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Shift and invert weighted Golub-Kahan-Lanczos bidiagonalization algorithm for linear response eigenproblem

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Abstract: Weighted Golub-Kahan-Lanczos bidiagonalization algorithm(wGKL$_u$) is used to solving the linear response eigenproblem. In this paper, we present an improvement to wGKL$_u$ based on the shift-and-invert strategy. Due to the interior eigenproblem being transformed to the exterior eigenproblem, our new algorithm saves lots of calculus. Numerical examples illustrates the behaviors.

Keywords: Linear response eigenproblem, Golub-Kahan-Lanczos, Shift and invert.
AMS classifications: 65F15, 15A18, 81Q15.

1 Introduction

In this paper, we consider the eigenvalue problem of the form

$$\mathbf{H}\mathbf{z} = \begin{bmatrix} 0 & K \\ M & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \mathbf{z}, \tag{1.1}$$

where $K, M \in \mathbb{C}^{n\times n}$, are hermitian positive definite.

Such a problem is referred as the linear response eigenvalue problem(LREP)[1, 14, 20]. It arises from linear response problem that computes excitation states (energies) of physical systems in the study of collective motion of many particle systems [3, 9, 11, 14, 8]. In the linear response problem, although there are cases that one of K and M may be indefinite [12], however, usually both of them are positive definite [14]. So in this paper, we consider the case that both of K and M are positive definite. There are a great deal of excellent work in developing efficient numerical algorithms for linear response problem [1, 2, 10, 15, 16, 18, 20].

As we all known, the classical Lanczos method is efficient and easy to execute for symmetric eigenvalue problem [13]. In order to take advantage of the classical Lanczos method, in [20], Tsiper proposed a Lanczos-type method for the linear response problem, and based on reducing both K and M to tridiagonal matrices. While in [18], Teng and Li presented another Lanczos-type method which can be viewed as a natural and elegant extension of the classical Lanczos method. It is based on reducing one of K and M to a tridiagonal matrix and the other to a diagonal matrix. We can see, both the above two methods reduce the original H to a unsymmetric matrix. Thus the calculation of its eigenpairs can not use any advantages from symmetric matrix eigenvalue calculation, consequently, it may generate extra computation and storage.

Recently, to avoid this problem, the weighted Golub-Kahan-Lanczos(wGKL) [21] was proposed for solving LREP, denoted by wGKL-LREP. It aims to generate a projection matrix $\mathbf{B}_k = \begin{bmatrix} 0 & B_k \\ B_k^T & 0 \end{bmatrix}$ of $\mathbf{H}$ at kth iteration, where B_k is an upper or lower bidiagonal matrix. Due

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to the symmetry of $\mathbf{B}_k$, the eigenpairs of $\mathbf{H}$ can be constructed just from B_k, not the whole $\mathbf{B}_k$. In the following discussion, we focus on B_k is an upper bidiagonal matrix, the corresponding algorithm of which is wGKL$_u$-LREP, the lower case can be similarly discussed.

Since often in linear response eigenvalue problem, the first l smallest positive eigenvalues λ_i for $i = 1, 2, \cdots, l$ are of interest. They lie in the middle of the spectrum of $\mathbf{H}$, and often crowd together, thus it is not easy to get them with the above algorithms. Fortunately, we can apply the preconditioning technique, the notion of which is better known for linear systems than for eigenvalue problems. A typical preconditioned iterative method for linear systems amounts to replacing the original linear system $Ax = b$ by the equivalent system $P^{-1}Ax = P^{-1}b$, where P is a matrix close to A in some sense. For eigenvalue problems, the best known preconditioning is the so-called shift-and-invert technique. If the shift σ is suitably chosen, the shifted and inverted matrix $P = (A - \sigma I)^{-1}$ will have a spectrum with much better separation properties than that of the original matrix A, and this will result in faster convergence. In this paper, we consider the shift-and-invert technique of weighted Golub-Kahan-Lanczos bidiagonalization algorithms. Since we are particularly interested in the smallest eigenvalues with the positive sign of $\mathbf{H}$, thus $\sigma = 0$ is often an obvious choice.

The paper is organized as follows. In section 2, we will give an outline of wGKL$_u$-LREP. The shift-and-invert version of wGKL$_u$-LREP will be described in section 3. In section 4, some numerical examples are illustrated the numerical behavior of our new algorithm. In the end, the conclusion will be given in section 5.

2 Preliminary

In this section, we will give some preliminary of the weighted Golub-Kahan-Lanczos upper bidiagonalization algorithm (wGKL$_u$) and its application algorithm (wGKL$_u$-LREP) for Linear response eigenvalue problem. Lemma 2.1 [21] is the basic theory of the above algorithms.

Lemma 2.1. *Suppose $0 < K, M \in \mathbb{C}^{n\times n}$. Then there exist an M-orthogonal matrix $X \in \mathbb{C}^{n\times n}$ and a K-orthogonal matrix $Y \in \mathbb{C}^{n\times n}$ such that*

$$MX = YB, \quad KY = XB^T, \tag{2.1}$$

where B is upper bidiagonal.

Let $X = [x_1, \cdots, x_n]$, $Y = [y_1, \cdots, y_n]$, and

$$B = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \alpha_2 & \ddots & \\ & & \ddots & \beta_{n-1} \\ & & & \alpha_n \end{bmatrix},$$

then from Lemma 2.1, wGKL$_u$ can be described as follows.

Algorithm 1 (wGKL$_u$)**.**

Choose x_1 satisfying $\|x_1\|_M = 1$, and set $\beta_0 = 1$, $y_0 = 0$. Compute $g_1 = Mx_1$.
For $j = 1, 2, \cdots$
$s_j = g_j/\beta_{j-1} - \beta_{j-1}y_{j-1}$

$f_j = Ks_j$
$\alpha_j = (s_j^T f_j)^{\frac{1}{2}}$
$y_j = s_j/\alpha_j$
$t_{j+1} = f_j/\alpha_j - \alpha_j x_j$
$g_{j+1} = Mt_{j+1}$
$\beta_j = (t_{j+1}^T g_{j+1})^{\frac{1}{2}}$
$x_{j+1} = t_{j+1}/\beta_j$
End

Suppose Algorithm 1 runs k iterations, we have the following relation

$$MX_k = Y_k B_k, \qquad KY_k = X_k B_k^T + \beta_k x_{k+1} e_k^T = X_{k+1} \begin{bmatrix} B_k & \beta_k e_k \end{bmatrix}^T, \tag{2.2}$$

and

$$X_k^H M X_k = I_k = Y_k^H K Y_k. \tag{2.3}$$

Define

$$\mathbf{X}_j = \begin{bmatrix} X_j & 0 \\ 0 & Y_j \end{bmatrix} \quad \mathbf{B}_j = \begin{bmatrix} 0 & B_j^T \\ B_j & 0 \end{bmatrix}.$$

Then from (2.2) and (2.3), we obtain

$$\mathbf{H}\mathbf{X}_k = \mathbf{X}_k \mathbf{B}_k + \beta_k \begin{bmatrix} x_{k+1} \\ 0 \end{bmatrix} e_{2k}^T \tag{2.4}$$

with $\mathbf{X}_k^H \mathbf{M} \mathbf{X}_k = I_k$, here $\mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}$.

Consequently, the first l smallest positive eigenvalues of $\mathbf{H}$ together with their corresponding eigenvectors can be approximately constructed from $\mathbf{B}_k$, which is obviously symmetric.

Since K and M are hermitian positive definite, all eigenvalues of KM (and MK) are real and positive. Denote these eigenvalues by λ_i^2 $(1 \le i \le n)$ in descending order, i.e.,

$$\lambda_1^2 \ge \lambda_2^2 \ge \cdots \ge \lambda_n^2 \ge 0,$$

where all $\lambda_i \ge 0$ and thus $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. From Theorem 2.1 [1], we know the eigenvalues of H are $\pm\lambda_i$, $1 \le i \le n$.

Suppose B_k has an SVD

$$B_k = \Phi_k \Sigma_k \Psi_k^T, \tag{2.5}$$

where $\Phi_k = [\phi_1, \cdots, \phi_k] \in \mathbb{R}^{k\times k}$, $\Psi_k = [\psi_1, \cdots, \psi_k] \in \mathbb{R}^{k\times k}$, $\Sigma_k = diag(\sigma_1, \cdots, \sigma_k)$, with $\sigma_1 \ge \cdots \ge \sigma_k > 0$, $\Phi_k^T \Phi_k = I_k$ and $\Psi_k^T \Psi_k = I_k$, then from (2.4), by using an orthogonal matrix $J = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ I_k & -I_k \end{bmatrix}$, the following equation is hold

$$\begin{aligned} \mathbf{H}\frac{1}{\sqrt{2}} \begin{bmatrix} X_k\Psi_k & X_k\Psi_k \\ Y_k\Phi_k & -Y_k\Phi_k \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} X_k\Psi_k & X_k\Psi_k \\ Y_k\Phi_k & -Y_k\Phi_k \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & -\Sigma_k \end{bmatrix} \\ &+ \frac{\beta_k}{\sqrt{2}} \begin{bmatrix} x_{k+1} \\ 0 \end{bmatrix} e_{2k}^T \begin{bmatrix} \Psi_k & \Psi_k \\ \Phi_k & -\Phi_k \end{bmatrix}. \end{aligned}$$

Thus we may take $\pm\sigma_1, \cdots, \pm\sigma_k$ as Ritz values of $\mathbf{H}$ and

$$\hat{\mathbf{z}}_j^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} X_k\psi_j \\ \pm Y_k\phi_j \end{bmatrix}, \quad j = 1, \ldots, k$$

as corresponding $\mathbf{M}$-orthonormal right Ritz vectors. Meanwhile, using the residual norm

$$\|\mathbf{H}\hat{\mathbf{z}}_j^{\pm} \pm \sigma_j \hat{\mathbf{z}}_j^{\pm}\|_{\mathbf{M}} = \frac{\beta_k|\phi_{jk}|}{\sqrt{2}}$$

as the stopping criterion, here ϕ_{jk} is the kth component of ϕ_j.

Algorithm 2 (wGKL$_u$-LREP)**.**

1. *Run k steps of Algorithm 1 with an initial x_1 satisfying $\|x_1\|_M = 1$ and an appropriate integer k to generate B_k, X_k, and Y_k;*
2. *Compute an SVD of B_k as in (2.5), select $l(\leq k)$ smallest singular value σ_j, and the associated left and right singular vector ϕ_j and ψ_j, $j = 1, \cdots, l$;*
3. *Form σ_j, $\hat{\mathbf{z}}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} X_k\psi_j \\ Y_k\phi_j \end{bmatrix}$, $j = 1, \cdots, l$;*
4. *If $\beta_k = 0$, break.*

3 Shift and invert weighted Golub-Kahan-Lanczos bidiagonalization algorithm

Usually, the first l smallest positive eigenvalues λ_i of $\mathbf{H}$ for $i = 1, 2, \cdots, l$ are of interest. They lie in the middle of the spectrum of $\mathbf{H}$, and often crowd together. Thus it is necessary to present an accelerating strategy for wGKL$_u$ when applying it for linear response eigenvalue problem. In this section, we will propose a shift-and-invert version of wGKL$_u$ for solving the eigenproblems of $\mathbf{H}$.

Choosing a shift σ, the shift-and-invert strategy is simply transformed the original problem $Ax = \lambda x$ into $(A-\sigma I)^{-1}x = \alpha x$. The simplest possible scheme is to run Arnoldi's method on the matrix $(A - \sigma I)^{-1}$. Thus, the eigenvalue of the original problem is $\lambda = \frac{1}{\alpha} + \sigma$, the eigenvectors of A and $(A - \sigma I)^{-1}$ are identical.

For linear response eigenvalue problem $\mathbf{Hz} = \lambda\mathbf{z}$, where $\mathbf{H}$ is from (1.1). As the above discussion, using the shift-and-invert strategy, is running the weighted Golub-Kahan-Lanczos upper bidiagonalization algorithm(wGKL$_u$) on matrix $(\mathbf{H} - \sigma I)^{-1}$. Since we are interested in the smallest eigenvalues with the positive sign of $\mathbf{H}$, thus $\sigma = 0$ is often an obvious choice. It is clear that the inverse matrix of $\mathbf{H}$ is $\mathbf{H}^{-1} = \begin{bmatrix} 0 & M^{-1} \\ K^{-1} & 0 \end{bmatrix}$. Because K^{-1} and M^{-1} are also both hermitian definite, thus we can directly apply wGKL$_u$ to $\mathbf{H}^{-1}$. Theorem 3.1 gives the theoretical relations of our new algorithm. Here, we still use the same denotation without misunderstanding.

Theorem 3.1. *Suppose $0 < K$, $M \in \mathbb{C}^{n\times n}$. Then there exist an M^{-1}-orthogonal matrix $X \in \mathbb{C}^{n\times n}$ and a K^{-1}-orthogonal matrix $Y \in \mathbb{C}^{n\times n}$ such that*

$$M^{-1}X = YB, \quad K^{-1}Y = XB^T, \tag{3.1}$$

where B is upper bidiagonal.

Proof. Since K, $M > 0$, then K^{-1}, $M^{-1} > 0$. Suppose $K^{-1} = LL^H$, $M^{-1} = RR^H$ are the Cholesky decomposition of K^{-1} and M^{-1}. From [7], we can assume

$$L^H R = UBV^H, \tag{3.2}$$

where $U, V \in \mathbb{C}^{n\times n}$ are unitary matrices, B is upper bidiagonal. Thus let $X = R^{-H}V$, $Y = L^{-H}U$, from (3.2), we have

$$L^H RR^H X = L^H YB, \quad R^H LL^H Y = R^H XB^T.$$

By multiplying L^{-H} and R^{-H}, respectively, and (3.1) holds obviously. Clearly, $X^H M^{-1} X = I$, $Y^H K^{-1} Y = I$. □

From Theorem 3.1, we can get the following algorithm.

Algorithm 3 (wGKL$_u$ on $\mathbf{H}^{-1}$).

Choose x_1 satisfying $\|x_1\|_{M^{-1}} = 1$, and set $\beta_0 = 1$, $y_0 = 0$. Compute $g_1 = M^{-1}x_1$.
For $j = 1, 2, \cdots$

$$\begin{aligned}
&s_j = g_j/\beta_{j-1} - \beta_{j-1}y_{j-1}\\
&f_j = K^{-1}s_j\\
&\alpha_j = (s_j^T f_j)^{\frac{1}{2}}\\
&y_j = s_j/\alpha_j\\
&t_{j+1} = f_j/\alpha_j - \alpha_j x_j\\
&g_{j+1} = M^{-1}t_{j+1}\\
&\beta_j = (t_{j+1}^T g_{j+1})^{\frac{1}{2}}\\
&x_{j+1} = t_{j+1}/\beta_j
\end{aligned}$$

End

Remark 1. *In Algorithm 3, we need to solve linear system $Kf = s$ and $Mg = t$. Here we use LU decomposition to solve it. After lots of experiments, we found it is not suitable to use iterative methods to solve these linear system, because iterative methods are not the exact methods generally. Even LU decomposition is an accurate method for linear system problems, but it will encounter some problems, such as more time and more memory, especially for large scale problems. Fortunately, because we transform the interior eigenproblem to the exterior eigenproblem, thus compared to the methods in the numerical examples, our algorithm still shows its superiority.*

Let X_k, Y_k, B_k be generated by Algorithm 3 after k iterations, we have

$$M^{-1}X_k = Y_k B_k, \qquad K^{-1}Y_k = X_k B_k^T + \beta_k x_{k+1} e_k^T = X_{k+1}\left[\begin{array}{cc} B_k & \beta_k e_k \end{array}\right]^T, \tag{3.3}$$

and

$$X_k^H M^{-1} X_k = I_k = Y_k^H K^{-1} Y_k. \tag{3.4}$$

Define

$$\mathbf{Y}_j = \begin{bmatrix} Y_j & 0 \\ 0 & X_j \end{bmatrix} \quad \mathbf{B}_j = \begin{bmatrix} 0 & B_j \\ B_j^T & 0 \end{bmatrix}.$$

Then from (3.3) and (3.4), one has

$$\mathbf{H}^{-1}\mathbf{Y}_k = \mathbf{Y}_k \mathbf{B}_k + \beta_k \begin{bmatrix} 0 \\ x_{k+1} \end{bmatrix} e_k^T, \tag{3.5}$$

with $\mathbf{Y}_k^H \mathbf{K} \mathbf{Y}_k = I_{2k}$, where $e_k = I_{2k}(:,k)$, $\mathbf{K} = \begin{bmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}$.

Similar as the discussion in section 2, suppose B_k has an SVD

$$B_k = \Phi_k \Sigma_k \Psi_k^T, \tag{3.6}$$

where $\Phi_k = [\phi_1, \cdots, \phi_k]$, $\Psi_k = [\psi_1, \cdots, \psi_k]$, $\Sigma_k = diag\{\sigma_1, \cdots, \sigma_k\}$, with $\sigma_1 \geq \cdots \geq \sigma_k > 0$, $\Phi_k^T \Phi_k = I_k$, $\Psi_k^T \Psi_k = I_k$. From (3.5), we may take $\pm\sigma_1, \ldots, \pm\sigma_k$ as Ritz values of $\mathbf{H}^{-1}$, i.e., approximate eigenvalues of $\mathbf{H}^{-1}$,

$$\hat{\mathbf{z}}_j^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} Y_k \phi_j \\ \pm X_k \psi_j \end{bmatrix} \quad j = 1, \ldots, k,$$

as corresponding $\mathbf{K}$-orthonormal Ritz vectors. Meanwhile, using the residual norm

$$\|\mathbf{H}^{-1} \hat{\mathbf{z}}_j^{\pm} \pm \sigma_j \hat{\mathbf{z}}_j^{\pm}\|_{\mathbf{K}} = \frac{\beta_k |\psi_{jk}|}{\sqrt{2}} \tag{3.7}$$

as the stopping criterion, here ψ_{jk} is the kth component of ψ_j. Consequently, $\pm\frac{1}{\sigma_1}, \ldots, \pm\frac{1}{\sigma_k}$ are approximate eigenvalues of $\mathbf{H}$, $\hat{\mathbf{z}}_j^{\pm}$, $j = 1, \ldots, k$, are the corresponding approximate eigenvectors.

The following is the shift-and-invert version of wGKL$_u$ for solving LREP of $\mathbf{H}$.

Algorithm 4 (Shift-and-invert-wGKL$_u$-LREP)**.**

1. *Run k steps of Algorithm 3 with an initial x_1 satisfying $\|x_1\|_{M^{-1}} = 1$ and an appropriate integer k to generate B_k, X_k, and Y_k;*
2. *Compute an SVD of B_k as in (3.6), select $l(\leq k)$ largest singular value σ_j, and the associated left and right singular vector ϕ_j and ψ_j, $j = 1, \cdots, l$;*
3. *Form $\frac{1}{\sigma_j}$, $\hat{\mathbf{z}}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} Y_k \phi_j \\ X_k \psi_j \end{bmatrix}$, $j = 1, \cdots, l$;*
4. *If $\beta_k = 0$, break.*

Remark 2. *Generally, (3.7) is hold for the approximate eigenpairs $(\sigma_j, \hat{\mathbf{z}}_j)$ of $\mathbf{H}^{-1}$, but not $\mathbf{H}$. While, we need to solve the approximate eigenpairs of $\mathbf{H}$. Thus for fairness and accuracy, we don't use (3.7) as the stopping criterion in actual algorithm, instead, we take normalized 1-norm of the residual. It will be elaborated in numerical examples.*

4 Convergence analysis

5 Numerical examples and results

In this section, we test Algorithm 2 (wGKL$_u$-LREP) and Algorithm 4 (Shift-and-invert-wGKL$_u$-LREP) with several numerical examples for solving the eigenvalue problem of $\mathbf{H}$, where the initial vector are $x_1/\|x_1\|_M$ and $x_1/\|x_1\|_{M^{-1}}$, respectively, here, x_1 is randomly selected. The numerical results are labeled with Alg-3 and Alg-4 respectively. In fact, Alg-4 is Alg-3 added with the precondition strategy, it's the accelerated version of Alg-3. For comparison we tested the first algorithm presented in [18] with the initial vector $x_1/\|x_1\|_2$. The numerical results are labeled with Alg-TL. We also tested the block Chebyshev-Davidson method (BChevbyDLR) presented in [19], and the locally optimal block preconditioned 4-D CG method (LOBP4DCG) in [2]. The experiments have been carried out in double precision (Digits=64) floating point arithmetic in Matlab R2014a with a PC-Intel(R)Core(TM)i5-6200U CPU 2.4GHz, 8GB RAM.

The same as in [19], for the LOBP4DCG method, we use the generic preconditioner

$$\Phi = \mathbf{H}^{-1} = \begin{pmatrix} 0 & K^{-1} \\ M^{-1} & 0 \end{pmatrix}.$$

The preconditioned search vectors q_i and p_i in [2] are computed by using the linear CG method [5] with maximal 5 iterations. The initial block size in BChevbyDLR and LOBP4DCG are chosen to be l, the methods are denoted by BChevbyDLR(l), and LOBP4DCG(l), respectively.

We only compute the approximate eigenvalues with positive sign. For illustrating the quality of computed approximations, we report the normalized residual 1-norms for the jth approximate eigenpair $(\sigma_j, \hat{\mathbf{z}}_j^+)$:

$$r(\sigma_j) := \frac{\|\mathbf{H}\hat{\mathbf{z}}_j^+ - \sigma_j \hat{\mathbf{z}}_j^+\|_1}{(\|\mathbf{H}\|_1 + \sigma_j)\|\hat{\mathbf{z}}_j^+\|_1},$$

if $r(\sigma_j) \leq tol = 10^{-8}$, the eigenpair $(\sigma_j, \hat{\mathbf{z}}_j^+)$ is considered as converged. The "exact" eigenvalues λ_j are computed with MATLAB code *eig*.

In this example, we tested the above algorithms with five problems. Table 1 lists the composed 5 problems. The matrices K and M of Test 1 come from the linear response analysis for Na2, which is generated by the turboTDDFT code in QUANTUM ESPRESSO-an electronic structure calculation code that implements density functional theory (DFT) using plane-waves as the basis set and pseudopotentials [6, 18]. The matrices K and M of the other test, are extracted from the University of Florida sparse matrix collection [4]. All K and M are symmetric positive definite.

We compute the first 10 smallest approximate eigenvalues with positive sign. For block size l of BChebyDLR(l), we choose l as 5 and 10. For LOBP4DCG, we set 10 as the initial block size. The two algorithms are both applied with a deflation procedure. We report the total number of matrix-vetor products (denoted by "MV"), iteration number (denoted by "iter"), and CPU time in seconds. And we count the $K^{-1}y$ or $M^{-1}x$ in Alg.4 as one matrix-vector products. The numerical results are listed in Table 1 and 2. "–" denotes the algorithm didn't converged in 1000 iterations.

From Table 2, we can see, since Alg-3 and Alg-TL didn't use any acceleration strategy, thus they can't converge within 1000 iterations. Alg-4 converged faster than the other algorithms, because of the least number of matrix-vector products, and this phenomenon also happens in some other tests not reported here, where the matrices K and M have a relatively large condition number. However, we also observe that for some other problems not reported here, where most of the K and M are both well-conditioned, even though Alg-4 used the least number of matrix-vector products, much lower than BChebyDLR and LOBP4DCG, it still converged slower than BChebyDLR and LOBP4DCG.

There are three main reasons for this phenomenon. The first is that BChebyDLR and LOBP4DCG are both block type methods, while Alg-4 is not. Usually block type methods with relatively small block sizes are more competitive than non-block versions, especially when the desired eigenvalues have clusters or even multiples. The second reason is that we use Cholesky decomposition of K and M to solve $K^{-1}y$ and $M^{-1}x$, while K and M are very sparse, their Cholesky factor may be a full lower triangular matrix, which will cost much time to solve. Thus in Alg-4, the CPU time used for one matrix-vector products must be more than the time used in BChebyDLR and LOBP4DCG. Consequently, it is necessary to consider a inverse free precondition strategy to accelerate Alg-3. The third reason is that BChebyDLR method refined the basis matrices at every step, which can make eigenvectors converge in fewer iterations [13, 17], since the refined basis matrices contains the information of the wanted eigenvectors. While in Alg-4, we don't use any refined restart. Above all, Further reseach is required to make Alg-4 more effective.

Table 1 Test problems

Problem	Test 1	Test 2	Test 3	Test 4	Test 5
n	1862	8032	9801	23052	73752
K	Na2	bcsstk38	fv2	bcsstk36	oilpan
M	Na2	msc23052	fv3	bcsstk36	oilpan

Table 2

		Alg-4	BChebyshev(5)	BChebyshev(10)	LOBP4DCG(10)	Alg-3	Alg-TL
	MV	240	4680	6760	4592	–	–
Test 1	iter	19	18	13	47	–	–
	CPU	0.905	3.3906	2.3696	8.7745	–	–
	MV	42	–	–	6824	–	–
Test 2	iter	10	–	–	40	–	–
	CPU	0.6080	–	–	3.5316	–	–
	MV	42	10920	24440	5114	–	–
Test 3	iter	10	42	41	50	–	–
	CPU	0.5531	1.5495	2.8754	2.2148	–	–
	MV	214	–	–	–	–	–
Test 4	iter	18	–	–	–	–	–
	CPU	14.3342	–	–	–	–	–
	MV	42	–	–	–	–	–
Test 5	iter	10	–	–	–	–	–
	CPU	45.4148	–	–	–	–	–

Example 2: The number of matrix-vector products (MV), number of iterations (iter), and CPU time in seconds for computing 10 smallest positive eigenpairs. For BChebyDLR(l) the filter degree used is 25, and the block size is $l = 5, 10$. For LOBP4DCG(l) the initial block size $l = 10$. Here "–" stands for the algorithm does not converge within 1000 iterations.

6 Conclusion

We propose a shift-and-invert weighted Golub-Kahan-Lanczos bidiagonal algorithm for solving the linear response eigenproblem(LREP). This algorithm can effectively calculate the smallest positive eigenvalues and associated eigenvectors of LREP. Numerical examples show that our new algorithm can appears faster than Alg.TL, BChebyDLR and LOBP4DCG, especially for the case of K and M have a relatively large condition number.

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Qualitative Study of Solution of Some Higher Order Difference Equations

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ABSTRACT

This paper is mindful with the solution of the nonlinear difference equation

$$x_{n+1}=\frac{x_{n-1}x_{n-6}}{x_{n-4}(\pm 1\pm x_{n-1}x_{n-6})},\quad n=0,1,...,$$

where the initial conditions $x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers and we study the behaviors of the solutions. Also, we gained the equilibrium points of the previous equations.

Keywords: stability, periodicity, solution of difference equation.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

In this paper we deal with the behavior of the solution of the following difference equations

$$x_{n+1}=\frac{x_{n-1}x_{n-6}}{x_{n-4}(\pm 1\pm x_{n-1}x_{n-6})},\quad n=0,1,..., \tag{1.1}$$

where the initial conditions $x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers.

Here, we display some basic definitions and some theorems which will be beneficial in our research.

Let I be some interval of real numbers and let $f: I^{k+1}\to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-k+1},...,\ x_0\in I$, the difference equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}),\quad n=0,1,..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [39].

Definition 1.1. (Equilibrium Point)

A point $\overline{x}\in I$ is called an equilibrium point of Eq.(1.2) if

$$\overline{x}=f(\overline{x},\overline{x},...,\overline{x}).$$

That is, $x_n=\overline{x}$ for $n\geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon>0$, there exists $\delta>0$ such that for all $x_{-k},\ x_{-k+1},...,\ x_{-1},\ x_0\in I$ with

$$|x_{-k}-\overline{x}|+|x_{-k+1}-\overline{x}|+...+|x_0-\overline{x}|<\delta,$$

we have

$$|x_n-\overline{x}|<\epsilon \quad \text{ for all } \quad n\geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(1.2) and there exists $\gamma>0$, such that for all $x_{-k},x_{-k+1},...,x_{-1},x_0\in I$ with

$$|x_{-k}-\overline{x}|+|x_{-k+1}-\overline{x}|+...+|x_0-\overline{x}|<\gamma,$$

we have

$$\lim_{n\to\infty} x_n=\overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k},x_{-k+1},...,x_{-1},x_0\in I$, we have

$$\lim_{n\to\infty} x_n=\overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).

(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1}=\sum_{i=0}^{k}\frac{\partial f(\overline{x},\overline{x},...,\overline{x})}{\partial x_{n-i}}y_{n-i}.$$

Theorem A [38]: Assume that $p,q\in R$ and $k\in\{0,1,2,...\}$. Then

$$|p|+|q|<1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1}+px_n+qx_{n-k}=0,\quad n=0,1,...\ .$$

Remark: Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k}+p_1x_{n+k-1}+...+p_kx_n=0,\ \ n=0,1,..., \tag{1.3}$$

where $p_1,p_2,...,p_k\in R$ and $k\in\{1,2,...\}$. Then Eq.(1.3) is asymptotically stable provided that

$$\sum_{i=1}^{k}|p_i|<1.$$

Definition 1.3. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.

In recent years, the study of difference equations has acquired a new significance, due in large part to their use in the formulation and analysis of discrete-time systems and the study of deterministic chaos.

However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. From the known work, one can see that it is so complicated to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [1], [5–14] for examples to illustrate this. Therefore, the study of rational difference equations of order greater than one is worth further consideration. The behavior of solutions differential equations has been studied by many researchers for example:

El-Metwally and Elsayed [9] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(\pm 1 \pm x_n x_{n-3})}.$$

Elsayed [13] studied the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm \alpha x_{n-1} x_{n-3} x_{n-5} x_{n-7}}.$$

Cinar [2]-[3] has got the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + a x_n x_{n-1}}.$$

In [4], Cinar and Yalcinkaya studied the behavior of the following difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Elabbasy et al. [6] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

In [29] Erdogan and Uslu investigated the global behavior of the following recursive sequence

$$x_{n+1} = \frac{1 - x_n}{A + \sum\limits_{i=1}^{k} x_{n-i}}.$$

Karatas et al. [35] gave that the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

See also [15]-[37]. Other related results on rational difference equations can be found in refs. [40]–[51].

2. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(1 + X_{N-1}X_{N-6}))$

In this section we realize a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1} x_{n-6}}{x_{n-4}(1 + x_{n-1} x_{n-6})}, \quad n = 0, 1, ..., \tag{2.1}$$

where the initial values are arbitrary positive real numbers.

Theorem 2.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(2.1). Then for $n = 0, 1, 2, ...$

$$x_{10n-6} = \frac{a^n f^n (\prod\limits_{i=1}^{n} [(5i)bg + 1])}{b^n g^{n-1} (\prod\limits_{i=1}^{n} [(5i-3)af + 1])}, \qquad x_{10n-5} = \frac{b^n g^n (\prod\limits_{i=0}^{n-1} [(5i)af + 1])}{a^n f^{n-1} (\prod\limits_{i=0}^{n-1} [(5i+3)bg + 1])},$$

$$x_{10n-4} = \frac{a^n f^n e (\prod\limits_{i=0}^{n-1} [(5i+1)bg + 1])}{b^n g^n (\prod\limits_{i=0}^{n-1} [(5i+3)af + 1])}, \qquad x_{10n-3} = \frac{b^n g^n d (\prod\limits_{i=0}^{n-1} [(5i+1)af + 1])}{a^n f^n (\prod\limits_{i=0}^{n-1} [(5i+4)bg + 1])},$$

$$x_{10n-2} = \frac{a^n f^n c\left(\prod_{i=1}^{n}[(5i-3)bg+1]\right)}{b^n g^n\left(\prod_{i=0}^{n-1}[(5i+4)af+1]\right)}, \qquad x_{10n-1} = \frac{g^n b^{n+1}\left(\prod_{i=1}^{n}[(5i-3)af+1]\right)}{a^n f^n\left(\prod_{i=1}^{n}[(5i)bg+1]\right)},$$

$$x_{10n} = \frac{a^{n+1} f^n\left(\prod_{i=1}^{n-1}[(5i+3)bg+1]\right)}{b^n g^n\left(\prod_{i=1}^{n}[(5i)af+1]\right)}, \qquad x_{10n+1} = \frac{b^{n+1} g^{n+1}\left(\prod_{i=0}^{n-1}[(5i+3)af+1]\right)}{a^n f^n e\left(\prod_{i=0}^{n}[(5i+1)bg+1]\right)},$$

$$x_{10n+2} = \frac{a^{n+1} f^{n+1}\left(\prod_{i=0}^{n-1}[(5i+4)bg+1]\right)}{b^n g^n d\left(\prod_{i=0}^{n}[(5i+1)af+1]\right)}, \qquad x_{10n+3} = \frac{b^{n+1} g^{n+1}\left(\prod_{i=0}^{n-1}[(5i+4)af+1]\right)}{a^n f^n c\left(\prod_{i=0}^{n}[(5i+2)bg+1]\right)},$$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$x_{10n-16} = \frac{a^{n-1} f^{n-1}\left(\prod_{i=1}^{n-2}[(5i)bg+1]\right)}{b^{n-1}\ g^{n-2}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)}, \qquad x_{10n-15} = \frac{b^{n-1} g^{n-1}\left(\prod_{i=1}^{n-2}[(5i)af+1]\right)}{a^{n-1}\ f^{n-2}\left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)},$$

$$x_{10n-14} = \frac{a^{n-1} f^{n-1}\ e\left(\prod_{i=0}^{n-2}[(5i+1)bg+1]\right)}{b^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+3)af+1]\right)}, \qquad x_{10n-13} = \frac{b^{n-1} g^{n-1}\ d\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)},$$

$$x_{10n-12} = \frac{a^{n-1} f^{n-1}\ c\left(\prod_{i=1}^{n-1}[(5i-3)bg+1]\right)}{b^{n-1}\ g^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}, \qquad x_{10n-11} = \frac{b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)}{a^{n-1} f^{n-1}\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)},$$

$$x_{10n-10} = \frac{a^n f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)}{b^{n-1} g^{n-1}\left(\prod_{i=1}^{n-1}[(5i)af+1]\right)}, \qquad x_{10n-9} = \frac{b^n g^n\left(\prod_{i=0}^{n-2}[(5i+3)af+1]\right)}{a^{n-1} f^{\ n-1} e\left(\prod_{i=0}^{n-1}[(5i+1)bg+1]\right)},$$

$$x_{10n-8} = \frac{a^n f^n\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}{b^{n-1} g^{n-1} d\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}, \qquad x_{10n-7} = \frac{b^n g^n\left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}{a^{n-1} f^{n-1} c\left(\prod_{i=0}^{n-1}[(5i+2)bg+1]\right)}.$$

Now, it follows from Eq.(2.1) that

$$\begin{aligned} x_{10n-6} &= \frac{x_{10n-8}x_{10n-13}}{x_{10n-11}(1+x_{10n-8}x_{10n-13})} \\ &= \frac{\left(\frac{a^n f^n\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}{b^{n-1}\ g^{n-1} d\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(\frac{b^{n-1} g^{n-1}\ d\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}\right)}{\left(\frac{b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}\right)\left(1+\frac{a^n f^n\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}{b^{n-1}\ g^{n-1} d\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\ \frac{b^{n-1} g^{n-1}\ d\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{a^{n-1}\ f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)bg+1]\right)}\right)} \end{aligned}$$

$$= \frac{\left(\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(a^{n-1}f^{n-1}\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)\right)}{\left(b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\right)\left(1+\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)}$$

$$= \frac{\left(\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(a^{n-1}f^{n-1}\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)\right)}{\left(b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\right)\left(\frac{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)+af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)}$$

$$= \frac{a^n f^n\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}{b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)\left([5(n-1)+1]af+1+af\right)}$$

$$= \frac{\left(\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)\left(a^{n-1}f^{n-1}\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)\right)}{\left(b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)\right)\left(1+\frac{af\left(\prod_{i=0}^{n-2}[(5i+1)af+1]\right)}{\left(\prod_{i=0}^{n-1}[(5i+1)af+1]\right)}\right)}$$

$$= \frac{a^n f^n\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}{b^n g^{n-1}\left(\prod_{i=1}^{n-1}[(5i-3)af+1]\right)((5n-3)af+1)} = \frac{a^n f^n\left(\prod_{i=1}^{n-1}[(5i)bg+1]\right)}{b^n g^{n-1}\left(\prod_{i=1}^{n}[(5i-3)af+1]\right)}.$$

Similarly

$$x_{10n-5} = \frac{x_{10-7}x_{10n-12}}{x_{10n-10}(1+x_{10-7}x_{10n-12})}$$

$$= \frac{\left(\frac{b^n g^n\left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}{a^{n-1}f^{n-1}c\left(\prod_{i=0}^{n-1}[(5i+2)bg+1]\right)}\right)\left(\frac{a^{n-1}f^{n-1}c\left(\prod_{i=1}^{n-1}[(5i-3)bg+1]\right)}{b^{n-1}g^{n-1}\left(\prod_{i=0}^{n-2}[(5i+4)af+1]\right)}\right)}{\left(\frac{a^n f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)}{b^{n-1}g^{n-1}\left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}\right)\left(1+\frac{bg\left(\prod_{i=1}^{n-1}[(5i-3)bg+1]\right)}{\left(\prod_{i=1}^{n}[(5i-3)bg+1]\right)}\right)}$$

$$= \frac{b^n g^n\left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\left(\prod_{i=0}^{n-2}[(5i+3)bg+1]\right)((5n-2)bg+1)} = \frac{b^n g^n\left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\left(\prod_{i=1}^{n-1}[(5i-2)bg+1]\right)((5n-2)bg+1)}$$

$$= \frac{b^n g^n\left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\left(\prod_{i=1}^{n}[(5i-2)bg+1]\right)} = \frac{b^n g^n\left(\prod_{i=0}^{n-1}[(5i)af+1]\right)}{a^n f^{n-1}\left(\prod_{i=0}^{n-1}[(5i+3)bg+1]\right)}.$$

The other relations can be proved similarly. Hence, the proof is completed.

Theorem 2.2. Eq.(2.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Proof: For the equilibrium points on Eq.(2.1), we can say

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}(1+\overline{x}^2)},$$

then, we get $\overline{x}^4 = 0$. Therefore, the equilibrium point of Eq.(2.1) is $\overline{x} = 0$. Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by $f(u,v,w) = \frac{uw}{v(1+uw)}$. We see that

$$f_u(u,v,w) = \frac{w}{v(1+uw)^2}, \quad f_v(u,v,w) = -\frac{uw}{v^2(1+uw)}, \quad f_w(u,v,w) = \frac{u}{v(1+uw)^2}.$$

Consequently,

$$f_u(\bar{x},\bar{x},\bar{x}) = 1, \ f_v(\bar{x},\bar{x},\bar{x}) = 1, \ f_w(u,v,w) = 1.$$

The proof follows by using Theorem A.

Numerical Examples:

For confirming the results of this section, we consider numerical examples which represent different type of solutions to Eq. (2.1).

Example 2.3. We take $x_{-6} = -7,\ x_{-5} = 1.5,\ x_{-4} = -3,\ x_{-3} = 2,\ x_{-2} = 12,\ x_{-1} = 2/7,\ x_0 = 9$. (See figure 1).

Example 2.4. See figure 2, since $x_{-6} = 2.1,\ x_{-5} = 4,\ x_{-4} = 3,\ x_{-3} = .8,\ x_{-2} = 1.2,\ x_{-1} = 7,\ x_0 = 4$.

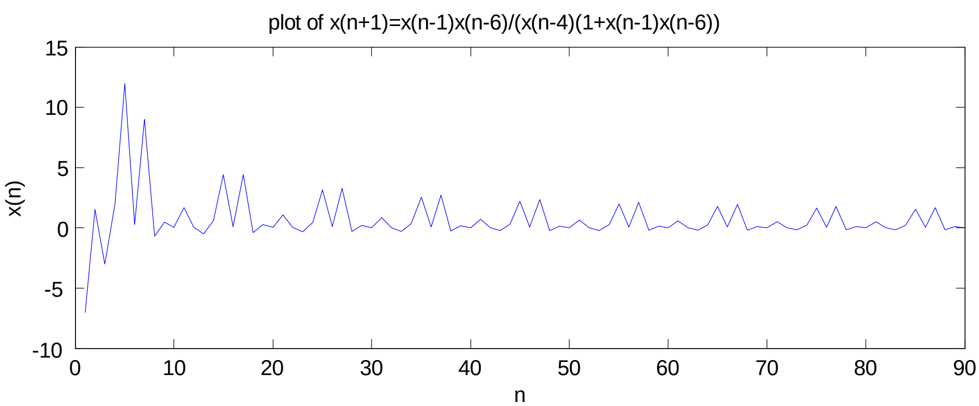

Figure 1.

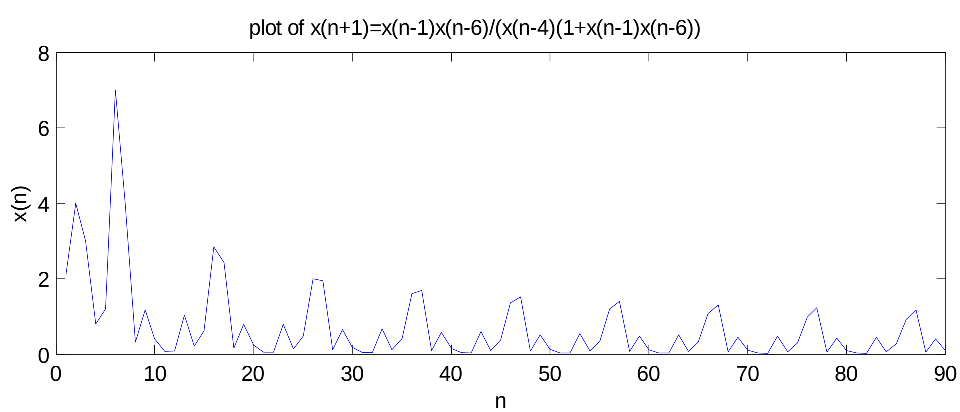

Figure 2.

3. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(1 - X_{N-1}X_{N-6}))$

In this section we obtain a specific form of the solution of the second equation in the following form :

$$x_{n+1} = \frac{x_{n-1}x_{n-6}}{x_{n-4}(1 - x_{n-1}x_{n-6})}, \quad n = 0, 1, ..., \tag{3.1}$$

where the initial values are arbitrary nonzero real numbers with $x_{-1}x_{-6} \neq 1$.

Theorem 3.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(3.1). Then for $n = 0, 1, 2, ...$

$$x_{10n-6} = \frac{a^n f^n (1 - \prod_{i=1}^{n-1}(5i)bg)}{b^n\ g^{n-1}\ (1 - \prod_{i=1}^{n}(5i-3)af)}, \qquad x_{10n-5} = \frac{b^n g^n (1 - \prod_{i=1}^{n-1}(5i)af)}{a^n\ f^{n-1}(\ 1 - \prod_{i=0}^{n-1}(5i+3)bg)},$$

$$x_{10n-4} = \frac{a^n f^n e\ (1 - \prod_{i=0}^{n-1}(5i+1)bg)}{b^n\ g^n\ (1 - \prod_{i=0}^{n-1}(5i+3)af)}, \qquad x_{10n-3} = \frac{b^n g^n d\ (1 - \prod_{i=1}^{n-1}(5i+1)af)}{a^n f^{\ n}(1 - \prod_{i=0}^{n-1}(5i+4)bg)},$$

$$x_{10n-2} = \frac{a^n f^n\ c\left(1 - \prod_{i=1}^{n}(5i-3)bg\right)}{b^n\ g^n\ \left(1 - \prod_{i=0}^{n-1}(5i+4)af\right)}, \qquad x_{10n-1} = \frac{g^n b^{n+1}\ \left(1 - \prod_{i=1}^{n}(5i-3)af\right)}{a^n\ f^n\ \left(1 - \prod_{i=1}^{n}(5i)bg\right)},$$

$$x_{10n} = \frac{a^{n+1} f^n\ \left(1 - \prod_{i=1}^{n-1}(5i+3)bg\right)}{b^n\ g^n\ \left(1 - \prod_{i=1}^{n}(5i)af\right)}, \qquad x_{10n+1} = \frac{b^{n+1} g^{n+1}\ \left([1 - \prod_{i=0}^{n-1}(5i+3)]af\right)}{a^n\ f^n\ e\left([1 - \prod_{i=0}^{n}(5i+1)]bg\right)},$$

$$x_{10n+2} = \frac{a^{n+1} f^{n+1}\ \left([1 - \prod_{i=0}^{n-1}[(5i+4)]bg\right)}{b^n\ g^n d\ \left([1 - \prod_{i=0}^{n}[(5i+1)]af\right)}, \qquad x_{10n+3} = \frac{b^{n+1} g^{n+1}\ \left([1 - \prod_{i=0}^{n-1}[(5i+4)]af\right)}{a^n\ f^n c\ \left([1 - \prod_{i=0}^{n}[(5i+2)]bg\right)}.$$

Proof: The proof as in the previous section so it will be left to the readers.

Theorem 3.2. Eq. (3.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Example 3.3. We put $x_{-6} = 1.7,\ x_{-5} = 8,\ x_{-4} = 3,\ x_{-3} = 9.8,\ x_{-2} = 1.2,\ x_{-1} = 7.2,\ x_0 = 3.5$. (See figure 3).

Example 3.4. See figure 4, since $x_{-6} = 7,\ x_{-5} = -2,\ x_{-4} = 3,\ x_{-3} = 2.5,\ x_{-2} = 12,\ x_{-1} = 5,\ x_0 = -7$.

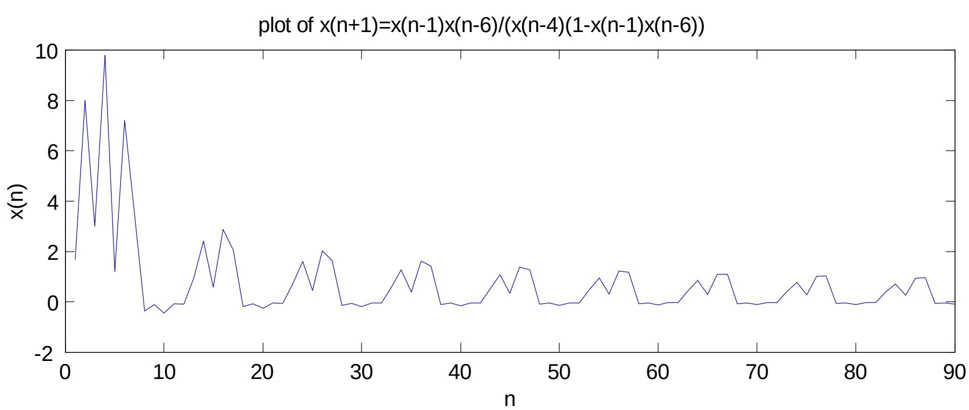

Figure 3.

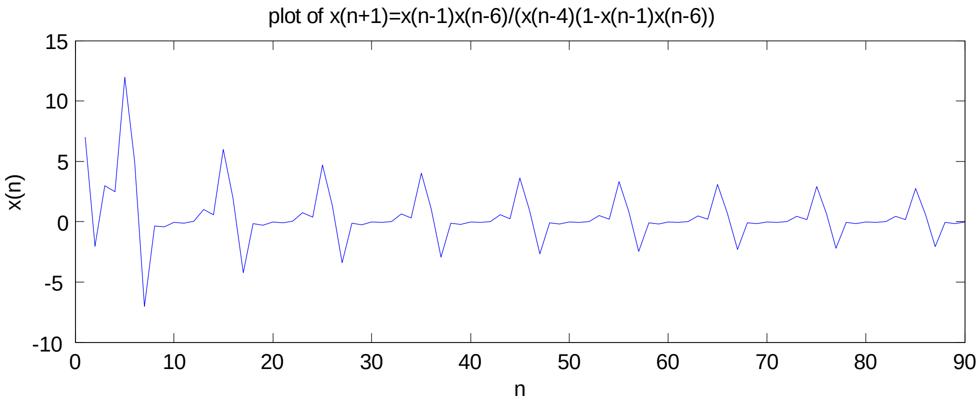

Figure 4.

4. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(-1 + X_{N-1}X_{N-6}))$

In this section we realize a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-6}}{x_{n-4}(-1 + x_{n-1}x_{n-6})}, \quad n = 0, 1, ..., \tag{4.1}$$

where the initial values are arbitrary positive real numbers with $x_{-1}x_{-6} \neq 1$.

Theorem 4.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(4.1). Then for $n = 0, 1, 2, ...$

$$\begin{aligned}
x_{20n-9} &= \frac{g^{2n}b^{2n}(af-1)^n}{a^{2n-1}f^{2n-1}e(bg-1)^n}, \quad x_{20n-8} = \frac{a^{2n}f^{2n}(bg-1)^{n-1}}{b^{2n-1}g^{2n-1}d(af-1)^n}, \quad x_{20n-7} = \frac{g^{2n}b^{2n}(af-1)^{n-1}}{a^{2n-1}f^{2n-1}c(bg-1)^n},\\
x_{20n-6} &= \frac{a^{2n}f^{2n}(bg-1)^n}{b^{2n}g^{2n-1}(af-1)^n}, \quad x_{20n-5} = \frac{g^{2n}b^{2n}(af-1)^n}{a^{2n}f^{2n-1}(bg-1)^n}, \quad x_{20n-4} = \frac{a^{2n}f^{2n}e(bg-1)^n}{b^{2n}g^{2n}(af-1)^n},\\
x_{20n-3} &= \frac{g^{2n}b^{2n}d(af-1)^n}{a^{2n}f^{2n}(bg-1)^n}, \quad x_{20n-2} = \frac{a^{2n}f^{2n}c(bg-1)^n}{b^{2n}g^{2n}(af-1)^n}, \quad x_{20n-1} = \frac{g^{2n}b^{2n+1}(af-1)^n}{a^{2n}f^{\ 2n}(bg-1)^n},\\
x_{20n} &= \frac{a^{2n+1}f^{2n}(bg-1)^n}{b^{2n}g^{2n}(af-1)^n}, \quad x_{20n+1} = \frac{g^{2n+1}b^{2n+1}(af-1)^n}{a^{2n}f^{2n}e(bg-1)^{n+1}}, \quad x_{20n+2} = \frac{a^{2n+1}f^{2n+1}(bg-1)^n}{b^{2n}\ g^{2n}d\ (af-1)^{n+1}},\\
x_{20n+3} &= \frac{g^{2n+1}b^{2n+1}(af-1)^n}{a^{2n}f^{2n}c(bg-1)^n}, \quad x_{20n+4} = \frac{a^{2n+1}f^{2n+1}(bg-1)^n}{b^{2n+1}g^{2n}(af-1)^n}, \quad x_{20n+5} = \frac{g^{2n+1}b^{2n+1}(af-1)^n}{a^{2n+1}f^{2n}(bg-1)^{n+1}},\\
x_{20n+6} &= \frac{a^{2n+1}f^{2n+1}e(bg-1)^{n+1}}{b^{2n+1}g^{2n+1}(af-1)^{n+1}}, \quad x_{20n+7} = \frac{g^{2n+1}b^{2n+1}d\ (af-1)^{n+1}}{a^{2n+1}f^{2n+1}(bg-1)^n}, \quad x_{20n+8} = \frac{a^{2n+1}f^{2n+1}c(bg-1)^n}{b^{2n+1}g^{2n+1}(af-1)^n},\\
x_{20n+9} &= \frac{g^{2n+1}b^{2n+2}(af-1)^n}{a^{2n+1}f^{2n+1}(bg-1)^{n+1}}, \quad x_{20n+10} = \frac{a^{2n+2}f^{2n+1}(bg-1)^{n+1}}{b^{2n+1}g^{2n+1}(af-1)^{n+1}}.
\end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{20n-17} &= \frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-2}f^{2n-2}c(bg-1)^{n-1}}, \quad x_{20n-16} = \frac{a^{2n-1}f^{2n-1}(bg-1)^{n-1}}{b^{2n-1}g^{2n-2}(af-1)^{n-1}}, \quad x_{20n-15} = \frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-1}f^{2n-2}(bg-1)^n},\\
x_{20n-14} &= \frac{a^{2n-1}f^{2n-1}e(bg-1)^n}{b^{2n-1}g^{2n-1}(af-1)^n}, \quad x_{20n-13} = \frac{g^{2n-1}b^{2n-1}d(af-1)^n}{a^{2n-1}f^{2n-1}(bg-1)^{n-1}}, \quad x_{20n-12} = \frac{a^{2n-1}f^{2n-1}c(bg-1)^{n-1}}{b^{2n-1}g^{2n-1}(af-1)^{n-1}},\\
x_{20n-11} &= \frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}f^{2n-1}(bg-1)^n}, \quad x_{20n-10} = \frac{a^{2n}f^{2n-1}(bg-1)^n}{b^{2n-1}g^{2n-1}(af-1)^n}.
\end{aligned}$$

Now, it follows from Eq. (4.1), we get:

$$x_{20n-9} = \frac{x_{20n-11}x_{20n-16}}{x_{20n-14}(-1 + x_{20n-11}x_{20n-16})}$$

$$
\begin{aligned}
&= \frac{\left(\frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n}}\right)\left(\frac{a^{2n-1}f^{\ 2n-1}(bg-1)^{n-1}}{b^{2n-1}\ g^{\ 2n-2}(af-1)^{n-1}}\right)}{\left(\frac{a^{2n-1}f^{\ \ 2n-1}e\ (bg-1)^{n}}{b^{2n-1}\ g^{\ 2n-1}\ (af-1)^{n}}\right)(-1+\left(\frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n}}\right)\left(\frac{a^{2n-1}f^{\ 2n-1}(bg-1)^{n-1}}{b^{2n-1}\ g^{\ 2n-2}(af-1)^{n-1}}\right))} \\
&= \frac{\left(\frac{g^{2n-1}b^{2n}(af-1)^{n-1}}{a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n}}\right)\left(\frac{a^{2n}f^{\ \ 2n-1}(bg-1)^{n}}{b^{2n-1}\ g^{\ 2n-1}\ (af-1)^{n}}\right)}{\left(\frac{a^{2n-1}f^{\ \ 2n-1}e\ (bg-1)^{n}}{b^{2n-1}\ g^{\ 2n-1}\ (af-1)^{n}}\right)(-1+\frac{bg}{(bg-1)})} = \frac{b^{2n}\ g^{\ 2n}\ (af-1)^{n}}{a^{2n-1}f^{\ \ 2n-1}e\ (bg-1)^{n}}.
\end{aligned}
$$

Also, we obtain

$$
\begin{aligned}
x_{20n-8} &= \frac{x_{20n-10}x_{20n-15}}{x_{20n-13}(-1+x_{20n-10}x_{20n-15})} = \frac{\left(\frac{a^{2n}f^{2n-1}(bg-1)^{n}}{b^{2n-1}g^{2n-1}(af-1)^{n}}\right)\left(\frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-1}f^{2n-2}(bg-1)^{n}}\right)}{\left(\frac{g^{2n-1}b^{2n-1}d(af-1)^{n}}{a^{2n-1}f^{2n-1}(bg-1)^{n-1}}\right)\left(-1+\left(\frac{a^{2n}f^{2n-1}(bg-1)^{n}}{b^{2n-1}g^{2n-1}(af-1)^{n}}\right)\left(\frac{g^{2n-1}b^{2n-1}(af-1)^{n-1}}{a^{2n-1}f^{2n-2}c(bg-1)^{n}}\right)\right)} \\
&= \frac{\left(\frac{af}{af-1}\right)a^{2n-1}\ f^{\ 2n-1}(bg-1)^{n-1}}{g^{2n-1}b^{2n-1}d\ (af-1)^{n}(-1+\frac{af}{af-1})} = \frac{a^{2n}\ f^{\ 2n}(bg-1)^{n-1}}{g^{2n-1}b^{2n-1}d\ (af-1)^{n}}.
\end{aligned}
$$

Thus, the proof of the other relations is similar.

