

$$\langle Li_r(1 - e^{-t}) \mid x^{n+1} \rangle = \sum_{m=1}^{n+1} (-1)^{m+n+1} \frac{m!}{m^r} S_2(n+1, m)$$

$$= \sum_{m=0}^{n} \binom{n+1}{m} B_m^{(r)}$$

$$= B_n^{(r-1)}$$

$$= (n+1)! \sum_{j_1, \cdots, j_{r-1} \geq 0, j_1 + \cdots + j_{r-1} = n} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)}.$$

The following can be seen also from (2.3), but here we will deduce it by using umbral calculus.

$$
\begin{aligned}
\gamma_n^{(r)}(\lambda, y) &= \left\langle \sum_{m=0}^{\infty} \gamma_m^{(r)}(\lambda, y) \frac{t^m}{m!} \mid x^n \right\rangle \\
&= \left\langle \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} \mid x^n \right\rangle \\
&= \left\langle \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \mid \sum_{l=0}^{\infty} \left(\frac{y}{\lambda}\right)^l \frac{(log(1 + \lambda t))^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \left(\frac{y}{\lambda}\right)^l \left\langle \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \mid \sum_{m=l}^{\infty} S_1(m, l) \frac{\lambda^m}{m!} t^m x^n \right\rangle \\
&= \sum_{l=0}^{n} \left(\frac{y}{\lambda}\right)^l \sum_{m=l}^{n} \binom{n}{m} \lambda^m S_1(m, l) \left\langle \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \mid x^{n-m} \right\rangle \\
&= \sum_{l=0}^{n} \left(\frac{y}{\lambda}\right)^l \sum_{m=l}^{n} \binom{n}{m} \lambda^m S_1(m, l) \gamma_{n-m}^{(r)}(\lambda).
\end{aligned}
$$

(3.1)

Here $(x|\lambda)_n = x(x - \lambda) \cdots (x - (n - 1)\lambda)$, for $n \geq 1$, and $(x|\lambda)_0 = 1$. Recalling that $(x|\lambda)_n = \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) x^m$, we have the following result.

Theorem 3.3. *For all integers $n \geq 0$, we have the following expressions.*

$$\gamma_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{l=0}^{m} \lambda^{m-l} S_1(m, l) x^l\right) \gamma_{n-m}^{(r)}(\lambda)$$

$$= \sum_{m=0}^{n} \binom{n}{m} \gamma_{n-m}^{(r)}(\lambda) (x|\lambda)_m.$$

Now, we would like to express the degenerate poly-Genocchi polynomials in terms of degenerate Euler polynomials, for this we need to observe the following.

$$
\begin{aligned}
\gamma_n^{(r)}(\lambda, y) &= \left\langle \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}(1 + \lambda t)^{\frac{y}{\lambda}} \mid x^n \right\rangle \\
&= \left\langle Li_r(1 - e^{-t}) \mid \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}(1 + \lambda t)^{\frac{y}{\lambda}} x^n \right\rangle \\
&= \left\langle Li_r(1 - e^{-t}) \mid \sum_{l=0}^{\infty} \mathcal{E}_l(\lambda, y) \frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_l(\lambda, y) \left\langle Li_r(1 - e^{-t}) \mid x^{n-l} \right\rangle.
\end{aligned}
\tag{3.2}
$$

From (3.2) and Theorem 2.2, we obtain the following explicit expressions for $\gamma_n^{(r)}(\lambda, x)$, as linear combinations of the degenerate Euler polynomials.

Theorem 3.4. *For all integers $n \geq 0$, we have the following identities.*

$$
\begin{aligned}
\gamma_n^{(r)}(\lambda, x) &= \sum_{l=1}^{n} \sum_{m=1}^{l} \binom{n}{l}(-1)^{l+m} \frac{m!}{m^r} S_2(l, m) \mathcal{E}_{n-l}(\lambda, x) \\
&= \sum_{l=1}^{n} \sum_{m=0}^{l-1} \binom{n}{l}\binom{l}{m} B_m^{(r)} \mathcal{E}_{n-l}(\lambda, x) \\
&= \sum_{l=1}^{n} \binom{n}{l} B_{l-1}^{(r-1)} \mathcal{E}_{n-l}(\lambda, x) \\
&= \sum_{l=1}^{n} \sum_{j_1,\cdots,j_{r-1} \geq 0, j_1 + \cdots + j_{r-1} = l-1} (n)_l \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)} \mathcal{E}_{n-l}(\lambda, x).
\end{aligned}
$$

Next, in order to express the degenerate poly-Genocchi polynomials in terms of degenerate Genocchi polynomials, we first observe the following.

$$
\begin{aligned}
\gamma_n^{(r)}(\lambda, y) &= \left\langle \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}(1 + \lambda t)^{\frac{y}{\lambda}} \mid x^n \right\rangle \\
&= \left\langle \frac{Li_r(1 - e^{-t})}{t} \mid \frac{2t}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}(1 + \lambda t)^{\frac{y}{\lambda}} x^n \right\rangle \\
&= \left\langle \frac{Li_r(1 - e^{-t})}{t} \mid \sum_{l=0}^{\infty} \gamma_l(\lambda, y) \frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \gamma_l(\lambda, y) \left\langle \frac{Li_r(1 - e^{-t})}{t} \mid x^{n-l} \right\rangle.
\end{aligned}
\tag{3.3}
$$

From (3.3) and Theorem 2.1, we obtain the following explicit expression for $\gamma_n^{(r)}(\lambda, x)$ as linear combinations of degenerate Genocchi polynomials.

