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An application of Binomial distribution series on certain analytic functions

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Abstract

In the present note we will introduce a Binomial distribution series and obtain necessary and sufficient conditions for this series belonging to the classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$. An integral operator related to this series is also considered.

2010 Mathematics Subject Classification: 30C45, 30C55

Key words and phrases: analytic function, binomial distribution, univalent

1 Introduction

Consider a class A consisting of functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Every $g \in A$ is analytic in the open unit disk $\mathbb{D}$ and satisfy the normalization condition $g(0) = g'(0) - 1 = 0$. Let S be a subclass of A consisting of functions of the form (1.1), which are also univalent in $\mathbb{D}$. Furthermore, consider T be the subclass of S containing the functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

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Let $T(\gamma, \delta)$ be a subclass of T having the functions which satisfy the following condition

$$Re\left\{\frac{zg'(z)}{\gamma zg'(z) + (1-\gamma)g(z)}\right\} > \delta \tag{1.3}$$

for all δ $(0 \leq \delta < 1)$, γ $(0 \leq \gamma < 1)$ and $z \in \mathbb{D}$. Also, we consider $C(\gamma, \delta)$ be an other subclass of T containing the functions which satisfy the following condition

$$Re\left\{\frac{g'(z) + zg''(z)}{g'(z) + \gamma zg''(z)}\right\} > \delta \tag{1.4}$$

for all δ $(0 \leq \delta < 1)$, γ $(0 \leq \gamma < 1)$ and $z \in \mathbb{D}$.

From (1.2) and (1.4) one can draw the following conclusion

$$g(z) \in C(\gamma, \delta) \iff zg'(z) \in T(\gamma, \delta). \tag{1.5}$$

Both $T(\gamma, \delta)$ and $C(\gamma, \delta)$ are extensively studied by Altinates and Owa [1] and certain conditions for hypergeometric function and generalized Bessel function g for these classes were studied by Mostafa [8] and Porwal and Dixit [11].

Let $g(l, p)$ be a binomial distribution defined by

$$g(l, p) = Pr(X = n) = \frac{l!}{(n-l)!n!}p^n(1-p)^{l-n}, \quad n = 0, 1, 2, \ldots, l$$

when $n > l$, then $f(l, p) = 0$.

Consider a power series defined as:

$$K(l, p, z) = z + \sum_{n=2}^{\infty} \frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n.$$

Now, we introduce the series

$$F(l, p, z) = z - \sum_{n=2}^{\infty} \frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n.$$

In [3], Carlson and Shaffer studied starlike and prestarlike hypergeometric functions. The sufficient condition for a (Gaussian) hypergeometric function to be uniformly convex of order δ, which is also necessary condition under additional restrictions is given by Cho et al. [4]. Starlike hypergeometric functions were studied by Merkes and Scott [6] and Carlson and Shaffer [3].

Motivated by results on connection between various subclasses of analytic functions by using the hypergeometric function by many author particularly the authors (see [3, 4, 6, 12, 13]) and generalized Bessel functions (see [2, 7]), Porrwal [10] obtained the necessary and sufficient conditions for a functions $F(l, p, z)$ defined by using the poisson distribution belong to the class $T(\delta, \gamma)$ and $C(\delta, \gamma)$.

In this article, we give the analogous conditions for the functions $F(l, p, z)$ and integral operator $H(l, p, z)$ defined by the binomial distribution belong to the $T(\delta, \gamma)$ and $C(\delta, \gamma)$.

To establish our main results, we will require the following lemmas due to Altintas and Owa [1].

Lemma 1.1. *A function $g(z)$ characterize by (1.2) belong to the class $T(\delta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}[n-\gamma\delta n-\delta+\gamma\delta]|a_n| \leq 1-\delta.$$

Lemma 1.2. *A function $g(z)$ characterize by (1.2) belong to the class $C(\delta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}n[n-\gamma\delta n-\delta+\gamma\delta]|a_n| \leq 1-\delta.$$

2 Main results

Theorem 2.1. *The function $F(k,p,z)$ belong to the class $T(\delta,\gamma)$ if and only if*

$$p(1-\delta\gamma)(l-1)+(1-\delta)A \leq 1-\delta,$$

where

$$A=\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}.$$

Proof. Since

$$F(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^{n},$$

according to the Lemma 1.1 we must show that

$$\sum_{n=2}^{\infty}[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n} \leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\gamma\delta)+(1-\delta)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n+1}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)p\sum_{n=0}^{\infty}\frac{(l-1)(l-2)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=p(1-\gamma\delta)(l-1)+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&\leq 1-\delta.
\end{aligned}$$

This completes the proof. □

Theorem 2.2. *The function $F(l,p,z)$ belong to the class $C(\gamma,\delta)$ if and only if*

$$p^2(1-\delta\gamma)(l-1)(l-2)+p(3-2\gamma\delta-\delta)(l-1)+(1-\delta)B\leq 1-\delta,$$

where

$$B=\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^n(1-p)^{l-n-1}.$$

Proof. As

$$F(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n,$$

therefore according to the Lemma 1.2 we must show that

$$\sum_{n=2}^{\infty}n[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(1-\gamma\delta)(n-1)(n-2)+(3-2\delta\gamma-\delta)(n-1)+(1-\delta)]\\
&\quad\times\frac{(l-1)!}{(l-n-2)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=3}^{\infty}\frac{(l-1)!}{(l-n)(n-3)!}p^{n-1}(1-p)^{l-n}+(3-2\delta\gamma-\delta)\\
&\quad\times\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-3)n!}p^{n+2}(1-p)^{l-n-3}+(3-2\delta\gamma-\delta)\\
&\quad\times\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n+1}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=p^2(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-3)n!}p^{n}(1-p)^{l-n-3}+p(3-2\delta\gamma-\delta)\\
&\quad\times\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n}(1-p)^{l-n}\\
&=p^2(1-\gamma\delta)(l-1)(l-2)\sum_{n=0}^{\infty}\frac{(l-3)!}{(l-n-3)n!}p^{n}(1-p)^{l-n-3}+p(3-2\delta\gamma-\delta)(l-1)\\
&\quad\times\sum_{n=0}^{\infty}\frac{(l-2)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=p^2(1-\gamma\delta)(l-1)(l-2)+p(3-2\delta\gamma-\delta)(l-1)+(1-\delta)B\\
&\leq 1-\delta.
\end{aligned}$$

This completes the proof. □

In the following theorem, we obtain the analogous results in connection with the particular integral operator $H(l,p,z)$ as follow:

$$H(l,p,z)=\int_0^z \frac{F(l,p,z)}{t}dt. \tag{2.1}$$

Theorem 2.3. *The operator $H(l,p,z)$ characterized by* (2.1) *is in the class $C(\gamma,\delta)$ if and only if*

$$p(1-\delta\gamma)(l-1)+(1-\delta)C\leq \delta-1,$$

where

$$C=\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^n(1-p)^{l-n-1}.$$

Proof. Since

$$H(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}z^n,$$

according to the Lemma 1.2 we must show that

$$\sum_{n=2}^{\infty}n[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)(n)!}p^{n-1}(1-p)^{l-n}\leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\gamma\delta)+(1-\delta)]\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n-1)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n+1}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)p\sum_{n=0}^{\infty}\frac{(l-1)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)p(1-l)\sum_{n=0}^{\infty}\frac{(l-2)!}{(l-n-2)n!}p^{n}(1-p)^{l-n-2}\\
&\quad+(1-\delta)\sum_{n=1}^{\infty}\frac{(l-1)!}{(l-n-1)n!}p^{n}(1-p)^{l-n-1}\\
&=(1-\gamma\delta)(1-l)p+(1-\delta)C\\
&\leq 1-\gamma.
\end{aligned}$$

This completes the proof. □

Theorem 2.4. *The operator $H(l,p,z)$ defined by (2.1) is in the class $T(\gamma,\delta)$ if and only if*

$$p(1-\delta\gamma)D+(1-\delta)E\leq\delta-1,$$

where

$$D=\sum_{n=0}^{\infty}\frac{(l-1)(l-2)!}{(l-n-2)(n+2)n!}p^{n}(1-p)^{l-n-2}$$

and

$$E=\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}.$$

Proof. As we know that

$$F(l,p,z)=z-\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}z^{n},$$

therefore according to the Lemma 1.1 we must show that

$$\sum_{n=2}^{\infty}[n-\gamma\delta n-\delta+\gamma\delta]\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\leq 1-\delta.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\gamma\delta)-\delta(1-\gamma)]\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\gamma\delta)+(1-\delta)]\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!(n-1)}{(l-n)n!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n(n-2)!}p^{n-1}(1-p)^{l-n}+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)p(l-1)\sum_{n=0}^{\infty}\frac{(l-2)!}{(l-n-2)(n+2)n!}p^{n}(1-p)^{l-n-2}\\
&\quad+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)(n)!}p^{n-1}(1-p)^{l-n}\\
&=(1-\gamma\delta)p\sum_{n=0}^{\infty}\frac{(l-1)(l-2)!}{(l-n-2)(n+2)n!}p^{n}(1-p)^{l-n-2}\\
&\quad+(1-\delta)\sum_{n=2}^{\infty}\frac{(l-1)!}{(l-n)n!}p^{n-1}(1-p)^{l-n}\\
&=p(1-\gamma\delta)D+(1-\delta)E\\
&\leq 1-\gamma.
\end{aligned}$$

This completes the proof. □

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Soft rough approximation operators via ideal

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Abstract

Soft rough approximation were introduced by Feng[7]. This paper extend soft rough approximation model by defining new soft rough approximation operators via ideal. An ideal on a set X is a non empty collection of subsets of X with heredity property which is also closed under finite unions. When I is the least ideal of $\wp(U)$, these two approximations coincide. We present the essential properties of new opertors via ideal and supported by illustrative examples. The notion of soft rough equal relations via ideal is proposed and related examples are examined. We also show that rough set via ideal [26] can be viewed as a special case of soft rough set via ideal, and these two notions will coincide provided that the underlaying soft set is a partition soft set. We obtain the structure of soft rough set via ideal, gives the structure of topologies induced by soft set and an ideal. Moreover, an example containing a comparative analysis between rough sets via ideal and soft rough sets via ideal is given. We show that soft rough approximation via ideal could provide a better approximation than rough set via ideal.

keywords: soft sets, rough approximations via ideal, soft rough sets via ideal, rough sets via ideal.

1. Introduction

In recent years vague concepts have been used in different areas as medical applications, pharmacology, economics, engineering since the classical mathematics methods are inadequate to solve many complex problems in these areas. Traditionally crisp (well-defined) property P(x) is used in mathematics, i.e., properties that are either true or false and each property defines a set: $\{x : x \textit{ has a property } P\}$[19]. Researchers have proposed many methods for vague notions. The most successful theoretical approach to the vagueness is undoubtedly fuzzy set theory [33] proposed by Zadeh in 1965. The basic idea of fuzzy set theory hinges on fuzzy membership function, which allows partial membership of elements to a set, i.e., it allows elements to belong to a set to a degree.

Rough set theory [20] is an extension of set theory for the analysis of a vague and inexact description of objects. Pawlak rough approximations are based on equivalence relation or their induced partition and subsystem, this requirement is not satisfied in many situations and thus limits the application domain of the rough set model. To solve this issue, generalizations of rough sets were considered. There are at least two approaches to generalize rough sets. One is to consider similarity, tolerance or general binary relation (see e.g.[30], [31],[32], Zhu [36]) rather than equivalence relation. The other is to extend the partition to cover (see e.g.[2, 3, 34, 36, 37]). Furthermore, as generalizations, rough sets were defined by fuzzy relation (see e.g.[5, 11, 12, 21, 22, 23, 24]) or a mapping [9, 26]. However, many of these generalizations have not been interconnected with each other.

All these theories have their own difficulties (see [23]). For example, theory of probabilities can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [16] proposed a completely new approach, which is called soft set theory, for modelling uncertainty. Molodtsov initiated a novel concept of soft set theory [16], which is a completely new approach for modeling vagueness in 1999. A soft set is a collection of approximate descriptions of an object. Molodtsov [16, 17] presented the fundamental results of the new theory and successfully applied it to several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration,theory of probability etc. He also showed that how soft set theory is free from the parametrization inadequacy syndrome of fuzzy set theory, rough set theory and etc. Soft systems provide a very general framework with the involvement of parameters. It has been found that fuzzy sets, rough sets and soft sets are closely related [1].

Maji et al. investigated the concept of fuzzy soft set in 2001 [13], a more generalized concept, which is a combination of fuzzy set and soft set and also studied some of its properties. This line of exploration was further investigated by several researchers [14, 28, 29]. Soft set and fuzzy soft set theories have rich potential for applications in several directions.

Feng et al. investigated the concept of soft rough set in 2010 [6] which is a combination of soft set and rough set. In [6, 7] basic properties of soft rough approximations were presented and supported by some illustrative examples. In fact, as soft set instead of an equivalence relation was used to granulate the universe of discourse. A new approach was introduced to soft rough sets which is called modified soft rough set (MSR-set) and some basic properties of MSR-sets were investigated in [25]. In [10] a new concept of soft class and soft class operations based on decision makers set are defined and some fundamental properties of soft class operations are investigated. In [18] soft rough sets and soft rough approximation operators on a complete atomic Boolean lattice are defined. Feng discussed soft set based group decision making in [8]. This study can be seen as a first attempt toward the possible application of soft rough approximations in multicriteria group decision making under vagueness.

It is well known that (fuzzy) ideal is an important tool for investigating rough sets (see e.g.[4, 27]). In Pawlak rough set model, any vague concept of a universe can be defined by a pair of precise concepts called the lower and upper approximations. Particularly, the empty set ϕ is a concept and the set $\{\phi\}$ is a special ideal. Hence, we have the following equivalent description of Pawlaks approximations. That is, the lower approximation contains all objects which the intersections between equivalence classes and the complement of the concept belong to $\{\phi\}$, and the upper approximation consists of all objects which the intersections between equivalence classes and the concept do not belong to $\{\phi\}$. It is a natural question to ask what does happen if we substitute a general ideal instead of the particular one. Here, the role of the ideal is to bring together some knowable and interrelated concepts of the universe, through which we can approximately obtain the imprecise concept. Since a given ideal has more concepts than that of $\{\phi\}$, the approximations based on ideals seem to enrich the Pawlaks approximations. In [27] we define new approximation operators in more general setting of complete atomic Boolean lattice by using an ideal.

The aim of this paper is to define new soft rough approximation operators in terms of an ideal. Our approach can be viewed as a generalization of several approaches that can be found in the literature. The reminder of this paper is organized as follows. In the following section, we recall some fundamental notions and propositions to be used in the present paper. In Section 3, the definition of soft rough approximations via ideal is proposed and basic properties are examined. These decrease the soft lower approximation and increase the soft upper approximation and hence increase the accuracy measure. We show by example that soft rough approximation via ideal reduce the soft boundary in comparison of soft rough approximation and the accuracy measure is better than the soft accuracy measure. So soft rough approximation via ideal could provide a better approximation than soft rough set. We also define soft rough equal relations in termes of soft rough approximation via ideal and explore some related properties. Finally, through an example we present a comparative analysis between rough set via ideal and soft rough set via ideal. In sction 4 we investigate the relationships between soft sets, topologies and an ideal, obtain the structure of topologies induced by a soft set and an ideal. In section 5 we investigate the relation between soft rough via ideal and rough set via ideal [27]. We show that rough set via ideal may be considered as a special case of soft rough set via ideal. Also, we define a new pair of soft rough approximation operators via ideal and giving the relationship between this pair and previous one.
Soft rough set approximation via ideal is a worth considering alternative to the soft rough set approximation and rogh approximation via ideal.

2. Preliminaries

In this section, we present the basic definitions and results of soft set theory which may found in earlier studies [15, 16, 17]. Throughout this paper, U refers to an initial universe, the complement of X in U is denoted by X', E is a set of parameters, $\wp(U)$ is the power set of X, and $A \subseteq E$.

Definition 2.1 [16] Let U be a universal set and E be a set of parameters. Let A be a non empty subset of E. A soft set over A, with support A ,denoted by f_A on U is defined by the set of ordered pairs

$$f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in \wp(U)\},$$

or is a function $f_A : E \to \wp(U)$ s.t

$$f_A(e) \neq \phi \quad \forall \quad e \in A \subseteq E \; and \; f_A(e) = \phi \; if \; e \notin A.$$

.

From now on, we will use $S(U, E)$ instead of all soft sets over U.

Definition 2.2 [16] The soft set $f_\phi \in S(U, E)$ is called null soft set, denoted by Φ, Here $F_\phi(e) = \phi, \forall e \in E$.

Definition 2.3 [15] Let $f_A \in S(U, E)$. If $f_A(e) = X, \forall e \in A$, then f_A is called A-absolute soft set, denoted by $\widetilde{A}$.
If $A = E$, then the A-absolute soft set is called absolute soft set denoted by $\widetilde{E}_U$.

Definition 2.4 [15] Let $f_A, g_B \in S(U, E)$. f_A is a soft subset of g_B, denoted $f_A \sqsubseteq g_B$ if $f_A(e) \subseteq g_B(e), \forall e \in E$.

Definition 2.5 [15] Let $f_A, g_B \in S(U, E)$. Union of f_A and g_B, is a soft set h_C defined by $h_C(e) = f_A(e) \bigcup g_B(e), \forall e \in E$, where $C = A \cup B$. That is,

$$h_C = f_A \sqcup g_B$$

Definition 2.6 [15] Let $f_A, g_B \in S(U, E)$. Intersection of f_A and g_B, is a soft set h_C defined by $h_C(e) = f_A(e) \bigcap g_B(e), \forall e \in E$ where $C = A \cap B$. That is

$$h_C = f_A \sqcap g_B.$$

Definition 2.7 [15] Let $f_A \in S(U, E)$. The complement of f_A, denoted by f'_A is defined by $f'_A(e) = (f(e))', \forall e \in E$.

Definition 2.8 [7] Let $f_A \in S(U, E)$.

i) f_E is called full, if $\bigcup_{a \in A} f(a) = U$;

iv) f_E is called partition of B if $\{f(a) : a \in A\}$ forms a partition of U.

Obviously, every partition soft set is full.

Definition 2.9 [35] Let $f_A \in S(U, E)$.

i) f_A is called keeping intersection, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cap f(b) = f(c)$;

ii) f_A is called keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \vee f(b) = f(c)$;

ii) f_A is called topological, if $\{f(a) : a \in A\}$ forms a topology on U.

Definition 2.10 [7]Let $f_A \in S(U, E)$. Then the Pair $P = (U, f_A)$ is called soft approximation space. We define a pair of operators $\underline{apr}_P, \overline{apr}_P : \wp(U) \to \wp(U)$ as follows:

$\underline{apr}_P(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a) \subseteq X\}$,

$\overline{apr}_P(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X \neq \emptyset\}$

The elements $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the **soft P-lower** and the **soft P-upper** approximations of X. Moreover, the sets

$$Pos_P(X) = \underline{apr}_P(X)$$
$$Neg_P(X) = (\overline{apr}_P(X))'$$
$$Bnd_P(X) = \overline{apr}_P(X) - \underline{apr}_P(X)$$

are called the soft P-positive region, the soft P-negative region and the soft P-boundary region of X, respectively. If $\underline{apr}_p(X) = \overline{apr}_P(X)$, X is said to be soft P-definable; otherwise X is called a soft P-rough set.

Definition 2.11[26] Let $\mathbf{B} = (B, \leq)$ be a bounded distributive lattice. A non empty subset I of B is called an ideal of B if for all $x,\ y \in B$

(i) $x,\ y \in I$ imply $x \vee y \in I$;

(ii) If $x \in I$ with $y \leq x$, then $y \in B$.

Definition 2.12[26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on B. For any element $x \in B$, let

$x^{\nabla I} = \bigvee \{x \wedge a : a \in A(B), \varphi(a) \wedge x' \in I \ \ and \ \ a \neq 0\}$, and

$x^{\triangle I} = \bigvee \{x \vee a : a \in A(B), \varphi(a) \wedge x \notin I \ \ and \ \ a \neq 1\}$.

The elements $x^{\nabla I}$ and $x^{\triangle I}$ are called the **lower** and the **upper** approximations of x via ideal I with respect to φ respectively. Two elements x and y are called equivalent via ideal I if they have the same upper and lower approximations via ideal I. The resulting equivalence classes are called rough sets via ideal I.

Proposition 2.13[26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on B, then for all $a \in A(B)$ and $x \in B$,

i) $a \leq x^{\nabla I} \iff \varphi(a) \wedge x' \in I$ and $a \leq x$;

ii) $a \leq x^{\triangle I} \iff \varphi(a) \wedge x \notin I$ or $a \leq x$.

Proposition 2.14 [26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on B, then

i) $0^{\triangle I}$=0 and $1^{\nabla I}$=1;

ii) $x \leq y$ implies $x^{\nabla I} \leq y^{\nabla I}$ and $x^{\triangle I} \leq y^{\triangle I}$.

Remark 2.15[26](1) In general, $x^{\nabla I} \leq x \leq x^{\triangle I}$.
(2) The two operations suggested in Definition 2.12 are suitable also for other operators based on binary relations. If U is any universal set, then $\wp(U)$ is a complete atomic boolean lattice whose atoms are singleton subsets of U. Let R and be a general relation on U and I any ideal on U. We define a mapping $\varphi : A(B) \longrightarrow B : U \longrightarrow \wp(U),\ x \longrightarrow R(x)$ where $R(x) = \{y \in U : xRy\}$. Then for any $X \subseteq U$, $X^{\nabla I} = \cup\{x \in U : R(x) \cap X' \in I\} \cap X$ and $X^{\triangle I} = \cup\{x \in U : R(x) \cap X \notin I\} \cup X$
If $X^{\nabla I} = X^{\triangle I}$, X is said to be R-I-definable; otherwise X is called R-I-rough set.

Table 1: Tabular representation of the soft set F_A

	u_1	u_2	u_3	u_4	u_5	u_6
e_1	0	1	1	0	0	0
e_2	0	0	0	0	1	0
e_3	1	0	0	1	0	0
e_4	0	1	0	0	0	1

3. Soft Rough Approximation operators via ideal

In this section we introduce soft rough approximations via ideal and soft rough set via ideal.

Definition 3.1 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. The triple (U, f_A, I) is called soft approximation space via ideal. We define a pair of operators $\underline{apr}_I, \overline{apr}_I : \wp(U) \to \wp(U)$ as follows:

$$\underline{apr}_I(X) = \{u \in X : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I\},$$

$$\overline{apr}_I(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X \notin I\}$$

The elements $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ are called the **soft I-lower** and the **soft I-upper** approximations of X via ideal. In general, we refer to $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ as soft rough approximations of X with respect to P via ideal. Moreover, the sets

$$Pos_I(X) = \underline{apr}_I(X)$$

$$Neg_I(X) = (\overline{apr}_I(X))'$$

$$Bnd_I(X) = \overline{apr}_I(X) - \underline{apr}_I(X)$$

are called the soft I-positive region, the soft I-negative region and the soft I-boundary region of X, respectively. If $\underline{apr}_I(X) = \overline{apr}_I(X)$, X is said to be soft I-definable; otherwise X is called a soft I-rough set.

Proposition 3.2 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$.

Proof: Let $u \in \underline{apr}_I(X)$, then $\exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I$. If $f(a) \cap X \in I$. So, $(f(a) \cap X) \cup (f(a) \cap X') \in I$ by properties of ideal. Thus $f(a) \cap (X \cup X') = f(a) \cap U = f(a) \in I$ a contradiction. Hence $f(a) \cap X \notin I$ and consequently $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$.

By Definition 3.1, we immediately have that $X \subseteq U$ is soft I-definable if the soft I-boundary region $Bnd_I(X)$ of X is empty. Also, By Proposition 3.2, we have $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$ for all $X \subseteq U$. Nevertheless, it is worth noticing that $X \subseteq \overline{apr}_I(X)$ does not hold in general.

Example 3.3 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let f_A be a soft over U given by Table 1. Let I be an ideal on U defined as follows $I = \{\phi, \{u_1\}, \{u_3\}, \{u_6\}, \{u_1, u_3\}, \{u_1, u_6\}, \{u_3, u_6\}, \{u_1, u_3, u_6\}\}$. Let $X = \{u_3, u_4, u_5\} \subseteq U$. So $X' = \{u_1, u_2, u_6\}$. Thus we have $\underline{apr}_I(X) = \{u_4, u_5\}$, and $\overline{apr}_I(X) = \{u_1, u_4, u_5\}$. So $\underline{apr}_I(X) \neq \overline{apr}_I(X)$ and X is soft I-rough set. In this case $X = \{u_3, u_4, u_5\} \nsubseteq \overline{apr}_I(X)$. Moreover, it is easy to see that $Pos_I(X) = \{u_4, u_5\}$, $Neg_I(X) = \{u_2, u_3, u_6\}$ and $Bnd_I(X) = \{u_1\}$. On the other hand, one can consider $X_1 = \{u_1, u_4, u_6\} \subseteq U$. Since $\underline{apr}_I(X_1) = \{u_1, u_4\} = \overline{apr}_I(X_1)$, then X_1 is a soft I-definable set.

Proposition 3.4 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for all $X \subseteq U$

i) $\underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\}$;

Table 2: Tabular representation of the soft set F_A

	u_1	u_2	u_3	u_4	u_5	u_6
e_1	1	0	0	0	0	1
e_2	0	0	0	0	1	0
e_3	0	0	0	1	0	0
e_4	1	1	0	0	1	0

ii) $\overline{apr}_I(X) = \bigcup\{f(a) : a \in A \text{ and } f(a) \cap X \notin I\}$.

Proof: i) Let $u \in \underline{apr}_I(X)$. So $u \in X$ and $\exists a \in A$, $s.t\ u \in f(a)$, $f(a) \cap X' \in I$. Hence $x \in X \cap \bigcup\{f(a) : a \in A \text{ and } f(a) \cap X' \in I\}$. The other inclusion can be proved similarly.

