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Journal of Computational Analysis and Applications

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Contribution to a Book

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FOURIER SERIES OF SUMS OF PRODUCTS OF ORDERED BELL AND EULER FUNCTIONS

TAEKYUN KIM[1], DAE SAN KIM[2], GWAN-WOO JANG[3], AND JIN-WOO PARK[4,*]

Abstract. In this paper, we will study three types of sums of products of ordered Bell and Euler functions and derive their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions.

1. Introduction

As a natural companion to *ordered Bell numbers*, the ordered Bell polynomials $b_n(x)$ were defined by the generating function (see [8])

$$\frac{1}{2-e^t}e^{xt}=\sum_{m=0}^{\infty}b_m(x)\frac{t^m}{m!}. \tag{1.1}$$

The first few ordered Bell polynomials are as follows:

$$\begin{gathered}b_0(x)=1,\ b_1(x)=x+1,\ b_2(x)=x^2+2x+3,\\ b_3(x)=x^3+3x^2+9x+13,\ b_4(x)=x^4+4x^3+18x^2+52x+75,\\ b_5(x)=x^5+5x^4+30x^3+130x^2+375x+541.\end{gathered}$$

The *ordered Bell numbers* $b_m=b_m(0)$ were introduced already in 1859 work of Cayley and have been studied in many counting problems in enumerative combinatorics and number theory (see [2-5,11,13,14]). They are all positive integers, as we can see, for example, from

$$b_m=\sum_{n=0}^{m}n!S_2(m,n)=\sum_{n=0}^{\infty}\frac{n^m}{2^{n+1}},\ (m\geq 0).$$

The ordered Bell polynomial $b_m(x)$ has degree m by (1.1) and is a monic polynomial with integral coefficients, as we see from

$$b_0(x)=1,\ b_m(x)=x^m+\sum_{l=0}^{m-1}\binom{m}{l}b_l(x),\ (m\geq 1).$$

From (1.1), we can derive

$$\begin{gathered}\frac{d}{dx}b_m(x)=mb_{m-1}(x),\ (m\geq 1),\\ -b_m(x+1)+2b_m(x)=x^m,\ (m\geq 0).\end{gathered}$$

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In turn, from these we obtain

$$\begin{aligned} -b_m(1) + 2b_m &= \delta_{m,0}, \ (m \geq 0), \\ \int_0^1 b_m(x)dx =& \frac{1}{m+1}(b_{m+1}(1) - b_{m+1}) \\ =& \frac{1}{m+1} b_{m+1}. \end{aligned}$$

The *Euler polynomials* $E_m(x)$ are given by the generating function

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_m(x) \frac{t^m}{m!}.$$

We recall here that the Euler polynomials satisfy

$$E_m(x+1) + E_m(x) = 2x^m, \ (m \geq 0),$$

and hence

$$E_m(1) + E_m = 2\delta_{m,0}, \ (m \geq 0).$$

The *Bernoulli polynomials* $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

For any real number x, we let

$$\langle x \rangle = x - \lfloor x \rfloor \in [0, \ 1)$$

denote the fractional part of x.

In this paper, we will study three types of sums of products of ordered Bell and Euler functions and derive their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions.

(1) $\alpha_m(\langle x \rangle) = \sum_{k=1}^{m} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), (m \geq 1)$;
(2) $\beta_m(\langle x \rangle) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), (m \geq 1)$;
(3) $\gamma_m(\langle x \rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), (m \geq 2)$.

The reader may refer to any book (for example, see [1,12,15]) for elementary facts about Fourier analysis.

For later use, we recall the following facts about Bernoulli functions $B_m(\langle x \rangle)$:

(a) for $m \geq 2$,

$$B_m(\langle x \rangle) = -m! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m},$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0 & \text{for } x \in \mathbb{Z}. \end{cases}$$

Finally, the reader may refer to the recent works [6,7,9,10] related with this paper.

2. Fourier series of functions of the first type

Let

$$\alpha_m(x)=\sum_{k=0}^{m} b_k(x)E_{m-k}(x),\ (m\geq 1).$$

Then we will investigate the function

$$\alpha_m(\langle x\rangle)=\sum_{k=0}^{m} b_k(\langle x\rangle)E_{m-k}(\langle x\rangle),\ (m\geq 1),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)}e^{2\pi inx},$$

where

$$\begin{aligned}A_n^{(m)}=&\int_0^1 \alpha_m(\langle x\rangle)e^{-2\pi inx}dx\\ =&\int_0^1 \alpha_m(x)e^{-2\pi inx}dx.\end{aligned}$$

Before proceeding further, we observe the following.

$$\begin{aligned}\alpha_m'(x)=&\sum_{k=0}^{m}\{kb_{k-1}(x)E_{m-k}(x)+(m-k)b_k(x)E_{m-k-1}(x)\}\\ =&\sum_{k=1}^{m}kb_{k-1}(x)E_{m-k}(x)+\sum_{k=0}^{m-1}(m-k)b_k(x)E_{m-k-1}(x)\\ =&\sum_{k=0}^{m-1}(k+1)b_k(x)E_{m-1-k}(x)+\sum_{k=0}^{m-1}(m-k)b_k(x)E_{m-1-k}(x)\\ =&(m+1)\alpha_{m-1}(x).\end{aligned}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)'=\alpha_m(x),$$

and

$$\int_0^1 \alpha_m(x)dx=\frac{1}{m+2}\left(\alpha_{m+1}(1)-\alpha_{m+1}(0)\right).$$

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For $m \geq 1$, we set

$$\begin{aligned}\Delta_m &= \alpha_m(1) - \alpha_m(0)\\ &= \sum_{k=0}^{m} (b_k(1)E_{m-k}(1) - b_k E_{m-k})\\ &= \sum_{k=0}^{m} \{(2b_k - \delta_{k,0})(-E_{m-k} + 2\delta_{m,k}) - b_k E_{m-k}\}\\ &= \sum_{k=0}^{m}(-3b_k E_{m-k} + 4b_k\delta_{m,k} + \delta_{k,0}E_{m-k} - 2\delta_{k,0}\delta_{m,k})\\ &= -3\sum_{k=0}^{m} b_k E_{m-k} + 4b_m + E_m\\ &= -3\sum_{k=0}^{m-1} b_k E_{m-k} + b_m + E_m.\end{aligned}$$

Now,

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0,$$

and

$$\begin{aligned}\int_0^1 \alpha_m(x)dx &= \frac{1}{m+2}\Delta_{m+1}\\ &= \frac{1}{m+2}\left(-3\sum_{k=0}^{m} b_k E_{m+1-k} + b_{m+1} + E_{m+1}\right).\end{aligned}$$

Next, we want to determine the Fourier coefficients $A_n^{(m)}$.

Case $1: n \neq 0$.

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m,\end{aligned}$$

from which by induction we can show that

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}.$$

Case $2: n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

$\alpha_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those positive integers with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers with $\Delta_m \neq 0$.

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Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\alpha_m(\langle x \rangle)$ converges uniformly to $\alpha_m(\langle x \rangle)$, and

$$\alpha_m(\langle x \rangle) = \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x}$$

$$= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} \left(-j! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right)$$

$$= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x \rangle)$$

$$+ \Delta_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Now, we can state our first theorem.

Theorem 2.1. *For each positive integer l, we let*

$$\Delta_l = -3 \sum_{k=0}^{l-1} b_k E_{l-k} + b_l + E_l.$$

Assume that $\Delta_m = 0$, for a positive integer m, Then we have the following.

(a) $\sum_{k=0}^{m} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle)$ *has the Fourier series expansion*

$$\sum_{k=0}^{m} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle)$$
$$= \frac{1}{m+2}\Delta_{m=1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=0}^{m} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle) = \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(\langle x \rangle),$$

for all $x \in \mathbb{R}$, where $B_j(\langle x \rangle)$ i s the Bernoulli function.

Assume next that $\Delta_m \neq 0$, for a positive integer m. Then $\alpha_m(0) \neq \alpha_m(1)$. Thus $\alpha_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x \rangle)$ converges pointwise to $\alpha_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m,$$

for $x \in \mathbb{Z}$.

Next, we can state our second theorem.

Theorem 2.2. *For each positive integer l, we let*

$$\Delta_l = -3\sum_{k=0}^{l-1} b_k E_{l-k} + b_l + E_l.$$

Assume that $\Delta_m \neq 0$, for a positive integer m, Then we have the following.

(a)

$$\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-\frac{1}{m+2}\sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1} \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} b_k(\langle x\rangle)E_{m-k}(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m} b_k E_{m-k} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m} \binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle)$$
$$= \sum_{k=0}^{m} b_k(\langle x\rangle)E_{m-k}(\langle x\rangle), \ \text{for } x \notin \mathbb{Z};$$

$$\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m} \binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle)$$
$$= \sum_{k=0}^{m} b_k E_{m-k} + \frac{1}{2}\Delta_m, \ \text{for } x \notin \mathbb{Z}.$$

3. Fourier series of functions of the second type

Let

$$\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(x)E_{m-k}(x), \ (m \geq 1).$$

Then we will consider the function

$$\beta_m(\langle x\rangle) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x\rangle)E_{m-k}(\langle x\rangle), \ (m \geq 1),$$

defined on $\mathbb{R}$, which is periodic with period 1.

Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(\langle x\rangle)e^{-2\pi i n x}dx$$
$$= \int_0^1 \beta_m(x)e^{-2\pi i n x}dx.$$

We need to note the following before proceeding further.

$$
\begin{aligned}
&\beta_m'(x) = \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}b_{k-1}(x)E_{m-k}(x) + \frac{m-k}{k!(m-k)!}b_k(x)E_{m-k-1}(x)\right\}\\
&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}b_{k-1}(x)E_{m-k}(x) + \sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}b_k(x)E_{m-k-1}(x)\\
&=\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}b_k(x)E_{m-1-k}(x) + \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}b_k(x)E_{m-1-k}(x)\\
&=2\beta_{m-1}(x).
\end{aligned}
$$

From this, we obtain

$$
\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x),
$$

and

$$
\int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\beta_{m+1}(1) - \beta_{m+1}(0)\right).
$$

For $m \geq 1$, we put

$$
\begin{aligned}
\Omega_m =&\beta_m(1) - \beta_m(0)\\
=&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(b_k(1)E_{m-k}(1) - b_kE_{m-k}\right)\\
=&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(\left(2b_k - \delta_{k,0}\right)\left(-E_{m-k} + 2\delta_{m,k}\right) - b_kE_{m-k}\right)\\
=&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(-3b_kE_{m-k} + 4b_k\delta_{m,k} + \delta_{k,0}E_{m-k} - 2\delta_{k,0}\delta_{m,k}\right)\\
=&-3\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_kE_{m-k} + \frac{4}{m!}b_m + \frac{1}{m!}E_m - \frac{2}{m!}\delta_{m,0}\\
=&-3\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}b_kE_{m-k} + \frac{1}{m!}b_m + \frac{1}{m!}E_m.
\end{aligned}
$$

From this, we get

$$
\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0,
$$

and

$$
\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.
$$

Next, we would like to determine the Fourier coefficients $B_n^{(m)}$.
Case $1: n \neq 0$.

$$
\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\beta_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m' e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\beta_m(1) - \beta_m(0)\right) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,
\end{aligned}
$$

from which by induction we can easily show

$$B_n^{(m)} = -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}.$$

Case 2 : $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise C^∞. Further, $\beta_m(\langle x\rangle)$ is continuous for those positive integers m with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers with $\Omega_m \neq 0$.

Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$
\begin{aligned}
\beta_m(\langle x\rangle) &= \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle) + \Omega_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}
$$

Now, we are ready to state our first result.

Theorem 3.1. *For each positive integer l,*

$$\Omega_l = -3\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!} b_k E_{l-k} + \frac{1}{l!} b_l + \frac{1}{l!} E_l.$$

Assume that $\Omega_m = 0$, for a positive integer m. Then we have the following.

(a)

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!} b_k(\langle x\rangle) E_{m-k}(\langle x\rangle)$$

has the Fourier series expansion

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle)$$
$$= \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the converges is uniform.

(b)

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle) = \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x \rangle),$$

for all $x \in \mathbb{R}$, where $B_j(\langle x \rangle)$ is the Bernoulli function.

Assume next that $\Omega_m \neq 0$, for a positive integer m. Then $\beta_m(0) \neq \beta_m(1)$. So $\beta_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(\langle x \rangle)$ converges pointwise to $\beta_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\beta_m(0) + \beta_m(1)\right) = \beta_m(0) + \frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

Now, we are ready to state our second result.

Theorem 3.2. *For each positive integer l, we let*

$$\Omega_l = -3\sum_{k=0}^{l-1} \frac{1}{k!(l-k)!} b_k E_{l-k} + \frac{1}{l!} b_l + \frac{1}{l!} E_l.$$

Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

(a)

$$\frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k E_{m-k} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x \rangle) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), \text{ for } x \notin \mathbb{Z};$$
$$\frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x \rangle) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k E_{m-k} + \frac{1}{2}\Omega_m, \text{ for } x \in \mathbb{Z}.$$

4. Fourier series of functions of the third type

Let

$$\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(x)E_{m-k}(x),\ (m\geq 2).$$

Then we will investigate the function

$$\gamma_m(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x\rangle)E_{m-k}(\langle x\rangle),\ (m\geq 2),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},$$

where

$$\begin{aligned}C_n^{(m)}&=\int_0^1\gamma_m(\langle x\rangle)e^{-2\pi inx}dx\\&=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.\end{aligned}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}\gamma_m'(x)=&\sum_{k=1}^{m-1}\frac{1}{m-k}b_{k-1}(x)E_{m-k}(x)+\sum_{k=1}^{m-1}\frac{1}{k}b_k(x)E_{m-k-1}(x)\\=&\sum_{k=0}^{m-2}\frac{1}{m-1-k}b_k(x)E_{m-1-k}(x)+\sum_{k=1}^{m-1}\frac{1}{k}b_k(x)E_{m-1-k}(x)\\=&\sum_{k=1}^{m-2}\left(\frac{1}{m-1-k}+\frac{1}{k}\right)b_k(x)E_{m-1-k}(x)+\frac{1}{m-1}E_{m-1}(x)+\frac{1}{m-1}b_{m-1}(x)\\=&(m-1)\gamma_{m-1}(x)+\frac{1}{m-1}E_{m-1}(x)+\frac{1}{m-1}b_{m-1}(x).\end{aligned}$$

Thus

$$\gamma_m'(x)=(m-1)\gamma_{m-1}(x)+\frac{1}{m-1}E_{m-1}(x)+\frac{1}{m-1}b_{m-1}.$$

From this, we see that

$$\left(\frac{1}{m}\left(\gamma_{m+1}(x)-\frac{1}{m(m+1)}E_{m+1}(x)-\frac{1}{m(m+1)}b_{m+1}(x)\right)\right)'=\gamma_m(x).$$

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$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
=&\frac{1}{m}\left[\gamma_{m+1}(x)-\frac{1}{m(m+1)}E_{m+1}(x)-\frac{1}{m(m+1)}b_{m+1)}(x)\right]_0^1 \\
=&\frac{1}{m}\bigg(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}\left(E_{m+1}(1)-E_{m+1}(0)\right) \\
&-\frac{1}{m(m+1)}\left(b_{m+1}(1)-b_{m+1}(0)\right)\bigg) \\
=&\frac{1}{m}\left(\gamma_{m+1}(1)-\gamma_{m+1}(0)+\frac{2}{m(m+1)}E_{m+1}-\frac{1}{m(m+1)}b_{m+1}\right).
\end{aligned}$$

For $m \geq 2$, we put

$$\begin{aligned}
\Lambda_m &= \gamma_m(1)-\gamma_m(0) \\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(b_k(1)E_{m-1}(1)-b_kE_{m-k}\right) \\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(\left(2b_k-\delta_{k,0}\right)\left(-E_{m-k}+2\delta_{m,k}\right)-b_kE_{m-k}\right) \\
&=-3\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_kE_{m-k}.
\end{aligned}$$

From this, we have

$$\gamma_m(0)=\gamma_m(1) \Longleftrightarrow \Lambda_m=0,$$

and

$$\int_0^1 \gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}E_{m+1}-\frac{1}{m(m+1)}b_{m+1}\right).$$

Next, we would like to determine the Fourier coefficients $C_n^{(m)}$. For this, we first note that

$$\int_0^1 E_l(x)e^{-2\pi inx}dx=\begin{cases} 2\sum_{k=1}^{l}\frac{(l)_{k-1}}{(2\pi in)^k}E_{l-k+1}, & \text{for } n\neq 0,\\ \frac{-2}{l+1}E_{l+1}, & \text{for } n=0,\end{cases}$$

$$\int_0^1 b_l(x)e^{-2\pi inx}dx=\begin{cases} -\sum_{k=1}^{l}\frac{(l)_{k-1}}{(2\pi in)^k}b_{l-k+1}, & \text{for } n\neq 0,\\ \frac{1}{l+1}b_{l+1}, & \text{for } n=0.\end{cases}$$

Case $1: \ n\neq 0$.

$$\begin{aligned}
&C_n^{(m)}=\int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
=&-\frac{1}{2\pi in}\left[\gamma_m(x)e^{-2\pi inx}\right]_0^1+\frac{1}{2\pi in}\int_0^1\gamma_m'(x)e^{-2\pi inx}dx \\
=&-\frac{1}{2\pi in}\left(\gamma_m(1)-\gamma_m(0)\right) \\
&+\frac{1}{2\pi in}\int_0^1\left\{(m-1)\gamma_{m-1}(x)+\frac{1}{m-1}E_{m-1}(x)+\frac{1}{m-1}b_{m-1}(x)\right\}e^{-2\pi inx}dx \\
=&\frac{m-1}{2\pi in}C_n^{(m-1)}-\frac{1}{2\pi in}\Lambda_m+\frac{2}{2\pi in(m-1)}\Theta_m-\frac{1}{2\pi in(m-1)}\Phi_m,
\end{aligned}$$

where

$$\Theta_m = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k} E_{m-k},$$

$$\Phi_m = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k} b_{m-k}.$$

From the recurrence relation

$$C_n^{(m)} = \frac{m-1}{2\pi in} C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Theta_m - \frac{1}{2\pi in(m-1)}\Phi_m,$$

and by induction on m, we can easily show that

$$C_n^{(m)} = -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} + 2\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1}.$$

We note here that

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
=&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)} \sum_{k=1}^{m-j} \frac{(m-j)_{k-1}}{(2\pi in)^k} E_{m-j-k+1} \\
=&\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}} E_{m-j-k+1} \\
=&\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{s=j+1}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s} E_{m-s+1} \\
=&\sum_{s=2}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s} E_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} \\
=&\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s} \frac{H_{m-1}-H_{m-s}}{m-s+1} E_{m-s+1}.
\end{aligned}$$

Putting everything altogether, we have

$$C_n^{(m)} = -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s}\left\{\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1}+b_{m-s+1})\right\}.$$

Case $2:\ n=0.$

$$\begin{aligned}
C_0^{(m)} =& \int_0^1 \gamma_m(x)dx \\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right).
\end{aligned}$$

$\gamma_m(\langle x\rangle)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(\langle x\rangle)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m \geq 2$ with $\Lambda_m \neq 0$.

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Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(\langle x \rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(\langle x \rangle)$ converges uniformly to $\gamma_m(\langle x \rangle)$, and

$$\begin{aligned}
&\gamma_m(\langle x \rangle) \\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right) \\
&+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right)\right\} e^{2\pi i n x} \\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right) \\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right)\left(-s!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right) \\
=&\frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right)B_s(\langle x \rangle) \\
&+ \Lambda_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we are ready to state our first result.

Theorem 4.1. *For each integer $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1}\frac{1}{k(l-k)}b_k E_{l-k},$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x \rangle)E_{m-k}(\langle x \rangle)$ *has Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x \rangle)E_{m-k}(\langle x \rangle) \\
=&\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right) \\
&+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right)\right\} e^{2\pi i n x},
\end{aligned}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x \rangle)E_{m-k}(\langle x \rangle) \\
=&\frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right)B_s(\langle x \rangle),
\end{aligned}$$

for all $x \in \mathbb{R}$, where $B_s(\langle x \rangle)$ is the Bernoulli function.

Assume next that m is an integer ≥ 2 with $\Lambda_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(\langle x \rangle)$ converges pointwise to $\gamma_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\gamma_m(0) + \gamma_m(1)\right) = \gamma_m(0) + \frac{1}{2}\Lambda_m,$$

for $x \in \mathbb{Z}$.

Now, we are ready to state our second result.

Theorem 4.2. *For each integer $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k E_{l-k},$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}E_{m+1} - \frac{1}{m(m+1)}b_{m+1}\right)$$
$$+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right)\right\} e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k E_{m-k} + \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m}\sum_{s=0}^{m} \binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right) B_s(\langle x \rangle)$$
$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) E_{m-k}(\langle x \rangle), \text{ for } x \notin \mathbb{Z};$$
$$\frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m} \binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(-2E_{m-s+1} + b_{m-s+1})\right) B_s(\langle x \rangle)$$
$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k E_{m-k} + \frac{1}{2}\Lambda_m, \text{ for } x \in \mathbb{Z}.$$

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FEKETE SZEGÖ PROBLEM FOR SOME SUBCLASSES OF MULTIVALENT NON-BAZILEVIČ FUNCTION USING DIFFERENTIAL OPERATOR

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Abstract. In this paper we derive the famous Fekete-Szegö inequality for the class of $p-$valent non-bazilevič function using differential operator.

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1. Introduction and preliminaries

Let $\mathcal{A}_p$ be the class of normalized analytic functions $f(z)$ in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$ is of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (z \in \mathbb{U}, p \in \mathbb{N} = 1, 2, \ldots). \tag{1.1}$$

Further, let $\mathcal{A}_1 = \mathcal{A}$, $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$.

For the two analytical functions $f(z)$ and $g(z)$ in $\mathbb{U}$, the function $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exits a Schwartz function $\omega(z)$, analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad and \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Mohammed and Darus[5] defined the operator,

$$\mathcal{D}_\lambda f(z) = (1 + p\lambda)f(z) + \lambda z f^{'}(z), \quad \lambda \geq -p, f \in \mathcal{A}_p.$$

$$\mathcal{D}_\lambda^0 f(z) = f(z)$$
$$\mathcal{D}_\lambda^1 f(z) = \mathcal{D}_\lambda f(z)$$
$$\mathcal{D}_\lambda^2 f(z) = \mathcal{D}_\lambda(\mathcal{D}_\lambda^1 f(z))$$

and in general,

$$\mathcal{D}^k_{\lambda,p}f(z) = z^p + \sum_{n=p+1}^{\infty}(1+\lambda p+n\lambda)^k a_n z^n, \quad \lambda > -p; k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}\, and\, p \in \mathbb{N}. \tag{1.2}$$

Obradovic[6] introduced the Non-Bazilevič type class of functions as

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\alpha+1}\right\} > 0, \quad z \in \mathbb{U}.$$

We can refer[1, 3, 7, 10] for the brief history of Non-Bazilevič type for the various subclasses of analytic functions.

Now, using the differential operator (1.2), we define the generalized $p-$valent Non-Bazilevič class of function as follows:

Definition 1. *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{N}^k_{\lambda,p}(\alpha,\mu,A,B)$ if it satisfies the inequality,*

$$(1-\alpha)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^{\mu} + \frac{\alpha}{p}\left(\frac{z(D^k_{\lambda,p}f(z))'}{D^k_{\lambda,p}f(z)}\right)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}, \tag{1.3}$$

where $\alpha \in \mathbb{C}; 0 < \mu < 1; -1 \leq B \leq 1, A \neq B, A \in \mathbb{R}$.

We note that, if $\lambda = 0, k = 0$ and $p = 1$ then the class $\mathcal{N}^k_{\lambda,p}(\alpha,\mu,A,B)$ will be reduced as the class defined by Wang el. at[10]. If $\alpha = 1, \lambda = 0, k = 0$ and $p = 1$ then the class $\mathcal{N}^k_{\lambda,p}(\alpha,\mu,A,B)$ reduced to the class defined by Obradovic[6]. If $\alpha = 1, \lambda = 0, k = 0, A = 1-\delta, B = -1$ and $p = 1$ then the class $\mathcal{N}^k_{\lambda,p}(\alpha,\mu,A,B)$ reduces to the class of non-Bazilevic functions of order $\delta, (0 \leq \delta < 1)$ which was studied by Tuneski and Darus[9].

By motivating the results of Goyal,Jiang and Seoudy[2, 3, 8], in this paper, we derive the classical Fekete Szegö results for the function $f(z)$ belongs to the subclass $\mathcal{N}^k_{\lambda,p}(\alpha,\mu,A,B)$. As a special consequences of our results, we derive some of the corollaries for various values of the parameters involving in this class.

We now giving the basic lemma which is essential to prove our main results.

Lemma 1. [4] *If suppose Ω denotes the class of analytic functions of the form*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \ldots$$

and satisfying the condition $\omega(0) = 0$ and $|\omega(z)| < 1, \quad z \in \mathbb{U}$ then for any complex number t,

$$|\omega_2 - t\omega_1^2| \leq max\,\{1, |t|\}.$$

the result is sharp for the functions $\omega(z) = z^2$ *and* $\omega(z) = z$.

2. MAIN RESULTS

Our main result is stated in this following theorem.

Theorem 1. *If the function* $f(z)$ *is given by* (1.1) *belongs to the class* $\mathcal{N}^k_{\lambda,p}(\alpha,\mu,A,B)$ *and* η *be the complex number, then*

$$|a_{p+2} - \eta a^2_{p+1}| \leq \frac{(A-B)p}{|2\alpha - p\mu|(1+\lambda p+(p+2)\lambda)^k} \\ max\left\{1, \left|B - \frac{(A-B)p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\left[\left(\frac{\mu+1}{2}\right)+\eta\frac{(1+\lambda p+(p+2)\lambda)^k}{(1+\lambda p+(p+1)\lambda)^{2k}}\right]\right|\right\}.$$

and the result is sharp.

Proof. if $f \in \mathcal{N}^k_{\lambda,p}(\alpha,\mu,A,B)$, then there exist a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ which is analytic in the open unit disk such that

$$(1-\alpha)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^\mu + \frac{\alpha}{p}\left(\frac{z(D^k_{\lambda,p}f(z))'}{D^k_{\lambda,p}f(z)}\right)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^\mu = \frac{1+A\omega(z)}{1+B\omega(z)} \qquad (2.1)$$

Now, it is a well known fact that

$$\frac{1+A\omega(z)}{1+B\omega(z)} = 1 + (A-B)\omega_1 z + [(A-B)\omega_2 - B(A-B)\omega_1^2]z^2 + \dots \qquad (2.2)$$

let us find,

$$(1-\alpha)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^\mu + \frac{\alpha}{p}\left(\frac{z(D^k_{\lambda,p}f(z))'}{D^k_{\lambda,p}f(z)}\right)\left(\frac{z^p}{D^k_{\lambda,p}f(z)}\right)^\mu = \\ 1 + \left(\frac{\alpha}{p}-\mu\right)(1+\lambda p+(p+1)\lambda)^k a_{p+1}z \\ + \left(\frac{2\alpha}{p}-\mu\right)\left[(1+\lambda p+(p+2)\lambda)^k a_{p+2} - \left(\frac{\mu+1}{2}\right)(1+\lambda p+(p+1)\lambda)^{2k}a^2_{p+1}\right]z^2 + \dots \qquad (2.3)$$

From equations (2.1),(2.2) and (2.3) we get,

$$a_{p+1} = \frac{(A-B)p\omega_1}{(\alpha-p\mu)(1+\lambda p+(p+1)\lambda)^k}$$

and

$$a_{p+2} = \frac{(A-B)p}{(2\alpha-p\mu)(1+\lambda p+(p+2)\lambda)^k}\left\{\omega_2 - \left[B - \left(\frac{\mu+1}{2}\right)\frac{(A-B)p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\right]\omega_1^2\right\}.$$

For any complex number η, we can derive

$$|a_{p+2} - \eta a_{p+1}^2| = \frac{(A-B)p}{|2\alpha - p\mu|(1+\lambda p + (p+2)\lambda)^k} |\omega_2 - t\omega_1^2|$$

where,

$$|t| = \left| B - \frac{(A-B)p(2\alpha - p\mu)}{B(\alpha - p\mu)^2} \left[\left(\frac{\mu+1}{2} \right) + \eta \frac{(1+\lambda p + (p+2)\lambda)^k}{(1+\lambda p + (p+1)\lambda)^{2k}} \right] \right|$$

Now the result is follow from Lemma 1,

$$|a_{p+2} - \eta a_{p+1}^2| \le \frac{(A-B)p}{|2\alpha - p\mu|(1+\lambda p + (p+2)\lambda)^k}$$

$$max \left\{ 1, \left| B - \frac{(A-B)p(2\alpha - p\mu)}{(\alpha - p\mu)^2} \left[\left(\frac{\mu+1}{2} \right) + \eta \frac{(1+\lambda p + (p+2)\lambda)^k}{(1+\lambda p + (p+1)\lambda)^{2k}} \right] \right| \right\}.$$

The result is sharp for the functions dened by

$$(1-\alpha) \left(\frac{z^p}{D_\lambda^k f(z)} \right)^\mu + \frac{\alpha}{p} \left(\frac{z(D_\lambda^k f(z))'}{D_\lambda^k f(z)} \right) \left(\frac{z^p}{D_\lambda^k f(z)} \right)^\mu = \frac{1+Az^2}{1+Bz^2}$$

or

$$(1-\alpha) \left(\frac{z^p}{D_\lambda^k f(z)} \right)^\mu + \frac{\alpha}{p} \left(\frac{z(D_\lambda^k f(z))'}{D_\lambda^k f(z)} \right) \left(\frac{z^p}{D_\lambda^k f(z)} \right)^\mu = \frac{1+Az}{1+Bz}$$

□

Now we are finding the coefficient bounds and Fekete Szegö results for different values of parameters in the following corollaries.

Corollary 1. *Let $\lambda = 0$, $k = 0$ and for any complex number η, we obtain*

$$a_{p+1} = \frac{(A-B)p\omega_1}{(\alpha - p\mu)},$$

$$a_{p+2} = \frac{(A-B)p}{(2\alpha - p\mu)} \left\{ \omega_2 - \left[B - \left(\frac{\mu+1}{2} \right) \frac{(A-B)p(2\alpha - p\mu)}{(\alpha - p\mu)^2} \right] \omega_1^2 \right\}$$

and

$$|a_{p+2} - \eta a_{p+1}^2| \le \frac{(A-B)p}{|2\alpha - p\mu|} max \left\{ 1, \left| B - \frac{(A-B)p(2\alpha - p\mu)}{(\alpha - p\mu)^2} \left[\frac{\mu + 1 + 2\eta}{2} \right] \right| \right\}.$$

Corollary 2. *Put $p = 1$ in corollary 1 and for any complex number η, we obtain*

$$a_2 = \frac{(A-B)\omega_1}{(\alpha - \mu)},$$

$$a_3 = \frac{(A-B)}{(2\alpha - \mu)} \left\{ \omega_2 - \left[B - \left(\frac{\mu+1}{2} \right) \frac{(A-B)(2\alpha - p\mu)}{(\alpha - \mu)^2} \right] \omega_1^2 \right\}$$

and

$$|a_3-\eta a_2^2|\leq \frac{(A-B)}{|2\alpha-\mu|}max\left\{1,\left|B-\frac{(A-B)(2\alpha-\mu)}{(\alpha-\mu)^2}\left[\frac{\mu+1+2\eta}{2}\right]\right|\right\}.$$

Corollary 3. *Put $\alpha=1$ in corollary 1 and for any complex number η, we obtain*

$$a_{p+1}=\frac{(A-B)p\omega_1}{(1-p\mu)},$$

$$a_{p+2}=\frac{(A-B)p}{(2-p\mu)}\left\{\omega_2-\left[B-\left(\frac{\mu+1}{2}\right)\frac{(A-B)p(2-p\mu)}{(1-p\mu)^2}\right]\omega_1^2\right\}$$

and

$$|a_{p+2}-\eta a_{p+1}^2|\leq \frac{(A-B)p}{|2-p\mu|}max\left\{1,\left|B-\frac{(A-B)p(2-p\mu)}{(1-p\mu)^2}\left[\frac{\mu+1+2\eta}{2}+\right]\right|\right\}.$$

Corollary 4. *Put $p=1$ in corollary 3 and for any complex number η, we obtain*

$$a_2=\frac{(A-B)\omega_1}{(1-\mu)},$$

$$a_3=\frac{(A-B)}{(2-\mu)}\left\{\omega_2-\left[B-\left(\frac{\mu+1}{2}\right)\frac{(A-B)(2-\mu)}{(1-\mu)^2}\right]\omega_1^2\right\}$$

and

$$|a_3-\eta a_2^2|\leq \frac{(A-B)}{|2-\mu|}max\left\{1,\left|B-\frac{(A-B)(2-\mu)}{(1-\mu)^2}\left[\frac{\mu+1+2\eta}{2}+\right]\right|\right\}.$$

Corollary 5. *Let $A=1$, $B=-1$ in corollary 1 and for any complex number η, we obtain*

$$a_{p+1}=\frac{2p\omega_1}{(\alpha-p\mu)},$$

$$a_{p+2}=\frac{2p}{(2\alpha-p\mu)}\left\{\omega_2+\left[1+\frac{(\mu+1)p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\right]\omega_1^2\right\}$$

and

$$|a_{p+2}-\eta a_{p+1}^2|\leq \frac{2p}{|2\alpha-p\mu|}max\left\{1,\left|1+(\mu+1-\eta)\frac{p(2\alpha-p\mu)}{(\alpha-p\mu)^2}\right|\right\}.$$

Corollary 6. *Let $p=1$ in corollary 5 and for any complex number η, we obtain*

$$a_2=\frac{2\omega_1}{(\alpha-\mu)},$$

$$a_3=\frac{2}{(2\alpha-\mu)}\left\{\omega_2+\left[1+\frac{(\mu+1)(2\alpha-\mu)}{(\alpha-\mu)^2}\right]\omega_1^2\right\}$$

and

$$|a_3-\eta a_2^2|\leq \frac{2}{|2\alpha-\mu|}max\left\{1,\left|1+(\mu+1-\eta)\frac{(2\alpha-\mu)}{(\alpha-\mu)^2}\right|\right\}.$$

Corollary 7. *Let $\alpha = 1$ in corollary 5 and for any complex number η, we obtain*

$$a_{p+1} = \frac{2p\omega_1}{(1-p\mu)},$$

$$a_{p+2} = \frac{2p}{(2-p\mu)}\left\{\omega_2 + \left[1 + \frac{(\mu+1)p(2-p\mu)}{(1-p\mu)^2}\right]\omega_1^2\right\}$$

and

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{2p}{|2-p\mu|}\, max\, \left\{1, \left|1 + (\mu+1-\eta)\frac{p(2-p\mu)}{(1-p\mu)^2}\right|\right\}.$$

Corollary 8. *Let $p = 1$ in corollary 7 and for any complex number η, we obtain*

$$a_2 = \frac{2\omega_1}{(1-\mu)},$$

$$a_3 = \frac{2}{(2-\mu)}\left\{\omega_2 + \left[1 + \frac{(\mu+1)(2-\mu)}{(1-\mu)^2}\right]\omega_1^2\right\}$$

and

$$|a_3 - \eta a_2^2| \leq \frac{2}{|2-\mu|}\, max\, \left\{1, \left|1 + (\mu+1-\eta)\frac{(2-p\mu)}{(1-\mu)^2}\right|\right\}.$$

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A NEW INTERPRETATION OF HERMITE-HADAMARD'S TYPE INTEGRAL INEQUALITIES BY THE WAY OF TIME SCALES

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ABSTRACT. The concept of convex functions has been generalized by using the Time Scales in [4] by C. Dinu which is unifying integral and differential calculus with the calculus of finite differences, offering a formalism for studying hybrid discretecontinuous dynamical systems. Cristaian Dinu in his article [5] established some Ostrowski type inequalities on Time Scales. R. P. Agarwal *et.al.* in [1] discussed inequalities on time scales. In this article, using the concept of time scale, we generalized some of the Hermite-Hadamard type integral inequalities.

1. INTRODUCTION

Let us rephrase some concept of Time scales already defined in [2].

A nonempty closed subset $\mathbb{T}$ of the set of real numbers $\mathbb{R}$ has been called a time scale by Stefan Hilger. Thus $\mathbb{R}$ itself, $\mathbb{Z}$ the set of integers , the set of non-negative integers $\mathbb{N}_o$, a singleton subset of $\mathbb{R}$, any finite subset of $\mathbb{R}$, any closed interval in $\mathbb{R}$ and are all the examples of time scales discussed in [11]. However, $\mathbb{Q}$, $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$, $\mathbb{C}$ and any open interval of $\mathbb{R}$ are not time scales. A neighborhood of a point $t_0 \in \mathbb{T}$ will be taken as the set $\mathbb{T} \cap]t_o - \delta, t_o + \delta[$ for any $\delta > 0$. If $\mathbb{T} = \mathbb{Z}$ then neighborhood of each $t \in \mathbb{T}$ is the point t itself. The mapping $\sigma : \mathbb{T} \to \mathbb{T}$ is called forward jump operator if it is defined as $\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = sup\{s \in \mathbb{T} : s < t\}$. The function $\mu : \mathbb{T} \to [0, \infty[$ defined by $\mu(t) = \sigma(t) - t$ is referred to as graininess function.

If $f : \mathbb{T} \to \mathbb{R}$ then the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t)) \forall t \in \mathbb{T}$, i.e, $f^\sigma = f \circ \sigma$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be continuous at $t_o \in \mathbb{T}$ if for every $\epsilon > 0$ there exists a

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$\delta > 0$ such that for all $t \in \mathbb{T} \cap \,]t_o - \delta, t_o + \delta[$

$$|f(t) - f(t_o)| < \epsilon$$

The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of the function $f : \mathbb{T} \to \mathbb{R}$ at a point $t_o \in \mathbb{T}^k$ if for every $\epsilon > 0$ there is a neighborhood $N = \mathbb{T} \cap \,]t_o - \delta, t_o + \delta[$ of t_o such that $\left|[f(t) - f^{\sigma}(t_o)] - f^{\Delta}(t_o)\,[t - \sigma(t_o)]\right| \leq \epsilon\,|t - \sigma(t_o)|, \forall t \in N$.
The function f is said to be delta (or Hilger) differentiable on $\mathbb{T}^k$ provided f^{Δ} exists for all $t \in \mathbb{T}^k$ [2].

Theorem 1 (Bohner, 2001)**.** *let $t \in \mathbb{T}$*
(1) If $f : \mathbb{T} \to \mathbb{R}$ is differentiable at t then f is continuous at t.
(2) If t is right-scattered and $f : \mathbb{T} \to \mathbb{R}$ is continuous at t, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}$$

Definition 1 (Bohner, 2001)**.** *The function $f : \mathbb{T} \to \mathbb{R}$ is refereed as an **rd-continuous** at every $t \in \mathbb{T}$, if f is continuous at right-dense point $t \in \mathbb{T}$. It is denoted by $f \in C_{rd}(\mathbb{T}, \mathbb{R})$*

Definition 2 (Bohner, 2001)**.** *Let $f \in C_{rd}$. Then $f : \mathbb{T} \to \mathbb{R}$ is known as **anti-derivative** of f on $\mathbb{T}$ if it s differentiable on $\mathbb{T}$ provided that $f^{\Delta}(t) = F(t)$ is valid for $t \in \mathbb{T}^k$, the integral of f is distinct by ;*

$$\int_a^b f(t)\Delta t = F(b) - F(a), \ \forall \ t \in \mathbb{T}$$

In recent years there have been many extensions, generalizations and similar results of the Hermite-Hadamard inequality studied in [3, 6, 7, 10, 11]. In this article, we obtain some new inequalities of Hermite-Hadamard type for functions on time scales which is actually a generalization of Hermite-Hadamard type inequalities. We also found some related results as well. Recent references that are available online are mentioned as well [8, 12, 13, 14].

2. Main Results

In [1], Barani et al. established inequalities for twice differentiable P-convex functions which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1. *Let $\mathbb{T}$ be a time scale and $I = [a, b]$, Let $f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentiable mapping on I^o (I^o is the interior of I) with $a < b$. If $f^{\Delta}(t) \in C_{rd}$ then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x = \frac{b-a}{2}\int_0^1 (1-2t)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.1)$$

Theorem 2. *Let $f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a differentiable mapping on I^0, $a, b \in I$ with $a < b$ and $f^{\Delta} \in C_{rd}$. If the mapping $|f^{\Delta}|$ is convex, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x\right| \leq \frac{b-a}{4}[f^{\Delta}(a)+f^{\Delta}(b)]\left(1-4h_2(\frac{1}{2},0)\right). \quad (2.2)$$

Proof. From lemma 1 , we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x\right| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||f^{\Delta}(ta+(1-t)b)|\Delta t$$

since $|f^{\Delta}|$ is convex , therefore

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x\right| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||tf^{\Delta}(a)+(1-t)f^{\Delta}(b)|\Delta t$$

Here

$$I = \int_0^1 |(1-2t)|\{|tf^{\Delta}(a)+(1-t)f^{\Delta}(b)|\}\Delta t$$

$$I = \int_0^{\frac{1}{2}} (1-2t)\{tf^{\Delta}(a)+(1-t)f^{\Delta}(b)\}\Delta t - \int_{\frac{1}{2}}^1 (1-2t)\{tf^{\Delta}(a)+(1-t)f^{\Delta}(b)\}\Delta t$$

using the following results

$$\int_0^{\frac{1}{2}} 1\Delta t = \int_{\frac{1}{2}}^1 \Delta t = \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} t\Delta t = h_2\left(\frac{1}{2},0\right)$$

$$\int_{\frac{1}{2}}^1 t\Delta t = \frac{1}{2} - h_2\left(\frac{1}{2},0\right)$$

we get

$$I = -f^{\Delta}(a)\left\{h_2\left(\frac{1}{2},0\right) - 2\int_0^{\frac{1}{2}} t^2\Delta t - \frac{1}{2} + h_2\left(\frac{1}{2},0\right) + 1 - 4h_2\left(\frac{1}{2},0\right) + 2\int_0^{\frac{1}{2}} t^2\Delta t\right\}$$

$$+f^{\Delta}(b)\left\{\frac{1}{2} - 3h_2\left(\frac{1}{2},0\right) + 2\int_0^{\frac{1}{2}} t^2\Delta t - \frac{1}{2} + \frac{3}{2} - 3h_2\left(\frac{1}{2},0\right) - 1 + 4h_2\left(\frac{1}{2},0\right) - 2\int_0^{\frac{1}{2}} t^2\Delta t\right\}$$

This leads to

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right|\leq\frac{b-a}{4}\left[f^\Delta(a)\left(1-4h_2\left(\frac{1}{2},0\right)\right)+f^\Delta(b)\left(1-4h_2\left(\frac{1}{2},0\right)\right)\right]$$

This completes the proof. □

Remark 1. If we consider $\mathbb{T}=\mathbb{R}$ then $\sigma(t)=t$ and

$$h_2\left(\frac{1}{2},0\right)=\int_0^{\frac{1}{2}}(t-0)\Delta t=\int_0^{\frac{1}{2}}(t-0)dt=\frac{t^2}{2}\Big|_0^{\frac{1}{2}}$$

Then from (2.2), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left|f'(a)\left(1-4\left(\frac{1}{8}\right)\right)+f'(b)\left(1-4\left(\frac{1}{8}\right)\right)\right|$$
$$=(b-a)\left[\frac{f'(a)+f'(b)}{8}\right]$$

This is a well-known result for Hermite-Hadamard inequality in $\mathbb{R}$

Lemma 2. *Let $f:\mathbb{T}\to\mathbb{R}$ be a differentiable mapping, $a,b\in\mathbb{T}$ with $a<b, f^\Delta\in C_{rd}$,then the following equality holds;*

$$f(a)\{1-h_2(1,0)\}+f(b)h_2(1,0)-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x$$
$$=\frac{b-a}{2}\int_0^1\int_0^1[f^\Delta(ta+(1-t)b)-f^\Delta(sa+(1-s)b)](s-t)\Delta t\Delta s \quad (2.3)$$

Proof. Consider

$$\frac{b-a}{2}\int_0^1\int_0^1\left[f^\Delta(ta+(1-t)b)-f^\Delta(sa+(1-s)b\right](s-t)\Delta t\Delta s \quad (2.4)$$

And let

$$I_1=\int_0^1\int_0^1[f^\Delta(ta+(1-t)b)](s-t)\Delta s\Delta t$$
$$I_2=\int_0^1\int_0^1[f^\Delta(sa+(1-s)b)](s-t)\Delta t\Delta s$$

Then by integrating and using the formula

$$\int_a^b f(t)g^\Delta(t)\Delta t=(fg)(a)-\int_a^b f^\Delta(t)g(\sigma(t))\Delta t$$

$$I_1=f(a)\{1-h_2(1,0)\}+f(b)h_2(1,0)-\int_0^1 f^\sigma(ta+(1-t)b)\Delta \quad (2.5)$$

$$I_2=f(a)h_2(1,0)-f(b)h_2(1,0)+\int_0^1 f^\sigma(sa+(1-s)b)\Delta s \quad (2.6)$$

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By putting the values of I_1 and I_2 from (2.5) and (2.6) in (2.4), we get (2.3). ∎

Remark 2. If we consider the case $\mathbb{T}=\mathbb{R}$ then $\sigma(x)=x$, and

$$h_2(1,0)\int_0^1(t-0)dt=\frac{1}{2}$$

thus (2.3) becomes

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1\int_0^1[f'(ta+(1-t)b)-f'(sa+(1-s)b)](s-t)dtds$$

Lemma 3. *Let $f:I^o\subseteq\mathbb{T}\to\mathbb{R}$ be delta differentiable on $I^o, a,b\in\mathbb{I}^{\rtimes}$ with $a<b$. If $f^{\Delta}\in C_{rd}([a,b],\mathbb{R})$, then the following equality holds :*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f^{\sigma}(x)\Delta x\\
&\quad=\frac{b-a}{2}\int_0^1\int_0^1\left[f^{\Delta}(ta+(1-t)b)-f^{\Delta}(sa+(1-s)b)\right](m(s)-m(t))\Delta t\Delta s \quad (2.7)
\end{aligned}$$

with

$$m(.):=\begin{cases}t, t\in\left[0,\frac{1}{2}\right]\\ t-1, t\in\left(\frac{1}{2},1\right]\end{cases}$$

Proof. By definition of $m(.)$, it follows that

$$\begin{aligned}
&\int_0^1\int_0^1\left(f^{\Delta}(ta+(1-t)b)-f^{\Delta}(sa+(1-s)b)\right)\times(m(t)-m(s))\Delta t\Delta s\\
&=\int_0^1\int_0^1 f^{\Delta}(ta+(1-t)b)(m(t)-m(s))\Delta t\Delta s-\int_0^1\int_0^1 f^{\Delta}(sa+(1-s)b)(m(t)-m(s))\Delta t\Delta s\\
&=\int_0^1\int_0^{\frac{1}{2}} f^{\Delta}((ta+(1-t)b)(t-m(s)))\Delta t\Delta s+\int_0^1\int_{\frac{1}{2}}^1 f^{\Delta}((ta+(1-t)b)(t-1-m(s))\Delta t)\Delta s\\
&\quad-\int_0^1\int_0^{\frac{1}{2}} f^{\Delta}(sa+(1-s)b)(t-m(s))\Delta t\Delta s+\int_0^1\int_{\frac{1}{2}}^1 f^{\Delta}(sa+(1-s)b)(t-1-m(s))\Delta t\Delta s\\
&=\int_0^{\frac{1}{2}}\left\{\int_0^{\frac{1}{2}}f^{\Delta}(ta+(1-t)b)(t-s)\Delta t\right\}\Delta s+\int_{\frac{1}{2}}^1\left\{\int_0^{\frac{1}{2}}f^{\Delta}(ta+(1-t)b)(t-s+1)\Delta t\right\}\Delta s\\
&+\int_{\frac{1}{2}}^1\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(ta+(1-t)b)(t-s)\Delta t\right\}\Delta s+\int_0^{\frac{1}{2}}\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(ta+(1-t)b)(t-s-1)\Delta t\right\}\Delta t\\
&+\int_0^{\frac{1}{2}}\left\{\int_0^{\frac{1}{2}}f^{\Delta}(sa+(1-s)b)(t-s)\Delta t\right\}\Delta s+\int_{\frac{1}{2}}^1\left\{\int_0^{\frac{1}{2}}f^{\Delta}(sa+(1-s)b)(t-s+1)\Delta t\right\}\Delta s\\
&+\int_0^{\frac{1}{2}}\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(sa+(1-s)b)(t-s-1)\Delta t\right\}\Delta s+\int_{\frac{1}{2}}^1\left\{\int_{\frac{1}{2}}^1 f^{\Delta}(sa+(1-s)b)(t-s)\Delta t\right\}\Delta s\\
&=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8
\end{aligned}$$

by integrating, we can state,

$$I_1 = \int_0^{\frac{1}{2}} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(ta + (1-t)(t-s)\Delta t) \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) + \frac{f(b)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_0^{\frac{a+b}{2}} f^{\sigma}(x)\Delta x$$

$$I_2 = \int_{\frac{1}{2}}^{1} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(ta + (1-t)b)(t-s+1)\Delta t \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f(b)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_b^{\frac{a+b}{2}} f^{\sigma}(x)\Delta x$$

$$I_3 = \int_0^{\frac{1}{2}} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(ta + (1-t)b)(t-s-1)\Delta t \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_4 = \int_{\frac{1}{2}}^{1} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(ta + (1-t)b)(t-s)\Delta s \right\} \Delta s$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_5 = \int_0^{\frac{1}{2}} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(sa + (1-s)b)(t-s)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f(b)}{a-b} h_2\left(\frac{1}{2}, 0\right) + \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_6 = \int_0^{1} \left\{ \int_0^{\frac{1}{2}} f^{\Delta}(sa + (1-s)b)(t-s+1)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_7 = \int_0^{\frac{1}{2}} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(sa(1-s)b)(t-s-1)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

$$I_8 = \int_{\frac{1}{2}}^{1} \left\{ \int_{\frac{1}{2}}^{1} f^{\Delta}(sa + (1-s)b)(t-s)\Delta t \right\} \Delta s$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} - \frac{f(a)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{f\left(\frac{a+b}{2}\right)}{a-b} h_2\left(\frac{1}{2}, 0\right) - \frac{1}{2(a-b)^2} \int_{\frac{a+b}{2}}^{a} f^{\sigma}(x)\Delta x$$

Adding $I_1, I_2, I_3, I_4, I_5, I_6, I_7$ and I_8 and rewriting we easily deduce,

$$= 2\frac{f\left(\frac{a+b}{2}\right)}{a-b} + \frac{2}{(a-b)^2}\left\{-\int_b^{\frac{a+b}{2}} f^\sigma(x)\Delta x - \int_{\frac{a+b}{2}}^a f^\sigma(x)\Delta x\right\}$$
$$= \frac{f\left(\frac{a+b}{2}\right)}{4(a-b)} + \frac{2}{(a-b)^2}\int_a^b f^\sigma(x)\Delta x$$

This leads to the required result.if we consider $\mathbb{T} = \mathbb{R}$ and $\sigma(t) = t$.Then we will come to a well-known result for Hermite-Hadamard inequality in $\mathbb{R}$. □

Theorem 3. *Let $f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentiability function I^o, where $a, b \in I$ with $a < b$ if $f^\Delta \in C_{rd}$ then the following inequality holds*

$$\frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f^\sigma(u)\Delta u$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f^\Delta(tx+(1-t)a)\Delta t + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f^\Delta(tx+(1-t)b)\Delta t \quad (2.8)$$

Proof. Let

$$I_1 = (t-1)\frac{f(tx+(1-t)a)}{x-a}\Big|_0^1 - \int_0^1 (1)\frac{f^\sigma(tx+(1-t)a)}{x-a}\Delta t$$
$$= \frac{f(a)}{x-a} - \frac{1}{x-a}\int_a^x \frac{f^\sigma(u)}{x-a}\Delta u$$
$$I_2 = (1-t)\frac{f(tx+(1-t)b)}{x-b}\Big|_0^1 - \int_0^1 (-1)\frac{f^\sigma(tx+(1-t)b)}{x-b}\Delta t$$
$$= -\frac{f(b)}{x-b} + \int_b^x \frac{f^\sigma(u)}{(x-b)^2}\Delta u$$

By substituting the values of I_1 and I_2 in (2.8) we get,

$$\frac{(x-a)^2}{b-a}\int_0^1 (t-1)f^\Delta(tx+(1-t)a)\Delta t + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f^\Delta(tx+(1-t)b)\Delta t$$
$$= \frac{1}{b-a}\left\{(x-a)f(a) - \int_a^x f^\sigma(u)\Delta u + (b-x)f(b) + \int_b^x f^\sigma(u)\Delta u\right\}$$
$$= \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a}\left\{\int_a^x f^\sigma(u)\Delta u + \int_b^x f6\sigma(u)\Delta u\right\}$$
$$= \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f^\sigma(u)\Delta u$$

which leads to the required result. □

Lemma 4. *let $f : I \subseteq \mathbb{T} \to R$ be delta differentiable on I^o, with $a, b \in I$ and $a < b$ and $\lambda, \mu \in \mathbb{R}$. If $f^\Delta \in C_{rd}$, then*

$$\begin{aligned}&(1-u)f(a) + \lambda f(b) + (\mu - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x \\ &= (b-a)\left[\int_0^{\frac{1}{2}} (\lambda - t) f^\Delta(ta + (1-t)b)\Delta t + \int_{\frac{1}{2}}^1 (\mu - t) f^\Delta(ta + (1-t)b)\Delta t\right] \quad (2.9)\end{aligned}$$

Proof. Choosing from R.H.S

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{2}} (\lambda - t) f^\Delta(ta + (1-t)b)\Delta t \\ &= \left(\lambda - \frac{1}{2}\right)\frac{f\left(\frac{a+b}{2}\right)}{a-b} - \frac{\lambda f(b)}{a-b} + \frac{1}{b-a}\int_0^{\frac{1}{2}} f^\sigma(ta + (1-t)b)\Delta t\end{aligned}$$

and

$$\begin{aligned}I_2 &= \int_{\frac{1}{2}}^1 (\mu - t) f^\Delta(ta + (1-t)b)\Delta t \\ &= (\mu - 1)\frac{f(a)}{a-b} - \left(\mu - \frac{1}{2}\right)\frac{f\left(\frac{a+b}{2}\right)}{a-b} + \frac{1}{b-a}\int_0^{\frac{1}{2}} f^\sigma(ta + (1-t)b)\Delta t\end{aligned}$$

Filling I_1 and I_2 in right hand side of (2.9) which completes the proof. □

Lemma 5. *Let$f : I \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentiable function on I^o, the interior of I where $a, b \in I$ with $a < b$. If $f^\Delta \in C_{rd}$ and $\lambda, \mu \in \mathbb{R}$ then the following inequality holds*

$$\begin{aligned}&\frac{\lambda f(a) + \mu f(b)}{2} + \frac{(2-\mu-\lambda)}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x \\ &= \frac{(b-a)}{4}\left[\int_0^1 (1-\lambda-t) f^\Delta\left(ta + (1-t)\frac{a+b}{2}\right) + (\mu - t) f^\Delta\left(t\frac{a+b}{2} + (1-t)b\right)\right]\Delta t \quad (2.10)\end{aligned}$$

Proof. Replacing λ and μ respectively by $\frac{\alpha}{2}$ and $1 - \frac{\beta}{2}$ in lemma 4 yields,

$$\begin{aligned}&\frac{1}{b-a}\left[\frac{\beta f(a) + \alpha f(b)}{2} + \frac{(2-\alpha-\beta)}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right] \\ &= \int_0^{\frac{1}{2}}\left(\frac{\alpha}{2} - t\right) f^\Delta(ta + (1-t)b)\Delta t + \int_{\frac{1}{2}}^1 \left(1 - \frac{\beta}{2} - t\right) f^\Delta(ta + (1-t)b)\Delta t \quad (2.11)\end{aligned}$$

simple calculations resulting

$$\begin{aligned}&\int_0^{\frac{1}{2}}\left(\frac{\alpha}{2} - t\right) f^\Delta(ta + (1-t)b)\Delta t = \frac{1}{4}\int_0^1 (\alpha - u) f^\Delta\left(\frac{u}{2}a + \frac{2-u}{2}b\right)\Delta u \\ &= \frac{1}{4}\int_0^1 (\alpha - u) f^\Delta\left(\frac{a+b}{2}u + (1-u)b\right)\Delta u \quad (2.12)\end{aligned}$$

$$\int_{\frac{1}{2}}^{1}(1-\frac{\beta}{2}-t)f^{\Delta}(ta+(1-t)b)\Delta t=\frac{1}{4}\int_{0}^{1}(1-\beta-u)f^{\Delta}(\frac{1+u}{2}a+\left(\frac{1-u}{2}b\right))\Delta u \quad (2.13)$$

utilizing (2.11), (2.12) and (2.13), leads to

$$\frac{\beta f(a)+\alpha f(b)}{2}+\frac{(2-\alpha-\beta)}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x$$

$$=\frac{b-a}{4}\int_{0}^{1}\left[(\alpha-u)f^{\Delta}\left(\frac{a+b}{2}u+(1-u)b\right)+(1-\beta-u)f^{\Delta}\left(\frac{1+u}{2}a+\left(\frac{1-u}{2}\right)b\right)\right]\Delta u$$

This is the required result. □

Corollary 1. *By taking* $\lambda=\frac{l}{m}$,$\mu=\frac{m-l}{m}$ *for* $m\neq 0$ *in lemma 5 , we have the following identities.*

$$\frac{l}{m}\left[f(a)+f(b)+\frac{(m-2l)}{m}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{1}\left(\frac{l}{m}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{m-l}{m}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.14)$$

In particular we have

$$\left[f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}(1-t)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}tf^{\Delta}(ta+(1-t)b)\Delta t \quad (2.15)$$

$$\left[f(a)+f(b)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]=(b-a)\int_{0}^{1}(1-2t)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.16)$$

$$\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}\left(\frac{1}{3}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{2}{3}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.17)$$

$$\frac{1}{2}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}\left(\frac{1}{4}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{3}{4}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.18)$$

$$\frac{1}{5}\left[f(a)+f(b)+3f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(x)\Delta x\right]$$

$$=(b-a)\int_{0}^{\frac{1}{2}}\left(\frac{1}{5}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^{1}\left(\frac{4}{5}-t\right)f^{\Delta}(ta+(1-t)b)\Delta t \quad (2.19)$$

$$\frac{1}{5}\left[2\{f(a)+f(b)\}+f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right]$$
$$=(b-a)\int_0^{\frac{1}{2}}\left(\frac{2}{5}-t\right)f^\Delta(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^1\left(\frac{3}{5}-t\right)f^\Delta(ta+(1-t)b)\Delta t \quad (2.20)$$

$$\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x\right]$$
$$=(b-a)\int_0^{\frac{1}{2}}\left(\frac{1}{6}-t\right)f^\Delta(ta+(1-t)b)\Delta t+\int_{\frac{1}{2}}^1\left(\frac{5}{6}-t\right)f^\Delta(ta+(1-t)b)\Delta t \quad (2.21)$$

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The general solution and Ulam stability of inhomogeneous Euler-Cauchy dynamic equations on time scales

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Abstract: In the present paper, we find the general solution of the inhomogeneous Euler-Cauchy dynamic equation

$$t\sigma(t)y^{\Delta\Delta}(t) + \alpha t y^{\Delta}(t) + \beta y(t) = f(t)$$

on the time scale with the constant graininess function and the linear variable graininess function, respectively. And then, we study the Ulam stability problem of the forgoing equation on different types of time scales. Our results can be viewed as a unfication and extension of the results of Mortici et al. [C. Mortici, T.M. Rassias, S.M. Jung, The inhomogeneous Euler equation and its Hyers-Ulam stability, Appl. Math. Lett. 40 (2015) 23-28].