Theorem 3.5. *For all integers $n \geq 0$, we have the following identities.*

$$\gamma_n^{(r)}(\lambda, x) = \sum_{l=0}^{n-1} \sum_{m=1}^{l+1} \frac{1}{l+1} \binom{n}{l} (-1)^{l+m-1} \frac{m!}{m^r} S_2(l+1, m) \gamma_{n-l}(\lambda, x)$$

$$= \sum_{l=0}^{n-1} \sum_{m=0}^{l} \frac{1}{l-m+1} \binom{n}{l} \binom{l}{m} B_m^{(r)} \gamma_{n-l}(\lambda, x)$$

$$= \sum_{l=0}^{n-1} \frac{1}{l+1} \binom{n}{l} B_l^{(r-1)} \gamma_{n-l}(\lambda, x)$$

$$= \sum_{l=0}^{n-1} \sum_{j_1, \cdots, j_{r-1} \geq 0, j_1 + \cdots + j_{r-1} = l} (n)_l \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)} \gamma_{n-l}(\lambda, x).$$

Here we apply the transfer formula (1.4) to

$$x^n \sim (1, t),$$

$$\frac{\frac{1}{\lambda}(e^{\lambda t} - 1)(e^t + 1)}{2 Li_r \left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)} \frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1} \sim \left(1, \frac{1}{\lambda}(e^{\lambda t} - 1)\right). \tag{3.4}$$

Then, for $n \geq 1$ we have

$$\frac{\frac{1}{\lambda}(e^{\lambda t} - 1)(e^t + 1)}{2 Li_r \left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)} \frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1} = x \left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^n x^{n-1}$$

$$= x \sum_{l=0}^{\infty} B_l^{(n)} \frac{\lambda^l}{l!} t^l x^{n-1} \tag{3.5}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} x^{n-l}.$$

Thus

$$\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} \frac{2 Li_r \left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{\frac{1}{\lambda}(e^{\lambda t} - 1)(e^t + 1)} x^{n-l}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} \sum_{m=0}^{\infty} \frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1} \frac{\left(\frac{1}{\lambda}(e^{\lambda t} - 1)\right)^m}{m!} x^{n-l}$$

$$= \sum_{l=0}^{n-1} \sum_{m=0}^{n-l} \binom{n-1}{l} \lambda^{l-m} B_l^{(n)} \frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1} \sum_{j=m}^{\infty} S_2(j, m) \frac{\lambda^j}{j!} t^j x^{n-l} \tag{3.6}$$

$$= \sum_{l=0}^{n} \sum_{m=0}^{n-l} \sum_{j=m}^{n-l} \binom{n-1}{l} \binom{n-l}{j} \lambda^{l-m+j} S_2(j, m) B_l^{(n)} \frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1} x^{n-l-j}$$

$$= \sum_{j=0}^{n} \left(\sum_{l=0}^{n-j} \sum_{m=0}^{n-j-l} \binom{n-1}{l} \binom{n-l}{j} \lambda^{n-m-j} S_2(n-j-l, m) B_l^{(n)} \frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\right) x^j, \quad (n \geq 1).$$

8   Umbral calculus approach to degenerate poly-Genocchi polynomials

Observing that

$$
\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}x^l = (x|\lambda)_n \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right),
$$

$$
\frac{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\left(e^t+1\right)}{2Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right),
$$

(3.7)

we have

$$
\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}
$$

$$
= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{2Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\left(e^t+1\right)}x^l
$$

$$
= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\sum_{m=0}^{\infty}\frac{\gamma_{m+1}^{(r)}(\lambda)\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^m}{m+1}\frac{}{m!}x^l
$$

$$
= \sum_{l=0}^{n}\sum_{m=0}^{l} S_1(n,l)\lambda^{n-l-m}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\sum_{j=m}^{\infty} S_2(j,m)\frac{\lambda^j}{j!}t^j x^l
$$

$$
= \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{j=m}^{l}\lambda^{n-l-m+j}\binom{l}{j} S_1(n,l)S_2(j,m)\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}x^{l-j}
$$

$$
= \sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{m=0}^{l-j}\binom{l}{j}\lambda^{n-m-j} S_1(n,l)S_2(l-j,m)\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}x^{j}.
$$

(3.8)

Hence, from (3.6) and (3.8), we get the following theorem.

Theorem 3.6. *We have the following expressions.*

$$
\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}
$$

$$
= \sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\sum_{m=0}^{n-j-l}\binom{n-1}{l}\binom{n-l}{j}\lambda^{n-m-j}S_2(n-j-l,m)B_l^{(n)}\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}\right)x^{j}, \quad (n\geq 1)
$$

$$
= \sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{m=0}^{l-j}\binom{l}{j}\lambda^{n-m-j}S_1(n,l)S_2(l-j,m)\frac{\gamma_{m+1}^{(r)}(\lambda)}{m+1}x^{j}, \quad (n\geq 0).
$$

The following Sheffer identity follows immediately from (1.5).

$$
\gamma_n^{(r)}(\lambda,x+y) = \sum_{j=0}^{n}\binom{n}{j}\gamma_j^{(r)}(\lambda,x)(y|\lambda)_{n-j}.
$$

(3.9)

Applying (1.6) to $\gamma_n^{(r)}(\lambda,x)$, here we get

$$
\frac{1}{\lambda}(e^{\lambda t}-1)\gamma_n^{(r)}(\lambda,x) = n\gamma_{n-1}^{(r)}(\lambda,x).
$$

(3.10)

Thus

$$
\gamma_n^{(r)}(\lambda,x+\lambda) - \gamma_n^{(r)}(\lambda,x) = n\lambda\gamma_{n-1}^{(r)}(\lambda,x).
$$

(3.11)

