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Stability of a within-host Chikungunya virus dynamics model with latency

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Abstract

This paper studies the stability of a mathematical model for within-host Chikungunya virus (CHIKV) infection. The model incorporates (i) two types of infected monocytes, latently infected monocytes which do not generate CHIKV until they have been activated and actively infected monocytes, (ii) antibody immune response, and (iii) saturated incidence rate. We derive a biological threshold number $\mathcal{R}_0$. Using the method of Lyapunov function, we established the global stability of the steady states of the model. We have proven that, when $\mathcal{R}_0 \leq 1$, then Q_0 is globally asymptotically stable and when $\mathcal{R}_0 > 1$, the endemic equilibrium Q_1 is globally asymptotically stable. The theoretical results have been supported by numerical simulations.

Keywords: Chikungunya virus infection; Latency; Lyapunov function; Global stability.

1 Introduction

In recent past, many mathematicians have been presented and developed mathematical models in order to describe the interaction between viruses (such as HIV, HCV, HBV, HTLV and Chikungunya virus) and human cells (see e.g. [1]-[22]) Mathematical models of human viruses can lead to develop antiviral drugs and to understand the virus-host interaction. Moreover it can help to predict the disease progression. Studying the stability analysis of the models is also important to understand the behavior of the virus.

Chikungunya virus (CHIKV) is an alphavirus and is transmitted to humans by Aedes aegypti and Aedes albopictus mosquitos. In the CHIKV literature, most of the mathematical models have been presented to describe the disease transmission in mosquito and human populations (see e.g. [23]-[30]). However, only few works have devoted for mathematical modeling of the dynamics of the CHIKV within host. In 2017, Wang and Liu [22] have presented a mathematical model for in host CHIKV infection model as:

$$\dot{S} = \mu - dS - bSV, \tag{1}$$

$$\dot{I} = bSV - \epsilon I, \tag{2}$$

$$\dot{V} = mI - rV - qBV, \tag{3}$$

$$\dot{B} = \eta + cBV - \delta B, \tag{4}$$

where S, I, V, and B are the concentrations of uninfected monocytes, infected monocytes, CHIKV particles and B cells, respectively. Parameters d and μ represent the death rate and birth rate constants of the uninfected monocytes, respectively. The uninfected monocytes become infected at rate bSV, where b is rate constant of the CHIKV-target incidence. The infected monocytes and free CHIKV particles die are rates ϵI and rV, respectively. An actively infected monocytes produces an average number m of CHIKV particles. The CHIKV particles are attacked by the B cells at rate qVB. The B cells are produced at constant rate η, proliferated at rate cBV and die at rate δB.

In system (1)-(4) it is assumed that when the CHIKV contacts the uninfected monocytes it becomes infected and viral producer in the same time. However, this is unrealistic assumption. Therefore our objective in the present paper is to incorporate such delay by adding latently infected monocytes as another compartment to model (1)-(4). Moreover, we replace the bilinear incidence by saturated incidence which is suitable to model the nonlinear dynamics of the CHIKV especially when its concentration is high. We investigate the nonnegativity and boundedness of the solutions of the CHIKV dynamics model. We show that the CHIKV dynamics is governed by one bifurcation parameter (the basic reproduction numbers $\mathcal{R}_0$).We use Lyapunov direct method to establish the global stability of the model's steady states.

2 The CHIKV dynamics model

We cosider the following within-host CHIKV dynamics model with latently infected monocytes and saturated incidence rate:

$$\dot{S} = \mu - dS - \frac{bSV}{1+\pi V}, \tag{5}$$

$$\dot{L} = (1-p)\frac{bSV}{1+\pi V} - (\theta+\lambda)L, \tag{6}$$

$$\dot{I} = p\frac{bSV}{1+\pi V} + \lambda L - \epsilon I, \tag{7}$$

$$\dot{V} = mI - rV - qBV, \tag{8}$$

$$\dot{B} = \eta + cBV - \delta B, \tag{9}$$

where L is the concentration of latently infected monocytes, while I is the concentration of the actively infected monocytes. A fraction $(1-p)$ of infected monocytes is assumed to be latently infected monocytes and the remaining p becomes actively infected monocytes, where $0 < p < 1$. The latently infected monocytes are transmitted to actively infected monocytes at rate λL and die at rate θL.

3 Properties of solutions

The nonnegativity and boundedness of the solutions of model (5)-(9) are established in the following lemma:

Lemma 1.

There exist $M_1, M_2, M_3 > 0$, such that the following compact set is positively invariant for system (5)-(9)

$$\Phi = \{(S,L,I,V,B) \in \mathbb{R}^5_{\geq 0} : 0 \leq S, L, I \leq M_1, 0 \leq V \leq M_2, 0 \leq B \leq M_3\}$$

Proof. Since

$$\dot{S}\Big|_{S=0} = \mu > 0, \qquad \dot{L}\Big|_{L=0} = (1-p)\frac{bSV}{1+\pi V} \geq 0 \ \text{ for all } S,V \geq 0,$$

$$\dot{I}\Big|_{I=0} = p\frac{bSV}{1+\pi V} + \lambda L \geq 0 \ \text{ for all } S,V.L \geq 0, \qquad \dot{V}\Big|_{V=0} = mI \geq 0 \ \text{ for all } I \geq 0,$$

$$\dot{B}\Big|_{B=0} = \eta > 0.$$

Then, $\mathbb{R}^5_{\geq 0} = \{(x_1, x_2, ..., x_5,) \in \mathbb{R}, x_i \geq 0, i = 1, 2, ..., 5\}$ is positively invariant for system (5)-(9).

We consider

$$T_1(t) = S(t) + L(t) + I(t),$$
$$T_2(t) = V(t) + \frac{q}{c}B(t), \tag{10}$$

then from Eqs. (5)-(9) we get

$$\dot{T}_1(t) = \mu - dS - \theta L - \epsilon I \leq \mu - \sigma_1 T_1$$

where $\sigma_1 = \min\{d, \theta, \epsilon\}$. Hence $T_1(t) \leq M_1$, if $T_1(0) \leq M_1$, where $M_1 = \frac{\mu}{\sigma_1}$. The non-negativity of $S(t), L(t)$ and $I(t)$ implies that $0 \leq S(t), L(t), I(t) \leq M_1$ if $0 \leq S(0) + L(0) + I(0) \leq M_1$. Moreover, we have

$$\dot{T}_2(t) = mI - rV + \frac{q}{c}\eta - \frac{\delta q}{c}B \leq mM_1 + \frac{q}{c}\eta - \sigma_2(V + \frac{q}{c}B) = mM_1 + \frac{q}{c}\eta - \sigma_2 T_2,$$

where $\sigma_2 = \min\{r, \delta\}$. Hence $T_2(t) \leq M_2$, if $T_2(0) \leq M_2$, where $M_2 = \frac{mM_1 + \frac{q}{c}\eta}{\sigma_2}$. We have $V(t) \geq 0$ and $B(t) \geq 0$, therefore, $0 \leq V(t) \leq M_2$ and $0 \leq B(t) \leq M_3$ if $0 \leq V(0) + \frac{q}{c}B(0) \leq M_2$, where $M_3 = \frac{cM_2}{q}$.□

3.1 Steady States

System (5)-(9) always admits a virus-free steady state $Q_0 = (S_0, L_0, I_0, V_0, B_0) = (\frac{\mu}{d}, 0, 0, 0, \frac{\eta}{\delta})$. To calculate the other steady states we let the R.H.S of system (5)-(9) be equal zero

$$0 = \mu - dS - \frac{bSV}{1 + \pi V}, \tag{11}$$
$$0 = (1 - p)\frac{bSV}{1 + \pi V} - (\theta + \lambda)L, \tag{12}$$
$$0 = \frac{pbSV}{1 + \pi V} + \lambda L - \epsilon I, \tag{13}$$
$$0 = mI - rV - qVB, \tag{14}$$
$$0 = \eta + cBV - \delta B. \tag{15}$$

From Eq. (11)-(15) we obtain

$$S = \frac{\mu\,(1 + \pi V)}{bV + d\,(1 + \pi V)},\ L = \frac{(1 - p)bSV}{(1 + \pi V)\,(\theta + \lambda)},\ I = \frac{bSV(\lambda + \theta p)}{\epsilon\,(1 + \pi V)\,(\theta + \lambda)},\ B = \frac{\eta}{\delta - cV}. \tag{16}$$

Substituting Eq. (16) into Eq. (14) we have

$$\left[\frac{mpb\mu}{\epsilon(bV + d\,(1 + \pi V))} + \frac{m\lambda(1 - p)b\mu}{\epsilon(bV + d\,(1 + \pi V))\,(\theta + \lambda)} - r - \frac{q\eta}{\delta - cV}\right] V = 0.$$

If $V \neq 0$, then

$$P_1 V^2 - P_2 V + P_3 = 0,$$

where

$$P_1 = r\epsilon c(\theta + \lambda)(b + \pi d),$$
$$P_2 = -\epsilon rcd(\theta + \lambda) + mb\mu c(\lambda + \theta p) + (r\epsilon\delta)\,(\theta + \lambda)(b + \pi d) + (q\epsilon\eta)(\theta + \lambda)(b + \pi d),$$
$$P_3 = mb\mu\delta(\lambda + \theta p) - \epsilon d\,(r\delta + q\eta)\,(\theta + \lambda).$$

P_1, P_2 and P_3 can be re-written as:

$$\begin{aligned}
P_1 &= (r\epsilon c)\,(\theta+\lambda)(b+\pi d),\\
P_2 &= \frac{\epsilon c d\,(r\delta+q\eta)\,(\theta+\lambda)}{\delta}(\mathcal{R}_0-1)+(r\epsilon\delta)\,(\theta+\lambda)(b+\pi d)+(q\epsilon\eta)(\theta+\lambda)(b+\pi d)\\
&\quad+\frac{cd\,(q\epsilon\eta)\,(\theta+\lambda)}{\delta},\\
P_3 &= \epsilon d\,(r\delta+q\eta)\,(\theta+\lambda)(\mathcal{R}_0-1),
\end{aligned}$$

where

$$\mathcal{R}_0=\frac{bm\delta\mu(\lambda+\theta p)}{\epsilon d(r\delta+q\eta)(\theta+\lambda)}.$$

Let

$$F(V)=P_1V^2-P_2V+P_3=0. \tag{17}$$

If $\mathcal{R}_0>1$, then we have

$$\begin{aligned}
F(0) &= \epsilon d\,(r\delta+q\eta)\,(\theta+\lambda)(\mathcal{R}_0-1)>0,\\
F\left(\frac{\delta}{c}\right) &= -(q\epsilon\eta)(\theta+\lambda)\left(\frac{(b+\pi d)\delta}{c}+d\right)<0,
\end{aligned}$$

$$F'(0)=\frac{\epsilon c d\,(r\delta+q\eta)\,(\theta+\lambda)}{\delta}(1-\mathcal{R}_0)-(r\epsilon\delta)\,(\theta+\lambda)(b+\pi d)-(q\epsilon\eta)(\theta+\lambda)(b+\pi d)-\left(\frac{cd\,(q\epsilon\eta)\,(\theta+\lambda)}{\delta}\right)<0.$$

Then, Eq. (17) has two positive roots

$$V_1=\frac{P_2-\sqrt{P_2^2-4P_1P_3}}{2P_1}<\frac{\delta}{c}\quad\text{and}\quad V_2=\frac{P_2+\sqrt{P_2^2-4P_1P_3}}{2P_1}>\frac{\delta}{c}.$$

If $V=V_2$, then from Eq. (16) we get $B_2=\frac{\eta}{\delta-cV_2}<0$. Thus, if $\mathcal{R}_0>1$, then system (5)-(9) has a unique endemic steady state $Q_1=(S_1,L_1,I_1,V_1,B_1)$, where

$$\begin{aligned}
S_1 &= \frac{\mu\,(1+\pi V_1)}{bV_1+d\,(1+\pi V_1)},\ L_1=\frac{(1-p)b\mu V_1}{(\theta+\lambda)(bV_1+d\,(1+\pi V_1))},\ I_1=\frac{(\lambda+\theta p)b\mu V_1}{\epsilon(\theta+\lambda)(bV_1+d\,(1+\pi V_1))},\\
V_1 &= \frac{P_2-\sqrt{P_2^2-4P_1P_3}}{2P_1},\ B_1=\frac{\eta}{\delta-cV_1}.
\end{aligned}$$

Therefore, $\mathcal{R}_0$ represents the basic reproduction number of system (5)-(9).

Clearly $Q_0\in\Phi$. From Eqs. (11)-(13) we have

$$dS_1+\theta L_1+\epsilon I_1=\mu.$$

$$\Rightarrow S_1<\frac{\mu}{d}\le M_1,\ \ L_1<\frac{\mu}{\theta}\le M_1,\ \ I_1<\frac{\mu}{\epsilon}\le M_1.$$

Moreover, from Eqs. (14)-(15) we have

$$mI_1-rV_1-qV_1B_1+\frac{q}{c}(\eta+cB_1V_1-\delta B_1)=0$$

$$\begin{aligned}
&\Rightarrow rV_1+\frac{\delta q}{c}B_1=mI_1+\frac{q}{c}\eta<mM_1+\frac{q}{c}\eta\\
&\Rightarrow V_1<\frac{mM_1+\frac{q}{c}\eta}{r}\le M_2,\quad B_1<\frac{c}{q}\frac{mM_1+\frac{q}{c}\eta}{\delta}\le\frac{cM_2}{q}=M_3.
\end{aligned}$$

It follows that $Q_1\in\mathring{\Phi}$, where $\mathring{\Phi}$ is the interior of the set Φ.

3.2 Global stability

In the following theorems we establish the global stability of the two steady states of system (5)-(9) by constructing suitable Lyapunov functions. Let us define

$$H(x) = x - \ln x - 1.$$

Clearly, $H(x) \geq 0$ for $x > 0$ and $H(1) = 0$.

Theorem 1. Suppose that $\mathcal{R}_0 \leq 1$, then Q_0 is globally asymptotically stable (GAS) in Φ.

Proof. Construct a Lyapunov function W_0 as:

$$W_0(S,L,I,V,B) = S_0 H\left(\frac{S}{S_0}\right) + \frac{\lambda}{\lambda+\theta p} L + \frac{\theta+\lambda}{\lambda+\theta p} I + \frac{\epsilon(\theta+\lambda)}{m(\lambda+\theta p)} V + \frac{\epsilon q(\theta+\lambda)}{mc(\lambda+\theta p)} B_0 H\left(\frac{B}{B_0}\right). \tag{18}$$

Note that, $W_0(S,L,I,V,B) > 0$ for all $S,L,I,V,B > 0$ and $W_0(S_0,0,0,0,B_0) = 0$. Calculating $\frac{dW_0}{dt}$ along the trajectories of (5)-(9) we get

$$\begin{aligned}
\frac{dW_0}{dt} &= \left(1-\frac{S_0}{S}\right)\left(\mu - dS - \frac{bSV}{1+\pi V}\right) + \frac{\lambda}{\lambda+\theta p}\left((1-p)\frac{bSV}{1+\pi V} - (\theta+\lambda)L\right) \\
&\quad + \frac{\theta+\lambda}{\lambda+\theta p}\left(\frac{pbSV}{1+\pi V} + \lambda L - \epsilon I\right) + \frac{\epsilon(\theta+\lambda)}{m(\lambda+\theta p)}(mI - rV - qVB) \\
&\quad + \frac{\epsilon q(\theta+\lambda)}{mc(\lambda+\theta p)}\left(1-\frac{B_0}{B}\right)(\eta + cBV - \delta B) \\
&= -d\frac{(S-S_0)^2}{S} + \frac{bS_0V}{1+\pi V} - \frac{\epsilon(\theta+\lambda)rV}{m(\lambda+\theta p)} - \frac{\epsilon(\theta+\lambda)qB_0V}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)}{mc(\lambda+\theta p)}\left(1-\frac{B_0}{B}\right)(\delta B_0 - \delta B) \\
&= -d\frac{(S-S_0)^2}{S} - \frac{\epsilon q(\theta+\lambda)\delta}{mc(\lambda+\theta p)}\frac{(B-B_0)^2}{B} + \frac{\epsilon(r\delta+q\eta)(\theta+\lambda)}{m\delta(\lambda+\theta p)}\left(\frac{bm\delta\mu(\lambda+\theta p)}{\epsilon d(r\delta+q\eta)(\theta+\lambda)(1+\pi V)} - 1\right)V \\
&= -d\frac{(S-S_0)^2}{S} - \frac{\epsilon q(\theta+\lambda)\delta}{mc(\lambda+\theta p)}\frac{(B-B_0)^2}{B} - \frac{(r\epsilon\delta+q\epsilon\eta)(\theta+\lambda)\mathcal{R}_0\pi V^2}{m\delta(\lambda+\theta p)(1+\pi V)} + \frac{\epsilon(r\delta+q\eta)(\theta+\lambda)}{m\delta(\lambda+\theta p)}(\mathcal{R}_0-1)V.
\end{aligned} \tag{19}$$

Therefore, $\frac{dW_0}{dt} \leq 0$ holds if $\mathcal{R}_0 \leq 1$. Furthermore, $\frac{dW_0}{dt} = 0$ if and only if $S = S_0$, $B = B_0$, $V = 0$. The solutions of system (5)-(9) converge to Γ, the largest invariant set of $\{(S,L,I,V,B) : \frac{dW_0}{dt} = 0\}$. For any element in Γ satisfies $V(t) = \dot{V}(t) = 0$. Then from Eq. (8) we have $I(t) = 0$, and from Eq. (7) we get $L(t) = 0$. By the LaSalle's invariance principle, Q_0 is GAS. □

Theorem 2. Suppose that $\mathcal{R}_0 > 1$, then Q_1 is GAS in $\mathring{\Phi}$.

Proof. Construct a Lyapunov function

$$\begin{aligned}
W_1(S,L,I,V,B) &= S_1 H\left(\frac{S}{S_1}\right) + \frac{\lambda}{\lambda+\theta p} L_1 H\left(\frac{L}{L_1}\right) + \frac{\theta+\lambda}{\lambda+\theta p} I_1 H\left(\frac{I}{I_1}\right) \\
&\quad + \frac{\epsilon(\theta+\lambda)}{m(\lambda+\theta p)} V_1 H\left(\frac{V}{V_1}\right) + \frac{\epsilon q(\theta+\lambda)}{mc(\lambda+\theta p)} B_1 H\left(\frac{B}{B_1}\right).
\end{aligned}$$

We have $W_1(S,L,I,V,B) > 0$ for all $S,L,I,V,B > 0$ and $W_1(S_1,L_1,I_1,V_1,B_1) = 0$. Calculating $\frac{dW_1}{dt}$ along the trajectories of (5)-(9) we get

$$\begin{aligned}
\frac{dW_1}{dt} &= \left(1-\frac{S_1}{S}\right)\left(\mu - dS - \frac{bSV}{1+\pi V}\right) + \frac{\lambda}{\lambda+\theta p}\left(1-\frac{L_1}{L}\right)\left((1-p)\frac{bSV}{1+\pi V} - (\theta+\lambda)L\right) \\
&\quad + \frac{\theta+\lambda}{\lambda+\theta p}\left(1-\frac{I_1}{I}\right)\left(\frac{pbSV}{1+\pi V} + \lambda L - \epsilon I\right) + \frac{\epsilon(\theta+\lambda)}{m(\lambda+\theta p)}\left(1-\frac{V_1}{V}\right)(mI - rV - qVB) \\
&\quad + \frac{\epsilon q(\theta+\lambda)}{mc(\lambda+\theta p)}\left(1-\frac{B_1}{B}\right)(\eta + cBV - \delta B).
\end{aligned} \tag{20}$$

Applying

$$\mu = dS_1 + \frac{bS_1V_1}{1+\pi V_1},\ \eta = \delta B_1 - cB_1V_1,$$

we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= \left(1-\frac{S_1}{S}\right)(dS_1 - dS) + \frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right) + \frac{bS_1V}{1+\pi V} - \frac{\lambda(1-p)bSVL_1}{(\lambda+\theta p)(1+\pi V)L} \\ &+ \frac{\lambda(\theta+\lambda)L_1}{(\lambda+\theta p)} - \frac{(\theta+\lambda)pbSVI_1}{(\lambda+\theta p)(1+\pi V)I} - \frac{\lambda(\theta+\lambda)LI_1}{(\lambda+\theta p)I} + \frac{\epsilon(\theta+\lambda)I_1}{(\lambda+\theta p)} - \frac{\epsilon(\theta+\lambda)IV_1}{(\lambda+\theta p)V} \\ &- \frac{r\epsilon(\theta+\lambda)V}{m(\lambda+\theta p)} + \frac{r\epsilon(\theta+\lambda)V_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)BV_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)}{mc(\lambda+\theta p)}\left(1-\frac{B_1}{B}\right)(\delta B_1 - \delta B) \\ &- \frac{\epsilon q(\theta+\lambda)B_1V}{m(\lambda+\theta p)} - \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}\left(\frac{B_1}{B}\right).\end{aligned}$$

Using the steady state conditions for Q_1:

$$(1-p)\frac{bS_1V_1}{1+\pi V_1} = (\theta+\lambda)L_1,\ \frac{pbS_1V_1}{1+\pi V_1} + \lambda L_1 = \epsilon I_1,\ mI_1 = rV_1 + qB_1V_1,$$

we get

$$\begin{aligned}\frac{\epsilon(\theta+\lambda)I_1}{(\lambda+\theta p)} &= \frac{bS_1V_1}{1+\pi V_1} = \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}, \\ \frac{r\epsilon(\theta+\lambda)V_1}{m(\lambda+\theta p)} &= \frac{bS_1V_1}{1+\pi V_1} - \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}.\end{aligned}$$

and

$$\begin{aligned}\frac{dW_1}{dt} &= -d\frac{(S-S_1)^2}{S} + \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right) + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right) \\ &+ \frac{bS_1V_1}{1+\pi V_1}\left(\frac{(1+\pi V_1)V}{(1+\pi V)V_1} - \frac{V}{V_1}\right) - \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{1+\pi V_1}\frac{SVL_1(1+\pi V_1)}{S_1V_1L(1+\pi V)} \\ &+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} - \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{1+\pi V_1}\frac{SVI_1(1+\pi V_1)}{S_1V_1I(1+\pi V)} - \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{1+\pi V_1}\frac{I_1L}{L_1I} \\ &+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)} - \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} \\ &- \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} + \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)} + \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)} \\ &- \frac{2\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)BV_1}{m(\lambda+\theta p)} + \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}\left(\frac{B_1}{B}\right) - \frac{\epsilon q(\theta+\lambda)\delta}{mc(\lambda+\theta p)}\frac{(B-B_1)^2}{B}.\end{aligned} \tag{21}$$

Eq. Eq.(21) can be simplified as

$$\begin{aligned}\frac{dW_1}{dt} &= -d\frac{(S-S_1)^2}{S} + \frac{bS_1V_1}{1+\pi V_1}\left(-1 + \frac{(1+\pi V_1)V}{(1+\pi V)V_1} - \frac{V}{V_1} + \frac{1+\pi V}{1+\pi V_1}\right) \\ &+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)}\left[5 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVL_1}{(1+\pi V)S_1V_1L} - \frac{I_1L}{L_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right] \\ &+ \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}\left[4 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVI_1}{(1+\pi V)S_1V_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right] \\ &- \frac{\epsilon q(\theta+\lambda)\delta}{mc(\lambda+\theta p)}\frac{(B-B_1)^2}{B} - \frac{\epsilon q(\theta+\lambda)B_1V_1}{m(\lambda+\theta p)}\left[2 - \frac{B}{B_1} - \frac{B_1}{B}\right],\end{aligned}$$

and then

$$\begin{aligned}\frac{dW_1}{dt} &= -d\frac{(S-S_1)^2}{S} - \frac{\pi b S_1 (V-V_1)^2}{(1+\pi V)(1+\pi V_1)^2} - \frac{\epsilon q(\theta+\lambda)\eta}{mc(\lambda+\theta p)B_1}\frac{(B-B_1)^2}{B} \\ &+ \frac{\lambda(1-p)}{(\lambda+\theta p)}\frac{bS_1V_1}{(1+\pi V_1)}\left[5 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVL_1}{(1+\pi V)S_1V_1L} - \frac{LI_1}{L_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right] \\ &+ \frac{(\theta+\lambda)}{(\lambda+\theta p)}\frac{pbS_1V_1}{(1+\pi V_1)}\left[4 - \frac{S_1}{S} - \frac{(1+\pi V_1)SVI_1}{(1+\pi V)S_1V_1I} - \frac{IV_1}{I_1V} - \frac{1+\pi V}{1+\pi V_1}\right]. \end{aligned} \tag{22}$$

The relation between the geometrical mean and the arithmetical mean implies that

$$\begin{aligned} 5 &\leq \frac{S_1}{S} + \frac{(1+\pi V_1)SVL_1}{(1+\pi V)S_1V_1L} + \frac{LI_1}{L_1I} + \frac{IV_1}{I_1V} + \frac{1+\pi V}{1+\pi V_1}, \\ 4 &\leq \frac{S_1}{S} + \frac{(1+\pi V_1)SVI_1}{(1+\pi V)S_1V_1I} + \frac{IV_1}{I_1V} + \frac{1+\pi V}{1+\pi V_1}. \end{aligned}$$

Then $\frac{dW_1}{dt} \leq 0$ and $\frac{dW_1}{dt} = 0$ if and only if $S = S_1$, $L = L_1$, $I = I_1$, $V = V_1$ and $B = B_1$. It follows from LaSalle's invariance principle, Q_1 is GAS in $\mathring{\Phi}$. $\square$

4 Numerical simulations

In order to illustrate our theoretical results, we perform numerical simulations for system (5)-(9) with parameters values given in Table 1. In the figures we show the evolution of the five states of the system S, L, I, V and B. We have used MATLAB for all computations.

• **Effect of** b **on the stability of steady states**: To show the global stability results we consider three different initial conditions as:

IC1: $S(0) = 2.0, L(0) = 0.2, I(0) = 0.4, V(0) = 0.4$ and $B(0) = 1.0$,

IC2: $S(0) = 1.7, L(0) = 0.4, I(0) = 0.6, V(0) = 0.6$ and $B(0) = 1.6$,

IC3: $S(0) = 1.4, L(0) = 0.6, I(0) = 0.8, V(0) = 0.8$ and $B(0) = 2.4$.

We fix the value $p = 0.5$ and consider two sets of the values of parameter b as follows:

Set (I): We choose $b = 0.1$. Using these data, we compute $\mathcal{R}_0 = 0.5469 < 1$, then the system has one steady state Q_0. From Figures 1-5 we can see that, the concentrations of the uninfected monocytes and B cells return to their values $S_0 = \frac{\mu}{d} = 2.2885$ and $B_0 = \frac{\eta}{\delta} = 1.1207$, respectively. On the other hand, the concentrations of latently infected monocytes, actively infected monocytes and CHIKV particles are decaying and approaching zero for all the three initial conditions IC1-IC3. It means that, Q_0 is GAS and the CHIKV will be removed. This result support the result of Theorem 1.

Set (II): We take $b = 0.5$. Then, we calculate $\mathcal{R}_0 = 2.7347 > 1$. Then the system has two positive steady states Q_0 and Q_1. It is clear from Figures 1-5 that, both the numerical results and the theoretical results given in Theorem 2 are consistent. It is seen that, the solutions of the system converge to the steady state $Q_1 = (1.67881, 0.405396, 0.638994, 0.6152, 2.77721)$ for all the three initial conditions IC1-IC3.

Table 1: The value of the parameters of model (5)-(9).

Parameter	Value	Parameter	Parameter
μ	1.826	m	2.02
π	varied	q	0.5964
c	1.2129	r	0.4418
d	0.79791	η	1.402
θ	0.5	δ	1.251
λ	0.1	b	varied
ϵ	0.4441	p	varied

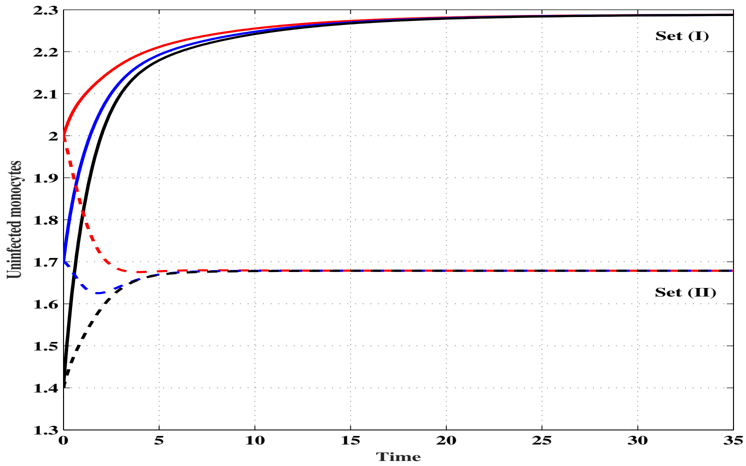

Figure 1: The Evolution of uninfected monocytes.

• **Effect of the saturation infection on the CHIKV dynamics**

In this case, we fix the values $p = 0.5$ and $b = 0.5$. We note that, the value of $\mathcal{R}_0$ does not depend on the value of the saturation parameter π. This means that, saturation can play a significant role in reducing the infection progress but do not play a role in clearing the CHIKV from the body. The simulation were performed using the initial condition IC2. Figures 6-10 show the effect of saturation infection. We observe that, as π is increased, the incidence rate of infection is decreased, and then the concentration of the uninfected monocytes are increased, while the concentrations of latently infected monocytes, actively infected monocytes, free CHIKV particles and B cells are decreased.

• **Effect of p on the basic reproduction number:**

In this case we take $\pi = 0.1$ and $b = 0.3$. From Figure 11, we can observed that as p is increased then $\mathcal{R}_0$ is increased. Let p^{cr} be the critical value of the parameter p, such that

$$\mathcal{R}_0 = \frac{bm\delta\mu(\theta p^{cr} + \lambda)}{\epsilon d(r\delta + q\eta)(\theta + \lambda)} = 1.$$

Using the data given in Table 1 we obtain $p^{cr} = 0.226612$, and we get the following:

(i) $0 < p \leq 0.226612$. Then the trajectory of the system will converge to Q_0 and this will suppress the CHIKV replication and clear the CHIKV from the body.

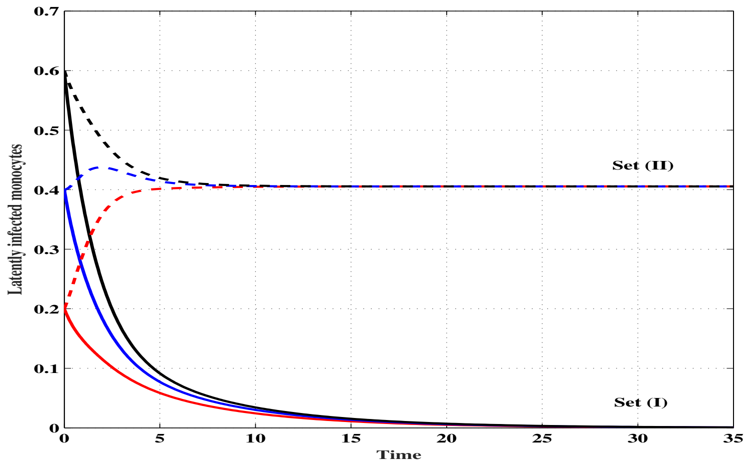

Figure 2: The Evolution of latently infected monocytes.

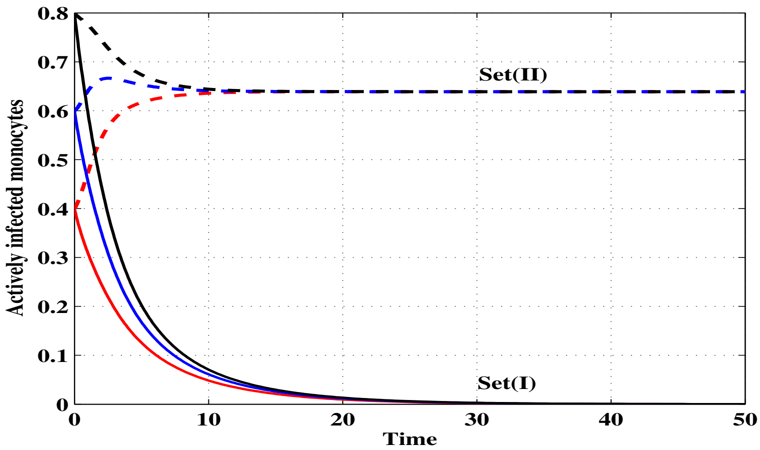

Figure 3: The Evolution of actively infected monocytes.

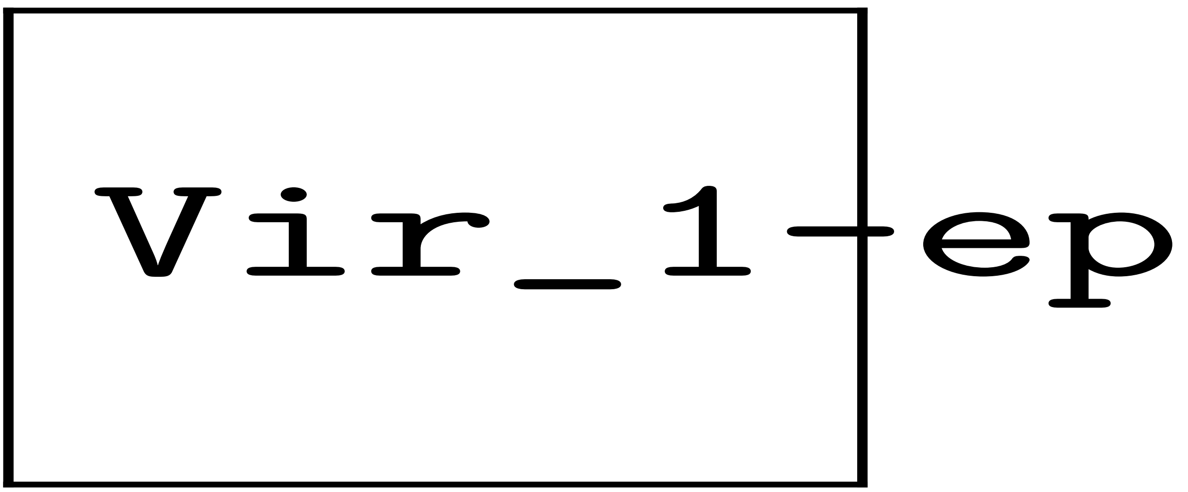

Figure 4: The Evolution of free CHIKV particles.

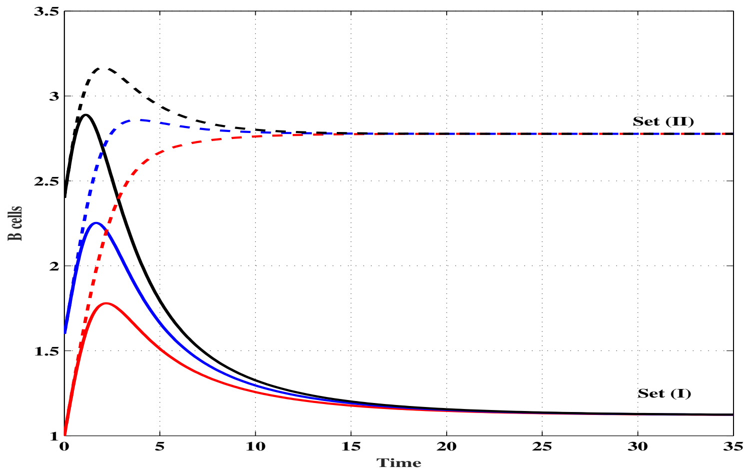

Figure 5: The Evolution of B cells.

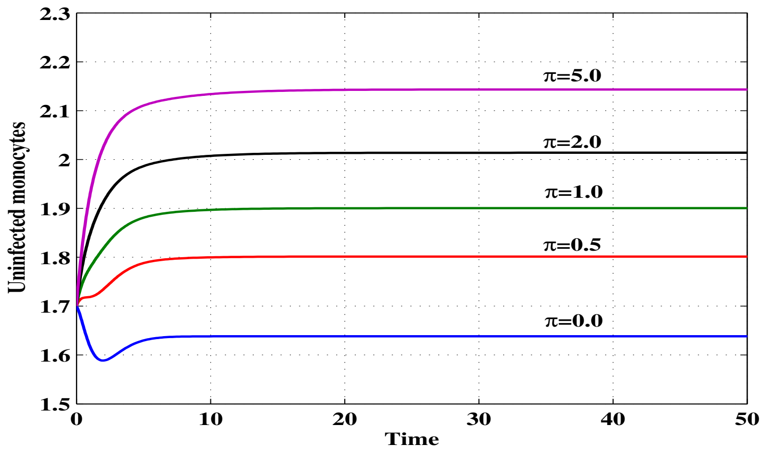

Figure 6: The concentration of uninfected monocytes.

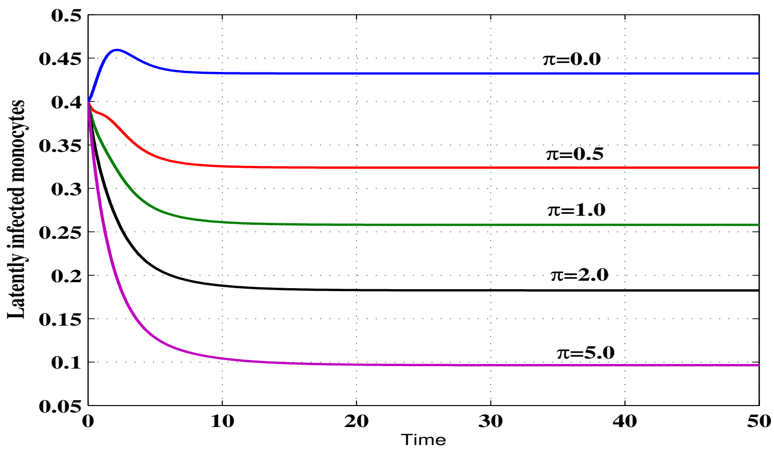

Figure 7: The concentration of latently infected monocytes.

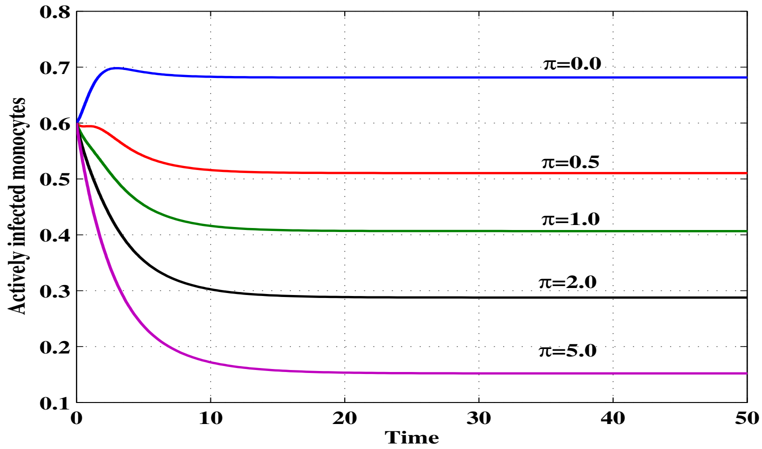

Figure 8: The concentration of actively infected monocytes.

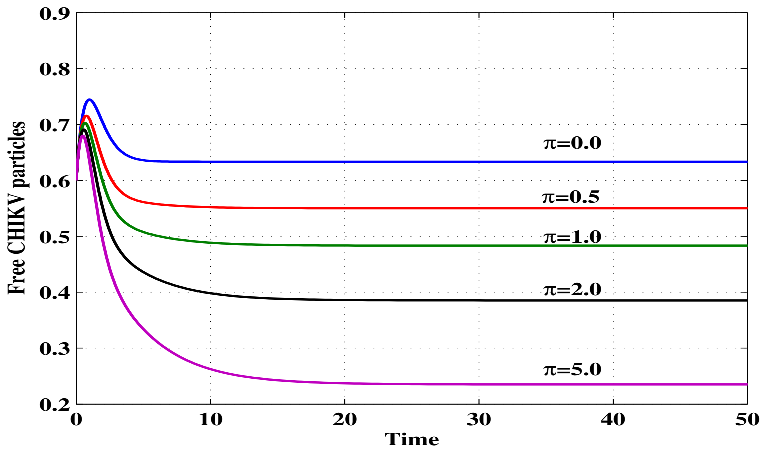

Figure 9: The concentration of free CHIKV particles.

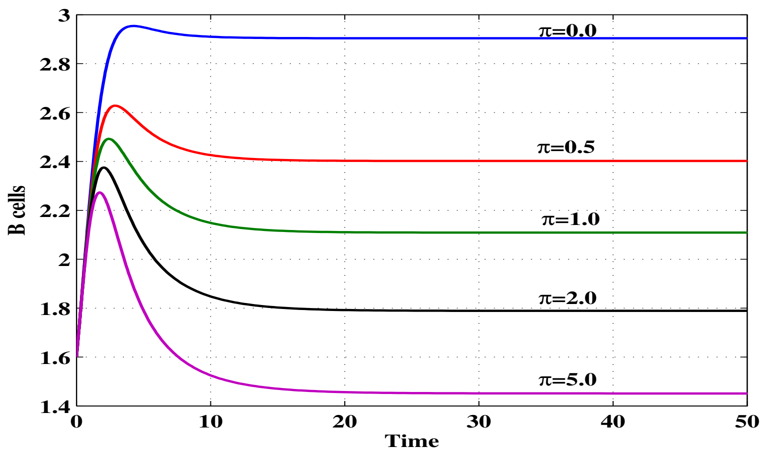

Figure 10: The concentration of B cells.

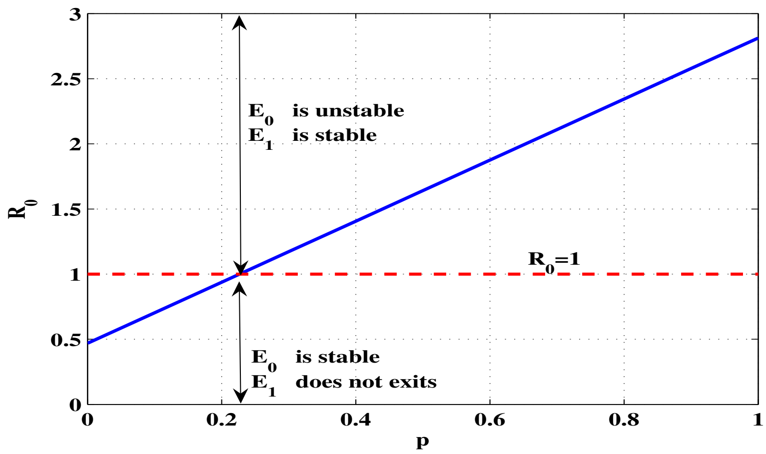

Figure 11: Effect of p on the basic reproduction number.

(ii) $0.226612 < p < 1$. Then the trajectory will converge to Q_1 and then the infection will be chronic. It means that, the factor $1-p$ plays the role of a controller which can be applied to stabilize the system around Q_0. From a biological point of view, the factor $1-p$ plays a similar role as the drug dose of antiviral treatment which can be used to eliminate the CHIKV. We observe that, sufficiently small p will suppress the CHIKV replication and clear the CHIKV. This gives us some suggestions on new drugs to decrease the fraction p.

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Quotient B-algebras induced by an int-soft normal subalgebra

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Abstract. The notions of an intersectional soft subalgebra and an intersectional soft normal subalgebra of a B-algebra are introduced, and related properties are investigated. A quotient structure of a B-algebra using an intersectional soft normal subalgebra is constructed. The fundamental homomorphism of a quotient B-algebra is established.

1. Introduction

Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [10] described the application of soft set theory to a decision making problem. Jun [5] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [3, 4, 14] for further information regarding algebraic structures/properties of soft set theory. On the while, Y. B. Jun, E. H. Roh and H. S. Kim [6] introduced a new notion, called a BH-algebra. J. Neggers and H. S. Kim [12] introduced a new notion, called a B-algebra. C. B. Kim and H. S. Kim [8] introduced the notion of a BG-algebra which is a generalization of B-algebras. S. S. Ahn and H. D. Lee [1] classified the subalgebras by their family of level subalgebras in BG-algebras.

In this paper, we discuss applications of the an intersectional soft set in a (normal) subalgebra of a B-algebra. We introduce the notion of an intersectional (normal) soft subalgebra of a B-algebra, and investigated related properties. We consider a new construction of a quotient B-algebra induced by an int-soft normal subalgebra. Also we establish the fundamental homomorphism of a quotient B-algebra.

2. Preliminaries

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A B-*algebra* ([12]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying axioms:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B) $(x * y) * z = x * (z * (0 * y))$

for any x, y, z in X. For brevity we call X a B-*algebra*. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BH-*algebra* if it satisfies (B1), (B2) and

(BH) $x * y = y * x = 0$ imply $x = y$ for any $x, y \in X$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BG-*algebra* if it satisfies (B1), (B2) and

(BG) $(x * y) * (0 * y) = x$ for any $x, y \in X$.

Proposition 2.1.([2, 12]) *Let $(X; *, 0)$ be a B-algebra. Then*

(i) *the left cancellation law holds in X, i.e., $x * y = x * z$ implies $y = z$,*
(ii) *if $x * y = 0$, then $x = y$ for any $x, y \in X$,*
(iii) *if $0 * x = 0 * y$, then $x = y$ for any $x, y \in X$,*
(iv) *$0 * (0 * x) = x$, for all $x \in X$,*
(v) *$x * (y * z) = (x * (0 * z)) * y$ for all $x, y, z \in X$.*

Theorem 2.2.([8]) *If $(X; *, 0)$ is a B-algebra, then it is a BG-algebra.*

Proposition 2.3.([8]) *Every BG-algebra is a BH-algebra.*

Let $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ be B-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for any $x, y \in X$. A homomorphism $\varphi : X \to Y$ is called an *isomorphism* if φ is a bijection, and denote it by $X \cong Y$. Let $\varphi : X \to Y$ be a homomorphism. Then the subset $\{x \in X | \varphi(x) = 0_Y\}$ of X is called the *kernel* of the homomorphism φ, and denote it by $Ker\ \varphi$. A non-empty subset S of X is called a *subalgebra* of X if $x * y \in S$ for any $x, y \in X$.

A non-empty subset N of X is said to be *normal* if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. Then any normal subset N of a B-algebra X is a subalgebra of X, but the converse need not be true ([13]). A non-empty subset X of a B-algebra X is a called a *normal subalgebra* of X if it is both a subalgebra and normal.

Let X be a B-algebra and let N be a normal subalgebra of X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if $x * y \in N$, where $x, y \in X$. Then it is a congruence relation on X ([13]). Denote the equivalence class containing x by $[x]_N$, i.e., $[x]_N := \{y \in X | x \sim_N y\}$ and let $X/N := \{[x]_N | x \in X\}$.

Theorem 2.4.([13]) *Let N be a normal subalgebra of a B-algebra X. Then X/N is a B-algebra.*

The B-algebra X/N is discussed in Theorem 2.4 is called the *quotient B-algebra* of X by N.

Theorem 2.5.([13]) *Let N be a normal subalgebra of a B-algebra X. Then the mapping $\gamma : X \to X/N$ given by $\gamma(x) := [x]_N$ is a surjective homomorphism, and $Ker\gamma = N$.*

Theorem 2.6.([13]) *Let $\varphi : X \to Y$ be a homomorphism of B-algebras. Then $Ker\varphi$ is a normal subalgebra of X.*

Theorem 2.7.([13]) *Let $\varphi : X \to Y$ be a homomorphism of B-algebras. Then $X/Ker\varphi \cong Im\varphi$. In particular, if φ is surjective, then $X/Ker\varphi \cong Y$.*

Molodtsov [12] defined the soft set in the following way: Let U be an initial universe set and let E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

A fair $(\tilde{f}, A)$ is called a *soft set* over U, where $\tilde{f}$ is a mapping given by $\tilde{f} : X \to \mathscr{P}(U)$. In other words, a soft set over U is parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the set $(\tilde{f}, A)$. A soft set over U can be represented by the set of ordered pairs:

$$(\tilde{f}, A) = \{(x, \tilde{f}(x)) | x \in A, \tilde{f}(x) \in \mathscr{P}(U)\},$$

where $\tilde{f} : X \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$. Clearly, a soft set is not a set.

For a soft set $(\tilde{f}, A)$ of X and a subset γ of U, the *γ-inclusive set* of $(\tilde{f}, A)$, defined to be the set

$$i_A(\tilde{f}; \gamma) := \{x \in A | \gamma \subseteq \tilde{f}(x)\}.$$

3. Int-soft subalgebra

In what follows let X denote a B-algebra X unless otherwise specified.

Definition 3.1. A soft set $(\tilde{f}, X)$ over U is called an *intersectional soft subalgebra* (briefly, *int-soft subalgebra*) of a B-algebra X if it satisfies:

(3.1) $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$ for all $x, y \in X$.

Proposition 3.2. *Every int-soft subalgebra $(\tilde{f}, X)$ of a B-algebra X satisfies the following inclusion:*

(3.2) *$\tilde{f}(x) \subseteq \tilde{f}(0)$ for all $x \in X$.*

Proof. Using (3.1) and (B1), we have $\tilde{f}(x) = \tilde{f}(x) \cap \tilde{f}(x) \subseteq \tilde{f}(x * x) = \tilde{f}(0)$ for all $x \in X$. □

Example 3.3. Let $(U = \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a B-algebra ([9]) with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Let $(\tilde{f}, X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0, \\ 3\mathbb{Z} & \text{if } x = 3, \\ 9\mathbb{Z} & \text{if } x \in \{1, 2\}. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over U.

Theorem 3.4. *A soft set $(\tilde{f}, X)$ of a B-algebra X over U is an int-soft subalgebra of X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is a subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

Proof. Assume that $(\tilde{f}, X)$ is an int-soft subalgebra over U. Let $x, y \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x, y \in i_X(\tilde{f}; \gamma)$. Then $\gamma \subseteq \tilde{f}(x)$ and $\gamma \subseteq \tilde{f}(y)$. It follows from (3.1) that $\gamma \subseteq \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$ Hence $x * y \in i_X(\tilde{f}; \gamma)$. Thus $i_X(\tilde{f}; \gamma)$ is a subalgebra of X.

Conversely, suppose that $i_X(\tilde{f}; \gamma)$ is a subalgebra X for all $\gamma \in \mathscr{P}(U)$ with $i_X(; \gamma) \neq \emptyset$. Let $x, y \in X$, be such that $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in i_X(\tilde{f}; \gamma)$ and so $x * y \in i_X(\tilde{f}; \gamma)$ by assumption. Hence $\tilde{f}(x) \cap \tilde{f}(y) = \gamma_x \cap \gamma_y = \gamma \subseteq \tilde{f}(x * y)$. Thus $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

Theorem 3.5. *Every subalgebra of a B-algebra can be represented as a γ-inclusive set of an int-soft subalgebra.*

Proof. Let A be a subalgebra of a B-algebra X. For a subset γ of U, define a soft set $(\tilde{f}, X)$ over U by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Obviously, $A = i_X(\tilde{f}; \gamma)$. We now prove that $(\tilde{f}; \gamma)$ is an int-soft subalgebra over U. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$ because A is a subalgebra of X. Hence $\tilde{f}(x) = \tilde{f}(y) = \tilde{f}(x * y) = \gamma$, and so $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. If $x \in A$ and $y \notin A$, then $\tilde{f}(x) = \gamma$ and $\tilde{f}(y) = \emptyset$ which imply that $\tilde{f}(x) \cap \tilde{f}(y) = \gamma \cap \emptyset = \emptyset \subseteq \tilde{f}(x * y)$. Similarly, if $x \notin A$ and $y \in A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Obviously, if $x \notin A$ and $y \notin A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Therefore $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

Any subalgebra of a B-algebra X may not be represented as a γ-inclusive set of an int-soft subalgebra $(\tilde{f}, X)$ over U in general (see Example 3.6).

Example 3.6. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a B-algebra with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Consider a soft set $(\tilde{f}, X)$ which is given by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{0, 2\} & \text{if } x = 0, \\ \{2\} & \text{if } x \in \{1, 2, 3\}. \end{cases}$$

It is easy to show that $(\tilde{f}, X)$ is an int-soft subalgebra over U. The γ-inclusive set of $(\tilde{f}, X)$ are described as follows:

$$i_X(\tilde{f}; \gamma) = \begin{cases} X & \text{if } \gamma \in \{\emptyset, \{2\}\}, \\ \{0\} & \text{if } \gamma \in \{\{0\}, \{0, 2\}\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 1\}$ cannot be a γ-inclusive set $i_X(\tilde{f}; \gamma)$ since there is no $\gamma \subseteq U$ such that $i_X(\tilde{f}; \gamma) = \{0, 1\}$.

Definition 3.7. A soft set $(\tilde{f}, X)$ over U is said to be *intersectional soft normal* (briefly, *int-soft normal*) of a B-algebra X if it satisfies:

(3.3) $\tilde{f}(x * y) \cap \tilde{f}(a * b) \subseteq \tilde{f}((x * a) * (y * b))$ for all $x, y, a, b \in X$.

A soft set $(\tilde{f}, X)$ over U is called an *intersectional soft normal subalgebra* (briefly, *int-soft normal subalgebra*) of a B-algebra X if it satisfies (3.1) and (3.3).

Example 3.8. Let $(U = \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a B-algebra as in Example 3.3. Let $(\tilde{f}, X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 3\}, \\ 7\mathbb{Z} & \text{if } x \in \{1, 2\}. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is int-soft normal over U.

Proposition 3.9. *Every int-soft normal $(\tilde{f}, X)$ of a B-algebra X is an int-soft subalgebra of X.*

Proof. Put $y := 0, b := 0$ and $a := y$ in (3.3). Then $\tilde{f}(x * 0) \cap \tilde{f}(y * 0) \subseteq \tilde{f}((x * y) * (0 * 0))$ for any $x, y \in X$. Using (B2) and (B1), we have $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Hence $(\tilde{f}, X)$ is an int-soft subalgebra of X. □

The converse of Proposition 3.9 may not be true in general (see Example 3.10).

Example 3.10. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set, where $X = \{0, 1, 2, 3, 4, 5\}$ is a B-algebra ([13]) with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Let $(\tilde{f}, X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = 5, \\ \gamma_1 & \text{if } x \in \{1, 2, 3, 4\}. \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over U. But it is not int-soft normal over U since $\tilde{f}(1 * 4) \cap \tilde{f}(3 * 2) = \tilde{f}(5) \cap \tilde{f}(5) = \gamma_2 \nsubseteq \gamma_1 = \tilde{f}(1) = \tilde{f}((1 * 3) * (4 * 2))$.

Theorem 3.11. *A soft set $(\tilde{f}, X)$ of a B-algebra X over U is an int-soft normal subalgebra of X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is a normal subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

Proof. Similar to Theorem 3.4. □

Proposition 3.12. *Let a soft set $(\tilde{f}, X)$ over U of a B-algebra X be int-soft normal. Then $\tilde{f}(x * y) = \tilde{f}(y * x)$ for any $x, y \in X$.*

Proof. Let $x, y \in X$. By (B1) and (B2), we have $\tilde{f}(x*y) = \tilde{f}((x*y)*(x*x)) \supseteq \tilde{f}(x*x) \cap \tilde{f}(y*x) = \tilde{f}(0) \cap \tilde{f}(y * x) = \tilde{f}(y * x)$. Interchanging x with y, we obtain $\tilde{f}(y * x) \supseteq \tilde{f}(x * y)$, which proves the proposition. □

Theorem 3.13. *Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a B-algebra X. Then the set*

$$X_{\tilde{f}} = \{x \in X | \tilde{f}(x) = \tilde{f}(0)\}$$

is a normal subalgebra of X.