Definition 3.5 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. For any $X \subseteq U$ measure of accuracy for soft set with respect to X denoted by $A_P(X)$ is defined by
$A_P(X) = \frac{|\underline{apr}_P(X)|}{|\overline{apr}_P(X)|}$
where $|\underline{apr}_P(X)|$ and $|\overline{apr}_P(X)|$, denotes the cardinalities of the sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ respectively. Also, measure of accuracy for soft set with respect to X via ideal denoted by $A_I(X)$ is defined by
$A_I(X) = \frac{|\underline{apr}_I(X)|}{|\overline{apr}_I(X)|}$
where $|\underline{apr}_I(X)|$ and $|\overline{apr}_I(X)|$, denotes the cardinalities of the sets $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ respectively
Now, we show in the next example that soft rough approximation via ideal provide a better approximation than soft rough approximation which provide a better approximation than rough sets.

Example 3.6 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let f_A be a soft over U given by Table 2. Let I be an ideal on U defined as follows
$I = \{\phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}\}$. Let $X = \{u_1, u_5\} \subseteq U$. So $X' = \{u_2, u_3, u_4, u_6\}$. Thus $\underline{apr}_P(X) = \{u_5\}$, $\underline{apr}_I(X) = \{u_1, u_5\} \cap \{u_1, u_2, u_5\} = \{u_1, u_5\}$, $\overline{apr}_P(X) = \{u_1, u_2, u_5, u_6\}$ and $\overline{apr}_I(X) = \{u_1, u_2, u_5\}$. So $\underline{apr}_P(X) \subseteq \underline{apr}_I(X) \subseteq X \subseteq \overline{apr}_I(X) \subseteq \overline{apr}_P(X)$. Therefore $A_P(X) = \frac{\underline{apr}_P(X)}{\overline{apr}_P(X)} = \frac{1}{4}$ and $A_I(X) = \frac{\underline{apr}_I(X)}{\overline{apr}_I(X)} = \frac{2}{3}$. Consequently, $A_I(X) > A_P(X)$. Consequently accuracy measure via ideal is better than accuracy measure for soft sets.

Proposition 3.7 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal.

i) $\underline{apr}_I(\phi) = \phi = \overline{apr}_I(\phi)$

ii) $\underline{apr}_I(U) = \overline{apr}_I(U) = \bigcup_{a \in A} f(a)$;

iii) $X \subseteq Y$ implies $\underline{apr}_I(X) \subseteq \underline{apr}_I(Y)$ and $\overline{apr}_I(X) \subseteq \overline{apr}_I(Y)$.

iv) $I \subseteq J$ implies $\underline{apr}_I(X) \subseteq \underline{apr}_J(X)$

Proof: (i)Clearly, $\underline{apr}_I(\phi) = \phi$. Also, $\overline{apr}_I(\phi) = \bigcup\{f(a) : a \in A \text{ and } f(a) \cap \phi \notin I\} = \bigcup\{f(a) : a \in A \text{ and } \phi \notin I\} = \phi$.
(ii) $\underline{apr}_I(U) = \bigcup\{f(a) : a \in A \text{ and } f(a) \cap \phi \in I\} = \bigcup\{f(a) : a \in A \text{ and } \phi \in I\} = \bigcup_{a \in A} f(a)$. Also, since $f(a) \notin I\ \forall a \in A$, then $\overline{apr}_I(U) = \bigcup_{a \in A} f(a)$
(iii) Assume that $X \subseteq Y$ and $u \in \underline{apr}_I(X)$. So $u \in X$ and $\exists a \in A$, $s.t\ u \in f(a)$, $f(a) \cap X' \in I$. Since $Y' \subseteq X'$, then $f(a) \cap Y' \in I$ by properties of ideal. Consequently, $u \in \underline{apr}_I(Y)$. The other part can be proved similarly.
(iv) Obvious

Proposition 3.8 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for all $X, Y \subseteq U$

i) $\underline{apr}_I(X \cup Y) \supseteq \underline{apr}_I(X) \cup \underline{apr}_I(Y)$

ii) $\underline{apr}_I(X \cap Y) \subseteq \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

iii) If f_A is keeping intersections, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

iv) If f_A is partition, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

v) $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$

vi) $\overline{apr}_I(X \cap Y) \subseteq \overline{apr}_I(X) \cup \overline{apr}_I(Y)$

Proof: (i) and (ii) follow immediately by Proposition 3.7.
(iii) By (i), $\underline{apr}_I(X \cap Y) \subseteq \underline{apr}_I(X) \cap \underline{apr}_I(Y)$. Let $u \in \underline{apr}_I(X) \cap \underline{apr}_I(Y)$, then $u \in X \cap Y$ and there exists $a, b \in A$ such that $u \in f(a),\ f(a) \cap X' \in I,\ u \in f(b),\ and\ f(b) \cap X' \in I$. Since f_A is keeping intersections, then there exists $c \in A,$ such that $f(a) \cap f(b) = f(c)$. By properties of ideal, $f(a) \cap f(b) \cap X' \in I$. So we prove that there exists $c \in A,$ such that $u \in f(c)$ and $f(c) \cap X' \in I$. Hence $u \in \underline{apr}_I(X \cap Y)$ and consequently, $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$.

(iv) Let $u \in \underline{apr}_I(X) \cap \underline{apr}_I(Y)$, then $u \in X \cap Y$ and there exists $a, b \in A$ such that $u \in f(a),\ f(a) \cap X' \in I,\ u \in f(b),\ and\ f(b) \cap X' \in I$. Since f_A is partition, then $f(a) = f(b)$. So, Therefore $u \in \underline{apr}_I(X \cap Y)$. Consequently, $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$.

(v)By Proposition 3.7, $\overline{apr}_I(X \cup Y) \supseteq \overline{apr}_I(X) \cup \overline{apr}_I(Y)$. On the other hand, let $u \in \overline{apr}_I(X \cup Y)$, then there exists $a \in A$ such that $u \in f(a),\ f(a) \cap (X \cup Y) = (f(a) \cap X) \cup (f(a) \cap Y) \notin I$. Hence either $f(a) \cap X \notin I$ or $f(a) \cap Y \notin I$ by properties of ideal. So $u \in \overline{apr}_I(X) \cup \overline{apr}_I(Y)$ and consequently, $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$.

(vi) Follows immediately by Proposition 3.7.

Proposition 3.9 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for all $X \subseteq U$

i) $\overline{apr}_I(X) = \underline{apr}_I(\overline{apr}_I(X))$

ii) $\underline{apr}_I(X) \subseteq \overline{apr}_I(\underline{apr}_I(X))$

iii) $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$

iv) $\overline{apr}_I(X) \subseteq \overline{apr}_I(\overline{apr}_I(X))$

Proof:(i) Let $Y = \overline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X \notin I$ for some $a \in A$. By Proposition 3.4(ii), $Y = \overline{apr}_I(X) = \bigcup\{f(a) : a \in A\ and\ f(a) \cap X \notin I\}$. So there exists $a \in A$ such that $u \in f(a) \subseteq Y$. Hence $f(a) \cap Y' = \phi \in I$ and consequently, $u \in \underline{apr}_I(Y)$. Therefore $Y \subseteq \underline{apr}_I(Y)$. On the other hand, since $\underline{apr}_I(Y) \subseteq Y$ for any $Y \subseteq U$, then $Y = \underline{apr}_I(Y)$ as required.

(ii)Let $Y = \underline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X' \in I$ for some $a \in A$. But $Y = \underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\}$. We deduce that $u \in f(a)$ and $f(a) \cap Y = f(a) \cap X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\} = f(a) \cap X$. If $f(a) \cap X \in I$, then $(f(a) \cap X) \cup (f(a) \cap X') \in I$ (by properties of ideal) i.e $f(a) \cap (X \cup' X) = f(a) \cap U = f(a) \in I$ a contradiction. Therefore, $f(a) \cap X = f(a) \cap Y \notin I$. Hence $u \in \overline{apr}_I(Y)$ and so $Y \subseteq \overline{apr}_I(Y)$.

(iii) Let $Y = \underline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X' \in I$ for some $a \in A$. But $Y = \underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A\ and\ f(a) \cap X' \in I\}$. We deduce that $f(a) \cap X \subseteq Y$. Hence $f(a) \cap X \cap Y' = (f(a) \cap Y') \cap X = \phi$. Hence $f(a) \cap Y' \subseteq X'$ and thus $f(a) \cap Y' \subseteq f(a) \cap X'$. Since $f(a) \cap X' \in I$, then $f(a) \cap Y' \in I$. Consequently, $u \in \underline{apr}_I(Y)$. So $Y \subseteq \underline{apr}_I(Y)$.

(iv) Let $Y = \overline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X \notin I$ for some $a \in A$. But $Y = \overline{apr}_I(X) = \bigcup\{f(a) : a \in A\ and\ f(a) \cap X \notin I\}$. It follows that $u \in f(a)$ and $f(a) \cap Y = f(a) \supseteq f(a) \cap X \notin I$ by properties of ideal. So $u \in \overline{apr}_I(Y)$ and hence $Y \subseteq \overline{apr}_I(Y)$.

Example 3.10 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let F_A be a soft over U given by Table 2. Let I be an ideal on U defined as follows $I = \{\phi, \{u_2\}, \{u_3\}, \{u_2, u_3\}\}$.Let $X = \{u_1, u_5, u_6\} \subseteq U$. So we have $X' = \{u_2, u_3, u_4\}$, and hence $\underline{apr}_I(X) = X \cap \{u_1, u_2, u_5, u_6\} = \{u_1, u_5, u_6\} = \{u_1, u_5, u_6\}$ and $\overline{apr}_I(X) = \{u_1, u_2, u_5, u_6\} = f(e_1) \cup f(e_2) \cup f(e_4)$. Let $Y = \overline{apr}_I(X)$. Then we have

$$\underline{apr}_I(\overline{apr}_I(X)) = \underline{apr}_I(Y) = f(e_1) \cup f(e_2) \cup f(e_4) = \overline{apr}_I(X) = Y.$$

Also, we have $\overline{apr}_I(\underline{apr}_I(X)) = \overline{apr}_I(X) = Y \supset_{\neq} X = \underline{apr}_I(X)$, which suggests that the inclusion (ii) in Proposition may hold strictly. Moreover, it is easy to see that $\underline{apr}_I(\underline{apr}_I(X)) = \underline{apr}_I(X)$.
Let $X_1 = \{u_4, u_6\}$, then $\overline{apr}_I(X_1) = \{u_1, u_4, u_6\}$. If $Y = \overline{apr}_I(X_1)$, then

$$\overline{apr}_{I_1}(\overline{apr}_I(X_1)) = \overline{apr}_I(Y_1) = \{u_1, u_2, u_4, u_5, u_6\} \supset_{\neq} Y_1 = \overline{apr}_I(X_1)$$

which indicates that the inclusion in Proposition may be strict.

Proposition 3.11 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then the following properties hold

i) If f_A is keeping union, then

- **a)** for any $X \subseteq U$, there exists $a \in A$ such that $\underline{apr}_I(X) = f(a) \cap X$
- **a)** for any $X \subseteq U$, there exists $a \in A$ such that $\overline{apr}_I(X) = f(a)$

ii) If f_A is full and keeping union, then

$$\overline{apr}_I(X) = U \text{ for any } X \subseteq U \text{ such that } X \notin I$$

Proof:i) This holds by Proposition 3.4.
ii) Since f_A is full and keeping union, then $U = \bigcup_{a \in A} f(a) = f(a^*)$ for some $a^* \in A$. For each $X \subseteq U$ such that $X \notin I$ and each $u \in U$, $u \in f(a^*)$ and $f(a^*) \cap X = X \notin I$. Therefore $\overline{apr}_I(X) = U$.

Proposition 3.12 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for any $X \subseteq U$, X is soft I-definable if and only if $\overline{apr}_I(X) \subseteq X$.

Proof: If X is soft I-definable, then $\overline{apr}_I(X) = \underline{apr}_I(X) \subseteq X$. Conversely, suppose that $\overline{apr}_I(X) \subseteq X$ for $X \subseteq U$. Since $f(a) \notin I\ \forall a \in A$, then $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$ by Proposition 3.2. To show that X is soft I-definable, it remains to prove that $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. Let $u \in \overline{apr}_I(X)$. Then $\exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X \notin I$. It follows that $u \in f(a) \subseteq \overline{apr}_I(X) \subseteq X$. So $u \in X$, $u \in f(a)$ and $f(a) \cap X' = \phi \in I$. Therefore $u \in \underline{apr}_I(X)$ and so $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$ as required.

Example 3.13 To illustrate the above result, we revisit Example 3.6. Let $X = \{u_2, u_4\} \subseteq U$. So $X' = \{u_1, u_3, u_5, u_6\}$, $\underline{apr}_I(X) = \{u_4\} = \overline{apr}_I(X)$. Hence $\overline{apr}_I(X) \subseteq X$ and X is soft I-definable set. On the other hand, for $X_1 = \{u_4, u_6\} \subseteq U$, $X_1{}' = \{u_1, u_2, u_3, u_5\}$, $\underline{apr}_I(X_1) = \{u_4, u_6\} \cap \{u_1, u_4, u_6\} = \{u_4, u_6\}$ and $\overline{apr}_I(X_1) = \{u_1, u_4, u_6\}$. Thus $\overline{apr}_I(X_1) \nsubseteq X$ and X_1 is soft I-rough set.

Proposition 3.14 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. The following conditions are equivalent

i) S is a full soft set.

ii) $\underline{apr}_I(U) = U$

iii) $\overline{apr}_I(U) = U$

Proof: $\underline{apr}_I(U) = U \cap (\bigcup\{f(a) : a \in A \; and \; f(a) \cap U' \in I\}) = \bigcup\{f(a) : a \in A \; and \; f(a) \cap \phi \in I\}) = \bigcup\{f(a) : a \in A \; and \; \phi \in I\}) = \bigcup_{a\in A} f(a)$.
Hence by definition, $S = (f, A)$ is a full soft set if and only if $\underline{apr}_I(U) = U$. That is, conditions (i) and (ii) are equivalent. Similarly, we can show that (i) and (iii) are equivalent conditions.

Proposition 3.15 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. The following conditions are equivalent

i) $X \subseteq \overline{apr}_I(X) \; \forall \; X \subseteq U$

ii) $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$

Proof: Assume that (i) holds, then $\{u\} \subseteq \overline{apr}_I(\{u\}) \; \forall \; u \in U$i.e, $\overline{apr}_I(\{u\}) \neq \phi$.
Assume that (ii) holds. Let $u \in X$, so by (ii)$\overline{apr}_I(\{u\}) \neq \phi$. Let $v \in \overline{apr}_I(\{u\})$, then $\exists a \in A, \; s.t \; v \in f(a) and \; f(a) \cap \{u\} \notin I$. So $f(a) \cap \{u\} \neq \phi$. It follows that $u = v \in f(a)$. Since $f(a) \cap \{u\} \notin I$ and $\{u\} \subseteq X$, then $f(a) \cap X \notin I$. Consequently, $u \in \overline{apr}_I(X)$.

Proposition 3.16 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. If $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then for any $X \subseteq U$

i) $(\underline{apr}_I(X))' \subseteq \overline{apr}_I(X')$

ii) $Neg_I(X) = (\overline{apr}_I(X))' \subseteq \underline{apr}_I(X')$

Proof: If $(\underline{apr}_I(X))'$ is empty, then clearly we have the inclusion (i). Suppose $(\underline{apr}_I(X))' \neq \phi$. Let $u \in (\underline{apr}_I(X))'$. Since f_A is full, then $\exists a_o \in A, \; s.t \; u \in f(a_o)$. Note also that
$(\underline{apr}_I(X))' = \{u \in U : \forall a \in A, \; u \in f(a) \;\Rightarrow\; f(a) \cap X' \notin I\} \cup X'$. Thus it follows that either $u \in X'$ or $f(a_o) \cap X' \notin I$ since $u \in f(a_o)$. If $u \in X'$, since $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then $X' \subseteq \underline{apr}_I(X')$ by Proposition 3.15. Therefore $u \in \overline{apr}_I(X')$. If $f(a_o) \cap X' \notin I$, then $u \in \overline{apr}_I(X')$. Consequently, $(\underline{apr}_I(X))' \subseteq \overline{apr}_I(X')$.
(ii)It is clear that the inclusion $Neg_I(X) = (\overline{apr}_I(X))' \subseteq \underline{apr}_I(X')$ holds when the set $(\overline{apr}_I(X))'$ is empty. So suppose that $(\overline{apr}_I(X))' \neq \phi$. Let $u \in (\overline{apr}_I(X))'$. Since $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then $X \subseteq \overline{apr}_I(X)$ by Proposition 3.15 and thus $u \in X'$. Since f_A is full, then $\exists a_o \in A, \; s.t \; u \in f(a_o)$. But we have that
$Neg_I(X) = (\overline{apr}_I(X))' = \{u \in U : \forall a \in A, \; u \in f(a) \;\Rightarrow\; f(a) \cap X \in I\}$. Thus it follows that $f(a_o) \cap (X')' \in I$ since $u \in f(a_o)$. Therefore $u \in \underline{apr}_I(X')$.

Definition 3.17 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let $X \subseteq U$, We define the following seven types of soft rough sets via ideal

i) X is roughly soft I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_I(X) \neq U$

ii) X is internally soft I-definable if $\underline{apr}_I(X) = \phi$ and $\overline{apr}_I(X) \neq U$

iii) X is externally soft I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_I(X) = U$

iv) X is totally soft I-definable if $\underline{apr}_I(X) = \phi$ and $\overline{apr}_I(X) = U$

iv) X is externally soft P-I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_P(X) = U$

iv) X is internally soft P-I-definable if $\underline{apr}_P(X) = \phi$ and $\overline{apr}_I(X) \neq U$

Proposition 3.18 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let $X \subseteq U$.

i) If X is roughly soft P-definable, then it is roughly soft I-definable.

ii) If X is totally soft I-definable, then it is totally soft P-definable.

Proof: Obvious.

Definition 3.19 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. For any $X, Y \subseteq U$ we define

i) $X \sim_I Y \Longleftrightarrow \underline{apr}_I(X) = \underline{apr}_I(Y)$

ii) $X \sim^I Y \Longleftrightarrow \overline{apr}_I(X) = \overline{apr}_I(Y)$

iii) $X \approx_I Y \Longleftrightarrow X \sim_I Y$ and $X \sim^I Y$

These binary relations are called the lower soft rough equal relation via ideal,the upper soft rough equal relation via ideal, and the soft rough equal relation via idea, respectively.
It is easy to verify that the relations defined above are all equivalence relations over $\wp(U)$.

Proposition 3.20 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. For any $X, Y \subseteq U$ we have

i) $X \sim^I Y \Longleftrightarrow X \sim^I (X \cup Y) \sim^I Y$

ii) $X \sim^I X_1, Y \sim^I Y_1 \Longrightarrow (X \cup Y) \sim^I (X_1 \cup Y_1)$

iii) $X \sim^I Y \Longrightarrow X \cup (Y') \sim^I U$

iv) $X \subseteq Y,\ Y \sim^I \phi \Longleftrightarrow X \sim^I \phi$

v) $X \subseteq Y,\ X \sim^I U \Longleftrightarrow Y \sim^I U$

Proof:(i)If $X \sim^I Y$, then $\overline{apr}_I(X) = \overline{apr}_I(Y)$. Since $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$, we deduce $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) = \overline{apr}_I(Y)$ and so $X \sim^I (X \cup Y) \sim^I Y$. Conversely, if $X \sim^I (X \cup Y) \sim^I Y$, then we immediately have that $X \sim^I Y$ by using the transitivity of the relation $\sim^I$.
(ii) Assume that $X \sim^I X_1$ and $Y \sim^I Y_1$. Then by definition, we know that $\overline{apr}_I(X) = \overline{apr}_I(X_1)$ and $\overline{apr}_I(Y) = \overline{apr}_I(Y_1)$. Since $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$ and $\overline{apr}_I(X_1 \cup Y_1) = \overline{apr}_I(X_1) \cup \overline{apr}_I(Y_1)$, we deduce that $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X_1 \cup Y_1)$, whence $(X \cup Y) \sim^I (X_1 \cup Y_1)$.
(iii) Suppose that $X \sim^I Y$. Then by definition, $\overline{apr}_I(X) = \overline{apr}_I(Y)$. Since $\overline{apr}_I(X \cup Y') = \overline{apr}_I(X) \cup \overline{apr}_I(Y')$ and $\overline{apr}_I(U) = \overline{apr}_I(Y) \cup \overline{apr}_I(Y')$, it follows that $\overline{apr}_I(X \cup Y') = \overline{apr}_I(U)$; hence $X \cup (Y') \sim^I U$ as required.
(iv) Let $X \subseteq Y$ and $Y \sim^I \phi$. Then we deduce $\overline{apr}_I(X) \subseteq \overline{apr}_I(Y) = \overline{apr}_I(\phi) = \phi$.
Hence $\overline{apr}_I(X) = \phi = \overline{apr}_I(\phi)$, and so we have that $X \sim^I \phi$.
(v) Suppose that $X \subseteq Y$ and $X \sim^I U$. Then we deduce $\overline{apr}_I(Y) \supseteq \overline{apr}_I(X) = \overline{apr}_I(U)$. Since $Y \subseteq U$, then $\overline{apr}_I(Y) \supseteq \overline{apr}_I(U)$. Therefore $\overline{apr}_I(Y) = \overline{apr}_I(U)$, and so $Y \sim^I Y$ as required.

Proposition 3.21 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. If f_A is keeping intersection, then for any $X, Y \subseteq U$ we have

i) $X \sim_I Y \Longleftrightarrow X \sim_I (X \cap Y) \sim_I Y$

ii) $X \sim_I X_1, Y \sim_I Y_1 \Longrightarrow (X \cap Y) \sim_I (X_1 \cap Y_1)$

iii) $X \sim_I Y \Longrightarrow X \cap (Y') \sim_I \phi$

iv) $X \subseteq Y,\ Y \sim_I \phi \Longrightarrow X \sim_I \phi$

v) $X \subseteq Y,\ X \sim_I U \Longleftrightarrow Y \sim_I U$

Proposition 3.22 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for any $X \subseteq U$

$$\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}$$

Table 3: An information table

	u_1	u_2	u_3	u_4	u_5	u_6
Sex	Woman	Woman	Man	Man	Man	Man
Age category	Young	Young	$Matureage$	Old	$Mature\ age$	$Baby$
Living area	City	City	$City$	$Village$	$City$	$Village$
Habits	NSND	NSND	$Smoke$	SD	$Smoke$	$NSND$

Proof: Let $u \in \underline{apr}_I(X)$. If $X \sim_I Y$, then by definition $\underline{apr}_I(X) = \underline{apr}_I(Y)$. But $\underline{apr}_I(Y) \subseteq Y$ for any $Y \subseteq U$. It follows that $u \in \underline{apr}_I(X) = \underline{apr}_I(Y) \subseteq Y$.
Hence $u \in \bigcap\{Y \subseteq U : X \sim_I Y\}$, and so $\underline{apr}_I(X) \subseteq \bigcap\{Y \subseteq U : X \sim_I Y\}$. Next, we show that the reverse inclusion $\bigcap\{Y \subseteq U : X \sim_I Y\} \subseteq \underline{apr}_I(X)$ also holds. Let $u \in \bigcap\{Y \subseteq U : X \sim_I Y\}$. Then by Proposition 3.9, we have $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$. Thus $X \sim_I \underline{apr}_I(X)$, and it follows that $u \in \underline{apr}_I(X)$. Consequently, we conclude that $\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}$.

Example 3.23 As in Example 3.6. Let $X = \{u_4, u_5, u_6\} \subseteq U$. So we have $X' = \{u_1, u_2, u_3\}$, and hence $\underline{apr}_I(X) = X \cap \{u_1, u_2, u_4, u_5, u_6\} = \{u_4, u_5, u_6\} = X$. It is easy to see that

$$\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}.$$

Example 3.24 Let us consider the following soft set $S = f_E$ which describes life expectancy. Suppose that the universe $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ consists of six persons and $E = \{e_1, e_2, e_3, e_4\}$ is a set of decision parameters. The e_i (i = 1,2,3,4) stands for "under stress", "young", "drug addict" and "healthy". Set $f(e_1) = \{u_1, u_6\}$, $f(e_2) = \{u_5\}$, $f(e_3) = \{u_4\}$; and $f(e_4) = \{u_1, u_2, u_6\}$. The soft set f_E can be viewed as the following collection of approximations:
$f_E = \{(understress, \{u_1, u_6\}); (young, \{u_5\}); (drugaddict, \{u_4\}); (healthy; \{u_1, u_2, u_6\})\}$.
On the other hand, "life expectancy" topic can also be described using rough sets as follows: The evaluation will be done in terms of attributes: "sex", "age category", "living area", "habits", characterized by the value sets "{man, woman}", "{baby, young, mature age, old}", "{village, city}", "{smoke, drinking, smoke and drinking, no smoke and no drinking}". We denote "smoke and drinking" by SD and "no smoke and no drinking" by NSND. The information will be given by Table 3, where the rows are labeled by attributes and the table entries are the attribute values for each person. From here we obtain the following equivalence classes, induced by the above mentioned attributes:
$[u_1]_R = [u_2]_R = \{u_1, u_2\}$, $[u_3]_R = [u_5]_R = \{u_3, u_5\}$, $[u_4]_R = \{u_4\}$, $[u_6]_R = \{u_6\}$.
Let I be an ideal on U defined as follows
$I = \{\phi, \{u_2\}, \{u_3\}, \{u_2, u_3\}\}$.
Let X be a target subset of U, that we wish to represent using the above equivalence classes. Hence we analyze the upper and lower approximations of X, in some particular cases:
1. Let $X = \{u_5\}$. It follows that
$X^{\nabla I} = \{u_5\}$, $X^{\triangle I} = \{u_3, u_5\}$. So X is R-I-rough.
Let us calculate now the soft I-lower and I-upper approximations of X. We obtain
$\underline{apr}_I(X) = \{u_5\} = X$, $\overline{apr}_I(X) = \{u_5\} = X$
hence X is soft I-definable.
2. Let $X = \{u_2, u_5\}$. It follows that $\underline{apr}_I(X) = \{u_5\} = \overline{apr}_I(X)$. So X is soft I-definable. On the other hand, $\underline{apr}_P(X) = \{u_5\}$, $\overline{apr}_P(X) = \{u_1, u_2, u_5, u_6\}$, hence X is soft P-rough.