By using (1.7), we obtain

$$
\begin{aligned}
\frac{\gamma_{n+1}^{(r)}(\lambda)}{n+1} &= \left(x - \frac{g'(t)}{g(t)} \right) e^{-\lambda t} \frac{\gamma_n^{(r)}(\lambda, x)}{n} \\
&= \frac{1}{n} x \gamma_n^{(r)}(\lambda, x - \lambda) - \frac{1}{n} e^{-\lambda t} \frac{g'(t)}{g(t)} \gamma_n^{(r)}(\lambda, x),
\end{aligned}
\tag{3.12}
$$

where $g(t) = \frac{(e^{\lambda t} - 1)(e^t + 1)}{2\lambda Li_r\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}$. Here,

$$
\begin{aligned}
\frac{g'(t)}{g(t)} &= (log g(t))' \\
&= \frac{1}{t} \left\{ \frac{\lambda t e^{\lambda t}}{e^{\lambda t} - 1} + \frac{t e^t}{e^t + 1} - \frac{t e^{\lambda t}}{Li_r\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)} \frac{Li_{r-1}\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t} - 1)} - 1} \right\}.
\end{aligned}
\tag{3.13}
$$

Note that $g(t)\gamma_n^{(r)}(\lambda, x) = (x|\lambda)_n = \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) x^m$, and that the expression in the curly bracket has order ≥ 1.

$$
\begin{aligned}
&e^{-\lambda t} \frac{g'(t)}{g(t)} \gamma_n^{(r)}(\lambda, x) \\
&= \frac{1}{t} \left\{ \frac{\lambda t}{e^{\lambda t} - 1} \frac{2\lambda Li_r\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{(e^{\lambda t} - 1)(e^t + 1)} + \frac{t e^{(1-\lambda)t}}{e^t + 1} \frac{2\lambda Li_r(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)})}{(e^{\lambda t} - 1)(e^t + 1)} \right. \\
&\qquad \left. - \frac{2}{e^t + 1} \frac{\lambda t}{e^{\lambda t} - 1} \frac{Li_{r-1}\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t} - 1)} - 1} \right\} \frac{(e^{\lambda t} - 1)(e^t + 1)}{2\lambda Li_r\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)} \gamma_n^{(r)}(\lambda, x) \\
&= \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) \frac{1}{t} \left\{ \frac{\lambda t}{e^{\lambda t} - 1} \frac{2\lambda Li_r\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{(e^{\lambda t} - 1)(e^t + 1)} + \frac{t e^{(1-\lambda)t}}{e^t + 1} \frac{2\lambda Li_r(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)})}{(e^{\lambda t} - 1)(e^t + 1)} \right. \\
&\qquad \left. - \frac{2}{e^t + 1} \frac{\lambda t}{e^{\lambda t} - 1} \frac{Li_{r-1}\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t} - 1)} - 1} \right\} x^m \\
&= \sum_{m=0}^{n} \frac{\lambda^{n-m}}{m+1} S_1(n, m) \left\{ \frac{\lambda t}{e^{\lambda t} - 1} \frac{2\lambda Li_r\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{(e^{\lambda t} - 1)(e^t + 1)} + \frac{t e^{(1-\lambda)t}}{e^t + 1} \frac{2\lambda Li_r(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)})}{(e^{\lambda t} - 1)(e^t + 1)} \right. \\
&\qquad \left. - \frac{2}{e^t + 1} \frac{\lambda t}{e^{\lambda t} - 1} \frac{Li_{r-1}\left(1 - e^{-\frac{1}{\lambda}(e^{\lambda t} - 1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t} - 1)} - 1} \right\} x^{m+1}
\end{aligned}
\tag{3.14}
$$

The next theorem follows from (3.12), (3.14), (3.15), (3.16), and (3.17).

Theorem 3.7. *For all integers $n \geq 0$, the following holds true.*

$$\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1} - \frac{x}{n}\gamma_n^{(r)}(\lambda, x - \lambda)$$

$$= -\frac{1}{n}\sum_{m=0}^{n}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{m+1}\binom{m+1}{j}\lambda^{n-m}S_1(n,m)S_2(j,l)\left\{\frac{1}{l+1}\lambda^{m+1-l}\gamma_{l+1}^{(r)}(\lambda)B_{m+1-j}\left(\frac{x}{\lambda}\right)\right.$$

$$\left. +\frac{1}{2}\frac{m+1-j}{l+1}\lambda^{j-l}\gamma_{l+1}^{(r)}(\lambda)E_{m-j}(x+1-\lambda) + \sum_{k=0}^{m+1-j}\binom{m+1-j}{k}\lambda^{j+k-l}B_l^{(r-1)}E_{m+1-j-k}B_k\left(\frac{x}{\lambda}\right)\right\}.$$

For the sake of completeness, we compute the three terms in (3.14) as follows:

$$\frac{\lambda t}{e^{\lambda t}-1}\frac{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{(e^{\lambda t}-1)(e^t+1)}x^{m+1}$$

$$=\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{\infty}\frac{\gamma_{l+1}^{(r)}(\lambda)\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^l}{l+1}\frac{1}{l!}x^{m+1}$$

$$=\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{m+1}\frac{\gamma_{l+1}^{(r)}(\lambda)\lambda^{-l}}{l+1}\sum_{j=l}^{\infty}S_2(j,l)\frac{\lambda^j}{j!}t^j x^{m+1} \qquad (3.15)$$

$$=\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)x^{m+1-j}$$

$$=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{l+1}\binom{m+1}{j}\lambda^{m+1-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)B_{m+1-j}\left(\frac{x}{\lambda}\right),$$

$$\frac{te^{(1-\lambda)t}}{e^t+1}\frac{2\lambda Li_r\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{(e^{\lambda t}-1)(e^t+1)}x^{m+1}$$

$$=\frac{te^{(1-\lambda)t}}{e^t+1}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{1}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)x^{m+1-j}$$

$$=\frac{e^{(1-\lambda)t}}{e^t+1}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{m+1-j}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)x^{m-j} \qquad (3.16)$$