Theorem 4.2. Eq.(4.1) has a periodic solution of period ten iff $af = bg = 2$ and will be taken the form $\{\frac{2}{e}, \frac{2}{d}, \frac{2}{c}, g, f, e, d, c, b, a, \frac{2}{e}, \frac{2}{d}, ...\}$.

Proof: First suppose that there exists a prime period twenty solution

$$\frac{2}{e}, \frac{2}{d}, \frac{2}{c}, g, f, e, d, c, b, a, \frac{2}{e}, \frac{2}{d}, ...,$$

of Eq.(4.1), we see from the form of the solution of Eq.(4.1) that

$$
\begin{aligned}
\frac{g^{2n}b^{2n}(af-1)^{n}}{a^{2n-1}f^{2n-1}e(bg-1)^{n}} &= \frac{2}{e},\ \frac{a^{2n}f^{2n}(bg-1)^{n-1}}{b^{2n-1}g^{2n-1}d(af-1)^{n}} = \frac{2}{d},\ \frac{g^{2n}b^{2n}(af-1)^{n-1}}{a^{2n-1}f^{2n-1}c(bg-1)^{n}} = \frac{2}{c}, \\
\frac{a^{2n}f^{2n}(bg-1)^{n}}{b^{2n}g^{2n-1}(af-1)^{n}} &= g\ ,\ ...\ ,\frac{a^{2n+2}f^{2n+1}(bg-1)^{n+1}}{b^{2n+1}g^{2n+1}(af-1)^{n+1}} = a.
\end{aligned}
$$

Then

$$af = bg = 2.$$

Second assume that $af = bg = 2$. Then we see from the form of the solution of Eq.(4.1) that

$$
\begin{aligned}
x_{20n-9} &= \frac{2}{e},\ x_{20n-8} = \frac{2}{d},\ x_{20n-7} = \frac{2}{c},\ x_{20n-6} = g,\ x_{20n-5} = f,\ x_{20n-4} = e,\ x_{20n-3} = d, \\
x_{20n-2} &= c,\ x_{20n-1} = b,\ x_{20n} = a,\ x_{20n+1} = \frac{2}{e},\ x_{20n+2} = \frac{2}{d},\ ...,\ x_{20n+9} = b,\quad x_{20n+10} = a.
\end{aligned}
$$

Thus we have a periodic solution of period ten and the proof is complete.

Theorem 4.3. Eq.(4.1) has a periodic solution of period twenty iff $af = bg = -2$ and will be taken the form $\{\frac{-2}{e}, \frac{-2}{d}, \frac{2}{3c}, g, f, e, d, c, b, a, \frac{2}{3e}, \frac{2}{3d}, \frac{-2}{c}, g, \frac{f}{-3}, e, -3d, c, \frac{b}{-3}, a, \frac{-2}{e}, \frac{-2}{d} ...\}$.

Proof: The proof as the proof of the previous theorem and so it will be omitted.

Theorem 4.4. Eq. (4.1) has three equilibrium points which are 0, $\pm\sqrt{2}$ and there equilibrium points are not locally asymptotically stable.

Example 4.5. See Figure 5 if we put $x_{-6} = 5,\ x_{-5} = -.4,\ x_{-4} = -3,\ x_{-3} = 4.6,\ x_{-2} = -6,\ x_{-1} = -2/5,\ x_0 = 5$. (See figure 5).

Example 4.6. Figure 6 shows the solutions where $x_{-6} = 2.1,\ x_{-5} = .4,\ x_{-4} = -3,\ x_{-3} = 4.6,\ x_{-2} = 1.2,\ x_{-1} = .6,\ x_0 = 9$.

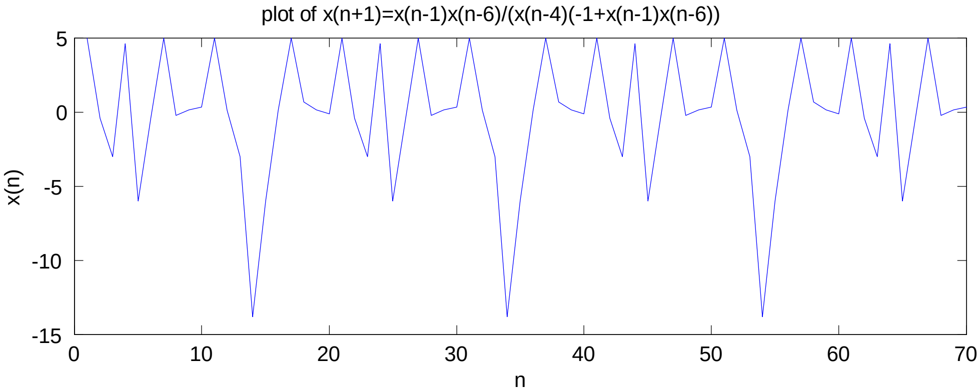

Figure 5.

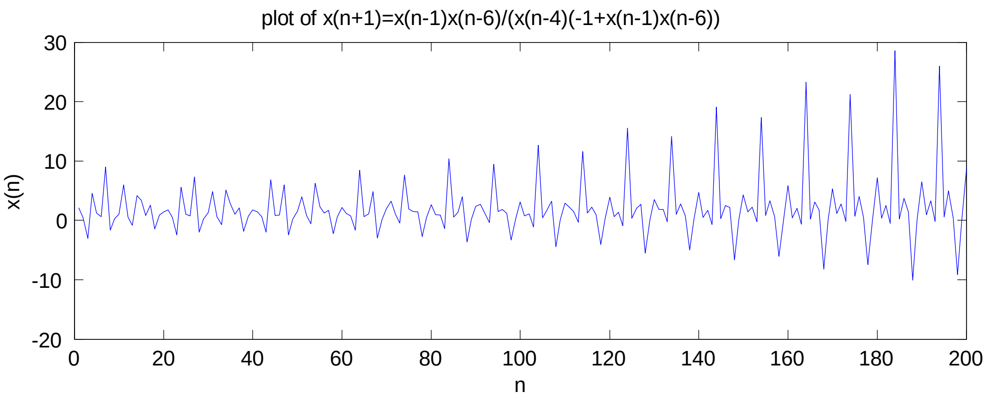

Figure 6.

The proof of the theorems in the following section as in this section so it will be left to the readers.

5. ON THE EQUATION $X_{N+1} = X_{N-1}X_{N-6}/(X_{N-4}(-1 - X_{N-1}X_{N-6}))$

In this section we realize a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-6}}{x_{n-4}(-1 - x_{n-1}x_{n-6})}, \quad n = 0, 1, ..., \tag{5.1}$$

where the initial values are arbitrary positive real numbers.

Theorem 5.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq.(5.1). Then for $n = 0, 1, 2, ...$

$$\begin{aligned}
x_{20n-9} &= \frac{g^{2n}b^{2n}(-1-af)^n}{a^{2n-1}f^{2n-1}e(-1-bg)^n}, \; x_{20n-8} = \frac{a^{2n}f^{2n}(-1-bg)^{n-1}}{b^{2n-1}g^{2n-1}d(-af-1)^n}, \; x_{20n-7} = \frac{g^{2n}b^{2n}(-1-af)^{n-1}}{a^{2n-1}f^{2n-1}c(-1-bg)^n}, \\
x_{20n-6} &= \frac{a^{2n}f^{\ 2n}(-1-bg)^n}{b^{2n}\ g^{\ 2n-1}(-1-af)^n}, x_{20n-5} = \frac{g^{2n}b^{2n}(-1-af)^n}{a^{2n}\ f^{\ 2n-1}(-1-bg)^n}, \; x_{20n-4} = \frac{a^{2n}f^{\ 2n}e(-1-bg)^n}{b^{2n}\ g^{\ 2n}\ (-1-af)^n}, \\
x_{20n-3} &= \frac{g^{2n}b^{2n}d\ (-1-af)^n}{a^{2n}\ f^{\ 2n}(-1-bg)^n}, \; x_{20n-2} = \frac{a^{2n}f^{\ 2n}c(-1-bg)^n}{b^{2n}\ g^{\ 2n}(-1-af)^n}, x_{20n-1} = \frac{g^{2n}b^{2n+1}(-1-af)^n}{a^{2n}\ f^{\ 2n}(-1-bg)^n}, \\
x_{20n} &= \frac{a^{2n+1}f^{\ 2n}(-1-bg)^n}{b^{2n}\ g^{\ 2n}\ (-1-af)^n}, \; x_{20n+1} = \frac{g^{2n+1}b^{2n+1}(-af-1)^n}{a^{2n}\ f^{\ 2n}e(-bg-1)^{n+1}}, \; x_{20n+2} = \frac{a^{2n+1}f^{\ 2n+1}(-bg-1)^n}{b^{2n}\ g^{\ 2n}d\ (-af-1)^{n+1}}, \\
x_{20n+3} &= \frac{g^{2n+1}b^{2n+1}(-1-af)^n}{a^{2n}\ f^{\ 2n}c(-1-bg)^n}, \; x_{20n+4} = \frac{a^{2n+1}f^{\ 2n+1}(-1-bg)^n}{b^{2n+1}\ g^{\ 2n}(-1-af)^n}, x_{20n+5} = \frac{g^{2n+1}b^{2n+1}(-1-af)^n}{a^{2n+1}\ f^{\ 2n}(-1-bg)^{n+1}}, \\
x_{20n+6} &= \frac{a^{2n+1}f^{2n+1}e(-1-bg)^{n+1}}{b^{2n+1}g^{2n+1}(-1-af)^{n+1}}, \; x_{20n+7} = \frac{g^{2n+1}b^{2n+1}d(-1-af)^{n+1}}{a^{2n+1}f^{2n+1}(-1-bg)^n}, \; x_{20n+8} = \frac{a^{2n+1}f^{2n+1}c(-1-bg)^n}{b^{2n+1}g^{2n+1}(-1-af)^n},
\end{aligned}$$

$$x_{20n+9} = \frac{g^{2n+1}b^{2n+2}(-1-af)^{n}}{a^{2n+1}\ f^{\ 2n+1}(-1-bg)^{n+1}}, \quad x_{20n+10} = \frac{a^{2n+2}f^{\ 2n+1}(-1-bg)^{n+1}}{b^{2n+1}\ g^{\ 2n+1}\ (-1-af)^{n+1}}.$$

Theorem 5.2. Eq.(5.1) has a periodic solution of period ten iff $af = bg = -2$ and will be taken the form $\{\frac{-2}{e}, \frac{-2}{d}, \frac{-2}{c}, g, f, e, d, c, b, a, \frac{-2}{e}, \frac{-2}{d}, ...\}$.

Theorem 5.3. Eq.(5.1) has a periodic solution of period twenty iff $af = bg = 2$ and will be taken the form $\{\frac{2}{e}, \frac{2}{d}, \frac{2}{-3c}, g, f, e, d, c, b, a, \frac{-2}{3e}, \frac{-2}{3d}, \frac{2}{c}, g, \frac{f}{-3}, e, -3d, c, \frac{b}{-3}, a, \frac{2}{e}, \frac{2}{d}...\}$.

Theorem 5.4. Eq. (5.1) has a unique equilibrium point which is the number zero, and this equilibrium point is not locally asymptotically stable.

Example 5.5. We take $x_{-6} = 3,\ x_{-5} = 7.4,\ x_{-4} = -2.3,\ x_{-3} = -13,\ x_{-2} = 6,\ x_{-1} = -2,\ x_0 = 2$. (See figure 7).

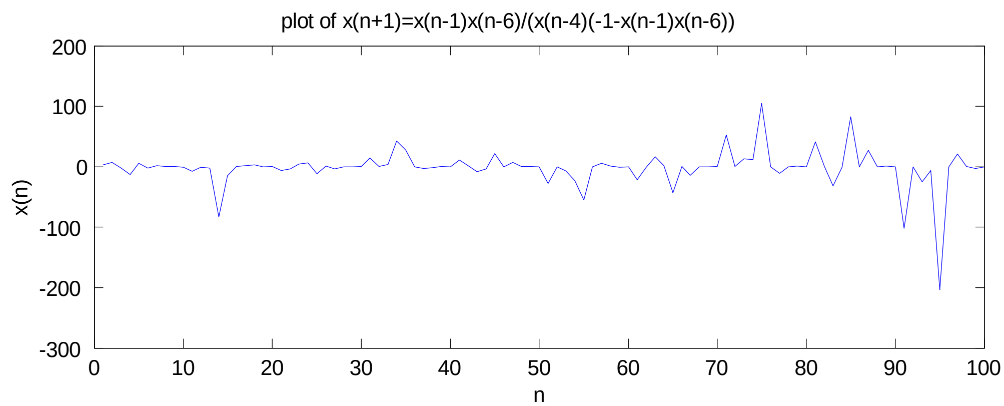

Figure 7.

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On some classes of nonlinear contractions in Fuzzy metric spaces

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Abstract

In this paper, we, motivated by Mihet [2], give the concept of two nonlinear contractions (($(\varphi, \varepsilon-\lambda)$-contraction and (φ, b_n)-contraction) in KM-fuzzy metric spaces, and obtained some fixed point theorems. We answer the open question posed by Mihet in [2, open question 2]. Finally, an example can be used to be exemplify our results.

Keywords: Fuzzy metric space; fixed point; fuzzy contraction

1 Introduction and preliminaries

In 1975, Kramosil and Michalek [6] gave a notion of fuzzy metric space (KM-fuzzy metric space), which was modified later by George and Veeramani [4]. Since then, many authors have contributed to the study of these concepts of fuzzy metric, fixed point theory is one of the most important topics of research. The first attempt to extend the well-known Banach contraction theorem to KM-fuzzy metrics was done by Grabiec in [8]. Later, Gregori and Sapena [5] gave another notion of fuzzy contractive mapping and studied its applicability to fixed point theory in both contexts of fuzzy metrics above mentioned. In their study, the authors needed to demand additional conditions to the completeness of the fuzzy metric in order to obtain a fixed point theorem, which constitutes a significant difference with the classical theory. Later, this notion of fuzzy contractive mapping and others that appeared in the literature were generalized by D. Mihet in [7] introducing the concept of fuzzy ψ-contractive mapping and he obtained a fixed point theorem for the class of complete non-Archimedean KM-fuzzy metrics.

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Recently, D. Wardowski [9] has provided a new contribution to the study of fixed point theory in fuzzy metric spaces. In [9], the author introduced the concept of fuzzy H-contractive mappings, which constitutes a generalization of the concept given by V. Gregori and A. Sapena, and he obtained the next fixed point theorem for complete fuzzy metric spaces in the sense of George and Veeramani.

In this paper, we, motivated by Mihet [2], give the definition of three nonlinear contractions (($\varphi, \varepsilon - \lambda$)-contraction and (φ, b_n)-contraction) in Km-fuzzy metric spaces, and obtained some fixed point theorems. Finally, an example can be used to be exemplify our main results.

Throughout this paper, let $\mathscr{R}^+ := [0, +\infty)$, $\mathscr{N}$ be the set of all positive integers, Φ_ω:={for each $t > 0$, there exists $r \geq t$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$}.

A mapping $F : \mathscr{R} \to \mathscr{R}^+$ is said to be a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$, $\sup_{t\in R} F(t) = 1$.

Let $\mathscr{D}^+$ the set of all distribution functions, while $H \in \mathscr{D}^+$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 1, \text{ if } t > 0. \end{cases}$$

A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied: $(a, 1) = a$; $(a, b) = (b, a)$; $a \geq b, c \geq d \Rightarrow (a, c) \geq (b, d)$; $(a,(b, c)) = ((a, b), c)$.

Definition 1.1 [11] A t-norm is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m\in\mathscr{N}}$ is equicontinuous at $t = 1$, where $\Delta^1(t) = \Delta(t, t)$, $\Delta^m(t) = \Delta(t, \Delta^{m-1}(t)).m = 1, 2, \cdots, t \in [0, 1](\Delta^0(t) = t)$.

Definition 1.2 [12] A fuzzy metric space in the sense of Kramosil and Michlek (briefly, a KM-fuzzy metric space) is a triple (X, M, Δ) where X is a nonempty set, Δ is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X \ ands, t > 0$:

(FM-1) $M(x, y, 0) = 0$;

(FM-2) $M(x, y, t) = 1$, for $t > 0$ if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$;

(FM-5) $M(x, y,) : \mathscr{R}^+ \to [0, 1]$ is left continuous.

Lemma 1.1 [1] If (X, M, Δ) is a KM-fuzzy metric space satisfying the condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$, then (X, F, Δ) is a Menger space, where F is defined by

$$F_{x,y}(t) = \begin{cases} M(x, y, t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{1.1}$$

On the other hand, if (X, M, Δ) is a Menger space, then (X, M, Δ) is a KM-fuzzy metric space with (FM-6), where M is defined by $M(x, y, t) = F_{x,y}(t)$ for $t \geq 0$.

Definition 1.3 [1] Let (X, M, Δ) be a complete KM-fuzzy metric space with a t-norm Δ of H-type, $T : X \to X$ be a mapping satisfied

$$M(Tx, Ty, \varphi(t)) \geq M(x, y, t) \;\; \forall\, x, y \in X \text{ and } t > 0 \tag{1.2}$$

where $\varphi \in \Phi_\omega$. Then T is said to be a fuzzy φ-contraction.

Lemma 1.2 [1] Let (X, M, Δ) be a complete KM-fuzzy metric space with a t-norm Δ of H-type, $T : X \to X$ be a mapping satisfied (1.2). Suppose that there exists some $x_0 \in X$ such that $\lim_{t\to\infty} M(x_0, Tx_0, t) = 1$. Then T has a unique fixed point x_* in $Y_0 = \{y \in X | \lim_{t\to\infty} M(x_0, y, t) = 1\}$.

In Fang [1] has given the definition of fuzzy φ-contraction and obtained some fixed point theorems in KM-fuzzy metric spaces. In this paper, we also obtain some fixed point results in KM-fuzzy metric spaces by cocerning nonliner contractions.

2 Fuzzy $(\varphi, \varepsilon - \lambda)$-contractions

In this section, we give the definition of fuzzy $(\varphi, \varepsilon - \lambda)$-contraction in KM-fuzzy metric spaces and obtain some fixed point theorems.

Definition 2.1 Let (X, M, Δ) be a KM-fuzzy metric spaces. A mapping $T : X \to X$ is called a fuzzy contraction of $(\varepsilon - \lambda)$-type, if for some $k \in (0, 1)$,

$$M(x, y, \varepsilon) > 1 - \lambda \Rightarrow M(Tx, Ty, k\varepsilon) > 1 - k\lambda, \quad \forall \varepsilon > 0, \forall \lambda \in (0, 1).$$

More generally one defines the concept of fuzzy $(\varphi, \varepsilon - \lambda)$-contraction.

Definition 2.2 Let (X, M, Δ) be a KM-fuzzy metric spaces and $\varphi \in \Phi_w$. A mapping $T : X \to X$ is said to be a fuzzy $(\varphi, \varepsilon - \lambda)$-contraction if the following implication holds:

$$M(x, y, \varepsilon) > 1 - \lambda \Rightarrow M(Tx, Ty, \varphi(\varepsilon)) > 1 - \varphi(\lambda), \; \forall \varepsilon > 0, \forall \lambda \in (0, 1). \tag{2.1}$$

Theorem 2.1 Let (X, M, Δ) be a KM-fuzzy metric space with Δ of H-type and $\varphi : [0, \infty) \to [0, \infty)$ be a function with the properties:

i) $\varphi((0,1)) \subset (0,1)$;

ii) $\lim_{n\to\infty} \varphi^n(t) = 0, \forall\ t > 0$.

Then every fuzzy $(\varphi, \varepsilon - \lambda)$-contraction on X have a unique fixed point.

Proof We show that every fuzzy $(\varphi, \varepsilon - \lambda)$-contraction $T : X \to X$ with φ satisfying i) and ii)is a fuzzy φ-contraction.

Indeed, let us assume by contradiction that T is a fuzzy $(\varphi, \varepsilon - \lambda)$-contraction, but it is not a fuzzy φ-contraction. Then $M(Tx, Ty, \varphi(t)) < M(x, y, t)$, for some $x, y \in X, t > 0$, and $\varphi(\lambda) > 1 - M(Tx, Ty, \varphi(t))$, for every $\lambda \in (1 - M(x, y, t), 1)$. In particular

$$\varphi(\lambda) > 1 - M(Tx, Ty, \varphi(t)),\ \forall \lambda \in (1 - M(Tx, Ty, \varphi(t)), 1).$$

Let $\alpha = 1 - M(Tx, Ty, \varphi(t))$. From $M(Tx, Ty, \varphi(t)) < M(x, y, t)$, it follows that $\alpha > 0$ and from i) we obtain $0 < \alpha < 1$. Hence $\varphi((0,1)) \subseteq (0,1)$, which contradicts ii).

By Lemma 1.2, it follows that T have a unique fixed point.

If the assumption $\varphi((0,1)) \subset (0,1)$ in Theorem 2.1 is replaced by the stronger condition $\varphi(t) < t, \forall t \in (0,1)$, we can consider $\varphi \in \Phi_\omega$.

Theorem 2.2 Let (X, M, Δ) be a KM-fuzzy metric space with Δ of H-type and $\varphi : [0,\infty) \to [0,\infty)$ be a function with the properties:

i) $\varphi : [0,\infty) \to [0,\infty)$;

ii) $\varphi \in \Phi_\omega$.

Then every fuzzy $(\varphi, \varepsilon - \lambda)$-contraction on X have a unique fixed point.

For the proof it suffices to see that any fuzzy $(\varphi, \varepsilon - \lambda)$-contraction T satisfying i) is a fuzzy φ-contraction: if we suppose that $M(Tx, Ty, \varphi(\varepsilon)) < M(x, y, \varepsilon)$ for some $x, y \in X,\ \varepsilon > 0$, then we reach a contradiction by choosing $\lambda \in (0,1)$ such that $M(Tx, Ty, \varphi(\varepsilon)) < 1 - \lambda < M(x, y, \varepsilon)$.

3 Fuzzy (φ, b_n)-contractions

Definition 3.1 Let (X, M, Δ) be a KM-fuzzy metric space and b_n be an increasing sequence in $(0,1)$ converging to 1. A mapping $T : X \to X$ is called a fuzzy b_n-contraction if

$$(\forall n \in \mathscr{N}, \exists k_n \in (0,1), \forall x, y \in X, t > 0)\ M(x, y, t) > b_n \Rightarrow M(Tx, Ty, k_n t) > b_n.$$

As a natural extension, we introduce the notion of fuzzy (φ, b_n)-contraction.

Definition 3.2 Let (X, M, Δ) be a KM-fuzzy metric space and b_n be an increasing sequence in $(0, 1)$ converging to 1, $\varphi : [0, \infty) \to [0, \infty)$ be a given function. A mapping $T : X \to X$ is said to be a fuzzy (φ, b_n)-contraction if

$$(\forall n \in \mathscr{N}, \forall x, y \in X, t > 0)\ M(x, y, t) \geq b_n \Rightarrow M(Tx, Ty, \varphi(t)) \geq b_n. \tag{3.1}$$

Lemma 3.1 Let (X, M, Δ) be a KM-fuzzy metric space and T be a fuzzy (φ, b_n)-contraction on X with $\varphi \in \Phi_\omega$. Let $x_0 \in X$ and $(x_n)_n \subset X$ be defined by $x_{n+1} = Tx_n$ for $n \in \mathscr{N}$. Then $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

Proof Let $t > 0$ and $\varepsilon \in (0, 1)$ be given, $m \in \mathscr{N}$ be such that $b_m > 1 - \varepsilon$.

By the definition of fuzzy metric spaces, there exists $s > 0$ such that $M(x_0, x_1, s) \geq b_m$. As $\varphi \in \Phi_\omega$, there exists $r \geq s$ with $\lim_{n\to\infty} \varphi^n(r) = 0$. By the monotonicity of $M(x, y, \cdot)$, we get $M(x_0, x_1, r) \geq b_m$ and, inductively,

$$M(x_n, x_{n+1}, \varphi^n(r)) \geq b_m,\ \forall n \in \mathscr{N}.$$

Let $n_0 \in \mathscr{N}$ such that $\varphi^n(r) < t$ for $n > n_0$. Then

$$M(x_n, x_{n+1}, t) \geq M(x_n, x_{n+1}, \varphi^n(r)) \geq b_m > 1 - \varepsilon,\ \forall n > n_0.$$

So $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$, concluding our proof.

In Theorem 3.3 of Mihet [2], the t-norm is releated to the sequence $(b_n)_n$. Now, we consider $\triangle$ is an arbitrary t-norm of H-type, whether the conclusion of Theorem 3.3 in [2] remain holds? we can see the following consequence.

Lemma 3.2 [2] Let (X, F, Δ) be a probabilistic metric space and T be a probabilistic (φ, b_n)-contraction on X with $\varphi \in \Phi_\omega$. Let $x_0 \in X$ and $(x_n)_n \subset X$ be defined by $x_{n+1} = Tx_n$ for $n \in \mathscr{N}$. Then $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$ for all $t > 0$.

Lemma 3.3 [1] Let (X, M, Δ) be a KM-fuzzy metric space, where the t-norm $\triangle$ is continuous at $(1, 1)$. Suppose that there exists $x_0, x_1 \in X$ such that $\lim_{t\to\infty} M(x_0, x_1, t) = 1$. Define $Y_0 = \{y \in X | \lim_{t\to\infty} M(x_0, y, t) = 1\}$. Then $(Y_0, F, \triangle)$ is a Menger space, where F defined by (1.1). If (X, M, Δ) is complete, then $(Y_0, F, \triangle)$ is a Menger space.

Theorem 3.1 Let $(X, F, \triangle)$ be a complete Menger PM space with a t-norm of H-type, and $T : X \to X$ be a probabilistic (φ, b_n)-contraction, where $b_n \in (0, 1)$ and $\lim_{n\to\infty} b_n = 1$,and $\varphi \in \Phi_\omega$. Then T is a Picard mapping.

Proof Because of in whole proof of Theorem 3.3 in [2], t-norm only be used to show that x_n is a *Cauchy* sequence, so we only need to prove x_n is a *Cauchy* sequence under the condition of Theorem 3.1.

For given $\varepsilon > 0$, by (PM-4) we get

$$F_{x_n,x_{n+p}}(t) \geq \triangle(F_{x_n,x_{n+1}}(\frac{\varepsilon}{p}), \triangle(F_{x_{n+1},x_{n+2}}(\frac{\varepsilon}{p}), \cdots, F_{x_{n+p-1},x_{n+p}}(\frac{\varepsilon}{p}))), \text{for } x \in X.$$

By Lemma 3.2, we know $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$, for $t > 0, n \in \mathscr{N}$. Therefore, $F_{x_n,x_{n+p}}(t) \to 1, n \to \infty$, for $n, p \in \mathscr{N}, t > 0$, so the sequence $(x_n)_n$ is a *Cauchy* sequence.

In fact, above Theorem improve the Theorem 3.3 in Mihet [2], at the same time, the reader could find in this Theorem a way of addressing the recent open question posed by Mihet in [2. open question 2].

Theorem 3.2 Let (X, M, Δ) be a complete KM-fuzzy metric space with a t-norm Δ of H-type, $T : X \to X$ be a fuzzy (φ, b_n)-contraction, where $(b_n)_n \subset (0, 1)$ and $\lim_{n\to\infty} b_n = 1$, $\varphi \in \Phi_\omega$. Suppose that there exists some $x_0 \in X$ such that $\lim_{t\to\infty} M(x_0, Tx_0, t) = 1$. Then T has a unique fixed point x_* in $Y_0 = \{y \in X | \lim_{t\to\infty} M(x_0, y, t) = 1\}$, and $\{T^n(y_0)\}$ converges to x_* for each $y_0 \in Y_0$. In particular, $\{T^n x_0\}$ converges to x_*.

Proof We define a mapping $F : Y_0 \times Y_0 \to \mathscr{D}^+$ by (1.1). Since (X, M, Δ) be a complete KM-fuzzy metric space and there exists some $x_0 \in X$ such that $\lim_{t\to\infty} M(x_0, Tx_0, t) = 1$, by Lemma 3.3 we know that $(Y_0, M, \triangle)$ is a complete Menger space.

We can prove that (3.1) implies that

$$M(Tx, Ty, t) \geq b_n. \tag{3.2}$$

In fact, since $\varphi \in \Phi_\omega$, for each $t > 0$, there exists $r \geq t$ such that $\varphi(r) < t$ and $M(x, y, r) \geq b_n$. By definition of fuzzy (φ, b_n)-contraction, we get

$$M(Tx, Ty, t) \geq M(Tx, Ty, \varphi(r)) \geq b_n.$$

It is not difficult to prove that T is a self-mapping on Y_0. In fact, if $y \in Y_0$, then $\lim_{t\to\infty} M(x_0, y, \frac{t}{2}) = 1$. By the hypothesis $\lim_{t\to\infty} M(x_0, Tx_0, \frac{t}{2}) = 1$. In addition, using (FM-4), we get

$$M(x_0, Ty, t) \geq \triangle(M(x_0, Tx_0, \frac{t}{2}), M(Tx_0, Ty, \frac{t}{2})) \geq \triangle(M(x_0, Tx_0, \frac{t}{2}), b_n).$$

Let $n \to \infty$, $t \to \infty$ in the above inequality. From the continuity of $\triangle$ at $(1, 1)$, we obtain $\lim_{t\to\infty} M(x_0, Ty, t) = 1$. i.e.,$Ty \in Y_0$. This show that T is a mapping of Y_0 into itself.

Clearly (3.1) implies that

$$F_{x,y}(t) \geq b_n \;\Rightarrow\; F_{Tx,Ty}(\varphi(t)) \geq b_n, \text{ for } x, y \in Y_0, t > 0 \text{ and } n \in \mathscr{N},$$

where F is defined by (1.1). This show that T is a probabilistic (φ, b_n)-contraction in $(Y_0, F, \triangle)$. Thus, by Theorem 3.1, we conclude that T has a unique fixed point x_* in Y_0, and $\{T^n(y_0)\}$ converges to x_* for each $y_0 \in Y_0$. In particular, $\{T^n x_0\}$ converges to x_*. This complete the proof.

4 An example

Example 4.1 Let $X = [0, \infty)$ and $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, \triangle_p)$ is a complete KM-fuzzy metric space. Let $Tx = \frac{x}{2}$ for $x \in X$, $\varphi(t) = \frac{t}{2}$ for $t > 0$. Define function $b_n = \frac{n-1}{n}$, $n \in \mathscr{N}$.

It is easy to see that $(b_n)_n \subset (0, 1)$, $\lim_{n\to\infty} b_n = 1$, $\varphi \in \Phi_\omega$. T is a fuzzy (φ, b_n)-contraction on X. In fact, since $M(x, y, t) \geq b_n$, so

$$M(Tx, Ty, \varphi(t)) = M(\frac{x}{2}, \frac{y}{2}, \frac{t}{2}) = \frac{\min\{\frac{x}{2}, \frac{y}{2}\}}{\max\{\frac{x}{2}, \frac{y}{2}\}} = \frac{\min\{x, y\}}{\max\{x, y\}} \geq b_n.$$

By the Theorem 3.2, we know T has a unique fixed point. And 0 is the unique fixed point of T.

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ON SUBCLASSES OF ANALYTIC FUNCTIONS WITH FIXED SECOND COEFFICIENTS

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ABSTRACT. Let A be the class of analytic functions in the open unit disc U with normalization $f(0) = f^{'}(0) - 1 = 0$. The purpose of the present paper is to obtain several sufficient conditions of starlikeness and strongly starlikeness for some subclasses of A with fixed second coefficients that satisfy certain conditions for the quotient of the representations of convexity and starlikeness.

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1. Introduction

Let A denote the class of all functions f which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f^{'}(0) - 1 = 0$. Further let

$$S^*(\alpha) := \left\{ f \in A : \Re \left(\frac{zf^{'}(z)}{f(z)} \right) > \alpha\,,\ \ 0 \le \alpha < 1,\ z \in U \right\},$$

and

$$S(\alpha) := \left\{ f \in A : \left| \arg \left(\frac{zf^{'}(z)}{f(z)} \right) \right| < \alpha\, \frac{\pi}{2},\ \ 0 \le \alpha < 1,\ z \in U \right\},$$

be the subspaces of A consisting of starlike functions of order α and strongly starlike functions of order α, respectively. Note that $S^*(0) = S(1) = S^*$ is the well-known space of normalized functions starlike (univalent) with respect to the origin. we denote by K, the family of all convex functions in U defined as:

$$K := \left\{ f \in A : f^{'}(0) \neq 0,\ \Re(1 + \frac{zf^{''}(z)}{f^{'}(z)}) > 0\,,\ \ z \in U \right\}$$

In [11] Silverman investigated an expression involving the quotient of the analytic representations of convex and starlike functions. Precisely, for $0 < b \le 1$ he considered the class

$$G_b := \left\{ f \in A : \ \left| \frac{1 + zf^{''}(z)/f^{'}(z)}{zf^{'}(z)/f(z)} - 1 \right| < b\,,\ \ z \in U \right\}$$

and proved that $G_b \subset S^*(2/(1+\sqrt{1 + 8b})$. Obradovic´ and Tuneski in [10] improved this result by showing $G_b \subseteq S^*(h(z)) \subseteq S^*(2/(1+\sqrt{1 + 8b})$, where $h(z) = 1/(1+bz)$. Tuneski in [14] gave a sufficient conditions for a function $f \in G_b$ to be in the class $S^*(\frac{1+Az}{1+Bz})$ and its subclasses, where $-1 \le B < A \le 1$. Sokol in [12] gave a generalization of main theorem contained in [14] . Further Obradovic and Owa [9],

1

Nunokawa [6, 7] and Kamali [3] obtained a sufficient conditions for starlikeness of functions which satisfies a certain conditions for the modulus of

$$\frac{1+zf^{''}(z)/f^{'}(z)}{zf^{'}(z)/f(z)}$$

Let $A(\beta)$ consists of analytic functions $f \in A$ of the form

$$f(z) = z + \beta z^2 + a_3 z^3 + ..., \tag{1.1}$$

where the second coefficient $\beta \in \mathbb{C}$ ($\mathbb{C}$ the complex plane) is fixed constant. Several authors have investigated functions with fixed second coefficient and these include, for example, by Ali et al. [1, 2] and Nagpal and Ravichandran [5]. In this paper, we prove several sufficient conditions for starlikeness and strongly starlikeness of some subclasses of A with fixed second coefficients that satisfy certain conditions for the quotient of the representations of convexity and starlikeness .

To derive our main theorem, we need the following lemma due to Kwon [4], which is an extension of a very popular lemma of Nunokawa [8].

Lemma 1. *Let $p(z) = 1 + \beta z + p_2 z^2 + ...$ be analytic in U, and $p(z) \neq 0$ $(z \in U)$. If there exists a point $z_0 \in U$, such that*

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha \ \ (\alpha > 0),$$

then we have

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik\alpha,$$

where

$$\begin{aligned} k &\geq \frac{2}{2+|\beta|}\left(a + \frac{1}{a}\right), \text{when } \arg(p(z_0)) = \frac{\pi}{2}\alpha, \\ k &\leq -\frac{2}{2+|\beta|}\left(a + \frac{1}{a}\right) \text{when } \arg(p(z_0)) = -\frac{\pi}{2}\alpha \end{aligned} \tag{1.2}$$

with $\{p(z_0)\}^{\frac{1}{\alpha}} = \pm ia$.

2. Main Results

Theorem 1. *If $f \in A(\beta)$ defined by (1.1) satisfies*

$$\left|\arg\left(\frac{1+zf^{''}(z)/f^{'}(z)}{zf^{'}(z)/f(z)}\right)\right| < \frac{\pi}{2}\delta\,,$$

where

$$\delta = \frac{2}{\pi}\arctan\left(\frac{4\eta\sin\ (\pi(1-\eta)/2)}{(2+|\beta|)\,(1-\eta)^{\frac{1}{2}(1-\eta)}(1+\eta)^{\frac{1}{2}(1+\eta)} + 4\eta\cos\ (\pi(1-\eta)/2)}\right)$$

then we have

$$\left|\arg\left(\frac{zf^{'}(z)}{f(z)}\right)\right| < \frac{\pi}{2}\eta\,.$$

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Proof. Let $p(z) = \frac{zf'(z)}{f(z)} = 1 + \beta z + p_2 z^2 + ...$, then we have

$$1 + \frac{zp'(z)}{p^2(z)} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}.$$

If there exists a point $z_0 \in U$, such that

$$|\arg(p(z))| < \frac{\pi}{2}\eta \quad \text{for } |z| < |z_0| \quad \text{and} \quad |\arg(p(z_0))| = \frac{\pi}{2}\eta \ \ (\eta > 0),$$

then from Lemma 1, for the case $\arg(p(z_0)) = \frac{\pi}{2}\eta$,

$$\begin{aligned}\arg\left(\frac{1 + z_0 f''(z_0)/f'(z_0)}{z_0 f'(z_0)/f(z_0)}\right) &= \arg\left(1 + \frac{z_0 p'(z_0)}{p^2(z_0)}\right) = \arg\left(1 + \frac{i\eta k}{(ia)^\eta}\right) \\ &= \arctan\left(\frac{\frac{\eta k}{a^\eta}\sin\left(\frac{\pi(1-\eta)}{2}\right)}{1 + \frac{\eta k}{a^\eta}\cos\left(\frac{\pi(1-\eta)}{2}\right)}\right).\end{aligned}$$

Since $\frac{\eta k}{a^\eta} \geq \frac{2\eta}{2+|\beta|}\left(a^{1-\eta} + a^{-1-\eta}\right)$. Now, we define a function $g : (0,\infty) \to \mathbb{R}$ by $g(a) = a^{1-\eta} + a^{-1-\eta}$, then $g'(a) = \frac{1-\eta}{2a^{\eta+2}}\left(a^2 - \frac{1+\eta}{1-\eta}\right)$. Hence $g(a)$ takes the minimum value at $a = \left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}}$. Therefore $\frac{\eta k}{a^\eta} \geq \frac{2\eta}{2+|\beta|}\left[\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)} + \left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}(1+\eta)}\right]$.Thus we have

$$\begin{aligned}&\arg\left(\frac{1 + z_0 f''(z_0)/f'(z_0)}{z_0 f'(z_0)/f(z_0)}\right) \\ \geq\ &\arctan\left(\frac{\frac{2\eta}{2+|\beta|}\left[\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)} + \left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}(1+\eta)}\right]\sin\left(\frac{\pi(1-\eta)}{2}\right)}{1 + \frac{2\eta}{2+|\beta|}\left[\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)} + \left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}(1+\eta)}\right]\cos\left(\frac{\pi(1-\eta)}{2}\right)}\right) \\ =\ &\arctan\left(\frac{4\eta\sin\left(\pi(1-\eta)/2\right)}{(2+|\beta|)(1-\eta)^{\frac{1}{2}(1-\eta)}(1+\eta)^{\frac{1}{2}(1+\eta)} + 4\eta\cos\left(\pi(1-\eta)/2\right)}\right) = \frac{\pi}{2}\delta\end{aligned}$$

This contradicts our condition in the theorem. For the case $p(z_0) = (-ia)^\eta (a > 0)$, using the same method, we can obtain a contradiction to the assumption. □

Theorem 2. *If $f \in A(\beta)$ defined by (1.1) satisfies*

$$\left|\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right| < \frac{2}{1+|\beta|}, \tag{2.1}$$

then we have

$$\left|f(z)/zf'(z) - 1\right| < 1 \quad \text{or} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > 0. \tag{2.2}$$

Proof. Letting

$$\frac{f(z)}{zf'(z)} - 1 = \frac{1-p(z)}{1+p(z)}, \tag{2.3}$$

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we know that $p(z) = 1+2\beta z+p_2z^2+...$, analytic in U, $P(0) = 1$, $P(z) \neq 0$ $(z \in U)$ and

$$\frac{f(z)}{zf'(z)} = \frac{2}{1+p(z)},\ z \in U.$$

Furthermore, we have

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} = 1 + \frac{2p}{(1+p(z))^2}\frac{zp'(z)}{p(z)}.$$

Suppose that there exists a point $z_0 \in U$, such that

$$\Re\left(p(z)\right) > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \Re\left(p(z_0)\right) = 0.$$

Then applying Lemma 1, we have,

$$\frac{z_0p'(z_0)}{p(z_0)} = ik,$$

where k is real number and

$$k \geq \frac{1}{1+|\beta|}\left(a+\frac{1}{a}\right), \quad \text{when } p(z_0) = ia\ (a > 0),$$

$$k \leq -\frac{1}{1+|\beta|}\left(a+\frac{1}{a}\right) \quad \text{when } p(z_0) = -ia\ (a > 0). \tag{2.4}$$

It follows that

$$\Re\left(\frac{1+z_0f''(z_0)/f'(z_0)}{z_0f'(z_0)/f(z_0)} - 1\right) = \Re\left(\frac{\mp 2ak}{(1 \pm ia)^2}\right) = \frac{\mp 2ak(1-a^2)}{(1+a^2)^2}.$$

Moreover, we have

$$\Im\left(\frac{z_0f''(z_0)/f'(z_0)}{z_0f'(z_0)/f(z_0)} - 1\right) = \frac{4a^2k}{(1+a^2)^2}.$$

Therefore by (2.4) we have,

$$\begin{aligned}\left|\frac{z_0f''(z_0)/f'(z_0)}{z_0f'(z_0)/f(z_0)} - 1\right|^2 &= \left(\Re\left(\frac{z_0f''(z_0)/f'(z_0)}{z_0f'(z_0)/f(z_0)} - 1\right)\right)^2 \\ &\quad + \left(\Im\left(\frac{z_0f''(z_0)/f'(z_0)}{z_0f'(z_0)/f(z_0)} - 1\right)\right)^2 \\ &= \frac{4a^2k^2}{(1+a^2)^2} > \frac{4}{(1+|\beta|)^2}.\end{aligned}$$

This contradicts the hypothesis (2.1) and therefore, we have $\Re\left\{p(z)\right\} > 0$ for $|z| < 1$. or

$$\left|\frac{1-p(z)}{1+p(z)}\right| < 1 \quad \text{for } |z| < 1.$$

Therefore, by (2.3) so we obtain (2.2). It completes the proof of Theorem 2. □

Theorem 3. *If* $f \in A(\beta)$ *defined by (1.1) satisfies*

$$\left|\frac{zf''(z)/f'(z)}{zf'(z)/f(z) - 1}\right| < \left(1 + \frac{1}{1+|\beta|}\right), \tag{2.5}$$

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then we have

$$\left| zf^{'}(z)/f(z) - 1 \right| < 1 \tag{2.6}$$

Proof. The following equation

$$\frac{zf^{'}(z)}{f(z)} - 1 = \frac{1-p(z)}{1+p(z)}. \tag{2.7}$$

Defines the function $p(z) = 1 - 2\beta z + p_2 z^2 + ...,$ analytic in U, $P(0) = 1$, $P(z) \neq 0$ $(z \in U)$. Then it follows that

$$\frac{zf^{'}(z)}{f(z)} = \frac{2}{1+p(z)},\ z \in U.$$

Furthermore, we have

$$\frac{zf^{''}(z)/f^{'}(z)}{zf'(z)/f(z) - 1} = 1 - \frac{p}{1-p(z)} \frac{zp^{'}(z)}{p(z)}.$$

If there exists a point $z_0 \in U$, such that

$$\Re\left(p(z)\right) > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \Re\left(p(z_0)\right) = 0,$$

then Lemma 1, gives that,

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik,$$

where the real number k is given by (2.4) and $p(z_0) = \pm ia$ $(a > 0)$. It follows that

$$\Re\left(\frac{z_0 f^{''}(z_0)/f^{'}(z_0)}{z_0 f'(z_0)/f(z_0) - 1}\right) = \Re\left(1 \pm \frac{ak}{(1 \mp ia)}\right) = \left(1 \pm \frac{ak}{1+a^2}\right),$$

and

$$\Im\left(\frac{z_0 f^{''}(z_0)/f^{'}(z_0)}{z_0 f'(z_0)/f(z_0) - 1}\right) = \frac{a^2 k}{1+a^2}.$$

Therefore,

$$\left|\frac{z_0 f^{''}(z_0)/f^{'}(z_0)}{z_0 f'(z_0)/f(z_0) - 1}\right|^2 = 1 \pm \frac{2ak}{1+a^2} + (1+a^2)\left(\frac{ak}{1+a^2}\right)^2.$$

By (2.4) we get

$$\begin{aligned}\left|\frac{z_0 f^{''}(z_0)/f^{'}(z_0)}{z_0 f'(z_0)/f(z_0) - 1}\right|^2 &> 1 + \frac{2}{1+|\beta|} + (1+a^2)\left(\frac{1}{1+|\beta|}\right)^2 \\ &> \left(1 + \frac{1}{1+|\beta|}\right)^2\end{aligned}$$

This contradicts the hypothesis (2.5). And the proof completed as in Theorem 2. $\square$

Theorem 4. *If $f \in A(\beta)$ defined by (1.1) satisfies*

$$\Re\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) > 2(1+|\beta|)\left|\frac{zf^{''}(z)}{f^{'}(z)}\right|, \tag{2.8}$$

then we have

$$\Re\left(f^{'}(z)\right) > 0 \tag{2.9}$$

Proof. We define the function $p(z)$ by

$$f^{'}(z) = \frac{2p(z)}{1+p(z)}. \tag{2.10}$$

Then we see that $p(z) = 1 + 4\beta z + p_2 z^2 + ...,$ is analytic in $U, P(0) = 1,\ P(z) \neq 0\ (z \in U)$. If there exists a point $z_0 \in U$, such that

$$\Re\left(p(z)\right) > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \Re\left(p(z_0)\right) = 0.$$

Then applying Lemma 1, for the case $p(z_0) = ia$ and $a > 0$, we have

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik, \tag{2.11}$$

where k is real number and

$$k \geq \frac{1}{1+2|\beta|}\left(a + \frac{1}{a}\right) . \tag{2.12}$$

The calculations give

$$\frac{1 + \frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}}{\left|\frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}\right|} = \frac{1 + \frac{z_0 p^{'}(z_0)}{p(z_0)}\left(1 - \frac{p(z_0)}{1+p(z_0)}\right)}{\left|\frac{z_0 p^{'}(z_0)}{p(z_0)}\left(1 - \frac{p(z_0)}{1+p(z_0)}\right)\right|}.$$

Therefore, by (2.11) and (2.12), we have

$$\begin{aligned}
\Re\left(\frac{1 + \frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}}{\left|\frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}\right|}\right) &= \Re\left(\frac{1 + ik(1 - \frac{ia}{1+ia})}{\left|ik(1 - \frac{ia}{1+ia})\right|}\right) \\
&= \frac{\sqrt{1+a^2}}{|k|}\left(\frac{(1+a^2) + ak}{1+a^2}\right) \qquad (k > 0) \\
&< \frac{\sqrt{1+a^2}}{\frac{1}{1+2|\beta|}\left(a + \frac{1}{a}\right)} + \frac{a}{\sqrt{1+a^2}} = \frac{2\left(1+|\beta|\right)a}{\sqrt{1+a^2}} < 2\left(1+|\beta|\right)
\end{aligned}$$

This contradicts the hypothesis (2.8) and therefore, we have

$$\Re\left(p(z)\right) > 0 \quad for\ |z| < 1. \tag{2.13}$$

Applying the same method as above, for the case $p(z_0) = -ia$ and $a > 0$, we can obtain,

$$\begin{aligned}
\Re\left(\frac{1 + \frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}}{\left|\frac{z_0 f^{''}(z_0)}{f^{'}(z_0)}\right|}\right) &= \frac{\sqrt{1+a^2}}{|k|}\left(\frac{(1+a^2) - ak}{1+a^2}\right) \qquad (k < 0) \\
&= \frac{\sqrt{1+a^2}}{|k|} + \frac{a}{\sqrt{1+a^2}} < 2\left(1+|\beta|\right).
\end{aligned}$$

This contradicts the hypothesis (2.8) , So we have (2.13). Furthermore,

$$\Re\left(\frac{zf^{'}(z)}{f(z)}\right) = \Re\left(\frac{2p(z)}{1+p(z)}\right) > 0 \qquad \text{(see [13])}$$

It completes the proof . □

Theorem 5. *If $f \in A(\beta)$ defined by (1.1) satisfies*

$$\Re\left(\frac{zf^{'}(z)}{f(z)}\right) > \left(\frac{2+|\beta|}{4}\right)\left|\frac{zf^{'}(z)}{f(z)} - 1\right|, \tag{2.14}$$

then we have

$$\Re\left(\frac{f(z)}{z}\right) > 0 \tag{2.15}$$

Proof. The following equation

$$\frac{f(z)}{z} = p(z). \tag{2.16}$$

Defines the function $p(z) = 1 + \beta z + p_2 z^2 + ...,$ is analytic in $U, P(0) = 1,\ P(z) \neq 0\ (z \in U)$. If there exists a point $z_0 \in U$, such that

$$\Re\left(p(z)\right) > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \Re\left(p(z_0)\right) = 0.$$

Then applying Lemma 1, for $p(z_0) = \pm ia$ and $a > 0$, we have

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik, \tag{2.17}$$

where the real number k is given by (1.2). The calculations give

$$\frac{\frac{z_0 f^{'}(z_0)}{f(z_0)}}{\left|\frac{z_0 f^{'}(z_0)}{f(z_0)} - 1\right|} = \frac{1 + \frac{z_0 p^{'}(z_0)}{p(z_0)}}{\left|\frac{z_0 p^{'}(z_0)}{p(z_0)}\right|} = \frac{1+ik}{|k|}.$$

Therefore, by (2.17) and (1.2), we have

$$\Re\left(\frac{\frac{z_0 f^{'}(z_0)}{f(z_0)}}{\left|\frac{z_0 f^{'}(z_0)}{f(z_0)} - 1\right|}\right) = \frac{1}{|k|} < \frac{a}{\frac{2}{2+|\beta|}\left(a^2+1\right)} < \frac{2+|\beta|}{4}$$

This contradicts the hypothesis (2.14) and therefore, we have

$$\Re\left(p(z)\right) > 0 \quad for\ |z| < 1.$$

□

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On strong convergence theorem of hybrid algorithm for a countable family of quasi-Lipschitz mappings

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Abstract

The purpose of this article is to establish a kind of non-convex hybrid iteration algorithms and to prove relevant strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. We establish a new non-convex hybrid algorithm and prove strong convergence theorem of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces.