The above results show that soft rough set approximation via ideal is a worth considering alternative to the rough set approximation via ideal. Soft rough sets via ideal could provide a better than rough sets via ideal do, depending on the structure of the equivalence approximation classes and of the subsets f(e), where $e \in E$.

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4. The relations among soft sets, ideal and topologies

In this section, we investigate the relationship between topological soft sets, topologies and an ideal.

Theorem 4.1 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. If f_A is full, then

i) $\tau_f = \{X \subseteq U : X = \underline{apr}_I(X)\}$ is a generalized topology on U.

ii) If f_A is keeping intersections, then τ_f is a topology on U.

Proof: Since $\underline{apr}_I(\phi) = \phi$, then $\phi \in \tau_f$. Let $\Im \subseteq \tau_f$. Denote $\Im = \{X_\alpha : \alpha \in \Gamma\}$ where Γ is an index set. Put $X = \bigcup\{X_\alpha : \alpha \in \Gamma\}$. Since $X_\alpha \subseteq X$ for each $\alpha \in \Gamma$, then $X_\alpha = \underline{apr}_I(X_\alpha) \subseteq \underline{apr}_I(X)$ by Proposition 3.7. So $X = \bigcup\{X_\alpha : \alpha \in \Gamma\} \subseteq \underline{apr}_I(X)$. Thus $\underline{apr}_I(X) = X$. This implies $\bigcup\{X_\alpha : \alpha \in \Gamma\} \in \tau_f$. Hence τ_f is a generalized topology on U.
(ii) By Propositions and $\underline{apr}_I(U) = U$ and thus $U \in \tau_f$. Let $X, Y \in \tau_f$, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y) = X \cap Y$ by Proposition 3.8. So $X \cap Y \in \tau_f$. By (i) τ_f is a generalized topology on U. Thus τ_f is a topology on U.

Definition 4.2 Let $f_A \in S(U,E)$ be full and keeping intersections and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then τ_f is called the topology induced by f_A and an ideal I on U.

The following Theorem gives the topological structure on soft sets and an ideal(i.e. the structure of topologies induced by soft sets and an ideal).

Theorem 4.3 Let $f_A \in S(U,E)$ be full and keeping intersections and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then

i) $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f = \{\underline{apr}_I(X) : X \subseteq U\}$

ii) $\tau_f \supseteq \{f(a) : a \in A\}$

iii) $\underline{apr}_I(X)$ is an interior operator of τ_f

Proof: (i) Since $\overline{apr}_I(X) = \underline{apr}_I(\overline{apr}_I(X))$ by Proposition 3.9, then $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f$.
Obviously,

$$\tau_f \subseteq \{\underline{apr}_I(X) : X \subseteq U\}$$

Let $Y \in \{\underline{apr}_I(X) : X \subseteq U\}$. Then $Y = \underline{apr}_I(X)$ for some $X \subseteq U$. By Proposition 3.9, $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$. So $Y \in \tau_f$. Thus $\tau_f \supseteq \{\underline{apr}_I(X) : X \subseteq U\}$. Hence $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f = \{\underline{apr}_I(X) : X \subseteq U\}$ as required.

(ii) For each $a \in A$, by Proposition 3.4 $\underline{apr}_I(f(a)) = f(a) \cap \bigcup\{f(a^*) : a^* \in A, f(a^*) \cap (f(a))' \in I\} \subseteq f(a)$. Since $f(a) \cap (f(a))' = \phi \in I$, then $f(a) \subseteq f(a) \cap \bigcup\{f(a^*) : a^* \in A, f(a^*) \cap (f(a))' \in I\} = \underline{apr}_I(f(a))$. Hence $f(a) = \underline{apr}_I(f(a))$ and so $f(a) \in \tau_f$. Therefore $\{f(a) : a \in A\} \subseteq \tau_f$.

(iii)It suffices to show that $\underline{apr}_I(X) = int(X) \; \forall X \subseteq U$. By (i) $\underline{apr}_I(X) \in \tau_f$ and since $\underline{apr}_I(X) \subseteq X$, then $\underline{apr}_I(X) \subseteq int(X)$. Conversely, let $Y \in int(X)$, then $Y \in \tau_f$ and $Y \subseteq X$. So $Y = \underline{apr}_I(Y) \subseteq \underline{apr}_I(X)$. Thus $int(X) = \bigcup\{Y : Y \in \tau_f, Y \subseteq X\} \subseteq \underline{apr}_I(X)$. Consequently, $\underline{apr}_I(X) = int(X)$.

Definition 4.4 Let τ be a topology on U and I be an ideal on U. Put $\tau = \{U_a : a \in A \; and \; U_a \notin I\}$ where A is the set of indexes. Define a mapping $f_\tau : A \to \wp(U)$ by $f_\tau(a) = U_a$ for each $a \in A$. Then, the soft set $(f_\tau)_A$ over U is called the soft set induced by τ on U and an ideal I on U.

Proposition 4.5 (1)Let τ be a topology on U and I be an ideal on U. Let $(f_\tau)_A$ be the soft set induced by τ and I on U. Then, $(f_\tau)_A$ is a full, keeping intersection, keeping union soft over U and

$(f_\tau)_A \notin I$ for each $a \in A$.

(2) Let τ_1 and τ_2 be two topologies on U and I_1 and I_2 be two ideals on U. Let $(f_{\tau_1})_{A_1}$ and $(f_{\tau_2})_{A_2}$ be two soft sets induced, respectively, by τ_1 and I_1 and, τ_2 and I_2 on U.
If $\tau_1 \subseteq \tau_2$, then

$$(f_{\tau_1})_{A_1} \supseteq (f_{\tau_2})_{A_2}$$

Proof: Obvious.

Proposition 4.6 Let τ be a topology on U, let I be an ideal on U such that $G \notin I$ $\forall G \in \tau$. Then there exists a full, keeping intersection, and keeping union soft set f_A with $f_A(a) \notin I$ for each $a \in A$ such that $\underline{apr}_I(X) \supseteq int(X)$ for each $X \in \wp(U)$ where (U, f_A, I) be a soft approximation space via ideal.

Proof: Put $\tau = \{U_a : a \in A\}$, where A is the set of indexes. Define a mapping $f : A \to \wp(U)$ by

$$f(a) = U_a \text{ for each } a \in A$$

By Proposition 4.5 f_A is full, keeping intersection, and keeping union and $f_A(a) \notin I$ for each $a \in A$. Now, we show that $\underline{apr}_I(X) \supseteq int(X)$ for each $X \in \wp(U)$. Let $X \in \wp(U)$ and $x \in int(X)$, then $\exists$ open neighbourhood W of x s.t $W \subseteq X$. So, $W = U_a$ for some $a \in A$. This implies $x \in U_a = f(a)$ and $f(a) \cap X' = \phi \in I$. Therefore $x \in \underline{apr}_I(X)$. Consequently, $\underline{apr}_I(X) \supseteq int(X)$.

Theorem 4.7 Let f_A be full and keeping intersections soft set over U and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let τ_f be the topology induced by f_A and I on U. Let $(f_{\tau_f})_B$ be the soft set induced by τ_f and I on U. Then

$$f_A \subseteq (f_{\tau_f})_B$$

Proof: By Theorem 4.3 $\tau_f \supseteq \{f(a) : a \in A\}$. Let $\tau_f = \{U_a : U_a \notin I, a \in B\}$, where $A \subseteq B$, $U_a = f(a)$ $\forall$ $a \in A$. Therefore $f_{\tau_f} : B \to \wp(U)$, where $f_{\tau_f}(a) = U_a$ for each $a \in B$. Hence $f_A \subseteq (f_{\tau_f})_B$.

5. The relations between soft rough approximation via ideal and rough approximation via ideal

In this section we will describe the relationship between rough sets via ideal and soft rough sets via ideal.

Definition 5.1 Let R be a binary relation on U and I be an ideal on U such that $R(a) \notin I$ $\forall a \in U$. . Define a mapping $f_R : U \to \wp(U)$ by

$$f_R(a) = R(a)$$

for each $a \in A$, where $A = U$. Then, $(f_R)_A$ is called the soft set induced by R and I on U.

Theorem 5.2 Let R be an equivalence relation on U, $(f_R)_A)$ be the soft set induced by R on U. Let I be an ideal on U and $P_R = (U, (f_R)_A, I)$ be a soft approximation space via ideal. If $\overline{apr}_I(\{u\}) \neq \phi$ $\forall$ $u \in U$, then for all $X \subseteq U$, $X^{\nabla I} = \underline{apr}_I(X)$ and $X^{\triangle I} = \overline{apr}_I(X)$.
Thus in this case,

i) $X \subseteq U$ is R-I-definable iff X is a soft I-definable set.

ii) $X \subseteq U$ is R-I-rough iff X is a soft I-rough set.

Proof: Let $X \subseteq U$ and $u \in U$. We show that $X^{\nabla I} = \underline{apr}_I(X)$. If $u \in \underline{R}_I(X) = \{x \in X : [x]_R \cap X' \in I\}$, then $[u]_R \cap X' \in I$. So, $\exists\ u \in X$ s.t $u \in [u]_R = f_R(u) \cap X' \in I$. Therefore $u \in \underline{apr}_I(X)$, and so $X^{\nabla I} \subseteq \underline{apr}_I(X)$. Conversely, assume that $u \in \underline{apr}_I(X)$. So,$u \in X$ and $\exists\ v \in U$ s.t $u \in f_R(v) = [v]_R$, $[v]_R \cap X' \in I$. It follows that $[u]_R = [v]_R$. Thus $[u]_R \cap X' = [v]_R \cap X' \in I$ and $u \in X^{\nabla I}$. Consequently, $X^{\nabla I} = \underline{apr}_I(X)$.
Now we show that $X^{\triangle I} = \overline{apr}_I(X)$. Let $u \in X^{\triangle I}$, then either $u \in X$ or $[u]_R \cap X \notin I$. If $u \in X$, then $u \in \overline{apr}_I(X)$ by Proposition 3.15 since $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$. If $[u]_R \cap X \notin I$, then $\exists\ u \in U$ s.t $u \in [u]_R = f_R(u) \cap X \notin I$ and therefore $u \in \overline{apr}_I(X)$. Therefore $X^{\triangle I} \subseteq \overline{apr}_I(X)$. Conversely, let $u \in \overline{apr}_I(X)$. Then $\exists\ v \in U$ s .t $u \in f_R(v) = [v]_R$, $[v]_R \cap X \notin I$. Thus $[u]_R = [v]_R$ and $[u]_R \cap X \notin I$. Hence $u \in X^{\triangle I}$ and consequently $X^{\triangle I} = \overline{apr}_I(X)$.

Definition 5.3 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$
(i) Define a binary relation R_f on U by

$$xR_f y \Leftrightarrow \exists\ a \in A\ , \{x, y\} \subseteq f(a)$$

for each $x, y \in U$. Then R_f is called the binary relation induced by f_A and I on U.
(ii) For each $x \in U$, define a successor neighbourhood $(R_f)_s(x) = \{y \in U : xR_f y\}$

Proposition 5.4 [35] Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let R_f be the binary relation induced by f_A on U. Then, the following properties hold.

i) R_f is a symmetric relation.

ii) If f_A is full, then R_f is a reflexive relation.

iii) If f_A is a partition, then R_f is an equivalence relation.

Proposition 5.5 [35] Let Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let R_f be the binary relation induced by f_A on U. Then, the following properties hold.

i) If $u \in f(a)$ for $a \in A$, then $f(a) \subseteq R_f(u)$.

ii) If f_A is a partition and $u \in f(a)$ for $a \in A$, then $f(a) = R_f(u)$.

iii) If f_A is keeping union, then for all $u \in U\ \exists a \in A,\ s.t\ R_f(u) = f(a)$.

Next, we define a new pair of soft rough approximation operators via ideal and giving the relationship between this pair and previous one.

Definition 5.6 Let Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. We define a pair of operators $\underline{apr}'_P, \overline{apr}'_P : \wp(U) \to \wp(U)$ as follows:

$\underline{apr}'_I(X) = \{x \in X : R_f(x) \cap X' \in I\}$,

$\overline{apr}'_I(X) = \{x \in U : R_f(x) \cap X \notin I\} \bigcup X$

Proposition 5.7 Let Let $f_A \in S(U, E)$ be partition and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let R_f be a binary relation induced by f_A on U. Then, the following properties hold for any $X \subseteq U$

i) If f_A is full, then

$$\underline{apr}_I(X) \supseteq \underline{apr}'_I(X)$$

ii) If f_A is full, keeping union and $X \notin I$, then

$$\overline{apr}_I(X) \supseteq \overline{apr}'_I(X)$$

iii) If f_A is partition, then

- **a)** $\underline{apr}_I(X) = \underline{apr}'_I(X)$
- **b)** If $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then $\overline{apr}_I(X) = \overline{apr}'_I(X)$

Proof: i) Suppose that $x \in \underline{apr}'_I(X)$. Then $x \in X$ and $R_f(x) \cap X' \in I$. Since f_A is full, then $x \in f(a)$ for some $a \in A$. By Proposition 5.5 $f(a) \subseteq R_f(x)$. Thus, $x \in f(a)$ and $f(a) \cap X' \in I$ by properties of ideal. Consequently, $x \in \underline{apr}_I(X)$. So,

$$\underline{apr}_I(X) \supseteq \underline{apr}'_I(X)$$

ii)Since $X \notin I$, then $X \neq \phi$. By Proposition 3.11(ii), $\overline{apr}_I(X) = U$. Thus

$$\overline{apr}_I(X) \supseteq \overline{apr}'_I(X)$$

iii) a) Suppose that $x \in \underline{apr}_I(X)$. Then, $x \in X$ and $\exists \; a \in A$ s.t $x \in f(a)$ and $f(a) \cap X' \in I$. Since f_A is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \underline{apr}'_I(X)$. Therefore

$$\underline{apr}_I(X) \subseteq \underline{apr}'_I(X)$$

Since every partition soft set is full, then by i)

$$\underline{apr}_I(X) = \underline{apr}'_I(X)$$

iii) b) Suppose that $x \in \overline{apr}_I(X)$. Then, $\exists \; a \in A$ s.t $x \in f(a)$ and $f(a) \cap X \notin I$. Since f_A is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \overline{apr}'_I(X)$. Therefore

$$\overline{apr}_I(X) \subseteq \overline{apr}'_I(X)$$

Suppose that $x \in \overline{apr}'_I(X)$. Then, either $x \in X$ or $R_f(x) \cap X \notin I$. If $x \in X$, since $\overline{apr}_I(\{u\}) \neq \phi \; \forall \; u \in U$, then $X \subseteq \overline{apr}_I(X)$ by Proposition 3.15 and therefore $x \in \overline{apr}_I(X)$. If $R_f(x) \cap X \notin I$, since f_A is full, then $x \in f(a)$ for some $a \in A$. Since f_A is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \overline{apr}_I(X)$. Therefore

$$\overline{apr}'_I(X) \subseteq \overline{apr}_I(X)$$

Hence $\overline{apr}_I(X) = \overline{apr}'_I(X)$.

Theorem 5.8 Let $f_A \in S(U,E)$ be partition and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let R_f be a binary relation induced by f_A on U. Then, for all $X \subseteq U$, $X^{\nabla I} = \underline{apr}_I(X) = \underline{apr}'_I(X)$ and $X^{\triangle I} = \overline{apr}_I(X) = \overline{apr}'_I(X)$.
where $X^{\nabla I_f}$ and $X^{\triangle I_f}$ are the rough approximations operators of X via ideal.
Proof: Follows immediately by Propositions 5.5 and 5.7.

Remark 5.9 Theorems 5.2 and 5.8 illustrate that rough set models via ideal can be viewed as a special case of soft rough sets via ideal.

Proposition 5.10 Let $f_A \in S(U,E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal and R_f be a binary relation induced by f_A on U.

i) If $X \subseteq U$ is R-I- definable, then X is soft I-definable.

ii) If $X \subseteq U$ is R-I- Rough, then X is soft I-Rough.

Proof: (i) If $X = \phi$, then X is soft I-definable by Proposition 3.7. Let $\phi \neq X \in \wp(U)$ be R-I-definable. by Proposition 3.2, $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$. It remains to show that $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. Let $u \in \overline{apr}_I(X)$, then there exists $a \in A$ such that $u \in f(a)$ and $f(a) \cap X \notin I$. By Proposition 5.5, $f(a) \subseteq R_f(u)$. Since $f(a) \cap X \notin I$, then $R_f(u) \cap X \notin I$ by Properties of ideal. But $u \in R_f(u)$, so $u \in X^{\triangle I} = X^{\nabla I}$. Hence $u \in X$ and $R_f(u) \cap X' \in I$. Therefore $f(a) \cap X' \in I$ by Properties of ideal and thus $u \in \underline{apr}_I(X)$. Consequently, $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. So X is soft I-definable.

(ii)Follows immediately by (i).

The following example shows that the converse of the above proposition is not true in general.

Example 5.11 Let $U = \{h_1, h_2, h_3, h_4, h_5\}$. Let I be an ideal on U and let R be a binary relation on U, defined as follows:

$I = \{\{h_1\}, \{h_2\}, \{h_1, h_2\}, \phi\}$ and let f_A be a soft set over U defined as follows

$f(a_1) = \{h_1, h_4\}$, $f(a_2) = \{h_4\}$, $f(a_3) = \{h_2, h_3, h_5\}$, $f(a_4) = \{h_1, h_2, h_4\}$.
Let R be the binary relation induced by f_A. Then

$R(h_1) = \{h_1, h_2, h_4\}$, $R(h_2) = \{h_1, h_2, h_3, h_4, h_5\}$, $R(h_3) = \{h_2, h_3, h_5\}$, $R(h_4) = \{h_1, h_2, h_4\}$, $R(h_5) = \{h_2, h_3, h_5\}$.
Let $X = \{h_2, h_3, h_5\} \subseteq U$. So $X' = \{h_1, h_4\}$. Thus $X^{\nabla I} = \{h_3, h_5\}$, and $X^{\triangle I} = \{h_2, h_3, h_5\}$. Also, $\underline{apr}_I(X) = \{h_2, h_3, h_5\}$, $\overline{apr}_I(X) = \{h_2, h_3, h_5\}$.

Then X is an R-I-rough set. But X is soft I-definable set.

6. Conclusion

In this paper, we have proposed the new concept of soft rough sets via ideal. We presented important properties of soft rough approximations via ideal based on soft approximation spaces via ideal, giving interesting examples. The accuracy measure is one of the ways of characterizing soft rough theory. Our approach makes the accuracy measures higher than the existing approximations. Soft rough relations via ideal were discussed. We researched relationships among soft sets, soft rough sets via ideal and topologies, obtained the structure of soft rough sets via ideal. Furthermore, we examined the relationship between soft rough sets via ideal and rough sets via ideal, and compared these two different models.

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Stability of C^*-ternary quadratic 3-homomorphisms

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Abstract. In this paper, we define C^*-ternary quadratic 3-homomorphisms associated with the quadratic mapping $f(x+y)+f(x-y)=2f(x)+2f(y)$, and prove the Hyers-Ulam stability of C^*-ternary quadratic 3-homomorphisms.

1. Introduction and preliminaries

As it is extensively discussed in [18], the full description of a physical system S implies the knowledge of three basic ingredients: the set of the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given statue. Originally the set of the observables were considered to be a C^*-algebra [10].

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [17] for linear mappings by considering an unbounded Cauchy difference.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called quadratic functional equation. In addition, every solution of the above equation is said to be a quadratic mapping. Czerwik [5] proved the Cauchy-Rassias stability of the quadratic functional equation. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for instances, [3, 7]).

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists (see [13]). As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [6] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc ([1, 20]). The comments on physical applications of ternary structures can be found in ([4, 8, 9, 12, 14, 15, 16]).

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Stability of C^*-ternary quadratic 3-homomorphisms

A ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $\Big[x, y, [z, u, v]\Big] = \Big[x, [y, z, u]v\Big] = \Big[[x, y, z], u, v\Big]$ and satisfies $\|[x, y, z]\| \le \|x\|\|y\|\|z\|$. A C^*-ternary algebra is a complex Banach space A equipped with a ternary product which is associative and $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and $\|[x, x, x]\| = \|x\|^3$ (see [21]). If a C^*-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $xoy := [x, e, y]$, $\quad x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if (A, o) is a unital C^*-algebra, then $[x, y, z] := xoy^*oz$ makes A into a C^*-ternary algebra.

Throughout this paper, let A and B be Banach ternary algebras.

A quadratic mapping $Q : A \to B$ is called a C^*-ternary quadratic homomorphism if

$$Q([x, y, z]) = [Q(x), Q(y), Q(z)]$$

for all $x, y, z \in A$.

Definition 1.1. Let A and B be C^*-ternary algebras. A quadratic mapping $Q : A \to B$ is called a C^*-ternary quadratic 3-homomorphism if it satisfies

$$Q([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) = [Q([x_1, x_2, x_3]), Q([y_1, y_2, y_3]), Q([z_1, z_2, z_3])]$$

for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in A$.

In this paper, we prove the Hyers-Ulam stability of C^*-ternary quadratic 3-homomorphisms in C^*-ternary algebras.

2. Stability of C^*-ternary quadratic 3-homomorphisms

In this section, we prove the Hyers-Ulam stability of C^*-ternary quadratic 3-homomorphisms for the quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y).$$

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^9 \to [0, \infty)$ such that*

$$\sum_{i=0}^{\infty} 4^{9i} \varphi(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{y_1}{2^i}, \frac{y_2}{2^i}, \frac{y_3}{2^i}, \frac{z_1}{2^i}, \frac{z_2}{2^i}, \frac{z_3}{2^i}) < \infty,$$

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \varphi(x, y, 0, 0, 0, 0, 0, 0, 0) \tag{2.1}$$

$$\Big\|f([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) - [f([x_1, x_2, x_3]), f([y_1, y_2, y_3]), f([z_1, z_2, z_3])]\Big\| \tag{2.2}$$
$$\le \varphi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$$

for all $x, y, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary quadratic 3-homomorphism $Q : A \to B$ such that

$$\|f(x) - Q(x)\| \le \widetilde{\varphi}(\frac{x}{2}, \frac{x}{2}, 0, 0, 0, 0, 0, 0, 0) \tag{2.3}$$

for all $x \in A$, where

$$\widetilde{\varphi}(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) := \sum_{i=0}^{\infty} 4^i \varphi(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \frac{x_3}{2^i}, \frac{y_1}{2^i}, \frac{y_2}{2^i}, \frac{y_3}{2^i}, \frac{z_1}{2^i}, \frac{z_2}{2^i}, \frac{z_3}{2^i})$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Proof. It follows from (2.1) that $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, x, 0, 0, 0, 0, 0, 0, 0) \tag{2.4}$$

for all $x \in A$. So

$$\|f(x) - 4f(\frac{x}{2})\| \leq \varphi(\frac{x}{2}, \frac{x}{2}, 0, 0, 0, 0, 0, 0, 0)$$

for all $x \in A$. Hence

$$\begin{aligned} \left\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\right\| &\leq \sum_{i=1}^{m-1} \left\|4^i f(\frac{x}{2^i}) - 4^{i+1} f(\frac{x}{2^{i+1}})\right\| \\ &\leq \sum_{i=0}^{m-1} 4^i \varphi\Big(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0, 0, 0, 0, 0, 0, 0\Big) \leq \sum_{i=0}^{m-1} 4^{9i} \varphi\Big(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0, 0, 0, 0, 0, 0, 0\Big) \end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.5) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

It follows from (2.1) that

$$\begin{aligned} \|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| &= \lim_{n\to\infty} 4^n \left\|f(\frac{x+y}{2^n}) + f(\frac{x-y}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n})\right\| \\ &\leq \lim_{n\to\infty} 4^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big) \leq \lim_{n\to\infty} 4^{9n} \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big) = 0 \end{aligned}$$

and so

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in A$.

It follows from (2.2) and the continuity of the ternary product that

$$\begin{aligned} &\Big\|Q([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) - [Q([x_1, x_2, x_3]), Q([y_1, y_2, y_3]), Q([z_1, z_2, z_3])]\Big\| \\ &= \lim_{n\to\infty} 4^{9n} \Big\|f([[\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}], [\frac{x_2}{2^n}, \frac{y_2}{2^n}, \frac{z_2}{2^n}], [\frac{x_3}{2^n}, \frac{y_3}{2^n}, \frac{z_3}{2^n}]]) - [f([\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}]), f([\frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}]), f([\frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n}])]\Big\| \\ &\leq \lim_{n\to\infty} 4^{9n} \varphi\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{y_1}{2^n}, \frac{y_2}{2^n}, \frac{y_3}{2^n}, \frac{z_1}{2^n}, \frac{z_2}{2^n}, \frac{z_3}{2^n}\Big) = 0 \end{aligned}$$

and so

$$Q([[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]) = [Q([x_1, x_2, x_3]), Q([y_1, y_2, y_3]), Q([z_1, z_2, z_3])]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Now, let $T : A \to B$ be another quadratic mapping satisfying (2.3). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= 4^n \left\|Q(\frac{x}{2^n}) - T(\frac{x}{2^n})\right\| \\ &\leq 4^n \Big(\left\|Q(\frac{x}{2^n}) - f(\frac{x}{2^n})\right\| + \left\|T(\frac{x}{2^n}) - f(\frac{x}{2^n})\right\|\Big) \\ &\leq 2 \cdot 4^n \varphi\Big(\frac{x}{2^n}, \frac{x}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big) \leq 2 \cdot 4^{9n} \varphi\Big(\frac{x}{2^n}, \frac{x}{2^n}, 0, 0, 0, 0, 0, 0, 0\Big), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $Q(x) = T(x)$ for all $x \in A$. This proves the uniqueness of Q. Thus the quadratic mapping $Q : A \to B$ is a unique C^*-ternary quadratic 3-homomorphism satisfying (2.3). □

Corollary 2.2. *Let r, θ be nonnegative real numbers with $r > 18$ and let $f : A \to B$ be a mapping satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \theta(\|x\|^r + \|y\|^r), \tag{2.6}$$

$$\begin{aligned}&\Big\|f([[x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]]) - [f([x_1,x_2,x_3]), f([y_1,y_2,y_3]), f([z_1,z_2,z_3])]\Big\| \\ &\le \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|y_1\|^r + \|y_2\|^r + \|y_3\|^r + \|z_1\|^r + \|z_2\|^r + \|z_3\|^r)\end{aligned} \tag{2.7}$$

for all $x, y, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary quadratic 3-homomorphism $Q : A \to B$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^r - 4}\|x\|^r$$

for all $x \in A$.