$$=\frac{1}{2}e^{(1-\lambda)t}\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\frac{m+1-j}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)\frac{2}{e^t+1}x^{m-j}$$

$$=\frac{1}{2}\sum_{j=0}^{m+1}\sum_{j=l}^{m+1}\frac{m+1-j}{l+1}\binom{m+1}{j}\lambda^{j-l}S_2(j,l)\gamma_{l+1}^{(r)}(\lambda)E_{m-j}(x+1-\lambda),$$

and

$$\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\frac{Li_{r-1}\left(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\right)}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)}-1}x^{m+1}$$

$$=\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{\infty}B_l^{(r-1)}\frac{(\frac{1}{\lambda}(e^{\lambda t}-1))^l}{l!}x^{m+1}$$

$$=\frac{2}{e^t+1}\frac{\lambda t}{e^{\lambda t}-1}\sum_{l=0}^{m+1}\lambda^{-l}B_l^{(r-1)}\sum_{j=l}^{\infty}S_2(j,l)\frac{\lambda^j}{j!}t^j x^{m+1}$$

$$=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\lambda^{j-l}\binom{m+1}{j}S_2(j,l)B_l^{(r-1)}\frac{\lambda t}{e^{\lambda t}-1}\frac{2}{e^t+1}x^{m+1-j}$$

$$=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\lambda^{j-l}\binom{m+1}{j}S_2(j,l)B_l^{(r-1)}\sum_{k=0}^{m+1-j}\binom{m+1-j}{k}E_{m+1-j-k}\lambda^k B_k\left(\frac{x}{\lambda}\right)$$

$$=\sum_{l=0}^{m+1}\sum_{j=l}^{m+1}\sum_{k=0}^{m+1-j}\binom{m+1}{j}\binom{m+1-j}{k}\lambda^{j+k-l}S_2(j,l)B_l^{(r-1)}E_{m+1-j-k}B_k\left(\frac{x}{\lambda}\right).$$

(3.17)

We note that

$$<\bar{f}(t)|x^{n-l}> = \left\langle\frac{1}{\lambda}log(1+\lambda t)\mid x^{n-l}\right\rangle$$

$$=\lambda^{-1}\left\langle\sum_{m=1}^{\infty}(-1)^{m-1}\lambda^m(m-1)!\frac{t^m}{m!}\mid x^{n-l}\right\rangle$$

$$=(-\lambda)^{n-l-1}(n-l-1)!.$$

(3.18)

Thus, from (1.11), we get

$$\frac{d}{dx}\left(\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1}\right)=n!\sum_{l=0}^{n-1}\frac{(-\lambda)^{n-l-1}}{l!(n-l)}\frac{\gamma_{l+1}^{(r)}(\lambda,x)}{l+1}.$$

(3.19)

Here we use (1.10) in order to get an expression of $\gamma_n^{(r)}(\lambda,x)$. For this, assume that $n\geq 1$.

$$\gamma_n^{(r)}(\lambda,y) = \left\langle\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}}\mid x^n\right\rangle$$

$$=\left\langle\left(\partial_t\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right\rangle$$

$$+\left\langle\frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\left(\partial_t(1+\lambda t)^{\frac{y}{\lambda}}\mid x^{n-1}\right)\right\rangle.$$

(3.20)

The second term of (3.20) is easily seen to be equal to $y\gamma_{n-1}^{(r)}(\lambda, y - \lambda)$. For the first term of (3.20), we observe that

$$\partial_t \left(\frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \right)$$

$$\frac{2\frac{Li_{r-1}(1-e^{-t})}{1-e^{-t}}e^{-t}((1 + \lambda t)^{\frac{1}{\lambda}} + 1) - 2Li_r(1 - e^{-t})(1 + \lambda t)^{\frac{1}{\lambda}-1}}{\left((1 + \lambda t)^{\frac{1}{\lambda}} + 1\right)^2} \tag{3.21}$$

$$= \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \frac{Li_{r-1}(1 - e^{-t})}{e^t - 1} - \frac{1}{1 + \lambda t} \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} + \frac{1}{2(1 + \lambda t)} \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}.$$

So, the first term can be written as three sums:

$$\left\langle \frac{Li_{r-1}(1 - e^{-t})}{e^t - 1} \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} \mid x^{n-1} \right\rangle$$

$$- \left\langle \frac{1}{1 + \lambda t} \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} \mid x^{n-1} \right\rangle \tag{3.22}$$

$$+ \left\langle \frac{1}{2(1 + \lambda t)} \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} \mid x^{n-1} \right\rangle.$$

Now, we compute the three terms in (3.22) as follows

$$\left\langle \frac{Li_{r-1}(1 - e^{-t})}{e^t - 1} \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} \mid x^{n-1} \right\rangle$$

$$= \left\langle \frac{Li_{r-1}(1 - e^{-t})}{e^t - 1} \mid \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} x^{n-1} \right\rangle$$

$$= \left\langle \frac{Li_{r-1}(1 - e^{-t})}{e^t - 1} \mid \sum_{l=0}^{\infty} \mathcal{E}_l(\lambda, y) \frac{t^l}{l!} x^{n-1} \right\rangle \tag{3.23}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \mathcal{E}_l(\lambda, y) \left\langle \frac{Li_{r-1}(1 - e^{-t})}{e^t - 1} \mid x^{n-1-l} \right\rangle$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \mathcal{E}_l(\lambda, y) B_{n-1-l}^{(r-1)},$$

$$\left\langle \frac{1}{1 + \lambda t} \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} \mid x^{n-1} \right\rangle$$

$$= \left\langle \frac{1}{1 + \lambda t} \mid \frac{2Li_r(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{y}{\lambda}} x^{n-1} \right\rangle$$