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1 Introduction

In mathematics, a fixed point theorem is a result saying that a function f will have at least one fixed point (a point x for which $f(x) = x$), under some conditions on f that can be stated in general terms [3]. Results of this kind are amongst the most generally useful in mathematics [7].

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The Banach fixed point theorem gives a general criterion guaranteeing that, if it is satisfied, the procedure of iterating a function yields a fixed point [6]. By contrast, the Brouwer fixed point theorem is a non-constructive result: it says that any continuous function from the closed unit ball in n-dimensional Euclidean space to itself must have a fixed point [24] but it doesn't describe how to find the fixed point (See also Sperner's lemma). For example, the cosine function is continuous in $[-1, 1]$ and maps it into $[-1, 1]$, and thus must have a fixed point. This is clear when examining a sketched graph of the cosine function; the fixed point occurs where the cosine curve $y = \cos(x)$ intersects the line $y = x$. Numerically, the fixed point is approximately $x = 0.73908513321516$ (thus $x = \cos(x)$ for this value of x). The Lefschetz fixed point theorem [11] (and the Fenchel-Nielsen fixed point theorem) [4] from algebraic topology is notable because it gives, in some sense, a way to count fixed points. There are a number of generalisations to Banach fixed point theorem and further; these are applied in partial differential equation theory. See fixed point theorems in infinite-dimensional spaces. The collage theorem in fractal compression proves that, for many images, there exists a relatively small description of a function that, when iteratively applied to any starting image, rapidly converges on the desired image [1].

Fixed point theory of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings is an active area of interest and finds applications in many related fields like image recovery, signal processing and geometry of objects [23]. From time to time, some versions of theorems relating to fixed points of functions of special nature keep on appearing in almost in all branches of mathematics. Consequently, we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. For example, a fixed point iteration scheme has been applied in intensity modulated radiation therapy optimization to pre-compute dose-deposition coefficient matrix, see [22]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present currently.

The Construction of fixed point theorems (e.g., Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map f that is contracting with respect to some (complete) metric on X will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions see [**?**]. But it only ensures weak convergence, see [5] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [2]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence, (see [10, 15, 17–20], and references therein).

First noticeable modification of Mann's Iteration process was suggested by Nakajo and Takahashi [16] in 2003. They introduced this modification for only one nonexpansive mapping in the context of Hilbert spaces where as Kim and Xu [9] introduced a variant for asymptotically nonexpansive mapping in the same context in 2006. In the same year Martinez-Yanes and Xu [14] introduced a variant of the Ishikawa Iteration process for a nonexpansive mapping. They also gave variant of Halpern iteration method. Su and Qin [21] proposed a monotone hybrid iteration process for nonexpansive mapping in a Hilbert space. Liu *et al.* [12] proposed a novel iteration method for finite family of quasi-asymptotically pseudo-contractive mapping in the realm of Hilbert spaces. Guan *et* *al.* [8] established the first non-convex hybrid algorithm and proved some strong convergence results relating to common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in H.

In this article, we establish a non-convex hybrid algorithms corresponding to Picard iteration scheme. Then we also establish strong convergence theorem of common fixed points for uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. Applications of this algorithm is also given.

2 Preliminaries

Let H be the fixed notation for a Hilbert space and C be a nonempty closed convex subset of H. First we recall some basic definitions that will accompany us throughout this paper.

Let $P_c(\cdot)$ be the metric projection onto C. A mapping $T : C \to C$ is said to be *nonexpensive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. And $T : C \to C$ is said to be *quasi-Lipschitz* if $Fix(T) \neq \phi$ and For all $p \in Fix(T)$, $\|Tx - p\| \leq L\|x - p\|$, where L is a constant $1 \leq L < \infty$.

If $L = 1$, then T is known as quasi-nonexpansive. It is well-known that T is said to be *closed* if for $n \to \infty$, $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. T is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$ as $n \to \infty$. It is admitted fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings having the nonempty fixed points set F. Then $\{T_n\}$ is defined to be *uniformly closed* if for all convergent sequences $\{z_n\} \subset C$ with conditions $\|Tnz_n - z_n\| \to 0$, $n \to \infty$ implies the limit of $\{z_n\}$ belongs to F.

Definition 2.1. Let C be a closed convex subset of a Hilbert space H and let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mapping from C into itself. Then $\{T_n\}$ is said to be asymptotic if $\lim_{n\to\infty} L_n = 1$.

Lemma 2.2. *Let C be a non-empty closed subset of a Hilbert space H. For $x \in H$ and $z \in C$, $z = P_C x$ if and only if we have $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Lemma 2.3. ([8]) *Let C be a closed convex subset of a Hibbert space H and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mapping from C into itself. Then the common fixed point set F is closed and convex.*

Lemma 2.4. *Let C be a closed convex subset of a Hilbert space H, for any given $x \in H$. Then we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$ for all $z \in C$.*

3 Main Results

This section contains main results.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H and let $\{T_n\}$ be uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Suppose that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated* *by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T_n z_n, \quad n \geq 0, \\ z_n = (1-\alpha_n)T_n x_n + \alpha_n T_n t_n, \quad n \geq 0, \\ t_n = (1-\beta_n) + \beta_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq L_n^2(1+(L_n-1)\alpha_n\beta_n)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$*, where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$ *and* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$*.*

Proof. We give our proof in following steps.

STEP 1. We know that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned} \|y_n - p\| &= \|T_n z_n - p\| \\ &= \|T_n[(1-\alpha_n)T_n x_n + \alpha_n T_n t_n] - p\| \\ &= \|T_n[(1-\alpha_n)T_n x_n + \alpha_n T_n((1-\beta_n) + \beta_n T_n x_n)] - p\| \\ &= \|(1-\alpha_n\beta_n)(T_n^2 x_n - p) + (\alpha_n\beta_n)(T_n^3 x_n)\| \\ &\leq (1-\alpha_n\beta_n)\|T_n^2 x_n - p\| + (\alpha_n\beta_n)\|T_n^3 x_n\| \\ &= L_n^2(1+(L_n-1)\alpha_n\beta_n)\|x_n - p\| \end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

STEP 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$ for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Lemma 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n$$

as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

STEP 3. We prove $\{x_n\}$ is bounded. Since F is a nonempty, closed, and convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

STEP 4. We show that $\{x_n\}$ converges strongly to a point of C (we show that $\{x_n\}$ is a cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Lemma 2.4), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0\rangle \leq 0$ and hence

$$\begin{aligned}\|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \quad n \to \infty\end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

STEP5. We show that $y_n \to q$, as $n \to \infty$. Let
$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + L_n^4(L_n - 1)(L_n + 1)\}$.
From the definition of D_n, we have

$$\begin{aligned}D_n &= \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle + L_n^4(L_n - 1)(L_n + 1)\} \\ &= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 \\ &\quad + L_n^4(L_n - 1)(L_n + 1)\} \\ &= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + L_n^4(L_n - 1)(L_n + 1)\}\end{aligned}$$

This shows that D_n is convex and closed, $n \in \mathbb{Z}^+ \cup \{0\}$.

Next, we want to prove that $C_n \subset D_n$, $n \geq 0$.

In fact, for any $z \in C_n$, we have

$$\begin{aligned}\|y_n - z\|^2 &\leq [L_n^2(1 + (L_n - 1)\alpha_n\beta_n)]^2\|x_n - z\|^2 \\ &= \|x_n - z\|^2 L_n^4 + L_n^4[2(L_n - 1)\alpha_n\beta_n + (L_n - 1)^2\alpha_n^2\beta_n^2]\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 L_n^4 + L_n^4[2(L_n - 1) + (L_n - 1)^2]\|x_n - z\|^2 \\ &= \|x_n - z\|^2 L_n^4 + L_n^4(L_n - 1)(L_n + 1)\|x_n - z\|^2.\end{aligned}$$

From

$$C_n = \{z \in C : \|y_n - z\| \leq [L_n^2(1 + (L_n - 1)\alpha_n\beta_n)]\|x_n - z\|\} \cap A, \quad n \geq 0,$$

we have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned}\|y_n - z\| &\leq \|x_n - z\|^2 L_n^4 + L_n^4(L_n - 1)(L_n + 1)\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + L_n^4(l_n - 1)(L_n + 1).\end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Since D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + L_n^4(L_n - 1)(L_n - 1) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

STEP 6. We show that $q \in F$. From the definition of y_n, we have

$$(1 + \alpha_n \beta_n T_n)\|T_n x_n - x_n\| = \|y_n - x_n\| \to 0$$

as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_n \to \infty \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\begin{aligned}\|z_0 - x_n\|^2 &= \|z_0 - x_n + x_n - x_0\|^2 \\ &= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.\end{aligned}$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. □

Now, we present an example of C_n which does not involve a convex subset.

Corollary 3.2. *Let C be a closed convex subset of a Hilbert space H, and let T be a closed quasi-nonexpansive mapping from C into itself. Assume that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T z_n, \quad n \geq 0, \\ z_n = (1 - \alpha_n) T x_n + \alpha_n T t_n, \quad n \geq 0, \\ t_n = (1 - \beta_n) + \beta_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$.

Proof. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 3.1, in this case, C_n is convex and closed and , for all $n \geq 0$, by using Theorem 3.1, we obtain Corollary 3.2. □

Corollary 3.3. *Let C be a closed convex subset of a Hilbert space H, and let T be a nonexpansive mapping from C into itself. Assume that $\alpha_n \in (0, 1]$, and $\beta_n \in [0, 1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T z_n, \quad n \geq 0, \\ z_n = (1 - \alpha_n) T x_n + \alpha_n T t_n, \quad n \geq 0, \\ t_n = (1 - \beta_n) + \beta_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$.

4 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x-p\|, \quad \forall x \in C,\, p \in F,$$

where F is common fixed point set of $\{T_n\}_{n=0}^{N-1}$, $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x-y\|, \quad \forall x, y \in C$$

for all $i \in \{0,1,2,...,N-1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 4.1. *Let C be a closed convex subset of a Hilbert space H, and $\{T_n\}_{n=0}^{N-1} : C \to C$ be finite uniformly L-Lipschitz family of asymptotically quasi-nonexpansive mappings with the nonempty common fixed point set F. Assume that $\alpha_n \in (0,1]$, and $\beta_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T_{i(n)}^{j(n)} z_n, \quad n \geq 0, \\ z_n = (1-\alpha_n)T_{i(n)}^{j(n)} x_n + \alpha_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ t_n = (1-\beta_n) + \beta_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq k_{i(n),j(n)} \\ \qquad (1 + (k_{i(n),j(n)} - 1)\alpha_n\beta)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$, $n = (j(n)-1)N + i(n)$ for all $n \geq 0$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We can drive the prove from the following two conclusions.

Conclusion 1 $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself.

Conclusion 2
$F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ denotes the fixed point set of the mappings T_n. ☐

Corollary 4.2. *Let C be a closed convex subset of a Hilbert space H, and $T : C \to C$ be a L-Lipschitz asymptotically quasi-nonexpansive mapping with the nonempty common fixed point set F. Assume that $\alpha_n \in (0,1]$, and $\beta_n \in [0,1]$ for all $n \in N$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T^n z_n, \quad n \geq 0, \\ z_n = (1-\alpha_n)T^n x_n + \alpha_n T^n z_n, \quad n \geq 0, \\ t_n = (1-\beta_n) + \beta_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq k_n(1 + (k_n - 1)\alpha_n\beta)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$, *where* $\overline{co}C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ in Theorem 4.1, we get the desired result. □

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SOME COMMON FIXED POINT THEOREMS IN ω-ORBITALLY COMPLETE MODULAR METRIC SPACES VIA C-CLASS FUNCTIONS AND APPLICATION

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Abstract. In this paper, some notions are introduced in modular metric spaces. Next some common fixed points are established in ω-orbitally complete modular metric spaces by employing C-class functions that extend and generalize the results of [10, 18]. Finally, for usibility of our results an application is provided to show the existence of solutions for certain system of integral equations.

1. Introduction

In 1976, Jungck [8] initiated a study of common fixed points of commuting mappings. On the other hand, in 1982, Sessa [17] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. After this, Jungck [7] gave the concept of weakly compatible mappings.

In 2008, Chistyakov [5] introduced the notion of modular metric spaces generated by F-modular and developed the theory of this space. In 2010, Chistyakov [6] defined the notion of modular on an arbitrary set and developed the theory of metric spaces generated by modular, which are called the modular metric spaces. Recently, Mongkolkeha *et al.* [11, 12] and Parya *et al.* [14] have introduced some notions and established some fixed point results in modular metric spaces. See [2, 4] for more information on fixed point results.

In this paper, some notions such as "ω-orbit, ω-orbitally complete modular metris space, ω-asymptotically regular mapping" are introduced. Continuation, existence and uniqueness results are proved for common fixed points of three self-mappings in ω-orbitally complete modular metric spaces via C-class functions. Also, suitable examples are provided to demonstrate the usability of the hypotheses of our results. Finally, these results are applied to prove the existence of solutions of a system of integral equations.

2. Basic notions

Definition 2.1. [14] Let X be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : X \to [0, \infty)$ is called a modular if it satisfies the following three conditions:

(i) $\rho(x) = 0$ if and only if $x = 0$;
(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$ and $x, y \in X$;
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

If we replace (iii) by

(iv) $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$ whenever $\alpha, \beta \geq 0$ and $\alpha^s + \beta^s = 1$ with an $s \in (0, 1]$,

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then the modular ρ is called an s-convex modular and if $s = 1$, then ρ is called a convex modular.

If ρ is modular in X, then the set, defined by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\},$$

is called a modular space. X_ρ is a vector subspace of X and it can be equipped with an F-norm defined by setting

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq \lambda\}, \quad x \in X.$$

In addition, if ρ is convex, then the modular space X_ρ coincides with

$$X_\rho^* = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \rho(\lambda x) < \infty\} \tag{2.1}$$

and the functional $\|x\|_\rho^* = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$ is an ordinary norm on X_ρ^* which is equivalent to $\|x\|_\rho$ (see [13]).

Let X be a nonempty set and $\lambda \in (0, \infty)$. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.2. [5] Let X be a nonempty set. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a modular metric on X if it satisfies the following three axioms:

(i) given $x, y \in X$, $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda > 0$ and $x, y, z \in X$.

If, instead of (i), we have the condition
(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudo modular on X. Assume that ω satisfies (i'), (iii) and
(i'') given $x, y \in X$, if there exists a number $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x, y) = 0$, then $x = y$. Then ω is called a strict modular metric on X.

A modular (pseudo modular, strict modular) on X is said to be convex if, instead of (iii), we replace the following condition:

(iv) $\omega_{\lambda+\mu}(x, y) \preceq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Clearly, if ω is a strict modular metric, then ω is a modular metric, which in turn implies that ω is a pseudo modular metric on X, and similar implications hold for convex ω. The essential property of a (pseudo) modular metric ω on a set X is as follows: given $x, y \in X$, the function $0 < \lambda \to \omega_\lambda(x, y) \in [0, \infty]$ is nonincreasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then we have

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

It follows that at each point $\lambda > 0$ the right limit $\omega_{\lambda+0}(x, y) := \lim_{\varepsilon\to+0} \omega_{\lambda+\varepsilon}(x, y)$ and the left limit $\omega_{\lambda-0}(x, y) := \lim_{\varepsilon\to+0} \omega_{\lambda-\varepsilon}(x, y)$ exist in $[0, \infty]$ and the following two inequalities hold:

$$\omega_{\lambda+0}(x, y) \leq \omega_\lambda(x, y) \leq \omega_{\lambda-0}(x, y).$$

It can be checked that if $x_0 \in X$, then the set

$$X_\omega = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x, x_0) = 0\}$$

is a metric space, called a modular space, whose metric is given by

$$d_\omega^0 = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\} \text{ for all } x, y \in X_\omega.$$

Moreover, if ω is convex, then the modular set X_ω is equal to

$$X_\omega^* = \{x \in X : \exists\ \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

and metrizable by

$$d_\omega^* = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\} \text{ for all } x, y \in X_\omega^*.$$

We know that if X is a real linear space, $\rho : X \to [0, \infty)$ and

$$\omega_\lambda(x, y) = \rho(\frac{x-y}{\lambda}) \text{ for all } \lambda > 0 \text{ and } x, y \in X,$$

then ρ is modular (convex modular) on X if and only if ω is modular metric (convex modular metric, respectively) on X.

On the other hand, assume that ω satisfies the following two conditions:

(i) $\omega_\lambda(\mu x, 0) = \omega_{\frac{\lambda}{\mu}}(x, 0)$ for all $\lambda, \mu > 0$ and $x \in X$;

(ii) $\omega_\lambda(x + z, y + z) = \omega_\lambda(x, y)$ for all $\lambda > 0$ and $x, y, z \in X$.

If we set $\rho(x) = \omega_1(x, 0)$ with (2.1), $x \in X$, then $X_\rho = X_\omega$ is a linear subspace of X and the functional $\|x\|_\rho = d^0_\omega(x, 0),\ x \in X_\rho$, is an F-norm on X_ρ. If ω is convex, then $X^*_\rho \equiv X^*_\omega = X_\rho$ is a linear subspace of X and the functional $\|x\|_\rho = d^*_\omega(x, 0),\ x \in X^*_\rho$, is a norm on X^*_ρ.

Similar assertions hold if we replace the word modular by pseudo modular. If ω is modular metric in X, then the set X_ω is called a modular metric space.

By the idea of property in metric spaces and modular spaces, we define the following:

Definition 2.3. Let X_ω be a modular metric space.

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$.
(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-Cauchy if $\omega_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$ for all $\lambda > 0$.
(3) A subset C of X_ω is said to be ω-closed with if the limit of a convergent sequence of C always belongs to C.
(4) A subset C of X_ω is said to be ω-complete if any ω-Cauchy sequence in C is a convergent sequence and its limit is in C.
(5) A subset C of X_ω is said to be ω-bounded if for all $\lambda > 0$ $\delta_\omega(C) = \sup\{\omega_\lambda(x, y); x, y \in C\} < \infty$.

Example 2.4. Let $(X, \|.\|)$ be a norm space. Then a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, defined by

$$\omega_\lambda(x, y) = \|x - y\|, \quad \text{for all } x, y \in X \text{ and } \lambda > 0,$$

is a modular metric.

Example 2.5. Let $(X, \|.\|)$ be a norm space. Then a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ defined by

$$\omega_\lambda(x, y) = \|\frac{x-y}{\lambda}\|^k, \quad \text{for all } x, y \in X,\ k \geq 1 \text{ and } \lambda > 0,$$

is a modular metric.

Example 2.6. Let

$$\rho(f) = \int_\Omega \varphi(v, |f(v)|)d\mu(v),$$

where μ is a σ-finite measure on Ω and $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ satisfies the following conditions:

(i) $\varphi(v, u)$ is a continuous even function of u which is nondecreasing for $u > 0$, such that $\varphi(v, 0) = 0, \varphi(v, u) > 0$ for $u \neq 0$ and $\varphi(v, u) \to \infty$ as $u \to \infty$.

(ii) $\varphi(v, u)$ is a measurable function of v for each $u \in \mathbb{R}$. The corresponding modular space is called a Musielak-Orlicz (or a generalized Orlicz) modular function space and is denoted by L^φ. If φ does not depend on the first variable, then L^φ is called an Orlicz space. Then L^φ is isomorphic to L^P.

An example of functions which satisfy the above conditions is given by

$$\varphi(u) = |u|^p, \text{ for } p > 0.$$

Now, if we define $\omega : (0,\infty) \times X \times X \to [0,\infty]$ by

$$\omega_\lambda(f,g) = \int_\Omega \varphi(v, |f(v) - g(v)|) d\mu(v),$$

where μ and φ satisfy the above coditions, then ω is a modular metric. Also, if $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is defined by

$$\omega_\lambda(f,g) = \int_\Omega \varphi(v, |\frac{f(v) - g(v)}{\lambda}|) d\mu(v),$$

then ω is a modular metric.

In the following, we give some useful notions in modular metric space that will be needed to prove our results.

Definition 2.7. Let X_ω be a modular metric space. Let f, g be self-mappings of X_ω. A point x in X_ω is called a coincidence point of f and g if and only if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Let $C(f,S)$ and $PC(f,S)$ denote the set of coincidence points and points of coincidence, respectively, of the pair (f,S).

Definition 2.8. Let X_ω be a modular metric space. Two self-mappings f and g of X_ω are said to be compatible if and only if $\lim_{n\to\infty} \omega_\lambda(fSx_n, Sfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X_ω such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X_\omega$.

Definition 2.9. Let X_ω be a modular metric space. Two self-mappings f and g of X_ω are said to be weakly compatible if they commute at coincidence points.

Lemma 2.10. *Let X_ω be a modular metric space and $\{y_n\}$ be a sequence in X_ω such that $\lim_{n\to\infty} \omega_\lambda(y_n, y_{n+1}) = 0$ for each $\lambda > 0$. If $\{y_n\}$ is not an ω-Cauchy sequence in X_ω, then there exist $\epsilon_0 > 0, \lambda_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that*

(i) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,
(ii) $\omega_{2\lambda_0}(y_{m_i}, y_{n_i}) > \epsilon_0$ and $\omega_{2\lambda_0}(y_{m_i-1}, y_{n_i}) \leq \epsilon_0$, $i = 1, 2, 3, \cdots$.

Proof. If $\{y_n\}$ is not an ω-Cauchy sequence in X_ω, then there exist $\epsilon_0 > 0, \lambda_0 > 0$ such that for each positive integers i, there exist positive integers m_i, n_i with $m_i > n_i$ such that

$$\omega_{2\lambda_0}(y_{m_i}, y_{n_i}) > \epsilon_0. \tag{2.2}$$

For $i = 1, 2, ...$, let m_i be the least positive integer exceeding n_i satisfying (2.2), that is, for $i = 1, 2, ...$,

$$\omega_{2\lambda_0}(y_{m_i}, y_{n_i}) > \epsilon_0, \quad \omega_{2\lambda_0}(y_{m_i-1}, y_{n_i}) \leq \epsilon_0.$$

Since $\lim_{i\to\infty} \omega_\lambda(y_{n_i}, y_{n_i+1}) = 0$ for all $\lambda > 0$, $\omega_{2\lambda_0}(y_{n_i}, y_{n_i+1}) \leq \epsilon_0$ and thus
$m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$. □

In the following, we present *C-class* functions and some examples of them.

Definition 2.11. [3] A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called a *C-class* function if it is continuous and satisfies the following axioms:
(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.
We denote the set of C-class functions by $\mathcal{C}$.

Example 2.12. [3] The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$:

(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;

(2) $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(4) $F(s,t) = \log(t + a^s)/(1 + t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(5) $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,t) = s \Rightarrow s = 0$;

(6) $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;

(7) $F(s,t) = s \log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$;

(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$, and is continuous, $F(s,t) = s \Rightarrow s = 0$;

(10) $F(s,t) = s - \frac{t}{k+t}, F(s,t) = s \Rightarrow t = 0$;

(11) $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12) $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$, here $h : [0,\infty)\times[0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t,s > 0$;

(13) $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$;

(14) $F(s,t) = \sqrt[n]{\ln(1 + s^n)}$, $F(s,t) = s \Rightarrow s = 0$;

(15) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$, here $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$;

(16) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$.

Definition 2.13. [9] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is nondecreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 2.14. We denote by Ψ the set of altering distance functions.

Definition 2.15. [3] An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

Remark 2.16. We denote by Φ_u the set of ultra altering distance functions.

Definition 2.17. A tripled (ψ, φ, F) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$, is said to be monotone if for all $x, y, z, t \in [0,\infty)$

$$x \leqslant y \Longrightarrow F(\psi(x), \varphi(x)) \leqslant F(\psi(y), \varphi(y)).$$

Example 2.18. Let $F(s,t) = s - t, \varphi(x) = \sqrt{x}$ and

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2 & \text{if x} > 1. \end{cases}$$

Then (ψ, φ, F) is monotone.

Example 2.19. Let $F(s,t) = s - t, \varphi(x) = x^2$ and

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2 & \text{if x} > 1. \end{cases}$$

Then (ψ, φ, F) is not monotone.

3. Main results

In this section, we present and introduce some notions in modular metric spaces which extend the same notions of Phaneendra [15], Sastry *et al.* [16], Aamri and Mountawaki [1]. Next by idea of Liu *et al.* [10] and Swatmaram *et al.* [18] and using C-class functions, some common fixed point theorems will be established in ω

Definition 3.1. Let X_ω be a modular metric space. For given $x_0 \in X_\omega$ and self-mappings f, S and T on X_ω, if there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X_ω such that

$$Sx_{2n} = fx_{2n+1},\ Tx_{2n+1} = fx_{2n+2},$$

then $O(S, T, f, x_0) = \{fx_n : n = 0, 1, 2, \cdots\}$ is called an (S, T)-ω-orbit at x_0 with respect to f.

Definition 3.2. The space X_ω is called ω-orbitally complete at x_0 if and only if every ω-Cauchy sequence in $O(S, T, f, x_0)$ converges in X_ω.

Definition 3.3. The pair (S, T) is ω-asymptotically regular at x_0 with respect to f if there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X_ω such that $Sx_{2n} = fx_{2n+1},\ Tx_{2n+1} = fx_{2n+2}$ and $\omega_\lambda(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$ for all $\lambda > 0$.

Definition 3.4. Self-mappings f and S satisfy property (E.A) if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X_ω such that $\lim_{n\to\infty} \omega_\lambda(fx_n, z) = \lim_{n\to\infty} \omega_\lambda(Sx_n, z) = 0$ for some $z \in X_\omega$ and all $\lambda > 0$.

Theorem 3.5. *Let f, S and T be self-mappings on a modular metric space X_ω satisfying the inequality*

$$\psi(\omega_\lambda(Sx, Ty) \le F\Big(\psi(M(x, y)), \varphi(W(M(x, y)))\Big), \quad \forall \lambda > 0, \tag{3.1}$$

for all $x, y \in X_\omega$, where $\psi \in \Psi, \varphi \in \Phi_u, F \in \mathcal{C}$,

$$M(x, y) = \max\{\omega_\lambda(fx, fy), \omega_\lambda(fx, Sx), \omega_\lambda(fy, Ty), \omega_\lambda(fx, Ty), \omega_\lambda(fy, Sx)\}$$

and $W : [0, \infty) \to [0, \infty)$ is a continuous mapping such that $W(t) < t$ for $t > 0$.
Suppose that

(a) *either (f, S) or (f, T) satisfies the property (E.A);*
(b) *$f(X_\omega)$ is an ω-orbitally complete subspace of X_ω;*
(c) *(f, S) or (f, T) is weakly compatible.*

Then f, S and T have a unique common fixed point.

Proof. By the property (E.A) for the pair (f, S), we have

$$\lim_{n\to\infty} \omega_\lambda(fx_n, z) = \lim_{n\to\infty} \omega_\lambda(Sx_n, z) = 0, \text{ for some } z \in X_\rho \text{ and all } \lambda > 0. \tag{3.2}$$

Let $\lim_{n\to\infty} \omega_\lambda(Tx_n, p) = 0$ for all $\lambda > 0$. Now we prove that $p = z$. By using (3.1) for $x = x_n$ and $y = x_n$, we have

$$\begin{aligned}\psi(\omega_\lambda(Sx_n, Tx_n) \le F\Big(&\psi(\max\{\omega_\lambda(fx_n, fx_n), \omega_\lambda(fx_n, Sx_n),\\ &\omega_\lambda(fx_n, Tx_n), \omega_\lambda(fx_n, Tx_n), \omega_\lambda(fx_n, Sx_n)\}),\\ &\varphi(W(\max\{\omega_\lambda(fx_n, fx_n), \omega_\lambda(fx_n, Sx_n),\\ &\omega_\lambda(fx_n, Tx_n), \omega_\lambda(fx_n, Tx_n), \omega_\lambda(fx_n, Sx_n)\}))\Big), \quad \forall \lambda > 0.\end{aligned}$$

Applying the limit as $n \to \infty$ and then using (3.2), we get

$$\begin{aligned}\psi(\omega_\lambda(z,p) &\le F\Big(\psi(\max\{0,0,\omega_\lambda(z,p),\omega_\lambda(z,p),0\}),\varphi(W(\max\{0,0,\omega_\lambda(z,p),\omega_\lambda(z,p),0)\})\Big)\\ &= F\Big(\psi(\omega_\lambda(z,p)),\varphi(W(\omega_\lambda(z,p))\Big)\\ &\le \psi(\omega_\lambda(z,p)),\ \forall\lambda>0\end{aligned}$$

and so, for all $\lambda > 0$, $\psi(\omega_\lambda(z,p)) = 0$ or $\varphi(W(\omega_\lambda(z,p))) = 0$. Thus $z = p$ and hence

$$\lim_{n\to\infty}\omega_\lambda(fx_n,z)=\lim_{n\to\infty}\omega_\lambda(Sx_n,z)=\lim_{n\to\infty}\omega_\lambda(Tx_n,z)=0,\quad \forall\lambda>0. \tag{3.3}$$

(3.3) can also be obtained in similar lines whenever (f,T) satisfies the property (E.A). From the ω-orbital completeness $f(X_\omega)$, we see that $z \in f(X_\omega)$ so that $z = fu$ for some $u \in X_\omega$. Now, taking $x = u$ and $y = x_n$ in (3.1), we get

$$\begin{aligned}\psi(\omega_\lambda(Su,Tx_n) \le F\Big(&\psi(\max\{\omega_\lambda(fu,fx_n),\omega_\lambda(fu,Su),\\ &\omega_\lambda(fx_n,Tx_n),\omega_\lambda(fu,Tx_n),\omega_\lambda(fx_n,Su)\}),\\ &\varphi(W(\max\{\omega_\lambda(fu,fx_n),\omega_\lambda(fu,Su),\\ &\omega_\lambda(fx_n,Tx_n),\omega_\lambda(fu,Tx_n),\omega_\lambda(fx_n,Su)\}))\Big),\quad\forall\lambda>0.\end{aligned}$$

Applying the limit as $n \to \infty$ and then using (3.3) and $fu = z$, we get

$$\begin{aligned}\psi(\omega_\lambda(Su,fu) \le F\Big(&\psi(\max\{0,\omega_\lambda(fu,Su),0,0,\omega_\lambda(fu,Su)\}),\\ &\varphi(W(\max\{0,\omega_\lambda(fu,Su),0,0,\omega_\lambda(fu,Su)\})\Big)\\ =\ & F\Big(\psi(\omega_\lambda(fu,Su)),\varphi(W(\omega_\lambda(fu,Su))\Big)\\ \le\ & \psi(\omega_\lambda(fu,Su)),\quad\forall\lambda>0\end{aligned}$$

and so, for all $\lambda > 0$, $\psi(\omega_\lambda(fu,Su)) = 0$ or $\varphi(W(\omega_\lambda(fu,Su))) = 0$. Therefore, $fu = Su = z$. Then from the weak compatibility of (f,S), we see that $fSu = Sfu$ or $fz = Sz$.
Again letting $x = y = z$ in (3.1) and using $fz = Sz$, we obtain

$$\begin{aligned}\psi(\omega_\lambda(Sz,Tz) &\le F\Big(\psi(\omega_\lambda(Sz,Tz)),\varphi(W(\omega_\lambda(Sz,Tz))\Big)\\ &\le \psi(\omega_\lambda(Sz,Tz)),\quad\forall\lambda>0.\end{aligned}$$

That is,

$$fz = Sz = Tz. \tag{3.4}$$

Again, taking $x = x_n, y = z$ in (3.1), we get

$$\begin{aligned}\psi(\omega_\lambda(Sx_n,Tz) \le F\Big(&\psi(\max\{\omega_\lambda(fx_n,fz),\omega_\lambda(fx_n,Sx_n),\\ &\omega_\lambda(fz,Tz),d(fx_n,Tz),\omega_\lambda(fz,Sx_n)\}),\\ &\varphi(W(\max\{\omega_\lambda(fx_n,fz),\omega_\lambda(fx_n,Sx_n),\\ &\omega_\lambda(fz,Tz),\omega_\lambda(fx_n,Tz),\omega_\lambda(fz,Sx_n)\}))\Big),\quad\forall\lambda>0.\end{aligned}$$

As $n \to \infty$, this along with (3.3) and (3.4) implies that

$$\begin{aligned}\psi(\omega_\lambda(z,Tz) &\le F\Big(\psi(\omega_\lambda(z,Tz)),\varphi(W(\omega_\lambda(z,Tz))\Big)\\ &\le \psi(\omega_\lambda(z,Tz)),\quad\forall\lambda>0.\end{aligned}$$

That is, $z = Tz$. Thus z is a common fixed point of self-mappings f, S and T.

On the other hand, with minor changes in the above proof, we can prove that $fu = Tu = z$. Suppose that the pair (f, T) is weakly compatible. Then $fTu = Tfu$ or $fz = Tz$. Proceeding as in the previous steps, we get that $fz = Tz = Sz = z$.

Let z, z' be two common fixed points of f, S and T. Then from (3.1) with $x = z$ and $y = z$, we get

$$\begin{aligned}\psi(\omega_\lambda(z, z') = \psi(\omega_\lambda(Sz, Tz') \leq F\Big(&\psi(\max\{\omega_\lambda(fz, fz'), \omega_\lambda(fz, Sz),\\ &\omega_\lambda(fz', Tz'), \omega_\lambda(fz, Tz'), \omega_\lambda(fz', Sz)\}),\\ &\varphi(W(\max\{\omega_\lambda(fz, fz'), \omega_\lambda(fz, Sz),\\ &\omega_\lambda(fz', Tz'), \omega_\lambda(fz, Tz'), \omega_\lambda(fz', Sz)\}))\Big), \quad \forall\lambda > 0,\end{aligned}$$

and thus

$$\begin{aligned}\psi(\omega_\lambda(z, z') &\leq F\Big(\psi(\omega_\lambda(z, z')), \varphi(W(\omega_\lambda(z, z'))\Big)\\ &\leq \psi(\omega_\lambda(z, z')), \quad \forall\lambda > 0,\end{aligned}$$

which implies that $z = z'$. Hence the fixed point is unique. □

With the same proof of Theorem 3.5, we have the following corollaries.

Corollary 3.6. *If in Theorem 3.5, we replace (3.1) with*

$$\psi(\omega_\lambda(Sx, Ty)) \leq F\Big(\psi(M(x, y) - W(M(x, y))), \varphi(M(x, y) - W(M(x, y)))\Big), \quad \forall\lambda > 0,$$

then f, S and T have a unique common fixed point.

Corollary 3.7. *If in Theorem 3.5, we replace (3.1) with*

$$\psi(\omega_\lambda(Sx, Ty)) \leq F\Big(\psi(M(x, y)), \varphi(M(x, y))\Big), \quad \forall\lambda > 0,$$

then f, S and T have a unique common fixed point.

Theorem 3.8. *Let f, S and T be self-mappings on a modular metric space X_ω satisfying the inequality*

$$\psi(\omega_\lambda(Sx, Ty) \leq F\Big(\psi(N(x, y)), \varphi(N(x, y))\Big), \quad \forall\lambda > 0, \tag{3.5}$$

for all $x, y \in X_\omega$, where $\psi \in \Psi, \varphi \in \Phi_u, F \in \mathcal{C}$, such that (ψ, φ, F) is monotone and

$$N(x, y) = \max\{\omega_{2\lambda}(fx, fy), \omega_{2\lambda}(fx, Sx), \omega_{2\lambda}(fy, Ty), \omega_{2\lambda}(fx, Ty), \omega_{2\lambda}(fy, Sx)\}.$$

Suppose that at some $x_0 \in X_\omega$,

(a) *the pair (S, T) is ω-asymptotically regular with respect to f;*
(b) *the space X_ω is ω-orbitally complete;*
(c) *(f, S) or (f, T) is a commuting pair.*

Then f, S and T have a unique common fixed point.

Proof. Since (S, T) is ω-asymptotically regular with respect to f at x_0, there exists a sequence $\{x_n\}$ in X_ω such that

$$Sx_{2n} = fx_{2n+1},\ Tx_{2n+1} = fx_{2n+2} \ \text{ for } n = 0, 1, 2, \cdots$$

and

$$\omega_{\lambda_n} = \omega_\lambda(fx_n, fx_{n+1}) \to 0 \ \text{ as } \ n \to \infty, \ \ \forall\lambda > 0. \tag{3.6}$$

We will show that $\{fx_n\}$ is an ω-Cauchy sequence. Suppose that the result is not true. Then there exist $\epsilon_0 > 0, \lambda_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

(A) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,
(B) $\omega_{2\lambda_0}(fx_{m_i}, fx_{n_i}) > \epsilon_0$ and $\omega_{2\lambda_0}(fx_{m_i-1}, fx_{n_i}) \le \epsilon_0$, $i = 1, 2, 3, \cdots$.

We have

$$\varepsilon_0 < \omega_{2\lambda_0}(fx_{m_i}, fx_{n_i}) \le \omega_{\lambda_0}(fx_{m_i}, fx_{n_i+1}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{n_i}). \tag{3.7}$$

Then

$$\varepsilon_0 \le \omega_{\lambda_0}(fx_{m_i}, fx_{n_i+1}), \quad \text{as } i \to \infty. \tag{3.8}$$

Now consider $\omega_{\lambda_0}(fx_{m_i}, fx_{n_i+1})$ in (3.7) and assume that both m_i and n_i are even. Then by (3.5), we get

$$\begin{aligned}
&\psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})) = \psi(\omega_{\lambda_0}(Sx_{n_i}, Tx_{m_i-1}))\\
&\le F\Big(\psi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, Sx_{n_i}), \omega_{2\lambda_0}(fx_{m_i-1}, Tx_{m_i-1}),\\
&\omega_{2\lambda_0}(fx_{n_i}, Tx_{m_i-1}), \omega_{2\lambda_0}(fx_{m_i-1}, Sx_{n_i})\}),\\
&\varphi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, Sx_{n_i}), \omega_{2\lambda_0}(fx_{m_i-1}, Tx_{m_i-1}),\\
&\omega_{2\lambda_0}(fx_{n_i}, Tx_{m_i-1}), \omega_{2\lambda_0}(fx_{m_i-1}, Sx_{n_i})\})\Big)\\
&= F\Big(\psi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{n_i+1})\}),\\
&\varphi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{n_i+1})\})\Big)\\
&\le F\Big(\psi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{\lambda_0}(fx_{n_i}, fx_{n_i+1}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{m_i-1}, fx_{m_i}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\}),\\
&\varphi(\max\{\omega_{2\lambda_0}(fx_{n_i}, fx_{m_i-1}), \omega_{2\lambda_0}(fx_{n_i}, fx_{n_i+1}), \omega_{2\lambda_0}(fx_{m_i-1}, fx_{m_i}),\\
&\omega_{\lambda_0}(fx_{n_i}, fx_{n_i+1}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{m_i-1}, fx_{m_i}) + \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\})\Big).
\end{aligned}$$

By (3.6), (B), (3.8) and taking limit as $i \to \infty$, we get

$$\begin{aligned}
&\lim_{i\to\infty} \psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}))\\
&\le \lim_{i\to\infty} F\Big(\psi(\max\{\varepsilon_0, 0, 0, \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\}),\\
&\varphi(\max\{\varepsilon_0, 0, 0, \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}), \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})\})\Big)\\
&\le \lim_{i\to\infty} F\Big(\psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})), \varphi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}))\Big)\\
&\le \lim_{i\to\infty} \psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})).
\end{aligned}$$

Thus

$$\lim_{i\to\infty} \psi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})) = 0 \quad \text{or} \quad \lim_{i\to\infty} \varphi(\omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i})) = 0$$

and so $\lim_{i\to\infty} \omega_{\lambda_0}(fx_{n_i+1}, fx_{m_i}) = 0$ and then by (3.8) we conclude that $\varepsilon_0 = 0$, which is a contradiction. Hence $\{fx_n\}$ is an ω-Cauchy sequence. Thus by the ω-orbital completeness of X_ω at x_0, we can find some $z \in X_\omega$ such that $\lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} fx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = z$, which immediately implies that the pairs (f, T) and (S, T) satisfy the property (E.A). Also every commuting pair is weakly compatible. Since the function $0<\lambda \to \omega_\lambda(x,y) \in [0,\infty]$ is nonincreasing on $(0,\infty)$, $N(x,y) \le M(x,y)$ for all $x,y \in X_\omega$ and hence

$$\begin{aligned}\psi(\omega_\lambda(Sx,Ty) &\le F\Big(\psi(N(x,y)),\varphi(N(x,y))\Big)\\ &\le F\Big(\psi(M(x,y)),\varphi(M(x,y))\Big),\quad \forall\lambda>0,\end{aligned}$$

for all $x,y \in X_\omega$. Therefore, by Corollary 3.7, f,S and T have a unique common fixed point. $\square$

Example 3.9. Let $X=[0,1)\cup\{2\}$ and $\omega:(0,\infty)\times X\times X\to[0,\infty]$ be defined by $\omega_\lambda(x,y)=\frac{|x-y|}{\lambda}$ for all $\lambda>0$.

Then X_ω is an ω-complete modular metric space.

Define $f,S,T:X_\omega\to X_\omega$ by $Sx=Tx=\frac{1}{3}x$, $fx=x$ for $x\in X$, $F(s,t)=s-t, \psi(t)=2t$ and $\varphi(t)=t$. Take $x_0=2$ and $w(t)=\frac{1}{2}t$ for $t\ge 0$. Then $O(x_0,S,T,f)=\{\frac{2}{3^n}:n=0,1,2,\cdots\}$, $f(X_\omega)=X_\omega$ is ω-orbitally complete at x_0, (f,S) or (f,T) satisfy the property (E.A), (f,S) or (f,T) is weakly compatible and for all $x,y\in X_\omega$, we have

$$\begin{aligned}\psi(\omega_\lambda(Sx,Ty)) = \frac{2}{3\lambda}|x-y| &\le \frac{3}{2\lambda}\max\{|x-y|,\frac{2}{3}x,\frac{2}{3}y,|x-\frac{1}{3}y|,|y-\frac{1}{3}x|\}\\ &= F\Big(\psi(M(x,y)),\varphi(W(M(x,y)))\Big).\end{aligned}$$

Therefore, all the conditions of Theorem 3.5 are satisfied and $x=0$ is the unique common fixed point of f,S and T.

4. Application to systems of integral equations

Consider the following system of integral equations:

$$\begin{cases} u(a)=\int_0^A k_1(a,b,u(b))db+q(a),\\ u(a)=\int_0^A k_2(a,b,u(b))db+q(a),\end{cases} \tag{4.1}$$

$a\in J=[0,A]$, where $A>0$. The purpose of this section is to give an existence theorem for a solution of the system (4.1) by using Theorem 3.5.

Let $\mathcal{X}:=C(J,R^n)$ with the usual supremum norm, i.e., $\|x\|_{\mathcal{X}}=\max_{a\in J}\|x(a)\|$ for $x\in C(J,R^n)$. Define $\omega:(0,\infty)\times\mathcal{X}\times\mathcal{X}\to[0,\infty]$ by $\omega_\lambda(x,y)=\max_{a\in J}\frac{\|x(a)-y(a)\|}{\lambda}$. Then it can be checked that $\mathcal{X}_\omega$ is an ω-complete modular metric space.
Define $f,S,T:\mathcal{X}_\omega\to\mathcal{X}_\omega$ by

$$fx(a)=x(a),\quad Sx(a)=\int_0^A k_1(a,b,x(b))db+q(a),\quad a\in[0,A],$$

and

$$Tx(a)=\int_0^A k_2(a,b,x(b))db+q(a),\quad a\in[0,A].$$

Theorem 4.1. *Consider the integral equations (4.1). Assume the following hypotheses:*

(i) $K_1,K_2:[0,A]\times[0,A]\times R^n\to R^n$ *and* $q:[0,A]\to R^n$ *are continuous;*

(ii) *There exists* $x\in\mathcal{X}$ *such that*

$$x(a)=\int_0^A k_1(a,b,x(b))db+q(a),\quad a,b\in[0,A],$$

or

$$x(a)=\int_0^A k_2(a,b,x(b))db+q(a),\quad a,b\in[0,A];$$

(iii) *There exists a sequence* $\{x_n\}$ *in* $\mathcal{X}$ *such that*

$$\lim_{n\to\infty} x_n(a) = \lim_{n\to\infty} \int_0^A k_1(a,b,x_n(b))db + q(a) = z, \quad a,b \in [0,A],\ z \in \mathcal{X},$$

or

$$\lim_{n\to\infty} x_n(a) = \lim_{n\to\infty} \int_0^A k_1(a,b,x_n(b))db + q(a) = z, \quad a,b \in [0,A],\ z \in \mathcal{X};$$

(iv) *For each* $a,b \in J$ *and* $u,v \in \mathcal{X}_\omega$,

$$\begin{aligned}&\int_0^A \|k_1(a,b,u(b)) - k_2(a,b,v(b))\|db\\ &\le \frac{3}{4}\max\{\|u(a)-v(a)\|, \|u(a)-Su(a)\|, \|v(a)-Tv(a)\|, \|u(a)-Tv(a)\|,\\ &\|v(a)-Su(a)\|\}.\end{aligned}$$

Then the system of integral equations (4.1) has a unique solution u^* *in* $C(J,R^n)_\omega$.

Proof. By (i), f, S and T are self-mappings on $\mathcal{X}_\omega$.

By (ii), (f,S) or (f,T) is weakly compatible, since f is the identity mapping on $\mathcal{X}_\omega$.

By (iii), either (f,S) or (f,T) satisfies the property (E.A).

Also for each $u,v \in \mathcal{X}_\omega$, $a,b \in J$, by (iv), we have

$$\begin{aligned}\|Su(a)-Tv(a)\| &\le \int_0^A \|k_1(a,b,u(b)) - k_2(a,b,v(b))\|db\\ &\le \frac{3}{4}\max\{\|u(a)-v(a)\|, \|u(a)-Su(a)\|, \|v(a)-Tv(a)\|, \|u(a)-Tv(a)\|,\\ &\|v(a)-Su(a)\|\}\end{aligned}$$

and so

$$\begin{aligned}&\frac{\|Su(a)-Tv(a)\|}{\lambda}\\ &\le \frac{3}{4}\max\{\frac{\|u(a)-v(a)\|}{\lambda}, \frac{\|u(a)-Su(a)\|}{\lambda}, \frac{\|v(a)-Tv(a)\|}{\lambda}, \frac{\|u(a)-Tv(a)\|}{\lambda},\\ &\frac{\|v(a)-Su(a)\|}{\lambda}\}, \quad \forall\lambda > 0.\end{aligned}$$

On routine calculations, we get

$$\psi(\omega_\lambda(Su,Tv)) \le F\Big(\psi(M(u,v)), \varphi(W(M(u,v)))\Big), \quad \forall\lambda > 0,$$

where $\psi(t) = 2t, \varphi(t) = t, F(s,t) = s-t$ and $W(t) = \frac{1}{2}t$.

Since $\mathcal{X}_\omega$ is an ω-complete modular metric space, every ω-Cauchy sequence in $O(S,T,f,x_0) = \{x_n : n = 0,1,2,\cdots\}$ (for some $x_0 \in \mathcal{X}_\omega$) converges in $\mathcal{X}_\omega$. Hence $f(\mathcal{X}_\omega) = \mathcal{X}_\omega$ is ω-orbitaly complete at x_0. Then Theorem 3.5 is applicable, where f is the identity mapping. So S and T have a common fixed point. Thus there exists a $u^* \in C(J,R^n)_\omega$, a common fixed point of S and T, that is, u^* is a unique solution to (4.1). □

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Trapezoidal interval type-2 hesitant fuzzy sets associated with new operations

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Abstract

This paper proposes the concept of trapezoidal interval type-2 hesitant fuzzy set (TIT2HFS), which is a generalization of trapezoidal interval type-2 and hesitant fuzzy set. Also, we study some of its operation laws and corresponding proprties are discussed.

Key words: Trapezoidal interval type-2 hesitant fuzzy set, Operation laws.

1 Introduction

Type-2 fuzzy set was proposed by Zadeh (1975) [19] which is an extension of Type-1 fuzzy set [18]. The principal difference between the two kinds of fuzzy sets is that the memberships of a type-1 fuzzy set are crisp numbers while the memberships of a type-2 fuzzy set are type-1 fuzzy sets [14]; hence, type-2 fuzzy sets include more vulnerabilities than type-1 fuzzy sets. Since its presentation, type-2 fuzzy sets are getting increasingly consideration. Since the computational multifaceted nature of using general type-2 fuzzy sets is very high, to date , interval type-2 fuzzy sets [8] are the most widely used type-2 fuzzy sets and have been effectively connected to numerous useful fields [1, 3, 6, 7, 9, 10, 15, 16]. IT2FS [6] can be viewed as a special case of general T2FS where all the values of secondary membership are equal to 1. In particular, interval type-2 trapezoidal fuzzy numbers, as a special case of interval type-2 fuzzy sets, can proficiently express subjective assessments or evaluations. The concept of Hesitant fuzzy set was proposed by Torra (2010) [12] and Torra and Narukawa (2009) [13] to deal with the problems where membership of element to a give set includes several different values. In this paper, by proposing the concept of TIT2HFS based on HFS and IT2TFS. Furthermore, we introduce some operation laws and their properties are investigated.

2 Preliminaries

In this subsection, we briefly describe some fundamental ideas and essential operation laws identified with HFSs that we need in our work.

2.1 Hesitant fuzzy set

Definition 1: [12, 13]

Let X be a reference set. A hesitant fuzzy set on X is defined in terms of a funcation h that returns a subset of $[0,1]$. To make it understood easily, a HFS can be represented by a mathematical symbol :

$$M := \{< x, h_M(x) >| \, x \in X\}$$

where $h_M(x)$ is a set of some valiues in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set M. For convenience, [17]call $h = h_M(x)$ a hesitant fuzzy element (HFE) and H the set of all HFEs.

Definition 2: [12, 13]

Let h, h_1 and h_2 be three HFEs then:

(1) $h^c = \underset{\gamma \in h}{\cup} \{1 - \gamma\}$.

(2) $h_1 \cup h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \max\{\gamma_1, \gamma_2\}$.

(3) $h_1 \cap h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \min\{\gamma_1, \gamma_2\}$.

Definition 3: [17]

Let h, h_1 and h_2 be three HFEs, and $\lambda > 0$ then:

(1) $h^\lambda = \underset{\gamma \in h}{\cup} \left\{\gamma^\lambda\right\}$.

(2) $\lambda h = \underset{\gamma \in h}{\cup} \left\{1 - (1-\gamma)^\lambda\right\}$.

(3) $h_1 \oplus h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$.

(4) $h_1 \otimes h_2 = \underset{\gamma_1 \in h_1, \gamma_2 \in h_2}{\cup} \{\gamma_1\gamma_2\}$.

2.2 Interval type-2 fuzzy set

The theory of type-1 fuzzy set interdused by Zadeh [18] where the membership value of an element is a real value between 0 and 1. A trapezoidal type-1 fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; H_1(A), H_2(A))$ in the universe of discourse, where $0 \leq H_1(A) \leq H_2(A) \leq 1$ is shown in Fig.1

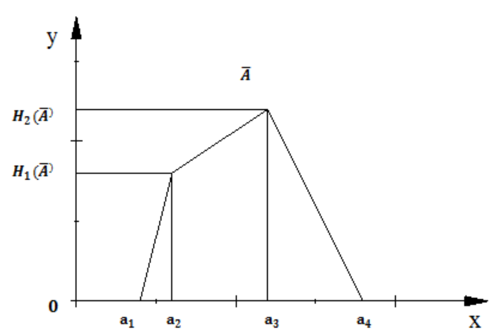

Fig.1 Atrapezoidal type-1 fuzzy number.

2

Type-2 fuzzy set were introduced as the extension of type-1 fuzzy set which is defined as follows.

Definition 4:[5, 7, 8]

A type-2 fuzzy set $\tilde{A}$ in the universe of discourse X can be represented by a type-2 membership function $\mu_{\tilde{A}}$, shown as follows:

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) \,|\forall x \in X, \forall u \in J_x \subseteq [0,1]\},$$

where $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$. The type-2 fuzzy set $\tilde{A}$ also can be represented as follows:

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} \mu_{\tilde{A}}(x,u)/(x,u) = \int_{x\in X}\left[\int_{u\in J_x} \mu_{\tilde{A}}(x,u)/u\right]/x),$$

where x is the primary variable, $J_x \subseteq [0,1]$ is the primary membership of x, u is the secondary variable and $\int_{u\in J_x} \mu_{\tilde{A}}(x,u)/u$ is the secondary membership function (MF) at x. $\int$ denotes union among all admissible x and u. For discrete universe of discourse, $\int$ is replaced by $\sum$.

Definition 5:[5, 8]

Let $\tilde{A}$ be a type-2 fuzzy set $\tilde{A}$ in the universe of discourse X represented by the type-2 membership function $\mu_{\tilde{A}}(x,u)$. If all $\mu_{\tilde{A}}(x,u) = 1$, then $\tilde{A}$ is called an interval type-2 fuzzy set. An interval type-2 fuzzy set $\tilde{A}$ can be regarded as a special case of a type-2 fuzzy set, shown as follows:

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x,u) = \int_{x\in X}\left[\int_{u\in J_x} 1/u\right]/x),$$

where x is the primary variable, $J_x \subseteq [0,1]$ is the primary membership of x, u is the secondary variable and $\int_{u\in J_x} 1/u$ is the secondary membership function (MF) at x.