Keywords: General solution; Ulam stability; Euler-Cauchy dynamic equations; Time scales; Graininess function

1 Introduction and preliminary

The Ulam stability originated from a question proposed by S.M. Ulam [12] in 1940, which was concerned with the stability of group homomorphisms. In the next year, Hyers [5] partially solved this question in a Banach space. Many years later, Ulam's question was generalized and partially solved by Rassias [10]. In 1993, Obloza [9] initiated the study of the Ulam stability of differential equations. Afterwards, Alsina and Ger [1] studied the Ulam stability of the differential equation $y' = y$ on any real interval. Soon after, Miura and Takahasi et al. [6, 7, 11] deeply investigated the Ulam stability of the differential equation $y' = \lambda y$ in various abstract spaces. Since then, the theory of Ulam stability of differential equations is gradually formed and extensively studied. In 2009, Jung and Min[4] discussed the general solution of inhomogeneous Euler equations by using the power series method. However, they only obtained the local Ulam stability of the Euler equation due to the limitation of the radius of convergence. Recently, Mortici et al. [8] obtained the general solution of inhomogeneous Euler equations by using the integration method. Meantime, they proved that the inhomogeneous Euler equation is Hyers-Ulam stable on a bounded domain. Undoubtedly, these results can be regarded as an extension of the results obtained by Jung and Min[4].

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Inspired by the idea of Mortici et al.[8], in this paper, we shall consider the general solution and Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation

$$t\sigma(t)y^{\Delta\Delta}(t) + \alpha t y^{\Delta}(t) + \beta y(t) = f(t) \tag{1}$$

on a time scale $\mathbb{T}$ with $\alpha, \beta \in \mathbb{R}$, where $f : \mathbb{T} \to \mathbb{R}$ is a rd-continuous function. Throughout this paper, we assume that $\mathbb{T} \subset (0, \infty)$ is a time scale with the constant graininess function $\mu(t) = \mu$ or the linear variable graininess function $\mu(t) = \eta t$, η is a constant. Indeed, several common time scales are included in these two cases (see Appendix A, Table 1).

Here, we briefly recall some basic notions related to the time scale. For more details, we recommend two excellent monographs [2, 3] written by Bohner and Peterson. Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real numbers and the set of all positive real numbers, respectively. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward jump operator σ and the back jump operator ρ are defined as $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \inf\{s \in \mathbb{T} : s < t\}$, respectively. Especially, $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$.

A point $t \in \mathbb{T}$ is said to be *right-scattered*, *right-dense*, *left-scattered* and *left-dense* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$ and $\rho(t) = t$, respectively. Given a time scale $\mathbb{T}$, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^{\kappa}$ is derived from the time scale $\mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum γ, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{\gamma\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. Successively, $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. Denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$.

2 The general solution of (1)

In this section, we shall solve the inhomogeneous Euler-Cauchy dynamic equation (1) based on the time scale with different graininess functions.

2.1 The constant graininess function $\mu(t) = \mu$

The associated characteristic equation of Eq.(1) is

$$r^2 + (\alpha - 1)r + \beta = 0. \tag{2}$$

Now, we assume that the following two regressivity conditions are satisfied:

$$t\sigma(t) - \alpha t\mu(t) + \beta\mu^2(t) \neq 0, \tag{3}$$

$$\sigma(t) + \lambda\mu(t) \neq 0, \tag{4}$$

where $t \in \mathbb{T}^{\kappa}$, λ is a characteristic root of Eq.(2). Under these two conditions, we know that $\frac{\lambda}{t}, \frac{\lambda}{\sigma(t)} \in \mathcal{R}$.

Let λ be a root of (2). Setting $x(t) = e_{\frac{\lambda}{t}}(t, 0)$. Replacing the unknown function $y(t)$ of Eq.(1) by $u(t)x(t)$. Then, we have

$$y^{\Delta}(t) = u(t)x^{\Delta}(t) + u^{\Delta}(t)x^{\sigma}(t). \tag{5}$$

Furthermore, we can obtain

$$y^{\Delta\Delta}(t) = u(t)x^{\Delta\Delta}(t) + u^{\Delta}(t)x^{\Delta\sigma}(t) + u^{\Delta}(t)x^{\sigma\Delta}(t) + u^{\Delta\Delta}(t)x^{\sigma\sigma}(t). \tag{6}$$

According to the definition of the exponential function $e_{\frac{\lambda}{t}}(t, t_0)$, we get

$$x^{\Delta}(t) = \frac{\lambda}{t} e_{\frac{\lambda}{t}}(t, t_0) = \frac{\lambda}{t} x(t). \tag{7}$$

Using the formula $x^{\sigma} = x + \mu x^{\Delta}$, it follows that

$$x^{\sigma}(t) = \Big(1 + \mu \frac{\lambda}{t}\Big) x(t). \tag{8}$$

Moreover, we can infer that

$$x^{\Delta\sigma}(t) = (x^{\Delta})^{\sigma}(t) = (x^{\Delta})(\sigma(t)) = \frac{\lambda}{\sigma(t)} x^{\sigma}(t) = \frac{\lambda}{\sigma(t)}\Big(1 + \mu\frac{\lambda}{t}\Big)x(t), \tag{9}$$

$$x^{\sigma\Delta}(t) = (x^{\sigma})^{\Delta}(t) = -\frac{\mu\lambda}{t\sigma(t)}x(t) + \Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x^{\Delta}(t) = -\frac{\mu\lambda}{t\sigma(t)}x(t) + \frac{\lambda}{t}\Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x(t), \tag{10}$$

$$x^{\sigma\sigma}(t) = (x^{\sigma})^{\sigma}(t) = \Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x^{\sigma}(t) = \Big(1 + \frac{\mu\lambda}{t}\Big)\Big(1 + \frac{\mu\lambda}{\sigma(t)}\Big)x(t), \tag{11}$$

$$x^{\Delta\Delta}(t) = -\frac{\lambda}{t\sigma(t)}x(t) + \frac{\lambda}{\sigma(t)}x^{\Delta}(t) = -\frac{\lambda}{t\sigma(t)}x(t) + \frac{\lambda^2}{t\sigma(t)}x(t). \tag{12}$$

Therefore, it follows from (5)-(12) that

$$\begin{aligned}
&t\sigma(t)y^{\Delta\Delta}(t) + \alpha t y^{\Delta}(t) + \beta y(t)\\
&= u(t)(\lambda^2 - \lambda)x(t) + u^{\Delta}(t)[\lambda(t + \mu\lambda)]x(t)\\
&\quad + u^{\Delta}(t)[\lambda(\sigma(t) + \mu\lambda) - \mu\lambda]x(t) + u^{\Delta\Delta}(t)[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]x(t)\\
&\quad + \alpha u^{\Delta}(t)(t + \mu\lambda)x(t) + \alpha\lambda u(t)x(t) + \beta u(t)x(t)\\
&= u^{\Delta\Delta}(t)[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]x(t) + u^{\Delta}(t)[(\alpha + 2\lambda)(t + \mu\lambda)]x(t)\\
&\quad + u(t)[\lambda^2 + (\alpha - 1)\lambda + \beta]x(t)\\
&= u^{\Delta\Delta}(t)[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]x(t) + u^{\Delta}(t)[(\alpha + 2\lambda)(t + \mu\lambda)]x(t)\\
&= f(t).
\end{aligned} \tag{13}$$

Multiplying both sides of the last equality of (13) by $e_{\ominus\frac{\lambda}{t}}(t, 0)$, we have

$$[(\sigma(t) + \mu\lambda)(t + \mu\lambda)]u^{\Delta\Delta}(t) + [(\alpha + 2\lambda)(t + \mu\lambda)]u^{\Delta}(t) = e_{\ominus\frac{\lambda}{t}}(t, 0)f(t). \tag{14}$$

Since $\frac{\lambda}{t}, \frac{\lambda}{\sigma(t)} \in \mathcal{R}$, we obtain that $(\sigma(t) + \mu\lambda)(t + \mu\lambda) \neq 0$. Dividing both sides of (14) by $(\sigma(t) + \mu\lambda)(t + \mu\lambda)$, we have that

$$u^{\Delta\Delta}(t) + \frac{\alpha + 2\lambda}{\sigma(t) + \mu\lambda}u^{\Delta}(t) = e_{\ominus\frac{\lambda}{t}}(t, 0)\frac{f(t)}{(\sigma(t) + \mu\lambda)(t + \mu\lambda)}. \tag{15}$$

Letting $u^{\Delta}(t) = z(t)$. From (15), we get

$$z^{\Delta}(t) = -\frac{\alpha + 2\lambda}{\sigma(t) + \mu\lambda} z(t) + e_{\ominus \frac{\lambda}{t}}(t, 0) \frac{f(t)}{(\sigma(t) + \mu\lambda)(t + \mu\lambda)}. \tag{16}$$

For simplicity, we put $m(t) = -\frac{\alpha+2\lambda}{\sigma(t)+\mu\lambda}$, $p(t) = \frac{\lambda}{t}$. By the regressivity condition (3), if λ_1 and λ_2 are two roots of the characteristic equation (2), then $\frac{\lambda_1}{t}, \frac{\lambda_2}{t} \in \mathcal{R}$. Thus, it is easy to verify that

$$1 + \mu \cdot m(t) = \frac{t + \mu(1 - \alpha - \lambda)}{t + \mu(\lambda + 1)} = \frac{t + \mu(1 - \alpha - \lambda)}{\sigma(t) + \mu\lambda} \neq 0,$$

since $\frac{\lambda}{\sigma(t)} \in \mathcal{R}$ and $1-\alpha-\lambda$ is another root of the characteristic equation (2). Then, the exponential function $e_m(t, t_0)$ ($t_0 = \inf \mathbb{T}$) is well-defined.

Note that the equation (16) is a first order linear dynamic equation, the general solution is given by

$$\begin{aligned} z(t) &= c_1 e_m(t, t_0) + \int_{t_0}^{t} e_m(t, \tau) \frac{1}{1 + \mu m(\tau)} \frac{e_{\ominus p}(\tau, 0) f(\tau)}{(\sigma(\tau) + \mu\lambda)(\tau + \mu\lambda)} \Delta\tau \\ &= c_1 e_m(t, t_0) + \int_{t_0}^{t} \frac{e_m(t, \tau) f(\tau)}{e_p(\tau, 0)(\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda)} \Delta\tau, \end{aligned} \tag{17}$$

where c_1 is an arbitrary constant. Integrating both sides of (17) from t_0 to t with respect to ω, we have

$$u(t) = c_2 + c_1 \int_{t_0}^{t} e_m(\omega, t_0) \Delta\omega + \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{e_m(\omega, \tau) f(\tau)}{e_p(\tau, 0)(\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda)} \Delta\tau \Delta\omega, \tag{18}$$

where c_2 is an arbitrary constant. Multiplying both sides of (18) by $e_p(t, 0)$, we conclude that

$$y(t) = c_2 e_p(t, 0) + c_1 e_p(t, 0) \int_{t_0}^{t} e_m(\omega, t_0) \Delta\omega + \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{e_p(t, \tau) e_m(\omega, \tau) f(\tau)}{(\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda)} \Delta\tau \Delta\omega. \tag{19}$$

Through the above argument, we can obtain the following result:

Theorem 2.1. *Let $\mathbb{T} \subset (0, \infty)$ be a time scale with the constant graininess function μ. Let $\alpha, \beta \in \mathbb{R}$ such that $(\alpha - 1)^2 - 4\beta \geq 0$. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a rd-continuous function. If λ is a root of the characteristic equation (2) and the regressivity conditions (3) and (4) are satisfied, then the function $y(t)$ defined by (19) is the general solution of the inhomogeneous Euler-Cauchy equation (1).*

2.2 The linear variable graininess function $\mu(t) = \eta t$

In fact, the formulas (5)-(12) are still valid except (10). In this case, the formula (8) is simplified as

$$x^{\sigma}(t) = (1 + \eta\lambda) x(t) \tag{20}$$

Then, we deduce that

$$x^{\sigma\Delta}(t) = (x^{\sigma})^{\Delta}(t) = (1 + \eta\lambda) x^{\Delta}(t) = \frac{\lambda(1 + \eta\lambda)}{t} x(t). \tag{21}$$

Analogously, we can infer that

$$\begin{aligned}
&t\sigma(t)y^{\Delta\Delta}(t) + \alpha t y^{\Delta}(t) + \beta y(t) \\
&= u^{\Delta\Delta}(t)[(\sigma(t) + \eta\lambda t)(t + \eta\lambda t)]x(t) + u^{\Delta}(t)[(1 + \eta\lambda)(\lambda t + \sigma(t)\lambda + \alpha t)]x(t) \\
&\quad + u(t)[\lambda^2 + (\alpha - 1)\lambda + \beta]x(t) \\
&= u^{\Delta\Delta}(t)[(\sigma(t) + \eta\lambda t)(t + \eta\lambda t)]x(t) + u^{\Delta}(t)[(1 + \eta\lambda)(\lambda t + \sigma(t)\lambda + \alpha t)]x(t) \\
&= f(t).
\end{aligned} \tag{22}$$

Notice that $\frac{\lambda}{\sigma(t)}, \frac{\lambda}{t} \in \mathcal{R}$ implies $(\sigma(t) + \eta\lambda t)(t + \eta\lambda t) \neq 0$. Thus, it follows that

$$u^{\Delta\Delta}(t) = -\frac{\lambda + \alpha + \eta + 1}{(\eta + 1 + \eta\lambda)t} u^{\Delta}(t) + \frac{e_{\ominus p}(t, 0) f(t)}{(\eta + 1 + \eta\lambda)(1 + \eta\lambda)t^2}. \tag{23}$$

Setting $n(t) = -\frac{\lambda+\alpha+\eta+1}{(\eta+1+\eta\lambda)t}$. If we assume that $\eta^2 - \alpha\eta - 1 \neq 0$, then we have

$$1 + \mu(t)n(t) = 1 - \frac{\eta(\lambda + \alpha + \eta + 1)}{\eta + 1 + \eta\lambda} = \frac{1 - \alpha\eta - \eta^2}{\eta + 1 + \eta\lambda} \neq 0.$$

Consequently, the exponential function $e_n(t, t_0)$ is well-defined. Letting $u^{\Delta}(t) = z(t)$. we know that (23) is a first order linear dynamic equation. And then, the general solution is given by

$$\begin{aligned}
z(t) &= c_1 e_n(t, t_0) + \int_{t_0}^{t} e_n(t, \tau) \frac{1}{1 + \mu(\tau)n(\tau)} \frac{e_{\ominus p}(\tau, 0) f(\tau)}{(\eta + 1 + \eta\lambda)(1 + \eta\lambda)\tau^2} \Delta\tau \\
&= c_1 e_n(t, t_0) + \int_{t_0}^{t} e_n(t, \tau) \frac{e_{\ominus p}(\tau, 0) f(\tau)}{(1 - \alpha\eta - \eta^2)(1 + \eta\lambda)\tau^2} \Delta\tau,
\end{aligned} \tag{24}$$

where c_1 is an arbitrary constant. Integrating both sides of (24) from t_0 to t with respect to ω, we can infer that

$$u(t) = c_2 + c_1 \int_{t_0}^{t} e_n(\omega, t_0)\Delta\omega + \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{e_n(\omega, \tau) f(\tau)}{e_p(\tau, 0)(1 - \alpha\eta - \eta^2)(1 + \eta\lambda)\tau^2} \Delta\tau\Delta\omega, \tag{25}$$

where c_2 is an arbitrary constant. Multiplying both sides of (25) by $e_p(t, 0)$, we have that

$$y(t) = c_2 e_p(t, 0) + c_1 e_p(t, 0) \int_{t_0}^{t} e_n(\omega, t_0)\Delta\omega + \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{e_p(t, \tau) e_n(\omega, \tau) f(\tau)}{(1 - \alpha\eta - \eta^2)(1 + \eta\lambda)\tau^2} \Delta\tau\Delta\omega. \tag{26}$$

Based on the foregoing analysis, the following theorem can be formulated.

Theorem 2.2. *Let $\mathbb{T} \subset (0, \infty)$ be a time scale with the linear variable graininess function $\mu(t) = \eta t$, η is a constant. Let $\alpha, \beta \in \mathbb{R}$ such that $(\alpha-1)^2 - 4\beta \geq 0$ and $\eta^2 - \alpha\eta - 1 \neq 0$. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a rd-continuous function. If λ is a root of the characteristic equation (2) and the regressivity conditions (3) and (4) are satisfied, then the function $y(t)$ defined by (26) is the general solution of the inhomogeneous Euler-Cauchy equation (1).*

3 Ulam stability of (1)

In this section, we shall prove the Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation (1) on the time scale with different graininess functions.

Theorem 3.1. *Let $\varphi : \mathbb{T} \to \mathbb{R}^+$ be a function such that the integral*

$$\int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\varphi(\tau)}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega \tag{27}$$

exists for any $t \in \mathbb{T}^{\kappa}$. Under the hypothesis of Theorem 2.1, if a twice rd-continuously differential function $y_\varphi : \mathbb{T} \to \mathbb{R}$ satisfies the following inequality

$$\Big|t\sigma(t)y_\varphi^{\Delta\Delta}(t) + \alpha t y_\varphi^{\Delta}(t) + \beta y_\varphi(t) - f(t)\Big| \le \varphi(t) \tag{28}$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that

$$|y_\varphi(t) - y(t)| \le \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\varphi(\tau)}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega \tag{29}$$

for all $t \in \mathbb{T}^{\kappa^2}$.

Proof. For the sake of convenience, we write

$$t\sigma(t)y_\varphi^{\Delta\Delta}(t) + \alpha t y_\varphi^{\Delta}(t) + \beta y_\varphi(t) := f_\varphi(t). \tag{30}$$

From (28), we get

$$|f_\varphi(t) - f(t)| \le \varphi(t) \tag{31}$$

for all $t \in \mathbb{T}^{\kappa^2}$. By Theorem 2.1 and (30), there exists $c_1, c_2 \in \mathbb{R}$ such that

$$y_\varphi(t) = c_2 e_p(t,0) + c_1 e_p(t,0)\int_{t_0}^{t} e_m(\omega,t_0)\Delta\omega + \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{e_p(t,\tau)e_m(\omega,\tau)f_\varphi(\tau)}{(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)}\Delta\tau\Delta\omega, \tag{32}$$

where m and p are given as in Section 2.1.

Define

$$y(t) := c_2 e_p(t,0) + c_1 e_p(t,0)\int_{t_0}^{t} e_m(\omega,t_0)\Delta\omega + \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{e_p(t,\tau)e_m(\omega,\tau)f(\tau)}{(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)}\Delta\tau\Delta\omega \tag{33}$$

for all $t \in \mathbb{T}^{\kappa^2}$. From (31), (32) and (33), it follows that

$$\begin{aligned}
|y_\varphi(t) - y(t)| &\le \Big|\int_{t_0}^{t}\int_{t_0}^{\omega} \frac{e_p(t,\tau)e_m(\omega,\tau)(f_\varphi(\tau)-f(\tau))}{(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)}\Delta\tau\Delta\omega\Big| \\
&\le \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\big|f_\varphi(\tau)-f(\tau)\big|}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega \\
&\le \int_{t_0}^{t}\int_{t_0}^{\omega} \frac{\big|e_p(t,\tau)e_m(\omega,\tau)\big|\varphi(\tau)}{\big|(\tau+\mu\lambda)(\sigma(\tau)+\mu\lambda)\big|}\Delta\tau\Delta\omega.
\end{aligned}$$

The proof of the theorem is now completed. □

In particular, Theorem 3.1 implies the Hyers-Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation (1) when the time scale is bounded and has a constant graininess function.

Corollary 3.2. *Let $\mathbb{T} \subset (0, \infty)$ be a bounded time scale with the constant graininess function μ and let $\inf \mathbb{T} = t_0$, $\sup \mathbb{T} = b$. Under the hypothesis of Theorem 2.1, for a given $\varepsilon > 0$, if a twice rd-continuously differential function $y_\varepsilon : \mathbb{T} \to \mathbb{R}$ satisfies the following inequality*

$$\left| t\sigma(t) y_\varphi^{\Delta\Delta}(t) + \alpha t y_\varphi^{\Delta}(t) + \beta y_\varphi(t) - f(t) \right| \le \varepsilon \tag{34}$$

for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that

$$|y_\varepsilon(t) - y(t)| \le K\varepsilon \tag{35}$$

for all $t \in \mathbb{T}^{\kappa^2}$, where

$$K = \int_{t_0}^{b} \int_{t_0}^{b} \frac{\left| e_p(t,\tau) e_m(\omega,\tau) \right|}{\left| (\tau + \mu\lambda)(\sigma(\tau) + \mu\lambda) \right|} \Delta\tau \Delta\omega.$$

.

Theorem 3.3. *Let $\varphi : \mathbb{T} \to \mathbb{R}^+$ be a function such that the integral*

$$\int_{t_0}^{t} \int_{t_0}^{\omega} \frac{\left| e_p(t,\tau) e_n(\omega,\tau) \right| \varphi(\tau)}{\left| (1 - \alpha\eta - \eta^2)(1 + \eta\lambda) \right| \tau^2} \Delta\tau \Delta\omega$$

exists for any $t \in \mathbb{T}^\kappa$. Under the hypothesis of Theorem 2.2, if a twice rd-continuously differential function $y_\varphi : \mathbb{T} \to \mathbb{R}$ satisfies the inequality (28) for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that

$$|y_\varphi(t) - y(t)| \le \int_{t_0}^{t} \int_{t_0}^{\omega} \frac{\left| e_p(t,\tau) e_n(\omega,\tau) \right| \varphi(\tau)}{\left| (1 - \alpha\eta - \eta^2)(1 + \eta\lambda) \right| \tau^2} \Delta\tau \Delta\omega$$

for all $t \in \mathbb{T}^{\kappa^2}$.

Proof. According to Theorem 2.2, this theorem can be proved by the same method as employed in Theorem 3.1. □

From Theorem 3.3, we can obtain the Hyers-Ulam stability of the inhomogeneous Euler-Cauchy dynamic equation (1) if the time scale is bounded and has a linear graininess function.

Corollary 3.4. *Let $\mathbb{T} \subset (0, \infty)$ be a bounded time scale with the linear variable graininess function $\mu(t) = \eta t$ and let $\inf \mathbb{T} = t_0$, $\sup \mathbb{T} = b$. Under the hypothesis of Theorem 2.2, for a given $\varepsilon > 0$, if a twice rd-continuously differential function $y_\varepsilon : \mathbb{T} \to \mathbb{R}$ satisfies the inequality (34) for all $t \in \mathbb{T}^{\kappa^2}$, then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the inhomogeneous Euler-Cauchy dynamic equation (1) such that*

$$|y_\varepsilon(t) - y(t)| \le L\varepsilon$$

for all $t \in \mathbb{T}^{\kappa^2}$, where

$$L = \int_{t_0}^{b} \int_{t_0}^{b} \frac{\left| e_p(t,\tau) e_n(\omega,\tau) \right|}{\left| (1 - \alpha\eta - \eta^2)(1 + \eta\lambda) \right| \tau^2} \Delta\tau \Delta\omega.$$

.

Appendix A.

Several common time scales and the corresponding graininess functions are given below (see Table 1).

$\mathbb{T}$	$\sigma(t)$	$\mu(t)$	Attribute of μ
$\mathbb{R}$	t	0	Constant
$\mathbb{Z}$	$t+1$	1	Constant
$h\mathbb{Z}$	$t+h$	h	Constant
$q^{\mathbb{N}}$	qt	$(q-1)t$	Linearity
$2^{\mathbb{N}}$	$2t$	t	Linearity

Table 1: Time scales and graininess functions

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Some existence theorems of generalized vector variational-like inequalities in fuzzy environment

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Abstract

In this paper, we establish two versions of the existence theorems of solutions set of generalized vector variational-like inequalities in fuzzy environment by using two different notions; the first one by using affineness and the second by using the notion of vector O-diagonally convexity. Moreover, an example is established in order to illustrate the main problem. The results of this paper can be viewed as a significant improvement and refinement of several other previously existing known results.

Keywords: Generalized vector variational-like inequality; KKM-mapping; Vector O-diagonally convex; Affine mapping; Fuzzy upper semicontinuous mapping

1. Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation, and structural analysis, see for instance, [5, 7, 20, 23] and the references therein. It seems this theory began by Browder [8] in 1966, by formulating and proving some basic existence theorems of solutions to a class of nonlinear variational inequalities. Since then, Liu et al. [29], Zhao et al. [26] and Ahmad et al. [1] extended Browder's results to more generalized nonlinear variational inequalities. In 2010, Xiao et al. [36] extended the results of Zhao et al. to generalized vector nonlinear variational-like inequalities with set-valued mappings.

In 1965, the concept of fuzzy sets were introduced by Zadeh [9] to manipulate data and information possessing nonstatistical uncertainties. The applications of fuzzy set theory can be found in many branches of mathematical and engineering sciences including artificial intelligence, management science, control engineering, computer science, see e.g. [37]. Heilpern [22] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mapping which is ananalogue of Nadler's fixed point theorem for multi-valued mappings. In 1989, Chang and Zhu [10] introduced the concept of variational inequalities for fuzzy mappings in abstract spaces and investigated the existence problem for solutions of some classes of inequalities for fuzzy mappings.

Recently Chang et al. [13] introduced and studied a new class of generalized vector variational-like inequalities in fuzzy environment and generalized vector variational inequalities in fuzzy environment. They obtained some existence results for the problems. Several kinds of variational

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inequalities and complementarity problems for fuzzy mapping were studied by Chang et al. [11], Chang and Salahuddin [12], Anastassiou and Salahuddin [4], Ahmad et al. [3], Verma and Salahuddin [35], Lee et al. [25, 28], Park et al. [31], Khan et al. [21], Ding et al. [18] and Lan and Verma [24].

Motivated and inspired by ongoing research in this direction, the purpose of this paper is to present two versions of the existence theorems for the generalized vector variational-like inequalities in fuzzy environment. The paper can be viewed as an alternative version which related to [13] by providing some new suitable conditions and methods for proving the main results.

2. Preliminaries

Let X be a nonempty set. We recall that a fuzzy set A in X is characterized by a function $\mu_A : X \to [0,1]$, called membership function of A, "which associates with each point x in X a real number in the interval $[0,1]$, with the value of μ_A at x representing the grade of membership of x in A": see [9, p.339]. Obviously, any crisp subset A of X can be viewed as a fuzzy set, where μ_A is such that $\mu_A(x) = 1$ when $x \in A$ and $\mu_A(x) = 0$ otherwise. Let E be a nonempty subset of a vector space V and D be a nonempty set. A mapping F from D into the collection $\mathfrak{F}(E)$, of all fuzzy sets of E, is called a *fuzzy* mapping. If $F : D \to \mathfrak{F}(E)$ is a fuzzy mapping, then $F(y)$, for each $y \in D$, is a fuzzy set in $\mathfrak{F}(E)$. So, the fuzzy mapping F can be identified with the function from $E \times D$ to $[0,1]$ which assigns with each $(x,y) \in E \times D$ the degree of membership of x in the fuzzy set $F(y)$, that is the number $F(x,y) = \mu_{F(y)}(x)$.

Let $A \in \mathfrak{F}(E)$ and $\alpha \in [0,1]$, then the set

$$(A)_\alpha = \{x \in E : A(x) \geq \alpha\}$$

is called an α-cut set of A.

In the sequel, we assume that Z and E are Hausdorff topological vector spaces. We denote by $L(E,Z)$ the space of all continuous linear operators from E into Z and $\langle l, x\rangle$, the evaluation of $l \in L(E,Z)$ at $x \in E$. We consider each topology on $L(E,Z)$ such that $L(E,Z)$ becomes a topological vector space and the bilinear mapping is continuous. Denote by $intA$ and coA the interior and convex hull of a set A, respectively. Let K be a nonempty convex subset of a Hausdorff topological vector space E and $C : K \to 2^Z$ be a set-valued mapping such that $C(x) \neq Z$ and $intC(x) \neq \emptyset$, for each $x \in K$. Let $\theta : K \times K \to E$ and $g : K \to K$ be the vector-valued mappings. Let $M, S, T : K \to \mathfrak{F}(L(E,Z))$ be the fuzzy mappings and $a, b, c : K \to [0,1]$ are the mappings. It is clear that the convex cone $C(x)$ of Z induces an ordering on Z which is denoted by $\leq_{C(x)}$ and defined as follows

$$y_1 \leq_{C(x)} y_2 \quad \text{if and only if} \quad y_2 - y_1 \in C(x), \text{ where } y_1, y_2 \in Z.$$

The rest of this section will deal with some definitions and basic results which are needed in the sequeul.

In this paper we are interested in studying the following problem.

Problem: [13] The "so called" *Generalized vector variational-like inequality problem in fuzzy environment* (GVVLIFE) (2.1) is to find an $x \in K, u \in (M(x))_{a(x)}, v \in (S(x))_{b(x)}$ and $w \in (T(x))_{c(x)}$ such that

$$\langle N(u,v,w), \theta(y, g(x))\rangle + \eta(g(x), y) \nsubseteq -intC(x), \ \ \forall y \in K, \tag{2.1}$$

where $M, S, T : K \to \mathfrak{F}(L(E,Z))$ are fuzzy mappings, $a, b, c : K \to [0,1]$, $\theta : K \times K \to E$, $\eta : K \times K \to 2^Z$, $g : K \to K$ and $N : L(E,Z) \times L(E,Z) \times L(E,Z) \to 2^{L(E,Z)}$ are mappings.

The following example is provided to illustrate Problem (2.1).

Example 2.1. Let $E = Z = \mathbb{R}$, $K = [0, +\infty)$, $C(x) = [0, +\infty)$, $\forall x \in K$. Define $M, S, T : K \to \mathfrak{F}(L(\mathbb{R}, \mathbb{R}) \equiv \mathbb{R})$ by

$$\mu_{M(x)}(u) = \begin{cases} \frac{1}{1+(u-1)^2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{1+x(u-2)^2}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$\mu_{S(x)}(v) = \begin{cases} \frac{1}{1+(v-1)^2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{2+x(v-2)^2}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$\mu_{T(x)}(w) = \begin{cases} \frac{1}{1+(w-1)^2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{3+x(w-2)^2}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

and $a, b, c : K \to [0, 1]$ as

$$a(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{1+x}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$b(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{2+x}, & \text{if} \quad x \in (1,+\infty), \end{cases}$$

$$c(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in [0,1], \\ \frac{1}{3+x}, & \text{if} \quad x \in (1,+\infty). \end{cases}$$

It is not hard to check that for any $x \in [0, 1]$, we have

$$(M(x))_{a(x)} = (M(x))_{\frac{1}{2}} = \left\{u \in \mathbb{R} \,\middle|\, \mu_{M(x)}(u) \geq \tfrac{1}{2}\right\} = \left\{u \in \mathbb{R} \,\middle|\, \tfrac{1}{1+(u-1)^2} \geq \tfrac{1}{2}\right\} = [0, 2],$$

$$(S(x))_{b(x)} = (S(x))_{\frac{1}{2}} = \left\{v \in \mathbb{R} \,\middle|\, \mu_{S(x)}(v) \geq \tfrac{1}{2}\right\} = \left\{v \in \mathbb{R} \,\middle|\, \tfrac{1}{1+(v-1)^2} \geq \tfrac{1}{2}\right\} = [0, 2],$$

$$(T(x))_{c(x)} = (T(x))_{\frac{1}{2}} = \left\{w \in \mathbb{R} \,\middle|\, \mu_{M(x)}(w) \geq \tfrac{1}{2}\right\} = \left\{w \in \mathbb{R} \,\middle|\, \tfrac{1}{1+(w-1)^2} \geq \tfrac{1}{2}\right\} = [0, 2],$$

whereas $x \in (1, \infty)$, we have

$$\begin{aligned} (M(x))_{a(x)} = (M(x))_{\frac{1}{1+x}} &= \left\{u \in \mathbb{R} \,\middle|\, \mu_{M(x)}(u) \geq \tfrac{1}{1+x}\right\} = \left\{u \in \mathbb{R} \,\middle|\, \tfrac{1}{1+x(u-2)^2} \geq \tfrac{1}{1+x}\right\} \\ &= \left\{u \in \mathbb{R} \,\middle|\, (u-2)^2 \leq 1\right\} = [1, 3], \end{aligned}$$

$$\begin{aligned} (S(x))_{b(x)} = (S(x))_{\frac{1}{2+x}} &= \left\{v \in \mathbb{R} \,\middle|\, \mu_{S(x)}(v) \geq \tfrac{1}{2+x}\right\} = \left\{v \in \mathbb{R} \,\middle|\, \tfrac{1}{2+x(v-2)^2} \geq \tfrac{1}{2+x}\right\} \\ &= \left\{v \in \mathbb{R} \,\middle|\, (v-2)^2 \leq 1\right\} = [1, 3], \end{aligned}$$

$$\begin{aligned} (T(x))_{c(x)} = (T(x))_{\frac{1}{3+x}} &= \left\{w \in \mathbb{R} \,\middle|\, \mu_{T(x)}(v) \geq \tfrac{1}{3+x}\right\} = \left\{w \in \mathbb{R} \,\middle|\, \tfrac{1}{3+x(w-2)^2} \geq \tfrac{1}{3+x}\right\} \\ &= \left\{w \in \mathbb{R} \,\middle|\, (w-2)^2 \leq 1\right\} = [1, 3], \end{aligned}$$

Now, we define $N : L(E, Z) \times L(E, Z) \times L(E, Z) \to 2^{L(E,Z)}$ by

$$N(u, v, w) = \{u + v + w\} \text{ for all } u, v, w \in L(E, Z)(= L(\mathbb{R}, \mathbb{R}) \equiv \mathbb{R}),$$

$g : K \to K$ by

$$g(x) = \tfrac{x}{2}, \forall\, x \in K,$$

$\theta : K \times K \to E$ by

$$\theta(x, y) = \tfrac{x}{2} - y, \forall x, y \in K,$$

and $\eta : K \times K \to 2^Z$ by

$$\eta(x, y) = \left\{ \tfrac{y}{2} - x \right\}, \forall x, y \in K.$$

Then, let us consider in the following 2 cases:

Case I, $x \in [0, 1]$, $u \in (M(x))_{\frac{1}{2}} = [0, 2]$, $v \in (S(x))_{\frac{1}{2}} = [0, 2]$ and $w \in (T(x))_{\frac{1}{2}} = [0, 2]$.

$$\begin{aligned}\langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) &= \left\langle u + v + w, \theta\left(y, \frac{x}{2}\right)\right\rangle + \eta\left(\frac{x}{2}, y\right)\\ &= (u + v + w)\left(\frac{y}{2} - \frac{x}{2}\right) + \left(\frac{y}{2} - \frac{x}{2}\right)\\ &= (u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}(u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right) \geq 0 &\Leftrightarrow y - x \geq 0\\ &\Leftrightarrow x \leq y, \quad \forall y \in K.\end{aligned}$$

This implies that $x = 0$ is a solution of the generalized vector variational-like inequality problem in fuzzy environment (GVVLIFE) (2.1).

Case II, $x \in (1, +\infty)$, $u \in (M(x))_{\frac{1}{1+x}} = [1, 3]$, $v \in (S(x))_{\frac{1}{2+x}} = [1, 3]$ and $w \in (T(x))_{\frac{1}{3+x}} = [1, 3]$.

$$\begin{aligned}\langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) &= \left\langle u + v + w, \theta\left(y, \frac{x}{2}\right)\right\rangle + \eta\left(\frac{x}{2}, y\right)\\ &= (u + v + w)\left(\frac{y}{2} - \frac{x}{2}\right) + \left(\frac{y}{2} - \frac{x}{2}\right)\\ &= (u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}(u + v + w + 1)\left(\frac{y}{2} - \frac{x}{2}\right) \geq 0 &\Leftrightarrow y - x \geq 0\\ &\Leftrightarrow x \leq y, \quad \forall y \in K.\end{aligned}$$

This implies that in the Case II, there is no solution for (GVVLIFE) (2.1). Therefore, from the Case I, we obtain that generalized vector variational-like inequality problem in fuzzy environment (GVVLIFE) (2.1) has a solution and a solution set is $\{0\}$.

Some special cases of GVVLIFE:

(i) Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be classical set-valued mappings. If the fuzzy sets $M(x)$, $S(x)$ and $T(x)$ as in the previous problem become the characteristic functions $\mathcal{X}_{\widetilde{M}(x)}$, $\mathcal{X}_{\widetilde{S}(x)}$ and $\mathcal{X}_{\widetilde{T}(x)}$, respectively. Together with $a(x) = b(x) = c(x) = 1$, for all $x \in K$ and $g : K \to K$ an identity mapping, then Problem (2.1) reduces to *generalized nonlinear vector variational-like* *inequality problems (GNVVLIP, in short)*: finding $x \in K$, $u \in \widetilde{M}(x)$, $v \in \widetilde{S}(x)$, $w \in \widetilde{T}(x)$ such that

$$\langle N(u,v,w), \theta(y,x)\rangle + \eta(x,y) \nsubseteq -intC(x), \ \ \forall y \in K. \tag{2.2}$$

This kind of problem was in considered and studied by Xiao et al. [36].

(ii) If $\theta(y, g(x)) = y - g(x)$, then (2.1) is equivalent to the problem of finding an $x \in K$, $u \in (M(x))_{a(x)}$, $v \in (S(x))_{b(x)}$, $w \in (T(x))_{c(x)}$ such that

$$\langle N(u,v,w), y - g(x)\rangle + \eta(g(x), y) \nsubseteq -intC(x), \ \ \forall y \in K. \tag{2.3}$$

This kind of problem was introduced and studied by Chang et al. [13].

(iii) If E is a Banach space and K is a nonempty convex subset of E, let $Z = \mathbb{R}, E^* = L(E,Z), b : K \times K \to \mathbb{R}$ be a real valued mapping and M, S, $T : K \to E^*$ be the single valued mappings. For a given $w^* \in E^*, N(u,v,w) = N(T(x), S(x)) - M(x) + w^*$, $\eta(x,y) = b(x,y) - b(x,x)$, $C(x) = \mathbb{R}^+$ for all $x \in K$, then (2.2) is equivalent to the problem of finding $x \in K$ such that

$$\langle N(T(x), S(x)) - M(x) + w^*, \theta(y,x)\rangle + b(x,y) - b(x,x) \geq 0, \ \ \forall y \in K.$$

This problem was considered by Zhao et al. [26].

(iv) Let E is a real Hilbert space and K is a nonempty convex subset of E. Let $Z = \mathbb{R}$, $C(x) = \mathbb{R}^+$ for all $x \in K$, $\eta(x,y) = \phi(y,x) - \phi(x,x)$ and $T(x) = \emptyset$ for all $x \in K$, then (2.2) is equivalent to finding $x \in K$, $u \in M(x)$ and $v \in S(x)$ such that

$$\langle N(u,v), \theta(y,x)\rangle + \phi(x,y) - \phi(x,x) \geq 0, \ \ \forall y \in K. \tag{2.4}$$

(v) If $N(u,v) = M(x) - S(x)$, where M, S are single valued mappings, then (2.4) collapses to finding $x \in K$ such that

$$\langle M(x) - S(x), \theta(y,x)\rangle + \phi(x,y) - \phi(x,x) \geq 0, \ \ \forall y \in K.$$

This kind of problem was introduced and studied by Ding [16].

(vi) If $N(u,v) = u$, then (2.4) reduces to the problem of finding $x \in K, u \in M(x)$ such that

$$\langle u, \theta(y,x)\rangle + \phi(x,y) - \phi(x,x) \geq 0, \ \ \forall y \in K. \tag{2.5}$$

This kind of problem was studied by Ding [17].

(vii) If $\phi \equiv 0$, then (2.5) reduces to the problem of finding $x \in K$ and $u \in M(x)$ such that

$$\langle u, \theta(y,x)\rangle \notin -intC(x), \ \ \forall y \in K. \tag{2.6}$$

This problem was considered by Ding et al. [19].
If, in addition, M is a single valued mapping, then it is equivalent to finding $x \in K$, such that

$$\langle M(x), \theta(y,x)\rangle \notin -intC(x), \ \ \forall y \in K,$$

which was studied by Salahuddin [32].

(viii) Moreover, if $\theta(y,x) = y - x$, then (2.6) reduces to finding $x \in K$ such that

$$\langle u, y - x\rangle \notin -intC(x), \ \ \forall y \in K,$$

which was studied by Lee et al. [27].

Clearly, generalized vector variational-like inequality problem in fuzzy environment includes many variational inequalities problems in the recent past.

Definition 2.2 ([13])**.** A mapping $f : K \to Z$ is $C(x)$-convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

that is,

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x).$$

Remark 2.3.

(i) In the case of $C(x) = C$, for all $x \in K$ where C is a convex in Z. Then Definition 2.2 reduces the usual definition of the vector convexity for the mapping f, i.e., $f : K \to Z$ is convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq_C tf(x_1) + (1-t)f(x_2),$$

that is

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C.$$

(ii) By taking $Z = \mathbb{R}$ and $C = [0, +\infty)$ in (i), Definition 2.2 reduces to the definition of the convex function, i.e., a mapping $f : K \to \mathbb{R}$ is convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \geq 0.$$

Definition 2.4 ([36])**.** Let X, Y be two topological spaces, $T : X \to 2^Y$ be a set-valued mapping. T is said to be:

(i) *Upper semicontinuous*, if for each $x \in X$ and each open set V in Y with $T(x) \subseteq V$, then there exists an open neighborhood U of x in X such that $T(u) \subseteq V$, for each $u \in U$.

(ii) *Closed*, if for any net $\{x_\alpha\}$ in X such that $x_\alpha \to x$ and any net $\{y_\alpha\}$ in Y such that $y_\alpha \to y$ and $y_\alpha \in T(x_\alpha)$ for any α, we have $y \in T(x)$, or equivalently, T is said to have a closed graph, if the graph of $T, Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Lemma 2.5 ([33])**.** *Let X, Y be two topological spaces and $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there exist $y_0 \in T(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 2.6 (Aubin [6])**.** *Let X and Y be two topological spaces. If $T : X \to 2^Y$ is an upper semicontinuous set-valued mapping with closed values, then T is closed.*

Definition 2.7 ([25])**.** Let X, Y be topological spaces and $T : X \to \mathfrak{F}(Y)$ be a fuzzy mapping. T is said to have fuzzy set-valued, if $T_x(y)$ is upper semicontinuous on $X \times Y$ as a real ordinary function.

Remark 2.8. If A is a closed subset of a topological space X, then the characteristic function $\mathcal{X}_A$ of A, $\mathcal{X}_A(x) = 1$ if $x \in A$ otherwise $\mathcal{X}_A(x) = 0$, is an upper semicontinuous function.

Lemma 2.9 ([21])**.** *Let K be a nonempty closed convex subset of a real Hausdorff topological space X, E be a nonempty closed convex subset of real Hausdorff topological space Y and $a : X \to [0, 1]$ be a lower semicontinuous function. Let $T : K \to \mathfrak{F}(E)$ be a fuzzy mapping with $(T(x))_{a(x)} \neq \emptyset$ for all $x \in X$ and $\widetilde{T} : K \to 2^E$ be a set-valued defined by $\widetilde{T}(x) = (T(x))_{a(x)}$. If T is a closed set-valued mapping, then $\widetilde{T}$ is a closed set-valued mapping.*

Definition 2.10 ([14, 30])**.** Let K be a convex subset of a topological vector space E, and Z be a topological vector space. Let $C : K \to 2^Z$ be a set-valued mapping. For any given finite subset $\Omega = \{x_1, x_2, ..., x_n\}$ of K, and any $x = \sum_{i=1}^{n} t_i x_i$ with $t_i \geq 0$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} t_i = 1$,

(i) a single valued mapping $h : K \times K \to Z$ is said to be *vector O-diagonally convex in the second variable*, if

$$\sum_{i=1}^{n} t_i h(x, x_i) \notin -intC(x),$$

(ii) a set-valued mapping $h : K \times K \to 2^Z$ is said to be *generalized vector O-diagonally convex in the second variable* if

$$\sum_{i=1}^{n} t_i u_i \notin -intC(x), \ \ \forall u_i \in h(x, x_i), \ \ i = 1, 2, ..., n.$$

Definition 2.11 ([2]). Let K be a nonempty of convex subset of a vector space X. A mapping $g : K \to K$ is said to be *affine* if for all $x_1, x_2, ..., x_m \in K$ and $\lambda_i \geq 0$ for all $i = 1, 2, ..., m$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$ such that

$$g\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i g(x_i).$$

The following examples show that notion of affine and vector O-diagonally convex are independent functions.

Example 2.12. Let $K = Z = \mathbb{R}$. Define the function $h : K \times K \to Z$ by

$$h(x, y) = \begin{cases} -1, & \text{if} \quad x \in \mathbb{Q}, \\ 0, & \text{if} \quad x \in \mathbb{Q}^c, \end{cases}$$

where $\mathbb{Q}$ and $\mathbb{Q}^c$ are rational numbers and irrational numbers respectively. It is clear that h is affine but it is not vector O-diagonally convex in the second variable.

Example 2.13. Let $K = Z = \mathbb{R}$. Define the function $h : K \times K \to Z$ by

$$h(x, y) = y^2.$$

It is easy to see that h is vector O-diagonally convex in the second variable but h is not affine.

From the above examples, it is noticed that

$$\boxed{\text{Affine} \nRightarrow \text{Vector O-diagonally convex in the second variable}}$$

and

$$\boxed{\text{Vector O-diagonally convex in the second variable} \nRightarrow \text{Affine}}.$$

In order to prove our main results we need the following.

Definition 2.14 ([15]). Let K be a subset of a topological vector space X. A set-valued mapping $T : K \to 2^X$ is called *Knaster-Kuratowski-Mazurkiewieg* mapping (KKM Mapping), if for each nonempty finite subset $\{x_1, x_2, ..., x_n\} \subseteq K$, we have $Co\{x_1, x_2, ..., x_n\} \subseteq \bigcup\limits_{i=1}^{n} T(x_i)$.

Lemma 2.15 ([31, 34], Maximal Element Lemma). *Let X be a nonempty convex subset of a Hausdorff topological vector space E. Let $S : X \to 2^X$ be a set-valued mapping satisfying the following conditions:*

(i) *for each $x \in X, x \notin coS(x)$ and for each $y \in X, S^{-1}(y)$ is open-valued in X;*
(ii) *there exist a nonempty compact subset A of X and a nonempty compact convex subset $B \subseteq X$ such that*

$$co(S(x)) \cap B \neq \emptyset, \ \ \forall x \in X \setminus A.$$

Then there exists $x_0 \in X$ such that $S(x_0) = \emptyset$.

3. Main results

In this section, two versions of the existence results of generalized vector variational-like inequalities in fuzzy environment are established by employing the Lemma 2.15. Before stating the main results, we need the following preliminary facts.

Lemma 3.1. *Let X be a topological vector space and $C \subseteq X$ be a cone. If $0 \in intC$, then $C = X$.*

Proof. Let $x \in X$ be an arbitrary element. Then there exists $t > 0$ such that $tx \in intC$, (note $0 \in intC$). Since C is a cone, we observe that $x = \frac{1}{t}(tx) \in C$. Thus $C = X$. ■

Lemma 3.2. *Let Z be a topological vector space, K be a nonempty convex subset of a Hausdorff topological vector space E. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M, S, T : K \to \mathfrak{F}(L(E,Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x),} \quad \widetilde{S}(x) = (S(x))_{b(x),} \quad \widetilde{T}(x) = (T(x))_{c(x),} \quad \forall x \in K.$$

Let $N : L(E,Z) \times L(E,Z) \times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K \times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings. Let $P : K \to 2^K$ be a multifunction defined by

$$\begin{aligned} P(x) = \{y \in K : &\langle N(u,v,w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),\\ &\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\}, \quad \forall x \in K, \end{aligned}$$

where η and θ are affine in second and first variable respectively. Then $P(x)$ is convex, for each $x \in K$.

Proof. Let $x \in K$ be an arbitrary element. If y_1, $y_2 \in P(x)$ and $\lambda \in (0,1)$, then

$$\langle N(u,v,w), \theta(y_i, g(x))\rangle + \eta(g(x), y_i) \subseteq -intC(x), \quad \forall i = 1, 2.$$

Hence

$$\langle N(u,v,w), \lambda\theta(y_1, g(x))\rangle + \lambda\eta(g(x), y_1) \subseteq \lambda(-intC(x)), \tag{3.1}$$

$$\langle N(u,v,w), (1-\lambda)\theta(y_2, g(x))\rangle + (1-\lambda)\eta(g(x), y_2) \subseteq (1-\lambda)(-intC(x)). \tag{3.2}$$

By (3.1), (3.2) and since $intC(x)$ is convex cone, we have

$$\langle N(u,v,w), \lambda\theta(y_1, g(x)) + (1-\lambda)\theta(y_2, g(x))\rangle + \lambda\eta(g(x), y_1) + (1-\lambda)\eta(g(x), y_2) \subseteq -intC(x).$$

Since θ is affine in the first variable and η is affine in the second variable, we have

$$\langle N(u,v,w), \theta(\lambda y_1 + (1-\lambda)y_2, g(x)) + \rangle + \eta(g(x), \lambda y_1 + (1-\lambda)y_2) \subseteq -intC(x).$$

So we get $\lambda y_1 + (1-\lambda)y_2 \in P(x)$. This completes the proof. ■

Now, we are ready to state the first version of the existence result for GVVLIFE (2.1).

Theorem 3.3. *Let Z be a topological vector space, K be a nonempty convex subset of a Hausdorff topological vector space E, and $L(E,Z)$ be a topological vector space. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M, S, T : K \to \mathfrak{F}(L(E,Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x)}, \quad \widetilde{S}(x) = (S(x))_{b(x)}, \quad \widetilde{T}(x) = (T(x))_{c(x)}, \quad \forall x \in K.$$

Let $N : L(E,Z) \times L(E,Z) \times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K \times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings. If the following conditions are satisfied:

(i$_a$) *η and θ are affine in second and first variable respectively, with $\eta(g(x), x) = 0$ and $\theta(x, g(x)) = 0$ for all $x \in K$;*

(ii$_a$) *For each $y \in K$, the set-valued mapping*

$$G_y(u, v, w, x) = \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \cap Z\backslash(-intC(x))$$

is upper semicontinuous with compact value;

(iii$_a$) *$C : K \to 2^Z$ is a set-valued mapping with convex values such that $C(x) \neq Z$ for all $x \in K$;*

(iv$_a$) *there exist a nonempty compact subset A of K and a nonempty compact convex subset B of K such that for each $x \in K\backslash A, \exists \bar{y} \in B$ such that*

$$\langle N(u, v, w), \theta(\bar{y}, g(x))\rangle + \eta(g(x), \bar{y}) \subseteq -intC(x),$$

$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)};$$

then the solution set of $GVVLIFE$ (2.1) is a nonempty compact subset of A.