Proof. Defining

$$\varphi(x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|y_1\|^r + \|y_2\|^r + \|y_3\|^r + \|z_1\|^r + \|z_2\|^r + \|z_3\|^r)$$

in Theorem 2.1, we get the desired result. □

Theorem 2.3. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^9 \to [0,\infty)$ satisfying (2.1) and (2.2) such that*

$$\widetilde{\varphi}(x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3) := \sum_{i=0}^{\infty} \frac{1}{4^i}\varphi(2^ix_1, 2^ix_2, 2^ix_3, 2^iy_1, 2^iy_2, 2^iy_3, 2^iz_1, 2^iz_2, 2^iz_3) < \infty$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary quadratic 3-homomorphisms $Q : A \to B$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4}\widetilde{\varphi}(x,x,0,0,0,0,0,0,0) \tag{2.8}$$

for all $x \in A$

Proof. It follows from (2.4) that

$$\|f(x) - \frac{1}{4}f(2x)\| \le \frac{1}{4}\varphi(x,x,0,0,0,0,0,0,0)$$

for all $x \in A$

$$\|\frac{1}{4^l}f(2^lx) - \frac{1}{4^m}f(2^mx)\| \le \sum_{j=l}^{m-1}\|\frac{1}{4^j}f(2^jx) - \frac{1}{4^{j+1}}f(2^{j+1}x)\| \le \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}\varphi(2^jx, 2^jx, 0,0,0,0,0,0,0) \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.9) that the sequence $\{(\frac{1}{4^n})f(2^nx)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{(\frac{1}{4^n})f(2^nx)\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^nx)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).

It follows from (2.2) and the continuity of the ternary product that

$$\begin{aligned}
&\Big\|Q([[x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]]) - [Q([x_1,x_2,x_3]),Q([y_1,y_2,y_3]),Q([z_1,z_2,z_3])]\Big\| \\
&= \lim_{n\to\infty} \frac{1}{4^{9n}} \Big\|f([[2^n x_1,2^n y_1,2^n z_1],[2^n x_2,2^n y_2,2^n z_2],[2^n x_3,2^n y_3,2^n z_3]]) \\
&\quad - [f([2^n x_1,2^n x_2,2^n x_3]),f([2^n y_1,2^n y_2,2^n y_3]),f([2^n z_1,2^n z_2,2^n z_3])]\Big\| \\
&\le \lim_{n\to\infty} \frac{1}{4^{9n}} \varphi\Big(2^n x_1,2^n x_2,2^n x_3,2^n y_1,2^n y_2,2^n y_3,2^n z_1,2^n z_2,2^n z_3\Big) \\
&\le \lim_{n\to\infty} \frac{1}{4^{n}} \varphi\Big(2^n x_1,2^n x_2,2^n x_3,2^n y_1,2^n y_2,2^n y_3,2^n z_1,2^n z_2,2^n z_3\Big) = 0
\end{aligned}$$

and so

$$Q([[x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]]) = [Q([x_1,x_2,x_3]),Q([y_1,y_2,y_3]),Q([z_1,z_2,z_3])]$$

for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$.

The rest of the proof is similar to the proof of Theorem 2.1 □

Corollary 2.4. *Let r,θ be nonnegative real numbers with $r < 2$ and let $f : A \to B$ be a mapping satisfying* (2.6) *and* (2.7)*. Then there exists a unique C^*-ternary quadratic* 3*-homomorphism $Q : A \to B$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4-2^r}\|x\|^r$$

for all $x \in A$.

Proof. Defining

$$\varphi(x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|y_1\|^r + \|y_2\|^r + \|y_3\|^r + \|z_1\|^r + \|z_2\|^r + \|z_3\|^r)$$

in Theorem 2.3, we get the desired result. □

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Stability of functional equations in Šerstnev probabilistic normed spaces

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Abstract: In this paper, we investigate the uniform version and non-uniform version of the Hyers-Ulam stability of the additive functional equation $f(3x+y)+f(x+3y) = 4f(x)+4f(y)$ in Šerstnev probabilistic normed spaces with a triangle function.

AMS Subject Classification: 39B22, 39B52, 39B82, 46S40.
Keywords: Hyers-Ulam stability, additive functional equation, probabilistic normed space.

1. Introduction

In 1940, Ulam gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms.

In 1941, Hyers [7] considered the case of approximately additive mappings $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and for some $\varepsilon > 0$, where X and Y are Banach spaces. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \varepsilon$$

for all $x \in X$.

Aoki [1] and Rassias [14] provided a generalization of the Hyers theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. ([14]) *Let $f : X \to Y$ be a mapping from a normed vector space X into a Banach space Y subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $A : X \to Y$ defined by $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in X$. If $p < 0$ then (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$.*

The above theorem has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept (see [4, 8]).

In 1994, a generalization of Rassias theorem was obtained by Găvruta [6] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and functions (see [2]–[13], [15]–[22] and [27, 28]).

A PN space wwas first defined by Šerstnev in 1963 (see [25]).

We recall the definition of probabilistic space given in [23].

Definition 1.2. ([23]) A probabilistic normed space (briefly, PN space) is a quadruple $(X, \nu, \tau, \tau*)$, where X is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν is a mapping (the probabilistic norm) from V into Δ^+ such that for every choice of p and q in V the following hold:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$ (θ is the null vector in X);
(N2) $\nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0, 1]$.

A PN space is called a Šerstnev space if it satisfies (N1), (N3) and the following condition:

$$\nu_{\alpha_p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right)$$

holds for every $\alpha \neq 0 \in \mathbb{R}$ and $x > 0$. When T is a continuous t-norm such that $\tau = \Pi_T$ and $\tau^* = \Pi_{T^*}$, the PN space (X, ν, τ, τ^*) is called a Menger PN space (briefly, MPN space), and is denoted by (X, ν, τ).

Let (X, ν, τ) be an MPN space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} \nu(x_n - x)(t) = 1$$

for all $t > 0$. In this case x is called the limit of $\{x_n\}$. The sequence $\{x_n\}$ in MPN space (X, ν, τ) is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$, there exists some n_0 such that $\nu(x_n - x_m)(\delta) > 1 - \varepsilon$ for all $m, n \geq n_0$. Clearly, every convergent sequence in an MPN space is Cauchy. If each Cauchy sequence is convergent in an MPN space (X, ν, τ), then (X, ν, τ) is called a Menger probabiistic Banach space (briefly, MPB space). Recently, the stability of functional equations in PN spaces and MPN spaces has been investigated by some authors; see [5, 24] and references therein.

In this paper, we investigate the stability of additive functional equations in Šerstnev probabilistic normed space endowed with Π_M triangle function.

2. Main results

We begin our work with uniform version of the Hyers-Ulam stability in Šerstnev PN spaces in which we uniformly approximate a uniform approximate additive mapping.

Theorem 2.1. *Let X be a linear space and (Y, ν, Π_M) be a Šerstnev PB space. Let $\varphi : X \times X \to [0, \infty)$ be a control function such that*

$$\tilde{\varphi_n}(x, y) = 3^{-n-1}\varphi(3^n x, 3^n y) \quad (x, y \in X) \tag{2.1}$$

converges to zero. Let $f : X \to Y$ be a uniformly approximately additive function with respect to φ in the sense that

$$\lim_{t\to\infty} \nu\left(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y)\right)(t\varphi(x, y)) = 1 \tag{2.2}$$

uniformly on $X \times X$. Then $A(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ for each $x \in X$ exists and defines an additive mapping $A : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$\nu\left(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y)\right)(\delta\varphi(x, y)) > \alpha \tag{2.3}$$

for all $x, y \in X$, then

$$\nu\left(A(x) - f(x)\right)(\delta\tilde{\varphi_n}(x, 0)) > \alpha$$

for all $x \in X$.

Proof. Given $\varepsilon > 0$, by (2.2), we can choose some t_0 such that

$$\nu\left(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y)\right)(t\varphi(x, y)) > 1 - \varepsilon \tag{2.4}$$

for all $x, y \in X$ and all $t \geq t_0$. Subsituting $y = 0$ in (2.4) , we obtain

$$\nu\left(f(3x) - 3f(x)\right)(t\varphi(x, 0)) > 1 - \varepsilon$$

and replacing x by $3^n x$, we get

$$\nu\left(3^{-n-1}f(3^{n+1}x) - 3^{-n}f(3^n x)\right)\left(t3^{-n-1}\varphi(3^n x, 0)\right) > 1 - \varepsilon.$$

Allowing to a nonincreasing subequence, if necessary, we assume that $\left\{3^{-n-1}\varphi(3^n x, 3^n y)\right\}$ is nonincreasing.

Thus for each $n > m$ we have

$$\begin{aligned}
&\nu\left(3^{-m}f(3^m x) - 3^{-n}f(3^n x)\right)\left(t3^{-m-1}\varphi(3^m x, 0)\right) \qquad (2.5)\\
&= \nu\left(\sum_{k=m}^{n-1}\left(3^{-k}f(3^k x) - 3^{-k-1}f(3^{k+1}x)\right)\right)\left(t3^{-m-1}\varphi(3^m x, 0)\right)\\
&\geq \Pi_M\left\{\nu\left(3^{-m}f(3^m x) - 3^{-n}f(3^n x)\right),\right.\\
&\qquad \left.\nu\left(\sum_{k=m+1}^{n-1}\left(3^{-k}f(3^k x) - 3^{-k-1}f(3^{k+1}x)\right)\right)\right\}(t3^{-m-1}\varphi(3^m x, 0))\\
&\geq \Pi_M\left\{1 - \varepsilon; \Pi_M\left\{\nu\left(3^{-m}f(3^m x) - 3^{-n}f(3^n x)\right),\right.\right.\\
&\qquad \left.\left.\nu\left(\sum_{k=m+2}^{n-1}\left(3^{-k}f(3^k x) - 3^{-k-1}f(3^{k+1}x)\right)\right)\right\}\left(t3^{-m-2}\varphi(3^{m+1}x, 0)\right)\right\}\\
&\geq 1 - \varepsilon
\end{aligned}$$

for all $x \in X$.

The convergence of (2.1) implies that for given $\delta > 0$ there is $n_0 \in N$ such that

$$t_0 3^{-n-1}\varphi(3^n x, 0) < \delta \quad \forall n \geq n_0.$$

Thus by (2.5) we deduce that

$$\nu(3^{-m}f(3^m x) - 3^{-n}f(3^n x))(\delta) \ge \nu(3^{-m}f(3^m x) - 3^{-n}f(3^n x))(t_0 3^{-m-1}\varphi(3^m x, 0)) \ge 1-\epsilon \tag{2.6}$$

for each $n \ge n_0$. Hence $\frac{1}{3^n}f(3^n x)$ is a Cauchy sequence in Y. Since (Y, ν, Π_M) is complete, this sequence converges to some $A(x) \in Y$. Therefore, we can define a mapping $A : X \to Y$ by $A(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$, namely, for each $t > 0$ and $x \in X$,

$$\nu(A(x) - 3^{-n}f(3^n x))(t) = 1.$$

Next, let $x, y \in X$. Temporarily fix $t > 0$ and $0 < \varepsilon < 1$. Since $\frac{1}{3^n}\varphi(3^n x, 0)$ converges to zero, there is some $n_1 > n_0$ such that $t_0\varphi(3^n x, 0) < t3^{n+1}$ for all $n > n_1$, we have

$$\begin{aligned}
&\nu\left(A(3x+y) + A(x+3y) - 4A(x) - 4A(y)\right)(t)\\
&\ge \Pi_M(\Pi_M(\nu(A(3x+y) - 3^{-n-1}f(3^{n+1}(3x+y)))(t),\\
&\nu(A(x+3y) - 3^{-n-1}f(3^{n+1}(x+3y)))(t), \nu 4(A(x) - 3^{n-1}f(3^{n+1}x))(t)\\
&\nu 4(A(y) - 3^{n-1}f(3^{n+1}y))(t), \nu(f(3^{n+1}(3x+y)) + f(3^{n+1}(x+3y)) - 4f(3^{n+1}x)\\
&-4f(3^{n+1}y)))(3^{n+1}t))
\end{aligned}$$

and so we have

$$\begin{aligned}
&\lim_{n\to\infty} \nu\left(A(3x+y) - 3^{-n-1}f(3^{n+1}(3x+y))\right)(t) = 1,\\
&\lim_{n\to\infty} \nu\left(A(x+3y) - 3^{-n-1}f(3^{n+1}(x+3y))\right)(t) = 1,\\
&\lim_{n\to\infty} 4\nu\left(A(x) - 3^{-n-1}f(3^{n+1}x)\right)(t) = 1,\\
&\lim_{n\to\infty} 4\nu\left(A(y) - 3^{-n-1}f(3^{n+1}y)\right)(t) = 1
\end{aligned}$$

and, by (2.4), for large enough n, we have

$$\begin{aligned}
&\nu\left(f(3^{n+1}(3x+y)) + f(3^{n+1}(x+3y)) - 4f(3^{n+1}x) - 4f(3^{n+1}y)\right)(3^{n+1}t)\\
&\ge \nu(f(3^{n+1}(3x+y)) + f(3^{n+1}(x+3y)) - 4f(3^{n+1}x) - 4f(3^{n+1}y))(t_0\varphi(3^n x, 0)) \ge 1-\epsilon.
\end{aligned}$$

Thus

$$\nu\left(A(3x+y) + A(x+3y) - 4A(x) - 4A(y)\right)(t) \ge 1-\epsilon \quad \forall t > 0, 0 < \epsilon < 1.$$

It follows that $\nu\left(A(3x+y) + A(x+3y) - 4A(x) - 4A(y)\right)(t) = 1$ for all $t > 0$ and by $N(1)$, we have $A(3x+y) + A(x+3y) = 4A(x) + 4A(y)$.

For some positive δ and α, let us assume that (2.3) holds. Let $x \in X$. Setting $m = 0$ and $\alpha = 1-\epsilon$ in (2.6), we get

$$\nu(f(3^n x) - 3^n f(x))(\delta) \ge \alpha$$

for all positive integers $n \ge n_0$. For large enough n, we have

$$\begin{aligned}
&\nu(f(x) - A(x))(\delta 3^{-n-1}\varphi(3^n x, 0))\\
&\ge \Pi_M\left\{\nu(f(x) - 3^{-n}f(3^n x)), \nu(3^{-n}f(3^n x) - A(x))\right\}(\delta 3^{-n-1}\varphi(3^n x, 0)) \ge \alpha,
\end{aligned}$$

which implies

$$\nu(A(x) - f(x))(\delta\tilde{\varphi}_n(x, 0)) > \alpha,$$

as desired. □

Corollary 2.2. *Let X be a linear space and (Y, ν, Π_M) a Šerstnev PB space. Let $\varphi : X \times X \to [0, \infty)$ be a control function satisfying (2.2). Let $f : X \to Y$ be a uniformly approximately additive function with respect to φ. Then there is a unique additive mapping $A : X \to Y$ such that*

$$\lim_{n\to\infty} \nu(f(x) - A(x))(t\tilde{\varphi_n}(x, 0)) = 1 \tag{2.7}$$

uniformly on X.

Proof. The existence of uniform limit (2.7) immediately follows from Theorem 2.1. It remans to prove the uniqueness assertion.

Let S be another additive mapping satisfying (2.7). Fix $c > 0$. Given $\epsilon > 0$, by (2.7), for T and S, we can find some $t_0 > 0$ such that

$$\nu(f(x) - A(x))(t\tilde{\varphi_n}(x, 0)) > 1 - \epsilon,$$
$$\nu(f(x) - S(x))(t\tilde{\varphi_n}(x, 0)) > 1 - \epsilon$$

for all $x \in X$ and $t \geq t_0$. Fix for some $x \in X$ and find some integer n_0 such that

$$t_0 3^{-n}\varphi(3^{n+1}x, 0) > c\forall n \geq n_0.$$

Then we have

$$\begin{aligned}
\nu\left(S(x) - A(x)\right)(c) &\geq \Pi_M \left\{\nu\left(3^{-n}f(3^n x) - A(x)\right), \nu\left(S(x) - 3^{-n}f(3^n x)\right)\right\}(c) \\
&= \Pi_M \left\{\nu\left(f(3^n x) - A(3^n x)\right), \nu\left(S(3^n x) - f(3^n x)\right)\right\}(3^n c) \\
&\geq \Pi_M \left\{\nu\left(f(3^n x) - A(3^n x)\right), \nu\left(S(3^n x) - f(3^n x)\right)\right\}\left(t_0\varphi\left(3^{n+1}x, 0\right)\right) \\
&\geq 1 - \epsilon.
\end{aligned}$$

It follows that $\nu\left(S(x) - A(x)\right)(c) = 1$ for all $c > 0$. Thus $A(x) = S(x)$ for all $x \in X$. □

Now we present a non-uniform version of the Hyers-Ulam theorem in Šerstnev PN spaces.

Theorem 2.3. *Let X be a linear space. Let (Z, ω, Π_M) be a Šerstnev MPN space. Let $\psi : X \times X \to Z$ be a function such that for all $0 < \alpha < 3$,*

$$\omega(\psi(3x, 3y))(t) \geq \omega(\psi(x, y))(t) \tag{2.8}$$

for all $x, y \in X$ and $t > 0$. Let (Y, ν, Π_M) be a Šerstnev PB space and let $f : X \to Y$ be a ψ-approximately additive mapping in the sense that

$$\nu(f(3x + y) + f(x + 3y) - 4f(x) - 4f(y))(t) \geq \omega(\psi(x, y))(t) \tag{2.9}$$

for each $t > 0$ and $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\nu(f(x) - A(x))(t) \geq \omega\left(\frac{1}{3}\psi(x, 0)(t)\right)$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (2.9), we get

$$\nu(f(3x) - 3f(x))(t) \geq \omega(\psi(x, 0))(t) \quad (x \in X, t > 0). \tag{2.10}$$

Using (2.8) and using induction on n, we obtain

$$\omega(\psi(3^n x, 3^n x))(t) \geq \omega(\alpha^n\psi(x, 0))(t) \tag{2.11}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^{n-1}x$ in (2.10) and using (2.11), we get

$$\nu(f(3^n x) - 3f(3^{n-1}x))(t) \geq \omega\left((\alpha^{n-1}\psi(x,0)\right)(t) \tag{2.12}$$

for all $x \in X$ and $t > 0$. It follows from (2.12) that

$$\nu(3^{-n}f(3^n x) - 3^{-n+1}f(3^{n-1}x))(3^{-n}t) \geq \omega\left(\left(\frac{1}{\alpha}\right)\psi(x,0)\right)(\alpha^{-n}t)$$

and so

$$\nu\left(3^{-n}f(3^n x) - 3^{-n+1}f(3^{n-1}x)\right)\left(\left(\frac{\alpha^n}{3^n}\right)t\right) \geq \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(t)$$

for all $n > m \geq 0, x \in X$ and $t > 0$. So

$$\begin{aligned}&\nu(3^{-n}f(3^n x) - 3^{-m}f(3^m x))\left(\left(\frac{\alpha^{m+1}}{3^{m+1}}\right)t\right)\\ &= \nu\left(\sum_{n}^{k=m+1} 3^{-k}f(3^k x) - 3^{-k+1}f(3^{k-1}x)\right)\left(\left(\frac{\alpha^{m+1}}{3^{m+1}}\right)t\right) \geq \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(t)\end{aligned}$$

and hence

$$\nu(3^{-n}f(3^n x) - 3^{-m}f(3^m x))(t) \geq \omega\left(\left(\frac{1}{\alpha}\right)\psi(x,0)\right)\left(\left(\frac{\alpha^{m+1}}{3^{m+1}}\right)t\right) \tag{2.13}$$

for all $n > m \geq 0, x \in X$ and $t > 0$. Fix $x \in X$. Since

$$\lim_{s\to\infty} \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(s) = 1,$$

$3^{-n}f(3^n x)$ is a Cauchy sequence in (Y, ν, Π_M). Since (Y, ν, Π_M) is complete, this sequence converges to some point $A(x) \in \gamma$. It follows from (2.9) that

$$\begin{aligned}\nu(f(3^n(3x+y)) + f(3^n(x+3y)) - 4f(3^n x) - 4f(3^n y))(t) &\geq \omega(\psi(3^n x, 3^n y))(t)\\ &\geq \omega(\alpha^n\psi(x,y))(t)\\ &\geq \omega(\psi(x,y))(\alpha^{-n}t)\end{aligned}$$

and hence

$$\begin{aligned}&\nu(3^{-n}f(3^n(3x+y)) + 3^{-n}f(3^n(x+3y)) - 3^{-n}4f(3^n x) - 3^{-n}4f(3^n y)\\ &\qquad \geq \omega(\psi(x,y))\left(\left(\frac{3}{\alpha}\right)^n t\right).\end{aligned} \tag{2.14}$$

So we have

$$\begin{aligned}&\nu\left(A(3x+y) + A(x+3y) - 4A(x) - 4A(y)\right)(t)\\ &\geq \Pi_M\left\{\Pi_M\left\{\nu(A(3x+y) - 3^{-n}f(3^n(3x+y))), \nu(A(x+3y) - 3^{-n}f(3^n(x+3y)))\right\}(t),\right.\\ &\quad \Pi_M\left\{4\nu(A(x) - 3^{-n}f(3^n x)), 4\nu(A(y) - 3^{-n}f(3^n y)),\right.\\ &\quad\quad \left.\left.\nu\left(3^{-n}f(3^n(3x+y)) + 3^{-n}f(3^n(x+3y)) - 3^{-n}f(3^n x) - 3^{-n}f(3^n y)\right)\right\}(t)\right\}.\end{aligned}$$

By (2.14) and the fact that

$$\lim_{n\to\infty} \nu(A(z) - 3^{-n}f(3^n z)) = 1$$

for all $z \in X$ and $r > 0$, each term on the rignt-hand side tends to 1 as $n \to \infty$. Hence

$$\nu(A(3x+y) + A(x+3y) - 4A(x) - 4T(y))(t) = 1.$$

By (N1), we have

$$A(3x+y) + A(x+3y) = 4A(x) + 4A(y).$$

Let $x \in X$ and $t > 0$. Using (2.13) with $m = 0$, we get

$$\begin{aligned} \nu(A(x) - f(x))(t) &\geq \Pi_M \left\{\nu(A(x) - 3^{-n} f(3^n x), \nu(3^{-n} f(3^n x) - f(x))\right\}(t) \\ &\geq \Pi_M \left\{\nu(A(x) - 3^{-n} f(3^n x), \omega\left(\frac{1}{3}\psi(x,0)\right)\right\}(t). \end{aligned}$$

Hence

$$\begin{aligned} \nu(A(x) - f(x))(t) &\geq \Pi_M \left\{\lim_{n\to\infty} \nu(A(x) - 3^{-n} f(3^n x), \omega\left(\frac{1}{3}\psi(x,0)\right)\right\}(t) \\ &\geq \omega\left(\frac{1}{3}\psi(x,0)\right)(t). \end{aligned}$$

The uniqueness of A can be proved in a similar manner as in the proof of Corollary 2.2. □

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New subclass of analytic functions in conic domains associated with q - Sãlãgean differential operator involving complex order

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Abstract

The main object of this article is to define a new class of analytic functions using q - Sãlãgean differential operator involving complex order. We obtain coefficient estimates and other useful properties for this new class.