If X is a set of real numbers, then a type-2 fuzzy set and an interval type-2 fuzzy set in X are called a type-2 fuzzy number and an interval type-2 fuzzy number, respectively.

Definition 6:[5]

Let $\tilde{A}_i$ be a trapezoidal interval type-2 fuzzy number in the universe of discourse X. It can represented by

$$\tilde{A}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right) = \left(\left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(A_i^U), H_2(A_i^U)\right), \left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(A_i^L), H_2(A_i^L)\right)\right)$$

where A_i^U and A_i^L are T1FSs, $a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U, a_{i1}^L, a_{i2}^L, a_{i3}^L$ and a_{i4}^L are the reference points of the IT2FSs $\tilde{A}_i, H_j(A_i^U)$ denotes the membership value of the element $a_{i(j+1)}^U$ in the upper trapezoidal membership function $A_i^U, 1 \leq j \leq$

3

$2, H_j(A_i^L)$ denotes the membership value of the element $a_{i(j+1)}^L$in the lower trapezoidal membership function $A_i^L, 1 \leq j \leq 2, H_1(A_i^U), H_2(A_i^U), H_1(A_i^L)$ and $H_2(A_i^L) \in [0,1], 1 \leq i \leq n$ as shown in Fig.2

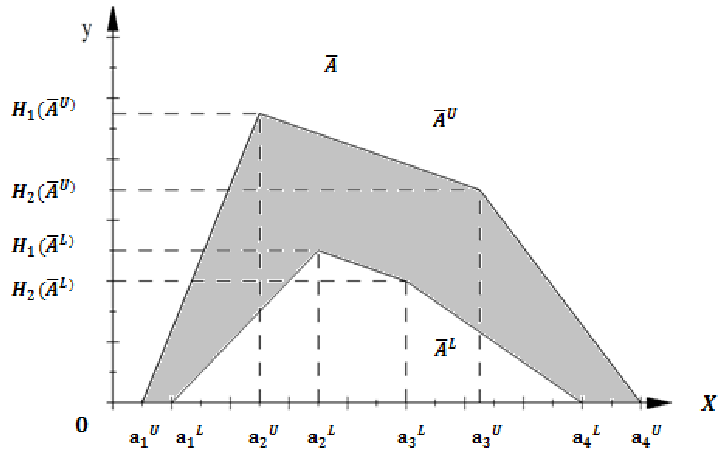

Fig.2 A trapezoidal interval type-2 fuzzy number.

3 Trapezoidal interval type-2 hesitant fuzzy set

3.1 The concept and operation laws of TIT2HFS

Definition 7:

Let X be a fixed set. A trapezoidal interval type-2 hesitant fuzzy set (TIT2HFS) on X is in terms of function that return of some trapezoidal interval type-2 fuzzy numbers (TIT2FNs) when applied to each x in X.

To make it easily understood, we express the TIT2HFS by a mathematical symbol:

$$E := \left\{ < x, \tilde{h}_E(x) > | x \in X \right\}$$

where $\tilde{h}_E(x)$ is a set of some TIT2FNs denoting the possible membership degrees of the element $x \in X$ to the set E. for convenience, we call $\tilde{h}_E(x) = \tilde{h} = \left\{ \tilde{A}i \in \tilde{h} | \tilde{A}_i = \left((a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(A_i^U), H_2(A_i^U)), (a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(A_i^L), H_2(A_i^L)) \right) \right\}$ an trapezoidal interval type-2 hesitant fuzzy element (TIT2HFE).

Example 8:

A hesitant among different TIT2FNs for a decision making, he / she proviedes a TIT2HFS $\tilde{h}_{ij} = \left\{ \begin{array}{c} (0.35, 0.45, 0.55, 0.65; 1, 1), (0.4, 0.5, 0.6, 0.7; 0.8, 0.8), \\ (0, 0, 0.2, 0.3; 0.8, 0.8), (0.72, 0.77, 0.78, 0.89; 1, 1) \end{array} \right\}$.

4

Definition 9:
Let $\tilde{h}_1=\left\{\tilde{A}\in\tilde{h}_1|\tilde{A}=\left(\left(a_1^U,a_2^U,a_3^U,a_4^U;H_1(A^U),H_2(A^U)\right),\left(a_1^L,a_2^L,a_3^L,a_4^L;H_1(A^L),H_2(A^L)\right)\right)\right\}$ and $\tilde{h}_2=\left\{\tilde{B}\in\tilde{h}_2|\tilde{B}=\left(\left(b_1^U,b_2^U,b_3^U,b_4^U;H_1(B^U),H_2(B^U)\right),\left(b_1^L,b_2^L,b_3^L,b_4^L;H_1(B^L),H_2(B^L)\right)\right)\right\}$ are two TIT2HFEs. Then, we introduce the follow operations:

(1) The union of $\tilde{h}_1$ and $\tilde{h}_2$which is denoted by $\tilde{h}_1\cup\tilde{h}_2$ can be defined as:

$$\tilde{h}_1\cup\tilde{h}_2=\bigcup_{\tilde{A}\in\tilde{h}_1,\tilde{B}\in\tilde{h}_2}\left\{\begin{array}{c}\left(\begin{array}{c}\max\left\{a_1^U,b_1^U\right\},\max\left\{a_2^U,b_2^U\right\},\max\left\{a_3^U,b_3^U\right\},\max\left\{a_4^U,b_4^U\right\};\\ \min\left\{H_1(A^U),H_1(B^U)\right\},\min\left\{H_2(A^U),H_2(B^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a_1^L,b_1^L\right\},\max\left\{a_2^L,b_2^L\right\},\max\left\{a_3^L,b_3^L\right\},\max\left\{a_4^L,b_4^L\right\};\\ \min\left\{H_1(A^L),H_1(B^L)\right\},\min\left\{H_2(A^L),H_2(B^L)\right\}\end{array}\right)\end{array}\right\}$$

(2) The intersection of $\tilde{h}_1$ and $\tilde{h}_2$which is denoted by $\tilde{h}_1\cap\tilde{h}_2$ can be defined as:

$$\tilde{h}_1\cap\tilde{h}_2=\bigcup_{\tilde{A}\in\tilde{h}_1,\tilde{B}\in\tilde{h}_2}\left\{\begin{array}{c}\left(\begin{array}{c}\min\left\{a_1^U,b_1^U\right\},\min\left\{a_2^U,b_2^U\right\},\min\left\{a_3^U,b_3^U\right\},\min\left\{a_4^U,b_4^U\right\};\\ \min\left\{H_1(A^U),H_1(B^U)\right\},\min\left\{H_2(A^U),H_2(B^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\min\left\{a_1^L,b_1^L\right\},\min\left\{a_2^L,b_2^L\right\},\min\left\{a_3^L,b_3^L\right\},\min\left\{a_4^L,b_4^L\right\};\\ \min\left\{H_1(A^L),H_1(B^L)\right\},\min\left\{H_2(A^L),H_2(B^L)\right\}\end{array}\right)\end{array}\right\}$$

(3) The complement of $\tilde{h}_1$ denoted by $\tilde{h}_1^c$ can be defined as:

$$\tilde{h}_1^c=\left\{\begin{array}{c}\tilde{A}\in\tilde{h}_1^c|\tilde{A}=\left(\left(1-a_1^U,1-a_2^U,1-a_3^U,1-a_4^U;H_1(A^U),H_2(A^U)\right),\right.\\ \left.\left(1-a_1^L,1-a_2^L,1-a_3^L,1-a_4^L;H_1(A^L),H_2(A^L)\right)\right)\end{array}\right\}.$$

we note that can be replaced max and min by $\vee$ and $\wedge$ respectively.

Example 10:
Let $\tilde{h}_1=\{(0.2,0.3,0.4,0.5;1,1),(0.25,0.35,0.35,0.45;0.8,0.8)\}$ and $\tilde{h}_2=\{(0.5,0.6,0.7,0.8;1,1),(0.55,0.65,0.65,0.75;0.8,0.8)\}$ are two TIT2HFEs, then:

(1) $\tilde{h}_1\cup\tilde{h}_2=\{(0.5,0.6,0.7,0.8;1,1),(0.55,0.65,0.65,0.75;0.8,0.8)\}.$

(2) $\tilde{h}_1\cap\tilde{h}_2=\{(0.2,0.3,0.4,0.5;1,1),(0.25,0.35,0.35,0.45;0.8,0.8)\}.$

(3) $\tilde{h}_1^c=\{(0.8,0.7,0.6,0.5;1,1),(0.75,0.65,0.65,0.55;0.8,0.8)\}$

Proposition 11: (De Morgan's laws in TIT2HFS)
Let $\tilde{h}_1$ and $\tilde{h}_2$ be two TIT2HFNs, then we have :

(1) $\left(\tilde{h}_1\cup\tilde{h}_2\right)^c=\tilde{h}_1^c\cap\tilde{h}_2^c.$

(2) $\left(\tilde{h}_1\cap\tilde{h}_2\right)^c=\tilde{h}_1^c\cup\tilde{h}_2^c.$

Proof:

(1) $$\left(\tilde{h}_1\cup\tilde{h}_2\right)^c=\bigcup_{\tilde{A}\in\tilde{h}_1,\tilde{B}\in\tilde{h}_2}\left\{\begin{array}{c}\left(\begin{array}{c}\max\left\{a_1^U,b_1^U\right\},\max\left\{a_2^U,b_2^U\right\},\max\left\{a_3^U,b_3^U\right\},\max\left\{a_4^U,b_4^U\right\};\\ \min\left\{H_1(A^U),H_1(B^U)\right\},\min\left\{H_2(A^U),H_2(B^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a_1^L,b_1^L\right\},\max\left\{a_2^L,b_2^L\right\},\max\left\{a_3^L,b_3^L\right\},\max\left\{a_4^L,b_4^L\right\};\\ \min\left\{H_1(A^L),H_1(B^L)\right\},\min\left\{H_2(A^L),H_2(B^L)\right\}\end{array}\right)\end{array}\right\}^c$$

$$=\bigcup_{\tilde{A}\in\tilde{h}_1,\tilde{B}\in\tilde{h}_2}\left\{\begin{array}{c}\left(\begin{array}{c}1-\max\left\{a_1^U,b_1^U\right\},1-\max\left\{a_2^U,b_2^U\right\},1-\max\left\{a_3^U,b_3^U\right\},1-\max\left\{a_4^U,b_4^U\right\};\\ \min\left\{H_1(A^U),H_1(B^U)\right\},\min\left\{H_2(A^U),H_2(B^U)\right\}\end{array}\right),\\ \left(\begin{array}{c}1-\max\left\{a_1^L,b_1^L\right\},1-\max\left\{a_2^L,b_2^L\right\},1-\max\left\{a_3^L,b_3^L\right\},1-\max\left\{a_4^L,b_4^L\right\};\\ \min\left\{H_1(A^L),H_1(B^L)\right\},\min\left\{H_2(A^L),H_2(B^L)\right\}\end{array}\right)\end{array}\right\}$$

5

$$= \bigcup_{\tilde{A}\in\tilde{h}_1,\tilde{B}\in\tilde{h}_2} \left\{ \begin{array}{c} \left(\begin{array}{c} \min\left\{1-a_1^U, 1-b_1^U\right\}, \min\left\{1-a_2^U, 1-b_2^U\right\}, \min\left\{1-a_3^U, 1-b_3^U\right\}, \\ \min\left\{1-a_4^U, 1-b_4^U\right\}; \min\left\{H_1(A^U), H_1(B^U)\right\}, \min\left\{H_2(A^U), H_2(B^U)\right\} \end{array} \right), \\ \left(\begin{array}{c} \min\left\{1-a_1^L, 1-b_1^L\right\}, \min\left\{1-a_2^L, 1-b_2^L\right\}, \min\left\{1-a_3^L, 1-b_3^L\right\}, \\ \min\left\{1-a_4^L, 1-b_4^L\right\}; \min\left\{H_1(A^L), H_1(B^L)\right\}, \min\left\{H_2(A^L), H_2(B^L)\right\} \end{array} \right) \end{array} \right\}$$

$= \tilde{h}_1^c \cap \tilde{h}_2^c$.

Similarly, we can prove that $\left(\tilde{h}_1 \cap \tilde{h}_2\right)^c = \tilde{h}_1^c \cup \tilde{h}_2^c$.■

Hu et al.(2015) [2] proposed the concept of interval type-2 hesitant fuzzy set (IT2HFS). Also, defined operation laws and corresponding properties are discussed. In this subsection, we briefly review some definitions of t-norm and t-conorm. Moreover, some other relationships can be established.

3.2 Operation laws of TIT2HFEs based on Archimedean t-norm and Archimedean t-conorm can be defined as follows:

Definition 12:[4, 11]

A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions:

(1) $T(1,x) = x$, for all $x \in [0,1]$.

(2) $T(x,y) = T(y,x)$, $\forall\ x,y \in [0,1]$.

(3) $T(x, T(y,z)) = T(T(x,y), z)$,$\forall\ x,y,z \in [0,1]$.

(4) If $x \le \grave{x}$ and $y \le \grave{y}$, then $T(x,y) \le T(\grave{x},\grave{y})$.

Definition 13:[4, 11]

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions:

(1) $S(0,x) = x$, for all $x \in [0,1]$.

(2) $S(x,y) = T(y,x)$, $\forall\ x,y \in [0,1]$.

(3) $S(x, S(y,z)) = S(S(x,y), z)$,$\forall\ x,y,z \in [0,1]$.

(4) If $x \le \grave{x}$ and $y \le \grave{y}$, then $S(x,y) \le S(\grave{x},\grave{y})$.

Definition 14:[4, 11]

A t-norm function $T(x,y)$ is called Archimedean t-norm if it is continuous and $T(x,x) < x$ for all $x \in (0,1)$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x,y \in (0,1)$.

Definition 15:[4, 11]

A t-conorm function $S(x,y)$ is called Archimedean t-conorm if it is continuous and $S(x,x) > x$ for all $x \in (0,1)$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x,y \in (0,1)$.

It is well known [11]that a strict Archimedean t-norm is expressed via its additive generator k as $T(x,y) = k^{-1}(k(x)+k(y))$, and similarly applied to the t-conorm $S(x,y) = l^{-1}(l(x)+l(y))$ with $l(t) = k(1-t)$. It is noted that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $k:[0,1]\to[0,\infty]$ such that $k(1)=0$.

Definition 16:[2]
Suppose
$\tilde{h}_1=\left\{\tilde{A}_1\in\tilde{h}_1|\tilde{A}_1=\left(\left(a_{11}^U,a_{12}^U,a_{13}^U,a_{14}^U;H_1(A_1^U),H_2(A_1^U)\right),\left(a_{11}^L,a_{12}^L,a_{13}^L,a_{14}^L;H_1(A_1^L),H_2(A_1^L)\right)\right)\right\}$
and $\tilde{h}_2=\left\{\tilde{A}_2\in\tilde{h}_2|\tilde{A}_2=\left(\left(a_{21}^U,a_{22}^U,a_{23}^U,a_{24}^U;H_1(A_2^U),H_2(A_2^U)\right),\left(a_{21}^L,a_{22}^L,a_{23}^L,a_{24}^L;H_1(A_2^L),H_2(A_2^L)\right)\right)\right\}$
are two IT2HFEs and $\lambda>0$. On the basis of **Definition 15,** we define the operation laws of IT2HFEs as follows :

(1) $\tilde{h}_1^\lambda=\bigcup_{\tilde{A}_1\in\tilde{h}_1}\left\{\begin{array}{c}\left(\left(k^{-1}(\lambda k(a_{11}^U),k^{-1}(\lambda k(a_{12}^U),k^{-1}(\lambda k(a_{13}^U),k^{-1}(\lambda k(a_{14}^U);H_1(A_1^U),H_2(A_1^U)\right),\right.\\ \left.\left(k^{-1}(\lambda k(a_{11}^L),k^{-1}(\lambda k(a_{12}^L),k^{-1}(\lambda k(a_{13}^L),k^{-1}(\lambda k(a_{14}^L);H_1(A_1^L),H_2(A_1^L)\right)\right)\end{array}\right\}$

(2) $\lambda\tilde{h}_1=\bigcup_{\tilde{A}_1\in\tilde{h}_1}\left\{\begin{array}{c}\left(\left(l^{-1}(\lambda l(a_{11}^U),l^{-1}(\lambda l(a_{12}^U),l^{-1}(\lambda l(a_{13}^U),l^{-1}(\lambda l(a_{14}^U);H_1(A_1^U),H_2(A_1^U)\right),\right.\\ \left.\left(l^{-1}(\lambda l(a_{11}^L),l^{-1}(\lambda l(a_{12}^L),l^{-1}(\lambda l(a_{13}^L),l^{-1}(\lambda l(a_{14}^L);H_1(A_1^L),H_2(A_1^L)\right)\right)\end{array}\right\}$.

(3) $\tilde{h}_1\oplus\tilde{h}_2=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\left\{\begin{array}{c}(l^{-1}(l(a_{11}^U)+l(a_{21}^U)),l^{-1}(l(a_{12}^U)+l(a_{22}^U)),l^{-1}(l(a_{13}^U)+l(a_{23}^U)),\\ l^{-1}(l(a_{14}^U)+l(a_{24}^U));\min(H_1(A_1^U),H_1(A_2^U)),\min(H_2(A_1^U),H_2(A_2^U)))\\ (l^{-1}(l(a_{11}^L)+l(a_{21}^L)),l^{-1}(l(a_{12}^L)+l(a_{22}^L)),l^{-1}(l(a_{13}^L)+l(a_{23}^L)),\\ l^{-1}(l(a_{14}^L)+l(a_{24}^L));\min(H_1(A_1^L),H_1(A_2^L)),\min(H_2(A_1^L),H_2(A_2^L)))\end{array}\right\}$.

(4) $\tilde{h}_1\otimes\tilde{h}_2=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\left\{\begin{array}{c}(k^{-1}(k(a_{11}^U)+k(a_{21}^U)),k^{-1}(k(a_{12}^U)+k(a_{22}^U)),k^{-1}(k(a_{13}^U)+k(a_{23}^U)),\\ k^{-1}(k(a_{14}^U)+k(a_{24}^U));\min(H_1(A_1^U),H_1(A_2^U)),\min(H_2(A_1^U),H_2(A_2^U)))\\ (k^{-1}(k(a_{11}^L)+k(a_{21}^L)),k^{-1}(k(a_{12}^L)+k(a_{22}^L)),k^{-1}(k(a_{13}^L)+k(a_{23}^L)),\\ k^{-1}(k(a_{14}^L)+k(a_{24}^L));\min(H_1(A_1^L),H_1(A_2^L)),\min(H_2(A_1^L),H_2(A_2^L)))\end{array}\right\}$.

Theorem 17:
Let $\tilde{h}_1=\left\{\tilde{A}_1\in\tilde{h}_1|\tilde{A}_1=\left(\left(a_{11}^U,a_{12}^U,a_{13}^U,a_{14}^U;H_1(A_1^U),H_2(A_1^U)\right),\left(a_{11}^L,a_{12}^L,a_{13}^L,a_{14}^L;H_1(A_1^L),H_2(A_1^L)\right)\right)\right\}$,
$\tilde{h}_2=\left\{\tilde{A}_2\in\tilde{h}_2|\tilde{A}_2=\left(\left(a_{21}^U,a_{22}^U,a_{23}^U,a_{24}^U;H_1(A_2^U),H_2(A_2^U)\right),\left(a_{21}^L,a_{22}^L,a_{23}^L,a_{24}^L;H_1(A_2^L),H_2(A_2^L)\right)\right)\right\}$
and $\tilde{h}_3=\left\{\tilde{A}_3\in\tilde{h}_3|\tilde{A}_3=\left(\left(a_{31}^U,a_{32}^U,a_{33}^U,a_{34}^U;H_1(A_3^U),H_2(A_3^U)\right),\left(a_{31}^L,a_{32}^L,a_{33}^L,a_{34}^L;H_1(A_3^L),H_2(A_3^L)\right)\right)\right\}$
are three TIT2HFEs, then the associative for operations $\oplus$ and $\otimes$ are vaild as follows:
(1) $\tilde{h}_1\oplus\left(\tilde{h}_2\oplus\tilde{h}_3\right)=\left(\tilde{h}_1\oplus\tilde{h}_2\right)\oplus\tilde{h}_3$
(2) $\tilde{h}_1\otimes\left(\tilde{h}_2\otimes\tilde{h}_3\right)=\left(\tilde{h}_1\otimes\tilde{h}_2\right)\otimes\tilde{h}_3$.

Proof:
we prove part (1), similarly we can be proven (2).

(1) $\tilde{h}_1\oplus\left(\tilde{h}_2\oplus\tilde{h}_3\right)=\tilde{h}_1\oplus$

$$\bigcup_{\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(l^{-1}(l(a_{21}^U+l(a_{31}^U)),l^{-1}(l(a_{22}^U+l(a_{32}^U)),\right.\\ l^{-1}(l(a_{23}^U+l(a_{33}^U)),l^{-1}(l(a_{24}^U+l(a_{34}^U));\\ \min\left(H_1(A_2^U),H_1(A_3^U)\right),\min\left(H_2(A_2^U),H_2(A_3^U)\right)),\\ \left(l^{-1}(l(a_{21}^L+l(a_{31}^L)),l^{-1}(l(a_{22}^L+l(a_{32}^L)),\right.\\ l^{-1}(l(a_{23}^L+l(a_{33}^L)),l^{-1}(l(a_{24}^L+l(a_{34}^L));\\ \min\left(H_1(A_2^L),H_1(A_3^L)\right),\min\left(H_2(A_2^L),H_2(A_3^L)\right))\end{array}\right\}$$

7

$$
= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \left\{ \begin{array}{c} l^{-1}(l(a_{11}^U) + l\left(l^{-1}(l(a_{21}^U + l(a_{31}^U))\right), l^{-1}(l(a_{12}^U) + l(l^{-1}(l(a_{22}^U + l(a_{32}^U))), \\ l^{-1}(l(a_{13}^U) + l(l^{-1}(l(a_{23}^U + l(a_{33}^U))), l^{-1}(l(a_{14}^U) + l(l^{-1}(l(a_{24}^U + l(a_{34}^U)); \\ \min((H_1(A_1^U), \min\left(H_1(A_2^U)\right), H_1(A_3^U))), \min((H_1(A_1^U), \min\left(H_2(A_2^U)\right), H_2(A_3^U))), \\ l^{-1}(l(a_{11}^L) + l(\left(l^{-1}(l(a_{21}^L + l(a_{31}^L))\right), l^{-1}(l(a_{12}^L) + l(l^{-1}(l(a_{22}^L + l(a_{32}^L))), \\ l^{-1}(l(a_{13}^L) + l(l^{-1}(l(a_{23}^L + l(a_{33}^L))), l^{-1}(l(a_{14}^U) + l(l^{-1}(l(a_{24}^L + l(a_{34}^L))); \\ \min((H_1(A_1^L), \min\left(H_1(A_2^L)\right), H_1(A_3^L))), \min((H_1(A_1^L), \min\left(H_2(A_2^L)\right), H_2(A_3^L))) \end{array} \right\}
$$

$$
= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \left\{ \begin{array}{c} l^{-1}(l(a_{11}^U) + l(a_{21}^U) + l(a_{31}^U)), l^{-1}(l(a_{12}^U) + l(a_{22}^U) + l(a_{32}^U)), \\ l^{-1}(l(a_{13}^U) + l(a_{23}^U) + l(a_{33}^U)), l^{-1}(l(a_{14}^U) + l(a_{24}^U) + l(a_{34}^U)); \\ \min(H_1(A_1^U), H_1(A_2^U), H_1(A_3^U)), \min(H_1(A_1^U), H_2(A_2^U), H_2(A_3^U)), \\ l^{-1}(l(a_{11}^L) + l(a_{21}^L) + l(a_{31}^L)), l^{-1}(l(a_{12}^L) + l(a_{22}^L) + l(a_{32}^L)), \\ l^{-1}(l(a_{13}^U) + l(a_{23}^L) + l(a_{33}^L)), l^{-1}(l(a_{14}^U) + l(a_{24}^L) + l(a_{34}^L)); \\ \min(H_1(A_1^L), H_1(A_2^L), H_1(A_3^L)), \min(H_1(A_1^L), H_2(A_2^L), H_2(A_3^L)) \end{array} \right\}
$$

$$
= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \left\{ \begin{array}{c} l^{-1}(l(l^{-1}l(a_{11}^U) + l(a_{21}^U))) + l(a_{31}^U)), l^{-1}(l(l^{-1}(l(a_{12}^U) + l(a_{22}^U))) + l(a_{32}^U)), \\ l^{-1}(l(l^{-1}(l(a_{13}^U) + l(a_{23}^U))) + l(a_{33}^U)), l^{-1}(l(l^{-1}(l(a_{14}^U) + l(a_{24}^U))) + l(a_{34}^U)); \\ \min(\min(H_1(A_1^U), H_1(A_2^U)), H_1(A_3^U)), \min(\min(H_1(A_1^U), H_2(A_2^U)), H_2(A_3^U)), \\ l^{-1}(l(l^{-1}(l(a_{11}^L) + l(a_{21}^L))) + l(a_{31}^L)), l^{-1}(l(l^{-1}(l(a_{12}^L) + l(a_{22}^L))) + l(a_{32}^L)), \\ l^{-1}(l(l^{-1}(l(a_{13}^U) + l(a_{23}^L))) + l(a_{33}^L)), l^{-1}(l(l^{-1}(l(a_{14}^U) + l(a_{24}^L))) + l(a_{34}^L)); \\ \min(\min(H_1(A_1^L), H_1(A_2^L)), H_1(A_3^L)), \min(\min(H_1(A_1^L), H_2(A_2^L)), H_2(A_3^L)) \end{array} \right\}
$$

$= \left(\tilde{h}_1 \oplus \tilde{h}_2\right) \oplus \tilde{h}_3.$ ■

Theorem 18:

Let $\tilde{h}_1 = \left\{\tilde{A}_1 \in \tilde{h}_1 | \tilde{A}_1 = \left(\left(a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1(A_1^U), H_2(A_1^U)\right), \left(a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1(A_1^L), H_2(A_1^L)\right)\right)\right\}$, $\tilde{h}_2 = \left\{\tilde{A}_2 \in \tilde{h}_2 | \tilde{A}_2 = \left(\left(a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U; H_1(A_2^U), H_2(A_2^U)\right), \left(a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; H_1(A_2^L), H_2(A_2^L)\right)\right)\right\}$ and $\tilde{h}_3 = \left\{\tilde{A}_3 \in \tilde{h}_3 | \tilde{A}_3 = \left(\left(a_{31}^U, a_{32}^U, a_{33}^U, a_{34}^U; H_1(A_3^U), H_2(A_3^U)\right), \left(a_{31}^L, a_{32}^L, a_{33}^L, a_{34}^L; H_1(A_3^L), H_2(A_3^L)\right)\right)\right\}$ are three TIT2HFEs, then:

(1) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \oplus \tilde{h}_3 = \left(\tilde{h}_1 \oplus \tilde{h}_3\right) \cup \left(\tilde{h}_2 \oplus \tilde{h}_3\right)$

(2) $\left(\tilde{h}_1 \cap \tilde{h}_2\right) \oplus \tilde{h}_3 = \left(\tilde{h}_1 \oplus \tilde{h}_3\right) \cap \left(\tilde{h}_2 \oplus \tilde{h}_3\right)$

(3) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \otimes \tilde{h}_3 = \left(\tilde{h}_1 \otimes \tilde{h}_3\right) \cup \left(\tilde{h}_2 \otimes \tilde{h}_3\right)$

(4) $\left(\tilde{h}_1 \cap \tilde{h}_2\right) \otimes \tilde{h}_3 = \left(\tilde{h}_1 \otimes \tilde{h}_3\right) \cap \left(\tilde{h}_2 \otimes \tilde{h}_3\right)$

(5) $\tilde{h}_1 \oplus \left(\tilde{h}_2 \cup \tilde{h}_3\right) = \left(\tilde{h}_1 \oplus \tilde{h}_2\right) \cup \left(\tilde{h}_1 \oplus \tilde{h}_3\right)$

(6) $\tilde{h}_1 \oplus \left(\tilde{h}_2 \cap \tilde{h}_3\right) = \left(\tilde{h}_1 \oplus \tilde{h}_2\right) \cap \left(\tilde{h}_1 \oplus \tilde{h}_3\right)$

(7) $\tilde{h}_1 \otimes \left(\tilde{h}_2 \cup \tilde{h}_3\right) = \left(\tilde{h}_1 \otimes \tilde{h}_2\right) \cup \left(\tilde{h}_1 \otimes \tilde{h}_3\right)$

(8) $\tilde{h}_1 \otimes \left(\tilde{h}_2 \cap \tilde{h}_3\right) = \left(\tilde{h}_1 \otimes \tilde{h}_2\right) \cap \left(\tilde{h}_1 \otimes \tilde{h}_3\right)$

Proof:

We prove (1) and (3). similarly, we can the others.

(1) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \oplus \tilde{h}_3 =$

8

$$\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2}\left\{\begin{array}{c}\left(\begin{array}{c}\max\left\{a^U_{11},a^U_{21}\right\},\max\left\{a^U_{12},a^U_{22}\right\},\max\left\{a^U_{13},a^U_{23}\right\},\max\left\{a^U_{14},a^U_{24}\right\};\\ \min\left\{H_1(A^U_1),H_1(A^U_2)\right\},\min\left\{H_2(A^U_1),H_2(A^U_2)\right\}\end{array}\right),\\ \left(\begin{array}{c}\max\left\{a^L_{11},a^L_{21}\right\},\max\left\{a^L_{12},a^L_{22}\right\},\max\left\{a^L_{13},a^L_{23}\right\},\max\left\{a^L_{14},a^L_{24}\right\};\\ \min\left\{H_1(A^L_1),H_1(A^L_2)\right\},\min\left\{H_2(A^L_1),H_2(A^L_2)\right\}\end{array}\right)\end{array}\right\}$$

$$\oplus\tilde{h}_3$$

$$=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(\begin{array}{c}l^{-1}(l(\max\left\{a^U_{11},a^U_{21}\right\})+l(a^U_{31})),l^{-1}(l(\max\left\{a^U_{12},a^U_{22}\right\})+l(a^U_{32})),\\ l^{-1}(l(\max\left\{a^U_{13},a^U_{23}\right\}+l(a^U_{33})),l^{-1}(l(\max\left\{a^U_{14},a^U_{24}\right\})+l(a^U_{34}));\\ \min(\min\left\{H_1(A^U_1),H_1(A^U_2)\right\},H_1(A^U_3)),\\ \min(\min\left\{H_2(A^U_1),H_2(A^U_2)\right\},H_2(A^U_3))\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(l(\max\left\{a^L_{11},a^L_{21}\right\})+l(a^L_{31})),l^{-1}(l(\max\left\{a^L_{12},a^L_{22}\right\})+l(a^L_{32})),\\ l^{-1}(l(\max\left\{a^L_{13},a^L_{23}\right\})+l(a^L_{33})),l^{-1}(l(\max\left\{a^L_{14},a^L_{24}\right\})+l(a^L_{34}));\\ \min(\min\left\{H_1(A^L_1),H_1(A^L_2)\right\},H_1(A^L_3)),\\ \min(\min\left\{H_2(A^L_1),H_2(A^L_2)\right\},H_2(A^L_3))\end{array}\right)\end{array}\right\}$$

$$=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(\begin{array}{c}l^{-1}(\max\{l(a^U_{11})+l(a^U_{31}),l(a^U_{21})+l(a^U_{31})\}),\\ l^{-1}(\max\{l(a^U_{12})+l(a^U_{32}),l(a^U_{22})+l(a^U_{32})\}),\\ l^{-1}(\max\{l(a^U_{13})+l(a^U_{33}),l(a^U_{23})+l(a^U_{33})\}),\\ l^{-1}(\max\{l(a^U_{14})+l(a^U_{34}),l(a^U_{24})+l(a^U_{34})\});\\ \min(\min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_1(A^U_2),H_1(A^U_3)\}),\\ \min(\min\{H_2(A^U_1),H_2(A^U_3)\},\min\{H_2(A^U_2),H_2(A^U_3)\})\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(\max\{l(a^L_{11})+l(a^L_{31}),l(a^L_{21})+l(a^L_{31})\}),\\ l^{-1}(\max\{l(a^L_{12})+l(a^L_{31}),l(a^L_{22})+l(a^L_{32})\}),\\ l^{-1}(\max\{l(a^L_{13})+l(a^L_{33}),l(a^L_{23})+l(a^L_{33})\}),\\ l^{-1}(\max l(a^L_{14})+l(a^L_{34}),l(a^L_{24})+l(a^L_{34})\});\\ \min(\min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_1(A^L_2),H_1(A^L_3)\}),\\ \min(\min\{H_2(A^L_1),H_2(A^L_3)\},\min\{H_2(A^L_2),H_2(A^L_3)\})\end{array}\right)\end{array}\right\}$$

$$=\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3}\left\{\begin{array}{c}\left(\begin{array}{c}\max\{l^{-1}(l(a^U_{11})+l(a^U_{31})),l^{-1}(l(a^U_{21})+l(a^U_{31}))\},\\ \max\{l^{-1}(l(a^U_{12})+l(a^U_{32})),l^{-1}(l(a^U_{22})+l(a^U_{32}))\},\\ \max\{l^{-1}(l(a^U_{13})+l(a^U_{33})),l^{-1}(l(a^U_{23})+l(a^U_{33}))\},\\ \max\{l^{-1}(l(a^U_{14})+l(a^U_{34})),l^{-1}(l(a^U_{24})+l(a^U_{34}))\};\\ \min(\min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_1(A^U_2),H_1(A^U_3)\}),\\ \min(\min\{H_2(A^U_1),H_2(A^U_3)\},\min\{H_2(A^U_2),H_2(A^U_3)\})\end{array}\right),\\ \left(\begin{array}{c}\max\{l^{-1}(l(a^L_{11})+l(a^L_{31})),l^{-1}(l(a^L_{21})+l(a^L_{31}))\},\\ \max\{l^{-1}(l(a^L_{12})+l(a^L_{31})),l^{-1}(l(a^L_{22})+l(a^L_{32}))\},\\ \max\{l^{-1}(l(a^L_{13})+l(a^L_{33})),l^{-1}(l(a^L_{23})+l(a^L_{33}))\},\\ \max\{l^{-1}(l(a^L_{14})+l(a^L_{34})),l^{-1}(l(a^L_{24})+l(a^L_{34}))\};\\ \min(\min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_1(A^L_2),H_1(A^L_3)\}),\\ \min(\min\{H_2(A^L_1),H_2(A^L_3)\},\min\{H_2(A^L_2),H_2(A^L_3)\})\end{array}\right)\end{array}\right\}$$

$$=\left\{\bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_3\in\tilde{h}_3}\begin{array}{c}\left(\begin{array}{c}l^{-1}(l(a^U_{11})+l(a^U_{31})),l^{-1}(l(a^U_{12})+l(a^U_{32})),\\ l^{-1}(l(a^U_{13})+l(a^U_{33})),l^{-1}(l(a^U_{14})+l(a^U_{34}));\\ \min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_2(A^U_1),H_2(A^U_3)\}\end{array}\right),\\ \left(\begin{array}{c}l^{-1}(l(a^L_{11})+l(a^L_{31})),l^{-1}(l(a^L_{12})+l(a^L_{31})),\\ l^{-1}(l(a^L_{13})+l(a^L_{33})),l^{-1}(l(a^L_{14})+l(a^L_{34}));\\ \min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_2(A^L_1),H_2(A^L_3)\}\end{array}\right)\end{array}\right\}$$

9

$$\cup \left\{ \bigcup_{\tilde{A}_2 \in \tilde{h}_2, \tilde{A}_3 \in \tilde{h}_3} \begin{array}{c} \left(\begin{array}{c} l^{-1}(l(a_{21}^U) + l(a_{31}^U)), l^{-1}(l(a_{22}^U) + l(a_{32}^U)), \\ l^{-1}(l(a_{23}^U) + l(a_{33}^U)), l^{-1}(l(a_{24}^U) + l(a_{34}^U)); \\ \min\{H_1(A_2^U), H_1(A_3^U)\}, \min\{H_2(A_2^U), H_2(A_3^U)\} \end{array} \right), \\ \left(\begin{array}{c} l^{-1}(l(a_{21}^L) + l(a_{31}^L)), l^{-1}(l(a_{22}^L) + l(a_{31}^L)), \\ l^{-1}(l(a_{23}^L) + l(a_{33}^L)), l^{-1}(l(a_{24}^L) + l(a_{34}^L)); \\ \min\{H_1(A_2^L), H_1(A_3^L)\}, \min\{H_2(A_2^L), H_2(A_3^L)\} \end{array} \right) \end{array} \right\}$$

$$= \left(\tilde{h}_1 \oplus \tilde{h}_3\right) \cup \left(\tilde{h}_2 \oplus \tilde{h}_3\right).$$

(3) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \otimes \tilde{h}_3 =$

$$\bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \left\{ \begin{array}{c} \left(\begin{array}{c} \max\left\{a_{11}^U, a_{21}^U\right\}, \max\left\{a_{12}^U, a_{22}^U\right\}, \max\left\{a_{13}^U, a_{23}^U\right\}, \max\left\{a_{14}^U, a_{24}^U\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{array} \right), \\ \left(\begin{array}{c} \max\left\{a_{11}^L, a_{21}^L\right\}, \max\left\{a_{12}^L, a_{22}^L\right\}, \max\left\{a_{13}^L, a_{23}^L\right\}, \max\left\{a_{14}^L, a_{24}^L\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{array} \right) \end{array} \right\}$$

$\otimes \tilde{h}_3$

$$= \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2, \tilde{A}_3 \in \tilde{h}_3} \left\{ \begin{array}{c} \left(\begin{array}{c} k^{-1}(k(\max\left\{a_{11}^U, a_{21}^U\right\}) + k(a_{31}^U)), k^{-1}(k(\max\left\{a_{12}^U, a_{22}^U\right\}) + k(a_{32}^U)), \\ k^{-1}(k(\max\left\{a_{13}^U, a_{23}^U\right\} + k(a_{33}^U)), k^{-1}(k(\max\left\{a_{14}^U, a_{24}^U\right\}) + k(a_{34}^U)); \\ \min(\min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, H_1(A_3^U)), \\ \min(\min\left\{H_2(A_1^U), H_2(A_2^U)\right\}, H_2(A_3^U)) \end{array} \right), \\ \left(\begin{array}{c} k^{-1}(k(\max\left\{a_{11}^L, a_{21}^L\right\}) + k(a_{31}^L)), k^{-1}(k(\max\left\{a_{12}^L, a_{22}^L\right\}) + k(a_{32}^L)), \\ k^{-1}(k(\max\left\{a_{13}^L, a_{23}^L\right\}) + k(a_{33}^L)), k^{-1}(k(\max\left\{a_{14}^L, a_{24}^L\right\}) + k(a_{34}^L)); \\ \min(\min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, H_1(A_3^L)), \\ \min(\min\left\{H_2(A_1^L), H_2(A_2^L)\right\}, H_2(A_3^L)) \end{array} \right) \end{array} \right\}$$

$$= \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2, \tilde{A}_3 \in \tilde{h}_3} \left\{ \begin{array}{c} \left(\begin{array}{c} k^{-1}(\max\{k(a_{11}^U) + k(a_{31}^U), k(a_{21}^U) + k(a_{31}^U)\}), \\ k^{-1}(\max\{k(a_{12}^U) + k(a_{32}^U), k(a_{22}^U) + k(a_{32}^U)\}), \\ k^{-1}(\max\{k(a_{13}^U) + k(a_{33}^U), k(a_{23}^U) + k(a_{33}^U)\}), \\ k^{-1}(\max\{k(a_{14}^U) + k(a_{34}^U), k(a_{24}^U) + k(a_{34}^U)\}); \\ \min(\min\{H_1(A_1^U), H_1(A_3^U)\}, \min\{H_1(A_2^U), H_1(A_3^U)\}), \\ \min(\min\{H_2(A_1^U), H_2(A_3^U)\}, \min\{H_2(A_2^U), H_2(A_3^U)\}) \end{array} \right), \\ \left(\begin{array}{c} k^{-1}(\max\{k(a_{11}^L) + k(a_{31}^L), k(a_{21}^L) + k(a_{31}^L)\}), \\ k^{-1}(\max\{k(a_{12}^L) + k(a_{31}^L), k(a_{22}^L) + k(a_{32}^L)\}), \\ k^{-1}(\max\{k(a_{13}^L) + k(a_{33}^L), k(a_{23}^L) + k(a_{33}^L)\}), \\ k^{-1}(\max k(a_{14}^L) + k(a_{34}^L), k(a_{24}^L) + k(a_{34}^L)\}); \\ \min(\min\{H_1(A_1^L), H_1(A_3^L)\}, \min\{H_1(A_2^L), H_1(A_3^L)\}), \\ \min(\min\{H_2(A_1^L), H_2(A_3^L)\}, \min\{H_2(A_2^L), H_2(A_3^L)\}) \end{array} \right) \end{array} \right\}$$

$$
= \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \left\{ \begin{array}{c} \left(\begin{array}{c} \max\{k^{-1}(k(a^U_{11})+k(a^U_{31})),k^{-1}(k(a^U_{21})+k(a^U_{31}))\}, \\ \max\{k^{-1}(k(a^U_{12})+k(a^U_{32})),k^{-1}(k(a^U_{22})+k(a^U_{32}))\}, \\ \max\{k^{-1}(k(a^U_{13})+k(a^U_{33})),k^{-1}(k(a^U_{23})+k(a^U_{33}))\}, \\ \max\{k^{-1}(k(a^U_{14})+k(a^U_{34})),k^{-1}(k(a^U_{24})+k(a^U_{34}))\}; \\ \min(\min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_1(A^U_2),H_1(A^U_3)\}), \\ \min(\min\{H_2(A^U_1),H_2(A^U_3)\},\min\{H_2(A^U_2),H_2(A^U_3)\}) \end{array} \right), \\ \left(\begin{array}{c} \max\{k^{-1}(k(a^L_{11})+k(a^L_{31})),k^{-1}(k(a^L_{21})+k(a^L_{31}))\}, \\ \max\{k^{-1}(k(a^L_{12})+k(a^L_{31})),k^{-1}(k(a^L_{22})+k(a^L_{32}))\}, \\ \max\{k^{-1}(k(a^L_{13})+k(a^L_{33})),k^{-1}(k(a^L_{23})+k(a^L_{33}))\}, \\ \max\{k^{-1}(k(a^L_{14})+k(a^L_{34})),k^{-1}(k(a^L_{24})+k(a^L_{34}))\}; \\ \min(\min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_1(A^L_2),H_1(A^L_3)\}), \\ \min(\min\{H_2(A^L_1),H_2(A^L_3)\},\min\{H_2(A^L_2),H_2(A^L_3)\}) \end{array} \right) \end{array} \right\}
$$

$$
= \left\{ \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_3\in\tilde{h}_3} \begin{array}{c} \left(\begin{array}{c} k^{-1}(k(a^U_{11})+k(a^U_{31})),k^{-1}(k(a^U_{12})+k(a^U_{32})), \\ k^{-1}(k(a^U_{13})+k(a^U_{33})),k^{-1}(k(a^U_{14})+k(a^U_{34})); \\ \min\{H_1(A^U_1),H_1(A^U_3)\},\min\{H_2(A^U_1),H_2(A^U_3)\} \end{array} \right), \\ \left(\begin{array}{c} k^{-1}(k(a^L_{11})+k(a^L_{31})),k^{-1}(k(a^L_{12})+k(a^L_{31})), \\ k^{-1}(k(a^L_{13})+k(a^L_{33})),k^{-1}(k(a^L_{14})+k(a^L_{34})); \\ \min\{H_1(A^L_1),H_1(A^L_3)\},\min\{H_2(A^L_1),H_2(A^L_3)\} \end{array} \right) \end{array} \right\}
$$

$$
\cup \left\{ \bigcup_{\tilde{A}_2\in\tilde{h}_2,\tilde{A}_3\in\tilde{h}_3} \begin{array}{c} \left(\begin{array}{c} l^{-1}(l(a^U_{21})+l(a^U_{31})),l^{-1}(l(a^U_{22})+l(a^U_{32})), \\ k^{-1}(k(a^U_{23})+k(a^U_{33})),k^{-1}(k(a^U_{24})+k(a^U_{34})); \\ \min\{H_1(A^U_2),H_1(A^U_3)\},\min\{H_2(A^U_2),H_2(A^U_3)\} \end{array} \right), \\ \left(\begin{array}{c} k^{-1}(k(a^L_{21})+k(a^L_{31})),k^{-1}(k(a^L_{22})+k(a^L_{31})), \\ k^{-1}(k(a^L_{23})+k(a^L_{33})),k^{-1}(k(a^L_{24})+k(a^L_{34})); \\ \min\{H_1(A^L_2),H_1(A^L_3)\},\min\{H_2(A^L_2),H_2(A^L_3)\} \end{array} \right) \end{array} \right\}
$$

$$
= \left(\tilde{h}_1 \otimes \tilde{h}_3\right) \cup \left(\tilde{h}_2 \otimes \tilde{h}_3\right). \blacksquare
$$

Theorem 19:
Let $\tilde{h}_1$and $\tilde{h}_2$ be two TIT2HFEs, then:
(1) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \oplus \left(\tilde{h}_1 \cap \tilde{h}_2\right) = \tilde{h}_1 \oplus \tilde{h}_2$.
(2) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \otimes \left(\tilde{h}_1 \cap \tilde{h}_2\right) = \tilde{h}_1 \otimes \tilde{h}_2$.

Proof:
(1) We know that for any two real numbers a and b, it follows that:

$$
\begin{gathered} \max\{a,b\} + \min\{a,b\} = a+b \\ \max\{a,b\} \, . \min\{a,b\} = a.b \end{gathered}
$$

Then we have:
(1) $\left(\tilde{h}_1 \cup \tilde{h}_2\right) \oplus \left(\tilde{h}_1 \cap \tilde{h}_2\right) =$

$$
\left\{ \bigcup_{\tilde{A}_1\in\tilde{h}_1,\tilde{A}_2\in\tilde{h}_2} \begin{array}{c} \left(\begin{array}{c} \max\left\{a^U_{11},a^U_{21}\right\},\max\left\{a^U_{12},a^U_{22}\right\},\max\left\{a^U_{13},a^U_{23}\right\},\max\left\{a^U_{14},a^U_{24}\right\}; \\ \min\left\{H_1(A^U_1),H_1(A^U_2)\right\},\min\left\{H_2(A^U_1),H_2(A^U_2)\right\} \end{array} \right), \\ \left(\begin{array}{c} \max\left\{a^L_{11},a^L_{21}\right\},\max\left\{a^L_{12},a^L_{22}\right\},\max\left\{a^L_{13},a^L_{23}\right\},\max\left\{a^L_{14},a^L_{24}\right\}; \\ \min\left\{H_1(A^L_1),H_1(A^L_2)\right\},\min\left\{H_2(A^L_1),H_2(A^L_2)\right\} \end{array} \right) \end{array} \right\}
$$

11

$$\oplus \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \left(\begin{array}{c} \min\left\{a_{11}^U, a_{21}^U\right\}, \min\left\{a_{12}^U, a_{22}^U\right\}, \min\left\{a_{13}^U, a_{23}^U\right\}, \min\left\{a_{14}^U, a_{24}^U\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{array} \right), \\ \left(\begin{array}{c} \min\left\{a_{11}^L, a_{21}^L\right\}, \min\left\{a_{12}^L, a_{22}^L\right\}, \min\left\{a_{13}^L, a_{23}^L\right\}, \min\left\{a_{14}^L, a_{24}^L\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{array} \right) \end{array} \right\}$$

$$= \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \left(\begin{array}{c} l^{-1}(l(\max\left\{a_{11}^U, a_{21}^U\right\}) + l(\min\left\{a_{11}^U, a_{21}^U\right\})), \\ l^{-1}(l(\max\left\{a_{12}^U, a_{22}^U\right\}) + l(\min\left\{a_{12}^U, a_{22}^U\right\})), \\ l^{-1}(l(\max\left\{a_{13}^U, a_{23}^U\right\}) + l(\min\left\{a_{13}^U, a_{23}^U\right\})), \\ l^{-1}(l(\max\left\{a_{14}^U, a_{24}^U\right\}) + l(\min\left\{a_{14}^U, a_{24}^U\right\})); \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{array} \right), \\ \left(\begin{array}{c} l^{-1}(l(\max\left\{a_{11}^L, a_{21}^L\right\}) + l(\min\left\{a_{11}^L, a_{21}^L\right\})), \\ l^{-1}(l(\max\left\{a_{12}^L, a_{22}^L\right\}) + l(\min\left\{a_{12}^L, a_{22}^L\right\})), \\ l^{-1}(l(\max\left\{a_{13}^L, a_{23}^L\right\}) + l(\min\left\{a_{13}^L, a_{23}^L\right\})) \\ , l^{-1}(l(\max\left\{a_{14}^L, a_{24}^L\right\}) + l(\min\left\{a_{14}^L, a_{24}^L\right\})); \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{array} \right) \end{array} \right\}$$

$$= \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \left(\begin{array}{c} l^{-1}(\max\left\{l(a_{11}^U), l(a_{21}^U)\right\} + \min\left\{l(a_{11}^U), l(a_{21}^U)\right\}), \\ l^{-1}(\max\left\{l(a_{12}^U), l(a_{22}^U)\right\} + \min\left\{l(a_{12}^U), l(a_{22}^U)\right\}), \\ l^{-1}(\max\left\{l(a_{13}^U), l(a_{23}^U)\right\} + \min\left\{l(a_{13}^U), l(a_{23}^U)\right\}), \\ l^{-1}(\max\left\{l(a_{14}^U), l(a_{24}^U)\right\} + \min\left\{l(a_{14}^U), l(a_{24}^U)\right\}); \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{array} \right), \\ \left(\begin{array}{c} l^{-1}(\max\left\{l(a_{11}^L), l\left(a_{21}^L\right)\right\} + \min\left\{l\left(a_{11}^L\right), l\left(a_{21}^L\right)\right\}), \\ l^{-1}(\max\left\{l\left(a_{12}^L\right), l\left(a_{22}^L\right)\right\} + \min\left\{l\left(a_{12}^L\right), l\left(a_{22}^L\right)\right\}), \\ l^{-1}(\max\left\{l\left(a_{13}^L\right), l\left(a_{23}^L\right)\right\} + \min\left\{l\left(a_{13}^L\right), l\left(a_{23}^L\right)\right\}), \\ l^{-1}(\max\left\{l\left(a_{14}^L\right), l\left(a_{24}^L\right)\right\} + \min\left\{l\left(a_{14}^L\right), l\left(a_{24}^L\right)\right\}); \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{array} \right) \end{array} \right\}$$

$$= \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \left(\begin{array}{c} l^{-1}\left\{l(a_{11}^U) + l(a_{21}^U)\right\}, l^{-1}\left\{l(a_{12}^U) + l(a_{22}^U)\right\}, \\ l^{-1}\left\{l(a_{13}^U) + l(a_{23}^U)\right\}, l^{-1}\left\{l(a_{14}^U) + l(a_{24}^U)\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{array} \right), \\ \left(\begin{array}{c} l^{-1}\left\{l(a_{11}^L) + l\left(a_{21}^L\right)\right\}, l^{-1}\left\{l\left(a_{12}^L\right) + l\left(a_{22}^L\right)\right\}, \\ l^{-1}\left\{l\left(a_{13}^L\right) + l\left(a_{23}^L\right)\right\}, l^{-1}\left\{l\left(a_{14}^L\right) + l\left(a_{24}^L\right)\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{array} \right) \end{array} \right\}$$

$= \tilde{h}_1 \oplus \tilde{h}_2$.
Similarly, we can proven (2) .■

Theorem 20:
Let $\tilde{h}_1$and $\tilde{h}_2$ be two TIT2HFEs and $\lambda > 0$, then:
(1) $\lambda\left(\tilde{h}_1 \cup \tilde{h}_2\right) = \lambda\tilde{h}_1 \cup \lambda\tilde{h}_2$.
(2) $\lambda\left(\tilde{h}_1 \cap \tilde{h}_2\right) = \lambda\tilde{h}_1 \cap \lambda\tilde{h}_2$.
(3) $\left(\tilde{h}_1 \cup \tilde{h}_2\right)^\lambda = \tilde{h}_1^\lambda \cup \tilde{h}_2^\lambda$.
(4) $\left(\tilde{h}_1 \cap \tilde{h}_2\right)^\lambda = \tilde{h}_1^\lambda \cap \tilde{h}_2^\lambda$.