Proof. Let $P : K \to 2^K$ be a set-valued mapping defined by

$$P(x) = \{y \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\} \ \forall x \in K.$$

Firstly, we wish to show that for all $x \in K, x \notin P(x)$. Suppose to the contrary, there is $\hat{x} \in K$ such that $\hat{x} \in P(\hat{x})$. Then

$$\{0\} = \langle N(u, v, w), \theta(\hat{x}, g(\hat{x}))\rangle + \eta(g(\hat{x}), \hat{x}) \subseteq -intC(\hat{x}).$$

We get $0 \in intC(\hat{x})$, and then Lemma 3.1 allows $C(\hat{x}) = Z$ which is contradicted by (iii$_a$). Hence for each $x \in K$, $x \notin P(x)$. By Lemma 3.2, $P(x)$ is convex, that is $P(x) = coP(x)$. Thus $x \notin coP(x)$ for all $x \in K$. Next, we intend to prove that for each $y \in K$, $P^{-1}(y)$ is an open set. To prove this goal, it is sufficient to prove that the complement $(P^{-1}(y))^c$ of $P^{-1}(y)$ is closed in K. It is not hard to verity that

$$P^{-1}(y) = \{x \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\},$$

and

$$(P^{-1}(y))^c = \{x \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \cap Z\backslash(-intC(x)) \neq \emptyset,$$
$$\exists u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\}.$$

Let $\{x_\alpha\}$ be a net in $(P^{-1}(y))^c$ such that $x_\alpha \to x^*$. We wish to show that $x^* \in (P^{-1}(y))^c$. Since $\{x_\alpha\} \subseteq (P^{-1}(y))^c$, there exist $u_\alpha \in \widetilde{M}(x_\alpha) = (M(x_\alpha))_{a(x_\alpha)}$, $v_\alpha \in \widetilde{S}(x_\alpha) = (S(x_\alpha))_{b(x_\alpha)}$, and $w_\alpha \in \widetilde{T}(x_\alpha) = (T(x_\alpha))_{c(x_\alpha)}$ such that

$$\langle N(u_\alpha, v_\alpha, w_\alpha), \theta(y, g(x_\alpha))\rangle + \eta(g(x_\alpha), y) \cap Z\backslash(-intC(x_\alpha)) \neq \emptyset.$$

Thus, we can let a net

$$\{z_\alpha\} \subseteq \langle N(u_\alpha, v_\alpha, w_\alpha), \theta(y, g(x_\alpha))\rangle + \eta(g(x_\alpha), y) \cap Z\backslash(-intC(x_\alpha)).$$

Notice that $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ are upper semicontinuous mappings with compact values. Thus, it follows from Lemma 2.5 that $\{u_\alpha\}, \{v_\alpha\}, \{w_\alpha\}$ have convergent subnets, $\{u_{\alpha_\beta}\}$, $\{v_{\alpha_\beta}\}$, $\{w_{\alpha_\beta}\}$, with limits say u^*, v^*, w^*, respectively, and $u^* \in \widetilde{M}(x^*)$, $v^* \in \widetilde{S}(x^*)$ and $w^* \in \widetilde{T}(x^*)$. Since

$G_y(\cdot,\cdot,\cdot,\cdot)$ is upper semicontinuous with compact values, it can be applied by Lemma 2.5 to produce a subnet $\{z_{\alpha_\beta}\}$ of $\{z_\alpha\}$ such that $z_{\alpha_\beta} \to z^*$ and

$$z^* \in G_y(u^*,v^*,w^*,x^*) = \langle N(u^*,v^*,w^*), \theta(y,g(x^*))\rangle + \eta(g(x^*),y) \cap Z\backslash(-intC(x^*)).$$

This shows that $x^* \in (P^{-1}(y))^c$. Therefore $(P^{-1}(y))^c$ contains all its limit points and then it is closed in K. Thus $P^{-1}(y)$ is an open for each $y \in K$. The desired result is proved.

Next, by employing Lemma 2.15 and condition (iv$_\mathrm{a}$) to ensure the existence of (GVVLIFE) (2.1). By condition (iv$_\mathrm{a}$), we assert that for each $x \in K\backslash A$ there exists a nonempty compact convex subset B of K such that $\bar{y} \in B$ and $\langle N(u,v,w), \theta(\bar{y},g(x))\rangle + \eta(g(x),\bar{y}) \subseteq -intC(x)$, $\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}$. This means that $\bar{y} \in B \cap P(x)$. We know from Lemma 3.2 that $P(x)$ is convex, so we have that $\bar{y} \in coP(x)$. This implies that $\bar{y} \in coP(x) \cap B$ and then $coP(x) \cap B \neq \emptyset$. This shows that P satisfies all the conditions of Lemma 2.15, so there exists $\bar{x} \in K$ such that $P(\bar{x}) = \emptyset$, this means there exists $\bar{x} \in K$, $u \in \widetilde{M}(\bar{x}) = (M(\bar{x}))_{a(\bar{x})}$, $v \in \widetilde{S}(\bar{x}) = (S(\bar{x}))_{b(\bar{x})}$, $w \in \widetilde{T}(\bar{x}) = (T(\bar{x}))_{c(\bar{x})}$ such that

$$\langle N(u,v,w), \theta(y,g(\bar{x}))\rangle + \eta(g(\bar{x}),y) \nsubseteq -intC(\bar{x}), \quad \forall y \in K.$$

Therefore $\bar{x} \in \Omega$ where Ω is the solution set of the generalized vector variational-like inequality in fuzzy environment (GVVLIFE) (2.1). Thus, $\Omega \neq \emptyset$.

To show that Ω is a subset of compact set A. Let $x \in \Omega$. Assume that $x \notin A$, by condition (iv$_\mathrm{a}$), there exists $\bar{y} \in B$ such that

$$\langle N(u,v,w), \theta(\bar{y},g(x))\rangle + \eta(g(x),\bar{y}) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)},\ v \in \widetilde{S}(x) = (S(x))_{b(x)},\ w \in \widetilde{T}(x) = (T(x))_{c(x)},$$

which means that x is not a solution of the problem, that is $x \notin \Omega$. This is a contradiction. Hence $x \in A$ and we obtain that $\Omega \subseteq A$.

Finally, we show that Ω is a compact subset of A. One can observe that $\Omega = (P^{-1}(y))^c$. In fact,

$$\begin{aligned}
\Omega = \{&x \in K : \langle N(u,v,w), \theta(y,g(x))\rangle + \eta(g(x),y) \nsubseteq -intC(\bar{x}),\\
&\exists u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\}\\
= \{&x \in K : \langle N(u,v,w), \theta(y,g(x))\rangle + \eta(g(x),y) \cap Z\backslash(-intC(x)) \neq \emptyset,\\
&\exists u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\}\\
= (&P^{-1}(y))^c.
\end{aligned}$$

Since we have already proved that $(P^{-1}(y))^c$ is closed in K, so we can conclude that Ω is a closed in K. Therefore Ω is a compact subset of A. This completes the proof of Theorem 3.3. ■

Remark 3.4. It can be observed that Theorem 3.3 is as an alternative version of Theorem 3.1 in [13] by replacing vector O-diagonally convexity with the affineness of η. Moreover, some assumptions are not necessary given in Theorem 3.3, for instance, continuity of θ, continuity and affineness of g.

Next, we will present the second version of the existence result of GVVLIFE (2.1). Before doing that we will provide the following lemma in order to be utilized in proving for the next version of the existence result.

Lemma 3.5. *Let Z be a topological vector space, K be a nonempty convex subset of a Hausdorff topological vector space E, and $L(E,Z)$ be a topological vector space. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M, S, T : K \to \mathfrak{F}(L(E,Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x)}, \quad \widetilde{S}(x) = (S(x))_{b(x)}, \quad \widetilde{T}(x) = (T(x))_{c(x)}, \quad \forall x \in K.$$

Let $N : L(E,Z)\times L(E,Z)\times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K\times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings and $P : K \to 2^K$ be a multifunction defined by

$$P(x) = \{y \in K : \langle N(u,v,w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\} \ \ \forall x \in K.$$

If the following conditions are satisfied:

($\mathrm{i_b}$) *η is generalized vector O-diagonally convex in the second argument;*
($\mathrm{ii_b}$) *θ is affine in the first variable with $\theta(x, g(x)) = 0$, $\forall x \in K$.*

Then for all $x \in K$, $x \notin coP(x)$.

Proof. We shall show that $x \notin coP(x)$ for all $x \in K$. Suppose to the contrary, there exists $\bar{x} \in K$ such that $\bar{x} \in coP(\bar{x})$. Then there exists a finite set $\{y_1, y_2, \cdots, y_n\} \subseteq P(\bar{x})$ such that $\bar{x} \in co\{y_1, y_2, \cdots, y_n\}$, hence we have

$$\langle N(u,v,w), \theta(y_i, g(\bar{x}))\rangle + \eta(g(\bar{x}), y_i) \subseteq -intC(\bar{x}), i = 1, 2, \cdots, n$$
$$\forall u \in \widetilde{M}(\bar{x}) = (M(\bar{x}))_{a(\bar{x})}, v \in \widetilde{S}(\bar{x}) = (S(\bar{x}))_{b(\bar{x})}, w \in \widetilde{T}(x) = (T(\bar{x}))_{c(\bar{x})}.$$

Since $intC(\bar{x})$ is a convex set and θ is affine in the first variable, for $\bar{x} = \sum_{i=1}^{n} t_i y_i \in K$, where $t_i \geq 0, i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} t_i = 1$, we have

$$\left\langle N(u,v,w), \theta\left(\sum_{i=1}^{n} t_i y_i, g(\bar{x})\right)\right\rangle + \sum_{i=1}^{n} t_i \eta(g(\bar{x}), y_i)$$
$$= \langle N(u,v,w), \theta(\bar{x}, g(\bar{x}))\rangle + \sum_{i=1}^{n} t_i \eta(g(\bar{x}), y_i) \subseteq -intC(\bar{x}).$$

Since $\theta(\bar{x}, g(\bar{x})) = 0$ by condition ($\mathrm{ii_b}$), we have

$$\sum_{i=1}^{n} t_i \eta(g(\bar{x}), y_i) \subseteq -intC(\bar{x}),$$

that is,

$$\sum_{i=1}^{n} t_i u_i \in -intC(\bar{x}), \quad \forall u_i \in \eta(g(\bar{x}), y_i), \ \ i = 1, 2, \cdots, n,$$

which contradicts condition ($\mathrm{i_b}$). Therefore $x \notin coP(x)$ for all $x \in K$. ∎

The following result is the second alternative version of Theorem 3.3 by applying the notion of O-diagonally convexity and uppersemicontinuity of the set-valued mapping G.

Theorem 3.6. *Let Z be a topological vector space, K be a nonempty convex subset of a Hausdorff topological vector space E, and $L(E,Z)$ be a topological vector space. Let $\widetilde{M}$, $\widetilde{S}$, $\widetilde{T} : K \to 2^{L(E,Z)}$ be upper semicontinuous set-valued mappings with nonempty compact values and induced by fuzzy mappings $M, S, T : K \to \mathfrak{F}(L(E,Z))$, respectively, i.e.,*

$$\widetilde{M}(x) = (M(x))_{a(x)}, \ \ \widetilde{S}(x) = (S(x))_{b(x)}, \ \ \widetilde{T}(x) = (T(x))_{c(x)}, \ \ \forall x \in K.$$

Let $N : L(E,Z) \times L(E,Z) \times L(E,Z) \to 2^{L(E,Z)}$ and $\eta : K \times K \to 2^Z$ be two set-valued mappings. Let $\theta : K \times K \to E$ and $g : K \to K$ be two single valued mappings. If the following conditions are satisfied:

($\mathrm{i_c}$) *η is generalized vector O-diagonally convex in the second argument;*
($\mathrm{ii_c}$) *θ is affine in the first variable with* $\theta(x, g(x)) = 0,\ \forall x \in K$;
($\mathrm{iii_c}$) *For each $y \in K$, the set-valued mapping*

$$G_y(u, v, w, x) = \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \cap Z\backslash(-intC(x))$$

is upper semicontinuous with compact value;
($\mathrm{iv_c}$) *$C : K \to 2^Z$ is a set-valued mapping with convex values;*
($\mathrm{v_c}$) *there exist a nonempty compact subset A of K and a nonempty compact convex subset B of K such that for each $x \in K\backslash A, \exists \bar{y} \in B$ such that*

$$\langle N(u, v, w), \theta(\bar{y}, g(x))\rangle + \eta(g(x), \bar{y}) \subseteq -intC(x),$$

$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)};$$

then the solution set of $GVVLIFE$ (2.1) is a nonempty compact subset of A.

Proof. Let $P : K \to 2^K$ be a set-valued mapping defined by

$$P(x) = \{y \in K : \langle N(u, v, w), \theta(y, g(x))\rangle + \eta(g(x), y) \subseteq -intC(x),$$
$$\forall u \in \widetilde{M}(x) = (M(x))_{a(x)}, v \in \widetilde{S}(x) = (S(x))_{b(x)}, w \in \widetilde{T}(x) = (T(x))_{c(x)}\} \quad \forall x \in K.$$

From Lemma 3.5, we obtain that $x \notin coP(x)$ for all $x \in K$. To show the remaining of the proof, one can show step by step based on the proof in Theorem 3.3. and then the desired results are obtained. ■

4. Conclusion

In this paper two versions of the existence theorems of generalized vector variational-like inequalities in fuzzy environment are proved by using two different notions, the first one by using affineness and the second one by using the notion of vector O-diagonally convexity. Moreover, an example is established to illustrate the main problem. The results presented in the paper can be viewed as alternative versions of [13] by providing a new method of proving the main theorems and an improvement of corresponding result given in Xiao et al. [36], Zhao et al. [26], Ding et al. [16, 17, 19], Salahuddin [32], Lee et al. [27, 28] and several authors.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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STRONG DIFFERENTIAL SUPERORDINATION AND SANDWICH THEOREM OBTAINED WITH SOME NEW INTEGRAL OPERATORS

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Abstract. In this paper we study certain strong differential superordinations, obtained by using a new integral operator introduced in [13].

Keywords. Analytic function, univalent function, convex function, strong differential superordination, best dominant, best subordinant.
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1. Introduction and preliminaries

The concept of differential subordination was introduced in [2], [3] and developed in [4], by S.S. Miller and P.T. Mocanu. The concept of differential superordination was introduced in [5], like a dual problem of the differential superordination by S.S. Miller and P.T. Mocanu. The concept of strong differential subordination was introduced in [1] by J.A. Antonino and S. Romaguera and developed in [7], [11], [12]. The concept of strong differential superordination was introduced in [8], like a dual concept of the strong differential subordination and developed in [9] and [10].

In [11] the author defines the following classes:

Let $\mathcal{H}(U \times \overline{U})$ denote the class of analytic function in $U \times \overline{U}$,

$$U = \{z \in \mathbb{C} : \ |z| < 1\}, \ \overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}, \ \partial U = \{z \in \mathbb{C} : \ |z| = 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$, let

$$H\zeta[a,n] = \{f(z,\zeta) \in \mathcal{H}(U \times \overline{U}) : f(z,\zeta) = a + a_n(\zeta)z^n + \ldots + a_{n+1}(\zeta)z^{n+1} + \ldots\}$$

with $z \in U$, $\zeta \in \overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$,

$$A\zeta_n = \{f(z,\zeta) \in \mathcal{H}(U \times \overline{U}) : f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + a_{n+2}(\zeta)z^{n+2} + \ldots\}$$

with $z \in U$, $\zeta \in \overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n+1$ so $A\zeta_1 = A\zeta$,

$$\mathcal{H}\zeta_u(U) = \{f(z,\zeta) \in \mathcal{H}\zeta[a,n](U \times \overline{U}) : f(z,\zeta) \text{ univalent in } U, \text{ for all } \zeta \in \overline{U}\},$$

$$S\zeta = \{f(z,\zeta) \in A\zeta, \ f(z,\zeta) \text{ univalent in } U, \text{ for all } \zeta \in \overline{U}\},$$

denote the class of univalent functions in $U \times \overline{U}$,

$$S^*\zeta = \left\{f(z,\zeta) \in A\zeta : \ \operatorname{Re} \frac{zf'(z,\zeta)}{f(z,\zeta)} > 0, \ z \in U, \text{ for all } \zeta \in \overline{U}\right\},$$

denote the class of normalized starlike functions in $U \times \overline{U}$,

$$K\zeta = \left\{f(z,\zeta) \in A\zeta : \ \operatorname{Re} \left[\frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1\right] > 0, \ z \in U, \text{ for all } \zeta \in \overline{U}\right\},$$

denote the class of normalized convex functions in $U \times \overline{U}$.

For $r \in \mathbb{N}$, let $A(r)\zeta$ denote the subclass of the functions $f(z,\zeta) \in \mathcal{H}(U \times \overline{U})$ of the form

$$f(z,\zeta) = z^r + \sum_{k=r+1}^{\infty} a_k(\zeta)z^k, \ r \in \mathbb{N}, \ z \in U, \ \zeta \in \overline{U} \text{ and set } A(1)\zeta = A\zeta.$$

To prove our main results, we need the following definitions and lemmas:

Definition 1.1. [9], [11] Let $f(z,\zeta)$ and $F(z,\zeta)$ be member of $\mathcal{H}(U \times \overline{U})$. The function $f(z,\zeta)$ is said to be strongly subordinated to $F(z,\zeta)$, or $F(z,\zeta)$ is said to be strongly superordinated to $f(z,\zeta)$, if there exists a function w analytic in $\overline{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z,\zeta) = F(w(z),\zeta)$. In such a case we write $f(z,\zeta) \prec\prec F(z,\zeta)$.

If $F(z,\zeta)$ is univalent then $f(z,\zeta) \prec\prec F(z,\zeta)$ if and only if $f(0,\zeta) = F(0,\zeta)$ and $f(U \times \overline{U}) \subset F(U \times \overline{U})$.

Remark 1.2. If $f(z,\zeta) \equiv f(z)$ and $F(z,\zeta) \equiv F(z)$, then the strong differential subordination or strong differential superordination becomes the usual notion of differential subordination or differential superordination.

Definition 1.3. [5], [11] We denote by Q_ζ the set of functions $q(z,\zeta)$ that are analytic and injective as functions of z on $\overline{U}\setminus E(q(z,\zeta))$, where

$$E(q(z,\zeta))=\left\{\xi\in\partial U:\ \lim_{z\to\xi}q(z,\zeta)=\infty\right\}$$

and are such that $q'(\xi,\zeta)\neq 0$, for $\xi\in\partial U\setminus E(q(z,\zeta))$.

The class of Q_ζ for which $q(0,\zeta)=a$, is denoted by $Q_\zeta(a)$.

We mention that all the derivatives which appear in this paper are considered with respect to variable z.

Let $\psi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ and let $h(z,\zeta)$ be univalent in U, for all $\zeta\in\overline{U}$. If $p(z,\zeta)$ is analytic in $U\times\overline{U}$ and satisfies the (second-order) strong differential subordination

$$\psi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)\prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U} \tag{1.1}$$

then $p(z,\zeta)$ is called a solution of the strong differential subordination.

The univalent function $q(z,\zeta)$ is called a dominant of the solutions of the strong differential subordination or simply a dominant, if $p(z,\zeta)\prec\prec q(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.1).

A dominant $\widetilde{q}(z,\zeta)$ that satisfies $\widetilde{q}(z,\zeta)\prec\prec q(z,\zeta)$ for all dominants $q(z,\zeta)$ of (1.1) is said to be the best dominant of (1.1). (Note that the best dominant is unique up to a rotation of U).

Let $\varphi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ and let $h(z,\zeta)$ be analytic in $U\times\overline{U}$.

If $p(z,\zeta)$ and $\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)$ are univalent in U, for all $\zeta\in\overline{U}$ and satisfy the (second-order) strong differential superordination

$$h(z,\zeta)\prec\prec\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta) \tag{1.2}$$

then $p(z,\zeta)$ is called a solution of the strong differential superordination. An analytic function $q(z,\zeta)$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q(z,\zeta)\prec\prec p(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.2). A univalent subordinant $\widetilde{q}(z,\zeta)$ that satisfies $q(z,\zeta)\prec\prec\widetilde{q}(z,\zeta)$ for all subordinants of (1.2) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U).

We rewrite the integral operators defined in [13] using the classes we have shown earlier.

Definition 1.4. [13] For $f(z,\zeta)\in A\zeta_n$, $n\in\mathbb{N}^*$, $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, let L_γ be the integral operator given by $L_\gamma:A\zeta_n\to A\zeta_n$

$$L_\gamma^0 f(z,\zeta)=f(z,\zeta),\dots$$
$$L_\gamma^m f(z,\zeta)=\frac{\gamma+1}{z^\gamma}\int_0^z L_\gamma^{m-1}f(z,\zeta)t^{\gamma-1}dt.$$

By using Definition 1.4, we can prove the following properties for this integral operator:

For $f(z,\zeta)\in A\zeta_n$, $n\in\mathbb{N}^*$, $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, we have

$$L_\gamma^m f(z,\zeta)=z+\sum_{k=n+1}^{\infty}\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^k,\ z\in U,\ \zeta\in\overline{U}\text{and} \tag{1.3}$$

$$z[L_\gamma^m f(z,\zeta)]'_z=(\gamma+1)L_\gamma^{m-1}f(z,\zeta)-\gamma L_\lambda^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U}. \tag{1.4}$$

Definition 1.5. [13] For $p\in\mathbb{N}$, $f(z,\zeta)\in A(p)\zeta$, let H be the integral operator given by $H:A(p)\zeta\to A(p)\zeta$

$$H^0 f(z,\zeta)=f(z,\zeta),\dots$$
$$H^m f(z,\zeta)=\frac{p+1}{z}\int_0^z H^{m-1}f(t,\zeta)dt,\ z\in U,\ \zeta\in\overline{U}.$$

From Definition 1.5 we have

$$H^m f(z,\zeta)=z^p+\sum_{k=p+1}^{\infty}\frac{(p+1)^m}{(p+k)^m}a_k(\zeta)z^k,\text{and} \tag{1.5}$$

$$z[H^m f(z,\zeta)]'_z=(p+1)H^{m-1}f(z,\zeta)-H^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U}. \tag{1.6}$$

We rewrite the following lemmas for the classes seen earlier in this paper. The proofs are similar to those given for the original lemmas which can be found in [4] and [5].

Lemma A. [5, Corollary 6.1] *Let $h_1(z,\zeta)$ and $h_2(z,\zeta)$ be convex in U, for all $\zeta\in\overline{U}$, with $h_1(0,\zeta)=h_2(0,\zeta)=a$. Let $\alpha\neq 0$, with $\operatorname{Re}\alpha\geq 0$, and let the functions $q_i(z,\zeta)$ be defined by $q_i(z,\zeta)=\dfrac{\alpha}{z^\alpha}\displaystyle\int_0^z h_i(t,\zeta)t^{\alpha-1}dt$ for $i=1,2$.*

If $p(z,\zeta)\in\mathcal{H}[a,1]\cap Q_\zeta$ and $p(z,\zeta)+\dfrac{zp'(z,\zeta)}{p(z,\zeta)}$ is univalent in U, for all $\zeta\in\overline{U}$, then

$$h_1(z,\zeta)\prec\prec p(z,\zeta)+\frac{zp'(z,\zeta)}{p(z,\zeta)}\prec\prec h_2(z,\zeta)$$

implies $q_1(z,\zeta) \prec\prec p(z,\zeta) \prec\prec q_2(z,\zeta)$.

The functions $q_1(z,\zeta)$ *and* $q_2(z,\zeta)$ *are convex and they are respectively the best subordinant and best dominant.*

Lemma B. [6, Theorem 2] *Let* $h_1(z,\zeta)$ *and* $h_2(z,\zeta)$ *be convex in* U, *for all* $\zeta \in \overline{U}$, *with* $h_1(0,\zeta) = h_2(0,\zeta) = a$ *and* $\theta, \varphi \in \mathcal{H}(D)$, *where* $D \subset \mathbb{C}$ *is a domain.*

Let $p(z,\zeta) \in \mathcal{H}[a,1] \cap Q_\zeta$ *and suppose that* $\theta(p(z,\zeta)) + zp'(z,\zeta)\phi(p(z,\zeta))$ *is univalent in* U, *for all* $\zeta \in \overline{U}$. *If the differential equations* $\theta(q_i(z,\zeta)) + zq_i'(z,\zeta)\phi(q_i(z,\zeta)) = h_i(z,\zeta)$, *have the univalent solutions* $q_i(z,\zeta)$ *that satisfy* $q_i(0,\zeta) = a$, $q_i(U \times \overline{U}) \subset D$, *and* $\theta(q_i(z,\zeta)) \prec\prec h_i(z,\zeta)$, *for* $i = 1,2$, *then*

$$h_1(z,\zeta) \prec\prec \theta(p(z,\zeta)) + zp'(z,\zeta)\phi(p(z,\zeta)) \prec\prec h_2(z,\zeta)$$

implies $q_1(z,\zeta) \prec\prec p(z,\zeta) \prec\prec q_2(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

The functions $q_1(z,\zeta)$ *and* $q_2(z,\zeta)$ *are the best subordinant and the best dominant respectively.*

Lemma C. [6, Corollary 9.2] *Let* $h_1(z,\zeta)$ *and* $h_2(z,\zeta)$ *be starlike in* U, *for all* $\zeta \in \overline{U}$ *and* $f(z,\zeta)$ *be univalent in* U, *for all* $\zeta \in \overline{U}$, *with* $h_1(0,\zeta) = h_2(0,\zeta) = f(0,\zeta) = 0$.

If $h_1(z,\zeta) \prec\prec f(z,\zeta) \prec\prec h_2(z,\zeta)$ *then*

$$\int_0^z \frac{h_1(t,\zeta)}{t}dt \prec\prec \int_0^z \frac{f(t,\zeta)}{t}dt \prec\prec \int_0^z \frac{h_2(t,\zeta)}{t}dt$$

when the middle integral is univalent.

2. Main results

Theorem 2.1. *Let* $h_1(z,\zeta) = \dfrac{2z}{\zeta - z}$ *and* $h_2(z,\zeta) = \dfrac{2z\zeta}{1-z}$ *be convex in* U, *for all* $\zeta \in \overline{U}$, *with* $h_1(0,\zeta) = h_2(0,\zeta) = 0$. *Let* $\alpha \neq 0$, *with* $\operatorname{Re}\alpha \geq 0$ *and let the functions* $q_1(z,\zeta) = \dfrac{\alpha}{z^\alpha}\displaystyle\int_0^z \frac{2t}{\zeta - t}t^{\alpha-1}dt = -2 + \frac{2\alpha\zeta}{z^\alpha}\cdot\sigma_1(z,\zeta)$, *where* $\sigma_1(z,\zeta)$ *given by*

$$\sigma_1(z,\zeta) = \int_0^z \frac{t^{\alpha-1}}{\zeta - t}dt \tag{2.1}$$

and $q_2(z,\zeta) = \dfrac{\alpha}{z^\alpha}\displaystyle\int_0^z \frac{2\zeta t}{1-t}t^{\alpha-1}dt = -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta)$, *where* $\sigma_2(z,\zeta)$ *given by*

$$\sigma_2(z,\zeta) = \int_0^z \frac{t^{\alpha-1}}{1-t}dt. \tag{2.2}$$

If $\dfrac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} \in \mathcal{H}[0,1] \cap Q_\zeta$ *and* $\dfrac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} + \dfrac{z[L_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]' - 1} - n + 1$ *is univalent in* U, *for all* $\zeta \in \overline{U}$, *then*

$$\frac{2z}{\zeta - z} \prec\prec \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} + \frac{zL_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]' - 1} - n + 1 \prec\prec \frac{2z\zeta}{1-z} \tag{2.3}$$

implies

$$-2 + \frac{2\alpha\zeta}{z^\alpha}\sigma_1(z,\zeta) \prec\prec \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} \prec\prec -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta),$$

where $\sigma_1(z,\zeta)$ *is given by (2.1) and* $\sigma_2(z,\zeta)$ *is given by (2.2).*

The functions $q_1(z,\zeta)$ *and* $q_2(z,\zeta)$ *are convex and they are respectively the best subordinant and the best dominant.*

Proof. We let

$$p(z,\zeta) = \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}}, \ z \in U, \ \zeta \in \overline{U}. \tag{2.4}$$

Using (1.3) în (2.4), we have

$$p(z,\zeta) = \frac{1 + \displaystyle\sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)kz^{k-1} - 1}{z^{n-1}} = \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)kz^{k-n}.$$

Since $p(0,\zeta) = 0$, we obtain $p(z,\zeta) \in \mathcal{H}[0,1]\zeta \cap Q_\zeta$.

Differentiating (2.4) and after a short calculus we obtain

$$p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)} = \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} + \frac{z[L_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]' - 1} - n + 1. \tag{2.5}$$

Using (2.5) in (2.3), we obtain

$$\frac{2z}{\zeta - z} \prec\prec p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)} \prec\prec \frac{2z\zeta}{1-z}, \ z \in U, \ \zeta \in \overline{U}. \tag{2.6}$$

Using Lemma A, we have

$$-2 + \frac{2\alpha\zeta}{z^\alpha}\sigma_1(z,\zeta) \prec\prec \frac{[L_\gamma^m f(z,\zeta)]' - 1}{z^{n-1}} \prec\prec -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta),$$

where $\sigma_1(z,\zeta)$ is given by (2.1) and $\sigma_2(z,\zeta)$ is given by (2.2).

The functions

$$q_1(z,\zeta) = -2 + \frac{2\alpha\zeta}{z^\alpha}\sigma_1(z,\zeta) \text{ and } q_2(z,\zeta) = -2\zeta + \frac{2\alpha\zeta}{z^\alpha}\sigma_2(z,\zeta)$$

are convex and they are respectively the best subordinant and the best dominant. □

Example 2.2. Let $\alpha = 2$, $\gamma = 2$, $m = 1$, $n = 2$, $f(z,\zeta) = z + \sum_{k=3}^\infty a_k(\zeta)z^k$,

$$L_2^1 f(z,\zeta) = \frac{3}{z^2}\int_0^z \left[t + \sum_{k=3}^\infty a_k(\zeta)t^k\right] tdt = z + \frac{3}{k+2}\sum_{k=3}^\infty a_k(\zeta)z^k,$$

$$p(z,\zeta) = \frac{[L_2^1 f(z,\zeta)]' - 1}{z} = \frac{3k}{k+2}\sum_{k=3}^\infty a_k(\zeta)z^{k-2},$$

$$q_1(z,\zeta) = \frac{2}{z^2}\int_0^z \frac{2t}{\zeta - t}tdt = \frac{2}{z^2}\int_0^z \left(-2t - 2\zeta + \frac{2\zeta^2}{\zeta - t}\right)dt = -2 - \frac{4\zeta}{z} - 4\zeta^2\ln(\zeta - z),$$

$$q_2(z,\zeta) = \frac{2}{z^2}\int_0^z \frac{2\zeta t^2}{1-t}dt = \frac{2}{z^2}\int_0^z \left(-2\zeta t - 2\zeta + \frac{2\zeta}{1-t}\right)dt = -2\zeta - \frac{4\zeta}{z} - \frac{4\zeta}{z^2}\ln(1-z).$$

Hence from the sharp form of Theorem 2.1 we obtain the following result.

$$\frac{2z}{\zeta - z} \prec\prec \frac{3}{k+2}\sum_{k=3}^\infty a_k(\zeta)kz^{k-2} + \frac{\sum_{k=3}^\infty \frac{3}{k+2}a_k(\zeta)k(k-1)z^{k-2}}{\sum_{k=3}^\infty \frac{3}{k+2}a_k(\zeta)kz^{k-2}} - 1 \prec\prec \frac{2z\zeta}{1-z}$$

implies

$$-2 - \frac{4\zeta}{z} - 4\zeta^2\ln(\zeta - z) \prec\prec \frac{3}{k+2}\sum_{k=3}^\infty a_k(\zeta)kz^{k-2} \prec\prec -2\zeta - \frac{4\zeta}{z} - \frac{4\zeta}{z^2}\ln(1-z), \ z \in U, \ \zeta \in \overline{U}.$$

Theorem 2.3. *Let $h_1(z,\zeta)$ and $h_2(z,\zeta)$ be convex for all $\zeta \in \overline{U}$, with $h_1(0,\zeta) = h_2(0,\zeta) = a = r-1$. Let $\frac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1 \in \mathcal{H}[r-1,1] \cap Q_\zeta$ and suppose that $\frac{z[H^m f(z,\zeta)]''}{[H^m f(z,\zeta)]'} + 1$ is univalent in U, for all $\zeta \in \overline{U}$.*

If the differential equations

$$\theta(q_i(z,\zeta)) + zq_i'(z,\zeta)\phi(q_i(z,\zeta)) = h_i(z,\zeta), \tag{2.7}$$

have the univalent solutions $q_i(z,\zeta)$ that satisfy $q_i(0,\zeta) = r-1$, $q_i(U \times \overline{U}) \subset D$, and $\theta(q_i(z,\zeta)) \prec\prec h_i(z,\zeta)$, for $i = 1,2$, then

$$h_1(z,\zeta) \prec\prec \frac{z[H^m f(z,\zeta)]''}{[H^m f(z,\zeta)]'} + 1 \prec\prec h_2(z,\zeta), \tag{2.8}$$

implies

$$q_1(z,\zeta) \prec\prec \frac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1 \prec\prec q_2(z,\zeta), \ z \in U, \ \zeta \in \overline{U}.$$

The functions $q_1(z,\zeta)$ and $q_2(z,\zeta)$ are the best subordinant and the best dominant respectively.

Proof. We let

$$p(z,\zeta) = \frac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1, \ z \in U, \ \zeta \in \overline{U}. \tag{2.9}$$

Using (1.5) in (2.9) we obtain

$$p(z,\zeta) = \frac{z\left[rz^{r-1} + \sum_{k=r+1}^\infty \frac{(r+1)^m}{(r+k)^m}a_k(\zeta)kz^{k-1}\right]}{z^r + \sum_{k=r+1}^\infty \frac{(r+1)^m}{(r+k)^m}a_k(\zeta)z^k} - 1.$$

Since $p(0,\zeta) = r-1$, we have $p(z,\zeta) \in \mathcal{H}[r-1,1]\zeta \cap Q_\zeta$.
Differentiating (2.9), and after a short calculus, we obtain

$$p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} = 1 + \frac{z[H^m f(z,\zeta)]''}{[H^m f(z,\zeta)]'}. \tag{2.10}$$

Using (2.10) în (2.8), we have

$$h_1(z,\zeta) \prec\prec p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} \prec\prec h_2(z,\zeta), \ z \in U, \ \zeta \in \overline{U}. \tag{2.11}$$

In order to prove the theorem, we shall use Lemma B. For that, we show that the necessary conditions are satisfied. Let the functions $\theta : \mathbb{C} \to \mathbb{C}$ and $\varphi : \mathbb{C} \to \mathbb{C}$, with

$$\theta(w) = w + 1, \text{ and} \tag{2.12}$$

$$\varphi(w) = \frac{1}{w+1}, \quad \varphi(w) \neq 0. \tag{2.13}$$

We check the conditions from the hypothesis of Lemma B.
Using (2.12), we have

$$\theta(p(z,\zeta)) = p(z,\zeta) + 1 \tag{2.14}$$

and

$$\theta(q_1(z,\zeta)) = q_1(z,\zeta) + 1, \quad \theta(q_2(z,\zeta)) = q_2(z,\zeta) + 1. \tag{2.15}$$

Using (2.13), we have

$$\varphi(p(z,\zeta)) = \frac{1}{p(z,\zeta)+1} \text{ and} \tag{2.16}$$

$$\varphi(q_1(z,\zeta)) = \frac{1}{q_1(z,\zeta)+1}, \quad \varphi(q_2(z,\zeta)) = \frac{1}{q_2(z,\zeta)+1}. \tag{2.17}$$

Using (2.14) and (2.16), we have

$$\theta(p(z,\zeta)) + zp'(z,\zeta)\varphi(p(z,\zeta)) = p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1}, \tag{2.18}$$

$$h_1(z,\zeta) = q_1(z,\zeta) + 1 + \frac{zq_1'(z,\zeta)}{q_1(z,\zeta)+1} \text{ and}$$

$$h_2(z,\zeta) = q_2(z,\zeta) + 1 + \frac{zq_2'(z,\zeta)}{q_2(z,\zeta)+1}.$$

Using (2.10) and (2.12), (2.8) becomes

$$q_1(z,\zeta) + 1 + \frac{zq_1'(z,\zeta)}{q_1(z,\zeta)+1} \prec\prec p(z,\zeta) + 1 + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} \prec\prec q_2(z,\zeta) + 1 + \frac{zq_2'(z,\zeta)}{q_2(z,\zeta)+1}, \ z \in U, \ \zeta \in \overline{U}. \tag{2.19}$$

We can apply Lemma B and we obtain $q_1(z,\zeta) \prec\prec p(z,\zeta) \prec\prec q_2(z,\zeta)$, i.e., $q_1(z,\zeta) \prec\prec \dfrac{z[H^m f(z,\zeta)]'}{H^m f(z,\zeta)} - 1 \prec\prec q_2(z,\zeta), \ z \in U, \ \zeta \in \overline{U}$.

The functions $q_1(z,\zeta)$ and $q_2(z,\zeta)$ are the best subordinant and the best dominant respectively.

Theorem 2.4. *Let $m \in \mathbb{N}$, $r \in \mathbb{N}$, $\gamma \in \mathbb{C}$, $h_1(z,\zeta) = \dfrac{\zeta z}{\zeta - z}$ and $h_2(z,\zeta) = \dfrac{z}{\zeta + z}$ be starlike in U, for all $\zeta \in \overline{U}$, with $h_1(0,\zeta) = h_2(0,\zeta) = 0$, $f(z,\zeta) \in A(r)\zeta$ with $f(0,\zeta) = 0$ and $z[H_\gamma^m f(z,\zeta)]' H_\gamma^m f(z,\zeta)$ be univalent in U for all $\zeta \in \overline{U}$.*

If

$$\frac{\zeta z}{\zeta - z} \prec\prec z[H_\gamma^m f(z,\zeta)]' H_\gamma^m f(z,\zeta) \prec\prec \frac{z}{\zeta + z} \tag{2.20}$$

then

$$\zeta \ln \frac{\zeta}{\zeta - z} \prec\prec \frac{[H_\gamma^m f(z,\zeta)]^2}{2} \prec\prec \ln \frac{\zeta + z}{\zeta} \tag{2.21}$$

when the function $\dfrac{[H_\gamma^m f(z,\zeta)]^2}{2}$ is univalent in U, for all $\zeta \in \overline{U}$.

Proof. In order to prove the theorem, we shall use Lemma C. We let

$$g(z,\zeta)=z[H_\gamma^m f(z,\zeta)]' H_\gamma^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U} \tag{2.22}$$

and (2.21) becomes

$$\frac{\zeta z}{\zeta-z}\prec\prec g(z,\zeta)\prec\prec\frac{z}{\zeta+z}, \tag{2.23}$$

where $h_1(z,\zeta)=\dfrac{\zeta z}{\zeta-z}$, $h_2(z,\zeta)=\dfrac{z}{\zeta+z}$ are starlike and $g(z,t)$ given by (2.22) is univalent in U, for all $\zeta\in\overline{U}$.

Using Lemma C, we have

$$\int_0^z\frac{\zeta}{\zeta-t}dt\prec\prec\int_0^z[H_\gamma^m f(t,\zeta)]' H_\gamma^m f(t,\zeta)dt\prec\prec\int_0^z\frac{1}{\zeta+t}dt$$

and after a short calculus we obtain

$$\zeta\ln\frac{\zeta}{\zeta-z}\prec\prec\frac{[H_\gamma^m f(z,\zeta)]^2}{2}\prec\prec\ln\frac{\zeta+z}{\zeta},\ z\in U,\ \zeta\in\overline{U}. \qquad \square$$

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Weighted composition operators from Zygmund-type spaces to weighted-type spaces

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Abstract. In this paper, we investigate the boundedness and compactness of weighted composition operators from Zygmund-type spaces to weighted-type spaces and little weighted-type spaces in the unit ball of $\mathbb{C}^n$.

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Keywords: Weighted composition operator, Zygmund-type space, weighted-type space.

1 Introduction

A positive continuous function μ on $[0, 1)$ is called normal if there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (see [13])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to1} \frac{\mu(r)}{(1-r)^a} = 0;$$

$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

For example, $\mu(r) = (1-r^2)^\alpha \Big(\log \frac{e^{\frac{\beta}{\alpha}}}{1-r^2}\Big)^\beta$ with $\alpha \in (0,\infty)$ and $\beta \in [0,\infty)$ is normal.

Let $\mathcal{B}$ be the unit ball of $\mathbb{C}^n$ and $H(\mathcal{B})$ the space of all holomorphic functions on $\mathcal{B}$. Let $A(\mathcal{B})$ denote the ball algebra consisting of all functions in $H(\mathcal{B})$ that are continuous up to the boundary of $\mathcal{B}$. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}, \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Let μ be normal on $[0, 1)$. The weighted-type space, denoted by $H^\infty_\mu = H^\infty_\mu(\mathcal{B})$, is the space of all $f \in H(\mathcal{B})$ such that (see, e.g., [15, 16]).

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathcal{B}} \mu(|z|)\,|f(z)| < \infty.$$

H^∞_μ is a Banach space with the norm $\|\cdot\|_{H^\infty_\mu}$. The little weighted-type space, denote by $H^\infty_{\mu,0}$, is the subspace of H^∞_μ consisting of those $f \in H^\infty_\mu$ such that

$$\lim_{|z|\to1} \mu(|z|)|f(z)| = 0.$$

When $\mu(r) = (1-r^2)^\alpha$, H_μ^∞ and $H_{\mu,0}^\infty$ will be denoted by H_α^∞ and $H_{\alpha,0}^\infty$, respectively. Let $H^\infty = H^\infty(\mathcal{B})$ denote the space of all bounded holomorphic functions on $\mathcal{B}$.

For $f \in H(\mathcal{B})$, let $\Re f$ denote the radial derivative of f, that is

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

We write $\Re^2 f = \Re(\Re f)$.

The Zygmund space, denote by $\mathscr{Z} = \mathscr{Z}(\mathcal{B})$, is the space consisting of all $f \in H(\mathcal{B})$ such that

$$\sup_{z\in\mathcal{B}} (1-|z|^2)|\Re^2 f(z)| < \infty.$$

It is well known that $f \in \mathscr{Z}$ if and only if $f \in A(\mathcal{B})$ and there exists a constant $C > 0$ such that

$$|f(\zeta + h) + f(\zeta - h) - 2f(\zeta)| < Ch,$$

for all $\zeta \in \partial\mathcal{B}$ and $\zeta \pm h \in \partial\mathcal{B}$ (see [19, p. 261]).

Let ω be normal on $[0, 1)$. An $f \in H(\mathcal{B})$ is said to belong to the Zygmund-type space, denoted by $\mathscr{Z}_\omega = \mathscr{Z}_\omega(\mathcal{B})$, if (see [10, 11, 17])

$$\|f\|_{\mathscr{Z}_\omega} = |f(0)| + \sup_{z\in\mathcal{B}} \omega(|z|) \left|\Re^2 f(z)\right| < \infty.$$

It is easy to check that $\mathscr{Z}_\omega$ is a Banach space under the norm $\|\cdot\|_{\mathscr{Z}_\omega}$. See [2, 3, 7, 8, 12] for more details on the Zygmund space in the unit disk.

Let φ be a holomorphic self-map of $\mathcal{B}$ and $u \in H(\mathcal{B})$. The weighted composition operator, denoted by uC_φ, is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathcal{B}), \qquad z \in \mathcal{B}.$$

When $u = 1$, the operator uC_φ is just the composition operator, denoted by C_φ. For more information about the theory of composition operator, see [1] and the references therein.

In the setting of $\mathcal{B}$, Stević studied weighted composition operators between H_α^∞ and mixed norm spaces in [14]. In [9], Li and Stević studied weighted composition operators between H^∞ and α-Bloch spaces. In [5], Gu studied weighted composition operators from generalized weighted Bergman spaces to H_μ^∞. In [20], Zhu studied weighted composition operators from $F(p, q, s)$ spaces to H_μ^∞. In [16], the operator norm of the weighted composition operator from the Bloch space to H_μ^∞ was studied. In [15], the essential norm of weighted composition operators from α-Bloch spaces to H_μ^∞ was studied. In [18], Yang studied weighted composition operators from Bloch type spaces with normal weight to H_μ^∞

In this paper, we study the boundedness and compactness of $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ and $uC_\varphi : \mathscr{Z}_\omega \to H_{\mu,0}^\infty$. Some necessary and sufficient conditions for uC_φ to be bounded or compact are provided.

Throughout this paper C will denote constants, they are positive and may differ from one occurrence to the other. $a \lesssim b$ means that there is a positive constant C such that $a \le Cb$. If both $a \lesssim b$ and $b \lesssim a$ hold, then one says that $a \approx b$.

2 Main results and proofs

In order to prove our main results, we need some auxiliary results which are incorporated in the following lemmas. The following lemma can be found in [17].

Lemma 1. *Assume that ω is normal on $[0,1)$. If $f \in \mathscr{Z}_\omega$, then*

$$|f(z)| \le C\bigg(1+\int_0^{|z|}\int_0^t \frac{ds}{\omega(s)}dt\bigg)\|f\|_{\mathscr{Z}_\omega}$$

or

$$|f(z)| \le C\bigg(1+\int_0^{|z|} \frac{|z|-t}{\omega(t)}dt\bigg)\|f\|_{\mathscr{Z}_\omega}$$

for some C independent of f.

Lemma 2. [20] *Assume that μ is normal on $[0,1)$. A closed set K in $H^\infty_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(|z|)|f(z)| = 0.$$

By standard arguments similar to those outlined in Proposition 3.11 of [1], the following lemma follows. We omit the details.

Lemma 3. *Assume that ω and μ are normal on $[0,1)$, $u \in H(\mathcal{B})$ and φ is a holomorphic self-map of $\mathcal{B}$. Then $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is compact if and only if $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{Z}_\omega$ which converges to zero uniformly on compact subsets of $\mathcal{B}$ as $k\to\infty$, we have $\|uC_\varphi f_k\|_{H^\infty_\mu} \to 0$ as $k \to \infty$.*

Lemma 4. [17] *Assume that ω is normal and $\int_0^1 \frac{1-t}{\omega(t)}dt < \infty$. Then for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset \mathscr{Z}_\omega$ converging to 0 uniformly on compact subsets of $\mathcal{B}$, we have that*

$$\lim_{k\to\infty}\sup_{z\in\mathcal{B}}|f_k(z)| = 0.$$

Lemma 5. [6] *Assume that ω is normal. Then exists a function g is holomorphic on the unit disk D, $g(r)$ is increasing on $[0,1)$ and*

$$0 < C_1 = \inf_{r\in[0,1)}\omega(r)g(r) \le \sup_{r\in[0,1)}\omega(r)g(r) \le C_2 < \infty.$$

Now we are in a position to state and prove our main results is this paper.

Theorem 1. *Assume that μ and ω are normal on $[0,1)$, $u\in H(\mathcal{B})$ and φ is a holomorphic self-map of $\mathcal{B}$. Then $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is bounded if and only if*

$$\sup_{z\in\mathcal{B}}\mu(|z|)|u(z)|\bigg(1+\int_0^{|\varphi(z)|}\frac{|\varphi(z)|-t}{\omega(t)}dt\bigg) < \infty. \tag{1}$$

Moreover, when $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded, then

$$\|uC_\varphi\|_{\mathscr{Z}_\omega \to H_\mu^\infty} \approx \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)|\bigg(1+\int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt\bigg). \tag{2}$$

Proof. Assume that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Taking $f(z) \equiv 1 \in \mathscr{Z}_\omega$, we get $u \in H_\mu^\infty$ and

$$\|u\|_{H_\mu^\infty} = \|uC_\varphi(1)\|_{H_\mu^\infty} \le \|uC_\varphi\|_{\mathscr{Z}_\omega\to H_\mu^\infty}. \tag{3}$$

Let $b \in \mathcal{B}$. Define

$$f_b(z) = \int_0^{\langle z,b\rangle}\int_0^\eta g(t)dtd\eta, \quad z\in\mathcal{B}, \tag{4}$$

where g is defined in Lemma 5. It is easy to check that there is a positive constant C such that $\sup_{b\in\mathcal{B}} \|f_b\|_{\mathscr{Z}_\omega} \le C$ and hence $f_b \in \mathscr{Z}_\omega$. Therefore, for every $w \in \mathcal{B}$,

$$\begin{aligned}\sup_{z\in\mathcal{B}} \mu(|z|)|f_{\varphi(w)}(\varphi(z))u(z)| &= \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f_{\varphi(w)})(z)| \\ &= \|uC_\varphi f_{\varphi(w)}\|_{H_\mu^\infty} \le C\|uC_\varphi\|_{\mathscr{Z}_\omega\to H_\mu^\infty}. \end{aligned}\tag{5}$$

By Lemma 5 we get

$$\sup_{w\in\mathcal{B}} \mu(|w|)|u(w)|\int_0^{|\varphi(w)|^2} \frac{|\varphi(w)|^2-t}{\omega(t)}dt \le C\|uC_\varphi\|_{\mathscr{Z}_\omega\to H_\mu^\infty} < \infty. \tag{6}$$

After a calculation, we get

$$\int_0^{|\varphi(w)|^2} \frac{|\varphi(w)|^2-t}{\omega(t)}dt \approx \int_0^{|\varphi(w)|} \frac{|\varphi(w)|-t}{\omega(t)}dt. \tag{7}$$

From (6), (7) and the fact that $u \in H_\mu^\infty$, we see that (1) holds.

Conversely, suppose that (1) holds. For any $f \in \mathscr{Z}_\omega$, by Lemma 1 we have

$$\begin{aligned}\|uC_\varphi f\|_{H_\mu^\infty} &= \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f)(z)| \\ &= \sup_{z\in\mathcal{B}} \mu(|z|)|f(\varphi(z))||u(z)| \\ &\le C\|f\|_{\mathscr{Z}_\omega} \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)|\bigg(1+\int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt\bigg). \end{aligned}\tag{8}$$

Therefore (1) implies that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Moreover

$$\|uC_\varphi\|_{\mathscr{Z}_\omega\to H_\mu^\infty} \le C\sup_{z\in\mathcal{B}} \mu(|z|)|u(z)|\bigg(1+\int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt\bigg). \tag{9}$$

From (3), (6), (7) and (9), (2) follows. $\square$

Theorem 2. *Assume that μ and ω are normal on $[0,1)$, $u \in H(\mathcal{B})$ and φ is a holomorphic self-map of $\mathcal{B}$. If $\int_0^1 \frac{1-t}{\omega(t)}dt < \infty$, then $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact if and only if $u \in H_\mu^\infty$.*

Proof. Assume that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact. Then it is clear that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Taking $f(z) \equiv 1$, we see that $u \in H_\mu^\infty$.

Conversely, suppose that $u \in H_\mu^\infty$. Since $\int_0^1 \frac{1-t}{\omega(t)}dt < \infty$, then

$$\sup_{z\in\mathcal{B}} \mu(|z|)|u(z)| \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt \le \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)| \int_0^1 \frac{1-t}{\omega(t)}dt < \infty. \tag{10}$$

For every $f \in \mathscr{Z}_\omega$, from (10) we obtain

$$\begin{aligned} \mu(|z|)|(uC_\varphi f)(z)| &= \mu(|z|)|f(\varphi(z))||u(z)| \\ &\le C\|f\|_{\mathscr{Z}_\omega} \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)| \left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt\right) \\ &\le C\|f\|_{\mathscr{Z}_\omega}\|u\|_{H_\mu^\infty} < \infty, \end{aligned} \tag{11}$$

which implies that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in $\mathscr{Z}_\omega$ and $f_k \to 0$ uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$. By Lemma 4 we obtain

$$\begin{aligned} \lim_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} &= \lim_{k\to\infty} \sup_{z\in\mathcal{B}} \mu(|z|)|f_k(\varphi(z))u(z)| \\ &\le \|u\|_{H_\mu^\infty} \lim_{k\to\infty} \sup_{z\in\mathcal{B}} |f_k(\varphi(z))| = 0. \end{aligned}$$

By Lemma 3, we see that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact. $\square$

Theorem 3. *Assume that μ and ω are normal on $[0,1)$, $u \in H(\mathcal{B})$, φ is a holomorphic self-map of $\mathcal{B}$. Assume that $\int_0^1 \frac{1-t}{\omega(t)}dt = \infty$. Then $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact if and only if $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|u(z)| \left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt\right) = 0. \tag{12}$$

Proof. Assume that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact. To prove (12), we only need to prove that

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|u(z)| \int_0^{|\varphi(z)|} \frac{|\varphi(z)|-t}{\omega(t)}dt = 0, \tag{13}$$

since they are equivalent. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then condition (12) is vacuously satisfied). For $k \in \mathbb{N}$, we define

$$f_k(z) = \left(\int_0^{|\varphi(z_k)|^2} \int_0^\eta g(t)dtd\eta\right)^{-1} \left(\int_0^{\langle z,\varphi(z_k)\rangle} \int_0^\eta g(t)dtd\eta\right)^2.$$

It is easy to see that $f_k \in \mathscr{Z}_\omega$ for every $k \in \mathbb{N}$, $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathscr{Z}_\omega} \le C$ and f_k converges to 0 uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$. By the assumption and Lemma 3 we see that $\lim_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} = 0$. Thus

$$\begin{aligned} & \lim_{k\to\infty} \mu(|z_k|)|u(z_k)| \int_0^{|\varphi(z_k)|^2} \frac{|\varphi(z_k)|^2 - t}{\omega(t)} dt \\ = & \lim_{k\to\infty} \mu(|z_k|)|u(z_k)|\, |f_k(\varphi(z_k))| \\ \le & \lim_{k\to\infty} \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f_k)(z)| = \lim_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} = 0, \end{aligned}$$

which implies

$$\lim_{k\to\infty} \mu(|z_k|)|u(z_k)| \int_0^{|\varphi(z_k)|} \frac{|\varphi(z_k)| - t}{\omega(t)} dt = 0.$$

From this we obtain (12).

Conversely, suppose that $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is bounded and (12) holds. Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathscr{Z}_\omega$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathscr{Z}_\omega} \le \Omega$ and $f_k \to 0$ uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$. By Lemma 3 we only need to show that $\lim_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} = 0$.

From (12), for every $\varepsilon > 0$, there is a constant $s \in (0,1)$, such that

$$\mu(|z|)|u(z)| \left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\right) < \varepsilon$$

when $s < |\varphi(z)| < 1$. By Lemma 1,

$$\begin{aligned} & \|uC_\varphi f_k\|_{H_\mu^\infty} = \sup_{z\in\mathcal{B}} \mu(|z|)|(uC_\varphi f_k)(z)| \\ = & \sup_{z\in\mathcal{B}} \mu(|z|)|u(z)||f_k(\varphi(z))| \\ \le & \sup_{|\varphi(z)|\le s} \mu(|z|)|u(z)||f_k(\varphi(z))| + C \sup_{|\varphi(z)|>s} \mu(|z|)|u(z)| \\ & \left(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\right) \|f_k\|_{\mathscr{Z}_\omega} \\ \le & \|u\|_{H_\mu^\infty} \sup_{|\varphi(z)|\le s} |f_k(\varphi(z))| + C\Omega\varepsilon. \end{aligned}$$

Since $f_k \to 0$ uniformly on compact subsets of $\mathcal{B}$ as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \sup_{|\varphi(z)|\le\eta} |f_k(\varphi(z))| = 0.$$

Hence $\limsup_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} \le C\Omega\varepsilon$. By the arbitrary of ε we obtain that

$$\lim_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} = 0.$$

Hence $uC_\varphi : \mathscr{Z}_\omega \to H_\mu^\infty$ is compact by Lemma 3. $\square$

Theorem 4. *Assume that μ and ω are normal on $[0,1)$, $u \in H(\mathcal{B})$ and φ is a holomorphic self-map of $\mathcal{B}$. Then $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) = 0. \tag{14}$$

Proof. Assume that $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$ is compact. Taking $f(z) \equiv 1$ and using the boundedness of $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$, we get

$$\lim_{|z|\to 1} \mu(|z|)|u(z)| = 0. \tag{15}$$

When $\int_0^1 \frac{1-t}{\omega(t)} dt < \infty$, then (14) follows by (15).