Proof. It is sufficient to show that $X_{\tilde{f}}$ is normal. Let $a, b, x, y \in X$ be such that $x * y \in X_{\tilde{f}}$ and $a * b \in X_{\tilde{f}}$. Then $\tilde{f}(x * y) = \tilde{f}(0) = \tilde{f}(a * b)$. Since $(\tilde{f}, X)$ is an int-soft normal subalgebra of X, we have $\tilde{f}((x * a) * (y * b)) \supseteq \tilde{f}(x * y) \cap \tilde{f}(a * b) = \tilde{f}(0)$. Using (3.2), we conclude that $\tilde{f}((x * a) * (y * b)) = \tilde{f}(0)$. Hence $(x * a) * (y * b) \in X_{\tilde{f}}$. This completes the proof. □

Theorem 3.14. *The intersection of any set of an int-soft normal subalgebra of a B-algebra X is also an int-soft normal subalgebra.*

Proof. Let $\{\tilde{f}_\alpha | \alpha \in \Lambda\}$ be a family of int-soft normal subalgebras of a B-algebra X and let $a, b, x, y \in X$. Then

$$\begin{aligned}\cap_{\alpha\in\Lambda}\tilde{f}_\alpha((x * a) * (y * b)) =& \inf_{\alpha\in\Lambda} \tilde{f}_\alpha((x * a) * (y * b))\\ \geq& \inf_{\alpha\in\Lambda}\{\tilde{f}_\alpha(x * y) \cap \tilde{f}_\alpha(a * b)\}\\ =&[\inf_{\alpha\in\Lambda} \tilde{f}_\alpha(x * y)] \cap [\inf_{\alpha\in\Lambda} \tilde{f}_\alpha(a * b)]\\ =&((\cap_{\alpha\in\Lambda}\tilde{f}_\alpha)(x * y)) \cap ((\cap_{\alpha\in\Lambda}\tilde{f}_\alpha)(a * b))\end{aligned}$$

which shows that $\cap_{\alpha\in\Lambda}\tilde{f}_\alpha$ is int-soft normal. By Proposition 3.9, $\cap_{\alpha\in\Lambda}\tilde{f}_\alpha$ is an int-soft normal subalgebra of X. □

The union of any set of int-soft normal subalgebra of a B-algebra X need not be an int-soft normal subalgebra of X.

Example 3.15. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a B-algebra as in Example 3.10. Let $(\tilde{f}, X)$ and $(\tilde{g}, X)$ be soft sets over $U := \mathbb{Z}$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 4\}, \\ 7\mathbb{Z} & \text{if } x \in \{1, 2, 3, 5\}, \end{cases}$$

and

$$\tilde{g} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 5\}, \\ 2\mathbb{Z} & \text{if } x \in \{1, 2, 3, 4\}. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ and $(\tilde{g}, X)$ are int-soft subalgebras over U. But $\tilde{f} \cup \tilde{g}$ is not an int-soft subalgebra of X because

$$\begin{aligned}(\tilde{f} \cup \tilde{g})(4) \cap (\tilde{f} \cup \tilde{g})(5) =& (\tilde{f}(4) \cup \tilde{g}(4)) \cap (\tilde{f}(5) \cup \tilde{g}(5))\\ =& (\mathbb{Z} \cup 2\mathbb{Z}) \cap (7\mathbb{Z} \cup \mathbb{Z}) = \mathbb{Z}\\ \not\subseteq& 7\mathbb{Z} \cup 2\mathbb{Z} = \tilde{f}(2) \cup \tilde{g}(2)\\ =& (\tilde{f} \cup \tilde{g})(2) = (\tilde{f} \cup \tilde{g})(4 * 5).\end{aligned}$$

Since every int-soft normal of a B-algebra X is an int-soft subalgebra of X, the union of int-soft normal subalgebra need not be an int-soft normal subalgebra of a B-algebra.

4. Quotient B-algebras induced by an int-soft normal subalgebra

Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a B-algebra X. For any $x, y \in X$, we define a binary operation " $\sim^{\tilde{f}}$ " on X as follows:

$$x \sim^{\tilde{f}} y \Leftrightarrow \tilde{f}(x * y) = \tilde{f}(0).$$

Lemma 4.1. *The operation " $\sim^{\tilde{f}}$ " is an equivalence relation on a B-algebra X.*

Proof. Obviously, it is reflexive. Let $x \sim^{\tilde{f}} y$. Then $\tilde{f}(x * y) = \tilde{f}(0)$. It follows from Proposition 3.12 that $\tilde{f}(0) = \tilde{f}(x*y) = \tilde{f}(y*x)$. Hence $\sim^{\tilde{f}}$ is symmetric. Let $x, y, z \in X$ be such that $x \sim^{\tilde{f}} y$ and $y \sim^{\tilde{f}} z$. Then $\tilde{f}(x * y) = \tilde{f}(0)$ and $\tilde{f}(y * z) = \tilde{f}(0)$. Using Proposition 3.12, (3.3), (B1), (B2) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) = \tilde{f}(x * y) \cap \tilde{f}(y * z) = \tilde{f}(x * y) \cap \tilde{f}(z * y) \\ \subseteq \tilde{f}((x * z) * (y * y)) \\ = \tilde{f}((x * z) * 0) = \tilde{f}(x * z) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}(x*z) = \tilde{f}(0)$, i.e., $\sim^{\tilde{f}}$ is transitive. Therefore " $\sim^{\tilde{f}}$ " is an equivalence relation on X. □

Lemma 4.2. *For any $x, y, p, q \in X$, if $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$, then $x * p \sim^{\tilde{f}} y * q$.*

Proof. Let $x, y, p, q \in X$ be such that $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$. Then $\tilde{f}(x * y) = \tilde{f}(y * x) = \tilde{f}(0)$ and $\tilde{f}(p * q) = \tilde{f}(q * p) = \tilde{f}(0)$. Using (3.3) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) = &\tilde{f}(x * y) \cap \tilde{f}(p * q) \\ \subseteq &\tilde{f}((x * p) * (y * q)) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}((x * p) * (y * q)) = \tilde{f}(0)$. By similar way, we get $\tilde{f}((y * q) * (x * p)) = \tilde{f}(0)$. Therefore $x * p \sim^{\tilde{f}} y * q$. Thus " $\sim^{\tilde{f}}$ " is a congruence relation on X. □

Denote by $\tilde{f}_x$ and $X/\tilde{f}$ the equivalent class containing x and the set of all equivalent classes of X, respectively, i.e.,

$$\tilde{f}_x := \{y \in X | y \sim^{\tilde{f}} x\} \text{ and } X/\tilde{f} := \{\tilde{f}_x | x \in X\}.$$

Define a binary relation $\bullet$ on $X/\tilde{f}$ as follows:

$$\tilde{f}_x \bullet \tilde{f}_y := \tilde{f}_{x*y}$$

for all $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$. Then this operation is well-defined by Lemma 4.2.

Theorem 4.3. *If $(\tilde{f}, X)$ is an int-soft normal subalgebra of a B-algebra X, then the quotient $X/\tilde{f} := (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a B-algebra.*

Proof. Straightforward. □

Proposition 4.4. *Let $\mu : X \to Y$ be a homomorphism of B-algebras. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of Y, then $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of X.*

Proof. For any $x, y, a, b \in X$, we have

$$\begin{aligned}(\tilde{f} \circ \mu)((x * a) * (y * b)) =& \tilde{f}(\mu((x * a) * (y * b)))\\ =& \tilde{f}((\mu(x) * \mu(a)) * (\mu(y) * \mu(b)))\\ \supseteq& \tilde{f}(\mu(x) * \mu(y)) \cap \tilde{f}(\mu(a) * \mu(b))\\ =& \tilde{f}(\mu(x * y)) \cap \tilde{f}(\mu(a * b))\\ =& (\tilde{f} \circ \mu)(x * y) \cap (\tilde{f} \circ \mu)(a * b).\end{aligned}$$

Hence $\tilde{f} \circ \mu$ is int-soft normal. By Proposition 3.9, $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of X. □

Proposition 4.5. *Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a B-algebra X. The mapping $\gamma : X \to X/\tilde{f}$, given by $\gamma(x) := \tilde{f}_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = \tilde{f}_0\} = X_{\tilde{f}}$.*

Proof. Let $\tilde{f}_x \in X/\tilde{f}$. Then there exists an element $x \in X$ such that $\gamma(x) = \tilde{f}_x$. Hence γ is surjective. For any $x, y \in X$, we have

$$\gamma(x * y) = \tilde{f}_{x*y} = \tilde{f}_x \bullet \tilde{f}_y = \gamma(x) \bullet \gamma(y).$$

Thus γ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = \tilde{f}_0\} = \{x \in X | x \sim^{\tilde{f}} 0\} = \{x \in X | \tilde{f}(x) = \tilde{f}(0)\} = X_{\tilde{f}}$. □

Example 4.6. Let $E = X$ be the set of parameters, and let $U := \mathbb{Z}$ be the initial universe set where $X = \{0, 1, 2, 3\}$ is a B-algebra ([7]) with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Let $(\tilde{f}, X)$ be a soft set over $U := \mathbb{Z}$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, 2\}, \\ 5\mathbb{Z} & \text{if } x \in \{1, 3\}. \end{cases}$$

It is easy to show that $X_{\tilde{f}} = \{x \in X | \tilde{f}(x) = \tilde{f}(0)\} = \{0, 2\}$. Define $x \sim^{\tilde{f}} y$ if and only if $\tilde{f}(x * y) = \tilde{f}(0)$. Then $\tilde{f}_0 = \{x \in X | x \sim^{\tilde{f}} 0\} = \{x \in X | \tilde{f}(x * 0) = \tilde{f}(0)\} = \{0, 2\}$ and $\tilde{f}_1 = \{x \in$

$X|x \sim^{\tilde{f}} 1\} = \{x \in X|\tilde{f}(x*1) = \tilde{f}(0)\} = \{1, 3\}$ Hence $X/\tilde{f} = \{\tilde{f}_0, \tilde{f}_1\}$. Let $\varphi : X \to X/\tilde{f}$ be a map defined by $\varphi(0) = \varphi(2) = \tilde{f}_0$ and $\varphi(1) = \varphi(3) = \tilde{f}_1$. It is easy to check that φ is a homomorphism and $Ker\varphi = \{x \in X|\varphi(x) = \tilde{f}_0\} = \{x \in X|x \sim^{\tilde{f}} 0\} = \{x \in X|\tilde{f}(x) = \tilde{f}(0)\} = X_{\tilde{f}}$.

Theorem 4.7. *Let $X := (X; *_X, 0_X)$ be a B-algebra and $Y := (Y; *_Y, 0_Y)$ be a B-algebra and let $\mu : X \to Y$ be an epimorphism. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of Y, then the quotient algebra $X/(\tilde{f} \circ \mu) := (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$.*

Proof. By Theorem 4.3 and Proposition 4.4, $X/\tilde{f} \circ \mu : (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is a B-algebra and $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$ is a B-algebra. Define a map

$$\eta : X/(\tilde{f} \circ \mu) \to Y/\tilde{f},\ (\tilde{f} \circ \mu)_x \mapsto \tilde{f}_{\mu(x)}$$

for all $x \in X$. Then the function η is well-defined. In fact, assume that $(\tilde{f} \circ \mu)_x = (\tilde{f} \circ \mu)_y$ for all $x, y \in X$. Then we have

$$\begin{aligned}\tilde{f}(\mu(x) *_Y \mu(y)) =&\tilde{f}(\mu(x *_X y)) = (\tilde{f} \circ \mu)(x *_X y)\\ =&(\tilde{f} \circ \mu)(0_X) = \tilde{f}(\mu(0_X)) = \tilde{f}(0_Y).\end{aligned}$$

Hence $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$, by Proposition 2.1(ii).

For any $(\tilde{f} \circ \mu)_x, (\tilde{f} \circ \mu)_y \in X/(\tilde{f} \circ \mu)$, we have

$$\begin{aligned}\eta((\tilde{f} \circ \mu)_x \bullet_X (\tilde{f} \circ \mu)_y) =&\eta((\tilde{f} \circ \mu)_{x*y}) = \tilde{f}_{\mu(x*_X y)}\\ =&\tilde{f}_{\mu(x)*_Y\mu(y)} = \tilde{f}_{\mu(x)} \bullet \tilde{f}_{\mu(y)}\\ =&\eta((\tilde{f} \circ \mu)_x) \bullet_Y \eta((\tilde{f} \circ \mu)_y)).\end{aligned}$$

Therefore η is a homomorphism.

Let $\tilde{f}_a \in Y/\tilde{f}$. Then there exists $x \in X$ such that $\mu(x) = a$ since μ is surjective. Hence $\eta((\tilde{f} \circ \mu)_x) = \tilde{f}_{\mu(x)} = \tilde{f}_a$ and so η is surjective.

Let $x, y \in X$ be such that $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$. Then we have

$$\begin{aligned}(\tilde{f} \circ \mu)(x *_X y) =&\tilde{f}(\mu(x *_X y)) = \tilde{f}(\mu(x) *_Y \mu(y))\\ =&\tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f} \circ \mu)(0_X).\end{aligned}$$

It follows that $(\tilde{f} \circ \mu)_x = (\tilde{f} \circ \mu)_y$. Thus η is injective. This completes the proof. ☐

The homomorphism $\pi : X \to X/\tilde{f},\ x \to \tilde{f}_X$, is called the *natural homomorphism* of X onto $X/\tilde{f}$. In Theorem 4.7, if we define natural homomorphisms $\pi_X : X \to X/\tilde{f} \circ \mu$ and $\pi_Y : Y \to Y/\tilde{f}$ then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ \mu$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle \pi_X}\downarrow & & {\scriptstyle \pi_Y}\downarrow \\ X/(\tilde{f}\circ\mu) & \xrightarrow{\eta} & Y/\tilde{f}. \end{array}$$

Proposition 4.8. *Let a soft set $(\tilde{f}, X)$ over U of a B-algebra X be an int-soft normal subalgebra of X. If J is a normal subalgebra of X, then $J/\tilde{f}$ is a normal subalgebra of $X/\tilde{f}$.*

Proof. Let a soft set $(\tilde{f}, X)$ over U of a B-algebra X be an int-soft normal subalgebra of X and let J be a normal subalgebra of X. Then for any $x, y \in J$, $x * y \in J$. Let $\tilde{f}_x, \tilde{f}_y \in J/\tilde{f}$. Then $\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_{x*y} \in J/\tilde{f}$. Hence $J/\tilde{f} = \{\tilde{f}_x | x \in J\}$ is a subalgebra of $X/\tilde{f}$.

For any $x * y, a * b \in J$, $(x * a) * (y * b) \in J$, so for any $\tilde{f}_x \bullet \tilde{f}_y, \tilde{f}_a \bullet \tilde{f}_b \in J/\tilde{f}$, we have $(\tilde{f}_x \bullet \tilde{f}_a) \bullet (\tilde{f}_y \bullet \tilde{f}_b) = \tilde{f}_{x*a} \bullet \tilde{f}_{y*b} = \tilde{f}_{(x*a)*(y*b)} \in J/\tilde{f}$. Thus $J/\tilde{f}$ is a normal subalgebra of $X/\tilde{f}$. □

Theorem 4.9. *If J^* is a normal subalgebra of a B-algebra $X/\tilde{f}$, then there exists a normal subalgebra $J = \{x \in X | \tilde{f}_x \in J^*\}$ in X such that $J/\tilde{f} = J^*$.*

Proof. Since J^* is a normal subalgebra of $X/\tilde{f}$, so $\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_{x*y} \in J^*$ for any $\tilde{f}_x, \tilde{f}_y \in J^*$. Thus $x * y \in J$ for any $x, y \in J$. And $\tilde{f}_{x*a} \bullet \tilde{f}_{y*b} = \tilde{f}_{(x*a)*(y*b)} \in J^*$ for any $\tilde{f}_{x*y}, \tilde{f}_{a*b} \in J^*$. Thus $(x * a) * (y * b) \in J$ for any $x * y, a * b \in J$. Therefore J is a normal subalgebra of X. By Proposition 4.5, we have

$$\begin{aligned} J/\tilde{f} =& \{\tilde{f}_j | j \in J\} \\ =& \{\tilde{f}_j | \exists \tilde{f}_x \in J^* \text{ such that } j \sim^{\tilde{f}} x\} \\ =& \{\tilde{f}_j | \exists \tilde{f}_x \in J^* \text{ such that } \tilde{f}_x = \tilde{f}_j\} \\ =& \{\tilde{f}_j | \tilde{f}_j \in J^*\} = J^*. \end{aligned}$$

□

Theorem 4.10. *Let a soft set $(\tilde{f}, X)$ over U be an int-soft normal subalgebra of a B-algebra X. If J is a normal subalgebra of X, then $\dfrac{X/\tilde{f}}{J/\tilde{f}} \cong X/J$.*

Proof. Note that $\dfrac{X/\tilde{f}}{J/\tilde{f}} = \{[\tilde{f}_x]_{J/\tilde{f}} | \tilde{f}_x \in X/\tilde{f}\}$. If we define $\varphi : \dfrac{X/\tilde{f}}{J/\tilde{f}} \to X/J$ by $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. Then $\tilde{f}_x \sim^{J/\tilde{f}} \tilde{f}_y$ and so $\tilde{f}_{x*y} = \tilde{f}_x \bullet \tilde{f}_y \in J/\tilde{f}$. Hence $x * y \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$.

Given $[\tilde{f}_x]_{J/\tilde{f}}, [\tilde{f}_y]_{J/\tilde{f}} \in \dfrac{X/\tilde{f}}{J/\tilde{f}}$, we have

$$\begin{aligned}\varphi([\tilde{f}_x]_{J/\tilde{f}} \bullet [\tilde{f}_y]_{J/\tilde{f}}) =&\varphi([\tilde{f}_x \bullet \tilde{f}_y]_{J/\tilde{f}})\\ =&[x * y]_J = [x]_J * [y]_J\\ =&\varphi([\tilde{f}_x]_{J/\tilde{f}}) * \varphi([\tilde{f}_y]_{J/\tilde{f}}).\end{aligned}$$

Hence φ is a homomorphism.

Obviously, φ is onto. Finally, we show that φ is one-to-one. If $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = \varphi([\tilde{f}_y]_{J/\tilde{f}})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $\tilde{f}_a \in [\tilde{f}_x]_{J/\tilde{f}}$, then $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_x$ and hence $\tilde{f}_{a*x} \in J/\tilde{f}$. It follows that $a * x \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y \in J$ and so $\tilde{f}_{a*y} \in J/\tilde{f}$. Therefore $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_y$. Hence $\tilde{f}_a \in [\tilde{f}_y]_{J/\tilde{f}}$. Thus $[\tilde{f}_x]_{J/\tilde{f}} \subseteq [\tilde{f}_y]_{J/\tilde{f}}$. Similarly, we obtain $[\tilde{f}_y]_{J/\tilde{f}} \subseteq [\tilde{f}_x]_{J/\tilde{f}}$. Therefore $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. It is completes the proof. □

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Fixed point theorems for rational type contractions in partially ordered S-metric spaces

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Abstract: In this paper, we develop some fixed point theorems by using auxiliary functions for maps satisfying a rational type contractive condition in partially ordered $S-$metric spaces. Conditions for uniqueness of fixed point are also discussed. Our results generalize some existing results in the literature of S-metric spaces.

MSC: 47H10; 54H25.

Keywords: Fixed point; rational type contraction; partially ordered set; S-metric space

1. Introduction and Preliminaries

Fixed point theory is one of the most powerful and most important tools in nonlinear analysis and applied sciences. Its core subject is concerned with the conditions for the existence of one or more fixed points of a mapping T from a nonempty set X into itself, that is, to find a point $x \in X$ such that $Tx = x$.

In 1922, Banach's contraction principle [1] ensures the existence and uniqueness of a unique fixed point for a self-mapping satisfying a contractive condition, which is called *Banach's contractive mapping*. After that, many authors have extended, improved and generalized Banach's contraction principle in several ways.

Especially, Banach's contractive mapping is continuous, which is used to prove Banach's contraction principle. Thus it is natural to consider the following question:

Do there exist some contractive conditions which do not force the mapping T to be continuous?

In 1968, Kannan [4] gave the positive answer for this question and he proved the following fixed point theorem for the following contractive condition:

Theorem K. *Let (E, d) be a complete metric space and $T : E \to E$ be a mapping such that there exists a number $h \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq h[d(Tx, x) + d(Ty, y)]$$

for all $x, y \in X$. Then T has a unique fixed point in E.

Also, some authors have introduced some kinds of contractive mappings, for example, Meir-Keeler

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contraction, Caristi's contraction, Hardy-Roages contractions, Chatterjea's contraction, Berinde's contraction, Reich's contraction, Ćirić's contraction and others (see [2]-[9]).

Another one to study Banach's contraction principle in metric spaces is to extend Banach's contraction principle to the classes of various kinds of metric spaces. Recently, some authors have introduced some extensions of metric spaces in several ways and have studied fixed point theory and its applications, for example, 2-metric spaces [10], D-metric spaces [11], G-metric spaces [12], D^*-metric spaces [13], S-metric spaces [14]-[17] and some others.

On the other hand, Ran and Reurings [18], Bhaskar and Lakshmikantham [19], Lakshmikantham and Ćirić [20], Neito and Lopéz [21], Harjani et al. [22], Harjani et al. [23] and Zhou et al. [24]-[25] studied fixed point problem in partially ordered sets.

Definition 1.1. **[14]** Let X be a nonempty set. A S*-metric* on X is a mapping $S : X^3 \mapsto [0, \infty)$ that satisfies the following conditions: for all $x, y, z, a \in X$,

(S1) $S(x, y, z) \geq 0$;

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$;

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S*-metric space*.

Immediate examples of such $S-$metric spaces are as follows:

(1) Let $\mathbb{R}$ be a real line and define $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Then S is an S-metric on $\mathbb{R}$. This S-metric is called the *usual* S*-metric* on $\mathbb{R}$.

(2) Let $X = \mathbb{R}^+$ with a norm $\|\cdot\|$ and define $S(x, y, z) = \|2x + y - 3z\| + \|x - z\|$ for all $x, y, z \in X$. Then S is an S-metric on X.

(3) Let X be a nonempty set and d be the ordinary metric on X. If we define $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$, then S is an S-metric on X.

Definition 1.2. **[14]** Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X is said to *convergent* to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \epsilon$.

(3) An S-metric space (X, S) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Lemma 1.1. **[14]** Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$, for all $x, y \in X$.

Lemma 1.2. **[14]** Let (X, S) be an S-metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

for all $x, y, z \in X$.

Lemma 1.3. **[14]** Let (X, S) be an S-metric space. If a sequence $\{x_n\}$ in X converges to a point $x \in X$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.4. Let (X, S) be an S-metric space. Then, for all $x, y, z \in X$, it follows that

(1) $S(x, y, y) \leq S(x, x, y)$;
(2) $S(x, y, x) \leq S(x, x, y)$;
(3) $S(x, y, z) \leq S(x, x, z) + S(y, y, z)$;
(4) $S(x, y, z) \leq S(y, y, z) + S(x, x, z)$;
(5) $S(x, y, z) \leq S(y, y, x) + S(z, z, x)$;
(6) $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$;
(7) $S(x, y, z) \leq \frac{2}{3}[S(x, x, y) + S(y, y, z) + S(z, z, x)]$.

Proof. It follows from $(S3)$ and Lemma 1.2, one can easily obtain (1)-(5).

Now, we prove (6) and (7) also hold. By Lemma 1.1 and Lemma 1.2, we have

$$\begin{aligned}2S(x, x, z) &= S(x, x, z) + S(z, z, x)\\ &\leq [2S(x, x, y) + S(y, y, z)] + [2S(z, z, y) + S(x, x, y)]\\ &= 3[S(y, y, z) + S(y, y, x)]\end{aligned}$$

and hence $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$. Thus (6) holds. By virtue of (3)-(5) and Lemma 1.2, we have

$$3S(x, y, z) = 2[S(x, x, y) + S(y, y, z) + S(z, z, x)],$$

which implies (7) holds. This completes the proof. □

Lemma 1.5. **[15]** Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0.$$

If $\{x_n\}$ is not a S-Cauchy sequence, then there exists $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that the following sequences tend to ϵ when $k \to \infty$:

$$S(x_{m_k}, x_{m_k}, x_{n_k}),\ \ S(x_{m_k}, x_{m_k}, x_{n_k+1}),\ \ S(x_{m_k-1}, x_{m_k-1}, x_{n_k}),$$

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}),\ \ S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}).$$

Definition 1.3. **[26]** A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a *C-class function* if it is continuous and satisfies the following conditions:

(C1) $F(s, t) \leq s$ for all $s, t \in [0, \infty)$;
(C2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Let $\mathcal{C}$ denote the set of C-class functions.

Example 1.1. **[26]** The following functions $F : [0, \infty)^2 \mapsto \mathbb{R}$ are elements of $\mathcal{C}$. For each $s, t \in [0, \infty)$,

1. $F(s, t) = s - t$.
2. $F(s, t) = ms$ for some $m \in (0, 1)$.
3. $F(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, \infty)$.
4. $F(s, t) = \log(t + a^s)/(1 + t)$ for some $a > 1$.

5. $F(s,t) = \ln(1+a^s)/2$ if $e > a > 1$. Indeed, $f(s,t) = s$ implies that $s = 0$.
6. $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$ if $l > 1$ and $r \in (0,\infty)$.
7. $F(s,t) = s\log_{t+a} a$ for all $a > 1$.
8. $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.
9. $F(s,t) = s\beta(s)$ if a function $\beta : [0,\infty) \to [0,1)$ and is continuous.
10. $F(s,t) = s - \frac{t}{k+t}$.
11. $F(s,t) = s - \varphi(s)$ if $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
12. $F(s,t) = sh(s,t)$ if $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$.
13. $F(s,t) = s - (\frac{2+t}{1+t})t$.
14. $F(s,t) = \sqrt[n]{\ln(1+s^n)}$.
15. $F(s,t) = \phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.
16. $F(s,t) = \frac{s}{(1+s)^r}$ for all $r \in (0,\infty)$.

Definition 1.4. **[27]** A function $\psi : [0,\infty) \to [0,\infty)$ is called an *altering distance function* if the following conditions are satisfied:

(AD1) ψ is strictly increasing and continuous,
(AD2) $\psi(t) = 0$ for all $t \in [0,\infty)$ if and only if $t = 0$.

Let Φ denote the class of all continuous and strictly increasing functions $\phi : [0,\infty) \mapsto [0,\infty)$ and Ψ the set of all functions such that $\lim\limits_{t\to r}\psi(t) > 0$ for all $r > 0$ and $\psi(t) = 0$ if and only if $t = 0$.

In [28], Mashina proved the following results:

Theorem 1.1. **[28]** Let $(X,\preceq)$ be a partial ordered set and (X,S) be a complete S-metric space. Let $T : X \mapsto X$ be a continuous and nondecreasing mapping with respect to $\preceq$ such that

$$S(Tx,Tx,Ty) \le \alpha \cdot \frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)} + \beta \cdot S(x,x,y) \tag{1}$$

for all $x, y \in X$ with $x \neq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. If there exists $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

Theorem 1.2. **[28]** Let $(X,\preceq)$ be a partial ordered set and (X,S) is a complete S-metric space. Assume that X satisfies the following condition:

(C1) If $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$ with $x^* = \sup_{n\ge1}\{x_n\}$ with respect to $\preceq$.

Let $T : X \mapsto X$ be a nondecreasing mapping with respect to $\preceq$ such that

$$S(Tx, Tx, Ty) \leq \alpha \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + \beta \cdot S(x, x, y)$$

for all $x, y \in X$ with $x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. If there exists $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

Also, Mashina [28] added the following assumption to Theorem 1.1 and Theorem 1.2 to guarantee the uniqueness of the fixed point of the given mapping.

(C2) For all $x, y \in X$, there exists $u \in X$ which is comparable to x and y.

The main aim of this paper is to generalize the results of Mashina [28] by using the auxiliary functions in the setting of S-metric spaces.

2. Main Results

Now, we give one definition for our main results in this paper.

Definition 2.1. Let $(X, \preceq)$ be a partially ordered set and $T : X \mapsto X$. We say that T is a nondecreasing mapping with respect to $\preceq$ if for $x, y \in X, x \preceq y \Rightarrow Tx \preceq Ty$.

Theorem 2.1. Let $(X, \preceq)$ be a partial ordered set and (X, S) is a complete S-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$\begin{aligned} & \phi(S(Tx, Tx, Ty)) \\ \leq \; & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + \beta \cdot S(x, x, y)\Big]\Big), \\ & \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + \beta \cdot S(x, x, y)\Big]\Big)\Big) \end{aligned} \tag{2}$$

for all $x, y \in X$ with $x \neq y$, for some $\alpha, \beta \in [0, \infty)$ with $\alpha + \beta > 0$ and $F \in \mathcal{C}, \phi \in \Phi, \psi \in \Psi$. If there exists $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof. Let $x_0 \in X$ such that $x_0 \preceq Tx_0$. Since T is nondecreasing with respect to $\preceq$, by induction, we obtain

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \cdots \preceq T^nx_0 \preceq T^{n+1}x_0 \preceq \cdots.$$

Let $x_{n+1} = Tx_n$ for each $n \geq 1$. If there exists $n_0 \geq 1$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0+1} = Tx_{n_0} = x_{n_0}$ and so x_{n_0} is a fixed point of T.

So, we assume that $x_{n+1} \neq x_n$ for each $n \in \{0\} \cup \mathbb{N}$. Putting $x = x_{n+1}$ and $y = x_n$ for each $n \geq 1$

in (2.1), we have

$$\begin{aligned}
& \phi(S(x_{n+1},x_{n+1},x_n)) \\
= \ & \phi(S(Tx_n,Tx_n,Tx_{n-1})) \\
\leq \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\alpha\Big[\frac{S(x_n,x_n,Tx_n)\cdot S(x_{n-1},x_{n-1},Tx_{n-1})}{S(x_n,x_n,x_{n-1})}+\beta S(x_n,x_n,x_{n-1})\Big]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}\alpha\Big[\frac{S(x_n,x_n,Tx_n)\cdot S(x_{n-1},x_{n-1},Tx_{n-1})}{S(x_n,x_n,x_{n-1})}+\beta S(x_n,x_n,x_{n-1})\Big]\Big)\Big) \\
= \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big)\Big) \\
\leq \ & \phi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big).
\end{aligned}$$

Since ϕ is strictly increasing, we have

$$S(x_{n+1},x_{n+1},x_n)\leq S(x_n,x_n,x_{n-1})$$

for all $n\geq 1$. Hence the sequence $\{S(x_n,x_n,x_{n+1})\}$ is a monotone decreasing and bounded below. Therefore, there exists $r\geq 0$ such that $\lim_{n\to\infty}S(x_n,x_n,x_{n-1})=r$.

Now, we prove that $r=0$. Assume that $r>0$. Using Definition 1.3, we know that, when $F(s,t)=s$, then $s=0$ or $t=0$ and $F(s,t)<s$ when $s>0$ and $t>0$. Using the properties of ϕ and ψ, we have $\phi(r)>\phi(0)\geq 0$ and $\lim_{n\to\infty}\phi(S(x_n,x_n,x_{n-1}))>0$. Therefore, by taking the limit as $n\to\infty$ and using the properties of F, we have

$$\begin{aligned}
\phi(r) \ \leq \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}[\alpha\cdot r+\beta\cdot r]\Big), \\
& \lim_{n\to\infty}\psi\Big(\frac{1}{\alpha+\beta}[\alpha S(x_n,x_n,x_{n+1})+\beta S(x_n,x_n,x_{n-1})]\Big)\Big) \\
< \ & \phi(r),
\end{aligned}$$

which is a contradiction. Thus we have $r=0$ and

$$\lim_{n\to\infty}S(x_n,x_n,x_{n-1})=0.$$

Next, we prove that $\{S(x_n,x_n,x_{n-1})$ is a Cauchy sequence. Suppose that a sequence $\{S(x_n,x_n,x_{n-1})\}$ is not a Cauchy sequence. From Lemma 1.5, there exists $\epsilon>0$ and $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$\lim_{k\to\infty}S(x_{m_k},x_{n_k},x_{n_k})=\epsilon.$$

Putting $x = x_{m_k}$ and $y = x_{n_k}$ for each $k \geq 1$ in (2.1), we have

$$\begin{aligned}
& \phi(S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1})) \\
= \ & \phi(S(Tx_{m_k}, Tx_{m_k}, Tx_{n_k})) \\
\leq \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, Tx_{m_k}) \cdot S(x_{n_k}, x_{n_k}, Tx_{n_k})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big), \\
& \quad \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, Tx_{m_k}) \cdot S(x_{n_k}, x_{n_k}, Tx_{n_k})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big)\Big) \\
= \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, x_{m_k+1}) \cdot S(x_{n_k}, x_{n_k}, x_{n_k+1})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, x_{m_k+1}) \cdot S(x_{n_k}, x_{n_k}, x_{n_k+1})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big)\Big).
\end{aligned}$$

Using the properties of ϕ and ψ, we have $\phi(\epsilon) > 0$ and

$$\begin{aligned}
& \lim_{k\to\infty} \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_{m_k}, x_{m_k}, x_{m_k+1}) \cdot S(x_{n_k}, x_{n_k}, x_{n_k+1})}{S(x_{m_k}, x_{m_k}, x_{n_k})} \\
& \quad +\beta \cdot S(x_{m_k}, x_{m_k}, x_{n_k})\Big]\Big) > 0.
\end{aligned}$$

Taking the limit $k \to \infty$ in the above inequality, we have

$$\phi(\epsilon) \leq F\Big(\phi\Big(\frac{\beta\varepsilon}{\alpha+\beta}\Big), \psi\Big(\frac{\beta\varepsilon}{\alpha+\beta}\Big)\Big) < \phi\Big(\frac{\beta\varepsilon}{\alpha+\beta}\Big) < \phi(\epsilon),$$

which is a contradiction. Hence the sequence $\{S(x_n, x_n, x_{n-1})\}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Also, the continuity of T implies

$$Tx^* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

which implies that x^* is a fixed point of T. This completes the proof. □

Remark 2.1. (1) If define $F(s,t) = (\alpha+\beta)s$ for some $\alpha, \beta \in [0,\infty)$ with $\alpha+\beta > 0$ and $\phi(t) = t$ for all $t > 0$ in Theorem 2.1, then Theorem 2.1 reduces to Theorem 1.1 of [28].

(2) In Theorem 2.1, we use the auxiliary function $F \in \mathcal{C}$ and $\mathcal{C}$ is a class of more general functions than the gauge function used in Theorem 2.1 and 2.2 of [23]. Indeed, the gauge function $F(s,t) = s-t$ in Theorem 2.1 and 2.2 of [23] is an element of $\mathcal{C}$.

(3) We note that, if ψ is an alerting distance function, then $\psi \in \Psi$. But the reverse is not true in general.

Taking $F(s,t) = s - t$ in Theorem 2.1, we obtain the following:

Corollary 2.1. Let $(X, \preceq)$ be a partial ordered set and (X, S) be a complete S-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$\begin{aligned}\phi(S(Tx,Tx,Ty)) &\leq \phi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)\Big)\\&\quad-\psi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)\Big)\end{aligned}$$

for all $x,y\in X$ with $x\neq y$, for some $a,b\in[0,1)$ with $a+b<1$ and $\phi\in\Phi,\psi\in\Psi$. If there exists $x_0\preceq Tx_0$, then T has a fixed point in X.

In addition, taking $\phi(t)=kt$ for all $t>0$ and $\psi(t)=(k-1)t$ for all $t>0$ with $k>1$ in Corollary 2.1, we have the following:

Corollary 2.2. Let $(X,\preceq)$ be a partial ordered set and (X,S) be a complete S-metric space. Let $T:X\mapsto X$ be a continuous and nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$S(Tx,Tx,Ty)\leq a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)$$

for all $x,y\in X$ with $x\neq y$, some $a,b\in[0,1)$ with $a+b<1$. If there exists $x_0\preceq Tx_0$. Then T has a fixed point in X.

Now, we present some examples to verify Theorem 2.1 and Corollary 2.2.

Example 2.1. Let $X=[0,\infty)$ with the S-metric defined by

$$S(x,y,z)=|x-z|+|y-z|$$

for all $x,y,z\in X$ and $\leq$ be the natural ordering of real numbers. Then X is a complete S-metric space. Let $T:X\to X$ be a mapping defined by $Tx=\frac{1}{8}(1+x)$ and $\phi\in\Phi,\psi\in\Psi$ be defined by

$$\phi(t)=t+\frac{1}{4},\quad \psi(t)=\frac{t}{2}.$$

Define a mapping $F\in\mathcal{C}$ by $F(s,t)=s-t$ and take $\alpha=3$ and $\beta=1$.

First, we note that, for all $x_0\in[0,\frac{1}{7}]$, we have $x_0\leq Tx_0$. Second, we verify the condition (2.1). Without loss of generality, we assume that $x>y$. Then we have

$$\begin{aligned}\phi(S(Tx,Tx,Ty)) &= \phi(2(Tx-Ty))\\&= \phi\Big(2\Big[\frac{1}{8}(1+x)-\frac{1}{8}(1+y)\Big]\Big)\\&= \phi\Big(\frac{1}{4}(x-y)\Big)\\&= \frac{1}{4}(x-y)+\frac{1}{4}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\cdot\frac{S(x,x,Tx)S(y,y,Ty)}{S(x,x,y)}+\beta S(x,x,y)\Big]\Big)\\&= \phi\Big(\frac{1}{4}\Big[3\frac{4[x-\frac{1}{8}(1+x)][y-\frac{1}{8}(1+y)]}{2(x-y)}+2(x-y)\Big]\Big)\\&= \phi\Big(\frac{1}{4}\Big[\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+2(x-y)\Big]\Big)\\&= \frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{4(x-y)}+\frac{1}{2}(x-y)+\frac{1}{4}.\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\cdot\frac{S(x,x,Tx)S(y,y,Ty)}{S(x,x,y)}+\beta S(x,x,y)\Big]\Big)\\
=\;&\psi\Big(\frac{1}{4}\Big[3\frac{4[x-\frac{1}{8}(1+x)][y-\frac{1}{8}(1+y)]}{2(x-y)}+2(x-y)\Big]\Big)\\
=\;&\psi\Big(\frac{1}{4}\Big[\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+2(x-y)\Big]\Big)\\
=\;&\frac{1}{8}\Big[\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+2(x-y)\Big].
\end{aligned}$$

Thus we have

$$F(\phi,\psi)=\frac{1}{8}\frac{6(\frac{7}{8}x-\frac{1}{8})(\frac{7}{8}y-\frac{1}{8})}{(x-y)}+\frac{1}{4}(x-y)+\frac{1}{4}.$$

Hence the condition (2.1) holds for $y<x\leq\frac{1}{7}$. Therefore, all the assumptions of Theorem 2.1 are satisfied and, further, $x=\frac{1}{7}$ is the fixed point of T.

Example 2.2. Let $X=[1,\infty)$ be an S-metric space with the S-metric defined by

$$S(x,y,z)=|x-y|+|y-z|$$

for all $x,y,z\in X$ and $\leq$ be the natural ordering of real numbers. Then (X,S) is a complete S-metric space. For $0<k<1$, consider the self-mapping $T:X\to X$ defined by $Tx=\frac{3x+2}{2x+3}$ for all $x\in X$. First, there exists $x_0=1\in X$ such that $x_0\leq Tx_0$. Second, we have

$$\begin{aligned}
S(Tx,Tx,Ty)&=\Big|\frac{3x+2}{2x+3}-\frac{3y+2}{2y+3}\Big|\\
&=\frac{5|x-y|}{(2x+3)(2y+3)}\\
&\leq\frac{|x-y|}{5}\\
&=\frac{1}{5}S(x,x,y).
\end{aligned}$$

So, we have

$$S(Tx,Tx,Ty)\leq a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)}+b\cdot S(x,x,y)$$

for all $x,y\in X$ with $x\neq y$ and $a\in[0,\frac{4}{5})$ and $b=\frac{1}{5}$. Hence all the assumptions of Corollary 2.2 are satisfied. Therefore, T has a fixed point in X and, further, $x=1$ is a fixed point of T.

In the next theorem, we omit the continuity of T and assume that the following condition, which has been stated in [22].

(C1) If $\{x_n\}$ is a nondecreasing sequence such that $x_n\to x^*$ with $x^*=\sup\{x_n\}$ with respect to $\preceq$.

Theorem 2.2. Let $(X,\preceq)$ be a partial ordered set and (X,S) be a complete S-metric space. Assume that X satisfies the condition $(C1)$. Let $T:X\to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the condition (2.1). If there exists $x_0\preceq Tx_0$, then T has a fixed point in X.

Proof. Following the proof of Theorem 2.1, we only need to verify $Tx^* = x^*$. Since $\{x_n\}$ is a nondecreasing sequence in X and $x_n \to x^*$, by the condition $(C1)$, it follows that $x_n \preceq x^*$. Since T is a nondecreasing mapping with respect to $\preceq$, we have $Tx_n = x_{n+1} \preceq Tx^*$ for all $n \in \mathbb{N}$. Moreover, since $x_0 \preceq Tx_0 \preceq Tx^*$ and $x^* = \sup\{x_n\}$, we have $x^* \preceq Tx^*$.

Using the similar arguments as in the proof of Theorem 2.1, for $x^* \preceq Tx^*$, it follows that $\{T^n x^*\}$ is a nondecreasing sequence and $\lim_{n\to\infty} T^n x^* = z$ for some $z \in X$. Again, using the condition $(C1)$, we have $z = \sup\{T^n x^*\}$. Moreover, from $x_0 \preceq x^*$, we have $x_n = T^n x_0 \preceq T^n x^*$ for each $n \geq 1$. Applying $x = x_n$ and $y = x^*$ for each $n \geq 1$ in (2.1), we have

$$\begin{aligned}
& \phi(S(x_{n+1}, x_{n+1}, T^{n+1}x^*)) \\
= \ & \phi(S(Tx_n, Tx_n, T(T^n x^*))) \\
\leq \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_n, x_n, Tx_n)\cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
& +\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_n, x_n, Tx_n)\cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
& +\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big)\Big) \\
= \ & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x_n, x_n, x_{n+1})\cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
& +\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}[\alpha \cdot \frac{S(x_n, x_n, x_{n+1})\cdot S(T^n x^*, T^n x^*, T(T^n x^*))}{S(x_n, x_n, T^n x^*)} \\
& +\beta \cdot S(x_n, x_n, T^n x^*)\Big]\Big)\Big).
\end{aligned}$$

Letting the limit $n \to \infty$ in the above inequality, by the properties of ϕ, ψ, F, we have

$$\phi(S(x^*, x^*, z)) \leq F\Big(\phi\Big(\frac{\alpha S(x^*, x^*, z)}{\alpha+\beta}\Big), \psi\Big(\frac{\beta S(x^*, x^*, z)}{\alpha+\beta}\Big)\Big) \leq \phi\Big(\frac{\beta S(x^*, x^*, z)}{\alpha+\beta}\Big),$$

which yields $\frac{\beta S(x^*, x^*, z)}{\alpha+\beta} = 0$ or $\psi\Big(\frac{\beta S(x^*, x^*, z)}{\alpha+\beta}\Big) = 0$. Thus we have $S(x^*, x^*, z) = 0$. Especially, $x^* = z = \sup\{x_n\}$ and so $Tx^* \preceq x^*$, which is a contradiction. Hence $x^* = Tx^*$. This completes the proof. □

Taking $F(s,t) = s - t$ in Theorem 2.2, we obtain the following:

Corollary 2.3. Let $(X, \preceq)$ be a partial ordered set and (X, S) be a complete S-metric space. Assume that X satisfies the condition $(C1)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$\begin{aligned}
\phi(S(Tx, Tx, Ty)) \ \leq \ & \phi\Big(a \cdot \frac{S(x, x, Tx)\cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)\Big) \\
& -\psi\Big(a \cdot \frac{S(x, x, Tx)\cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)\Big)
\end{aligned}$$

for all $x, y \in X$ with $x \neq y$, for some $a, b \in [0, 1)$ with $a + b < 1$ and $\phi \in \Phi, \psi \in \Psi$. If there exists $x_0 \preceq Tx_0$, then T has a fixed point in X.

In addition, taking $\phi(t) = kt$ for all $t > 0$ and $\psi(t) = (k-1)t$ for all $t > 0$ with $k > 1$ in Corollary 2.3, we have the following:

Corollary 2.4. Let $(X, \preceq)$ be a partial ordered set and (X, S) be a complete S-metric space. Assume that X satisfies the condition $(C1)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$S(Tx, Tx, Ty) \leq a \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)$$

for all $x, y \in X$ with $x \neq y$ and for some $a, b \in [0, 1)$ with $a + b < 1$. If there exists $x_0 \preceq Tx_0$, then T has a fixed point in X.

Now, we give an example to illustrate Theorem 2.2.

Example 2.3. Let $X = [0, \infty)$ with the S-metric S defined by

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in X$ and $\leq$ be the natural ordering of real numbers. Then X is a complete S-metric space. Let $T : X \to X$ be a mapping defined by $Tx = 4 - \frac{1}{2x}$ for all $x \in X$ and $\phi \in \Phi, \psi \in \Psi$ be defined by

$$\phi(t) = t + \frac{1}{4}, \quad \psi(t) = \frac{t}{2},$$

respectively. Define a mapping $F \in \mathcal{C}$ by $F(s, t) = s - t$ and take $\alpha = 3$ and $\beta = 1$.

First, we note that there exists $x_0 \in [0, \frac{\sqrt{2}+1}{2}] \subseteq [0, \infty)$ such that $x_0 \leq Tx_0$. It is easily to verify that the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ with $x_0 = \frac{\sqrt{2}+1}{2}$ is nondecreasing and converges to $x^* = \frac{3+\sqrt{14}}{2}$ with $x^* = \sup_{n \geq 1}\{x_n\}$ with respect to $\leq$. Second, we verify the condition (2.1). Without loss of generality, we assume that $x > y$. Then we have

$$\begin{aligned} \phi(S(Tx, Tx, Ty)) &= \phi(2(Tx - Ty)) \\ &= \phi\Big(2\Big[\Big(4 - \frac{1}{2x}\Big) - \Big(4 - \frac{1}{2y}\Big)\Big]\Big) \\ &= \phi\Big(\frac{1}{y} - \frac{1}{x}\Big) \\ &= \frac{1}{y} - \frac{1}{x} + \frac{1}{4}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx)S(y, y, Ty)}{S(x, x, y)} + \beta S(x, x, y)\Big]\Big) \\ = \;& \phi\Big(\frac{1}{4}\Big[\frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{(x - y)} + 2(x - y)\Big]\Big) \\ = \;& \frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{4(x - y)} + \frac{1}{2}(x - y) + \frac{1}{4}, \end{aligned}$$

$$\begin{aligned} &\psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \cdot \frac{S(x, x, Tx)S(y, y, Ty)}{S(x, x, y)} + \beta S(x, x, y)\Big]\Big) \\ = \;& \psi\Big(\frac{1}{4}\Big[\frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{(x - y)} + 2(x - y)\Big]\Big) \\ = \;& \frac{1}{8}\frac{6(x + \frac{1}{2x} - 4)(y + \frac{1}{2y} - 4)}{(x - y)} + \frac{1}{4}(x - y) \end{aligned}$$

and

$$F(\phi,\psi)=\frac{1}{8}\frac{6(x+\frac{1}{2x}-4)(y+\frac{1}{2y}-4)}{(x-y)}+\frac{1}{4}(x-y)+\frac{1}{4}.$$

Hence the condition (2.1) holds for $y<x\in[0,\frac{\sqrt{2}+1}{2}]$. Therefore, all the assumptions of Theorem 2.2 are satisfied and, further, $x=\frac{3+\sqrt{14}}{2}$ is the fixed point of T.

For the uniqueness of the fixed point, we consider the following condition stated in [22].

(C2) For all $x,y\in X$, there exists $u\in X$ which is comparable to x and y.

Theorem 2.3. If you give the condition $(C2)$ to the hypotheses of Theorem 2.1 (or Theorem 2.2), then the fixed point of the mapping T is unique.

Proof. Suppose that x^* and $y^*\in X$ are fixed points of the mapping T. Then we consider two cases.

Case 1: If x^* and y^* are comparable and $x^*\neq y^*$, then, using the condition (2.1), we have

$$\begin{aligned}
&\phi(S(x^*,x^*,y^*))\\
=\ &\phi(S(Tx^*,Tx^*,Ty^*))\\
\leq\ &F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(x^*,x^*,Tx^*)\cdot S(y^*,y^*,Ty^*)}{S(x^*,x^*,y^*)}+\beta S(x^*,x^*,y^*)\Big]\Big),\\
&\quad\psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(x^*,x^*,Tx^*)\cdot S(y^*,y^*,Ty^*)}{S(x^*,x^*,y^*)}+\beta S(x^*,x^*,y^*)\Big]\Big)\Big)\\
=\ &F\Big(\phi\Big(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)\Big),\psi\Big(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)\Big)\Big)\\
\leq\ &\phi\Big(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)\Big),
\end{aligned}$$

which yields $\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*)=0$ or $\psi(\frac{\beta}{\alpha+\beta}S(x^*,x^*,y^*))=0$. Thus we have $S(x^*,x^*,y^*)=0$. Therefore, $x^*=y^*$.

Case 2: If x^* is not comparable to y^*, then, by the condition $(C2)$, there exists $u\in X$ comparable to x^* and y^*. The monotonicity implies that T^nu is comparable to $T^nx^*=x^*$ and $T^ny^*=y^*$ for each $n\geq 0$. If there exists $n_0\geq 1$ such that $T^{n_0}u=x^*$, then, since x^* is a fixed point of T, the sequence $\{T^nu:n\geq n_0\}$ is constant and so $\lim_{n\to\infty}T^nu=x^*$.

On the other hand, if $T^nu\neq x^*$ for each $n\geq 1$, then, using the condition (2.1), it follows that, for each $n \geq 2$,

$$\begin{aligned}
& \phi(S(T^n u, T^n u, x^*)) \\
= \; & \phi(S(T^n u, T^n u, T^n x^*)) \\
\leq \; & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(T^{n-1}x^*, T^{n-1}x^*, T^n x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& +\beta S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)\Big]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(T^{n-1}x^*, T^{n-1}x^*, T^n x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& +\beta S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)\Big]\Big)\Big) \\
= \; & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(x^*, x^*, x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& +\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big]\Big), \\
& \psi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\frac{S(T^{n-1}u, T^{n-1}u, T^n u)\cdot S(x^*, x^*, x^*)}{S(T^{n-1}u, T^{n-1}u, T^{n-1}x^*)} \\
& +\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big]\Big)\Big) \\
= \; & F\Big(\phi\Big(\frac{1}{\alpha+\beta}\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big), \psi\Big(\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big)\Big) \\
\leq \; & \phi\Big(\frac{1}{\alpha+\beta}\beta S(T^{n-1}u, T^{n-1}u, x^*)\Big) \\
< \; & \phi(S(T^{n-1}u, T^{n-1}u, x^*)),
\end{aligned}$$

which implies that $S(T^n u, T^n u, x^*) < S(T^{n-1}u, T^{n-1}u, x^*)$. Therefore, the sequence $\{S(T^n u, T^n u, x^*)\}$ is monotone decreasing, bounded below and converges to $d \geq 0$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\phi(d) \leq F\Big(\phi\Big(\frac{\beta}{\alpha+\beta}d\Big), \phi\Big(\frac{\beta}{\alpha+\beta}d\Big)\Big) < \phi(d),$$

which yields $\frac{\beta}{\alpha+\beta}d = 0$ or $\phi\Big(\frac{\beta}{\alpha+\beta}d\Big) = 0$. Thus we have $d = 0$ and

$$\lim_{n\to\infty} T^n u = x^*.$$

It can be shown that $\lim_{n\to\infty} T^n u = y^*$ by the similar arguments mentioned above. Thus we can conclude that $x^* = y^*$ and hence fixed point of the mapping T is unique. This completes the proof. □

Taking $F(s,t) = s - t$ in Theorem 2.3, we obtain the following:

Corollary 2.5. Let $(X, \preceq)$ be a partial ordered set and (X, S) be a complete S-metric space. Assume that X satisfies the condition $(C2)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$\begin{aligned}
\phi(S(Tx, Tx, Ty)) \leq \; & \phi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)} + b\cdot S(x,x,y)\Big) \\
& -\psi\Big(a\cdot\frac{S(x,x,Tx)\cdot S(y,y,Ty)}{S(x,x,y)} + b\cdot S(x,x,y)\Big)
\end{aligned}$$

for all $x, y \in X$ with $x \neq y$, for some $a, b \in [0,1)$ with $a + b < 1$ and $\phi \in \Phi, \psi \in \Psi$. If there exists $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

In addition, taking $\phi(t) = kt$ for all $t > 0$ and $\psi(t) = (k-1)t$ for all $t > 0$ with $k > 1$ in Corollary 2.5, we have the following:

Corollary 2.6. Let $(X, \preceq)$ be a partial ordered set and (X, S) be a complete S-metric space. Assume that X satisfies the condition $(C2)$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$ satisfying the following condition:

$$S(Tx, Tx, Ty) \leq a \cdot \frac{S(x, x, Tx) \cdot S(y, y, Ty)}{S(x, x, y)} + b \cdot S(x, x, y)$$

for all $x, y \in X$ with $x \neq y$ and for some $a, b \in [0, 1)$ with $a + b < 1$. If there exists $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

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On stochastic pantograph differential equations in the G-framework

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Abstract

The purpose of this research is to study the stochastic pantograph differential equations (SPDEs) in the G-framework. We determine that any solution $Z(t)$ of stochastic pantograph differential equation in the G-framework is bounded i.e., in particular $Z(t) \in M_G^2([0,T];\mathbb{R}^n)$. Subject to growth and Lipschitz conditions, we prove that SPDEs in the G-framework admit unique solution. Some useful inequalities, such as the Hölder's inequality, Doobs martingale's inequality, Burkholder-Davis-Gundy's (BDG) inequalities and Gronwall's inequality are utilized to derive our results. In addition, we obtain the asymptotic estimates for the solutions to SPDEs in the G-framework.

Keywords: Existence, uniqueness, asymptotic estimates, G-Brownian motion, stochastic pantograph differential equations.

MSC2010 Classification: 60G10, 60G17, 60G20, 60H05, 60H10, 60H20.