Proof:
In the following, we prove (1) and (3), the rest can be proven analogously:

12

(1) $\lambda\left(\tilde{h}_1 \cup \tilde{h}_2\right) =$

$$\lambda \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} \max\left\{a_{11}^U, a_{21}^U\right\}, \max\left\{a_{12}^U, a_{22}^U\right\}, \max\left\{a_{13}^U, a_{23}^U\right\}, \max\left\{a_{14}^U, a_{24}^U\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} \max\left\{a_{11}^L, a_{21}^L\right\}, \max\left\{a_{12}^L, a_{22}^L\right\}, \max\left\{a_{13}^L, a_{23}^L\right\}, \max\left\{a_{14}^L, a_{24}^L\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right\}$$

$$= l^{-1} \left\{ \lambda l \left(\bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} \max\left\{a_{11}^U, a_{21}^U\right\}, \max\left\{a_{12}^U, a_{22}^U\right\}, \max\left\{a_{13}^U, a_{23}^U\right\}, \max\left\{a_{14}^U, a_{24}^U\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} \max\left\{a_{11}^L, a_{21}^L\right\}, \max\left\{a_{12}^L, a_{22}^L\right\}, \max\left\{a_{13}^L, a_{23}^L\right\}, \max\left\{a_{14}^L, a_{24}^L\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right) \right\}$$

$$= \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} l^{-1}(\lambda l(\max\left\{a_{11}^U, a_{21}^U\right\}), l^{-1}(\lambda l(\max\left\{a_{12}^U, a_{22}^U\right\}), \\ l^{-1}(\lambda l(\max\left\{a_{13}^U, a_{23}^U\right\}), l^{-1}(\lambda l(\max\left\{a_{14}^U, a_{24}^U\right\}); \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} l^{-1}(\lambda l(\max\left\{a_{11}^L, a_{21}^L\right\}), l^{-1}(\lambda l(\max\left\{a_{12}^L, a_{22}^L\right\}), \\ l^{-1}(\lambda l(\max\left\{a_{13}^L, a_{23}^L\right\}), l^{-1}(\lambda l(\max\left\{a_{14}^L, a_{24}^L\right\}); \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right\}$$

$$= \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} \max\left\{l^{-1}(\lambda l(a_{11}^U), l^{-1}(\lambda l(a_{21}^U)\right\}, \max\left\{l^{-1}(\lambda l(a_{12}^U), l^{-1}(\lambda l(a_{22}^U)\right\}, \\ \max\left\{l^{-1}(\lambda l(a_{13}^U), l^{-1}(\lambda l(a_{23}^U)\right\}, \max\left\{l^{-1}(\lambda l(a_{14}^U), l^{-1}(\lambda l(a_{24}^U)\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} \max\left\{l^{-1}(\lambda l(a_{11}^L), l^{-1}(\lambda l(a_{21}^L)\right\}, \max\left\{l^{-1}(\lambda l(a_{12}^L), l^{-1}(\lambda l(a_{22}^L)\right\}, \\ \max\left\{l^{-1}(\lambda l(a_{13}^L), l^{-1}(\lambda l(a_{23}^L)\right\}, \max\left\{l^{-1}(\lambda l(a_{14}^L), l^{-1}(\lambda l(a_{24}^L)\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right\}$$

$= \lambda\tilde{h}_1 \cup \lambda\tilde{h}_2.$

(3) $\left(\tilde{h}_1 \cup \tilde{h}_2\right)^\lambda =$

$$\left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} \max\left\{a_{11}^U, a_{21}^U\right\}, \max\left\{a_{12}^U, a_{22}^U\right\}, \max\left\{a_{13}^U, a_{23}^U\right\}, \max\left\{a_{14}^U, a_{24}^U\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} \max\left\{a_{11}^L, a_{21}^L\right\}, \max\left\{a_{12}^L, a_{22}^L\right\}, \max\left\{a_{13}^L, a_{23}^L\right\}, \max\left\{a_{14}^L, a_{24}^L\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right\}^\lambda$$

$$= k^{-1} \left\{ \lambda k \left(\bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} \max\left\{a_{11}^U, a_{21}^U\right\}, \max\left\{a_{12}^U, a_{22}^U\right\}, \max\left\{a_{13}^U, a_{23}^U\right\}, \max\left\{a_{14}^U, a_{24}^U\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} \max\left\{a_{11}^L, a_{21}^L\right\}, \max\left\{a_{12}^L, a_{22}^L\right\}, \max\left\{a_{13}^L, a_{23}^L\right\}, \max\left\{a_{14}^L, a_{24}^L\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right) \right\}$$

$$= \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} k^{-1}(\lambda k(\max\left\{a_{11}^U, a_{21}^U\right\}), k^{-1}(\lambda k(\max\left\{a_{12}^U, a_{22}^U\right\}), \\ k^{-1}(\lambda k(\max\left\{a_{13}^U, a_{23}^U\right\}), k^{-1}(\lambda k(\max\left\{a_{14}^U, a_{24}^U\right\}); \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} k^{-1}(\lambda k(\max\left\{a_{11}^L, a_{21}^L\right\}), k^{-1}(\lambda k(\max\left\{a_{12}^L, a_{22}^L\right\}), \\ k^{-1}(\lambda k(\max\left\{a_{13}^L, a_{23}^L\right\}), k^{-1}(\lambda k(\max\left\{a_{14}^L, a_{24}^L\right\}); \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right\}$$

$$= \left\{ \bigcup_{\tilde{A}_1 \in \tilde{h}_1, \tilde{A}_2 \in \tilde{h}_2} \begin{array}{c} \begin{pmatrix} \max\left\{k^{-1}(\lambda k(a_{11}^U), k^{-1}(\lambda k(a_{21}^U)\right\}, \max\left\{k^{-1}(\lambda k(a_{12}^U), k^{-1}(\lambda k(a_{22}^U)\right\}, \\ \max\left\{k^{-1}(\lambda k(a_{13}^U), k^{-1}(\lambda k(a_{23}^U)\right\}, \max\left\{k^{-1}(\lambda k(a_{14}^U), k^{-1}(\lambda k(a_{24}^U)\right\}; \\ \min\left\{H_1(A_1^U), H_1(A_2^U)\right\}, \min\left\{H_2(A_1^U), H_2(A_2^U)\right\} \end{pmatrix}, \\ \begin{pmatrix} \max\left\{k^{-1}(\lambda k(a_{11}^L), k^{-1}(\lambda k(a_{21}^L)\right\}, \max\left\{k^{-1}(\lambda k(a_{12}^L), k^{-1}(\lambda k(a_{22}^L)\right\}, \\ \max\left\{k^{-1}(\lambda k(a_{13}^L), k^{-1}(\lambda k(a_{23}^L)\right\}, \max\left\{k^{-1}(\lambda k(a_{14}^L), k^{-1}(\lambda k(a_{24}^L)\right\}; \\ \min\left\{H_1(A_1^L), H_1(A_2^L)\right\}, \min\left\{H_2(A_1^L), H_2(A_2^L)\right\} \end{pmatrix} \end{array} \right\}$$

$= \tilde{h}_1^{\lambda} \cup \tilde{h}_2^{\lambda}.\blacksquare$

4 Conclusions

We introduced the notions of Trapezoidal interval type-2 hesitant fuzzy set. At the same time, some operation laws of TIT2HFS were provided to complete its theory.

Conflict of Interests
The authors declare that there is no conflict of interest regarding the publication of this paper.

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MENGER PROBABILISTIC NORMED RIESZ SPACES AND STABILITY OF LATTICE PRESERVING FUNCTIONAL EQUATION

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Abstract. The purpose of this paper is to introduce the concept of a Menger probabilistic normed Riesz space. We study some properties of these spaces and compare normed Riesz spaces with Menger probabilistic normed Riesz spaces. Next, we investigate the Hyers-Ulam stability of lattice homomorphisms in Menger probabilistic normed Riesz spaces.

1. Introduction

Riesz spaces are named after Frigyes Riesz who first defined them in 1930 [20]. Riesz spaces are real vector spaces equipped with a partial order. Under this partial order the Riesz space must satisfy some axioms, including the axiom that it is a lattice.

The theory of probabilistic normed spaces (briefly, PN spaces) was born as a "natural consequence of the theory of probabilistic metric spaces. For the basic theory of vector lattices (Riesz spaces) and Banach lattices and for unexplained terminology we refer to [2, 17, 27].

The theory of probabilistic metric spaces was introduced in 1951 by Menger [11]. He replaced the number $d(p, q)$, which gives the distance between two points p and q in a nonempty set S, by a distribution function F_{pq} whose value $F_{p,q}(t)$ at $t \in [0, +\infty)$ is interpreted as the probability that the distance between the points p and q is smaller than t. Menger's idea was developed by the authors in [6, 7, 10].

The theory of PN spaces was introduced by Serstnev [23]. It were redefined by Alsina, Schweizer and Sklar [3, 4].

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D? If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940. In 1941, Hyers [8] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. The result of Hyers was generalized by Rassias [18] for linear mapping by considering an unbounded Cauchy difference. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([1, 9]). Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated in [12, 13, 14, 15, 16, 19, 24, 25].

In this paper, Riesz fuzzy normed spaces are defined and the stability conditions are verified.

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A nonempty set V with a relation "$\leq$" is said to be an ordered set whenever the following conditions are satisfied:

1. $x \leq x$ for all $x \in V$.
2. $x \leq y$ and $y \leq x$ imply that $x = y$.
3. $x \leq y$ and $y \leq z$ imply that $x \leq z$.

If, in addition, for all $x, y \in V$ either $x \leq y$ or $y \leq x$, then V is called a totally ordered set. Let A be subset of an ordered set V. $x \in V$ is called an upper bound of A if $y \leq x$ for all $y \in A$. $z \in V$ is called a lower bound of A if $y \geq z$ for all $y \in A$. Moreover, if there is an upper bound of A, then A is said to be bounded from above. If there is a lower bound of A, then A is said to be bounded from below. If A is bounded from above and from below, then we will briefly say that A is order bounded.

An order set $(V, \leq)$ is called a lattice if any two elements $x, y \in V$ have a least upper bound denoted by $x \vee y = \sup\{x, y\}$ and a greatest lower bound denoted by $x \wedge y = \inf\{x, y\}$.

A real vector space V which is also an order set is an order vector space if the order and the vector space structure are compatible in the following sense:

1. If $x, y \in V$ such that $x \leq y$ then $x + z \leq y + z$ for all $z \in V$.
2. If $x, y \in V$ such that $x \leq y$, then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

$(V, \leq)$ is called a Riesz space if $(V, \leq)$ is a lattice and an order vector space.

A norm $\|\cdot\|$ on a Riesz space V is called a lattice norm if $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. In the latter case, $(V, \|\cdot\|)$ is called a normed Riesz space.

$(V, \|\cdot\|)$ is called a Banach lattice if for all $x, y \in V$

1. $(V, \|\cdot\|)$ is a Banach space;
2. V is a Riesz space;
3. $\|\cdot\|$ is a lattice norm.

Let V be a Riesz space and the positive cone V^+ of V consist of all $x \in V$ such that $x \geq 0$. For every $x \in V$, let

$$x^+ = x \vee 0, \quad x^- = -x \vee 0, \quad |x| = x \vee -x.$$

Let V be a Riesz space. For all $x, y, z \in V$, the following assertions hold:

1. $x + y = x \vee y + x \wedge y$, $-(x \vee y) = -x \wedge -y$;
2. $x + (y \vee z) = (x + y) \vee (x + z)$, $x + (y \wedge z) = (x + y) \wedge (x + z)$;
3. $|x| = x^+ + x^-$, $|x + y| \leq |x| + |y|$;
4. $x \leq y$ is equivalent to $x^+ \leq y^+$ and $y^- \leq x^-$;
5. $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z)$, $(x \wedge y) \vee z = (x \vee y) \wedge (y \vee z)$.

A Riesz space V is Archimedean if $x \leq 0$ holds whenever the set $\{nx : n \in N\}$ is bounded from above.

Definition 1.1. [17] *Let V be a Riesz space. The sequence $\{x_n\}$ is called uniformly bounded if there exist $e \in V^+$ and $\{a_n\} \in l^1$ such that $x_n \leq a_n \cdot e$.*

Definition 1.2. [17] *A Riesz space V is called uniformly complete if* $\sup\{\sum_{i=1}^n x_i \ : \ n \in \mathbf{N}\}$ *exists for every uniformly bounded sequence $\{x_n\}$, where $x_n \in V^+$.*

Definition 1.3. [17] *Let V, W be Archimedean Riesz spaces. The function $P : V \to W$ is called positive if $P(V^+) = \{P(|x|) : x \in V\} \subset W^+$.*

Theorem 1.1. [2] *For a function $P : V \to W$ between two Riesz spaces, the following statements are equivalent:*

1. *P is a lattice homomorphism;*

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2. $P(x^+) = P(x)^+$ *for all* $x \in V$*;*
3. $P(x \wedge y) = P(x) \wedge P(y)$*;*
4. *if* $x \wedge y = 0$ *in* V*, then* $P(x) \wedge P(y) = 0$ *holds in* W*;*
5. $P(|x|) = |P(x)|$.

Definition 1.4. [1] *Let* V *and* W *be Banach lattices and* $P : V \to W$ *a positive mapping. We define*

(P_1) *a lattice homomorphism functional equation:*

$$P(|x| \vee |y|) = P(|x|) \vee P(|y|);$$

(P_2) *a semi-homogeneity: for all* $x \in V$ *and every number* $\alpha \in \mathbb{R}^+$

$$P(\alpha|x|) = \alpha P(|x|).$$

Remark 1.1. [1] *Given two Banach lattices* V *and* W *and* $P : V \to W$ *be a positive function satisfying the property* (P_1)*. Then the following statements are valid.*

1. $P(|x \vee y|) \le P(|x|) \vee P(|y|)$ *for all* $x, y \in V$*.*
2. *The semi-homogeneity implies that* $P(0) = 0$*.*
3. P *is an increasing operator, in the sense that if* $x, y \in V$ *are such that* $|x| \le |y|$*, then* $P(|x|) \le P(|y|)$*.*

A *distance distribution function* (briefly, d.d.f.) is a non-decreasing function F defined on $\mathbb{R}^+$ that satisfies $F(0) = 0$ and $F(+\infty) = 1$, and is left continuous on $(0, \infty)$. The set of all d.d.f's will be denoted by Δ^+; and the set of all F in Δ^+ for which $\lim_{x \to +\infty^-} F(x) = 1$ by D^+. The elements of Δ^+ are partially ordered via $F \le G$ if and only if $F(x) \le G(x)$ for all $x \in \mathbb{R}^+$.

The space Δ^+ has both maximal element ϵ_0 and a minimal element ϵ_∞ defined by

$$\epsilon_0(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \\ 1 & \text{if } x > 0, \end{cases} \qquad \epsilon_\infty(x) = \begin{cases} 0 & \text{if } x < +\infty \\ \\ 1 & \text{if } x = \infty. \end{cases}$$

Let $[F, G; h]$ denote the condition

$$G(x) \le F(x + h) + h \quad \forall x \in \left(0, \frac{1}{h}\right).$$

For any $F, G \in \Delta^+$ and h in $(0, 1]$, the function d_L defined on $\Delta^+ \times \Delta^+$ by

$$d_L(F, G) = \inf \{h \mid \text{both } [F, G; h] \text{ and } [G, F; h] \text{ hold } \}$$

is called the modified levy metric on Δ^+. Convergence with respect to this metric is to week convergence of distribution function, i.e., for any sequence $\{F_n\}$ in Δ^+ and any F in Δ^+, we have $d_L(F_n, F) \to 0$ if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at each continuity point x of F. Moreover, the metric space (Δ^+, d_L) is compact. If F and G are in Δ^+ and $F \le G$, then $d_L(G, \epsilon_0) \le d_L(F, \epsilon_0)$. The supremum of any set of d.d.f.'s in Δ^+ is in Δ^+ (see [5]).

Definition 1.5. [5] *A triangle norm (t-norm, for short) is a binary operation on the unit interval* $[0, 1]$*, i.e., a function* $T : [0, 1] \times [0, 1] \to [0, 1]$ *such that for all* $x, y, z \in [0, 1]$ *the following four axioms are satisfied:*

$(T1)$ $T(x, y) = T(y, x)$;
$(T2)$ $T(x, T(y, z)) = T(T(x, y), z)$;
$(T3)$ $T(x, y) \le T(x, z)$ *whenever* $y \le z$*;*
$(T4)$ $T(x, 1) = x$.

A t-norm T is continuous if and only if it is continuous in the first component, i.e., if for each $y \in [0,1]$ the one place function

$$T(\cdot, y) : [0,1] \to [0,1], \qquad x \longmapsto T(x,y),$$

is continuous. A continuous t-conorm T^* is a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x, 1-y)$. A continuous t-norm T is *Archimedean* if $T(x,x) < x$ for all $x \in (0,1)$ (see [21]).

Definition 1.6. *A triangle function is a binary operation on Δ^+, namely, a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing in each argument and which has ϵ_0 as unit, viz, for all $F, G, H \in \Delta^+$,*
1. $\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$;
2. $\tau(F,G) = \tau(G,F)$;
3. $F \leq G \Rightarrow \tau(F,H) \leq \tau(G,H)$;
4. $\tau(F,\epsilon_0) = F$.

A triangle function τ is *Archimedean* on Δ^+ if $\tau(F,G) < F$ for all $F, G \in \Delta^+$ and $F \neq \epsilon_\infty$, $G \neq \epsilon_0$. Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_L). Typical continuous triangle functions are

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t)) \quad \tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s), G(t)),$$

where T and T^* are t-norm and t-conorm respectively. If T and T^* are continuous t-norm and t-conorm, respectively, then τ_T and τ_{T^*} are uinformly continuous on (Δ^+, d_L) (see [21]).

Theorem 1.2. [21] *Let T be an Archmidean continuous t-norm. Then τ_T is a triangle function having no nontrivial idempotent in Δ^+, that is, τ_T is Archimedean triangle function (there is a similar theorem for τ_{T^*}).*

Definition 1.7. [5] *A probabilistic normed space, which will henceforth be called briefly a PN space, is a quadruple (V, ν, τ, τ^*), where V is a linear space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$, and the mapping $\nu : V \to \Delta^+$ satisfies, for all p and q in V, the conditions*
$(N1)$ $\nu_p = \epsilon_0$ *if and only if* $p = \theta$ *(θ is the null vector in X);*
$(N2)$ $\nu_{-p} = \nu_p$;
$(N3)$ $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
$(N4)$ $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ *for every* $\alpha \in [0,1]$.

The function ν is called the probabilistic norm, a PN space is called a Serstnev space if it satisfies $(N1)$, $(N3)$ and the following condition:

$$\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right)$$

holds for all $\alpha \in \mathbb{R} \setminus \{0\}$ and $x > 0$. If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its t-conorm T^* then (V, ν, τ, τ^*) is denoted by (V, ν, T) and is a *Menger PN space*. For $p \in V$ and $t > 0$, the strong t-neighbourhood of p is defined by the set

$$\mathcal{N}_p(t) = \{q \in V : d_L(\nu_{p-q}, \epsilon_0) < t\} \quad = \quad \{q \in V \ : \ \nu_{p-q}(t) > 1 - t\}.$$

Since τ is continuous, the system of neighbourhood $\{\mathcal{N}_p(t) : p \in V \text{ and } t > 0\}$ determines a Hausdorff and first countable topology on V, called a strong topology.

A sequence $\{p_n\}$ in (V, ν, τ, τ^*) is said to be strongly convergent (convergent with respect to the probabilistic norm) to a point p in V, and we will write $p_n \xrightarrow{PN} p$, if for any $t > 0$, there is a positive integer N such that p_n is in $\mathcal{N}_p(t)$ whenever $n \geq N$. Thus $p_n \xrightarrow{PN} p$ if and only if $\lim_{n\to\infty} d_L(\nu_{p_n - p}, \epsilon_0) = 0$. We will call p the strong limit of $\{p_n\}$.

A sequence $\{p_n\}$ in (V, ν, τ, τ^*) is said to be strong Cauchy if for any $t > 0$, there is an integer N such that p_n is in $\mathcal{N}_{p_m}(t)$ whenever $n, m \geq N$. If every strong Cauchy sequence is strongly convergent to a point p in V, then we say that (V, ν, τ, τ^*) is complete in the strong topology.

Theorem 1.3. [5] *Let (V, ν, τ, τ^*) be a PN space in which τ^* is Archimedean and $\nu_p \neq \epsilon_\infty$ for all $p \in V$. Then for every $p \in V$, the mapping $R \ni \alpha \longmapsto \alpha p$ is uniformly continuous.*

Theorem 1.4. [5] *Let (V, ν, τ, τ^*) be a PN space with τ continuous. If V is endowed with the strong topology and Δ^+ with the topology of levy metric d_L, then the probabilistic norm $\nu : V \to \Delta^+$ is uniformly continuous.*

Note that if T is an Archmidean continuous t-norm, we use the above theorems in Menger PN space (V, ν, T).

Definition 1.8. [22] *Let $(V, \leq)$ be a (real) Riesz space equipped with a probabilistic norm ν, and continuous triangle functions τ and τ^* such that $\tau \leq \tau^*$. The probabilistic norm on V is a probabilistic Riesz norm provided that $|x| \leq |y|$ in V implies $\nu_x \geq \nu_y$. Any Riesz space, equipped with probabilistic Riesz norm is a probabilistic normed Riesz space (PNR space, briefly). If a PNR space V is complete with respect to the strong topology, then V is a probabilistic Banach lattice (PBL, in short).*

Remark 1.2. *In classical Riesz space theory, it is known that every normed Riesz space is Archimedean. In general, a PNR space V need not be Archimedean (see* [22]*). Nevertheless, if the condition that the triangle function τ^* of the PNR space V is Archimedean and $\nu_p \neq \epsilon_\infty$ for all $p \in V$ is satisfied, then V is also Archimedean (see* [5]*).*

2. Main results

Definition 2.1. *A Menger probabilistic normed Riesz space (MPNR- space, for short) is a quaternary $(V, \nu, T, \leq)$ where $(V, \leq)$ is a real Riesz space, T is a continuous t-norm and $\nu : V \to D^+$ (for $x \in V$ the distribution function $\nu(x)$ is denoted by ν_x and $\nu_x(t)$ is the value of ν_x at $t \in \mathbb{R}$) satisfies the following conditions:*

(M1) *$\nu_x(0) = 0$ for all $x \in V$;*
(M2) *$\nu_x = \epsilon_0$ if and only if $x = \theta$ (θ is the null vector in V);*
(M3) *$\nu_{\alpha x}(t) = \nu_x(\frac{t}{|\alpha|})$ for all $x \in V$ and $\alpha \in \mathbb{R} \setminus \{0\}$;*
(M4) *$\nu_{x+y}(t_1 + t_2) \geq T(\nu_x(t_1), \nu_y(t_2))$, for all $x, y \in V$ and $t_1, t_2 \in \mathbb{R}^+$;*
(M5) *norm Riesz Menger property: $\nu_x(t) \geq \nu_y(t)$ whenever $|x| \leq |y|$ for all $x, y \in V$ and $t \in \mathbb{R}^+$.*

Example 2.1. *Let $(V, \|.\|, \leq)$ be a normed Riesz space. Define $\nu : V \to D^+$ by*

$$\nu_x(t) = \begin{cases} \dfrac{t}{t + \|x\|} & \text{if } t > 0, \\ \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then $(V, \nu, T, \leq)$ is a Menger PN space. It is clear that $(M1) - (M4)$ hold. Suppose that $|x| \leq |y|$ for all $x, y \in V$. Then $\|x\| \leq \|y\|$ since $(V, \| \cdot \|, \leq)$ is a normed Riesz space. Therefore,

$$\frac{t}{t+\|x\|} \geq \frac{t}{t+\|y\|}$$

and so $\nu_x(t) \geq \nu_y(t)$ for all $t > 0$.

Lemma 2.1. *If $(\mathbb{R}, \nu, T)$ is a Menger PN-space, then $(\mathbb{R}, \nu, T, \leq)$ is a Menger probabilistic normed Riesz space.*

We show that norm Riesz Menger property is satisfied in $(\mathbb{R}, \nu, T, \leq)$. Let $|x| \leq |y|$ for $x, y \in \mathbb{R} \setminus \{0\}$. Then

$$\nu_x(t) = \nu_{\frac{x}{y}.y}(t) = \nu_y\left(\frac{t}{|\frac{x}{y}|}\right) \geq \nu_y(t)$$

for all $t \in \mathbb{R}^+$.

Definition 2.2. *Let $(V, \nu, T, \leq)$ be an Menger probabilistic normed Riesz space. Let $\{x_n\}$ be a sequence in V. Then $\{x_n\}$ is said to be convergent if there exists $x \in V$ such that*

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1.$$

In this case, x is called the limit of $\{x_n\}$.

Definition 2.3. *The sequence $\{x_n\}$ in a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$ is called Cauchy if for each $\epsilon > 0$ and $\delta > 0$, there exists some n_0 such that*

$$\nu_{x_n - x_m}(\delta) > 1 - \epsilon$$

for all $m, n \geq n_0$.

Clearly, every convergent sequence in a Menger probabilistic normed Riesz space is Cauchy. If each Cauchy sequence is convergent in a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$, then $(V, \nu, T, \leq)$ is called a Menger probabilistic Banach Riesz space (briefly, MPBR- space).

Definition 2.4. *A sequence $\{x_n\}$ in a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$ is called order Menger convergent to x as $n \to \infty$ if there exists a sequence $\{y_n\} \downarrow 0$ as $n \to \infty$ and $\nu_{x_n - x}(t) \geq \nu_{y_n}(t)$ for all $n \in \mathbb{N}$ and $t > 0$. We write $x = OM - \lim_{n\to\infty} x_n$.*

Theorem 2.1. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. Then each lattice operator is continuous.*

Proof. Assume that

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1 \quad \& \quad \lim_{n\to\infty} \nu_{y_n - y}(s) = 1$$

for all $t, s > 0$. Then

$$\begin{aligned} \nu_{x_n \wedge y_n - x \wedge y}(t+s) &= \nu_{x_n \wedge y_n - x_n \wedge y + x_n \wedge y - x \wedge y}(t+s) \\ &\geq T\left(\nu_{x_n \wedge y_n - x_n \wedge y}(t), \nu_{x_n \wedge y - x \wedge y}(s)\right) \\ &\geq T\left(\nu_{y_n - y}(t), \nu_{x_n - x}(s)\right). \end{aligned}$$

As $n \to \infty$, we have

$$\lim_{n\to\infty} \nu_{x_n \wedge y_n - x \wedge y}(t+s) = 1.$$

So

$$\lim_{n\to\infty} x_n \wedge y_n = x \wedge y.$$

It is easy to see that the other lattice operations are continuous. □

Theorem 2.2. *Let $(V, \nu, T, \leq)$ be a Menger PNR space and T be an Archimedean continuous t-norm and $\nu_x \neq \epsilon_\infty$ for all $x \in V$. Then V is Archimedean Menger PNR space.*

Proof. Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. Consider $x, y \in V^+$ such that $nx \leq y$ for all $n \in \mathbb{N}$. Then

$$\nu_{nx}(t) \geq \nu_y(t), \qquad \forall t > 0$$

and so

$$\nu_x\left(\frac{t}{n}\right) \geq \nu_y(t), \qquad \forall t > 0.$$

Replacing t by nt, we get

$$\nu_x(t) \geq \nu_y(nt) = \nu_{\frac{y}{n}}(t) \qquad \forall t > 0.$$

Since T is an Archimedean continuous t-norm and $\nu_x \neq \epsilon_\infty$, the probabilistic norm ν is continuous (see Theorem 1.3) and we have $x = 0$. Hence V has Archimedean property (see Theorems 1.4 and 1.2). $\square$

Throughout this article we will assume that Menger PN space $(V, \nu, T, \leq)$ has an Archimedean continuous t-norm T and $\nu_x \neq \epsilon_\infty$.

Proposition 2.1. *Assume that $\{x_n\}$ and $\{y_n\}$ are sequences in Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$ such that $x_n \to x$ and $y_n \to y$ in order Menger as $n \to \infty$. Then*

$$OM - \lim_{n\to\infty}(x_n + y_n) = x + y,$$
$$OM - \lim_{n\to\infty}(x_n \vee y_n) = x \vee y,$$
$$OM - \lim_{n\to\infty}(x_n \wedge y_n) = x \wedge y.$$

Theorem 2.3. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. If $x_n \to x$ (in order Menger or in norm) and $x_n \geq y$ for all n, then $x \geq y$. If $x_n \to x$ and $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x \geq 0$. This shows that the positive cone V^+ is closed.*

Proof. It may be assumed that $y = 0$. Since $|x^- - x_n^-| \leq |x - x_n|$,

$$\nu_{x^- - x_n^-}(t) \quad \geq \quad \nu_{x - x_n}(t)$$

and so the sequence $\{x_n\}$ converges to x as $n \to \infty$. Thus $\nu_{x^- - x_n^-}(t) \geq 1$, which means that $x^- = 0$ and hence $x \geq 0$. $\square$

Theorem 2.4. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. Every increasing convergent sequence $\{x_n\} \subset V$ is convergent to $u = \sup\{x_n \ : \ n \in \mathbb{N}\}$.*

Proof. Suppose that $\{x_n\}$ is an increasing convergent sequence and

$$\lim_{n\to\infty} \nu_{x_n - x}(t) = 1 \quad \text{for all } t > 0 \text{ for all } n \in \mathbb{N}.$$

Since for every $m \geq n$, we have $x_m - x_n \in V^+$, it follows from Theorem 2.3 that $x \geq x_n$ and $x_n \leq u \leq x$ for all $n \in N$. So by $(M4)$

$$\nu_{u - x_n}(t) \geq \nu_{x - x_n}(t) \text{ for all } t > 0.$$

Therefore, we have

$$\lim_{n\to\infty} \nu_{x_n - u}(t) = 1 \text{ for all } t > 0.$$

Hence $u = x$. $\square$

Theorem 2.5. *Every Menger probabilistic Banach Riesz space is uniformly complete.*

Proof. Let $(V, \nu, T, \leq)$ be a Menger probabilistic Banach Riesz space and $\{x_n\} \subset V^+$ be a sequence such that $x_n \leq a_n e$ for a suitable sequence $\{a_n\} \in l^1$ and some $e \in V^+$. We show that $\sup\{\sum_{i=1}^n x_i \ : \ n \in N\}$ exists. Let

$$y_n = x_1 + x_2 + ... + x_n \quad \text{and} \quad b_n = \sum_{j=n+1}^{\infty} a_j.$$

By Theorem 2.1 and $(PN4)$, we have

$$\nu_{y_{n+p}-y_n}(t) = \nu_{x_{n+1}+...+x_{n+p}}(t) \geq \nu_{\sum_{j=1}^{\infty} a_{n+j}.e}(t) = \nu_{b_n \cdot e}(t)$$

for all $t > 0$. As $n \to \infty$, we get

$$\lim_{n\to\infty} \nu_{y_{n+p}-y_n}(t) = 1.$$

So $\{y_n\}$ is a Cauchy sequence in Menger probabilistic Banach Riesz space and therefore there exists $y \in V$ such that $y_n \to y$. Since y_n is increasing and convergence sequence, by Theorem 2.4, we have

$$\lim_{n\to\infty} \nu_{y_n - \vee y_n}(t) = 1,$$

that is, $y_n \to \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in N\}$. Using a unique limit, we have

$$y = \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in N\}.$$

Thus the proof is complete. □

Definition 2.5. *(i) Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. The subset A of V is said to be solid if the following conditions hold:*
(1) *$x \in A$ if and only if $|x| \in A$;*
(2) *$0 \leq x \in A$ and $y \in V^+$ imply that $x \wedge y \in A$.*

(ii) The subset A of V is called an ideal in V if A is a solid linear subspace of V.
(iii) An order Menger closed ideal A of V is called a band.

Theorem 2.6. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. The closure solid subset of V is solid.*

Proof. Suppose that $A \subseteq V$ is a solid and $x \in \overline{A}$. Assume that $\{x_n\} \subseteq A$ is a sequence such that $x_n \to x$ as $n \to \infty$. It follows from $(M5)$ that

$$\nu_{|x_n|-|x|}(t) \geq \nu_{|x_n-x|}(t) = \nu_{x_n-x}(t).$$

Therefore $|x_n| \to |x|$ as $n \to \infty$ and so $|x| \in \overline{A}$, since A is a solid.

On the other hand, suppose that $|x| \in \overline{A}$. Then there exists $x_n \subset A^+$ such that $x_n \to |x|$. It follows from Theorem 2.1 that

$$x_n \wedge x \to x \wedge |x| = x,$$

as $n \to \infty$ and hence $x \in \overline{A}$.

Finally, suppose that $0 \leq x \in \overline{A}$ and $y \in V^+$. Then there exists $x_n \subset A^+$ such that $x_n \to x$ as $n \to \infty$. It follows from Theorem 2.1 that

$$x_n \wedge y \to x \wedge y.$$

Therefore, $x \wedge y \in \overline{A}$. Thus the proof is complete. □

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Theorem 2.7. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. Then every band in V is closed.*

Proof. Suppose that B is a band and assume that $\{x_n\} \subset B$ is a sequence such that $x_n \longrightarrow x$ for some $x \in V$. It follows from Theorem 2.1 that

$$|x_n| \wedge |x| \longrightarrow |x|$$

as $n \to \infty$. For every $n \in \mathbb{N}$, let

$$y_n = (|x_n| \vee ... \vee |x_1|) \wedge |x|.$$

Then $\{y_n\}$ is an increasing sequence and

$$y_n = (|x_n| \wedge |x|) \vee ... \vee (|x_1| \wedge |x|)$$

and so $|x_n| \wedge |x| \leq y_n \leq |x|$. By $(M4)$, we have

$$\nu_{|x|-y_n}(t) \geq \nu_{|x|-|x_n| \wedge |x|}(t)$$

for all $t > 0$. Hence $y_n \longrightarrow |x|$ as $n \to \infty$. Theorem 2.4 implies that $|x| = \sup\{y_n : n \in N\} \in B$. Hence $x \in B$. □

Theorem 2.8. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. We define the function $\|\cdot\|$ by*

$$\|x\| = \inf\{t \geq 0, \nu_x(t) = 1\} \quad \textit{for all } x \in V.$$

Then $\|\cdot\|$ is a lattice norm on V and $(E, \|\cdot\|, \leq)$ is a normed Riesz space.

Proof. It suffices to show that $\|\cdot\|$ satisfies the lattice norm conditions.
(1) From $(M1)$ and $(M2)$ it is easy to see that $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
(2) From $(M3)$, for any $\alpha \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} \|\alpha x\| = \inf\{t \geq 0, \nu_{\alpha x}(t) = 1\} &= \inf\left\{t \geq 0, \nu_x\left(\frac{t}{\alpha}\right) = 1\right\} \\ &= |\alpha| \inf\{t \geq 0, \nu_x(t) = 1\} \\ &= |\alpha| \cdot \|x\|, \end{aligned}$$

and if $\alpha = 0$, then the above equality still holds.
(3) By definition of $\|\cdot\|$, for any $\epsilon > 0$, we have

$$\exists\ t_1 \in A \text{ such that } t_1 \leq \|x\| + \frac{\epsilon}{2},$$

where $A = \{t \geq 0;\ \nu_x(t) = 1\}$. Therefore

$$\nu_x\left(\|x\| + \frac{\epsilon}{2}\right) = 1\ ,\ \nu_y\left(\|y\| + \frac{\epsilon}{2}\right) = 1.$$

Hence from $(M4)$ it follows that

$$\nu_{x+y}\left(\|x\| + \|y\| + \epsilon\right) = 1 \Rightarrow \|x\| + \|y\| + \epsilon \in A$$

for all $x, y \in V$. By definition of A,

$$\|x + y\| \leq \|x\| + \|y\| + \epsilon.$$

Letting $\epsilon \to 0$, we have

$$\|x + y\| \leq \|x\| + \|y\|.$$

So $\|\cdot\|$ is a norm on V.

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(4) Finally, assume that $|x| \leq |y|$ for all $x, y \in V$. Then $\nu_x(t) \geq \nu_y(t)$. We define

$$\|x\| = \inf A_2 = \inf\{t \geq 0; \nu_x(t) = 1\};$$
$$\|y\| = \inf A_1 = \inf\{t \geq 0; \nu_y(t) = 1\}.$$

If $t_1 \in A_1$, then $\nu_x(t_1) = 1$ and so $A_1 \subseteq A_2$. Therefore $\|y\| \geq \|x\|$. Thus the proof is complete. □

Theorem 2.9. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic normed Riesz space. We define the function $\|\cdot\|_\alpha$ by*

$$\|x\|_\alpha = \inf\{t \geq 0,\ \nu_x(t) > 1 - \alpha\} \quad \text{for all } x \in V\ ,\ \alpha \in (0, 1).$$

Then $\|\cdot\|_\alpha$ is a lattice semi-norm.

Proof. The proof is the same as in the proof of the above theorem. □

Theorem 2.10. *Let $(E, \|\cdot\|_\alpha, \leq)$ be a normed Riesz space. We define the function $\nu_x(t)$ by*

$$\nu_x(t) = \sup\{\alpha \in (0, 1)\ :\ \|x\|_\alpha \leq t\}.$$

Then $(V, \nu, T, \leq)$ is a Menger probabilistic normed Riesz space, where T is a t-norm.

Proof. The proof is the same as in the prrof of Theorem 2.8. □

Corollary 2.1. *Let $(V, \nu, T, \leq)$ be a Menger probabilistic Banach Riesz space, and $\|\cdot\|$ be defined in Theorem 2.8. If $P : E \to E$ is a positive linear operator then P is continuous.*

Proof. Assume that P fails to be continuous. Hence for every $n \in \mathbb{N}$ there exists $x_n \in V$ such that $\|x_n\| \leq 2^{-n}$ and $n \leq \|Px_n\|$, i.e., $x_n \to \theta$ but $Px_n \nrightarrow \theta$, where θ is a null vector in V. Since P is a positive linear operator, $Px \leq P|x|$ then $\nu_{Px}(t) \geq \nu_{P|x|}(t)$. So

$$\|P|x|\| \ \ = \ \ \inf\{t \geq 0,\ \nu_{P|x|}(t) = 1\} \geq \inf\{t \geq 0,\ \nu_{Px}(t) = 1\} = \|Px\|$$

for all $x \in V$. We may assume that $x_n \geq 0$. Let

$$x = \sum_n x_n \in V^+.$$

Then $x \geq x_n$ and so $\|Px\| \geq \|Px_n\| \geq n$ for all $n \in \mathbb{N}$. This is a contradiction. □

3. Hyers-Ulam stability of lattice homomorphisms in Menger PNR spaces

Using the direct method, we investigate the Hyers-Ulam stability of lattice homomorphisms in Menger probabilistic normed Riesz spaces.

Theorem 3.1. *Let f be a positive function from a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$ to a Menger probabilistic Banach Riesz space $(W, \mu, T, \leq)$, where T is an Archimedean continuous t-norm and $\nu_p, \mu_q \neq \epsilon_\infty$, for all $p \in V$ and $q \in W$. Let*

$$\mu_{f(\tau x \vee \eta y) - \tau f(x) \vee \eta f(y)}(t) \geq \nu_{\varphi(\tau x \vee \eta y, \tau x \wedge \eta y)}(t) \tag{3.1}$$

for all $x, y \in V$ and $t > 0$. Here $\varphi : V \times V \to V$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi(\tfrac{x}{\tau}, \tfrac{y}{\eta}) \tag{3.2}$$

for all $\tau, \eta \geq 1$ and for some $\alpha \in [0, 1)$. Then there exists a unique positive function $\mathbf{T} : V \to W$ which satisfies the properties $(P1)$,$(P2)$ and inequality

$$\mu_{\mathbf{T}(x) - f(x)}(t) \geq \nu_{\varphi(x,x)}\left(\frac{\tau - \tau^\alpha}{\tau^\alpha} t\right)$$

for all $x \in V^+$.

Proof. Putting $y = x$ and $\tau = \eta$ in (3.1), we have

$$\mu_{f(\tau x)-\tau f(x)}(t) \geq \nu_{\varphi(\tau x,\tau x)}(t).$$

By $(M5)$ and (3.2), we obtain

$$\mu_{\frac{1}{\tau}f(\tau x)-f(x)}(\tau^{\alpha-1}t) \geq \nu_{\varphi(x,x)}(t). \tag{3.3}$$

Replacing x by τx in (3.3) and using (3.2) and $(M5)$, we have

$$\mu_{\frac{1}{\tau}f(\tau^2 x)-f(\tau x)}(\tau^{\alpha-1}t) \geq \nu_{\varphi(\tau x,\tau x)}(t) \geq \nu_{\tau^\alpha\varphi(x,x)}(t) = \nu_{\varphi(x,x)}(\tfrac{t}{\tau^\alpha}).$$

Hence

$$\mu_{\frac{1}{\tau^2}f(\tau^2 x)-\frac{1}{\tau}f(\tau x)}(\tau^{2\alpha-2}t) \geq \nu_{\varphi(x,x)}(t). \tag{3.4}$$

By comparing (3.3) and (3.4) and using $(M4)$, we have

$$\mu_{\frac{1}{\tau^2}f(\tau^2 x)-f(x)}\left((\tau^{\alpha-1}+\tau^{2(\alpha-1)})t\right) \geq \nu_{\varphi(x,x)}(t). \tag{3.5}$$

Again, replacing x by τx in (3.5), we get

$$\mu_{\frac{1}{\tau^2}f(\tau^3 x)-f(\tau x)}\left((\tau^{\alpha-1}+\tau^{2(\alpha-1)})t\right) \geq \nu_{\varphi(\tau x,\tau x)}(t) \geq \nu_{\tau^\alpha\varphi(x,x)}(t) \geq \nu_{\varphi(x,x)}\left(\tfrac{t}{\tau^\alpha}\right)$$

and so

$$\mu_{\frac{1}{\tau^3}f(\tau^3 x)-\frac{1}{\tau}f(\tau x)}\left((\tau^{2(\alpha-1)}+\tau^{3(\alpha-1)})t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t). \tag{3.6}$$

By comparing (3.3) and (3.6), we obtain

$$\mu_{\frac{1}{\tau^3}f(\tau^3 x)-f(x)}\left((\tau^{(\alpha-1)}+\tau^{2(\alpha-1)}+\tau^{3(\alpha-1)})t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t).$$

With this process, we have

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-f(x)}\left(\sum_{k=1}^{n}\tau^{k(\alpha-1)}t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t) \tag{3.7}$$

for all $n \in \mathbb{N}$. If $m \in \mathbb{N}$ and $n > m$, then $n - m \in \mathbb{N}$. Replacing n by $n - m$ in (3.7), we get

$$\mu_{\frac{1}{\tau^{n-m}}f(\tau^{n-m}x)-f(x)}\left(\sum_{k=1}^{n-m}\tau^{k(\alpha-1)}t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t). \tag{3.8}$$

Replacing x by $\tau^m x$ in (3.8) and using $(M5)$, we obtain

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x)}\left(\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t\right) \quad \geq \quad \nu_{\varphi(x,x)}(t). \tag{3.9}$$

Let $c > 0$ and $\epsilon > 0$ be given. Since $\nu_{\varphi(x,x)}(t) \in D^+$, $\lim_{t\to\infty}\nu_{\varphi(x,x)}(t) = 1$. Therefore, there is some $t_0 > 0$ such that

$$\nu_{\varphi(x,x)}(t_0) \geq 1 - \epsilon.$$

Fix some $t \geq t_0$. The convergence of $\sum_{k=1}^{\infty}\tau^{k(\alpha-1)}t$ guarantees that there exists some $n_0 \geq 0$ such that for each $n > m > n_0$, the inequality

$$\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t < c$$

holds. It follows that

$$\begin{aligned} \mu_{\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x)}(c) &\geq \mu_{\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x)}\left(\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t_0\right) \\ &\geq \nu_{\varphi(x,x)}(t_0) \\ &\geq 1-\epsilon. \end{aligned}$$

So $\left\{\frac{1}{\tau^n}f(\tau^n x)\right\}$ is a Cauchy sequence in the Menger probabilistic Banach Riesz space $(W,\mu,T,\leq)$ and thus this sequence converges to $\mathbf{T}(x)\in W$. It means that

$$\lim_{n\to\infty}\mu_{\frac{1}{\tau^n}f(\tau^n x)-\mathbf{T}(x)}(t)=1.$$

Furthermore, by putting $m=0$ in (3.9), we obtain

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-f(x)}\left(\sum_{k=1}^{n}\tau^{k(\alpha-1)}t\right)\geq\nu_{\varphi(x,x)}(t).$$

So

$$\mu_{\frac{1}{\tau^n}f(\tau^n x)-f(x)}(t)\geq\nu_{\varphi(x,x)}\left(\frac{t}{\sum_{k=1}^{n}\tau^{k(\alpha-1)}}\right).$$

Since $\nu_p,\mu_q\neq\epsilon_\infty$ and $\mathbf{T}$ is an Archimedean continuous t-norm, norm probabilistic is continuous (see Theorems 1.3 and 1.4). Thus we have

$$\mu_{\mathbf{T}(x)-f(x)}(t)\geq\nu_{\varphi(x,x)}\left(\frac{\tau-\tau^\alpha}{\tau^\alpha}t\right).$$

Next, we show that $\mathbf{T}$ satisfies $(P1)$. Putting $\tau=\eta=\tau^n$ in (3.1), we get

$$\mu_{f(\tau^n x\vee\tau^n y)-\tau^n f(x)\vee\tau^n f(y)}(t)\geq\nu_{\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y)}(t)\geq\nu_{\varphi(x\vee y,x\wedge y)}\left(\frac{t}{\tau^{n\alpha}}\right).$$

Replacing x by $\tau^n x$ and y by $\tau^n y$ in the last inequality, one can get

$$\begin{aligned} \mu_{f(\tau^n(\tau^n x\vee\tau^n y))-\tau^n f(\tau^n x)\vee\tau^n f(\tau^n y)}(t) &\geq \nu_{\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y)}\left(\frac{t}{\tau^{n\alpha}}\right) \\ &\geq \nu_{\varphi(x\vee y,x\wedge y)}\left(\frac{t}{\tau^{2n\alpha}}\right), \end{aligned}$$

which implies

$$\mu_{\frac{f(\tau^{2n}(x\vee y))}{\tau^{2n}}-\frac{f(\tau^n x)}{\tau^n x}\vee\frac{f(\tau^n y)}{\tau^n}}(t)\geq\nu_{\tau^{2n(\alpha-1)}\varphi(x\vee y,x\wedge y)}(t).$$

Since norm probabilistic is continuous, the term on the right-hand side of the above inequality tends to 1 as $n\to\infty$. By Theorem 2.1, we obtain

$$\mu_{\mathbf{T}(x\vee y)-\mathbf{T}(x)\vee\mathbf{T}(y)}(t)\geq 1$$

for all $x,y\in V$. This means that

$$\mathbf{T}(x\vee y)=\mathbf{T}(x)\vee\mathbf{T}(y).$$

Consequently, the property $(P1)$ holds. We show that $\mathbf{T}(\tau x)=\tau\mathbf{T}(x)$ for all $x\in V^+$ and $\tau\geq 1$. In fact, in the inequality (3.1), we choose $\eta=\tau$ and $y=0$ and substitute $2^n\tau$ for τ and consider Remark 1.1. Then

$$\mu_{(f(2^n\tau x)-2^n\tau f(x))}(t)\geq\nu_{\varphi(2^n\tau x,0)}(t) \tag{3.10}$$

for all $x \in V^+$. Now, replacing x by $2^n x$ in (3.10), we obtain

$$\mu_{\left(\frac{f(4^n\tau x)}{4^n} - \frac{\tau f(2^n x)}{2^n}\right)}\left(\frac{t}{4^n}\right) \geq \nu_{\varphi(4^n\tau x,0)}(t) \geq \nu_{4^{n\alpha}\tau^\alpha\varphi(x,0)}(t).$$

Therefore,

$$\mu_{\frac{f(4^n\tau x)}{4^n} - \frac{\tau f(2^n x)}{2^n}}(t) \geq \nu_{4^{n(\alpha-1)}\tau^\alpha\varphi(x,0)}(t).$$

Since norm probabilistic is continuous, the term on the right-hand side of the above inequality tends to 1 as $n \to \infty$. Thus

$$\mathbf{T}(\tau x) = \tau\mathbf{T}(x),$$

as desired. □

Corollary 3.1. *Let f be a positive function from a Menger probabilistic normed Riesz space $(V, \nu, T, \leq)$ to a Menger probabilistic Banach Riesz space $(W, \mu, T, \leq)$, where T is an Archimedean continuous t-norm and $\nu_p, \mu_q \neq \epsilon_\infty$, for all $p \in V$ and $q \in W$. Let $\rho : [0, \infty) \to [0, \infty)$ be a continuous function, for which there are numbers $\eta \in \mathbb{R}$ and $0 \leq r < 1$ such that*

$$\mu_{\left(f(\alpha|x|\vee\beta|y|) - \frac{\alpha\rho(\alpha)f(|x|)\vee\beta\rho(\beta)f(|y|)}{\rho(\alpha)\vee\rho(\beta)}\right)}(t) \geq \nu_{(\eta(x^r\vee y^r))}(t) \tag{3.11}$$

for all $x, y \in V$ and $\alpha, \beta \in \mathbb{R}^+$. Then there exists an unique positive mapping $\mathbf{T} : V \to W$ which satisfies the properties $(P_1), (P_2)$ and the inequality

$$\mu_{(F(|x|)-\mathbf{T}(|x|)}(t) \quad \geq \quad \nu_{\left(\frac{2\eta x}{2-2^r}\right)}(t)$$

for all $x \in V^+$.

Proof. Putting $\alpha = \beta = 2$ and $x = y$ in (3.11), we get

$$\mu_{\left(f(2|x|) - \frac{2\rho(2)f(|x|)\vee 2\rho(2)f(|x|)}{\rho(2)\vee\rho(2)}\right)}(t) \geq \nu_{(\eta x^r)}(t)$$

for all $x \in \mathcal{X}$ and $r \in [0, 1)$. Therefore,

$$\mu_{(f(2|x|)-2f(|x|))}(t) \quad \geq \quad \nu_{(\eta x^r)}(t),$$

$$\mu_{\left(\frac{1}{2}f(2|x|)-f(|x|)\right)}(t) \quad \geq \quad \nu_{(\eta x^r)}(2t).$$

The rest of the proof is similar to the previous one. □

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FOURIER SERIES OF SUMS OF PRODUCTS OF POLY-GENOCCHI AND POLY-BERNOULLI FUNCTIONS

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Abstract. In this paper, we consider three types of functions given by the sums of products of poly-Genocchi and poly-Bernoulli functions and derive their Fourier series expansions. Moreover, we will express each of them in terms of Bernoulli functions.

1. Introduction

Let r be any integer. The following series

$$Li_r(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^r} \tag{1.1}$$

is the rth polylogarithm function for $r \geq 1$, and a rational function for $r \leq 0$. Then it is easy to see that

$$\frac{d}{dx}(Li_{r+1}(x)) = \frac{1}{x} Li_r(x). \tag{1.2}$$

The poly-Bernoulli polynomials $B_m^{(r)}(x)$ of index r are given by (see [5–7])

$$\frac{Li_r(1-e^{-t})}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{t^m}{m!}. \tag{1.3}$$

When $x = 0$, $B_m^{(r)} = B_m^{(r)}(0)$ are called poly-Bernoulli numbers of index r. In particular, if $r = 1$, $B_m(x) = B_m^{(1)}(x)$ are the Bernoulli polynomials defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}. \tag{1.4}$$

We note here, in passing, that this definition of poly-Bernoulli polynomials are slightly different from the original definition (see [4–6]). As to poly-Bernoulli polynomials, we need to note the following:

$$\begin{aligned} &B_0^{(r)}(x) = 1,\ B_m^{(0)}(x) = x^m,\ B_m^{(0)} = \delta_{m,0}, \\ &\frac{d}{dx} B_m^{(r)}(x) = m B_{m-1}^{(r)}(x), B_m^{(r+1)}(1) - B_m^{(r+1)} = B_{m-1}^{(r)},\ (m \geq 1). \end{aligned} \tag{1.5}$$

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The poly-Genocchi polynomials $G_m^{(r)}(x)$ of index r were introduced in [3] as an analogy to poly-Bernoulli polynomials and defined by (see [8–11])

$$\frac{2Li_r(1-e^{-t})}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!}. \tag{1.6}$$

When $x=0$, $G_m^{(r)} = G_m^{(r)}(0)$ are called poly-Genocchi numbers of index r. In the special case of $r=1$, $G_m(x) = G_m^{(1)}(x)$ are the Genocchi polynomials given by

$$\frac{2t}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} G_m(x)\frac{t^m}{m!}. \tag{1.7}$$

We would like to mention here that the poly-Genocchi polynomials were named as poly-Euler polynomials in [3] and denoted by $\mathbf{E}_m^{(r)}$. However, for the obvious reason it seems more appropriate to call them poly-Genocchi polynomials rather than poly-Euler polynomials. In fact, there are other definitions for poly-Euler numbers and polynomials. For these, the interested reader may refer to the papers [1,16,17].