Now we consider the case $\int_0^1 \frac{1-t}{\omega(t)} dt = \infty$. From the assumption, it is obvious that $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_\mu$ is compact. By Theorem 2, we get

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) = 0. \tag{16}$$

By (16), for every $\varepsilon > 0$, there exists a $\eta \in (0,1)$, such that

$$\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) < \varepsilon$$

when $\eta < |\varphi(z)| < 1$. By (15), for the above ε, there is a $s \in (0,1)$, such that

$$\mu(|z|)|u(z)| < \bigg(1 + \int_0^{\eta} \frac{\eta - t}{\omega(t)} dt\bigg)^{-1} \varepsilon$$

when $s < |z| < 1$.

Hence, if $s < |z| < 1$ and $\eta < |\varphi(z)| < 1$, we obtain

$$\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) < \varepsilon. \tag{17}$$

If $s < |z| < 1$ and $|\varphi(z)| \le \eta$, we get

$$\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg) \le \bigg(1 + \int_0^{\eta} \frac{\eta - t}{\omega(t)} dt\bigg)\mu(|z|)|u(z)| < \varepsilon. \tag{18}$$

From (17) and (18), we see that (14) holds.

Conversely, assume that (14) holds. To prove that $uC_\varphi : \mathscr{Z}_\omega \to H^\infty_{\mu,0}$ is compact, by Lemma 2 we only need to prove that

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathscr{Z}_\omega}\le 1} \mu(|z|)|(uC_\varphi f)(z)| = 0. \tag{19}$$

Applying Lemma 1, we obtain

$$\mu(|z|)|(uC_\varphi f)(z)| \le C\mu(|z|)|u(z)|\bigg(1 + \int_0^{|\varphi(z)|} \frac{|\varphi(z)| - t}{\omega(t)} dt\bigg)\|f\|_{\mathscr{Z}_\omega}. \tag{20}$$

Taking the supremum in (20) over the the unit ball in the space $\mathscr{Z}_\omega$, then letting $|z| \to 1$ and applying (14) we get the desired result. $\square$

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Positive solutions for a singular semipositone boundary value problem of nonlinear fractional differential equations *

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Abstract: In this paper, we consider the existence of positive solutions to a singular semipositone boundary value problem of nonlinear fractional differential equations. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are obtained. In addition, two examples are presented to demonstrate the application of our main results.

Keywords: Fractional differential equation, Singular semipositone boundary value problem, Positive solution, fixed point theorem, Eigenvalue.

2010 Mathematics Subject Classification: 34B15, 34B16, 34B18

1 Introduction

In this paper, we discuss the following singular semipositone boundary value problem (BVP for short):

$$\begin{cases} D_{0+}^{\alpha}u(t) = \lambda f\left(t, u(t), v(t)\right), & 0 < t < 1, \\ D_{0+}^{\alpha}v(t) = \mu g\left(t, u(t), v(t)\right), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases} \tag{1.1}$$

where $3 < \alpha \leq 4$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, λ, μ are positive parameters, and $f, g : (0,1) \times [0, +\infty) \times [0, +\infty) \to (-\infty, +\infty)$ are given continuous functions. f, g may be singular at $t = 0$ and/or $t = 1$ and may take negative values. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are established.

Singular boundary value problems arise from many fields in physics, biology, chemistry and economics, and play a very important role in both theoretical development and application. Recently, some work has been done to study the existence of solutions or positive solutions of nonlinear singular semipositone boundary value problems by the use of techniques of nonlinear analysis such as Leray-Schauder theory, fixed point index theorem, etc[1, 3, 4, 8, 10, 11].

In [8], Wang, Liu and Wu have discussed the existence of positive solutions of the following nonlinear fractional differential equation boundary value problem with changing sign nonlinearity:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f\left(t, u(t)\right) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, λ is a positive parameter, f may change sign and may be singular at $t = 0$ and/or $t = 1$ and may take negative values.

In [6], Henderson and Luca have considered the existence of positive solutions for the system of nonlinear fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\beta}v(t) + \mu g\left(t, u(t), v(t)\right) = 0, & t \in (0,1), \end{cases}$$

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with the coupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u'(1) = \int_0^1 v(s)\mathrm{d}H(s), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, & v'(1) = \int_0^1 u(s)\mathrm{d}K(s), \end{cases}$$

where $\alpha \in (n-1, n], \beta \in (m-1, m], n, m \in \mathbb{N}, n, m \geq 3$, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ denote the standard Riemann-Liouville fractional derivatives, f, g are sign-changing continuous functions and may be nonsingular or singular at $t = 0$ and/or $t = 1$.

Motivated by the above work, we consider the existence of positive solutions for the system of fractional order singular semipositone BVP (1.1).

This paper is organized as follows. In Section 2, we present some basic definitions and properties from the fractional calculus theory. In Section 3, based on the Krasnoselskii's fixed point theorem, we prove existence theorems of the positive solutions for boundary value problem (1.1). In section 4, two examples are presented to illustrate the main results.

2 Preliminaries

In this section, we present here the necessary definitions and properties from fractional calculus theory. These definitions and properties can be found in the recent literature [2, 5, 7, 9, 10, 12].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \qquad t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left(I_{0+}^{n-\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, \quad t > 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.1. Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation

$$D_{0+}^{\alpha} u(t) = 0$$

has solutions $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, C_i \in \mathbb{R}, i = 1, 2, \cdots, n,\ n = [\alpha] + 1$.

Lemma 2.2. Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha(\alpha > 0)$ that belongs to $C(0, 1) \cap L(0, 1)$, then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

for some $C_i \in \mathbb{R}, i = 1, 2, \cdots, n,\ n = [\alpha] + 1$.

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 2.3. ([9]) Let $y \in C(0, 1) \cap L(0, 1)$ and $3 < \alpha \leq 4$, the unique solution of problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = y(t), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t, s) y(s)\mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2} \left[(s-t) + (\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-2}(1-s)^{\alpha-2} \left[(s-t) + (\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Here $G(t,s)$ is called the Green's function of BVP (2.1).

Lemma 2.4. ([9, 10]) The function $G(t,s)$ defined by (2.2) possesses the following properties:

(1)$G(t,s) > 0$, for $t,s \in (0,1)$;

(2)$G(t,s) = G(1-s, 1-t)$, for $t,s \in (0,1)$;

(3)$t^{\alpha-2}(1-t)^2 q(s) \le G(t,s) \le (\alpha-1)q(s)$, for $t,s \in (0,1)$;

(4)$t^{\alpha-2}(1-t)^2 q(s) \le G(t,s) \le ((\alpha-1)(\alpha-2)/\Gamma(\alpha))\, t^{\alpha-2}(1-t)^2$, for $t,s \in (0,1)$,
where $q(s) = ((\alpha-2)/\Gamma(\alpha))\, s^2(1-s)^{\alpha-2}$.

Lemma 2.5. The function $q(1-t)$ has the property:

$$\max_{t\in(0,1)} q(1-t) = q\left(\frac{2}{\alpha}\right) = \frac{4(\alpha-2)^{\alpha-1}}{\Gamma(\alpha)\alpha^\alpha}.$$

Proof. From Lemma 2.4, we can easily get $q(1-t) = \frac{\alpha-2}{\Gamma(\alpha)} t^{\alpha-2}(1-t)^2$. Let $F(t) = t^{\alpha-2}(1-t)^2$, then $F'(t) = (1-t)t^{\alpha-3}\left[-\alpha t + (\alpha-2)\right]$, for $t \in (0,1)$. Let $F'(t) = 0$, we get $t_0 = \dfrac{\alpha-2}{\alpha}$.

Since $3 < \alpha \le 4$, we can know $0 < t_0 < 1$. So, the function $F(t)$ achieve the maximum when $t = \dfrac{\alpha-2}{\alpha}$.

Therefore $\max\limits_{t\in(0,1)} F(t) = F\left(\dfrac{\alpha-2}{\alpha}\right) = \dfrac{4(\alpha-2)^{\alpha-2}}{\alpha^\alpha}$, thus, $\max\limits_{t\in(0,1)} q(1-t) = q\left(\dfrac{2}{\alpha}\right) = \dfrac{4(\alpha-2)^{\alpha-1}}{\Gamma(\alpha)\alpha^\alpha}$.

Lemma 2.6. Let $p_i \in C(0,1) \cap L(0,1)$ with $p_i(t) \ge 0, i = 1,2$, then the boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) = p_i(t), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2.3}$$

has a unique solution $w_i(t) = \int_0^1 G(t,s)p_i(s)\mathrm{d}s$ with

$$w_i(t) \le (\alpha-1)q(1-t)\int_0^1 p_i(s)\mathrm{d}s, t \in [0,1], \quad i = 1,2. \tag{2.4}$$

Proof. By Lemma 2,3 and Lemma 2.4, we have $w_i(t) = \int_0^1 G(t,s)p_i(s)\mathrm{d}s$ is the unique solution of (2.3) and

$$w_i(t) = \int_0^1 G(t,s)p_i(s)\mathrm{d}s \le (\alpha-1)q(1-t)\int_0^1 p_i(s)\mathrm{d}s, \quad i = 1,2.$$

The proof is completed.

For any $x \in C[0,1]$, we define a function $[x(\cdot)]^* : [0,1] \to [0,+\infty)$ by

$$[x(\cdot)]^* = \begin{cases} x(t), & x(t) \ge 0, \\ 0, & x(t) < 0. \end{cases}$$

In order to overcome the difficulty associated with semipositone, we consider the following approximately singular nonlinear differential system:

$$\begin{cases} D_{0+}^\alpha u(t) = \lambda\left[f\left(t, [u(t)-\lambda w_1(t)]^*, [v(t)-\mu w_2(t)]^*\right) + p_1(t)\right], & 0 < t < 1, \\ D_{0+}^\alpha v(t) = \mu\left[g\left(t, [u(t)-\lambda w_1(t)]^*, [v(t)-\mu w_2(t)]^*\right) + p_2(t)\right], & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases} \tag{2.5}$$

where $w_i(t) (i = 1,2)$ are defined in Lemma 2.6.

It is well-known that the problem (2.5) can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases} u(t) = \lambda\displaystyle\int_0^1 G(t,s)\left[f\left(s, [u(s)-\lambda w_1(s)]^*, [v(s)-\mu w_2(s)]^*\right) + p_1(t)\right]\mathrm{d}s, 0 \le t \le 1, \\ v(t) = \mu\displaystyle\int_0^1 G(t,s)\left[g\left(s, [u(s)-\lambda w_1(s)]^*, [v(s)-\mu w_2(s)]^*\right) + p_2(t)\right]\mathrm{d}s, 0 \le t \le 1. \end{cases} \tag{2.6}$$

We consider the Banach space $X = C[0,1]$ with the norm $\|u\| = \max\limits_{0\le t\le 1}|u(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$.

We define the cone $P \subset Y$ by

$$P = \left\{(u,v) \in Y | u(t) \ge \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|, v(t) \ge \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|, t \in [0,1]\right\}.$$

3

For $\lambda, \mu > 0$, we define the operators $T_1, T_2 : Y \to X$ and $T : Y \to Y$ as follows:

$$\begin{cases} T_1(u,v)(t) = \lambda \int_0^1 G(t,s) \left[f\left(s, [u(s) - \lambda w_1(s)]^*, [v(s) - \mu w_2(s)]^*\right) + p_1(t) \right] \mathrm{d}s, 0 \le t \le 1, \\ T_2(u,v)(t) = \mu \int_0^1 G(t,s) \left[g\left(s, [u(s) - \lambda w_1(s)]^*, [v(s) - \mu w_2(s)]^*\right) + p_2(t) \right] \mathrm{d}s, 0 \le t \le 1, \end{cases}$$

and $T(u,v) = (T_1(u,v), T_2(u,v)), (u,v) \in Y$. Thus, the solutions of our problem (2.5) are the fixed points of the operator T.

Lemma 2.7. ([5]) Let E be a Banach space, and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 be two open subsets of E with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \to P$ be a completely continuous operator such that either

(i) $\|Tw\| \le \|w\|, w \in P \cap \partial\Omega_1, \|Tw\| \ge \|w\|, w \in P \cap \partial\Omega_2$, or

(ii) $\|Tw\| \ge \|w\|, w \in P \cap \partial\Omega_1, \|Tw\| \le \|w\|, w \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap \overline{\Omega}_2 \backslash \Omega_1$.

3 Main results and proof

For convenience, throughout the rest of the paper, we make the following assumptions:

(H_1) $f, g \in C((0,1) \times [0,+\infty) \times [0,+\infty), (-\infty,+\infty))$ and there exist functions $p_i, a_i, k \in L((0,1),[0,+\infty)) \cap C((0,1),[0,+\infty))$ and $h \in C([0,+\infty) \times [0,+\infty),[0,+\infty))$ such that

$$a_1(t)h(x,y) \le f(t,x,y) + p_1(t) \le k(t)h(x,y),$$
$$a_2(t)h(x,y) \le g(t,x,y) + p_2(t) \le k(t)h(x,y),$$

where $a_i(t) \ge c_i k(t)$ a.e. $t \in (0,1), 0 < c_i \le 1, i = 1,2, \forall (t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty)$.

(H_2) There exists $(a,b) \subset [0,1]$ such that

$$\lim_{x \to +\infty} \min_{t \in [a,b]} \frac{f(t,x,y)}{x} = +\infty, \text{ or}$$
$$\lim_{x \to +\infty} \min_{t \in [a,b]} \frac{g(t,x,y)}{x} = +\infty.$$

(H_3) There exists $(c,d) \subset [0,1]$ such that

$$\lim_{x \to +\infty} \min_{t \in [c,d]} f(t,x,y) > \frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1 c^2 (1-d)^2 \Gamma(\alpha) \int_c^d q(s)\mathrm{d}s}, \text{ or}$$
$$\lim_{x \to +\infty} \min_{t \in [c,d]} g(t,x,y) > \frac{2(\alpha-1)^2(\alpha-2)r_2}{c_2 c^2 (1-d)^2 \Gamma(\alpha) \int_c^d q(s)\mathrm{d}s},$$

where $r_1 = \int_0^1 p_1(s)\mathrm{d}s, r_2 = \int_0^1 p_2(s)\mathrm{d}s$, and

$$\lim_{x,y \to +\infty} \frac{h(x,y)}{x} = 0.$$

Lemma 3.1. $T : P \to P$ is a completely continuous operator.

Proof. Let $(u,v) \in P$ be an arbitrary element. From Lemma 2.4 and (H_1), we can get

$$\begin{aligned} \|T_1(u,v)\| &= \max_{0 \le t \le 1} |T_1(u,v)(t)| \\ &\le \int_0^1 (\alpha-1)q(s) \left[f\left(s, [u(s) - \lambda w_1(s)]^*, [v(s) - \mu w_2(s)]^*\right) + p_1(t) \right] \mathrm{d}s \\ &\le (\alpha-1) \int_0^1 q(s)k(s)h\left([u(s) - \lambda w_1(s)]^*, [v(s) - \mu w_2(s)]^*\right) \mathrm{d}s, \end{aligned}$$

$$\begin{aligned}\|T_2(u,v)\| =& \max_{0\le t\le 1}|T_2(u,v)(t)|\\ \le& \int_0^1 (\alpha-1)q(s)\left[g\left(s,[u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)+p_2(t)\right]\mathrm{d}s\\ \le&(\alpha-1)\int_0^1 q(s)k(s)h\left([u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)\mathrm{d}s,\end{aligned}$$

Hence, we obtain

$$\|T(u,v)\|\le(\alpha-1)\int_0^1 q(s)\left[k(s)h\left([u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)\right]\mathrm{d}s. \tag{3.1}$$

By (H_1) and (3.1), we have

$$\begin{aligned}T_1(u,v)(t)\ge& t^{\alpha-2}(1-t)^2\int_0^1 q(s)\left[f\left(s,[u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)+p_1(t)\right]\mathrm{d}s\\ \ge& t^{\alpha-2}(1-t)^2\int_0^1 q(s)a_1(s)h\left([u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)\mathrm{d}s\\ \ge& c_1t^{\alpha-2}(1-t)^2\int_0^1 q(s)k(s)h\left([u(s)-\lambda w_1(s)]^*,[v(s)-\mu w_2(s)]^*\right)\mathrm{d}s\\ \ge&\frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}\|T(u,v)\|.\end{aligned}$$

In the similar manner, we deduce $T_2(u,v)(t)\ge\dfrac{c_2t^{\alpha-2}(1-t)^2}{\alpha-1}\|T(u,v)\|$.

Thus $T(u,v)\in P$, that is $T(P)\subset P$.

According to the Arzela-Ascoli theorem, we can easily get that $T:P\to P$ is a completely continuous operator. The proof is completed.

Theorem 3.1. If (H_1) and (H_2) hold, then there exists $\eta>0$ such that the BVP (1.1) has at least one positive solution for any $\lambda,\mu\in(0,\eta)$.

Proof. Choose $R_1=\max\left\{\frac{(\alpha-1)^2(\alpha-2)r_i}{c_i\Gamma(\alpha)},i=1,2\right\}$. Let

$$\eta=\min\left\{1,\frac{\Gamma(\alpha)\alpha^\alpha R_1}{4(\alpha-1)(\alpha-2)^{\alpha-1}h^*(R_1)\int_0^1 k(s)\mathrm{d}s}\right\}$$

where

$$h^*(R_1)=\max_{x,y\in[0,R_1]}h(x,y). \tag{3.2}$$

Suppose $\lambda,\mu\in(0,\eta)$, let $P_{R_1}=\{(u,v)\in P,\|(u,v)\|<R_1\}$, for any $(u,v)\in\partial P_{R_1}$, that is $\|(u,v)\|=R_1$. Noticing that

$$\begin{aligned}u(t)\ge&\frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|=\frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}R_1,\quad t\in[0,1],\\ v(t)\ge&\frac{c_2t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\|=\frac{c_2t^{\alpha-2}(1-t)^2}{\alpha-1}R_1,\quad t\in[0,1],\end{aligned}$$

and

$$\begin{aligned}w_1(t)\le(\alpha-1)q(1-t)\int_0^1 p_1(s)\mathrm{d}s=(\alpha-1)q(1-t)r_1,\\ w_2(t)\le(\alpha-1)q(1-t)\int_0^1 p_2(s)\mathrm{d}s=(\alpha-1)q(1-t)r_2,\end{aligned}$$

for any $t\in[0,1]$, we get that

$$\begin{aligned}0\le\left[\frac{c_1\Gamma(\alpha)R_1}{(\alpha-1)(\alpha-2)}-(\alpha-1)r_1\right]q(1-t)\le u(t)-\lambda w_1(t)\le R_1,\\ 0\le\left[\frac{c_2\Gamma(\alpha)R_1}{(\alpha-1)(\alpha-2)}-(\alpha-1)r_2\right]q(1-t)\le v(t)-\mu w_2(t)\le R_1.\end{aligned} \tag{3.3}$$

Then from (H_1) and Lemma 2.5, we have

$$\begin{aligned}
T_1(u,v)(t) =&\lambda \int_0^1 G(t,s)\left[f\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_1(s)\right]\mathrm{d}s\\
\leq&\lambda(\alpha-1)q(1-t)\int_0^1 k(s)h\left(u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\\
\leq&\lambda(\alpha-1)q(1-t)h^*(R_1)\int_0^1 k(s)\mathrm{d}s\\
\leq&\frac{4\lambda(\alpha-1)(\alpha-2)^{\alpha-1}h^*(R_1)}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s\\
\leq&R_1.
\end{aligned}$$

In the similar manner, we deduce

$$\begin{aligned}
T_2(u,v)(t) =&\mu \int_0^1 G(t,s)\left[g\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_2(s)\right]\mathrm{d}s\\
\leq&\mu(\alpha-1)q(1-t)\int_0^1 k(s)h\left(u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\\
\leq&\mu(\alpha-1)q(1-t)h^*(R_1)\int_0^1 k(s)\mathrm{d}s\\
\leq&\frac{4\mu(\alpha-1)(\alpha-2)^{\alpha-1}h^*(R_1)}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s\\
\leq&R_1.
\end{aligned}$$

Thus

$$\|T(u,v)\|\leq\|(u,v)\|,\forall(u,v)\in\partial P_{R_1}.$$

On the other hand, choose a constant $L>0$ such that

$$L\geq\frac{6}{c_1\lambda a^4(1-b)^4\int_a^b q(s)\mathrm{d}s}. \tag{3.4}$$

By (H_2), there exists a constant $N>0$ such that for any $t\in[a,b],x\geq N$, we have

$$\frac{f(t,x,y)}{x}>L. \tag{3.5}$$

Select

$$R_2>\max\left\{2R_1,\frac{6N}{c_1a^2(1-b)^2}\right\}.$$

Then for any $(u,v)\in\partial P_{R_2}$, we have $u(t)-\lambda w_1(t)\geq 0, v(t)-\mu w_2(t)\geq 0, t\in[0,1]$. Moreover, by $R_2>2R_1$, we have

$$\frac{(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}<\frac{c_1R_2}{2(\alpha-1)},$$

thus for any $t\in[a,b]$, noticing $2<\alpha-1\leq 3$,

$$\begin{aligned}
u(t)-\lambda w_1(t)\geq&\frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}R_2-\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}t^{\alpha-2}(1-t)^2r_1\\
\geq&t^{\alpha-2}(1-t)^2\left[\frac{c_1R_2}{(\alpha-1)}-\frac{(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}\right]\\
\geq&a^{\alpha-2}(1-b)^2\left[\frac{c_1R_2}{(\alpha-1)}-\frac{c_1R_2}{2(\alpha-1)}\right]\\
>&\frac{a^{\alpha-2}(1-b)^2c_1R_2}{2(\alpha-1)}\\
\geq&\frac{a^2(1-b)^2c_1R_2}{6}.
\end{aligned}$$

noticing $R_2 > \frac{6N}{c_1 a^2(1-b)^2}$, we have

$$u\,(t) - \lambda w_1(t) \geq \frac{a^2(1-b)^2 c_1 R_2}{6} > N.$$

Hence from (3.5) and Lemma 2.5, we get

$$\begin{aligned}
T_1(u,v)(t) =&\lambda \int_0^1 G(t,s)\left[f\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)+p_1(s)\right]\mathrm{d}s\\
\geq&\lambda \int_a^b G(t,s) f\left(s,u(s)-\lambda w_1(s),v(s)-\mu w_2(s)\right)\mathrm{d}s\\
>&\lambda L \int_a^b G(t,s)\left[u(s)-\lambda w_1(s)\right]\mathrm{d}s\\
>&\frac{c_1 a^2(1-b)^2\lambda L R_2}{6}\int_a^b G(t,s)\mathrm{d}s\\
\geq&\frac{c_1 a^2(1-b)^2\lambda L R_2}{6} t^{\alpha-2}(1-t)^2\int_a^b q(s)\mathrm{d}s\\
\geq&\frac{c_1 a^2(1-b)^2\lambda L R_2}{6}\int_a^b q(s)\mathrm{d}s \min_{t\in[a,b]}\left\{t^{\alpha-2}(1-t)^2\right\}\\
>&\frac{c_1 a^4(1-b)^4\lambda L R_2}{6}\int_a^b q(s)\mathrm{d}s\\
\geq&R_2
\end{aligned}$$

Thus

$$\|T(u,v)\| \geq \|(u,v)\|\,, \forall (u,v)\in \partial P_{R_2}.$$

In the similar manner, we can get the same result when $\lim\limits_{x\to+\infty}\min\limits_{t\in[a,b]}\dfrac{g(t,x,y)}{x}=+\infty$.

By using Lemma 2.7, we conclude that T has a fixed point (u,v) such that $R_1 \leq \|(u,v)\| \leq R_2$. Notice that $(u(t),v(t))$ is a solution of system (2.5) and $w_i(t)(i=1,2)$ are solutions of system (2.3). Thus $(u(t)-\lambda w_1(t), v(t)-\mu w_2(t))$ is a positive solution of the singular semipositone BVP (1.1).

Theorem 3.2. If (H_1) and (H_3) hold, then there exists $\overline{\eta}>0$ such that BVP (1.1) has at least one positive solution for any $\lambda,\mu\in(\overline{\eta},+\infty)$.

Proof. By the first of (H_3), we have that there exists a constant $\overline{N}>0$ such that for any $t\in[c,d], u\geq \overline{N}$, we have

$$f(t,u,v)\geq \frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s}.$$

Select

$$\overline{\eta}=\frac{\overline{N}\Gamma(\alpha)}{c^2(1-d)^2(\alpha-1)(\alpha-2)r_1}.$$

In the following of the proof, we suppose $\lambda,\mu>\overline{\eta}$.

Let

$$R_3=\frac{2\lambda(\alpha-1)^2(\alpha-2)r_1}{c_1\Gamma(\alpha)}.$$

$P_{R_3}=\{(u,v)\in P, \|(u,v)\|<R_3\}$, for any $(u,v)\in\partial P_{R_3}$, that is $\|(u,v)\|=R_3$. Then

$$\begin{aligned}
u(t)-\lambda w_1(t) \geq& \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1}R_3 - \frac{\lambda(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}t^{\alpha-2}(1-t)^2 r_1\\
\geq& t^{\alpha-2}(1-t)^2\left[\frac{c_1R_3}{(\alpha-1)}-\frac{\lambda(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}\right]\\
\geq& t^{\alpha-2}(1-t)^2\frac{\lambda(\alpha-1)(\alpha-2)r_1}{\Gamma(\alpha)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{t^{\alpha-2}(1-t)^2}{c^{\alpha-2}(1-d)^2}\overline{N} \\
&> \overline{N}.
\end{aligned}
$$

Hence for $(u,v) \in \partial P_{R_3}, t \in [c,d]$, we have

$$
\begin{aligned}
T_1(u,v)(t) =& \lambda \int_0^1 G(t,s)\left[f\left(s, u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right)+p_1(s)\right] \mathrm{d}s \\
\geq& \lambda \int_c^d G(t,s) f\left(s, u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right) \mathrm{d}s \\
\geq& \lambda \frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1 c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s} \int_c^d G(t,s)\mathrm{d}s \\
\geq& \lambda \frac{2(\alpha-1)^2(\alpha-2)r_1}{c_1 c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s} t^{\alpha-2}(1-t)^2 \int_c^d q(s)\mathrm{d}s \\
=& \frac{R_3}{c^2(1-d)^2} t^{\alpha-2}(1-t)^2 \\
>& R_3.
\end{aligned}
$$

Thus

$$
\|T(u,v)\| \geq \|(u,v)\|, \forall (u,v) \in \partial P_{R_3}.
$$

In the similar manner, we can get the same result when

$$
\lim_{x\to+\infty} \min_{t\in[c,d]} g(t,x,y) > \frac{2(\alpha-1)^2(\alpha-2)r_2}{c_2 c^2(1-d)^2\Gamma(\alpha)\int_c^d q(s)\mathrm{d}s}.
$$

On the other hand, $h(t)$ is continuous on $[0,+\infty)\times[0,+\infty)$, from the limit of (H_3), we known

$$
\lim_{z\to+\infty} \frac{h^*(z)}{z} = 0, \tag{3.6}
$$

where $h^*(z)$ is defined by (3.2). For

$$
\varepsilon = \frac{\Gamma(\alpha)\alpha^\alpha}{4(\alpha-1)(\alpha-2)^{\alpha-1}\max\{\lambda,\mu\}\int_0^1 k(s)\mathrm{d}s}
$$

there exists $\widetilde{N} > 0$ such that when $z \geq \widetilde{N}$, we have $h^*(z) \leq \varepsilon z$.

Select $R_4 \geq \max\left\{R_3, \widetilde{N}\right\}$, then for $(u,v) \in \partial P_{R_4}$, we get

$$
\begin{aligned}
T_1(u,v)(t) =& \lambda \int_0^1 G(t,s)\left[f\left(s, u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right)+p_1(s)\right] \mathrm{d}s \\
\leq& \lambda(\alpha-1)q(1-t)\int_0^1 k(s) h\left(u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right) \mathrm{d}s \\
\leq& \lambda(\alpha-1)q(1-t)h^*(R_4)\int_0^1 k(s)\mathrm{d}s \\
\leq& \frac{4\lambda(\alpha-1)(\alpha-2)^{\alpha-1}\varepsilon R_4}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s \\
\leq& R_4.
\end{aligned}
$$

In the similar manner, we deduce

$$
\begin{aligned}
T_2(u,v)(t) =& \mu \int_0^1 G(t,s)\left[g\left(s, u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right)+p_2(s)\right] \mathrm{d}s \\
\leq& \mu(\alpha-1)q(1-t)\int_0^1 k(s) h\left(u(s)-\lambda w_1(s), v(s)-\mu w_2(s)\right) \mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}&\leq \lambda(\alpha-1)q(1-t)h^*(R_4)\int_0^1 k(s)\mathrm{d}s\\&\leq \frac{4\mu(\alpha-1)(\alpha-2)^{\alpha-1}\varepsilon R_4}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\mathrm{d}s\\&\leq R_4.\end{aligned}$$

Thus

$$\|T(u,v)\| \leq \|(u,v)\|, \forall (u,v) \in \partial P_{R_4}.$$

Therefore, applying Lemma 2.7, we conclude that T has a fixed point (u,v) such that $R_3 \leq \|(u,v)\| \leq R_4$. Notice that $(u(t),v(t))$ is a solution of system (2.5) and $w_i(t)(i=1,2)$ are solutions of system (2.3). Thus $(u(t)-\lambda w_1(t), v(t)-\mu w_2(t))$ is a positive solution of the singular semipositone BVP (1.1).

Remark 3.1. The conclusion of Theorem 3.1 is valid if (H_2) is replaced by

(H_2^*) There exists $(a,b) \subset [0,1]$ such that

$$\begin{aligned}&\lim_{y\to+\infty}\min_{t\in[a,b]}\frac{f(t,x,y)+p_1(t)}{y} \geq \overline{L}, \text{ or}\\&\lim_{y\to+\infty}\min_{t\in[a,b]}\frac{g(t,x,y)+p_1(t)}{y} \geq \overline{L}.\end{aligned}$$

where $\overline{L} \geq \dfrac{6}{c_2\mu a^4(1-b)^4\int_a^b q(s)\mathrm{d}s}$.

Remark 3.2. The conclusion of Theorem 3.2 is valid if (H_3) is replaced by

(H_3^*) There exists $(c,d) \subset [0,1]$ such that

$$\begin{aligned}&\lim_{x\to+\infty}\min_{t\in[c,d]} f(t,x,y) = +\infty, \text{ or}\\&\lim_{x\to+\infty}\min_{t\in[c,d]} g(t,x,y) = +\infty,\end{aligned}$$

and

$$\lim_{x,y\to+\infty}\frac{h(x,y)}{x} = 0.$$

4 Examples

Now, we present two examples to illustrate the main results.

Example 4.1. Consider the following system of fractional differential equations

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) = \frac{1}{6}t^{-\frac{1}{3}}\left(u^2+v^2\right) - \frac{1}{8}t^{-\frac{1}{4}}, & 0<t<1,\\ D_{0+}^{\frac{7}{2}}v(t) = \frac{1}{3}t^{-\frac{1}{3}}\left(u^2+v^2\right) - \frac{1}{2}t^{-\frac{1}{2}}, & 0<t<1,\\ u(0)=u(1)=u'(0)=u'(1)=v(0)=v(1)=v'(0)=v'(1)=0. \end{cases} \tag{4.1}$$

In BVP (4.1), $\alpha = \frac{7}{2}$ and

$$\begin{aligned}f(t,u,v) &= \frac{1}{6}t^{-\frac{1}{3}}\left(u^2+v^2\right) - \frac{1}{8}t^{-\frac{1}{4}},\\ g(t,u,v) &= \frac{1}{3}t^{-\frac{1}{3}}\left(u^2+v^2\right) - \frac{1}{2}t^{-\frac{1}{2}},\end{aligned}$$

for $t \in [0,1], u,v \geq 0$.

We deduce $p_1(t) = \frac{1}{8}t^{-\frac{1}{4}}, p_2(t) = \frac{1}{2}t^{-\frac{1}{2}}, k(t) = \frac{1}{3}t^{-\frac{1}{3}}, a_i(t) = \frac{1}{9}t^{-\frac{1}{3}}, c_i = \frac{1}{3}, i=1,2$. $h(u,v) = u^2+v^2$, and

$$\lim_{u\to+\infty}\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\frac{f(t,u,v)}{u} = +\infty.$$

So all conditions of Theorem 3.1 are satisfied. Hence it follows from Theorem 3.1 that BVP (4.1) has at least one positive solution.

Example 4.2. Consider the following system of fractional differential equations

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) = \frac{1}{10}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{16}t^{-\frac{1}{8}}, \quad 0 < t < 1, \\ D_{0+}^{\frac{7}{2}}v(t) = \frac{1}{5}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{4}t^{-\frac{1}{4}}, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0. \end{cases} \tag{4.2}$$

In BVP (4.2), $\alpha = \frac{7}{2}$ and

$$f(t,u,v) = \frac{1}{10}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{16}t^{-\frac{1}{8}},$$
$$g(t,u,v) = \frac{1}{5}t^{-\frac{1}{5}}\left(\ln(1+u) + \frac{1}{v+1}\right) - \frac{1}{4}t^{-\frac{1}{4}},$$

for $t \in [0,1], u, v \geq 0$.

We deduce $p_1(t) = \frac{1}{16}t^{-\frac{1}{8}}, p_2(t) = \frac{1}{4}t^{-\frac{1}{4}}, k(t) = \frac{1}{5}t^{-\frac{1}{5}}, a_i(t) = \frac{1}{15}t^{-\frac{1}{5}}, c_i = \frac{1}{3}, i = 1,2$. $h(u,v) = \ln(1+u) + \frac{1}{v+1}$, and

$$\lim_{u\to+\infty} \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} f(t,u,v) = +\infty,$$
$$\lim_{u,v\to+\infty} \frac{h(u,v)}{u} = 0.$$

So all conditions of Remark 3.2 are satisfied. Hence it follows from Corollary 3.2 that BVP (4.2) has at least one positive solution.

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Existence and uniqueness of positive solutions of fractional differential equations with infinite-point boundary value conditions*

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Abstract

In this work, we consider the following nonlinear fractional differential equation with infinite-point boundary value condition

$$\begin{cases} \mathcal{D}^{\alpha}x(t)+r(t)f(t,x(t))+q(t)=0, t\in(0,1),\\ x(0)=x'(0)=\cdots=x^{n-2}(0)=0,\\ x^{i}(1)=\sum\limits_{j=1}^{\infty}\alpha_j x(\xi_j), \end{cases} \tag{0.1}$$

where $\alpha>2$, $n-1<\alpha<n$, $i\in[0,n-2]$ is a fixed integer, $\alpha_j\geq 0$,

$$0<\xi_1<\xi_2<\cdots<\xi_{j-1}<\xi_j<\cdots<1 (j=1,2,\ldots),$$

$\Delta-\sum\limits_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-1}>0$ and

$$\Delta=\begin{cases} 1, i=0,\\ (\alpha-1)(\alpha-2)\cdots(\alpha-i), i\in(0,n-2]. \end{cases} \tag{0.2}$$

By the Lipschitz constant related to the first eigenvalue corresponding to the relevant operator and a μ_0-bounded positive operator, we prove the existence and uniqueness of the positive solution of the fractional differential equation(0.1). Finally an example is given to illustrate the effectiveness of our result.

Keywords: fractional differential equations; μ_0-bounded positive operators; the first eigenvalues; Green functions; completely continuous operators

1 Introduction

In recent years, boundary value problems of nonlinear fractional differential equations have been studied extensively in resent works [1–8]. Most of the results have at least one and multiple positive solutions by the theory of nonlinear analysis. For example, the authors [1] considered the existence of multiple positive solutions of the following fractional differential equation

$$\begin{cases} \mathcal{D}^{\alpha}x(t)+q(t)f(t,x(t))=0, t\in(0,1),\\ x(0)=x'(0)=\cdots=x^{n-2}(0)=0,\\ x^{i}(1)=\sum\limits_{j=1}^{\infty}\alpha_j x(\xi_j), \end{cases} \tag{1.1}$$

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where $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative $\alpha > 2$, $n-1 < \alpha < n$ and $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1 (j = 1, 2, \ldots)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$. They established the existence results by introducing height function and Guo-Krasnosel'skii fixed point theorem of cone expansion-compression and obtained several local existence and multiplicity of positive solutions. In [2] the authors studied the existence of solutions of the following fractional differential equation:

$$\begin{cases} -\mathcal{D}^\alpha x(t) = q(t)f(t, x(t)) - p(t), 0 < t < 1, \\ x(0) = x'(0) = x(1) = 0, \end{cases} \tag{1.2}$$

where $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative, $2 < \alpha \leq 3$ is a real number, $p : (0, 1) \to [0, +\infty)$ is Lebesgue integrable and may be singular at some zero measure set of $(0, 1)$. They obtained that the existence and multiplicity of positive solutions by Krasnosel'skii fixed point theorem. In [3] the authors studied the fractional differential equation

$$\begin{cases} -\mathcal{D}^\alpha x(t) = q(t)f(t, x(t)) + p(t), 0 < t < 1, \\ x(0) = x'(0) = x(1) = 0, \end{cases} \tag{1.3}$$

where $2 < \alpha \leq 3$ is a real number, and got the uniqueness of solution under the assumption that $f(t, x)$ is a Lipschitz continuous function. Some similar results of the existence and multiplicity of positive solutions can refer to [5, 7–10, 12, 13]. But the uniqueness of positive solutions of fractional differential equations are seldom considered in recent works. Motivated by the above results, we study the existence and uniqueness of the positive solution of the fractional differential equation (0.1) under the assumption that $f(t, x)$ is a Lipschitz continuous function. Then we obtain some results by the basic properties of μ_0-bounded positive operators. Our results extend the corresponding results of [1, 3, 4].

For the sake of description, we list three conditions as follows:

$(L1)$ $q : (0, 1) \to \mathbb{R}$ is continuous and Lebesgue integrable;

$(L2)$ $r : (0, 1) \to [0, +\infty)$ is a continuous function which does not vanish identically on any subinterval of $(0, 1)$ and satisfies

$$0 < \int_0^1 r(s)ds < +\infty;$$

$(L3)$ $f : [0, 1] \times \mathbb{R} \to [0, +\infty)$ is continuous.

2 Preliminaries

For the convenience of the reader, we present the necessary definitions and lemmas from fractional calculus theory. These definitions and lemmas can be found in monographs [1–6, 10].

Definition 2.1. *([10]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} f(s)ds,$$

provided that the right-hand side is point wise defined on $(0, +\infty)$.

Definition 2.2. *([10]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{(n-\alpha-1)} f(s)ds,$$

where $n - 1 \leq \alpha < n$, provided that the right-hand side is point wise defined on $(0, +\infty)$.

Lemma 2.1. *([10]) Assume that $x \in C(0,1)\bigcap L(0,1)$ with a fractional derivative of order $\alpha > 0$, then*

$$I^{\alpha}\mathcal{D}^{\alpha}x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}(i = 1, 2, \cdots, n)$, n is the smallest integer greater than or equal to α.

In this paper the norm of $E = C[0,1]$ is defined by $\|x\| = \max\limits_{t\in[0,1]} |x(t)|$ and $P = \{x \in E | x(t) \geq 0, t \in [0,1]\}$ is a cone of E. The following conceptions come from Krasnosel'skill [12] and [1].

Definition 2.3. *([4]) A bounded linear operator $T : E \to E$ is called a μ_0-bounded positive operator if there exists $\mu_0 \in E \setminus (-P)$ such that for each $x \in E \setminus (-P)$, there exist a natural number n and positive constants $\alpha(x)$, $\beta(x)$ such that*

$$\alpha(x)\mu_0 \leq T^n x \leq \beta(x)\mu_0.$$

Lemma 2.2. *([4]) Suppose that $T : E \to E$ is a completely continuous μ_0-bounded positive operator and $T(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and T has only one positive eigenfunction φ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$, i.e. $\varphi = \lambda_1 T\varphi$.*

Lemma 2.3. *Given $y \in C[0,1]\bigcap L[0,1]$, then the unique solution of the following equation:*

$$\begin{cases} \mathcal{D}^{\alpha}x(t) + y(t) = 0, t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{n-2}(0) = 0, \\ x^i(1) = \sum\limits_{j=1}^{\infty} \alpha_j x(\xi_j), \end{cases} \tag{2.1}$$

is

$$x(t) = \int_0^1 G(t,s)y(s)ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1}, 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}, 0 \leq t \leq s \leq 1, \end{cases} \tag{2.2}$$

here $p(s) = \Delta - \sum_{s\leq\xi_j} \alpha_j (\frac{\xi_j - s}{1-s})^{\alpha-1}(1-s)^i$.

Proof. The proof is similar to that of Lemma 2.2 of [4], so we omit the details. □

Lemma 2.4. *(1)The function $p(s)$ in Lemma 2.3 satisfies that $p(s) > 0$ and $p(s)$ is increasing on $[0,1]$;*
(2)For each $s \in [0,1]$, we have $m_1 s + p(0) \leq p(s) \leq M_1 + p(0)$, where

$$M_1 = \sup_{0<s\leq 1} \frac{p(s) - p(0)}{s}, \quad m_1 = \inf_{0<s\leq 1} \frac{p(s) - p(0)}{s};$$

(3) $G(t,s) > 0, \quad \forall t, s \in (0,1)$;
(4)$m_1 s(1-s)^{\alpha-1-i}t^{\alpha-1} \leq p(0)\Gamma(\alpha)G(t,s) \leq [M_1 + p(0)n]s(1-s)^{\alpha-1-i}, \quad \forall t, s \in (0,1)$;
(5)$m_1 s(1-s)^{\alpha-1-i}t^{\alpha-1} \leq p(0)\Gamma(\alpha)G(t,s) \leq [M_1 + p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}, \ \forall t, s \in (0,1)$.

Proof. We only prove (4) and (5) since the proofs of (1), (2) and (3) are easy.

When $0 \le s \le t \le 1$, we have

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1} \\ &= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)[t^{\alpha-1}(1-s)^{\alpha-1-i} - (t-s)^{\alpha-1}] \\ &\ge m_1 s t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}[(1-s)^{\alpha-1-i} - (1-\frac{s}{t})^{\alpha-1}] \\ &\ge m_1 s t^{\alpha-1}(1-s)^{\alpha-1-i}, \end{aligned}$$

and

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1} \\ &= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)[t^{\alpha-1}(1-s)^{\alpha-1-i} - (t-s)^{\alpha-1}] \\ &\le M_1 s t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i}[1-(1-\frac{s}{t})^{\alpha-1}] \\ &\le M_1 s t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i}[1-(1-\frac{s}{t})][1+(1-\frac{s}{t})+\cdots+(1-\frac{s}{t})^{n-1}] \\ &\le M_1 s t^{\alpha-1}(1-s)^{\alpha-1-i} + np(0)t^{\alpha-1}(1-s)^{\alpha-1-i}\frac{s}{t} \\ &\le [M_1+p(0)n]s(1-s)^{\alpha-1-i}. \end{aligned}$$

When $0 \le t \le s \le 1$, we have

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} \\ &= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i} \\ &\ge m_1 s t^{\alpha-1}(1-s)^{\alpha-1-i} \end{aligned}$$

and

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} \\ &= [p(s)-p(0)]t^{\alpha-1}(1-s)^{\alpha-1-i} + p(0)t^{\alpha-1}(1-s)^{\alpha-1-i} \\ &\le M_1 s t^{\alpha-1}(1-s)^{\alpha-1-i} + np(0)t^{\alpha-1}(1-s)^{\alpha-1-i} \\ &\le [M_1+p(0)n]s(1-s)^{\alpha-1-i}. \end{aligned}$$

So (4) is proved. Now we prove (5). We only prove

$$p(0)\Gamma(\alpha)G(t,s) \le [M_1+p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}, \forall t,s \in (0,1).$$

When $0 \le s \le t \le 1$, from the proof process of (4) we have

$$p(0)\Gamma(\alpha)G(t,s) \le M_1 s t^{\alpha-1}(1-s)^{\alpha-1-i} + np(0)t^{\alpha-1}(1-s)^{\alpha-1-i}\frac{s}{t},$$

So

$$p(0)\Gamma(\alpha)G(t,s) \le [M_1+p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}.$$

When $0 \le t \le s \le 1$, we have similarly that

$$p(0)\Gamma(\alpha)G(t,s) \le [M_1+p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}.$$

□

Now let $P_1 = \{x \in E | x(t) \ge \frac{m_1 t^{\alpha-1}}{M_1+p(0)n}\}$ and two operators T and A be defined, respectively, by

$$(Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds, t \in [0,1], x \in C[0,1]$$

and

$$(Ax)(t) = \int_0^1 G(t,s)[r(s)f(s,x(s)) + q(s)]ds, t \in [0,1], x \in C[0,1].$$

4

Lemma 2.5. *$T : P_1 \to P_1$ is a linear completely continuous operator and a μ_0-bounded positive operator with $\mu_0(t) = t^{\alpha-1}$.*

Proof. According to Lemma 2.4 for any k_1, $k_2 \in \mathbb{R}$ and x_1, x_2, $x \in E$ we have

$$\begin{aligned} T(k_1x_1 + k_2x_2)(t) &= \int_0^1 G(t,s)r(s)(k_1x_1 + k_2x_2)(s)ds \\ &= k_1 \int_0^1 G(t,s)r(s)x_1(s)ds + k_2 \int_0^1 G(t,s)r(s)x_2(s)ds \\ &= k_1(Tx_1)(t) + k_2(Tx_2)(t), \end{aligned}$$

$$\begin{aligned} \|Tx\| = \max_{t\in[0,1]} |(Tx)(t)| &= \max_{t\in[0,1]} \int_0^1 G(t,s)r(s)x(s)ds \\ &\le \frac{M_1 + p(0)n}{p(0)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1-i}r(s)x(s)ds, \end{aligned}$$

and

$$\begin{aligned} (Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds &\ge \frac{m_1 t^{\alpha-1}}{p(0)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1-i}r(s)x(s)ds \\ &\ge \frac{m_1 t^{\alpha-1}}{M_1 + p(0)n} \|Tx\| \in P_1. \end{aligned}$$

Notice the continuity of $G(t,s)$, by a standard argument it is not difficult to prove that $T : P_1 \to P_1$ is linear completely continuous.

Now we prove that T is a μ_0-bounded positive operator with $\mu_0(t) = t^{\alpha-1}$. According to Lemma 2.4 we have

$$\begin{aligned} (Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds &\le \frac{M_1 + p(0)n}{p(0)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1-i}r(s)x(s)dst^{\alpha-1}, \\ (Tx)(t) = \int_0^1 G(t,s)r(s)x(s)ds &\ge \frac{m_1}{p(0)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1-i}r(s)x(s)dst^{\alpha-1}. \end{aligned}$$

From Definition 2.3, T is a μ_0-bounded positive operator with $\mu_0(t) = t^{\alpha-1}$. □

According to Lemma 2.4 we can easily get that $A : P_1 \to P_1$ is a completely continuous operator. And it is not hard to see that A is a solution of the equation (0.1) if and only if A has a fixed point in P_1. This is crucial for the proof of the following Theorem 3.1.

3 Main results

Theorem 3.1. *Suppose that $(L1) - (L3)$ hold and there exists $k \in [0,1)$ such that*

$$|f(t,u) - f(t,v)| \le k\lambda_1|u - v|, \forall t \in [0,1], u, v \in E,$$

where λ_1 is the first eigenvalue of T. Then the equation (0.1) has a unique solution x^* in E and for each $x_0 \in E$, the iterative sequence $x_n = Ax_{n-1}(n = 1, 2, \cdots)$ converges to x^*.

Proof. For any given $x_0 \in E$, let $x_n = Ax_{n-1}(n = 1, 2, \cdots)$, according to Lemma 2.5 and Definition 2.3, there exists $\beta = \beta(|x_1 - x_0|) > 0$ such that

$$T(|x_1 - x_0|)(t) \leq \beta\mu_0(t), \forall t \in [0, 1].$$

For all $m \in \mathbb{N}$ we have

$$\begin{aligned}
|x_{m+1} - x_m| &= |Ax_m(t) - Ax_{m-1}(t)| \\
&= \left|\int_0^1 G(t,s)[r(s)f(s, x_m(s)) + q(s)]ds - \int_0^1 G(t,s)[r(s)f(s, x_{m-1}(s)) + q(s)]ds\right| \\
&\leq \int_0^1 G(t,s)r(s)|f(s, x_m(s) - f(s, x_{m-1}(s))|ds \leq k\lambda_1 T(|x_m - x_{m-1}|)(t) \\
&\leq \cdots \leq k^m\lambda_1^m T^m(|x_1 - x_0|)(t) \leq k^m\lambda_1^m \beta T^{m-1}\mu_0 = k^m\lambda_1\beta\mu_0.
\end{aligned}$$

Then for any $n \geq m \in \mathbb{N}$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_{m-1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m-1} - x_m| \\
&\leq \beta\lambda_1[k^{n-1} + k^{n-2} + \cdots + k^m]\mu_0 \leq \beta\lambda_1 \frac{k^m}{1-k}\mu_0.
\end{aligned}$$

So $\|x_n - x_m\| \leq \beta\lambda_1 \frac{k^m}{1-k}\|\mu_0\| \to 0(m \to \infty)$. By the completeness of E, there exists $x^* \in E$ such that $\lim_{n\to\infty} x_n = x^*$. Due to $x_n = Ax_{n-1}$ and noting that A is continuous, we obtain that $x^* = Ax^*(n \to \infty)$. In other words, x^* is a fixed point of A.

Suppose y^* is another fixed point of A and $x^* \neq y^*$. From Lemma 2.5 and Definition 2.3, there exists $\beta = \beta(|x^* - y^*|) > 0$ such that

$$T(|x^* - y^*|)(t) \leq \beta\mu_0, \forall t \in [0, 1].$$

For all $n \in \mathbb{N}$ we have

$$|x^*(t) - y^*(t)| = |A^n x^*(t) - A^n y^*(t)| \leq k^n\beta\lambda_1\mu_0,$$

so $\|x^*(t) - y^*(t)\| \leq k^n\beta\lambda_1\|\mu_0\| \to 0(n \to \infty)$ which implies $x^* = y^*$. This means that A has a unique fixed point. □

Theorem 3.2. *Suppose that $(L1) - (L3)$ hold and there exist $k \in [0, 1)$ and $x_0 \in E$ such that*

(1) $\mathcal{D}^\alpha x_0(t) + r(t)f(t, x_0(t)) + q(t) \geq 0, \quad t \in (0, 1)$;

(2) $x_0(0) = x_0'(0) = \cdots = x_0^{n-2}(0) \geq 0$;

(3) $x_0^i(1) \geq \sum_{j=1}^{\infty} \alpha_j x_0(\xi_j)$ and

(4) $|f(t, u) - f(t, v)| \leq k\lambda_1|u - v|$, $\forall t \in [0, 1]$, $u(t), v(t) \in \Omega$,

where $f(t, x)$ is non-descending in x, $\Omega = \{x \in E | x \leq x_0\}$ and λ_1 is the first eigenvalue of T. Then the equation (0.1) has a unique positive solution x^* in Ω.

Proof. According to Lemma 2.3 we can get that A is decreasing on Ω, $Ax_0 \leq x_0$ and $A(\Omega) \subset \Omega$. Let $x_n = Ax_{n-1}(n = 1, 2, \cdots)$, then we have

$$x_0 \geq x_1 \geq \cdots x_n \geq \cdots.$$

According to Definition 2.3, there exists $\beta > 0$ such that $T(x_0 - x_1) \leq \beta\mu_0(t)$. Then for each $n \in \mathbb{N}$ and $t \in [0, 1]$,

$$\begin{aligned}
0 &\leq x_n(t) - x_{n+1}(t) = Ax_{n-1}(t) - Ax_n(t) \\
&\leq k\lambda_1 T(x_{n-1} - x_n)(t) \leq \cdots \leq (k\lambda_1 T)^n(x_0 - x_1)(t) \\
&\leq \beta k^n\lambda_1\mu_0(t).
\end{aligned}$$

Then for every $n \geq m \in \mathbb{N}$,

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_{m-1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m-1} - x_m| \\ &\leq \beta\lambda_1[k^{n-1} + k^{n-2} + \cdots + k^m]\mu_0 \leq \beta\lambda_1 \frac{k^m}{1-k}\mu_0. \end{aligned}$$

So $\|x_n - x_m\| \leq \beta\lambda_1 \frac{k^m}{1-k}\|\mu_0\| \to 0(m \to \infty)$. By the completeness of E, there exists $x^* \in E$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Furthermore, x^* is a fixed point of A in Ω.

Suppose $y^* \in \Omega$ is another fixed point of A. By Lemma 2.5 and Definition 2.3, there exists $\beta_1 = \beta_1(x_0 - y^*) > 0$ such that

$$T(x_0 - y^*)(t) \leq \beta_1\mu_0(t), \forall t \in [0, 1].$$

For all $n \in \mathbb{N}$ we have $y^* \leq x_n \leq x_0$, so $y^* \leq x^* \leq x_n \leq x_0$. Then we have

$$\begin{aligned} |y^*(t) - x^*(t)| &\leq |y^*(t) - x_n(t)| + |x_n(t) - x^*(t)| \leq 2|y^*(t) - x_n(t)| \\ &= |A^n y^*(t) - A^n x_0(t)| \leq 2k^n\beta_1\mu_0(t). \end{aligned}$$

Thus $y^* = x^*$ which implies that A has a unique fixed point in Ω. □

Theorem 3.3. *Suppose that $(L1) - (L3)$ hold and there exist $k \in [0, 1)$ and $x_0 \in E$ such that*

(1) $\mathcal{D}^\alpha x_0(t) + r(t)f(t, x_0(t)) + q(t) \leq 0$, $t \in (0, 1)$;

(2) $x_0(0) = x_0'(0) = \cdots = x_0^{n-2}(0) \leq 0$;

(3) $x_0^i(1) \leq \sum\limits_{j=1}^{\infty} \alpha_j x_0(\xi_j)$ and

(4) $|f(t, u) - f(t, v)| \leq k\lambda_1|u - v|$, $\forall t \in [0, 1]$, $u(t), v(t) \in \Omega$,

where $f(t, x)$ is non-decreasing in x, $\Omega = \{x \in E | x \geq x_0\}$ and λ_1 is the first eigenvalue of T. Then the equation (0.1) has a unique positive solution x^* in Ω.

Proof. The proof is similar to that of Theorem 3.2, so we omit it. □

Example 3.1 Consider the following equation

$$\begin{cases} \mathcal{D}^{\frac{7}{2}} x(t) + \lambda(1 - t)^2(\frac{2}{5}x(t) + 1 - sinx(t)) + t^2 = 0, t \in [0, 1] \\ x(0) = x'(0) = x''(0) = 0, \\ x'(1) = \sum\limits_{j=1}^{\infty} (\frac{1}{2})^j x(1 - (\frac{1}{2})^j), \end{cases} \tag{3.1}$$

where $0 \leq \lambda \leq \lambda_1$, λ_1 is the first eigenvalue of T, $\alpha = \frac{7}{2}$, $n = 4$, $i = 1$, $\Delta = \frac{5}{2}$, $r(t) = (1 - t)^2$, $f(t, x) = (\frac{2}{5}x(t) + 1 - sinx(t))$, $q(t) = t^2$, $\alpha_j = (\frac{1}{2})^j$, $\xi_j = 1 - (\frac{1}{2})^j$. By a careful calculation we get $\Delta - \sum\limits_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$ and $|f(t, u) - f(t, v)| \leq \frac{9}{10}\lambda_1|u - v|$. From Theorem 3.1, equation (3.1) has a unique solution.