1 Introduction

The stochastic differential equations (SDEs) theory is used in different disciplines of engineering and sciences. For instance, in physics, SDEs are used to study and model the influence of random changes on various physical phenomena. These equations describe the transport of cosmic rays in space. The percolation of fluid through absorbent structures and water catchment can be modeled by SDEs [15]. They are used to find out the problems of stochastic volatility and risk measures in finance and economics. In biology, they model the accomplishment of stochastic changes in reproduction on populations procedures [32, 33]. The weather and climate can also be modeled by these equations. A huge literature is available on the applications of SDEs in various discipline of engineering such as mechanical engineering [25, 27, 28], wave processes [26], stability theory [24] and random vibrations [3, 23]. In general, we can not find the explicit solutions for non-linear SDEs,

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so we have to present and study the analysis for the solutions of these equations. By virtue of the Lipschitz and growth conditions, the existence theory for solutions to SDEs in the G-framework was given by Peng [20, 21] and later by Gao [14]. The said theory with integral Lipschitz coefficients was developed by Bai and Lin [1]. While Faizullah generalized the existence of solutions for SDEs in the G-framework with discontinuous coefficients [10]. In view of the Picard approximation technique, the existence-uniqueness results for stochastic functional differential equations (SFDEs) in the G-framework were commenced by Ren, Bi and Sakthivel [22]. The stated theory with Caratheodory approximation scheme was developed by Faizullah [9]. He presented the pth moment estimates for the solutions to SFDEs in the G-framework [6, 7]. Recently, Faizullah generalized the existence theory for SFDEs in the G-framework with non-Lipschitz conditions [5]. The pantograph differential equations arise in different fields such as quantum mechanics, number theory, dynamical systems, electrodynamics and probability. These equations were utilized by Taylor and Ockendon to investigate the collection of electric current [19]. The stochastic version of pantograph differential equations were introduced by Backer and Buckwar [2]. They studied the existence theory for linear stochastic pantograph differential equations (SPDEs). While Xiao, Song and Liu determined that the Euler scheme for linear SPDEs is convergent [30]. The existence theory for solutions to non-linear SPDEs were developed by Fan, Liu and Cao [11], in which the convergence of Euler scheme was established by Xiao and Zhang [31]. However, up to the best of our knowledge, no one has studied SPDEs in the G-framework. The current paper will fill the mentioned gap. Consider an m-dimensional G-Brownian motion $W(t) = (W_1(t)), W_2(t)), W_3(t)), ..., W_m(t))^T$ defined on a complete probability space $(\Omega, \mathcal{F}_t, P)$. Let $W(t)$ is adopted to the filtration $\{\mathcal{F}_t; t \geq 0\}$ and fulfilling the usual conditions. Assume $0 \leq t_0 \leq t \leq T < \infty$. Suppose the coefficients κ, λ and μ be Borel measurable such that $\kappa : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$, $\lambda : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $\mu : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$. We study the following d-dimensional stochastic pantograph differential equation in the G-framework

$$dZ(t) = \kappa(t, Z(t), Z(qt))dt+\lambda(t, Z(t), Z(qt))d\langle W, W\rangle(t)+\mu(t, Z(t), Z(qt))dW(t),\ 0 \leq t \leq T, \quad (1.1)$$

where $q \in (0, 1)$, the initial condition $Z_0 \in \mathbb{R}^d$ is given and κ, λ, μ are given mappings satisfying $\kappa, \lambda, \mu \in M^2_G([0,T]; \mathbb{R}^d)$. We denote the quadratic variation process of G-Brownian motion $\{W(t)\}_{t\geq 0}$ by $\{\langle W, W\rangle(t)\}_{t\geq 0}$. The integral form of equation (1.1) is given as the following

$$Z(t) = Z_0 + \int_0^t \kappa(s, Z(s), Z(qs))ds + \int_0^t \lambda(s, Z(s), Z(qs))d\langle W, W\rangle(s) + \int_0^t \mu(s, Z(s), Z(qs))dW(s). \quad (1.2)$$

Definition 1.1. Let $t \in [0,T]$. A stochastic process $Z(t) \in \mathbb{R}^d$ is known as solution of problem (1.1) if the below characteristics hold.

(i) $\{Z(t)\}_{0\leq t\leq T}$ is $\mathcal{F}_t$-adapted and continuous.

(ii) The coefficients $\kappa(t, Z(t), Z(qt)) \in \mathcal{L}^1([0,T]; \mathbb{R}^d)$, $\lambda(t, Z(t), Z(qt)) \in \mathcal{L}^2([0,T]; \mathbb{R}^{d\times m})$ and $\mu(t, Z(t), Z(qt)) \in \mathcal{L}^2([0,T]; \mathbb{R}^{d\times m})$.

(iii) For each $t \in [0,T]$, equation (1.2) holds q.s.

A solution $Z(t)$ of problem (1.1) is said to be unique if for any other solution $Y(t)$ of (1.1) we have

$$E[\sup_{0\leq t\leq T} \mid Z(t)-Y(t)\mid^2]=0,$$

which means that $Z(t)$ and $Y(t)$ are identical. For all $t\in[t_0,T]$ and all $z,y,u,v\in\mathbb{R}^n$, throughout the current paper the following two conditions are assumed.

$$|\kappa(t,z,y)|^2+|\lambda(t,z,y)|^2+|\mu(t,z,y)|^2\leq C(1+|z|^2+|y|^2), \tag{1.3}$$

where C is a positive constant. This condition (1.3) is known as a linear growth condition and the below (1.4) is called the Lipschitz condition.

$$\begin{aligned}&|\kappa(t,z,y)-\kappa(t,u,v)|^2+|\lambda(t,z,y)-\lambda(t,u,v)|^2+|\mu(t,z,y)-\mu(t,u,v)|^2\\&\leq C(\mid z-u\mid^2+\mid y-v\mid^2),\end{aligned} \tag{1.4}$$

where C is a positive constant. We organize the present article in the forthcoming fashion. Section 2 presents several fundamental notions, definitions and results, which are required for our research work. In section 3 we determine that $Z(t)$ is bounded and belongs to the space $M_G^2([0,T];\mathbb{R}^n)$. This section also contains the existence and uniqueness theorem for the solutions to SPDEs in the G-framework. Finally, we derive the path-wise estimates for the solutions to the said equations in section 4.

2 Preliminaries

Building on the previous notions of G-Brownian motion theory, this section presents the fundamental definitions and results required for the further discussion of the subject. For more details on the concepts briefly discussed, readers are suggested to consult the more depth oriented papers [8, 13, 17, 20, 21]. Let Ω be a given fundamental non-empty set. Suppose $\mathcal{H}$ be a space of linear real functions defined on Ω satisfying that (i) $1\in\mathcal{H}$ (ii) for every $d\geq 1$, $X_1,X_2,...,X_d\in\mathcal{H}$ and $\varphi\in C_{b.Lip}(\mathbb{R}^d)$ it holds $\varphi(X_1,X_2,...,X_d)\in\mathcal{H}$ i.e., with respect to Lipschitz bounded functions, $\mathcal{H}$ is stable. Then $(\Omega,\mathcal{H},E)$ is a sub-expectation space, where E is a sub-expectation defined as the following.

Definition 2.1. A functional $E:\mathcal{H}\to\mathbb{R}$ satisfying the below four features is known as a sub-expectation. Let $Z,Y\in\mathcal{H}$, then

(**1**) $E[Z]\leq E[Y]$ if $Z\leq Y$.

(**2**) $E[K]=K$, for all $K\in\mathbb{R}$.

(**3**) $E[\alpha Z]=\alpha E[Z]$, for all $\alpha\in\mathbb{R}^+$.

(**4**) $E[Z]+E[Y]\geq E[Z+Y]$.

The above properties $(1),(2),(3)$ and (4) are known as monotonicity, constant preserving, positive homogeneity and sub-additivity respectively. Moreover, let Ω be the space of all $\mathbb{R}^d$-valued continuous paths $(w_t)_{t\geq 0}$ starting from zero. Also, suppose that associated with the below distance, Ω is a metric space

$$\rho(w^1,w^2)=\sum_{i=1}^{\infty}\frac{1}{2^i}(\max_{t\in[0,k]}|w_t^1-w_t^2|\wedge 1).$$

Fix $T\geq 0$ and set

$$L_{ip}^0(\Omega_T)=\{\phi(B_{t_1},B_{t_2},...,B_{t_m}):m\geq 1,t_1,t_2,...,t_m\in[0,T],\phi\in C_{b.Lip}(\mathbb{R}^{m\times d}))\},$$

where B is the canonical process, $L_{ip}^0(\Omega_t)\subseteq L_{ip}^0(\Omega_T)$ for $t\leq T$ and $L_{ip}^0(\Omega)=\cup_{n=1}^{\infty}L_{ip}^0(\Omega_n)$. The completion of $L_{ip}^0(\Omega)$ under the Banach norm $E[|.|^p]^{\frac{1}{p}}$, $p\geq 1$ is denoted by $L_G^p(\Omega)$, where $L_G^p(\Omega_t)\subseteq L_G^p(\Omega_T)\subseteq L_G^p(\Omega)$ for $0\leq t\leq T<\infty$. We indicate the filtration generated by the canonical process $\{W(t)\}_{t\geq 0}$, as $\mathcal{F}_t=\sigma\{W_s,0\leq s\leq t\}$ and $\mathcal{F}=\{\mathcal{F}_t\}_{t\geq 0}$. Suppose $\pi_T=\{t_0,t_1,...,t_N\}$, $0\leq t_0\leq t_1\leq ...\leq t_N\leq\infty$ be a partition of $[0,T]$. Set $p\geq 1$, then $M_G^{p,0}(0,T)$ indicates a collection of the below type processes

$$\alpha_t(w)=\sum_{i=0}^{N-1}\beta_i(w)I_{[t_i,t_{i+1}]}(t), \tag{2.1}$$

where $\beta_i\in L_G^p(\Omega_{t_i})$, $i=0,1,...,N-1$. Furthermore, the completion of $M_G^{p,0}(0,T)$ with the below given norm is indicated by $M_G^p(0,T)$, $p\geq 1$

$$\|\alpha\|=\{\int_0^T E[|\alpha_s|^p]ds\}^{1/p}.$$

Definition 2.2. A stochastic process $\{W(t)\}_{t\geq 0}$ of d-dimensional satisfying the below properties is called a G-Brownian motion

(**1**) $W(0)=0$.

(**2**) For any $t,m\geq 0$, the increment $W_{t+m}-W_t$ is G-normally distributed and independent from $W_{t_1},W_{t_2},........W_{t_n}$, for $n\in N$ and $0\leq t_1\leq t_2\leq,...,\leq t_n\leq t$,

Definition 2.3. Let $\alpha_t\in M_G^{2,0}(0,T)$ having the form (2.1). Then the G-quadratic variation process $\{\langle W\rangle_t\}_{t\geq 0}$ and G-Itô's integral $I(\alpha$ are respectively defined by

$$\langle W\rangle_t=W_t^2-2\int_0^t W_s dW_s,$$
$$I(\alpha)=\int_0^T\alpha_s dW_s=\sum_{i=0}^{N-1}\beta_i(W_{t_{i+1}}-W_{t_i}).$$

The below two results are taken from the book [18]. They are called as Hölder's and Gronwall's inequalities respectively, .

Lemma 2.4. *Assume $m, n > 1$ such that $\frac{1}{m} + \frac{1}{n} = 1$ and $\xi \in L^2$ then $\eta\xi \in L^1$ and*

$$\int_a^b \eta\xi \leq \left(\int_a^b |\eta|^m\right)^{\frac{1}{m}} \left(\int_a^b |\xi|^n\right)^{\frac{1}{n}}.$$

Lemma 2.5. *Let $\eta(t) \geq 0$ and $\xi(t)$ be continuous real functions defined on $[a, b]$. If for all $t \in [a, b]$,*

$$\xi(t) \leq K + \int_a^b \eta(s)\xi(s)ds,$$

where $K \geq 0$, then

$$\xi(t) \leq K e^{\int_a^t \eta(s)ds},$$

for all $t \in [a, b]$.

Definition 2.6. Suppose that the group of entire probability measures on $(\Omega, \mathcal{B}(\Omega)$ is indicated by $\mathcal{P}$. The capacity is denoted by $\hat{C}$ and is given by

$$\hat{C}(D) = \sup_{P \in \mathcal{P}} P(D),$$

where $D \in \mathcal{B}(\Omega)$ is Borel σ-algebra of Ω.

Definition 2.7. A set $D \in \mathcal{B}(\Omega)$ is called polar if

$$\hat{C}(D) = 0.$$

A characteristic fulfills quasi-surely (in short $q.s.$) if it fulfills outer a polar set.

Now we state the following result [4].

Theorem 2.8. *Let $Z \in L^2$. Then for every $\epsilon > 0$,*

$$\hat{C}(|Z|^2 > \epsilon) \leq \frac{E[|Z|^2]}{\epsilon}.$$

The following lemma, known as Doob's martingale inequality, can be found in [14].

Lemma 2.9. *Assume $[a, b]$ be a bounded interval of $\mathbb{R}_+$. Consider an $\mathbb{R}^d$ valued G-martingale $\{Z(t)\}_{t \geq 0}$. Then*

$$E[\sup_{a \leq t \leq b} |Z(t)|^p] \leq (\frac{p}{p-1})^p E[|Z(b)|^P],$$

where $p > 1$ and $Z(t) \in L_G^p(\Omega, \mathbb{R}^d)$. In particular, if $p = 2$ then $E[\sup_{a \leq t \leq b} |Z(t)|^2] \leq 4E[|Z(b)|^2]$.

The following lemma, known as Banach contraction mapping principle, is borrowed from the book [12].

Lemma 2.10. *Assume Z is a complete metric space. Let $L : Z \to Z$ is a contraction mapping. Then L holds a unique fixed point in Z.*

3 Existence and uniqueness results

Firstly, we demonstrate a useful lemma. This lemma will be utilized in the upcoming existence-uniqueness result. This will also be used in the proof of path wise asymptotic estimates for the solutions to SPDEs in the G-framework.

Lemma 3.1. *Let equation* (1.1) *admits a solution* $Z(t)$. *Suppose* (1.3) *holds. Then*

$$E[\sup_{0\leq s\leq T}|Z(s)|^2] \leq \left(1+4E|Z_0|^2\right)e^{16C(T+2)T},$$

where the constant $C>0$ *is already defined.*

Proof. Let $k\geq 1$ be an arbitrary integer. Set the following stopping time

$$\tau_k = T\wedge \inf\{t\in[0,T] :\parallel Z(t)\parallel \geq k\} \;\; and \;\; Z^k(t) = Z(t\wedge\tau_k).$$

Clearly, $\tau_k \uparrow T$ a.s. as $k\to\infty$ and $Z^k(t)$ satisfies the following equation

$$\begin{aligned} Z^k(t) = Z_0 + \int_0^t \kappa(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}ds + \int_0^t \lambda(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}d\langle W,W\rangle(s) \\ + \int_0^t \mu(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}dW(s). \end{aligned}$$

By virtue of the basic inequality $|\sum_{i+1}^4 c_i|^2 \leq 4\sum_{i+1}^4 |c_i|^2$, we have

$$\begin{aligned} |Z^k(t)|^2 \leq 4|Z_0|^2 + 4\left|\int_0^t \kappa(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}ds\right|^2 + 4\left|\int_0^t \lambda(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}d\langle W,W\rangle(s)\right|^2 \\ + 4\left|\int_0^t \mu(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}dW(s)\right|^2. \end{aligned}$$

Taking sub-expectation on both sides, we have

$$\begin{aligned} E[\sup_{0\leq s\leq t}|Z^k(s)|^2] \leq 4E|Z_0|^2 + 4E[\sup_{0\leq s\leq t}\left|\int_0^t \kappa(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}ds\right|^2] \\ + 4E[\sup_{0\leq s\leq t}\left|\int_0^t \lambda(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}d\langle W,W\rangle(s)\right|^2] \\ + 4E[\sup_{0\leq s\leq t}\left|\int_0^t \mu(s, Z^k(s), Z^k(qs))I_{[0,\tau_k]}dW(s)\right|^2]. \end{aligned}$$

Use the Hölder's, Doob's martingale's and Burkholder-Davis-Gundy's (BDG) inequalities [14].

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Then by applying condition (1.4) we get

$$\begin{aligned} E[\sup_{0\le s\le t}|z^k(s)|^2] &\le 4E|Z_0|^2 + 4TC\int_0^t \Big(1+E|Z^k(s)|^2+E|Z^k(qs)|^2\Big)\,ds \\ &+ 4TC\int_0^t \Big(1+E|Z^k(s)|^2+E|Z^k(qs)|^2\Big)\,ds \\ &+ 16C\int_0^t \Big(1+E|Z^k(s)|^2+E|Z^k(qs)|^2\Big)\,ds \\ &\le 4E|Z_0|^2 + 8C(T+2)\int_0^t \Big(1+2E[\sup_{0\le r\le s}E|Z^k(r)|^2]\Big)\,ds, \end{aligned}$$

which yields

$$\begin{aligned} 1+E[\sup_{0\le s\le t}|Z^k(s)|^2] &\le 1+4E|Z_0|^2 + 8C(T+2)\int_0^t \Big(1+2E[\sup_{0\le r\le s}E|Z^k(r)|^2]\Big)\,ds \\ &\le 1+4E|Z_0|^2 + 16C(T+2)\int_0^t \Big(1+E[\sup_{0\le r\le s}E|Z^k(r)|^2]\Big)\,ds. \end{aligned}$$

In view of the Gronwall inequality we obtain

$$1+E[\sup_{0\le s\le T}|Z^k(s)|^2] \le \left(1+4E|Z_0|^2\right)e^{16C(T+2)T}.$$

Consequently,

$$E[\sup_{0\le s\le T}|Z(s)|^2] \le \left(1+4E|Z_0|^2\right)e^{16C(T+2)T}.$$

The proof stands completed. □

Remark 3.2. Lemma 3.1 indicates that if problem (1.1) admits a solution $Z(t)$, then it must be bounded i.e. in particular $Z(t) \in M_G^2([0,T];\mathbb{R}^n)$.

Theorem 3.3. *Let* (1.3) *and* (1.4) *hold. Then equation* (1.1) *admits at most one solution* $Z(t) \in M_G^2([0,T];\mathbb{R}^d)$.

Proof. Assume $T > 0$, $12KT(T+2) < 1$ and $Z(t) \in M_G^2([0,T];\mathbb{R}^d)$. Define the mapping

$$\begin{aligned} (LZ)(t) = Z_0 &+ \int_0^t \kappa(s,Z(s),Z(qs))ds + \int_0^t \lambda(s,Z(s),Z(qs))d\langle W,W\rangle(s) \\ &+ \int_0^t \mu(s,Z(s),Z(qs))dW(s), \end{aligned}$$

$t \in [0,T]$. It is clear that LZ is a continuous measurable $\{\mathcal{F}_t\}$-adapted process. Taking sub-expectation on both sides

$$\begin{aligned}
E[\sup_{0\le t\le T}|(LZ)(t)|^2] &= E|Z_0 + \sup_{0\le t\le T}(\int_0^t \kappa(s,Z(s),Z(qs))ds) \\
&+ \sup_{0\le t\le T}(\int_0^t \lambda(s,Z(s),Z(qs))d\langle W,W\rangle(s)) \\
&+ \sup_{0\le t\le T}(\int_0^t \mu(s,Z(s),Z(qs))dW(s))|^2 \\
&\le 4E|Z_0|^2 + 4E[\sup_{0\le t\le T}|\int_0^t \kappa(s,Z(s),Z(qs))ds|^2] \\
&+ 4E[\sup_{0\le t\le T}|\int_0^t \lambda(s,Z(s),Z(qs))d\langle W,W\rangle(s)|^2] \\
&+ 4E[\sup_{0\le t\le T}|\int_0^t \mu(s,Z(s),Z(qs))dW(s)|^2].
\end{aligned}$$

Use Hölder's, Doob martingale and BDG [14] inequalities. Then apply (1.4) to obtain

$$\begin{aligned}
E[\sup_{0\le t\le T}|(LZ)(t)|^2] &\le 4E|Z_0|^2 + 4TC\int_0^t \left(1+E|Z(s)|^2+E|Z(qs)|^2\right)ds \\
&+ 4TC\int_0^t \left(1+E|Z(s)|^2+E|Z(qs)|^2\right)ds \\
&+ 16C\int_0^t \left(1+E|Z(s)|^2+E|Z(qs)|^2\right)ds \\
&\le 4E|Z_0|^2 + 8C(T+2)\int_0^t \left(1+2\sup_{0\le t\le T}E|Z(t)|^2\right)ds \\
&\le 4E|Z_0|^2 + 8C(T+2)T + 8C(T+2)\int_0^t E[\sup_{0\le t\le T}|Z(t)|^2]ds \\
&\le 4E|Z_0|^2 + 4CT(2T+1) + 8CT(2T+1)\left(1+4E|Z_0|^2\right)e^{8C(T+2)T} < \infty.
\end{aligned}$$

Thus $\|LZ\| < \infty$ and $LZ \in M_G^2([0,T];\mathbb{R}^d)$. This shows that L is a function from $M_G^2([0,T];\mathbb{R}^d)$ to itself. Now we have to derive that L is a contraction function. Let $Z, Y \in M_G^2([0,T];\mathbb{R}^d)$, then

$$\begin{aligned}
E[\sup_{0\le t\le T}|(LY)(t)-(LZ)(t)|^2] &= E(\sup_{0\le t\le T}|\int_0^t [\kappa(s,Y(s),Y(qs))-\kappa(s,Z(s),Z(qs))]ds \\
&+ \int_0^t [\lambda(s,Y(s),Y(qs))-\lambda(s,Z(s),Z(qs))]d\langle W,W\rangle(s) \\
&+ \int_0^t [\mu(s,Y(s),Y(qs))-\mu(s,Z(s),Z(qs))]dW(s)|^2)
\end{aligned}$$

By the basic inequality $|\sum_{i=1}^3 c_i|^2 \le 3\sum_{i=1}^3 |c_i|^2$ and monotonic property of sub-expectation we

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obtain

$$E[\sup_{0\le t\le T}|(LY)(t)-(LZ)(t)|^2]\le 3E\left(\sup_{0\le t\le T}\left|\int_0^t[\kappa(s,Y(s),Y(qs))-\kappa(s,Z(s),Z(qs))]ds\right|^2\right)$$
$$+3E\left(\sup_{0\le t\le T}\left|\int_0^t[\lambda(s,Y(s),Y(qs))-\lambda(s,Z(s),Z(qs))]d\langle W,W\rangle(s)\right|^2\right)$$
$$+3E\left(\sup_{0\le t\le T}\left|\int_0^t[\mu(s,Y(s),Y(qs))-\mu(s,Z(s),Z(qs))]dW(s)\right|^2\right).$$

Next we use the Hölder's inequality, BDG inequalities [14], Doob's martingale inequality and Lipschitz condition (1.4) as follows

$$\begin{aligned}E[\sup_{0\le t\le T}|(LY)(t)-(LZ)(t)|^2]&\le\le 3TE\left(\int_0^t|\kappa(s,Y(s),Y(qs))-\kappa(s,Z(s),Z(qs))|^2ds\right)\\&+3TE\left(\int_0^t|\lambda(s,Y(s),Y(qs))-\lambda(s,Z(s),Z(qs))|^2ds\right)\\&+12E\left(\int_0^t|\mu(s,Y(s),Y(qs))-\mu(s,Z(s),Z(qs))|^2ds\right)\\&\le 3TK\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\&+3TK\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\&+12K\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\&=6K(T+2)\int_0^t E(|Y(s)-Z(s)|^2+|Y(qs)-Z(qs)|^2)ds\\&\le 12K(T+2)\int_0^t E(\sup_{0\le t\le T}|Y(t)-Z(t)|^2)ds\\&\le 12KT(T+2)E(\sup_{0\le t\le T}|Y(t)-Z(t)|^2)\end{aligned}$$

In view of $12KT(T+2)<1$ and lemma 2.10, the function L admits a unique fixed point in $M_G^2([0,T];\mathbb{R}^d)$, i.e., there is a unique stochastic process $Z(t,w)$, which fulfills

$$E(\sup_{0\le t\le T}|Y(t)-Z(t)|^2)=0.$$

Thus problem (1.1) admits a unique solution $Z(t)$ in $[0,T]$. Assume $T_0=T$, $T_j=\min\{T+T_{j-1},\frac{T_{j-1}}{q}\}$, where $j=1,2,3,...$ Then it is clear that $T_j\to\infty$ as $j\to\infty$ and $M_G^2([T_{j-1},T_j];\mathbb{R}^d)$ is a Banach space. Now suppose that (1.1) admits a unique solution $\psi_{j-1}(t)$ in $[0,T_{j-1}]$, let $Z\in M_G^2([T_{j-1},T_j];\mathbb{R}^d)$ and define

$$(LZ)(t)=\psi_{j-1}(T_{j-1})+\int_{T_{j-1}}^t\kappa(s,Z(s),\psi_{j-1}(qs))ds+\int_{T_{j-1}}^t\lambda(s,Z(s),\psi_{j-1}(qs))d\langle W,W\rangle(s)$$
$$+\int_{T_{j-1}}^t\mu(s,Z(s),\psi_{j-1}(qs))dW(s),$$

$t \in [T_{j-1}, T_j]$. Obviously, $LZ \in M_G^2([T_{j-1}, T_j]; \mathbb{R}^d)$. Using identical arguments as above one can derive that $E[\sup_{0\leq t\leq T} |(LY)(t) - (LZ)(t)|^2] \leq 12KT(T+2)E(\sup_{0\leq t\leq T} |Y(t) - Z(t)|^2)$ i.e., the mapping L admits a fixed point Z in $M_G^2([T_{j-1}, T_j]; \mathbb{R}^d)$ and $Z(T_{j-1}) = \psi_{j-1}(T_{j-1})$. Thus

$$\psi_j(t) = \begin{cases} \psi_{j-1}(t), & \text{if } t \in [0, T_{j-1}) \ ; \\ Z(t), & \text{if } t \in [T_{j-1}, T_j] \end{cases}$$

is the solution of problem (1.1) in $[0, T_j]$. Hence by induction, the proof stands completed. □

4 Path-wise asymptotic estimate

This section presents the path-wise asymptotic estimate for the solution to problem (1.1). We use lemma 3.1 to determine that the second moment of Lyapunov exponent $\lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)|$ [16] is bounded.

Theorem 4.1. *Let the linear growth condition* (1.3) *is satisfied. Then*

$$\lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)| \leq 8C(T+2), \quad q.s.$$

Proof. Using lemma 3.1, for each $j = 1, 2, ...,$

$$E(\sup_{j-1\leq t\leq j} |Z(t)|^2) \leq K_1 e^{K_2 j},$$

where $K_1 = \big(1 + 4E|X_0|^2\big)$ and $K_2 = 16C(T+2)$. For any arbitrary $\epsilon > 0$, in view of theorem 2.8 we obtain

$$\begin{aligned} \hat{C}(w : \sup_{j-1\leq t\leq j} |Z(t)|^2 > e^{(K_2+\epsilon)j}) &\leq \frac{E[\sup_{j-1\leq t\leq j} |Z(t)|^2]}{e^{(K_2+\epsilon)j}} \\ &\leq \frac{K_1 e^{K_2 j}}{e^{(K_2+\epsilon)j}} \\ &= K_1 e^{-\epsilon j}. \end{aligned}$$

For almost all $w \in \Omega$, the Borel-Cantelli lemma follows that a random integer $j_0 = j_0(w)$ exists such that

$$\sup_{j-1\leq t\leq j} |Z(t)|^2 \leq e^{(K_2+\epsilon)j} \quad \textit{whenever} \ \ j \geq j_0,$$

which yields,

$$\begin{aligned} \lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)| &\leq \frac{K_2+\epsilon}{2} \\ &= \frac{1}{2}[16C(T+2)] + \frac{\epsilon}{2} \\ &= 8C(T+2) + \frac{\epsilon}{2}, \quad q.s. \end{aligned}$$

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But ϵ is arbitrary, so

$$\lim_{t\to\infty}\sup\frac{1}{t}log|Z(t)|\leq 8C(T+2),\quad q.s.$$

The proof stands completed. □

5 Conclusion

The current investigation presents the study of stochastic pantograph differential equations in the G-framework. The Gronwall's, Burkholder-Davis-Gundy's (in short BDG), Doobs martingale and Hölder's inequalities are utilized to obtain the results. By virtue of the growth condition, it is revealed that solutions of the stated equations are bounded. The existence and uniqueness results for G-SPDEs are derived. In addition, the path-wise asymptotic estimates for the solutions to SPDEs in the G-framework are determined. The results of the current paper open several new research directions. For example, what are the p-moment estimates for SPDEs in the G-framework? How to develop the existence-uniqueness results with non-linear and non-Lipschitz conditions? What about the stability of solutions for these equations? etc. We hope this article will play a key role to provide framework for the concepts briefly discussed.

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On dual partial metric topology and a fixed point theorem

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Abstract. In this paper, we present some properties of dual partial metric (abbreviation, pmetric) topology and investigate a fixed point result for self mappings in dual pmetric space. This result generalizes Banach contraction principle in a different way than in the known results from the literature. The article includes an example which shows the validity of our result.

1. Introduction

Metric spaces are inevitably Hausdorff and so cannot, for example, be used to study non-Hausdorff topologies such as those required in the Tarskian approach to programming language semantics. Matthews [3] presented a symmetric generalized metric for such topologies, an approach which sheds new light on how metric tools such as Banach's Theorem can be extended to non-Hausdorff topologies. Matthews [3] defined the partial metric (pmetric) p on nonempty set X ($p : X \times X \to [0, \infty)$) and generalized Banach fixed point theorem (see [2, 7]). Essentially, the partial metric generalization is that the distance of a point from itself is not necessarily zero anymore. The axioms were first introduced in [3], where the range of a pmetric was restricted to $[0, \infty)$. Neill [5] extended the range to $(-\infty, \infty)$ and called this functional a dual partial metric denoted by p^*, since this is both natural (in that there is no difficulty in extending the results from [3]) and essential for a natural dual pmetric. The natural context in which to view a partial metric space (X, p) is as a bitopological space $(X, \tau(p), \tau(d))$. Neill [5] showed that successive conditions on a valuation can ensure that the pmetric topology is first of all order consistent (with the underlying poset), then equivalent to the Scott topology, and finally that the induced metric topology is equivalent to the patch topology. Neill also established some topological properties of functional p^* but did not give any fixed point result in p^*. However, Oltra *et al.* [4] established the criteria of convergence of sequences and completeness in p^* and generalized the fixed point result presented by Matthews.

In this paper, we present some more topological properties of p^* and establish fixed point results for self mappings in dual pmetric space. These results generalize Banach contraction principle in a different way than in the known results from the literature. The article includes an example which shows the validity of our results.

2. Valuation and dual pmetric

Throughout this paper the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, real numbers and natural numbers, respectively.

Definition 1. (Consistent Semilattice) Let $(X, \preceq)$ be a poset such that

(1) for all $x, y \in X$ $x \wedge y \in X$,

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(2) if $\{x, y\} \subseteq X$ is consistent, then $x \vee y \in X$.
Then $(X, \preceq)$ with (1) and (2) is called a consistent semilattice.

Definition 2. (Valuation Space) A valuation space is a consistent semilattice $(X, \preceq)$ and a function $\mu : X \to \mathbb{R}$, called valuation, such that
(1) if $x \preceq y$ and $x \neq y$, $\mu(x) < \mu(y)$ and
(2) if $\{x, y\} \subseteq X$ is consistent, then

$$\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y).$$

Matthews pmetric is defined as follws.

Definition 3. [3] Let X be a nonempty set and $p : X \times X \to \mathbb{R}_0^+$ satisfy the following properties: for all $x, y, z \in X$
(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$.
Then p is called a pmetric.

Definition 4. Let p be a pmetric defined on a nonempty set X. The functional $p^* : X \times X \to \mathbb{R}$ defined by

$$p^*(x, y) = p(x, y) - p(x, x) - p(y, y) \text{ for all } x, y \in X$$

is called a dual partial metric (dual pmetric) on X and (X, p^*) is known as a dual partial metric space. Moreover, it can easily be proved that the expression

$$d^*(x, y) = 2p^*(x, y) - p^*(x, x) - p^*(y, y)$$

defines a metric on X.

Note that the function $p : X \times X \to \mathbb{R}_0^+$ satisfies $(p_1) - (p_4)$, that is,
(p_1^*) $x = y \Leftrightarrow p^*(x, x) = p^*(x, y) = p^*(y, y)$,
(p_2^*) $p^*(x, x) \leq p^*(x, y)$,
(p_3^*) $p^*(x, y) = p^*(y, x)$,
(p_4^*) $p^*(x, z) + p^*(y, y) \leq p^*(x, y) + p^*(y, z)$.

Unlike other generalized metrics (such as the quasimetrics) this duality is not a consequence of a lack of symmetry in the axioms. Indeed it is perhaps one of the strengths of the partial metric generalization that symmetry is preserved as an axiom.

Remark 1. We observe that, as in the metric case, if p^* is a dual pmetric then $p^*(x, y) = 0$ implies $x = y$ but converse may not be true. $p^*(x, x)$ referred to as the size or weight of x, is a feature used to describe the amount of information contained in x. It is obvious that if p is a partial metric then p^* is a dual partial metric but converse is not true. Note that $p^*(x, x) \leq p^*(x, y)$ does not imply $p(x, x) \leq p(x, y)$. Nevertheless, the restriction of p^* to $\mathbb{R}_0^+$ is a partial metric.

Lemma 1. *Suppose that $(X, \preceq, \mu)$ is a valuation space. Then $p^*(x, y) = \mu(x \vee y)$ defines a dual pmetric on X.*

Proof. The axioms (p_2^*) and (p_3^*) are immediate. For (p_1^*), we proceed as

$$\text{if } p^*(x, x) = p^*(x, y) = p^*(y, y), \text{ then } \mu(x \vee y) = \mu(x) = \mu(y) \text{ implies } x = y.$$

The converse is obvious. We prove (p_4^*):

$$\begin{aligned} p^*(x, z) + p^*(y, y) &= \mu(x \vee z) + \mu(y) \\ &\leq \mu(x \vee y \vee z) + \mu[(x \vee y) \wedge (y \vee z)] \\ &= \mu(x \vee y \vee z) + \mu(x \vee y) + \mu(y \vee z) - \mu(x \vee y \vee z) \\ &= \mu(x \vee y) + \mu(y \vee z) = p^*(x, y) + p^*(y, z), \end{aligned}$$

as desired. □

Example 1. *Let p be a pmetric defined on a nonempty set $X = \{[a,b]\,; a \leq b\}$. The functional p^* : $X \times X \to \mathbb{R}$ defined by*

$$p^*([a,b],[c,d]) = \begin{cases} c-d & \text{if } \max\{b,d\} = b, \min\{a,c\} = a \\ a-b & \text{if } \max\{b,d\} = d, \min\{a,c\} = c \end{cases}$$

defines a daul pmetric on X.

Example 2. *Let d be a metric and p be a pmetric defined on a nonempty set X and $c > 0$ be a real number. The functional $p^* : X \times X \to \mathbb{R}$ defined by*

$$p^*(x,y) = d(x,y) - c \text{ for all } x, y \in X$$

is a dual pmetric on X.

For a partial metric space (X,p), we immediately have a natural definition (although slightly different from the one given in [3]) for the open balls:

$$B_\epsilon(x;p) = \{y \in X | p(x,y) < p(x,x) + \epsilon\} \text{ for all } x \in X.\epsilon > 0. \tag{2.1}$$

The set $\mathcal{T}[p] = \{B_\epsilon(x;p),\ x \in X.\epsilon > 0\}$ defines a pmetric topology on X. It can easily be seen that $\mathcal{T}[p]$ is a T_0 topology. The equation (2.1) naturally implies that

$$B^*_\epsilon(x;p^*) = \{y \in X | p^*(x,y) < p^*(x,x) + \epsilon\} \text{ for all } x \in X, \epsilon > 0,$$

which gives a structure for open balls in dual pmetric space (X,p^*). Unlike their metric counterpart, some dual pmetric open balls may be empty. For example, if $p^*(x,x) \neq 0$, then

$$\begin{aligned} B^*_{p^*(x,x)}(x;p^*) &= \{y \in X | p^*(x,y) < 2p^*(x,x)\} \\ &= \{y \in X | p(x,y) - p(x,x) - p(y,y) < -2p(x,x)\} \\ &= \{y \in X | p(x,y) + p(x,x) < p(y,y)\} = \Phi. \end{aligned}$$

We prove that the set $\{B^*_\epsilon(x;p^*);$ for all $x \in X,\ \epsilon > 0\}$ of open balls forms the basis for dual pmetric topology denoted by $\mathcal{T}[p^*]$. Each dual pmetric topology is T_0 topology and every open ball in a dual pmetric space is an open set.

Theorem 1. *The set $\{B^*_\epsilon(x;p^*);$ for all $x \in X,\ \epsilon > 0\}$ of open balls forms the basis for dual pmetric topology denoted by $\mathcal{T}[p^*]$.*

Proof. It is obvious that

$$X = \cup_{x \in X} B^*_\epsilon(x;p^*)$$

and for any two open balls $B^*_\epsilon(x;p^*)$ and $B^*_\delta(y;p^*)$, we note that

$$B^*_\epsilon(x;p^*) \cap B^*_\delta(y;p^*) = \cup\{B^*_\kappa(c;p^*) |\, c \in B^*_\epsilon(x;p^*) \cap B^*_\delta(y;p^*)\}$$

$$\text{where, } \kappa = p^*(c,c) + \min\{\epsilon - p^*(x,c), \delta - p^*(y,c)\},$$

as desired. □

Theorem 2. *Each dual pmetric topology is a T_0 topology.*

Proof. Suppose $p^* : X \times X \to \mathbb{R}$ is a dual pmetric and $x \neq y$. Then without loss of generality, we have $p^*(x,x) < p^*(x,y)$ for all $x, y \in X$. Choose $\epsilon = p^*(x,y) - p^*(x,x)$. Since

$$p^*(x,x) < p^*(x,x) + \epsilon = p^*(x,y),$$

$x \in B^*_\epsilon(x;p^*)$ and $y \notin B^*_\epsilon(x;p^*)$ because otherwise we obtain an absurdity $(p^*(x,y) < p^*(x,y))$. □

Theorem 3. *Every open ball in a dual pmetric space is an open set.*

Proof. Let (X, p^*) be a dual pmetric space and $B^*_\epsilon(v; p^*)$ be an open ball, centered at v, of radius $\epsilon > 0$. We show that for $x \neq v$,

$$x \in B^*_\delta(x; p^*) \subseteq B^*_\epsilon(v; p^*).$$

Suppose that $x \in B^*_\epsilon(v; p^*)$. Using (p^*_1) and (p^*_2), we have

$$p^*(x,x) < p^*(x,v) < p^*(v,v) + \epsilon. \tag{2.2}$$

Take $\delta = \epsilon + p^*(v,v) - p^*(x,x)$. (2.2) implies $p^*(x,x) < p^*(x,x) + \delta$. Thus $x \in B^*_\delta(x; p^*)$.

Next we show that

$$B^*_\delta(x; p^*) \subseteq B^*_\epsilon(v; p^*).$$

Suppose that $y \in B^*_\delta(x; p^*)$. Then

$$\begin{aligned} p^*(x,y) &< p^*(x,x) + \delta, \\ p^*(x,y) &< p^*(x,x) + \epsilon + p^*(v,v) - p^*(x,x) = \epsilon + p^*(v,v), \end{aligned}$$

which implies that $y \in B^*_\epsilon(v; p^*)$. □

Remark 2. (1) To see in what sense p^* is dual to p, we recall that the specialization order induced by a T_0-topology $\mathcal{T}$, is defined by

$$x \preceq_{\mathcal{T}} y \text{ if and only if for all } O \in \mathcal{T},\ x \in O \text{ implies } y \in O.$$

Then, for a partial metric space (X, p), it is not difficult to check that:

$$\begin{aligned} x \preceq_{\mathcal{T}[p]} y &\Leftrightarrow p(x,y) = p(x,x) \\ &\Leftrightarrow p^*(x,y) = p^*(x,x) \\ &\Leftrightarrow y \preceq_{\mathcal{T}[p^*]} x. \end{aligned}$$

It is also clear that $p^{**} = p$. Now if (X, p) is a partial metric space, then

$$d(x,y) = p(x,y) + p^*(x,y), \text{ for all } x, y \in X,$$

defines a metric on X, which we call the induced metric. If we denote the metric topology by $\mathcal{T}[d]$, then $\mathcal{T}[d] = \mathcal{T}[p] \vee \mathcal{T}[p^*]$.

(2) For complete valuation space $\mathcal{T}[p] = \sigma_p =$ Scott topology, moreover, if the valuation space is compact then $\mathcal{T}[p^*] = \sigma^*_p =$ dual Scott topology.

If (X, p^*) is a dual pmetric space, then the function $d_{p^*} : X \times X \to \mathbb{R}^+_0$ defined by

$$d_{p^*}(x,y) = p^*(x,y) - p^*(x,x), \tag{2.3}$$

is a quasi metric on X such that $\mathcal{T}[p^*] = \mathcal{T}[d_{p^*}]$ where $B_\epsilon(x; d_{p^*}) = \{y \in X | d_{p^*}(x,y) < \epsilon\}$. In this case, $d^s_{p^*}(x,y) = \max\{d_{p^*}(x,y), d_{p^*}(y,x)\}$ defines a metric on X, known as induced metric.

A dual pmetric p^* can quantify the amount of information in an object x using the numerical measure $p^*(x,x)$ and also that p^* has an open ball topology. This would not be of much use in Computer Science without a partial ordering. Therefore, we define a partial ordering and obtain some related results.

Definition 5. Let (X, p^*) be a dual pmetric space. We define the relation $\preceq_{p^*}$ on X^2 such that

$$\forall x, y \in X,\ \ x \preceq_{p^*} y \text{ if and only if } p^*(x,x) = p^*(x,y).$$

Lemma 2. *For each dual pmetric p^*, $\preceq_{p^*}$ is a partial ordering.*

Proof. We prove that $\preceq_{p^*}$ is reflexive, antisymmetric and transitive.

(O1) Since, $p^*(x,x) = p^*(x,x)$ for all $x \in X$, $x \preceq_{p^*} x$.

(O2) Suppose that $x \preceq_{p^*} y$ and $y \preceq_{p^*} x$. Then

$$p^*(x,x) = p^*(x,y) \text{ and } p^*(y,y) = p^*(y,x).$$

Using (p^*_3), we have $p^*(x,x) = p^*(x,y) = p^*(y,y)$ and then by (p^*_1) we obtain $x = y$.

(O3) For all $x, y, z \in X$, assume that $x \preceq_{p^*} y$ and $y \preceq_{p^*} z$ then
$$p^*(x,x) = p^*(x,y) \text{ and } p^*(y,y) = p^*(y,z).$$
Due to (p_4^*) we have
$$\begin{aligned} p^*(x,z) &\leq p^*(x,y) + p^*(y,z) - p^*(y,y) \\ &= p^*(x,x) + p^*(y,y) - p^*(y,y), \\ p^*(x,z) &\leq p^*(x,x), \end{aligned}$$
but also due to (p_2^*) we have $p^*(x,x) \leq p^*(x,z)$. Thus $p^*(x,x) = p^*(x,z)$ which implies that $x \preceq_{p^*} z$.

Hence (O1), (O2) and (O3) ensure that $\preceq_{p^*}$ defines a partial order on X. □

Theorem 4. *For each dual pmetric p^*, $\mathcal{T}[p^*]$ is weakly order consistent topology over $\preceq_{p^*}$.*

Proof. We show that $\mathcal{T}[p^*] \subseteq \mathcal{T}[\preceq_{p^*}]$. For this purpose it is sufficient to show that for all $x \in X$ and $\epsilon > 0$
$$B_\epsilon^*(x;p^*) = \cup \left\{ \{z | y \preceq_{p^*} z\} | y \in B_\epsilon^*(x;p^*) \right\}.$$
Suppose that $x, y, z \in X$ and $\epsilon > 0$ are such that $y \preceq_{p^*} z$ and $y \in B_\epsilon^*(x;p^*)$. Consider
$$\begin{aligned} p^*(x,z) &\leq p^*(x,y) + p^*(y,z) - p^*(y,y) \text{ by } (p_4^*) \\ &= p^*(x,y), \text{ since } y \preceq_{p^*} z, \\ &< p^*(x,x) + \epsilon, \text{ since } y \in B_\epsilon^*(x;p^*). \end{aligned}$$
This shows that $z \in B_\epsilon^*(x;p^*)$, which completes the proof. □

Thus $\mathcal{T}[p^*]$ is a dual Scott-like topology over $\preceq_{p^*}$ if each chain X has a least upper bound l and if
$$\lim_{n\to\infty} p^*(x_n, x_n) = p^*(l,l).$$
Now we present a theorem containing conditions under which $\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}]$.

Theorem 5. *Let $p^* : X^2 \to \mathbb{R}$ be a dual pmetric. Then*
$$\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}] \Leftrightarrow \forall\ x \in X, \exists\ \epsilon > 0 \text{ such that } B_\epsilon^*(x;p^*) = \{y | x \preceq_{p^*} y\}.$$

Proof. Suppose that $B_\epsilon^*(x;p^*) = \{y|x \preceq_{p^*} y\}$ for all $x \in X$, $\epsilon > 0$ and for all $\mathcal{U} \in \mathcal{T}[\preceq_{p^*}]$, we have
$$\mathcal{U} = \cup_{x\in\mathcal{U}} \{y|x \preceq_{p^*} y\} = \cup_{x\in\mathcal{U}} B_\epsilon^*(x;p^*) \in \mathcal{T}[p^*].$$
Thus $\mathcal{T}[\preceq_{p^*}] \subseteq \mathcal{T}[p^*]$. Using Theorem 4, we conclude that $\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}]$.

Conversely, suppose that $\mathcal{T}[p^*] = \mathcal{T}[\preceq_{p^*}]$. Then for all $x \in X$ $\{y|x \preceq_{p^*} y\} \in \mathcal{T}[p^*]$. Thus there exists $\epsilon > 0$ such that $x \in B_\epsilon^*(x;p^*) \subseteq \{y|x \preceq_{p^*} y\}$. Now if $x \in B_\epsilon^*(x;p^*)$, then $\{y|x \preceq_{p^*} y\} \subseteq B_\epsilon^*(x;p^*)$. As a result, $B_\epsilon^*(x;p^*) = \{y|x \preceq_{p^*} y\}$. □

3. Convergence criteria in dual pmetric space

The following definition and lemma describe the convergence criteria established by Oltra *et al.* [4].

Definition 6. [4] Let (X, p^*) be a dual partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p^*) is called a Cauchy sequence if $\lim_{n,m\to\infty} p^*(x_n, x_m)$ exists and is finite.

(2) A dual partial metric space (X, p^*) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}[p^*]$, to a point $\upsilon \in X$ such that
$$p^*(x,x) = \lim_{n,m\to\infty} p^*(x_n, x_m).$$

Lemma 3. [4]

(1) *Every Cauchy sequence in $(X, d_{p^*}^s)$ is also a Cauchy sequence in (X, p^*).*

(2) *A dual partial metric (X, p^*) is complete if and only if the metric space $(X, d_{p^*}^s)$ is complete.*

(3) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *converges to a point* $\upsilon \in X$ *with respect to* $\mathcal{T}[(d^s_{p^*})]$ *if and only if*

$$\lim_{n\to\infty} p^*(\upsilon, x_n) = p^*(\upsilon,\upsilon) = \lim_{n\to\infty} p^*(x_n,\upsilon).$$

(4) *If* $\lim_{n\to\infty} x_n = \upsilon$ *such that* $p^*(\upsilon,\upsilon) = 0$*, then* $\lim_{n\to\infty} p^*(x_n, k) = p^*(\upsilon, k)$ *for every* $k \in X$*.*

4. Fixed point theorem

In this section, by establishing Theorem 8 in dual pmetric space, we show that Banach's contraction mapping theorem can be generalized to many T_0 topologies for applications in program verification and domain theory. Let $\mathcal{B}$ denote the set of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

The following generalization of Banach's contraction principle, proved in 1973, is due to Geraghty [1].

Theorem 6. [1] Let (M, d) be a complete metric space and $T : M \to M$ be a mapping. If there exists $\beta \in \mathcal{B}$ such that, for all $j, k \in M$,

$$d(T(j), T(k)) \le \beta(d(j,k))d(j,k).$$

Then T has a unique fixed point $\upsilon \in M$ and, for any choice of the initial point $j_0 \in M$, the sequence $\{j_n\}$ defined by $j_n = T(j_{n-1})$ for each $n \ge 1$ converges to the point υ.

In [6], La Rosa and Vetro extended the notion of Geraghty contraction mappings to the context of partial metric spaces and proved partial metric version of Theorem 6, stated below:

Theorem 7. [6, Theorem 3.5] Let (M, p) be a complete partial metric space. If the self mapping $T : M \to M$ is a Ciríc type Geraghty contraction, then T has a unique fixed point $j \in M$ and the Picard iterative sequence $\{T^n(j_0)\}_{n\in\mathbb{N}}$ converges to υ with respect to $\tau(p^s)$, for any $j_0 \in M$. Moreover, $p(\upsilon,\upsilon) = 0$.

We prove the same in dual pmetric space.

Theorem 8. *Let* (M, p^*) *be a complete dual pmetric space and* $T : M \to M$ *be a mapping such that for all* $j, k \in M$ *and* $\beta \in \mathcal{B}$

$$|p^*(T(j), T(k))| \le \beta\left(\mathcal{Q}(j,k)\right)\mathcal{Q}(j,k), \tag{4.1}$$

where

$$\mathcal{Q}(j,k) = \max\left\{|p^*(j,k)|, |p^*(j,T(j))|, |p^*(k,T(k))|\right\}.$$

Then T *has a unique fixed point* υ^* *in* M*.*

Proof. Let j_0 be an initial point in M and $j_n = T(j_{n-1})$, $n \ge 1$, an iterative sequence starting with j_0. If there exists a positive integer r such that $j_{r+1} = j_r$, then j_r is the fixed point of T and it completes the proof. Suppose that $j_n \ne j_{n+1}$ for all $n \in \mathbb{N}$ and by (4.1), we have

$$\begin{aligned} |p^*(j_{n+1}, j_{n+2})| &= |p^*(T(j_n), T(j_{n+1}))| \\ &\le \beta\left(\mathcal{Q}(j_n, j_{n+1})\right)\mathcal{Q}(j_n, j_{n+1}) \qquad (4.2)\\ &= \beta\left(|p^*(j_n, j_{n+1})|\right)|p^*(j_n, j_{n+1})|, \\ |p^*(j_{n+1}, j_{n+2})| &< |p^*(j_n, j_{n+1})|, \text{ since } \beta \in \mathcal{B}, \qquad (4.3) \end{aligned}$$

where

$$\mathcal{Q}(j_n, j_{n+1}) = \max\left\{|p^*(j_n, j_{n+1})|, |p^*(j_n, j_{n+1})|, |p^*(j_{n+1}, j_{n+2})|\right\} = |p^*(j_n, j_{n+1})|.$$

For if $\mathcal{Q}(j_n, j_{n+1}) = |p^*(j_{n+1}, j_{n+2})|$ then we get a contradiction from (4.2). The inequality (4.3) implies that $\{|p^*(j_n, j_{n+1})|\}_{n=1}^{\infty}$ is a monotone and bounded above sequence and hence it is convergent and converges to a point α_1, that is, $\lim_{n\to\infty} |p^*(j_n, j_{n+1})| = \alpha_1 \ge 0$. If $\alpha_1 = 0$, then we have done but if $\alpha_1 > 0$, then from (4.3) we have

$$|p^*(j_{n+1}, j_{n+2})| \le \beta(|p^*(j_n, j_{n+1})|)|p^*(j_n, j_{n+1})|,$$

which implies that

$$\left|\frac{p^*(j_{n+1},j_{n+2})}{p^*(j_n,j_{n+1})}\right| \leq \beta(|p^*(j_n,j_{n+1})|).$$

Taking limit we have

$$\lim_{n\to\infty}\beta(|p^*(j_n,j_{n+1})|)=1.$$

Since $\beta\in\mathcal{B}$, $\lim_{n\to\infty}|p^*(j_n,j_{n+1})|=0$ entails $\alpha_1=0$. Hence

$$\lim_{n\to\infty}p^*(j_n,j_{n+1})=0.$$

Similarly, using (4.1) we can prove that

$$\lim_{n\to\infty}p^*(j_n,j_n)=0.$$

Now since $d_{p^*}(j_n,j_{n+1})=p^*(j_n,j_{n+1})-p^*(j_n,j_n)$, we deduce that $\lim_{n\to\infty}d_{p^*}(j_n,j_{n+1})=0$ for all $n\geq 1$. Now, we show that the sequence $\{j_n\}$ is a Cauchy sequence in $(M,d^s_{p^*})$. Suppose on contrary that $\{j_n\}$ is not a Cauchy sequence. Then given $\epsilon>0$, we will construct a pair of subsequences $\{j_{m_r}\}$ and $\{j_{n_r}\}$ violating the following condition for least integer n_r such that $m_r>n_r>r$ where $r\in\mathbb{N}$

$$d_{p^*}(j_{m_r},j_{n_r})\geq\epsilon. \tag{4.4}$$

In addition, upon choosing the smallest possible m_r, we may assume that

$$d_{p^*}(j_{m_r},j_{n_{r-1}})<\epsilon.$$

By the triangle inequality, we have

$$\begin{aligned}\epsilon &\leq d_{p^*}(j_{m_r},j_{n_r})\\ &\leq d_{p^*}(j_{m_r},j_{n_{r-1}})+d_{p^*}(j_{n_{r-1}},j_{n_r})\\ &< \epsilon+d_{p^*}(j_{n_{r-1}},j_{n_r}).\end{aligned}$$

That is,

$$\epsilon<\epsilon+d_{p^*}(j_{n_{r-1}},j_{n_r}) \tag{4.5}$$

for all $r\in\mathbb{N}$. In the view of (4.5) and (2.3), we have

$$\lim_{r\to\infty}d_{p^*}(j_{m_r},j_{n_r})=\epsilon. \tag{4.6}$$

Again using the triangle inequality, we have

$$d_{p^*}(j_{m_r},j_{n_r})\leq d_{p^*}(j_{m_r},j_{m_{r+1}})+d_{p^*}(j_{m_{r+1}},j_{n_{r+1}})+d_{p^*}(j_{n_{r+1}},j_{n_r})$$

and

$$d_{p^*}(j_{m_{r+1}},j_{n_{r+1}})\leq d_{p^*}(j_{m_{r+1}},j_{m_r})+d_{p^*}(j_{m_r},j_{n_r})+d_{p^*}(j_{n_r},j_{n_{r+1}}).$$

Taking limit as $r\to+\infty$ and using (2.3) and (4.6), we obtain

$$\lim_{r\to+\infty}d_{p^*}(j_{m_{r+1}},j_{n_{r+1}})=\epsilon.$$

Now from contractive condition (4.1), we have

$$\begin{aligned}|p^*(j_{n_{r+1}},j_{m_{r+2}})| &= |p^*(T(j_{n_r}),T(j_{m_{r+1}}))|,\\ &\leq \beta(|p^*(j_{n_r},j_{m_{r+1}})|)|p^*(j_{n_r},j_{m_{r+1}})|.\end{aligned}$$

We conclude that

$$\left|\frac{p^*(j_{n_{r+1}},j_{m_{r+2}})}{p^*(j_{n_r},j_{m_{r+1}})}\right|\leq\beta(|p^*(j_{n_r},j_{m_{r+1}})|).$$

By using (2.3), letting $r\to+\infty$ in the above inequality, we obtain

$$\lim_{r\to\infty}\beta(|p^*(j_{n_r},j_{m_{r+1}})|)=1.$$

Since $\beta\in\mathcal{B}$, $\lim_{r\to\infty}|p^*(j_{n_r},j_{m_{r+1}})|=0$ and hence $\lim_{r\to\infty}d_{p^*}(j_{n_r},j_{m_{r+1}})=0<\epsilon$ which contradicts our assumption (4.4). Arguing like above, we can have $\lim_{r\to\infty}d_{p^*}(j_{m_r},j_{n_{r+1}})=0<\epsilon$. Hence $\{j_n\}$ is a Cauchy sequence in $(M, d^s_{p^*})$, that is, $\lim_{n,m\to\infty} d^s_{p^*}(j_n, j_m) = 0$. Since $(M, d^s_{p^*})$ is a complete metric space, $\{j_n\}$ converges to a point υ in M, i.e., $\lim_{n\to\infty} d^s_{p^*}(j_n, \upsilon) = 0$. Then from Lemma 3, we get

$$\lim_{n\to\infty} p^*(\upsilon, j_n) = p^*(\upsilon, \upsilon) = \lim_{n,m\to\infty} p^*(j_n, j_m) = 0. \tag{4.7}$$

We are left to prove that υ is a fixed point of T. For this purpose, using contractive condition (4.2) and (4.7), we get

$$\begin{aligned} |p^*(j_{n+1}, T(\upsilon))| &= |p^*(T(j_n), T(\upsilon))| \\ &\leq \beta(|p^*(j_n, \upsilon)|)|p^*(j_n, \upsilon)|, \\ \lim_{n\to\infty} |p^*(j_{n+1}, T(\upsilon))| &\leq \lim_{n\to\infty} \beta(p^*(j_n, \upsilon))p^*(j_n, \upsilon). \end{aligned}$$

This shows that $p^*(\upsilon, T(\upsilon)) = 0$. So from (p^*_1) and (p^*_2) we deduce that $\upsilon = T(\upsilon)$ and hence υ is a fixed point of T. Uniqueness is obvious. □

Corollary 1. *Let (M, p) be a complete partial metric space and $T : M \to M$ be a mapping. If for any $j, k \in M$ and $\beta \in \mathcal{B}$, T satisfies the condition*

$$p(T(j), T(k)) \leq \beta\left(\mathcal{Q}(j, k)\right)\mathcal{Q}(j, k), \tag{4.8}$$

where $\mathcal{Q}(j, k) = \max\{p(j, k), p(j, T(j)), p(k, T(k))\}$, then T has a unique fixed point υ^* in M.