$$= \left\langle \frac{1}{1 + \lambda t} \mid \sum_{l=0}^{\infty} \gamma_l^{(r)}(\lambda, y) \frac{t^l}{l!} x^{n-1} \right\rangle \tag{3.24}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \gamma_l^{(r)}(\lambda, y) \left\langle \frac{1}{1 + \lambda t} \mid x^{n-1-l} \right\rangle$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \gamma_l^{(r)}(\lambda, y)(-\lambda)^{n-1-l}(n - 1 - l)!,$$

and

$$\left\langle \frac{1}{2(1+\lambda t)} \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} \mid x^{n-1} \right\rangle$$

$$= \left\langle \frac{1}{2(1+\lambda t)} \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \frac{2Li_r(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{y}{\lambda}} x^{n-1} \right\rangle$$

$$= \left\langle \frac{1}{2(1+\lambda t)} \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \mid \sum_{l=0}^{\infty} \gamma_l^{(r)}(\lambda,y)\frac{t^l}{l!} x^{n-1} \right\rangle$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \gamma_l^{(r)}(\lambda,y) \left\langle \frac{1}{2(1+\lambda t)} \mid \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} x^{n-1-l} \right\rangle$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \gamma_l^{(r)}(\lambda,y) \left\langle \frac{1}{2(1+\lambda t)} \mid \sum_{m=0}^{\infty} \mathcal{E}_m(\lambda)\frac{t^m}{m!} x^{n-1-l} \right\rangle$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \gamma_l^{(r)}(\lambda,y) \sum_{m=0}^{n-1-l} \binom{n-1-l}{m} \mathcal{E}_m(\lambda) \left\langle \frac{1}{2}\frac{1}{1+\lambda t} \mid x^{n-1-l-m} \right\rangle \qquad (3.25)$$

$$= \frac{1}{2} \sum_{l=0}^{n-1} \sum_{m=0}^{n-1-l} \binom{n-1}{l}\binom{n-1-l}{m} \gamma_l^{(r)}(\lambda,y) \, \mathcal{E}_m(\lambda)(-\lambda)^{n-1-l-m}(n-1-l-m)!$$

Putting (3.20), (3.22), (3.23), (3.24), (3.25) altogether, we obtain the following result.

Theorem 3.8. *For all integers $n \geq 0$, the following holds true.*

$$\gamma_n^{(r)}(\lambda,x) = x\gamma_{n-1}^{(r)}(\lambda, x-\lambda) + \sum_{l=0}^{n-1} \frac{(n-1)!}{l!} \left\{ \frac{1}{(n-1-l)!} B_{n-1-l}^{(r-1)} \mathcal{E}_l(\lambda,x) \right.$$

$$\left. -(-\lambda)^{n-1-l}\gamma_l^{(r)}(\lambda,x) + \frac{1}{2} \sum_{m=0}^{n-1-l} \frac{1}{m!}\mathcal{E}_m(\lambda)(-\lambda)^{n-1-l-m}\gamma_l^{(r)}(\lambda,x) \right\}.$$

From now on, we will exploit (1.9) in order to express degenerate poly-Genocchi polynomials as linear combinations of well known families of polynomials. For this, we remind the reader that

$$\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} \sim \left(\frac{(e^{\lambda t}-1)(e^t+1)}{2\lambda Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t}-1) \right). \qquad (3.26)$$

We let $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} = \sum_{k=0}^{n} C_{n,k}\mathcal{E}_k(\lambda,x)$, with noting that

$$\mathcal{E}_n(\lambda,x) \sim \left(\frac{e^t+1}{2}, \frac{1}{\lambda}(e^{\lambda t}-1) \right). \qquad (3.27)$$

Then

$$
\begin{aligned}
C_{n,k} &= \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}+1}{2} \frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)} t^k \mid x^n \right\rangle \\
&= \frac{1}{k!} \left\langle \frac{Li_r(1-e^{-t})}{t} \mid t^k x^n \right\rangle \\
&= \binom{n}{k} \left\langle \frac{Li_r(1-e^{-t})}{t} \mid x^{n-k} \right\rangle \\
&= \frac{1}{n-k+1} \binom{n}{k} B_{n-k}^{(r-1)}.
\end{aligned}
\tag{3.28}
$$

Thus we have shown

$$
\gamma_{n+1}^{(r)}(\lambda, x) = \sum_{k=0}^{n} \binom{n+1}{k} B_{n-k}^{(r-1)} \mathcal{E}_k(\lambda, x).
\tag{3.29}
$$

To express degenerate poly-Genocchi polynomials as linear combinations of Euler polynomials, write $\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1} = \sum_{k=0}^{n} C_{n,k} E_k(x)$, with noting that

$$
E_n(\lambda, x) \sim \left(\frac{e^t+1}{2}, t \right).
\tag{3.30}
$$

Then

$$
\begin{aligned}
C_{n,k} &= \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}}+1}{2} \frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \left(\frac{1}{\lambda} log(1+\lambda t) \right)^k \mid x^n \right\rangle \\
&= \lambda^{-k} \left\langle \frac{Li_r(1-e^{-t})}{t} \mid \frac{1}{k!} (log(1+\lambda t))^k x^n \right\rangle \\
&= \lambda^{-k} \left\langle \frac{Li_r(1-e^{-t})}{t} \mid \sum_{l=k}^{\infty} S_1(l,k) \frac{\lambda^l}{l!} t^l x^n \right\rangle \\
&= \lambda^{-k} \sum_{l=k}^{n} \binom{n}{l} \lambda^l S_1(l,k) \left\langle \frac{Li_r(1-e^{-t})}{t} \mid x^{n-l} \right\rangle \\
&= \lambda^{-k} \sum_{l=k}^{n} \frac{1}{n-l+1} \binom{n}{l} \lambda^l S_1(l,k) B_{n-l}^{(r-1)}.
\end{aligned}
\tag{3.31}
$$

Thus we have derived the following theorem.