Example 3.2 Consider the following equation

$$\begin{cases} \mathcal{D}^{\frac{9}{2}} x(t) + \dfrac{\lambda}{\lambda + 1}(1 - t)^{\frac{5}{4}}(\frac{1}{2}x(t) + 1 + \frac{9}{20}cosx(t)) + t^{\frac{7}{2}} = 0, t \in [0, 1] \\ x(0) = x'(0) = x''(0) = x'''(0), \\ x(1) = \sum\limits_{j=1}^{\infty} (2j - 1)(\frac{1}{2})^{j+1} x(1 - (\frac{1}{2})^j), \end{cases} \tag{3.2}$$

where $0 \leq \lambda \leq \lambda_1$, λ_1 is the first eigenvalue of T, $\alpha = \frac{9}{2}$, $n = 5$, $i = 0$, $\Delta = 1$, $r(t) = \frac{(1-t)^{\frac{5}{4}}}{1+\lambda}$, $f(t,x) = \lambda(\frac{1}{2}x(t) + 1 + \frac{9}{20}cosx(t))$, $q(t) = t^{\frac{7}{2}}$, $\alpha_j = (2j-1)(\frac{1}{2})^{j+1}$, $\xi_j = 1 - (\frac{1}{2})^j$. By a careful calculation we get $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$ and $|f(t,u) - f(t,v)| \leq \frac{19}{20}\lambda_1|u-v|$. From Theorem 3.1, equation (3.2) has a unique solution.

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DYNAMICAL ANALYSIS OF A NON-LINEAR DIFFERENCE EQUATION

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Abstract

In this article, we investigate the dynamics of the solutions of the following non-linear difference equation

$$x_{n+1} = x_{n-2}x_{n-3} - 1, \; n \in \mathbb{N}_0$$

with arbitrary initial conditions x_{-2}, x_{-1}, x_0. Besides, we have studied periodic behaviours of related difference equation especially asymptotic periodicity and eventually periodicity. Then, we have researched unbounded solutions of difference equation.

Key Words : Difference equation, equilibrium point, periodicity, asymptotic periodicity, unbounded.

Mathematics Subject Classification : 39A10, 39A23.

1 Introduction

Recently, the difference equations became a very popular topic among mathematicians. Difference equations have applications in many fields of science such as biology in [12], [8] and [10], economics in [1] and so forth.

Up to the present, many authors investigated to dynamics of various forms of difference equation $x_{n+1} = x_{n-k}x_{n-l} - 1$, $n \in \mathbb{N}_0$ such as $k = 0$, $l = 1$ in [4]; $k = 0$, $l = 2$ in [6]; $k = 1$, $l = 2$ in [5]; $k = 0$, $l = 3$ in [7]. Besides, Stević and Iričanin have obtained some results regarding the general form of the related difference equation in [18].

In this work we will study dynamic behaviours of the difference equation

$$x_{n+1} = x_{n-2}x_{n-3} - 1, \; n \in \mathbb{N}_0. \tag{1}$$

The Diff. Eq.(1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1, \; n \in \mathbb{N}_0, \tag{2}$$

with specific selection of k and l, where $k, l \in \mathbb{N}_0$.

This work can be considered as a continuance of our systematic analysis of Diff. Eq.(2).

There are two equilibrium points of Diff. Eq.(1) respectively:

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2},\ \bar{x}_2 = \frac{1+\sqrt{5}}{2}. \tag{3}$$

Note that this equilibrium points are the Golden Number and its conjugate.

2 Existence of Periodicity of Diff. Eq.(1)

In this section, we show that Diff. Eq.(1) has minimal prime periodic solutions with period seven. Also Diff. Eq.(1) has eventually periodic solutons with period seven.

Theorem 1 *Diff. Eq.(1) has no eventually constant solutions.*
Proof. *If $\{x_n\}_{n=-3}^{\infty}$ is eventually constant solutions of Diff. Eq.(1), hence $x_N = x_{N+1} = x_{N+2} = x_{N+3} = \bar{x}$, for some $N \in \mathbb{N}_0$, where $\bar{x}$ is an equilibrium point. However, Diff. Eq.(1) gives $x_{N+3} = x_N x_{N-1} - 1$, which implies*

$$x_{N-1} = \frac{x_{N+3}+1}{x_N} = \frac{\bar{x}+1}{\bar{x}} = \bar{x}.$$

Repetition the procedure, we get that $x_n = \bar{x}$ for $-3 \le n \le N+3$. Then, the proof is completed. ■

Theorem 2 *There are no nontrivial nor eventually period-two solutions of Diff. Eq.(1).*

Proof. Suppose that $x_N = x_{N+2k}$ and $x_{N+1} = x_{N+2k+1}$, for all $k \in \mathbb{N}_0$, and some $N \ge -1$, with $x_N \ne x_{N+1}$. Therefore, we have

$$\begin{aligned} x_{N+4} &= x_{N+1}x_N - 1 && (4)\\ &= x_{N-1}x_N - 1 = x_{N+3} && (5)\\ &= x_{N-1}x_{N-2} - 1 = x_{N+2} && (6)\\ &= x_{N-3}x_{N-2} - 1 = x_{N+1} && (7) \end{aligned}$$

From (5)-(7) and since $x_{N+4} = x_N$ we arrive a contradiction, as desired. ■

Theorem 3 *Diff. Eq.(1) has no minimal prime period-three solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-three solution of Diff. Eq.(1). Then, $x_{3n-3} = a$, $x_{3n-2} = b$, $x_{3n-1} = c$ and $x_{3n} = a$ for all $n \in \mathbb{N}_0$ and a, b and $c \in \mathbb{R}$ such that at least two are different from each other. From Diff. Eq.(1), we have

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = ba - 1 = b && (8)\\ x_2 &= x_{-1}x_{-2} - 1 = cb - 1 = c && (9)\\ x_3 &= x_0 x_{-1} - 1 = ac - 1 = a && (10) \end{aligned}$$

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From (8)-(10) we obtain that

$$a = b = c = \bar{x}_1$$

or

$$a = b = c = \bar{x}_2.$$

Thus, the proof is completed. ■

Theorem 4 *Diff. Eq.(1) has no minimal prime period-four solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-four solution of Diff. Eq.(1). Then, $x_{4n-3} = a$, $x_{4n-2} = b$, $x_{4n-1} = c$ and $x_{4n} = d$ for all $n \in \mathbb{N}_0$ and a, b, c and $d \in \mathbb{R}$ such that at least two of them are different. From Diff. Eq.(1), we have

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = ba - 1 = a && (11)\\ x_2 &= x_{-1}x_{-2} - 1 = cb - 1 = b && (12)\\ x_3 &= x_0x_{-1} - 1 = dc - 1 = c && (13)\\ x_4 &= x_1x_0 - 1 = ad - 1 = d. && (14) \end{aligned}$$

From (11)-(14) we obtain that

$$a = b = c = d = \bar{x}_1$$

or

$$a = b = c = d = \bar{x}_2$$

as desired. ■

Theorem 5 *Diff. Eq.(1) has no minimal prime period-five solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a periodic solution of Diff. Eq.(1) with minimal prime period-five. Then, $x_{5n-3} = a$, $x_{5n-2} = b$, $x_{5n-1} = c$, $x_{5n} = d$ and $x_{5n+1} = e$ for all $n \in \mathbb{N}_0$ and a, b, c, d and $e \in \mathbb{R}$ such that at least two of them are different. From Diff. Eq.(1), we obtain

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = ba - 1 = e && (15)\\ x_2 &= x_{-1}x_{-2} - 1 = cb - 1 = a && (16)\\ x_3 &= x_0x_{-1} - 1 = dc - 1 = b && (17)\\ x_4 &= x_1x_0 - 1 = ed - 1 = c && (18)\\ x_5 &= x_2x_1 - 1 = ae - 1 = d. && (19) \end{aligned}$$

From (15)-(19) we have

$$a = b = c = d = e = \bar{x}_1$$

or

$$a = b = c = d = e = \bar{x}_2$$

as desired. ■

Theorem 6 *Diff. Eq.(1) has no period solutions with minimal prime period-six.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-six solution of Diff. Eq.(1). Then, $x_{6n-3} = a$, $x_{6n-2} = b$, $x_{6n-1} = c$, $x_{6n} = d$, $x_{6n+1} = e = ac - 1$ and $x_{6n+2} = f = bd - 1$ for all $n \in \mathbb{N}_0$ and a, b, c, d, e and $f \in \mathbb{R}$ such that at least two of them are different. We have

$$x_1 = x_{-2}x_{-3} - 1 = ab - 1 = e \tag{20}$$
$$x_2 = x_{-1}x_{-2} - 1 = bc - 1 = f \tag{21}$$
$$x_3 = x_0x_{-1} - 1 = cd - 1 = a \tag{22}$$
$$x_4 = x_1x_0 - 1 = de - 1 = b \tag{23}$$
$$x_5 = x_2x_1 - 1 = ef - 1 = c \tag{24}$$
$$x_6 = x_3x_2 - 1 = fa - 1 = d. \tag{25}$$

From (20)-(25) we obtain

$$a = b = c = d = e = f = \bar{x}_1$$

or

$$a = b = c = d = e = f = \bar{x}_2$$

as desired. ■

Theorem 7 *There are periodic solutions of Diff. Eq.(1) with minimal prime period-seven if and only if*

(i) $x_{-3} = 0, x_{-2} = m, x_{-1} = -1, x_0 = -1;$

(ii) $x_{-3} = -1, x_{-2} = m, x_{-1} = 0, x_0 = 0;$

(iii) $x_{-3} = -1, x_{-2} = -1, x_{-1} = -1, x_0 = m;$

(iv) $x_{-3} = m, x_{-2} = -1, x_{-1} = -1, x_0 = -1;$

(v) $x_{-3} = -1, x_{-2} = -1, x_{-1} = m, x_0 = 0;$

where m is arbitrary.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a periodic solution of Diff. Eq.(1) with minimal prime period-seven. Then, $x_{7n-3} = a$, $x_{7n-2} = b$, $x_{7n-1} = c$, $x_{7n} = d$, $x_{7n+1} = e = ac-1$, $x_{7n+2} = f = bc-1$ and $x_{7n+3} = g = cd-1$ for all $n \in \mathbb{N}_0$ and a, b, c, d, e, f and $g \in \mathbb{R}$ such that at least two are different from each other. We have

$$\begin{aligned}
x_1 &= x_{-2}x_{-3} - 1 = ab - 1 = e \\
x_2 &= x_{-1}x_{-2} - 1 = bc - 1 = f \\
x_3 &= x_0x_{-1} - 1 = cd - 1 = g \\
x_4 &= x_1x_0 - 1 = de - 1 = a \\
x_5 &= x_2x_1 - 1 = ef - 1 = b \\
x_6 &= x_3x_2 - 1 = fg - 1 = c \\
x_7 &= x_4x_3 - 1 = ga - 1 = d.
\end{aligned}$$

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Thus, the following equalities are obtained:

$$\begin{aligned} x_4 &= d(ab-1)-1 = a && (26)\\ x_5 &= (ab-1)(bc-1)-1 = b && (27)\\ x_6 &= (bc-1)(cd-1)-1 = c && (28)\\ x_7 &= (cd-1)a-1 = d. && (29) \end{aligned}$$

From (26)-(29), then by direct calculation we have

Case 1 $a=0, c=-1, d=-1$;

Case 2 $a=-1, c=0, d=0$;

Case 3 $a=-1, b=-1, c=-1$;

Case 4 $b=-1, c=-1, d=-1$;

Case 5 $a=-1, b=-1, d=0$;

and so,

$$x_{-3}=0, x_{-2}=m, x_{-1}=-1, x_0=-1$$

$$x_{-3}=-1, x_{-2}=m, x_{-1}=0, x_0=0$$

$$x_{-3}=-1, x_{-2}=-1, x_{-1}=-1, x_0=m$$

$$x_{-3}=m, x_{-2}=-1, x_{-1}=-1, x_0=-1$$

$$x_{-3}=-1, x_{-2}=-1, x_{-1}=m, x_0=0$$

where m is arbitrary as desired. ■

Consequently, all minimal prime period-seven solutions are of the forms;

Case 1 If $x_{-3}=0, x_{-2}=m, x_{-1}=-1, x_0=-1$, then $(-1,-m-1,0,0,m,-1,-1,...)$,

Case 2 If $x_{-3}=-1, x_{-2}=m, x_{-1}=0, x_0=0$, then $(-m-1,-1,-1,-1,m,0,0,...)$,

Case 3 If $x_{-3}=-1, x_{-2}=-1, x_{-1}=-1, x_0=m$, then $(0,0,-m-1,-1,-1,-1,m,...)$,

Case 4 If $x_{-3}=m, x_{-2}=-1, x_{-1}=-1, x_0=-1$, then $(-m-1,0,0,m,-1,-1,-1,...)$,

Case 5 If $x_{-3}=-1, x_{-2}=-1, x_{-1}=m, x_0=0$, then $(0,-m-1,-1,-1,-1,m,0,...)$.

From now on, we will refer to any one of these seven periodic solution of Diff. Eq.(1) as

$$...,-1,-1,-1,m,0,0,-m-1,... \qquad (30)$$

where m is arbitrary.

Theorem 8 *There are eventually periodic solutions with minimal period-seven and they have two forms, respectively:*

Form 1: $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, -1, -1, -1, m, 0, 0, -m-1, ...)$

where, $N \geq -3$, $x_{N+1}x_N = 0, x_{N+2}x_{N+1} = 0, x_{N+3}x_{N+2} = 0$, *and, if* $N \neq -3$, $x_{n-2} = (x_{n+1}+1)/x_{n-3}$ *for* $0 \leq n \leq N$.

Form 2: $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, 0, 0, -m-1, -1, -1, -1, m, ...)$

where, $N \geq -3$, $x_{N+1}x_N = 1, x_{N+2}x_{N+1} = 1$, *and, if* $N \neq -3$, $x_{n-2} = (x_{n+1}+1)/x_{n-3}$ *for* $0 \leq n \leq N$.

Proof. Form 1: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Diff. Eq.(1) that is eventually periodic with prime period-seven. Then by Theorem 7, there is an $N \geq -3$ such that $x_{N+4} = -1$, $x_{N+5} = -1$ and $x_{N+6} = -1$. Then, $-1 = x_{N+4} = x_N x_{N+1} - 1$ and consequently $x_N x_{N+1} = 0$. Hence, $-1 = x_{N+5} = x_{N+2}x_{N+1} - 1$ and then $x_{N+2}x_{N+1} = 0$. Hence, $-1 = x_{N+6} = x_{N+3}x_{N+2} - 1$ and so $x_{N+3}x_{N+2} = 0$. Therefore,

$$\begin{aligned}
x_{N+7} &= x_{N+4}x_{N+3} - 1 = m \\
x_{N+8} &= x_{N+5}x_{N+4} - 1 = 0 \\
x_{N+9} &= x_{N+6}x_{N+5} - 1 = 0 \\
x_{N+10} &= x_{N+7}x_{N+6} - 1 = -m-1 \\
x_{N+11} &= x_{N+8}x_{N+7} - 1 = -1.
\end{aligned}$$

From Diff. Eq.(1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1}+1)/x_{n-3}$, for $0 \leq n \leq N$, as desired.

Form 2: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Diff. Eq.(1) that is eventually periodic with prime period-seven. Then by Theorem 7, there is an $N \geq -3$ such that $x_{N+4} = 0, x_{N+5} = 0$ and $x_{N+6} = -m-1$. Then, $0 = x_{N+4} = x_N x_{N+1} - 1$ and consequently $x_N x_{N+1} = 1$. Hence, $0 = x_{N+5} = x_{N+2}x_{N+1} - 1$ and then $x_{N+2}x_{N+1} = 1$. Hence, $-m-1 = x_{N+6} = x_{N+3}x_{N+2} - 1$ and so $x_{N+3}x_{N+2} = -m$. Therefore,

$$\begin{aligned}
x_{N+7} &= x_{N+4}x_{N+3} - 1 = -1 \\
x_{N+8} &= x_{N+5}x_{N+4} - 1 = -1 \\
x_{N+9} &= x_{N+6}x_{N+5} - 1 = -1 \\
x_{N+10} &= x_{N+7}x_{N+6} - 1 = m \\
x_{N+11} &= x_{N+8}x_{N+7} - 1 = 0.
\end{aligned}$$

From Diff. Eq.(1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1}+1)/x_{n-3}$, for $0 \leq n \leq N$, as desired. ■

Remark 9 *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of Diff. Eq.(1). If* $x_{-3}x_{-2} = 1$ *and* $x_{-1}x_0 = 1$, *then* x_n *converges to period-seven cycle as*

$$\cdots, 0, 0, 0, -1, -1, -1, -1, \cdots. \tag{31}$$

Proof. Let $x_{-3} = a, x_{-2} = 1/a, x_{-1} = b$ and $x_0 = 1/b$ for $a \neq 0$ and $b \neq 0$. From Eq.(1),

$$\begin{aligned} x_1 &= x_{-2}x_{-3} - 1 = 0 \\ x_2 &= x_{-1}x_{-2} - 1 = \frac{b}{a} - 1 \\ x_3 &= x_0 x_{-1} - 1 = 0 \\ x_4 &= x_1 x_0 - 1 = -1. \end{aligned}$$

Hence, by induction Diff. Eq.(1) converges to period-seven cycle as (31). The proof is completed. ■

3 Asymptotically Periodic Solution of Diff. Eq.(1)

In this section, we study the existence of asymptotic periodic solutions of Diff. Eq.(1).

Diff. Eq.(1) has the seven-periodic solutions as (30) for the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in (-1, 0)$. Focus on the asymptotically seven-periodic solutions, we get $u_k^{(0)} = x_{n+7k}$, $u_k^{(1)} = x_{n+7k-1}$, $u_k^{(2)} = x_{n+7k-2}$, $u_k^{(3)} = x_{n+7k-3}$, $u_k^{(4)} = x_{n+7k-4}$, $u_k^{(5)} = x_{n+7k-5}$ and $u_k^{(6)} = x_{n+7k-6}$. Now, we make the ansatz as in [11]:

$$u_k^{(0)} = \sum_{v=0}^{\infty} a_v p^v t^{vk}, \ a_0 = m; \tag{32}$$

$$u_k^{(1)} = \sum_{v=0}^{\infty} b_v p^v t^{vk}, \ b_0 = 0; \tag{33}$$

$$u_k^{(2)} = \sum_{v=0}^{\infty} c_v p^v t^{vk}, \ c_0 = 0; \tag{34}$$

$$u_k^{(3)} = \sum_{v=0}^{\infty} d_v p^v t^{vk}, \ d_0 = -m - 1; \tag{35}$$

$$u_k^{(4)} = \sum_{v=0}^{\infty} e_v p^v t^{vk}, \ e_0 = -1; \tag{36}$$

$$u_k^{(5)} = \sum_{v=0}^{\infty} f_v p^v t^{vk}, \ f_0 = -1; \tag{37}$$

$$u_k^{(6)} = \sum_{v=0}^{\infty} g_v p^v t^{vk}, \ g_0 = -1; \tag{38}$$

with arbitrary p and $m \in (-1, 0)$. We choose $p > 0$ and from Eq.(1), it follows that:

$$\begin{aligned}
u_k^{(0)} &= u_k^{(3)}u_k^{(4)} - 1 \\
u_{k+1}^{(6)} &= u_k^{(2)}u_k^{(3)} - 1 \\
u_{k+1}^{(5)} &= u_k^{(1)}u_k^{(2)} - 1 \\
u_{k+1}^{(4)} &= u_k^{(0)}u_k^{(1)} - 1 \\
u_{k+1}^{(3)} &= u_{k+1}^{(6)}u_k^{(0)} - 1 \\
u_{k+1}^{(2)} &= u_{k+1}^{(5)}u_{k+1}^{(6)} - 1 \\
u_{k+1}^{(1)} &= u_{k+1}^{(4)}u_{k+1}^{(5)} - 1.
\end{aligned}$$

Substitution of (32)-(38) into these equations. Hence, when we compare the coefficients, we obtain that

$$\begin{aligned}
a_1 &= b_1 = c_1 = d_1 = e_1 = f_1 = g_1 = 0 \\
a_2 &= b_2 = c_2 = d_2 = e_2 = f_2 = g_2 = 0
\end{aligned}$$

and by induction,

$$a_n = b_n = c_n = d_n = e_n = f_n = g_n = 0, \text{ for all } n > 0.$$

Therefore $x_{n+7k} = u_k^{(0)} = \sum_{v=0}^{\infty} a_v p^v t^{vk} = m + 0 + 0 + ...$, so x_{n+7k} converges to m. Similarly, x_{n+7k-1} converges to 0, x_{n+7k-2} converges to 0, x_{n+7k-3} converges to $-m-1$, x_{n+7k-4} converges to -1, x_{n+7k-5} converges to -1 and x_{n+7k-6} converges to -1. Hence, the proof is complete.

4 Stability of Diff. Eq.(1)

In this section, we examine the stability of the two equilibria of Diff. Eq.(1).

Theorem 10 *The positive equilibrium point of Diff. Eq.(1), $\bar{x}_2$, is unstable.*

Proof. The characteristic equation of equilibria of Diff. Eq.(1) is the following:

$$\lambda^4 - \bar{x}_2\lambda - \bar{x}_2 = 0$$

with eigenvalues

$$\begin{aligned}
\lambda_1 &\approx -0,7756, \\
\lambda_2 &\approx 1,4044, \\
\lambda_3, \lambda_4 &\approx -0,3142 \pm 1,1773i.
\end{aligned}$$

Therefore, $|\lambda_1| < 1$ and $|\lambda_2|, |\lambda_3|, |\lambda_4| > 1$. Herewith, $\bar{x}_2$ is unstable, which is a saddle point. ■

Theorem 11 *The negative equilibrium point of Diff. Eq.(1), $\bar{x}_1$, is unstable.*

Proof. The characteristic equation of equilibria of Eq.(1) is the following:

$$\lambda^4 - \bar{x}_1\lambda - \bar{x}_1 = 0$$

with eigenvalues

$$\begin{aligned}\lambda_1, \lambda_2 &\approx -0,6412 \pm 0,4125i \\ \lambda_3, \lambda_4 &\approx -0,6412 \pm 0,8075i.\end{aligned}$$

Therefore, $|\lambda_1|, |\lambda_2| < 1$ and $|\lambda_3|, |\lambda_4| > 1$. So, $\bar{x}_1$ is unstable and which is a saddle point. ■

5 Existence of Unbounded Solutions of Diff. Eq.(1)

Now, we work the existence of unbounded solutions of Diff. Eq.(1).

Theorem 12 *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Diff. Eq.(1). If $x_{-3}, x_{-2}, x_{-1}, x_0 > \bar{x}_2 = \frac{1+\sqrt{5}}{2}$, the following statements hold true:*

(i) $x_{-2} < x_1 < x_4 < \cdots$, $x_{-1} < x_2 < x_5 < \cdots$ and $x_0 < x_3 < x_6 < \cdots$;

(ii) the solutions tends to $+\infty$.

Proof. *(i)* Since $x_{-2} > \frac{1+\sqrt{5}}{2}$, we obtain $\frac{1}{x_{-2}} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_{-2}} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-3}.$$

Then, $x_{-3} > 1 + \frac{1}{x_{-2}}$. Hence, $x_{-3}x_{-2} > x_{-2} + 1$. Therefore, $x_{-3}x_{-2} - 1 > x_{-2}$. Thus, $x_{-2} < x_1$.

Since $x_{-1} > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_{-1}} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_{-1}} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-1}.$$

Then, $x_{-2} > 1 + \frac{1}{x_{-1}}$. Hence, $x_{-1}x_{-2} > x_{-1} + 1$. Therefore, $x_{-1}x_{-2} - 1 > x_{-1}$. Thus, $x_{-1} < x_2$.

Since $x_0 > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_0} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_0} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-1}.$$

Then, $x_{-1} > 1 + \frac{1}{x_0}$. Hence, $x_0x_{-1} > x_0 + 1$. Therefore, $x_0x_{-1} - 1 > x_0$. Thus, $x_0 < x_3$.

Since $x_1 > x_{-2} > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_1} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_1} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_0.$$

Then, $x_0 > 1 + \frac{1}{x_1}$. Hence, $x_1 x_0 > x_1 + 1$. Therefore, $x_1 x_0 - 1 > x_1$. Thus, $x_1 < x_4$.

Hence, by induction it easily follows that

$$x_{-2} < x_1 < x_4 < \cdots \quad (39)$$

$$x_{-1} < x_2 < x_5 < \cdots \quad (40)$$

$$x_0 < x_3 < x_6 < \cdots . \quad (41)$$

(ii) Suppose one of (39)-(41) subsequences given in (i) is bounded. Hence, from Diff. Eq.(1), we obtain

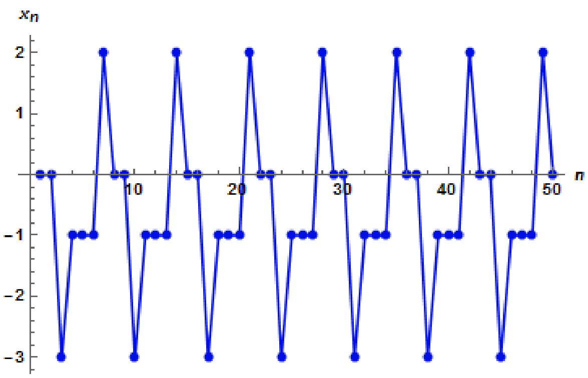
$$x_{n-3} = \frac{1 + x_{n+1}}{x_{n-2}},\ n \in \mathbb{N}_0.$$

Therefore, the subsequences $(x_{3n})_{n=0}^{\infty}$, $(x_{3n-1})_{n=0}^{\infty}$ and $(x_{3n-2})_{n=0}^{\infty}$ must be convergent. Thereby, there are two situations for whole solution of Diff. Eq.(1). Then, in the first case, all solution of Diff. Eq.(1) converges to a periodic solution with period three. But this is not possible. Because, there are not nontrivial period three solution of Diff. Eq.(1). In the other case, all solution of Diff. Eq.(1) converge to an equilibria. Unfortunately, this is impossible. Because the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are bigger then the largest equilibria. This is a contradiction, as desired. ■

6 Numerical Examples

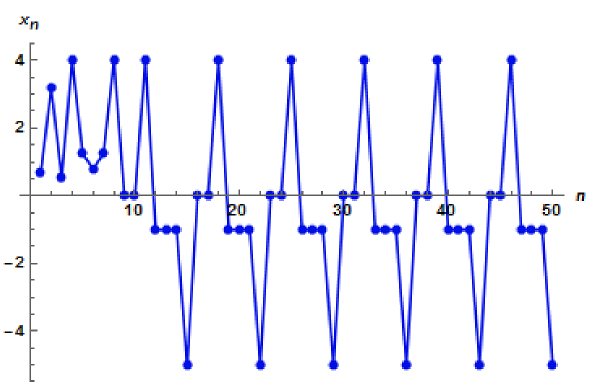
In this section, we present graphs of the some results.

Example 13 *If the initial conditions are $x_{-3} = -1, x_{-2} = -1, x_{-1} = -1$, $x_0 = m$ and $m = 2$, then Diff. Eq.(1) has periodic solutions with minimal* *prime period-seven as (30). The following graph shows this status.*



Graph 1: The initial conditions are $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = -1$, $x_0 = m$ *and* $m = 2$ *for Diff. Eq.(1).*

Example 14 *If the initial conditions are* $x_{-3} = \frac{122625}{1376256}$, $x_{-2} = \frac{21504}{1125}$, $x_{-1} = \frac{225}{1024}$ *and* $x_0 = \frac{64}{9}$, *then Diff. Eq.(1) has eventually seven-periodic solutions as Theorem 8. The next graph illustrates this condition.*



Graph 2: The initial conditions are $x_{-3} = \frac{122625}{1376256}$, $x_{-2} = \frac{21504}{1125}$, $x_{-1} = \frac{225}{1024}$ *and* $x_0 = \frac{64}{9}$ *for Diff. Eq.(1).*

Example 15 *If the initial conditions are* $x_{-3} = -\frac{2}{3}$, $x_{-2} = -\frac{3}{2}$, $x_{-1} = \frac{1}{5}$ *and* $x_0 = 5$, *then Diff. Eq.(1) converges to seven-periodic solutions as Remark 9.*

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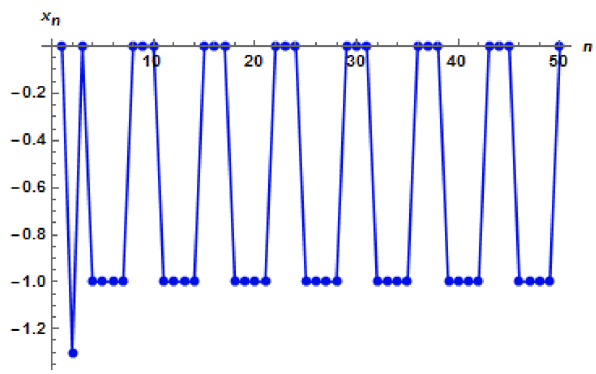
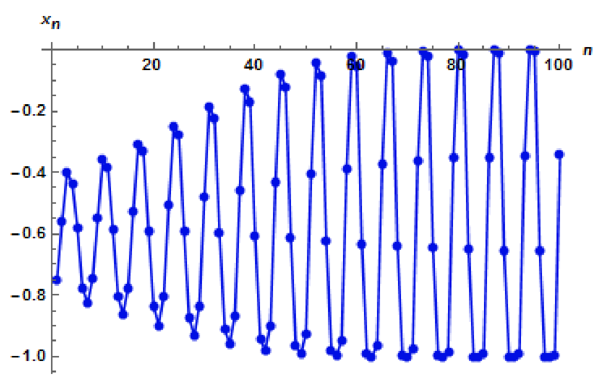
The following graph shows this situation.



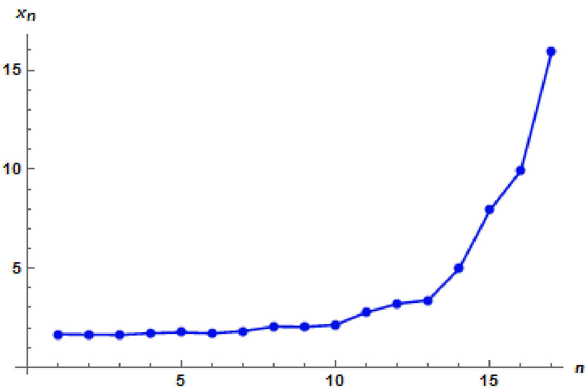
Graph 3: The initial conditions are $x_{-3} = -\frac{2}{3}$, $x_{-2} = -\frac{3}{2}$, $x_{-1} = \frac{1}{5}$ *and* $x_0 = 5$ *for Diff. Eq.(1).*

Example 16 *If the initial conditions are* $x_{-3} = -0.45$, $x_{-2} = -0.55$, $x_{-1} = -0.7$ *and* $x_0 = -0.75$, *then Diff. Eq.(1) has asymptotically seven-periodic solutions. The next graph illustrates this condition.*



Graph 4: The initial conditions are $x_{-3} = -0.45$, $x_{-2} = -0.55$, $x_{-1} = -0.7$ *and* $x_0 = -0.75$ *for Diff. Eq.(1).*

Example 17 *If the initial conditions are* $x_{-3} = 1.63$, $x_{-2} = 1.64$, $x_{-1} = 1.62$ *and* $x_0 = 1.63$, *then Diff. Eq.(1) has unbounded solutions as Theorem 12. The* *following graph illustrates this case.*



Graph 5: The initial conditions are $x_{-3} = 1.63$, $x_{-2} = 1.64$, $x_{-1} = 1.62$ *and* $x_0 = 1.63$ *for Diff. Eq.(1).*

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A new fixed point theorem in cones and applications to elastic beam equations

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Abstract

In this paper, we first establish a new fixed point theorem in cones of Banach spaces. Then, we apply the fixed point theorem to study the existence and uniqueness of monotone positive solutions for an elastic beam equation $u^{(4)}(t) = f(t, u(t), u'(t))$ with superlinear boundary conditions. An example is given to illustrate our main result. Compared with some earlier results (cf. [10]), the biggest differences are that we consider such equation with *superlinear* boundary conditions and remove some restrictive conditions.

Keywords: cone, fixed point theorem, monotone positive solutions, elastic beam equations.

1 Introduction and preliminaries

In this paper, we consider the existence and uniqueness of monotone positive solutions for the following fourth-order two-point boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0,\ u^{(3)}(1) = g(u(1)), \end{cases} \tag{1.1}$$

where $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ and $g : [0, +\infty) \to (-\infty, 0]$ are continuous (for full assumptions on f and g, see Section 2).

In fact, equation (1.1) models an elastic beam problem (for more details and backgrounds, we refer to reader to [1,3] and references therein. Recently, there has been of great interest for many authors to study fourth-order boundary value problems such as (1.1) and related problems (see, e.g., [1-5,9-14]). Especially, several authors utilize fixed point theorems on cones to investigate the existence and uniqueness of monotone positive solutions for equation (1.1). For example, Li and Zhang [9] utilized a fixed point theorem of generalized concave operators to study problem (1.1) and established the existence and uniqueness of monotone positive solutions. In [10], Li and Zhai obtain the existence and uniqueness of monotone positive solutions for a fourth-order boundary value problem via two fixed point theorems of mixed monotone operators with perturbation.

However, in most of works using fixed point theorems on cones to study equation (1.1), the following assumption on g is assumed:

$$\text{(H0)} \quad g(\lambda x) \leq \lambda g(x), \quad \lambda \in (0,1),\ x \geq 0.$$

In this paper, we aim to consider equation (1.1) without the assumption (H0). That is the main motivation of this work.

Next, Let us recall some basic notations about cone (for more details, we refer the reader to [6]).

Let E be a real Banach space, and θ be the zero element in E. A closed and convex set P in E is called a cone if the following two conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for every $\lambda \geq 0$;

(ii) if $x \in P$ and $-x \in P$, then $x = \theta$.

A cone P induces a partial ordering $\leq$ in E by

$$x \leq y \quad if\, and\, only\, if \quad y - x \in P.$$

If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$.

For any given $u, v \in P$ with $u \leq v$,

$$[u, v] := \{x \in X | u \leq x \leq v\}.$$

A cone P is called normal if there exists a constant $k > 0$ such that

$$\theta \leq x \leq y \quad implies\, that \quad ||x|| \leq k||y||.$$

We denote by P^o the interior of P. A cone P is called a solid cone if $P^o \neq \emptyset$.

An operator $T : P \to P$ is called increasing if $\theta \leq x \leq y$ *implies* $Tx \leq Ty$, and is called decreasing if $\theta \leq x \leq y$ *implies* $Tx \geq Ty$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$, we denote by

$$P_h = \{x \in E : x \sim h\}.$$

It is easy to see that $P_h \subset P$ is convex and $rP_h = P_h$ for all $r > 0$.

Definition 1.1. *(see [7,8]) An operator $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i,\ v_i (i = 1,\ 2) \in P,\ u_1 \leq u_2,\ v_1 \geq v_2$ implies $A(u_1, v_1) \leq A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x, x) = x$.*

Definition 1.2. *Let $n \geq 1$. An operator $D : P \to P$ is said to be n-superlinear if it satisfies*

$$D(tx) \geq t^n Dx, \quad t > 0,\, x \in P. \tag{1.2}$$

2 Main results

2.1 Cone and fixed point theorems

In order to study equation (1.1), we first consider the following operator equation on an ordered Banach space:

$$B(x, x) + Dx = x, \tag{2.1}$$

where B is a mixed monotone operator, D is an increasing and superlinear operator. If there is no special statements, we always assume that E is a real Banach space with a partial order introduced by a normal cone P of E, $h \in P$ is a nonzero element, and P_h is given as in the preliminaries.

Lemma 2.1. [13] *Let P be a normal cone in E. Assume that $T : P \times P \to P$ is a mixed monotone operator and satisfies:*

(A1) there exists $h \in P$ with $h \neq \theta$ such that $T(h, h) \in P_h$;

(A2) for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that $T(tu, t^{-1}v) \geq \varphi(t)T(u, v)$.

Then (1) $T : P_h \times P_h \to P_h$;

(2) there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 < v_0$, $u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0$;

(3) T has a unique fixed point x^* in P_h;

(4) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \qquad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

By using the above lemma, we establish a new fixed point theorem in the following :

Theorem 2.2. *Let $n \geq 1$, $B : P \times P \to P$ be a mixed monotone operator, and $D : P \to P$ be an increasing and n-superlinear operator. Assume that*

(D1) there exists $h_0 \in P_h$ such that $B(h_0, h_0) \in P_h$ and $Dh_0 \in P_h$;

(D2) there exists a constant $\delta_0 > 0$ such that $B(x, y) \geq \delta_0 Dx$ for all $x, y \in P$;

(D3) there exists a function $\phi : (0, 1) \to (0, +\infty)$ such that for all $x, y \in P$ and $t \in (0, 1)$,

$$B(tx, t^{-1}y) \geq \phi(t)B(x, y), \tag{2.2}$$

and

$$\phi(t) > t + \frac{1}{\delta_0}(t - t^n). \tag{2.3}$$

Then (1) $B : P_h \times P_h \to P_h$ and $D : P_h \to P_h$;

(2) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that

$$rv_0 \le u_0 < v_0, \quad u_0 \le B(u_0, v_0) + Du_0 \le B(v_0, u_0) + Dv_0 \le v_0;$$

(3) the operator equation $B(x,x) + Dx = x$ has a unique fixed point x^* in P_h;

(4) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = B(x_{n-1}, y_{n-1}) + Dx_{n-1}, \quad y_n = B(y_{n-1}, x_{n-1}) + Dy_{n-1}, \qquad n = 1, 2, \dots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Proof. It follows from (1.2) and (2.2) that for all $t \in (0,1)$ and $x, y \in P$,

$$B\left(\frac{1}{t}x, ty\right) \le \frac{1}{\phi(t)} B(x,y) \quad \textit{and} \quad D\left(\frac{1}{t}x\right) \le \frac{1}{t^n} Dx. \tag{2.4}$$

Since $h_0 \in P_h$ and $B(h_0, h_0) \in P_h$, there exist constants $\lambda, \alpha \in (0,1)$ such that

$$\lambda h \le h_0 \le \frac{1}{\lambda} h \quad \textit{and} \quad \alpha h \le B(h_0, h_0) \le \frac{1}{\alpha} h.$$

Since B is a mixed monotone operator, combing (2.2) and (2.4), we have

$$B(h,h) \le B\left(\frac{h_0}{\lambda}, \lambda h_0\right) \le \frac{1}{\phi(\lambda)} B(h_0, h_0) \le \frac{1}{\phi(\lambda)} \cdot \frac{1}{\alpha} h,$$

and

$$B(h,h) \ge B\left(\lambda h_0, \frac{h_0}{\lambda}\right) \ge \phi(\lambda) B(h_0, h_0) \ge \phi(\lambda) \cdot \alpha h.$$

Thus, $B(h,h) \in P_h$.

Taking $x, y \in P_h$, there exist $\gamma_1, \gamma_2 \in (0,1)$ such that

$$\gamma_1 h \le x \le \frac{1}{\gamma_1} h \quad \textit{and} \quad \gamma_2 h \le y \le \frac{1}{\gamma_2} h.$$

Let $\gamma = \min\{\gamma_1, \gamma_2\}$. Then $\gamma \in (0,1)$. It follows from (2.2) and (2.4) that

$$B(x,y) \le B\left(\frac{1}{\gamma_1} h, \gamma_2 h\right) \le B\left(\frac{1}{\gamma} h, \gamma h\right) \le \frac{1}{\phi(\gamma)} B(h,h),$$

and

$$B(x,y) \ge B\left(\gamma_1 h, \frac{1}{\gamma_2} h\right) \ge B\left(\gamma h, \frac{1}{\gamma} h\right) \ge \phi(\gamma) B(h,h).$$

Then, we have $B(x,y) \in P_h$ since $B(h,h) \in P_h$. This completes the proof of $B : P_h \times P_h \to P_h$.

Since $Dh_0 \in P_h$, there exists $\beta \in (0,1)$ such that

$$\beta h \leq Dh_0 \leq \frac{1}{\beta} h.$$

Next we show $D : P_h \to P_h$. For any $x' \in P_h$, we can choose a sufficiently small number $\gamma' \in (0,1)$ such that

$$\gamma' h \leq x' \leq \frac{1}{\gamma'} h.$$

Since D is increasing, by using (1.2) and (2.4), we have

$$Dx' \leq D\left(\frac{1}{\gamma'} h\right) \leq \frac{1}{(\gamma')^n} Dh \leq \frac{1}{(\gamma')^n} D\left(\frac{h_0}{\lambda}\right) \leq \frac{1}{(\gamma')^n \lambda^n} Dh_0 \leq \frac{1}{(\gamma')^n \lambda^n} \cdot \frac{1}{\beta} h,$$

and

$$Dx' \geq D(\gamma' h) \geq (\gamma')^n Dh \geq (\gamma')^n D(\lambda h_0) \geq (\gamma')^n \lambda^n Dh_0 \geq (\gamma')^n \lambda^n \cdot \beta h.$$

which means that $Dx' \in P_h$, and thus $D : P_h \to P_h$. So the conclusion (1) holds.

Now, we define an operator T by

$$T(x,y) = B(x,y) + Dx, \quad x \in P.$$

Then, $T : P \times P \to P$ is a mixed monotone operator and $T(h,h) \in P_h$. Moreover, By using (D2) and (D3), for all $t \in (0,1)$ and $x, y \in P$,

$$\begin{aligned}
T(tx, t^{-1}y) &= B(tx, t^{-1}y) + D(tx) \\
&\geq \phi(t) B(x,y) + t^n Dx \\
&= tT(x,y) + [\phi(t) - t] B(x,y) + (t^n - t) Dx \\
&\geq tT(x,y) + [\phi(t) - t] B(x,y) + \frac{1}{\delta_0}(t^n - t) B(x,y) \\
&= tT(x,y) + \left[\phi(t) - t - \frac{1}{\delta_0}(t - t^n)\right] B(x,y) \\
&\geq tT(x,y) + \frac{\delta_0}{1+\delta_0}\left[\phi(t) - t - \frac{1}{\delta_0}(t - t^n)\right] T(x,y) \\
&= \varphi(t) T(x,y),
\end{aligned}$$

where φ is defined by

$$\varphi(t) = t + \frac{\delta_0}{1+\delta_0}\left[\phi(t) - t - \frac{1}{\delta_0}(t - t^n)\right], \quad t \in (0,1).$$

By (2.3), we have $\varphi(t) > t$ for all $t \in (0,1)$. In addition,

$$T(h,h) \geq T(th, t^{-1}h) \geq \varphi(t) T(h,h), \quad t \in (0,1)$$

yields that $\varphi(t) \leq 1$ for all $t \in (0,1)$. Hence the conclusion (A2) in Lemma 2.1 is satisfied. Then, the conclusions (2)-(4) follows from Lemma 2.1. $\square$

In the proof of our existence result, we will use the following corollary of Theorem 2.2:

Corollary 2.3. *Let $n > 1$, $B : P \to P$ be an increasing operator, and $D : P \to P$ be an increasing and n-superlinear operator. Assume that the following conditions hold:*

(B1) there is $h_0 \in P_h$ such that $Bh_0 \in P_h$ and $Dh_0 \in P_h$;

(B2) there exists a constant $\delta_0 > 0$ such that $Bx \geq \delta_0 Dx$ for all $x \in P$;

(B3) there exists a function $\varphi : (0,1) \to (0,+\infty)$ such that for all $x \in P$ and $\lambda \in (0,1)$,

$$B(\lambda x) \geq \varphi(\lambda) Bx, \tag{2.5}$$

and

$$\varphi(\lambda) > \lambda + \frac{1}{\delta_0}(\lambda - \lambda^n). \tag{2.6}$$

Then (1) $B : P_h \to P_h$ and $D : P_h \to P_h$;

(2) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that

$$rv_0 \leq u_0 < v_0, \quad u_0 \leq Bu_0 + Du_0 \leq Bv_0 + Dv_0 \leq v_0;$$

(3) the operator equation $Bx + Dx = x$ has a unique fixed point x^* in P_h;

(4) for any initial value $x_0 \in P_h$, constructing successively the sequence

$$x_n = Bx_{n-1} + Dx_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ as $n \to \infty$.

2.2 Existence and uniqueness

Firstly, In order to use Corollary 2.3 to study problem (1.1), we need to clarify some symbols. In this section, we denote the Banach space $E = C^1[0,1]$ equipped with the norm

$$||u|| = \max\{\max_{0\leq t\leq 1} |u(t)|,\ \max_{0\leq t\leq 1} |u'(t)|\}.$$

Let

$$P = \{u \in E : u(t) \geq 0,\ u'(t) \geq 0,\ \forall\, t \in [0,1]\}.$$

It is not difficult to verify that P is a normal cone in E. Also, P induces an order relation $\dot{\leq}$ in E by defining $u \dot{\leq} v$ if and only if $v - u \in P$.

Let $G(t,s)$ be the Green function of the linear problem $u^{(4)}(t) = 0$ with the boundary conditions in problem (1.1). It follows from [3] that

$$G(t,s) = \begin{cases} \frac{s^2(3t-s)}{6}, & 0 \leq s \leq t \leq 1, \\ \frac{t^2(3s-t)}{6}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.7}$$

Thus, equation (1.1) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds - g\left(u(1)\right) \phi(t), \quad t \in [0,1],$$

where $\phi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3$ for all $t \in [0,1]$.

The following properties of the Green function $G(t,s)$ and $\phi(t)$ will be used in our proof.

Lemma 2.4. [9, 10] *For all $t, s \in [0,1]$, we have*

$$\frac{1}{3}s^2t^2 \leq G(t,s) \leq \frac{1}{2}st^2, \qquad \frac{1}{3}t^2 \leq \phi(t) \leq \frac{1}{2}t^2,$$

and

$$\frac{1}{2}s^2 t \leq \frac{\partial G(t,s)}{\partial t} \leq st, \qquad \frac{1}{2}t \leq \phi'(t) \leq 2t.$$

Now, we are ready to present our existence and uniqueness theorem.

Theorem 2.5. *Let $n \geq 1$. Assume that*

(H1) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous, $g : [0,+\infty) \to (-\infty, 0]$ is continuous, and

$$\inf_{t\in[0,1],x,y\geq 0} f(t,x,y) > 0, \quad \inf_{x\geq 0} g(x) > -\infty;$$

(H2) g is decreasing on $[0,+\infty)$, for every $t \in [0,1]$ and $x \geq 0$, $f(t,x,\cdot)$ is increasing on $[0,+\infty)$, and for every $t \in [0,1]$ and $y \geq 0$ $f(t,\cdot,y)$ is increasing on $[0,+\infty)$;

(H3) $g(\lambda x) \leq \lambda^n g(x)$ for all $\lambda \in (0,1)$ and $x \in [0,+\infty)$; moreover, there exists a function $\varphi : (0,1) \to (0,+\infty)$ such that

$$f(t,\lambda x, \lambda y) \geq \varphi(\lambda) f(t,x,y), \quad t \in [0,1],\ \lambda \in (0,1),\ x, y \in [0,+\infty),$$

and

$$\varphi(\lambda) > \lambda + \frac{\sup\limits_{x\geq 0} -g(x)}{\inf\limits_{t\in[0,1],x,y\geq 0} f(t,x,y)} \cdot 3(\lambda - \lambda^n), \quad \lambda \in (0,1). \tag{2.8}$$

Then (1) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \dot{\leq} u_0 \dot{<} v_0$ and

$$u_0(t) \leq \int_0^1 G(t,s) f\left(s, u_0(s), u_0'(s)\right) ds - g\left(u_0(1)\right) \phi(t), \quad t \in [0,1],$$

$$u_0'(t) \leq \int_0^1 G_t(t,s) f\left(s, u_0(s), u_0'(s)\right) ds - g\left(u_0(1)\right) \phi'(t), \quad t \in [0,1],$$

$$v_0(t) \geq \int_0^1 G(t,s) f\left(s, v_0(s), v_0'(s)\right) ds - g\left(v_0(1)\right) \phi(t), \quad t \in [0,1],$$

$$v_0'(t) \geq \int_0^1 G_t(t,s) f\left(s, v_0(s), v_0'(s)\right) ds - g\left(v_0(1)\right) \phi'(t), \quad t \in [0,1],$$

where $h(t) = t^2$ for all $t \in [0,1]$ and $G(t,s)$ is given as in (2.7);

(2) equation (1.1) has a unique monotone positive solution u^* in P_h;

(3) for every $x_0 \in P_h$, constructing successively the sequence

$$x_n(t) = \int_0^1 G(t,s) f\left(s, x_{n-1}(s), x_{n-1}'(s)\right) ds - g\left(x_{n-1}(1)\right) \phi(t), \quad n = 1, 2, \ldots,$$

we have $||x_n - u^*|| \to 0$ as $n \to \infty$.

Proof. Recall that equation (1.1) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds - g\left(u(1)\right) \phi(t), \quad t \in [0,1],$$

where $\phi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3$ for all $t \in [0,1]$. So, we define two operators B, D on P by

$$Bu(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds, \quad Du(t) = -g\left(u(1)\right) \phi(t), \quad u \in P,\ t \in [0,1].$$

Then, equation (1.1) is transformed into the operator equation $u = Bu + Du$.

Next, we will verify all the assumptions of Corollary 2.3. We divide the remaining proof by four steps.

Step 1. B is an increasing operator from P to P, and D is an increasing and n-superlinear operator from P to P.

It is easy to see that

$$(Bu)'(t) = \int_0^1 G_t(t,s) f\left(s, u(s), u'(s)\right) ds, \quad (Du)'(t) = -g\left(u(1)\right) \phi'(t), \quad u \in P,\ t \in [0,1].$$

For every $u \in P$, since $u(t) \geq 0$ and $u'(t) \geq 0$ for all $t \in [0,1]$, by (H1) and Lemma 2.4, we have

$$Bu(t) \geq 0,\ Du(t) \geq 0,\ (Bu)'(t) \geq 0,\ (Du)'(t) \geq 0, \quad t \in [0,1].$$

Therefore, $Bu \in P$ and $Du \in P$, i.e., $B : P \to P$ and $D : P \to P$. Moreover, for every $\lambda \in (0,1)$ and $u \in P$, by (H3) we have

$$D(\lambda u)(t) = -g\left(\lambda u(1)\right) \phi(t) \geq -\lambda^n g\left(u(1)\right) \phi(t) = \lambda^n Du(t), \quad t \in [0,1],$$

and

$$\left(D(\lambda u)\right)'(t) = -g\left(\lambda u(1)\right) \phi'(t) \geq -\lambda^n g\left(u(1)\right) \phi'(t) = \lambda^n (Du)'(t), \quad t \in [0,1].$$

which means that $D(\lambda u) \dot{\geq} \lambda^n Du$.

It remains prove that B, D are two increasing operators. Taking any $u, v \in P$ with $u \dot{\leq} v$, we know that

$$u(t) \leq v(t),\ u'(t) \leq v'(t), \quad t \in [0,1].$$

Combining this with (H1) and (H2), we have

$$\begin{aligned} Bu(t) &= \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds \\ &\leq \int_0^1 G(t,s) f\left(s, v(s), v'(s)\right) ds \\ &= Bv(t), \qquad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} (Bu)'(t) &= \int_0^1 G_t(t,s) f\left(s, u(s), u'(s)\right) ds \\ &\leq \int_0^1 G_t(t,s) f\left(s, v(s), v'(s)\right) ds \\ &= (Bv)'(t), \qquad t \in [0,1]. \end{aligned}$$

Thus, $Bu \dot{\leq} Bv$. Moreover, we have

$$\begin{aligned} Du(t) &= -g\left(u(1)\right) \phi(t) \\ &\leq -g\left(v(1)\right) \phi(t) \\ &= Dv(t), \qquad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} (Du)'(t) &= -g\left(u(1)\right) \phi'(t) \\ &\leq -g\left(v(1)\right) \phi'(t) \\ &= (Dv)'(t), \qquad t \in [0,1]. \end{aligned}$$

That is, $Du \dot{\leq} Dv$.

Step 2. The assumption (B1) of Corollary 2.3 holds.

It suffices to show that $Bh \in P_h$ and $Dh \in P_h$. Combining (H1), (H2) and Lemma 2.4, for all $t \in [0,1]$, we have

$$\begin{aligned} Bh(t) &= \int_0^1 G(t,s) f\left(s, h(s), h'(s)\right) ds \\ &= \int_0^1 G(t,s) f\left(s, s^2, 2s\right) ds \\ &\leq \int_0^1 \frac{1}{2} s t^2 f\left(s, s^2, 2s\right) ds \end{aligned}$$

$$\leq \frac{1}{2}\int_0^1 sf(s,1,2)\,ds\cdot h(t),$$

and

$$\begin{aligned}
Bh(t) &= \int_0^1 G(t,s)f\left(s,h(s),h'(s)\right)ds\\
&= \int_0^1 G(t,s)f\left(s,s^2,2s\right)ds\\
&\geq \int_0^1 \frac{1}{3}s^2t^2f\left(s,s^2,2s\right)ds\\
&\geq \frac{1}{3}\int_0^1 s^2f(s,0,0)\,ds\cdot h(t).
\end{aligned}$$

In addition, also from (H1), (H2) and Lemma 2.4, for all $t\in[0,1]$, we have

$$\begin{aligned}
(Bh)'(t) &= \int_0^1 G_t(t,s)f\left(s,h(s),h'(s)\right)ds\\
&= \int_0^1 G_t(t,s)f\left(s,s^2,2s\right)ds\\
&\leq \int_0^1 stf\left(s,s^2,2s\right)ds\\
&\leq \frac{1}{2}\int_0^1 sf(s,1,2)\,ds\cdot h'(t),
\end{aligned}$$

and

$$\begin{aligned}
(Bh)'(t) &= \int_0^1 G_t(t,s)f\left(s,h(s),h'(s)\right)ds\\
&= \int_0^1 G_t(t,s)f\left(s,s^2,2s\right)ds\\
&\geq \int_0^1 \frac{1}{2}s^2tf\left(s,s^2,2s\right)ds\\
&\geq \frac{1}{4}\int_0^1 s^2f(s,0,0)\,ds\cdot h'(t).
\end{aligned}$$

Let

$$c_1=\frac{1}{4}\int_0^1 s^2f(s,0,0)\,ds,\quad c_2=\frac{1}{2}\int_0^1 sf(s,1,2)\,ds.$$

By (H1) and (H2), we have

$$c_2\geq c_1\geq \frac{\inf_{t\in[0,1],x,y\geq 0}f(t,x,y)}{12}>0.$$

Noting that

$$c_1 h(t) \leq Bh(t) \leq c_2 h(t), \quad t \in [0,1],$$

and

$$(c_1 h)'(t) = c_1 h'(t) \leq (Bh)'(t) \leq c_2 h'(t) = (c_2 h)'(t), \quad t \in [0,1],$$

we conclude $c_1 h \dot{\leq} Bh \dot{\leq} c_2 h$. Thus, $Bh \in P_h$.