Proof. Since the restriction of p^* to $\mathbb{R}^+_0$, that is, $p^*|_{\mathbb{R}^+_0}$, is a partial metric p, the result is obvious. □

The following example illustrates Theorem 8 and shows that condition (4.1) in dual pmetric space is more general than contractivity condition (4.8) in partial metric space. This example also emphasis the use of absolute value function in contractive condition (4.1).

Example 3. *Let $M = [-1, 0]$ and define the functional $p^*{}_\vee : M \times M \to M$ by $p^*{}_\vee(j, k) = \max\{j, k\}$ for all $j, k \in M$. Then $(X, p^*{}_\vee)$ is a complete dualistic partial metric space. Define the mapping $T : X \to X$ and β by*

$$T(j) = \frac{j}{2} \text{ and } \beta(|j|) = \frac{9}{10}, \text{ for all } j \in M.$$

Without loss of generality we may assume that $j \geq k$ and then,

$$\begin{aligned} |p^*{}_\vee(T(j), T(k))| &= \left|\frac{j}{2} \vee \frac{k}{2}\right| = \left|\frac{j}{2}\right|, \\ |p^*{}_\vee(j, k)| &= |j|, \\ |p^*{}_\vee(j, T(j))| &= \left|j \vee \frac{j}{2}\right| = \left|\frac{j}{2}\right|, \\ |p^*{}_\vee(k, T(k))| &= \left|k \vee \frac{k}{2}\right| = \left|\frac{k}{2}\right|. \end{aligned}$$

Thus $\mathcal{Q}(j, k) = \max\left\{|j|, \left|\frac{k}{2}\right|\right\}$ and consider

$$\begin{aligned} |p^*{}_\vee(T(j), T(k))| &\leq \beta\left(\mathcal{Q}(j, k)\right)\mathcal{Q}(j, k) \\ \left|\frac{j}{2}\right| &\leq \beta(|j|)|j| \text{ if } \mathcal{Q}(j, k) = |j| \\ \left|\frac{j}{2}\right| &\leq \beta\left(\left|\frac{k}{2}\right|\right)\left|\frac{k}{2}\right| \text{ if } \mathcal{Q}(j, k) = \left|\frac{k}{2}\right|. \end{aligned}$$

The above inequalities are true for all $j, k \in X$. Therefore, the contractive condition (4.1) holds true. Thus all the conditions of Theorem 8 are satisfied by the mapping T. Note that $j = 0$ is the unique fixed point of T. Moreover, the contractive condition (4.8) in the statement of Corollary 1 does not exist for this particular case and hence the contractive condition (4.1) cannot be replaced with contractive condition (4.8) in Theorem 8 and as a result, Corollary 1 fails to ensure the fixed point of T.

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The approximation on analytic functions of infinite order represented by Laplace-Stieltjes transforms convergent in the half plane *

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Abstract

One purpose of this paper is to investigate the growth of analytic function represented by Laplace-Stieltjes transform which is of infinite order and converges in the half plane, and a necessary and sufficient conditions on the growth of Laplace-Stieltjes transforms of finite X_U-order was obtained. Besiders, we further investigate the error in approximating on Laplace-Stieltjes transform of finite X_U-order, and obtained some relations between the error and growth of Laplace-Stieltjes transforms of finite X_U-order.

Key words: approximation, X_U-order, Laplace-Stieltjes transform.

2010 Mathematics Subject Classification: 44A10, 30E10.

1 Introduction and basic notes

Laplace-Stieltjes transform

$$G(s)=\int_0^{+\infty} e^{-sx}d\alpha(x), \qquad s=\sigma+it, \tag{1}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0,Y](0<Y<+\infty)$, and σ and t are real variables, named for Pierre-Simon Laplace and Thomas Joannes Stieltjes, is an integral transform similar to the Laplace transform. It can be used in many fields of mathematics, such as functional analysis, and certain areas of theoretical and applied probability.

For Laplace-Stieltjes transform (1), Widder in [18] pointed out that $G(s)$ can become the classical Laplace integral form

$$G(s)=\int_0^{\infty} e^{-st}\varphi(t)dt,$$

when $\alpha(t)$ is absolutely continuous. Moreover, if $\alpha(t)$ is a step-function, we can choose a sequence $\{\lambda_n\}_0^\infty$ satisfying

$$0\le\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty \ \ as \ \ n\to\infty, \tag{2}$$

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and

$$\alpha(x) = \begin{cases} a_1 + a_2 + \cdots + a_n, & \lambda_n \le x < \lambda_{n+1}; \\ 0, & 0 \le x < \lambda_1; \\ \dfrac{\alpha(x+) + \alpha(x-)}{2}, & x > 0, \end{cases}$$

by Theorem 1 in [18, Page 36], then $G(s)$ becomes a Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \qquad s = \sigma + it. \tag{3}$$

(σ, t are real variables), a_n are nonzero complex numbers.

Yu J. R. in 1963 [25] first investigated the growth and value distribution of Laplace-Stieltjes transform (1), and obtained the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence and uniform convergence and the Borel line of Laplace-Stieltjes transforms. After his works, many mathematicians further studied some properties on the growth and value distribution of Laplace-Stieltjes transforms, and there were a number of results about this subject, such as: Batty C. J. K., M. N. Sheremeta, Kong Y. Y., Sun D. C., Huo Y. Y. and Xu H. Y. investigated the growth of analytic functions with kinds of order defined by Laplace-Stieltjes transforms (see [1, 3, 4, 5, 6, 19, 22]), and Yu J. R., Shang L. N., Gao Z. S., and Xu H. Y. investigated the value distribution of such functions (see [11, 20, 21, 25]). Moreover, as for Dirichlet series (3), a special form of Laplace-Stieltjes transform, considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series and lots of interesting results can be founded in (see [2, 9, 10, 12, 15, 16, 17, 23, 24]).

Luo and Kong [7, 8] in 2012 and 2014 studied the growth of the following form of Laplace-Stieltjes transform

$$F(s) = \int_0^{+\infty} e^{sx} d\alpha(x), \qquad s = \sigma + it, \tag{4}$$

where $\alpha(x)$ is stated as in (1), and $\{\lambda_n\}$ satisfy (2) and

$$\limsup_{n\to+\infty}(\lambda_{n+1} - \lambda_n) = h < +\infty, \qquad \limsup_{n\to\infty} \frac{n}{\lambda_n} = E < +\infty. \tag{5}$$

Set

$$A_n^* = \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{ity} d\alpha(y) \right|,$$

by using the same argument as in [25], we can get the similar results about the abscissa of uniformly convergent of $F(s)$ easily. If

$$\limsup_{n\to+\infty} \frac{\log A_n^*}{\lambda_n} = 0, \tag{6}$$

by (2), (5), (6) and Ref. [25], one can get that $\sigma_u^F = 0$, *i.e.*, $F(s)$ is analytic in the half plane.

Set

$$M_u(\sigma, F) = \sup_{0<x<+\infty, -\infty<t<+\infty} \left| \int_0^x e^{(\sigma+it)y} d\alpha(y) \right|, \quad \mu(\sigma, F) = \max_{n\in N}\{A_n^* e^{\lambda_n \sigma}\}(\sigma < +\infty),$$

and

Definition 1.1 *If the Laplace-Stieltjes transform (4) satisfies $\sigma_u^F = 0$ (the sequence $\{\lambda_n\}$ satisfy (2),(5) and (6)) and*

$$\limsup_{\sigma\to+\infty} \frac{\log^+ \log^+ M_u(\sigma, F)}{-\log(-\sigma)} = \rho,$$

we call $F(s)$ is of order ρ in the half plane; If $\rho \in (0, +\infty)$, the type of $F(s)$ is defined by

$$\tau = \limsup_{\sigma \to 0^-} \frac{\log^+ M_u(\sigma, F)}{(-\frac{1}{\sigma})^\rho},$$

where $\log^+ x = \begin{cases} \log x, & x \geq 1 \\ 0, & x < 1. \end{cases}$

Remark 1.1 *For $\rho = 0, 0 < \rho < \infty, \rho = \infty$, $F(s)$ can be called Laplace-Stieltjes transform of zero order, finite order, infinite order, respectively.*

For $\rho = \infty$, we will give the definition of the X-order of Laplace-Stieltjes transform (4) as follows.

Definition 1.2 *If Laplace-Stieltjes transform (4) of infinite order satisfies*

$$\limsup_{\sigma \to 0^-} \frac{X(\log^+ M_u(\sigma, F))}{-\log(-\sigma)} = \rho_X,$$

where $X(x) \in \mathfrak{F}$, then ρ^* is called the X-order of $F(s)$, and $\mathfrak{F}$ is the class of all functions $X(x)$ satisfies the following conditions:

(i) $X(x)$ is defined on $[a, +\infty), a > 0$, and positive, strictly increasing, differentiable and tends to $+\infty$ as $x \to +\infty$;

(ii) $xX'(x) = o(1)$ as $x \to +\infty$.

We investigate the growth of Laplace-Stieltjes transform $F(s)$ with finite X-order, and obtain the following theorem.

Theorem 1.1 *Let $F(s) \in L$ be of X-order $\rho_X (0 < \rho_X < \infty)$, then*

$$\limsup_{\sigma \to 0^-} \frac{X(\log \mu(\sigma, F))}{-\log(-\sigma)} = \limsup_{\sigma \to 0^-} \frac{X(\log M_u(\sigma, F))}{-\log(-\sigma)}, \tag{7}$$

and

$$\limsup_{n \to \infty} \frac{X(\lambda_n)}{\log \lambda_n - \log^+ \log^+ A_n^*} = \rho_X = \limsup_{\sigma \to 0^-} \frac{X(\log M_u(\sigma, F))}{-\log(-\sigma)}. \tag{8}$$

Thus, a question arises naturally: *what may happen when $\rho_X = \infty$ in Theorem 1.1?* Inspired by this question, we will further investigate the growth of Laplace-Stieltjes transform (4) by using the type function of Sun [15], and obtain the following results.

Theorem 1.2 *If Laplace-Stieltjes transform $F(s) \in L$ is of infinite X-order, then*

$$\limsup_{\sigma \to 0^-} \frac{X(\log^+ M_u(\sigma, F))}{\log U\left(-\frac{1}{\sigma}\right)} = \tau_X \Longleftrightarrow \limsup_{\sigma \to 0^-} \frac{X(\log^+ \mu(\sigma, F))}{\log U\left(-\frac{1}{\sigma}\right)} = \tau_X.$$

where $0 < \tau_X < \infty$ and $U(x) = x^{\rho(x)}$ satisfies the following conditions

(i) $\rho(x)$ is monotone and $\lim_{x \to \infty} \rho(x) = \infty$;

(ii) $\lim_{x \to \infty} \frac{U(x')}{U(x)} = 1$, where $x' = x + \frac{x \log x}{\log U(x) \log^2 \log U(x)}$.

Remark 1.2 *If Laplace-Stieltjes transform $F(s)$ of infinite order has infinite X-order and satisfies*

$$\limsup_{\sigma \to 0^-} \frac{X(\log^+ M_u(\sigma, F))}{\log U\left(-\frac{1}{\sigma}\right)} = \tau_X,$$

then τ_X is called the X_U-order of Laplace-Stieltjes transform $F(s)$.

Theorem 1.3 *Let $F(s)\in L$ are of infinite X-order, then*

$$\limsup_{\sigma\to 0^-}\frac{X(\log M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})}=\tau_X \Longleftrightarrow \limsup_{n\to\infty}\frac{X(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})}=\tau_X, \tag{9}$$

where $0<\tau_X<\infty$ and $U(x)$ is stated as in Theorem 1.2.

We denote $\overline{L}_\beta$ to be the class of all the functions $F(s)$ of the form (4) which are analytic in the half plane $\Re s<\beta(-\infty<\alpha<\infty)$ and the sequence $\{\lambda_n\}$ satisfies (2) and (5), and denote L to be the class of all the functions $F(s)$ of the form (4) which are analytic in the half plane $\Re s<0$ and the sequence $\{\lambda_n\}$ satisfies (2), (5) and (6). Thus, if $-\infty<\beta<0$ and $F(s)\in L$, then $F(s)\in\overline{L}_\beta$; if $0<\beta<+\infty$ and $F(s)\in\overline{L}_\beta$, then $F(s)\in L$. If $A_n^*=0$ for $n\geq k+1$, and $A_n^*\neq 0$, then $F(s)$ will be called an exponential polynomial of degree k usually denoted by p_k, *i.e.*, $p_k(s)=\int_0^{\lambda_k}\exp(sy)d\alpha(y)$. When we choice a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in tems of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^k b_i\exp(s\lambda_i)$ and we use Π_n to denote the class of exponential polynomials of degree n. For $F(s)\in\overline{L}_\beta,-\infty<\beta<+\infty$, we denote $E_n(F,\alpha)$ to be the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F,\beta)=\inf_{p\in\Pi_n}\parallel F-p\parallel_\beta,\quad n=1,2,\dots,$$

where

$$\parallel F-p\parallel_\beta=\max_{-\infty<t<+\infty}|F(\beta+it)-p(\beta+it)|.$$

In 2015 and 2017, C. Singhal and G. S. Srivastava [13, 14] investigated the approximation of analytic functions defined by Laplace-Stieltjes transforms of finite order, and obtained the following results.

Theorem 1.4 *(see [13, Theorem 3.5]). Let $F(s)\in L$ be of the order ρ and $-\infty<\beta<0$. Then*

$$\rho=\limsup_{n\to+\infty}\frac{\log^+\log^+[E_n(F,\beta)\exp(-\beta\lambda_{n+1})]}{\log\lambda_{n+1}-\log^+\log^+[E_n(F,\beta)\exp(-\beta\lambda_{n+1})]}.$$

Theorem 1.5 *(see [13, Theorem 3.6]). Let $F(s)\in L$, belongs to the class $L(-\infty<\beta<0)$ having order $\rho(0<\rho<\infty)$. Then $F(s)$ is of type τ if and only if*

$$\tau=\frac{\rho^\rho}{(\rho+1)^{\rho+1}}\limsup_{n\to+\infty}\frac{\{\log^+[E_n(F,\beta)\exp(-\beta\lambda_{n+1})]\}^{\rho+1}}{(\lambda_{n+1})^\rho}.$$

In this paper, we further investigated the approximation of analytic function defined by Laplace-Stieltjes transform and obtained the relations between the error $E_n(F,\beta)$ and the growth order of $F(s)$ when $F(s)$ is of infinite order as follows.

Theorem 1.6 *Let $F(s)\in L$ be of finite X-order ρ_X, then for any real number $-\infty<\beta<0$, we have*

$$\limsup_{n\to+\infty}\frac{X(\lambda_n)}{\log\lambda_n-\log^+\log^+[E_{n-1}(F,\beta)e^{-\beta\lambda_n}]}=\rho_X.$$

Theorem 1.7 *If $F(s)\in L$ is of infinite X-order, then for any fixed real number $-\infty<\beta<0$, we have*

$$\limsup_{\sigma\to 0^-}\frac{X(\log^+ M(\sigma,F))}{\log U\left(-\frac{1}{\sigma}\right)}=\tau_X \Longleftrightarrow \limsup_{n\to+\infty}\Psi_n(F,\beta,\lambda_n)=\tau_X;$$

where

$$\Psi_n(F,\beta,\lambda_n)=\frac{X\left(\log^+[E_{n-1}(F,\beta)e^{-\beta\lambda_n}]\right)}{\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)e^{-\beta\lambda_n}]}\right)}.$$

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2 Proofs of Theorems 1.1-1.3

To prove Theorems 1.1-1.3, we require some lemmas as follows.

Lemma 2.1 *Let $X(x)\in\mathfrak{F}$ and c be a constant, and $\varphi(x)$ be the function such that*

$$\limsup_{x\to+\infty}\frac{\log^+\varphi(x)}{\log x}=\varrho,\quad (0\le\varrho<\infty),$$

and if the real function $M(x)$ satisfies $\limsup_{x\to+\infty}\frac{X(\log M(x))}{\log x}=\nu(>0)$. Then we have

$$\limsup_{x\to+\infty}\frac{X(\log M(x)+c)}{\log x}=\nu,\quad \limsup_{x\to+\infty}\frac{X(\varphi(x)\log M(x))}{\log x}=\nu.$$

Proof: From the properties of $X(x)$, we can easily prove

$$\limsup_{x\to+\infty}\frac{X(\log M(x)+c)}{\log x}=\nu.$$

Next, we will divide into two cases to prove

$$\limsup_{x\to+\infty}\frac{X(\varphi(x)\log M(x))}{\log x}=\nu.$$

Case 1. If $\varphi(x)$ is not a constant. From the assumptions of Lemma 2.1, it follows that $\varphi(x)\to\infty$ as $x\to\infty$. Then, for sufficiently large x, we have $\varphi(x)>1$. From $X(x)\in\mathfrak{F}$, we have $\lim_{x\to+\infty}\log M(x)=\infty$. Then from the Cauchy mean value theorem, there exists $\xi(\log M(x)<\xi<X(x)\log M(x))$ satisfying

$$\frac{X(\varphi(x)\log M(x))-X(\log M(x))}{\log(\varphi(x)\log M(x))-\log\log M(x)}=\frac{X'(\xi)}{(\log\xi)'}=\xi X'(\xi),$$

that is,

$$X(\varphi(x)\log M(x))=X(\log M(x))+\log\varphi(x)\xi X'(\xi). \tag{10}$$

Since $xX'(x)=o(1)$ as $x\to+\infty$ and $\limsup_{x\to+\infty}\frac{\log\varphi(x)}{\log x}=\varrho,(0\le\varrho<\infty)$, by (10), we can get the conclusion of Lemma 2.1.

Case 2. If $\varphi(x)$ is a constant. By using the same argument as in Case 1, we can prove that the conclusion of Lemma 2.1 is true.

Thus, this completes the proof of Lemma 2.1. □

2.1 The proof of Theorem 1.1

Set

$$I(x;\sigma+it)=\int_{\lambda_n}^{x}\exp\{(\sigma+it)y\}d\alpha(y).$$

From (5), there exists $\eta>0$ satisfying $0<\lambda_{n+1}-\lambda_n\le\eta$ $(n=1,2,3,\ldots)$. When σ is sufficiently close to $0-$, it follows $e^{-\eta\sigma}<2$. When $x>\lambda_n$, it follows

$$\begin{aligned}\int_{\lambda_n}^{x}\exp\{ity\}d\alpha(y)&=\int_{\lambda_n}^{x}e^{-\sigma y}d_yI(y;\sigma+it)\\&=I(y;\sigma+it)e^{-\sigma y}|_{\lambda_n}^{x}+\sigma\int_{\lambda_n}^{x}e^{-\sigma y}I(y;\sigma+it)dy.\end{aligned}$$

Then, for $\sigma < 0$, it follows

$$\left|\int_{\lambda_n}^{x} \exp\{ity\}d\alpha(y)\right| \leq M_u(\sigma, F)\left[|e^{-\sigma x} + e^{-\sigma\lambda_n}| + |e^{-\sigma x} - e^{-\sigma\lambda_n}|\right] \leq 2M_u(\sigma, F)e^{-\sigma x}.$$

Thus, for any $\sigma < 0$ and any $x \in (\lambda_n, \lambda_{n+1}]$, we have

$$\left|\int_{\lambda_n}^{x} \exp\{ity\}d\alpha(y)\right| \leq 2M_u(\sigma, F)e^{-\sigma\lambda_n}e^{-\sigma\eta} \leq 4M_u(\sigma, F)e^{-\sigma\lambda_n},$$

that is,

$$\mu(\sigma, F) \leq 4M_u(\sigma, F). \tag{11}$$

Let $I_k(x; it) = \int_{\lambda_k}^{x} \exp(ity)d\alpha(y)(\lambda_k < x \leq \lambda_{k+1})$, then for $\lambda_k < x \leq \lambda_{k+1}, -\infty < t < +\infty$, we have $|I_k(x; it)| \leq A_k^* \leq \mu(\sigma, F)e^{-\lambda_k\sigma}$. Thus, for $\varepsilon > 0, \lambda_n < x \leq \lambda_{n+1}$ and $\sigma < 0$, we have

$$\begin{aligned}
\int_{0}^{x} \exp\{(\sigma + it)y\}d\alpha(y) &= \sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \exp\{(\sigma + it)y\}d\alpha(y) + \int_{\lambda_n}^{x} \exp\{(\sigma + it)y\}d\alpha(y) \\
&= \sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \exp\{\sigma y\}d_yI_k(y; it) + \int_{\lambda_n}^{x} \exp\{\sigma y\}d_yI_n(y; it) \\
&= \sum_{k=1}^{n-1}\left[\exp(\lambda_{k+1}\sigma)I_k(\lambda_{k+1}; it) - \sigma\int_{\lambda_k}^{\lambda_{k+1}} \exp\{\sigma y\}I_k(y; it)dy\right] \\
&\quad + \exp(x\sigma)I_n(x; it) - \sigma\int_{\lambda_n}^{x} \exp\{\sigma y\}I_n(y; it)dy.
\end{aligned}$$

Hence,

$$\left|\int_{0}^{x} \exp\{(\sigma + it)y\}d\alpha(y)\right| \leq \sum_{k=1}^{n} A_k^* e^{\lambda_k\sigma} \leq \mu((1-\varepsilon)\sigma, F)\sum_{k=1}^{\infty} e^{\lambda_k\varepsilon\sigma}.$$

From the second equation of (5), for the above $\varepsilon > 0$, there exists a positive integer N such that $\lambda_n \geq \frac{n}{E+\varepsilon}$ for $n \geq N$. Thus, for sufficiently small $\sigma < 0$, we have

$$M_u(\sigma, F) \leq \mu((1-\varepsilon)\sigma, F)\left(N + \sum_{k=N+1}^{\infty} \exp\left[\frac{k\varepsilon}{E+\varepsilon}\sigma\right]\right) \leq K(\varepsilon)\mu((1-\varepsilon)\sigma, F)(-\frac{1}{\sigma}). \tag{12}$$

From (11)-(12) and by Lemma 2.1, we can prove (7) easily.

By using the same argument as in [20, Theorem 4], we prove (8) easily.

Therefore, this completes the proof of Theorem 1.1.

2.2 The Proof of Theorem 1.2

By Lemma 2.1 and from (11)-(12), the conclusion of Theorem 1.2 can be proved easily.

2.3 The proof of Theorem 1.3

We firstly prove the sufficiency of Theorem 1.3. Let $W(x)$ be the inverse function of $X(x)$, and $V(x)$ be the inverse function of $U(x)$. Next, we will divide into two steps as follows.

Step One. Suppose that

$$\limsup_{n\to+\infty} \frac{X(\log^+ A_n^*)}{\log U\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} = \tau_X, \tag{13}$$

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thus for any small $\varepsilon(>0)$ and sufficiently large n, we have

$$\log^+ A_n^* < W\left[(\tau_X+\varepsilon)\log U\left(\frac{\lambda_n}{\log^+ A_n^*}\right)\right],$$

it follows

$$\frac{\lambda_n}{\log^+ A_n^*} > V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right], \qquad \log^+ A_n^* < \frac{\lambda_n}{V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right]}.$$

Thus, we have

$$\log^+ A_n^* e^{\lambda_n\sigma} \le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right]\right)^{-1}+\sigma\right). \tag{14}$$

For $\sigma\to 0^-$, set

$$\begin{aligned} I &= W\left[(\tau_X+\varepsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right] \qquad (15)\\ &< W\left[(\tau_X+\varepsilon)\log U\left(-\frac{1}{\sigma}+\frac{-\frac{1}{\sigma}\log(-\frac{1}{\sigma})}{\log U(-\frac{1}{\sigma})\log^2\log U\left(-\frac{1}{\sigma}\right)}\right)\right], \end{aligned}$$

then it follows

$$-\frac{1}{\sigma}-\frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)} = V\left(\exp\left\{\frac{1}{\tau_X+\varepsilon}X(I)\right\}\right). \tag{16}$$

If $\log^+ A_n^*\le I$, then for $\sigma\to 0^-$, it follows from (14)-(16) and the properties of $U(x)$ that

$$\begin{aligned} \log^+ A_n^* e^{\lambda_n\sigma} &\le \log^+ A_n^* \le I = W\left[(\tau_X+\varepsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right]\\ &\le W\left[(\tau_X+2\varepsilon)\log U(-\frac{1}{\sigma})\right]. \qquad (17) \end{aligned}$$

If $\log^+ A_n^* > I$, then from (14)-(16), we have

$$\begin{aligned} \log^+ A_n^* e^{\lambda_n\sigma} &\le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(\log^+ A_n^*)\right\}\right]\right)^{-1}+\sigma\right)\\ &\le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{\tau_X+\varepsilon}X(I)\right\}\right]\right)^{-1}+\sigma\right)\\ &= \lambda_n\frac{\sigma}{1+\log^2 U(-\frac{1}{\sigma})} < 0. \qquad (18) \end{aligned}$$

Hence, it follows from (17) and (18) that

$$\log\mu(\sigma,F)\le W\left[(\tau_X+2\varepsilon)\log U(-\frac{1}{\sigma})\right]. \tag{19}$$

From (19) and Theorem 1.2, and let $\varepsilon\to 0$, it follows

$$\limsup_{\sigma\to 0^-}\frac{X(\log M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})} \le \limsup_{n\to+\infty}\frac{X(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})} = \tau_X.$$

Step Two. Suppose that

$$\limsup_{\sigma\to 0^-}\frac{X(\log M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})}=J<\limsup_{n\to+\infty}\frac{X(\log^+ A_n^*)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})}=\tau_X. \tag{20}$$

Take $\eta>0$ such that $\tau_X=J+5\eta$, then for any $n\in N_+$ and sufficiently small $\sigma(<0)$, from (11) and (20), and by Lemma 2.1 we have

$$\log^+ A_n^* e^{\lambda_n\sigma}\le \log M_u(\sigma,F)+2\log 2<W\left((J+\eta)\log U(-\frac{1}{\sigma})\right), \tag{21}$$

and from (20), there exists a subsequence $\{n(\nu)\}$ such that

$$X(\log^+ A_{n(\nu)}^*)>(\tau_X-\eta)\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}). \tag{22}$$

Take the sequence $\{\sigma_\nu\}$ such that

$$W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)=\frac{\log^+ A_{n(\nu)}^*}{1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})}. \tag{23}$$

Thus, it follows form (21)-(23) that

$$\log^+ A_{n(\nu)}^* e^{\lambda_{n(\nu)}\sigma_\nu}<\frac{\log^+ A_{n(\nu)}^*}{1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})},$$

$$\Longrightarrow -\frac{1}{\sigma_\nu}\le\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\left(1+\frac{1}{\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})}\right),$$

$$\begin{aligned}\Longrightarrow U(-\frac{1}{\sigma_\nu})&\le U\left(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\left(1+\frac{1}{\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})}\right)\right) \\ &\le(1+\eta)U\left(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\right).\end{aligned} \tag{24}$$

From (23), we have

$$\log^+ A_{n(\nu)}^*=W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\right).$$

Thus, from the Cauchy mean value theorem and (24), there exists a real number ξ between $W((J+\eta)\log U(-\frac{1}{\sigma_\nu}))$ and $W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\right)$ such that

$$\begin{aligned}X\left(\log^+ A_{n(\nu)}^*\right)&=X\left(1+\log^2 U\left(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*}\right)W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\right)\\ &=X\left(W\left((J+\eta)\log U(-\frac{1}{\sigma_\nu})\right)\right)\\ &\quad+\log\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A_{n(\nu)}^*})\right)\xi X'(\xi),\end{aligned}$$

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and since

$$\lim_{\nu\to+\infty}\frac{\log\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}})\right)}{\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}})}=0,$$

then for sufficiently large ν and from (24), it follows

$$\begin{aligned} X\left(\log^+ A^*_{n(\nu)}\right) &= (J+\eta)\log U(-\frac{1}{\sigma_\nu})+K\xi X'(\xi)\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}}) \\ &= (J+3\eta)\log U(\frac{\lambda_{n(\nu)}}{\log^+ A^*_{n(\nu)}}), \end{aligned} \tag{25}$$

where K is a constant.

From (20) and (25), we obtain a contradiction with the condition $\eta=\frac{\tau_X-J}{5}>0$. Thus, we have

$$\limsup_{\sigma\to 0^-}\frac{X(\log M_u(\sigma,f))}{\log U(-\frac{1}{\sigma})}=\limsup_{n\to+\infty}\frac{X(\log^+ A^*_n)}{\log U(\frac{\lambda_n}{\log^+ A^*_n})}=\tau_X.$$

Therefore, this completes the proof of the sufficiency of Theorem 1.3.

By using the similar argument as in the above discussion, we can prove the necessity of Theorem 1.3.

Hence, this completes the proof of Theorem 1.3.

3 Proofs of Theorem 1.6 and Theorem 1.7

Here we only give the proof of Theorem 1.7 because the proof of Theorem 1.6 is similarly.

3.1 The Proof of Theorem 1.7

First of all, we prove " $\Longleftarrow$ " of Theorem 1.7. Next, we will divide into two steps as follows.

Step One. For convenience, hereinafter let $E_{n-1}:=E_{n-1}(F,\beta)$. Suppose that

$$\limsup_{n\to+\infty}\Psi_n(F,\beta,\lambda_n)=\limsup_{n\to+\infty}\frac{X(\log^+[E_{n-1}e^{-\beta\lambda_n}])}{\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}e^{-\beta\lambda_n}]}\right)}=\tau_X. \tag{26}$$

Then for sufficiently large positive integer n and any positive real number $\epsilon>0$, we have

$$\log^+[E_{n-1}e^{-\beta\lambda_n}]<W\left((\tau_X+\epsilon)\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}e^{-\beta\lambda_n}]}\right)\right).$$

By using the same argument as in the proof of Theorem 1.3, we have

$$\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}]\le\lambda_n\left(\left(V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(\log^+[E_{n-1}e^{-\beta\lambda_n}])\right\}\right)\right)^{-1}+\sigma\right). \tag{27}$$

For any fixed and sufficiently small $\sigma<0$, set

$$G=W\left((\tau_X+\epsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma}+\frac{1}{-\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}=V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(G)\right\}\right). \tag{28}$$

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If $\log^+[E_{n-1}e^{-\beta\lambda_n}] \le G$, for sufficiently large positive integer n, let

$$V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(\log^+[E_{n-1}e^{-\beta\lambda_n}])\right\}\right) \ge 1,$$

since $\sigma < 0$, and from (27),(28) and the definition of $U(x)$, we have

$$\begin{aligned}\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}] &\le \lambda_n\left(\left(V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(\log^+[E_{n-1}e^{-\beta\lambda_n}])\right\}\right)\right)^{-1}+\sigma\right)\\ &\le G = W\left((\tau_X+\epsilon)\log U\left(-\frac{1}{\sigma}-\frac{1}{\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right)\\ &\le W\left((\tau_X+\epsilon)\log\left[(1+o(1))U\left(-\frac{1}{\sigma}\right)\right]\right). \qquad (29)\end{aligned}$$

If $\log^+[E_{n-1}e^{-\beta\lambda_n}] > G$, it follows from (27) and (28) that

$$\begin{aligned}\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}] &\le \lambda_n\left(\left(V\left(\exp\left\{\frac{1}{\tau_X+\epsilon}X(G)\right\}\right)\right)^{-1}+\sigma\right)\\ &\le \lambda_n\left(\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log^2 U\left(-\frac{1}{\sigma}\right)}\right)^{-1}+\sigma\right) < 0. \qquad (30)\end{aligned}$$

Hence, it follows from (29) and (30) that for sufficiently large positive integer n

$$\log^+[E_{n-1}e^{(\sigma-\beta)\lambda_n}] \le W\left((\tau_X+\epsilon)\log\left[(1+o(1))U\left(-\frac{1}{\sigma}\right)\right]\right). \qquad (31)$$

For any $\beta < 0$, then from the definition of $E_k(F,\beta)$, there exists $p_1 \in \Pi_{n-1}$ satisfying

$$\|F-p_1\| \le 2E_{n-1}. \qquad (32)$$

And since

$$\begin{aligned}A_n^*\exp\{\beta\lambda_n\} &= \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^x \exp\{ity\}d\alpha(y)\right|\exp\{\beta\lambda_n\}\\ &\le \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^x \exp\{(\beta+it)y\}d\alpha(y)\right|\\ &\le \sup_{-\infty<t<+\infty}\left|\int_{\lambda_n}^\infty \exp\{(\beta+it)y\}d\alpha(y)\right|,\end{aligned}$$

then for any $p \in \Pi_{n-1}$, we have

$$A_n^*\exp\{\beta\lambda_n\} \le |F(\beta+it)-p(\beta+it)| \le \|F-p\|_\beta. \qquad (33)$$

Hence for any $\beta < 0$ and $F(s) \in L$, it follows from (32) and (33) that

$$A_n^*\exp\{\beta\lambda_n\} \le 2E_{n-1}, \qquad A_n^* \le 2E_{n-1}\exp\{-\beta\lambda_n\}.$$

that is,

$$A_n^*e^{\sigma\lambda_n} \le 2E_{n-1}e^{(\sigma-\beta)\lambda_n}. \qquad (34)$$

Thus, from (31) and (34), and by Lemma 2.1 and Theorem 1.2, let $\varepsilon \to 0$ we have

$$\limsup_{\sigma\to 0^-}\frac{X(\log^+ M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})} \le \tau_X.$$

Step Two. Suppose that

$$\limsup_{\sigma\to 0^-}\frac{X(\log^+ M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})}=\vartheta<\tau_X.$$

Then there exists any real number $\varepsilon(0<\varepsilon<\frac{\tau_X-\vartheta}{4})$, and for any sufficiently small $\sigma<0$ we have

$$\log M_u(\sigma,F)\le W\left((\vartheta+\varepsilon)\log U(-\frac{1}{\sigma})\right). \tag{35}$$

Since

$$\begin{aligned} E_{n-1}(F,\beta) &\le \| F-p_{n-1} \|_\beta \le |F(\beta+it)-p_{n-1}(\beta+it)| \\ &\le \left|\int_0^{+\infty}\exp\{(\beta+it)y\}d\alpha(y)-\int_0^{\lambda_n}\exp\{(\beta+it)y\}d\alpha(y)\right| \\ &= \left|\int_{\lambda_n}^{\infty}\exp\{(\beta+it)y\}d\alpha(y)\right|, \end{aligned} \tag{36}$$

for $\beta<\sigma<0$, and

$$\left|\int_{\lambda_k}^{\infty}\exp\{(\beta\gamma+it)y\}d\alpha(y)\right|=\lim_{b\to+\infty}\left|\int_{\lambda_k}^{b}\exp\{(\beta+it)y\}d\alpha(y)\right|.$$

Set

$$I_{j+k}(b;it)=\int_{\lambda_{j+k}}^{b}\exp\{ity\}d\alpha(y),\quad(\lambda_{j+k}<b\le\lambda_{j+k+1}),$$

then we have $|I_{j+k}(b;it)|\le A^*_{j+k}$. Thus, it follows

$$\begin{aligned} &\left|\int_{\lambda_k}^{b}\exp\{(\beta+it)y\}d\alpha(y)\right| \\ &=\left|\sum_{j=k}^{n+k-1}\int_{\lambda_j}^{\lambda_{j+1}}\exp\{\beta y\}d_yI_j(y;it)+\int_{\lambda_{n+k}}^{b}\exp\{\beta y\}d_yI_{n+k}(y;it)\right| \\ &=\left|\left[\sum_{j=k}^{n+k-1}e^{\lambda_{j+1}\beta}I_j(\lambda_{j+1};it)-\beta\int_{\lambda_j}^{\lambda_{j+1}}e^{\beta y}I_j(y;it)dy\right]\right. \\ &\quad\left.+e^{\beta b}I_{n+k}(b;it)-\beta\int_{\lambda_{n+k}}^{b}e^{\beta y}I_j(y;it)dy\right| \\ &\le\sum_{j=k}^{n+k-1}\left[A^*_je^{\lambda_{j+1}\beta}+A^*_j(e^{\lambda_{j+1}\beta}-e^{\lambda_j\beta})\right]+2e^{\beta\lambda_{n+k+1}}A^*_{n+k}-e^{\beta\lambda_{n+k}}A^*_{n+k} \\ &\le 2\sum_{j=k}^{n+k}A^*_ne^{\lambda_{n+1}\beta}. \end{aligned}$$

Because $b\to+\infty$ as $n\to+\infty$, thus it follows

$$\left|\int_{\lambda_k}^{\infty}\exp\{(\beta+it)y\}d\alpha(y)\right|\le 2\sum_{n=k}^{+\infty}A^*_n\exp\{\beta\lambda_{n+1}\}. \tag{37}$$

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Hence from (11), (36) and (37), we have

$$E_{n-1} \leq 2\sum_{k=n}^{\infty} A_{k-1}^{*} \exp\{\beta\lambda_k\} \leq 8M_u(\sigma, F)\sum_{k=n}^{\infty} \exp\{(\beta-\sigma)\lambda_k\}. \tag{38}$$

From (5), we can take $h'(0 < h' < h)$ such that $(\lambda_{n+1} - \lambda_n) \geq h'$ for $n \geq 0$. Then from (38), for $\sigma \geq \frac{\beta}{2}$, we have

$$\begin{aligned} E_{n-1} &\leq 8M_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}\sum_{k=n}^{\infty}\exp\{(\lambda_k-\lambda_n)(\beta-\sigma)\} \\ &\leq 8M_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}\exp\{-\frac{\beta}{2}h'n\}\sum_{k=n}^{\infty}(\exp\{\frac{\beta}{2}h'k\}) \\ &= 8M_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}\left(1-\exp\{\frac{\beta}{2}h'\}\right)^{-1}, \end{aligned}$$

that is,

$$E_{n-1} \leq KM_u(\sigma, F)\exp\{\lambda_n(\beta-\sigma)\}, \tag{39}$$

where K is a constant. Then for sufficiently small $\sigma < 0$ and $-\infty < \beta < \sigma < 0$, we have

$$M_u(\sigma, F) \geq K_1 E_{n-1}(F, \beta)e^{-\lambda_n(\beta-\sigma)} = K_1 E_{n-1}\exp\{-\beta\lambda_n\}e^{\lambda_n\sigma}, \tag{40}$$

where $K_3 = 1 - e^{\frac{\beta}{2}h'}$. Hence it follows from (35) and (40) that

$$\log^{+}\left[K_1 E_{n-1}\exp\{-\beta\lambda_n\}e^{\lambda_n\sigma}\right] \leq \log M_u(\sigma, F) \leq W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma})\right). \tag{41}$$

From the assumption, there exists a subsequence $\{\lambda_{n(p)}\}$ such that for sufficiently large p

$$X(\log^{+}[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]) > (\tau_X - \varepsilon)\log U\left(\frac{\lambda_{n(p)}}{\log^{+}[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]}\right). \tag{42}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$\begin{aligned} &W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma_p})\right) \\ =&\frac{\log^{+}[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]}{1+\log U(\frac{\lambda_{n(p)}}{\log^{+}[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^{+}[E_{n(p-1)}\exp\{-\beta\lambda_{n(p)}\}]})}. \end{aligned} \tag{43}$$

From (41) and (43), by using the same argument as in the proof of Theorem 1.3, we get

$$\begin{aligned} &\log^{+}[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}] \\ =&W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma_p})\right)\left(1+\frac{1}{\log U(\frac{\lambda_{n(p)}}{\log^{+}[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^{+}[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})}\right). \end{aligned}$$

Then by applying the Cauchy mean value theorem, there exists a real number $\xi \in (\zeta_1, \zeta_2)$ where

$$\zeta_1 = W\left((\vartheta+2\varepsilon)\log U(-\frac{1}{\sigma_p})\right),$$

and

$$\zeta_2=\zeta_1\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right),$$

such that

$$\begin{aligned}&X\left(\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]\right)\\=&X\left(W\left((\vartheta+2\varepsilon)(1+o(1))\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right)\right)\\&+\log\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right)\xi X'(\xi),\end{aligned}$$

Since

$$\lim_{p\to\infty}\frac{\log\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})\right)}{\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})}=0,$$

then for $p\to+\infty$ and let $\sigma\to 0^-$, it follows

$$X\left(\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]\right)=(\vartheta+2\varepsilon)(1+o(1))\log U(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}e^{-\beta\lambda_{n(p)}}]})+o(1). \tag{44}$$

From (41) and (44), by applying Lemma 2.1, we can obtain a contradiction with the assumption $0<\varepsilon<\frac{\tau_X-\vartheta}{4}$. Hence

$$\limsup_{\sigma\to 0^-}\frac{X(\log^+ M_u(\sigma,F))}{\log U(-\frac{1}{\sigma})}=\tau_X.$$

Hence, we complete the proof of the sufficiency of Theorem 1.7. By using the similar argument as in the above, we can prove the necessity of Theorem 1.7.

Therefore, this completes the proof of Theorem 1.7.

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q-ANALOGUE OF MODIFIED DEGENERATE CHANGHEE POLYNOMIALS AND NUMBERS

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Abstract. The Changhee polynomials and numbers are introduced in [3], and some interesting identities and properties of these polynomials are found by many researcher. In this paper, we consider the q-analogue of modified degenerated Changhee polynomials and derive some new and interesting identities and properties of those polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The *p-adic norm* $|\cdot|_p$ is defined normally as $|p|_p = \frac{1}{p}$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined by T. Kim as follows :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [3-8, 10-12, 16, 17, 19])}. \tag{1.1}$$

If we put $f_n(x) = f(x+n)$, then, by (1.1), we can derive the following very useful integral identity;

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) q^l, \tag{1.2}$$

where

$$[x]_{-q} = \frac{1-(-q)^x}{1-(-q)} \text{ and } [x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to1}[x]_q = x$ for each $x \in \mathbb{Z}_p$. In particular, if $n = 0$, then

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.3}$$

The Stirling numbers of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l \ (x \geq 0), \tag{1.4}$$

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and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}$$

(see [1, 20]). Note that

$$(\log(x+1))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!},\ (n\geq 0),$$

(see [1, 20]).

As is well-known, *q-Euler polynomials of order r* are defined by the generating function to be

$$\left(\frac{[2]_q}{1+q^de^{dt}}\right)^r\sum_{a=0}^{d-1}(-1)^aq^ae^{at} = \sum_{n=0}^{\infty} E_{n,q}^{(r)}\frac{t^n}{n!},,\ \text{(see [3-6, 9, 15-17, 19]).} \tag{1.5}$$

In the special case, $x=0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the *Euler numbers of order r*.

From (1.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{2}{e^t+1}\right)^r e^{xt}\\ &= e^{xt}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),\end{aligned} \tag{1.6}$$

and by (1.6), we have

$$E_n^{(r)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),\ (n\geq 0), \tag{1.7}$$

(see [3-6, 9, 15-17, 19]).

In [3], authors defined the *Changhee polynomials* as follows:

$$\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{2}{2+t}(1+t)^x,$$

and, in [17], authors defined the *modified degenerate Euler of order r* polynomials as follows:

$$\sum_{n=0}^{\infty}\xi_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right)^r(1+\lambda)^{\frac{t}{\lambda}x}. \tag{1.8}$$

Recently, Changhee numbers and polynomials are introduced by Kim et. al. in [3], and by many mathematicians, which are generalized and obtained many new and interesting properties(see [2, 9-14, 16, 18, 19]). In this paper, we consider the modified degenerate Changhee polynomials and numbers by using the p-adic invariant integral, and derive some new and interesting identities and properties of those polynomials.

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2. q-analogue of Modified degenerate Changhee polynomials and numbers

From now on, we assume that $t \in \mathbb{C}$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$.

The *modified degenerate q-Changhee polynomials* are defined by the generating function to be

$$\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} = \sum_{n=0}^{\infty} MCh_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

In the special case, $x=0$, $MCh_{n,\lambda,q} = MCh_{n,\lambda,q}(0)$ are called *q-modified degenerate Changhee numbers*.

Note that

$$\begin{aligned}\lim_{\lambda\to 0}\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} =& \frac{[2]_q}{q(1+t)+1}(1+t)^x \\ =& \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}.\end{aligned}$$

Since

$$\begin{aligned}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)} =& e^{\log(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}} \\ =& \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (x+y)^n(\log(1+t))^n\frac{1}{n!} \\ =& \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (x+y)^n\frac{1}{n!}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{t^l}{l!} \\ =& \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} MCh_{n,\lambda,q}(x)\frac{t^n}{n!} =& \frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} \\ =& \left(\sum_{n=0}^{\infty} MCh_{n,\lambda,q}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)x^l\frac{t^m}{m!}\right) \\ =& \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)x^l MCh_{n-m,\lambda,q}\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$MCh_{n,\lambda,q}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)MCh_{n-m,\lambda,q}x^l.$$

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Note that, by (1.3), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-q}(y) =&\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\ =&\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{t^n}{n!},\end{aligned} \tag{2.4}$$

and, by (2.2) and (1.7),

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-q}(y)\\ =&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m(x+y)^m S_1(n,m)d\mu_{-q}(y)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_{m,q}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.2. *For each* $n\in\mathbb{N}\cup\{0\}$,

$$\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-q}(y),$$

and

$$MCh_{n,\lambda}(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_{m,q}(x).$$

By replacing t as e^t-1 in (2.1), we have

$$\begin{aligned}\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{\lambda}}}(1+\lambda)^{\frac{t}{\lambda}x}=&\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{1}{n!}\left(e^t-1\right)^m\\ =&\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\ =&\sum_{n=0}^{\infty}\sum_{m=0}^{n}MCh_{m,\lambda,q}(x)S_2(n,m)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+(e^t-1))}}(1+\lambda)^{\frac{x}{\lambda}\log\left(1+(e^t-1)\right)}=&\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{x}}}(1+\lambda)^{\frac{xt}{\lambda}}\\ =&\sum_{n=0}^{\infty}\xi_{n,\lambda,q}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we obtain the following corollary.

Corollary 2.3. *For each nonnegative integer* n,

$$\xi_{n,\lambda,q}(x)=\sum_{m=0}^{n}MCh_{m,\lambda}(x)S_2(n,m).$$

By (1.3), we note that

$$\begin{aligned}[2]_q =&q\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+1}{\lambda}\log(1+t)}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&q\sum_{n=0}^{\infty}MCh_{n,\lambda,q}(1)\frac{t^n}{n!}+\sum_{n=0}^{\infty}MCh_{n,\lambda,q}\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\left(qMCh_{n,\lambda,q}(1)+MCh_{n,\lambda,q}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.8}$$

By (2.8), we obtain the following theorem.

Theorem 2.4. *For each positive integer n, we have*

$$MCh_{0,\lambda,q}=1,\ qMCh_{n,\lambda}(1)+MCh_{n,\lambda}=[2]_q\delta_{0,n},$$

where $\delta_{i,j}$ is the Kronecker's symbols.

For each $n\in\mathbb{N}$ with $n\equiv 1 \pmod 2$, by (1.2), we have

$$\begin{aligned}&q^n\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+n}{\lambda}\log(1+t)}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&[2]_q\sum_{a=0}^{n-1}(-1)^aq^a(1+\lambda)^{\frac{a}{\lambda}\log(1+t)}\\ =&\sum_{l=0}^{\infty}\left([2]_q\sum_{m=0}^{l}\sum_{a=0}^{n-1}(-1)^aq^aa^m\left(\frac{\log(1+\lambda)}{\lambda}\right)^mS_1(l,m)\right)\frac{t^l}{l!}\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}&q^n\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+n}{\lambda}\log(1+t)}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&\sum_{l=0}^{\infty}\left(q^nMCh_{l,\lambda,q}(n)+MCh_{l,\lambda,q}\right)\frac{t^l}{l!}.\end{aligned}\tag{2.10}$$

Hence, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.5. *For each nonnegative odd integer n and each nonnegative integer l, we have*

$$q^nMCh_{l,\lambda,q}(n)+MCh_{l,\lambda,q}=[2]_q\sum_{m=0}^{l}\sum_{a=0}^{n-1}(-1)^aq^aa^m\left(\frac{\log(1+\lambda)}{\lambda}\right)^mS_1(l,m).$$

From now on, we consider the *modified degenerate q-Changhee polynomials of order r* are defined as by the generating function to be

$$\sum_{n=0}^{\infty}MCh_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-q}(x_q)\cdots d\mu_{-q}(x_r).\tag{2.11}$$

When $x=0$, $MCh_{n,\lambda,q}^{(r)}=MCh_{n,\lambda,q}^{(r)}(0)$ are called *modified degenerate q-Changhee numbers of order r.*

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Note that, by (1.1) and (2.2),

$$\begin{aligned}
&\sum_{n=0}^{\infty} MCh_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}\\
&=\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r (1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\
&=\left(\sum_{n=0}^{\infty} MCh_{n,\lambda,q}\frac{t^n}{n!}\right)^r \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m x^m S_1(n,m)\frac{t^n}{n!}\right)\\
&=\left(\sum_{n=0}^{\infty}\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=n}}^{\infty} MCh_{n_1,\lambda,q}\cdots MCh_{n_r,\lambda,q}\frac{t^{n_1}}{n_1!}\cdots\frac{t^{n_r}}{n_r!}\right)\\
&\quad\times\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m x^m S_1(n,m)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=m}}\sum_{k=0}^{n-m}\binom{m}{n_1,\cdots,n_r}\binom{n}{m} MCh_{n_1,\lambda,q}\cdots MCh_{n_r,\lambda,q}\right.\\
&\quad\left.\times\left(\frac{\log(1+\lambda)}{\lambda}\right)^k x^k S_1(n-m,k)\right)\frac{t^n}{n!},
\end{aligned} \tag{2.12}$$

where $\binom{m}{n_1,\dots,n_r}$ are the multinomial coefficients.

In addition, by (1.1) and (2.2), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^m d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_{m,q}^{(r)}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

By (2.11), (2.12) and (2.13), we obtain the following theorem.

Theorem 2.6. *For each nonnegative integer n, we have*

$$\begin{aligned}
MCh_{n,\lambda,q}^{(r)}(x)=&\sum_{m=0}^{n}\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=m}}\sum_{k=0}^{n-m}\binom{m}{n_1,\cdots,n_r}\binom{n}{m} MCh_{n_1,\lambda,q}\cdots MCh_{n_r,\lambda,q}\\
&\times\left(\frac{\log(1+\lambda)}{\lambda}\right)^k S_1(n-m,k)x^k,
\end{aligned}$$

and

$$MCh_{n,\lambda,q}^{(r)}(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_{m,q}^{(r)}(x).$$

By replacing t as e^t-1 in (2.12), we get

$$\begin{aligned}\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{\lambda}}}\right)^r(1+\lambda)^{\frac{t}{\lambda}x}&=\sum_{n=0}^{\infty}MCh_{n,\lambda,q}^{(r)}(x)\frac{1}{n!}\left(e^t-1\right)^m\\&=\sum_{n=0}^{\infty}MCh_{n,\lambda,q}^{(r)}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}MCh_{m,\lambda,q}^{(r)}(x)S_2(n,m)\frac{t^n}{n!},\end{aligned}\tag{2.14}$$

and

$$\begin{aligned}&\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+(e^t-1))}}\right)^r(1+\lambda)^{\frac{x}{\lambda}\log\left(1+(e^t-1)\right)}\\&=\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{t}{x}}}\right)^r(1+\lambda)^{\frac{xt}{\lambda}}\\&=\sum_{n=0}^{\infty}\xi_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.15}$$

By (2.14) and (2.15), we obtain the following theorem.

Theorem 2.7. *For each $n\geq 0$, we have*

$$\xi_{n,\lambda,q}^{(r)}(x)=\sum_{m=0}^{n}MCh_{m,\lambda,q}^{(r)}(x)S_2(n,m).$$

By (2.12), we observe that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(qMCh_{n,\lambda}^{(r)}(x+1)+MCh_{n,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}\\&=q\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r(1+\lambda)^{\frac{x+1}{\lambda}\log(1+t)}+\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\&=[2]_q\left(\frac{[2]_q}{1+q(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^{r-1}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\&=[2]_q\sum_{n=0}^{\infty}MCh_{n,\lambda,q}^{(r-1)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.8. *For each $n\geq 0$ and $r\in\mathbb{N}$, we have*

$$qMCh_{n,\lambda}^{(r)}(x+1)+MCh_{n,\lambda,q}^{(r)}(x)=[2]_qMCh_{n,\lambda,q}^{(r-1)}(x).$$

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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES

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ABSTRACT. In this paper, we solve the following quadratic ρ-functional inequalities

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ &\qquad\qquad -f(x)-f(y)-f(z)\| \\ &\le \|\rho(f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) \\ &\qquad\qquad -4f(x)-4f(y)-4f(z))\|, \end{aligned} \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{8}$, and

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) \\ &\qquad\qquad -4f(x)-4f(y)-4f(z)\| \\ &\le \left\| \rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right.\right. \\ &\qquad\qquad -f(x)-f(y)-f(z))\|, \end{aligned} \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 4$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces and prove the Hyers-Ulam stability of quadratic ρ-functional equations associated with the quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [20] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{1.1}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional

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equation was proved by Skof [19] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

The functional equation

$$2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

is called a *Jensen type quadratic equation.* See [8, 9, 10, 11, 12, 15, 16, 17, 18] for more information on functional equations and their stability.

In [5], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\| \tag{1.2}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [14]. Gilányi [6] and Fechner [3] proved the Hyers-Ulam stability of the functional inequality (1.2). Park, Cho and Han [10] proved the Hyers-Ulam stability of additive functional inequalities.

In Section 3, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ-functional equation associated with the quadratic ρ-functional inequality (0.1) in complex Banach spaces.

In Section 4, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ-functional equation associated with the quadratic ρ-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Quadratic functional equation

Theorem 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$f\left(\frac{x+y+z}{2} + \frac{x-y-z}{2} + \frac{y-x-z}{2} + \frac{z-x-y}{2}\right) = f(x) + f(y) + f(z) \tag{2.1}$$

if and only if the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Sufficiency. Assume that $f : X \to Y$ satisfies (2.1)

Letting $x = y = z = 0$ in (2.1), we have $4f(0) = 3f(0)$. So $f(0) = 0$.

Letting $y = z = 0$ in (2.1), we get

$$2f\left(\frac{x}{2}\right) + 2f\left(-\frac{x}{2}\right) = f(x), \tag{2.2}$$

$$2f\left(-\frac{x}{2}\right) + 2f\left(\frac{x}{2}\right) = f(-x)$$

for all $x \in X$, which imply that $f(x) = f(-x)$ for all $x \in X$.

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From this and (2.2), we obtain $4f\left(\frac{x}{2}\right) = f(x)$ or $f(2x) = 4f(x)$ for all $x \in X$.