Theorem 3.9. *For all $n \geq 0$, we have the following.*

$$
\gamma_{n+1}^{(r)}(\lambda, x) = \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{n+1}{l} \lambda^{l-k} S_1(l,k) B_{n-l}^{(r-1)} E_k(x).
$$

Let $\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1} = \sum_{k=0}^{n} C_{n,k} \beta_k^{(r)}(\lambda, x)$, with

$$
\beta_n^{(r)}(\lambda, x) \sim \left(\frac{e^t-1}{Li_r(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t}-1) \right).
\tag{3.32}
$$

Then

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{Li_r(1 - e^{-t})} \frac{2\lambda Li_r(1 - e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}} + 1)} \,\middle|\, t^k x^n \right\rangle$$

$$= \binom{n}{k} \left\langle \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} \,\middle|\, \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} x^{n-k} \right\rangle$$

$$= \binom{n}{k} \left\langle \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} \,\middle|\, \sum_{l=0}^{\infty} \frac{(1|\lambda)_{l+1}}{l+1} \frac{t^l}{l!} x^{n-k} \right\rangle \tag{3.33}$$

$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{(1|\lambda)_{l+1}}{l+1} \left\langle \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} \,\middle|\, x^{n-k-l} \right\rangle$$

$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{(1|\lambda)_{l+1}}{l+1} \mathcal{E}_{n-k-l}(\lambda).$$

Thus we have shown the following result.

Theorem 3.10. *For $n \geq 0$, we have the following.*

$$\gamma_{n+1}^{(r)}(\lambda, x) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n+1}{l+1} \binom{n-l}{k} (1|\lambda)_{l+1} \mathcal{E}_{n-k-l}(\lambda) \beta_k^{(r)}(\lambda, x).$$

Write $\frac{\gamma_{n+1}^{(r)}(\lambda, x)}{n+1} = \sum_{k=0}^{n} C_{n,k} \beta_k(\lambda, x)$, with noting that

$$\beta_n(\lambda, x) \sim \left(\frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}, \frac{1}{\lambda}(e^{\lambda t} - 1) \right). \tag{3.34}$$

Then

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{\lambda((1+\lambda t)^{\frac{1}{\lambda}} - 1)}{\lambda t} \frac{2\lambda Li_r(1 - e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}} + 1)} \,\middle|\, t^k x^n \right\rangle$$

$$= \binom{n}{k} \left\langle \frac{2Li_r(1 - e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}} + 1)} \,\middle|\, \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} x^{n-k} \right\rangle$$

$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{(1|\lambda)_{l+1}}{l+1} \left\langle \frac{2Li_r(1 - e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}} + 1)} \,\middle|\, x^{n-k-l} \right\rangle \tag{3.35}$$

$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{(1|\lambda)_{l+1}}{l+1} \frac{\gamma_{n-k-l+1}^{(r)}(\lambda)}{n-k-l+1}.$$

Thus we obtain the following theorem.

Theorem 3.11. *For all $n \geq 0$, the following holds true.*

$$\gamma_{n+1}^{(r)}(\lambda, x) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{1}{l+1} \binom{n+1}{l} \binom{n-l+1}{k} (1|\lambda)_{l+1} \gamma_{n-k-l+1}^{(r)}(\lambda) \beta_k(\lambda, x).$$

Lastly, we would like to express the degenerate poly-Genocchi polynomials in terms of Bernoulli polynomials, with noting that

$$B_n(x) \sim \left(\frac{e^t - 1}{t}, t \right), \tag{3.36}$$

we let $\frac{\gamma_{n+1}^{(r)}(\lambda,x)}{n+1} = \sum_{k=0}^{n} C_{n,k} B_k(x)$. Then

$$
\begin{aligned}
C_{n,k} &= \frac{1}{k!} \left\langle \frac{\lambda((1+\lambda t)^{\frac{1}{\lambda}}-1)}{log(1+\lambda t)} \frac{2\lambda Li_r(1-e^{-t})}{\lambda t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \ \Big| \ \left(\frac{1}{\lambda}log(1+\lambda t)\right)^k x^n \right\rangle \\
&= \lambda^{-k} \left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} \ \Big| \ \frac{1}{k!}(log(1+\lambda t))^k x^n \right\rangle \\
&= \lambda^{-k} \left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} \ \Big| \ \sum_{l=k}^{\infty} S_1(l,k)\frac{\lambda^l}{l!}t^l x^n \right\rangle \\
&= \lambda^{-k} \sum_{l=0}^{n} \binom{n}{l} \lambda^l S_1(l,k) \left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \ \Big| \ \frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t} x^{n-l} \right\rangle \\
&= \lambda^{-k} \sum_{l=0}^{n} \binom{n}{l} \lambda^l S_1(l,k) \left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \frac{\lambda t}{log(1+\lambda t)} \ \Big| \ \sum_{m=0}^{\infty} \frac{(1|\lambda)_{m+1}}{m+1}\frac{t^m}{m!}x^{n-l} \right\rangle \\
&= \lambda^{-k} \sum_{l=0}^{n} \binom{n}{l} \lambda^l S_1(l,k) \sum_{m=0}^{n-l} \binom{n-l}{m} \frac{(1|\lambda)_{m+1}}{m+1} \left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \ \Big| \ \frac{\lambda t}{log(1+\lambda t)} x^{n-l-m} \right\rangle \\
&= \lambda^{-k} \sum_{l=k}^{n} \binom{n}{l} \lambda^l S_1(l,k) \sum_{m=0}^{n-l} \binom{n-l}{m} \frac{(1|\lambda)_{m+1}}{m+1} \left\langle \frac{2Li_r(1-e^{-t})}{t((1+\lambda t)^{\frac{1}{\lambda}}+1)} \ \Big| \ \sum_{j=0}^{\infty} b_j \frac{\lambda^j}{j!} t^j x^{n-l-m} \right\rangle \\
&= \lambda^{-k} \sum_{l=k}^{n} \binom{n}{l} \lambda^l S_1(l,k) \sum_{m=0}^{n-l} \binom{n-l}{m} \frac{(1|\lambda)_{m+1}}{m+1} \sum_{j=0}^{n-l-m} \binom{n-l-m}{j} \lambda^j b_j \frac{\gamma_{n-l-m-j+1}^{(r)}}{n-l-m-j+1}.
\end{aligned}
$$
(3.37)