Similarly, it follows from (H1), (H2) and Lemma 2.4 that for all $t \in [0,1]$, there hold

$$Dh(t) = -g\left(h(1)\right)\phi(t) \leq -g(1) \cdot \frac{1}{2}t^2 = -\frac{1}{2}g(1) \cdot h(t),$$

$$Dh(t) = -g\left(h(1)\right)\phi(t) \geq -g(1) \cdot \frac{1}{3}t^2 = -\frac{1}{3}g(1) \cdot h(t),$$

$$(Dh)'(t) = -g\left(h(1)\right)\phi'(t) \leq -g(1) \cdot 2t = -g(1) \cdot h'(t),$$

and

$$(Dh)'(t) = -g\left(h(1)\right)\phi'(t) \geq -g(1) \cdot \frac{1}{2}t = -\frac{1}{4}g(1) \cdot h'(t).$$

Combing the above four inequalities, we can obtain $Dh \in P_h$.

Step 3. The assumption (B2) of Corollary 2.3 holds.

For every $u \in P$ and $t \in [0,1]$, we have

$$\begin{aligned} Bu(t) &= \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds \\ &\geq \int_0^1 G(t,s) ds \cdot \inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \\ &\geq \frac{\phi(t)}{3} \cdot \inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \\ &\geq \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)} \cdot \phi(t) \sup_{x\geq 0} -g(x) \\ &\geq \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)} \cdot -g[u(1)]\phi(t) \\ &= \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)} \cdot Du(t), \end{aligned}$$

where

$$\begin{aligned} \int_0^1 G(t,s) ds &= \int_0^t G(t,s) ds + \int_t^1 G(t,s) ds \\ &= \int_0^t \frac{s^2(3t-s)}{6} ds + \int_t^1 \frac{t^2(3s-t)}{6} ds \\ &= \frac{1}{4}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 \end{aligned}$$

$$\geq \frac{\frac{1}{2}t^2 - \frac{1}{6}t^3}{3} = \frac{\phi(t)}{3}, \quad t \in [0,1].$$

In addition, we have

$$\begin{aligned}
(Bu)'(t) &= \int_0^1 G_t(t,s) f\left(s, u(s), u'(s)\right) ds \\
&\geq \int_0^1 G_t(t,s) ds \cdot \inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \\
&\geq \frac{\phi'(t)}{3} \cdot \inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \\
&\geq \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)} \cdot -g[u(1)]\phi'(t) \\
&= \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)} \cdot (Du)'(t),
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 G_t(t,s) ds &= \int_0^t \frac{s^2}{2} ds + \int_t^1 (st - \frac{t^2}{2}) ds \\
&= \frac{1}{2}t - \frac{1}{2}t^2 + \frac{1}{6}t^3 \\
&\geq \frac{t - \frac{1}{2}t^2}{3} = \frac{\phi'(t)}{3}, \quad t \in [0,1].
\end{aligned}$$

Let

$$\delta_0 = \frac{\inf_{t\in[0,1],x,y\geq 0} f(t,x,y)}{3\sup_{x\geq 0} -g(x)}.$$

Then

$$Bu(t) \geq \delta_0 Du(t),\ (Bu)'(t) \geq \delta_0 (Du)'(t), \quad t \in [0,1],\ u \in P,$$

i.e., $Bu \dot{\geq} \delta_0 Du$ for all $u \in P$.

Step 4. The assumption (B3) of Corollary 2.3 holds.

For every $\lambda \in (0,1)$, $t \in [0,1]$ and $u \in P$, by (H3), we have

$$\begin{aligned}
B(\lambda u)(t) &= \int_0^1 G(t,s) f\left(s, \lambda u(s), \lambda u'(s)\right) ds \\
&\geq \int_0^1 G(t,s)\varphi(\lambda) f\left(s, u(s), u'(s)\right) ds \\
&= \varphi(\lambda) Bu(t),
\end{aligned}$$

and

$$(B(\lambda u))'(t) = \int_0^1 G_t(t,s) f\left(s, \lambda u(s), \lambda u'(s)\right) ds$$

$$\begin{aligned} &\geq \int_0^1 G_t(t,s)\varphi(\lambda)f\left(s,u(s),u'(s)\right)ds \\ &= \varphi(\lambda)(Bu)'(t). \end{aligned}$$

Thus, $B(\lambda u)\dot{\geq}\varphi(\lambda)Bu$ for all $\lambda \in (0,1)$ and $u \in P$. Moreover, it follows from (2.8) that

$$\varphi(\lambda) > \lambda + \frac{1}{\delta_0}(\lambda - \lambda^n), \quad \lambda \in (0,1).$$

Now, we have verified all the assumptions of Corollary 2.3. Then, the conclusions (1)-(3) follows from Corollary 2.3. This completes the proof. □

Remark 2.6. Compared with some earlier results (see, e.g., [10]), the biggest difference are that we consider equation $u^{(4)}(t) = f(t,u(t),u'(t))$ with *superlinear* boundary conditions, and remove some restrictive conditions, for example, we do not assume that

$$\inf_{t\in[0,1],x,y\geq 0} f(t,x,y) \geq \sup_{x\geq 0} -g(x).$$

Moreover, in Theorem 2.5, for convenience, we only consider the case of $f(t,x,y)$ being increasing about the second and the third argument. In fact, by a similar proof to that of Theorem 2.5, one can also consider the case of $f(t,x,y)$ being increasing about the second argument and decreasing about the third argument. In addition, Theorem 2.2 can also be applied to other problems (see, e.g., [15]).

2.3 Example

In this section, we give an example to illustrate how Theorem 2.5 can be used.

Example 2.7. Let

$$n = \frac{33}{32}, \quad f(t,x,y) = \frac{\sqrt{x}}{1+\sqrt{x}} + \frac{\sqrt{y}}{1+\sqrt{y}} + 1, \quad g(x) = -\frac{2x^{\frac{33}{32}}}{1+x^{\frac{33}{32}}} - \varepsilon,$$

where

$$\varepsilon = \frac{\frac{1}{2}}{10^{32}-1}. \tag{2.9}$$

It is easy to verify that (H1) and (H2) hold. Moreover,

$$\inf_{t\in[0,1],x,y\geq 0} f(t,x,y) = 1, \quad \sup_{x\geq 0} -g(x) = 2+\varepsilon.$$

It remains to verify the assumption (H3).

For every $\lambda \in (0,1)$, $t \in [0,1]$, and $x, y \in [0,+\infty)$, we have

$$g(\lambda x) = -\frac{2(\lambda x)^{\frac{33}{32}}}{1+(\lambda x)^{\frac{33}{32}}} - \varepsilon$$

$$
\begin{aligned}
&\leq -\frac{2\lambda^{\frac{33}{32}}x^{\frac{33}{32}}}{1+x^{\frac{33}{32}}}-\varepsilon \\
&\leq -\frac{2\lambda^{\frac{33}{32}}x^{\frac{33}{32}}}{1+x^{\frac{33}{32}}}-\lambda^{\frac{33}{32}}\varepsilon=\lambda^{\frac{33}{32}}g(x),
\end{aligned}
$$

and

$$f(t,\lambda x,\lambda y)=\frac{\sqrt{\lambda x}}{1+\sqrt{\lambda x}}+\frac{\sqrt{\lambda y}}{1+\sqrt{\lambda y}}+1\geq\sqrt{\lambda}f(t,x,y)=\varphi(\lambda)f(t,x,y),$$

where $\varphi(\lambda):=\sqrt{\lambda}$ for $\lambda\in(0,1)$. We claim that

$$\varphi(\lambda)=\sqrt{\lambda}>\lambda+16(\lambda-\lambda^{\frac{33}{32}}),\quad \lambda\in(0,1).$$

In fact, for every $\lambda\in(0,1)$, we have

$$
\begin{aligned}
\frac{\lambda^{\frac{1}{2}}-\lambda}{\lambda-\lambda^{\frac{33}{32}}} &= \frac{\lambda^{\frac{1}{2}}\left(1-\lambda^{\frac{1}{2}}\right)}{\lambda\left(1-\lambda^{\frac{1}{32}}\right)} \\
&= \frac{1}{\lambda^{\frac{1}{2}}}\cdot\frac{\left[1-\left(\lambda^{\frac{1}{32}}\right)^{16}\right]}{1-\lambda^{\frac{1}{32}}} \\
&= \frac{1}{\lambda^{\frac{1}{2}}}\left[1+\lambda^{\frac{1}{32}}+\left(\lambda^{\frac{1}{32}}\right)^{2}+\cdots+\left(\lambda^{\frac{1}{32}}\right)^{15}\right] \\
&= \frac{1}{\lambda^{\frac{1}{2}}}+\frac{1}{\lambda^{\frac{15}{32}}}+\frac{1}{\lambda^{\frac{14}{32}}}+\cdots+\frac{1}{\lambda^{\frac{1}{32}}} \\
&> 16.
\end{aligned}
$$

Combining this with

$$\frac{3\sup_{x\geq 0}-g(x)}{\inf_{t\in[0,1],x,y\geq 0}f(t,x,y)}=3(2+\varepsilon)<16,$$

we know that (2.8) holds. This shows that (H3) holds.

Then, by applying Theorem 2.5, the following fourth-order boundary value problem:

$$
\begin{cases}
u^{(4)}(t)=\frac{\sqrt{u(t)}}{1+\sqrt{u(t)}}+\frac{\sqrt{u'(t)}}{1+\sqrt{u'(t)}}+1, \quad 0<t<1,\\
u(0)=u'(0)=0,\\
u''(1)=0,\ u^{(3)}(1)=-\frac{2[u(1)]^{\frac{33}{32}}}{1+[u(1)]^{\frac{33}{32}}}-\varepsilon,
\end{cases}
\tag{2.10}
$$

admits a monotone positive solution.

Remark 2.8. In Example 2.7, the function g does not satisfy the (H0) condition:

$$g(\lambda x)\leq\lambda g(x),\quad \lambda\in(0,1),\ x\geq 0.$$

In fact, letting $\lambda_0 = \left(\frac{1}{10}\right)^{32}$ and $x_0 = 1$, we have

$$g(\lambda_0 x_0) = -\frac{2}{10^{33}+1} - \varepsilon,$$

and

$$\lambda_0 g(x_0) = -\frac{1+\varepsilon}{10^{32}}.$$

Then, by a direct calculation, we can obtain

$$g(\lambda_0 x_0) > \lambda_0 g(x_0).$$

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A SEPTENDECIC FUNCTIONAL EQUATION IN MATRIX NORMED SPACES

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Abstract. In this paper, we study the septendecic functional equation and prove the Hyers-Ulam stability for the septendecic functional equation in matrix normed spaces by using the fixed point technique.

1. Introduction and preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [28] implies that quotients, mapping spaces and various tensor products of operator spaces may be treated as operator spaces. Owing this result, the theory of operator spaces is having a increasingly significant effect on operator algebra theory (see [9]).

The proof given in [28] appealed to the theory of ordered operator spaces [6]. Effros and Ruan [10] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [22] and Haagerup [12] (as modified in [8]).

We will use the following notations:
$e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$;
E_{ij} is that (i, j)-component is 1 and the other components are zero;
$E_{ij} \otimes x$ is that (i, j)-component is x and the other components are zero;
For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}$, $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

In 1940, an interesting topic was presented by S. M. Ulam [30] triggered the study of stability problems for various functional equations. He addressed a question concerning the stability of homomorphism. In the following year, 1941, D. H. Hyers [13] was able to give a partial solution to Ulam's question. The result of Hyers was then generalized by Aoki [1] for additive mappings.

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In 1978, Th. M. Rassias [25] succeeded in extending the result of Hyers theorem by weakening the condition for the Cauchy difference.

The stability phenomenon that was presented by Th. M. Rassias is called the Hyers-Ulam stability. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [11] by replacing the unbounded Cauchy difference by a general control function.

The result of Rassias has furnished a lot of influence during the past thirty eight years in the development of the Hyers-Ulam cocepts. Further, the generalized Hyers-Ulam stability of functional equations and inequalities in matrix normed spaces has been studied by number of authors [15, 16, 17, 18, 21, 31].

Now, we introduce the following new functional equation

$$\begin{aligned}
&f(x+9y)-17f(x+8y)+136f(x+7y)-680f(x+6y)+2380f(x+5y)-6188f(x+4y)\\
&\quad+12376f(x+3y)-19448f(x+2y)+24310f(x+y)-24310f(x)\\
&\quad+19448f(x-y)-12376f(x-2y)+6188f(x-3y)-2380f(x-4y)\\
&\quad+680f(x-5y)-136f(x-6y)+17f(x-7y)-f(x-8y)=17!f(y), \qquad (1.1)
\end{aligned}$$

where $17! = 355687428100000$ in matrix normed spaces. The above functional equation is said to be septendecic functional equation since the function $f(x) = cx^{17}$ is its solution.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1. [3, 7] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y^*) \le \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [2, 4, 5, 20, 23, 24, 26, 27, 29, 32]).

In Section 2, we study the septendecic functional equation (1.1).

In Section 3, using the fixed point technique, we prove the Hyers-Ulam stability of the functional equation (1.1) in matrix normed spaces.

2. Septendecic functional equation (1.1)

In this section, we study the septendecic functional equation (1.1). For this, let us consider $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces.

Theorem 2. *If a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies the functional equation (1.1) for all $x, y \in \mathcal{A}$, then $f(2x) = 2^{17} f(x)$ for all $x \in \mathcal{A}$.*

Proof. Letting $(x, y) = (0, 0)$ in (1.1), we get $f(0) = 0$.

Replacing (x, y) by $(0, x)$ in (1.1) and using $f(0) = 0$, we get

$$f(9x)-17f(8x)+136f(7x)-680f(6x)+2380f(5x)-6188f(4x)$$

$$
\begin{aligned}
&+ 12376f(3x) - 19448f(2x) + 24310f(x) - 24310f(0)\\
&+ 19448f(-x) - 12376f(-2x) + 6188f(-3x) - 2380f(-4x)\\
&+ 680f(-5x) - 136f(-6x) + 17f(-7x) - f(-8x) = 17!f(x) \qquad (2.1)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(-x, x)$ in (1.1) and using $f(0) = 0$, we get

$$
\begin{aligned}
&f(-8x) - 17f(-7x) + 136f(-6x) - 680f(-5x) + 2380f(-4x)\\
&\quad - 6188f(-3x) + 12376f(-2x) - 19448f(-x) + 24310f(0)\\
&\quad - 24310f(x) + 19448f(2x) - 12376f(3x) + 6188f(4x) - 2380f(5x)\\
&\quad + 680f(6x) - 136f(7x) + 17f(8x) - f(9x) = 17!f(-x) \qquad (2.2)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

By (2.1) and (2.2), we get

$$f(-x) = -f(x)$$

for all $x \in \mathcal{A}$. So f is an odd mapping.

Replacing (x, y) by $(0, 2x)$ in (1.1), we get

$$
\begin{aligned}
&f(18x) - 16f(16x) + 119f(14x) - 544f(12x) + 1700f(10x)\\
&\quad - 3808f(8x) + 6188f(6x) - 7072f(4x) + (4862 - 17!)f(2x) = 0 \qquad (2.3)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(9x, x)$ in (1.1), we obtain

$$
\begin{aligned}
&f(18x) - 17f(17x) + 136f(16x) - 680f(15x) + 2380f(14x)\\
&\quad - 6188f(13x) + 12376f(12x) - 19448f(11x) + 24310f(10x)\\
&\quad - 24310f(9x) + 19448f(8x) - 12376f(7x) + 6188f(6x) - 2380f(5x)\\
&\quad + 680f(4x) - 136f(3x) + 17f(2x) - (1 + 17!)f(x) = 0 \qquad (2.4)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Subtracting from (2.3) to (2.4), we obtain

$$
\begin{aligned}
&17f(17x) - 152f(16x) + 680f(15x) - 2261f(14x)\\
&\quad + 6188f(13x) - 12920f(12x) + 19448f(11x) - 22610f(10x)\\
&\quad + 24310f(9x) - 23256f(8x) + 12376f(7x) + 2380f(5x)\\
&\quad -7752f(4x) + 136f(3x) + (4845 - 17!)f(2x) + 17!f(x) = 0 \qquad (2.5)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(8x, x)$ in (1.1), we obtain

$$
\begin{aligned}
&f(17x) - 17f(16x) + 136f(15x) - 680f(14x) + 2380f(13x)\\
&\quad - 6188f(12x) + 12376f(11x) - 19448f(10x) + 24310f(9x)\\
&\quad - 24310f(8x) + 19448f(7x) - 12376f(6x) + 6188f(5x) - 2380f(4x)\\
&\quad + 680f(3x) - 136f(2x) + (17 - 17!)f(x) = 0 \qquad (2.6)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Multiplying (2.6) by 17, we get

$$
\begin{aligned}
&17f(17x) - 289f(16x) + 2312f(15x) - 11560f(14x) + 40460f(13x)\\
&\quad - 105196f(12x) + 210392f(11x) - 330616f(10x) + 413270f(9x)\\
&\quad - 413270f(8x) + 330616f(7x) - 210392f(6x) + 105196f(5x) - 40460f(4x)\\
&\quad +11560f(3x) - 2312f(2x) - 17(17!)f(x) = 0 \qquad (2.7)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Subtracting from (2.5) to (2.7), we obtain

$$137f(16x) - 1632f(15x) + 9299f(14x) - 34272f(13x)$$

$$
\begin{aligned}
&- 190944f(11x) + 308006f(10x) - 388960f(9x) + 390014f(8x)\\
&- 318240f(7x) + 210392f(6x) - 102816f(5x) + 32708f(4x)\\
&- 11424f(3x) + 92276f(12x) + (7157 - 17!)f(2x) + 18(17!)f(x) = 0 \qquad (2.8)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(7x, x)$ in (1.1), we get

$$
\begin{aligned}
&f(16x) - 17f(15x) + 136f(14x) - 680f(13x) + 2380f(12x) - 6188f(11x)\\
&+ 12376f(10x) - 19448f(9x) + 24310f(8x) - 24310f(7x) + 19448f(6x)\\
&- 12376f(5x) + 6188f(4x) - 2380f(3x) + 680f(2x) - (135 + 17!)f(x) = 0 \qquad (2.9)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Multiplying (2.9) by 137 , we get

$$
\begin{aligned}
&137f(16x) - 2329f(15x) + 18632f(14x) - 93160f(13x) + 326060f(12x)\\
&- 847756f(11x) + 1695512f(10x) - 2664376f(9x) + 3330470f(8x)\\
&- 3330470f(7x) + 2664376f(6x) - 1695512f(5x) + 847756f(4x)\\
&-326060f(3x) + 93160f(2x) - 137(17!)f(x) = 0 \qquad (2.10)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Subtracting from (2.8) to (2.10), we obtain

$$
\begin{aligned}
&697f(15x) - 9333f(14x) + 58888f(13x) - 233784f(12x) + 656812f(11x)\\
&- 1387506f(10x) + 2275416f(9x) - 2940456f(8x) + 3012230f(7x)\\
&- 2453984f(6x) + 1592696f(5x) - 815048f(4x) + 314636f(3x)\\
&- (86003 + 17!)f(2x) + 155(17!)f(x) = 0 \qquad (2.11)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(6x, x)$ in (1.1), we get

$$
\begin{aligned}
&f(15x) - 17f(14x) + 136f(13x) - 680f(12x) + 2380f(11x) - 6188f(10x)\\
&+ 12376f(9x) - 19448f(8x) + 24310f(7x) - 24310f(6x)\\
&+19448f(5x) - 12376f(4x) + 6188f(3x) - 2379f(2x) + (663 - 17!)f(x) = 0 \qquad (2.12)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Multiplying (2.12) by 697, we get

$$
\begin{aligned}
&697f(15x) - 11849f(14x) + 94792f(13x) - 473960f(12x) + 1658860f(11x)\\
&- 4313036f(10x) + 8626072f(9x) - 13555256f(8x) + 16944070f(7x)\\
&- 16944070f(6x) + 13555256f(5x) - 8626072f(4x) + 4313036f(3x)\\
&- 1658163f(2x) - 697(17!)f(x) = 0 \qquad (2.13)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Subtracting from (2.11) to (2.13), we get

$$
\begin{aligned}
&2516f(14x) - 35904f(13x) + 240176f(12x) - 1002048f(11x)\\
&+ 2925530f(10x) - 6350656f(9x) + 10614800f(8x) - 13931840f(7x)\\
&+ 14490086f(6x) - 11962560f(5x) + 7811024f(4x)\\
&- 3998400f(3x) + (1572160 - 17!)f(2x) + 852(17!)f(x) = 0 \qquad (2.14)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(5x, x)$ in (1.1), we obtain

$$
\begin{aligned}
&f(14x) - 17f(13x) + 136f(12x) - 680f(11x) + 2380f(10x) - 6188f(9x)\\
&+ 12376f(8x) - 19448f(7x) + 24310f(6x) - 24310f(5x)\\
&+19448f(4x) - 12375f(3x) + 6171f(2x) - (2244 + 17!)f(x) = 0 \qquad (2.15)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Multiplying (2.15) by 2516, we get

$$\begin{aligned} &2516f(14x) - 42772f(13x) + 342176f(12x) - 1710880f(11x) + 5988080f(10x) \\ &\quad - 15569008f(9x) + 31138016f(8x) - 48931168f(7x) + 61163960f(6x) \\ &\quad - 61163960f(5x) + 48931168f(4x) - 31135500f(3x) + 15526236f(2x) \\ &\quad - 2516(17!)f(x) = 0 \end{aligned} \tag{2.16}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.14) to (2.16), we obtain

$$\begin{aligned} &6868f(13x) - 102000f(12x) + 708832f(11x) - 3062550f(10x) + 9218352f(9x) \\ &\quad - 20523216f(8x) + 34999328f(7x) - 46673874f(6x) \\ &\quad + 49201400f(5x) - 41120144f(4x) + 27137100f(3x) \\ &\quad - (13954076 + 17!)f(2x) + 3368(17!)f(x) = 0 \end{aligned} \tag{2.17}$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(4x, x)$ in (1.1), we get

$$\begin{aligned} &f(13x) - 17f(12x) + 136f(11x) - 680f(10x) + 2380f(9x) - 6188f(8x) \\ &\quad + 12376f(7x) - 19448f(6x) + 24310f(5x) - 24309f(4x) \\ &\quad + 19431f(3x) - 12240f(2x) + (5508 - 17!)f(x) = 0 \end{aligned} \tag{2.18}$$

for all $x \in \mathcal{A}$.

Multiplying (2.18) by 6868, we obtain

$$\begin{aligned} &6868f(13x) - 116756f(12x) + 934048f(11x) - 4670240f(10x) + 16345840f(9x) \\ &\quad - 42499184f(8x) + 84998368f(7x) - 133568864f(6x) + 166961080f(5x) \\ &-166954212f(4x) + 133452108f(3x) - 84064320f(2x) - 6868(17!)f(x) = 0 \end{aligned} \tag{2.19}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.17) to (2.19), we get

$$\begin{aligned} &14576f(12x) - 225216f(11x) + 1607690f(10x) - 7127488f(9x) + 21975968f(8x) \\ &\quad - 49999040f(7x) + 86894990f(6x) - 117759680f(5x) + 125834068f(4x) \\ &-106315008f(3x) + (70110244 - 17!)f(2x) + 10236(17!)f(x) = 0 \end{aligned} \tag{2.20}$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(3x, x)$ in (1.1), we get

$$\begin{aligned} &f(12x) - 17f(11x) + 136f(10x) - 680f(9x) + 2380f(8x) - 6188f(7x) \\ &\quad + 12376f(6x) - 19447f(5x) + 24293f(4x) - 24174f(3x) \\ &\quad + 18768f(2x) - (9996 + 17!)f(x) = 0 \end{aligned} \tag{2.21}$$

for all $x \in \mathcal{A}$.

Multiplying (2.21) by 14756, we obtain

$$\begin{aligned} &14756f(12x) - 250852f(11x) + 2006816f(10x) - 10034080f(9x) + 35119280f(8x) \\ &\quad - 91310128f(7x) + 182620256f(6x) - 286959932f(5x) + 358467508f(4x) \\ &-356711544f(3x) + 276940608f(2x) - 14756(17!)f(x) = 0 \end{aligned} \tag{2.22}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.20) to (2.22), we get

$$\begin{aligned} &25636f(11x) - 399126f(10x) + 2906592f(9x) - 13143312f(8x) + 41311088f(7x) \\ &\quad - 95725266f(6x) + 169200252f(5x) - 232633440f(4x) + 250396536f(3x) \\ &\quad - (206830364 + 17!)f(2x) + 24992(17!)f(x) = 0 \end{aligned} \tag{2.23}$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(2x, x)$ in (1.1), we get

$$f(11x) - 17f(10x) + 136f(9x) - 680f(8x) + 2380f(7x) - 6187f(6x) + 12359f(5x)$$
$$-19312f(4x) + 23630f(3x) - 21930f(2x) + (13260 - 17!)f(x) = 0 \tag{2.24}$$

for all $x \in \mathcal{A}$.

Multiplying (2.24) by 25636, we obtain

$$\begin{aligned}25636f(11x) &- 435812f(10x) + 3486496f(9x) - 17432480f(8x) + 61013680f(7x)\\ &- 158609932f(6x) + 316835324f(5x) - 495082432f(4x) + 605778680f(3x)\\ &- 562197480f(2x) - 25636(17!)f(x) = 0\end{aligned} \tag{2.25}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.23) to (2.25), we get

$$\begin{aligned}36686f(10x) &- 579904f(9x) + 4289168f(8x) - 19702592f(7x) + 62884666f(6x)\\ &- 147635072f(5x) + 262448992f(4x) - 355382144f(3x)\\ &+ (355367116 - 17!)f(2x) + 50628(17!)f(x) = 0\end{aligned} \tag{2.26}$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by (x, x) in (1.1), we get

$$f(10x) - 17f(9x) + 136f(8x) - 679f(7x) + 2363f(6x) - 6052f(5x)$$
$$+11696f(4x) - 17068f(3x) + 18122f(2x) - (11934 + 17!)f(x) = 0 \tag{2.27}$$

for all $x \in \mathcal{A}$.

Multiplying (2.27) by 36686, we obtain

$$\begin{aligned}36686f(10x) &- 623662f(9x) + 4989296f(8x) - 24909794f(7x) + 86689018f(6x)\\ &- 222023672f(5x) + 429079456f(4x) - 626156648f(3x) + 664823692f(2x)\\ &- 36686(17!)f(x) = 0\end{aligned} \tag{2.28}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.26) to (2.28), we get

$$43758f(9x) - 700128f(8x) + 5207202f(7x) - 23804352f(6x) + 74388600f(5x)$$
$$-166630464f(4x) + 270774504f(3x) - (309456576 + 17!)f(2x) + 87314(17!)f(x) = 0 \tag{2.29}$$

for all $x \in \mathcal{A}$.

Replacing (x, y) by $(0, x)$ in (1.1), we get

$$f(9x) - 16f(8x) + 119f(7x) - 544f(6x) + 1700f(5x) - 3808f(4x)$$
$$+ 6188f(3x) - 7072f(2x) + (4862 - 17!)f(x) = 0 \tag{2.30}$$

for all $x \in \mathcal{A}$.

Multiplying (2.30) by 43758, we obtain

$$\begin{aligned}43758f(9x) &- 700128f(8x) + 5207202f(7x) - 23804352f(6x)\\ &+ 74388600f(5x) - 166630464f(4x) + 270774504f(3x)\\ &- 309456576f(2x) - 43758(17!)f(x) = 0\end{aligned} \tag{2.31}$$

for all $x \in \mathcal{A}$.

Subtracting from (2.29) to (2.31), we get

$$-17!f(2x) + 131072(17!)f(x) = 0$$

and so $f(2x) = 2^{17}f(x)$ for all $x \in \mathcal{A}$. □

3. Stability of the septendecic functional equation in matrix normed spaces

Throughout this section, let $(X, \|.\|_n)$ be a matrix normed space, $(Y, \|.\|_n)$ be a matrix Banach space and let n be a fixed non-negative integer.

In this section, we prove the stability of the septendecic functional equation (1.1) in matrix normed spaces by using the fixed point method.

For a mapping $f : X \to Y$, define $\mathcal{G}f : X^2 \to Y$ and $\mathcal{G}f_n : M_n(X^2) \to M_n(Y)$ by

$$\begin{aligned}\mathcal{G}f(a,b) = f(a+9b) &- 17f(a+8b) + 136f(a+7b) - 680f(a+6b) + 2380f(a+5b)\\ &- 6188f(a+4b) + 12376f(a+3b) - 19448f(a+2b) + 24310f(a+b)\\ &- 24310f(a) + 19448f(a-b) - 12376f(a-2b) + 6188f(a-3b)\\ &- 2380f(a-4b) + 680f(a-5b) - 136f(a-6b) + 17f(a-7b)\\ &\qquad - f(a-8b) - 17!f(b),\end{aligned}$$

$$\begin{aligned}\mathcal{G}f_n([x_{ij}],[y_{ij}]) = f_n([x_{ij}+9y_{ij}]) &- 17f_n([x_{ij}+8y_{ij}]) + 136f_n([x_{ij}+7y_{ij}])\\ &- 680f_n([x_{ij}+6y_{ij}]) + 2380f_n([x_{ij}+5y_{ij}]) - 6188f_n([x_{ij}+4y_{ij}])\\ &+ 12376f_n([x_{ij}+3y_{ij}]) - 19448f_n([x_{ij}+2y_{ij}]) + 24310f_n([x_{ij}+y_{ij}])\\ &- 24310f_n([x_{ij}]) + 19448f_n([x_{ij}-y_{ij}]) - 12376f_n([x_{ij}-2y_{ij}])\\ &+ 6188f_n([x_{ij}-3y_{ij}]) - 2380f_n([x_{ij}-4y_{ij}]) + 680f_n([x_{ij}-5y_{ij}])\\ &- 136f_n([x_{ij}-6y_{ij}]) + 17f_n([x_{ij}-7y_{ij}]) - f_n([x_{ij}-8y_{ij}]) - 17!f_n([y_{ij}])\end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3. *Assume that $l = \pm 1$ be fixed and let $\psi : X^2 \to [0, \infty)$ be a function such that there exists an $\eta < 17$ with*

$$\psi(a,b) \le 2^{17l}\eta\psi(\frac{a}{2^l}, \frac{b}{2^l}) \tag{3.1}$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|\mathcal{G}f_n([x_{ij}],[y_{ij}])\|_n \le \sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}) \tag{3.2}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique septendecic mapping $\mathcal{S_D} : X \to Y$ such that

$$\|f_n([x_{ij}]) - \mathcal{S_D}_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{\eta^{\frac{1-l}{2}}}{2^{17}(1-\eta)}\overline{\psi}(x_{ij}), \tag{3.3}$$

where

$$\begin{aligned}\overline{\psi}(x_{ij}) &= \frac{1}{17!}[\psi(0,2x_{ij}) + \psi(9x_{ij},x_{ij}) + 17\psi(8x_{ij},x_{ij}) + 137\psi(7x_{ij},x_{ij})\\ &+ 697\psi(6x_{ij},x_{ij}) + 2516\psi(5x_{ij},x_{ij}) + 6868\psi(4x_{ij},x_{ij}) + 14756\psi(3x_{ij},x_{ij})\\ &+ 25636\psi(2x_{ij},x_{ij}) + 36686\psi(x_{ij},x_{ij}) + 43758\psi(0,x_{ij})]\end{aligned}$$

Proof. Letting $n = 1$ in (3.2), we obtain

$$\|\mathcal{G}f(a,b)\| \le \psi(a,b) \tag{3.4}$$

Replacing (a, b) by $(0, 2a)$ in (3.4), we get

$$\begin{aligned}&\|f(18a) - 16f(16a) + 119f(14a) - 544f(12a) + 1700f(10a)\\ &-3808f(8a) + 6188f(6a) - 7072f(4a) + (4862 - 17!)f(2a)\| \le \psi(0,2a) \quad (3.5)\end{aligned}$$

for all $a \in X$.

Replacing (a,b) by $(9a,a)$ in (3.4), we obtain

$$\begin{aligned}&\|f(18a)-17f(17a)+136f(16a)-680f(15a)+2380f(14a)\\&\quad-6188f(13a)+12376f(12a)-19448f(11a)+24310f(10a)\\&\quad-24310f(9a)+19448f(8a)-12376f(7a)+6188f(6a)-2380f(5a)\\&\quad+680f(4a)-136f(3a)+17f(2a)-(1+17!)f(a)\|\le\psi(9a,a)\end{aligned}\tag{3.6}$$

for all $a\in X$.

It follows from (3.5) and (3.6) that

$$\begin{aligned}&\|17f(17a)-152f(16a)+680f(15a)-2261f(14a)\\&\quad+6188f(13a)-12920f(12a)+19448f(11a)-22610f(10a)\\&\quad+24310f(9a)-23256f(8a)+12376f(7a)+2380f(5a)\\&\quad-7752f(4a)+136f(3a)+(4845-17!)f(2a)+17!f(a)\|\le\psi(0,2a)+\psi(9a,a)\end{aligned}\tag{3.7}$$

for all $a\in X$.

Replacing (a,b) by $(8a,a)$ in (3.4), we obtain

$$\begin{aligned}&\|f(17a)-17f(16a)+136f(15a)-680f(14a)+2380f(13a)\\&\quad-6188f(12a)+12376f(11a)-19448f(10a)+24310f(9a)\\&\quad-24310f(8a)+19448f(7a)-12376f(6a)+6188f(5a)-2380f(4a)\\&\quad+680f(3a)-136f(2a)+(17-17!)f(a)\|\le\psi(8a,a)\end{aligned}\tag{3.8}$$

for all $a\in X$.

Multiplying (3.8) by 17, we get

$$\begin{aligned}&\|17f(17a)-289f(16a)+2312f(15a)-11560f(14a)+40460f(13a)\\&\quad-105196f(12a)+210392f(11a)-330616f(10a)+413270f(9a)\\&\quad-413270f(8a)+330616f(7a)-210392f(6a)+105196f(5a)-40460f(4a)\\&\quad+11560f(3a)-2312f(2a)-17(17!)f(a)\|\le17\psi(8a,a)\end{aligned}\tag{3.9}$$

for all $a\in X$.

It follows from (3.7) and (3.9) that

$$\begin{aligned}&\|137f(16a)-1632f(15a)+9299f(14a)-34272f(13a)\\&\quad-190944f(11a)+308006f(10a)-388960f(9a)+390014f(8a)\\&\quad-318240f(7a)+210392f(6a)-102816f(5a)+32708f(4a)\\&\quad-11424f(3a)+92276f(12a)+(7157-17!)f(2a)+18(17!)f(a)\|\\&\le\psi(0,2a)+\psi(9a,a)+17\psi(8a,a)\end{aligned}\tag{3.10}$$

for all $a\in X$.

Replacing (a,b) by $(7a,a)$ in (3.4), we get

$$\begin{aligned}&\|f(16a)-17f(15a)+136f(14a)-680f(13a)+2380f(12a)\\&\quad-6188f(11a)+12376f(10a)-19448f(9a)+24310f(8a)\\&\quad-24310f(7a)+19448f(6a)-12376f(5a)+6188f(4a)\\&\quad-2380f(3a)+680f(2a)-(135+17!)f(a)\|\le\psi(7a,a)\end{aligned}\tag{3.11}$$

for all $a\in X$.

Multiplying (3.11) by 137, we get

$$\begin{aligned}&\|137f(16a)-2329f(15a)+18632f(14a)-93160f(13a)+326060f(12a)\\&\quad-847756f(11a)+1695512f(10a)-2664376f(9a)+3330470f(8a)\\&\quad-3330470f(7a)+2664376f(6a)-1695512f(5a)+847756f(4a)\\&\quad-326060f(3a)+93160f(2a)-137(17!)f(a)\|\le137\psi(7a,a)\end{aligned}\tag{3.12}$$

for all $a\in X$.

It follows from (3.10) and (3.12) that

$$\begin{aligned}\|697f(15a) &- 9333f(14a) + 58888f(13a) - 233784f(12a) + 656812f(11a)\\ &-1387506f(10a) + 2275416f(9a) - 2940456f(8a) + 3012230f(7a)\\ &-2453984f(6a) + 1592696f(5a) - 815048f(4a)\\ &+314636f(3a) - (86003 + 17!)f(2a) + 155(17!)f(a)\|\\ &\le \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a) \qquad (3.13)\end{aligned}$$

for all $a \in X$.

Replacing (a, b) by $(6a, a)$ in (3.4), we get

$$\begin{aligned}\|f(15a) &- 17f(14a) + 136f(13a) - 680f(12a) + 2380f(11a) - 6188f(10a)\\ &+12376f(9a) - 19448f(8a) + 24310f(7a) - 24310f(6a) + 19448f(5a)\\ &-12376f(4a) + 6188f(3a) - 2379f(2a) + (663 - 17!)f(a)\| \le \psi(6a, a) \qquad (3.14)\end{aligned}$$

for all $a \in X$.

Multiplying (3.14) by 697, we get

$$\begin{aligned}\|697f(15a) &- 11849f(14a) + 94792f(13a) - 473960f(12a) + 1658860f(11a)\\ &-4313036f(10a) + 8626072f(9a) - 13555256f(8a) + 16944070f(7a)\\ &-16944070f(6a) + 13555256f(5a) - 8626072f(4a) + 4313036f(3a)\\ &-1658163f(2a) - 697(17!)f(a)\| \le 697\psi(6a, a) \qquad (3.15)\end{aligned}$$

for all $a \in X$.

It follows from (3.13) and (3.15) that

$$\begin{aligned}\|2516f(14a) &- 35904f(13a) + 240176f(12a) - 1002048f(11a) + 2925530f(10a)\\ &-6350656f(9a) + 10614800f(8a) - 13931840f(7a) + 14490086f(6a)\\ &-11962560f(5a) + 7811024f(4a) - 3998400f(3a)\\ &+(1572160 - 17!)f(2a) + 852(17!)f(a)\|\\ &\le \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a) + 697\psi(6a, a) \qquad (3.16)\end{aligned}$$

for all $a \in X$.

Replacing (a, b) by $(5a, a)$ in (3.4), we obtain

$$\begin{aligned}\|f(14a) &- 17f(13a) + 136f(12a) - 680f(11a) + 2380f(10a) - 6188f(9a)\\ &+12376f(8a) - 19448f(7a) + 24310f(6a) - 24310f(5a) + 19448f(4a)\\ &-12375f(3a) + 6171f(2a) - (2244 + 17!)f(a)\| \le \psi(5a, a) \qquad (3.17)\end{aligned}$$

for all $a \in X$.

Multiplying (3.17) by 2516, we get

$$\begin{aligned}\|2516f(14a) &- 42772f(13a) + 342176f(12a) - 1710880f(11a) + 5988080f(10a)\\ &-15569008f(9a) + 31138016f(8a) - 48931168f(7a) + 61163960f(6a)\\ &-61163960f(5a) + 48931168f(4a) - 31135500f(3a) + 15526236f(2a)\\ &-2516(17!)f(a)\| \le 2516\psi(5a, a) \qquad (3.18)\end{aligned}$$

for all $a \in X$.

It follows from (3.16) and (3.18) that

$$\begin{aligned}\|6868f(13a) &- 102000f(12a) + 708832f(11a) - 3062550f(10a) + 9218352f(9a)\\ &-20523216f(8a) + 34999328f(7a) - 46673874f(6a) + 49201400f(5a)\\ &-41120144f(4a) + 27137100f(3a) - (13954076 + 17!)f(2a)\\ &+3368(17!)f(a)\| \le \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a)\\ &+ 137\psi(7a, a) + 697\psi(6a, a) + 2516\psi(5a, a) \qquad (3.19)\end{aligned}$$

for all $a \in X$.

Replacing (a, b) by $(4a, a)$ in (3.4), we get

$$\begin{aligned}\|f(13a) - 17f(12a) + 136f(11a) - 680f(10a) + 2380f(9a) - 6188f(8a)\\ +12376f(7a) - 19448f(6a) + 24310f(5a) - 24309f(4a)\\ +19431f(3a) - 12240f(2a) + (5508 - 17!)f(a)\| \leq \psi(4a, a)\end{aligned} \tag{3.20}$$

for all $a \in X$.

Multiplying (3.20) by 6868, we obtain

$$\begin{aligned}\|6868f(13a) - 116756f(12a) + 934048f(11a) - 4670240f(10a) + 16345840f(9a)\\ -42499184f(8a) + 84998368f(7a) - 133568864f(6a) + 166961080f(5a)\\ -166954212f(4a) + 133452108f(3a) - 84064320f(2a)\\ -6868(17!)f(a)\| \leq 6868\psi(4a, a)\end{aligned} \tag{3.21}$$

for all $a \in X$.

It follows from (3.19) and (3.21) that

$$\begin{aligned}\|14576f(12a) - 225216f(11a) + 1607690f(10a) - 7127488f(9a) + 21975968f(8a)\\ -49999040f(7a) + 86894990f(6a) - 117759680f(5a) + 125834068f(4a)\\ -106315008f(3a) + (70110244 - 17!)f(2a) + 10236(17!)f(a)\|\\ \leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a)\\ + 697\psi(6a, a) + 2516\psi(5a, a) + 6868\psi(4a, a)\end{aligned} \tag{3.22}$$

for all $a \in X$.

Replacing (a, b) by $(3a, a)$ in (3.4), we get

$$\begin{aligned}\|f(12a) - 17f(11a) + 136f(10a) - 680f(9a) + 2380f(8a) - 6188f(7a)\\ +12376f(6a) - 19447f(5a) + 24293f(4a) - 24174f(3a)\\ +18768f(2a) - (9996 + 17!)f(a)\| \leq \psi(3a, a)\end{aligned} \tag{3.23}$$

for all $a \in X$.

Multiplying (3.23) by 14756, we obtain

$$\begin{aligned}\|14756f(12a) - 250852f(11a) + 2006816f(10a) - 10034080f(9a) + 35119280f(8a)\\ -91310128f(7a) + 182620256f(6a) - 286959932f(5a) + 358467508f(4a)\\ -356711544f(3a) + 276940608f(2a) - 14756(17!)f(a)\| \leq 14756\psi(3a, a)\end{aligned} \tag{3.24}$$

for all $a \in X$.

It follows from (3.22) and (3.24) that

$$\begin{aligned}\|25636f(11a) - 399126f(10a) + 2906592f(9a) - 13143312f(8a)\\ +41311088f(7a) - 95725266f(6a) + 169200252f(5a) - 232633440f(4a)\\ +250396536f(3a) - (206830364 + 17!)f(2a) + 24992(17!)f(a)\|\\ \leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a) + 697\psi(6a, a)\\ + 2516\psi(5a, a) + 6868\psi(4a, a) + 14756\psi(3a, a)\end{aligned} \tag{3.25}$$

for all $a \in X$.

Replacing (a, b) by $(2a, a)$ in (3.4), we get

$$\begin{aligned}\|f(11a) - 17f(10a) + 136f(9a) - 680f(8a) + 2380f(7a) - 6187f(6a) + 12359f(5a)\\ -19312f(4a) + 23630f(3a) - 21930f(2a) + (13260 - 17!)f(a)\| \leq \psi(2a, a)\end{aligned} \tag{3.26}$$

for all $a \in X$.

Multiplying (3.26) by 25636, we obtain

$$\begin{aligned}\|25636f(11a) - 435812f(10a) + 3486496f(9a) - 17432480f(8a)\\ +61013680f(7a) - 158609932f(6a) + 316835324f(5a) - 495082432f(4a)\\ +605778680f(3a) - 562197480f(2a) - 25636(17!)f(a)\| \leq 25636\psi(2a, a)\end{aligned} \tag{3.27}$$

for all $a \in X$.

It follows from (3.25) and (3.27) that

$$\begin{aligned}&\|36686f(10a) - 579904f(9a) + 4289168f(8a) - 19702592f(7a) + 62884666f(6a)\\ &\quad -147635072f(5a) + 262448992f(4a) - 355382144f(3a) + 50628(17!)f(a)\\ &\quad +(355367116 - 17!)f(2a)\| \leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a)\\ &\quad +697\psi(6a, a) + 2516\psi(5a, a) + 6868\psi(4a, a) + 14756\psi(3a, a) + 25636\psi(2a, a) \qquad (3.28)\end{aligned}$$

for all $a \in X$.

Replacing (a, b) by (a, a) in (3.4), we get

$$\begin{aligned}&\|f(10a) - 17f(9a) + 136f(8a) - 679f(7a) + 2363f(6a) - 6052f(5a)\\ &\quad +11696f(4a) - 17068f(3a) + 18122f(2a) - (11934 + 17!)f(a)\| \leq \psi(a, a) \qquad (3.29)\end{aligned}$$

for all $a \in X$.

Multiplying (3.29) by 36686, we obtain

$$\begin{aligned}&\|36686f(10a) - 623662f(9a) + 4989296f(8a) - 24909794f(7a) + 86689018f(6a)\\ &\quad -222023672f(5a) + 429079456f(4a) - 626156648f(3a) + 664823692f(2a)\\ &\quad -36686(17!)f(a)\| \leq 36686\psi(a, a) \qquad (3.30)\end{aligned}$$

for all $a \in X$.

It follows from (3.28) and (3.30) that

$$\begin{aligned}&\|43758f(9a) - 700128f(8a) + 5207202f(7a) - 23804352f(6a) + 74388600f(5a)\\ &\quad -166630464f(4a) + 270774504f(3a) - (309456576 + 17!)f(2a)\\ &\quad +87314(17!)f(a)\| \leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a) + 697\psi(6a, a)\\ &\quad +2516\psi(5a, a) + 6868\psi(4a, a) + 14756\psi(3a, a) + 25636\psi(2a, a) + 36686\psi(a, a) \qquad (3.31)\end{aligned}$$

for all $a \in X$.

Replacing (a, b) by $(0, a)$ in (3.4), we get

$$\begin{aligned}&\|f(9a) - 16f(8a) + 119f(7a) - 544f(6a) + 1700f(5a) - 3808f(4a)\\ &\quad +6188f(3a) - 7072f(2a) + (4862 - 17!)f(a)\| \leq \psi(0, a) \qquad (3.32)\end{aligned}$$

for all $a \in X$.

Multiplying (3.32) by 43758, we obtain

$$\begin{aligned}&\|43758f(9a) - 700128f(8a) + 5207202f(7a) - 23804352f(6a) + 74388600f(5a)\\ &\quad -166630464f(4a) + 270774504f(3a) - 309456576f(2a)\\ &\quad -43758(17!)f(a)\| \leq 43758\psi(0, a) \qquad (3.33)\end{aligned}$$

for all $a \in X$.

It follows from (3.31) and (3.33) that

$$\begin{aligned}&\|-17!f(2a) + 131072(17!)f(a)\| \leq \psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a)\\ &\quad + 137\psi(7a, a) + 697\psi(6a, a) + 2516\psi(5a, a) + 6868\psi(4a, a)\\ &\quad +14756\psi(3a, a) + 25636\psi(2a, a) + 36686\psi(a, a) + 43758\psi(0, a) \qquad (3.34)\end{aligned}$$

for all $a \in X$. By (3.34)

$$\left\|2^{17}f(a) - f(2a)\right\| \leq \overline{\psi}(a) \qquad (3.35)$$

for all $a \in X$, where

$$\begin{aligned}\overline{\psi}(a) &= \frac{1}{17!}[\psi(0, 2a) + \psi(9a, a) + 17\psi(8a, a) + 137\psi(7a, a) + 697\psi(6a, a)\\ &+ 2516\psi(5a, a) + 6868\psi(4a, a) + 14756\psi(3a, a) + 25636\psi(2a, a)\\ &+ 36686\psi(a, a) + 43758\psi(0, a)].\end{aligned}$$

Thus

$$\left\|f(a)-\frac{1}{2^{17l}}f(2^l a)\right\| \le \frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}}\overline{\psi}(a) \qquad \forall\ a\in X. \tag{3.36}$$

We consider the set $\mathcal{M}=\{f : X\to Y\}$ and introduce the generalized metric on $\mathcal{M}$ as follows:

$$\rho(f,g)=\inf\left\{\mu\in\mathbb{R}_+ : \|f(a)-g(a)\|\le\mu\overline{\psi}(a), \forall a\in X\right\},$$

It is easy to check that $(\mathcal{M},\rho)$ is complete (see the proof of [[19], Lemma 2.1]).
Define the mapping $\mathcal{P}:\mathcal{M}\to\mathcal{M}$ by

$$\mathcal{P}f(a)=\frac{1}{2^{17l}}f(2^l a) \qquad \forall\ a\in X.$$

Let $f,g\in\mathcal{M}$ be an arbitrary constant with $\rho(f,g)=\nu$. Then

$$\|f(a)-g(a)\|\le\nu\overline{\psi}(a) \quad \text{for all } a\in X.$$

Utilizing (3.1), we find that

$$\|\mathcal{P}f(a)-\mathcal{P}g(a)\|=\left\|\frac{1}{2^{17l}}f(2^l a)-\frac{1}{2^{17l}}g(2^l a)\right\|\le\eta\nu\overline{\psi}(a) \quad \text{for all } a\in X.$$

Hence it holds that $\rho(\mathcal{P}f,\mathcal{P}g)\le\eta\nu$, that is, $\rho(\mathcal{P}f,\mathcal{P}g)\le\eta\rho(f,g)$ for all $f,g\in\mathcal{M}$.

It follows from (3.36) that $\rho(f,\mathcal{P}f)\le\dfrac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}}$.

According to [3, Theorem 2.2], there exists a mapping $\mathcal{S_D}:X\to Y$ which satisfying:

(1) $\mathcal{S_D}$ is a unique fixed point of $\mathcal{P}$ in the set $\mathcal{S}=\{g\in\mathcal{M}:\rho(f,g)<\infty\}$, which is satisfied

$$\mathcal{S_D}(2^l a)=2^{17l}\mathcal{S_D}(a) \qquad \forall\ a\in X.$$

In other words, there exists a μ satisfying

$$\|f(a)-g(a)\|\le\mu\overline{\psi}(a) \quad \forall\ a\in X.$$

(2) $\rho(\mathcal{P}^k f,\mathcal{S_D})\to 0$ as $k\to\infty$. This implies that

$$\lim_{k\to\infty}\frac{1}{2^{17kl}}f(2^{kl}a)=\mathcal{S_D}(a) \qquad \forall\ a\in X.$$

(3) $\rho(f,\mathcal{S_D})\le\dfrac{1}{1-\eta}\rho(f,\mathcal{P}f)$, which implies the inequality $\rho(f,\mathcal{S_D})\le\dfrac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}(1-\eta)}$.

$$\text{So} \qquad \|f(a)-\mathcal{S_D}(a)\|\le\frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}(1-\eta)}\overline{\psi}(a) \qquad \forall\ a\in X. \tag{3.37}$$

It follows from (3.1) and (3.4) that

$$\begin{aligned}
\lim_{k\to\infty}\frac{1}{2^{17kl}}\Big\| & f(2^{kl}(a+9b))-17f(2^{kl}(a+8b))+136f(2^{kl}(a+7b))\\
&-680f(2^{kl}(a+6b))+2380f(2^{kl}(a+5b))-6188f(2^{kl}(a+4b))\\
&+12376f(2^{kl}(a+3b))-19448f(2^{kl}(a+2b))+24310f(2^{kl}(a+b))\\
&-24310f(2^{kl}(a))+19448f(2^{kl}(a-b))-12376f(2^{kl}(a-2b))\\
&+6188f(2^{kl}(a-3b))-2380f(2^{kl}(a-4b))+680f(2^{kl}(a-5b))\\
&-136f(2^{kl}(a-6b))+17f(2^{kl}(a-7b))-f(2^{kl}(a-8b))-17!f(2^{kl}(b))\Big\|\\
&\le\lim_{k\to\infty}\frac{1}{2^{17kl}}\psi(2^{kl}a,2^{kl}b)=0
\end{aligned}$$

and so

$$\begin{aligned}
&\mathcal{S}_{\mathcal{D}}(a+9b)-17\mathcal{S}_{\mathcal{D}}(a+8b)+136\mathcal{S}_{\mathcal{D}}(a+7b)-680\mathcal{S}_{\mathcal{D}}(a+6b)+2380\mathcal{S}_{\mathcal{D}}(a+5b)\\
&\quad-6188\mathcal{S}_{\mathcal{D}}(a+4b)+12376\mathcal{S}_{\mathcal{D}}(a+3b)-19448\mathcal{S}_{\mathcal{D}}(a+2b)+24310\mathcal{S}_{\mathcal{D}}(a+b)\\
&\quad-24310\mathcal{S}_{\mathcal{D}}(a)+19448\mathcal{S}_{\mathcal{D}}(a-b)-12376\mathcal{S}_{\mathcal{D}}(a-2b)+6188\mathcal{S}_{\mathcal{D}}(a-3b)\\
&\quad-2380\mathcal{S}_{\mathcal{D}}(a-4b)+680\mathcal{S}_{\mathcal{D}}(a-5b)-136\mathcal{S}_{\mathcal{D}}(a-6b)+17\mathcal{S}_{\mathcal{D}}(a-7b)\\
&\qquad\qquad\qquad\qquad-\mathcal{S}_{\mathcal{D}}(a-8b)=17!\mathcal{S}_{\mathcal{D}}(b)
\end{aligned}$$

for all $a,b\in X$. Therefore, the mapping $\mathcal{S}_{\mathcal{D}}:X\to Y$ is septendecic mapping.

It follows from [17, Lemma 2.1] and (3.37) that

$$\|f_n([x_{ij}])-\mathcal{S}_{\mathcal{D}n}([x_{ij}])\|_n\le\sum_{i,j=1}^{n}\|f(x_{ij})-\mathcal{S}_{\mathcal{D}}(x_{ij})\|\le\sum_{i,j=1}^{n}\frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{17}(1-\eta)}\overline{\psi}(x_{ij})$$

for all $x=[x_{ij}]\in M_n(X)$, where

$$\begin{aligned}
\overline{\psi}(x_{ij}) &= \frac{1}{17!}[\psi(0,2x_{ij})+\psi(9x_{ij},x_{ij})+17\psi(8x_{ij},x_{ij})+137\psi(7x_{ij},x_{ij})\\
&+ 697\psi(6x_{ij},x_{ij})+2516\psi(5x_{ij},x_{ij})+6868\psi(4x_{ij},x_{ij})+14756\psi(3x_{ij},x_{ij})\\
&+ 25636\psi(2x_{ij},x_{ij})+36686\psi(x_{ij},x_{ij})+43758\psi(0,x_{ij})]
\end{aligned}$$

for all $x=[x_{ij}]\in M_n(X)$.

Thus $\mathcal{S}_{\mathcal{D}}:X\to Y$ is a unique septendecic mapping satisfying (3.3). □

Corollary 1. *Assume that $l=\pm1$ be fixed and let t,ϵ be positive real numbers with $t\neq17$. Let $f:X\to Y$ be a mapping such that*

$$\|\mathcal{G}f_n([x_{ij}],[y_{ij}])\|_n\le\sum_{i,j=1}^{n}\epsilon(\|x_{ij}\|^t+\|y_{ij}\|^t) \tag{3.38}$$

for all $x=[x_{ij}],y=[y_{ij}]\in M_n(X)$. Then there exists a unique septendecic mapping $\mathcal{S}_{\mathcal{D}}:X\to Y$ such that

$$\|f_n([x_{ij}])-\mathcal{S}_{\mathcal{D}n}([x_{ij}])\|_n\le\sum_{i,j=1}^{n}\frac{\epsilon_s}{l(2^{17}-2^t)}\|x_{ij}\|^t$$

for all $x=[x_{ij}]\in M_n(X)$, where

$$\begin{aligned}
\epsilon_s &= \frac{\epsilon}{17!}[43758+36687(2^t)+25636(3^t)+14756(4^t)+6868(5^t)\\
&+ 2516(6^t)+697(7^t)+137(8^t)+17(9^t)+(10)^t)].
\end{aligned}$$

Proof. The proof follows from Theorem 3 by taking $\psi(a,b)=\epsilon(\|a\|^t+\|b\|^t)$ for all $a,b\in X$. Then we can choose $\eta=2^{l(t-17)}$, and we can obtain the required result. □

Now we will give an example to illustrate that the functional equation (1.1) is not stable for $t=17$ in Corollary 1.