As to poly-Genocchi polynomials, we need to note the following properties.

$$\begin{aligned} &\frac{d}{dx}G_m^{(r)}(x) = mG_{m-1}^{(r)}(x), G_m^{(r+1)}(1) + G_m^{(r+1)} = 2B_{m-1}^{(r)},\ (m \geq 1),\\ &G_0^{(r)}(x) = 0,\ G_1^{(r)}(x) = 1,\ \deg G_m^{(r)}(x) = m-1,\ (m \geq 1). \end{aligned} \tag{1.8}$$

The properties in (1.8) immediately follow from the identity

$$\sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!} = \sum_{m=1}^{\infty}\left(\sum_{l=0}^{m-1}\binom{m}{l}a_{m-l}E_l(x)\right)\frac{t^m}{m!}, \tag{1.9}$$

where $Li_r(1-e^{-t}) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n \frac{t^n}{n!}$, and $E_m(x)$ are the Euler polynomials given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!}. \tag{1.10}$$

For any real number x, we let

$$< x > = x - [x] \in [0,1) \tag{1.11}$$

denote the fractional part of x.

We also need the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \tag{1.12}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.13}$$

Here we will consider the following three types of sums of products of poly-Genocchi and poly-Bernoulli functions $\alpha_m(< x >), \beta_m(< x >)$, and $\gamma_m(< x >)$, and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

(1) $\alpha_m(< x >) = \sum_{k=0}^{m-1} B_k^{(r+1)}(< x >)G_{m-k}^{(s+1)}(< x >),\ (m \geq 2)$,
(2) $\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >)G_{m-k}^{(s+1)}(< x >),\ (m \geq 2)$,
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(< x >)G_{m-k}^{(s+1)}(< x >),\ (m \geq 2)$.
For related recent works, one may refer to the papers (see [2, 12–15]).

2. The function $\alpha_m(< x >)$

Let $\alpha_m(x) = \sum_{k=0}^{m-1} B_k^{(r+1)}(x)G_{m-k}^{(s+1)}(x),\ (m \geq 2)$.

Then we now consider the function

$\alpha_m(< x >) = \sum_{k=0}^{m-1} B_k^{(r+1)}(< x >)G_{m-k}^{(s+1)}(< x >),\ (m \geq 2)$,

defined on $\mathbb{R}$, which is periodic with period 1.
The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.1}$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx. \tag{2.2}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}
\alpha_m'(x) &= \sum_{k=0}^{m-1} \left(kB_{k-1}^{(r+1)}(x)G_{m-k}^{(s+1)}(x) + (m-k)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x)\right) \\
&= \sum_{k=1}^{m-1} kB_{k-1}^{(r+1)}(x)G_{m-k}^{(s+1)}(x) + \sum_{k=0}^{m-2}(m-k)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x) \\
&= \sum_{k=0}^{m-2}(k+1)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x) + \sum_{k=0}^{m-2}(m-k)B_k^{(r+1)}(x)G_{m-k-1}^{(s+1)}(x) \\
&= (m+1)\sum_{k=0}^{m-2} B_k^{(r+1)}(x)G_{m-1-k}^{(s+1)}(x) \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.3}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.4}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.5}$$

For $m \geq 2$, we put

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m-1} \left(B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&= \sum_{k=1}^{m-1} \left(B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) + G_m^{(s+1)}(1) - G_m^{(s+1)} \\
&= \sum_{k=1}^{m-1} \left((B_k^{(r+1)} + B_{k-1}^{(r)})(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&- G_m^{(s+1)} + 2B_{m-1}^{(s)} - G_m^{(s+1)} \\
&= \sum_{k=0}^{m-1} 2B_k^{(r+1)}(-G_{m-k}^{(s+1)} + B_{m-k-1}^{(s)}) + \sum_{k=1}^{m-1} B_{k-1}^{(r)}(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}).
\end{aligned} \tag{2.6}$$

Clearly, we have

$$\alpha_m(1) = \alpha_m(0) \iff \Delta_m = 0, \tag{2.7}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.8}$$

We are now going to determine the Fourier coefficients $A_n^{(m)}$.

Case $1 : n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m
\end{aligned} \tag{2.9}$$

from which by induction on m we can show that

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}.$$

Case $2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.10}$$

$\alpha_m(< x >), (m \geq 2)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for an integer $m \geq 2$. Then $\alpha_m(0) = \alpha_m(1)$. So $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}\alpha_m(< x >) = &\frac{1}{m+2}\Delta_{m+1} \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} \\ &\times \left(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned} \tag{2.11}$$

We can now state our first result.

Theorem 2.1. *For each integer $l \geq 2$, let*

$$\begin{aligned}\Delta_l = &2 \sum_{k=0}^{l-1} B_k^{(r+1)} (-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\ &+ \sum_{k=1}^{l-1} B_{k-1}^{(r)} (-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).\end{aligned} \tag{2.12}$$

Assume that $\Delta_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >) \\ =&\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x},\end{aligned}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >),\end{aligned}$$

for all $x \in \mathbb{R}$.

Assume next that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.13}$$

for $x \in \mathbb{Z}$.

Now, we can state our second result.

Theorem 2.2. *For each integer* $l \geq 2$*, let*

$$\begin{aligned}\Delta_l &= 2\sum_{k=0}^{l-1} B_k^{(r+1)}(-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\ &+ \sum_{k=1}^{l-1} B_{k-1}^{(r)}(-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).\end{aligned} \tag{2.14}$$

Assume that $\Delta_m \neq 0$*, for an integers* $m \geq 2$*. Then we have the following.*

$$\begin{aligned}&\text{(a)} \frac{1}{m+2}\Delta_{m+1} \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >), \textit{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m-1} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Delta_m, \textit{for } x \in \mathbb{Z}. \end{cases}\end{aligned}$$

$$\begin{aligned}&\text{(b)} \frac{1}{m+2} \sum_{j=0}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &= \sum_{k=0}^{m-1} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >), \ \textit{for } x \notin \mathbb{Z},\end{aligned}$$

$$\begin{aligned}&\frac{1}{m+2} \sum_{j=0, j\neq 1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &= \sum_{k=0}^{m-1} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Delta_m, \ \textit{for } x \in \mathbb{Z}.\end{aligned}$$

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3. The function $\beta_m(< x >)$

Let $\beta_m(x) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(x) G_{m-k}^{(s+1)}(x),\ (m \geq 2)$.

Then we consider the function

$\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >),\ (m \geq 2)$,
defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.1}$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \tag{3.2}$$

Before continuing our discussion, we need to note the following.

$$\begin{aligned}
\beta_m'(x) &= \sum_{k=0}^{m-1} \Big(\frac{k}{k!(m-k)!} B_{k-1}^{(r+1)}(x) G_{m-k}^{(s+1)}(x) + \frac{m-k}{k!(m-k)!} B_k^{(r+1)}(x) G_{m-k-1}^{(s+1)}(x) \Big) \\
&= \sum_{k=1}^{m-1} \frac{1}{(k-1)!(m-k)!} B_{k-1}^{(r+1)}(x) G_{m-k}^{(s+1)}(x) \\
&+ \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} B_k^{(r+1)}(x) G_{m-k-1}^{(s+1)}(x) \\
&= \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&+ \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= 2\beta_{m-1}(x).
\end{aligned} \tag{3.3}$$

From this, we have

$$\Big(\frac{\beta_{m+1}(x)}{2} \Big)' = \beta_m(x), \tag{3.4}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \Big(\beta_{m+1}(1) - \beta_{m+1}(0) \Big). \tag{3.5}$$

For $m \geq 2$, we set

$$
\begin{aligned}
&\Omega_m = \beta_m(1) - \beta_m(0) \\
&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left(B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(B_k^{(r+1)}(1) G_{m-k}^{(s+1)}(1) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&+ \frac{1}{m!} G_m^{(s+1)}(1) - \frac{1}{m!} G_m^{(s+1)} \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left((B_k^{(r+1)} + B_{k-1}^{(r)})(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}) - B_k^{(r+1)} G_{m-k}^{(s+1)} \right) \\
&+ \frac{1}{m!}(-G_m^{(s+1)} + 2B_{m-1}^{(s)}) - \frac{1}{m!} G_m^{(s+1)} \\
&= \sum_{k=0}^{m-1} \frac{2}{k!(m-k)!} B_k^{(r+1)} (-G_{m-k}^{(s+1)} + B_{m-k-1}^{(s)}) \\
&+ \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} B_{k-1}^{(r)} (-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}).
\end{aligned} \tag{3.6}
$$

Now,

$$\beta_m(0) = \beta_m(1) \Leftrightarrow \Omega_m = 0, \tag{3.7}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \Omega_{m+1}. \tag{3.8}$$

We are now ready to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$
\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\beta_m(x) e^{-2\pi i n x} \right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left(\beta_m(1) - \beta_m(0) \right) + \frac{2}{2\pi i n} \int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n} \Omega_m,
\end{aligned} \tag{3.9}
$$

from which by induction on m we can easily deduce that

$$B_n^{(m)} = -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1}. \tag{3.10}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) = \frac{1}{2}\Omega_{m+1}. \tag{3.11}$$

$\beta_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$, and discontinuous at integers with jump discontinuities for those integers $m \geq 2$ with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for an integer $m \geq 2$. Then $\beta_m(0) = \beta_m(1)$. So $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
\beta_m(< x >) &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(- \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \Big) e^{2\pi i n x} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m-j+1} \times \Big(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \Big) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) \\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.12}$$

Now, we can state our first theorem.

Theorem 3.1. *For each integer $l \geq 2$, let*

$$\begin{aligned}
\Omega_l &= \sum_{k=0}^{l-1} \frac{2}{k!(l-k)!} B_k^{(r+1)} (-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\
&+ \sum_{k=1}^{l-1} \frac{1}{k!(l-k)!} B_{k-1}^{(r)} (-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).
\end{aligned} \tag{3.13}$$

Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(- \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \Big) e^{2\pi i n x},
\end{aligned} \tag{3.14}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}(b) &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>),\end{aligned} \tag{3.15}$$

for all $x \in \mathbb{R}$.

Assume next that $\Omega_m \neq 0$, for all integer $m \geq 2$. Then $\beta_m(0) \neq \beta_m(1)$. Thus $\beta_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.16}$$

for $x \in \mathbb{Z}$.

We can now state our second theorem.

Theorem 3.2. *For each integer* $l \geq 2$, *let*

$$\begin{aligned}\Omega_l = &\sum_{k=0}^{l-1} \frac{2}{k!(l-k)!} B_k^{(r+1)}(-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\ &+ \sum_{k=1}^{l-1} \frac{1}{k!(l-k)!} B_{k-1}^{(r)}(-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).\end{aligned} \tag{3.17}$$

Assume that $\Omega_m \neq 0$, *for an integer* $m \geq 2$. *Then we have the following.*

$$(a) \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>), \textit{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Omega_m, \textit{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\begin{aligned}&\sum_{j=0}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>), \quad \textit{for } x \notin \mathbb{Z},\end{aligned}$$

$$\begin{aligned}&\sum_{j=0, j\neq 1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Omega_m, \quad \textit{for } x \in \mathbb{Z}.\end{aligned}$$

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4. The function $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(x) G_{m-k}^{(s+1)}(x)$, $(m \geq 2)$.
Then we are going to consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(< x >) G_{m-k}^{(s+1)}(< x >),\ (m \geq 2), \tag{4.1}$$

defined on $\mathbb{R}$, which is periodic with period 1.
The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \tag{4.3}$$

Before going further, we need to observe the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left(k B_{k-1}^{(r+1)}(x) G_{m-k}^{(s+1)}(x) + (m-k) B_k^{(r+1)}(x) G_{m-k-1}^{(s+1)}(x) \right) \\
&= \sum_{k=0}^{m-2} \frac{1}{m-1-k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&+ \sum_{k=1}^{m-1} \frac{1}{k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1} G_{m-1}^{(s+1)}(x) + \sum_{k=1}^{m-2} \frac{1}{m-1-k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&+ \sum_{k=1}^{m-2} \frac{1}{k} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1} G_{m-1}^{(s+1)}(x) + (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} B_k^{(r+1)}(x) G_{m-1-k}^{(s+1)}(x) \\
&= \frac{1}{m-1} G_{m-1}^{(s+1)}(x) + (m-1)\gamma_{m-1}(x).
\end{aligned} \tag{4.4}$$

From this, we immediately see that

$$\left(\frac{1}{m} (\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}^{(s+1)}(x)) \right)' = \gamma_m(x), \tag{4.5}$$

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$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}[\gamma_{m+1}(x) - \frac{1}{m(m+1)}G_{m+1}^{(s+1)}(x)]_0^1 \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(G_{m+1}^{(s+1)}(1) - G_{m+1}^{(s+1)}(0))\right) \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)}(G_{m+1}^{(s+1)} - B_m^{(s)})\right).
\end{aligned} \tag{4.6}$$

For $m \geq 2$, we let

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(B_k^{(r+1)}(1)G_{m-k}^{(s+1)}(1) - B_k^{(r+1)}G_{m-k}^{(s+1)}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left((B_k^{(r+1)} + B_{k-1}^{(r)})(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)}) - B_k^{(r+1)}G_{m-k}^{(s+1)}\right) \\
&= \sum_{k=1}^{m-1} \frac{2}{k(m-k)}B_k^{(r+1)}(-G_{m-k}^{(s+1)} + B_{m-k-1}^{(s)}) \\
&+ \sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_{k-1}^{(r)}(-G_{m-k}^{(s+1)} + 2B_{m-k-1}^{(s)})
\end{aligned} \tag{4.7}$$

Now,

$$\gamma_m(0) = \gamma_m(1) \Leftrightarrow \Lambda_m = 0, \tag{4.8}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}(G_{m+1}^{(s+1)} - B_m^{(s)})\right). \tag{4.9}$$

Now, we want to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$
\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\gamma_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\
&+ \frac{1}{2\pi i n}\int_0^1 \Big\{\frac{1}{m-1} G_{m-1}^{(s+1)}(x) + (m-1)\gamma_{m-1}(x)\Big\} e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} + \frac{1}{2\pi i n (m-1)}\int_0^1 G_{m-1}^{(s+1)}(x) e^{-2\pi i n x} dx \\
&= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m + \frac{2}{2\pi i n(m-1)}\Phi_m,
\end{aligned} \tag{4.10}
$$

where

$$
\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi i n)^k}\left(G_{m-k}^{(s+1)} - B_{m-k-1}^{(s)}\right). \tag{4.11}
$$

Here we can show that

$$
\begin{aligned}
&\int_0^1 G_l^{(s+1)}(x) e^{-2\pi i n x} dx \\
&= \begin{cases} 2\sum_{k=1}^{l-1} \frac{(l)_{k-1}}{(2\pi i n)^k}\left(G_{l-k+1}^{(s+1)} - B_{l-k}^{(s)}\right), \text{for } n \neq 0, \\ \frac{-2}{l+1}\left(G_{l+1}^{(s+1)} - B_l^{(s)}\right), \text{for } n = 0. \end{cases}
\end{aligned}
$$

From this, by induction on m we can show that

$$
\begin{aligned}
C_n^{(m)} &= \frac{(m-1)!}{(2\pi i n)^{m-2}} C_n^{(2)} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{m-j+1} \\
&+ \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi i n)^j (m-j)}\Phi_{m-j+1}.
\end{aligned} \tag{4.12}
$$

Further, we can easily show that $C_n^{(2)} = -\frac{1}{2\pi i n}\Lambda_2$.
Thus we deduce that

$$
\begin{aligned}
C_n^{(m)} &= -\frac{1}{m}\sum_{j=1}^{m-1} \frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1} \\
&+ \frac{1}{m}\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi i n)^j (m-j)}\Phi_{m-j+1}.
\end{aligned} \tag{4.13}
$$

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Here we note that

$$
\begin{aligned}
&\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= \sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi in)^j(m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1}}{(2\pi in)^k}(G_{m-j-k+1}^{(s+1)} - B_{m-j-k}^{(s)}) \\
&= \sum_{j=1}^{m-2} \sum_{k=1}^{m-j-1} \frac{2(m)_{j+k-1}}{(2\pi in)^{j+k}(m-j)}(G_{m-j-k+1}^{(s+1)} - B_{m-j-k}^{(s)}) \\
&= 2\sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{a=j+1}^{m-1} \frac{(m)_{a-1}}{(2\pi in)^a}(G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}) \\
&= 2\sum_{a=2}^{m-1} \frac{(m)_{a-1}}{(2\pi in)^a}(G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}) \sum_{j=1}^{a-1} \frac{1}{m-j} \\
&= 2\sum_{a=2}^{m-1} \frac{(m)_{a-1}}{(2\pi in)^a}(G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)})(H_{m-1} - H_{m-a}) \\
&= 2\sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi in)^a} \frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1}(H_{m-1} - H_{m-a}).
\end{aligned} \tag{4.14}
$$

Putting everything altogether, we obtain

$$
\begin{aligned}
C_n^{(m)} &= -\frac{1}{m}\sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi in)^a}\Lambda_{m-a+1} \\
&\quad + \frac{2}{m}\sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi in)^a} \frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1}(H_{m-1} - H_{m-a}). \\
&= -\frac{1}{m}\sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi in)^a} \\
&\quad \times \left(\Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1}(H_{m-1} - H_{m-a})\right).
\end{aligned} \tag{4.15}
$$

Case 2: $n = 0$.

$$
\begin{aligned}
C_0^{(m)} &= \int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}(G_{m+1}^{(s+1)} - B_m^{(s)})\right).
\end{aligned} \tag{4.16}
$$

$\gamma_m(< x >), (m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. So $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$
\begin{aligned}
&\gamma_m(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}(\sum_{a=1}^{m-1}\frac{(m)_a}{(2\pi in)^a}\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)e^{2\pi inx}\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)+\frac{1}{m}\sum_{a=1}^{m-1}\binom{m}{a}\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)\left(-a!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^a}\right)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)+\frac{1}{m}\sum_{a=2}^{m-1}\binom{m}{a}\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)B_a(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}
\tag{4.17}
$$

Now, we can state our first result.

Theorem 4.1. *For each integer $l\geq 2$, let*

$$
\begin{aligned}
\Lambda_l=&\sum_{k=1}^{l-1}\frac{2}{k(l-k)}B_k^{(r+1)}(-G_{l-k}^{(s+1)}+B_{l-k-1}^{(s)})\\
&+\sum_{k=1}^{l-1}\frac{1}{k(l-k)}B_{k-1}^{(r)}(-G_{l-k}^{(s+1)}+2B_{l-k-1}^{(s)}).
\end{aligned}
\tag{4.18}
$$

Assume that $\Lambda_m=0$, for an integers $m\geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>)$ *has the Fourier series expansion*

$$
\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k^{(r+1)}(<x>)G_{m-k}^{(s+1)}(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}(G_{m+1}^{(s+1)}-B_m^{(s)})\right)-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}(\sum_{a=1}^{m-1}\frac{(m)_a}{(2\pi in)^a}\\
&\times\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right))e^{2\pi inx}
\end{aligned}
\tag{4.19}
$$

for all $x\in\mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}
&\text{(b)} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>) \\
&= \frac{1}{m} \sum_{a=0, a\neq 1}^{m-1} \binom{m}{a} \left(\Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1} (H_{m-1} - H_{m-a}) \right) \\
&\times B_a(<x>),
\end{aligned} \tag{4.20}$$

for all $x \in \mathbb{R}$.

Assume next that $\Lambda_m \neq 0$, for an integers $m \geq 2$. Then $\gamma_m(0) \neq \gamma_m(1)$. So $\gamma_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Hence the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.21}$$

for $x \in \mathbb{Z}$.

Next, we can state our second result.

Theorem 4.2. *For each integer* $l \geq 2$, *let*

$$\begin{aligned}
\Lambda_l &= \sum_{k=1}^{l-1} \frac{2}{k(l-k)} B_k^{(r+1)}(-G_{l-k}^{(s+1)} + B_{l-k-1}^{(s)}) \\
&+ \sum_{k=1}^{l-1} \frac{1}{k(l-k)} B_{k-1}^{(r)}(-G_{l-k}^{(s+1)} + 2B_{l-k-1}^{(s)}).
\end{aligned} \tag{4.22}$$

Assume that $\Lambda_m \neq 0$, *for an integers* $m \geq 2$. *Then we have the following.*

$$\begin{aligned}
&\text{(a)} \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}(G_{m+1}^{(s+1)} - B_m^{(s)}) \right) - \frac{1}{m} \sum_{n=-\infty, n\neq 0}^{\infty} \Big(\sum_{a=1}^{m-1} \frac{(m)_a}{(2\pi i n)^a} \\
&\times \big(\Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1}(H_{m-1} - H_{m-a}) \big) \Big) e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>), \textit{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)} G_{m-k}^{(s+1)} + \frac{1}{2}\Lambda_m, \textit{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
&\frac{1}{m} \sum_{a=0}^{m-1} \binom{m}{a} \left(\Lambda_{m-a+1} - 2\frac{G_{m-a+1}^{(s+1)} - B_{m-a}^{(s)}}{m-a+1}(H_{m-1} - H_{m-a}) \right) \\
&\times B_a(<x>) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k^{(r+1)}(<x>) G_{m-k}^{(s+1)}(<x>), \ \textit{for } x \notin \mathbb{Z},
\end{aligned}$$

$$\frac{1}{m}\sum_{a=0,a\neq 1}^{m-1}\binom{m}{a}\left(\Lambda_{m-a+1}-2\frac{G_{m-a+1}^{(s+1)}-B_{m-a}^{(s)}}{m-a+1}(H_{m-1}-H_{m-a})\right)$$
$$\times B_a(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k^{(r+1)}G_{m-k}^{(s+1)}+\frac{1}{2}\Lambda_m,\ \textit{for } x\in\mathbb{Z}.$$

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ADDITIVE-QUADRATIC ρ-FUNCTIONAL EQUATIONS IN β-HOMOGENEOUS F-SPACES

SUNGSIK YUN

Abstract. Let

$$\begin{aligned} M_1 f(x,y): &= \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ &\quad + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \end{aligned}$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic ρ-functional equations

$$M_1 f(x,y) = \rho M_2 f(x,y) \tag{0.1}$$

and

$$M_2 f(x,y) = \rho M_1 f(x,y), \tag{0.2}$$

where ρ is a fixed nonzero number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equations (0.1) and (0.2) in β-homogeneous F-spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25]).

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN$_1$) $\|x\| = 0$ if and only if $x = 0$;

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(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(FN_3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
(FN_5) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.
Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [16]).

In Section 2, we solve the additive-quadratic ρ-functional equation (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (0.1) in β-homogeneous F-spaces.

In Section 3, we solve the additive-quadratic ρ-functional equation (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (0.2) in β-homogeneous F-spaces.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1-homogeneous real or complex F^*-space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous complex F-space with norm $\|\cdot\|$.

Let ρ be a nonzero number with $\rho \neq 1$.

2. Additive-quadratic ρ-functional equation (0.1) in β-homogeneous F-spaces

We solve and investigate the additive-quadratic ρ-functional equation (0.1) in β-homogeneous F^*-spaces.

Lemma 2.1.
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof. (i)

$$M_1 f_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x, y) = \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x, y) = 2 f_o\left(\frac{x + y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0, 0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x, y) = 2 f_e\left(\frac{x + y}{2}\right) + 2 f_e\left(\frac{x - y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0, 0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

From now on, for a given mapping $f : X \to Y$, define $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then f_o is an odd mapping and f_e is an even mapping.

Lemma 2.2. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$M_1 f(x,y) = \rho M_2 f(x,y) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is the sum of the Cauchy additive mapping f_o and the quadratic mapping f_e.

Proof. Letting $y = x$ in (2.1) for f_o, we get $f_o(2x) - 2f_o(x) = 0$ and so $f_o(2x) = 2f_o(x)$ for all $x \in X$. Thus

$$f_o\left(\frac{x}{2}\right) = \frac{1}{2} f_o(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f_o(x+y) - f_o(x) - f_o(y) &= \rho\left(2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y)\right) \\ &= \rho(f_o(x+y) - f_o(x) - f_o(y)) \end{aligned}$$

and so

$$f_o(x+y) = f_o(x) + f_o(y)$$

for all $x, y \in X$.

Letting $y = x$ in (2.1) for f_e, we get $\frac{1}{2} f_e(2x) - 2f_e(x) = 0$ and so $f_e(2x) = 4f_e(x)$ for all $x \in X$. Thus

$$f_e\left(\frac{x}{2}\right) = \frac{1}{4} f_e(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} &\frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y) \\ &= \rho\left(2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y)\right) \\ &= \rho\left(\frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y)\right) \end{aligned}$$

and so

$$f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y)$$

for all $x, y \in X$.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping f_o and the quadratic mapping f_e. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2.1) in β-homogeneous F-spaces.

Theorem 2.3. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|M_1 f(x,y) - \rho M_2 f(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.4}$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_o(x) - A(x)\| \le \frac{4\theta}{2^{\beta_2}(2^{\beta_1 r} - 2^{\beta_2})} \|x\|^r, \tag{2.5}$$

$$\|f_e(x) - Q(x)\| \le \frac{4\theta}{2^{\beta_1 r} - 4^{\beta_2}}\|x\|^r \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.4) for f_o, we get

$$\|f_o(2x) - 2f_o(x)\| \le \frac{4\theta}{2^{\beta_2}}\|x\|^r \tag{2.7}$$

for all $x \in X$. So

$$\left\|f_o(x) - 2f_o\left(\frac{x}{2}\right)\right\| \le \frac{4\theta}{2^{\beta_2+\beta_1 r}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|2^l f_o\left(\frac{x}{2^l}\right) - 2^m f_o\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f_o\left(\frac{x}{2^j}\right) - 2^{j+1} f_o\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{4\theta}{2^{\beta_2+\beta_1 r}}\sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \end{aligned}\tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^k f_o(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f_o(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f_o\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f_o is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.5).

It follows from (2.4) that

$$\begin{aligned}&\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\ &= \lim_{n\to\infty}\left\|2^n\left(f_o\left(\frac{x+y}{2^n}\right) - f_o\left(\frac{x}{2^n}\right) - f_o\left(\frac{y}{2^n}\right)\right)\right. \\ &\left.-2^n\rho\left(2f_o\left(\frac{x+y}{2^{n+1}}\right) - f_o\left(\frac{x}{2^n}\right) - f_o\left(\frac{y}{2^n}\right)\right)\right\| \le \frac{4\theta}{2^{\beta_2}}\lim_{n\to\infty}\frac{2^{\beta_2 n}}{2^{\beta_1 r n}}\|x\|^r = 0\end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f_o\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f_o\left(\frac{x}{2^q}\right)\right\| \\ &\le \frac{8\theta}{2^{\beta_2}(2^{\beta_1 r} - 2^{\beta_2})}\frac{2^{\beta_2 q}}{2^{\beta_1 r q}}\|x\|^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

Letting $y = x$ in (2.4) for f_e, we get

$$\left\|\frac{1}{2}f_e(2x) - 2f_e(x)\right\| \le \frac{4\theta}{2^{\beta_2}}\|x\|^r \tag{2.9}$$

for all $x \in X$. So

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\| \le \frac{4\theta}{2^{\beta_1 r}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f_e\left(\frac{x}{2^l}\right) - 4^m f_e\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f_e\left(\frac{x}{2^j}\right) - 4^{j+1} f_e\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{4\theta}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \end{aligned} \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{4^k f_e(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^k f_e(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f_e\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f_e is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.6).

It follows from (2.4) that

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right. \\ &\left. - \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\ &= \lim_{n\to\infty}\left\|4^n\left(\frac{1}{2}f_e\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f_e\left(\frac{x-y}{2^n}\right) - f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{y}{2^n}\right)\right)\right. \\ &\left. - 4^n\rho\left(2f_e\left(\frac{x+y}{2^{n+1}}\right) + 2f_e\left(\frac{x-y}{2^{n+1}}\right) - f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \frac{4\theta}{2^{\beta_2}}\lim_{n\to\infty}\frac{4^{\beta_2 n}}{2^{\beta_1 r n}}\|x\|^r = 0\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}&\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \\ &= \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.6). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f_e\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f_e\left(\frac{x}{2^q}\right)\right\| \\ &\le \frac{8\theta}{2^{\beta_1 r} - 4^{\beta_2}}\frac{4^{\beta_2 q}}{2^{\beta_1 r q}}\|x\|^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q, as desired. □

Theorem 2.4. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.4). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_o(x) - A(x)\| \le \frac{4\theta}{2^{\beta_2}(2^{\beta_2} - 2^{\beta_1 r})}\|x\|^r, \tag{2.11}$$

$$\|f_e(x) - Q(x)\| \le \frac{4\theta}{4^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{2.12}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f_o(x) - \frac{1}{2}f_o(2x)\right\| \le \frac{4\theta}{4^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f_o(2^l x) - \frac{1}{2^m}f_o(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f_o\left(2^j x\right) - \frac{1}{2^{j+1}}f_o\left(2^{j+1}x\right)\right\| \\ &\le \frac{4\theta}{4^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{2^{\beta_2 j}}\|x\|^r \end{aligned} \tag{2.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.13) that the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f_o(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.11).

It follows from (2.9) that

$$\left\|f_e(x) - \frac{1}{4}f_e(2x)\right\| \le \frac{4\theta}{4^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f_e(2^l x) - \frac{1}{4^m}f_e(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f_e\left(2^j x\right) - \frac{1}{4^{j+1}}f_e\left(2^{j+1}x\right)\right\| \\ &\le \frac{4\theta}{4^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{4^{\beta_2 j}}\|x\|^r \end{aligned} \tag{2.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.14) that the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f_e(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.12).

The rest of the proof is similar to the proof of Theorem 2.3. □

3. Additive-quadratic ρ-functional equation (0.2) in β-homogeneous F-spaces

We solve and investigate the additive-quadratic ρ-functional equation (0.2) in β-homogeneous F^*-spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$M_2 f(x, y) = \rho M_1 f(x, y) \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is the sum of the Cauchy additive mapping f_o and the quadratic mapping f_e.

Proof. Letting $y = 0$ in (3.1) for f_o, we get

$$f_o\left(\frac{x}{2}\right) = \frac{1}{2} f_o(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f_o(x+y) - f_o(x) - f_o(y) &= 2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) \\ &= \rho(f_o(x+y) - f_o(x) - f_o(y)) \end{aligned}$$

and so

$$f_o(x+y) = f_o(x) + f_o(y)$$

for all $x, y \in X$.

Letting $y = 0$ in (3.1) for f_e, we get

$$f_e\left(\frac{x}{2}\right) = \frac{1}{4} f_e(x) \tag{3.3}$$

for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned} &\frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y) \\ &= 2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) \\ &= \rho\left(\frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y)\right) \end{aligned}$$

and so

$$f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in β-homogeneous F-spaces.

Theorem 3.2. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|M_2 f(x, y) - \rho M_1 f(x, y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.4}$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_o(x) - A(x)\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(2^{\beta_1 r} - 2^{\beta_2})}\|x\|^r, \tag{3.5}$$

$$\|f_e(x) - Q(x)\| \leq \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(2^{\beta_1 r} - 4^{\beta_2})}\|x\|^r \tag{3.6}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4) for f_o, we get

$$\left\|f_o(x) - 2f_o\left(\frac{x}{2}\right)\right\| = \left\|2f_o\left(\frac{x}{2}\right) - f_o(x)\right\| \leq \frac{2\theta}{2^{\beta_2}}\|x\|^r \tag{3.7}$$

for all $x \in X$. So

$$\begin{aligned}\left\|2^l f_o\left(\frac{x}{2^l}\right) - 2^m f_o\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|2^j f_o\left(\frac{x}{2^j}\right) - 2^{j+1} f_o\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{2\theta}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \end{aligned}\tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{2^k f_o(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f_o(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f_o\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f_o is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.5).

Letting $y = 0$ in (3.4) for f_e, we get

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\| = \left\|4f_e\left(\frac{x}{2}\right) - f_e(x)\right\| \leq \frac{2\theta}{2^{\beta_2}}\|x\|^r \tag{3.9}$$

for all $x \in X$. So

$$\begin{aligned}\left\|4^l f_e\left(\frac{x}{2^l}\right) - 4^m f_e\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|4^j f_e\left(\frac{x}{2^j}\right) - 4^{j+1} f_e\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{2\theta}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\|x\|^r \end{aligned}\tag{3.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{4^k f_e(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^k f_e(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f_e\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f_e is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.6).

The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 3.3. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.4). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_o(x) - A(x)\| \leq \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(2^{\beta_2} - 2^{\beta_1 r})}\|x\|^r, \tag{3.11}$$

$$\|f_e(x) - Q(x)\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{2^{\beta_2}(4^{\beta_2} - 2^{\beta_1 r})}\|x\|^r \tag{3.12}$$

for all $x \in X$.

Proof. It follows from (3.7) that

$$\left\|f_o(x) - \frac{1}{2}f_o(2x)\right\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{4^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f_o(2^l x) - \frac{1}{2^m}f_o(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f_o\left(2^j x\right) - \frac{1}{2^{j+1}}f_o\left(2^{j+1}x\right)\right\| \\ &\le \frac{2 \cdot 2^{\beta_1 r}\theta}{4^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{2^{\beta_2 j}}\|x\|^r \end{aligned} \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f_o(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f_o(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.11).

It follows from (3.9) that

$$\left\|f_e(x) - \frac{1}{4}f_e(2x)\right\| \le \frac{2 \cdot 2^{\beta_1 r}\theta}{8^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f_e(2^l x) - \frac{1}{4^m}f_e(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f_e\left(2^j x\right) - \frac{1}{4^{j+1}}f_e\left(2^{j+1}x\right)\right\| \\ &\le \frac{2 \cdot 2^{\beta_1 r}\theta}{8^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{4^{\beta_2 j}}\|x\|^r \end{aligned} \tag{3.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f_e(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f_e(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.14), we get (3.12).

The rest of the proof is similar to the proof of Theorem 2.3. □

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DIFFERENTIAL SUBORDINATION FOR ANALYTIC FUNCTIONS ASSOCIATED WITH LEAF-LIKE DOMAINS

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Abstract

In our present investigation, we obtain several differential subordination results involving leaf-like domains. Moreover, certain sharp coefficient estimates are investigated when the class of functions lies in leaf-like domains.

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Keywords and Phrases: Analytic function, Starlike function, Convex function, Leaf-like domain.

1. **Introduction and Definitions**

Let $\mathcal{A}$ denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. A function f is subordinate to the function g, written as $f \prec g$ or $f(z) \prec g(z)$, provided that there is an analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g[w(z)]$ for $z \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. For two functions $f, g \in \mathcal{A}$, the Hadamard product is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (z \in \mathbb{U}),$$

where a_n and b_n are the coefficients of f and g, respectively.

Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ such that $\Re(p(z)) > 0$ in $\mathbb{U}$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ be the class of all starlike functions of order γ and convex functions of order $\gamma (0 \le \gamma < 1)$, respectively. A function $f \in \mathcal{A}$ is in the class $\mathcal{R}(\gamma)$, if it satisfies the inequality:

$$\Re(f'(z)) > \gamma \ \ (z \in \mathbb{U}, \ 0 \le \gamma < 1).$$

We write $\mathcal{R}(0) = \mathcal{R}$, the familiar class of functions in $\mathcal{A}$ which are of bounded turning in $\mathbb{U}$. It is well known that $\mathcal{S}^* \not\subset \mathcal{R}$ and $\mathcal{R} \not\subset \mathcal{S}^*$ (see [13]).

The class of k-*starlike* functions is introduced and studied by Kanas and Wiśniowska ([6], [7])(For more details, see [5],[8],[9],[10]) as defined by $f \in k - \mathcal{ST}$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k \left|\frac{zf'(z)}{f(z)} - 1\right| \ \ (0 \le k < \infty, z \in \mathbb{U}). \tag{1.2}$$

One may be easily see that the conditions (1.2) may be rewritten into the form

$$\Re(p(z)) > k|p(z) - 1| \ \ (z \in \mathbb{U}).$$

Also, it is easy to see that $p(\mathbb{U})$ is a conical domain

$$\Omega_k = \{\omega \in \mathbb{C} : \Re(\omega) > k|\omega - 1|\},$$

or

$$\Omega_k = \left\{\omega = u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\},$$

where $0 \le k < \infty$. For $k > 1$, the curve $\partial\Omega_k$ is the ellipse defined by

$$\partial\Omega_k = \left\{\omega = u + iv : u^2 = k^2(u-1)^2 + k^2 v^2\right\}.$$

For $k \ge 2 + \sqrt{2}$, this ellipse lies entirely inside $\mathcal{L}$, where $\mathcal{L} = \left\{\omega \in \mathbb{C} : \left|\omega^2 - 1\right| < 1\right\}$ is the interior of the right half of the lemniscate of Bernoulli $(u^2 + v^2)^2 = 2(u^2 - v^2)$. Therefore $k - \mathcal{ST} \subset \mathcal{SL}^*$ for $k \ge 2 + \sqrt{2}$.

Recently, Sokół and Stankiewicz [18] defined the class $\mathcal{SL}^*$ given by

$$\mathcal{SL}^* = \left\{f \in \mathcal{S} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, \ z \in \mathbb{U}\right\}. \tag{1.3}$$

It is easy to see that

$$f \in \mathcal{SL}^* \Leftrightarrow \frac{zf'(z)}{f(z)} \prec q_0(z) = \sqrt{1+z} \quad (q_0(0) = 1)$$

and $\mathcal{L} \subset \left\{\omega : \left|\omega - \sqrt{2}/2\right| < \sqrt{2}/2\right\}$.

Analogous to the class $\mathcal{SL}^*$, recently Patel and Sahoo [16] defined a class $\tilde{\mathcal{R}}$. A function $f \in \mathcal{S}$ is said to be in the class $\tilde{\mathcal{R}}$, if it satisfies the condition

$$\tilde{\mathcal{R}} = \left\{f \in \mathcal{S} : \left|(f'(z))^2 - 1\right| < 1 \quad (z \in \mathbb{U})\right\}. \tag{1.4}$$

It follows from (1.4) and the definition of subordination that a function $f \in \tilde{\mathcal{R}}$ satisfies the subordinate relation

$$f'(z) \prec \sqrt{1+z} \qquad (z \in \mathbb{U}). \tag{1.5}$$

Sokół and Paprocki [14] studied the class of analytic and univalent functions defined by

$$\mathcal{S}^*(\alpha,b)=\left\{f\in\mathcal{S}:\left|\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}-b\right|<b,\ \left(\frac{zf'(z)}{f(z)}\right)^{\alpha}_{z=0}=1\quad(z\in\mathbb{U})\right\},\tag{1.6}$$

where $\alpha\geq 1, b\geq\frac{1}{2}$. For the choice of $\alpha=1$, the class of $\mathcal{S}^*(1,b)$ investigated by Janowski [3]. For the choice of $\alpha=2, b=1$, the class $\mathcal{S}^*(2,1)$ investigated by Sokół [14]. It is easy to see that $f\in\mathcal{S}^*(\alpha,b)$ if and only if

$$\frac{zf'(z)}{f(z)}\prec q_0(z)=\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}\quad(q_0(0)=1).\tag{1.7}$$

Note that the set,

$$\Omega(\alpha,b)=\left\{\omega\in\mathbb{C}:|\omega^{\alpha}-b|<b,|\arg(\omega)|\leq\frac{\pi}{2\alpha},\alpha\geq 1,b\geq\frac{1}{2}\right\}\tag{1.8}$$

is connected with the class $\mathcal{S}^*(\alpha,b)$ and is a leaf-like set. The concept of leaf-like domain was investigated by Sokół and Paprocki [14]. For more details related to the leaf-like domain, one may refer to the recent papers (see [1, 4, 17, 18, 19, 20, 21, 22, 23]).

Motivated essentially by the work of Sokół and Paprocki [14] and Sahoo and Patel [16], we introduce the class $\tilde{\mathcal{R}}(\alpha,b)$ related to the concept of leaf-like domain as given below.

A function $f\in\mathcal{S}$ is said to be in the class $\tilde{\mathcal{R}}(\alpha,b)$, if it satisfies the condition

$$\left|(f'(z))^{\alpha}-b\right|<b\qquad(z\in\mathbb{U}).\tag{1.9}$$

Let

$$\mathcal{Q}=\left\{\omega\in\mathbb{C}:0<\Re(\omega),|\omega^{\alpha}-b|<b\text{ for }z\in\mathbb{U},\alpha\geq 1,b\geq\frac{1}{2}\right\}.$$

It is easy to see that, the set $\mathcal{Q}$ represents all points on the right half plane such that the product of the distances from each point to the end points $-b$ and b is less than b. It follows from (1.9) and the definition of subordination that a function $f\in\tilde{\mathcal{R}}(\alpha,b)$ satisfies the subordinate relation

$$f'(z)\prec\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}\quad\left(\alpha\geq 1,b\geq\frac{1}{2}\right).\tag{1.10}$$

All powers are principle one. In the present investigation, the authors obtain several differential subordination results involving in the classes $\tilde{\mathcal{R}}(\alpha,b)$ and $\mathcal{S}^*(\alpha,b)$. Apart from the differential subordination results, certain sharp coefficient estimates are obtained for the class of functions $\tilde{\mathcal{R}}(\alpha,b)$ and $\mathcal{S}^*(\alpha,b)$.

2. Main Results

To prove main results, we need the following lemmas.

Lemma 2.1. [12] *Let q be univalent in $\mathbb{U}$ and let φ be analytic in a domain containing $q(\mathbb{U})$. Let $zq'(z)\varphi(q(z))$ be starlike. If p is analytic in $\mathbb{U}$, $p(0)=q(0)$ and satisfies*

$$zp'(z)\varphi(p(z))\prec zq'(z)\varphi(q(z)),\tag{2.1}$$

then $p(z)\prec q(z)$ and q is the best dominant.

Lemma 2.2. [2] *If a function ω is analytic for $|z| \le |z_0| < 1$, $\omega(0) = 0$, and $|\omega(z_0)| = \max\{|\omega(z)| : |z| \le |z_0|\}$, then*

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} \ge 1. \tag{2.2}$$

Theorem 2.1. *Let function $f \in \mathcal{A}$. Then*

$$\Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1}{4} \Rightarrow \frac{zf''(z)}{f'(z)} \prec q_0(z) = \sqrt{1+z}. \tag{2.3}$$

Proof. Let us denote $Q(f,z) = f'(z)$. Suppose that $Q(f,z) \not\prec q_0(z)$. The function q_0 is univalent in $\mathbb{U}$ so there exist z_0, ζ_0 such that $|z_0| = r_0 < 1$, $|\zeta_0| = 1$, $Q(f,z)\,(|z| < r_0) \subset q_0(\mathbb{U})$ and $Q(f,z_0) \prec q_0(\zeta_0)$. Then the function $\omega(z) = q_0^{-1}(Q(f,z))$ is analytic in $|z| < r_0$ and $\omega(0) = 0, \omega(z_0) = \zeta_0$. Thus $|\omega(z)|$ assumes at z_0 its maximum in $|z| \le |z_0|$ and by Lemma 2.2, $z_0\omega'(z_0) = m\omega(z_0)$, $m \ge 1$. Differentiating $q_0(\omega(z)) = Q(f,z)$ we obtain

$$\frac{z\omega'(z)}{\omega(z)}\frac{\omega(z)}{2(1+\omega(z))} = \frac{zf''(z)}{f'(z)}. \tag{2.4}$$

Then we have

$$\frac{z_0 f''(z_0)}{f'(z_0)} = \frac{z_0\omega'(z_0)}{\omega(z_0)}\frac{\omega(z_0)}{2(1+\omega(z_0))} = \frac{m}{4} \ge \frac{1}{4}, \tag{2.5}$$

which contradicts the hypothesis of the theorem. Hence $\frac{zf''(z)}{f'(z)} \prec q_0(z) = \sqrt{1+z}$. □

Theorem 2.2. *A function $f \in \tilde{\mathcal{R}}(\alpha, b)$ if and only if there exist an analytic function q with $q(0) = 1$ and $q(\mathbb{U}) \subset \Omega(\alpha, b)$ such that*

$$f(z) = \int_0^z q(t)dt \quad (z \in \mathbb{U}). \tag{2.6}$$

Proof. Let $f \in \tilde{\mathcal{R}}(\alpha, b)$ and let $q(z) = f'(z)$. If f is given by (2.6) with an analytic q satisfying $q(0) = 1$ and $q(\mathbb{U}) \subset \Omega(\alpha, b)$, then

$$q(z) \prec q_0(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

Now differentiating (2.6), we obtain $f'(z) = q(z)$. Therefore

$$f'(z) \prec q_0(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

and hence $f \in \tilde{\mathcal{R}}(\alpha, b)$. □

Next we determine the lower bound for β so that

$$1 + \frac{\beta z p'(z)}{p(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

implies that

$$p(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

Lemma 2.3. *Let*

$$\beta_0 = \frac{2\alpha}{(2b-1)}\left[(2b)^{\frac{1}{\alpha}} - 1\right] \quad \left(\alpha \geq 1, b \geq \frac{1}{2}\right).$$

If

$$1 + \frac{\beta z p'(z)}{p(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.7}$$

then

$$p(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}. \tag{2.8}$$

The lower bound β_0 is the best possible.

Proof. Define the function $q : \mathbb{U} \to \mathbb{C}$ by

$$q(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

with $q(0) = 1$. Since

$$q(\mathbb{U}) = \left\{\omega \in \mathbb{C} : |\omega^\alpha - b| < b,\ |\arg(\omega)| \leq \frac{\pi}{2\alpha}\right\}$$

is the right half of leaf-like set, $q(\mathbb{U})$ is a convex set and hence q is a convex. Let us take the subordination,

$$1 + \frac{\beta z p'(z)}{p(z)} \prec 1 + \frac{\beta z q'(z)}{q(z)}. \tag{2.9}$$

Performing a calculation, one can find that

$$\frac{\beta z p'(z)}{p(z)} = \frac{\beta z(2b-1)}{\alpha}\left[\frac{1}{(1+z)(b+(1-b)z)}\right] \tag{2.10}$$

is convex in $\mathbb{U}$ (and hence starlike). Thus, in view of Lemma 2.1, it follows that $p(z) \prec q(z)$. To conclude the proof, it is left to show that,

$$q(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \prec 1 + \frac{\beta z q'(z)}{q(z)} = 1 + \frac{\beta z(2b-1)}{\alpha}\left[\frac{1}{(1+z)(b+(1-b)z)}\right] =: h(z). \tag{2.11}$$

Since

$$h(\mathbb{U}) = \left\{\omega : \Re(\omega) < 1 + \frac{\beta(2b-1)}{2\alpha}\right\}$$

and

$$q(\mathbb{U}) = \{\omega : |\omega^\alpha - b| < b\} \subset \left\{\omega : \Re(\omega) < (2b)^{\frac{1}{\alpha}}\right\},$$

it follows that $q(\mathbb{U}) \subset h(\mathbb{U})$ if

$$(2b)^{\frac{1}{\alpha}} \leq 1 + \frac{\beta(2b-1)}{2\alpha}.$$

Thus $q(z) \prec p(z)$ for

$$\beta \geq \frac{2\alpha}{2b-1}\left[(2b)^{\frac{1}{\alpha}} - 1\right]$$

and this completes the proof of Lemma 2.3. □

Theorem 2.3. *Let β_0 be given in Lemma 2.3 and $f \in \mathcal{A}$. If f satisfies*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.12}$$

then $f \in \mathcal{S}^*(\alpha, b)$.

Proof. Define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}). \tag{2.13}$$

Then the analytic function p satisfies p(0)= 1. A simple calculation yields,

$$\frac{zp'(z)}{p(z)} = 1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}. \tag{2.14}$$

Therefore an application of Lemma 2.3 gives Theorem 3. □

Similarly by taking $p(z) = z^2 f'(z)/f^2(z)$ and $p(z) = f'(z)$ in Lemma 2.3, we have the following results, respectively.

Theorem 2.4. *Let β_0 be given in Lemma 2.3 and $f \in \mathcal{A}$. If f satisfies*

$$1+\beta\left(1+\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.15}$$

then

$$\frac{z^2 f'(z)}{f^2(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

Theorem 2.5. *Let β_0 be given in Lemma 2.3 and $f \in \mathcal{A}$. If f satisfies*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.16}$$

then

$$f'(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

Lemma 2.4. *Let*

$$\beta_0 = \frac{\alpha(3-2b)}{(2b-1)}\left[(2b)^{\frac{1}{\alpha}}-1\right]\left(\alpha \geq 1,\ b \geq \frac{1}{2}\right).$$

If

$$1+\frac{\beta z p'(z)}{p^{1-\alpha}(z)} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \geq \beta_0), \tag{2.17}$$

then

$$p(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}. \tag{2.18}$$

The lower bound β_0 is the best possible.

Proof. Let q be a convex function given by

$$q(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (z \in \mathbb{U}).$$

Then we obtain

$$1 + \frac{\beta z p'(z)}{p^{1-\alpha}(z)} \prec 1 + \frac{\beta z q'(z)}{q^{1-\alpha}(z)}. \tag{2.19}$$

A simple computation implies that

$$\frac{\beta z p'(z)}{p^{1-\alpha}(z)} = \frac{\beta z(2b-1)}{\alpha b}\left[\frac{1}{1+((\frac{1-b}{b})\,z)^2}\right] \tag{2.20}$$

is convex in $\mathbb{U}$(and hence starlike). Thus, in view of Lemma 2.1, it follows that $p(z) \prec q(z)$. To conclude the proof, it is left to show that,

$$q(z) = \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \prec 1 + \frac{\beta z q'(z)}{q^{1-\alpha}(z)} = 1 + \frac{\beta z(2b-1)}{\alpha b}\left[\frac{1}{1+((\frac{1-b}{b})\,z)^2}\right] =: h(z). \tag{2.21}$$

Since

$$h(\mathbb{U}) = \left\{\omega : \Re(\omega) < 1 + \frac{\beta(2b-1)}{\alpha(3-2b)}\right\},$$

and

$$q(\mathbb{U}) = \{\omega : |\omega^\alpha - b| < b\} \subset \left\{\omega : \Re(\omega) < (2b)^{\frac{1}{\alpha}}\right\},$$

it follows that $q(\mathbb{U}) \subset h(\mathbb{U})$ if

$$(2b)^{\frac{1}{\alpha}} \le 1 + \frac{\beta(2b-1)}{\alpha(3-2b)}.$$

Thus $q(z) \prec p(z)$ for

$$\beta \ge \frac{\alpha(3-2b)}{2b-1}\left[(2b)^{\frac{1}{\alpha}} - 1\right]$$

and this completes the proof of Lemma 2.4. □

By taking $p(z) = zf'(z)/f(z)$ and $p(z) = f'(z)$ in Lemma 2.4, we state the following Theorems 2.6 and 2.7, respectively as below.

Theorem 2.6. *Let β_0 be given in Lemma 2.4 and $f \in \mathcal{A}$. If*

$$1 + \frac{\beta\left(\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} - (\frac{zf'(z)}{f(z)})^2\right)}{\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \ge \beta_0), \tag{2.22}$$

then $f \in \mathcal{S}^*(\alpha, b)$

Theorem 2.7. *Let β_0 be given in Lemma 2.4 and $f \in \mathcal{A}$. If*

$$1 + \frac{\beta z f''(z)}{(f'(z))^{1-\alpha}} \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}} \quad (\beta \ge \beta_0), \tag{2.23}$$

then

$$f'(z) \prec \left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}.$$

For the function

$$q(z)=\left(\frac{1+z)}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}=1+\frac{2b-1}{\alpha b}z+\frac{2b-1}{2\alpha b}\left(\frac{2b-1}{\alpha b}-\frac{1}{b}\right)z^2+\cdots, \tag{2.24}$$

we have $q(\mathbb{U})=\Omega(\alpha,b)$ and from (2.6) we can obtain a function f_0, related to q of the form

$$f_0(z)=\int_0^z q(t)dt \quad (z\in\mathbb{U}) \tag{2.25}$$

$$=z+\frac{2b-1}{\alpha b}z^2+\frac{2b-1}{2\alpha b}\left(\frac{2b-1}{\alpha b}-\frac{1}{b}\right)z^3+\cdots, \tag{2.26}$$

It is easy to see that

$$\mathcal{P}(\alpha,b)=\{p\in\mathcal{P}:p(z)\prec q(z)\}. \tag{2.27}$$

Corollary 2.1. *A function f belongs to the class $\tilde{\mathcal{R}}(\alpha,b)$ $\left(\alpha\geq 1, b\geq\frac{1}{2}\right)$ if and only if*

$$f'(z)\prec q(z). \tag{2.28}$$

Theorem 2.8. *A function $f\in\tilde{\mathcal{R}}(\alpha,b)$ $\left(\alpha\geq 1, b\geq\frac{1}{2}\right)$ if and only if there exist an analytic function p satisfying*

$$p(z)\prec p_{\alpha,b}:=\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

such that

$$f(z)=\int_0^z p(t)dt \quad (p(0)=1,\ z\in\mathbb{U}). \tag{2.29}$$

Moreover, if for the function $f_{\alpha,b}\in\tilde{\mathcal{R}}(\alpha,b)$, it takes the form

$$f_{\alpha,b}(z)=\frac{\left[\sqrt{b}+\left(\sqrt{1+b}-1\right)z\right]^{\frac{2+\alpha}{\alpha}}-\left(\sqrt{b}\right)^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b}-1\right)\left(\frac{2+\alpha}{\alpha}\right)} \quad (z\in\mathbb{U}), \tag{2.30}$$

then

$$\frac{f(z)}{z}\prec\frac{f_{\alpha,b}(z)}{z}. \tag{2.31}$$

Proof. Let $f\in\tilde{\mathcal{R}}(\alpha,b)$ and let $p(z)=f'(z)$. Integration of this equation yields (2.29). If f is given by (2.29) with an analytic function then $p(z)\prec p_{\alpha,b}(z)$. Now differentiating (2.29), we obtain $f'(z)=p(z)$. Therefore

$$f'(z)\prec\left(\frac{1+z}{1+\left(\frac{1-b}{b}\right)z}\right)^{\frac{1}{\alpha}}$$

and consequently $f\in\tilde{\mathcal{R}}(\alpha,b)$.