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1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, denote by $\mathcal{S}$, the class of all univalent functions in $\mathcal{A}$. Also, let $\mathcal{S}^*, \mathcal{K}, \mathcal{S}_p$ and $\mathcal{UCV}$ denote the subclasses of $\mathcal{S}$ which are starlike, convex, parabolic starlike and uniformly convex functions respectively. (For more details see [3], [17]). Kanas and Wiśniowska [6] introduced the subclasses of univalent functions called k- uniformly convex functions and k-starlike functions with $0 \leq k < \infty$, and denoted by $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ respectively. The analytic characterization of these classes are following(for more details one may refer to [5], [7], [8], [9], [10], [11]), [20]

$$k-\mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathbb{U}) \right\} \tag{1.2}$$

$$k-\mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathbb{U}) \right\}. \tag{1.3}$$

A function f is subordinate to the function g, written as $f \prec g$, provided that there is an analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g[w(z)]$ for $z \in \mathbb{U}$. In particular if the function g is univalent in $\mathbb{U}$ then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. For any non-negative integer n , the *q-integer number n* denoted by $[n]_q$, (See for example [2], [4],[13], [15]) is defined as

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \quad [0]_q = 0. \tag{1.4}$$

The q-number shifted factorial is defined by $[0]_q! = 1$ and $[n]_q! = [1]_q[2]_q[3]_q \cdots [n]_q$. We have, $\lim_{q\to 1^-}[n]_q = n$ and $\lim_{q\to 1^-}[n]_q! = n!$. The q-derivative operator or q- difference operator is defined as

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathbb{U}, \text{ where } \mathbb{U} = \{z \in \mathbb{C} \text{ and } |z| < 1\}. \tag{1.5}$$

It is easy to see that

$$\partial_q z^z = [n]_q z^{n-1}, \ \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \tag{1.6}$$

One can easily verify that $\partial_q f(z) \to f'(z)$ as $q \to 1^-$. In general, for a non-integer number t, $[t]$ is defined by $[t] = \dfrac{1-q^t}{1-q}$. Throughout this paper, we will assume q to be a fixed number between 0 and 1. For $f \in \mathcal{A}$, let the Sãlãgean q-differential operator ([2], [4],[13], [15], [19]) be defined by

$$\mathcal{S}_q^0 f(z) = f(z), \ \mathcal{S}_q^1 f(z) = z\partial_q f(z), \ \mathcal{S}_q^m f(z) = z\partial_q(\mathcal{S}_q^{m-1} f(z)).$$

A simple calculation yields,

$$\mathcal{S}_q^m f(z) = f(z) * G_{q,m}(z) \quad (z \in \mathbb{U}, m \in \mathbb{N} \cup \{0\} = \mathbb{N}_0), \tag{1.7}$$

where,

$$G_{q,m}(z) = z + \sum_{n=2}^{\infty} [n]_q^m z^n \quad (z \in \mathbb{U}, m \in \mathbb{N}_0). \tag{1.8}$$

Making use of (1.7) and (1.8), the power series of $\mathcal{S}_q^m f(z)$ for f of the form (1.1) is given by

$$\mathcal{S}_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n \quad (z \in \mathbb{U}). \tag{1.9}$$

Note that $\lim_{q\to 1^-} G_{q,m}(z) = z + \sum_{n=2}^{\infty} n^m z^n$ and $\lim_{q\to 1^-} \mathcal{S}_q^m f(z) = f(z) * (z + \sum_{n=2}^{\infty} n^m z^n)$,
which is the familiar Sãlãgean derivative operator [18]. Motivated by the works of Mahmood and Sokol [15] and Kanas and Yaguchi [12], we define the following class of functions using the theory of q-calculus.

Definition 1. *Let $0 \le k < \infty, \gamma \in \mathbb{C} \setminus 0, q \in (0,1)$ and $m \in \mathbb{N}_0$. A function $f \in \mathcal{A}$ is the class $\mathcal{S}_q(k,\gamma,m)$, if it satisfies the condition*

$$\Re \left\{ 1 + \frac{1}{\gamma} \left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right) \right\} > k \left| \frac{1}{\gamma} \left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right) \right|, \ (z \in \mathbb{U}). \tag{1.10}$$

Geometric Interpretation

A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_q(k,\gamma,m)$ if and only if $\dfrac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)}$ takes all values in the conic domain $\Omega_{k,\gamma} = p_{k,\gamma}(\mathbb{U})$ such that$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\gamma)$, where $\Omega_k = \{u + iv : u^2 > k^2(u-1)^2 + k^2v^2\}$ or equivalently

$$\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} \prec p_{k,\gamma}(z), \ \Omega_{k,\gamma} = p_{k,\gamma}\ (\mathbb{U}). \tag{1.11}$$

The boundary $\partial\Omega_{k,\gamma}$ of the above set becomes the imaginary axis when $k = 0$, while hyperbolic when $0 < k < 1$. In this case $0 \le k < 1$, we have $p_{k,\gamma}(z) = 1 + \frac{2\gamma}{1-k^2} \sinh^2 \left\{ \left(\frac{2}{\pi} \arccos k \operatorname{arctanh} \sqrt{z} \right) \right\}$
$(z \in \mathbb{U})$. For $k = 1$, the boundary $\partial\Omega_{k,\gamma}$, becomes a parabola and $p_{k,\gamma}(z) = 1 + \frac{2\gamma}{\pi^2} \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2$ $(z \in \mathbb{U})$. It is an ellipse when $k > 1$ and in this case $p_{k,\gamma}(z) = 1 + \frac{\gamma}{k^2-1} \sin \left(\frac{\pi}{2\kappa(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}} \right) + \frac{\gamma}{k^2-1})$, with $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$ $(0 < t < 1, z \in \mathbb{U})$, where t is chosen such that $k = \cosh \frac{\pi\kappa'(t)}{4\kappa(t)}$, and $\kappa(t)$ is Legendre's complete elliptic integral of the first kind and $\kappa'(t)$ complementary integral of $\kappa(t)$. Moreover , $p_{k,\gamma}(z)(\mathbb{U})$ is convex univalent in $\mathbb{U}$ [see [6], [8], [13]]. All of these curves have the vertex at the point $\dfrac{(k+\gamma)}{(k+1)}$. Therefore the domain $\Omega_{k,\gamma}$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and right half plane when $k = 0$, symmetric with respect to real axis. Because $p_{k,\gamma}(\mathbb{U}) = \Omega_{k,\gamma}$, the functions $p_{k,\gamma}$ play the role of extremal functions for several problems in this class $\mathcal{S}_q(k,\gamma,m)$.

2 Preliminary Lemmas

In the present investigation, we also need the following lemmas.

Lemma 1. *[16] Let $p(z) = \sum_{n=1}^{\infty} p_n z^n \prec F(z) = \sum_{n=1}^{\infty} d_n z^n$ in $\mathbb{C}$. If $F(z)$ is convex univalent in $\mathbb{U}$, then*

$$|p_n| \leq |d_1|, \quad (n \geq 1). \tag{2.1}$$

Lemma 2. *[5] Let $0 \leq k < \infty$ be fixed and $p_{k,\gamma}$ be the Riemann map of $\mathbb{U}$ onto $\Omega_{k,\gamma}$. If*

$$p_{k,\gamma}(z) = 1 + Q_1 z + Q_2 z^2 + \cdots \quad (z \in \mathbb{U}), \tag{2.2}$$

then

$$Q_1 = \begin{cases} \dfrac{2\gamma A^2}{1-k^2} & 0 \leq k < 1, \\ \dfrac{8\gamma}{\pi^2} & k = 1, \\ \frac{\pi^2 \gamma}{4(k^2-1)\kappa^2(t)\sqrt{t}(1+t)} & k > 1, \end{cases} \tag{2.3}$$

and

$$Q_2 = \begin{cases} \dfrac{(A^2+2)}{3} Q_1 & 0 \leq k < 1, \\ \dfrac{2}{3} Q_1 & k = 1, \\ \dfrac{(4\kappa^2(t)(t^2+6t+1) - \pi^2)}{24\kappa^2(t)\sqrt{t}(1+t)} Q_1 & k > 1, \end{cases} \tag{2.4}$$

where

$$A = \frac{2}{\pi} \arccos k,$$

and $\kappa(t)$ is the complete elliptic integral of the first kind(for details see [1]).

3 Properties of the class $\mathcal{S}_q(k, \gamma, m)$

In this section, we discuss certain sufficient condition for a class of functions f to be in the class $\mathcal{S}_q(k, \gamma, m)$.

Theorem 1. *Let $f \in \mathcal{A}$ be given by* (1.1)*. If the inequality*

$$\sum_{n=2}^{\infty} \left\{ [n]_q^m ((k+1)([n]_q - 1) + |\gamma|) \right\} |a_n| < |\gamma|, \tag{3.1}$$

holds true for some k $(0 \leq k < \infty), m \in \mathbb{N}_0$ and $\gamma \in \mathbb{C} \setminus 0$, then $f \in \mathcal{S}_q(k, \gamma, m)$.

Proof. In view of definition (1.10), it suffices to prove that

$$\frac{k}{\gamma} \left| \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right| - \Re \left\{ \frac{1}{\gamma} \left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right) \right\} < 1.$$

We have,

$$\begin{aligned} \frac{k}{\gamma} \left| \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right| - \Re \left\{ \frac{1}{\gamma} \left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right) \right\} &\leq \frac{(k+1)}{|\gamma|} \left| \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1 \right| \\ &= \frac{(k+1)}{|\gamma|} \left| \frac{\sum_{n=2}^{\infty} [n]_q^m ([n]_q - 1) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} [n]_q^m a_n z^{n-1}} \right| \\ &< \frac{(k+1)}{|\gamma|} \frac{\sum_{n=2}^{\infty} [n]_q^m ([n]_q - 1) |a_n|}{1 - \sum_{n=2}^{\infty} [n]_q^m |a_n|}. \end{aligned}$$

The last expression is bounded by 1, if inequality (3.1) holds. ☐

The next few corollaries can be easily obtained from Theorem 1.

Corollary 1. *Let* $f(z) = z + a_n z^n$. *If*

$$|a_n| \leq \frac{|\gamma|}{[n]_q^m((k+1)([n]_q - 1) + |\gamma|)} z^n \quad (n \geq 2),$$

then $f \in \mathcal{S}_q(k, \gamma, m)$.

For the choice of $m = 0$, Theorem 1 reduces to the following.

Corollary 2. *A function* $f \in \mathcal{A}$ *of the form* (1.1) *is in the class* $\mathcal{S}_q(k, \gamma, 0)$,*if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{(k+1)([n]_q - 1) + |\gamma|\} |a_n| < |\gamma|. \tag{3.2}$$

For the choices of $m = 0$ and $k = 0$, Theorem 1 reduces to the following.

Corollary 3. *A function* $f \in \mathcal{A}$ *of the form* (1.1) *is in the class* $\mathcal{S}_q(0, \gamma, 0)$,*if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{([n]_q - 1) + |\gamma|\} |a_n| < |\gamma|. \tag{3.3}$$

Theorem 2. *Let* $f \in \mathcal{S}_q(k, \gamma, m)$. *Then*

$$\mathcal{S}_q^m f(z) \prec \int_0^z \frac{p_{k,\gamma}(\omega(\xi)) - 1}{\xi} d\xi, \tag{3.4}$$

where $\omega(z)$ *is analytic in* $\mathbb{U}$ *with* $\omega(0) = 0$ *and* $|\omega(z)| < 1$. *Moreover, for* $|z| = \rho$, *we have*

$$\exp\left(\int_0^z \frac{p_{k,\gamma}(-\rho) - 1}{\rho} d\rho\right) \leq \left|\frac{\mathcal{S}_q^m f(z)}{z}\right| \leq \exp\left(\int_0^z \frac{p_{k,\gamma}(\rho) - 1}{\rho} d\rho\right),$$

where $p_{k,\gamma}(z)$ *is given by* (1.11).

Proof. Let $f \in \mathcal{S}_q(k, \gamma, m)$, then using the relation (1.11), we obtain

$$\frac{\partial_q \mathcal{S}_q^m f(z)}{\mathcal{S}_q^m f(z)} - \frac{1}{z} = \frac{p_{k,\gamma}(\omega(z)) - 1}{z}, \tag{3.5}$$

for some function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. Integrating (3.5), we have

$$\mathcal{S}_q^m f(z) \prec z \exp \int_0^z \frac{p_{k,\gamma}(\omega(\xi)) - 1}{\xi} d\xi. \tag{3.6}$$

This proves (3.4). Noting that the univalent function $p_{k,\gamma}(z)$ maps the disk $|z| < \rho$ $(0 < \rho \leq 1)$ onto a region which is convex and symmetric with respect to the real axis, we get

$$p_{k,\gamma}(-\rho|z|) \leq \Re\{p_{k,\gamma}(\omega(\rho z))\} \leq p_{k,\gamma}(\rho|z|) \ (0 < \rho \leq 1, z \in \mathbb{U}). \tag{3.7}$$

Using (3.7), we have

$$\int_0^z \frac{p_{k,\gamma}(-\rho|z|) - 1}{\rho} d\rho \leq \Re \int_0^z \frac{p_{k,\gamma}(\omega(\rho z)) - 1}{\rho} d\rho \leq \int_0^z \frac{p_{k,\gamma}(\rho|z|) - 1}{\rho} d\rho.$$

Consequently, the subordination (3.6) leads to

$$\int_0^z \frac{p_{k,\gamma}(-\rho|z|) - 1}{\rho} d\rho \leq \log\left|\frac{\mathcal{S}_q^m f(z)}{z}\right| \leq \int_0^z \frac{p_{k,\gamma}(\rho|z|) - 1}{\rho} d\rho,$$

which implies that

$$\exp\left(\int_0^z \frac{p_{k,\gamma}(-\rho) - 1}{\rho} d\rho\right) \leq \left|\frac{\mathcal{S}_q^m f(z)}{z}\right| \leq \exp\left(\int_0^z \frac{p_{k,\gamma}(\rho) - 1}{\rho} d\rho\right).$$

This completes the proof. □

Theorem 3. *If $f \in \mathcal{S}_q(k, \gamma, m)$, then*

$$|a_2| \leq \frac{\sigma}{[2]_q^m}, \ |a_n| \leq \frac{\sigma}{[n-1]_q [n]^m} \prod_{\mu=1}^{n-2} \left(1 + \frac{\sigma}{[\mu]_q}\right), \ (n \geq 3) \tag{3.8}$$

where $\sigma = |Q_1|/q$ with Q_1 is given by (2.3).

Proof. Let

$$\frac{z\partial_q \mathcal{S}_q^m f(z)}{\mathcal{S}_q^m f(z)} = p(z),$$

where $p(z)$ is analytic in $\mathbb{U}$. This can be written as

$$z\partial_q \mathcal{S}_q^m f(z) = p(z)\mathcal{S}_q^m f(z). \tag{3.9}$$

Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $\mathcal{S}_q^m f(z)$ be given by (1.9) . Then (3.9) becomes

$$z + \sum_{n=2}^{\infty} [n]_q^{m+1} a_n z^n = \left(\sum_{n=0}^{\infty} p_n z^n\right)\left(z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n\right).$$

Now comparing the coefficients of z^n, we obtain

$$[n]_q^{m+1} a_n = [n]_q^m a_n + \sum_{\mu=1}^{n-1} [\mu]_q^m a_\mu p_{n-\mu},$$

which implies that

$$a_n = \frac{1}{q[n-1]_q [n]_q^m} \sum_{\mu=1}^{n-1} [\mu]_q^m a_\mu p_{n-\mu}.$$

Using Lemma [16], we obtain,

$$|a_n| \leq \frac{|Q_1|}{q[n-1]_q [n]_q^m} \sum_{\mu=1}^{n-1} [\mu]_q^m |a_\mu|.$$

Now take $\sigma = \frac{|Q_1|}{q}$. Then, we have

$$|a_n| \leq \frac{\sigma}{[n-1]_q [n]_q^m} \sum_{\mu=1}^{n-1} [\mu]_q^m |a_\mu|. \tag{3.10}$$

So for $n = 2$, we have from (3.10)

$$|a_2| \leq \frac{\sigma}{[2]_q^m}, \tag{3.11}$$

which shows that (3.8) holds for $n = 2$. To prove (3.8), we apply mathematical inductions for $n = 3$. We have from (3.10)

$$|a_3| \leq \frac{\sigma}{[3]_q^m [2]_q} \left(1 + [2]_q^m |a_2|\right),$$

Using (3.11), we have

$$|a_3| \leq \frac{\sigma}{[3]_q^m [2]_q} (1 + \sigma) = \frac{\sigma([1]_q + \sigma)}{[3]_q^m [2]_q},$$

which shows that (3.8) holds for $n = 3$. Assume that (3.8) is true for $n \leq t$, that is

$$|a_t| \leq \frac{\sigma}{[t-1]_q [t]^m} \prod_{\mu=1}^{t-2} \left(1 + \frac{\sigma}{[\mu]_q}\right).$$

Consider

$$\begin{aligned}|a_{t+1}| &\leq \frac{\sigma}{[t]_q[t+1]_q^m}\left[1+[1]_q^m|a_2|+[2]_q^m|a_3|+\cdots+[t-1]_q^m|a_t|\right]\\ &\leq \frac{\sigma}{[t]_q[t+1]_q^m}\left[1+\sigma+\sigma\left(1+\frac{\sigma}{[1]_q}\right)+\sigma\left(1+\frac{\sigma}{[1]_q}\right)\left(1+\frac{\sigma}{[2]_q}\right)\right.\\ &\left.+\cdots+\sigma\prod_{\mu=1}^{t-2}\left(1+\frac{\sigma}{[\mu]_q}\right)\right]\\ &\leq \frac{\sigma}{[t]_q[t+1]_q^m}\prod_{\mu=1}^{t-1}\left(1+\frac{\sigma}{[\mu]_q}\right).\end{aligned}$$

Therefore the result is true for $n=t+1$. Consequently, using mathematical induction, we have proved that (3.8) holds true for all $n,\ n\geq 2$. This completes the proof of the theorem. □

Theorem 4. *Let $f(z)\in\mathcal{S}_q(k,\gamma,m)$. Then $f(\mathbb{U})$ contains an open disk of radius*

$$\frac{q[2]_q^m}{2q[2]_q^m+|Q_1(k)|}, \tag{3.12}$$

where $Q_1(k)$ is defined by (2.3).

Proof. Let $\omega_0\neq 0$ be a complex number such that $f(z)\neq\omega_0$ for $z\in\mathbb{U}$. Then

$$f_1(z)=\frac{\omega_0 f(z)}{\omega_0-f(z)}=z+\left(a_2+\frac{1}{\omega_0}\right)z^2+\cdots. \tag{3.13}$$

Since $f_1(z)$ is univalent, so

$$\left|a_2+\frac{1}{\omega_0}\right|\leq 2.$$

Now using Theorem 3, we have

$$\left|\frac{1}{\omega_0}\right|\leq 2+\frac{|Q_1(k)|}{q[2]_q^m}. \tag{3.14}$$

Therefore,

$$|\omega_0|\geq\frac{q[2]_q^m}{2q[2]_q^m+|Q_1(k)|}. \tag{3.15}$$

□

4 A coefficient inequality for the class $\mathcal{S}_q(k,\gamma,m)$

To obtain the coefficient inequality over the class $\mathcal{S}_q(k,\gamma,m)$, we need the following lemmas.

Lemma 3. *[14] If $q(z)=1+c_1z+c_2z^2+\cdots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2-vc_1^2|\leq 2\max\{1;|2v-1|\}. \tag{4.1}$$

In particular, if v is a real number, then

$$|c_2-vc_1^2|\leq\begin{cases}-4v+2 & \text{if } v\leq 0\\ 2 & \text{if } 0\leq v\leq 1\\ 4v-2 & \text{if } v\geq 1.\end{cases} \tag{4.2}$$

when $v<0$ or $v>1$,the equality holds true if and only if $q(z)=\dfrac{1+z}{1-z}$ or one of its rotations. If $0<v<1$, then the equality holds true if and only if $q(z)=\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v=0$, the the equality holds true if and only if

$$g(z)=\left(\frac{1}{2}+\frac{\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\lambda}{2}\right)\frac{1-z}{1+z}\quad(0\leq\lambda\leq 1)$$

or one of its rotations. If $v = 1$,then the equality is true if $q(z)$ is a reciprocal of one of the functions such that the equality is true in the case, when $v = 0$.

Theorem 5. *Let $0 \le k < \infty, \gamma \in \mathbb{C} \setminus 0, q \in (0,1)$ and $m \in \mathbb{N}_0$. Suppose that the function f of the form* (1.1) *belongs to the class $\mathcal{S}_q(k,\gamma,m)$. Then, for a complex number μ*

$$|a_3 - \mu a_2^2| \le \frac{\gamma Q_1}{q(1+q)(1+q+q^2)^m} \max\left\{1; \left|\frac{\gamma\mu Q_1(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} - \frac{Q_2}{Q_1} - \frac{\gamma Q_1}{q}\right|\right\}. \tag{4.3}$$

Proof. If $f \in \mathcal{S}_q(k,\gamma,m)$, then there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$1 + \frac{1}{\gamma}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right) = p_{k,\gamma}(\omega(z)) \ (z \in \mathbb{C}). \tag{4.4}$$

Define the function $h(z)$, by $h(z) = \dfrac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots$. Since $\omega(z)$ is a Schwarz function, we see that $\Re(h(z)) > 0$ and $h(0) = 1$. We also have,

$$\omega(z) = \frac{h(z)-1}{h(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right].$$

This gives,

$$p_{k,\gamma}(\omega(z)) = 1 + \frac{1}{2}c_1 Q_1 z + \left(\frac{1}{2}c_2 Q_1 + \frac{1}{4}c_1^2(Q_2 - Q1)\right) z^2 + \cdots . \tag{4.5}$$

From (4.4), we get,

$$\begin{aligned} 1 + \frac{1}{\gamma}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right) &= 1 + \frac{1}{\gamma}\left[q(1+q)^m a_2 z\right. \\ &+ \left.\left\{q(1+q)(1+q+q^2)^m a_3 - q(1+q)^{2m} a_2^2\right\} z^2 + \cdots\right]. \end{aligned} \tag{4.6}$$

Comparing the coefficients of z and z^2 in (4.5) and (4.6), we get

$$a_2 = \frac{\gamma c_1 Q_1}{2q(1+q)^m}. \tag{4.7}$$

$$a_3 = \frac{\gamma}{2q(1+q)(1+q+q^2)^m}\left(c_2 Q_1 + \frac{c_1^2 Q_2}{2} - \frac{c_1^2 Q_1}{2} + \frac{\gamma c_1^2 Q_1^2}{2q}\right). \tag{4.8}$$

This implies that,

$$a_3 - \mu a_2^2 = \frac{\gamma Q_1}{2q(1+q)(1+q+q^2)^m}\left[c_2 - v c_1^2\right],$$

where

$$v = \frac{1}{2}\left(1 + \frac{\gamma\mu Q_1(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} - \frac{Q_2}{Q_1} - \frac{\gamma Q_1}{q}\right).$$

□

It is easy to see that Theorem 6 directly follows from (4.2).

Theorem 6. *Let $0 \le k < \infty, \gamma \in \mathbb{C} \setminus 0, q \in (0,1)$ and $m \in \mathbb{N}_0$. Suppose that the function f of the form* (1.1) *belongs to the class $\mathcal{S}_q(k,\gamma,m)$. Then, for a real number μ,*

$$|a_3 - \mu a_2^2| \le \frac{\gamma}{q(1+q)(1+q+q^2)^m}\begin{cases} P_2 + \dfrac{\gamma P_1^2}{q} - \dfrac{\gamma\mu P_1^2(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} & \text{if } \mu \le \sigma_1 \\ P_1 & \text{if } \sigma_1 \le \mu \le \sigma_2 \\ -P_2 - \dfrac{\gamma P_1^2}{q} + \dfrac{\gamma\mu P_1^2(1+q)(1+q+q^2)^m}{q(1+q)^{2m}} & \text{if } \mu \ge \sigma_2, \end{cases} \tag{4.9}$$

where

$$\sigma_1 = \frac{q(1+q)^{2m}}{\gamma P_1^2(1+q)(1+q+q^2)^m}\left(P_2 + \frac{\gamma P_1^2}{q} - P_1\right)$$

$$\sigma_2 = \frac{q(1+q)^{2m}}{\gamma P_1^2(1+q)(1+q+q^2)^m}\left(P_2 + \frac{\gamma P_1^2}{q} + P_1\right).$$

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FOURIER SERIES OF SUMS OF PRODUCTS OF ORDERED BELL AND GENOCCHI FUNCTIONS

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Abstract. In this paper, we will study three types of sums of products of ordered Bell and Genocchi functions and derive their Fourier series expansions. Further, we will express those functions in terms of Bernoulli functions.

1. Introduction

The Genocchi polynomials $G_m(x)$ are given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!},\quad \text{(see [2, 6, 11, 12, 17, 21])}. \tag{1.1}$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned}&G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,\\&G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2+1,\\&G_5(x)=5x^4-10x^3+5x,\ G_6(x)=6x^5-15x^4+15x^2-3.\end{aligned} \tag{1.2}$$

The Genocchi polynomials are related to the Euler polynomials as

$$G_m(x)=mE_{m-1}(x)\quad (m\geq 1). \tag{1.3}$$

From this, we have

$$\begin{aligned}&\deg G_m(x)=m-1\ (m\geq 1),\ G_m=mE_{m-1}\ (m\geq 1),\\&G_0=0,\ G_1=1,\ G_{2m+1}=0\ (m\geq 1),\ \text{and } G_{2m}\neq 0\ (m\geq 1).\end{aligned} \tag{1.4}$$

In addition, by (1.1) we obtain

$$\begin{aligned}&\frac{d}{dx}G_m(x)=mG_{m-1}(x)\ (m\geq 1),\\&G_m(x+1)+G_m(x)=2mx^{m-1}\ (m\geq 0).\end{aligned} \tag{1.5}$$

From these, we also get

$$G_m(1)+G_m(0)=2\delta_{m,1},\quad (m\geq 0), \tag{1.6}$$

and

$$\begin{aligned}\int_0^1 G_m(x)dx&=\frac{1}{m+1}(G_{m+1}(1)-G_{m+1}(0))\\&=\begin{cases}0, & \text{if } m \text{ is even},\\ -\frac{2}{m+1}G_{m+1}, & \text{if } m \text{ is odd}.\end{cases}\end{aligned} \tag{1.7}$$

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The ordered Bell polynomials $b_m(x)$ are a natural companion to ordered Bell numbers and defined by the generating function

$$\frac{1}{2-e^t}e^{xt}=\sum_{m=0}^{\infty}b_m(x)\frac{t^m}{m!}. \tag{1.8}$$

The first few ordered Bell polynomials are as follows:

$$\begin{aligned}&b_0(x)=1,\ b_1(x)=x+1,\ b_2(x)=x^2+2x+3,\\ &b_3(x)=x^3+3x^2+9x+13,\ b_4(x)=x^4+4x^3+18x^2+52x+75,\\ &b_5(x)=x^5+5x^4+30x^3+130x^2+375x+541.\end{aligned} \tag{1.9}$$

The ordered Bell numbers $b_m=b_m(0)$ have been studied in many counting problems in enumerative combinatorics and number theory, the first appearance of which goes back to as early as 1859,, (see [3-5,7-8,13,16,19,20]). The ordered Bell polynomials are monic polynomials with integral coefficients as we can see from

$$b_0(x)=1,\ b_m(x)=x^m+\sum_{l=0}^{m-1}\binom{m}{l}b_l(x),\quad (m\geq 1). \tag{1.10}$$

Also, the ordered Bell numbers are positive integers, as we can notice from

$$b_m=\sum_{n=0}^{m}n!S_2(m,n)=\sum_{n=0}^{\infty}\frac{n^m}{2^{n+1}},\quad (m\geq 0). \tag{1.11}$$

From (1.8), we can derive

$$\begin{aligned}&\frac{d}{dx}b_m(x)=mb_{m-1}(x),\ \ (m\geq 1),\\ &-b_m(x+1)+2b_m(x)=x^m,\quad (m\geq 0).\end{aligned} \tag{1.12}$$

In turn, from these we obtain

$$-b_m(1)+2b_m=\delta_{m,0},\quad (m\geq 0), \tag{1.13}$$

and

$$\int_0^1 b_m(x)dx=\frac{1}{m+1}(b_{m+1}(1)-b_{m+1}(0))=\frac{1}{m+1}b_{m+1}. \tag{1.14}$$

For any real number x, we let $<x>=x-[x]\in[0,1)$ denote the fractional part of x. We recall the following facts about Bernoulli functions $B_m(<x>)$:

(a) for $m\geq 2$,

$$B_m(<x>)=-m!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.15}$$

(b) for $m=1$,

$$-\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{2\pi in}=\begin{cases}B_1(<x>), & \text{for}\quad x\in\mathbb{Z}^c,\\ 0, & \text{for}\quad x\in\mathbb{Z},\end{cases} \tag{1.16}$$

where $\mathbb{Z}^c=\mathbb{R}-\mathbb{Z}$.