Example 4. Let $\psi:\mathbb{R}\to\mathbb{R}$ be a function defined by

$$\psi(x)=\begin{cases}\epsilon x^{17}, & if\ |x|<1,\\ \epsilon, & otherwise,\end{cases}$$

where $\epsilon > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{17n}}$$

for all $x \in \mathbb{R}$. Then f satisfies the inequality

$$\begin{aligned}
&\|f(x+9y) - 17f(x+8y) + 136f(x+7y) - 680f(x+6y) + 2380f(x+5y) \\
&\quad -6188f(x+4y) + 12376f(x+3y) - 19448f(x+2y) + 24310f(x+y) \\
&\quad -24310f(x) + 19448f(x-y) - 12376f(x-2y) + 6188f(x-3y) \\
&\quad -2380f(x-4y) + 680f(x-5y) - 136f(x-6y) + 17f(x-7y) \\
&\quad -f(x-8y) - 17!f(y)\| \le \frac{(355687428200000)}{131071}(131072)^2\epsilon(|x|^{17} + |y|^{17})
\end{aligned} \tag{3.39}$$

for all $x, y \in \mathbb{R}$. Then there do not exist a septendecic function $\mathcal{S}_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ such that

$$|f(x) - \mathcal{S}_{\mathcal{D}}(x)| \le \lambda |x|^{17} \tag{3.40}$$

for all $x \in \mathbb{R}$.

Solution. Now

$$|f(x)| \le \sum_{n=0}^{\infty} \frac{|\psi(2^n x)|}{|2^{17n}|} = \sum_{n=0}^{\infty} \frac{\epsilon}{2^{17n}} = \frac{131072\epsilon}{131071}.$$

Thus f is bounded. Next we show that f satisfies (3.39). If $x = y = 0$, then (3.39) is trivial. If $|x|^{17} + |y|^{17} \ge \frac{1}{2^{17}}$, then L.H.S of (3.39) is less than $\frac{(355687428200000)(131072)\epsilon}{131071}$.
Suppose that $0 < |x|^{17} + |y|^{17} < \frac{1}{2^{17}}$. Then there exists a non-negative integer k such that

$$\frac{1}{2^{17(k+1)}} \le |x|^{17} + |y|^{17} < \frac{1}{2^{17k}}. \tag{3.41}$$

So $2^{17(k-1)}|x|^{17} < \frac{1}{2^{17}}, 2^{17(k-1)}|y|^{17} < \frac{1}{2^{17}}$, and
$2^n(x), 2^n(y), 2^n(x+9y), 2^n(x+8y), 2^n(x+7y),$
$2^n(x+6y), 2^n(x+5y), 2^n(x+4y), 2^n(x+3y), 2^n(x+2y),$
$2^n(x+y), 2^n(x-y), 2^n(x-2y), 2^n(x-3y), 2^n(x-4y),$
$2^n(x-5y), 2^n(x-6y), 2^n(x-7y), 2^n(x-8y) \in (-1, 1)$
for all $n = 0, 1, 2, ..., k-1$. Hence

$$\begin{aligned}
&\psi(2^n(x+9y)) - 17\psi(2^n(x+8y)) + 136\psi(2^n(x+7y)) - 680\psi(2^n(x+6y)) \\
&\quad +2380\psi(2^n(x+5y)) - 6188\psi(2^n(x+4y)) + 12376\psi(2^n(x+3y)) \\
&\quad -19448\psi(2^n(x+2y)) + 24310\psi(2^n(x+y)) - 24310\psi(2^n(x)) \\
&\quad +19448\psi(2^n(x-y)) - 12376\psi(2^n(x-2y)) + 6188\psi(2^n(x-3y)) \\
&\quad -2380\psi(2^n(x-4y)) + 680\psi(2^n(x-5y)) - 136\psi(2^n(x-6y)) \\
&\quad +17\psi(2^n(x-7y)) - \psi(2^n(x-8y)) - 17!\psi(2^n(y)) = 0
\end{aligned}$$

for $n = 0, 1, 2, \cdots, k-1$. From the definition of f and (3.41), it follows that

$$\begin{aligned}
&|f(x+9y) - 17f(x+8y) + 136f(x+7y) - 680f(x+6y) + 2380f(x+5y) \\
&\quad -6188f(x+4y) + 12376f(x+3y) - 19448f(x+2y) + 24310f(x+y) \\
&\quad -24310f(x) + 19448f(x-y) - 12376f(x-2y) + 6188f(x-3y) \\
&\quad -2380f(x-4y) + 680f(x-5y) - 136f(x-6y) + 17f(x-7y) \\
&\quad -f(x-8y) - 17!f(y)| \\
&\le \textstyle\sum_{n=0}^{\infty} \frac{1}{2^{17n}} |\psi(2^n(x+9y)) - 17\psi(2^n(x+8y)) + 136\psi(2^n(x+7y)) \\
&\quad -680\psi(2^n(x+6y)) + 2380\psi(2^n(x+5y)) - 6188\psi(2^n(x+4y)) \\
&\quad +12376\psi(2^n(x+3y)) - 19448\psi(2^n(x+2y)) + 24310\psi(2^n(x+y))
\end{aligned}$$

$$-24310\psi(2^n(x)) + 19448\psi(2^n(x-y)) - 12376\psi(2^n(x-2y))$$
$$+6188\psi(2^n(x-3y)) - 2380\psi(2^n(x-4y)) + 680\psi(2^n(x-5y))$$
$$-136\psi(2^n(x-6y)) + 17\psi(2^n(x-7y)) - \psi(2^n(x-8y)) - 17!\psi(2^n(y))|$$
$$\leq \sum_{n=k}^{\infty} \frac{(355687428200000)\epsilon}{2^{17n}} = \frac{(131072)(355687428200000)\epsilon}{2^{17k}(131071)}$$
$$\leq \frac{(355687428200000)}{131071}(131072)^2\epsilon(|x|^{17} + |y|^{17}).$$

Hence f satisfies (3.39) for all $x, y \in \mathbb{R}$ with $0 < |x|^{17} + |y|^{17} < \frac{1}{2^{17}}$. Now, we prove that the septendecic functional equation (1.1) is not stable for $t = 17$ in Corollary 1.

Suppose that there exists a septendecic function $\mathcal{S}_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying (3.40). Since f is bounded and continuous for all $x \in \mathbb{R}$, $\mathcal{S}_{\mathcal{D}}$ is bounded on any open interval containing the origin and continuous at origin.

In view of Theorem 3, $\mathcal{S}_{\mathcal{D}}$ must have the form $\mathcal{S}_{\mathcal{D}}(x) = cx^{17}$ for any $x \in \mathbb{R}$. Thus we obtain that

$$|f(x)| \leq (\lambda + |c|)\,|x|^{17}. \tag{3.42}$$

But we can choose a non-negative integer m with $m\epsilon > \lambda + |c|$.

If $x \in (0, \frac{1}{2^{m-1}})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \cdots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{17n}} \geq \sum_{n=0}^{m-1} \frac{\epsilon(2^n x)^{17}}{2^{17n}} = m\epsilon x^{17} > (\lambda + |c|)\,|x|^{17},$$

which contradicts to (3.42). Thus the septendecic functional equation (1.1) is not stable for $t = 17$.

4. Conclusions

In this investigation, we identified the septendecic functional equation and establised the Ulam-Hyers stability of this functional equation in matrix normed spaces by using the fixed point method and also provided an example for non-stability.

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A novel similarity measure for pseudo-generalized fuzzy rough sets

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Abstract: Various fuzzy generalizations of rough approximations have been made over the years. In this paper, the pseudo-generalized fuzzy rough sets are presented and some properties of the pseudo fuzzy rough approximation operators are investigated. It is necessary to measure the similarity between two pseudo-generalized fuzzy rough sets in some practical cases, such as pattern recognition, image processing and fuzzy reasoning. A novel similarity measure between two pseudo-generalized fuzzy rough sets is proposed in this paper. At the same time, we show that the similarity measure between two pseudo-generalized fuzzy rough sets can be given according to the pseudo-operation.

Keywords: Pseudo-operations; Fuzzy rough sets; Approximation operators; Similarity measure

1. Introduction

The theory of rough set[27] as a mathematical approach to handle imprecision, vagueness and uncertainty in data analysis. However, in Pawlak's rough set model[27], the equivalence relation is a key and primitive notion. This equivalence relation may limit the application domain of the rough set model. Generalizations of rough set theory were considered by scholars in order to deal with complex practical problems [6,13,32,36,38,43].

There are at least two approaches for the development of definitions of lower and upper approximation operators, namely, the constructive and axiomatic approaches. In the constructive approach, some authors have extended equivalence relation to tolerance relations [21,33], similarity relations [34], ordinary binary relations [42,43], and others [16,28,48]. Meanwhile, some authors have relaxed the partition of universe to the covering and obtain the covering-based rough sets [29,32,40,45-47]. In addition, generalizations of rough sets to the fuzzy environment have also been made [5,6,9,12,36]. By introducing the lower and upper approximations in fuzzy set theory, Dubois and Prade [4] formulated rough fuzzy sets and fuzzy rough sets, they constructed a pair of lower and upper approximation operators for fuzzy sets with respect to fuzzy similarity relation by using the t-norm Min and its dual conorm Max. By using a residual implication (for short, R-implication) to define the lower approximation operator, Morsi and Yakout [19] generalized the fuzzy

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rough sets in the sense of Dubois and Prade. Later, Radzikowska and Kerre [30] proposed a more general approach to the fuzzification of a rough set. This approach is based on a border implication $\mathcal{I}$ (not necessarily a R-implication) and a triangular norm $\mathcal{T}$. In the axiomatic approaches, a set of axioms is used to characterize the approximations. Lin and Liu [14] proposed six axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. Under these axioms, there exists an equivalence relation such that the lower and upper approximations are the same as the abstract operators. The most important axiomatic studies for crisp rough sets were made by Yao [41-43]. Recently, the research of the axiomatic approach has also been extended to approximation operators in the fuzzy environment [15,18,19,31,37,39].

In some problems with uncertainty in the theory of probabilistic metric spaces, fuzzy logics and fuzzy measures, the pseudo-operations such as pseudo-additions and pseudo-multiplications are usually used [7,11,24]. Pseudo-analysis [7,8,10,11,22-26,35] has been applied in different fields, e.g., measure theory, integration, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc. Interestingly, by using the Aczel's theorem [1], the pseudo-additions and pseudo-multiplications could be transferred into the corresponding results of reals such as the addition operator and multiplication operator. This can bring us the convenience of calculation.

We note that there are some literatures about pseudo integrals [7,8,10,25,35], but little literatures about rough set model based on pseudo-operations. In order to present the rough set model based on pseudo-operations, a general framework for the study of fuzzy rough approximation operators based on pseudo-operations are studied by Shi and Gong[31]. In [31], by using the pseudo-operations, the pseudo-lower and pseudo-upper approximation operators are defined. Meanwhile, some properties of the proposed pseudo fuzzy rough approximation operators are investigated. Compared with the previous rough set models based on triangular norms [18,19,30,39], the pseudo-generalized fuzzy rough sets[31] have its advantages to calculate its lower and upper approximations conveniently.

In recent years, various similarity measure between generalized fuzzy sets are given[2,3,17,20]. It is necessary to measure the similarity between two pseudo-generalized fuzzy rough sets in some practical cases, such as pattern recognition, image processing and fuzzy reasoning. In this paper, we will present a novel similarity measure between two pseudo-generalized fuzzy rough sets. We show that the similarity measure between two pseudo-generalized fuzzy rough sets can be given according to the pseudo-operation.

The remainder of this paper is organized as follows. In section 2, we recall some basic concepts of rough sets, fuzzy sets, fuzzy relation and pseudo-operations. In section 3, the pseudo-generalized fuzzy rough sets are presented. Some properties of the proposed pseudo fuzzy rough approximation operators are also investigated in this section. In Section 4, the similarity measure between pseudo-generalized fuzzy rough sets is proposed. Section 5 presents conclusions.

2. Preliminaries

2.1 Pawlak rough sets

In traditional Pawlak rough set theory, the pair (U, R) is called an approximation space (it is also called Pawlak approximation space), where U is a finite and non-empty set called the universe and R is an equivalence relation on U, i.e., R is reflexive, symmetrical and transitive. The relation R decomposes the set U into a disjoint class in such a way that two elements x and y are in the same class iff $(x, y) \in R$.

Suppose R is an equivalence relation on U. With respect to R, we can define an equivalence class of an element x in U as follows:

$$[x]_R = \{y|\ (x, y) \in R\}.$$

The quotient set of U by the relation R is denoted by U/R, and

$$U/R = \{X_1, X_2, \cdots, X_m\}.$$

where X_i $(i = 1, 2, \cdots, m)$ is an equivalence class of R.

Given an arbitrary set $X \subseteq U$, it may not be possible to describe X precisely in the approximation space (U, R). One may characterize X by a pair of lower and upper approximations defined as follows:

$\underline{R}X = \{x \in U|\ [x]_R \subseteq X\} = \bigcup\{Y \in U/R|\ Y \subseteq X\}$;

$\overline{R}X = \{x \in U|\ [x]_R \cap X \neq \emptyset\} = \bigcup\{Y \in U/R|\ Y \cap X \neq \emptyset\}$.

The pair $(\underline{R}X, \overline{R}X)$ is referred to as a rough set of X.

2.2 Fuzzy sets

Let U be a universe. Fuzzy set A is a mapping from U into the unit interval $[0, 1]$:

$$A : U \to [0, 1],$$

where for each $x \in U$, we call $A(x)$ the membership degree of x in A.

If $U = \{x_1, x_2, \cdots, x_n\}$, then the fuzzy set A on U can be expressed by $\sum\limits_{i=1}^{n} A(x_i)/x_i$. Additionally, the fuzzy power set, i.e., the set of all fuzzy sets in the universe U is denoted by $\mathcal{F}(U)$ [44].

For fuzzy sets $A, B \in \mathcal{F}(U)$,

$A \subseteq B \Leftrightarrow A(x) \leq B(x)$;

$(A \cap B)(x) = A(x) \wedge B(x) = \min\{A(x), B(x)\}$;

$(A \cup B)(x) = A(x) \vee B(x) = \max\{A(x), B(x)\}$;

$(\sim A)(x) = 1 - A(x)$, where $\sim A$ is the complement of A.

2.3 Fuzzy relation

Let U and W be two nonempty sets. The Cartesian product of U and W is denoted by $U \times W$. A fuzzy relation R from U to W is a fuzzy subset of $U \times W$, i.e., $R \in \mathcal{F}(U \times W)$, and $R(x, y)$ is called the degree of relation between x and y. In particular, if $U = W$, we call R a fuzzy relation on U. Usually, a fuzzy relation can be expressed by a fuzzy matrix.

2.4 Pseudo-operations

Throughout this paper, we only consider the case of pseudo-addition and present the fuzzy generalized rough sets using pseudo-addition. For the case of pseudo-multiplication, the discussion can be given similarly.

Definition 2.1 An operation $\oplus : [0, \infty]^2 \to [0, \infty]$ is called a pseudo-addition if it satisfies the following axioms:

(1) Associativity: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in [0, \infty]$.

(2) Monotonicity: $a \oplus b \le c \oplus d$ whenever $0 \le a \le c \le \infty, 0 \le b \le d \le \infty$.

(3) 0 is neutral element: $a \oplus 0 = 0 \otimes a = a$ for all $a \in [0, \infty]$.

(4) Continuity: for any sequences $(a_n)_{n \in N}$, $(b_n)_{n \in N}$ in $[0, \infty]^N$ such that $\lim\limits_{n \to \infty} a_n = a$ and $\lim\limits_{n \to \infty} b_n = b$ it holds $\lim\limits_{n \to \infty} a_n \oplus b_n = a \oplus b$.

From [11], we know that each pseudo-addition is also commutative, i.e., it satisfies

(5) Commutativity: $a \oplus b = b \oplus a$ for all $a, b \in [0, \infty]$.

Lemma 2.1 (Aczel's theorem) Let g be a positive strictly monotone function defined on $[a, b] \subseteq (-\infty, +\infty)$ such that $0 \in Ran(g)$. The generalized generated pseudo-addition $\oplus$ and the generalized generated pseudo-multiplication $\odot$ are given by

$$x \oplus y = g^{-1}(g(x) + g(y)),$$

$$x \odot y = g^{-1}(g(x)g(y)),$$

where g^{-1} is pseudo-inverse function for function g: $g^{-1}(y) = \sup\{x \in [a, b] | g(x) < y\}$ if g is a non-decreasing function and $g^{-1}(y) = \sup\{x \in [a, b] | g(x) > y\}$ if g is a non-increasing function.

Example 2.1 Suppose that $g(x) = 1 - x$ $(x \in [0, 1])$, then its pseudo-inverse is

$$g^{-1}(x) = \begin{cases} 1 - x, & x \in [0, 1], \\ 0, & x \in [1, +\infty). \end{cases}$$

And $x \oplus y = g^{-1}(g(x) + g(y)) = \max\{0, x + y - 1\}$, this is Lukasiewicz t-norm.

3. Construction of pseudo fuzzy rough approximation operators

Definition 3.1 Let (U, W, R) be a fuzzy approximation space, where U and W are two nonempty sets, R is a fuzzy relation from U to W. $g : [0, 1] \to [0, +\infty)$ is a strictly decreasing function such that $g(1) = 0$ and $g(x) + g(y) \in Ran(g) \cup [g(0^+), +\infty)$ for all $(x, y) \in [0, 1]^2$. Then for any $A \in \mathcal{F}(W)$, the pseudo-lower approximation $\underline{R}_{\oplus}(A)$ and the pseudo-upper approximation $\overline{R}_{\oplus}(A)$ of A are defined as follows, respectively:

$\underline{R}_{\oplus}(A)(x) = \bigwedge\limits_{y\in W}\{1-R(x,y)\oplus(1-A(y))\} = \bigwedge\limits_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\}, x \in U$;

$\overline{R}_{\oplus}(A)(x) = \bigvee\limits_{y\in W}\{R(x,y)\oplus A(y)\} = \bigvee\limits_{y\in W}\{g^{-1}(g(R(x,y))+g(A(y)))\}, x \in U.$

The pair $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ is called a pseudo-generalized fuzzy rough set. $\underline{R}_{\oplus}$ and $\overline{R}_{\oplus}$ are referred to as the pseudo-lower and pseudo-upper fuzzy rough approximation operators, respectively.

Remark 3.1 If R is a crisp binary relation from U to W, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [37]. That is, for every $A \in \mathcal{F}(W),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y)|y \in R_s(x)\},\ \ \underline{R}_{\oplus}(A)(x) = \inf\{A(y)|y \in R_s(x)\},$$

where $R_s(x) = \{y \in W|(x,y) \in R\}$.

In fact,

$$\begin{aligned}
&\overline{R}_{\oplus}(A)(x)\\
&= \bigvee_{y\in W}\{g^{-1}(g(R(x,y))+g(A(y)))\}\\
&= \sup\{g^{-1}(g(1)+g(A(y)))|y \in R_s(x)\} \bigvee \sup\{g^{-1}(g(0)+g(A(y)))|y \notin R_s(x)\}\\
&= \sup\{g^{-1}(g(1)+g(A(y)))|y \in R_s(x)\}\\
&= \sup\{g^{-1}(0+g(A(y)))|y \in R_s(x)\}\\
&= \sup\{A(y)|y \in R_s(x)\},
\end{aligned}$$

$$\begin{aligned}
&\underline{R}_{\oplus}(A)(x)\\
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\}\\
&= \inf\{1-g^{-1}(g(1)+g(1-A(y)))|y \in R_s(x)\} \bigwedge \inf\{1-g^{-1}(g(0)+g(1-A(y)))|y \notin R_s(x)\}\\
&= \inf\{1-g^{-1}(g(1)+g(1-A(y)))|y \in R_s(x)\}\\
&= \inf\{1-g^{-1}(0+g(1-A(y)))|y \in R_s(x)\}\\
&= \inf\{A(y)|y \in R_s(x)\}.
\end{aligned}$$

Remark 3.2 If R is a crisp binary relation on U and A is a crisp set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [43]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U|R_s(x) \cap A \neq \phi\},\ \ \underline{R}_{\oplus}(A) = \{x \in U|R_s(x) \subseteq A\}.$$

where $R_s(x) = \{y \in U|(x,y) \in R\}$.

In fact, by Remark 3.2, we know that if $A \in P(U)$ then for any $x \in U$,

$x \in \overline{R}_{\oplus}(A) \Leftrightarrow \overline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow \exists\ y \in R_s(x)$ such that $A(y) = 1$, i.e., $y \in A \Leftrightarrow R_s(x) \cap A \neq \phi$,

$x \in \underline{R}_{\oplus}(A) \Leftrightarrow \underline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow A(y) = 1$ for every $y \in R_s(x)$, i.e., $y \in A \Leftrightarrow R_s(x) \subseteq A$.

Remark 3.3 If R is a crisp equivalence relation on U and A is a fuzzy set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [4]. That is, for every $A \in \mathcal{F}(U),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y)|y \in [x]_R\},\quad \underline{R}_{\oplus}(A)(x) = \inf\{A(y)|y \in [x]_R\}.$$

In fact, if R is a crisp equivalence relation on U, then $R_s(x) = [x]_R$.

Remark 3.4 If R is a crisp equivalence relation on U and A is a crisp set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [27]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U|[x]_R \cap A \neq \phi\},\quad \underline{R}_{\oplus}(A) = \{x \in U|[x]_R \subseteq A\}.$$

Theorem 3.1 Let R be a fuzzy relation from U to W. Then the pseudo-lower fuzzy rough approximation operator $\underline{R}_{\oplus}$ and the pseudo-upper fuzzy rough approximation operator $\overline{R}_{\oplus}$ satisfy the following properties: for any $A, B \in \mathcal{F}(W)$, $x \in U$, $y \in W$,

(1) $\underline{R}_{\oplus}(A) = \sim \overline{R}_{\oplus}(\sim A),\ \overline{R}_{\oplus}(A) = \sim \underline{R}_{\oplus}(\sim A)$;

(2) $\underline{R}_{\oplus}(W) = U,\ \overline{R}_{\oplus}(\phi) = \phi$;

(3) $\underline{R}_{\oplus}(A \cap B) = \underline{R}_{\oplus}(A) \cap \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cup B) = \overline{R}_{\oplus}(A) \cup \overline{R}_{\oplus}(B)$;

(4) $A \subseteq B \Rightarrow \underline{R}_{\oplus}(A) \subseteq \underline{R}_{\oplus}(B),\ A \subseteq B \Rightarrow \overline{R}_{\oplus}(A) \subseteq \overline{R}_{\oplus}(B)$;

(5) $\underline{R}_{\oplus}(A \cup B) \supseteq \underline{R}_{\oplus}(A) \cup \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cap B) \subseteq \overline{R}_{\oplus}(A) \cap \overline{R}_{\oplus}(B)$.

Proof

(1)
$$\begin{aligned}
\overline{R}_{\oplus}(\sim A)(x) &= \bigvee_{y \in W} \{g^{-1}(g(R(x,y)) + g(1 - A(y)))\} \\
&= 1 - \bigwedge_{y \in W} \{1 - g^{-1}(g(R(x,y)) + g(1 - A(y)))\} \\
&= 1 - \underline{R}_{\oplus}(A)(x) \\
&= \sim \underline{R}_{\oplus}(A)(x).
\end{aligned}$$

It follows that $\underline{R}_{\oplus}(A) = \sim \overline{R}_{\oplus}(\sim A)$.

Similarly, $\overline{R}_{\oplus}(A) = \sim \underline{R}_{\oplus}(\sim A)$ can be verified.

(2)
$$\begin{aligned}
\underline{R}_{\oplus}(W)(x) &= \bigwedge_{y \in W} \{1 - g^{-1}(g(R(x,y)) + g(1 - W(y)))\} \\
&= \bigwedge_{y \in W} \{1 - g^{-1}(g(R(x,y) + 1))\} \\
&= 1.
\end{aligned}$$

Therefore, $\underline{R}_{\oplus}(W) = U$.

$\overline{R}_{\oplus}(\phi) = \phi$ can be verified in a similar way.

(3)

$$\begin{aligned}
\underline{R}_{\oplus}(A\cap B)(x) &= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-\min\{A(y),B(y)\}))\}\\
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+\min\{g(A(y)),g(B(y))\})\}\\
&= \bigwedge_{y\in W}\{1-g^{-1}(\min\{g(R(x,y)+g(A(y))),g(R(x,y)+g(B(y)))\})\}\\
&= \min\{\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(A(y))),\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))\\
&\quad +g(B(y)))\}\\
&= \min\{\underline{R}_{\oplus}(A)(x),\underline{R}_{\oplus}(B)(x)\}.
\end{aligned}$$

That is, $\underline{R}_{\oplus}(A\cap B)=\underline{R}_{\oplus}(A)\cap\underline{R}_{\oplus}(B)$.
Similarly, $\overline{R}_{\oplus}(A\cup B)=\overline{R}_{\oplus}(A)\cup\overline{R}_{\oplus}(B)$ is also hold.
(4) $A\subseteq B\Leftrightarrow A(y)\leq B(y)\Leftrightarrow 1-A(y)\geq 1-B(y)$, it implies that

$$\begin{aligned}
\underline{R}_{\oplus}(A)(x) &= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\}\\
&\leq \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-B(y)))\}\\
&= \underline{R}_{\oplus}(B)(x).
\end{aligned}$$

That is, $A\subseteq B\Rightarrow\underline{R}_{\oplus}(A)\subseteq\underline{R}_{\oplus}(B)$. Similarly, $A\subseteq B\Rightarrow\overline{R}_{\oplus}(A)\subseteq\overline{R}_{\oplus}(B)$.
(5) $\underline{R}_{\oplus}(A\cup B)(x)$

$$\begin{aligned}
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-\max\{A(y),B(y)\}))\}\\
&= \bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+\max\{g(1-A(y)),g(1-B(y))\})\}\\
&\geq \max\{\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-A(y)))\},\bigwedge_{y\in W}\{1-g^{-1}(g(R(x,y))+g(1-B(y)))\}\}\\
&= \max\{\underline{R}_{\oplus}(A)(x),\underline{R}_{\oplus}(B)(x)\}
\end{aligned}$$

Thus $\underline{R}_{\oplus}(A\cup B)\supseteq\underline{R}_{\oplus}(A)\cup\underline{R}_{\oplus}(B)$. Similarly, $\overline{R}_{\oplus}(A\cap B)\subseteq\overline{R}_{\oplus}(A)\cap\overline{R}_{\oplus}(B)$. □

4. Similarity measure between pseudo-generalized fuzzy rough sets

It is necessary to measure the similarity between two pseudo-generalized fuzzy rough sets in some practical cases, such as pattern recognition, image processing and fuzzy reasoning. In this section, we will show that in a fuzzy approximation space, similarity measure between two pseudo-generalized fuzzy rough sets can be given according to the pseudo-operation.

Let (U,R) be a fuzzy approximation space, where R is a fuzzy relation on U. Suppose there are two pseudo-generalized fuzzy rough sets $(\underline{R}_{\oplus}(A),\overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B),\overline{R}_{\oplus}(B))$.

Definition 4.1 Let U be a universe of discourse. A real function $D : \mathcal{F}(U)\times\mathcal{F}(U) \to [0,1]$ is called an inclusion degree on $\mathcal{F}(U)$ if for any $A, B, C \in \mathcal{F}(U)$, D satisfies the following properties:

(1)$0 \leq D(B/A) \leq 1$;

(2)$A \subseteq B \Rightarrow D(B/A) = 1$;

(3)$A \subseteq B \subseteq C \Rightarrow D(A/C) \leq D(A/B)$.

In particular, let $(X, \leq)$ be a partially ordered set, a real function $D : X\times X \to [0,1]$ is called an inclusion degree on X if for any $x, y, z \in X$, D satisfies the following properties:

(1)$0 \leq D(y/x) \leq 1$;

(2)$x \leq y \Rightarrow D(y/x) = 1$;

(3)$x \leq y \leq z \Rightarrow D(x/z) \leq D(x/y)$.

Theorem 4.1 Let g be a strictly decreasing function on $[0,1]$ such that $g(1) = 0$. For any $a, b \in [0,1]$, we define

$$\theta^{'}(b/a) = \sup\{c \in [0,1] |\ a \oplus c \leq b\}.$$

Then $\theta^{'}$ is an inclusion degree on $[0,1]$.

Proof

It follows immediately from Definition 4.1. □

Theorem 4.2 Let (U, R) be a fuzzy approximation space. For any $A, B \in \mathcal{F}(U)$, $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B), \overline{R}_{\oplus}(B))$ are two pseudo-generalized fuzzy rough sets on U. Then

$$\underline{\theta}(B/A) = \frac{1}{n}\sum_{i=1}^{n}\theta^{'}(\underline{R}_{\oplus}(B)(x_i)/\underline{R}_{\oplus}(A)(x_i)) \qquad (4.1)$$

and

$$\overline{\theta}(B/A) = \frac{1}{n}\sum_{i=1}^{n}\theta^{'}(\overline{R}_{\oplus}(B)(x_i)/\overline{R}_{\oplus}(A)(x_i)) \qquad (4.2)$$

are inclusion degree on $\mathcal{F}(U)$.

Proof

We need only to prove that $\underline{\theta}$ determined by formula (4.1) is an inclusion degree on $\mathcal{F}(U)$.

(1) By the definition of $\theta^{'}$, $0 \leq \theta^{'}(\underline{R}_{\oplus}(B)(x_i)/\underline{R}_{\oplus}(A)(x_i)) \leq 1$ is obvious. Therefore

$$0 \leq \underline{\theta}(B/A) \leq 1.$$

(2) If $A \subseteq B$, by Theorem 3.1, we know that

$$\underline{R}_{\oplus}(A) \subseteq \underline{R}_{\oplus}(B),$$

i.e.,

$$\underline{R}_{\oplus}(A)(x) \leq \underline{R}_{\oplus}(B)(x),\ x \in U.$$

Thus,

$$\theta^{'}(\underline{R}_{\oplus}(B)(x_i)/\underline{R}_{\oplus}(A)(x_i)) = 1.$$

Therefore $\underline{\theta}(B/A)=1$.

(3) If $A\subseteq B\subseteq C$ $(A,B,C\in\mathcal{F}(U))$, then by Theorem 3.1, $\underline{R}_{\oplus}(A)\subseteq\underline{R}_{\oplus}(B)\subseteq\underline{R}_{\oplus}(C)$, i.e.,

$$\underline{R}_{\oplus}(A)(x)\leq\underline{R}_{\oplus}(B)(x)\leq\underline{R}_{\oplus}(C)(x)$$

for every $x\in U$.

Thus, we can obtain $\underline{\theta}(A/C)\leq\underline{\theta}(A/B)$. $\square$

Definition 4.2 A real function $S:\mathcal{F}(U)\times\mathcal{F}(U)\to[0,1]$ is called a similarity measure on $\mathcal{F}(U)$ if for any $A,B,C\in\mathcal{F}(U)$, S satisfies the following properties:

(1) $0\leq S(A,B)\leq 1, S(A,A)=1$;

(2) $S(A,B)=S(B,A)$;

(3) $A\subseteq B\subseteq C\Rightarrow S(A,C)\leq S(A,B)$.

Theorem 4.3 Let (U,R) be a fuzzy approximation space. For any $A,B\in\mathcal{F}(U)$, $(\underline{R}_{\oplus}(A),\overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B),\overline{R}_{\oplus}(B))$ are two pseudo-generalized fuzzy rough sets on U. Then

$$S(A,B)=\frac{1}{2}[\underline{\theta}(B/A)\oplus\underline{\theta}(A/B)+\overline{\theta}(B/A)\oplus\overline{\theta}(A/B)]$$

is a similarity measure between $(\underline{R}_{\oplus}(A),\overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B),\overline{R}_{\oplus}(B))$, where $x\oplus y=g^{-1}(g(x)+g(y))$ and $g:[0,1]\to[0,+\infty)$ is a strictly decreasing function such that $g(1)=0$.

Proof

(1) By $g^{-1}:[0,+\infty)\to[0,1]$, we have

$$0\leq\underline{\theta}(B/A)\oplus\underline{\theta}(A/B)\leq 1,$$

$$0\leq\overline{\theta}(B/A)\oplus\overline{\theta}(A/B)\leq 1.$$

Thus, $0\leq S(A,B)\leq 1$. And by $\underline{\theta}(A/A)=1$ and $\overline{\theta}(A/A)=1$, we get $S(A,A)=1$.

(2) By $x\oplus y=y\oplus x$, we have $S(A,B)=S(B,A)$.

(3) If $A\subseteq B\subseteq C$ $(A,B,C\in\mathcal{F}(U))$, by $\underline{\theta}$ and $\overline{\theta}$ are inclusion degree on $\mathcal{F}(U)$, we obtain that

$$\underline{\theta}(A/C)\leq\underline{\theta}(A/B),$$
$$\overline{\theta}(A/C)\leq\overline{\theta}(A/B).$$

On the other hand,

$$\begin{aligned}S(A,C)&=\frac{1}{2}[\underline{\theta}(C/A)\oplus\underline{\theta}(A/C)+\overline{\theta}(C/A)\oplus\overline{\theta}(A/C)]\\&=\frac{1}{2}[1\oplus\underline{\theta}(A/C)+1\oplus\overline{\theta}(A/C)]\\&=\frac{1}{2}[\underline{\theta}(A/C)+\overline{\theta}(A/C)],\end{aligned}$$

$$\begin{aligned}
S(A,B) &= \frac{1}{2}[\underline{\theta}(B/A)\oplus\underline{\theta}(A/B)+\overline{\theta}(B/A)\oplus\overline{\theta}(A/B)] \\
&= \frac{1}{2}[1\oplus\underline{\theta}(A/B)+1\oplus\overline{\theta}(A/B)] \\
&= \frac{1}{2}[\underline{\theta}(A/B)+\overline{\theta}(A/B)].
\end{aligned}$$

Hence $S(A,C)\le S(A,B)$.

This completes the proof. □

Example 4.1 Let $U=\{x_1,x_2,x_3\}$ be a universe of discourse, R be a fuzzy relation on U (see Table 1).

Table 1: A fuzzy relation on U

U	x_1	x_2	x_3
x_1	1	0.4	0.6
x_2	0.4	1	0.7
x_3	0.6	0.7	1

Suppose that

$$A=0.3/x_1+0.4/x_2+0.8/x_3,$$

$$B=0.2/x_1+0.7/x_2+0.8/x_3,$$

and

$$g(x)=1-x\ (x\in[0,1]).$$

Then the pseudo-lower and pseudo-upper approximations of A and B can be computed as follows:

In one hand,

$\underline{R}_{\oplus}(A)(x_1)=\min\{1-g^{-1}(0+0.3),1-g^{-1}(0.6+0.4),1-g^{-1}(0.4+0.8)\}=0.3;$

$\underline{R}_{\oplus}(A)(x_2)=\min\{1-g^{-1}(0.6+0.3),1-g^{-1}(0+0.4),1-g^{-1}(0.3+0.8)\}=0.4;$

$\underline{R}_{\oplus}(A)(x_3)=\min\{1-g^{-1}(0.4+0.3),1-g^{-1}(0.3+0.4),1-g^{-1}(0+0.8)\}=0.7;$

$\overline{R}_{\oplus}(A)(x_1)=\max\{g^{-1}(0+0.7),g^{-1}(0.6+0.6),g^{-1}(0.4+0.2)\}=0.4;$

$\overline{R}_{\oplus}(A)(x_2)=\max\{g^{-1}(0.6+0.7),g^{-1}(0+0.6),g^{-1}(0.3+0.2)\}=0.5;$

$\overline{R}_{\oplus}(A)(x_3)=\max\{g^{-1}(0.4+0.7),g^{-1}(0.3+0.6),g^{-1}(0+0.2)\}=0.8.$

That is,

$$\underline{R}_{\oplus}(A)=0.3/x_1+0.4/x_2+0.7/x_3,$$

$$\overline{R}_{\oplus}(A)=0.4/x_1+0.5/x_2+0.8/x_3.$$

On the other hand,

$\underline{R}_{\oplus}(B)(x_1)=\min\{1-g^{-1}(0+0.2),1-g^{-1}(0.6+0.7),1-g^{-1}(0.4+0.8)\}=0.2;$

$\underline{R}_{\oplus}(B)(x_2)=\min\{1-g^{-1}(0.6+0.2),1-g^{-1}(0+0.7),1-g^{-1}(0.3+0.8)\}=0.7;$

$\underline{R}_{\oplus}(B)(x_3)=\min\{1-g^{-1}(0.4+0.2),1-g^{-1}(0.3+0.7),1-g^{-1}(0+0.8)\}=0.6;$

$\overline{R}_{\oplus}(B)(x_1)=\max\{g^{-1}(0+0.8),g^{-1}(0.6+0.3),g^{-1}(0.4+0.2)\}=0.4;$

$\overline{R}_{\oplus}(B)(x_2) = \max\{g^{-1}(0.6+0.8), g^{-1}(0+0.3), g^{-1}(0.3+0.2)\} = 0.7;$
$\overline{R}_{\oplus}(B)(x_3) = \max\{g^{-1}(0.4+0.8), g^{-1}(0.3+0.3), g^{-1}(0+0.2)\} = 0.8.$
That is,

$$\underline{R}_{\oplus}(B) = 0.2/x_1 + 0.7/x_2 + 0.6/x_3,$$

$$\overline{R}_{\oplus}(B) = 0.4/x_1 + 0.7/x_2 + 0.8/x_3.$$

Since $g(x) = 1 - x$, so $\theta^{'}(b/a) = \sup\{c \in [0,1] |\ a \oplus c \leq b\} = 1 \wedge (1 - a + b)$.
Therefore

$$\underline{\theta}(B/A) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\underline{R}_{\oplus}(B)(x_i)/\underline{R}_{\oplus}(A)(x_i)) = \frac{1}{3}(0.9+1+0.9) = \frac{28}{30},$$

$$\underline{\theta}(A/B) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\underline{R}_{\oplus}(A)(x_i)/\underline{R}_{\oplus}(B)(x_i)) = \frac{1}{3}(1+0.7+1) = \frac{27}{30},$$

$$\overline{\theta}(B/A) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\overline{R}_{\oplus}(B)(x_i)/\overline{R}_{\oplus}(A)(x_i)) = \frac{1}{3}(1+1+1) = 1,$$

$$\overline{\theta}(A/B) = \frac{1}{3}\sum_{i=1}^{3}\theta^{'}(\overline{R}_{\oplus}(A)(x_i)/\overline{R}_{\oplus}(B)(x_i)) = \frac{1}{3}(1+0.8+1) = \frac{28}{30},$$

and

$$\underline{\theta}(B/A) \oplus \underline{\theta}(A/B) = g^{-1}[g(\underline{\theta}(B/A)) + g(\underline{\theta}(A/B))] = g^{-1}[1 - \frac{28}{30} + 1 - \frac{27}{30}] = \frac{5}{6},$$

$$\overline{\theta}(B/A) \oplus \overline{\theta}(A/B) = g^{-1}[g(\overline{\theta}(B/A)) + g(\overline{\theta}(A/B))] = g^{-1}[1 - 1 + 1 - \frac{28}{30}] = \frac{14}{15}.$$

Thus, the similarity measure between $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ and $(\underline{R}_{\oplus}(B), \overline{R}_{\oplus}(B))$ can be given as follows:

$$S(A,B) = \frac{1}{2}[\underline{\theta}(B/A) \oplus \underline{\theta}(A/B) + \overline{\theta}(B/A) \oplus \overline{\theta}(A/B)] = \frac{1}{2}(\frac{5}{6} + \frac{14}{15}) = \frac{53}{60}.$$

5. Conclusions

It is interesting to combine pseudo-operations and rough set in order to expand the application domain of pseudo-analysis and rough set. In this paper, we presented a generalized fuzzy rough set model based on pseudo-operation, constructed pseudo fuzzy rough approximation operations. Because it is necessary to measure the similarity between two fuzzy rough sets in some practical cases, using the pseudo-operations, the similarity measure between pseudo-generalized fuzzy rough sets are given in this paper. The results of this paper may be applied to some practical problems about pattern recognition or fuzzy reasoning.

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FOURIER SERIES OF FUNCTIONS INVOLVING HIGHER-ORDER EULER POLYNOMIALS

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ABSTRACT. In this paper, we consider three types of functions involving higher-order Euler polynomials and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

1. Introduction

For each positive integer r, Euler polynomials $E_m^{(r)}(x)$ of order r are given by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{m=0}^{\infty} E_m^{(r)}(x)\frac{t^m}{m!}, \quad \text{(see } [2-4, 11-13, 17, 19]), \tag{1.1}$$

When $x = 0$, $E_m^{(r)} = E_m^{(r)}(0)$ are called Euler numbers of order r. For $r = 1$, $E_m(x) = E_m^{(1)}(x)$ and $E_m = E_m^{(1)}$ are called Euler polynomials and numbers, respectively. From (1.1), we see that

$$\begin{aligned} &\frac{d}{dx}E_m^{(r)}(x) = mE_{m-1}^{(r)}(x),\ (m \geq 0),\\ &E_m^{(r)}(x+1) + E_m^{(r)}(x) = 2E_m^{(r-1)}(x),\ (m \geq 0). \end{aligned} \tag{1.2}$$

In turn, these imply that

$$E_m^{(r)}(1) = 2E_m^{(r-1)} - E_m^{(r)}, \quad (m \geq 0). \tag{1.3}$$

and

$$\int_0^1 E_m^{(r)}(x)dx = \frac{2}{m+1}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)}), \quad (m \geq 0). \tag{1.4}$$

For any real number x, we let $< x >= x - [x] \in [0, 1)$ denote the fractional part of x.

The Bernoulli polynomials $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}, \quad \text{(see } [2-4, 11, 17]). \tag{1.5}$$

We will need the following facts about Bernoulli functions $B_m(< x >)$ for later use:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \tag{1.6}$$

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(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(<x>), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \tag{1.7}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$.

In this paper, we will consider the following three types of functions $\alpha_m(<x>), \beta_m(<x>)$, and $\gamma_m(<x>)$ involving higher-order Euler polynomials and derive their Fourier series expansions. Further, we will express each of them in terms of Bernoulli functions:

(1) $\alpha_m(<x>) = \sum_{k=0}^{m} E_k^{(r)}(<x>) <x>^{m-k}, (m \geq 1)$;
(2) $\beta_m(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}, (m \geq 1)$;
(3) $\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>) <x>^{m-k}, (m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1,16,20]). As to $\gamma_m(<x>)$, we note that the polynomial identity (1.8) follows immediately from Theorems 4.1 and 4.2 which is in turn derived from the Fourier series expansion of $\gamma_m(<x>)$.

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(x) x^{m-k} \\ &= \frac{1}{m} \sum_{s=0}^{m} \binom{m}{s} \left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} \left(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \right) \right) B_s(x), \end{aligned} \tag{1.8}$$

where $\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)} (2E_k^{(r-1)} - E_k^{(r)})$, for $l \geq 2$, with $\Lambda_1 = 0$, and $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers. The obvious polynomial identities can be derived also for $\alpha_m(<x>)$ and $\beta_m(<x>)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(<x>) B_{m-k}(<x>)$ we can derive the Faber-Pandharipande-Zagier identity (see [6-9]) and the Miki's identity (see [5,7-9,18]). For recent related works, we refer the reader to [10,14,15].

2. Fourier series of functions of the first type involving higher-order Euler polynomials

In this section, we will study the Fourier series of functions of the first type involving higher-order Euler polynomials. Let $\alpha_m(x) = \sum_{k=0}^{m} E_k^{(r)}(x) x^{m-k}, \ (m \geq 1)$. Then we will consider the function

$$\alpha_m(<x>) = \sum_{k=0}^{m} E_k^{(r)}(<x>) <x>^{m-k}, \ (m \geq 1). \tag{2.1}$$

defined on $\mathbb{R}$ which is periodic with period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(<x>) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{2.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned}\alpha'_m(x) &= \sum_{k=0}^{m}\{kE_{k-1}^{(r)}(x)x^{m-k} + (m-k)E_k^{(r)}(x)x^{m-k-1}\}\\ &= \sum_{k=1}^{m} kE_{k-1}^{(r)}(x)x^{m-k} + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)x^{m-k-1}\\ &= \sum_{k=0}^{m-1}(k+1)E_k^{(r)}(x)x^{m-1-k} + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)x^{m-1-k}\\ &= (m+1)\sum_{k=0}^{m-1}E_k^{(r)}(x)x^{m-1-k}\\ &= (m+1)\alpha_{m-1}(x).\end{aligned} \tag{2.4}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{2.6}$$

For $m \geq 1$, we set

$$\begin{aligned}\Delta_m &= \alpha_m(1) - \alpha_m(0)\\ &= \sum_{k=0}^{m}\left(E_k^{(r)}(1) - E_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=0}^{m}\left(2E_k^{(r-1)} - E_k^{(r)} - E_k^{(r)}\delta_{m,k}\right)\\ &= \sum_{k=0}^{m}(2E_k^{(r-1)} - E_k^{(r)}) - E_m^{(r)}.\end{aligned} \tag{2.7}$$

We now note that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0, \tag{2.8}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

We are now ready to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m,
\end{aligned} \tag{2.10}$$

from which by induction we can deduce

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}. \tag{2.11}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = -\frac{1}{m+2}\Delta_{m+1}. \tag{2.12}$$

$\alpha_m(< x >)$, $(m \geq 1)$ is piecewise C^{∞}. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers m with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m \neq 0$.

Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^{∞}, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}
&\alpha_m(< x >) \\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right) e^{2\pi i n x} \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1} B_j(< x >) \\
&\quad + \Delta_m \times \begin{cases} B_1(< x >), & \text{for } \; x \in \mathbb{Z}^c, \\ 0, & \text{for } \; x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{2.13}$$

Now, we can state our first result.

Theorem 2.1. *For each positive integer l, we put*

$$\Delta_l = \sum_{k=0}^{l}(2E_k^{(r-1)} - E_k^{(r)}) - E_l^{(r)}. \tag{2.14}$$

Assume that $\Delta_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=1}^{m} E_k^{(r)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} E_k^{(r)}(<x>) <x>^{m-k}\\&=\frac{1}{m+2}\Delta_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned}\tag{2.15}$$

for all $x\in\mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=0}^{m} E_k^{(r)}(<x>) <x>^{m-k}\\&=\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>),\end{aligned}\tag{2.16}$$

for all $x\in\mathbb{R}$, where $B_j(<x>)$ is the Bernoulli function.

Assume next that $\Delta_m\neq 0$, for a positive integer m. Then $\alpha_m(1)\neq\alpha_m(0)$. Hence $\alpha_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x\in\mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0)+\alpha_m(1))=\alpha_m(0)+\frac{1}{2}\Delta_m,\tag{2.17}$$

for $x\in\mathbb{Z}$. We can now state our second result.

Theorem 2.2. *For each positive inetger l, we set*

$$\Delta_l=\sum_{k=0}^{l}(2E_k^{(r-1)}-E_k^{(r)})-E_l^{(r)}.\tag{2.18}$$

Assume that $\Delta_m\neq 0$, for a positive integer m, Then we have the following.

(a)

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}E_k^{(r)}(<x>) <x>^{m-k}, & \text{for } \ x\in\mathbb{Z}^c,\\ E_m^{(r)}+\frac{1}{2}\Delta_m, & \text{for } \ x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{2.19}$$

(b)

$$\begin{aligned}&\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=\sum_{k=0}^{m}E_k^{(r)}(<x>) <x>^{m-k}, x\in\mathbb{Z}^c;\\&\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=E_m^{(r)}+\frac{1}{2}\Delta_m,\ \text{ for } x\in\mathbb{Z}.\end{aligned}\tag{2.20}$$

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3. Fourier series of functions of the second type involving higher-order Euler polynomials

Let $\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(x) x^{m-k}$, $(m \geq 1)$. Then we will consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) < x >^{m-k}, \tag{3.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{3.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned} \beta_m'(x) &= \sum_{k=0}^{m} \left\{ \frac{k}{k!(m-k)!} E_{k-1}^{(r)}(x) x^{m-k} + \frac{m-k}{k!(m-k)!} E_k^{(r)}(x) x^{m-k-1} \right\} \\ &= \sum_{k=1}^{m} \frac{1}{(k-1)!(m-k)!} E_{k-1}^{(r)}(x) x^{m-k} + \sum_{k=0}^{m-1} \frac{1}{k!(m-k-1)!} E_k^{(r)}(x) x^{m-k-1} \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k^{(r)}(x) x^{m-1-k} + \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k^{(r)}(x) x^{m-1-k} \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.4}$$

From this, we obtain

$$\left(\frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x), \tag{3.5}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} (\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.6}$$

From $m \geq 1$, we set

$$\begin{aligned} \Omega_m = \beta_m(1) - \beta_m(0) &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left(E_k^{(r)}(1) - E_k^{(r)} \delta_{m,k} \right) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} (2E_k^{(r-1)} - E_k^{(r)} - E_k^{(r)} \delta_{m,k}) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} (2E_k^{(r-1)} - E_k^{(r)}) - \frac{1}{m!} E_m^{(r)}. \end{aligned} \tag{3.7}$$

From this, we now see that,

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.8}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.9}$$

We now would like to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\beta_m(1)-\beta_m(0)) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{2}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m, \end{aligned} \tag{3.10}$$

from which by induction we can derive

$$B_n^{(m)} = -\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}. \tag{3.11}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.12}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise C^{∞}. Moreover, $\beta_m(< x >)$ is continuous for those positive integers m with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for a positive integer m. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^{∞}, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} &\beta_m(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.13}$$

We are now ready to state our first result.

Theorem 3.1. *For each positive integer l, we let*

$$\Omega_l = \sum_{k=0}^{l}\frac{1}{k!(l-k)!}(2E_k^{(r-1)} - E_k^{(r)}) - \frac{1}{l!}E_l^{(r)}. \tag{3.14}$$

Assume that $\Omega_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x}, \end{aligned} \tag{3.15}$$

for all $x \in \mathbb{R}$, *where the convergence is uniform.*

(b)

$$\begin{aligned} &\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k} \\ &= \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>), \end{aligned} \tag{3.16}$$

for all $x \in \mathbb{R}$, *where* $B_j(<x>)$ *is the Bernoulli function.*

Assume next that $\Omega_m \neq 0$, for a positive integer m. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.17}$$

for $x \in \mathbb{Z}$. Now we are ready to state our second result.

Theorem 3.2. *For each positive integer* l, *we let*

$$\Omega_l = \sum_{k=0}^{l} \frac{1}{k!(l-k)!} (2E_k^{(r-1)} - E_k^{(r)}) - \frac{1}{l!} E_l^{(r)}. \tag{3.18}$$

Assume that $\Omega_m \neq 0$, *for a positive integer* m. *Then we have the following.*

(a)

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{1}{m!} E_m^{(r)} + \frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.19}$$

(b)

$$\sum_{j=0}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>) <x>^{m-k}, \tag{3.20}$$

for $x \in \mathbb{Z}^c$;

$$\sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>) = \frac{1}{m!} E_m^{(r)} + \frac{1}{2}\Omega_m, \tag{3.21}$$

for $x \in \mathbb{Z}$.

4. Fourier series of functions of the third type involving higher-order Euler polynomials

Let $\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(x)x^{m-k},\ (m\geq 2)$. Then we will consider the function

$$\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)<x>^{m-k}, \tag{4.1}$$

defined on $\mathbb{R}$, which is periodic of period 1. The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty,n\neq 0}^{\infty}C_n^{(m)}e^{2\pi inx}, \tag{4.2}$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx. \tag{4.3}$$

We need to observe the following to proceed further.

$$\begin{aligned}
\gamma_m'(x)&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left\{kE_{k-1}^{(r)}(x)x^{m-k}+(m-k)E_k^{(r)}(x)x^{m-k-1}\right\}\\
&=\sum_{k=0}^{m-2}\frac{1}{m-1-k}E_k^{(r)}(x)x^{m-1-k}+\sum_{k=1}^{m-1}\frac{1}{k}E_k^{(r)}(x)x^{m-1-k}\\
&=\sum_{k=1}^{m-2}\left(\frac{1}{m-1-k}+\frac{1}{k}\right)E_k^{(r)}(x)x^{m-1-k}+\frac{1}{m-1}x^{m-1}+\frac{1}{m-1}E_{m-1}^{(r)}(x)\\
&=(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}E_k^{(r)}(x)x^{m-1-k}+\frac{1}{m-1}x^{m-1}+\frac{1}{m-1}E_{m-1}^{(r)}(x)\\
&=(m-1)\gamma_{m-1}(x)+\frac{1}{m-1}x^{m-1}+\frac{1}{m-1}E_{m-1}^{(r)}(x).
\end{aligned} \tag{4.4}$$

Thus,

$$\gamma_m'(x)=(m-1)\gamma_{m-1}(x)+\frac{1}{m-1}x^{m-1}+\frac{1}{m-1}E_{m-1}^{(r)}(x), \tag{4.5}$$

from which we see that

$$\left(\frac{1}{m}\left(\gamma_{m+1}(x)-\frac{1}{m(m+1)}x^{m+1}-\frac{1}{m(m+1)}E_{m+1}^{(r)}(x)\right)\right)'=\gamma_m(x). \tag{4.6}$$

This implies that

$$\begin{aligned}
&\int_0^1\gamma_m(x)dx\\
&=\frac{1}{m}\left(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}-\frac{1}{m(m+1)}(E_{m+1}^{(r)}(1)-E_{m+1}^{(r)})\right)\\
&=\frac{1}{m}\left(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}-\frac{2}{m(m+1)}(E_{m+1}^{(r-1)}-E_{m+1}^{(r)})\right).
\end{aligned} \tag{4.7}$$

For $m \geq 2$, we put

$$\begin{aligned}\Lambda_m = \gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (E_k^{(r)}(1) - E_k^{(r)} \delta_{m,k}) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (2E_k^{(r-1)} - E_k^{(r)}).\end{aligned} \tag{4.8}$$

We now notice that

$$\gamma_m(1) = \gamma_m(0) \Longleftrightarrow \Lambda_m = 0, \tag{4.9}$$

and

$$\int_0^1 \gamma_m(x) dx = \frac{1}{m} \left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)} (E_{m+1}^{(r-1)} - E_{m+1}^{(r)}) \right). \tag{4.10}$$

We are now ready to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} [\gamma_m(x) e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n} \int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n} (\gamma_m(1) - \gamma_m(0)) + \frac{1}{2\pi i n} \int_0^1 \left\{ (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} x^{m-1} + \frac{1}{m-1} E_{m-1}^{(r)}(x) \right\} e^{-2\pi i n x} dx \\ &= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m + \frac{1}{2\pi i n (m-1)} \int_0^1 x^{m-1} e^{-2\pi i n x} dx \\ &\quad + \frac{1}{2\pi i n (m-1)} \int_0^1 E_{m-1}^{(r)}(x) e^{-2\pi i n x} dx. \end{aligned} \tag{4.11}$$

We can show that

$$\int_0^1 x^l e^{-2\pi i n x} dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k}, & \text{for} \quad n \neq 0, \\ \frac{1}{l+1}, & \text{for} \quad n = 0. \end{cases} \tag{4.12}$$

Also, from [ ], we have

$$\int_0^1 E_l^{(r)}(x) e^{-2\pi i n x} dx = \begin{cases} \sum_{k=1}^{l} \frac{2(l)_{k-1}}{(2\pi i n)^k} (E_{l-k+1}^{(r)} - E_{l-k+1}^{(r+1)}), & \text{for} \quad n \neq 0, \\ \frac{2}{l+1} (E_{l+1}^{(r-1)} - E_{l+1}^{(r)}), & \text{for} \quad n = 0. \end{cases} \tag{4.13}$$

From (4.11), (4.12), and (4.13), we get

$$C_n^{(m)} = \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m - \frac{1}{2\pi i n (m-1)} \Phi_m - \frac{1}{2\pi i n (m-1)} \Theta_m, \tag{4.14}$$

where

$$\begin{aligned} \Phi_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \\ \Theta_m &= \sum_{k=1}^{m-1} \frac{2(m-1)_{k-1}}{(2\pi i n)^k} (E_{m-k}^{(r-1)} - E_{m-k}^{(r)}). \end{aligned} \tag{4.15}$$

Thus we have shown that

$$C_n^{(m)} = \frac{m-1}{2\pi in} C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_m - \frac{1}{2\pi in(m-1)}\Theta_m, \tag{4.16}$$

from which by induction on m we can easily show that

$$C_n^{(m)} = -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}. \tag{4.17}$$

Here we note that

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&= \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)} \sum_{k=1}^{m-j} \frac{2(m-j)_{k-1}}{(2\pi in)^k}(E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
&= \sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k}}(E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
&= \sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{s=j+1}^{m} \frac{2(m-1)_{s-2}}{(2\pi in)^s}(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \\
&= \sum_{s=2}^{m} \frac{2(m-1)_{s-2}}{(2\pi in)^s}(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \sum_{j=1}^{s-1} \frac{1}{m-j} \\
&= \frac{2}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s}(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}) \frac{H_{m-1}-H_{m-s}}{m-s+1},
\end{aligned} \tag{4.18}$$

where $H_m = \sum_{j=1}^{m}\frac{1}{j}$ are the harmonic numbers. Similarly, we can show that

$$\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} = \frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}. \tag{4.19}$$

Putting everything altogether,

$$C_n^{(m)} = -\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left\{\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\left(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\}. \tag{4.20}$$

Case 2: $n = 0$.

$$\begin{aligned}
C_0^{(m)} &= \int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right).
\end{aligned} \tag{4.21}$$

$\gamma_m(<x>), (m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(<x>)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$, for an integer $m \geq 2$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right)\\
&+ \sum_{n=-\infty, n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\}e^{2\pi inx}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right)\\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\Big(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\Big)\right)\left(-s!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right) \quad (4.22)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right)\\
&+ \frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\Big(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\Big)\right)B_s(< x >)\\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c,\\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we are going to state our first result.