Putting $z = 0$ in (2.1), we obtain

$$\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) = f(x) + f(y)$$

for all $x, y \in X$, which means that $f : X \to Y$ is a quadratic mapping.

Necessity. Assume that $f : X \to Y$ is quadratic.

By $f(x+y) + f(x-y) = 2f(x) + 2f(y)$, one can easily get $f(0) = 0$, $f(x) = f(-x)$ and $f(2x) = 4f(x)$ for all $x \in X$. So

$$\begin{aligned} f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) \\ = \Big[2f\Big(\frac{x}{2}\Big) + 2f\Big(\frac{y+z}{2}\Big)\Big] + \Big[2f\Big(-\frac{x}{2}\Big) + 2f\Big(\frac{y-z}{2}\Big)\Big] \\ = 4f\left(\frac{x}{2}\right) + f\Big(\frac{y+z+y-z}{2}\Big) + f\Big(\frac{y+z-y+z}{2}\Big) \\ = f(x) + f(y) + f(z) \end{aligned}$$

for all $x, y, z \in X$, which is the functional equation (2.1) and the proof is complete. □

Corollary 2.2. *Let X and Y be vector spaces. An even mapping $f : X \to Y$ satisfies*

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) = 4f(x) + 4f(y) + 4f(z) \quad (2.3)$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Assume that $f : X \to Y$ satisfies (2.3)

Letting $x = y = z = 0$ in (2.3), we have $4f(0) = 12f(0)$. So $f(0) = 0$.

Letting $z = 0$ in (2.3), we get

$$2f(x+y) + 2f(x-y) = 4f(x) + 4f(y)$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

3. Quadratic ρ-functional inequality (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{8}$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.1) in complex normed spaces.

Lemma 3.1. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} \Big\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ -f(x) - f(y) - f(z) \| \\ \le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ -4f(x) - 4f(y) - 4f(z))\| \end{aligned} \quad (3.1)$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (3.1), we get $\|f(0)\| \le |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.1), we get $\|4f\left(\frac{x}{2}\right) - f(x)\| \le 0$ and so $4f(\frac{x}{2}) = f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \\
&\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| \\
&= |\rho| \left\|4f\left(\frac{x+y+z}{2}\right) + 4f\left(\frac{x-y-z}{2}\right) + 4f\left(\frac{y-x-z}{2}\right) + 4f\left(\frac{z-x-y}{2}\right)\right. \\
&\qquad\qquad \left. -4f(x) - 4f(y) - 4f(z)\right\| \\
&\le 4|\rho| \left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right. \\
&\qquad\qquad \left. -f(x) - f(y) - f(z)\right\|
\end{aligned}$$

and so

$$f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) = f(x) + f(y) + f(z)$$

for all $x, y, z \in X$.

The converse is obviously true. □

Corollary 3.2. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned}
&f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \\
&= \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))
\end{aligned} \tag{3.3}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

The functional equation (3.3) is called a *quadratic ρ-functional equation.*

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (3.1) in complex Banach spaces.

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping such that*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{3.4}$$

$$\begin{aligned}
&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \\
&\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\
&\qquad -4f(x) - 4f(y) - 4f(z))\| + \varphi(x, y, z)
\end{aligned} \tag{3.5}$$

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for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \Psi(x, 0, 0) \tag{3.6}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.5), we get $\|f(0)\| \leq |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.5), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0, 0) \tag{3.7}$$

for all $x \in X$. So

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0, 0\right) \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.6).

It follows from (3.4) and (3.5) that

$$\begin{aligned}
&\left\|h\left(\frac{x+y+z}{2}\right) + h\left(\frac{x-y-z}{2}\right) + h\left(\frac{y-x-z}{2}\right) + h\left(\frac{z-x-y}{2}\right) - h(x) - h(y) - h(z)\right\| \\
&= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right) + f\left(\frac{z-x-y}{2^{n+1}}\right)\right. \\
&\qquad\qquad \left. -f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} 4^n|\rho| \left\|f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right) + f\left(\frac{z-x-y}{2^n}\right)\right. \\
&\qquad\qquad \left. -4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&= \|\rho(h(x+y+z) + h(x-y-z) + h(y-x-z) + h(z-x-y) \\
&\qquad\qquad -4h(x) - 4h(y) - 4h(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
&\left\|h\left(\frac{x+y+z}{2}\right) + h\left(\frac{x-y-z}{2}\right) + h\left(\frac{y-x-z}{2}\right) + h\left(\frac{z-x-y}{2}\right) - h(x) - h(y) - h(z)\right\| \\
&\leq \|\rho(h(x+y+z) + h(x-y-z) + h(y-x-z) + h(z-x-y) - 4h(x) - 4h(y) - 4h(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.6). Then we have

$$\begin{aligned}\|h(x) - T(x)\| &= \left\|4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2 \cdot 4^q \Psi\left(\frac{x}{2^q}, 0, 0\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (3.6). □

Corollary 3.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}&\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \\ &\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ &\qquad -4f(x) - 4f(y) - 4f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)\end{aligned} \tag{3.9}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{2^r\theta}{2^r - 4}\|x\|^r \tag{3.10}$$

for all $x \in X$.

Theorem 3.5. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (3.5) *and*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z) < \infty \tag{3.11}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \Psi(x, 0, 0) \tag{3.12}$$

for all $x \in X$.

Proof. It follows from (3.7) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(2x, 2x, 2x)$$

for all $x \in X$. Hence

$$\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \le \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}\varphi(2^{j+1}x, 0, 0) \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.12).

The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 3.6. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.9)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2^r \theta}{4 - 2^r} \|x\|^r \tag{3.14}$$

for all $x \in X$.

By the triangle inequality, we have

$$\begin{aligned}
&\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right\| \\
&- \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| \\
&\leq \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\
&\quad - f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\
&\quad\quad \left. + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\right\|.
\end{aligned}$$

As corollaries of Theorems 3.3 and 3.5, we obtain the Hyers-Ulam stability results for the quadratic ρ-functional equation (3.3) in complex Banach spaces.

Corollary 3.7. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying* (3.4) *and*

$$\begin{aligned}
&\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\
&\quad - f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\
&\quad \left. + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\right\| \leq \varphi(x, y, z)
\end{aligned} \tag{3.15}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (3.6)*.*

Corollary 3.8. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}
&\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\
&\quad - f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\
&\quad \left. + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\right\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned} \tag{3.16}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (3.10)*.*

Corollary 3.9. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (3.11) *and* (3.15)*. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.12)*.*

Corollary 3.10. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.16)*. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.14)*.*

Remark 3.11. If ρ is a real number such that $-\frac{1}{8} < \rho < \frac{1}{8}$ and Y is a real Banach space, then all the assertions in this section remain valid.

4. Quadratic ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 4$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 4.1. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z)\| \\ &\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right.\right. \\ &\qquad\qquad \left.\left. -f(x)-f(y)-f(z)\right)\right\| \end{aligned} \tag{4.1}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (4.1).

Letting $x = y = z = 0$ in (4.1), we get $\|8f(0)\| \le |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.1), we get

$$\|2f(2x) - 8f(x)\| \le 0 \tag{4.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (4.1) and (4.2) that

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z)\| \\ &\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right.\right. \\ &\qquad\qquad \left.\left. -f(x)-f(y)-f(z)\right)\right\| \\ &= \left\|\rho\left(\frac{1}{4}f(x+y+z)+\frac{1}{4}f(x-y-z)+\frac{1}{4}f(y-x-z)+\frac{1}{4}f(z-x-y)\right.\right. \\ &\qquad\qquad \left.\left. -f(x)-f(y)-f(z)\right)\right\| \\ &= \frac{|\rho|}{4}\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) \\ &\qquad\qquad -4f(x)-4f(y)-4f(z)\| \end{aligned}$$

and so

$$f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) = 4f(x)+4f(y)+4f(z)$$

for all $x, y, z \in X$. So f is quadratic.

The converse is obviously true. □

Corollary 4.2. *An even mapping $f : X \to Y$ satisfies*

$$f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z)$$

$$= \rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right. \qquad (4.3)$$
$$\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)$$

for all $x, y, z \in X$ and only if $f : X \to Y$ is quadratic.

The functional equation (4.3) is called a *quadratic ρ-functional equation.*

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (4.1) in complex Banach spaces.

Theorem 4.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \qquad (4.4)$$

$$\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z)\|$$
$$\leq \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right. \qquad (4.5)$$
$$\left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\|+\varphi(x, y, z)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{1}{8}\Psi(x, x, 0) \qquad (4.6)$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (4.5), we get $\|8f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.5), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \qquad (4.7)$$

for all $x \in X$. So

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|$$
$$\leq \sum_{j=l}^{m-1} 2^{2j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \qquad (4.8)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.6).

The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 4.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ &\leq \left\| \rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right.\right. \\ &\quad \left.\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{4.9}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{1}{2^r - 4}\theta\|x\|^r \tag{4.10}$$

for all $x \in X$.

Theorem 4.5. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ and let $f : X \to Y$ be an even mapping satisfying (4.5) and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z) < \infty \tag{4.11}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{1}{8}\Psi(x, x, 0) \tag{4.12}$$

for all $x \in X$.

Proof. It follows from (4.7) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{1}{8}\varphi(x, x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right) \right\| \\ &\leq \frac{1}{8}\sum_{j=l}^{m-1} \frac{1}{4^j}\varphi(2^j x, 2^j x, 0) \end{aligned} \tag{4.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.13) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.13), we get (4.12).

The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 4.6. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying (4.9). Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{1}{4 - 2^r}\theta\|x\|^r \tag{4.14}$$

for all $x \in X$.

By the triangle inequality, we have

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z)\| \\ &\qquad -\left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right. \\ &\qquad\qquad \left.\left. +f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\| \\ &\le \|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z) \\ &\qquad -\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right. \\ &\qquad\qquad \left. +f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\Bigg\|. \end{aligned}$$

As corollaries of Theorems 4.3 and 4.5, we obtain the Hyers-Ulam stability results for the quadratic ρ-functional equation (4.3) in complex Banach spaces.

Corollary 4.7. *Let* $\varphi : X^3 \to [0,\infty)$ *be a function and let* $f : X \to Y$ *be an even mapping satisfying* (4.4) *and*

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z) \\ &\quad -\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right. \\ &\qquad \left. -f(x)-f(y)-f(z)\right)\| \le \varphi(x,y,z) \end{aligned} \tag{4.15}$$

for all $x, y, z \in X$. *Then there exists a unique quadratic mapping* $h : X \to Y$ *satisfying* (4.6).

Corollary 4.8. *Let* $r > 2$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be an even mapping such that*

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z) \\ &\quad -\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right. \\ &\qquad \left. -f(x)-f(y)-f(z)\right)\| \le \theta(\|x\|^r+\|y\|^r+\|y\|^r) \end{aligned} \tag{4.16}$$

for all $x, y, z \in X$. *Then there exists a unique quadratic mapping* $h : X \to Y$ *satisfying* (4.10).

Corollary 4.9. *Let* $\varphi : X^3 \to [0,\infty)$ *be a function with* $\varphi(0,0,0) = 0$ *and let* $f : X \to Y$ *be an even mapping satisfying* (4.11) *and* (4.15). *Then there exists a unique quadratic mapping* $h : X \to Y$ *satisfying* (4.12).

Corollary 4.10. *Let* $r < 2$ *and* θ *be positive real numbers, and let* $f : X \to Y$ *be an even mapping satisfying* (4.16). *Then there exists a unique quadratic mapping* $h : X \to Y$ *satisfying* (4.14).

Remark 4.11. If ρ is a real number such that $-4 < \rho < 4$ and Y is a real Banach space, then all the assertions in this section remain valid.

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ON A SUBCLASS OF p-VALENT ANALYTIC FUNCTIONS OF COMPLEX ORDER INVOLVING A LINEAR OPERATOR

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ABSTRACT. Using the linear operator $\mathcal{L}_p(a,c)$, we introduce a class $R^b_{p,n}(\mu,a,c,A,B)$ of multivalent analytic functions with complex order. For this class, a sufficient condition in terms of the coefficients for f is obtained, the Fekete-Szego problem and determination of sharp upper bound for the second Hankel determinant is completely solved. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

1. Introduction and preliminaries

We denote by $\mathrm{A}_p(n)$ the family of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (p, n \in \mathbb{N} = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $n = 1$ and $n = 1,\ p = 1$, we symbolise the above class by A_p and A, respectively.

For the functions f_1 and f_2 analytic in $\mathbb{U}$, we say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$ or $f_1(z) \prec f_2(z)$ $(z \in \mathbb{U})$ if there exists a Schwarz function ω, which (by defintion) is analytic in $\mathbb{U}$ with $\omega(0) = 0,\ |\omega(z)| < 1$ and $f_1(z) = f_2(\omega(z))$ for $z \in \mathbb{U}$. If the function f_2 is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [23]; see also [24]).

$$f_1(z) \prec f_2(z) \Longleftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{U}) \subset f_2(\mathbb{U}).$$

If we have two functions $h_j(z) = \sum_{k=0}^{\infty} a_{k,j} z^k$ $(j = 1, 2)$ which are analytic in $\mathbb{U}$, we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(h_1 \star h_2)(z) = \sum_{k=0}^{\infty} a_{k,1} a_{k,2} z^k = (h_2 \star h_1)(z) \quad (z \in \mathbb{U}).$$

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The classes $\mathcal{S}^*_{p,n}(b,\rho)$ and $C_{p,n}(b,\rho)$ are called p-valently starlike and convex of complex order b and type ρ which consists f of $\mathrm{A}_p(n)$ and f satisfies the following inequalities, respectively:

$$\mathrm{Re}\left\{p+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right\}>\rho \quad (b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}, 0\le\rho<p; z\in\mathbb{U}), \tag{1.2}$$

$$\mathrm{Re}\left\{p+\frac{1}{b}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right\}>\rho \quad (b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}, 0\le\rho<p; z\in\mathbb{U}). \tag{1.3}$$

From (1.1) and (1.3), it follows that

$$f\in\ C_{p,n}(b,\rho)\Longleftrightarrow\frac{zf'(z)}{p}\in\mathcal{S}^*_{p,n}(b,\rho).$$

In particular, for $p=n=1$, the classes $\mathcal{S}^*_{p,n}(b,\rho)$ and $C_{p,n}(b,\rho)$ reduces to the classes $\mathcal{S}^*(b,\rho)$ and $C(b,\rho)$ of starlike functions of complex order b and type ρ, and convex function of complex order b and type $\rho\,(b\in\mathbb{C}^*; 0\le\rho<p)$, respectively, which were introduced by Frasin [8].

Setting $\rho=0$ in $\mathcal{S}^*(b,\rho)$ and $C(b,\rho)$, we get the classes $\mathcal{S}^*(b)$ and $C(b)$. These classes of starlike and convex functions of order b were considered earlier by Nasr and Aouf [27] and Wiatrowski [37], respectively (see also [5] and [36]). We further observe that $\mathcal{S}^*_{p,1}(1,\rho)=\mathcal{S}^*_p(\rho)$ and $C_{p,1}(1,\rho)=\ C_p(\rho)$ are, respectively, the classes of p-valently starlike and p-valently convex functions of order $\rho\,(0\le\rho<p)$ in $\mathbb{U}$. Also, we note that $\mathcal{S}^*_1(\rho)=\mathcal{S}^*(\rho)$ and $C_1(\rho)=\ C(\rho)$ are the usual classes of starlike and convex functions of order $\rho\,(0\le\rho<1)$ in $\mathbb{U}$. In the special cases, $\mathcal{S}^*(0)=\mathcal{S}^*$ and $C(0)=C$ are the familiar classes of starlike and convex functions in $\mathbb{U}$.

Furthermore, let $R_{p,n}(b,\rho)$ denote the class of functions in $A_p(n)$ satisfying the condition:

$$\mathrm{Re}\left\{p+\frac{1}{b}\left(\frac{f'(z)}{z^{p-1}}-p\right)\right\}>\rho \quad (b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}, 0\le\rho<p; z\in\mathbb{U}).$$

We note that $R_{p,n}(1,\rho)$ is a subclass of p-valently close-to-convex functions of order $\rho\,(0\le\rho<p)$ in the unit disk $\mathbb{U}$.

Let φ_p be the incomplete beta function defined by

$$\varphi_p(a,c;z)=z^p+\sum_{k=1}^{\infty}\frac{(a)_k}{(c)_k}z^{p+k} \quad (z\in\mathbb{U}), \tag{1.4}$$

where $a\in\mathbb{R}, c\in\mathbb{R}\setminus\mathbb{Z}_0^-, \mathbb{Z}_0^-=\{0,-1,-2,\dots\}$ and the symbol $(x)_k$ denotes the Pochhammer symbol (or shifted factorial) given by

$$(x)_k=\begin{cases}1, & (k=0, x\in\mathbb{C}^*=\mathbb{C}\setminus\{0\})\\ x(x+1)\cdots(x+k-1), & (k\in\mathbb{N}, x\in\mathbb{C}).\end{cases}$$

With the aid of the function φ_p, given by (1.4) and the Hadamard product, we consider the linear operator $\mathcal{L}_p(a,c):\mathcal{A}_p(n)\longrightarrow\mathcal{A}_p(n)$ defined by

$$\mathcal{L}_p(a,c)f(z)=\varphi_p(a,c;z)\star f(z) \qquad (z\in\mathbb{U}). \tag{1.5}$$

If f is given by (1.1), then from (1.5), it readily follows that

$$\mathcal{L}_p(a,c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^{p+k} \qquad (z \in \mathbb{U}). \tag{1.6}$$

The linear operator $\mathcal{L}_p(a,c)$ on the class A_p was introduced and studied by Saitoh [33], which generalizes the linear operator $\mathcal{L}_1(a,c) = \mathcal{L}(a,c)$ introduced by Carlson and Shaffer [4] in their systematic investigations of certain interesting subclasses of starlike, convex and prestarlike hypergeometric functions.

We also note that for $f \in A_p$,

(i) $\mathcal{L}_p(a,a)f(z) = f(z)$;

(ii) $\mathcal{L}_p(p+1,p)f(z) = z^2 f''(z) + 2zf'(z)/p(p+1)$;

(iii) $\mathcal{L}_p(p+2,p)f(z) = zf'(z)/p$;

(iv) $\mathcal{L}_p(m+p,1)f(z) = D^{m+p-1}f(z)$ $(m \in \mathbb{Z}, m > -p)$, the operator studied by Goel and Sohi [9]. In the case $p=1$, $D^m f$ is the familiar Ruscheweyh derivative [32] of $f \in A$.

(v) $\mathcal{L}_p(\nu+p,1)f(z) = D^{\nu,p}f(z)$ $(\nu > -p)$, the extended linear derivative operator of Rusheweyh type introduced by Raina and Srivastava [31]. In particular, when $\nu = m$, we get operator $D^{m+p-1}f(z)$ $(m \in \mathbb{Z}, m > -p)$, studied by Goel and Sohi [9].

(vi) $\mathcal{L}_p(p+1,m+p)f(z) = \mathcal{I}_{m,p}f(z)$ $(m \in \mathbb{Z}, m > -p)$, the extended Noor integral operator considered by Liu and Noor [19].

(vii) $\mathcal{L}_p(p+1,p+1-\lambda)f(z) = \Omega_z^{(\lambda,p)}f(z)$ $(-\infty < \lambda < p+1)$, the extended fractional differintegral operator considered by Patel and Mishra [30].

Note that

$$\Omega_z^{0,p}f(z) = f(z),\ \Omega_z^{1,p}f(z) = \frac{zf'(z)}{p} \text{ and } \Omega_z^{2,p}f(z) = \frac{z^2 f''(z)}{p(p-1)}\ (p \geq 2;\ \ z \in \mathbb{U}).$$

Now, by using the operator $\mathcal{L}_p(a,c)$, we introduce the following new subclasses of p-valent analytic functions in the unit disk $\mathbb{U}$.

Definition 1.1. $R^b_{p,n}(\mu,a,c,A,B)$ is the subclass of analytic $p-$ valent functions, which consists of f given in the form of (1.1) and satisfies the subordination condition:

$$1 + \frac{1}{b}\left\{p(1-\mu)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \mu\frac{(L_p(a,c)f)'(z)}{z^{p-1}} - p\right\} \prec \frac{1+Az}{1+Bz}, \tag{1.7}$$

where $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$, $b \in \mathbb{C}^*, 0 \leq \mu \leq 1$ and $z \in \mathbb{U}$. Equivalently, we say $f \in A_p(n)$ is a member of $R^b_{p,n}(\mu,a,c,A,B)$, if

$$\left|\frac{p(1-\mu)\mathcal{L}_p(a,c)f(z) + \mu z(L_p(a,c)f)'(z) - pz^p}{b(A-B)z^p - B\left\{p(1-\mu)\mathcal{L}_p(a,c)f(z) + \mu z(L_p(a,c)f)'(z) - pz^p\right\}}\right| < 1 \quad (z \in \mathbb{U}). \tag{1.8}$$

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For $n=1$ we denote the class by $R_p^b(\mu,a,c,A,B)$. It may be noted that by suitably choosing the parameters involved in Definition 1.1, the class $R_{p,n}^b(a,c,\lambda,\rho)$ extends several subclasses of p-valent analytic functions in $\mathbb{U}$.

Example 1.1. *For $n=1,\ b=pe^{-i\theta}\cos\theta,\ A=1-2\rho/p,\ B=-1$ in Definition 1.1, we get*

$$R_p^{pe^{-i\theta}\cos\theta}\left(\mu,a,c,1-\frac{2\rho}{p},-1\right) = R_p(\mu,a,c,\theta,\rho)$$

$$=\left\{f\in A_p : Re\left[e^{i\theta}\left(p(1-\mu)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\mu\frac{(\mathcal{L}_p(a,c)f)'(z)}{z^{p-1}}\right)\right]>\rho\cos\theta\right\},$$

where $0\le\rho<p, |\theta|<\pi/2$ *and* $z\in\mathbb{U}$.

• Putting $\mu=0,\ p=1, a=\alpha$ and $c=\beta$ in Example 1.1, we get the class $R_{\alpha,\beta}(\theta,\rho)$ considered by Mishra and Kund [26].
• Taking $a=c$ in Example 1.1, we get

$$R_p(\mu,a,c,\theta,\rho) = R_p(\mu,\theta,\rho)=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(p(1-\mu)\frac{f(z)}{z^p}+\mu\frac{f'(z)}{z^{p-1}}\right)\right]>\rho\cos\theta\right\}.$$

We write

$$R_p(0,\theta,\rho)=R_{p,\theta}(\rho)=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(\frac{f(z)}{z^p}\right)\right]>\frac{\rho}{p}\cos\theta\right\}$$

and

$$R_p(1,\theta,\rho)= R_{p,\theta}(\rho)=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(\frac{(f)'(z)}{z^{p-1}}\right)\right]>\frac{\rho}{p}\cos\theta\right\},$$

where $(0\le\rho<p, |\theta|<\pi/2,\ z\in\mathbb{U})$ which reduces to the class R (see, MacGregor [21]) for $p=1$ and $\theta=\rho=0$.
• Taking $a=p+1,\ c=p+1-\lambda$ in Example 1.1, we obtain

$$R_p^{pe^{-i\theta}\cos\theta}\left(\mu,\ p+1,p+1-\lambda,1-\frac{2\rho}{p},-1\right) = R_{p,\lambda}(\mu,\theta,\rho)$$

$$=\left\{f\in A_p : \mathrm{Re}\left[e^{i\theta}\left(p(1-\mu)\frac{\Omega_z^{\lambda,p}(a,c)f(z)}{z^p}+\mu\frac{(\Omega_z^{\lambda,p}(a,c)f)'(z)}{z^{p-1}}\right)\right]>\rho\cos\theta\right\},$$

where $0\le\rho<p,\ -\infty<\lambda<p+1,\ |\theta|<\pi/2$ and $z\in\mathbb{U}$. We write $R_{p,\lambda}(0,\theta,\rho)=R_{p,\lambda}(\theta,\rho)$ and the class $R_{1,\lambda}(\theta,\rho)=R_\lambda(\theta,\rho)$ was investigated by Mishra and Gochhayat [25].
•

$$R_p^{\frac{2p\beta\left(1-\frac{\alpha}{p}\right)e^{-i\theta}\cos\theta}{1+\beta}}(\mu,\ p+1,p,1,-\beta) = R_{p,\alpha,\beta}^{\theta,\,\mu}\ (0\le\alpha<p, 0\le\beta<1, |\theta|<\pi/2)$$

$$=\left\{f\in A_p : \left|\frac{(1-\mu+\frac{\mu}{p})f'(z)+\frac{\mu}{p}zf''(z)-pz^{p-1}}{(1-\mu+\frac{\mu}{p})f'(z)+\frac{\mu}{p}zf''(z)-pz^{p-1}+2(p-\alpha)e^{-i\theta}\cos\theta\, z^{p-1}}\right|<\beta; z\in\mathbb{U}\right\}.$$

We note that $R_{1,\alpha,\beta}^{\theta,0}= R_{\alpha,\beta}^{\theta}$ is the subclass of A investigated by Makowka [22], $R_{1,\alpha,\beta}^{0,0}= R(\alpha,\beta)$ is the class studied by Juneja and Mogra [12] and $R_{1,0,\beta}^{0,0}= R(\beta)$ is the class considered by Padmanabhan [29] (see also [3]).

Example 1.2. *For $\mu = 0,\ n = 1$ and replacing b by bp, we get subclass $R_p^b(a, c, A, B)$ of A_p which satisfies the following subordination condition:*

$$1 + \frac{1}{b}\left(\frac{\mathcal{L}_p(a, c)f(z)}{z^p} - 1\right) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{1.9}$$

where $a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\left(Z_0^- = \{..., -2, -1, 0\}\right)$ and $0 \neq b \in \mathbb{C}$.

The sub class of $R_p^b(a, c, A, B)$ is recently studied by Sahoo and Patel [35].

Recently, Janteng *et al.* [11], Mishra and Gochhayat [25] and Mishra and Kund [26] have obtained sharp upper bounds to the second Hankel determinant $H_2(2)$ for the families $R,\ R_\lambda(\theta, \rho)$ and $R_{\alpha,\beta}(\theta, \rho)$, respectively.

Further, taking $A = p - \rho, \quad B = 0$ in Definition 1.1, we get the following subclass $R_{p,n}^b(\mu, a, c, \rho)$ of $\mathrm{A}_p(n)$ studied by Sahoo and Patel [34]

• A function $f \in A_p(n)$ is said to be in the class $R_{p,n}^b(\mu, a, c, \rho)$, if it satisfies the following inequality:

$$\left|\frac{1}{b}\left\{p(1-\mu)\frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \mu\frac{\left(L_p(a, c)f\right)'(z)}{z^{p-1}} - p\right\}\right| < p - \rho \tag{1.10}$$

$$(b \in \mathbb{C}^*, 0 \leq \mu \leq 1, 0 \leq \rho < p; z \in \mathbb{U}).$$

• $R_{p,n}^b(\mu, p+1, p+1-\lambda, \rho) = R_{p,n}^b(\mu, \lambda, \rho)\,(b \in \mathbb{C}^*, -\infty < \lambda < p, 0 \leq \mu)$, which yields the class considered by Aouf [2] for $\rho = p - \beta\,(0 < \beta \leq 1,\ 0 \leq \rho < p)$.

Special cases of the parameters p, λ and ρ in the class $R_{p,n}^b\ (\mu, \lambda, \rho)$ yields the following subclasses of A_p.

(i) $R_{p,n}^b(\mu, 0, \rho) = R_{p,n}^b(\mu, \rho)$

$$= \left\{f \in\ A_p : \left|\frac{1}{b}\left(p(1-\mu)\frac{f(z)}{z^p} + \mu\frac{f'(z)}{z^{p-1}} - p\right)\right| < p - \rho, \mu \geq 0, 0 \leq \rho < p; z \in \mathbb{U}\right\}.$$

(ii) $R_{p,n}^b(\mu, 1, \rho) = \mathcal{P}_{p,n}^b(\mu, \rho)$

$$= \left\{f \in\ A_p : \left|\frac{1}{b}\left((\mu + \mu(1-p))\frac{f'(z)}{pz^{p-1}} + \mu\frac{f''(z)}{pz^{p-2}} - p\right)\right| < p - \rho,\ \mu \geq 0, 0 \leq \rho < p; z \in \mathbb{U}\right\}.$$

(iii) $R_{1,n}^b(\mu, 1, 1-\beta) = R_n^b(\mu, \beta)$

$$= \left\{f \in\ A_p : \left|\frac{1}{b}\left(f'(z) + \mu z f''(z) - 1\right)\right| < \beta, \mu \geq 0, 0 < \beta \leq 1; z \in \mathbb{U}\right\}.$$

The class $R_n^b(\mu, \beta)$ was studied by Altintas *et al.* [1].

Let $\mathscr{P}$ denote the class of analytic functions ϕ normalized by

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}) \tag{1.11}$$

such that $\mathrm{Re}\{\phi(z)\} > 0$ in $\mathbb{U}$.

Noonan and Thomas [28] defined the q-th Hankel determinant of a sequence $a_n, a_{n+1}, a_{n+2}, \cdots$ of real or complex numbers by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (n \in \mathbb{N}, q \in \mathbb{N} \setminus \{1\}).$$

This determinant has been studied by several authors with the subject of inquiry ranging from the rate of growth of $H_q(n)$ (as $n \to \infty$) to the determination of precise bounds with specific values of n and q for certain subclasses of analytic functions in the unit disk $\mathbb{U}$. Ehrenborg [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [16].

In particular, when $n = 1, q = 2, a_1 = 1$ and $n = q = 2$, the Hankel determinant simplifies to

$$H_2(1) = |a_3 - a_2^2| \quad \text{and} \quad H_2(2) = |a_2 a_4 - a_3^2|.$$

We refer to $H_2(2)$ as the second Hankel determinant. It is known [5] that if

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \tag{1.12}$$

is analytic and univalent in $\mathbb{U}$, then the sharp inequality $H_2(1) = |a_3 - a_2^2| \leq 1$ holds. For a family $\mathfrak{F}$ of analytic functions of the form (1.7), the more general problem of finding the sharp upper bounds for the functionals $|a_3 - \mu\, a_2^2|$ $(\mu \in \mathbb{R}/\mathbb{C})$ is popularly known as Fekete-Szegö problem for the class $\mathfrak{F}$. The Fekete-Szegö problem for the known classes of univalent functions, starlike functions, convex functions and close-to-convex functions has been completely settled [7, 10, 13, 14, 15].

Recently, Janteng *et al.* [11], Mishra and Gochhayat [25] and Mishra and Kund [26] have obtained sharp upper bounds on the second Hankel determinant $H_2(2)$ for the families R, $R_\lambda(\theta, \rho)$ and $R_{\alpha,\beta}(\theta, \rho)$, respectively.

In our present investigation, by following the techniques devised by Libera and Zlotkiewicz [17, 18], we derive sharp upper bound for the Fekete-Szegö problem and for the second Hankel determinant as well of the functions belonging to the class $R_p^b(\mu, a, c, A, B)$. Relevant connections of the results obtained here with some earlier known work are also pointed out.

To establish our main results, we shall need the followings lemmas.

Lemma 1.1. [5, 17, 18, 20] *Let the function ϕ, given by (1.2) be a member of the class $\mathscr{P}$. Then*

(i) $|p_k| \leq 2$ $(k \geq 1)$ *and the estimate is sharp for the function*

$$t(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

(ii) $\left|p_2 - \nu\, p_1^2\right| \leq 2 \max\{1, |2\nu - 1|\}$, *where* $\nu \in \mathbb{C}$ *and the result is sharp for the functions*

given by

$$q(z) = \frac{1+z^2}{1-z^2} \quad and \quad s(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

(iii)

$$p_2 = \frac{1}{2}\left\{p_1^2 + (4-p_1^2)x\right\}$$

and

$$p_3 = \frac{1}{4}\left\{p_1^3 + 2(4-p_1^2)p_1x - (4-p_1^2)p_1x^2 + 2(4-p_1^2)(1-|x|^2)z\right\}$$

for some complex numbers x, z satisfying $|x| \le 1$ and $|z| \le 1$.

2. Main results

Unless otherwise mentioned, we assume throughout the sequel that

$$b \in \mathbb{C}^*, 0 \le \mu \le 1, p \in \mathbb{N}, a > 0, c > 0, -1 \le B < A \le 1, z \in \mathbb{U}$$

and the powers appearing in different expression are understood as principal values.

At the outset, we obtain a sufficient condition for a function $f \in A_p$ to be in the class $R^b_{p,n}(\mu, a, c, A, B)$.

Theorem 2.1. *If f given by* (1.1) *satisfies*

$$\sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k) \le \frac{|b|(A-B)}{(1+|B|)}, \tag{2.1}$$

then $f \in R^b_{p,n}(\mu, a, c, A, B)$.

Proof. To prove that f given by (1.1) is a member of $R^b_{p,n}(\mu, a, c, A, B)$, it need to satisfy (1.8). For $|z| = 1$, we have

$$\left|\frac{p(1-\mu)\mathcal{L}_p(a,c)f(z) + \mu z(L_p(a,c)f)'(z) - pz^p}{b(A-B)z^p - B\left\{p(1-\mu)\mathcal{L}_p(a,c)f(z) + \mu z(L_p(a,c)f)'(z) - pz^p)\right\}}\right|$$

$$= \frac{\left|\sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k}(p+\mu k)z^k\right|}{\left|b(A-B) - B\sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)z^k\right|}$$

$$\le \frac{\sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)z^k}{|b|(A-B) - |B|\sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)z^k} \quad (z \in \mathbb{U}).$$

The last expression is needed to be bounded above by 1, which requires

$$\sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k) \le \frac{|b|(A-B)}{(1+|B|)}$$

Thus by maximum modulus theorem the assertion (1.8) is satisfied for $z \in \mathbb{U}$ and the proof of Theorem 2.1 is completed. ☐

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Remark 2.1. *Putting* $n=1,\ \mu=0$ *in Theorem 2.1, we get Theorem 1 of Sahoo and Patel* [35].

Taking $n=1,\ b=pe^{-i\theta}$, we get following result.

Corollary 2.1. *For* $f\in \mathrm{A}_p,\ |\theta|<\frac{\pi}{2},\ 0\le\rho<p,\ \sum_{k=1}^{\infty}\dfrac{(a)_k}{(c)_k}|a_{p+k}|(p+\mu k)\le(p-\rho)\cos\theta$ *is the sufficient condition to be a member of* $R_p(\mu,\theta,a,c,\rho)$.

Theorem 2.2. *If the function* f, *given by* (1.1) *belongs to the class* $R_{p,n}^b(\mu,a,c,A,B)$, *then*

$$|a_{p+k}|\le\frac{|b|(A-B)(c)_k}{(p+\mu k)(a)_k}\quad(k\ge n\in\mathbb{N}).\tag{2.2}$$

The estimate (2.2) *is sharp.*

Proof. Since $f\in R_{p,n}^b(\mu,a,c,A,B)$, we have

$$\frac{p(1-\mu)\mathcal{L}_p(a,c)f(z)+\mu z(\mathcal{L}_p(a,c)f)'(z)-pz^p}{z^p}=\frac{b(A-B)\omega(z)}{1+B\omega(z)}\quad(z\in\mathbb{U}),\tag{2.3}$$

where $\omega(z)=w_1z+w_2z^2+\cdots$ is analytic in $\mathbb{U}$ satisfying the condition $|\omega(z)|\le|z|$ for $z\in\mathbb{U}$. Substituting the series expansion of f and ω in (2.3) followed by simplification, we deduce that

$$\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}a_{p+k}(k\mu+p)z^k=\left\{b(A-B)-B\sum_{k=n}^{\infty}\frac{(a)_k}{(c)_k}a_{p+k}(k\mu+p)z^k\right\}\sum_{k=1}^{\infty}w_kz^k\quad(z\in\mathbb{U}).\tag{2.4}$$

Equating the corresponding coefficient on both side of (2.4), we find that the coefficient a_{p+k} on the left hand side of (2.4) depends only on $a_{p+n},a_{p+(n+1)},\cdots,a_{p+k-1},\ k\ge n\in\mathbb{N}$ on the right hand side of (2.4). Hence, for $k\ge n$, it follows from (2.4) that

$$\sum_{k=n}^{t}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k+\sum_{k=t+1}^{\infty}d_kz^k=\left\{b(A-B)-B\sum_{k=n}^{t-1}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k\right\}\omega(z),$$

where the series $\sum_{k=t+1}^{\infty}d_kz^k$ converges in $\mathbb{U}$. Since $|\omega(z)|<1$ for $z\in\mathbb{U}$, we get

$$\left|\sum_{k=n}^{t}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k+\sum_{k=t+1}^{\infty}d_kz^k\right|\le\left|\left\{b(A-B)-B\sum_{k=n}^{t-1}\frac{(a)_k}{(c)_k}(k\mu+p)a_{p+k}z^k\right\}\right|.\tag{2.5}$$

Writing $z=re^{i\theta}$ $(r<1)$, squaring both sides of (2.5) and then integrating, we obtain

$$\sum_{k=n}^{t}\frac{(a)_k^2}{(c)_k^2}(k\mu+p)^2|a_{p+k}|^2r^{2k}+\sum_{k=t+1}^{\infty}|d_k|^2r^{2k}\le|b|^2(A-B)^2+|B|^2\sum_{k=n}^{t-1}\frac{(a)_k^2}{(c)_k^2}(k\mu+p)^2|a_{p+k}|^2r^{2k}.$$

Letting $r\to1^-$ in the above inequality, we get

$$\frac{(a)_t^2}{(c)_t^2}(t\mu+p)^2|a_{p+t}|^2\le|b|^2(A-B)^2-(1-|B|^2)\sum_{k=1}^{t-1}\frac{(a)_k^2}{(c)_k^2}(k\mu+p)^2|a_{p+k}|^2\le|b|^2(A-B)^2,$$

where we have used the fact that $|B|\le1$. Thus, it follows that

$$|a_{p+t}|\le\frac{|b|(A-B)(c)_t}{(t\mu+p)(a)_t}\quad(t\ge n\in\mathbb{N}).\tag{2.6}$$

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It is easily seen that the estimate (2.6) is sharp for the functions

$$f_k(z) = \phi_p(c, a; z) \star z^p \left[\frac{(k\mu + p) + \{B(k\mu + p) + b(A - B)\}z^k}{(k\mu + p)(1 + Bz^k)}\right] \quad (k \in \mathbb{N}; z \in \mathbb{U}).$$

□

From the above theorem 2.2 we can draw the following result.

Corollary 2.2.

$$R^b_{p,n}(\mu, a + 1, c, A, B) \subset R^b_{p,n}(\mu, a, c, A, B)$$

and

$$R^b_{p,n}(\mu, a, c, A, B) \subset R^b_{p,n}(\mu, a, c + 1, A, B).$$

Letting $b = pe * -i\theta,\ A = 1 - 2\rho/p,\ B = -1$ in Theorem 2.1, we get

Corollary 2.3. *If the function $f \in \mathrm{A}_p$ is in the class $R_p(\mu, a, c, \theta, \rho)$, then*

$$|a_{p+k}| \leq \frac{2p(1 - \frac{\rho}{p})(c)_k}{(p + \mu k)(a)_k} \quad (k \geq n \in \mathbb{N}).$$

3. Hankel determinant

In this section, we solve the Fekete-Szegö problem and also determine the sharp upper bound to the second Hankel determinant for the class $R^b_p(\mu, a, c, A, B)$.

We first prove

Theorem 3.1. *If the function $f \in A_p$, belongs to the class $R^b_p(\mu, a, c, A, B)$, then for any $\lambda \in \mathbb{C}$*

$$|a_{p+2} - \lambda a^2_{p+1}| \leq \frac{|b|(A - B)}{(p + 2\mu)}\frac{(c)_2}{(a)_2} \max\left\{1, \left|B + \frac{\lambda b(A - B)(p + 2\mu)}{(p + \mu)^2}\frac{c(a + 1)}{a(c + 1)}\right|\right\}. \tag{3.1}$$

The estimate (3.1) *is sharp.*

Proof. Since $f \in R^b_p(\mu, a, c, A, B)$, we can find $\varphi \in \mathscr{P}$ of the form (1.4) such that

$$p(1 - \mu)\frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \mu\frac{(\mathcal{L}_p(a, c)f)'(z)}{z^{p-1}} - p = \frac{b(A - B)(\varphi(z) - 1)}{(1 - B) + (1 + B)\varphi(z)} \quad (z \in \mathbb{U}). \tag{3.2}$$

Writing the series expansion of both sides, we obtain

$$\left(\sum_{k=1}^{\infty}\frac{(a)_{p+k}}{(b)_{p+k}}(p + \mu k)a_{p+k}z^k\right)\left(2 + (1 + B)\sum_{k=1}^{\infty} q_k z^k\right) = b(A - B)\sum_{k=1}^{\infty} q_k z^k. \tag{3.3}$$

Equating coefficient of $z,\ z^2$ and z^3, we get

$$a_{p+1} = \frac{c}{a}\frac{b(A - B)q_1}{2(p + \mu)}, \tag{3.4}$$

$$a_{p+2} = \frac{(c)_2}{(a)_2}\frac{b(A - B)}{2(p + 2\mu)}\left\{q_2 - \left(\frac{1 + B}{2}\right)q_1^2\right\}, \tag{3.5}$$

and

$$a_{p+3} = \frac{(c)_3}{(a)_3}\frac{b(A-B)}{2(p+3\mu)}\left\{q_3 - \left(\frac{1+B}{2}\right)q_1q_2 + \left(\frac{1+B}{2}\right)^2 q_1^3\right\}. \tag{3.6}$$

Now for any $\mu \in \mathbb{C}$, we have

$$a_{p+2} - \lambda\, a_{p+1}^2 = \frac{b(A-B)}{2(p+2\mu)}\frac{(c)_2}{(a)_2}\left\{q_2 - \left[\frac{1+B}{2} + \frac{\lambda b(A-B)(p+2\mu)}{2(p+\mu)^2}\frac{c(a+1)}{a(c+1)}\right]q_1^2\right\}.$$

From the above expression with the aid of Lemma 1.1, we get

$$2\gamma - 1 = B + \frac{\lambda b(A-B)(p+2\mu)}{(p+\mu)^2}\frac{c(a+1)}{a(c+1)},$$

which yields the required estimate (3.1). Equality in (3.1) is attained for the function f, defined in $\mathbb{U}$ by

$$f(z) = \begin{cases} \phi_p(c,a;z)\star z^p\left\{\dfrac{1+\left(B+b\dfrac{(A-B)}{p+2\mu}\right)z^2}{1+Bz^2}\right\}, & \text{if } \left|B+\lambda\dfrac{b(A-B)(p+2\mu)(a+1)c}{(p+\mu)^2a(c+1)}\right| \le 1 \\ \phi_p(c,a;z)\star z^p\left\{\dfrac{(p+2\mu)+(B(p+\mu)+b(A-B))\,z}{(p+2\mu)+B(p+\mu)z}\right\}, & \text{if } \left|B+\lambda\dfrac{b(A-B)(p+2\mu)(a+1)c}{(p+\mu)^2a(c+1)}\right| > 1. \end{cases}$$

This completes the proof of Theorem 3.1. □

For λ to be real, we get the following result.

Corollary 3.1. *If the function $f \in A_p$, belongs to the class $R_p^b(\mu,a,c,A,B)$, then for any $\lambda \in \mathbb{R}$*

$$|a_{p+2}-\lambda a_{p+1}^2| \le \begin{cases} \dfrac{|b|(A-B)}{(p+2\mu)}\dfrac{(c)_2}{(a)_2}, & \textit{for } \dfrac{-(1+B)(p+\mu)^2a(c+1)}{b(A-B)(p+2\mu)c(a+1)} \le \lambda \le \dfrac{(1-B)(p+\mu)^2a(c+1)}{b(A-B)(p+2\mu)} \\ \dfrac{|b|(A-B)}{(p+2\mu)}\dfrac{(c)_2}{(a)_2}\left\{B+\dfrac{\lambda b(A-B)(p+2\mu)}{(p+\mu)^2}\dfrac{c(a+1)}{a(c+1)}\right\}, & \textit{Otherwise.} \end{cases}$$

Remark 3.1. *Taking $\mu = 0$ and substituting b by bp in Theorem 3.1, we get Theorem 3 of Sahoo and Patel* [35].

Putting $b = pe^{-i\theta}\cos\theta,\ A = 1-2\rho/p,\ B = -1$ in Theorem3.1, we get the following result.

Corollary 3.2. *If $f \in R_{p,}^{pe^{-i\theta}\cos\theta}(\mu,a,c,1-\dfrac{2\rho}{p},-1)$, then*

$$|a_{p+2}-\lambda a_{p+1}^2| \le \frac{2(p-\rho)\cos\theta}{(p+2\mu)}\frac{(c)_2}{(a)_2}\max\left\{1, \left|\frac{2\lambda e^{-i\theta}\cos\theta(p-\rho)(p+2\mu)}{(p+\mu)^2}\frac{c(a+1)}{a(c+1)} - 1\right|\right\}.$$

The estimate is sharp.

Theorem 3.2. *If $f \in R_p^b(\mu,a,c,A,B)$ and $a \ge c > 0$, then*

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{\frac{|b|(A-B)(c)_2}{(p+2\mu)(a)_2}\right\}^2. \tag{3.7}$$

Proof. Using equation (3.4), (3.5) and (3.6), we get

$$\begin{aligned} a_{p+3}a_{p+1}-a_{p+2}^2 =& \frac{b^2(A-B)^2}{4}\frac{c(c)_2}{a(a)_2}\left\{\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}q_1q_3-\frac{(c+1)}{(a+1)(p+2\mu)^2}q_2^2\right.\\ &+\left[\frac{(c+1)}{(a+1)(p+2\mu)^2}-\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}\right](1+B)q_1^2q_2\\ &\left.+\left[\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}-\frac{(c+1)}{(a+1)(p+2\mu)^2}\right]\left(\frac{1+B}{2}\right)^2q_1^4\right\}.\end{aligned}$$

Also,from Lemma 1.1, we get

$$\begin{aligned} &a_{p+3}a_{p+1}-a_{p+2}^2 =\\ &\frac{b^2(A-B)^2}{4}\frac{c(c)_2}{a(a)_2}\left\{\frac{1}{4(p+3\mu)(p+\mu)}\frac{c+2}{a+2}\left[q_1^4+2(4-q_1^2)q_1^2x-(4-q_1^2)q_1^2x^2+2q_1(4-q_1^2)(1-|x|^2z)\right]\right.\\ &-\frac{(c+1)}{(a+1)(p+2\mu)^2}\left[q_1^4+2(4-q_1^2)q_1^2x+(4-q_1^2)x^2\right]\\ &+\left[\frac{(c+1)}{(a+1)(p+2\mu)^2}-\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}\right]\frac{(1+B)}{2}\left[q_1^4+(4-q_1^2)q_1^2x\right]\\ &\left.+\left[\frac{1}{(p+3\mu)(p+\mu)}\frac{c+2}{a+2}-\frac{(c+1)}{(a+1)(p+2\mu)^2}\right]\left(\frac{1+B}{2}\right)^2q_1^4\right\}.\end{aligned}$$

For simplicity in the expression, we put

$$\alpha=\frac{b^2(A-B)^2}{4}\frac{c(c)_2}{a(a)_2},\quad \beta=\frac{c+2}{4(p+3\mu)(p+\mu)(a+2)}$$

and

$$\Gamma=\frac{(c+1)}{4(a+1)(p+2\mu)^2}.$$

Then by simple calculation, it can be observed that $0<\Gamma<\beta<2\Gamma$. Using above notation and triangle inequality, we can write

$$\begin{aligned} |a_{p+3}a_{p+1}-a_{p+2}^2|\leq |\alpha|&\left\{\frac{1}{8}\left[(\beta-\Gamma)(8+B(1+B))\right]q_1^4\right.\\ &+\frac{1}{8}\left[(\beta-\Gamma)(15-B)\right](4-q_1^2)q_1^2x\\ &+\left(\beta q_1^2+\Gamma(4-q_1^2)\right)(4-q_1^2)x^2\\ &\left.+\left(2\beta q_1(4-q_1^2)(1-x^2)\right)\right\}. \end{aligned}\tag{3.8}$$

Since the functions $\phi(z)$ and $\phi(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ belong to the class $\mathcal{P}$, we can assume $q_1>0$, by which generality is not lost. Taking $x=v,\ q_1=u$ in (3.8), we get the the function $T(u,v)$ (say)

$$\begin{aligned} T(u,v)=&|\alpha|\left\{\frac{1}{8}\left[(\beta-\Gamma)(8+B(1+B))\right]u^4+\frac{1}{8}\left[(\beta-\Gamma)(15-B)\right](4-u^2)u^2v\right.\\ &\left.+\left(\beta u^2+\Gamma(4-u^2)\right)(4-u^2)v^2+\left(2\beta u(4-u^2)(1-v^2)\right)\right\}.\end{aligned}$$

We need to find maximum value of $T(u.v)$ in the interval $0 \leq u \leq 2,\ 0 \leq v \leq 1$. We can see by using the fact $0 < \Gamma < \beta < 2\Gamma$,

$$\frac{\partial T}{\partial v} = |\alpha|(4-u^2)\left\{\frac{1}{8}\left[(\beta-\Gamma)(15-B)\right] + 2(\beta-\Gamma)u^2 v + 4(2\Gamma-\beta u)v\right\} > 0 \ \ (0 \leq u \leq 2,\ 0 \leq v \leq 1).$$

So $T(u,v)$ can not attain its maximum value within $0 < u < 2,\ 0 < v < 1$. Moreover, for fixed $u \in [0,2]$,

$$M(u) = \max_{0\leq v\leq 1} T(u,v) = T(u,1) = |\alpha|\left\{\frac{1}{8}\left[(\beta-\Gamma)(8+B(1+B))\right]u^4 + \frac{1}{8}\left[(\beta-\Gamma)(15-B)\right](4-u^2)u^2 + \left(\beta u^2 + \Gamma(4-u^2)\right)(4-u^2)\right\}$$

and

$$M'(u) = |\alpha|\left\{\frac{1}{2}(\beta-\Gamma)\left[B^2+2B-15\right]u^3 + \left((\beta-\Gamma)(23-B)-8\Gamma\right)u\right\}.$$

Since $M'(u) > 0$, the maximum value occurs at $u = 0,\ v = 1$. Therefore

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \leq \left\{\frac{|b|(A-B)(c)_2}{(p+2\mu)(a)_2}\right\}^2.$$

□

Taking $b = pe^{-i\theta}\cos\theta,\ A = 1-2\rho/p,\ B = -1$ in Theorem3.2 we get the following result.

Corollary 3.3. *If* $f \in R_{p,}^{pe^{-i\theta}\cos\theta}(\mu, a, c, 1-2\rho/p, -1)$, *then*

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \leq \left\{\frac{2\cos\theta(p-\rho)(c)_2}{(p+2\mu)(a)_2}\right\}^2. \tag{3.9}$$

The estimate (3.9) *is sharp.*

Remark 3.2. *Taking* $\mu = 0,\ p = 1, a = \alpha,\ b = \beta$ *in Corollary3.3, we get the result of Theorem 3.1 of Mishra and Kund* [26].

Putting $a = p+1,\ c = p+1+\lambda$ in Corollary 3.3, we get following result.

Corollary 3.4. *If* $f \in R_{p,\lambda}(\mu, \theta,\ \rho)$, *then*

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \leq \left\{\frac{2\cos\theta(p-\rho)(p+1-\lambda)_2}{(p+2\mu)(p+1)_2}\right\}^2. \tag{3.10}$$

The estimate (3.10) *is sharp.*

Remark 3.3. *Putting* $\mu = 0,\ p = 1,\ \theta = \alpha$ *in Corollary 3.4, we get the result obtained in theorem 3.1 of Mishra and Gochhayat* [25]

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A GENERALIZATION OF SOME RESULTS FOR APPELL POLYNOMIALS TO SHEFFER POLYNOMIALS

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Abstract. Recently, Mihoubi and Taharbouchet gave some interesting method of obtaining certain identities for Appell polynomials of arbitrary orders starting from the given identities for Appell polynomials of fixed orders. In addition, they illustrated their method with several examples. The purpose of this paper is to note that their method can be generalized so as to include any Sheffer polynomials. Also, we will provide many examples that illustrate our results.

1. Introduction and Preliminaries

Here we will go over very briefly some basic facts about umbral calculus. The reader is advised to refer to [12] for a complete treatment. Let $\mathfrak{F}$ be the algebra of all formal power series in the variable t with the coefficients in the field $\mathbb{C}$ of complex numbers:

$$\mathfrak{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}. \tag{1}$$

Let $\mathbb{P} = \mathbb{C}[x]$ be the ring of polynomials in x with coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$. For $L \in \mathbb{P}^*$, $p(x) \in \mathbb{P}$, $< L|p(x) >$ denotes the action of the linear functional L on $p(x)$. The linear functional $< f(t)|\cdot >$ on $\mathbb{P}$ is defined by

$$< f(t)|x^n >= a_n, \ \ (n \geq 0), \tag{2}$$

where $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathfrak{F}$. For $L \in \mathbb{P}^*$, let us set $f_L(t) = \sum_{k=0}^{\infty} < L|x^k > \frac{t^k}{k!} \in \mathfrak{F}$. Then we see that $< f_L(t)|x^n >=< L|x^n >$, and the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathfrak{F}$. Thus $\mathfrak{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in t, and so an element $f(t) \in \mathfrak{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathfrak{F}$ is called the umbral algebra, the study of which is the umbral calculus(see [1-12]).

The order $\circ(f(t))$ of $0 \neq f(t) \in \mathfrak{F}$ is the smallest integer k such that the coefficient of t^k does not vanish. In particular, $0 \neq f(t) \in \mathfrak{F}$ is called an invertible series if $\circ(f(t)) = 0$ and a delta series if $\circ(f(t)) = 1$. For $f(t), g(t) \in \mathfrak{F}$ with $\circ(g(t)) = 0, \circ(f(t)) = 1$, there exists a unique sequence $s_n(x)$ ($deg\ s_n(x) = n$) such that $< g(t)f(t)^k|s_n(x) >= n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. Further, it is known that $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\overline{f}(t))} e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \text{(see[1-12])}, \tag{3}$$

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where $\overline{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\overline{f}(t)) = \overline{f}(f(t)) = t$. In particular, $s_n(x)$ is called the Appell sequence for $g(t)$ if $s_n(x) \sim (g(t), t)$.

Assume now that $s_n(x) \sim (g(t), f(t))$. Thus $s_n(x)$ is the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, and

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} = \frac{1}{g(\overline{f}(t))}e^{x\overline{f}(t)}, \text{ (see [12, 13])}. \tag{4}$$

Here we will assume that $g(0) = 1$, though it is not necessary. So, for any $\alpha \in \mathbb{C}$ and

$$g(t) = 1 + \sum_{k=1}^{\infty} a_k \frac{x^k}{k!}, \tag{5}$$

$$g(t)^{\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \frac{\left(\sum_{k=1}^{\infty} a_k \frac{t^k}{k!}\right)^n}{n!}, \tag{6}$$

where $(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$ for $n \geq 1$, and $(\alpha)_0 = 1$. Let $s_n^{(\alpha)}(x) \sim (g(t)^{\alpha}, f(t))$. Then

$$\sum_{n=0}^{\infty} s_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha} e^{x\overline{f}(t)}. \tag{7}$$

Also, we set

$$\widetilde{s}_n(x) \sim (g(t), t), \;\; \widetilde{s}_n^{(\alpha)}(x) = (g(t)^{\alpha}, t). \tag{8}$$

Thus $\widetilde{s}_n(x)$ and $\widetilde{s}_n^{(\alpha)}(x)$ are Appell polynomials and

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{s}_n(x)\frac{t^n}{n!} &= \frac{1}{g(t)}e^{xt}, \\ \sum_{n=0}^{\infty} \widetilde{s}_n^{(\alpha)}(x)\frac{t^n}{n!} &= \left(\frac{1}{g(t)}\right)^{\alpha} e^{xt}. \end{aligned} \tag{9}$$

We observe here that

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{s}_n(x)\frac{(\overline{f}(t))^n}{n!} &= \frac{1}{g(\overline{f}(t))}e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \\ \sum_{n=0}^{\infty} \widetilde{s}_n^{(\alpha)}(x)\frac{(\overline{f}(t))^n}{n!} &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha} e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n^{(\alpha)}(x)\frac{t^n}{n!}. \end{aligned} \tag{10}$$

Adopting the conventional notation used in [10], we let $\frac{1}{g(t)} = e^{At}$. So if $\frac{1}{g(t)} = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$, then $a_n = A^n$. Moreover,

$$\sum_{n=0}^{\infty} \widetilde{s}_n(x)\frac{t^n}{n!} = e^{(A+x)t} = \sum_{n=0}^{\infty} (A+x)^n \frac{t^n}{n!}, \tag{11}$$

so that $\widetilde{s}_n(x) = (A+x)^n$.