In this paper, we will study three types of sums of products of oredered Bell and Genocchi functions, and derive their Fourier expansions. Further, we will express those functions in terms of Bernoulli functions as follows:

(1) $\alpha_m(<x>)=\sum_{k=0}^{m-1}b_k(<x>)G_{m-k}(<x>),\ \ (m\geq 2)$;

(2) $\beta_m(<x>) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>) G_{m-k}(<x>), \ \ (m \geq 2)$;
(3) $\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(<x>) G_{m-k}(<x>), \ \ (m \geq 2)$.

For elementary facts on Fourier analysis and some related recent works, the reader may refer to [1,8,22]) and [9,10,14,15], respectively.

2. Fourier series of functions of the first type

In this section, we will derive the Fourier series of sums of products of oredered Bell and Genocchi functions of the first type. Let

$$\alpha_m(x) = \sum_{k=0}^{m-1} b_k(x) G_{m-k}(x), \ \ (m \geq 2). \tag{2.1}$$

Then we will consider the function $\alpha_m(<x>) = \sum_{k=0}^{m-1} b_k(<x>) G_{m-k}(<x>), \ \ (m \geq 2)$ defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(<x>) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{2.3}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m-1} \{ k b_{k-1}(x) G_{m-k}(x) + (m-k) b_k(x) G_{m-k-1}(x) \} \\ &= \sum_{k=1}^{m-1} k b_{k-1}(x) G_{m-k}(x) + \sum_{k=0}^{m-2} (m-k) b_k(x) G_{m-k-1}(x) \\ &= \sum_{k=0}^{m-2} (k+1) b_k(x) G_{m-1-k}(x) + \sum_{k=0}^{m-2} (m-k) b_k(x) G_{m-1-k}(x) \\ &= (m+1) \sum_{k=0}^{m-2} b_k(x) G_{m-1-k}(x) \\ &= (m+1) \alpha_{m-1}(x). \end{aligned} \tag{2.4}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+2} \right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} (\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{2.6}$$

For $m \geq 2$, we put $\Delta_m = \alpha_m(1) - \alpha_m(0)$. Then we have

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m-1} (b_k(1)G_{m-k}(1) - b_k G_{m-k}) \\
&= \sum_{k=0}^{m-1} ((2b_k - \delta_{k,0})(-G_{m-k} + 2\delta_{m-1,k}) - b_k G_{m-k}) \\
&= \sum_{k=0}^{m-1} (-3b_k G_{m-k} + 4b_k \delta_{m-1,k} + \delta_{k,0} G_{m-k} - 2\delta_{k,0}\delta_{m-1,k}) \\
&= -3\sum_{k=0}^{m-1} b_k G_{m-k} + 4b_{m-1} + G_m - 2\delta_{m,1} \\
&= -3\sum_{k=0}^{m-2} b_k G_{m-k} + b_{m-1} + G_m.
\end{aligned} \tag{2.7}$$

Note that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0, \tag{2.8}$$

and

$$\begin{aligned}
\int_0^1 \alpha_m(x)dx &= \frac{1}{m+2}\Delta_{m+1} \\
&= \frac{1}{m+2}(-3\sum_{k=0}^{m-1} b_k G_{m+1-k} + b_m + G_{m+1}).
\end{aligned} \tag{2.9}$$

We are now ready to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m.
\end{aligned} \tag{2.10}$$

From this by induction on m we can deduce

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}. \tag{2.11}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.12}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise C^∞. In addition, $\alpha_m(< x >)$ is continuous for those integers $(m \geq 2)$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those integers $(m \geq 2)$ with

$\Delta_m \neq 0$. Assume first that m is an integer $m \geq 2$ with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Thus, the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned} &\alpha_m(< x >) \\ &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} \left(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n}}{(2\pi i n)^j} \right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &\quad + \Delta_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

We now state our first result.

Theorem 2.1. *For each integer l, with $l \geq 2$, we put*

$$\Delta_l = -3 \sum_{k=0}^{l-2} b_k G_{l-k} + b_{l-1} + G_l. \tag{2.14}$$

Assume that $\Delta_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=0}^{m-1} b_k(< x >)G_{m-k}(< x >)$ has the Fourier series expansion

$$\begin{aligned} &\sum_{k=0}^{m-1} b_k(< x >)G_{m-k}(< x >) \\ &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x}, \end{aligned} \tag{2.15}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned} &\sum_{k=0}^{m-1} b_k(< x >)G_{m-k}(< x >) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >), \end{aligned} \tag{2.16}$$

for all $x \in \mathbb{R}$, where $B_j(< x >)$ is the Bernoulli function.

Assume next that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then $\alpha_m(0) \neq \alpha_m(1)$. Thus $\alpha_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.17}$$

for $x \in \mathbb{Z}$. We can now state our second result.

Theorem 2.2. *For each integer l, with $l \geq 2$, we let*

$$\Delta_l = -3\sum_{k=0}^{l-2} b_k G_{l-k} + b_{l-1} + G_l. \tag{2.18}$$

Assume that $\Delta_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \begin{cases} \sum_{k=0}^{m-1} b_k(<x>)G_{m-k}(<x>), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} b_k G_{m-k} + \frac{1}{2}\Delta_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.19}$$

(b)

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\ &= \sum_{k=0}^{m-1} b_k(<x>)G_{m-k}(<x>), \quad \text{for } x \in \mathbb{Z}^c; \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\ &= \sum_{k=0}^{m-1} b_k G_{m-k} + \frac{1}{2}\Delta_m, \ x \in \mathbb{Z}. \end{aligned} \tag{2.21}$$

3. Fourier series of functions of the second type

Let $\beta_m(x) = \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}b_k(x)G_{m-k}(x)$, $(m \geq 2)$. Then we will investigate the function

$$\beta_m(<x>) = \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}b_k(<x>)G_{m-k}(<x>), \ (m \geq 2), \tag{3.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(<x>)e^{-2\pi inx}dx \\ &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx. \end{aligned} \tag{3.3}$$

Before proceeding further, we need to notice the following.

$$\begin{aligned}\beta_m'(x) &= \sum_{k=0}^{m-1}\left\{\frac{k}{k!(m-k)!}b_{k-1}(x)G_{m-k}(x)+\frac{m-k}{k!(m-k)!}b_k(x)G_{m-k-1}(x)\right\}\\ &= \sum_{k=1}^{m-1}\frac{1}{(k-1)!(m-k)!}b_{k-1}(x)G_{m-k}(x)+\sum_{k=0}^{m-2}\frac{1}{k!(m-k-1)!}b_k(x)G_{m-k-1}(x)\\ &= \sum_{k=0}^{m-2}\frac{1}{k!(m-1-k)!}b_k(x)G_{m-1-k}(x)+\sum_{k=0}^{m-2}\frac{1}{k!(m-1-k)!}b_k(x)G_{m-1-k}(x)\\ &= 2\beta_{m-1}(x).\end{aligned} \tag{3.4}$$

From this, we note that

$$\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x), \tag{3.5}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}(\beta_{m+1}(1)-\beta_{m+1}(0)). \tag{3.6}$$

For $m \geq 2$, we set

$$\begin{aligned}\Omega_m &= \beta_m(1)-\beta_m(0)\\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}(b_k(1)G_{m-k}(1)-b_kG_{m-k})\\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}((2b_k-\delta_{k,0})(-G_{m-k}+2\delta_{m-1,k})-b_kG_{m-k})\\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}(-3b_kG_{m-k}+4b_k\delta_{m-1,k}+\delta_{k,0}G_{m-k}-2\delta_{k,0}\delta_{m-1,k})\\ &= -3\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}b_kG_{m-k}+\frac{4}{(m-1)!}b_{m-1}+\frac{1}{m!}G_m-\frac{2}{m!}\delta_{m,1}\\ &= -3\sum_{k=0}^{m-2}\frac{1}{k!(m-k)!}b_kG_{m-k}+\frac{1}{(m-1)!}b_{m-1}+\frac{1}{m!}G_m.\end{aligned} \tag{3.7}$$

From this, we see that

$$\beta_m(0)=\beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.8}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.9}$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left[\beta_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}(\beta_m(1) - \beta_m(0)) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\ &= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m, \end{aligned} \tag{3.10}$$

from which by induction we have

$$B_n^{(m)} = -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}. \tag{3.11}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}. \tag{3.12}$$

$\beta_m(< x >), (m \geq 2)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that m is an integer $m \geq 2$ with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} \beta_m(< x >) &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x} \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j! \sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.13}$$

We are now ready to state our first result.

Theorem 3.1. *For each integer $l \geq 2$, we let*

$$\Omega_l = -3\sum_{k=0}^{l-2}\frac{1}{k!(l-k)!} b_k G_{l-k} + \frac{1}{(l-1)!} b_{l-1} + \frac{1}{l!} G_l. \tag{3.14}$$

Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!} b_k(< x >) G_{m-k}(< x >)$ has the Fourier series expansion

$$\begin{aligned} &\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!} b_k(< x >) G_{m-k}(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x}, \end{aligned} \tag{3.15}$$

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for all $x \in \mathbb{R}$, *where the convergence is uniform.*

(b)

$$\begin{aligned}&\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>) G_{m-k}(<x>)\\&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>),\end{aligned} \tag{3.16}$$

for all $x \in \mathbb{R}$, *where* $B_j(<x>)$ *is the Bernoulli function.*

Assume next that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then $\beta_m(0) \neq \beta_m(1)$. Thus $\beta_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.17}$$

for $x \in \mathbb{Z}$. Now, we are ready to state our second result.

Theorem 3.2. *For each integer* l, *with* $l \geq 2$, *we let*

$$\Omega_l = -3\sum_{k=0}^{l-2} \frac{1}{k!(l-k)!} b_k G_{l-k} + \frac{1}{(l-1)!} b_{l-1} + \frac{1}{l!} G_l. \tag{3.18}$$

Assume that $\Omega_m \neq 0$, *for an integer* $m \geq 2$. *Then we have the following.*

(a)

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi in)^j} \Omega_{m-j+1} \right) e^{2\pi inx}\\&= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>) G_{m-k}(<x>), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k G_{m-k} + +\frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases}\end{aligned} \tag{3.19}$$

(b)

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>)\\&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k(<x>) G_{m-k}(<x>),\end{aligned} \tag{3.20}$$

for $x \in \mathbb{Z}^c$;

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(<x>)\\&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k G_{m-k} + \frac{1}{2}\Omega_m,\end{aligned} \tag{3.21}$$

for $x \in \mathbb{Z}$.

4. Fourier series of functions of the third type

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(x) G_{m-k}(x),\ (m \geq 2)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(< x >) G_{m-k}(< x >),\ (m \geq 2), \tag{4.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{4.3}$$

Before proceeding further, we would like to observe the following.

$$\begin{aligned} \gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{m-k} b_{k-1}(x) G_{m-k}(x) + \sum_{k=1}^{m-1} \frac{1}{k} b_k(x) G_{m-k-1}(x) \\ &= \sum_{k=0}^{m-2} \frac{1}{m-1-k} b_k(x) G_{m-1-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} b_k(x) G_{m-1-k}(x) \\ &= \sum_{k=1}^{m-2} \left(\frac{1}{m-1-k} + \frac{1}{k} \right) b_k(x) G_{m-1-k}(x) + \frac{1}{m-1} G_{m-1}(x) \\ &= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} b_k(x) G_{m-1-k}(x) + \frac{1}{m-1} G_{m-1}(x) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}(x). \end{aligned} \tag{4.4}$$

Thus we have $\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}(x)$, and from this, we see that

$$\left(\frac{1}{m} \left(\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \right) \right)' = \gamma_m(x), \tag{4.5}$$

and

$$\begin{aligned} &\int_0^1 \gamma_m(x) dx \\ &= \frac{1}{m} \left[\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \right]_0^1 \\ &= \frac{1}{m} \left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} (G_{m+1}(1) - G_{m+1}(0)) \right) \\ &= \frac{1}{m} \left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} (-2G_{m+1}(0) + 2\delta_{m,0}) \right) \\ &= \frac{1}{m} \left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)} G_{m+1} \right). \end{aligned} \tag{4.6}$$

For $m \geq 2$, we put

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (b_k(1)G_{m-k}(1) - b_k G_{m-k}) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} ((2b_k - \delta_{k,0})(-G_{m-k} + 2\delta_{m-1,k}) - b_k G_{m-k}) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (-3b_k G_{m-k} + 4b_k \delta_{m-1,k} + \delta_{k,0} G_{m-k} - 2\delta_{k,0}\delta_{m-1,k}) \\
&= -3 \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k G_{m-k} + \frac{4}{m-1} b_{m-1}.
\end{aligned} \tag{4.7}$$

Then

$$\gamma_m(0) = \gamma_m(1) \Longleftrightarrow \Lambda_m = 0, \tag{4.8}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\Big(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\Big). \tag{4.9}$$

Now, we are going to determine the Fourier coefficients $C_n^{(m)}$. For this, we first observe that, for $l \geq 2$,

$$\int_0^1 G_l(x) e^{-2\pi i n x} dx = \begin{cases} 2\sum_{k=1}^{l-1} \frac{(l)_{k-1} G_{l-k+1}}{(2\pi i n)^k}, & \text{for} \quad n \neq 0, \\ -\frac{2G_{l+1}}{l+1}, & \text{for} \quad n = 0. \end{cases} \tag{4.10}$$

Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} [\gamma_m(x) e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n} \int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} (\gamma_m(1) - \gamma_m(0)) + \frac{1}{2\pi i n} \int_0^1 \Big\{ (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}(x) \Big\} e^{-2\pi i n x} dx \\
&= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m + \frac{2}{2\pi i n (m-1)} \Theta_m,
\end{aligned} \tag{4.11}$$

where $\Theta_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1} G_{m-k}}{(2\pi i n)^k}$. From the recurrence relation

$$C_n^{(m)} = \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m + \frac{2}{2\pi i n (m-1)} \Theta_m, \tag{4.12}$$

by induction we can show that

$$C_n^{(m)} = -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{m-j+1} + 2\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi i n)^j (m-j)} \Theta_{m-j+1}. \tag{4.13}$$

We note here that

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Theta_{m-j+1} \\
&= \sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{k=1}^{m-j-1} \frac{(m-1)_{j+k-2} G_{m-j-k+1}}{(2\pi in)^{j+k}} \\
&= \sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{s=j+1}^{m-1} \frac{(m-1)_{s-2} G_{m-s+1}}{(2\pi in)^s} \\
&= \sum_{s=2}^{m-1} \frac{(m-1)_{s-2} G_{m-s+1}}{(2\pi in)^s} \sum_{j=1}^{s-1} \frac{1}{m-j} \\
&= \frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}).
\end{aligned} \tag{4.14}$$

Putting everything altogether, we have

$$C_n^{(m)} = -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \left\{ \Lambda_{m-s+1} - 2\frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \right\}. \tag{4.15}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\Big(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\Big). \tag{4.16}$$

$\gamma_m(< x >), (m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\right) \\
&\quad + \sum_{n=-\infty, n\neq 0}^{\infty} \left\{ -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \right) \right\} e^{2\pi inx} \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\right) \\
&\quad + \frac{1}{m} \sum_{s=1}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \right) \left(-s! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^s} \right) \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\right) \\
&\quad + \frac{1}{m} \sum_{s=2}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \right) B_s(< x >) \\
&\quad + \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.17}$$

We are now ready to state our first result.

Theorem 4.1. *For each integer l, with $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k G_{l-k} + \frac{4}{l-1} b_{l-1}. \tag{4.18}$$

Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(<x>) G_{m-k}(<x>)$ has the Fourier series expansion

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(<x>) G_{m-k}(<x>) \\ &= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)} G_{m+1}\right) \\ &\quad + \sum_{n=-\infty, n\neq 0}^{\infty} \left\{ -\frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) \right) \right\} e^{2\pi i n x}, \end{aligned} \tag{4.19}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(<x>) G_{m-k}(<x>) \\ &= \frac{1}{m} \sum_{s=0, s\neq 1}^{m-1} \binom{m}{s} \left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) \right) B_s(<x>), \end{aligned} \tag{4.20}$$

for all $x \in \mathbb{R}$, where $B_s(<x>)$ is the Bernoulli function.

Assume next that m is an integer ≥ 2 with $\Lambda_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.21}$$

for $x \in \mathbb{Z}$. Now, we are ready to state our second result.

Theorem 4.2. *For each integer l, with $l \geq 2$, we let*

$$\Lambda_l = -3\sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k G_{l-k} + \frac{4}{l-1} b_{l-1}. \tag{4.22}$$

Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

(a)

$$\begin{aligned}&\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\&\quad+\sum_{n=-\infty,n\neq 0}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\right\}e^{2\pi inx}\\&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(<x>)G_{m-k}(<x>), & \text{for} \quad x\in\mathbb{Z}^c,\\ \sum_{k=0}^{m-1}\frac{1}{k(m-k)}b_kG_{m-k}+\frac{1}{2}\Delta_m, & \text{for} \quad x\in\mathbb{Z}.\end{cases}\end{aligned} \tag{4.23}$$

(b)

$$\begin{aligned}&\frac{1}{m}\sum_{s=0}^{m-1}\binom{m}{s}(2\pi in)^s\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\\&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(<x>)G_{m-k}(<x>), \quad \text{for } x\in\mathbb{Z}^c;\end{aligned} \tag{4.24}$$

$$\begin{aligned}&\frac{1}{m}\sum_{s=0,s\neq 1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\\&\sum_{k=0}^{m-1}\frac{1}{k(m-k)}b_kG_{m-k}+\frac{1}{2}\Delta_m, \ x\in\mathbb{Z}.\end{aligned} \tag{4.25}$$

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TWO TRANSFORMATION FORMULAS ON THE BILATERAL BASIC HYPERGEOMETRIC SERIES

QING ZOU

ABSTRACT. In this paper, the author first proves a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_3\psi_3$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_4\phi_3$ series. Then, the author proves a transformation formula for the well-poised bilateral basic hypergeometric ${}_4\psi_4$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_8\phi_7$ series.

1. INTRODUCTION

One of the main parts of the theory of basic hypergeometric series is bilateral series. The general bilateral basic hypergeometric series in base q with r numerator and s denominator parameters is defined by

$$ {}_r\psi_s \left[\begin{array}{cccc} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{array} ; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{s-r} z^n, $$

where the denominator factors are never zero, $q \neq 0$ if $s < r$, and $z \neq 0$ if the power of z is negative.

To understand this definition better, we need to define the following notations. Assume $|q| < 1$. Define

$$ (x)_0 := (x; q)_0 = 1, $$

$$ (x)_n := (x; q)_n := \prod_{k=0}^{n-1} (1 - xq^k), $$

$$ (x_1, \cdots, x_m)_n := (x_1, \cdots, x_m; q)_n := (x_1; q)_n \cdots (x_m; q)_n, $$

$$ (x; q)_{-k} = \frac{(-q/x)^k q^{\binom{n}{2}}}{(q/x; q)_k}. $$

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1

By some algebraic computations of the terms with negative n, we can obtain

$$
{}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}; q, z\right] = \sum_{0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n}[(-1)^n q^{\binom{n}{2}}]^{s-r} z^n + \sum_{n=1}^{\infty} \frac{(q/b_1, q/b_2, \cdots, q/b_s; q)_n}{(q/a_1, q/a_2, \cdots, q/a_r; q)_n}\left(\frac{b_1 b_2 \cdots b_s}{a_1 a_2 \cdots a_r z}\right)^n. \tag{1.1}
$$

The convergence of each series in (1.1) can be seen in [1].

An ${}_r\psi_r$ is said to be well-poised if

$$a_1 b_1 = a_2 b_2 = \cdots = a_r b_r,$$

and very-well-poised if it is well-poised and

$$a_1 = -a_2 = q b_1 = -q b_2.$$

When it comes to basic hypergeometric series, it is unavoidable to talk about basic hypergeometric series because they are closely related. So, let us introduce the basic hypergeometric series next. Generally speaking, basic hypergeometric series are series $\sum c_n$ with c_{n+1}/c_n a rational function of q^n for a fixed parameter q, which is usually taken to satisfy $|q| < 1$, but at other times is a power of a prime. More precisely, we can define an ${}_r\phi_s$ basic hypergeometric series as

$$
{}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, b_2, \cdots, b_s; q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r} z^n,
$$

where $q \neq 0$ when $r > s + 1$. This definition is an extension of Heine's series (cf. [2, 3, 4]).

We say a basic hypergeometric series ${}_{r+1}\phi_r$ is well-poised if

$$q a_1 = a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} br,$$

and very-well-poised if it is well-poised and

$$a_2 = q a^{1/2}, \;\; a_3 = -q a_1^{1/2}.$$

An ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-m} with $m = 0, 1, 2, \cdots$ and $q \neq 0$. Basic hypergeometric series is very useful. Case in point [1], Gauss used a basic hypergeometric series identity in his first proof of the determination of the sign of the Gauss sum, and Jacobi used some to determine the number of ways an integer can be written as the sum of two, four, six and eight squares.

From the definition of ${}_r\psi_s$ and ${}_r\phi_s$, we can easily deduce that the results of these two series have nothing to do with the orders of $a_1, a_2, \cdots, a_r$ and $b_1, b_2, \cdots, b_s$. This point is very important. Furthermore, in the second appendix of [1], Gasper and Rahman showed several sums of bilateral basic series, namely, Ramanujan's ${}_1\psi_1$ sum, the sum of a well-poised ${}_2\psi_2$ series, Bailey's sum of a well-poised ${}_3\psi_3$, and etc.. In [5], Zhang and Hu provided two transformation formulas on the bilateral series ${}_5\psi_5$. In this paper, we would like to show a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_3\psi_3$ series and a a transformation formula for the well-poised bilateral basic hypergeometric ${}_4\psi_4$ series.

2. Main Lemmas

In order to prove the main results of this paper, we need to introduce the following two lemmas first.

Lemma 2.1. *Let b, c, d, e and f be indeterminate. Then*

$$\begin{aligned}&{}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & f \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f\end{matrix} ;q,\frac{q^4}{bcdef}\right] \qquad (2.1)\\ =&(1-q)\ {}_8\phi_7\left[\begin{matrix} & q, & q^{3/2}, & -q^{3/2}, & b, & c, & d, & e, & f \\ q^{1/2}, & -q^{1/2}, & q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f & \end{matrix} ;q,\frac{q^4}{bcdef}\right].\end{aligned}$$

provided $|\frac{q^4}{bcdef}|<1.$

The proof of this lemma can be seen in [5].

Lemma 2.2. *For $def - cq^{4-n}$ and $\frac{c}{f}=-\frac{1}{q^2}, n\in\mathbb{N}$, we have*

$$\begin{aligned}&{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & c, & dq^n, & eq^n \\ -q^{1/2}, & q^{1/2}, & q^2/c, & e, & d\end{matrix} ;q,-\frac{q^{5/2}}{d}\right]\\ =&\frac{(q^2,q^5/f^2;q^2)_\infty}{(1-q/f)(1-q^2/f)(1-q^3/f)(q^8/f^2;q^2)_\infty}\ {}_4\phi_3\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & c, & q^{-n} \\ & d, & e, & f\end{matrix} ;q,q\right],\end{aligned}$$

provided $|q|<1.$

Proof. According to Lemma 2.1 and [1, Appendix III (III.20)], we can infer that

$$\begin{aligned}&{}_5\psi_5\left[\begin{matrix} a, & b, & c, & dq^n, & eq^n \\ q^2/a, & q^2/b, & q^2/c, & e, & d\end{matrix} ;q,\frac{efq^n}{bc}\right]\\ =&\frac{(1-q)(aq/f,bq/f,cq/f,q^2;q)_\infty}{(abq/f,acq/f,bcq/f,q/f;q)_\infty}\ {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b, & c \\ & d, & e, & f\end{matrix} ;q,q\right], \qquad (2.2)\end{aligned}$$

where $abcq^{1-n}=def$ and $\frac{abc}{f}=q.$

Let $a=-q^{3/2}, b=q^{3/2}$ in (2.2) and simplfy the result, we can obtain our conclusion. □

These two lemmas are very useful. Let us give two examples to illustrate this point.