Theorem 4.1. *For each integer $l \geq 2$, we let*

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{1}{k(l-k)}(2E_k^{(r-1)} - E_k^{(r)}), \tag{4.23}$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m = 0$, for an integer $m \geq 2$, Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(< x >) < x >^{m-k}$ *has the Fourier expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(< x >) < x >^{m-k}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right) \quad (4.24)\\
&+ \sum_{n=-\infty, n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\}e^{2\pi inx},
\end{aligned}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(< x >) < x >^{m-k}\\
&= \frac{1}{m}\sum_{s=0, s\neq 1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(1+2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)}))\right)B_s(< x >),
\end{aligned} \tag{4.25}$$

for all $x \in \mathbb{R}$, where $B_s(< x >)$ is the Bernoulli function.

Assume next that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.26}$$

for $x \in \mathbb{Z}$. Next, we are going to state our second result.

Theorem 4.2. *For each integer $l \geq 2$, we let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)}(2E_k^{(r-1)} - E_k^{(r)}), \tag{4.27}$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\begin{aligned} &\frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)})\right) \\ &+ \sum_{n=-\infty, n\neq 0}^{\infty} \left\{ -\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right\} e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(< x >) < x >^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{1}{2}\Lambda_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{4.28}$$

(b)

$$\begin{aligned} &\frac{1}{m}\sum_{s=0}^{m} \binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}\left(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right) B_s(< x >) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(< x >) < x >^{m-k}, \end{aligned} \tag{4.29}$$

for $x \in \mathbb{Z}^c$;

$$\begin{aligned} &\frac{1}{m}\sum_{s=0, s\neq 1}^{m} \binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}\left(1 + 2(E_{m-s+1}^{(r-1)} - E_{m-s+1}^{(r)})\right)\right) B_s(< x >) \\ &= \frac{1}{2}\Lambda_m, \end{aligned} \tag{4.30}$$

for $x \in \mathbb{Z}$.

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FERMAT TYPE EQUATIONS OR SYSTEMS WITH COMPOSITE FUNCTIONS

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Abstract. In this paper, we give some necessary conditions on the existence of meromorphic solutions on Fermat type difference equations. We also consider the properties of transcendental entire solutions on the systems of Fermat type differential-difference equations.

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Keywords: Fermat type equations; meromorphic solutions; composite functions.

1. Introduction and Results

Fermat type equations in functional field

$$f^n + g^n = 1 \tag{1.1}$$

and its generalizations have been considered by many mathematicians in the last century, where n is an integer. We recall the following results. Iyer [10] proved (1.1) has no entire solutions when $n \geq 3$, Gross [6] obtained that (1.1) has no meromorphic solutions when $n \geq 4$. Some related results on (1.1) also can be found in [9]. For the case of $n = 2$, Iyer [10] concluded the following result.

Theorem A. If $n = 2$, then (1.1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, no other solutions exist.

Recent investigations on (1.1) are to explore the precise expressions on $f(z)$ when $g(z)$ has a special relationship with $f(z)$. We mainly recall the following different references on the meromorphic solutions when $n = 2$ in (1.1).

$\star$ Some results on $g(z)$ takes a differential operator of $f(z)$ can be found in [21, 20, 24].

$\star$ Some results on $g(z)$ is a shift operator that is $g(z) = f(z + c)$ or difference operator that is $g(z) = f(z + c) - f(z)$ can be seen in [12, 13, 11, 16].

$\star$ The case that $g(z) = f(qz)$ was considered in [15].

$\star$ The case that $g(z)$ is a differential-difference operator such as $g(z) = f^{(k)}(z+c)$ was considered in [14, 5].

We agree to say that a meromorphic function $f(z)$ in the complex plane is properly meromorphic if $f(z)$ has at least one pole. Fermat type differential equations, for example $f(z)^2 + f^{(k)}(z)^2 = 1$ has no properly meromorphic solutions, it means that all meromorphic solutions are transcendental entire. In addition, the same conclusion is valid for $f(z)^2 + f^{(k)}(z+c)^2 = 1$, where c is a non-zero constant. However, the situation is different for Fermat type difference equations. There exist properly meromorphic solutions with finite order or infinite order for $f(z)^2 + f(z + c)^2 = 1$ and $f(z)^2 + f(qz)^2 = 1$, we cite the examples [16] as follows for the readers.

Example 1. *Let* $c=\frac{\pi}{2}$*. The function* $f(z)=\frac{\frac{1}{\sqrt{i}\tan z}+\sqrt{i}\tan z}{2}$ *is a finite order properly meromorphic solution of* $f(z)^2+f(z+\frac{\pi}{2})^2=1$.

Example 2. *Let* $c=\frac{\pi}{2}$*. The function* $f(z)=\frac{\sqrt{-i}\tan(e^{4zi}+z)+\frac{1}{\sqrt{-i}\tan(e^{4zi}+z)}}{2}$ *is an infinite order properly meromorphic solution of* $f(z)^2+f(z+\frac{\pi}{2})^2=1$.

Example 3. *If* $q=-i$*, then* $f(z)=\frac{\frac{z}{e^{z^{4n}}-1}+\frac{e^{z^{4n}}-1}{z}}{2}$ *is a finite order properly meromorphic solution of* $f(z)^2+f(-iz)^2=1$.

Example 4. *If* $q=-i$*, then* $f(z)=\frac{\frac{z}{e^{e^{z^{4n}}}-1}+\frac{e^{e^{z^{4n}}}-1}{z}}{2}$ *is an infinite order properly meromorphic solution of* $f(z)^2+f(-iz)^2=1$.

We assume that the reader is familiar with the basic notations and results on Nevanlinna theory [8] as well as the uniqueness theory of entire and meromorphic functions [23]. Some necessary conditions for the existence of meromorphic solutions on Fermat differential-difference equations of certain types can be found in Section 2. Section 2 also includes the discussions on composite function with Fermat type equations. In Section 3, we mainly explore the entire solutions on the systems of Fermat type differential-difference equations. In Section 4, we will discuss the meromorphic solutions on the systems of Fermat type difference equations.

2. Necessary conditions for the existence

Let $L(f)$ be a differential-difference polynomial of $f(z)$ with rational coefficients. From the cited references and examples in Section 1, a basic fact is when considering the existence of meromorphic solutions on the equations

$$f(z)^2+\{L[f(g(z))]\}^2=1, \tag{2.1}$$

then $g(z)$ always has the form $g(z)=Az+B$, where A is a non-zero constant and B is a constant. We first to explain the reasons below. We will consider an improvement of (2.1) as follows

$$a(z)f(z)^n+\{L[f(g(z))]\}^n=c(z), \tag{2.2}$$

where $a(z),c(z)$ are rational functions. Yang [22] investigated a generalization of the Fermat type functional equation (1.1) as

$$a(z)f(z)^m+b(z)g(z)^n=1, \tag{2.3}$$

where $T(r,a(z))=S(r,f),T(r,b(z))=S(r,g)$ and obtained the following result.

Theorem B. If $a(z),b(z),f(z),g(z)$ are meromorphic functions, $m\geq 3,n\geq 3$ are integers, then (2.3) cannot hold unless $m=n=3$. If $\frac{1}{m}+\frac{1}{n}<1$, then there are no transcendental entire solutions $f(z)$ and $g(z)$ satisfy (2.3).

Theorem B shows that $n\leq 3$ in (2.2) provided that (2.2) admits meromorphic solutions.

Theorem 2.1. *Let* $g(z)$ *be an entire function in* (2.2)*. The necessary condition of the existence of transcendental entire solutions on* (2.2) *is* $g(z)=Az+B$*, where* $|A|=1$ *and* B *is a constant.*

For the proof of Theorem 2.1, we need the following lemmas on the properties of composite functions. We recall the following result [4, Corollary 1].

Lemma 2.2. *Assume that $f(z)$ is a transcendental meromorphic function, and $g(z)$ is a transcendental entire function, then*

$$\limsup_{r\to+\infty}\frac{T(r,f(g))}{T(r,f)}=+\infty.$$

The proof of the following lemma is included in the proof of Lemma 4 in [7].

Lemma 2.3. *Let $g(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_1z+a_0$, $a_k\neq 0$ be a non-constant polynomial of degree k and let f be a transcendental meromorphic function. Given $0<\varrho<|a_k|$, denote $\zeta=|a_k|+\varrho$ and $\eta=|a_k|-\varrho$. Then, given $\varepsilon>0$, we have*

$$(1-\varepsilon)T(\eta r^k,f)\leq T(r,f(g))\leq(1+\varepsilon)T(\zeta r^k,f)$$

for all r large enough.

Combining the above two lemmas on composite functions with the definitions of order and type of meromorphic functions, we have the following result.

Lemma 2.4. *Let $f(z)$ be a transcendental function and $g(z)$ be a polynomial of degree k and the leading coefficient $a_k\neq 0$. Let $F=f(g)$. Then $\rho(F)=k\rho(f)$ and $\tau(F)=|a_k|^{\rho(f)}\tau(f)$, where $\rho(f)$ is the order of $f(z)$ and $\tau(f)$ is the type of $f(z)$.*

Proof of Theorem 2.1. Assume that $f(z)$ is a transcendental meromorphic solution on (2.2), then we see that

$$T(r,L(f(g(z))))=T(r,f(z))+O(1).$$

From Lemma 2.2, we get $g(z)$ should be a polynomial. Since $L(f)$ is a differential-difference polynomial of $f(z)$, then it implies that at least one of $f^{(k)}(g(z+c))$ (c, k are constants, may take zero) satisfies

$$T(r,f^{(k)}(g(z+c)))=T(r,f(z))+S(r,f),$$

we have $g(z)$ must be a polynomial with degree one and $g(z)=Az+B$, where $|A|=1$ by Lemma 2.4.

We proceed to consider Fermat type equation with composite functions such as

$$f(h(z))^2+f(g(z))^2=1, \tag{2.4}$$

where $h(z)$ and $g(z)$ are two non-constant polynomials. Based on Theorem 2.1, we guess that $g(z)=Ah(z)+B$ provided that there exist meromorphic solutions on (2.4). However, the above result is false by Remark 2.7 below. We need the following lemmas on factorization theory [2, 3].

Lemma 2.5. [3] *If $f(z)$ is a non-constant entire function, and $p(z),q(z)$ are non-constant polynomials satisfying $f(p(z))=f(q(z))$, then one of the following cases holds:*

(i) there exist a root of unity λ and a constant β such that $p(z)=\lambda q(z)+\beta$;

(ii) there exist a polynomial $r(z)$ and constants c,k such that $p(z)=(r(z))^2+k$, $q(z)=(r(z)+c)^2+k$.

Lemma 2.6. [2] *Let f be non-constant meromorphic and $p(z),q(z)$ non-constant polynomials such that $f(p(z))=f(q(z))$. Then there exist a constant k, a positive integer m, a polynomial $r(z)$ and a linear map $L(z)=\lambda z+\beta$ where λ is a root of unit, such that $p(z)=(L(r(z)))^m+k, q(z)=r(z)^m+k$.*

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Remark 2.7. *Let* $G(z) = \left(f(z)^2 - \frac{1}{2}\right)^2$*. From* (2.4)*, we have* $G(h(z)) = G(g(z))$*. Using Lemma 2.5, we have* $h(z) = \lambda g(z) + \beta$ *or there exist a polynomial* $r(z)$ *and constants* c, k *such that* $h(z) = (r(z))^2 + k$ *and* $g(z) = (r(z) + c)^2 + k$*. The second case may happen, for example, the entire function* $f(z) = \cos\sqrt{z}$ *with order* $\frac{1}{2}$*, then* $f(z)$ *solves*

$$f(r(z)^2)^2 + f((r(z) + c)^2)^2 = 1,$$

where $c = \frac{\pi}{2}$*.*

In the following, we will focus on complex difference equations

$$a(z)f(z)^2 + b(z)f(Az + B)^2 = c(z) \tag{2.5}$$

where $a(z), b(z), c(z)$ are non-zero polynomials, and A, B are constants.

Theorem 2.8. *The necessary condition of the existence on transcendental entire solutions with finite order on* (2.5) *is* $\frac{c(z)}{a(z)} = \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$*.*

Proof. Let $G(z) = f(z)^2$. Thus $G(Az + B) = f(Az + B)^2$ and

$$a(z)G(z) + b(z)G(Az + B) = c(z). \tag{2.6}$$

So we have

$$a\left(\frac{z-B}{A}\right)G\left(\frac{z-B}{A}\right) = c\left(\frac{z-B}{A}\right) - b\left(\frac{z-B}{A}\right)G(z). \tag{2.7}$$

From the expression of $G(z)$ and (2.6), (2.7), we have the zeros of $G(z)$, $G(z) - \frac{c(z)}{a(z)}$, $G(z) - \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$ are multiple except possibly finite many zeros. If 0, $\frac{c(z)}{a(z)}$, $\frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$ are distinct, using the second main theorem for small functions, then

$$\begin{aligned}
2\ T(r,G) &\le \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}\left(r, \frac{1}{G(z) - \frac{c(z)}{a(z)}}\right) + \overline{N}\left(r, \frac{1}{G(z) - \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}}\right) \\
&\quad + S(r,G) \\
&\le \frac{1}{2}N(r,\frac{1}{G}) + \frac{1}{2}N\left(r, \frac{1}{G(z) - \frac{c(z)}{a(z)}}\right) + \frac{1}{2}N\left(r, \frac{1}{G(z) - \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}}\right) + S(r,G) \\
&\le \frac{3}{2}T(r,G) + S(r,G),
\end{aligned}$$

which is a contradiction. Thus, $\frac{c(z)}{a(z)} = \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$. □

Remark 2.9. (1) *If* $a(z) = b(z)$ *are non-zero constants and* $A = 1, B \neq 0$*, then* $c(z)$ *reduces to a constant* c*. Thus* (2.5) *reduces to* $f(z)^2 + f(z+B)^2 = c$*, obviously,* $f(z) = \sqrt{c}\sin z$ *and* $B = \frac{\pi}{2}$ *satisfies the above equation.*

If $a(z) = b(z)$ *are non-zero constants and* $|A| = 1, A \neq 1, B = 0$*, then* $c(z)$ *can be an even polynomial. For example*

$$f(z) = \frac{ze^{z+\frac{\pi}{4}i} + ze^{-z-\frac{\pi}{4}i}}{2}, \tag{2.8}$$

solves $f(z)^2 + f(-z)^2 = z^2$*.*

(2) *Consider* $f(z)^2 + f(Az+B)^2 = 1$, *where* $|A| = 1$ *and* B *is a constant, Theorem A shows that* $f(z) = \sin h(z)$ *and* $f(Az+B) = \cos h(z)$, *thus* $h(Az+B) = h(z) + \frac{\pi}{2} + 2k\pi$ *or* $h(Az+B) = -h(z) + \frac{\pi}{2} + 2k\pi$ *where* k *is an integer. If* $f(z)$ *is of finite order, then* $h(z)$ *is a polynomial. Combining Lemma 2.10 below, if* $h(Az+B) = h(z) + \frac{\pi}{2} + 2k\pi$, *we have the following two cases:*

Case 1: If $|A| = 1$ *and* $A \neq 1$, *then* $B = 0$. *There is no any polynomial* $h(z)$ *satisfy* $h(Az+B) = h(z) + \frac{\pi}{2} + 2k\pi$;

Case 2: If $A = 1$ *and* $B \neq 0$, *then* $h(z)$ *is a linear polynomial.*

If $h(Az+B) = -h(z) + \frac{\pi}{2} + 2k\pi$, *we have the following two cases:*

Case 1: If $|A| = 1$ *and* $A \neq 1$, *then* $B = 0$, $h(z)$ *must be a polynomial with* $h(z) = a_{n_1}z^{n_1} + a_{n_2}z^{n_2} + \cdots + a_{n_t}z^{n_t} + \frac{\pi}{4} + k\pi$ *where* $A^{n_t} = -1$;

Case 2: If $A = 1$ *and* $B \neq 0$, *then there is no any polynomial* $h(z)$ *satisfy* $h(Az+B) = -h(z) + \frac{\pi}{2} + 2k\pi$.

Lemma 2.10. *Let* $h(z)$ *be a non-constant polynomial with degree* n *and* a, b, c *be constants,* $a \neq 0$.

(1) *The equation* $h(az+b) = h(z) + c$ *is valid for two cases as follows:*

($1a$) $b \neq 0$, $a = 1$ *and* $h(z)$ *is a linear polynomial.*

($1b$) $b = 0$, $c = 0$ *and* $h(az) = h(z)$, *thus* $h(z) = a_{m_1}z^{m_1} + a_{m_2}z^{m_2} + \cdots + a_{m_k}z^{m_k}$, *where* $a^{m_j} = 1$.

(2) *The equation* $h(az+b) + h(z) = c$ *is valid for two cases as follows:*

($2a$) $b \neq 0$, $a = -1$ *and* $h(z)$ *is a linear polynomial.*

($2b$) $b = 0$, $c = 2h(0)$, *thus* $h(z) = a_{n_1}z^{n_1} + a_{n_2}z^{n_2} + \cdots + a_{n_k}z^{n_k} + a_0$, *where* $a^{n_j} = -1$.

Proof. Let $h(z) = a_n z^n + \cdots + a_1 z + a_0$, where $a_n \neq 0$. It is easy to see ($1b$) is true, we next prove ($1a$). We have

$$a_n(az+b)^n + a_{n-1}(az+b)^{n-1} + \cdots + a_1(az+b) + a_0 = (a_n z^n + \cdots + a_0) + c.$$

Thus, $a^n = 1$. If $a_{n-1} \neq 0$, then

$$a_n n a^{n-1} b + a_{n-1} a^{n-1} = a_{n-1} = a_{n-1} a^n,$$

thus $a = 1 + \frac{a_n n b}{a_{n-1}}$. Since $|a| = 1$, then $b = 0$ follows, which is a contradiction. Thus, $a_{n-1} = 0$. Using the similar method as the above, we get $a_{n-k} = 0$, $k = 2, \cdots, n-1$. So $h(z) = a_n z^n + a_0$, then $n = 1$ follows, thus $a = 1$.

It is easy to see (2b) can happen. Next we prove ($2a$). We have

$$a_n(az+b)^n + a_{n-1}(az+b)^{n-1} + \cdots + a_1(az+b) + a_0 + (a_n z^n + \cdots + a_0) = c.$$

Thus, $a^n = -1$. If $a_{n-1} \neq 0$, then

$$a_n n a^{n-1} b + a_{n-1} a^{n-1} = -a_{n-1} = -a_{n-1} a^n,$$

thus $a = -1 - \frac{a_n n b}{a_{n-1}}$. Since $|a| = 1$, then $b = 0$ follows, which is a contradiction. Thus, $a_{n-1} = 0$. Using the similar method as the above, we get $a_{n-k} = 0$, $k = 2, \cdots, n-1$. So $h(z) = a_n z^n + a_0$, then $n = 1$ follows, thus $a = -1$. □

Using the similar method as the proof of Theorem 2.8, we get the following result.

Theorem 2.11. *The necessary condition on the existence of transcendental meromorphic solutions on*

$$a(z)f(z)^3 + b(z)f(Az+B)^3 = c(z) \tag{2.9}$$

is $\frac{c(z)}{a(z)} = \frac{c\left(\frac{z-B}{A}\right)}{b\left(\frac{z-B}{A}\right)}$.

Baker [1] proved an important result as follows.

Theorem C. Any functions $F(z)$ and $G(z)$, which are meromorphic in the plane and satisfy $F^3 + G^3 = 1$, have the form

$$F(z) = f(h(z)), G(z) = \eta g(h(z)) = \eta f(-h(z)) = f(-\eta^2 h(z)),$$

where $f(z) = \frac{1}{2}\frac{1+\frac{\varphi'(z)}{\sqrt{3}}}{\varphi(z)}$ and $g(z) = \frac{1}{2}\frac{1-\frac{\varphi'(z)}{\sqrt{3}}}{\varphi(z)}$, $h(z)$ is an entire function of z and η is a cube-root of unity, where $\varphi(z)$ is the Weierstrass φ-function that satisfies the differential equation

$$(\varphi'(z))^2 = 4\varphi^3(z) - 1. \tag{2.10}$$

Recently, Lü and Han [17] proved that if $a(z) = b(z) = c(z)$ and $A = 1$ in (2.9), then the equation $f(z)^3 + f(z+c)^3 = 1$ has no transcendental meromorphic solutions with finite order. We will discuss the meromorphic solutions for

$$f(z)^3 + f(Az+B)^3 = 1. \tag{2.11}$$

From Theorem C, if there exist meromorphic solutions on (2.11), then $A = -\eta^2$, $B = 0$. It means that (2.11) reduces to $f(z)^3 + f(-\eta^2 z)^3 = 1$. However, we are interested into another equations as follows. If $\varphi(z)$ is the Weierstrass function, can we give more details for a polynomial $h(z)$ satisfies

$$\frac{1 + \frac{\varphi'(h(Az+B))}{\sqrt{3}}}{\varphi(h(Az+B))} = \frac{1 + \frac{\varphi'(-\eta^2 h(z))}{\sqrt{3}}}{\varphi(-\eta^2 h(z))} \tag{2.12}$$

which is from (2.11) and Theorem C. We affirm that the polynomial $h(z)$ should be a linear polynomial in (2.12).

From Lemma 2.6, we have (i) $h(Az+B) \equiv -\lambda\eta^2 h(z) + \beta$, (ii)$h(Az+B) = r(z)^m + k$ and $-\eta^2 h(z) = (\lambda r(z) + \beta)^m + k$, where $m \geq 2$.

If (i) happens, since $h(z)$ is a polynomial, assume that $h(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n \neq 0$. If $n \geq 2$, we have

$$a_n(Az+B)^n + a_{n-1}(Az+B)^{n-1} + \cdots + a_1(Az+B) + a_0 = -\lambda\eta^2(a_n z^n + \cdots + a_0) + \beta.$$

So, we have $A^n = -\lambda\eta^2$. If $a_{n-1} \neq 0$, we have

$$a_n n A^{n-1} B + a_{n-1} A^{n-1} = -\lambda\eta^2 a_{n-1} = A^n a_{n-1},$$

so

$$A = 1 + \frac{a_n n B}{a_{n-1}},$$

since $|A| = 1$, thus $B = 0$ and $A = 1$. If $a_{n-1} = 0$, using the same method, we have $a_{n-k} = 0$. Thus $h(z) = a_n z^n + a_0$, then we have $h(z)$ must be a linear polynomial. We get $A = -\lambda\eta^2$.

If (ii) happens, then we see that $[r(\frac{z-B}{A})]^m - [\frac{r(z)+\beta}{c}]^m = t$, where $c^m = -\eta^2$ and $t = k(-1 - \frac{1}{\eta^2})$. The above equation is impossible when $m \geq 2$.

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3. The entire solutions on differential-difference systems

Differential-difference equations always can not solved easily. For some linear differential-difference equations, the properties are not known clearly, for example the existence on entire solutions with infinite order of $f'(z) = f(z+c)$ is not clear, where c is a non-zero constant. Naftalevich [19] ever obtained partial results on differential-difference equations using operator theory. Recently, Fermat type differential-difference equations or systems also be investigated using Nevanlinna theory. Liu, Cao and Cao [13] considered the transcendental entire solutions on

$$f'(z)^2 + f(z+c)^2 = 1 \tag{3.1}$$

and obtained the following result.

Theorem D. The transcendental entire solutions with finite order of (3.1) must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant and $c = 2k\pi$ or $c = 2k\pi + \pi$.

Gao [18] considered the systems of complex differential-difference equations

$$\begin{cases} f_1'(z)^2 + [f_2(z+c)]^2 = 1 \\ f_2'(z)^2 + [f_1(z+c)]^2 = 1. \end{cases} \tag{3.2}$$

Assume that there exists a properly meromorphic solution on (3.2), let z_0 be a pole of $f_1(z)$ with multiplicity k. Thus we have $z_0 + 2mc$ is also a pole of $f_1(z)$ with multiplicity $k + 2m$, m is a positive integer, so $\lambda(\frac{1}{f}) \geq 2$. Unfortunately, we can not give examples to show the existence of meromorphic solutions. Considering the transcendental entire solutions of finite order, Gao [18] obtained the following result.

Theorem D. Let $(f_1(z), f_2(z))$ be the transcendental entire solution with finite order of (3.2), then $(f_1(z), f_2(z)) = (\sin(z - bi), \sin(z - b_1 i))$ and $c = k\pi$, where b, b_1 are constants.

If $g(z)$ is a non-constant polynomial and

$$\begin{cases} f_1'(z)^2 + [f_2(g(z))]^2 = 1 \\ f_2'(z)^2 + [f_1(g(z))]^2 = 1 \end{cases} \tag{3.3}$$

admits transcendental meromorphic solutions, then $g(z)$ should be a linear polynomial $g(z) = Az + c$ and $|A| = 1$, which can be proved by Lemma 2.2 and Lemma 2.4 and the following basic fact. From (3.3), we have

$$T(r, f_1(g(z))) \leq 2T(r, f_2(z)) + S(r, f_2(z))$$

and

$$\begin{aligned} T(r, f_2(g(g(z)))) &= T(r, f_1'(g(z))) + O(1) \\ &\leq 2T(r, f_1(g(z))) + S(r, f_1(g(z))) \\ &\leq 4T(r, f_2(z)) + S(r, f_2(z)). \end{aligned}$$

We proceed to consider

$$\begin{cases} f_1'(z)^2 + [f_2(Az+c)]^2 = 1 \\ f_2'(z)^2 + [f_1(Az+c)]^2 = 1 \end{cases} \tag{3.4}$$

and obtain the following result.

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Theorem 3.1. *Let $(f_1(z), f_2(z))$ be a transcendental entire solution with finite order of* (3.4)*, then we have two cases:*

Case 1: If $A^2 = 1$, then $(f_1(z), f_2(z)) = (\sin(z + b'), \sin(z + b''))$,

Case 2: If $A^2 = -1$, then $(f_1(z), f_2(z)) = (\sin(iz + b'), \sin(iz + b''))$, where b', b'' are constants may different values at different occasions.

Corollary 3.2. *The finite order transcendental entire solutions of* (3.4) *should have order one.*

For the proof of Theorem 3.1, we need the following lemmas.

Lemma 3.3. [23, Theorem 1.56] *Let $f_j(z), (j = 1, 2, 3)$ be meromorphic functions, f_1 be not a constant. If $\sum_{j=1}^{3} f_j = 1$ and*

$$\sum_{j=1}^{3} N(r, \frac{1}{f_j}) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$, $T(r) = \max_{1\leq j\leq 3}\{T(r, f_j)\}$, then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Lemma 3.4. *If $\sin(h_1(z)) = p(z)\sin(h_2(z))$ holds, then $p(z)$ should be a constant p and $p^2 = 1$, where $h_1(z), h_2(z)$ are non-constant polynomials.*

Proof. From $\sin(h_1(z)) = p(z)\sin(h_2(z))$, we have

$$e^{ih_1(z)} - e^{-ih_1(z)} = p(z)e^{ih_2(z)} - p(z)e^{-ih_2(z)},$$

thus,

$$\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} + \frac{e^{ih_2(z)-ih_1(z)}}{p(z)} + e^{2ih_2(z)} = 1.$$

Obviously, $e^{2ih_2(z)} \not\equiv 1$, Lemma 3.3 implies $\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} \equiv 1$ or $\frac{e^{ih_2(z)-ih_1(z)}}{p(z)} \equiv 1$, so we have $p(z)$ should be a constant. Furthermore, if $\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} \equiv 1$, then $\frac{e^{ih_2(z)-ih_1(z)}}{p(z)} + e^{2ih_2(z)} = 0$ follows, thus $p(z)^2 = 1$.

If $\frac{e^{ih_2(z)-ih_1(z)}}{p(z)} \equiv 1$, then $\frac{e^{ih_1(z)+ih_2(z)}}{-p(z)} + e^{2ih_2(z)} = 0$ follows, thus $p(z)^2 = 1$. □

Proof of Theorem 3.1. From Theorem A, we obtain

$$\begin{cases} f_1'(z) = \sin h_1(z) \\ f_2(Az + c) = \cos h_1(z) \end{cases}$$

and

$$\begin{cases} f_2'(z) = \sin h_2(z) \\ f_1(Az + c) = \cos h_2(z). \end{cases}$$

If $f_1(z)$ and $f_2(z)$ are transcendental entire functions with finite order, then $h_1(z), h_2(z)$ are polynomials. Combining with the above two systems, we have

$$\begin{cases} f_1'(Az + c) = \sin h_1(Az + c) \\ f_1'(Az + c) = \frac{-h_2'(z)}{A} \sin h_2(z) \end{cases}$$

and

$$\begin{cases} f_2'(Az + c) = \sin h_2(Az + c) \\ f_2'(Az + c) = \frac{-h_1'(z)}{A} \sin h_1(z). \end{cases}$$

Thus, we have

$$\sin h_1(Az + c) = \frac{-h_2'(z)}{A} \sin h_2(z).$$

and

$$\sin h_2(Az+c) = \frac{-h_1'(z)}{A}\sin h_1(z). \tag{3.5}$$

The above two equations imply to

$$\sin h_1(A^2z+Ac+c) = \frac{h_2'(Az+c)h_1'(z)}{A^2}\sin h_1(z). \tag{3.6}$$

From Lemma 3.4, we have $h_2'(Az+c)h_1'(z) = \pm A^2$. Since $h_1(z), h_2(z)$ are non-constant polynomials, so $h_1(z) = a_1z+b_1$ and $h_2(z) = a_2z+b_2$.

If $h_2'(Az+c)h_1'(z) = A^2 = a_1a_2A$, since (3.6) implies that

$$\sin h_1(A^2z+Ac+c) = \sin h_1(z),$$

so $h_1(A^2z+Ac+c) - h_1(z) = 2k\pi$. We have $A^2 = 1$ by $h_1(z) = a_1z+b_1$.

Case 1: If $Ac+c \neq 0$, it implies that $A=1$, $c=k\pi$. Then (3.5) reduces to

$$\sin h_2(z+c) = -a_1\sin h_1(z),$$

we have $a_1^2 = 1$ follows by Lemma 3.4.

Subcase (1): If $a_1 = 1$, then $h_1(z) = z+b_1$ and $h_2(z) = z+b_2$ and $c = k\pi$. So

$$f_1(z+c) = \cos(z+b_2) = \sin(z+b_2+\frac{\pi}{2}+2k_1\pi) \Rightarrow f_1(z) := \sin(z+b'),$$

where $b' = b_2 + \frac{\pi}{2} + 2k_1\pi - k\pi$. Also

$$f_2(z+c) = \cos(z+b_1) = \sin(z+b_1+\frac{\pi}{2}+2k_2\pi) \Rightarrow f_2(z) := \sin(z+b''),$$

where $b' = b_1 + \frac{\pi}{2} + 2k_2\pi - k\pi$.

Subcase (2): If $a_1 = -1$, then $h_1(z) = -z+b_1$ and $h_2(z) = -z+b_2$ and $c = k\pi$. One can get $f_1(z) := \sin(z+b'), f_2(z) := \sin(z+b'')$ also only modify the value b', b'' by $\cos z$ is even.

Case 2: If $Ac+c = 0$, thus two cases happen.

Subcase (1): $A = -1$, c is any non-zero constant. Thus, $a_1a_2 = -1$. We also can get $f_1(z) := \sin(z+b'), f_2(z) := \sin(z+b'')$ using the similar discussions as the above with $\cos z$ is even.

Subcase (2): $A = -1$ and $c = 0$, from (3.5), we also have a_1, a_2 take 1 or -1, then we can get $f_1(z) := \sin(z+b'), f_2(z) := \sin(z+b'')$.

If $h_2'(Az+c)h_1'(z) = -A^2 = a_1a_2A$, since (3.6) implies that

$$\sin h_1(A^2z+Ac+c) = -\sin h_1(z),$$

so $h_1(A^2z+Ac+c) + h_1(z) = 2k\pi$. It implies that $A^2 = -1$ by $h_1(z) = a_1z+b_1$.

Case 1: If $A = i$, then $c = \frac{2k\pi - 2b_1}{a_1(i+1)}$. Then (3.5) reduces to

$$\sin h_2(z+c) = \frac{-a_1}{A}\sin h_1(z),$$

we have $a_1^2 = -1$ follows by Lemma 3.4.

Subcase (1): If $a_1 = i$, then $h_1(z) = iz+b_1$ and $h_2(z) = -iz+b_2$ so

$$f_1(z+c) = \cos(iz+b_2) = \sin(iz+b_2+\frac{\pi}{2}+2k_1\pi) \Rightarrow f_1(z) := \sin(iz+b'),$$

where $b' = b_2 + \frac{\pi}{2} + 2k_1\pi - k\pi$. Also

$$f_2(z+c) = \cos(-iz+b_1) = \sin(iz-b_1+\frac{\pi}{2}+2k_2\pi) \Rightarrow f_2(z) := \sin(iz+b''),$$

where $b' = -b_1 + \frac{\pi}{2} + 2k_2\pi - k\pi$.

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Subcase (2): If $a_1 = -i$, then $h_1(z) = -iz + b_1$ and $h_2(z) = iz + b_2$. One can get $f_1(z) := \sin(iz + b')$, $f_2(z) := \sin(iz + b'')$ also by modifying the value b', b'' with $\cos z$ is even.

Case 2: If $A = -i$, one can get $f_1(z) := \sin(iz + b')$, $f_2(z) := \sin(iz + b'')$ also by modifying the value b', b'' with $\cos z$ is even.

4. Meromorphic solutions on Fermat type difference system

We will consider the meromorphic solutions on Fermat type difference system

$$\begin{cases} f_1(z)^2 + [f_2(Az+c)]^2 = 1 \\ f_2(z)^2 + [f_1(Az+c)]^2 = 1 \end{cases} \tag{4.1}$$

where A is a non-zero constant and c is a constant.

Firstly, if f_1 is transcendental meromorphic and satisfy $f_1(z)^2 + f_1(Az+c)^2 = 1$, we see that $f_1(z) = \pm f_2(z)$ is the solution on (4.1). From the introduction of the paper, we know that the transcendental meromorphic solutions are exist indeed.

Secondly, considering the transcendental entire solutions with finite order, we get the following properties.

Theorem 4.1. *The finite order transcendental entire solutions on* (4.1) *have order one except that* $c = 0$ *and* $A^{2m_j} = -1$, m_j *are integers.*

Proof. Using Theorem A, we have

$$\begin{cases} f_1(z) = \sin(h_1(z)) \\ f_2(Az+c) = \cos(h_1(z)) \end{cases}$$

and

$$\begin{cases} f_2(z) = \sin(h_2(z)) \\ f_1(Az+c) = \cos(h_2(z)) \end{cases}$$

where $h_1(z)$ and $h_2(z)$ are non-constant polynomials.

Combining with the above two systems, we obtain

$$f_1(Az+c) = \sin(h_1(Az+c)) = \cos(h_2(z)) = \sin(\pm h_2(z) + \frac{\pi}{2}).$$

Thus, we have $h_1(Az+c) = \pm h_2(z) + \frac{\pi}{2} + 2k\pi$, where k is an integer. We also can get

$$f_2(Az+c) = \sin(h_2(Az+c)) = \cos(h_1(z)) = \sin(\pm h_1(z) + \frac{\pi}{2}),$$

thus $h_2(Az+c) = \pm h_1(z) + \frac{\pi}{2} + 2n\pi$, where n is an integer, hence

$$h_1(A^2z + Ac + c) = \pm h_1(z) + \pi + 2m\pi,$$

where m is an integer. Since $h_1(z)$ is a polynomial, using Lemma 2.10, we have two cases as follows.

Case 1: $h_1(A^2z + Ac + c) = h_1(z) + \pi + 2m\pi$. If $c = 0$, the above equation is impossible. If $c \neq 0$ and $A = -1$, the above equation is also impossible. if $c \neq 0$ and $A \neq -1$, then we should have $h_1(z) = az + b$, where a is a non-zero constant and b is a constant.

Case 2: $h_1(A^2z + Ac + c) = -h_1(z) + \pi + 2m\pi$. In this case, if $c = 0$, $h(z)$ is a polynomial $h(z) = a_{m_1}z^{m_1} + a_{m_2}z^{m_2} + \cdots + a_{m_k}z^{m_k} + \frac{\pi}{2} + k\pi$, where $A^{2m_j} = -1$. If $c \neq 0$ and $A = -1$, then $h(z)$ is a constant, which is a contradiction. If $c \neq 0$ and $A \neq -1$, we have $h(z)$ should be a linear polynomial. □

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ON SHARED VALUE PROPERTIES OF DIFFERENCE PAINLEVÉ EQUATIONS

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ABSTRACT. In this paper, we study some shared value properties for finite order meromorphic solutions of difference Painlevé I-III equations.

Keywords: Meromorphic functions; Difference Painlevé equation; Value sharing.

MSC 2010: Primary 39A05; Secondary 30D35

1. INTRODUCTION

A century ago, Painlevé [9, 10], Fuchs [3] and Gambier [4] classified a large class of second order differential equations of the Painlevé type of the form

$$w''(z) = F(z, w, w'),$$

where F is rational in w and w' and (locally) analytic in z. In the past two decades, the interest in nonlinear analytic difference equations has increased, especially in response to programme of finding some kind of an analogue of Painlevé property of differential equations for difference equations. Recently, Halburd and Korhonen [5], Ronkainen [11] studied the following complex difference equations

$$f(z+1) + f(z-1) = R(z, f) \tag{1.1}$$

and

$$f(z+1)f(z-1) = R(z, f), \tag{1.2}$$

where $R(z, f)$ is rational in f and meromorphic in z. They obtained that if (1.1) or (1.2) has an admissible meromorphic solution of finite order(or hyper order less than 1), then either f satisfies a difference Riccati equation, or (1.1) and (1.2) can be transformed by a linear change in f to some difference equations, which include the difference Painlevé I-III equations

$$\begin{aligned} f(z+1) + f(z-1) &= \frac{az+b}{f} + \frac{c}{f^2}, \quad (P_I) \\ f(z+1) + f(z-1) &= \frac{(az+b)f+c}{1-f^2}, \quad (P_{II}) \end{aligned} \tag{1.3}$$

$$\begin{aligned} f(z+1)f(z-1) &= \frac{af^2 - bf + c}{(f-1)(f-d)}, \quad (P_{III}) \\ f(z+1)f(z-1) &= \frac{af^2 - bf}{f-1}, \quad (P_{III}) \end{aligned} \tag{1.4}$$

where a, b, c, d are small functions of $f(z)$. Some results about properties of finite order transcendental meromorphic solutions of (1.3) and (1.4), can

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be found in [1, 2, 13]. In 2007, Lin and Tohge [7] studied some shared value properties of the first, the second and the fourth Painlevé differential equations

$$\begin{aligned} &f'' = z + 6f^2 \\ &f'' = 2f^3 + zf + a, \quad \alpha \in \mathbb{C} \\ &2ff'' = (f')^2 + 3f^4 + 8zf^3 = 4(z^2 - \alpha)f^2 + \beta, \quad \alpha, \beta \in \mathbb{C} \end{aligned} \tag{1.5}$$

They obtained the following result

Theorem A. *Let $f(z)$ be an arbitrary nonconstant solution of one of the equations (1.5), and $g(z)$ be a nonconstant meromorphic function which shares four distinct values a_j IM with $f(z)$, where $j = 1, 2, 3, 4$. Then $f(z) \equiv g(z)$.*

Remark. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [12]. As usual, the abbreviation CM stands for "counting multiplicities", while IM means "ignoring multiplicities".

A natural question is: what is the uniqueness result for finite order meromorphic solutions of difference Painlevé equations. Corresponding to this question, we consider shared value properties of equations (1.3) and (1.4).

Set

$$\Theta_1(z, f) = (f(z+1) + f(z-1))f^2 - (az + b)f - c$$

and

$$\Theta_2(z, f) = (f(z+1) + f(z-1))(1 - f^2) - (az + b)f - c.$$

Then we can get a uniqueness theorem for finite order meromorphic solutions of difference P_I, P_{II} equations.

Theorem 1.1. *Let $f(z)$ be a finite order transcendental meromorphic solution of (1.3), let e_1, e_2 be two distinct finite numbers such that $\Theta_i(z, e_1) \not\equiv 0$, $\Theta_i(z, e_2) \not\equiv 0$, $i = 1, 2$. If $f(z)$ and another meromorphic function $g(z)$ share the values e_1, e_2 and ∞ CM, then $f(z) \equiv g(z)$.*

Regarding shared value properties of difference P_{III} equations, we have

Theorem 1.2. *Let $f(z)$ be a finite order transcendental meromorphic solution of*

$$f(z+1)f(z-1) = \frac{af^2 - bf + c}{(f-1)(f-d)}. \tag{1.6}$$

And let e_1, e_2 be two distinct finite numbers such that $\Phi(z, e_1) \not\equiv 0$, $\Phi(z, e_2) \not\equiv 0$, where $\Phi(z, f) = f(z+1)f(z-1)(f-1)(f-d) - af^2 + bf - c$. If $f(z)$ and another meromorphic function $g(z)$ share the values e_1, e_2 and ∞ CM, then $f(z) \equiv g(z)$.

Theorem 1.3. *Let $f(z)$ be a finite order transcendental meromorphic solution of*

$$f(z+1)f(z-1) = \frac{af^2 - bf}{f-1}.$$

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And let e_1, e_2 be two distinct finite numbers such that $\Psi(z, e_1) \not\equiv 0$, $\Psi(z, e_2) \not\equiv 0$, where $\Psi(z, f) = f(z+1)f(z-1)(f-1) - af^2 + bf$. If $f(z)$ and another meromorphic function $g(z)$ share the values e_1, e_2 and ∞ CM, then $f(z) \equiv g(z)$.

Remarks. (1) By Lemma 2.4 below, we can get Theorem 1.1 easily. And using similar methods as the proof of Theorem 1.2, we can prove Theorem 1.3. Here, we omit the details.

(2) Some ideas of this paper come from [8].

2. Some Lemmas

Lemma 2.1. [6, Theorem 2.2] *Let $f(z)$ be a transcendental meromorphic solution with finite order $\sigma(f)$ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f)$ is a difference product of total degree n in $f(z)$ and its shifts, and where $P(z, f)$, $Q(z, f)$ are difference polynomials such that the total degree of $Q(z, f)$ is at most n. Then for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O(r^{\sigma(f)-1+\varepsilon}) + o(T(r, f))$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.2. [6, Theorem 2.4] *Let $f(z)$ be a transcendental meromorphic solution with finite order $\sigma(f)$ of the difference equation*

$$L(z, f) = 0,$$

where $L(z, f)$ is a difference polynomial in $f(z)$ and its shifts. If $L(z, a) \not\equiv 0$ for slowly moving target $a(z)$. Then for each $\varepsilon > 0$,

$$m(r, \frac{1}{f-a}) = O(r^{\sigma(f)-1+\varepsilon}) + o(T(r, f))$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.3. [12, Theorem 1.51] *Suppose that $f_j(z)$ $(j = 1, \dots n)$ $(n \geq 2)$ are meromorphic functions and $g_j(z)$ $(j = 1, \dots, n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.

(2) $1 \leq j < k \leq n$, *$g_j(z) - g_k(z)$ are not constants for* $1 \leq j < k \leq n$.

(3) *For* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, r \notin E,$$

where $E \subset (1, \infty)$ is of finite linear measure.

Then $f_j(z) \equiv 0$.

Lemma 2.4. [8, Theorem 1.1] *Let $f(z)$ be a finite order transcendental meromorphic solution of*

$$\sum_{i=1}^{n} a_i f(z + c_i) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{j=0}^{p} b_j f^j}{\sum_{k=0}^{q} d_k f^k},$$

where $a_i (\not\equiv 0)$, b_j, d_k, are small functions of f , $c_j (\neq 0)$ are pairwise distinct constants. And let e_1, e_2 be two distinct finite numbers such that $\Theta(z, e_1) \not\equiv$

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0, $\Theta(z,e_2)\not\equiv 0$, $p\le q=n$, where $\Theta(z,f)=\sum_{i=1}^{n}a_i f(z+c_i)Q(z,f)-P(z,f)$. If $f(z)$ and another meromorphic function $g(z)$ share the values e_1, e_2 and ∞ CM, then $f(z)\equiv g(z)$.

3. Proof of Theorem 1.2

Suppose that $f(z)$ is a finite order transcendental meromorphic solution of Eq. (1.6). Then we get

$$f^2f(z+1)f(z-1)=(d+1)ff(z+1)f(z-1)-df(z+1)f(z-1)+af^2-bf+c.$$

Applying Lemma 2.1, we obtain

$$m(r,f)=S(r,f). \tag{3.1}$$

From Lemma 2.2 and the assumption that $\Phi(z,e_1)\not\equiv 0$, $\Phi(z,e_2)\not\equiv 0$, we know

$$m(r,\frac{1}{f-e_1})=S(r,f),\quad m(r,\frac{1}{f-e_2})=S(r,f). \tag{3.2}$$

By the assumption that $f(z)$ and $g(z)$ share the values e_1, e_2 and ∞ CM, we have that

$$\begin{aligned}T(r,f)&\le N(r,f)+N(r,\frac{1}{f-e_1})+N(r,\frac{1}{f-e_2})+S(r,f)\\&\le N(r,g)+N(r,\frac{1}{g-e_1})+N(r,\frac{1}{g-e_2})+S(r,f)\\&\le 3T(r,g)+S(r,f).\end{aligned}$$

Similarly, we can get $T(r,g)\le 3T(r,f)+T(r,f)$. Hence,

$$T(r,g)=T(r,f)+S(r,f). \tag{3.3}$$

Moreover, from the assumption that $f(z)$ and $g(z)$ share the values e_1, e_2 and ∞ CM, we see

$$\frac{f-e_1}{g-e_1}=e^{A(z)},\quad \frac{f-e_2}{g-e_2}=e^{B(z)}, \tag{3.4}$$

where $A(z)$ and $B(z)$ are two polynomials. Clearly, when $e^{A(z)}=1$, or $e^{B(z)}=1$, or $e^{B(z)-A(z)}=1$, The conclusion $f(z)\equiv g(z)$ holds. In the following, we suppose that $e^{A(z)}\ne 1$, $e^{B(z)}\ne 1$ and $e^{B(z)-A(z)}\ne 1$ at the same time. Combining (3.3) and (3.4), we obtain

$$\begin{aligned}T(r,e^A)&\le 2T(r,f)+S(r,f),\\T(r,e^B)&\le 2T(r,f)+S(r,f).\end{aligned} \tag{3.5}$$

Rewrite above Eq. (3.4) as the following forms

$$f(z)=e_1+(e_2-e_1)\frac{e^{B(z)}-1}{e^{C(z)}-1}, \tag{3.6}$$

or

$$f(z)=e_2+(e_2-e_1)\frac{e^{A(z)}-1}{e^{C(z)}-1}e^{C(z)}, \tag{3.7}$$

where $C(z)=B(z)-A(z)$.

Next we prove that $\deg A(z) = \deg B(z) = \deg C(z) > 0$. Assume that the largest common factor of $e^{B(z)} - 1$ and $e^{C(z)} - 1$ is $D(z)$, hence

$$e^{B(z)} - 1 = D(z)B_1(z), \quad e^{C(z)} - 1 = D(z)C_1(z),$$

where $B_1(z)$, $C_1(z)$ and $D(z)$ are entire functions. Substituting above equations into (3.6), we conclude that

$$f(z) = e_1 + (e_2 - e_1)\frac{B_1(z)}{C_1(z)}.$$

This, together with (3.1) and (3.2), it follows that

$$T(r,f) = m(r, \frac{1}{f-e_1}) + N(r, \frac{1}{f-e_1}) + S(r,f) = N(r, \frac{1}{B_1}) + S(r,f)$$

and

$$T(r,f) = m(r,f) + N(r,f) = N(r, \frac{1}{C_1}) + S(r,f).$$

Furthermore, we have

$$T(r,e^B) = N(r, \frac{1}{e^B - 1}) + S(r,f) = N(r, \frac{1}{B_1}) + N(r, \frac{1}{D}) + S(r,f)$$

and

$$T(r,e^C) = N(r, \frac{1}{e^C - 1}) + S(r,f) = N(r, \frac{1}{C_1}) + N(r, \frac{1}{D}) + S(r,f).$$

Observing four equations above, we see

$$T(r,e^C) = T(r,e^B) + S(r,f). \tag{3.8}$$

Using the same way to deal with Eq. (3.7), we get

$$T(r,e^C) = T(r,e^A) + S(r,f). \tag{3.9}$$

This, together with (3.7) and (3.8),

$$\deg A(z) = \deg B(z) = \deg C(z) = k > 0$$

follows. On the other hand, Substituting (3.6) into (1.6), we have

$$\begin{aligned}
&(e_1 + (e_2 - e_1)\frac{e^{B(z+1)} - 1}{e^{C(z+1)} - 1})(e_1 + (e_2 - e_1)\frac{e^{B(z-1)} - 1}{e^{C(z-1)} - 1}) \\
&(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1} - 1)(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1} - d) \\
&= a(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1})^2 - b(e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1}) + c.
\end{aligned} \tag{3.10}$$

Both sides of Eq. (3.10) multiplied by $(e^{C(z+1)} - 1)(e^{C(z-1)} - 1)(e^C - 1)^2$, we get

$$\begin{aligned}
&\left(e_1(e^{C(z+1)} - 1) + (e_2 - e_1)(e^{B(z+1)} - 1)\right)\left(e_1(e^{C(z-1)} - 1) + (e_2 - e_1)(e^{B(z-1)} - 1)\right) \\
&\left((e_1 - 1)(e^C - 1) + (e_2 - e_1)(e^B - 1)\right)\left((e_1 - d)(e^C - 1) + (e_2 - e_1)(e^B - 1)\right) \\
&= (e^{C(z+1)} - 1)(e^{C(z-1)} - 1)(a(e_1(e^C - 1) + (e_2 - e_1)(e^B - 1))^2 \\
&- b(e_1(e^C - 1)^2 + (e_2 - e_1)(e^B - 1)(e^C - 1)) + c(e^C - 1)^2).
\end{aligned} \tag{3.11}$$

Set

$$B(z+1) = B(z) + s_1(z), \quad B(z-1) = B(z) + s_2(z),$$
$$C(z+1) = C(z) + t_1(z), \quad C(z-1) = C(z) + t_2(z),$$

where s_i, t_i are polynomials of degrees at most $k-1$. Then Eq. (3.11) can be represented as the following form:

$$\sum_{\mu=0}^{4}\sum_{\lambda=0}^{4} M_{\mu,\lambda} e^{\mu B+\lambda C} = 0, \tag{3.12}$$

where $M_{\mu,\lambda}$ is either 0 or polynomial in a, b, c, d, e_1, e_2 and e^{s_i}, e^{t_i}. Especially, we have

$$M_{0,0} = e_2^2(e_2-1)(e_2-d) - (ae_2^2 - be_2 + c) = \Phi(z, e_2) \not\equiv 0. \tag{3.13}$$

Finally, we prove that

$$\deg(\mu^* B + \lambda^* C) = \deg(\mu^* B - \lambda^* C) = k, \quad 1 \le \mu^* \le 4, 1 \le \lambda^* \le 4.$$

Suppose, contrary to the assertion, that $\deg(\mu^* B + \lambda^* C) < k$ or $\deg(\mu^* B - \lambda^* C) < k$.

If $\deg(\mu^* B + \lambda^* C) < k$, then $e^{\mu^* B+\lambda^* C}$ is a small function of e^A and $f(z)$ by (3.5), (3.8) and (3.9). Hence,

$$T(r, e^{\mu^* B+\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{-\mu^* A}) = \mu^* T(r, e^A) + S(r, f).$$

Moreover,

$$T(r, e^{\mu^* B+\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{(\mu^*+\lambda^*)C}) = (\mu^* + \lambda^*)T(r, e^A) + S(r, f).$$

Since $\lambda^* \neq 0$, comparing two equations above, we get a contradiction.

If $\deg(\mu^* B + \lambda^* C) < k$, then we have

$$T(r, e^{\mu^* B-\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{-\mu^* A}) = \mu^* T(r, e^A) + S(r, f),$$

and

$$T(r, e^{\mu^* B-\lambda^* C} \cdot e^{-\mu^* A}) = T(r, e^{(\mu^*-\lambda^*)C}) = (\mu^* - \lambda^*)T(r, e^A) + S(r, f).$$

As $\lambda^* \neq 0$, we can get a contradiction as well. Therefore, we know

$$T(r, M_{\mu,\lambda}) = S(r, e^{\pm(\mu^* B+\lambda^* C)}), \quad T(r, M_{\mu,\lambda}) = S(r, e^{\pm(\mu^* B-\lambda^* C)}),$$

where μ^* and λ^* are not equal to zero at the same time. This, together with Lemma 2.3, it follows that $M_{\mu,\lambda} \equiv 0$, which contradicts Eq. (3.13), and the conclusion follows.

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