Recently, Mihoubi and Taharbouchet [10] gave some interesting method of obtaining certain identities for Appell polynomials of arbitrary orders starting from the given identities for Appell polynomials of fixed orders. In addition, they illustrated their method with several examples. The purpose of this paper is to note that their method can be generalized so as to include any Sheffer polynomials. Also, we will provide many examples that illustrate our results.

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2. Main results

We will prove Theorem 2, which includes Propositions 2 and 3 in [10] as special cases, after showing a lemma corresponding to Lemma 1 in [10].

Lemma 2.1. *Let $s_n(x) \sim (g(t), f(t))$, and let $\alpha \in \mathbb{C}$.*

(a) $s_n^{(\alpha)}(A+x) = s_n^{(\alpha+1)}(x)$,

(b) $$(\alpha+1)(A+x)s_n^{(\alpha)}(A+x) = \sum_{l=0}^{n}\binom{n}{l}\theta_{n-l}s_{l+1}^{(\alpha+1)}(x) + \alpha x s_n^{(\alpha+1)}(x), \tag{12}$$

where $\frac{1}{\overline{f}'(t)} = \sum_{n=0}^{\infty}\theta_n\frac{t^n}{n!}$, *with* $\overline{f}'(t) = \frac{d}{dt}\overline{f}(t)$.

Proof. (a)

$$\begin{aligned}\sum_{n=0}^{\infty}s_n^{(\alpha)}(A+x)\frac{t^n}{n!} &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha}e^{(A+x)\overline{f}(t)}\\ &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha}e^{A\overline{f}(t)}e^{x\overline{f}(t)}\\ &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha}\frac{1}{g(\overline{f}(t))}e^{x\overline{f}(t)}\\ &= \left(\frac{1}{g(\overline{f}(t))}\right)^{\alpha+1}e^{x\overline{f}(t)}\\ &= \sum_{n=0}^{\infty}s_n^{(\alpha+1)}\frac{t^n}{n!}. \end{aligned} \tag{13}$$

(b) Using Lemma 1 of [10] and replacing t by $\overline{f}(t)$, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}(A+x)\widetilde{s}_n^{(\alpha)}(A+x)\frac{(\overline{f}(t))^n}{n!}\\ =\ &\sum_{n=0}^{\infty}\left\{\frac{1}{\alpha+1}\widetilde{s}_{n+1}^{(\alpha+1)}(x) + \frac{\alpha x}{\alpha+1}\widetilde{s}_n^{(\alpha+1)}(x)\right\}\frac{(\overline{f}(t))^n}{n!}.\end{aligned} \tag{14}$$

The LHS of (14) is obviously equal to

$$\sum_{n=0}^{\infty}(A+x)s_n^{(\alpha)}(A+x)\frac{t^n}{n!}. \tag{15}$$

Applying $\frac{d}{dt}$ on both sides of

$$\sum_{n=0}^{\infty}\widetilde{s}_n^{(\alpha+1)}(x)\frac{(\overline{f}(t))^n}{n!} = \sum_{n=0}^{\infty}s_n^{(\alpha+1)}(x)\frac{t^n}{n!}, \tag{16}$$

we get

$$\sum_{n=0}^{\infty}\widetilde{s}_{n+1}^{(\alpha+1)}(x)\frac{(\overline{f}(t))^n}{n!}\left(\frac{d}{dx}\overline{f}(t)\right) = \sum_{n=0}^{\infty}s_{n+1}^{(\alpha+1)}(x)\frac{t^n}{n!}. \tag{17}$$

Noting that $\overline{f}'(t)$ is invertiable, we have

$$\sum_{n=0}^{\infty}\widetilde{s}_{n+1}^{(\alpha+1)}(x)\frac{(\overline{f}(t))^n}{n!} = \frac{1}{\overline{f}'(t)}\sum_{l=0}^{\infty}s_{l+1}^{(\alpha+1)}(x)\frac{t^l}{l!}$$

$$\begin{aligned} &= \left(\sum_{m=0}^{\infty}\theta_m\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}s_{l+1}^{(\alpha+1)}(x)\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\theta_{n-l}s_{l+1}^{(\alpha+1)}(x)\right)\frac{t^n}{n!}. \end{aligned} \tag{18}$$

In view of (18), we now see that the RHS of (14) is

$$\sum_{n=0}^{\infty}\left\{\frac{1}{\alpha+1}\sum_{l=0}^{n}\binom{n}{l}\theta_{n-l}s_{l+1}^{(\alpha+1)}(x)+\frac{\alpha x}{\alpha+1}s_n^{(\alpha+1)}(x)\right\}\frac{t^n}{n!}. \tag{19}$$

□

For the next theorem, we keep the notations in Proposition 2 of [8].

Theorem 2.2. *Let $n,a,b\in\mathbb{Z}_{\geq 0}$, $s_n(x)\sim(g(t),f(t))$, and let $(u_k),(v_k),(U(n,k):0\leq k\leq n),(V(n,k):0\leq k\leq n)$ be sequences of complex numbers. Assume that*

$$\sum_{k=0}^{n}U(n,k)s_k^{(a)}(x+u_k)=\sum_{k=0}^{n}V(n,k)s_k^{(b)}(x+v_k). \tag{20}$$

Then, for any $\alpha\in\mathbb{C}$, we have
(a)

$$\sum_{k=0}^{n}U(n,k)s_k^{(\alpha+a-b)}(x+u_k)=\sum_{k=0}^{n}V(n,k)s_k^{(\alpha)}(x+v_k). \tag{21}$$

(b)

$$\begin{aligned} &\sum_{k=0}^{n}U(n,k)\left\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha+a-b)}(x+u_k)+((a-b-1)x-\alpha u_k)s_k^{(\alpha+a-b)}(x+u_k)\right\} \\ &=\sum_{k=0}^{n}V(n,k)\left\{(\alpha+a-b)\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+v_k)-(x+(\alpha+a-b)v_k)s_k^{(\alpha)}(x+v_k)\right\}, \end{aligned} \tag{22}$$

where $\frac{1}{\overline{f}'(t)}=\sum_{n=0}^{\infty}\theta_n\frac{t^n}{n!}$, with $\overline{f}'(t)=\frac{d}{dt}\overline{f}(t)$.

Proof. (a) As was shown in [1], $\tilde{s}_n^{(\alpha)}(x)$ is a polynomial in α of degree $\alpha\leq n$. Since $\sum_{n=0}^{\infty}\tilde{s}_n^{(\alpha)}(x)=\frac{(\overline{f}(t))^n}{n!}=\sum_{n=0}^{\infty}s_n^{(\alpha)}(x)\frac{t^n}{n!}$, $s_n^{(\alpha)}$ is also a polynomial in α of degree $\leq n$. Let

$$\Phi(\alpha)=\sum_{k=0}^{n}U(n,k)s_k^{(\alpha+a-b)}(x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(\alpha)}(x+v_k). \tag{23}$$

By assumption, $\Phi(b)=0$. In $0=\Phi(b)=\sum_{k=0}^{n}U(n,k)s_k^{(a)}(x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(b)}(x+v_k)$, replace x by $A+x$. Then

$$\begin{aligned} 0&=\sum_{k=0}^{n}U(n,k)s_k^{(a)}(A+x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(b)}(A+x+v_k) \\ &=\sum_{k=0}^{n}U(n,k)s_k^{(a+1)}(x+u_k)-\sum_{k=0}^{n}V(n,k)s_k^{(b+1)}(x+v_k). \end{aligned} \tag{24}$$

Thus $\Phi(b+1)=0$. Proceeding inductively, we see that $\Phi(m)=0$, for all integers $m \geq b$. As $\Phi(\alpha)$ is a polynomial in α of degree $\leq n$, $\Phi(\alpha)$ is identically zero as a polynomial in α. This shows (a).

(b) Replacing α by $\alpha-1$ in (a), multiplying both sides by x, substituting $A+x$ for x, and multiplying the resulting equation by $\alpha(\alpha+a-b)$, we obtain

$$\begin{aligned}&\alpha(\alpha+a-b)\sum_{k=0}^{n}U(n,k)\left\{(A+x+u_k)s_k^{(\alpha+a-b-1)}(A+x+u_k)-u_k s_k^{(\alpha+a-b-1)}(A+x+u_k)\right\}\\&=\alpha(\alpha+a-b)\sum_{k=0}^{n}V(n,k)\left\{(A+x+v_k)s_k^{(\alpha-1)}(A+x+v_k)-v_k s_k^{(\alpha-1)}(A+x+v_k)\right\}.\end{aligned} \tag{25}$$

Using (a) and (b) of Lemma 1, (26) becomes

$$\begin{aligned}&\sum_{k=0}^{n}U(n,k)\left\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha+a-b)}(x+u_k)\right.\\&\quad\left.+\alpha(\alpha+a-b-1)(x+u_k)s_k^{(\alpha+a-b)}(x+u_k)-\alpha(\alpha+a-b)u_k s_k^{(\alpha+a-b)}(x+u_k)\right\}\\=&\sum_{k=0}^{n}V(n,k)\left\{(\alpha+a-b)\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+v_k)\right.\\&\quad\left.+(\alpha+a-b)(\alpha-1)(x+v_k)s_k^{(\alpha)}(x+v_k)-\alpha(\alpha+a-b)v_k s_k^{(\alpha)}(x+v_k)\right\}.\end{aligned} \tag{26}$$

Substracting

$$\left\{\alpha^2+(\alpha-1)(a-b-1)\right\}x\sum_{k=0}^{n}U(n,k)s_k^{(\alpha+a-b)}(x+u_k) \tag{27}$$

$$=\left\{\alpha^2+(\alpha-1)(a-b-1)\right\}x\sum_{k=0}^{n}V(n,k)s_k^{(\alpha)}(x+v_k) \tag{28}$$

from both sides of (27), we get the desired result. □

Remark 2.3. *When $a=b=0$, the assumption in Theorem 2*

$$\sum_{k=0}^{n}U(n,k)s_k^{(0)}(x+u_k)=\sum_{k=0}^{n}V(n,k)s_k^{(0)}(x+v_k) \tag{29}$$

depends only on $f(t)$, since

$$\sum_{n=0}^{\infty}s_n^{(0)}(x)\frac{t^n}{n!}=e^{x\bar{f}(t)}. \tag{30}$$

Thus we have, for any $s_n(x)\sim(g(t),f(t))$, with any $g(t)$ but with the same $f(t)$,

$$\sum_{k=0}^{n}U(n,k)s_k^{(\alpha)}(x+u_k)=\sum_{k=0}^{n}V(n,k)s_k^{(\alpha)}(x+v_k), \tag{31}$$

$$\begin{aligned}&\sum_{k=0}^{n} U(n,k)\left\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+u_k)-(x+\alpha u_k)s_k^{(\alpha)}(x+u_k)\right\}\\&=\sum_{k=0}^{n} V(n,k)\left\{\alpha\sum_{l=0}^{k}\binom{k}{l}\theta_{k-l}s_{l+1}^{(\alpha)}(x+v_k)-(x+\alpha v_k)s_k^{(\alpha)}(x+v_k)\right\}.\end{aligned} \tag{32}$$

3. Examples

Here we will illustrate our results with many interesting examples.

Example 3.1. Let $s_n(x) \sim (g(t), f(t) = e^t - 1)$, for some invertible series $g(t)$. Here $\overline{f}(t) = log(1+t)$, and hence $\frac{1}{\overline{f}'(t)} = 1+t$. So, $\theta_0 = \theta_1 = 1$, and $\theta_m = 0$, for $m \geq 2$. Observe here that $s_n^{(0)}(x) = (x)_n$. This applies to many special polynomials.

• Bernoulli polynomials of the second kind $b_n(x)$ given by (see [9])

$$\frac{t}{log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, b_n(x) \sim \left(\frac{t}{e^t-1}, e^t-1\right). \tag{33}$$

.

• Daehee polynomials of the first kind $D_n(x)$ given by (see [5])

$$\frac{log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, D_n(x) \sim \left(\frac{e^t-1}{t}, e^t-1\right). \tag{34}$$

• Daehee polynomials of the second kind $\widehat{D}_n(x)$ given by (see [5])

$$\frac{(1+t)log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n(x)\frac{t^n}{n!}, \widehat{D}_n(x) \sim \left(\frac{e^t-1}{te^t}, e^t-1\right). \tag{35}$$

• Boole polynomials $Bl_{n,\lambda}(x)$ given by (see [6])

$$(1+(1+t)^\lambda)^{-1}(1+t)^x = \sum_{n=0}^{\infty} Bl_{n,\lambda}(x)\frac{t^n}{n!}, Bl_{n,\lambda}(x) \sim \left(1+e^{\lambda t}, e^t-1\right). \tag{36}$$

Note here that the higher-order Boole polynomials $Bl_{n,\lambda}^{(\alpha)}(x)$ are called Peters polynomials.

• Korobov polynomials of the first kind $K_n(\lambda, x)$ given by (see [2])

$$\frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n=0}^{\infty} K_n(\lambda,x)\frac{t^n}{n!}, K_n(\lambda,x) \sim \left(\frac{e^{\lambda t}-1}{\lambda(e^t-1)}, e^t-1\right). \tag{37}$$

• degenerate poly-Bernoulli polynomials of the second kind $\mathbb{B}_{n,k}(\lambda, x)$ with the index k given by (see [3])

$$\frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n=0}^{\infty} \mathbb{B}_{n,k}(\lambda,x)\frac{t^n}{n!}, \mathbb{B}_{n,k}(\lambda,x) \sim \left(\frac{e^{\lambda t}-1}{\lambda Li_k(1-e^{-(e^t-1)})}, e^t-1\right), \tag{38}$$

where $Li_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}$ is the kth polylogarithmic function for $k \geq 1$ and a rational function for $k \leq 0$.

- λ-Daehee polynomials of the first kind $D_{n,\lambda}(x)$ given by (see [8])

$$\frac{\lambda log(1+t)}{(1+t)^{\lambda}-1}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}, D_{n,\lambda}(x)\sim\left(\frac{e^{\lambda t}-1}{\lambda t}, e^t-1\right). \tag{39}$$

- The polynomials $IA_n(x)$ given by (see [12])

$$(1+t)^{-1}(1+t)^x=\sum_{n=0}^{\infty}IA_n(x)\frac{t^n}{n!}, IA_n(x)\sim\left(e^t, e^t-1\right). \tag{40}$$

Note here that $IA_n^{(\alpha)}(x)$ is the inverse, under umbral composition, of $a_n^{(\alpha)}(-x)$, where $a_n^{(\alpha)}(x)$ is the actuarial polynomial with $a_n^{(\alpha)}(x)\sim((1-t)^{-\alpha}, log(1-t))$.

(a) We recall Gould's identity 640 from [11], page 10:

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!}(x)_k=\frac{(-1)^n}{n!}(x-1)_n. \tag{41}$$

From Theorem 2, we have the following identities

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!}s_k^{(\alpha)}(x)=\frac{(-1)^n}{n!}s_n^{(\alpha)}(x-1), \tag{42}$$

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(-1)^k}{k!}(\alpha s_{k+1}^{(\alpha)}(x)-(x-\alpha k)s_k^{(\alpha)}(x))\\ =&\frac{(-1)^n}{n!}(\alpha s_{n+1}^{(\alpha)}(x-1)-(x-(n+1)\alpha)s_n^{(\alpha)}(x-1)), (n\geq 0).\end{aligned} \tag{43}$$

(b) The Vandermonde convolution formula can be written as

$$\sum_{k=0}^{n}\binom{n}{k}(y)_{n-k}(x)_k=(x+y)_n. \tag{44}$$

Then Theorem 2 implies the following identities

$$\sum_{k=0}^{n}\binom{n}{k}(y)_{n-k}s_k^{(\alpha)}(x)=s_k^{(\alpha)}(x+y), \tag{45}$$

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(y)_{n-k}(\alpha s_{k+1}^{(\alpha)}(x)-(x-\alpha k)s_k^{(\alpha)}(x))\\ =&\alpha s_{n+1}^{(\alpha)}(x+y)-(x+(y-n)\alpha)s_n^{(\alpha)}(x+y), (n\geq 0).\end{aligned} \tag{46}$$

(c) For any $s_n(x)\sim(g(t), e^t-1)$, the Sheffer identity says

$$s_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}s_{n-k}(y)(x)_k. \tag{47}$$

From Theorem 2 with $a=1, b=0,$, we obtain the following identities

$$s_n^{(\alpha+1)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}s_{n-k}(y)s_k^{(\alpha)}(x), \tag{48}$$

$$\alpha s_{n+1}^{(\alpha+1)}(x+y)+\alpha(n-y)s_n^{(\alpha+1)}(x+y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(y)((\alpha+1)s_{k+1}^{(\alpha)}(x) + ((\alpha+1)k - x)s_k^{(\alpha)}(x)),\ (n \geq 0). \tag{49}$$

(d) Let $A(n,k)(0 \leq k \leq n)$ be the Eulerian numbers determined by

$$\frac{1-t}{e^{(t-1)x} - t} = \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}, A_n(t) = \sum_{k=0}^{n} A(n,k)t^k, \tag{50}$$

Worpitzky's identity is given by

$$x^n = \sum_{k=0}^{n-1} A(n,k)\binom{x+k}{n}, \tag{51}$$

which can be rewritten as

$$\sum_{k=0}^{n} S_2(n,k)(x)_k = \sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{j=0}^{n-1} A(n,j)\binom{j}{n-k}\right)(x)_k, \tag{52}$$

with $S_2(n,k)$ denoting the Stirling numbers of the second kind. Now, Theorem 2 yields the following identities

$$\sum_{k=0}^{n} S_2(n,k)s_k^{(\alpha)}(x) = \sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{n-1} A(n,j)\binom{j}{n-k}s_k^{(\alpha)}(x), \tag{53}$$

$$\sum_{k=0}^{n} S_2(n,k)(\alpha s_{k+1}^{(\alpha)}(x) - (x-\alpha k)s_k^{(\alpha)}(x))$$

$$= \sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{n-1} A(n,j)\binom{j}{n-k}(\alpha s_{k+1}^{(\alpha)}(x) - (x-\alpha k)s_k^{(\alpha)}(x)),\ (n \geq 0). \tag{54}$$

Example 3.2. Let $s_n(x) \sim (g(t), \frac{1}{\lambda}(e^{\lambda t} - 1))$, for some invertiable series $g(t)$. Here $\overline{f}(t) = \frac{1}{\lambda}log(1+\lambda t)$, and hence $\frac{1}{\overline{f}'(t)} = 1 + \lambda t$. Thus $\theta_0 = 1, \theta_1 = \lambda$, and $\theta_m = 0$, for $m \geq 2$. Observe here that $s_n^{(0)}(x) = (x|\lambda)_n$, where $(x|\lambda)_n = x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n \geq 1$, and $(x|\lambda)_0 = 1$. This includes many special polynomials:

- degenerate Bernoulli polynomials $\beta_n(\lambda, x)$ given by (see [1])

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_n(\lambda,x)\frac{t^n}{n!}, \beta_n(\lambda,x) \sim \left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{55}$$

- degenerate Euler polynomials $\mathcal{E}_n(\lambda, x)$ given by (see [1])

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\mathcal{E}_n(\lambda,x)\frac{t^n}{n!}, \mathcal{E}_n(\lambda,x) \sim \left(\frac{e^t+1}{2}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{56}$$

- degenerate poly-Bernoulli polynomials $\beta_{n,k}(\lambda, x)$ given by (see [7])

$$\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_{n,k}(\lambda,x)\frac{t^n}{n!},$$

$$\beta_{n,k}(\lambda,x) \sim \left(\frac{e^t-1}{Li_k(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}(e^{\lambda t}-1)\right). \tag{57}$$

(a) For any $s_n(x) \sim (g(t), \frac{1}{\lambda}(e^{\lambda t} - 1))$, the Sheffer identity says

$$s_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}s_{n-k}(y)(x|\lambda)_k \tag{58}$$

From Theorem 2, we get the following identities.

$$s_n^{(\alpha+1)}(x+y) = \sum_{k=0}^n \binom{n}{k} s_{n-k}(y) s_k^{(\alpha)}(x),$$

$$\alpha s_{n+1}^{(\alpha+1)}(x+y) + \alpha(n\lambda - y) s_n^{(\alpha+1)}(x+y)$$
$$= \sum_{k=0}^n \binom{n}{k} s_{n-k}(y)((\alpha+1)s_{k+1}^{(\alpha)}(x) + ((\alpha+1)k\lambda - x)s_k^{(\alpha)}(x)), (n \geq 0). \tag{59}$$

(b) From the identity $(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{y}{\lambda}} = (1+\lambda t)^{\frac{x+y}{\lambda}}$, we have the convolution formula

$$\sum_{k=0}^n \binom{n}{k} (y|\lambda)_{n-k} (x|\lambda)_k = (x+y|\lambda)_n \tag{60}$$

We can deduce the following identities from Theorem 2.

$$\sum_{k=0}^n \binom{n}{k} (y|\lambda)_{n-k} s_k^{(\alpha)}(x) = s_n^{(\alpha)}(x+y), \tag{61}$$

$$\sum_{k=0}^n \binom{n}{k} (y|\lambda)_{n-k} (\alpha s_{k+1}^{(\alpha)}(x) - (x - \alpha k\lambda) s_k^{(\alpha)}(x))$$
$$= \alpha s_{n+1}^{(\alpha)}(x+y) - (x + (y - n\lambda)\alpha) s_n^{(\alpha)}(x+y), \quad (n \geq 0). \tag{62}$$

(c) In [4], Hsu and Shiue introdued Stirling-type pair $\{S(n,k;\alpha,\beta,r), S(n,k;\beta,\alpha,-r)\}$ by the inverse relations

$$(x|\alpha)_n = \sum_{k=0}^n S(n,k;\alpha,\beta,r)(x-r|\beta)_k, \tag{63}$$

$$(x|\beta)_n = \sum_{k=0}^n S(n,k;\beta,\alpha,-r)(x+r|\alpha)_k. \tag{64}$$

They showed that $S(n,k) = S(n,k;\alpha,\beta,r)$ satisfies the recurrence relation

$$S(n+1,k) = S(n,k-1) + (k\beta - n\alpha + r)S(n,k), \ (n \geq k \geq 1), \tag{65}$$

which together with the obvious facts $S(n,0) = (r|\alpha)_n, S(n,n) = 1, \ (n \geq 0)$, completely determines $S(n,k)$. Clearly, $S_1(n,k) = S(n,k;1,0,0)$, $S_2(n,k) = S(n,k;0,1,0)$, $\binom{n}{k} = S(n,k;0,0,1)$, and hence the Stirling-type pair are nothing but far-reaching generalization of the classical Stirling numbers of the first kind and of the second kind.

Remark 3.1. *We now apply Theorem 2 by choosing $\alpha = \beta = \lambda$. Then*

$$(x|\lambda)_n = \sum_{k=0}^n S(n,k;\lambda,\lambda,r)(x-r|\lambda)_k, \tag{66}$$

where $S(n,k) = S(n,k;\lambda,\lambda,r)$ satisfies the relation

$$S(n+1,k) = S(n,k-1) + ((k-n)\lambda + r)S(n,k), \ (n \geq k \geq 1), \tag{67}$$

$$S(n,0) = (r|\lambda)_n, \ S(n,n) = 1, \ (n \geq 0). \tag{68}$$

Applying Theorem 2 to (66), we obtain the following identities

$$s_n^{(\alpha)}(x) = \sum_{k=0}^n S(n,k;\lambda,\lambda,r) s_k^{(\alpha)}(x-r),$$

$$\alpha s_{n+1}^{(\alpha)}(x) - (x - n\lambda\alpha) s_n^{(\alpha)}(x)$$
$$= \sum_{k=0}^n S(n,k;\lambda,\lambda,r)(\alpha s_{k+1}^{(\alpha)}(x-r) - (x - (r+k\lambda)\alpha) s_k^{(\alpha)}(x-r)), \ (n \geq 0). \tag{69}$$

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New Two-step Viscosity Approximation Methods of Fixed Points for Set-valued Nonexpansive Mappings Associated with Contraction Mappings in CAT(0) Spaces

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Abstract. The purpose of this paper is to introduce and study a class of new two-step viscosity iteration methods for approximating fixed points of set-valued nonexpansive mappings in CAT(0) spaces. Here, the fixed point is unique solution of a variational inequality with a contraction mapping. Further, we prove strong convergence theorem of the two-step viscosity iterations with some general conditions in a complete CAT(0) space. The presented results improve and unify the corresponding results in the literature.

Key Words and Phrases: New two-step viscosity approximation method, fixed point, strong convergence, set-valued nonexpansive mapping, CAT(0) space.

AMS Subject Classification: 47H09, 47H10, 54E70.

1 Introduction

As all we know, Kirk [1] first introduced and studied fixed point theory in CAT(0) spaces, and showed that every (single-valued) nonexpansive mapping on a bounded closed convex subset of a complete CAT(0) space (called also Hadamard space) always has a fixed point. On the other hand, fixed point theory for set-valued mappings has many useful applications in applied sciences, game theory and optimization theory. Since then, fixed point theory of single-valued and set-valued mapping in CAT(0) spaces has been rapidly developed, and it is natural and particularly meaningful to extend research of the known fixed point results for single-valued mappings to the setting of set-valued mappings.

Recalled that a mapping $f : X \to X$ on a metric space (X, d) is said to be a *contraction* if there exists a constant $k \in (0, 1]$ such that

$$d(f(x), f(y)) \leq kd(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Here, f is called *nonexpansive* when $k = 1$ in (1.1). Denote by $Fix(f)$ the set of all fixed points of f, i.e., $Fix(f) = \{x| x = f(x)\}$. Further, a set-valued mapping $T : E \to BC(X)$ is said to be nonexpansive if and only if

$$H(Tx, Ty) \leq d(x, y),$$

where E is a nonempty subset of X, $BC(X)$ is the family of nonempty bounded closed subsets of X, and $H(\cdot, \cdot)$ is Hausdorff distance on $BC(X)$, i.e., for any $A, B \in BC(X)$,

$$H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(b, a)\}.$$

If $x \in Tx$ for all $x \in E$, then x is called a fixed point of set-valued mapping T. We shall denote by $F(T)$ the set of all fixed points of T. A set-valued mapping T is said to satisfy endpoint condition

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$\mathbb{C}$ (see [2]) if $F(T) \neq \emptyset$ and $Tx = \{x\}$ for any $x \in F(T)$. We note that Panyanak and Suantai [3] pointed out "the condition $\mathbb{C}$ must be needed for set-valued mapping in the CAT(0) spaces".

Indeed, using contractions to approximate nonexpansive mappings is a classical way for studying a nonexpansive mapping $g : X \to X$. More precisely, take $\alpha \in (0,1)$ and define a contraction $g_\alpha : E \to E$ by

$$g_\alpha(x) = \alpha u + (1-\alpha)g(x), \quad \forall x \in E,$$

where $u \in E \subseteq X$ is an arbitrary fixed element. By $Banach's$ contraction mapping principle, g_α has a unique fixed point $x_\alpha \in E$. It is unclear, in general, what the behavior of x_α is as $\alpha \to 0$, even if g has a fixed point. However, in the case of g having a fixed point, Browder [4] proved that x_α converges strongly to a fixed point of g, which is nearest to u in the frame work of Hilbert spaces. Further, Reich [5] extended $Browder's$ result in [4] to the setting of Banach spaces and proved that x_α converges strongly to a fixed point of g in a uniformly smooth Banach space, and the limit defines the unique sunny nonexpansive retraction from E onto $Fix(g)$. Halpern [6] introduced and investigated the following explicit iterative scheme $\{x_n\}$ for a nonexpansive mapping g on a nonempty subset E of a Hilbert space: for any taken points $u, x_1 \in E$, and every $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)g(x_n). \tag{1.2}$$

In 2010, Saejung [7] studied some convergence theorems of the following $Halpern's$ iterations for a nonexipansive mapping $g : E \to E$ in a Hadamard space:

$$x_\alpha = \alpha u \oplus (1-\alpha)g(x_\alpha) \tag{1.3}$$

and

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)g(x_n), \ n \geq 1, \tag{1.4}$$

where u is an any fixed element, $x_1 \in E$ are arbitrarily chosen and $\alpha_n \in (0,1)$, and $x_\alpha \in E$ is called the unique fixed point of the contraction $x \mapsto \alpha u \oplus (1-\alpha)g(x)$ for all $\alpha \in (0,1)$. In [7], Saejung showed that $\{x_\alpha\}$ and $\{x_n\}$ converges strongly to $\tilde{x} \in Fix(g)$ as $\alpha \to 0$ and $n \to \infty$ under certain appropriate conditions on $\{\alpha_n\}$, respectively. Here, $\tilde{x}$ is nearest to u, i.e. $\tilde{x} = P_{Fix(g)}u$, here $P_E : X \to E$ is a metric projection from X onto E, i.e.,

$$P_E(x) = x_0 \in E,$$

where x_0 is satisfied with $d(x, x_0) < d(x, y)$ for any $y \in E$ and $y \neq x_0$ and E is a nonempty closed convex subset of (X, d).

Moreover, Shi and Chen [8] first studied convergence theorems of the following $Moudafi's$ viscosity iterative methods for a nonexpansive mapping $g : E \to E$ with $Fix(g) \neq \emptyset$ and a contraction mapping $f : E \to E$ in CAT(0) space X:

$$x_\alpha = \alpha f(x_\alpha) \oplus (1-\alpha)g(x_\alpha), \tag{1.5}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)g(x_n), \ n \geq 1, \tag{1.6}$$

where $\alpha \in (0,1)$, $\alpha_n \in (0,1)$, x_1 is an any given element in a nonempty closed convex subset $E \subseteq X$. $x_\alpha \in E$ is called unique fixed point of contraction $x \mapsto \alpha f(x) \oplus (1-\alpha)g(x)$. We remark that (1.5) and (1.6) is a extension case of (1.3) and (1.4), respectively. Shi and Chen [8] proved that $\{x_\alpha\}$ defined by (1.5) converges strongly as $\alpha \to 0$ to $\tilde{x} \in Fix(g)$ such that $\tilde{x} = P_{Fix(g)}f(\tilde{x})$ in the framework of CAT(0) space (X, d) satisfying the following property $\mathbb{P}$: For every $x, u, y_1, y_2 \in X$,

$$d(x, m_1)d(x, y_1) \leq d(x, m_2)d(x, y_2) + d(x, u)d(y_1, y_2),$$

where $m_i = P_{[x,y_i]}u$ for $i = 1, 2$. Furthermore, the authors also found that the sequence $\{x_n\}$ generated by (1.6) converges strongly to $\tilde{x} \in Fix(g)$ under certain appropriate conditions imposing on $\{\alpha_n\}$. By using the concept of quasi-linearization due to Berg and Nikolaev [9], Wangkeeree and Preechasilp [10] studied strong convergence theorems for (1.5) and (1.6) in CAT(0) spaces without the property $\mathbb{P}$, and presented that the iterative processes (1.5) and (1.6) converge strongly to $\tilde{x} \in Fix(g)$, where $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ is unique solution of variational inequality

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \;\; x \in Fix(g).$$

Recently, Panyanak and Suantai [3] extended (1.5) and (1.6) to T being a set-valued nonexpansive mapping from E to $BC(X)$. That is, for each $\alpha \in (0,1)$, let a set-valued contraction G_α on E define by

$$G_\alpha(x) = \alpha f(x) \oplus (1-\alpha)Tx, \quad \forall x \in E.$$

By $Nadler's$ theorem [11], one can easy to see that G_α has a (not necessarily unique) fixed point $x_\alpha \in E$ such that

$$x_\alpha \in \alpha f(x_\alpha) \oplus (1-\alpha)Tx_\alpha,$$

i.e., for each x_α, there exists $y_\alpha \in Tx_\alpha$ such that

$$x_\alpha = \alpha f(x_\alpha) \oplus (1-\alpha)y_\alpha. \tag{1.7}$$

Correspondingly, there is an explicit approximation method. More precisely, let $T : E \to C(E)$ be a nonexpansive mapping, where $C(E)$ denotes the family of nonempty compact subsets of E, $f : E \to E$ be a contraction and $\{\alpha_n\} \subseteq (0,1)$. For any given $x_1 \in E$ and $y_1 \in Tx_1$, let

$$x_2 = \alpha_1 f(x_1) \oplus (1-\alpha_1)y_1.$$

By the definition of Hausdorff distance and the nonexpansiveness of T, one can choose $y_2 \in Tx_2$ such that $d(y_1, y_2) \leq d(x_1, x_2)$. Inductively, we obtain

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)y_n, \; y_n \in T(x_n), \tag{1.8}$$

and $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then, Panyanak and Suantai [3] proved strong convergence of one-step viscosity approximation method defined by (1.7) and (1.8) for set-valued nonexpansive mapping T in CAT(0) spaces when the contraction constant coefficient of f is $k \in [0, \frac{1}{2})$ and $\{\alpha_n\} \subset (0, \frac{1}{2-k})$ satisfying some suitable conditions. Further, Chang et al.[12] affirmatively answered the open question [3, Question 3.6] proposed by Panyanak and Suantai:"If $k \in [0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfying the same conditions, does $\{x_n\}$ converge to $\tilde{x} = P_{F(T)} f(\tilde{x})$?"

Moreover, Kaewkhao et al. [13] proved strong convergence of a two-step viscosity iteration method in complete CAT(0) spaces defined as follows:

$$\begin{aligned} y_n &= \alpha_n f(x_n) \oplus (1-\alpha_n)g(x_n), \\ x_{n+1} &= \beta_n x_n \oplus (1-\beta_n)y_n, \;\; \forall\, n \geq 1, \end{aligned} \tag{1.9}$$

where $x_1 \in E$ is an arbitrary fixed element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. (1.9) is also considered and studied by Chang et al.[14] when the property $\mathbb{P}$ is not satisfied and $k \in [0,1)$, which dues to the open questions in [13].

Motivated and inspired mainly by Panyanak and Suantai [3] and Kaewkhao et al. [13], The purpose of this paper is to consider the following two-step viscosity iteration approximation for set-valued nonexpansive mapping $T : E \to C(E)$ on a nonempty closed convex subset E of a complete CAT(0) space (X, d):

$$\begin{aligned} x_{n+1} &= \beta_n x_n \oplus (1-\beta_n)y_n, \\ y_n &= \alpha_n f(x_n) \oplus (1-\alpha_n)z_n, \;\; \forall\, n \geq 1, \end{aligned} \tag{1.10}$$

where $x_1 \in E$ is an arbitrary fixed element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$, $f : E \to E$ is a contraction mapping and $z_n \in T(x_n)$ satisfying $d(z_n, z_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, which can be inducted from the definition of Hausdorff distance and the nonexpansiveness of T (see [11]). We shall prove the sequence $\{x_n\}$ proposed by (1.10) converges strongly to fixed points $\tilde{x} \in F(T)$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$ is unique solution of the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \;\; \forall x \in F(T).$$

Remark 1.1. (i) When T is a nonexpansive single-valued mapping g, then (1.10) is equivalent to (1.9).

(ii) However, (1.9) can not becomes (1.8), unless $\beta_n = 0$.

2 Preliminaries

In the sequel, (X, d) delegates a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map ξ from a closed interval $[0, l] \subseteq \mathbb{R}$ to X such that $\xi(0) = x, \xi(l) = y$, and $d(\xi(s), \xi(t)) = |s - t|$ for any $s, t \in [0, l]$. In particular, ξ is a isometry and $d(x, y) = l$. The image of ξ is said to be a geodesic segment (or metric) joining x and y if unique is denoted by $[x, y]$. The space (X, d) is called a geodesic space when every two points in X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for all $x, y \in X$. A subset E of X is said to be convex if E includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(p, q, r)$ in a geodesic space (X, d) consists of three points p, q, r in X (vertices of $\triangle$) and a choice of three geodesic segments $[p, q], [q, r], [r, p]$ (edge of $\triangle$) joining them. A comparison triangle for geodesic triangle $\triangle(p, q, r)$ in X is a triangle $\overline{\triangle}(\bar{p}, \bar{q}, \bar{r})$ in Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{p}, \bar{q}) = d(p, q),\ d_{\mathbb{R}^2}(\bar{q}, \bar{r}) = d(q, r),\ d_{\mathbb{R}^2}(\bar{r}, \bar{p}) = d(r, p).$$

A point $\bar{u} \in [\bar{p}, \bar{q}]$ is said to be a comparison point for $u \in [p, q]$ if $d(p, u) = d_{\mathbb{R}^2}(\bar{p}, \bar{u})$. Similarly, we can give the definitions to comparison points on $[\bar{q}, \bar{r}]$ and $[\bar{r}, \bar{p}]$.

Recalled that a geodesic space is called CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom: Let $\triangle$ be a geodesic triangle in (X, d) and $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy CAT(0) inequality if for any $u, v \in \triangle$ and for their comparison points $\bar{u}, \bar{v} \in \overline{\triangle}$,

$$d(u, v) \leq d_{\mathbb{R}^2}(\bar{u}, \bar{v}).$$

Complete CAT(0) spaces are often called Hadamard spaces (see [15]). For other equivalent definitions and basic properties of CAT(0) spaces, we refer to [16]. It is well known that every CAT(0) space is uniquely geodesic and any complete, simply connected Riemannina manifold having non-positive sectional curvature is a CAT(0) space. Other examples for CAT(0) spaces include Pre-Hilbert spaces [16], $\mathbb{R}-$trees [17], Euclidean buildings [18] and complex Hilbert ball with a hyperbolic metric [19] as special case .

Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d). It follows from Proposition 2.4 of [16] that for each $x \in X$, there exists a unique point $x_0 \in E$ such that

$$d(x, x_0) = \inf\{d(x, y) : y \in E\}.$$

In this case, x_0 is called *unique nearest point* of x in E.

Let (X, d) be a CAT(0) space. For each $x, y \in X$ and $t \in [0, 1]$, by Lemma 2.1 of Phompongsa and Panyanak [20], there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(y, z) = td(x, y). \tag{2.1}$$

We shall denote by $tx \oplus (1 - t)y$ unique point z satisfying (2.1). Now, we collect some elementary facts about CAT(0) spaces which will be used in proof of our main results.

Lemma 2.1. ([1, 20]) Assume that (X, d) is a CAT(0) space. Then for any $x, y, z \in X$ and $\alpha \in [0, 1]$,

$$\begin{aligned} &d(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d(x, z) + (1 - \alpha)d(y, z), \\ &d^2(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d^2(x, z) + (1 - \alpha)d^2(y, z) - \alpha(1 - \alpha)d^2(x, y), \\ &d(\alpha x \oplus (1 - \alpha)z, \alpha y \oplus (1 - \alpha)z) \leq \alpha d(x, y). \end{aligned}$$

Lemma 2.2. ([21]) Let (X, d) be a CAT(0) space. If for any $x, y \in X$ and $\alpha, \beta \in [0, 1]$, then

$$d(\alpha x \oplus (1 - \alpha)y, \beta x \oplus (1 - \beta)y) \leq |\alpha - \beta| d(x, y).$$

Lemma 2.3. ([22]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space (X, d) and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$. If $x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)y_n$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \leq 0,$$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.4. ([23, Lemma 2.1]) Let $\{u_n\}$ be a sequence of non-negative real numbers satisfying

$$u_{n+1} \leq (1-\alpha_n)u_n + \alpha_n\beta_n, \ \forall\, n \geq 1,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\beta_n\} \subset \mathbb{R}$ such that (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (ii) $\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$. Then $\{u_n\}$ converges to zero as $n \to \infty$.

Lemma 2.5. ([24, Lemma 3.1]) Let E be a closed convex subset of a complete CAT(0) space (X, d) and $T : E \to BC(X)$ be a nonexpansive mapping. If T satisfies endpoint condition $\mathbb{C}$, then $F(T)$ is closed and convex.

The concept of quasi-linearization was introduced by Berg and Nikolaev [9]. Let us denote a pair (a, b) in $X\times X$ by $\overrightarrow{ab}$ and call it a vector. The quasi-linearization is a map $\langle\cdot,\cdot\rangle$: $(X\times X)\times(X\times X) \to \mathbb{R}$ defined by

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \frac{1}{2}\left[d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right] \text{ for all } a,b,c,d \in X.$$

It is easy to see that $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle\overrightarrow{cd}, \overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = -\langle\overrightarrow{ba}, \overrightarrow{cd}\rangle$, $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle + \langle\overrightarrow{ad}, \overrightarrow{bc}\rangle = \langle\overrightarrow{ac}, \overrightarrow{bd}\rangle$ and $\langle\overrightarrow{ax}, \overrightarrow{cd}\rangle + \langle\overrightarrow{xb}, \overrightarrow{cd}\rangle = \langle\overrightarrow{ab}, \overrightarrow{cd}\rangle$ for all $a,b,c,d,x \in X$. We say that a geodesic metric space (X, d) satisfies Cauchy-Schwarz inequality if

$$\left|\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle\right| \leq d(a,b)d(c,d) \text{ for all } a,b,c,d \in X.$$

It is known from [9, Corollary 3] that a geodesic space (X, d) is a CAT(0) space if and only if X satisfies Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as follows.

Lemma 2.6. ([25, Theorem 2.4]) Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d), $u \in X$ and $x \in E$. Then

$$x = P_E u \text{ if and only if } \langle\overrightarrow{xu}, \overrightarrow{yx}\rangle \geq 0, \ \forall y \in E.$$

Lemma 2.7. ([10, Lemma 2.9]) Let (X, d) be a CAT(0) space. Then

$$d^2(x,u) \leq d^2(y,u) + 2\langle\overrightarrow{xy}, \overrightarrow{xu}\rangle, \quad \forall u,x,y \in X.$$

Lemma 2.8. ([10, Lemma 2.10]) Let u and v be two points in a CAT(0) space (X, d). For each $\alpha \in [0,1]$, setting $u_\alpha = \alpha u \oplus (1-\alpha)v$, then, for each $x, y \in X$, we have

(i) $\langle\overrightarrow{u_\alpha x}, \overrightarrow{u_\alpha y}\rangle \leq \alpha\langle\overrightarrow{ux}, \overrightarrow{u_\alpha y}\rangle + (1-\alpha)\langle\overrightarrow{vx}, \overrightarrow{u_\alpha y}\rangle$;

(ii) $\langle\overrightarrow{u_\alpha x}, \overrightarrow{uy}\rangle \leq \alpha\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-\alpha)\langle\overrightarrow{vx}, \overrightarrow{uy}\rangle$ and $\langle\overrightarrow{u_\alpha x}, \overrightarrow{vy}\rangle \leq \alpha\langle\overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-\alpha)\langle\overrightarrow{vx}, \overrightarrow{vy}\rangle$.

Lemma 2.9. ([13, Lemma 2.10]) Let (X, d) be a CAT(0) space. If for any $x, y, z \in X$ and $\alpha \in [0,1]$, then

$$d^2(\alpha x \oplus (1-\alpha)y, z) \leq \alpha^2 d^2(x,z) + (1-\alpha)^2 d^2(y,z) + 2\alpha(1-\alpha)\langle\overrightarrow{xz}, \overrightarrow{yz}\rangle.$$

Recalled that a continuous linear functional μ is said to be Banach limit on ℓ_∞, if $\|\mu\| = \mu(1,1,\cdots) = 1$ and $\mu_n(u_n) = \mu_n(u_{n+1})$ for all $\{u_n\} \in \ell_\infty$.

Lemma 2.10. ([26, Proposition 2]) Let α be a real number and let $(u_1, u_2, \cdots) \in \ell_\infty$ satisfy $\mu_n(u_n) \leq \alpha$ for all Banach limits μ and $\limsup_n(u_{n+1} - u_n) \leq 0$. Then $\limsup_n u_n \leq \alpha$.

3 Main theorem

In this section, we will prove strong convergence theorem of a class of new two-step viscosity iterations for approximating fixed points of set-valued nonexpansive mappings with some general conditions in a complete CAT(0) space.

Lemma 3.1. ([3, Theorem 3.1]) Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d), $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$, and $f : E \to E$ be a contraction with $k \in [0, 1)$. Then the following statements hold:

(i) $\{x_\alpha\}$ defined by (1.7) converges strongly to $\tilde{x}$ as $\alpha \to 0$, where $\tilde{x} = P_{F(T)}f(\tilde{x})$.

(ii) If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, Then for any Banach limits μ_n,

$$d^2(f(\tilde{x}), \tilde{x}) \leq \mu_n d^2(f(\tilde{x}), x_n).$$

Now, we are ready to prove our main theorem.

Theorem 3.1. Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d), $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$. Let $f : E \to E$ be a contraction with $k \in [0, 1)$, and $\{\alpha_n\}$ be a sequence in $(0, 1-k)$, and $\{\beta_n\}$ be a sequences in $(0, 1)$ satisfying the following conditions:

(C_1) $\lim_{n\to\infty} \alpha_n = 0$;

(C_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (1.10) converges strongly to $\tilde{x}$, which satisfies

$$\tilde{x} = P_{F(T)}f(\tilde{x}), \quad \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \forall x \in F(T).$$

Proof. We divide proof into three steps.

Step 1. We show that $\{x_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{f(x_n)\}$ are bounded sequences. Let $p \in F(T)$. By Lemma 2.1, we have

$$\begin{aligned} d(y_n, p) \quad & \leq \alpha_n d(f(x_n), p) + (1-\alpha_n) dist(z_n, T(p)) \\ & \leq \alpha_n d(f(x_n), p) + (1-\alpha_n) H(T(x_n), T(p)) \\ & \leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(x_n, p) \\ & \leq \alpha_n d(f(x_n), f(p)) + \alpha_n d(f(p), p) + (1-\alpha_n) d(x_n, p) \\ & \leq [1-(1-k)\alpha_n] d(x_n, p) + \alpha_n d(f(p), p), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) \quad & \leq \beta_n d(x_n, p) + (1-\beta_n) d(y_n, p) \\ & \leq [1-(1-k)(1-\beta_n)\alpha_n] d(x_n, p) + (1-k)(1-\beta_n)\alpha_n \frac{d(f(p), p)}{1-k} \\ & \leq \max\left\{ d(x_n, p), \frac{d(f(p), p)}{1-k} \right\}. \end{aligned}$$

By induction, we also have

$$d(x_n, p) \leq \max\left\{ d(x_1, p), \frac{d(f(p), p)}{1-k} \right\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{z_n\}$, $\{y_n\}$ and $\{f(x_n)\}$.

Step 2. $\lim_{n\to\infty} dist(x_n, T(x_n)) = \lim_{n\to\infty} d(z_n, x_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. In fact, by applying Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} d(y_n, y_{n+1}) \quad & \leq \quad d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_n, \alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1}) z_{n+1}) \\ & \leq \quad d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_n, \alpha_n f(x_n) \oplus (1-\alpha_n) z_{n+1}) \\ & \quad\quad + d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_{n+1}, \alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}) \\ & \quad\quad + d(\alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}, \alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1}) z_{n+1}) \\ & \leq \quad \alpha_n d(f(x_n), f(x_{n+1})) + (1-\alpha_n) d(z_n, z_{n+1}) \\ & \quad\quad + |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \\ & \leq \quad \alpha_n k d(x_n, x_{n+1}) + (1-\alpha_n) d(x_n, x_{n+1}) \\ & \quad\quad + |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \\ & \leq \quad (1-\alpha_n(1-k)) d(x_n, x_{n+1}) + |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}), \end{aligned}$$

which implies

$$d(y_n, y_{n+1}) - d(x_n, x_{n+1}) \leq |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) - (1-k)\alpha_n d(x_n, x_{n+1}).$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\limsup_{n\to\infty} [d(y_{n+1}, y_n) - d(x_{n+1}, x_n)] \leq 0$. By Lemma 2.3, we know that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Thus,

$$dist(x_n, T(x_n)) \leq d(x_n, z_n) \leq d(x_n, y_n) + \alpha_n d(f(x_n), z_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1}$$

By (3.1), now we know that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{3.2}$$

Moreover,

$$d(x_n, x_{n+1}) = (1-\beta_n) d(x_n, y_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Step 3. $\{x_n\}$ converges strongly to $\tilde{x}$ which satisfies $\tilde{x} = P_{F(T)} f(\tilde{x})$ and

$$\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \;\; x \in F(T).$$

Above all, since $T(x)$ is compact for any $x \in E$, $T(x) \in BC(X)$. It follows from Lemma 2.5 that $F(T)$ is closed and convex. This implies that $P_{F(T)}u$ is well defined for any $u \in X$. By Lemma 3.1 (i), we know that $\{x_\alpha\}$ defined by (1.7) converges strongly to $\tilde{x}$ as $\alpha \to 0$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$. Thus applying Lemma 2.6, one can see that $\tilde{x}$ is unique solution of the following variational inequatity

$$\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \;\; x \in F(T).$$

Next, by using Lemma 3.1 (ii), we have

$$d^2(f(\tilde{x}), \tilde{x}) \leq \mu_n d^2(f(\tilde{x}), x_n) \quad \text{for each Banach limit } \mu_n,$$

and so

$$\mu_n(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n)) \leq 0.$$

Moreover, since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$,

$$\limsup_{n\to\infty} [(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_{n+1})) - (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n))] = 0.$$

It follows from Lemma 2.10 that

$$\limsup_{n\to\infty} (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n)) \leq 0. \tag{3.3}$$

Finally, we show $x_n \to \tilde{x}$ as $n \to \infty$. It follows from Lemma 2.1 and Lemmas 2.7-2.9 that

$$\begin{aligned}
d^2(x_{n+1}, \tilde{x}) \quad & \leq \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}) \\
& \leq \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) \left[\alpha_n^2 d^2(f(x_n), \tilde{x}) + (1-\alpha_n)^2 d^2(z_n, \tilde{x})\right] \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{z_n\tilde{x}} \rangle \\
& \leq \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n)(1-\alpha_n)^2 dist^2(z_n, T(\tilde{x})) \\
& \quad +\alpha_n^2(1-\beta_n) \left[d^2(x_{n+1}, f(x_n)) + 2\langle \overrightarrow{\tilde{x} x_{n+1}}, \overrightarrow{\tilde{x} f(x_n)} \rangle\right] \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \left[\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{z_n x_n} \rangle + \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle\right] \\
& \leq \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n)(1-\alpha_n)^2 H^2(T(x_n), T(\tilde{x})) \\
& \quad +\alpha_n^2(1-\beta_n) d^2(x_{n+1}, f(x_n)) + 2\alpha_n^2(1-\beta_n) \langle \overrightarrow{\tilde{x} x_{n+1}}, \overrightarrow{\tilde{x} f(x_n)} \rangle \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{z_n x_n} \rangle \\
& \quad +2\alpha_n(1-\alpha_n)(1-\beta_n) \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n d^2(x_n,\tilde{x}) + (1-\beta_n)(1-\alpha_n)^2 d(x_n,\tilde{x}) \\
&\quad +\alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n)) \\
&\quad +2\alpha_n^2(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\right] \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{z_n x_n}\rangle \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_n\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\right] \\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n)) \\
&\quad +2\alpha_n^2(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\right] \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{z_n x_n}\rangle \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\left[\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_n\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\right] \\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n)) \\
&\quad +2\alpha_n^2(1-\beta_n)d(f(x_n),f(\tilde{x}))d(x_{n+1},\tilde{x}) \\
&\quad +2\alpha_n^2(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),\tilde{x})d(z_n,x_n) \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),f(\tilde{x}))d(x_n,\tilde{x}) \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle \\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n)) \\
&\quad +2k\alpha_n^2(1-\beta_n)d(x_n,\tilde{x})d(x_{n+1},\tilde{x}) \\
&\quad +\alpha_n^2(1-\beta_n)\left[d^2(x_{n+1},\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_{n+1})\right] \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),\tilde{x})d(z_n,x_n) \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n)d^2(x_n,\tilde{x}) \\
&\quad +\alpha_n(1-\alpha_n)(1-\beta_n)\left[d^2(x_n,\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_n)\right] \\
&\leq \left[\beta_n + (1-\beta_n)(1-\alpha_n)^2\right] d^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)d^2(x_{n+1},f(x_n)) \\
&\quad +k\alpha_n^2(1-\beta_n)\left[d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\right] \\
&\quad +\alpha_n^2(1-\beta_n)\left[d^2(x_{n+1},\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_{n+1})\right] \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)d(f(x_n),\tilde{x})d(z_n,x_n) \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n)d^2(x_n,\tilde{x}) \\
&\quad +\alpha_n(1-\alpha_n)(1-\beta_n)\left[d^2(x_n,\tilde{x}) + d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_n)\right].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d^2(x_{n+1},\tilde{x}) \quad &\leq \left[\frac{\beta_n + (1-\beta_n)(1-\alpha_n) + k\alpha_n(1-\beta_n)(2-\alpha_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}\right] d^2(x_n,\tilde{x}) \\
&\quad + \frac{\alpha_n^2(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)} d^2(x_{n+1},f(x_n)) \\
&\quad + \frac{2\alpha_n(1-\alpha_n)(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)} d(f(x_n),\tilde{x})d(z_n,x_n) \\
&\quad + \frac{\alpha_n^2(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)} (d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_{n+1})) \\
&\quad + \frac{\alpha_n(1-\alpha_n)(1-\beta_n)}{1-(1+k)\alpha_n^2(1-\beta_n)} (d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_n)).
\end{aligned}
$$

Thus,

$$
d^2(x_{n+1},\tilde{x}) \leq (1-\alpha_n')d^2(x_n,\tilde{x}) + \alpha_n'\beta_n', \tag{3.4}
$$

8

where $\alpha_n' = \frac{2\alpha_n(1-\beta_n)(1-k-\alpha_n)}{1-(1+k)\alpha_n^2(1-\beta_n)}$ and

$$\begin{aligned}\beta_n' &= \frac{\alpha_n}{2(1-k-\alpha_n)} d^2(x_{n+1}, f(x_n)) + \frac{1-\alpha_n}{1-k-\alpha_n} d(f(x_n), \tilde{x}) d(z_n, x_n) \\ &+ \frac{\alpha_n}{2(1-k-\alpha_n)} (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_{n+1})) \\ &+ \frac{1-\alpha_n}{2(1-k-\alpha_n)} (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n)).\end{aligned}$$

Since $k \in [0,1)$ and $\alpha_n \in (0, 1-k)$, then $\alpha_n' \in (0,1)$. Applying Lemma 2.4 to the inequality (3.4) (also combining (3.2) and (3.3)), we have $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

From Theorem 3.1, we have the following result.