Now we proceed to prove that $f_{\alpha,b}\in\tilde{\mathcal{R}}(\alpha,b)$. For this purpose we will show that the inclusion relation

$$\mathcal{Q}_{\alpha,b}=\left\{\omega\in\mathbb{C}:0<\Re(\omega),\left|\omega^{\frac{\alpha}{2}}-b^{\frac{1}{2}}\right|<\sqrt{1+b}-1,\alpha\geq 1,b\geq\frac{1}{2}\right\}\subset\mathcal{Q}. \tag{2.32}$$

Let $\omega\in\mathcal{Q}_{\alpha,b}$. Then

$$\left|\omega^{\frac{\alpha}{2}}-b^{\frac{1}{2}}\right|<\sqrt{1+b}-1\Rightarrow\left|\omega^{\frac{\alpha}{2}}+b^{\frac{1}{2}}\right|<\sqrt{1+b}+1. \tag{2.33}$$

By multiplying these inequalities, we obtain

$$|\omega^{\alpha} - b| < b \Rightarrow \omega \in \mathcal{Q}. \tag{2.34}$$

Denoting

$$q_{\alpha,b}(z) = \left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2}{\alpha}},$$

we pose that

$$\omega^{\frac{\alpha}{2}} := [q_{\alpha,b}(z)] = \sqrt{b} + \left(\sqrt{1+b} - 1\right) z. \tag{2.35}$$

Then

$$q_{\alpha,b}(\mathbb{U}) = \left\{\omega \in \mathbb{C} : 0 < \Re(\omega), \left|\omega^{\frac{\alpha}{2}} - b^{\frac{1}{2}}\right| < \sqrt{1+b} - 1, \alpha \geq 1, b \geq \frac{1}{2}\right\} \subset \mathcal{Q}. \tag{2.36}$$

Hence $q_{\alpha,b}(z) \prec p_{\alpha,b}(z)$, by putting $q_{\alpha,b}(z)$ in (2.29) implies (2.30). To prove the subordination relation (2.31), firstly we show that $f_{\alpha,b}(z)/z$ is convex univalent function. We observe that

$$\begin{aligned}
f_{\alpha,b}(z) &= \frac{\left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2+\alpha}{\alpha}} - (\sqrt{b})^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b} - 1\right)\left(\frac{2+\alpha}{\alpha}\right)} \\
&= \frac{(\sqrt{b})^{\frac{2+\alpha}{\alpha}}\left[1 + \left(\frac{2+\alpha}{\alpha}\right)\frac{\left(\sqrt{1+b}-1\right)z}{\sqrt{b}} + \left(\frac{2+\alpha}{\alpha}\right)\frac{\left(\sqrt{1+b}-1\right)^2 z^2}{\alpha b} + \cdots\right] - (\sqrt{b})^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b} - 1\right)\left(\frac{2+\alpha}{\alpha}\right)(\sqrt{b})^{\frac{2}{\alpha}}} \\
&= z + \frac{\left(\sqrt{1+b} - 1\right)}{\alpha\sqrt{b}} z^2 + \cdots \\
&= z + \sum_{n=2}^{\infty} \lambda(\alpha, b) z^n \in \mathcal{A}.
\end{aligned}$$

Let us consider the function

$$F_{\alpha,b}(z) = \frac{\alpha\sqrt{b}}{\sqrt{1+b} - 1}\left[\frac{f_{\alpha,b}(z)}{z} - 1\right] \in \mathcal{A}. \tag{2.37}$$

A simple computation gives,

$$F'_{\alpha,b}(z) = \frac{\alpha\sqrt{b}}{\sqrt{1+b} - 1}\left[\frac{f'_{\alpha,b}(z)}{z} - \frac{f_{\alpha,b}(z)}{z^2}\right] \tag{2.38}$$

and

$$F''_{\alpha,b}(z) = \frac{\alpha\sqrt{b}}{\sqrt{1+b} - 1}\left[\frac{z f''_{\alpha,b}(z) - f'_{\alpha,b}(z)}{z^2} - \frac{z^2 f'_{\alpha,b}(z) - 2z f_{\alpha,b}(z)}{z^4}\right]. \tag{2.39}$$

Then we obtain

$$\begin{aligned}
f_{\alpha,b}(z) &= \frac{\left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2+\alpha}{\alpha}} - (\sqrt{b})^{\frac{2+\alpha}{\alpha}}}{\left(\sqrt{1+b} - 1\right)\left(\frac{2+\alpha}{\alpha}\right)}, \\
f'_{\alpha,b}(z) &= \left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2}{\alpha}}, \\
f''_{\alpha,b}(z) &= \frac{2\left(\sqrt{1+b} - 1\right)}{\alpha}\left[\sqrt{b} + \left(\sqrt{1+b} - 1\right) z\right]^{\frac{2}{\alpha} - 1}.
\end{aligned} \tag{2.40}$$

The aim of our calculation is to show that $1 + zF''_{\alpha,b}(z)/F'_{\alpha,b}(z)$ has a positive real part in the unit disk. Let $z \in \mathcal{Q}_{\alpha,b}$, that is, $\Re(z) > 0$. Since $0 < \sqrt{1+b} - 1 < 1$, then by using (2.40), we have

$$\Re\left(1 + \frac{zF''_{\alpha,b}(z)}{F'_{\alpha,b}(z)}\right) = \Re\left(\frac{z^2 f''_{\alpha,b}(z)}{zf'_{\alpha,b}(z) - f_{\alpha,b}(z)} - 1\right) > 0. \tag{2.41}$$

Hence for choosing suitable parameter $\alpha, \beta(\alpha \geq 1, b \geq \frac{1}{2})$, we have

$$\Re\left(1 + \frac{zF''_{\alpha,b}(z)}{F'_{\alpha,b}(z)}\right) > 0. \tag{2.42}$$

Consequently, we obtain that $F_{\alpha,b} \in \mathcal{K}$, where $\mathcal{K}$ is the class of convex functions. Therefore $f_{\alpha,b}(z)/z$ is a convex function. Now by using the fact that if for $F, G \in \mathcal{K}$, satisfy $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ and $k(z) = z/(1-z)$ is a convex function, then we immediately establish (2.31). This completes proof of the theorem. $\square$

As a consequence of the subordination (2.31), we obtain the following result.

Theorem 2.9. *If $f \in \tilde{\mathcal{R}}(\alpha, b)$ and $|z| = r$, then*

$$|f_{\alpha,b}(-r)| \leq |f(z)| \leq |f_{\alpha,b}(r)| \tag{2.43}$$

and

$$|f'_{\alpha,b}(-r)| \leq |f'(z)| \leq |f'_{\alpha,b}(r)|. \tag{2.44}$$

Let $f \in \tilde{\mathcal{R}}(\alpha, b)$. Then

$$f'(z) = \left(\frac{1+\omega(z)}{1+B\omega(z)}\right)^{\frac{1}{\alpha}} \quad B = \frac{b-1}{b} \in [-1, 1) \quad (z \in \mathbb{U}), \tag{2.45}$$

where ω satisfies Schwarz's Lemma, so $\omega(0) = 0$ and $|\omega(z)| < |z|$ $(z \in \mathbb{U})$ and

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2} \quad (z \in \mathbb{U}). \tag{2.46}$$

Then from (2.45) and (2.46), we get

$$\begin{aligned}
\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &= 1 + \Re\left\{\frac{(1+B)z\omega'(z)}{\alpha(1+\omega(z))(1-B\omega(z))}\right\} \\
&\geq 1 - \frac{(1+B)|z|(1-|\omega(z)|^2)}{\alpha(1-|z|^2)(1-|\omega(z)|)(1-|B||\omega(z)|)} \\
&\geq 1 - \frac{(1+B)|z|(1+|\omega(z)|}{\alpha(1-|z|^2)(1-|B||\omega(z)|)} \\
&\geq 1 - \frac{(1+B)|z|}{\alpha(1-|z|)(1-|B||z|)} \\
&\geq 1 - \frac{(1+B)r}{\alpha(1-r)(1-|B|r)}.
\end{aligned}$$

Therefore we can easily obtain the following result.

Theorem 2.10. *Let r_0 denotes the smallest positive root of the equation*

$$1 - \frac{(1+B)r}{\alpha(1-r)(1-|B|r)} = 0. \tag{2.47}$$

If the function belongs to the class $\tilde{\mathcal{R}}(\alpha, b)$, then it maps disc $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$ onto a convex set. For $B = 0$, this result is sharp.

Proof. The function

$$h(r) = 1 - \frac{(1+B)r}{\alpha(1-r)(1-|B|r)}$$

with $h(0) = 1$ and $h(r) \to \infty$ as $r \to 1$ is an decreasing function in $[0, 1)$. Therefore (2.47) has positive solution in $[0, 1)$. If $B = 0$, then (2.47) has form

$$1 - \frac{r}{\alpha(1-r)} = 0. \tag{2.48}$$

and for the function

$$f(z) = \int_0^z (1+t)^{\frac{1}{\alpha}} dt \quad (z \in \mathbb{U}),$$

we have

$$\begin{aligned} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &= 1 + \Re\left\{\frac{z}{\alpha(1+z)}\right\} \\ &\geq 1 - \left|\frac{z}{\alpha(1+z)}\right| \\ &\geq 1 - \frac{|z|}{\alpha(1-|z|)}. \end{aligned}$$

□

3. Coefficient inequalities

Lemma 3.1. [11] *Let the function $\omega \in \mathcal{B}_0$ be given by*

$$\omega(z) = d_1 z + d_2 z^2 + \cdots \quad (z \in \mathbb{U}), \tag{3.1}$$

where

$$\mathcal{B}_0 = \{\omega \in \mathcal{A} : \omega(0) = 0,\ |\omega(z)| < 1 \quad (z \in \mathbb{U})\} \tag{3.2}$$

Then for every complex number s,

$$|d_2 - s d_1^2| \leq 1 + (|s| - 1)|d_1|^2. \tag{3.3}$$

Now we determine an sharp upper bound for the class $\tilde{\mathcal{R}}(\alpha, b)$.

Theorem 3.1. *If the function f given by (1.1) belong to the class $\tilde{\mathcal{R}}(\alpha, b)$, then for $-\infty < \mu < \infty$*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\left(\frac{2b-1}{6\alpha b}\right)\left[\left(\frac{3\mu}{2} - 1\right)\left(\frac{2b-1}{\alpha b}\right) + \frac{1}{b}\right], & \mu < \sigma_1(\alpha, b) \\ \frac{2b-1}{3\alpha b}, & \sigma_1(\alpha, b) \leq \mu \leq \sigma_2(\alpha, b) \\ \left(\frac{2b-1}{3\alpha b}\right)\left[\left(\frac{3\mu}{2} - 1\right)\left(\frac{2b-1}{\alpha b}\right) + \frac{1}{b}\right], & \mu > \sigma_2(\alpha, b), \end{cases} \tag{3.4}$$

where $\sigma_1(\alpha,b)$ *and* $\sigma_2(\alpha,b)$ *is given by*

$$\sigma_1(\alpha,b) = \frac{4\alpha b}{3(2b-1)}\left[\left(\frac{2b-1}{2\alpha b}\right) - \frac{1}{2b} - 1\right] \tag{3.5}$$

and

$$\sigma_2(\alpha,b) = \frac{4\alpha b}{3(2b-1)}\left[\left(\frac{2b-1}{2\alpha b}\right) - \frac{1}{2b} + 1\right] \tag{3.6}$$

The estimates in (3.4) *are sharp.*

Proof. By the definition of subordination, there exists a function $\omega \in \mathcal{B}_0$ such that

$$f'(z) = \left(\frac{1+\omega(z)}{1+\left(\frac{1-b}{b}\right)\omega(z)}\right)^{\frac{1}{\alpha}} \qquad (z \in \mathbb{U}),$$

Suppose that $\omega(z)$ is given by the series (3.1). A simple calculation shows that

$$a_2 = \left(\frac{2b-1}{\alpha b}\right)\frac{d_1}{2} \tag{3.7}$$

and

$$a_3 = \left(\frac{2b-1}{3\alpha b}\right)\left[\left(\frac{2b-1}{2\alpha b} - \frac{1}{2b}\right)d_1^2 + d_2\right]. \tag{3.8}$$

Then, by using (3.7) and (3.8), easily we get

$$a_3 - \mu a_2^2 = \left(\frac{2b-1}{3\alpha b}\right)\left[d_2 + \left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right)d_1^2\right]. \tag{3.9}$$

Suppose that $\mu < \sigma_1(\alpha,b)$, then (3.9) gives

$$|a_3 - \mu a_2^2| \le \left(\frac{2b-1}{3\alpha b}\right)\left[|d_2| + \left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right)|d_1|^2\right].$$

Applying the estimates $|d_2| \le 1 - |d_1|^2$ of Lemma 3.1 and the well known estimate $|d_1| \le 1$ of the Schwarz lemma, we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \left(\frac{2b-1}{3\alpha b}\right)\left[1 + \left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b} - 1\right)\right] \\ &\le \left(\frac{2b-1}{3\alpha b}\right)\left(\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right) \end{aligned} \tag{3.10}$$

which proves the first inequality in (3.4).
From (3.9) we have,

$$|a_3 - \mu a_2^2| = \left(\frac{2b-1}{3\alpha b}\right)\left|d_2 - d_1^2 + \left\{\frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b}\right\}d_1^2\right|.$$

On the other hand if $\mu > \sigma_2(\alpha,b)$; then using the estimates $|d_2 - d_1^2| \le 1$ from Lemma 3.1 and $|p_1| \le 1$, we get

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \left(\frac{2b-1}{3\alpha b}\right)\left[1 + \left\{\frac{3\mu}{4}\frac{2b-1}{\alpha b} + \frac{1}{2b} - \frac{2b-1}{2\alpha b} - 1\right\}\right] \\ &= \left(\frac{2b-1}{3\alpha b}\right)\left[\frac{3\mu}{4}\frac{2b-1}{\alpha b} + \frac{1}{2b} - \frac{2b-1}{2\alpha b}\right] \end{aligned} \tag{3.11}$$

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which is precisely the last inequality in (3.4).

Finally , if $\sigma_1(\alpha, b) \leq \mu \leq \sigma_2(\alpha, b)$, then

$$\left| \frac{2b-1}{2\alpha b} - \frac{1}{2b} - \frac{3\mu}{4}\frac{2b-1}{\alpha b} \right| \leq 1.$$

Therefore we obtain

$$|a_3 - \mu a_2^2| \leq \left(\frac{2b-1}{3\alpha b} \right), \tag{3.12}$$

which proves the middle inequality in (3.4).

Next, we discuss sharpness of the inequality (3.4). Suppose $\mu < \sigma_1(\alpha, b)$. Then equality holds in (3.4), that is, in (3.10) if $|d_1| = 1$ (and hence $d_2 = 0$). Thus $\omega(z)$ is a rotation of z and the extremal function is a rotation of $q_{\alpha,b}(z)$. Next, if $\mu > \sigma_2(\alpha, b)$, equality holds in (3.4), that is, in (3.11) if $d_1^2 = -1$ and hence $|d_2 - d_1^2| = 1$. Therefore $\omega(z) = iz$ and the extremal function is $q_{\alpha,b}(iz)$. Lastly, if $\sigma_1(\alpha, b) \leq \mu \leq \sigma_2(\alpha, b)$, then equality holds in (3.4) if $d_1 = 0$ and $|d_2| = 1$. Therefore $\omega(z)$ is a rotation of z^2 and $f'(z) = q_{\alpha,b}\left(e^{i\theta} z^2\right)$. This completes proof of Theorem 3.1. $\square$

Letting $\mu = 0$ (or $\mu = 1$, respectively) in Theorem 3.1, we get the following result.

Corollary 3.1. *If the function f given by* (1.1) *belong to the class $\tilde{\mathcal{R}}(\alpha, b)$, then*

$$|a_3| \leq \left(\frac{2b-1}{3\alpha b} \right) \quad \textit{and} \quad |a_3 - a_2^2| \leq \left(\frac{2b-1}{3\alpha b} \right) \quad (\alpha \geq 1, b \geq \frac{1}{2}) \tag{3.13}$$

The estimates in (3.13) *are sharp for the function $f_0 \in \mathcal{A}$ defined by*

$$f_0'(z) = \left(\frac{1+z^2}{1+\left(\frac{1-b}{b}\right) z^2} \right)^{\frac{1}{\alpha}} \quad (z \in \mathbb{U}). \tag{3.14}$$

For the choice of $\alpha = 2,\ b = 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *If the function f given by* (1.1) *belong to the class $\tilde{\mathcal{R}}(2, 1)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\dfrac{3\mu+2}{48}, & \mu < \frac{-10}{3} \\ \dfrac{1}{6}, & \frac{-10}{3} \leq \mu \leq 2 \\ \dfrac{3\mu+2}{48}, & \mu > 2. \end{cases} \tag{3.15}$$

If we take $\alpha = 2,\ b = 1$ and $\mu = 0$, and $\alpha = 1, b = 1$ and $\mu = 1$ in Theorem 3.1, then we have the following corollaries, respectively.

Corollary 3.3. *If the function f given by* (1.1) *belong to the class $\tilde{\mathcal{R}}(2, 1)$, then*

$$|a_3| \leq \frac{1}{6}. \tag{3.16}$$

Corollary 3.4. *If the function f given by* (1.1) *belong to the class $\tilde{\mathcal{R}}(1, 1)$, then*

$$|a_3 - a_2^2| \leq \frac{1}{6}. \tag{3.17}$$

Next we prove a sharp coefficient inequalities for the class $\mathcal{S}^*(\alpha, b)$.

Theorem 3.2. *If the function f given by (1.1) belong to the class $\mathcal{S}^*(\alpha, b)$ $\left(\alpha \geq 1,\ b \geq \frac{1}{2}\right)$, then for $-\infty < \mu < \infty$,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\left(\dfrac{2b-1}{2\alpha b}\right)\left[(2\mu - \dfrac{3}{2})\left(\dfrac{2b-1}{\alpha b}\right) + \dfrac{1}{2b}\right], & \mu < \sigma_3(\alpha, b) \\ \dfrac{2b-1}{2\alpha b}, & \sigma_3(\alpha, b) \leq \mu \leq \sigma_4(\alpha, b) \\ \left(\dfrac{2b-1}{2\alpha b}\right)\left[(2\mu - \dfrac{3}{2})\left(\dfrac{2b-1}{\alpha b}\right) + \dfrac{1}{2b}\right], & \mu > \sigma_4(\alpha, b), \end{cases} \tag{3.18}$$

where $\sigma_3(\alpha, b)$ and $\sigma_4(\alpha, b)$ is given by

$$\sigma_3(\alpha, b) = \frac{\alpha b}{2(2b-1)}\left[\frac{3}{2}\left(\frac{2b-1}{\alpha b}\right) - \frac{1}{2b} - 1\right] \tag{3.19}$$

and

$$\sigma_4(\alpha, b) = \frac{\alpha b}{2(2b-1)}\left[\frac{3}{2}\left(\frac{2b-1}{\alpha b}\right) - \frac{1}{2b} + 1\right]. \tag{3.20}$$

The estimates in (3.18) are sharp.

Proof. From (1.7), it follows that,

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\omega(z)}{1+\left(\frac{1-b}{b}\right)\omega(z)}\right)^{\frac{1}{\alpha}} \quad (z \in \mathbb{U}), \tag{3.21}$$

where $\omega(z)$ is given by (3.1). From (3.21), we have

$$\frac{zf'(z)}{f(z)} = 1 + \left(\frac{2b-1}{\alpha b}\right)d_1 z + \left\{\left(\frac{2b-1}{\alpha b}\right)d_2 + \left(\frac{2b-1}{2\alpha b}\right)\left(\frac{2b-1}{\alpha b} - \frac{1}{b}\right)d_1^2\right\}z^2 + \cdots. \tag{3.22}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3 a_2)z^3 + \ldots, \tag{3.23}$$

comparing the coefficients of z and z^2 in (3.22) and (3.23), we reduce that

$$a_2 = \left(\frac{2b-1}{\alpha b}\right)d_1 \tag{3.24}$$

and

$$a_3 = \left(\frac{2b-1}{2\alpha b}\right)\left[d_2 + \left(\frac{3}{2}\frac{2b-1}{\alpha b} - \frac{1}{2b}\right)d_1^2\right] \tag{3.25}$$

Following a similar method adopted for Theorem 3.1, one can easily show that inequality (3.18) is satisfied and is sharp for the functions as in similar lines mentioned in Theorem 3.1. □

Letting $\mu = 0$ (or $\mu = 1$, respectively) in Theorem 3.2, we get the following result.

Corollary 3.5. *If the function f given by (1.1) belong to the class $\mathcal{S}^*(\alpha, b)$, then*

$$|a_3| \leq \left(\frac{2b-1}{2\alpha b}\right) \text{ and } |a_3 - a_2^2| \leq \left(\frac{2b-1}{2\alpha b}\right) \quad (\alpha \geq 1, b \geq \frac{1}{2}). \tag{3.26}$$

The estimates in (3.26) are sharp for the function $f_0 \in \mathcal{A}$ defined by

$$f_0'(z) = \left(\frac{1+z^2}{1+\left(\frac{1-b}{b}\right)z^2}\right)^{\frac{1}{\alpha}} \quad (z \in \mathbb{U}). \tag{3.27}$$

For the choice of $\alpha = 2,\ b = 1$ in Theorem 3.2, we have the following result.

Corollary 3.6. *If the function f given by* (1.1) *belong to the class* $\mathcal{S}^*(2,1)$, *then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{1-4\mu}{16}, & \mu < \frac{-3}{4} \\ \dfrac{1}{4}, \frac{-3}{4} \le \mu \le \frac{5}{4} & \\ \dfrac{4\mu-1}{16}, & \mu > \frac{5}{4}. \end{cases} \tag{3.28}$$

If we take $\alpha = 2, b = 1$ and $\mu = 0$, and $\alpha = 2, b = 1$ and $\mu = 1$ in Theorem 3.2, then we have the following corollaries, respectively.

Corollary 3.7. *If the function f given by* (1.1) *belong to the class* $\mathcal{S}^*(2,1)$, *then*

$$|a_3| \le \frac{1}{4}. \tag{3.29}$$

Corollary 3.8. *If the function f given by* (1.1) *belong to the class* $\mathcal{S}^*(2,1)$, *then*

$$|a_3 - a_2^2| \le \frac{1}{4}. \tag{3.30}$$

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On iterative approach to common fixed points of nonexpansive mappings in Hilbert spaces

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Abstract

In this paper, we introduce a viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces. The strong convergence of this technique is proved under certain assumptions imposed on the sequence of parameters.

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1 Introduction

Fixed points of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings has become a field of interest and has a variety of applications in related fields like image recovery, signal processing and geometry of objects. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. A fixed-point iteration scheme has been applied in intensity modulated radiation therapy optimization to pre-compute dose-deposition coefficient matrix, see [15]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now. Constructive fixed point theorems (e.g., Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map f that is contracting with respect to some (complete) metric on X will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and

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is invariably used in most of the occasions see [6]. But it only ensures weak convergence, see [2] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence. For literature review we refer to the readers (see [3, 4, 8–12], and references therein).

In this paper, we shall take H as a real Hilbert space, $\langle \cdot, \cdot \rangle$ as inner product, $\| \cdot \|$ as the induced norm, and C as a nonempty closed subset of H.

Definition 1.1. Let $T : H \to H$ be a mapping. Then T is called *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Definition 1.2. A mapping $f : H \to H$ is called a *contraction* if for all $x, y \in H$ and $\theta \in [0, 1)$

$$\|f(x) - f(y)\| \leq \theta \|x - y\|.$$

Definition 1.3. $P_c : H \to C$ is called a *metric projection* if for every $x \in H$ there exists a unique nearest point in C, denoted by $P_c x$, such that

$$\|x - P_c x\| \leq \|x - y\|, \forall y \in C.$$

In order to verify the weak convergence of an algorithm to a fixed point of a non-expansive mapping we need the demiclosedness principle:

Theorem 1.4. ([5]) (*The demiclosedness principle*) *Let C be a nonempty closed convex subset of the real Hilbert space H and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \text{and} \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$. Here $\to$ and $\rightharpoonup$) denotes strong and weak convergence, respectively.

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

Theorem 1.5. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$.

(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $a_n \to 0$.

The following strong convergence theorem, which is also called the *viscosity approximation method*, for non-expansive mappings in real Hilbert spaces is given by Moudafi [7] in 2000.

Theorem 1.6. ([7]) *Let C be a non-empty closed convex subset of the real Hilbert space H. Let T be a non-expansive mapping of C into itself such that $F(T)$ is nonempty. Let f be a contraction of C into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1 + \epsilon_n} f(x_n) + \frac{1}{1 + \epsilon_n} T(x_n), \quad n \geq 0,$$

where the sequence $\{\epsilon_n\}$ in $(0, 1)$ satisfies

(1) $\lim_{n\to\infty} \epsilon_n = 0$,

(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$, *and*

(3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the non-expansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I-f)x, y-x\rangle \geq 0, \quad \forall \in F(T).$$

In 2015, Xu et al. [13] applied viscosity method on the midpoint rule for nonexpansive mappings and give the generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of T. Ke et al. [14], motivated and inspired by the idea of Xu et al. [13], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(s_n x_n + (1-s_n)x_{n+1}\right),$$

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}).$$

In this paper, we give a viscosity approximation method for common fixed point of two nonexpansive mappings in Hilbert spaces. Our contribution in this direction is the following viscosity rule

$$\epsilon_{n+1} = \alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n). \tag{1.1}$$

We prove strong convergence of (1.1) under certain assumptions. We also solve some examples to check the validity of (1.1).

2 Main result

Following Theorem 2.1 is about convergence of our proposed viscosity technic.

Theorem 2.1. *Let S and T be two non-expansive mappings from a closed convex subset X of real Hilbert space H into X with $U := F(T) \cap F(S) \neq \emptyset$. Also let that $f : X \to X$ be a contraction with coefficient $\theta \in [0,1)$. Assume that the sequence $\{\epsilon_n\}$ in X is generated by (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying*

(1) $\alpha_n + \beta_n + \gamma_n = 1$,

(2) $\lim_{n\to\infty} \alpha_n = 0$,

(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(4) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(5) $\lim_{n\to\infty} \|T(\epsilon_n) - S(\epsilon_n)\| = 0$.

Then $\{\epsilon_n\}$ converges strongly to $\epsilon^* \in U$, which satisfy the variational inequality

$$\langle \epsilon^* - f(\epsilon^*), y - \epsilon^*\rangle \geq 0, \quad \forall y \in U.$$

Proof. We will prove this theorem into the following five steps.

STEP 1. In this step, we show ϵ_n is bounded. Take $\zeta \in U$ arbitrarily, we have

$$\begin{aligned}
\|\epsilon_{n+1} - \zeta\| &= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - \zeta\| \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - (\alpha_n + \beta_n + \gamma_n)\zeta\| \\
&\leq \alpha_n \|f(\epsilon_n) - \zeta\| + \beta_n \|S(\epsilon_n) - \zeta\| + \gamma_n \|T(\epsilon_n) - \zeta\| \\
&= \alpha_n \|f(\epsilon_n) - f(\zeta) + f(\zeta) - \zeta\| + \beta_n \|S(\epsilon_n) - \zeta\| + \gamma_n \|T(\epsilon_n) - \zeta\| \\
&\leq \alpha_n \|f(\epsilon_n) - f(\zeta)\| + \alpha_n \|f(\zeta) - \zeta\| + \beta_n \|\epsilon_n - \zeta\| + \gamma_n \|\epsilon_n - \zeta\| \\
&\leq \theta\alpha_n \|\epsilon_n - \zeta\| + \alpha_n \|f(\zeta) - \zeta\| + (\beta_n + \gamma_n)\|\epsilon_n - \zeta\| \\
&= \theta\alpha_n \|\epsilon_n - \zeta\| + \alpha_n \|f(\zeta) - \zeta\| + (1 - \alpha_n)\|\epsilon_n - \zeta\| \\
&= (1 - \alpha_n + \alpha_n\theta)\|\epsilon_n - \zeta\| + \alpha_n \|f(\zeta) - \zeta\| \\
&= [1 - \alpha_n(1-\theta)]\|\epsilon_n - \zeta\| + \alpha_n(1-\theta)\left[\frac{1}{(1-\theta)}\|f(\zeta) - \zeta\|\right].
\end{aligned}$$

Thus,

$$\|\epsilon_{n+1} - \zeta\| \leq \max\left\{\|\epsilon_n - \zeta\|, \frac{1}{1-\theta}\|f(\zeta) - \zeta\|\right\}.$$

Similarly

$$\|\epsilon_n - \zeta\| \leq \max\left\{\|\epsilon_{n-1} - \zeta\|, \left(\frac{1}{1-\theta}\|f(\zeta) - \zeta\|\right)\right\}.$$

From this

$$\begin{aligned}
\|\epsilon_{n+1} - \zeta\| &\leq \max\left\{\|\epsilon_n - \zeta\|, \left(\frac{1}{1-\theta}\|f(\zeta) - \zeta\|\right)\right\} \\
&\leq \max\left\{\|\epsilon_{n-1} - \zeta\|, \left(\frac{1}{1-\theta}\|f(\zeta) - \zeta\|\right)\right\} \\
&\ \ \vdots \\
&\leq \max\left\{\|\epsilon_0 - \zeta\|, \left(\frac{1}{1-\theta}\|f(\zeta) - \zeta\|\right)\right\}.
\end{aligned}$$

We obtain

$$\|\epsilon_{n+1} - \zeta\| \leq \max\left\{\|\epsilon_0 - \zeta\|, \frac{1}{1-\theta}\|f(\zeta) - \zeta\|\right\}.$$

Hence, we concluded that $\{\epsilon_n\}$ is a bounded sequence. Consequently, $\{f(\epsilon_n)\}$, $\{S(\epsilon_n)\}$ and $\{T(\epsilon_n)\}$ are bounded.

STEP 2. Now, we prove that $\|\epsilon_{n+1} - \epsilon_n\| \to 0$ as $n \to \infty$

$$\begin{aligned}
&\|\epsilon_{n+1} - \epsilon_n\| \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - \{\alpha_{n-1} f(\epsilon_{n-1}) + \beta_{n-1} S(\epsilon_{n-1}) + \gamma_{n-1} T(\epsilon_{n-1})\}\| \\
&= \|\alpha_n \{f(\epsilon_n) - f(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1}) f(\epsilon_{n-1}) + \beta_n (S(\epsilon_n) - S(\epsilon_{n-1})) \\
&\quad + (\beta_n - \beta_{n-1}) S(\epsilon_{n-1}) + \gamma_n \{T(\epsilon_n) - T(\epsilon_{n-1})\} + (\gamma_n - \gamma_{n-1}) T(\epsilon_{n-1})\|
\end{aligned}$$

$$
\begin{aligned}
&= \|\alpha_n\{f(\epsilon_n) - f(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1})f(\epsilon_{n-1}) + \beta_n\{S(\epsilon_n) - S(\epsilon_{n-1})\} \\
&\quad + (\beta_n - \beta_{n-1})S(\epsilon_{n-1}) + \gamma_n\{T(\epsilon_n) - T(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1} + \beta_n - \beta_{n-1})T(\epsilon_{n-1})\| \\
&= \|\alpha_n\{f(\epsilon_n) - f(\epsilon_{n-1})\} + (\alpha_n - \alpha_{n-1})\{f(\epsilon_{n-1}) - T(\epsilon_{n-1})\} + \beta_n\{S(\epsilon_n) - S(\epsilon_{n-1})\} \\
&\quad + (\beta_n - \beta_{n-1})\{S(\epsilon_{n-1}) - T(\epsilon_{n-1})\} + \gamma_n\{T(\epsilon_n) - T(\epsilon_{n-1})\}\| \\
&\leq \alpha_n\|f(\epsilon_n) - f(\epsilon_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(\epsilon_{n-1}) - T(\epsilon_{n-1})\| + \beta_n\|S(\epsilon_n) - S(\epsilon_{n-1})\| \\
&\quad + |\beta_n - \beta_{n-1}|\|S(\epsilon_{n-1}) - T(\epsilon_{n-1})\| + \gamma_n\|T(\epsilon_n) - T(\epsilon_{n-1})\| \\
&\leq \alpha_n\theta\|\epsilon_n - \epsilon_{n-1}\| + \beta_n\|\epsilon_n - \epsilon_{n-1}\| + \gamma_n\|\epsilon_n - \epsilon_{n-1}\| \\
&\quad + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2 \\
&= (\alpha_n\theta + \beta_n + \gamma_n)\|\epsilon_n - \epsilon_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2 \\
&= (\alpha_n\theta + 1 - \alpha_n)\|\epsilon_n - \epsilon_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2 \\
&= (1 - \alpha_n(1-\theta))\|\epsilon_n - \epsilon_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2,
\end{aligned}
$$

where

$$M_2 \geq \max\left\{\sup_{n\geq 0}\|f(\epsilon_n) - T(\epsilon_n)\|, \sup_{n\geq 0}\|S(\epsilon_n) - T(\epsilon_n)\|\right\}.$$

Note that $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Using Theorem 1.5, we have $\lim_{n\to\infty}\|\epsilon_{n+1} - \epsilon_n\| = 0$.

STEP 3. Now, we will show that $\lim_{n\to\infty}\|\epsilon_n - S(\epsilon_n)\| = 0$ and $\lim_{n\to\infty}\|\epsilon_n - T(\epsilon_n)\| = 0$.

Consider

$$
\begin{aligned}
\|\epsilon_n - S(\epsilon_n)\| &= \|\epsilon_n - \epsilon_{n+1} + \epsilon_{n+1} - S(\epsilon_n)\| \\
&\leq \|\epsilon_n - \epsilon_{n+1}\| + \|\epsilon_{n+1} - S(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - S(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \gamma_n T(\epsilon_n) - (1-\beta_n)S(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \gamma_n T(\epsilon_n) - (\alpha_n + \gamma_n)S(\epsilon_n)\| \\
&\leq \|\epsilon_{n+1} - \epsilon_n\| + \alpha_n\|f(\epsilon_n) - S(\epsilon_n)\| + \gamma_n\|T(\epsilon_n) - S(\epsilon_n)\|.
\end{aligned}
$$

Then by $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\|T(\epsilon_n) - S(\epsilon_n)\| = 0$, and $\lim_{n\to\infty}\|\epsilon_{n+1} - \epsilon_n\| \to 0$, we get $\|\epsilon_n - S(\epsilon_n)\| \to 0$ as $n \to \infty$.

Now, consider

$$
\begin{aligned}
\|\epsilon_n - T(\epsilon_n)\| &= \|\epsilon_n - \epsilon_{n+1} + \epsilon_{n+1} - T(\epsilon_n)\| \\
&\leq \|\epsilon_n - \epsilon_{n+1}\| + \|\epsilon_{n+1} - T(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - T(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) - (1-\gamma_n)T(\epsilon_n)\| \\
&= \|\epsilon_n - \epsilon_{n+1}\| + \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) - (\alpha_n + \beta_n)T(\epsilon_n)\| \\
&\leq \|\epsilon_{n+1} - \epsilon_n\| + \alpha_n\|f(\epsilon_n) - T(\epsilon_n)\| + \beta_n\|T(\epsilon_n) - S(\epsilon_n)\|.
\end{aligned}
$$

Then by $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\|T(\epsilon_n) - S(\epsilon_n)\| = 0$, and $\lim_{n\to\infty}\|\epsilon_{n+1} - \epsilon_n\| \to 0$, we get $\|\epsilon_n - T\epsilon_n\| \to 0$ as $n \to \infty$.

STEP 4. In this step, we will show that $\limsup_{n\to\infty}\langle\epsilon^* - f(\epsilon^*), \epsilon^* - \epsilon_n\rangle \leq 0$, where $\epsilon^* = P_U f(\epsilon^*)$.

Indeed, we take a subsequence $\{\epsilon_{n_i}\}$ of $\{\epsilon_n\}$ which converges weakly to a fixed point $\zeta \in U = F(T) \cap F(S)$. From $\lim_{n\to\infty}\|\epsilon_n - S(\epsilon_n)\| = 0$, $\lim_{n\to\infty}\|\epsilon_n - T(\epsilon_n)\| = 0$ and Theorem 1.4 we have $\zeta = S\zeta$ and $\zeta = T\zeta$. This together with the property of the metric projection implies that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle\epsilon^* - f(\epsilon^*), \epsilon^* - \epsilon_n\rangle &= \limsup_{n\to\infty}\langle\epsilon^* - f(\epsilon^*), \epsilon^* - \epsilon_{n_i}\rangle \\
&= \langle\epsilon^* - f(\epsilon^*), \epsilon^* - \zeta\rangle \leq 0.
\end{aligned}
$$

STEP 5. Finally, we show that $\lim_{n\to\infty} \epsilon_n = \epsilon^*$ as. Now we again take $\epsilon^* \in U$ is the unique fixed point of the contraction $P_U f$.

Consider

$$
\begin{aligned}
&\|\epsilon_{n+1} - \epsilon_n\|^2 \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - \epsilon^*\|^2 \\
&= \|\alpha_n f(\epsilon_n) + \beta_n S(\epsilon_n) + \gamma_n T(\epsilon_n) - (\alpha_n + \beta_n + \gamma_n)\epsilon^*\|^2 \\
&= \|\alpha_n\{f(\epsilon_n) - \epsilon^*\} + \beta_n\{S(\epsilon_n) - \epsilon^*\} + \gamma_n\{T(\epsilon_n) - \epsilon^*\}\|^2 \\
&= \alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + \beta_n^2\|S(\epsilon_n) - \epsilon^*\|^2 + \gamma_n^2\|T(\epsilon_n) - \epsilon^*)\|^2 \\
&\quad + 2\alpha_n\beta_n\langle f(\epsilon_n) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\gamma_n\langle f(\epsilon_n) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\beta_n\gamma_n\langle S(\epsilon_n) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\le \alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + \beta_n^2\|\epsilon_n - \epsilon^*\|^2 + \gamma_n^2\|\epsilon_n - \epsilon^*)\|^2 \\
&\quad + 2\alpha_n\beta_n\langle f(\epsilon_n) - f(\epsilon^*), S(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\alpha_n\gamma_n\langle f(\epsilon_n) - f(\epsilon^*), T(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\beta_n\gamma_n\langle S(\epsilon_n) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle \\
&\le (\beta_n^2 + \gamma_n^2)\|\epsilon_n - \epsilon^*)\|^2 + 2\alpha_n\beta_n\|f(\epsilon_n) - f(\epsilon^*)\| \cdot \|S(\epsilon_n) - \epsilon^*\| \\
&\quad + 2\alpha_n\gamma_n\|f(\epsilon_n) - f(\epsilon^*)\| \cdot \|T(\epsilon_n) - \epsilon^*\| + 2\beta_n\gamma_n\|S(\epsilon_n) - \epsilon^*\| \cdot \|T(\epsilon_n) - \epsilon^*\| + L_n \\
&\le (\beta_n^2 + \gamma_n^2)\|\epsilon_n - \epsilon^*\|^2 + 2\alpha_n\beta_n\theta\|\epsilon_n - \epsilon^*\| \cdot \|\epsilon_n - \epsilon^*\| \\
&\quad + 2\alpha_n\gamma_n\theta\|\epsilon_n - \epsilon^*\| \cdot \|\epsilon_n - \epsilon^*\| + 2\beta_n\gamma_n\|\epsilon_n - \epsilon^*\| \cdot \|\epsilon_n - \epsilon^*\| + L_n \\
&= (\beta_n^2 + \gamma_n^2 + 2\beta_n\gamma_n + 2\alpha_n\gamma_n\theta + 2\alpha_n\beta_n\theta)\|\epsilon_n - \epsilon^*\|^2 + L_n \\
&= [(\beta_n + \gamma_n)^2 + 2\alpha_n\theta(\gamma_n + \beta_n)]\|\epsilon_n - \epsilon^*\|^2 + L_n \\
&= (\beta_n + \gamma_n)[\beta_n + \gamma_n + 2\alpha_n\theta]\|\epsilon_n - \epsilon^*\|^2 + L_n \\
&= (1 - \alpha_n)[1 - \alpha_n + 2\alpha_n\theta]\|\epsilon_n - \epsilon^*\|^2 + L_n,
\end{aligned}
$$

where

$$L_n = \alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + 2\alpha_n\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle + 2\alpha_n\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle.$$

Note that since $\alpha_n\theta < 1$ $(2\alpha_n\theta < 2)$, $1 - \alpha_n + 2\alpha_n\theta < 2 + 1 - \alpha_n < 3$, using this in (1.1) we have

$$\|\epsilon_{n+1} - \epsilon^*\|^2 < 3(1 - \alpha_n)\|\epsilon_n - \epsilon^*\|^2 + L_n. \tag{2.1}$$

Also we get

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{L_n}{\alpha_n} &= \limsup_{n\to\infty} \frac{1}{\alpha_n}\big[\alpha_n^2\|f(\epsilon_n) - \epsilon^*\|^2 + 2\alpha_n\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle \\
&\quad +2\alpha_n\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle\big] \\
&= \limsup_{n\to\infty}\big[\alpha_n\|f(\epsilon_n) - \epsilon^*\|^2 + 2\beta_n\langle f(\epsilon^*) - \epsilon^*, S(\epsilon_n) - \epsilon^*\rangle \\
&\quad + 2\gamma_n\langle f(\epsilon^*) - \epsilon^*, T(\epsilon_n) - \epsilon^*\rangle\big] \\
&\le 0.
\end{aligned} \tag{2.2}
$$

From (2.1), (2.2) and Theorem 1.5 we have

$$\lim_{n\to\infty} \|\epsilon_{n+1} - \epsilon^*\|^2 = 0,$$

which implies that $\epsilon_n \to \epsilon^*$ as $n \to \infty$. This completes the proof. □

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BEST PROXIMITY POINTS INVOLVING F-CONTRACTION ON A CLOSED BALL

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Abstract. In this paper, we introduce a new idea of best proximity point of F-contraction on a closed ball and obtain new theorems in a complete metric space. That is why this outcome becomes useful for contraction of a mapping on a closed ball instead of the whole space. At the same time, some comparative examples are constructed which establish the superiority of our results. Our results that have come into being give a proof of extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

1. Introduction and preliminaries

Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to B$. A point $x \in A$ is said to be a fixed point of T provided that $Tx = x$. A point $x^* \in A$, where $\inf\{d(x, Tx^*) : x \in A\}$ is attained, is a best approximation to $Tx^* \in B$ in A. Such a point is called an approximate fixed point of T.

Clearly, $T(A) \cap A \neq \emptyset$ is a necessary but not sufficient condition for the existence of a fixed point of T. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$ and hence an operator equation $Tx = x$ does not admit a solution. In such situations, it is a reasonable demand to settle down with a point x^* in A which is closest to Tx^* in B. Thus instead of having $d(x^*, Tx^*) = 0$, one finds a point x^* in A such that $d(x^*, Tx^*) \leq d(x, Tx^*)$ holds for all x in A. Such point is called a best approximate point of T or approximate fixed point of T. The study of conditions that assure existence and uniqueness of approximate fixed point of a mapping T is an active area of research.

Suppose that $d(A, B) = \inf(\{d(a, b) : a \in A, b \in B\})$ is the measure of a distance between two sets A and B. A point x^* is called a best proximity point of T if $d(x^*, Tx^*) = d(A, B)$. Thus a best proximity point problem defined by a mapping T and a pair of sets (A, B) is to find a point x^* in A such that $d(x^*, Tx^*) = d(A, B)$. As $d(x, Tx) \geq d(A, B)$ holds for all $x \in A$, so the global minimum of the mapping $x \to d(x, Tx)$ is attained at a best proximity point. If we take $A = B$, then a best proximity point problem reduces to fixed point problem. From this perspective, best proximity point problem can be viewed as a natural generalization of fixed point problem. The aim of best proximity point theory is to study sufficient conditions that assure the existence of best proximity points of mappings satisfying certain contractive conditions on its domain equipped with some distance structure. For more results in this direction, we refer to [1, 2, 4, 5, 6, 7, 9, 20] and references therein.

Fixed point results of mappings satisfying certain contractive conditions on the entire domain have been at the centre of rigorous research activity and it has a wide range of applications in

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different areas such as nonlinear and adaptive control systems, and parameterized estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium and convergence of recurrent networks. From the application point of view, the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X. Arshad *et al.* [3] established fixed point results of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space. Hussain *et al.* [10] introduced the concept of an α-admissible mappings with respect to η and modified (α, ψ)-contractive condition for a pair of mappings and established common fixed point results of four mappings on a closed ball in complete dislocated metric space.

Jleli *et al.* [12] obtained best proximity point results of (α, ψ)-proximal contractive type mappings in complete metric space. For more work in this direction, we refer to [11, 14, 16, 17, 18, 19].

In this paper, we obtain best proximity point results of α-η-proximal F-contractive mappings on a closed ball in complete metric spaces. Our results extend, unify and generalize various comparable results in [5, 6, 12].

In the sequel, the letter $\mathbb{N}$ will denote the set of all natural numbers. The following definitions, notations and results will also be needed in the sequel.

Let (X, d) be a metric space and A and B be nonempty subsets of X. For $x_0 \in X$ and $\varepsilon > 0$, the set $\overline{B(x_0, \varepsilon)} = \{y \in X : d(x_0, y) \leq \varepsilon\}$ is a closed ball in X.

In 2012, Wardowski [21] introduced a concept of F-contraction as follows:

Definition 1. [21] *Let (X, d) be a metric space. A self mapping T is said to be an F-contraction if there exists $\tau > 0$ such that*

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right),$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:
(F1) *F is strictly increasing, i.e., for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;*
(F2) *For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers,* $\lim_{n\to\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$*;*
(F3) *There exists $\kappa \in (0, 1)$ such that* $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

We denote by Δ_F the set of all functions satisfying the conditions (F1)-(F3).

Suppose that

$$\begin{aligned} A_0 &: = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}, \\ B_0 &: = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}, \end{aligned}$$

and $CB(B)$ is the set of all nonempty closed and bounded subsets of B. A point $x \in X$ is said to be a best proximity point of $T : A \to CB(B)$ if $d(x, Tx) = dist(A, B)$. The set B is said to be approximatively compact with respect to the set A if each $\{v_n\}$ in B with $d(x, v_n) \to d(x, B)$ for some x in A has a convergent subsequence [8].

Definition 2. *Let $\alpha, \eta : A \times A \to [0, \infty)$. A mapping $T : A \to B$ is $(\alpha$-$\eta)$-proximal admissible if for any $x_1, x_2, u_1, u_2 \in A$,*

$$\left\{\begin{array}{l} \alpha(x_1, x_2) \geq \eta(x_1, x_2) \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right. \quad \textit{imply that } \alpha(u_1, u_2) \geq \eta(u_1, u_2),$$

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Note that if $A = B$ and T is (α-η)-proximal admissible then T is α-admissible with respect to η.

Definition 3. [13] *A mapping $T : A \to CB(B)$ is said to be an α_F- proximal contraction of Ciric type if there exist two functions $\alpha : A \times A \to [0, \infty)$, $F \in \Delta_F$ and $\tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1, v_2 \in Tx_2$ with $\alpha(x_1, x_2) \geq 1$ and $d(u_1, v_1) = dist(A, B) = d(u_2, v_2)$ we have*

$$\alpha(u_1, u_2) \geq 1 \text{ and } \tau + F(d(u_1, u_2)) \leq F(M(x_1, x_2)),$$

whenever $\min\{d(u_1, u_2), M(x_1, x_2)\} > 0$, *where*

$$M(x_1, x_2) = \max\left\{d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), \frac{d(x_1, u_2) + d(x_2, u_1)}{2}\right\}.$$

Definition 4. *A mapping $T : A \to CB(B)$ is said to be an α-η-proximal F-contraction of Ciric type on a closed ball if there exist two functions $\alpha : A \times A \to [0, \infty)$, $F \in \Delta_F$ and $r > 0, \tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1, v_2 \in Tx_2$ with $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(u_1, v_1) = dist(A, B) = d(u_2, v_2)$ we have*

$$\alpha(u_1, u_2) \geq \eta(u_1, u_2) \text{ and } \tau + F(d(u_1, u_2)) \leq F(kM(x_1, x_2)) \tag{1.1}$$

for all $x_1, x_2 \in Y = \overline{B(x_1, r)}$ and

$$d(x_1, Tx_1) < (1 - k)r, \text{ where } 0 \leq k < 1, \tag{1.2}$$

whenever $\min\{d(u_1, u_2), M(x_1, x_2)\} > 0$, *where*

$$M(x_1, x_2) = \max\left\{d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), \frac{d(x_1, u_2) + d(x_2, u_1)}{2}\right\}.$$

2. Main results

We start with the following result.

Theorem 5. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T : A \to CB(B)$ is an α-η-proximal F-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$;

(iii) T is continuous;

(iv) B is approximatively compact with respect to A.

Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d(x^*, Tx^*) = dist(A, B)$.

Proof. From (ii), there exist x_1, x_2 in A_0 and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$. Since $v_2 \in Tx_2 \subseteq B_0$, there exists $x_3 \in A_0$ satisfying $d(x_3, v_2) = dist(A, B)$. From (1.1), we have $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$ and

$$\begin{aligned} \tau + F(d(x_2, x_3)) &\leq F\left(k \max\left\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), \frac{d(x_1, x_3) + d(x_2, x_2)}{2}\right\}\right) \\ &\leq F(k \max\{d(x_1, x_2), d(x_2, x_3)\}) \\ &= F(kd(x_1, x_2)). \end{aligned} \tag{2.1}$$

Otherwise we have a contradiction. From above we get $x_2, x_3 \in A_0$ and $v_2 \in Tx_2$ satisfying $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$ and $d(x_3, v_2) = dist(A, B)$.
Since $v_3 \in Tx_3 \subseteq B_0$, there exists $x_4 \in A_0$ satisfying $d(x_4, v_3) = dist(A, B)$.
From (1.1), we can obtain $\alpha(x_3, x_4) \geq \eta(x_3, x_4)$ and

$$\begin{aligned} \tau + F\left(d(x_3, x_4)\right) &\leq F\left(k \max\left\{d(x_2, x_3), d(x_2, x_3), d(x_3, x_4), \frac{d(x_2, x_4) + d(x_3, x_3)}{2}\right\}\right) \\ &\leq F\left(k \max\left\{d(x_2, x_3), d(x_3, x_4)\right\}\right) \\ &= F\left(kd(x_2, x_3)\right). \end{aligned} \tag{2.2}$$

Otherwise we have a contradiction. From (2.1) and (2.2), we have

$$\tau + F\left(d(x_3, x_4)\right) \leq F\left(k^2 d(x_1, x_2)\right) - 2\tau.$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in A_0 and v_3 in B_0 such that $v_n \in Tx_n, \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, $d(x_{n+1}, v_n) = dist(A, B)$ and it satisfies

$$F(d(x_n, x_{n+1})) \leq F\left(k^n d(x_1, x_2)\right) - n\tau \text{ for each } n \in \mathbb{N} \setminus \{1\},$$

which implies

$$F(d(x_n, x_{n+1})) \leq F\left(d(x_1, x_2)\right) - n\tau \text{ for each } n \in \mathbb{N} \setminus \{1\}. \tag{2.3}$$

Now we show that $x_n \in \overline{B(x_1, r)}$ for all $n \in \mathbb{N}$. By (1.2), we have $d(x_1, Tx_1)) \leq r$ and hence $x_1 \in \overline{B(x_0, r)}$. Let $x_2, \cdots, x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. Note that $\alpha(x_{i-1}, x_i) \geq \eta(x_{i-1}, x_{i-1})$ and T is an α-η-proximal F-contraction of Ciric type mapping on a closed ball. Since F is strictly increasing,

$$\begin{aligned} d(x_1, x_{j+1}) &= d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + ... + d(x_j, x_{j+1}) \\ &\leq (1-k)r + (1-k)kr + (1-k)k^2 r + ... + (1-k)k^{j-1} r \\ &= (1-k)r\left[1 + k + k^2 + \cdots + k^{j-1}\right] \\ &= (1-k)r\frac{(1-k^j)}{(1-k)} \leq r, \end{aligned}$$

which implies that $x_{j+1} \in \overline{B(x_1, r)}$ and hence $x_n \in \overline{B(x_1, r)}$ for all $n \in \mathbb{N} \setminus \{1\}$. From (2.3), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$. Since $F \in \Delta_F$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

From $(F3)$, there exists $K \in (0, 1)$ such that

$$\lim_{n\to\infty} \left(\left(d(x_n, x_{n+1})\right)^K F\left(d(x_n, x_{n+1})\right)\right) = 0. \tag{2.5}$$

From (2.3), for all $n \in \mathbb{N}$, we obtain

$$\left(d(x_n, x_{n+1})\right)^K \left(F\left(d(x_n, x_{n+1})\right) - F\left(d(x_0, x_1)\right)\right) \leq -\left(d(x_n, x_{n+1})\right)^K n\tau \leq 0. \tag{2.6}$$

Using (2.4), (2.5) and letting $n \to \infty$ in (2.6), we have

$$\lim_{n\to\infty} \left(n\left(d(x_n, x_{n+1})\right)^K\right) = 0. \tag{2.7}$$

By (2.7), there exists $n_1 \in \mathbb{N}$ such that $n\left(d(x_n, x_{n+1})\right)^K \leq 1$ for all $n \geq n_1$. So we get

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{K}}} \text{ for all } n \geq n_1. \tag{2.8}$$

Now, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then by the triangle inequality and from (2.8) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \dots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{2.9}$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{K}}}$ is convergent. By taking limit as $n \to \infty$ in (2.9), we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in A. Since A is closed subset of a complete metric space, there exists x^* in A and $x^* \in \overline{B(x_1, r)}$ such that $x_n \to x^*$ as $n \to \infty$. As $d(x_{n+1}, v_n) = dist(A, B)$ we have $\lim_{n\to\infty} d(x^*, v_n) = dist(A, B)$. Since B is approximatively compact with respect to A, we get a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_{n_k} \in Tv_{n_k}$ that converges to v^*.
Thus

$$d(x^*, v^*) = \lim_{n\to\infty} d(x_{n_k}, v_{n_k}) = dist(A, B).$$

By (iii), when T is continuous, we get $v^* \in Tx^*$ and hence $dist(A, B) \leq d(x^*, Tx^*) \leq d(x^*, v^*) = dist(A, B)$. Therefore, $d(x^*, Tx^*) = dist(A, B)$. □

In the following theorem, the assumption of continuity is replaced with the following suitable condition:

(H) If $\{x_n\}$ is a sequence in A such that $x_n \to x^* \in A_0$ as $n \to \infty$, and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all n, then we have $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ for all n.