Corollary 2.1. *Let d, e and f be indeterminate. Then*

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & d, & e, & f \\ -q^{1/2}, & q^{1/2}, & q^2/d, & q^2/e, & q^2/f \end{matrix} ;q, -\frac{q}{def}\right]
$$

$$
=\frac{(q, q^2/de, q^2/df, q^2/ef; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def; q)_\infty}\, {}_4\phi_3\left[\begin{matrix} -1/q, & d, & e, & f \\ -q^{1/2}, & q^{1/2}, & def/q \end{matrix} ;q, q\right],
$$

provided $\max\{|q|, |\frac{q}{def}|\} < 1$ *and* ${}_4\phi_3$ *terminates.*

Proof. In [6], Watson showed the Watson's transformation formula (a new proof of this formula can be seen in [7]),

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & f \\ a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix} ;q, \frac{a^2q^2}{bcdef}\right]
$$

$$
=\frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}\, {}_4\phi_3\left[\begin{matrix} aq/bc, & d, & e, & f \\ aq/b, & aq/c, & def/a \end{matrix} ;q, q\right], \tag{2.3}
$$

whenever the ${}_8\phi_7$ series converges and the ${}_4\phi_3$ series terminates.

By Lemma 2.1 and (2.3), we derive that

$$
{}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & f \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f \end{matrix} ;q, \frac{q^4}{bcdef}\right]
$$

$$
=\frac{(q, q^2/de, q^2/df, q^2/ef; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def; q)_\infty}\, {}_4\phi_3\left[\begin{matrix} q^2/bc, & d, & e, & f \\ q^2/b, & q^2/c, & def/q \end{matrix} ;q, q\right].
$$

Sunstituting b and c by $-q^{3/2}$ and $q^{3/2}$, respectively, the conclusion follows. This completes the proof. □

If we let $f = q^{-n}, n \in \mathbb{N}$ in Corollary 2.1 (a new proof of $f = q^{-n}$ of the q-analogue of Watson's ${}_3F_2$ summation formula can also be found in [7]), we will arrive at

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & d, & e, & q^{-n} \\ -q^{1/2}, & q^{1/2}, & q^2/d, & q^2/e, & q^{n+2} \end{matrix} ;q, -\frac{q^{n+1}}{de}\right]
$$

$$
=\frac{(q; q)_{n+1}(q^2/de; q)_n}{(q^2/d, q^2/e; q)_n}\, {}_4\phi_3\left[\begin{matrix} -1/q, & d, & e, & q^{-n} \\ -q^{1/2}, & q^{1/2}, & deq^{-n-1} \end{matrix} ;q, q\right].
$$

Or equivalently,

$$
{}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & f, & g, & q^{-n} \\ -q^{1/2}, & q^{1/2}, & q^2/d, & q^2/e, & q^{n+2} \end{matrix} ;q, -\frac{q^{n+1}}{fg}\right]
$$

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$$=\frac{(1-q)(h,h/fg;q)_n}{(h/f,h/g;q)_n}\,{}_4\phi_3\left[\begin{matrix}-1/q, & f, & g, & q^{-n}\\ -q^{1/2}, & q^{1/2}, & h\end{matrix};q,q\right],$$

where $h=fgq^{-n-1}$.

With Lemma 2.2 in hand, we can obtain the following transformation formula for ${}_5\psi_5$ by using Sears' transformations of terminating balanced ${}_4\phi_3$ series [8], [1, Appendix III (III.15), (III.16)] (for the generalization of [1, Appendix III (III.15)], cf. [9])

$$\begin{aligned}
&{}_5\psi_5\left[\begin{matrix}-q^{3/2}, & q^{3/2}, & c, & dq^n, & eq^n\\ -q^{1/2}, & q^{1/2}, & q^2/c, & e, & d\end{matrix};q,-\frac{q^{5/2}}{d}\right]\\
=&\frac{(1-q)(aq/f,bq/f,cq/f,q^2;q)_\infty}{(abq/f,acq/f,bcq/f,q/f;q)_\infty}\frac{(-eq^{-3/2},-fq^{-3/2};q)_n}{(e,f;q)_n}(-q^{3/2})^n\\
&\times{}_4\phi_3\left[\begin{matrix}-q^{3/2}, & dq^{-3/2}, & d/c, & q^{-n}\\ d, & -q^{5/2-n}/e, & -q^{5/2-n}/f\end{matrix};q,q\right]\\
=&\frac{(1-q)(aq/f,bq/f,cq/f,q^2;q)_\infty}{(abq/f,acq/f,bcq/f,q/f;q)_\infty}\frac{(-q^{3/2},-efq^{-3},-q^{-3/2}ef/c;q)_n}{(e,f,-q^{-3}ef/c;q)_n}\\
&\times{}_4\phi_3\left[\begin{matrix}-q^{3/2}e, & q^{-3/2}f, & -q^{-3}ef/c, & q^{-n}\\ -efq^{-3}, & -q^{3/2}ef/c, & -q^{3/2}def\end{matrix};q,q\right],
\end{aligned}$$

where d,e and f are indeterminate and $|q|<1$.

With these two lemmas in hand, we are ready to show our main results.

3. TRANSFORMATION FORMULA FOR THE VERY-WELL-POISED ${}_3\psi_3$

In this section, we would like to prove a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_3\psi_3$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_4\phi_3$ series. The main conclusion can be summarized as the following conclusion.

Theorem 3.1. *For $n\in\mathbb{N}$ and $|q|<1$,*

$$\begin{aligned}
&{}_3\psi_3\left[\begin{matrix}-q^{3/2}, & q^{3/2}, & q^{\frac34+\frac n2}\\ -q^{1/2}, & q^{1/2}, & q^{\frac54-\frac n2}\end{matrix};q,q\right]\\
=&\frac{(q^2,q^{n+\frac52};q^2)_\infty}{(1+q^{\frac n2-\frac14})(1+q^{\frac n2+\frac34})(1+q^{\frac n2+\frac74})(q^{n+\frac{11}{2}};q^2)_\infty}\\
&\times{}_4\phi_3\left[\begin{matrix}-q^{3/2}, & q^{3/2}, & q^{\frac34-\frac n2}, & q^{-n}\\ q^{\frac54-\frac n2}, & q^{\frac34-\frac n2}, & -q^{\frac54-\frac n2}\end{matrix};q,q\right]
\end{aligned}$$

$$=\frac{(q^2,q^{n+\frac{5}{2}};q^2)_\infty}{(1+q^{\frac{n}{2}-\frac{1}{4}})(1+q^{\frac{n}{2}+\frac{3}{4}})(1+q^{\frac{n}{2}+\frac{7}{4}})(q^{n+\frac{11}{2}};q^2)_\infty}$$
$$\times{}_4\phi_3\left[\begin{matrix} q^3, & q^{3/2}, & q^{-\frac{3}{2}-n}, & q^{-n} \\ q^{\frac{5}{2}-n}, & q^{\frac{3}{4}-\frac{n}{2}}, & -q^{\frac{7}{4}-\frac{n}{2}} & \end{matrix};q^2,q^2\right].$$

Proof. Let
$$c=q^{\frac{3}{4}-\frac{n}{2}},\quad d=q^{\frac{5}{4}-\frac{n}{2}},\quad e=q^{\frac{3}{4}-\frac{n}{2}},\quad f=-q^{\frac{5}{4}-\frac{n}{2}}$$
in Lemma 2.2, we get that
$${}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}+\frac{n}{2}} \\ -q^{1/2}, & q^{1/2}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}-\frac{n}{2}} \end{matrix};q,q\right]$$
$$=\frac{(q^2,q^{n+\frac{5}{2}};q^2)_\infty}{(1+q^{\frac{n}{2}-\frac{1}{4}})(1+q^{\frac{n}{2}+\frac{3}{4}})(1+q^{\frac{n}{2}+\frac{7}{4}})(q^{n+\frac{11}{2}};q^2)_\infty}$$
$$\times{}_4\phi_3\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{-n} \\ q^{\frac{5}{4}-\frac{n}{2}}, & q^{\frac{3}{4}-\frac{n}{2}}, & -q^{\frac{5}{4}-\frac{n}{2}} & \end{matrix};q,q\right]$$

Note that
$${}_5\psi_5\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}+\frac{n}{2}} \\ -q^{1/2}, & q^{1/2}, & q^{\frac{5}{4}+\frac{n}{2}}, & q^{\frac{3}{4}-\frac{n}{2}}, & q^{\frac{5}{4}-\frac{n}{2}} \end{matrix};q,q\right]={}_3\psi_3\left[\begin{matrix} -q^{3/2}, & q^{3/2}, & q^{\frac{3}{4}+\frac{n}{2}} \\ -q^{1/2}, & q^{1/2}, & q^{\frac{5}{4}-\frac{n}{2}} \end{matrix};q,q\right].$$

Thus the first equation holds.

Askey and Wilson [10] proved
$${}_4\phi_3\left[\begin{matrix} a^2, & b^2, & c, & d \\ abq^{1/2}, & -abq^{1/2}, & -cd & \end{matrix};q,q\right]={}_4\phi_3\left[\begin{matrix} a^2, & b^2, & c^2, & d^2 \\ a^2b^2q, & -cd, & -cdq & \end{matrix};q^2,q^2\right] \quad (3.1)$$
provided that both series terminate. This formula is called Singh's quadratic transformation formula since this formula can be traced back to [11], which was written by Singh.

Let
$$a=q^{-\frac{n}{2}},\quad b=q^{\frac{3}{4}},\quad c=q^{-\frac{3}{4}-\frac{n}{2}},\quad d=-q^{\frac{3}{2}}$$
in (3.1), we can arrive at the second equation.

This completes the proof. □

4. TRANSFORMATION FORMULA FOR THE WELL-POISED ${}_4\psi_4$

In this section, we would like to prove a transformation formula for the very-well-poised bilateral basic hypergeometric ${}_4\psi_4$ series by using the relationship between the bilateral basic hypergeometric ${}_5\psi_5$ series and basic hypergeometric ${}_8\phi_7$ series. The main conclusion can be summarized as the following conclusion.

Theorem 4.1. *For $|q| < 1$, we have*

$$
{}_4\psi_4\left[\begin{matrix} a, & q/a, & -d, & -q/d \\ q^2/a, & qa, & -q^2/d, & -qd \end{matrix}; q, -q\right]
= \frac{(q;q)_\infty^2(-adq, -aq^2/d, -dq^2/a, -q^3/ad; q^2)_\infty}{(-qd, -q^2/d, aq, q^2/a; q)_\infty}.
$$

Proof. According to Lemma 2.1, we have

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} a, & q/a, & -d, & -q/d \\ q^2/a, & qa, & -q^2/d, & -qd \end{matrix}; q, -q\right] \\
=&{}_5\psi_5\left[\begin{matrix} a, & q/a, & -q, & -d, & -q/d \\ q^2/a, & qa, & -q, & -q^2/d, & -qd \end{matrix}; q, -q\right] \\
=&(1-q)\ {}_8\phi_7\left[\begin{matrix} q, & q^{3/2}, & -q^{3/2}, & a, & q/a, & -q, & -d, & -q/d \\ q^{1/2}, & -q^{1/2}, & q^2/a, & qa, & -q, & -q^2/d, & -qd \end{matrix}; q, -q\right],
\end{aligned} \tag{4.1}
$$

provided $|q| < 1$.

In [12, 3.4(1)], Bailey showed Whipples ${}_3F_2$ formula. In [13], Gasper and Rahman proved the following q-analogue of Whipples formula as follows,

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} -c, & q(-c)^{1/2}, & -q(-c)^{1/2}, & a, & q/a, & c, & -d, & -q/d \\ (-c)^{1/2}, & -(-c)^{1/2}, & -cq/a, & -ac, & -q, & cq/d, & cd \end{matrix}; q, c\right] \\
=&\frac{(-c, -cq; q)_\infty(acd, acq/d, cdq/a, cq^2/ad; q^2)_\infty}{(cd, cq/d, -ac, -cq/a; q)_\infty}.
\end{aligned} \tag{4.2}
$$

Note that

$$
(1-q)\cdot(q, q^2; q)_\infty = (q;q)_\infty^2.
$$

Then let $c = -q$ in (4.2) and then substitute it into (4.1), the conclusion can be obtained. □

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The p-moment exponential estimates for neutral stochastic functional differential equations in the G-framework

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Abstract

The neutral stochastic functional differential equations have attracted much attention because of their practical applications in various fields such as biology, physics, medicine, finance, telecommunication networks and population dynamics. In this note, we investigate the p-moment estimates of solutions to neutral stochastic functional differential equations (NSFDEs) in the framework of G-Brownian motion. Under non-linear growth condition, the L^p estimates of solutions to NSFDEs in the G-framework are given. The Gronwall's inequality, Hölder's inequality, G-Itô's formula and Burkholder-Davis-Gundy (BDG) inequalities are utilized to establish the above stated theory. Moreover, the asymptotic estimates for the solutions to these equations are studied and the Lyapunov exponent is estimated for NSFDEs in the G-framework.

Key words: G-Brownian motion, p-moment estimates, neutral stochastic functional differential equations, non-linear growth condition, Lyapunov exponent.

1 Introduction

The multifaceted usage of stochastic dynamical models has proved to be tantamount to indispensable due to their reliability and authenticity in natural sciences, engineering and economics. The ever-developing field of medical science, which is always on the lookout for such mathematically accurate tools for the investigation of a variety of maladies, is no exception in using these models. Among others, the efficacy of these models has been established to generate optimal dynamic health policies for controlling spreads of infectious diseases [15]. Such is the quantitative accuracy and efficiency of stochastic differential equation (SDE) models that the prediction of the growth of bacterial populations from a small number of pathogens [1] can be calculated through these models. Besides, these models have the highly-cherished reliability to the extent that control and navigation systems are also using them as must-have tool. Various kinds of disturbances in telecommunications systems and the effect of thermal noise in electrical circuits are modeled by SDEs. Moreover, stock prices can also be modeled using stochastic differential equations. Stochastic differential equations in the framework of G-Brownian motion were instigated by Peng [11, 12]. Afterward, SDEs in the G-frame were studied by Bai and Li with integral Lipschitz coefficients [2] and then with discontinuous coefficients by Faizullah [4]. The stochastic functional differential equations

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(SFDEs) in the framework of G-Brownian motion were initiated by Ren, Bi and Sakthivel [14]. Later on, Faizullah developed the existence and uniqueness theory for SFDEs in the framework of G-Brownian motion with Cauchy-Maruyama approximation scheme [5]. Recently, the existence theory for neutral stochastic functional differential equations (NSFDEs) in the G-framework were established by Faizullah [7]. Moment estimate is a useful and fundamental method of analyzing and exploring dynamic behavior of NSFDEs in the G-framework. However, the pth moment estimates for the solutions of NSFDEs in the framework of G-Brownian motion with non-linear growth condition have not been utterly investigated, which remains a motivating and attractive research theme. This paper will fill the stated gap. The topic of our analysis is neutral stochastic functional differential equations in the G-framework of the form

$$d(Z(t) - D(Z_t)) = \kappa(t, Z_t)dt + \lambda(t, Z_t)d\langle B, B\rangle(t) + \mu(t, Z_t)dB(t), \tag{1.1}$$

with initial data $Z_{t_0} = \zeta = \{\zeta(s) : -\tau \leq s \leq 0\}$ such that $\zeta(s)$ is $\mathcal{F}_0$- measurable, $BC([-\tau, 0]; \mathbb{R}^n)$-valued random variable and belongs to $M_G^2([-\tau, 0]; \mathbb{R})$. The coefficients $\kappa, \lambda, \mu \in M_G^2([-\tau, T]; \mathbb{R})$, $Z(t)$ is the value of stochastic process at time t and $Z_t = \{Z(t + \theta) : -\tau \leq \theta \leq 0, \tau > 0\}$ is a bounded continuous real valued stochastic process defined on $[-\rho, 0]$ [6]. An $\mathcal{F}_t$-adapted process $Z = \{Z(t) : -\tau \leq t \leq T\}$ is called the solution of NSFDE (1.1) if it satisfies the above initial data and for all $t \geq 0$ the following integral equation holds q.s.

$$Z(t) - D(Z_t) = \zeta(0) - D(Z_{t_0}) + \int_0^t \kappa(v, Z_v)dv + \int_0^t \lambda(v, Z_v)d\langle B, B\rangle(v) + \int_0^t \mu(v, Z_v)dB(v). \tag{1.2}$$

All through this article, we suppose that the following non-linear growth condition satisfies. Assume that $\Upsilon(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave and increasing function in such a way that $\Upsilon(z) > 0$ for $z > 0$, $\Upsilon(0) = 0$ and

$$\int_{0+} \frac{dz}{\Upsilon(z)} = \infty. \tag{1.3}$$

Then for each $\chi \in BC([-\tau, 0]; \mathbb{R})$,

$$|\kappa(t, \chi)|^2 + |\lambda(t, \chi)|^2 + |\mu(t, \chi)|^2 \leq \Upsilon(1 + |\chi|^2), \ \ t \in [0, T]. \tag{1.4}$$

Since $\Upsilon(0) = 0$ and the function Υ is concave so for all $z \geq 0$ we have

$$\Upsilon(z) \leq \alpha + \beta z, \tag{1.5}$$

where α and β are positive constants. The remaining article is arranged in the following manner. In section 2, preliminaries are given. In section 3, the p-moment estimates for the solutions to neutral stochastic functional differential equations in the G-framework are studied. In section 4, asymptotic estimates for the solutions to NSFDEs in the G-framework are obtained.

2 Preliminaries

This section presents some basic notions and results of G-expectation and G-Brownian motion [3, 6, 13]. They are used in the forthcoming research work of this article.

Definition 2.1. Assume Ω be a nonempty basic space. Let $\mathcal{H}$ be a space of linear real valued functions defined on Ω. Then a real valued functional E defined on $\mathcal{H}$ fulfilling the following characteristics is called a sub-linear expectation

(a) If $X \geq Y$ then $E[X] \geq E[Y]$, where $X, Y \in \mathcal{H}$.

(b) $E[\alpha] = \alpha$, where α is a real constant.

(c) $E[\beta X] = \beta E[X]$, where $\beta > 0$.

(d) $E[X+Y] \leq E[X] + E[Y]$, for all $X, Y \in \mathcal{H}$.

Let $C_{b.Lip}(\mathbb{R}^{l\times d})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{l\times d}$ and

$$L_G^p(\Omega_T) = \{\phi(B_{t_1}, B_{t_2}, ..., B_{t_l}/l \geq 1, t_1, t_2, ..., t_l \in [0,T], \phi \in C_{b.Lip}(\mathbb{R}^{l\times d}))\}.$$

Let $\rho_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N-1$ then the collection of the following kind of processes is expressed by $M_G^0(0,T)$

$$\eta_t(w) = \sum_{i=0}^{N-1} \rho_i(w) I_{[t_i, t_{i+1}]}(t),$$

where the above process is defined on a partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0,T]$. Associated with norm $\|\eta\| = \{\int_0^T E[|\eta_u|^p]du\}^{1/p}$, $M_G^p(0,T)$, $p \geq 1$, is the completion of $M_G^0(0,T)$. For all $\eta_t \in M_G^{2,0}(0,T)$, the G-Itô's integral $I(\eta)$ and G-quadratic variation process $\{\langle B\rangle_t\}_{t\geq 0}$ are respectively given by

$$I(\eta) = \int_0^T \eta_u dB_u = \sum_{i=0}^{N-1} \rho_i(B_{t_{i+1}} - B_{t_i}),$$

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_u dB_u.$$

The book [10] is a good reference for the following six lemmas. The first two inequalities are known as Hölder's inequality and Gronwall's inequality respectively.

Lemma 2.2. *If $\frac{1}{p} + \frac{1}{q} = 1$ for all $p, q > 1$, $g \in L^2$ and $h \in L^2$ then $gh \in L^1$ and*

$$\int_c^d gh \leq (\int_c^d |g|^p)^{\frac{1}{p}} (\int_c^d |h|^q)^{\frac{1}{q}}. \tag{2.1}$$

Lemma 2.3. *Let $K \geq 0$, $H(t) : [c,d] \to \mathbb{R}$ be a continuous function, $h(t) \geq 0$ and for all $t \in [c,d]$, $H(t) \leq K + \int_c^d h(s)H(s)ds$, then*

$$H(t) \leq K e^{\int_c^t h(s)ds},$$

for all $c \leq t \leq d$.

Lemma 2.4. *Let $\delta \in (0,1)$ and $c, d \geq 0$. Then*

$$(c+d)^2 \leq \frac{c^2}{\delta} + \frac{d^2}{1-\delta}.$$

Lemma 2.5. *Let $p \geq 1$ and let $|D(\zeta)| \leq \delta\|\zeta\|$. Then for $\zeta \in CB([-\tau, 0]; \mathbb{R}^n)$,*

$$|\zeta(0) - D(\zeta)|^p \leq (1+\delta)^p \|\zeta\|^p.$$

Lemma 2.6. *Let $\hat{\delta}, c, d > 0$ and $p \geq 2$. Then the below results hold*

(i) $c^{p-1}d \leq \frac{(p-1)\hat{\delta}c^p}{p} + \frac{d^p}{p\hat{\delta}^{p-1}}$.

(ii) $c^{p-2}d^2 \leq \frac{(p-2)\hat{\delta}c^p}{p} + \frac{2d^p}{p\hat{\delta}^{\frac{p-2}{2}}}$.

Lemma 2.7. *Let $p \geq 1$ and $|D(\zeta)| \leq \delta\|\zeta\|$, $\delta \in (0,1)$. Then*

$$\sup_{0\leq u\leq t}|X(u)|^p \leq \frac{\delta}{1-\delta}\|\zeta\|^p + \frac{1}{(1-\delta)^p}\sup_{0\leq u\leq t}|X(u) - D(X_u)|^p.$$

Theorem 2.8. *Let $Z \in L^p$. Then for every $\epsilon > 0$,*

$$\hat{C}(|Z|^p > \epsilon) \leq \frac{E[|Z|^p]}{\epsilon},$$

where $\hat{C}$ is called the capacity.

The capacity is defined by $\hat{C}(A) = \sup_{P\in\mathcal{P}} P(A)$. A collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$ is denoted by $\mathcal{P}$ and $A \in \mathcal{B}(\Omega)$, which is Borel σ-algebra of Ω. Set A is known as a polar set if $\hat{C}(A) = 0$. A property holds quasi-surely ($q.s.$ in short) if it holds outside a polar set.

3 The pth moment estimates for NSFDEs in the G-framework

This section discusses the exponential estimate of the solution to NSFDE in the framework of G-Brownian motion (1.1) with the given initial data. Let equation (1.1) admit a unique solution $Z(t)$. Suppose the non-linear growth condition (1.4) holds. In addition, assume that $|D(\zeta)| \leq \delta\|\zeta\|$, where $\delta \in (0,1)$.