Theorem 3.2. Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d), $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$. Suppose that $u, x_1 \in E$ are arbitrarily given elements and $\{x_n\}$ is defined by

$$y_n = \alpha_n u \oplus (1-\alpha_n) z_n, x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n, \ \ \forall\, n \geq 1,$$

where $z_n \in T(x_n)$ such that $d(z_n, z_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$ satisfying (C_1), (C_2) and (C_3) in Theorem 3.1. Then the sequence $\{x_n\}$ converges strongly to unique nearest point $\tilde{x}$ of u in $F(T)$; i.e., $\tilde{x} = P_{F(T)} u$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}u}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in F(T).$$

Proof. We define $f : E \to E$ by $f(x) = u$ for all $x \in E$, then f is a contrction with $k = 0$. The conclusion follows immediately from Theorem 3.1. □

If $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$, then, replacing by $g : E \to E$ be a nonexpansive singie-valued mapping with $Fix(g) \neq \emptyset$, and we have the following two corollaries.

Corollary 3.1. Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d), $g : E \to E$ be a nonexpansive mapping with $Fix(g) \neq \emptyset$. Let $f : E \to E$ be a contraction with $k \in [0,1)$, and $\{\alpha_n\}$ be a sequence in $(0, 1-k)$, and $\{\beta_n\}$ be a sequences in $(0,1)$ satisfying (C_1), (C_2) and (C_3) in Theorem 3.1. Then sequence $\{x_n\}$ defined by (1.9) converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in Fix(g).$$

Corollary 3.2. ([3, Theorem 3.3]) Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d), $T : E \to C(E)$ be a nonexpansive mapping satisfying endpoint condition $\mathbb{C}$. Let $f : E \to E$ be a contraction with $k \in \left[0, \frac{1}{2}\right)$, and $\{\alpha_n\}$ be a sequence in $\left(0, \frac{1}{2-k}\right)$ satisfying (C_1) and (C_2) in Theorem 3.1 and the following condition:

(C_4) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$. Then sequence $\{x_n\}$ defined by (1.8) converges strongly to $\tilde{x}$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in F(T).$$

By corollary 3.1, the following result can be obtained.

Corollary 3.3. Let E be a nonempty closed convex subset of a complete $\mathbb{R}-$tree (X, d), and $T : E \to BCC(E)$ be a nonexpansive mapping with $F(T) \neq \emptyset$, where $BCC(E)$ is the family of nonempty bounded closed convex subsets of E. Let $f : E \to E$ be a contraction with $k \in [0,1)$, and $\{\alpha_n\}$ be a sequence in $(0, 1-k)$, and $\{\beta_n\}$ be a sequences in $(0,1)$ satisfying (C_1), (C_2) and (C_3) in Theorem 3.1.Then sequence $\{x_n\}$ defined by (1.10) converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)} f(\tilde{x})$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \ x \in F(T).$$

Proof. By Theorem 4.1 given by Aksoy and Khamsi [27], there exists a single-valued nonexpansive mapping $h : E \to E$ such that $h(x) \in T(x)$ and $d(h(x), h(y)) \leq H(T(x), T(y))$ for all $x, y \in E$. Hence, $z_n = h(x_n) \in T(x)$ for (1.10). Again, it follows from [27, Theorem 4.2] (also Theorem 4.2 in [3]) that $Fix(h) = F(T) \neq \emptyset$. The conclusion follows from Corollary 3.1. □

Remark 3.1. The results presented in this paper improve and unify corresponding results in Panyanak and Suantai [3], Kaewkhao et al. [13] and many others. In this regard, we show as follows:

(i) Corollary 3.1 extends Theorem 3.2 of [13] from $k \in [0, \frac{1}{2})$ to $k \in [0, 1)$.

(ii) When T in Theorem 3.1 is a single-value mapping, then our main results in Theorem 3.1 become to corresponding results of Theorem 3.3 in [3] for a contraction f from $k \in \left[0, \frac{1}{2}\right)$ to $k \in [0, 1)$, and $\alpha_n \in \left(0, \frac{1}{2-k}\right)$ to $\alpha_n \in (0, 1-k)$. Further, the condition (C_4) is not needed.

(iii) If we add condition (C_4), and change $\alpha_n \in \left(0, \frac{1}{2-k}\right)$ as $\alpha_n \in (0, 1-k)$, then Theorem 4.2 of [3] happens to be Corollary 3.3.

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Generalized Partial ToDD's Difference Equation in n-dimensional space

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Abstract

In this paper we introduce a generalized form of the well known ToDD's difference equation and give the closed form expressions for this generalized form . In other words , we have the following nonlinear rational partial difference equation

$$T\langle X_1, X_2, X_3, ..., X_n\rangle$$
$$= \frac{1 + T\langle X_1 - 1, X_2 - 1, ..., X_n - 1\rangle + T\langle X_1 - 2, X_2 - 2, ..., X_n - 2\rangle}{T\langle X_1 - 3, X_2 - 3, X_3 - 3, ..., X_n - 3\rangle}$$

where $X_1, X_2, ..., X_n \in \mathbb{N}$,and the initial values $T\langle p_1, p_2, ..., p_n\rangle$,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle$,$T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle$,... ...,$T\langle p_2, p_3, p_4, ...p_1, p_n\rangle$,$T\langle p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\rangle$ are real numbers with $p_1 \in \{0, -1, -2\}$ and $p_2, p_3, ..., p_n \in \mathbb{N}$ such that $T\langle p_1, p_2, ..., p_n\rangle \neq 0$,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle \neq 0$, $T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle \neq 0$,...,$T\langle p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\rangle \neq 0$.

We will use a novel technique to prove the results by using what we call 'piecewise n-dimensional mathematical induction' which we introduce here for the first time . We will obvious that this new concept represents generalized form for many types of mathematical induction . As a direct consequences , we investigate and drive the explicit solutions for the well known ordinary ToDD's difference Equation .

AMS Subject Classification: 39A10, 39A14.
Key Words and Phrases: (partial)difference equations, solutions , piecewise n-dimesional mathematical induction.

1 Introduction

We know that the studying of ordinary difference equations has been widely treated in the past . However , partial difference equations (PΔEs) have not received the same full attentiveness . Both of ordinary and partial difference equations may be found in the study of dynamics ,probability and other branches of mathematical physics .Moreover,partial difference equations arise in applications involving finite difference schemes ,population dynamics with spatial migrations and chemical reactions . Indeed Lagrange and Laplace took into consideration the solution of partial difference equations in their treatises of dynamics and probability.

An example can get if we suppose initially, the probability of finding a particle at one of the integral coordinates j of the x-axis is $P(j, 0)$. At the end of each time interval, the particle makes a decision to stay at its present position or move one unit in the positive direction along the x-axis. Assume that the probability that the particle does not move in a given unit of time is p, and the probability that the particle moves in a given unit of time is q. Let is $P(j, t)$be the probability that the particle is at the point is $x = j$ at the end of the t-th interval of time. Then by Bayes' formula, it is easy to see that the following partial difference equation holds:

$$P(j, t) = pP(j, t - 1) + P(j - 1, t - 1)$$

An another example of a partial difference equation is the following well known relation

$$B_m^{(n)} = B_{m-1}^{(n-1)} + B_m^{(n-1)} \quad , 1 \leq m < n.$$

The solution of this equation is the celebrated binomial coefficient function $B_m^{(n)}$ defined by

$$B_m^{(n)} = \frac{n!}{m!(n - m)!} \quad , 0 \leq m < n.$$

Some authors investigate the closed form solutions for certain partial difference equations .
For instance , Heins [[2]] considered the solution of the partial difference equation

$$y(p + 1, q) + y(p - 1, q) = 2y(p, q + 1)$$

under some conditions

Ibrahim in [[10]] studied the closed form solution for higher order nonlinear rational partial difference equation in the form

$$S\{n, m\} = \frac{S\{n - r, m - r\}}{\Psi + \prod_{i=1}^{r} S\{n - i, m - i\}}$$

where $n, m \in \mathbb{N}$ and the initial values $S\{n,t\}$,$S\{t,m-r\}$ are real numbers with $t \in \{0,-1,-2,......,-r+1\}$ such that $\prod_{j=0}^{r-1} S\{j-r+1, i+j-r+1\} \neq -\Psi$ and $\prod_{j=0}^{r-1} S\{i+j-r+2, j-r+1\} \neq -\Psi$, $i \in \mathbb{N}_0$.

For more results about partial difference equations we refer to ([1], [3],[4], [5]-[9],[11]-[15]).

In this paper we introduce a generalized form of the well known ToDD's difference equation and give the closed form expressions for this generalized form. In other words , we have the following nonlinear rational partial difference equation

$$T\langle X_1, X_2, X_3, ..., X_n\rangle = \frac{1+T\langle X_1-1, X_2-1, ..., X_n-1\rangle + T\langle X_1-2, X_2-2, ..., X_n-2\rangle}{T\langle X_1-3, X_2-3, X_3-3, ..., X_n-3\rangle} \quad (1)$$

where $X_1, X_2, ..., X_n \in \mathbb{N}$,and the initial values $T\langle p_1, p_2, ..., p_n\rangle$, $T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle$,$T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle$,... ...,$T\langle p_2, p_3, p_4, ...p_1, p_n\rangle$,$T\langle p_2-3, p_3-3, p_4-3, ...p_n-3, p_1\rangle$ are real numbers with $p_1 \in \{0,-1,-2\}$ and $p_2, p_3, ..., p_n \in \mathbb{N}$ such that $T\langle p_1, p_2, ..., p_n\rangle \neq 0$,$T\langle p_2, p_1, p_3, p_4, ..., p_n\rangle \neq 0$, $T\langle p_2, p_3, p_1, p_4, ..., p_n\rangle \neq 0$,...,$T\langle p_2-3, p_3-3, p_4-3, ...p_n-3, p_1\rangle \neq 0$.

We,ll use a novel technique to prove the results by using what we call 'piecewise n-dimesional mathematical induction' which we introduce here for the first time . We'll obvious that this new concept represents generalized form for many types of mathematical induction . As a direct consequences , we investigate and drive the explicit solutions for the well known ToDD's ordinary Difference Equation .

Now let us firstly introduce some important concepts .
Ibrahim [10] constructed a new concept who call it "'piecewise double mathematical induction' which represented a generalization for some kinds of inductions. The definition was formulated as the following form:

Definition 1. (Piecewise Double Mathematical Induction of r-pieces) Let $S(m,n)$ be a statement involving two positive integer variables m and n. Beside , we suppose that the statement $S(m,n)$ is piecewise with r-pieces . Then the statement $S(m,n)$ holds if

1. $S(k_1+\alpha, k_2+\beta)$
2. If $S(m, k_2+\beta)$, then $S(m+r, k_2+\beta)$
3. If $S(m,n)$, then $S(m, n+r)$
   where $\alpha, \beta \in \{0,1,2,.......r-1\}$ and k_1 and k_2 are the smallest values of m and n .

We briefly call this concept “r-double mathematical induction” .We can call this concept “’piecewise two-dimesional mathematical induction’

Here we will construct an another notion which we call it ‘piecewise triple mathematical induction’ or ‘piecewise three-dimensional mathematical induction’ which offer an another generalization for some kinds of inductions .

Definition 2. (Piecewise Triple Mathematical Induction of r-pieces)
Let $H(n, m, l)$ be a statement involving three positive integer variables n, m and l. Beside , we suppose that the statement $H(n, m, l)$ is piecewise with r-pieces . Then the statement $H(n, m, l)$ holds if

1. $H(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3)$
2. If $H(\alpha_1 + \beta_1, m, l)$, then $H(\alpha_1 + \beta_1, m + r, l)$
   If $H(n, m, \alpha_3 + \beta_3)$, then $H(n + r, m, \alpha_3 + \beta_3)$
3. If $H(n, m, l)$, then $H(n, m, l + r)$
   where $\beta_1, \beta_2, \beta_3 \in \{0, 1, 2, ...r - 1\}$ and α_1,α_2 and α_3 are the smallest values of n , m and l respectively .

We briefly call this concept “r-triple mathematical induction”

Remark 1. *We can see that the previous concept contains many types of mathematical induction. For instances ,*

1. *If $r = 1$, we have $\beta_1 = \beta_2 = \beta_3 = 0$, thus we have a triple mathematical induction .*
2. *If $r = 2$, we have $\beta_1, \beta_2, \beta_3 \in \{0, 1\}$, thus we have the odd-even triple mathematical induction .*
3. *If we put $n = m = l$ we have a special case of the above definition which introduce an another new concept. This type of mathematical induction called “Piecewise single Mathematical Induction of r-pieces” . In this case , if we put $r = 1$ with $n = m = l$ we easily get the basic mathematical induction . Also if we put $r = 2$ with $n = m = l$,we get easily the odd-even mathematical induction .*

Finally we can introduce a generalized concept ‘piecewise n-dimesional mathematical induction’ as a generalization for the above definitions .

Definition 3. (Piecewise n-dimesional Mathematical Induction of r-pieces)
Let $H(N_1, N_2, ..., N_n)$ be a statement involving positive integer variables $N_1, N_2, ..., N_n$. Beside , we suppose that the statement $H(N_1, N_2, ..., N_n)$ is piecewise with r-pieces . Then the statement $H(N_1, N_2, ..., N_n)$ holds if

1. $H(\alpha_1+\beta_1,\alpha_2+\beta_2,......,\alpha_n+\beta_n)$

2. If $H(\alpha_1+\beta_1,N_2,N_3,...,N_n)$, then $H(\alpha_1+\beta_1,N_2+r,N_3,...,N_n)$
   If $H(N_1,\alpha_2+\beta_2,N_3,...,N_n)$, then $H(N_1,\alpha_2+\beta_2,,N_3+r,...,N_n)$
   .
   .
   .
   If $H(N_1,N_2,...,\alpha_{n-1}+\beta_{n-1},N_n)$, then $H(N_1,N_2,...,\alpha_{n-1}+\beta_{n-1},N_n+r)$

3. If $H(N_1,N_2,...,N_n)$, then $H(N_1+r,N_2,...,N_n)$
   where $\beta_i\in\{0,1,2,.......r-1\}$,$i\in\{1,2,.......n\}$ and α_i are the smallest values of $N_1,N_2,...,N_n$ respectively .

We briefly call this concept "(r,n)-dimensional mathematical induction"

Remark 2. *We can easy see that both of r-double mathematical induction and r-triple mathematical induction are special cases of "(r,n)-dimensional mathematical induction" .*

2 Forms of Solutions

In this section we shall give explicit forms of solutions of the partial difference equation (1) of order three .

2.1 Form of Solutions for PΔE (1) when $n=2$

In this subsection we introduce a generalized form of ToDD's difference equation with two discrete variables $X_1$and X_2 and give the closed form expressions for this generalized form . In other words , we have the following nonlinear rational partial difference equation

$$T\langle X_1,X_2\rangle=\frac{1+T\langle X_1-1,X_2-1\rangle+T\langle X_1-2,X_2-2\rangle}{T\langle X_1-3,X_2-3\rangle} \tag{2}$$

Here we give the closed form solution of the partial difference equation (2).

Theorem 4. *Let $\{T\langle X_1,X_2\rangle\}_{X_1,X_2=-k}^{\infty}$ be a solution of the partial difference equation (2) , where $X_1,X_2\in\mathbb{N}$,and the initial values $T\langle p,q\rangle$ and $T\langle q,p-3\rangle$ are real numbers with $q\in\{0,-1,-2\}$ and $p\in\mathbb{N}$ such that $T\langle p,q\rangle\neq 0$ and $T\langle q,p-3\rangle\neq 0$. Then, the form of solutions of (2) ,for $X_1\leq X_2$ are as follows:*

$$T\left\langle X_1, X_2\right\rangle = \begin{cases} \frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle}{T\langle -2,X_2-(X_1+2)\rangle}, X_1 = L_1; \\ \frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle+T\langle -2,X_2-(X_1+2)\rangle(1+T\langle 0,X_2-X_1\rangle)}{T\langle -1,X_2-(X_1+1)\rangle T\langle -2,X_2-(X_1+2)\rangle}, X_1 = L_2; \\ \frac{(1+T\langle -1,X_2-(X_1+1)\rangle+T\langle -2,X_2-(X_1+2)\rangle)(1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle)}{T\langle 0,X_2-X_1\rangle)T\langle -1,X_2-(X_1+1)\rangle T\langle -2,X_2-(X_1+2)\rangle}, X_1 = L_3; \\ \frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle 0,X_2-X_1\rangle+T\langle -2,X_2-(X_1+2)\rangle(1+T\langle 0,X_2-X_1\rangle)}{T\langle -1,X_2-(X_1+1)\rangle T\langle 0,X_2-X_1\rangle}, X_1 = L_4; \\ \frac{1+T\langle -1,X_2-(X_1+1)\rangle+T\langle -2,X_2-(X_1+2)\rangle}{T\langle 0,X_2-X_1\rangle}, X_1 = L_5; \\ T\left\langle -2, X_2-(X_1+2)\right\rangle, X_1 = L_6; \\ T\left\langle -1, X_2-(X_1+1)\right\rangle, X_1 = L_7; \\ T\left\langle 0, X_2-X_1\right\rangle, X_1 = L_8; \end{cases} \tag{3}$$

$$T\left\langle X_2, X_1\right\rangle = \begin{cases} \frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle}{T\langle X_2-(X_1+2),-2\rangle}, X_1 = L_1; \\ \frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle+T\langle X_2-(X_1+2),-2\rangle(1+T\langle X_2-X_1,0\rangle)}{T\langle X_2-(X_1+1),-1\rangle T\langle X_2-(X_1+2),-2\rangle}, X_1 = L_2; \\ \frac{(1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-(X_1+2),-2\rangle)(1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle)}{T\langle X_2-X_1,0\rangle)T\langle X_2-(X_1+1),-1\rangle T\langle X_2-(X_1+2),-2\rangle}, X_1 = L_3; \\ \frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-X_1,0\rangle+T\langle X_2-(X_1+2),-2\rangle(1+T\langle X_2-X_1,0\rangle)}{T\langle X_2-(X_1+1),-1\rangle T\langle X_2-X_1,0\rangle}, X_1 = L_4; \\ \frac{1+T\langle X_2-(X_1+1),-1\rangle+T\langle X_2-(X_1+2),-2\rangle}{T\langle X_2-X_1,0\rangle}, X_1 = L_5; \\ T\left\langle X_2-(X_1+2), -2\right\rangle, X_1 = L_6; \\ T\left\langle X_2-(X_1+1), -1\right\rangle, X_1 = L_7; \\ T\left\langle X_2-X_1, 0\right\rangle, X_1 = L_8; \end{cases} \tag{4}$$

where $L_i = 8k + i$, $1 \leq i \leq 8$, $i \in \mathbb{N}$.

Proof. We shall use the principle of piecewise double mathematical induction defined in definition (1) . Firstly , we shall prove that the relations (3) and (4) hold for $T\langle p,q\rangle$. where $p,q\in\{1,2,,...8\}$. From equation (2)we can see

$$T\langle 1,1\rangle=\frac{1+T\langle 0,0\rangle+T\langle -1,-1\rangle}{T\langle -2,-2\rangle}=\frac{1+T\langle 0,1-1\rangle+T\langle -1,1-(1+1)\rangle}{T\langle -2,1-(1+2)\rangle}$$

$$T\langle 2,2\rangle=\frac{1+T\langle 1,1\rangle+T\langle 0,0\rangle}{T\langle -1,-1\rangle}$$

$$=\frac{1+T\langle 0,0\rangle+T\langle -1,-1\rangle+T\langle -2,-2\rangle(1+T\langle 0,0\rangle)}{T\langle -2,-2\rangle T\langle -1,-1\rangle}$$

$$=\frac{1+T\langle 0,2-2\rangle+T\langle -1,2-(2+1)\rangle+T\langle -2,2-(2+2)\rangle(1+T\langle 0,2-2\rangle)}{T\langle -2,2-(2+2)\rangle T\langle -1,2-(2+1)\rangle}$$

$$T\langle 1,2\rangle=\frac{1+T\langle 0,1\rangle+T\langle -1,0\rangle}{T\langle -2,-1\rangle}=\frac{1+T\langle 0,2-1\rangle+T\langle -1,2-(1+1)\rangle}{T\langle -2,2-(1+2)\rangle}$$

$$T\langle 2,3\rangle=\frac{1+T\langle 1,2\rangle+T\langle 0,1\rangle}{T\langle -1,0\rangle}$$

$$=\frac{1+T\langle 0,1\rangle+T\langle -1,0\rangle+T\langle -2,-1\rangle(1+T\langle 0,1\rangle)}{T\langle -2,-1\rangle T\langle -1,0\rangle}$$

$$=\frac{1+T\langle 0,3-2\rangle+T\langle -1,3-(2+1)\rangle+T\langle -2,3-(2+2)\rangle(1+T\langle 0,3-2\rangle)}{T\langle -2,3-(2+2)\rangle T\langle -1,3-(2+1)\rangle}$$

Similarly we can prove the remaining values for p and q .
Now suppose that the relations (3) and (4) hold for $X_1=1,2,..,8$ with $X_2\in\mathbb{N}$.
We try to prove that relations (3) and (4) hold for $X_1=1,2,...,8$ with X_2+8.

$$T\langle X_2+8,1\rangle=\frac{1+T\langle X_2+8-1,1-1\rangle+T\langle X_2+8-2,1-2\rangle}{T\langle X_2+8-3,1-3\rangle}$$

$$=\frac{1+T\langle X_2+8-(1),0\rangle+T\langle X_2+8-(1+1),-1\rangle}{T\langle X_2+8-(1+2),-2\rangle}$$

$$T\langle X_2+8,2\rangle=\frac{1+T\langle X_2+8-1,2-1\rangle+T\langle X_2+8-2,2-2\rangle}{T\langle X_2+8-3,2-3\rangle}$$

$$=\frac{1+T\langle X_2+7,1\rangle+T\langle X_2+6,0\rangle}{T\langle X_2+5,-1\rangle}$$

$$=\frac{1+(\frac{1+T\langle X_2+5,-1\rangle+T\langle X_2+6,0\rangle}{T\langle X_2+4,-2\rangle})+T\langle X_2+6,0\rangle}{T\langle X_2+5,-1\rangle}$$

$$\frac{1+T\langle X_2+8-(2+1),-1\rangle+T\langle X_2+8-2,0\rangle}{T\langle X_2+8-(2+1),-1\rangle T\langle X_2+8-(2+2),-2\rangle}$$

$$+\frac{T\langle X_2+8-(2+2),-2\rangle(1+T\langle X_2+8-2,0\rangle)}{T\langle X_2+8-(2+1),-1\rangle T\langle X_2+8-(2+2),-2\rangle}$$

Similarly we can prove the other cases for for $X_1 = 3, ..., 8$ with $X_2 + 8$. .

Finally , we suppose that relations (3) and (4) hold for $X_2, X_1 \in \mathbb{N}$.

We shall prove that relations (3) and (4) hold for $X_2, X_1 + 8 \in \mathbb{N}$.

From equation (2)we have

$$T\langle X_2, X_1+8\rangle = \frac{1+T\langle X_2-1, X_1+8-1\rangle + T\langle X_2-2, X_1+8-2\rangle}{T\langle X_2-3, X_1+8-3\rangle}$$

$$= \frac{1+T\langle X_2-1, X_1+7\rangle + T\langle X_2-2, X_1+6\rangle}{T\langle X_2-3, X_1+5\rangle}$$

There are sixteen cases :

(1) When $X_2 > 8(k+1)+i$, $i = 1, 2, ..., 8$.
We take the cases when $i = 3$ and $i = 7$ .The other cases for $i = 1, 2, 4, 5, 6, 8$ can be given by the same way .
In order to simplify the calculations we consider the following notations:
$T\langle X_2 - X_1 - 8, 0\rangle = T\langle 0\rangle$, $T\langle X_2 - X_1 - 9, -1\rangle = T\langle -1\rangle$,
$T\langle X_2 - X_1 - 10, -2\rangle = T\langle -2\rangle$,
Now if $X_2 > 8(k+1)+3$:

$$T\langle X_2, X_1+8\rangle = \frac{1+T\langle X_2-1, X_1+7\rangle + T\langle X_2-2, X_1+6\rangle}{T\langle X_2-3, X_1+5\rangle}$$

$$= \frac{1+\frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle -2\rangle}+\frac{1+T\langle -1\rangle+T\langle 0\rangle}{T\langle -2\rangle}}{T\langle 0\rangle}$$

$$= \frac{(1+T\langle -1\rangle)^2+T\langle 0\rangle(1+T\langle -1\rangle)+T\langle -2\rangle(1+T\langle 0\rangle+T\langle -1\rangle)}{T\langle 0\rangle T\langle -1\rangle T\langle -2\rangle}$$

$$= \frac{(1+T\langle -1\rangle+T\langle -2\rangle)(1+T\langle -1\rangle+T\langle 0\rangle)}{T\langle 0\rangle)T\langle -1\rangle T\langle -2\rangle}$$

If $X_2 > 8(k+1)+7$ we have :

$$T\langle X_2, X_1+8\rangle = \frac{1+T\langle X_2-1, X_1+7\rangle + T\langle X_2-2, X_1+6\rangle}{T\langle X_2-3, X_1+5\rangle}$$

$$= \frac{1+T\langle -2\rangle+\frac{1+T\langle -1\rangle+T\langle -2\rangle}{T\langle 0\rangle}}{\frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle 0\rangle}} = T\langle -1\rangle$$

(2) When $X_2 < 8(k+1)+i$, $i = 1, 2, ..., 8$. We take the cases when $i = 4$ and $i = 6$ .The other cases for $i = 1, 2, 3, 5, 7, 8$ can be given by the same way.
In order to simplify the calculations we consider the following notations:
$T\langle 0, X_1 - X_2 + 8\rangle = T\langle 0\rangle^*$, $\quad T\langle -1, X_1 - X_2 + 7\rangle = T\langle -1\rangle^*$,
$T\langle -2, X_1 - X_2 + 6\rangle = T\langle -2\rangle^*$,
Now if $X_2 < 8(k+1)+4$:

$$T\langle X_2, X_1 + 8\rangle = \frac{1 + T\langle X_2 - 1, X_1 + 7\rangle + T\langle X_2 - 2, X_1 + 6\rangle}{T\langle X_2 - 3, X_1 + 5\rangle}$$

$$= \frac{1 + \frac{(1+T\langle -1\rangle^* + T\langle -2\rangle^*)(1+T\langle -1\rangle^* + T\langle 0\rangle^*)}{T\langle 0\rangle^* T\langle -1\rangle^* T\langle -2\rangle^*} + \frac{1+T\langle -1\rangle^* + T\langle 0\rangle^* + T\langle -2\rangle^*(1+T\langle 0\rangle^*)}{T\langle -1\rangle^* T\langle -2\rangle^*}}{\frac{1+T\langle -1\rangle^* + T\langle 0\rangle^*}{T\langle -2\rangle^*}}$$

$$= \frac{(1 + T\langle -1\rangle^* + T\langle 0\rangle^*)(1 + T\langle -1\rangle^* + T\langle 0\rangle^* + T\langle -2\rangle^*(1 + T\langle 0\rangle^*))}{(1 + T\langle -1\rangle^* + T\langle 0\rangle^*)(T\langle -1\rangle^* T\langle 0\rangle^*)}$$

$$= \frac{1 + T\langle -1\rangle^* + T\langle 0\rangle^* + T\langle -2\rangle^*(1 + T\langle 0\rangle^*)}{T\langle -1\rangle^* T\langle 0\rangle^*}$$

If $X_2 < 8(k+1)+8$ we have :

$$T\langle X_2, X_1 + 8\rangle = \frac{1 + T\langle X_2 - 1, X_1 + 7\rangle + T\langle X_2 - 2, X_1 + 6\rangle}{T\langle X_2 - 3, X_1 + 5\rangle}$$

$$= \frac{1 + T\langle -1\rangle^* + T\langle -2\rangle^*}{\frac{1+T\langle -1\rangle^* + T\langle -2\rangle^*}{T\langle 0\rangle^*}} = T\langle 0\rangle^*$$

□

Remark 3. *If we take into account that $X_1 = X_2 = n$ in equation (2), we have the ordinary ToDD's difference equation in the form*

$$T\langle n\rangle = \frac{1 + T\langle n-1\rangle + T\langle n-2\rangle}{T\langle n-3\rangle} \tag{5}$$

We can obtain the solutions for equation (5) from theorem (4) and we will formulate the closed form solutions in the following corollary .

Corollary 5. *Let $\{T\langle n\rangle\}_{n=-k}^{\infty}$ be a solution of the ordinary difference equation (5) , where $n \in \mathbb{N}$,and the initial values $T\langle q\rangle$ and are real numbers with $q \in \{0, -1, -2\}$ such that $T\langle q\rangle \neq 0$. Then, the form of solutions of (5)* *are as follows:*

$$T\langle n\rangle = \begin{cases} \frac{1+T\langle -1\rangle+T\langle 0\rangle}{T\langle -2\rangle}, X_1 = L_1; \\ \frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle -2\rangle}, X_1 = L_2; \\ \frac{(1+T\langle -1\rangle+T\langle -2\rangle)(1+T\langle -1\rangle+T\langle 0\rangle)}{T\langle 0\rangle)T\langle -1\rangle T\langle -2\rangle}, X_1 = L_3; \\ \frac{1+T\langle -1\rangle+T\langle 0\rangle+T\langle -2\rangle(1+T\langle 0\rangle)}{T\langle -1\rangle T\langle 0\rangle}, X_1 = L_4; \\ \frac{1+T\langle -1\rangle+T\langle -2\rangle}{T\langle 0\rangle}, X_1 = L_5; \\ T\langle -2\rangle, X_1 = L_6; \\ T\langle -1\rangle, X_1 = L_7; \\ T\langle 0\rangle, X_1 = L_8; \end{cases}$$

where $L_i = 8k + i$, $1 \leq i \leq 8$, $i \in \mathbb{N}$.

Remark 4. *It is easy to see that all solutions of* (5) *are periodic with period eight.*

2.2 Form of Solutions for P∆E (1) when $n = 3$

In this subsection we introduce a generalized form of ToDD's difference equation with three discrete variables X_1 ,X_2 and X_3 and give the closed form expressions for this generalized form . In other words , we have the following nonlinear rational partial difference equation

$$T\langle X_1, X_2, X_3\rangle = \frac{1 + T\langle X_1 - 1, X_2 - 1, X_3 - 1\rangle + T\langle X_1 - 2, X_2 - 2, X_3 - 2\rangle}{T\langle X_1 - 3, X_2 - 3, X_3 - 3\rangle} \tag{6}$$

where $X_1, X_2, X_3 \in \mathbb{N}$.

Here we give the closed form solution of the partial difference equation (6).

Theorem 6. *Let* $\{T\langle X_1, X_2, X_3\rangle\}_{X_1,X_2,X_3=-k}^{\infty}$ *be a solution of the partial difference equation* (6) *,where* $X_1, X_2, X_3 \in \mathbb{N}$*,and the initial values* $T\langle p_1, p_2, p_3\rangle$ *,*$T\langle p_2, p_3, p_1\rangle$ *and* $T\langle p_2 - 3, p_1, p_3 - 3\rangle$ *are real numbers with* $p_1 \in \{0, -1, -2\}$ *and* $p_2, p_3 \in \mathbb{N}$ *such that* $T\langle p_1, p_2, p_3\rangle \neq 0$ *,*$T\langle p_2, p_1, p_3\rangle \neq 0$ *and* $T\langle p_2 - 3, p_3 - 3, p_1\rangle \neq 0$ *. Then, the form of solutions of* (6) *, for* $X_1 \leq X_2 \leq X_3$ *are as follows:*

$$T\left\langle X_1,X_2,X_3\right\rangle=\begin{cases}\frac{1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle}{T_3\langle(-2)23\rangle},X_1=L_1;\\ \frac{1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle+T_3\langle(-2)23\rangle(1+T_3\langle(0)23\rangle)}{T_3\langle(-1)23\rangle T_3\langle(-2)23\rangle},X_1=L_2;\\ \frac{(1+T_3\langle(-1)23\rangle+T_3\langle(-2)23\rangle)(1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle)}{T_3\langle(0)23\rangle)T_3\langle(-1)23\rangle T_3\langle(-2)23\rangle},X_1=L_3;\\ \frac{1+T_3\langle(-1)23\rangle+T_3\langle(0)23\rangle+T_3\langle(-2)23\rangle(1+T_3\langle(0)23\rangle)}{T_3\langle(-1)23\rangle T_3\langle(0)23\rangle},X_1=L_4;\\ \frac{1+T_3\langle(-1)23\rangle+T_3\langle(-2)23\rangle}{T_3\langle(0)23\rangle},X_1=L_5;\\ T_3\left\langle(-2)23\right\rangle,X_1=L_6;\\ T_3\left\langle(-1)23\right\rangle,X_1=L_7;\\ T_3\left\langle(0)23\right\rangle,X_1=L_8;\end{cases}$$

$$T\left\langle X_1,X_3,X_2\right\rangle=\begin{cases}\frac{1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle}{T_3\langle(-2)32\rangle},X_1=L_1;\\ \frac{1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle+T_3\langle(-2)32\rangle(1+T_3\langle(0)32\rangle)}{T_3\langle(-1)32\rangle T_3\langle(-2)32\rangle},X_1=L_2;\\ \frac{(1+T_3\langle(-1)32\rangle+T_3\langle(-2)32\rangle)(1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle)}{T_3\langle(0)32\rangle)T_3\langle(-1)32\rangle T_3\langle(-2)32\rangle},X_1=L_3;\\ \frac{1+T_3\langle(-1)32\rangle+T_3\langle(0)32\rangle+T_3\langle(-2)32\rangle(1+T_3\langle(0)32\rangle)}{T_3\langle(-1)32\rangle T_3\langle(0)32\rangle},X_1=L_4;\\ \frac{1+T_3\langle(-1)32\rangle+T_3\langle(-2)32\rangle}{T_3\langle(0)32\rangle},X_1=L_5;\\ T_3\left\langle(-2)32\right\rangle,X_1=L_6;\\ T_3\left\langle(-1)32\right\rangle,X_1=L_7;\\ T_3\left\langle(0)32\right\rangle,X_1=L_8;\end{cases}$$

$$T\left\langle X_3,X_1,X_2\right\rangle=\begin{cases}\frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle}{T_3\langle 3(-2)2\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle+T_3\langle 3(-2)2\rangle(1+T_3\langle 3(0)2\rangle)}{T_3\langle 3(-1)2\rangle T_3\langle 3(-2)2\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 3(-1)2\rangle+T_3\langle 3(-2)2\rangle)(1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle)}{T_3\langle 3(0)2\rangle)T_3\langle 3(-1)2\rangle T_3\langle 3(-2)2\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(0)2\rangle+T_3\langle 3(-2)2\rangle(1+T_3\langle 3(0)2\rangle)}{T_3\langle 3(-1)2\rangle T_3\langle 3(0)2\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 3(-1)2\rangle+T_3\langle 3(-2)2\rangle}{T_3\langle 3(0)2\rangle}, X_1=L_5;\\ T_3\left\langle 3(-2)2\right\rangle, X_1=L_6;\\ T_3\left\langle 3(-1)2\right\rangle, X_1=L_7;\\ T_3\left\langle 3(0)2\right\rangle, X_1=L_8;\end{cases}$$

$$T\left\langle X_3,X_2,X_1\right\rangle=\begin{cases}\frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle}{T_3\langle 32(-2)\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle+T_3\langle 32(-2)\rangle(1+T_3\langle 32(0)\rangle)}{T_3\langle 32(-1)\rangle T_3\langle 32(-2)\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 32(-1)\rangle+T_3\langle 32(-2)\rangle)(1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle)}{T_3\langle 32(0)\rangle)T_3\langle 32(-1)\rangle T_3\langle 32(-2)\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(0)\rangle+T_3\langle 32(-2)\rangle(1+T_3\langle 32(0)\rangle)}{T_3\langle 32(-1)\rangle T_3\langle 32(0)\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 32(-1)\rangle+T_3\langle 32(-2)\rangle}{T_3\langle 32(0)\rangle}, X_1=L_5;\\ T_3\left\langle 32(-2)\right\rangle, X_1=L_6;\\ T_3\left\langle 32(-1)\right\rangle, X_1=L_7;\\ T_3\left\langle 32(0)\right\rangle, X_1=L_8;\end{cases}$$

$$T\langle X_2,X_1,X_3\rangle=\begin{cases}\frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle}{T_3\langle 2(-2)3\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle+T_3\langle 2(-2)3\rangle(1+T_3\langle 2(0)3\rangle)}{T_3\langle 2(-1)3\rangle T_3\langle 2(-2)3\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 2(-1)3\rangle+T_3\langle 2(-2)3\rangle)(1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle)}{T_3\langle 2(0)3\rangle)T_3\langle 2(-1)3\rangle T_3\langle 2(-2)3\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(0)3\rangle+T_3\langle 2(-2)3\rangle(1+T_3\langle 2(0)3\rangle)}{T_3\langle 2(-1)3\rangle T_3\langle 2(0)3\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 2(-1)3\rangle+T_3\langle 2(-2)3\rangle}{T_3\langle 2(0)3\rangle}, X_1=L_5;\\ T_3\langle 2(-2)3\rangle, X_1=L_6;\\ T_3\langle 2(-1)3\rangle, X_1=L_7;\\ T_3\langle 2(0)3\rangle, X_1=L_8;\end{cases}$$

$$T\langle X_2,X_3,X_1\rangle=\begin{cases}\frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle}{T_3\langle 23(-2)\rangle}, X_1=L_1;\\ \frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle+T_3\langle 23(-2)\rangle(1+T_3\langle 23(0)\rangle)}{T_3\langle 23(-1)\rangle T_3\langle 23(-2)\rangle}, X_1=L_2;\\ \frac{(1+T_3\langle 23(-1)\rangle+T_3\langle 23(-2)\rangle)(1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle)}{T_3\langle 23(0)\rangle)T_3\langle 23(-1)\rangle T_3\langle 23(-2)\rangle}, X_1=L_3;\\ \frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(0)\rangle+T_3\langle 23(-2)\rangle(1+T_3\langle 23(0)\rangle)}{T_3\langle 23(-1)\rangle T_3\langle 23(0)\rangle}, X_1=L_4;\\ \frac{1+T_3\langle 23(-1)\rangle+T_3\langle 23(-2)\rangle}{T_3\langle 23(0)\rangle}, X_1=L_5;\\ T_3\langle 23(-2)\rangle, X_1=L_6;\\ T_3\langle 23(-1)\rangle, X_1=L_7;\\ T_3\langle 23(0)\rangle, X_1=L_8;\end{cases}$$

where
$T_3\langle(0)23\rangle=T\langle 0,X_2-X_1,X_3-X_1\rangle$,
$T_3\langle(-1)23\rangle=T\langle -1,X_2-(X_1+1),X_3-(X_1+1)\rangle$,
$T_3\langle(-2)23\rangle=T\langle -2,X_2-(X_1+2),X_3-(X_1+2)\rangle$,

$T_3\left<(0)32\right> = T\left<0, X_3 - X_1, X_2 - X_1\right>$,
$T_3\left<(-1)32\right> = T\left<-1, X_3 - (X_1+1), X_2 - (X_1+1)\right>$,
$T_3\left<(-2)32\right> = T\left<-2, X_3 - (X_1+2), X_2 - (X_1+2)\right>$,
$T_3\left<3(0)2\right> = T\left<X_3 - X_1, 0, X_2 - X_1\right>$,
$T_3\left<3(-1)2\right> = T\left<X_3 - (X_1+1), -1, X_2 - (X_1+1)\right>$,
$T_3\left<3(-2)2\right> = T\left<X_3 - (X_1+2), -2, X_2 - (X_1+2)\right>$,
$T_3\left<32(0)\right> = T\left<X_3 - X_1, X_2 - X_1, 0\right>$,
$T_3\left<32(-1)\right> = T\left<X_3 - (X_1+1), X_2 - (X_1+1), -1\right>$,
$T_3\left<32(-2)\right> = T\left<X_3 - (X_1+2), X_2 - (X_1+2), -2\right>$,
$T_3\left<2(0)3\right> = T\left<X_2 - X_1, 0, X_3 - X_1\right>$,
$T_3\left<2(-1)3\right> = T\left<X_2 - (X_1+1), -1, X_3 - (X_1+1)\right>$,
$T_3\left<2(-2)3\right> = T\left<X_2 - (X_1+2), -2, X_3 - (X_1+2)\right>$,
$T_3\left<23(0)\right> = T\left<X_2 - X_1, X_3 - X_1, 0\right>$,
$T_3\left<23(-1)\right> = T\left<X_2 - (X_1+1), X_3 - (X_1+1), -1\right>$,
$T_3\left<23(-2)\right> = T\left<X_2 - (X_1+2), X_3 - (X_1+2), -2\right>$,
,$L_i = 8k + i$, $1 \leq i \leq 8$, $i \in \mathbb{N}$.

Proof. We can prove this theorem by using the concept of piecewise triple mathematical induction which stated in definition (2) similar to what has been done in theorem (4) by using piecewise double mathematical induction stated in definition (1) . □

2.3 Form of Solutions for P∆E (1) for any value n

In this subsection we introduce the generalized form of ToDD's difference equation with n discrete variables $X_1, X_2, ..., X_n$ and give the closed form expressions for it .

Theorem 7. *Let* $\{T\left<X_1, X_2, ..., X_n\right>\}_{X_1,X_2,...X_n=-k}^{\infty}$ *be a solution of the partial difference equation* (1) *,where* $X_1, X_2, ..., X_n \in \mathbb{N}$*,and the initial values* $T\left<p_1, p_2, ..., p_n\right>$,$T\left<p_2, p_1, p_3, p_4, ..., p_n\right>$,$T\left<p_2, p_3, p_1, p_4, ..., p_n\right>$,...
...,$T\left<p_2, p_3, p_4, ...p_1, p_n\right>$,$T\left<p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\right>$ *are real numbers with* $p_1 \in \{0, -1, -2\}$ *and* $p_2, p_3, ..., p_n \in \mathbb{N}$ *such that* $T\left<p_1, p_2, ..., p_n\right> \neq 0$,$T\left<p_2, p_1, p_3, p_4, ..., p_n\right> \neq 0$,$T\left<p_2, p_3, p_1, p_4, ..., p_n\right> \neq 0$,...,
$T\left<p_2 - 3, p_3 - 3, p_4 - 3, ...p_n - 3, p_1\right> \neq 0$ *. Then, the form of solutions of* (1) *,for* $X_1 \leq X_2 \leq X_3 \leq ... \leq X_n$ *are as follows:*

$$
{}_p^qT_n = \begin{cases}
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)}}{{}_p^qT_n^{(-2)}}, X_1 = L_1; \\
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)}+{}_p^qT_n^{(-2)}(1+{}_p^qT_n^{(0)})}{{}_p^qT_n^{(-1)}.{}_p^qT_n^{(-2)}}, X_1 = L_2 \\
\frac{(1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(-2)})(1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)})}{{}_p^qT_n^{(0)}.{}_p^qT_n^{(-1)}.{}_p^qT_n^{(-2)}}, X_1 = L_3 \\
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(0)}+{}_p^qT_n^{(-2)}(1+{}_p^qT_n^{(0)})}{{}_p^qT_n^{(-1)}.{}_p^qT_n^{(0)}}, X_1 = L_4; \\
\frac{1+{}_p^qT_n^{(-1)}+{}_p^qT_n^{(-2)}}{{}_p^qT_n^{(0)}}, X_1 = L_5; \\
{}_p^qT_n^{(-2)}, X_1 = L_6; \\
{}_p^qT_n^{(-1)}, X_1 = L_7; \\
{}_p^qT_n^{(0)}, X_1 = L_8;
\end{cases}
$$

where

$$ {}_p^qT_n = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., X_1}_{p-times}, .., X_{i_n} \right\rangle $$

$$ {}_p^qT_n^{(0)} = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., 0}_{p-times}, .., X_{i_n} \right\rangle $$

$$ {}_p^qT_n^{(-1)} = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., (-1)}_{p-times}, .., X_{i_n} \right\rangle $$

$$ {}_p^qT_n^{(-2)} = T\left\langle \underbrace{X_{i_1}, X_{i_2}, .., (-2)}_{p-times}, .., X_{i_n} \right\rangle $$

$i_1, i_2, i_3, ..., i_n \in \{1, 2, 3....n\}$, $\quad , p = 1, 2, ...n$, $q = 1, 2, ...n-1$
$,L_i = 8k + i$, $1 \le i \le 8$, $i \in \mathbb{N}$.

Proof. We can prove this theorem by using the concept of piecewise n-dimensional mathematical induction which stated in definition (3) similar to what has been done in theorem (4) by using piecewise double mathematical induction stated in definition (1) . □

Remark 5. *we can note that the the number of equations for solutions* ${}_p^qT_n$ *is* $n!$ *. For example , if* $n = 2$ *we find that* $p = 1, 2$ *,* $q = 1$ *and then the number of equations for solutions is 2!=2 (see theorem* (4) *). That is* ${}_1^1T_2 = T\langle X_1, X_2\rangle$ *and* ${}_2^1T_2 = T\langle X_2, X_1\rangle$ *. So if we put* $n = 2$ *in theorem* (7) *we can get the solutions of equation* (2)
Another example , if $n = 3$ *we find that* $p = 1, 2, 3$ *,* $q = 1, 2$ *and then the number of equations for solutions is 3!=6 (see theorem* (6) *). That is* ${}_1^1T_3 = T\langle X_1, X_2, X_3\rangle$, ${}_2^1T_3 = T\langle X_3, X_1, X_2\rangle$, ${}_3^1T_3 = T\langle X_3, X_2, X_1\rangle$ *,* ${}_1^2T_3 = T\langle X_1, X_3, X_2\rangle$, ${}_2^2T_3 = T\langle X_2, X_1, X_3\rangle$ *and* ${}_3^2T_3 = T\langle X_2, X_3, X_1\rangle$ *. So if we put* $n = 3$ *in theorem* (7) *we can get the solutions of equation* (6).

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Meromorphic Solutions of Some Types of Systems of Complex Differential-Difference Equations *

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Abstract: Using Nevanlinna theory of the value distribution of meromorphic functions, we investigate the problem of the existence of meromorphic solutions of some types of systems of complex differential-difference equations and some properties of meromorphic solutions, and we obtain some results, which are the improvements and extensions of some results in references. Example shows that our results are precise.

Key words: value distribution; meromorphic solutions; systems of complex differential-difference equation

2010 MR Subject Classification: 30D35.

1 Introduction and Notation

Throughout the article, we assume that the reader is familiar with the standard notation and basic results of the Nevanlinna theory of meromorphic functions, see, for example [1-3].

Let $w(z)$ be a non-constant meromorphic function of finite order, if meromorphic function $g(z)$ satisfies $T(r,g) = o\{T(r,w)\} = S(r,w)$, for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$, then $g(z)$ is called small function of $w(z)$.

Using the Nevanlinna theory of the distribution of meromophic functions, many authors investigate solutions of some types of complex differential equations, and obtain some results, see [4-8]. Especially, J Malmquist has investigated the problem of existence of complex differential equation and has obtained a result as follows.

Theorem A (Malmquist Theorem) (see [1]) Let $P(z,w(z))$ and $Q(z,w(z))$ are relatively prime polynomials in $w(z)$. If the complex differential equation

$$\frac{dw}{dz} = R(z,w) = \frac{P(z,w)}{Q(z,w)} = \frac{\sum_{k=0}^{p} a_k(z)w^k}{\sum_{j=0}^{q} b_j(z)w^j}$$

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with coefficients of rational functions $a_0(z), \dots, a_p(z), b_0(z), \dots, b_q(z)$, admits a transcendental meromorphic solution, then

$$q = 0, \ p \leq 2.$$

Theorem B (see [1]) Let $\Omega(z,w) = \sum_{(i)\in I} a_{(i)}(z) w^{i_0}(w')^{i_1} \cdots (w^{(n)})^{i_n}$, $P(z, w(z))$ and $Q(z, w(z))$ are relatively prime polynomials in $w(z)$. If $w(z)$ is a transcendental meromorphic solution of the complex differential equation

$$\Omega(z,w) = R(z,w) = \frac{P(z,w)}{Q(z,w)} = \frac{\sum_{k=0}^{p} a_k(z) w^k}{\sum_{j=0}^{q} b_j(z) w^j}$$

with coefficients $a_{(i)}(z)((i) \in I)$, $a_k(z)(k = 0, 1, \dots, p)$ and $b_j(z)(j = 0, 1, \dots, q)$, which are rational functions, where I is a finite index set, then

$$q = 0, p \leq \min\{\triangle, \lambda + \overline{\mu}(1 - \Theta(\infty))\},$$

where $\Delta = \max\{\sum_{\alpha=0}^{n} (\alpha+1) i_\alpha\}$, $\lambda = \max\{\sum_{\alpha=0}^{n} i_\alpha\}$, $\overline{\mu} = \max\{\sum_{\alpha=1}^{n} \alpha i_\alpha\}$, $\Theta(\infty) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,w)}{T(r,w)}$.

Recently, meromorphic solutions of complex difference equations have become a subject of great interest. Many authors, such as I Laine, R Korhonen, Chiang Y M, Chen Zongxuan and Gao Lingyun, investigate complex difference equations, and obtain many results, see [9-24]. Especially, in 2000, M J Ablowitz, R Halburd and B Herbst have investigated the problem of existence of meromorphic solutions of complex difference equations and have obtained a result as follows.

Theorem C (see [9]) If the complex difference equation

$$w(z+1) + w(z-1) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)},$$

with polynomial coefficients $a_i(z)(i = 0, 1, \dots, p)$ and $b_j(z)(j = 0, 1, \dots, q)$, admits a transcendental meromorphic solution of finite order, then

$$d = \max\{p, q\} \leq 2.$$

I Laine, J Rieppo and H Silvennoinen generalized the above result, and obtained the following result.

Theorem D (see [22]) Let $c_1, c_2, \dots, c_n$ be distinct nonzero complex numbers. If $w(z)$ is a finite order transcendental meromorphic solution of the following complex difference equation

$$\sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} w(z+c_j)\right) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)},$$

with coefficients $\alpha_J(z)$, $a_i(z)(i=0,1,\dots,p)$ and $b_j(z)(j=0,1,\dots,q)$, which are small functions relative to $w(z)$, where J is a collection of all subsets of $\{1,2,\dots,n\}$, then

$$d=\max\{p,q\}\leq n.$$

In [22], I Laine, J Rieppo and H Silvennoinen also obtained the following result.

Theorem E (see [22]) Suppose that $c_1,c_2,\dots,c_n$ are distinct, non-zero complex numbers, and that $w(z)$ is a transcendental meromorphic solution of

$$\sum_{j=1}^{n}\alpha_j(z)w(z+c_j)=R(z,w(z))=\frac{P(z,w(z))}{Q(z,w(z))},$$

where the coefficients $\alpha_j(z)$ are non-vanishing small functions relative to $w(z)$, and where $P(z,w(z)),Q(z,w(z))$ are relatively prime polynomials in $w(z)$ over the field of small functions relative to $w(z)$. Moreover, we assume that $q=\deg_w^Q>0$,

$$n=\max\{p,q\}:=\max\{\deg_w^P,\deg_w^Q\},$$

and that, without restricting generality, $Q(z,w(z))$ is a monic polynomial. If there exists $\alpha\in[0,n)$ such that for all r sufficiently large,

$$\overline{N}(r,\sum_{j=1}^{n}\alpha_j(z)w(z+c_j))\leq\alpha\overline{N}(r+c,w(z))+S(r,w),$$

where $c=\max\{|c_1|,|c_2|,\dots,|c_n|\}$, then either the order $\rho(w)=+\infty$, or

$$Q(z,w(z))\equiv(w(z)+h(z))^q,$$

where $h(z)$ is a small meromorphic function relative to $w(z)$.

Further, I Laine, J Rieppo and H Silvennoinen also obtained the following Theorem.

Theorem F (see [22]) Suppose that $w(z)$ is a transcendental meromorphic solution of the equation

$$\sum_{\{J\}}\alpha_J(z)(\prod_{j\in J}w(z+c_j))=w(p(z))$$

where $p(z)$ is a polynomial of degree $k\geq 2$, J is a collection of all subsets of $\{1,2,\dots,n\}$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to $w(z)$ and that $n\geq k$. Then

$$T(r,w)=O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha=\dfrac{\log n}{\log k}$, $\varepsilon>0$ is arbitrarily small.

After some authors investigate complex difference equations, solutions of system of complex difference equations are also investigated, naturally, see [13].

Let $c_1,c_2,\dots,c_n$ are distinct non-zero complex numbers, differential-difference polynomials $\Omega_1(z,w_1),\Omega_2(z,w_1),\Omega_3(z,w_2),\Omega_4(z,w_2)$ can be expressed as

$$\Omega_1(z,w_1)=\sum_{i_1\in I_1}a_{i_1}(z)(w_1^{(t)}(z+c_1))^{l_{i_1,1}}(w_1^{(t)}(z+c_2))^{l_{i_1,2}}\dots(w_1^{(t)}(z+c_n))^{l_{i_1,n}},\ t\geq 1,t\in\mathbf{N},$$

$$\Omega_2(z,w_1)=\sum_{j_1\in J_1} b_{j_1}(z)(w_1^{(t)}(z+c_1))^{m_{j_1,1}}(w_1^{(t)}(z+c_2))^{m_{j_1,2}}\dots(w_1^{(t)}(z+c_n))^{m_{j_1,n}},\ t\geq 1, t\in \mathbf{N},$$

$$\Omega_3(z,w_2)=\sum_{i_2\in I_2} c_{i_2}(z)(w_2^{(t)}(z+c_1))^{l_{i_2,1}}(w_2^{(t)}(z+c_2))^{l_{i_2,2}}\dots(w_2^{(t)}(z+c_n))^{l_{i_2,n}},\ t\geq 1, t\in \mathbf{N},$$

$$\Omega_4(z,w_2)=\sum_{j_2\in J_2} d_{j_2}(z)(w_2^{(t)}(z+c_1))^{m_{j_2,1}}(w_2^{(t)}(z+c_2))^{m_{j_2,2}}\dots(w_2^{(t)}(z+c_n))^{m_{j_2,n}},\ t\geq 1, t\in \mathbf{N},$$

where coefficients $\{a_{i_1}(z)\}, \{b_{j_1}(z)\}$ are small functions relative to w_1, coefficients $\{c_{i_2}(z)\}$, $\{d_{j_2}(z)\}$ are small functions relative to w_2. $I_1=\{i_1=(l_{i_1,1},l_{i_1,2},\dots,l_{i_1,n}):l_{i_1,k}\in\mathbf{N},k=1,2,\dots,n\}$, $J_1=\{j_1=(m_{j_1,1},m_{j_1,2},\dots,m_{j_1,n}):m_{j_1,k}\in\mathbf{N},k=1,2,\dots,n\}$, $I_2=\{i_2=(l_{i_2,1},l_{i_2,2},\dots,l_{i_2,n}):l_{i_2,k}\in\mathbf{N},k=1,2,\dots,n\}$, $J_2=\{j_2=(m_{j_2,1},m_{j_2,2},\dots,m_{j_2,n}):m_{j_2,k}\in\mathbf{N},k=1,2,\dots,n\}$ are four finite index sets.

Existence of solutions of complex differential-difference equations is investigated, see[16].

In this article, we will investigate the problem of the existence of solutions of some types of systems of complex differential-difference equations.

The remainder of the article is organized as follows. In §2, we study meromorphic solutions of systems of complex differential-difference equations, and obtain three theorems. Example that we give shows that our results in §2 are precise. In §3, we give a series of lemmas for the proof of theorems 2.1-2.3. In §4, we prove theorems 2.1-2.3 for systems of complex differential-difference equations by lemma given in §3.

2 Main results

We obtain the following results about systems of complex differential-difference equations.