Here b_j are the Bernoulli numbers of the second kind given by $\frac{t}{log(1+t)} = \sum_{j=0}^{\infty} b_j \frac{t^j}{j!}$. Thus we have derived the following result.

Theorem 3.12. *For all integers $n \geq 0$, the following holds true.*

$$
\begin{aligned}
&\gamma_{n+1}^{(r)}(\lambda, x) \\
&= \sum_{k=0}^{n} \sum_{l=k}^{n} \sum_{m=0}^{n-l} \sum_{j=0}^{n-l-m} \frac{1}{m+1} \binom{n+1}{j} \binom{n-j+1}{m} \binom{n-j-m+1}{l} \lambda^{l+j-k} S_1(l,k) \\
&\quad \times (1|\lambda)_{m+1} b_j \gamma_{n-l-m-j+1}^{(r)} B_k(x).
\end{aligned}
$$

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QUADRATIC (ρ_1, ρ_2)-FUNCTIONAL INEQUALITY IN FUZZY NORMED SPACES

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ABSTRACT. In this paper, we introduce and solve the following quadratic (ρ_1, ρ_2)-functional inequality

$$N(f(x + y) + f(x - y) - 2f(x) - 2f(y), t) \tag{0.1}$$
$$\leq min\left(N\left(\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right), \right.$$
$$\left. N\left(\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x - y) - 2f(x) - 2f(y)\right), t\right) \right)$$

in fuzzy normed spaces, where ρ_1 and ρ_2 are fixed nonzero real numbers with $\frac{1}{|\rho_1|} + \frac{1}{2|\rho_2|} < 1$, and $f(0) = 0$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.1) in fuzzy Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Katsaras [16] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [10, 20, 43]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [19]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 24, 25] to investigate the Hyers-Ulam stability of quadratic (ρ_1, ρ_2)-functional inequality in fuzzy Banach spaces.

Definition 1.1. [2, 24, 25, 26] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$.

(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

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The pair (X, N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [23, 24].

Definition 1.2. [2, 24, 25, 26] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 24, 25, 26] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [42] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [35] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 13, 15, 17, 18, 21, 31, 32, 33, 36, 37, 38, 39, 40, 41]).

Park [29, 30] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 9] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

QUADRATIC (ρ_1, ρ_2)-FUNCTIONAL INEQUALITY IN FUZZY NORMED SPACES

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 23, 27, 28, 33, 34]).

In Section 2, we solve the quadratic (ρ_1, ρ_2)-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that ρ_1 and ρ_2 are fixed nonzero real numbers with $\frac{1}{|\rho_1|} + \frac{1}{2|\rho_2|} < 1$.

2. QUADRATIC (ρ_1, ρ_2)-FUNCTIONAL INEQUALITY (0.1)

In this section, we solve and investigate the quadratic (ρ_1, ρ_2)-functional inequality in fuzzy normed spaces.

Lemma 2.1. *Let X be a real vector space and (Y, N) be a fuzzy normed vector space. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \tag{2.1}$$
$$\leq min\left(N\left(\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right), t \right), \right.$$
$$\left. N\left(\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right), t \right) \right)$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = 0$ in (2.1), we get

$$1 \leq min\left(N\left(\rho_1 \left(4f\left(\frac{x}{2}\right) - f(x) \right), t \right), \left(\rho_2 \left(4f\left(\frac{x}{2}\right) - f(x) \right), t \right) \right)$$
$$\leq N\left((\rho_1 + \rho_2)\left(4f\left(\frac{x}{2}\right) - f(x) \right), 2t \right)$$

Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.2}$$

for all $x \in X$.

Now we consider $P : X \to Y$ that

$$P(x, y) = f(x+y) + f(x-y) - 2f(x) - 2f(y)$$

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and we consider

$$\alpha = \frac{1}{|\rho_1|} + \frac{1}{2|\rho_2|}.$$

It follows from (2.1) and (2.2) that

$$
\begin{aligned}
N(P(x,y),t) &\leq\ min\left(N\left(\frac{\rho_1}{2}P(x,y),t\right), N\left(\rho_2 P(x,y),t\right)\right)\\
&=\ min\left(N\left(\frac{1}{2}P(x,y),\frac{t}{|\rho_1|}\right), N\left(\frac{1}{2}P(x,y),\frac{t}{2|\rho_2|}\right)\right)\\
&\leq\ N\left(P(x,y),\left(\frac{1}{|\rho_1|}+\frac{1}{2|\rho_2|}\right)t\right) = N\left(P(x,y),\alpha t\right)
\end{aligned}
$$

for all $t > 0$. By (N_5) and (N_6),

$$P(x,y) = f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0$$

for all $x, y \in X$, since $\alpha < 1$. So $f : X \to Y$ is quadratic. $\qquad\square$

We prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (2.1) in fuzzy Banach spaces.