Theorem 6. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T : A \to CB(B)$ is an α-η-proximal F-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$;

(iii) (H) holds;

(iv) B is approximatively compact with respect to A.

Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d(x^*, Tx^*) = dist(A, B)$.

Proof. The proof follows from similar lines of Theorem 5. From the condition (H), assume that we have

$$\alpha(x_n, x^*) \geq \eta(x_n, x^*)$$

for all $n \in \mathbb{N} \cup \{1\}$ and $x_n \to x^* \in \overline{B(x_0, r)}$ as $n \to \infty$. For each $x^* \in A_0$, we have $Tx^* \subseteq B_0$. This implies that for $z^* \in Tx^*$, we have $w^* \in A_0$ such that $d(w^*, z^*) = dist(A, B)$. Further note that $d(x_{n+1}, v_n) = dist(A, B)$. We claim that $d(w^*, z^*) = 0$. On contrary assume that $d(w^*, z^*) \neq 0$. Now from (1.1), we get

$$\tau + F(d(x_{n+1}, w^*)) \leq F\left(k \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, w^*), \frac{d(x_n, w^*) + d(x_{n+1}, x^*)}{2}\right\}\right).$$

Letting $n \to \infty$, we obtain

$$\tau + F(d(x^*, w^*)) \leq F(kd(x^*, w^*)) \leq F(d(x^*, w^*)).$$

This implies

$$\tau + F(d(x^*, w^*)) \leq F(d(x^*, w^*)),$$

which is not possible. Hence $d(x^*, w^*) = 0$. Thus we get

$$dist(A, B) \leq d(x^*, Tx^*) \leq d(x^*, z^*) = dist(A, B)$$

and hence $d(x^*, Tx^*) = d(A, B)$. □

If we take $\eta(x, y) = 1$ for all $x, y \in X$ in Theorems 5 and 6, then we obtain the following results.

Corollary 7. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T : A \to CB(B)$ is an α_F-proximal F-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*
(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;
(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq 1$ and $d(x_2, v_1) = dist(A, B)$;
(iii) T is continuous;
(iv) B is approximatively compact with respect to A.
Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d(x^*, Tx^*) = dist(A, B)$.

Corollary 8. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T : A \to CB(B)$ is an α_F-proximal F-contraction of Ciric type mapping on a closed ball satisfying the following assertion:*
(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;
(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$;
(iii) (H) holds;
(iv) B is approximatively compact with respect to A.
Then there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d(x^*, Tx^*) = dist(A, B)$.

Example 9. *Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in \overline{B(x_1, r)} \subset X$. Define the mapping $T : A \to CB(B)$ by*

$$T(0, x) = \begin{cases} \left(1, \frac{x}{3}\right), \left(1, \frac{x}{2}\right) & \text{if } x \geq 0 \\ (1, x), (1, x^2) & \text{otherwise,} \end{cases}$$

where $A = \{(0, x) : -1 \leq x \leq 1\}$ and $B = \{(1, x) : -1 \leq x \leq 1\}$, and $\alpha, \eta : A \times A \to \mathbb{R}^+$

$$\alpha((0, x), (0, y)) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } \eta((0, x), (0, y)) = \begin{cases} \frac{1}{2} & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}.$$

Take $F(x) = \ln x$ for each $x \in \mathbb{R}^+$ and $\tau = \frac{2}{3}$. It is easy to see that T is an α-η-proximal F-contraction of Ciric type mapping on a closed ball. For each $x \in A_0$, we have $Tx \subseteq B_0$. Also for $x_1 = (0, \frac{1}{2}) \in A_0$ and $v_1 = (1, \frac{1}{4}) \in Tx_1$, we have $x_2 = (0, \frac{1}{4})$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(x_2, v_1) = dist(A, B)$.Moreover $\{x_n\}$ is a sequence in A such that $x_n \to x \in A_0$ as $n \to \infty$, and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all n, we have $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ for all n. Further note that B is approximatively compact with respect to A. Therefore, all the conditions of Theorems 5 and 6 hold. Hence T has a best proximity point.

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Asymptotic lines of a discrete Lotka-Volterra competition model

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Abstract

The Euler difference scheme for a two-dimensional Lotka-Volterra competition model is considered. Recently, we have shown that the difference scheme has positive and bounded solutions, and that the solutions of the scheme converge to the equilibrium points under some sufficient conditions. In this paper, we find asymptotic lines of the solutions of the Euler discrete scheme in two categories of partitions of domain. We present sufficient conditions under which the line between the two equilibrium points of the scheme is the asymptotic line of the solutions of the scheme in each category. Numerical examples are given to verify the results.

Keywords: Euler difference scheme, competition model, asymptotic line

1. Introduction

The two-dimensional Lokta-Volterra competition model is given by

$$\frac{dx}{dt} = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)),\ \frac{dy}{dt} = y(t)(r_2 - a_{21}x(t) - a_{22}y(t)), \tag{1}$$

where $r_i > 0$ and $a_{ij} > 0$. Here $x(t)$ and $y(t)$ denote the population sizes or population density in two species x and y at time t, which are competing for a common resource. The parameters r_i are the intrinsic growth rates and a_{ii} $(i = 1, 2)$ measure the inhibiting effect on the two species x and y, respectively, where a_{12} and a_{21} are the interspecific acting coefficients.

The dynamics of the model (1) is well-known [1–4]. Many reseachers have studied the Lokta-Volterra models; the solutions of (1) are positive and bounded, and the system (1) is stable. There are a number of works on investigating continuous time models [5–10]. But relatively few theoretical papers are published on their discretized models [11–14].

Recently, we have studied the global stability of the discrete-time Lokta-Volterra model. In [15], Choo has introduced a method to present global stability in the discrete Lokta-Volterra predator-prey model for the case that all species coexist at a unique equilibrium. In [16], we have shown the global stability of the Euler difference scheme for a three-dimensional predator-prey model using a new approach.

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In this paper, we consider the Euler difference scheme for the two-dimensional Lokta-Volterra competition model given by

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\}, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\}, \tag{2}$$

where

$$f(x, y) = r_1 - a_{11}x - a_{12}y, \quad g(x, y) = r_2 - a_{21}x - a_{22}y, \tag{3}$$

and Δt is a time step size, $x_n = x_0 + n\Delta t$ and $y_n = y_0 + n\Delta t$ with $(x_0, y_0) = (x(0), y(0))$.

In [17], we have shown the Euler difference scheme has positive and bounded solutions, and have presented sufficient conditions for the global stability of the fixed points of the discrete competition model with two species. The main idea of our approach has been to divide the domain used for the boundedness of solutions of the discrete model and to describe how to trace the trajectories with respect to each partition. We have obtained the following global convergences to $(0, r_2a_{22}^{-1})$ in Figure 1-(a) and $(r_1a_{11}^{-1}, 0)$ in Figure 1-(b). In the numerical results the line between the two points $(0, r_2a_{22}^{-1})$ and $(r_1a_{11}^{-1}, 0)$ looks like the asymptotic line in the two cases: one is $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$ as in Figure 1-(a), and the other is $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$ as in Figure 1-(b).

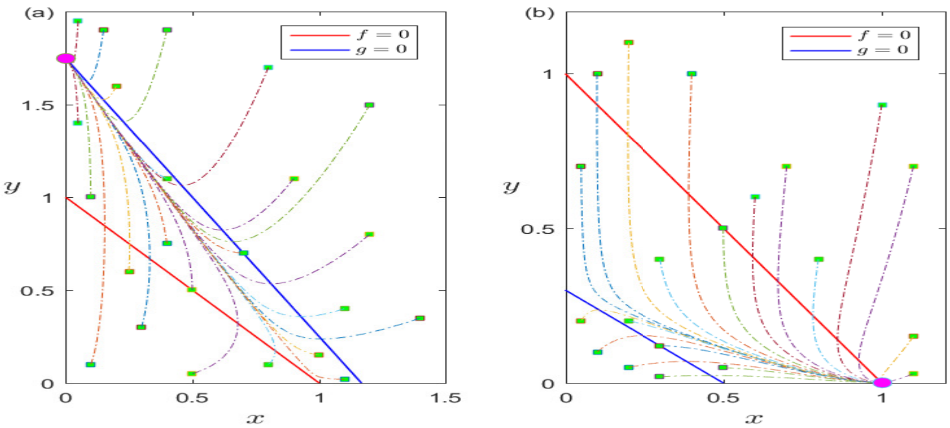

Figure 1: Trajectories for different initial points. (a) $r_1 = 1, a_{11} = 1, a_{12} = 2, r_2 = 3.5, a_{21} = 3, a_{22} = 2$. (b) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.5, a_{21} = 3, a_{22} = 5$. The box and circle symbols denote initial and equilibrium points, respectively.

Therefore the goal of this paper is to find some conditions under which the line between the two points plays a role as the boundary dividing the convergence region surrounded by the four lines $f(x, y) = 0$, $g(x, y) = 0$, $x = 0$ and $y = 0$.

The paper is organized as follows. In Section 2, we give some conditions under which the solutions of (2) are positive and bounded, and converge to equilibrium points of (2) starting in the partitioned regions of the domain. In Section 3, we have sufficient conditions under which the line between the two equilibrium points of the scheme (2) is the asymptotic line of the solutions of the scheme. In Section 4, some numerical examples are presented to verify our results.

2. Positivity, boundedness and stability of the discrete solutions

For the positivity and boundedness of the solutions (x_n, y_n) of (2), we assume

$$\Delta t < 1/\max\{r_1, r_2\} \tag{4}$$

and consider constants x^* and y^* such that

$$r_1 a_{11}^{-1} \le x^* \le U_1(y^*),\ r_2 a_{22}^{-1} \le y^* \le U_2(x^*), \tag{5}$$

where

$$U_1(\tau_2) = \frac{1 + r_1\Delta t - a_{12}\tau_2\Delta t}{2a_{11}\Delta t},\quad U_2(\tau_1) = \frac{1 + r_2\Delta t - a_{21}\tau_1\Delta t}{2a_{22}\Delta t}. \tag{6}$$

Then we have the positivity and boundedness of (x_n, y_n) using x^* and y^* in (5) as follows (see [17]).

Theorem 1. *Let (x_n, y_n) be the solution of (2). Assume that (4) and (5) hold.*

If $(x_0, y_0) \in (0, x^*) \times (0, y^*)$, then $(x_n, y_n) \in (0, x^*) \times (0, y^*)$ for all n.

Let $\mathcal{D} = (0, x^*) \times (0, y^*)$ for x^* and y^* defined in (5). To discuss the stability of the Euler difference scheme (2) for each initial position (x_0, y_0) contained in $\mathcal{D}$, we partition $\mathcal{D}$ by two lines $f(x, y) = 0$ and $g(x, y) = 0$ into the four regions

$$\begin{aligned} \mathrm{I} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \ge 0,\ g(\mathbf{x}) > 0\},\quad \mathrm{II} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) < 0,\ g(\mathbf{x}) \ge 0\}, \\ \mathrm{III} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \le 0,\ g(\mathbf{x}) < 0\},\quad \mathrm{IV} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) > 0,\ g(\mathbf{x}) \le 0\}, \end{aligned} \tag{7}$$

where $\mathbf{x} = (x, y)$, and $f(x, y)$ and $g(x, y)$ are given in (3).

Since the location of the regions depends on the x and y-intercepts of the two lines, there are four categories $\mathcal{C}_i (1 \le i \le 4)$ of partition in $\mathcal{D}$ as in Figure 2; we use the symbol $\mathcal{C}_1$ for the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$, the symbol $\mathcal{C}_2$ for $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$, the symbol $\mathcal{C}_3$ for $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$, and finally the symbol $\mathcal{C}_4$ for $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$. The magenta circles in Figure 2 denote the stable points of the difference model (2) in the categories.

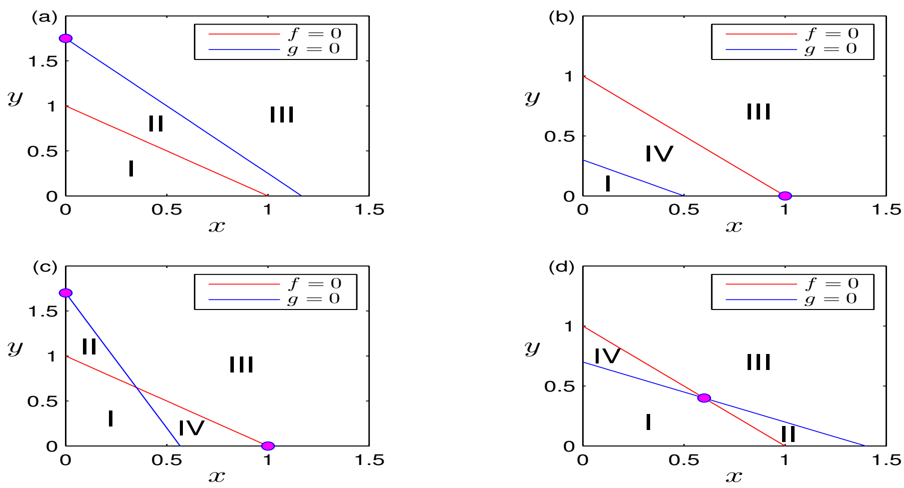

Figure 2: Two lines $f = 0$ and $g = 0$ and regions with stable points. The values of the parameters are (a) $r_2 = 3.5, a_{21} = 3.0, a_{22} = 2$ in the category $\mathcal{C}_1$. (b) $r_2 = 1.5, a_{21} = 3, a_{22} = 5$ in the category $\mathcal{C}_2$. (c) $r_2 = 1.7, a_{21} = 3, a_{22} = 1$ in the category $\mathcal{C}_3$. (d) $r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$ in the category $\mathcal{C}_4$.

For the stability we assume

$$1 > \Delta t \left(a_{11}x^* + a_{22}y^* + x^*y^*|a_{12}a_{21} - a_{11}a_{22}|\Delta t\right). \tag{8}$$

Then we have the following lemma (see [17]).

Lemma 1. *Let (x_n, y_n) be the solution of (2). Assume that (4), (5) and (8) hold. Then we have*

(i) *If $(x_k, y_k) \in \mathrm{I}$ for some k, then (x_{k+1}, y_{k+1}) is not contained in* III.
(ii) *If $(x_k, y_k) \in \mathrm{III}$ for some k, then (x_{k+1}, y_{k+1}) is not contained in* I.
(iii) *If $(x_k, y_k) \in \mathrm{II}$ for some k, then $(x_n, y_n) \in \mathrm{II}$ for all $n \geq k$.*
(iv) *If $(x_k, y_k) \in \mathrm{IV}$ for some k, then $(x_n, y_n) \in \mathrm{IV}$ for all $n \geq k$.*

In the following theorem, we have the global stability of the solutions of (2) for the category $\mathcal{C}_1$ and $\mathcal{C}_2$ as in Figure 2-(a) and Figure 2-(b), respectively (see [17]).

Theorem 2. *Let (x_n, y_n) be the solution of (2). Assume that (4), (5) and (8) hold. Then we have*

(i) *If $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$, then $\left(0, r_2a_{22}^{-1}\right)$ is globally stable.*
(ii) *If $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, then $\left(r_1a_{11}^{-1}, 0\right)$ is globally stable.*

Remark 1. Under the same conditions as in Theorem 2, we have the convergence of the solutions (x_n, y_n) of (2) for the category $\mathcal{C}_3$ as in Figure 2-(c). If $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$, then the solutions converge with the limit $(r_1a_{11}^{-1}, 0)$ or $(0, r_2a_{22}^{-1})$. We have the global stability of the solutions for the category $\mathcal{C}_4$ as in Figure 2-(d) where each component of the equilibrium point is positive. If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, then (θ_1, θ_2) is globally stable, where $(\theta_1, \theta_2) = (a_{11}a_{22} - a_{12}a_{21})^{-1}(r_1a_{22} - r_2a_{12}, -r_1a_{21} + r_2a_{11})$ with $f(\theta_1, \theta_2) = g(\theta_1, \theta_2) = 0$. See [17] in detail.

Remark 2. Using the results in this section, we present the asymptotic lines of the discrete solutions in $\mathcal{C}_1$ and $\mathcal{C}_2$ in the next section. In the case of $\mathcal{C}_3$ and $\mathcal{C}_4$, the corresponding asymptotic lines will be treated in the future work.

3. Asymptotic lines of the discrete solutions in $\mathcal{C}_1$ and $\mathcal{C}_2$

In this section, we give sufficient conditions under which the line between the two equilibrium points of the scheme (2) is the asymptotic line of the solutions of the scheme in the two categories $\mathcal{C}_1$ and $\mathcal{C}_2$.

First, we consider the category $\mathcal{C}_1$ as in Figure 1-(a), which is the case

$$r_1a_{11}^{-1} < r_2a_{21}^{-1}, \; r_1a_{12}^{-1} < r_2a_{22}^{-1}. \tag{9}$$

By Theorem 2, $(0, r_2a_{22}^{-1})$ is the unique equibrium point in this case.

Denote the line between the two points $(r_1a_{11}^{-1}, 0)$ and $(0, r_2a_{22}^{-1})$ as $h(x, y) = 0$, where

$$h(x, y) = r_1r_2 - r_2a_{11}x - r_1a_{22}y. \tag{10}$$

The condition that (x_k, y_k) is located between two lines $h(x, y) = 0$ and $g(x, y) = 0$ is equivalent that

$$r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k < 0 \tag{11}$$

and

$$r_2 - a_{21}x_k - a_{22}y_k > 0. \tag{12}$$

The equation (12) implies $(x_k, y_k) \in \mathrm{II}$, which gives $(x_{k+1}, y_{k+1}) \in \mathrm{II}$ due to Lemma 1-(iii). Therefore $r_2 - a_{21}x_{k+1} - a_{22}y_{k+1} > 0$. In this case, we have the following lemma.

Lemma 2. *Assume that for some $\alpha_k > 0$*

$$r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k = -\alpha_k^2 < 0. \tag{13}$$

Then we have

$$\begin{aligned} r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} &= \alpha_k^2 \{-1 + \Delta t \big[\alpha_k^2 \frac{1}{r_1} + p(x_k)\big]\} \\ &+ x_k \Delta t \big(\frac{r_2}{r_1 a_{22}}\big) \{r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22})\}(x_k a_{11} - r_1), \end{aligned} \tag{14}$$

where

$$p(x) = \big(\frac{r_2 a_{11} a_{12}}{r_1 a_{22}} - \frac{2 r_2 a_{11}}{r_1} + a_{21}\big) x + r_2. \tag{15}$$

Proof. We have from (2) and (3) that

$$\begin{aligned} r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} &= r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k \\ &- r_2 a_{11} x_k \Delta t (r_1 - a_{11} x_k - a_{12} y_k) - r_1 a_{22} y_k \Delta t (r_2 - a_{21} x_k - a_{22} y_k). \end{aligned} \tag{16}$$

Then, by (13) and (15), we have from (16) that

$$\begin{aligned} &r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} \\ &= -\alpha_k^2 - r_2 a_{11} x_k \Delta t \big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k + \alpha_k^2}{r_1 a_{22}}\big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k + \alpha_k^2) \Delta t \big(- a_{21} x_k - \frac{-r_2 a_{11} x_k + \alpha_k^2}{r_1}\big) \\ &= \alpha_k^4 (\Delta t \frac{1}{r_1}) + \alpha_k^2 \{-1 - r_2 a_{11} x_k \Delta t \big(- \frac{a_{12}}{r_2 a_{22}}\big) - (r_1 r_2 - r_2 a_{11} x_k) \cdot \big(- \Delta t \frac{1}{r_1}\big) \\ &\quad - \Delta t (-a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1})\} + \alpha_k^0 \{-r_2 a_{11} x_k \Delta t \big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k}{r_1 a_{22}}\big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k) \Delta t (-a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1})\} \\ &= \alpha_k^2 \{-1 + \Delta t \big[\alpha_k^2 \frac{1}{r_1} + p(x_k)\big]\} + G(x_k). \end{aligned} \tag{17}$$

Here the last term in (17) is

$$\begin{aligned} G(x_k) &= - r_2 a_{11} x_k \Delta t \big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k}{r_1 a_{22}}\big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k) \Delta t \big(- a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1}\big) \\ &= x_k^2 \{(-r_2 a_{11}) \Delta t \big(- a_{11} + \frac{r_2 a_{11} a_{12}}{r_1 a_{22}}\big) - (-r_2 a_{11}) \Delta t (-a_{21} + \frac{r_2 a_{11}}{r_1})\} \\ &\quad + x_k \{-r_2 a_{11} \Delta t \big(r_1 - \frac{r_2 a_{12}}{a_{22}}\big) - (r_1 r_2) \Delta t \big(- a_{21} + \frac{r_2 a_{11}}{r_1}\big)\} \\ &= x_k^2 \{(-r_2 a_{11}) \Delta t \big(\frac{a_{11}}{r_1 a_{22}}\big) + r_2 a_{11} \Delta t \frac{1}{r_1} (-r_1 a_{21} + a_{11} r_2)\} \\ &\quad - x_k \Delta t \big(\frac{r_2 a_{11}}{a_{22}}\big) \{a_{11}(r_1 a_{22} - r_2 a_{12}) + a_{22}(-r_1 a_{21} + r_2 a_{11})\} \\ &= x_k \Delta t \big(\frac{r_2}{r_1 a_{22}}\big) \{r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22})\}(x_k a_{11} - r_1). \end{aligned} \tag{18}$$

Hence we obtain the result. □

In the following lemma, we consider the case that the point (x_k, y_k) is located between two lines $h(x,y) = 0$ and $f(x,y) = 0$. It is equivalent to the case that (x_k, y_k) satisfies

$$r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k > 0 \tag{19}$$

and

$$r_1 - a_{11} x_k - a_{12} y_k < 0. \tag{20}$$

The equation (19) and (20) implie $(x_k, y_k) \in \text{II}$, which gives $(x_{k+1}, y_{k+1}) \in \text{II}$ due to Lemma 1-(iii). Therefore $r_1 - a_{11}x_{k+1} - a_{12}y_{k+1} < 0$. We have the following result in this case.

Lemma 3. *Assume that for some $\alpha_k > 0$*

$$r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k = \alpha_k^2 > 0. \tag{21}$$

Then we have

$$\begin{aligned} r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} &= \alpha_k^2 \{1 + \Delta t[\alpha_k^2 \frac{1}{r_1} + q(x_k)]\} \\ &+ x_k \Delta t (\frac{r_2}{r_1 a_{22}}) \{r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22})\}(x_k a_{11} - r_1), \end{aligned} \tag{22}$$

where

$$q(x) = \big(- \frac{r_2 a_{11} a_{12}}{r_1 a_{22}} + a_{21} \big) x - r_2. \tag{23}$$

Proof. By a similar way in the proof of Lemma 2, we have from (16), (21), (23) and (18) that

$$\begin{aligned} & r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} \\ &= \alpha_k^2 - r_2 a_{11} x_k \Delta t \big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k - \alpha_k^2}{r_1 a_{22}} \big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k - \alpha_k^2) \Delta t \big(- a_{21} x_k + \frac{r_2 a_{11} x_k + \alpha_k^2}{r_1} \big) \\ &= \alpha_k^4 \big(\Delta t \frac{1}{r_1} \big) - \alpha_k^2 \{ -1 - r_2 a_{11} x_k \Delta t \big(- \frac{a_{12}}{r_2 a_{22}} \big) - (r_1 r_2 - r_2 a_{11} x_k) \cdot \big(- \Delta t \frac{1}{r_1} \big) \\ &\quad - \Delta t \big(- a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1} \big) \} + \alpha_k^0 \{ -r_2 a_{11} x_k \Delta t \big(r_1 - a_{11} x_k - a_{12} \frac{r_1 r_2 - r_2 a_{11} x_k}{r_1 a_{22}} \big) \\ &\quad - (r_1 r_2 - r_2 a_{11} x_k) \Delta t \big(- a_{21} x_k + \frac{r_2 a_{11} x_k}{r_1} \big) \} \\ &= \alpha_k^2 \{ 1 + \Delta t \big[\alpha_k^2 \frac{1}{r_1} + q(x_k) \big] \} + G(x_k). \end{aligned} \tag{24}$$

Hence we obtain the result. ☐

Since the solution (x_k, y_k) of (2) and $\alpha_k^2 = |r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k|$ in (13) and (21) are bounded by Theorem 1, it is possible to take Δt so small, which satisfies the inequalities

$$\Delta t \{ \alpha_k^2 \frac{1}{r_1} + p(x_k) \} < 1, \quad 1 + \Delta t \{ \alpha_k^2 \frac{1}{r_1} + q(x_k) \} > 0. \tag{25}$$

We divide the region II based on the two lines $h(x,y)=0$ and $x=r_1a_{11}^{-1}$, and then the region is partitioned into three parts II^0, II^{u} and II^{d} (see Figure 3).

II^0 is the region with the three boundaries $g(x,y)=0,\ y=0$ and $x=r_1a_{11}^{-1}$.

II^{u} is the region with the three boundaries $g(x,y)=0,\ h(x,y)=0$ and $x=r_1a_{11}^{-1}$.

II^{d} is the region with the three boundaries $f(x,y)=0,\ h(x,y)=0$ and $x=0$.

In the following theorems, we have the results that if the solution (x_n,y_n) of (2) starts at II^{u} or II^{d}, it remains in the same region.

Theorem 3. *Let the conditions (4), (5), (8) and (25) hold. Let (x_n,y_n) be the solution of (2) with $r_1a_{11}^{-1}<r_2a_{21}^{-1}$, $r_1a_{12}^{-1}<r_2a_{22}^{-1}$ and*

$$r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22})\geq 0. \tag{26}$$

If for some k

$$(x_k,y_k)\in \mathrm{II}^{\mathrm{u}},$$

then for all $i\geq k$

$$(x_i,y_i)\in \mathrm{II}^{\mathrm{u}},$$

where II^{u} is the the region with the three boundaries

$$g(x,y)=0,\ h(x,y)=0 \text{ and } x=r_1a_{11}^{-1}.$$

Proof. Since $x_n>0$ and $y_n>0$ for all n in Theorem 1, $g(x,y)=r_2-a_{21}x-a_{22}y$ and $h(x,y)=r_1r_2-r_2a_{11}x-r_1a_{22}y$, the inclusion $(x_k,y_k)\in\mathrm{II}^{\mathrm{u}}$ is equivalent to

$$r_2-a_{21}x_k-a_{22}y_k>0,\ r_1r_2-r_2a_{11}x_k-r_1a_{22}y_k<0.$$

Then it is enough to show that for all $i\geq k$

$$r_2-a_{21}x_i-a_{22}y_i>0, \tag{27}$$
$$r_1r_2-r_2a_{11}x_i-r_1a_{22}y_i<0. \tag{28}$$

Note that due to Lemma 1-(iii)

$$\text{if } (x_k,y_k)\in \mathrm{II}, \text{then } (x_i,y_i)\in\mathrm{II} \text{ for all } i\geq k. \tag{29}$$

Since $(x_k,y_k)\in\mathrm{II}^{\mathrm{u}}$ and $\mathrm{II}^{\mathrm{u}}\subset\mathrm{II}$, we have $(x_i,y_i)\in\mathrm{II}$ for all $i\geq k$ due to (29), so that the definition of II yields the inequality (27).

Now it remains to show the inequality (28), which can be proved using the equality (14) in Lemma 2:

$$\begin{aligned} r_1r_2-r_2a_{11}x_{k+1}-r_1a_{22}y_{k+1} &= \alpha_k^2\{-1+\Delta t[\alpha_k^2\frac{1}{r_1}+p(x_k)]\} \\ &\quad + x_k\Delta t(\frac{r_2}{r_1a_{22}})\{r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22})\}(x_ka_{11}-r_1), \end{aligned} \tag{30}$$

where

$$p(x)=\big(\frac{r_2a_{11}a_{12}}{r_1a_{22}}-\frac{2r_2a_{11}}{r_1}+a_{21}\big)x+r_2$$

and

$$\alpha_k^2 = -(r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k) > 0$$

due to $r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k < 0$. Applying both (25) and (26) into (30) with $x_i < r_1 a_{11}^{-1}$ for all $i \geq k$ obtained from (29), we have that

$$r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} < 0.$$

Using mathematical induction, we can obtain the desired result. □

Theorem 4. *Let the conditions (4), (5), (8) and (25) hold. Let (x_n, y_n) be the solution of (2) with $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and*

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) \leq 0. \tag{31}$$

If for some k

$$(x_k, y_k) \in \mathrm{II}^{\mathrm{d}},$$

then for all $i \geq k$

$$(x_i, y_i) \in \mathrm{II}^{\mathrm{d}},$$

where II^{d} is the the region with the three boundaries

$$f(x,y) = 0, \;\; h(x,y) = 0 \;\; \textit{and} \;\; x = 0.$$

Proof. Since $x_n > 0$ and $y_n > 0$ for all n in Theorem 1, $f(x,y) = r_1 - a_{11}x - a_{12}y$ and $h(x,y) = r_1 r_2 - r_2 a_{11} x - r_1 a_{22} y$, the inclusion $(x_k, y_k) \in \mathrm{II}^{\mathrm{d}}$ is equivalent to

$$r_1 - a_{11} x_k - a_{12} y_k < 0, \;\; r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k > 0.$$

Then it is enough to show that for all $i \geq k$

$$r_1 - a_{11} x_i - a_{12} y_i < 0, \tag{32}$$

$$r_1 r_2 - r_2 a_{11} x_i - r_1 a_{22} y_i > 0. \tag{33}$$

Since $(x_k, y_k) \in \mathrm{II}^{\mathrm{d}}$ and $\mathrm{II}^{\mathrm{d}} \subset \mathrm{II}$, we have $(x_i, y_i) \in \mathrm{II}$ for all $i \geq k$ due to (29), so that the definition of II yields the inequality (32).

Now it remains to show the inequality (33), which can be proved using the equality (22) in Lemma 3:

$$\begin{aligned} r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} &= \alpha_k^2 \{1 + \Delta t[\alpha_k^2 \frac{1}{r_1} + q(x_k)]\} \\ &+ x_k \Delta t (\frac{r_2}{r_1 a_{22}}) \{r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22})\}(x_k a_{11} - r_1), \end{aligned} \tag{34}$$

where

$$q(x) = \big(-\frac{r_2 a_{11} a_{12}}{r_1 a_{22}} + a_{21} \big) x - r_2$$

and

$$\alpha_k^2 = r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k > 0$$

due to $r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k > 0$. Applying both (25) and (31) into (34) with $x_i < r_1 a_{11}^{-1}$ for all $i \geq k$ obtained from (29), we have that

$$r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} > 0.$$

Using mathematical induction, we can obtain the desired result. □

Remark 3. In $\mathcal{C}_1$, we have from Theorem 3 that if $r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22}) \geq 0$ in (3), then the sequence (x_k, y_k) in II^u defined by (2) remains in II^u as follows:

(i) If $(x_k, y_k) \in \mathrm{I} \cup \mathrm{III}$ for some k, then there exists a positive integer l such that $(x_{k+l}, y_{k+l}) \in \mathrm{II}$.

(ii) If $(x_k, y_k) \in \mathrm{II}$ for some k, then $(x_{k+i}, y_{k+i}) \in \mathrm{II}$ for all $i \geq 1$ and $\lim_{k\to\infty}(x_k, y_k) = (0, r_2a_{22}^{-1})$ by Lemma 1-(iii) and Theorem 2-(i).

(iii) By (ii), if $(x_k, y_k) \in \mathrm{II}$, then there exists a nonnegative integer l such that $(x_{k+l}, y_{k+l}) \in \mathrm{II}^u \cup \mathrm{II}^d$. If there exists m such that $(x_{k+l+m}, y_{k+l+m}) \in \mathrm{II}^u$, then $(x_{k+l+i}, y_{k+l+i}) \in \mathrm{II}^u$ $(i \geq m)$ by Theorem 3. Otherwise, $(x_{k+l+i}, y_{k+l+i}) \in \mathrm{II}^d$ for all $i \geq 1$.

Also we have from Theorem 4 that if $r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22}) \leq 0$ in (3), then the sequence (x_k, y_k) in II^d defined by (2) remains in II^d.

In the case of $\mathcal{C}_2$, we divide the region IV into two parts $\mathrm{IV^u}$ and $\mathrm{IV^d}$ by the line $h(x, y) = 0$ (see Figure 4).

$\mathrm{IV^u}$ is the region with the three boundaries $f(x, y) = 0, h(x, y) = 0$ and $x = 0$.

$\mathrm{IV^d}$ is the region with the three boundaries $g(x, y) = 0, h(x, y) = 0$ and $y = 0$.

In the following theorems, we have the result that if the solution (x_n, y_n) of (2) starts at any part of IV, it remains in the same part.

Theorem 5. *Let the conditions (4), (5), (8) and (25) hold. Let (x_n, y_n) be the solution of (2) with $r_1a_{11}^{-1} > r_2a_{21}^{-1}$, $r_1a_{12}^{-1} > r_2a_{22}^{-1}$ and*

$$r_1a_{22}(a_{11}-a_{21})-r_2a_{11}(a_{12}-a_{22}) \geq 0. \tag{35}$$

If for some k

$$(x_k, y_k) \in \mathrm{IV^u},$$

then for all $i \geq k$

$$(x_i, y_i) \in \mathrm{IV^u},$$

where $\mathrm{IV^u}$ is the the region with the three boundaries

$$f(x, y) = 0,\ h(x, y) = 0 \text{ and } x = 0.$$

Proof. Since $x_n > 0$ and $y_n > 0$ for all n in Theorem 1, $f(x, y) = r_1 - a_{11}x - a_{12}y$ and $h(x, y) = r_1r_2 - r_2a_{11}x - r_1a_{22}y$, the inclusion $(x_k, y_k) \in \mathrm{IV^u}$ is equivalent to

$$r_1 - a_{11}x_k - a_{12}y_k > 0,\ r_1r_2 - r_2a_{11}x_k - r_1a_{22}y_k < 0.$$

Then it is enough to show that for all $i \geq k$

$$r_1 - a_{11}x_i - a_{12}y_i > 0, \tag{36}$$

$$r_1r_2 - r_2a_{11}x_i - r_1a_{22}y_i < 0. \tag{37}$$

Note that due to Lemma 1-(iv)

$$\text{if } (x_k, y_k) \in \mathrm{IV}, \text{then } (x_i, y_i) \in \mathrm{IV} \text{ for all } i \geq k. \tag{38}$$

Since $(x_k, y_k) \in \mathrm{IV^u}$ and $\mathrm{IV^u} \subset \mathrm{IV}$, we have $(x_i, y_i) \in \mathrm{IV}$ for all $i \geq k$ due to (38), so that the definition of IV yields the inequality (36).

As in the proof of Theorem 3, we use the equality (14) in Lemma 2 with $\alpha_k^2 > 0$ to show the inequality (37). Applying both (25) and (35) into (14) with $x_i < r_1 a_{11}^{-1}$ for all $i \geq k$ obtained from (38), we have that

$$r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} < 0.$$

Using mathematical induction, we can obtain the desired result. □

Theorem 6. *Let the conditions (4), (5), (8) and (25) hold. Let (x_n, y_n) be the solution of (2) with $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ and*

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) \leq 0. \tag{39}$$

If for some k

$$(x_k, y_k) \in \mathrm{IV}^{\mathrm{d}},$$

then for all $i \geq k$

$$(x_i, y_i) \in \mathrm{IV}^{\mathrm{d}},$$

where IV^{d} is the the region with the three boundaries

$$g(x,y) = 0,\ \ h(x,y) = 0 \ \textit{and}\ y = 0.$$

Proof. Since $x_n > 0$ and $y_n > 0$ for all n in Theorem 1, $g(x, y) = r_2 - a_{21}x - a_{22}y$ and $h(x, y) = r_1 r_2 - r_2 a_{11} x - r_1 a_{22} y$, the inclusion $(x_k, y_k) \in \mathrm{IV}^{\mathrm{d}}$ is equivalent to

$$r_2 - a_{21}x_k - a_{22}y_k < 0,\ \ r_1 r_2 - r_2 a_{11} x_k - r_1 a_{22} y_k > 0.$$

Then it is enough to show that for all $i \geq k$

$$r_1 - a_{11}x_i - a_{12}y_i < 0, \tag{40}$$

$$r_1 r_2 - r_2 a_{11} x_i - r_1 a_{22} y_i > 0. \tag{41}$$

Since $(x_k, y_k) \in \mathrm{IV}^{\mathrm{d}}$ and $\mathrm{IV}^{\mathrm{d}} \subset \mathrm{IV}$, we have $(x_i, y_i) \in \mathrm{IV}$ for all $i \geq k$ due to (38), so that the definition of IV yields the inequality (40).

As in the proof of Theorem 4, we use the equality (22) in Lemma 3 with $\alpha_k^2 > 0$ to show the inequality (41). Applying both (25) and (39) into (22) with $x_i < r_1 a_{11}^{-1}$ for all $i \geq k$ obtained from (38), we have that

$$r_1 r_2 - r_2 a_{11} x_{k+1} - r_1 a_{22} y_{k+1} > 0.$$

Using mathematical induction, we can obtain the desired result. □

Remark 4. We have similar results as Remark 3. In the case of $\mathcal{C}_2$, we have from Theorem 5 that if $r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) \geq 0$ in (3), then the sequence (x_k, y_k) defined by (2) remains in IV^u as follows:

(i) If $(x_k, y_k) \in \mathrm{I} \cup \mathrm{III}$ for some k, then there exists l such that$(x_{k+l}, y_{k+l}) \in \mathrm{IV}$.
(ii) If $(x_k, y_k) \in \mathrm{IV}$ for some k, then $(x_{k+i}, y_{k+i}) \in \mathrm{IV}$ for all $i \geq 1$ and $\lim_{k\to\infty}(x_k, y_k) = (r_1 a_{11}^{-1}, 0)$ by Lemma 1-(iv) and Theorem 2-(ii).
(iii) By (ii), if $(x_k, y_k) \in \mathrm{IV}$, then $(x_k, y_k) \in \mathrm{IV}^u \cup \mathrm{IV}^d$. If there exists m such that $(x_{k+m}, y_{k+m}) \in \mathrm{IV}^u$, then $(x_{k+i}, y_{k+i}) \in\mathrm{IV}^u$ $(i \geq m)$ by Theorem 5. Otherwise, $(x_{k+i}, y_{k+i}) \in \mathrm{IV}^d$ for all $i \geq 1$.

As a similar way, we have from Theorem 6 that if $r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) \leq 0$ in (3), then the sequence (x_k, y_k) in IV^d defined by (2) remains in IV^d.

4. Numerical examples

In this section, we provide simulations that illustrate our results in Theorem 3-Theorem 6 for the difference scheme (2) with $\Delta t = 0.001$ and $(x^*, y^*) = (r_1 a_{11}^{-1}+50, r_2 a_{22}^{-1}+50)$. The values of parameters used in the following examples satisfy the conditions in (4), (5), (8) and (25). From the following examples, we verify the result that the line $h(x,y) = r_1 r_2 - r_2 a_{11} x - r_1 a_{22} y = 0$ is the asymptotic line of the solutions (x_n, y_n) of (2).

Example 1. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 0.5, 1, 4, 1, 2)$, which satisfies the three conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = 1 > 0$$

in Theorem 3. Then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1} = 2)$ as displayed in Figure 3-(a). The sequence of the solutions in II^u remains in II^u.

Example 2. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 5, 4, 2)$, which satisfies the conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = -1 < 0$$

in Theorem 4. Then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1} = 2.5)$ as displayed in Figure 3-(b). The sequence of the solutions in II^d remains in II^d.

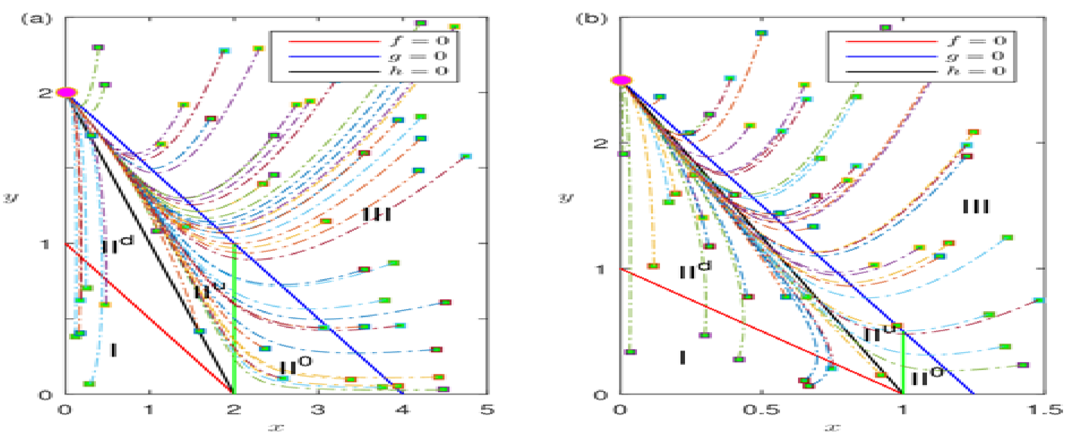

Figure 3: Trajectories for different initial points in the regions I, II, III in the category $\mathcal{C}_1$ with (a) $r_1 = 1, a_{11} = 0.5, a_{12} = 1, r_2 = 4, a_{21} = 1, a_{22} = 2$. (b) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 5, a_{21} = 4, a_{22} = 2$. The box and circle symbols denote initial and equilibrium points, respectively. The green line segment is $x = r_1 a_{11}^{-1}$ in the region II.

Example 3. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (3, 1, 1.5, 1, 0.5, 1)$, which satisfies the three conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = 1 > 0$$

in Theorem 5. Then the solutions (x_n, y_n) of (2) converge to $(r_1 a_{11}^{-1} = 3, 0)$ as displayed in Figure 4-(a). The sequence of the solutions in IV^u remains in IV^u.

Example 4. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (4, 1, 2, 1, 1, 1)$, which satisfies the conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = -1 < 0$$

in Theorem 6. Then the solutions (x_n, y_n) of (2) converge to $(r_1 a_{11}^{-1} = 4, 0)$ as displayed in Figure 4-(b). The sequence of the solutions in IV^d remains in IV^d.

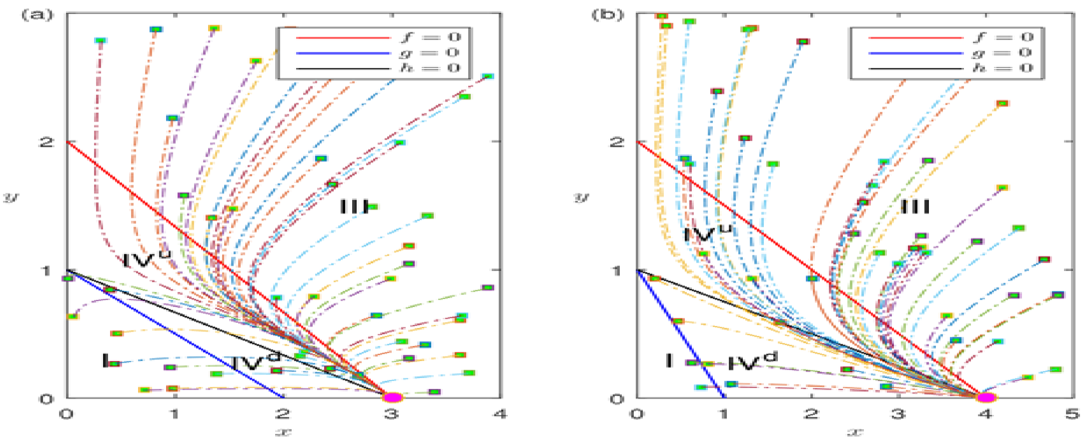

Figure 4: Trajectories for different initial points in the regions I, III, IV in the category $\mathcal{C}_2$ with (a) $r_1 = 3, a_{11} = 1, a_{12} = 1.5, r_2 = 1, a_{21} = 0.5, a_{22} = 1$, (b) $r_1 = 4, a_{11} = 1, a_{12} = 2, r_2 = 1, a_{21} = 1, a_{22} = 1$. The box and circle symbols denote initial and equilibrium points, respectively.

Example 5. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 2.5, 1, 1)$, which satisfies the three conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = 0$$

in Theorem 3 and Theorem 4. Then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1} = 2.5)$ as displayed in Figure 5-(a). For the trajectory of the solutions from III to II, if (x_k, y_k) in II^u, then $(x_{k+i}, y_{k+i}) \in$ for all $i \geq 0$ remains in II^u. Also for the trajectory of the solutions (x_k, y_k) from I to II, if (x_k, y_k) in II^d, then $(x_{k+i}, y_{k+i}) \in$ for all $i \geq 0$ remains in II^d. Therefore the line $h(x, y) = 0$ is the asymptotic line of the solutions.

Example 6. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (2.5, 1, 1, 1, 1, 1)$, which satisfies the conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$, $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ and

$$r_1 a_{22}(a_{11} - a_{21}) - r_2 a_{11}(a_{12} - a_{22}) = 0$$

in Theorem 5 and Theorem 6. Then the solutions (x_n, y_n) of (2) converge to $(r_1 a_{11}^{-1} = 2.5, 0)$ as displayed in Figure 5-(b). For the trajectory of the solutions from III to IV, the sequence of the solutions in IV^u does not cross the line $h(x, y) = 0$, which is the asymptotic line of the solutions. Also for the trajectory of the solutions (x_n, y_n) from I to IV, the sequence of the solutions in IV^d remains in IV^d.

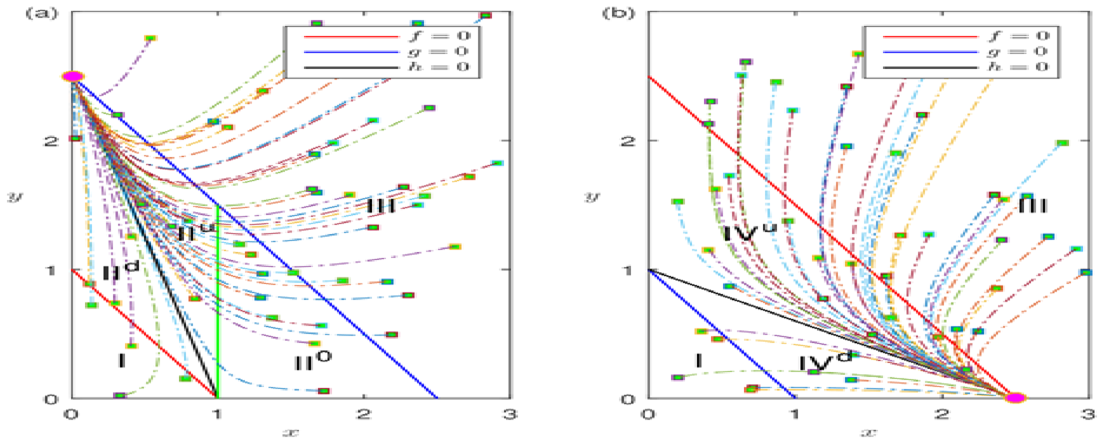

Figure 5: (a) Trajectories for different initial points in the regions I, II, III with $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 2.5, a_{21} = 1, a_{22} = 1$ in the category $\mathcal{C}_1$. The green line segment is $x = r_1 a_{11}^{-1}$ in the region II. (b) Trajectories for different initial points in the regions I, III, IV with $r_1 = 2.5, a_{11} = 1, a_{12} = 1, r_2 = 1, a_{21} = 1, a_{22} = 1$ in the category $\mathcal{C}_2$. The box and circle symbols denote initial and equilibrium points, respectively.

5. Conclusions

In this paper, we have found sufficient conditions under which the line $h(x, y) = 0$ between the two equilibrium points of the scheme (2) is the asymptotic line of the solutions of the scheme in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. In these conditions, the line $h(x, y) = 0$ plays a role as the boundary dividing the convergence region surrounded by the four lines $f(x, y) = 0$, $g(x, y) = 0$, $x = 0$ and $y = 0$, and the sequence of the solutions of (2) starting in the partitioned regions of the domain does not cross the line $h(x, y) = 0$. Some numerical examples are presented to verify our results. We have obtained the results in the two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, but the methods used in this paper can be applied to find the asymptotic lines of the solutions of (2) in the other categories $\mathcal{C}_3$ and $\mathcal{C}_4$, which will be shown in the future work.

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TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 7, 2019

Shift and invert weighted Golub-Kahan-Lanczos bidiagonalization algorithm for linear response eigenproblem, Hong-xiu Zhong, Guo-liang Chen, and Wan-qiang Shen,……………………1169

Qualitative Study of Solution of Some Higher Order Difference Equations, E. M. Elsayed, K. N. Alshabi, and Faris Alzahrani,…………………………………………………………………1179

On some classes of nonlinear contractions in Fuzzy metric spaces, Dangdang Wang, Chuanxi Zhu, and Zhaoqi Wu,……………………………………………………………………………1192

On subclasses of analytic functions with fixed second coefficients, A. Y. Lashin and F.Z. El-Emam,………………………………………………………………………………………1200

On strong convergence theorem of hybrid algorithm for a countable family of quasi-Lipschitz mappings, Muhammad Saeed Ahmad, Waqas Nazeer, Mobeen Munir, Shin Min Kang, and Samina Kausar,…………………………………………………………………………………1208

Some common fixed point theorems in ω-orbitally complete modular metric spaces via C-class functions and application, Bahman Moeini, Arslan Hojat Ansari, and Choonkil Park,……1217

Trapezoidal interval type-2 hesitant fuzzy sets associated with new operations, N. O. Alshehri and H. A. Alshehri,……………………………………………………………………………1229

Menger probabilistic normed Riesz spaces and stability of lattice preserving functional equation, Seyed Mohammad Sadegh Modarres Mosadegh, Ehsan Movahednia, Jung Rye Lee, and Choonkil Park,…………………………………………………………………………………1244

Fourier series of sums of products of poly-Genocchi and poly-Bernoulli functions, Taekyun Kim, Dae San Kim, Dmitry V. Dolgy, and Jongkyum Kwon,……………………………………1258

Additive-quadratic ρ-functional equations in β-homogeneous F-spaces, Sungsik Yun,….1276

Differential subordination for analytic functions associated with leaf-like domains, S. Sivasubramanian, M. Govindaraj, G. Murugusundaramoorthy, and N. E. Cho,………….1286

On iterative approach to common fixed points of nonexpansive mappings in Hilbert spaces, Muhammad Saeed Ahmad, Waqas Nazeer, Shin Min Kang, Syeed Fakhar Abbas Naqvi,1302

Best proximity points involving F-contraction on a closed ball, Aftab Hussain and Choonkil Park,……………………………………………………………………………………1309

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO. 7, 2019

(continued)

Asymptotic lines of a discrete Lotka-Volterra competition model, Young-Hee Kim and Sang-Mok Choo,……………………………………………………………………………………1316