Theorem 2.1. Let $(w_1(z),w_2(z))$ be a finite order transcendental meromorphic solution of

$$\begin{cases} \dfrac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)}=R_1(z,w_2)=\dfrac{P_1(z,w_2)}{Q_1(z,w_2)},\\ \dfrac{\Omega_3(z,w_2)}{\Omega_4(z,w_2)}=R_2(z,w_1)=\dfrac{P_2(z,w_1)}{Q_2(z,w_1)},\end{cases}\tag{2.1}$$

where $P_1(z,w_2),Q_1(z,w_2)$ are relatively prime polynomials in w_2 over the field of small functions relative to w_2, $P_2(z,w_1),Q_2(z,w_1)$ are relatively prime polynomials in w_1 over the field of small functions relative to w_1. Then

$$\max\{p_1,q_1\}\max\{p_2,q_2\}\leq(t+1)^2\lambda_1\lambda_2,$$

where $\lambda_{1k}=\max\limits_{i_1\in I_1,j_1\in J_1}\{l_{i_1,k},m_{j_1,k}\}$, $k=1,2,\dots,\ n$. $\lambda_{2k}=\max\limits_{i_2\in I_2,j_2\in J_2}\{l_{i_2,k},m_{j_2,k}\}$, $k=1,2,\dots,n$. $\lambda_1=\sum\limits_{k=1}^{n}\lambda_{1k}$, $\lambda_2=\sum\limits_{k=1}^{n}\lambda_{2k}$, $p_1=\deg_{w_2}^{P_1}$, $q_1=\deg_{w_2}^{Q_1}$, $p_2=\deg_{w_1}^{P_2}$, $q_2=\deg_{w_1}^{Q_2}$.

Example 2.1 shows the upper in Theorem 2.1 can be reached.

Example 2.1. $(w_1(z), w_2(z)) = (e^{-z} + z^2, e^z + z)$ is a finite order transcendental meromorphic solution of the following system of complex differential-difference equations

$$\begin{cases} w_1'(z+1) = \dfrac{P_1(z, w_2)}{Q_1(z, w_2)}, \\ w_2'(z+1) = \dfrac{P_2(z, w_1)}{Q_2(z, w_1)}, \end{cases}$$

where

$$P_1(z, w_2) = (2z+2)w_2^2(z) - (8z^2+8z+e^{-1})w_2(z) - z^2(2z+2) + z(8z^2+8z+e^{-1}) + 2ze^{-1},$$

$$Q_1(z, w_2) = w_2^2(z) - 4zw_2(z) + 3z^2,$$

$$P_2(z, w_1) = w_1^2(z) - [2z^2 - e - 3z + 1]w_1(z) + z^4 - z^2(e + 3z - 1) + (3z - 1)e,$$

$$Q_2(z, w_1) = w_1^2(z) - [2z^2 - 3z + 1]w_1(z) + z^4 - z^2(3z - 1).$$

In this case

$$\max\{p_1, q_1\} = 2, \max\{p_2, q_2\} = 2, t = 1, \lambda_1 = \lambda_2 = 1.$$

Thus

$$\max\{p_1, q_1\} \max\{p_2, q_2\} = 4 = (t+1)^2 \lambda_1 \lambda_2.$$

Theorem 2.2. Suppose that $(w_1(z), w_2(z))$ is a transcendental meromorphic solution of the following system of complex differential-difference equations

$$\begin{cases} \dfrac{\Omega_1(z, w_1)}{\Omega_2(z, w_1)} = R_1(z, w_2) = \dfrac{P_1(z, w_2)}{Q_1(z, w_2)}, \\ \dfrac{\Omega_3(z, w_2)}{\Omega_4(z, w_2)} = R_2(z, w_1) = \dfrac{P_2(z, w_1)}{Q_2(z, w_1)}, \end{cases} \tag{2.1}$$

where $P_1(z, w_2), Q_1(z, w_2)$ are relatively prime polynomials in w_2 over the field of small functions relative to w_2, $P_2(z, w_1), Q_2(z, w_1)$ are relatively prime polynomials in w_1 over the field of small functions relative to w_1. Moreover, we assume that $q_1 = \deg_{w_2}^{Q_1} > 0$, $q_2 = \deg_{w_1}^{Q_2} > 0$, $p_1 = \deg_{w_2}^{P_1}$, $p_2 = \deg_{w_1}^{P_2}$, $Q_1(z, w_2)$ and $Q_2(z, w_1)$ are respectively monic polynomials. $\lambda_1(t+1) = \max\{p_1, q_1\}$, $\lambda_2(t+1) = \max\{p_2, q_2\}$, $\lambda' = \min\{\lambda_1, \lambda_2\}$, $c = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. If there exists $\alpha, \beta \in [0, \lambda'(t+1))$, such that for all r sufficiently large,

$$\begin{cases} \overline{N}(r, \dfrac{\Omega_1(z, w_1)}{\Omega_2(z, w_1)}) \leq \alpha \overline{N}(r + c, w_1(z)) + S(r, w_1), \\ \overline{N}(r, \dfrac{\Omega_3(z, w_2)}{\Omega_4(z, w_2)}) \leq \beta \overline{N}(r + c, w_2(z)) + S(r, w_2), \end{cases} \tag{2.2}$$

and

$$\begin{cases} \displaystyle\sum_{k=1}^{n} \lambda_{1k}(t+1)\overline{N}(r, w_1(z + c_k)) \leq \alpha \overline{N}(r + c, w_1(z)) + S(r, w_1), \\ \displaystyle\sum_{k=1}^{n} \lambda_{2k}(t+1)\overline{N}(r, w_2(z + c_k)) \leq \beta \overline{N}(r + c, w_2(z)) + S(r, w_2). \end{cases} \tag{2.3}$$

Then, either at least one of

$$\rho(w_1) = +\infty, \rho(w_2) = +\infty$$

will be true, or at least one of

$$Q_1(z, w_2) \equiv (w_2(z) + h_2(z))^{q_1}, Q_2(z, w_1) \equiv (w_1(z) + h_1(z))^{q_2}$$

will be true, where $h_1(z)$ is a small meromorphic function relative to $w_1(z)$, $h_2(z)$ is a small meromorphic function relative to $w_2(z)$.

Theorem 2.3. Suppose that $(w_1(z), w_2(z))$ is a transcendental meromorphic solution of the following system of complex differential-difference equations

$$\begin{cases} \dfrac{\Omega_1(z, w_1)}{\Omega_2(z, w_1)} = w_2(p(z)), \\ \dfrac{\Omega_3(z, w_2)}{\Omega_4(z, w_2)} = w_1(p(z)), \end{cases} \tag{2.4}$$

where $p(z)$ is a polynomial of degree $\overline{d} \geq 2$. $\lambda_{1k} = \max\limits_{i_1 \in I_1, j_1 \in J_1} \{l_{i_1,k}, m_{j_1,k}\}$, $k = 1, 2, \ldots, n$. $\lambda_{2k} = \max\limits_{i_2 \in I_2, j_2 \in J_2} \{l_{i_2,k}, m_{j_2,k}\}$, $k = 1, 2, \ldots, n$. $\lambda_1 = \sum\limits_{k=1}^{n} \lambda_{1k}$, $\lambda_2 = \sum\limits_{k=1}^{n} \lambda_{2k}$, $\overline{\lambda} = \max\{\lambda_1, \lambda_2\}$. Moreover, we assume that $\overline{\lambda}(t+1)^2 \geq \overline{d}$. Then

$$T(r, w_1) = O((\log r)^{\alpha+\varepsilon}),$$

$$T(r, w_2) = O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha = \dfrac{\log \overline{\lambda}(t+1)^2}{\log \overline{d}}$, and $\varepsilon > 0$ is arbitrarily small.

3 Some Lemmas for the Proof of Theorems

We need the following lemmas to proof theorems.

Lemma 3.1 (see [23]) Let

$$R(z, w(z)) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)}$$

be an irreducible rational function in $w(z)$ with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$. If $w(z)$ is a meromorphic function, then

$$T(r, R(z, w(z))) = \max\{p, q\}T(r, w(z)) + O\{\sum T(r, a_i(z)) + \sum T(r, b_j(z))\}.$$

Lemma 3.2 (see [3]) Let $w(z)$ be a transcendental meromorphic function, then

$$T(r, w^{(k)}) \leq (k+1)T(r, w) + S(r, w).$$

Lemma 3.3 (see [11]) Let $w(z)$ be a non-constant meromorphic function of finite order, c is a non-zero complex constant, then

$$T(r, w(z+c)) = T(r, w) + S(r, w),$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Lemma 3.4 (see [6]) Let $f_1, f_2, \ldots, f_p$ be distinct meromorphic functions and

$$F(z) = \frac{P(z)}{Q(z)} = \frac{\sum\limits_{K \in K_0} f_1^{k_1} f_2^{k_2} \cdots f_p^{k_p}}{\sum\limits_{I \in I_0} f_1^{i_1} f_2^{i_2} \cdots f_p^{i_p}}.$$

If $s_v = \max\{\max\limits_{K \in K_0} k_v, \max\limits_{I \in I_0} i_v\}$, $v = 1, 2, \ldots, p$. Then

$$m(r, F) \leq \sum_{v=1}^{p} s_v m(r, f_v) + N(r, Q) - N(r, \frac{1}{Q}) + O(1),$$

$$T(r, F) \leq \sum_{v=1}^{p} s_v T(r, f_v) + O(1),$$

where $Q(z) \neq 0$, $K_0 = \{K = (k_1, k_2, \ldots, k_p) : k_v \in N \bigcup \{0\}, v = 1, 2, \ldots, p\}$, $I_0 = \{I = (i_1, i_2, \ldots, i_p) : i_v \in N \bigcup \{0\}, v = 1, 2, \ldots, p\}$ are two finite index sets.

Lemma 3.5 (see [24]) Let $w(z)$ be a meromorphic function and let Φ be given by

$$\Phi = w^n + a_{n-1} w^{n-1} + \cdots + a_0,$$

$$T(r, a_j) = S(r, w), j = 0, 1, ..., n-1.$$

Then either

$$\Phi \equiv (w + \frac{a_{n-1}}{n})^n,$$

or

$$T(r, w) \leq \overline{N}(r, \frac{1}{\Phi}) + \overline{N}(r, w) + S(r, w).$$

Lemma 3.6 (see [22]) Let $w(z)$ be a non-constant meromorphic function and let $P(z, w), Q(z, w)$ be two polynomials in $w(z)$ with meromorphic coefficients small relative to $w(z)$. If $P(z, w)$ and $Q(z, w)$ have no common factors of positive degree in $w(z)$ over the field of small functions relative to $w(z)$, then

$$\overline{N}(r, \frac{1}{Q(z, w)}) \leq \overline{N}(r, \frac{P(z, w)}{Q(z, w)}) + S(r, w).$$

Lemma 3.7 (see [21]) Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function, $\delta \in (0, 1), s \in (0, +\infty)$. If T is of finite order, i.e

$$\lim_{r \to \infty} \frac{\log T(r)}{\log r} = \rho < \infty,$$

then

$$T(r + s) = T(r) + o(\frac{T(r)}{r^{\delta}}),$$

outside an exceptional set of finite logarithmic measure.

Lemma 3.8 (see [14]) Let $w(z)$ be a transcendental meromorphic function, and $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$, $a_k \neq 0$, be a non-constant polynomial of degree

k. Given $0 < \delta < |a_k|$, denote $\lambda = |a_k| + \delta$ and $\mu = |a_k| - \delta$. Then given $\varepsilon > 0$ and $a \in \mathbf{C} \cup \{\infty\}$, we have

$$kn(\mu r^k, a, w) \le n(r, a, w(p(z))) \le kn(\lambda r^k, a, w),$$

$$N(\mu r^k, a, w) + O(\log r) \le N(r, a, w(p(z))) \le N(\lambda r^k, a, w) + O(\log r),$$

$$(1-\varepsilon)T(\mu r^k, w) \le T(r, w(p(z))) \le (1+\varepsilon)T(\lambda r^k, w).$$

Lemma 3.9 (see [2]) Let $g : (0, +\infty) \to \mathbf{R}$, $h : (0, +\infty) \to \mathbf{R}$ be monotone increasing functions such that $g(r) \le h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.

Lemma 3.10 (see [15]) Let $\phi_i : [r_0, +\infty) \to (0, +\infty)(i = 1, 2)$ be positive and bounded in every finite interval, and suppose that

$$\phi_1(\mu r^m) \le A_1\phi_1(r) + B_1\phi_2(r) + d_1,$$

$$\phi_2(\mu r^m) \le A_2\phi_1(r) + B_2\phi_2(r) + d_2,$$

holds for all r large enough, where $\mu > 0, m > 1, A_i > 1, B_i > 1, (i = 1, 2)$, and d_1, d_2 are real constants. Then

$$\phi_1(r) = O((\log r)^\alpha), \quad \phi_2(r) = O((\log r)^\alpha)$$

where $\alpha = \dfrac{\log 2A}{\log m}, A = \max\limits_{i=1,2}\{A_i, B_i\}$.

4 Proof of Theorems 2.1-2.3

Proof of Theorem 2.1. Suppose that $(w_1(z), w_2(z))$ is a set of finite order transcendental meromorphic solution of system of complex differential-difference equations (2.1). Using Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4, we obtain

$$\begin{aligned}
\max\{p_1, q_1\}T(r, w_2) &= T(r, R_1(z, w_2)) + S(r, w_2) \\
&= T(r, \frac{\Omega_1(z, w_1)}{\Omega_2(z, w_1)}) + S(r, w_2) \\
&\le \sum_{k=1}^{n} \lambda_{1k} T(r, w_1^{(t)}(z + c_k)) + S(r, w_1) + S(r, w_2) \\
&\le \sum_{k=1}^{n} \lambda_{1k}(t+1) T(r, w_1(z + c_k)) + S(r, w_1) + S(r, w_2) \\
&= \sum_{k=1}^{n} \lambda_{1k}(t+1) T(r, w_1(z)) + S(r, w_1) + S(r, w_2) \\
&= \lambda_1(t+1)T(r, w_1) + S(r, w_1) + S(r, w_2).
\end{aligned}$$

Thus, we have

$$\max\{p_1, q_1\}T(r, w_2) \le \lambda_1(t+1)T(r, w_1) + S(r, w_1) + S(r, w_2). \tag{4.1}$$

Similarly, we obtain

$$\max\{p_2, q_2\}T(r, w_1) \le \lambda_2(t+1)T(r, w_2) + S(r, w_1) + S(r, w_2). \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\max\{p_1,q_1\}\max\{p_2,q_2\}\leq (t+1)^2\lambda_1\lambda_2.$$

Theorem 2.1 is proved.

Proof of Theorem 2.2. Suppose that $(w_1(z),w_2(z))$ is a set of transcendental meromorphic solution of (2.1) and the second alternative of the conclusion is not true. It follows from Lemma 3.5, Lemma 3.6, (2.1) and (2.2) that

$$\begin{aligned} T(r,w_2) &\leq \overline{N}(r,\frac{1}{Q_1(z,w_2)})+\overline{N}(r,w_2)+S(r,w_2)\\ &\leq \overline{N}(r,\frac{P_1(z,w_2)}{Q_1(z,w_2)})+\overline{N}(r,w_2)+S(r,w_2)\\ &= \overline{N}(r,\frac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)})+\overline{N}(r,w_2)+S(r,w_2)\\ &\leq \alpha\overline{N}(r+c,w_1)+\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{aligned}$$

Thus, we obtain

$$T(r,w_2)-\overline{N}(r,w_2)\leq \alpha\overline{N}(r+c,w_1)+S(r,w_1)+S(r,w_2). \tag{4.3}$$

where $\alpha\in[0,\lambda'(t+1))$, $\lambda'=\min\{\lambda_1,\lambda_2\}$, $\lambda_1(t+1)=\max\{p_1,q_1\}$, $\lambda_2(t+1)=\max\{p_2,q_2\}$.

Similarly, we have

$$T(r,w_1)-\overline{N}(r,w_1)\leq \beta\overline{N}(r+c,w_2)+S(r,w_1)+S(r,w_2). \tag{4.4}$$

where $\beta\in[0,\lambda'(t+1))$, $\lambda'=\min\{\lambda_1,\lambda_2\}$, $\lambda_1(t+1)=\max\{p_1,q_1\}$, $\lambda_2(t+1)=\max\{p_2,q_2\}$.

Assuming, contrary to the assertion, that $\rho(w_i)<+\infty, i=1,2$. Then it implies that

$$S(r,w_i(z+c_k))=S(r,w_i(z)), i=1,2,k=1,2,\ldots,n.$$

By (4.3) and (4.4), we obtain

$$T(r,w_2(z+c_k))-\overline{N}(r,w_2(z+c_k))\leq \alpha\overline{N}(r+c,w_1(z+c_k))+S(r,w_1)+S(r,w_2). \tag{4.5}$$

$$T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))\leq \beta\overline{N}(r+c,w_2(z+c_k))+S(r,w_1)+S(r,w_2). \tag{4.6}$$

where $k=1,2,\ldots,n$.

Applying Lemma 3.1, Lemma 3.2, Lemma 3.4 and Lemma 3.7, and using (2.3) and

(4.6), we conclude that

$$\begin{aligned}
\lambda_1(t+1)T(r,w_2) &= T(r,\frac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)})+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}T(r,w_1^{(t)}(z+c_k))+S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)T(r,w_1(z+c_k))+S(r,w_1)+S(r,w_2)\\
&= \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))]+\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))\\
&\quad +S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)\beta\overline{N}(r+c,w_2(z+c_k))+\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)\beta\overline{N}(r+2c,w_2(z))+\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)\beta\overline{N}(r+2c,w_2(z))+\alpha\overline{N}(r+2c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \lambda_1(t+1)\beta\overline{N}(r+2c,w_2(z))+\alpha\overline{N}(r+2c,w_1(z))+S(r,w_1)+S(r,w_2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
T(r,w_2)-\overline{N}(r,w_2) &\leq \beta\overline{N}(r+2c,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r+2c,w_1)\\
&\quad -\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2).
\end{aligned}\tag{4.7}$$

Similarly, applying Lemma 3.1, Lemma 3.2, Lemma 3.4 and Lemma 3.7, and using (2.3) and (4.5), we conclude that

$$\begin{aligned}
T(r,w_1)-\overline{N}(r,w_1) &\leq \alpha\overline{N}(r+2c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+2c,w_2)\\
&\quad -\overline{N}(r,w_1)+S(r,w_1)+S(r,w_2).
\end{aligned}\tag{4.8}$$

Applying Lemma 3.7, and using (4.8), we obtain

$$\begin{aligned}
\lambda_1(t+1)T(r,w_2) &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))]+\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))\\
&\quad +S(r,w_1)+S(r,w_2)\\
&\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[\alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r-c,w_1)]\\
&\quad +\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2)\\
&\leq \lambda_1\alpha(t+1)\overline{N}(r+3c,w_1)+\frac{\lambda_1\beta}{\lambda_2}\overline{N}(r+3c,w_2)-\lambda_1(t+1)\overline{N}(r-c,w_1)\\
&\quad +\alpha\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).
\end{aligned}$$

Namely,

$$\begin{aligned}
T(r,w_2) &\leq \alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r,w_1)\\
&\quad +\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned} T(r,w_2)-\overline{N}(r,w_2) &\leq \alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r,w_1) \\ &\quad +\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{aligned}$$

Similarly, applying Lemma 3.7, and using (4.8), we have

$$\begin{aligned} T(r,w_1)-\overline{N}(r,w_1) &\leq \beta\overline{N}(r+3c,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r+3c,w_1)-\overline{N}(r,w_1) \\ &\quad +\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{aligned}$$

This implies that

$$\begin{cases} T(r,w_2)-\overline{N}(r,w_2) \leq \alpha\overline{N}(r+3c,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+3c,w_2)-\overline{N}(r,w_1) \\ \qquad +\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2), \\ T(r,w_1)-\overline{N}(r,w_1) \leq \beta\overline{N}(r+3c,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r+3c,w_1)-\overline{N}(r,w_1) \\ \qquad +\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{cases} \tag{4.9}$$

We now proceed, inductively, to prove

$$\begin{cases} T(r,w_2)-\overline{N}(r,w_2) \leq \alpha\overline{N}(r+(2m+1)c,w_1)+\frac{m\beta}{\lambda_2(t+1)}\overline{N}(r+(2m+1)c,w_2)-m\overline{N}(r,w_1) \\ \qquad +\frac{m\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-m\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2), \\ T(r,w_1)-\overline{N}(r,w_1) \leq \beta\overline{N}(r+(2m+1)c,w_2)+\frac{m\alpha}{\lambda_1(t+1)}\overline{N}(r+(2m+1)c,w_1)-m\overline{N}(r,w_1) \\ \qquad +\frac{m\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-m\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2). \end{cases} \tag{4.10}$$

The case $m=1$ has been proved. We assume that (4.10) holds when $m=l$.

$$\begin{aligned} \lambda_1(t+1)T(r,w_2) &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))] \\ &\quad +\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))+S(r,w_1)+S(r,w_2) \\ &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)[\beta\overline{N}(r+(2l+1)c,w_2(z+c_k))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+1)c,w_1(z+c_k)) \\ &\quad -l\overline{N}(r,w_1(z+c_k))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z+c_k))-l\overline{N}(r,w_2(z+c_k))] \\ &\quad +\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2) \\ &\leq \lambda_1(t+1)[\beta\overline{N}(r+(2l+2)c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+2)c,w_1(z)) \\ &\quad -l\overline{N}(r-c,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r+c,w_2(z))-l\overline{N}(r-c,w_2(z))] \\ &\quad +\alpha\overline{N}(r+c,w_1(z))+S(r,w_1)+S(r,w_2). \end{aligned}$$

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Therefore

$$\begin{aligned}T(r,w_2) \ \leq\ & \beta\overline{N}(r+(2l+2)c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+2)c,w_1(z))\\ &-l\overline{N}(r,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-l\overline{N}(r,w_2(z))\\ &+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).\end{aligned}$$

Namely,

$$\begin{aligned}T(r,w_2)-\overline{N}(r,w_2) \ \leq\ & \beta\overline{N}(r+(2l+2)c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+(2l+2)c,w_1(z))\\ &-l\overline{N}(r,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-l\overline{N}(r,w_2(z))\\ &-\overline{N}(r,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))+S(r,w_1)+S(r,w_2).\end{aligned}$$

Similarly,

$$\begin{aligned}T(r,w_1)-\overline{N}(r,w_1) \ \leq\ & \alpha\overline{N}(r+2(l+1)c,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r+2(l+1)c,w_2(z))\\ &-l\overline{N}(r,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-l\overline{N}(r,w_1(z))\\ &-\overline{N}(r,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))+S(r,w_1)+S(r,w_2).\end{aligned}$$

$$\begin{aligned}\lambda_1(t+1)T(r,w_2) \ \leq\ & \sum_{k=1}^{n}\lambda_{1k}(t+1)[T(r,w_1(z+c_k))-\overline{N}(r,w_1(z+c_k))]\\ &+\sum_{k=1}^{n}\lambda_{1k}(t+1)\overline{N}(r,w_1(z+c_k))+S(r,w_1)+S(r,w_2)\\ \leq\ & \sum_{k=1}^{n}\lambda_{1k}(t+1)[\alpha\overline{N}(r+2(l+1)c+c,w_1(z))+\frac{l\beta}{\lambda_2(t+1)}\overline{N}(r+2(l+1)c+c,w_2(z))\\ &-l\overline{N}(r-c,w_2(z))+\frac{l\alpha}{\lambda_1(t+1)}\overline{N}(r+c,w_1(z))-l\overline{N}(r-c,w_1(z))-\overline{N}(r-c,w_1(z))\\ &+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r+c,w_2(z))]+\alpha\overline{N}(r+c,w_1)+S(r,w_1)+S(r,w_2).\end{aligned}$$

This implies that

$$\begin{aligned}T(r,w_2) \ \leq\ & \alpha\overline{N}(r+[2(l+1)+1]c,w_1)+\frac{(l+1)\beta}{\lambda_2(t+1)}\overline{N}(r+[2(l+1)+1]c,w_2)\\ &-l\overline{N}(r,w_2)+\frac{(l+1)\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-(l+1)\overline{N}(r,w_1(z))\\ &+S(r,w_1)+S(r,w_2).\end{aligned}$$

Thus

$$\begin{aligned}T(r,w_2)-\overline{N}(r,w_2) \ \leq\ & \alpha\overline{N}(r+[2(l+1)+1]c,w_1)+\frac{(l+1)\beta}{\lambda_2(t+1)}\overline{N}(r+[2(l+1)+1]c,w_2)\\ &-(l+1)\overline{N}(r,w_2)+\frac{(l+1)\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1(z))-(l+1)\overline{N}(r,w_1(z))\\ &+S(r,w_1)+S(r,w_2).\end{aligned}$$

Similarly,

$$\begin{aligned} T(r,w_1)-\overline{N}(r,w_1) &\le \beta\overline{N}(r+[2(l+1)+1]c,w_2)+\frac{(l+1)\alpha}{\lambda_1(t+1)}\overline{N}(r+[2(l+1)+1]c,w_1)\\ &\quad -(l+1)\overline{N}(r,w_2)+\frac{(l+1)\beta}{\lambda_2(t+1)}\overline{N}(r,w_2(z))-(l+1)\overline{N}(r,w_1(z))\\ &\quad +S(r,w_1)+S(r,w_2). \end{aligned}$$

The above two inequalities shows that (4.10) holds for $m=l+1$. We complete the induction.

Applying Lemma 3.7, and using (4.10), we obtain

$$\begin{cases} \overline{N}(r,w_1)+\overline{N}(r,w_2)\le(\frac{\alpha}{m}+\frac{\alpha}{\lambda_1(t+1)})\overline{N}(r,w_1)+\frac{\beta}{\lambda_2(t+1)}\overline{N}(r,w_2)+S(r,w_1)+S(r,w_2),\\ \overline{N}(r,w_1)+\overline{N}(r,w_2)\le(\frac{\beta}{m}+\frac{\beta}{\lambda_2(t+1)})\overline{N}(r,w_2)+\frac{\alpha}{\lambda_1(t+1)}\overline{N}(r,w_1)+S(r,w_1)+S(r,w_2). \end{cases} \tag{4.11}$$

Noting that $\alpha,\beta\in[0,\lambda'(t+1)),\lambda'=\min\{\lambda_1,\lambda_2\}$. Let m be large enough such that

$$\frac{1}{\eta_1}:=\frac{\alpha}{m}+\frac{\alpha}{\lambda_1(t+1)}=\alpha(\frac{1}{m}+\frac{1}{\lambda_1(t+1)})<1,\quad \frac{\beta}{\lambda_2(t+1)}<1.$$

$$\frac{1}{\eta_2}:=\frac{\beta}{m}+\frac{\beta}{\lambda_2(t+1)}=\beta(\frac{1}{m}+\frac{1}{\lambda_2(t+1)})<1,\quad \frac{\alpha}{\lambda_1(t+1)}<1.$$

By (4.11), we have

$$\begin{cases} (1-\frac{1}{\eta_1})\overline{N}(r,w_1)+(1-\frac{\beta}{\lambda_2(t+1)})\overline{N}(r,w_2)\le S(r,w_1)+S(r,w_2),\\ (1-\frac{1}{\eta_2})\overline{N}(r,w_2)+(1-\frac{\alpha}{\lambda_1(t+1)})\overline{N}(r,w_1)\le S(r,w_1)+S(r,w_2). \end{cases} \tag{4.12}$$

Using (4.12), for m large enough, we conclude that

$$\overline{N}(r,w_1)=S(r,w_1)+S(r,w_2).$$

$$\overline{N}(r,w_2)=S(r,w_1)+S(r,w_2).$$

Applying Lemma 3.7, and using (4.3) and (4.4), we have

$$T(r,w_1)=S(r,w_1)+S(r,w_2).$$

$$T(r,w_2)=S(r,w_1)+S(r,w_2).$$

Thus

$$[1+o(1)]T(r,w_1)=S(r,w_2).$$

$$[1+o(1)]T(r,w_2)=S(r,w_1).$$

Therefore

$$[1+o(1)]T(r,w_1)T(r,w_2)=S(r,w_1)S(r,w_2).$$

Then we obtain $1=0$, which is a contradiction. Therefore, we conclude that at least one of $\rho(w_1)=+\infty,\rho(w_2)=+\infty$ will be true.

This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. Suppose that $(w_1(z), w_2(z))$ is a set of transcendental meromorphic solution of (2.4). Applying Lemma 3.2, Lemma 3.4 and Lemma 3.8, and the first equation of (2.4), we have

$$\begin{aligned}(1-\varepsilon_2)T(\mu r^{\overline{d}}, w_2) &\leq T(r, w_2(pz)) \\ &= T(r, \frac{\Omega_1(z,w_1)}{\Omega_2(z,w_1)}) \\ &\leq \sum_{k=1}^{n}\lambda_{1k}T(r, w_1^{(t)}(z+c_k)) + S(r,w_1) \\ &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)T(r, w_1(z+c_k)) + S(r,w_1) \\ &\leq \sum_{k=1}^{n}\lambda_{1k}(t+1)T(r+c, w_1(z)) + S(r,w_1),\end{aligned}$$

where $\varepsilon_2 > 0$ is arbitrarily small.

For every $\beta_1 > 1$, and for r large enough, we obtain

$$T(r+c, w_1) \leq T(\beta_1 r, w_1).$$

Suppose that r to be large enough, outside of a possible exceptional set with finite logarithmic measure, we conclude that

$$(1-\varepsilon_2)T(\mu r^{\overline{d}}, w_2) \leq \lambda_1(t+1)(1+\overline{\varepsilon}_1)T(\beta_1 r, w_1),$$

where $\overline{\varepsilon}_1 > 0$ is arbitrarily small.

By Lemma 3.9, whenever $\gamma_1 > 1$, for all r large enough, we obtain

$$(1-\varepsilon_2)T(\mu r^{\overline{d}}, w_2) \leq \lambda_1(t+1)(1+\overline{\varepsilon}_1)T(\beta_1\gamma_1 r, w_1). \tag{4.13}$$

Similarly,

$$(1-\varepsilon_1)T(\mu r^{\overline{d}}, w_1) \leq \lambda_2(t+1)(1+\overline{\varepsilon}_2)T(\beta_2\gamma_2 r, w_2). \tag{4.14}$$

Denote $\overline{\beta} = \max\{\beta_1, \beta_2\}$, $\overline{\gamma} = \max\{\gamma_1, \gamma_2\}$, $\overline{\varepsilon} = \max\{\varepsilon_1, \varepsilon_2, \overline{\varepsilon}_1, \overline{\varepsilon}_2\}$. Then (4.13), (4.14) may become

$$(1-\overline{\varepsilon})T(\mu r^{\overline{d}}, w_2) \leq \lambda_1(t+1)(1+\overline{\varepsilon})T(\overline{\beta}\overline{\gamma} r, w_1). \tag{4.15}$$

$$(1-\overline{\varepsilon})T(\mu r^{\overline{d}}, w_1) \leq \lambda_2(t+1)(1+\overline{\varepsilon})T(\overline{\beta}\overline{\gamma} r, w_2). \tag{4.16}$$

Let $\overline{t} = \overline{\beta}\overline{\gamma} r$, then the above two inequalities become

$$T(\frac{\mu}{(\overline{\beta}\overline{\gamma})^{\overline{d}}}\overline{t}^{\overline{d}}, w_2) \leq \frac{\lambda_1(t+1)(1+\overline{\varepsilon})}{1-\overline{\varepsilon}}T(\overline{t}, w_1). \tag{4.17}$$

$$T(\frac{\mu}{(\overline{\beta}\overline{\gamma})^{\overline{d}}}\overline{t}^{\overline{d}}, w_1) \leq \frac{\lambda_2(t+1)(1+\overline{\varepsilon})}{1-\overline{\varepsilon}}T(\overline{t}, w_2). \tag{4.18}$$

Noting that $\overline{\lambda} = \max\{\lambda_1, \lambda_2\}$, by means of Lemma 3.10, we obtain

$$T(r, w_1) = O((\log r)^s),$$

$$T(r, w_2) = O((\log r)^s),$$

where $s = \dfrac{\log \frac{(t+1)^2\overline{\lambda}(1+\overline{\varepsilon})}{1-\overline{\varepsilon}}}{\log \overline{d}} = \dfrac{\log(t+1)^2\overline{\lambda}}{\log \overline{d}} + o(1)$. Let $\alpha = \dfrac{\log(t+1)^2\overline{\lambda}}{\log \overline{d}}$.

This completes the proof of Theorem 2.3.

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A note on a certain kind of nonlinear difference equations

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Abstract: In this paper, we mainly investigate a certain type of difference equation of the form $f^n(z) + p(z)(\Delta f)^m = r(z)e^{q(z)}$, where $p(z), r(z), q(z)$ are nonzero polynomials and n, m are two positive integers satisfying $n > m$. Some examples are also structured to show that our results are sharp.

Key words and phrases: meromorphic; difference equation; small function.

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1 Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane **C**. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory (see [5, 7, 11, 12]):

$$T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \cdots.$$

And we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. A polynomial $Q(z, f)$ is called a difference polynomial in f if Q is a polynomial in f, its derivatives and shifts with small meromorphic coefficients, say $\{a_\lambda | \lambda \in I\}$, such that $T(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. We define the difference operator $\Delta f = f(z + 1) - f(z)$.

One of the most important results in the value distribution theory is the following theorem due to Hayman.

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Theorem 1 *If g is a transcendental meromorphic function, then either g itself assumes every finite complex value infinitely often, or $g^{(k)}$ assumes every finite non-zero value infinitely often.*

As a consequence of Theorem 1, we have

Theorem 2 *If f is a transcendental entire function, then f^2+af' has infinitely many zeros for each finite non-zero complex value a.*

In fact, if f is an entire function, then $g=\frac{1}{f}$ has not any zero. It follows from Theorem 1 that $g'-\frac{1}{a}$ has infinitely many zeros, namely f^2+af' has infinitely many zeros.

It is well known that Δf can be considered as the difference counterpart of f'. The difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been established recently (see [1, 2, 3, 4, 6], which brings about a number of papers focusing on difference topics. And so here one nature question arise, that is what can be said if we replace f^2+af' with $f^2+a\Delta f$ in Theorem 2? Here we shall deal with this problem and obtain the following main result.

Theorem 3 *If f is a transcendental entire solution of finite order of the following non-linear difference equation*

$$f^2(z)+p(z)\Delta f=r(z)e^{q(z)}, \tag{1}$$

where $p(z), r(z), q(z)$ are nonzero polynomials such that $\deg p(z)\le 1$, then

$$\Delta f\equiv 0,$$

and f must be of the form

$$f(z)=ce^{2k\pi iz},$$

where $c\neq 0$ and $k\in Z$.

Example 1 *For the following non-linear difference equation*

$$f^2(z)+(z-1)^2\Delta f=\big(z(z-1)\big)^2e^{4\pi iz},$$

it admits a finite order transcendental entire solution

$$f(z)=z(z-1)e^{2\pi iz}-(z-1).$$

But $\Delta f\not\equiv 0$.

This example shows that the assumption $\deg p(z)\le 1$ is necessary for our result in Theorem 3. And from Theorem 3, we also obtain the following corollary corresponding to Theorem 2.

Corollary 1 *Let f be a transcendental entire function of finite order and $\Delta f \not\equiv 0$, then $f^2(z) + p(z)\Delta f$ has infinitely many zeros, where $p(z)$ is a nonzero polynomial whose degree is at most 1.*

This corollary can be regarded as the general case of the following result (see Theorem 1.1 in [9]) due to Liu and Laine in some sense.

Theorem 4 *[9] Let f be a transcendental entire function of finite order ρ, not of period c, where c is a nonzero complex constant. Then the difference polynomial $f^n(z) + f(z+c) - f(z)$ has infinitely many zeros in the complex plane, provided that $n \geq 2$.*

In 1970, C. C. Yang [13] obtained the following well known theorem.

Theorem 5 *Let m, n be two positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there are no transcendental entire solutions $f(z)$ and $g(z)$ satisfying the equation*

$$a(z)f^n(z) + b(z)g^m(z) = 1$$

with $a(z), b(z)$ being small functions of $f(z)$.

People have obtained quite a number of results by considering special functions f, g in Theorem 5. For example, J. Zhang [14] obtained the following result.

Theorem 6 *For the following difference equation*

$$f^n(z) + f^m(z+1) = p(z),$$

where p(z) is a nonzero polynomial with $\deg p(z) = k$, suppose it admits a transcendental entire function $f(z)$ of finite order. Then holds
(i) $m = n = 2, p(z)$ is a nonzero constant and $f(z)$ has form of $f(z) = ae^{Az} + be^{-Az}$, where $e^A = -i$ and a, b are two constants such that $4ab = p$.
(ii) $m = n = 1$ and $f^{(k+1)}(z)$ is a periodic entire function with period 2.

Here we consider the non-linear difference equation of the following form

$$f^n(z) + p(z)(\Delta f)^m = r(z)e^{q(z)}, \tag{2}$$

where $p(z), r(z), q(z)$ are nonzero polynomials and $n > m$, and obtain the following theorem, which can be considered as the more general case in Theorem 3.

Theorem 7 *If equation (2) admits a transcendental entire solution f with finite order such that $\Delta f \not\equiv 0$, then $n = 2$ and $m = 1$.*

Example 2 *For the following non-linear difference equation*

$$f(z) + \Delta f = ee^z,$$

it admits a finite order transcendental entire solution

$$f(z) = e^z.$$

But $\Delta f \not\equiv 0$.

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Example 3 *For the following non-linear difference equation*

$$f(z) - \frac{1}{4}(\Delta f)^2 = e^{\pi i z},$$

it admits a finite order transcendental entire solution

$$f(z) = e^{2\pi i z} + e^{\pi i z}.$$

But $\Delta f \not\equiv 0$.

Examples 2-3 show that the assumption $n > m$ is necessary for our result in Theorem 7. Combining Theorem 3 and Theorem 7, we can obtain the following corollary.

Corollary 2 *For the non-linear difference equation of the form*

$$f^n(z) + p(z)(\Delta f)^m = r(z)e^{q(z)}, \tag{3}$$

where $p(z), r(z), q(z)$ *are nonzero polynomials satisfying* $\deg p(z) \leq 1$ *and* n, m *are two positive integers satisfying* $n > m$, *the equation (3) admits no finite order transcendental entire solution* f *such that* $\Delta f \not\equiv 0$.

2 Some lemmas

To prove our results, we need some lemmas as follows.

Lemma 1 *(see[1]) Let* $f(z)$ *be a transcendental meromorphic function with finite order* σ. *Then for each* $\varepsilon > 0$, *we have*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 1 has another form as follows.

Lemma 2 *(see [3]) Let* f *be a meromorphic function with a finite order* σ, *and* η *be a nonzero constant. Then*

$$m(r, \frac{f(z+\eta)}{f(z)}) = S(r, f).$$

Lemma 3 *(see [10]) Let* f *be a transcendental meromorphic function and*

$$F = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0 \ (a_n \not\equiv 0)$$

be a polynomial in f *with coefficients being small functions of* f. *Then either*

$$F = a_n(f + \frac{a_{n-1}}{na_n})^n \quad or \quad T(r, f) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, f) + S(r, f).$$

Lemma 4 *(see[8]) Let $f(z)$ be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$H(z,f)P(z,f) = Q(z,f),$$

where $H(z,f), P(z,f), Q(z,f)$ are difference polynomials in $f(z)$ such that the total degree of $H(z,f)$ in $f(z)$ and its shifts is n and that the corresponding total degree of $Q(z,f)$ is at most n. If $H(z,f)$ just contains one term of maximal total degree, then for any $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\sigma-1+\varepsilon}) + S(r,f)$$

holds possibly outside of an exceptional set of finite logarithmic measure.

Remark 1 *From Lemmas 1-2, we can obtain $m\big(r, P(z,f)\big) = S(r,f)$ in Lemma 4.*

3 The proofs of main theorems

1. Proof of theorem 3.
First of all, suppose equation (1) admits a transcendental entire solution f with finite order. We may assume $q(z)$ is not any constant. Otherwise if $q(z)$ is a constant, then we rewrite equation (1) as the following form

$$f^2 = re^q - p\Delta f.$$

By Lemma 2, we see

$$2T(r,f) = m(r,f^2) = m(r,\Delta f) + S(r,f) \leq m(r,f) + S(r,f),$$

which is impossible. By differentiating equation (1) and eliminating $e^{q(z)}$, we obtain

$$f[2f' - (\frac{r'}{r} + q')f] + p\Delta f' + p'\Delta f - p(\frac{r'}{r} + q')\Delta f = 0. \quad (4)$$

Set $H = 2f' - Bf$, where $B = \frac{r'}{r} + q'$. Since $q(z)$ is not any constant, we see B is a nonzero rational function with $\deg_\infty B \geq 0$, specially, $\lim_{z\to\infty} B(z)$ is nonzero constant or ∞. Thus we rewrite equation (4) as the following form.

$$fH + p\Delta f' + (p' - Bp)\Delta f = 0. \quad (5)$$

By applying Lemma 4 to equation (5), we see

$$m(r,H) = S(r,f),$$

which means $T(r,H) = S(r,f)$. From the definition of H, we get

$$f' = \frac{1}{2}(H + Bf). \quad (6)$$

It follows equation (6) that

$$\Delta f' = \frac{1}{2}[\Delta H + B(z+1)\Delta f + \Delta B \cdot f]. \tag{7}$$

From equation (5), we see

$$\Delta f' = -\frac{1}{p}[Hf + (p' - pB)\Delta f]. \tag{8}$$

If $pB(z+1) + 2(p' - pB) \equiv 0$, then

$$1 \leftarrow \frac{B(z+1)}{B} = 2(1 - \frac{p'}{p}\frac{1}{B}) \to 2, \text{ as } z \to \infty,$$

which is impossible. Thus we can assume $pB(z+1) + 2(p' - pB) \not\equiv 0$. By eliminating $\Delta f'$ in equations (7)- (8), we get

$$\Delta f = a_1 f + a_0, \tag{9}$$

where

$$a_1 = -\frac{2H + p\Delta B}{pB(z+1) + 2(p' - pB)} \text{ and } a_0 = -\frac{p\Delta H}{pB(z+1) + 2(p' - pB)}$$

are two small functions of f. Substituting equation (9) into equation (1), we get

$$f^2(z) + p(z)a_1(z)f + p(z)a_0(z) = r(z)e^{q(z)}.$$

That is to say $f^2(z) + p(z)a_1(z)f + p(z)a_0(z)$ has just only finitely many zeros. It follows from Lemma 3 that there exists a small function β with respect to f such that

$$f^2(z) + p(z)a_1(z)f + p(z)a_0(z) = (f + \beta)^2 = r(z)e^{q(z)}. \tag{10}$$

From equation (10), we get $pa_1 = 2\beta$, $pa_0 = \beta^2$ and

$$f = Re^Q - \beta, \tag{11}$$

where $R = \sqrt{r}$ and $Q = \frac{q}{2}$ are two nonzero polynomials. Thus from (11), we get β is an entire function and

$$\Delta f = [R(z+1)e^{\Delta Q} - R]e^Q - \Delta\beta. \tag{12}$$

Thus from (9), (11) and (12), we obtain

$$[R(z+1)e^{\Delta Q} - R - \frac{2\beta R}{p}]e^Q = \Delta\beta - \frac{\beta^2}{p}. \tag{13}$$

It is obvious that $T(r, f) = T(r, e^Q) + S(r, f)$ from equation (11), which means $R(z+1)e^{\Delta Q} - R - \frac{2\beta R}{p}$ and $\Delta\beta - \frac{\beta^2}{p}$ are small functions of e^Q. Therefore from equation (13), we see

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$$\Delta\beta - \frac{\beta^2}{p} = R(z+1)e^{\Delta Q} - R - \frac{2\beta R}{p} = 0. \tag{14}$$

Thus $p\Delta\beta = \beta^2$, where $\beta = Re^Q - f$ is an entire function. If β is a transcendental entire function, then from Lemma 2, we see

$$2T(r,\beta) = m(r,\beta^2) = m(r,\Delta\beta) + S(r,\beta) \le m(r,\beta) + S(r,\beta),$$

which is impossible. If β is a polynomial, then

$$2\deg\beta = \deg\beta^2 = \deg(p\Delta\beta) = \deg p + \deg\Delta\beta = \deg p + \deg\beta - 1,$$

which implies $\deg\beta = \deg p - 1 \le 0$. Thus it follows from equation (14) that $\beta \equiv 0$ and $R(z+1)e^{\Delta Q} = R$. It means $e^{\Delta Q}$ is a constant, which leads to $Q(z) = mz + n$. Then

$$e^m = e^{\Delta Q} = \frac{R}{R(z+1)} \to 1, \text{ as } z \to \infty.$$

Therefore $R(z) = R(z+1)$, that is to say R is a constant. By $pa_1 = 2\beta$, $pa_0 = \beta^2$, we see $a_1 = a_0 = 0$, which means $\Delta f = 0$ from equation (9). Thus we have $f = e^{mz+n} = ce^{mz}$ and then $\Delta f = c(e^m - 1)e^{mz}$, which implies $m = 2k\pi i, k \in Z$.
The proof of Theorem 3 is completed.

2. The Proof of Theorem 7.
First of all, suppose equation (2) admits a transcendental entire solution f with finite order. We may assume $q(z)$ is not any constant. Otherwise if $q(z)$ is a constant, then we rewrite equation (2) as the form

$$f^n = re^q - p(\Delta f)^m.$$

By Lemma 2, we see

$$nT(r,f) = m(r,f^n) = mm(r,\Delta f) + S(r,f) \le mm(r,f) + S(r,f),$$

which is impossible when $n > m$. By differentiating equation (2) and eliminating $e^{q(z)}$, we obtain

$$f^{n-1}[nf' - Bf] = (Bp - p')(\Delta f)^m - mp(\Delta f)^{m-1}\Delta f', \tag{15}$$

where B is defined as same as in Theorem 3. Set $H = nf' - Bf$. If $H \equiv 0$, then f must be form of

$$f(z) = cR(z)e^{Q(z)}, \tag{16}$$

where $R = \sqrt[n]{r}$ and $Q = \frac{q}{n}$ are two polynomials. From equation (16), we see

$$\Delta f = Ae^{Q(z)}, \tag{17}$$

where $A = c\big(R(z+1)e^{\Delta Q} - R\big)$. It is obvious that $T(r, A) = S(r, e^Q)$. By our assumption $\Delta f \not\equiv 0$, we see $A \not\equiv 0$. Substituting equations (16)-(17) into equation (2), we see

$$(c^n - 1)R^n = -pA^m e^{(m-n)Q},$$

which contradicts our assumption that q is a nonconstant polynomial. Thus $H \not\equiv 0$. Next we shall consider the following two cases separately to our discussion.

Case 1 $n > m+1$. By applying Lemma 4 to equation (15), we see

$$m(r, H) = S(r, f)$$

and

$$m(r, Hf) = S(r, f),$$

From the two equations above, we obtain

$$T(r, f) = m(r, f) \leq m(r, Hf) + m(r, \frac{1}{H}) \leq S(r, f) + m(r, H) = S(r, f),$$

which is impossible.

Case 2 n=m+1. We rewrite equation (2) as the following form

$$\frac{1}{r}\big(fe^{-\frac{q}{n}}\big)^n + \frac{p}{r}\big(e^{-\frac{q}{m}}\Delta f\big)^m = 1.$$

If $m > 1$, then

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{m} + \frac{1}{m+1} \leq \frac{1}{2} + \frac{1}{3} < 1.$$

From Theorem 4, we obtain that $fe^{-\frac{q}{n}}$ and $e^{-\frac{q}{m}}\Delta f$ are two polynomials. Thus

$$f = se^{\frac{q}{n}} \tag{18}$$

and

$$\Delta f = te^{\frac{q}{m}}, \tag{19}$$

where s, t are two nonzero polynomials. From equation (18), we see

$$\Delta f = \big(s(z+1)e^{\frac{\Delta q}{n}} - s\big)e^{\frac{q}{n}}. \tag{20}$$

It follows from equations (19)-(20) that

$$s(z+1)e^{\frac{\Delta q}{n}} - s = te^{(\frac{1}{m} - \frac{1}{n})q},$$

which is impossible. Thus $m = 1$ and $n = 2$.

The proof of Theorem 7 is completed.

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A FIXED POINT APPROACH TO THE STABILITY OF QUADRATIC (ρ_1, ρ_2)-FUNCTIONAL INEQUALITIES IN MATRIX BANACH SPACES

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Abstract. By using the fixed point method, we solve the Hyer-Ulam stability of the following quadratic (ρ_1, ρ_2)-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+|\rho_2|<1$, and

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\right\|, \end{aligned} \tag{0.2}$$

where ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+2|\rho_2|<1$, in matrix Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [30] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [29] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

Park [17, 18] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 7, 10, 16, 19, 20, 23, 24, 25, 26, 27, 28, 31, 32]).

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 9] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$,*

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either

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 21]).

We will use the following notations:
$M_n(X)$ is the set of all $n \times n$-matrices in X;
$e_j \in M_{1,n}(\mathbb{C})$ is that j-th component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{C})$ is that (i, j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that (i, j)-component is x and the other components are zero;
For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space. A matrix Banach space $(X, \{\|\cdot\|_n\}$ is called a matrix Banach algebra if X is an algebra.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an L^∞-*matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$ (see [14]).

Lemma 1.2. ([14]) *Let* $(X, \{\|.\|_n\})$ *be a matrix normed space.*

(1) $\|E_{kl} \otimes x\|_n = \|x\|$ for $x \in X$.
(2) $\|x_{kl}\| \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^n \|x_{ij}\|$ for $[x_{ij}] \in M_n(X)$.
(3) $lim_{n\to\infty} x_n = x \ if \ and \ only \ if \ lim_{n\to\infty} x_{nij} = x_{ij} \ \ for \ x_n = [x_{nij}], x = [x_{ij}] \in M_k(X)$.

In Section 2, we solve the quadratic (ρ_1, ρ_2)-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.1) in matrix Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic (ρ_1, ρ_2)-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.2) in matrix Banach spaces by using the fixed point method.

Throughout this paper, let X be a real or complex matrix normed space with norm $\|\cdot\|_n$ and Y a complex matrix Banach space with norm $\|\cdot\|_n$.

2. Quadratic (ρ_1, ρ_2)-functional inequality (0.1) in matrix normed spaces

Throughout this section, assume that ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + |\rho_2| < 1$.

In this section, we solve and investigate the quadratic (ρ_1, ρ_2)-functional inequality (0.1) in matrix Banach spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \end{aligned} \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &= \left\|\frac{\rho_1}{2} (f(x+y) + f(x-y) - 2f(x) - 2f(y))\right\| \\ &+ \|\rho_2 (f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \\ &= \left(\frac{|\rho_1|}{2} + |\rho_2|\right) \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

for all $x, y \in X$. Since $\frac{|\rho_1|}{2} + |\rho_2| < 1$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Thus f is quadratic. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.1) in matrix Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4} \varphi(x, y) \tag{2.3}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\ &\leq \left\|\rho_1 \left(2f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij} - y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\ &+ \left\|\rho_2 \left(4f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + f_{ij}([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\right)\right\|_n + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}) \end{aligned} \tag{2.4}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{L}{4(1-L)} \varphi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Putting $n = 1$ in (2.4), we get

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| + \varphi(x,y) \end{aligned} \tag{2.5}$$

for all $x, y \in X$.

Letting $y = x$ in (2.5), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x,x) \tag{2.6}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x,x), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [15]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \leq 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq 4\varepsilon\frac{L}{4}\varphi(x,x) = L\varepsilon\varphi(x,x)$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.6) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{4}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{2.7}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \quad \leq \quad \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{4(1-L)}\varphi(x, x) \tag{2.8}$$

for all $x \in X$.

It follows from (2.3) and (2.5) that

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\
&\quad = \lim_{m\to\infty} 4^m \left\|f\left(\frac{x+y}{2^m}\right) + f\left(\frac{x-y}{2^m}\right) - 2f\left(\frac{x}{2^m}\right) - 2f\left(\frac{y}{2^m}\right)\right\| \\
&\quad \leq \lim_{m\to\infty} 4^m |\rho_1| \left\|2f\left(\frac{x+y}{2^{m+1}}\right) + 2f\left(\frac{x-y}{2^{m+1}}\right) - f\left(\frac{x}{2^m}\right) - f\left(\frac{y}{2^m}\right)\right\| \\
&\quad + \lim_{m\to\infty} 4^m |\rho_2| \left\|4f\left(\frac{x+y}{2^{m+1}}\right) + f\left(\frac{x-y}{2^m}\right) - 2f\left(\frac{x}{2^m}\right) - 2f\left(\frac{y}{2^m}\right)\right\| + \lim_{m\to\infty} 4^m \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \\
&\quad = \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&\quad + \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\
&\quad \leq \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&\quad + \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

It follows from Lemma 1.2 and (2.8) that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \|f(x_{ij}) - Q(x_{ij})\| \leq \sum_{i,j=1}^{n} \frac{L}{4(1-L)}\varphi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. □

Corollary 2.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|f_n([x_{ij}+y_{ij}]) + f_n([x_{ij}-y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \qquad (2.9)\\ &\le \left\|\rho_1\left(2f_n\left(\frac{[x_{ij}+y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij}-y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\ &+ \left\|\rho_2\left(4f_n\left(\frac{[x_{ij}+y_{ij}]}{2}\right) + f_n([x_{ij}-y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\right)\right\|_n + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \end{aligned}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 4}\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{1}{4(1-L)}\varphi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r}\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

Remark 2.6. If ρ is a real number such that $\frac{|\rho_1|}{2} + |\rho_2| < 1$ and Y is a real matrix Banach algebra, then all the assertions in this section remain valid.

3. Quadratic (ρ_1, ρ_2)-functional inequality (0.2) in matrix normed spaces

Throughout this section, assume that ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$.

In this section, we solve and investigate the quadratic (ρ_1, ρ_2)-functional inequality (0.2) in matrix Banach spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \|\rho_2 \left(2f\left(x+y\right) + 2f\left(x-y\right) - f(2x) - f(2y)\right)\| \end{aligned} \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Letting $y = x$ in (3.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &\leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \|\rho_2 \left(2f\left(x+y\right) + 2f\left(x-y\right) - f(2x) - f(2y)\right)\| \\ &= \left\|\frac{\rho_1}{2} \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &+ \|2\rho_2 \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \\ &= \left(\frac{|\rho_1|}{2} + 2|\rho_2|\right) \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

for all $x, y \in X$. Since $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. ☐

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.2) in matrix Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4}\varphi\left(x, y\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\ &\leq \left\|\rho_1 \left(2f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij} - y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\ &+ \|\rho_2 \left(2f_n\left([x_{ij} + y_{ij}]\right) + 2f_n\left([x_{ij} - y_{ij}]\right) - f_n(2[x_{ij}]) - f_n(2[y_{ij}])\right)\|_n + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}) \end{aligned} \tag{3.3}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{L}{4(1-L)}\varphi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Putting $n = 1$ in (3.3), we get

$$\begin{aligned}
&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\
&+ \|\rho_2\left(2f(x+y) + 2f(x-y) - f(2x) - f(2y)\right)\| + \varphi(x,y)
\end{aligned} \tag{3.4}$$

for all $x, y \in X$.

Letting $y = x$ in (3.4), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x,x) \tag{3.5}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\
&\le \left\|\rho_1\left(2f_n\left(\frac{[x_{ij} + y_{ij}]}{2}\right) + 2f_n\left(\frac{[x_{ij} - y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right)\right\|_n \\
&+ \left\|\rho_2\left(2f_n([x_{ij} + y_{ij}]) + 2f_n([x_{ij} - y_{ij}]) - f_n(2[x_{ij}]) - f_n(2[y_{ij}])\right)\right\|_n + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)
\end{aligned} \tag{3.6}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 4}\|x_{ij}\|_n^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{1}{4(1-L)} \varphi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

It follows from (3.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{4} \varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r} \|x_{ij}\|_n^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

Remark 3.6. If ρ is a real number such that $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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