Theorem 2.2. *Let X be a real vector space and (Y, N) be a fuzzy normed vector space. Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$
min\left(N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t), \frac{t}{t + \varphi(x,y)}\right) \tag{2.3}
$$
$$
\leq min\left(N\left(\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right),\right.
$$
$$
\left. N\left(\rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right), t\right)\right)
$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N - \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $C : X \to Y$ such that

$$
N(f(x) - Q(x), t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \beta\varphi(x, 0)} \tag{2.4}
$$

for all $x \in X$ and all $t > 0$ while $\beta = \frac{1}{|\rho_1|} + \frac{1}{|\rho_2|}$.

Proof. Letting $y = 0$ in (2.3), we get

$$
\begin{aligned}
\frac{t}{t + \varphi(x,0)} &\leq\ min\left(N\left(\rho_1\left(4f\left(\frac{x}{2}\right) - f(x)\right), t\right), N\left(\rho_2\left(4f\left(\frac{x}{2}\right) - f(x)\right), t\right)\right) \tag{2.5}\\
&\leq\ N\left(4f\left(\frac{x}{2}\right) - f(x), \left(\frac{1}{|\rho_1|} + \frac{1}{|\rho_2|}\right)\frac{t}{2}\right)\\
&=\ N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{\beta}{2}t\right)
\end{aligned}
$$

QUADRATIC (ρ_1, ρ_2)-FUNCTIONAL INEQUALITY IN FUZZY NORMED SPACES

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}, \ \forall x \in X, \forall t > 0 \right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [22, Lemma 2.1]). Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(Jg(x) - Jh(x), L\epsilon t) = N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\epsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L\epsilon t}{4}\right)$$

$$\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{\beta}{2}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{\beta}{2}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{\beta}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$min \left(N \left(4^n \left(f \left(\frac{x+y}{2^n} \right) + f \left(\frac{x-y}{2^n} \right) - f \left(\frac{x}{2^n} \right) - 2f \left(\frac{y}{2^n} \right) \right), 4^n t \right), \frac{t}{t + \varphi \left(\frac{x}{2^n}, \frac{y}{2^n} \right)} \right)$$
$$\leq min \left(N \left(4^n \rho_1 \left(2f \left(\frac{x+y}{2^{n+1}} \right) + 2f \left(\frac{x-y}{2^{n+1}} \right) - f \left(\frac{x}{2^n} \right) - f \left(\frac{y}{2^n} \right) \right), 4^n t \right), \right.$$
$$\left. N \left(4^n \rho_2 \left(4f \left(\frac{x+y}{2^{n+1}} \right) + f \left(\frac{x-y}{2^n} \right) - 2f \left(\frac{x}{2^n} \right) - 2f \left(\frac{y}{2^n} \right) \right), 4^n t \right) \right)$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$min \left(N \left(4^n \left(f \left(\frac{x+y}{2^n} \right) + f \left(\frac{x-y}{2^n} \right) - f \left(\frac{x}{2^n} \right) - 2f \left(\frac{y}{2^n} \right) \right), t \right), \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n} \varphi (x, y)} \right)$$
$$\leq min \left(N \left(4^n \rho_1 \left(2f \left(\frac{x+y}{2^{n+1}} \right) + 2f \left(\frac{x-y}{2^{n+1}} \right) - f \left(\frac{x}{2^n} \right) - f \left(\frac{y}{2^n} \right) \right), t \right), \right.$$
$$\left. N \left(4^n \rho_2 \left(4f \left(\frac{x+y}{2^{n+1}} \right) + f \left(\frac{x-y}{2^n} \right) - 2f \left(\frac{x}{2^n} \right) - 2f \left(\frac{y}{2^n} \right) \right), t \right) \right)$$

Since $\lim_{n \to \infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n} \varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y), t)$$
$$\leq min \left(N \left(\rho_1 \left(2Q \left(\frac{x+y}{2} \right) + 2Q \left(\frac{x-y}{2} \right) - Q(x) - Q(y) \right), t \right), \right.$$
$$\left. N \left(\rho_2 \left(4Q \left(\frac{x+y}{2} \right) + Q(x - y) - 2Q(x) - 2Q(y) \right), t \right) \right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. $\square$

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\| \cdot \|$ and (Y, N) be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$min \left(N(f(x + y) + f(x - y) - 2f(x) - 2f(y), t), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \right) \quad (2.7)$$
$$\leq min \left(N \left(\rho_1 \left(2f \left(\frac{x+y}{2} \right) + 2f \left(\frac{x-y}{2} \right) - f(x) - f(y) \right), t \right), \right.$$
$$\left. N \left(\rho_2 \left(4f \left(\frac{x+y}{2} \right) + f(x - y) - 2f(x) - 2f(y) \right), t \right) \right)$$

for all $x, y \in X$ *and all* $t > 0$. *Then* $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{2(2^p - 4)t}{2(2^p - 4)t + \beta\theta\|2x\|^p}$$

for all $x \in X$ *and all* $t > 0$ *while* $\beta = \frac{1}{|\rho_1|} + \frac{1}{|\rho_2|}$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. $\square$

Theorem 2.4. *Let* X *be a real vector space and* (Y, N) *be a fuzzy normed vector space. Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* (2.3). *Then* $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(8 - 8L)t}{(8 - 8L)t + \beta\varphi(x, 0)} \tag{2.8}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{\beta}{8}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$. Then $d(f, Jf) \leq \frac{\beta}{8}$. Hence

$$d(f, Q) \leq \frac{\beta}{8 - 8L},$$

which implies that the inequality (2.8) holds.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 2$. *Let* X *be a normed vector space with norm* $\|\cdot\|$ *and* (Y, N) *be a fuzzy normed vector space. Let* $f : X \to Y$ *be a mapping satisfying* (2.7). *Then* $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{2(4 - 2^p)t}{2(4 - 2^p)t + \beta\theta\|x\|^p}$$

for all $x \in X$ *and all* $t > 0$ *while* $\beta = \frac{1}{|\rho_1|} + \frac{1}{|\rho_2|}$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. $\square$

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