Theorem 3.1. *Let the non-linear growth condition holds. Let $p \geq 2$ and $E\|\zeta\|^p < \infty$. Then*

$$E[\sup_{-\tau\leq s\leq t}|Z(s)|^p] \leq K_1 e^{K_2 T},$$

where $K_1 = \frac{1}{(1-\delta)^p}[(1-\delta)^p + \epsilon(1-\delta)^{p-1} + 2(1+\delta)^p]E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T$, $K_2 = \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$, $\gamma_1 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{p-1}}$, $\gamma_2 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{p-1}}]$, $\gamma_3 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}$, $\gamma_4 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}]$ *and c_2, c_3 are positive constants.*

Proof. Apply the G-Itô's formula to $U(t, Z(t)) = |Z(t) - D(Z_t)|^p,\ p \geq 2$, we obtain

$$U(t,Z(t)) = U(0,Z(0)) + \int_0^t [U_u(u,Z(u)) + U_Z(u,Z(u))\kappa(Z_u,u)]du + \int_0^t U_Z(u,Z(u))\mu(Z_u,u)dB(u)$$
$$+ \int_0^t [U_Z(u,Z(u))\lambda(Z_u,u) + \frac{1}{2}trace\mu^T(Z_u,u)U_{ZZ}(u,Z(u))\mu(Z_u,u)]d\langle B,B\rangle(u),$$

Next we apply G-expectation on both side and use lemma 2.5. We also use the Hölder's (2.1) and BDG inequalities [8] to get

$$\begin{aligned}E[\sup_{0\leq u\leq t}|Z(u)-D(Z_u)|^p] &\leq E|\zeta(0)-D(\zeta)|^p + E[\sup_{0\leq u\leq t} p\int_0^t |Z(u)-D(Z_u)|^{p-1}|\kappa(u,Z_u)|]du \\ &+ E[\sup_{0\leq u\leq t}\int_0^t p|Z(u)-D(Z_u)|^{p-1}|\mu(u,Z_u)|dB(u)] \\ &+ E[\sup_{0\leq u\leq t}\int_0^t [p|Z(u)-D(Z_u)|^{p-1}|\lambda(u,Z_u)| \\ &+ \frac{p(p-1)}{2}|Z(u)-D(Z_u)|^{p-2}|\mu(u,Z_u)|^2]d\langle B,B\rangle(u)] \\ &\leq (1+\delta)^p E\|\zeta\|^p + J_i + J_{ii} + J_{iii},\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}J_i &= E[\sup_{0\leq u\leq t}\int_0^t p|Z(u)-D(Z_u)|^{p-1}|\kappa(u,Z_u)|du], \\ J_{ii} &= E[\sup_{0\leq u\leq t}\int_0^t p|Z(u)-D(Z_u)|^{p-1}|\mu(u,Z_u)|dB(u)], \\ J_{iii} &= E[\sup_{0\leq u\leq t}\int_0^t [p|Z(u)-D(Z_u)|^{p-1}|\lambda(u,Z_u)| + \frac{p(p-1)}{2}|Z(u)-D(Z_u)|^{p-2}|\mu(u,Z_u)|^2]d\langle B,B\rangle(u)].\end{aligned} \tag{3.2}$$

We use lemma 2.5, Lemma 2.6 and the non-linear growth condition (1.4), for any $\hat{\delta} > 0$,

$$\begin{aligned}p|Z(t)-D(Z_t)|^{p-1}|\kappa(t,Z_t)| &\leq (p-1)\hat{\delta}|Z(t)-D(Z_t)|^p + \frac{|\kappa(t,Z_t)|^p}{\hat{\delta}^{p-1}} \\ &\leq (p-1)\hat{\delta}(1+\delta)^p\|Z\|^p + \frac{[\Upsilon(1+\|Z\|^2)]^{\frac{p}{2}}}{\hat{\delta}^{p-1}} \\ &\leq (p-1)\hat{\delta}(1+\delta)^p\|Z\|^p + \frac{[\alpha+\beta(1+\|Z\|^2)]^{\frac{p}{2}}}{\hat{\delta}^{p-1}} \\ &\leq (p-1)\hat{\theta}(1+\delta)^p\|Z\|^p + \frac{(2)^{\frac{p}{2}-1}[(\alpha+\beta)^{\frac{p}{2}} + \beta^{\frac{p}{2}}\|Z\|^p]}{\hat{\delta}^{p-1}} \\ &= \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{p-1}} + [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{p-1}}]\|Z\|^p.\end{aligned}$$

So,

$$p|Z(t)-D(Z_t)|^{p-1}|\kappa(t,Z_t)| \leq \gamma_1 + \gamma_2\|Z\|^p, \tag{3.3}$$

where $\gamma_1 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{p-1}}$ and $\gamma_2 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{p-1}}]$. In a similar way as above,

$$\begin{aligned}p|Z(t)-D(Z_t)|^{p-1}|\lambda(t,Z_t)| &\leq \gamma_1 + \gamma_2\|Z\|^p, \\ p|Z(t)-D(Z_t)|^{p-1}|\mu(t,Z_t)| &\leq \gamma_1 + \gamma_2\|Z\|^p, \\ p|Z(t)-D(Z_t)|^{p-2}|\mu(t,Z_t)|^2 &\leq \gamma_3 + \gamma_4\|Z\|^p,\end{aligned} \tag{3.4}$$

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where $\gamma_3 = \frac{(2)^{\frac{p}{2}-1}(\alpha+\beta)^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}$ and $\gamma_4 = [(p-1)\hat{\delta}(1+\delta)^p + \frac{(2)^{\frac{p}{2}-1}\beta^{\frac{p}{2}}}{\hat{\delta}^{\frac{p}{2}-1}}]$. By the inequality (3.3) we obtain

$$\begin{aligned} J_i &\leq \int_0^t [\gamma_1 + \gamma_2 \|Z\|^p] du \\ &\leq \gamma_1 T + \gamma_2 \int_0^t \|Z\|^p du. \end{aligned}$$

By using lemma 2.6, inequality (3.4), second mean value theorem, BDG inequalities [8] and fundamental inequality $|c||d| \leq \frac{c^2}{2} + \frac{d^2}{2}$ we proceed as follows

$$\begin{aligned} J_{ii} &= pE[\sup_{0\leq u\leq t} | \int_0^t |Z(u) - D(Z_u)|^{p-1} |\mu(u, Z_u)| dB(u)|] \\ &\leq pc_3 E[\sup_{0\leq u\leq t} \int_0^t |Z(u) - D(Z_u)|^{2p-2} |\mu(u, Z_u)|^2 du]^{\frac{1}{2}} \\ &\leq pc_3 E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p \int_0^t |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2 du]^{\frac{1}{2}} \\ &\leq \frac{1}{2} E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p] + \frac{p^2 c_3^2}{2} E[\sup_{0\leq u\leq t} \int_0^t |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2 du] \\ &\leq \frac{1}{2} E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p] + \frac{pc_3^2}{2} E[\sup_{0\leq u\leq t} \int_0^t (\gamma_1 + \gamma_2 \|Z_u\|^p)] du \\ &= \frac{1}{2} E[\sup_{0\leq u\leq t} |Z(u) - D(Z_u)|^p] + \frac{pc_3^2}{2} \gamma_1 T + \frac{pc_3^2}{2} \gamma_2 \int_0^t E[\sup_{0\leq u\leq t} |Z_u|^p] du. \end{aligned}$$

By using the BDG inequalities [8], inequality (3.4) and lemma 2.6 we get

$$\begin{aligned} J_{iii} &= E[\sup_{0\leq u\leq t} | \int_0^t [p|Z(u) - D(Z_u)|^{p-1} |\lambda(u, Z_u)| + \frac{p(p-1)}{2} |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2] d\langle B, B\rangle(u)]| \\ &\leq c_2 \int_0^t E \sup_{0\leq u\leq t} [p|Z(u) - D(Z_u)|^{p-1} |\lambda(u, Z_u)| + \frac{p(p-1)}{2} |Z(u) - D(Z_u)|^{p-2} |\mu(u, Z_u)|^2] du \\ &\leq c_2 \int_0^t E \sup_{0\leq u\leq t} [\gamma_1 + \gamma_2 \|Z_u\|^p + \frac{(p-1)}{2} (\gamma_3 + \gamma_4 \|Z_u\|^p)] du \\ &\leq c_2 (\gamma_1 + \frac{1}{2}(p-1)\gamma_3) T + c_2 (\gamma_2 + \frac{1}{2}(p-1)\gamma_4) \int_0^t E[\sup_{0\leq u\leq t} |Z_u|^p] du. \end{aligned}$$

Using the values of J_i, J_{ii} and J_{iii} in (3.1), we have

$$\begin{aligned}
E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p] &\le (1+\delta)^p E\|\zeta\|^p + \gamma_1 T + \gamma_2\int_0^t E[\sup_{0\le u\le t}|Z_u|^p]du \\
&+ \frac{1}{2}E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p] + \frac{pc_3^2}{2}\gamma_1 T + \frac{pc_3^2}{2}\gamma_2\int_0^t E[\sup_{0\le u\le t}|Z_u|^p]du \\
&+ c_2(\gamma_1+\frac{1}{2}(p-1)\gamma_3)T + c_2(\gamma_2+\frac{1}{2}(p-1)\gamma_4)\int_0^t E[\sup_{0\le u\le t}|Z_u|^p]du \\
&= (1+\delta)^p E\|\zeta\|^p + (1+\frac{1}{2}pc_3^2+c_2)\gamma_1 T + \frac{1}{2}c_2(p-1)\gamma_3 T \\
&+ \frac{1}{2}E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p] \\
&+ [(1+\frac{1}{2}pc_3^2+c_2)\gamma_2 + \frac{1}{2}(p-1)\gamma_4]\int_0^t E[\sup_{0\le u\le t}\|Z_u\|^p]du,
\end{aligned}$$

simplification follows that

$$\begin{aligned}
E[\sup_{0\le u\le t}|Z(u)-D(Z_u)|^p] &\le 2(1+\epsilon)^p E\|\zeta\|^p + [(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T \\
&+ [(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}\|Z(r)\|^p]du.
\end{aligned}$$

By using lemma (2.7), it yields

$$\begin{aligned}
E[\sup_{0\le u\le t}|Z(u)|^p] &\le \frac{\delta}{1-\delta}E\|\zeta\|^p + 2\frac{(1+\delta)^p}{(1-\delta)^p}E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T \\
&+ \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}\|Z(r)\|^p]du
\end{aligned}$$

Noting the fact $\sup_{-\tau\le u\le t}|Z(u)|^p \le \|\zeta\|^p + \sup_{0\le u\le t}|Z(u)|^p$, we have

$$\begin{aligned}
E[\sup_{-\tau\le u\le t}|Z(u)] &\le E\|\zeta\|^p + \frac{\delta}{1-\delta}E\|\zeta\|^p + 2\frac{(1+\delta)^p}{(1-\delta)^p}E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T \\
&+ \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}|Z(r)|^p]du \\
&= \frac{1}{(1-\delta)^p}[(1-\delta)^p + \delta(1-\delta)^{p-1} + 2(1+\delta)^p]E\|\zeta\|^p \\
&+ \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T \\
&+ \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]\int_0^t E[\sup_{-\tau\le r\le u}|Z(r)|^p]du \\
&= K_1 + K_2\int_0^t E[\sup_{-\tau\le r\le u}|Z(r)|^p]du,
\end{aligned}$$

where $K_1 = \frac{1}{(1-\delta)^p}[(1-\delta)^p + \delta(1-\delta)^{p-1} + 2(1+\delta)^p]E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T$ and $K_2 = \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$. Consequently, the Grownwall's inequality gives

$$E[\sup_{-\tau\le u\le T}|Z(u)|^p] \le K_1 e^{K_2 T}.$$

The proof stands completed. □

4 Asymptotic estimates for NSFDEs in the G-framework

We now present the asymptotic estimate for the solution to NSFDE in the frame of G-Brownian motion (1.1). Recall that $\lim_{t\to\infty}\sup\frac{1}{t}log|Z(t)|$ is known as the Lyapunov exponent [9]. We show that $\frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$ is the upper bound for the Lyapunov exponent.

Theorem 4.1. *Suppose that the non-linear growth condition* (1.4) *satisfies. Then*

$$\lim_{t\to\infty}\sup\frac{1}{t}log|Z(t)| \le \frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4] \quad q.s.$$

Proof. By theorem 3.1 for each $l = 1, 2, ...$, the following inequality holds.

$$E(\sup_{l-1\le t\le l}|Z(t)|^p) \le K_1 e^{K_2 l},$$

where $K_1 = \frac{1}{(1-\delta)^p}[(1-\delta)^p + \delta(1-\delta)^{p-1} + 2(1+\delta)^p]E\|\zeta\|^p + \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_1 + c_2(p-1)\gamma_3]T$ and $K_2 = \frac{1}{(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4]$. Thus by theorem 2.8 for any arbitrary $\delta > 0$,

$$\begin{aligned}\hat{C}(w : \sup_{l-1\le t\le l}|Z(t)|^p > e^{(K_2+\epsilon)l}) &\le \frac{E[\sup_{l-1\le t\le l}|Z(t)|^p]}{e^{(K_2+\epsilon)l}} \\ &\le \frac{K_1 e^{K_2 l}}{e^{(K_2+\epsilon)l}} \\ &= K_1 e^{-\epsilon l}.\end{aligned}$$

For almost all $w \in \Omega$, the Borel-Cantelli lemma follows that there is a random integer $l_0 = l_0(w)$ so that

$$\sup_{l-1\le t\le l}|Z(t)|^p \le e^{(K_2+\epsilon)l} \qquad whenever \quad l \ge l_0,$$

it yields,

$$\begin{aligned}\lim_{t\to\infty}\sup\frac{1}{t}log|Z(t)| &\le \frac{K_2+\epsilon}{p} \\ &= \frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4] + \frac{\epsilon}{p}, \quad q.s.\end{aligned}$$

Since ϵ is arbitrary therefore

$$\lim_{t\to\infty}\sup\frac{1}{t}log|Z(t)| \le \frac{1}{p(1-\delta)^p}[(2+pc_3^2+2c_2)\gamma_2 + (p-1)\gamma_4], \quad q.s.$$

The proof stands completed. □

Remark 4.2. If $p = 2$, then we have

$$\lim_{t\to\infty} \sup \frac{1}{t} log|Z(t)| \leq \frac{1}{2(1-\delta)^2}[(2+2c_3^2+2c_2)\gamma_2+\gamma_4],$$

which shows that the Lyapunov exponent will not be greater than $\frac{1}{2(1-\delta)^2}[(2+2c_3^2+2c_2)\gamma_2+\gamma_4]$.

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Generalized contractions with triangular α-orbital admissible mappings with respect to η on partial rectangular metric spaces

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Abstract

In this paper, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

Keywords: Partial rectangular metric spaces, triangular α-orbital admissible mappings with respect to η, α-orbital attractive mappings with respect to η.

1 Introduction and preliminaries

In 2000, Branciari [2] presented a class of generalized (rectangular) metric spaces and proved the interesting topological properties in such spaces. The author [2] also assured the Banach contraction principle in the setting of complete rectangular metric spaces. After that, many authors extended and improved the existence of fixed point theorems in complete rectangular metric spaces, see [4, 5, 6, 7, 8, 9, 10, 11, 15] and the references contained therein.

Recently, Arshad et al. [1] extended the results proved by Jleli et al. [6, 7] in the setting of complete rectangular metric spaces. On the other hand, Matthew [12] presented the concept of partial metric spaces as a part of the study of denotational semantics of data flow network. In this space, the usual metric is replaced by a partial metric with an interesting property that the self-distance of

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any point of a space may not be zero. Later on, Shukla [16] introduced the partial rectangular metric spaces as a generalization of the concept of rectangular metric spaces and extended the concept of partial metric spaces.

In this paper, we introduce a notion of generalized contractions appeared in [1] in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

We now recall some definitions, lemmas and propositions that will be used in the sequel.

Definition 1.1 [2] Let X be a nonempty set. We say that a mapping $d : X \times X \to \mathbb{R}$ is a Branciari metric on X if d satisfies the following:

(d1) $0 \leq d(x, y)$, for all $x, y \in X$;

(d2) $d(x, y) = 0$ if and only if $x = y$;

(d3) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d4) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$, for all $x, y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$.

If d is a Branciari metric on X, then a pair (X, d) is called a Branciari metric space (or for short BMS). As mentioned before, Branciari metric spaces are also called rectangular metric spaces in the literature. A sequence $\{x_n\}$ in X converges to $x \in X$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. A sequence $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. A rectangular metric space (X, d) is called complete if every Cauchy sequence in X converges in X.

Shukla [16] introduced a concept of the partial rectangular metric spaces as the following:

Definition 1.2 [16] Let X be a nonempty set. We say that a mapping $p : X \times X \to \mathbb{R}$ is a partial rectangular metric on X if p satisfies the following:

(p1) $p(x, y) \geq 0$, for all $x, y \in X$;

(p2) $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$, for all $x, y \in X$;

(p3) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;

(p4) $p(x, y) = p(y, x)$, for all $x, y \in X$;

(p5) $p(x, y) \leq p(x, w) + p(w, z) + p(z, y) - p(w, w) - p(z, z)$, for all $x, y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$.

If p is a partial rectangular metric on X, then a pair (X, p) is called a partial rectangular metric space.

Remark 1.3 [16] In a partial rectangular metric space (X, p), if $x, y \in X$ and $p(x, y) = 0$, then $x = y$ but the converse may not be true.

Remark 1.4 [16] It is clear that every rectangular metric space is a partial rectangular metric space with zero self-distance. However, the converse of this fact need not hold.

Example 1.5 [16] Let $X = [0, d], \alpha \geq d \geq 3$ and define a mapping $p : X \times X \to [0, \infty)$ by

$$p(x,y) = \begin{cases} x & \text{if } x = y; \\ \frac{3\alpha+x+y}{2} & \text{if } x, y \in \{1,2\},\ x \neq y; \\ \frac{\alpha+x+y}{2} & \text{otherwise.} \end{cases}$$

Then (X, p) is a partial rectangular metric space but it is not a rectangular metric space. Moreover, (X, p) is not a partial metric space.

Proposition 1.6 [16] *For each partial rectangular metric space (X, p), the pair (X, d_p) is a rectangular metric space where*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$.

Definition 1.7 [16] Let (X, p) be a partial rectangular metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then,

(i) the sequence $\{x_n\}$ is said to converges to $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$;

(ii) the sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, p) if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite;

(iii) (X, p) is said to be a complete partial rectangular metric space if for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

Lemma 1.8 [16] *Let (X, p) be a partial rectangular metric space and let $\{x_n\}$ be a sequence in X. Then $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x)$.*

Lemma 1.9 [16] *Let (X, p) be a partial rectangular metric space and let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in (X, d_p).*

Lemma 1.10 [16] *A partial rectangular metric space (X, p) is complete if and only if a rectangular metric space (X, d_p) is complete.*

In 2014, Popescu [13] introduced the definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings including α-orbital attractive mappings.

Definition 1.11 [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be α-orbital admissible if for all $x \in X$,

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1.$$

Definition 1.12 [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be triangular α-orbital admissible if:

(T3) T is α-orbital admissible;

(T4) for all $x, y \in X, \alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

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Definition 1.13 [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be α-orbital attractive if for all $x \in X$,
$\alpha(x, Tx) \geq 1$ implies $\alpha(x, y) \geq 1$ or $\alpha(y, Tx) \geq 1$ for all $y \in X$.

We denote by Θ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ1) θ is non-decreasing;

(Θ2) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+;$$

(Θ3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Example 1.14 [6] The following functions $\theta : (0, \infty) \to (1, \infty)$ are in Θ:

(1) $\theta(t) = e^{\sqrt{t}}$;

(2) $\theta(t) = e^{\sqrt{te^t}}$;

(3) $\theta(t) = 2 - \frac{2}{\pi} \arctan(\frac{1}{t^\gamma})$ where $0 < \gamma < 1$.

Very recently Jleli et al. [6, 7] established the following generalization of the Banach fixed point theorem in the setting of complete rectangular metric spaces.

Theorem 1.15 [6] *Let (X, d) be a complete rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \text{ implies } \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^\lambda.$$

Then T has a unique fixed point.

Theorem 1.16 [7] *Let (X, d) be a complete rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \text{ implies } \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^\lambda,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then T has a unique fixed point.

Later, Arshad et al. [1] extended the results proved by Jleli et al. [6, 7] by using the concept of triangular α-orbital admissible mappings.

Theorem 1.17 [1] *Let (X, d) be a complete rectangular metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that the*

following conditions hold :
(1) there exist $\theta \in \Theta$ *and* $\lambda \in (0,1)$ *such that for all* $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y)\cdot\theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\};$$

(2) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$ *and* $\alpha(x_1, T^2x_1) \geq 1$*;*
(3) T *is a triangular* α*-orbital admissible mapping;*
(4) if $\{T^n x_1\}$ *is a sequence in* X *such that* $\alpha(T^n x_1, T^{n+1}x_1) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{T^{n(k)}x_1\}$ *of* $\{T^n x_1\}$ *such that* $\alpha(T^{n(k)}x_1, x) \geq 1$ *for all* $k \in \mathbb{N}$*;*
(5) θ *is continuous;*
(6) if z *is a periodic point* T, *then* $\alpha(z, Tz) \geq 1$.
Then T *has a fixed point.*

Theorem 1.18 [1] *Let* (X, d) *be a complete rectangular metric space,* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0,\infty)$ *be a function. Suppose that the following conditions hold :*
(1) there exist $\theta \in \Theta$ *and* $\lambda \in (0,1)$ *such that for all* $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y)\cdot\theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\};$$

(2) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$ *and* $\alpha(x_1, T^2x_1) \geq 1$*;*
(3) T *is an* α*-orbital admissible mapping;*
(4) T *is an* α*-orbital attractive mapping;*
(5) θ *is continuous;*
(6) if z *is a periodic point* T, *then* $\alpha(z, Tz) \geq 1$.
Then T *has a fixed point.*

In 2016, Chuadchawna [3] introduced the notion of triangular α-orbital admissible mappings with respect to η and proved the key lemma which will be used for proving our main results.

Definition 1.19 [3] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Then T is said to be α-orbital admissible with respect to η if for all $x \in X$,

$$\alpha(x,Tx) \geq \eta(x,Tx) \text{ implies } \alpha(Tx,T^2x) \geq \eta(Tx,T^2x).$$

Definition 1.20 [3] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Then T is said to be triangular α-orbital admissible with respect to η if

(T1) T is α-orbital admissible with respect to η;

(T2) for all $x, y \in X, \alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$ imply

$$\alpha(x,Ty) \geq \eta(x,Ty).$$

Remark 1.21 If we suppose that $\eta(x,y) = 1$ for all $x, y \in X$, then Definition 1.19 and Definition 1.20 reduces to Definition 1.11 and Definition 1.12, respectively.

Lemma 1.22 [3] *Let $T : X \to X$ be a triangular α-orbital admissible mapping with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Definition 1.23 Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Then T is said to be α-orbital attractive with respect to η if for all $x \in X$,

$\alpha(x,Tx) \geq \eta(x,Tx)$ implies $\alpha(x,y) \geq \eta(x,y)$ or $\alpha(y,Tx) \geq \eta(y,Tx)$ for all $y \in X$.

2 Main results

We now prove the following lemma needed in proving our result. The idea comes from [10] but the proof is slightly different.

Lemma 2.1 *Let (X,p) be a partial rectangular metric space and $\{x_n\}$ be a sequence in (X,p) such that $p(x_n,x) \to p(x,x)$ as $n \to \infty$ for some $x \in X$, $p(x,x) = 0$ and $\lim_{n\to\infty} p(x_n,x_{n+1}) = 0$. Then $p(x_n,y) \to p(x,y)$ as $n \to \infty$ for all $y \in X$.*

Proof. Suppose that $x \neq y$. If $x_n = y$ for arbitrarily large n, then $p(y,x) = p(x,x) = p(y,y)$. Therefore $x = y$. Assume that $y \neq x_n$ for all $n \in \mathbb{N}$. We also suppose that $x_n \neq x$ for infinitely many n. Otherwise, the result is complete. It follows that we may assume that $x_n \neq x_m \neq x$ and $x_n \neq x_m \neq y$ for all $m, n \in \mathbb{N}$ with $m \neq n$. By the partial rectangular inequality, we have

$$\begin{aligned} p(y,x) &\leq p(y,x_n) + p(x_n,x_{n+1}) + p(x_{n+1},x) - p(x_n,x_n) - p(x_{n+1},x_{n+1}) \\ &\leq p(y,x_n) + p(x_n,x_{n+1}) + p(x_{n+1},x) \end{aligned}$$

and

$$\begin{aligned} p(y,x_n) &\leq p(y,x) + p(x,x_{n+1}) + p(x_{n+1},x_n) - p(x,x) - p(x_{n+1},x_{n+1}) \\ &\leq p(y,x) + p(x,x_{n+1}) + p(x_{n+1},x_n). \end{aligned}$$

Since $\lim_{n\to\infty} p(x_n,x_{n+1}) = 0$ and taking the limit as $n \to \infty$ in the above inequalities, we have

$$\limsup_n p(y,x_n) \leq p(y,x) \leq \liminf_n p(y,x_n).$$

Hence the proof is complete. ■

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Theorem 2.2 *Let (X,p) be a complete partial rectangular metric space, $T : X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :*
(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda}, \tag{2.1}$$

where

$$R(x,y) = \max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\Big\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;
(3) T is a triangular α-orbital admissible mapping with respect to η;
(4) if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1)$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq \eta(T^{n(k)}x_1, x)$ for all $k \in \mathbb{N}$;
(5) θ is continuous;
(6) if z is a periodic point T, then $\alpha(z, Tz) \geq \eta(z, Tz)$.
Then T has a fixed point.

Proof. By (2), there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_1$ for all $n \in \mathbb{N}$. By Lemma 1.22, we obtain that

$$\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

If $T^n x_1 = T^{n+1}x_1$ for some $n \in \mathbb{N}$, then $T^n x_1$ is a fixed point of T. Thus we suppose that $T^n x_1 \neq T^{n+1}x_1$ for all $n \in \mathbb{N}$. That is $p(T^n x_1, T^{n+1}x_1) > 0$. Applying (2.1), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \theta(p(T^n x_1, T^{n+1}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^n x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^n x_1))]^{\lambda}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R(T^{n-1}x_1, T^n x_1) &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^n x_1, T(T^n x_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^n x_1, T(T^n x_1))}{1+p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^n x_1, T^{n+1}x_1)}{1+p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1)\}. \end{aligned}$$

If $R(T^{n-1}x_1, T^n x_1) = p(T^n x_1, T^{n+1}x_1)$. By (2.3), we have

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^n x_1, T^{n+1}x_1))]^{\lambda}.$$

This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1}x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1}x_1))],$$

which is a contradiction with $\lambda \in (0,1)$. This implies that $R(T^{n-1}x_1, T^n x_1) = p(T^{n-1}x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (2.3), we obtain that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^{\lambda} \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1}x_1)) \leq \cdots \leq [\theta(p(x_1, Tx_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+1}x_1)) = 1. \tag{2.5}$$

By using condition (Θ2), we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+1}x_1) = 0. \tag{2.6}$$

From condition (Θ3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} = \ell.$$

Assume that $\ell < \infty$. Let $B = \frac{\ell}{2} > 0$. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\Big|\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} - \ell\Big| \leq B \quad \text{for all } n \geq n_0.$$

This implies that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Thus we have

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Assume that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq B \quad \text{for all } n \geq n_0.$$

This implies that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. From the above two cases, there exist $A > 0$ and $n_0 \in N$ such that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0.$$

Using (2.4), we have

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An([\theta(p(x_1, Tx_1))]^{\lambda^n} - 1) \quad \text{for all } n \geq n_0. \tag{2.7}$$

Taking the limit as $n \to \infty$ in the above inequality, we get that

$$\lim_{n\to\infty} n[p(T^n x_1, T^{n+1}x_1)]^r = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{2.8}$$

We now prove that T has a periodic point. Suppose that T does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using Lemma 1.22 and (2.1), we get that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+2}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^{n+1}x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^{n+1}x_1))]^\lambda,\end{aligned}$$

where

$$\begin{aligned}R(T^{n-1}x_1, T^{n+1}x_1) &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^{n+1}x_1, T(T^{n+1}x_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^{n+1}x_1, T(T^{n+1}x_1))}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^{n+1}x_1, T^{n+2}x_1)}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\}.\end{aligned}$$

Thus we have

$$\theta(p(T^n x_1, T^{n+2}x_1)) \leq [\theta(\max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\})]^\lambda.$$

It follows that

$$\theta(p(T^n x_1, T^{n+2}x_1)) \leq [\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^\lambda. \tag{2.9}$$

Let I be the set of $n \in \mathbb{N}$ such that
$u_n := \max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}$

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$= \theta(p(T^{n-1}x_1, T^{n+1}x_1))$.

If $|I| < \infty$, then there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}$$
$$= \max\{\theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}.$$

For all $n \geq N$, from (2.9), we obtain that

$$1 \leq \theta(p(T^n x_1, T^{n+2}x_1)) \leq [\max\{\theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^{\lambda}.$$

Taking the limit as $n \to \infty$ in the above inequality and using (2.5), we get that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1.$$

If $|I| = \infty$, then we can find a subsequence of $\{u_n\}$, denoted by $\{u_n\}$, such that $u_n = \theta(p(T^{n-1}x_1, T^{n+1}x_1))$ for large n. From (2.9), we have

$$1 \leq \theta(p(T^n x_1, T^{n+2}x_1)) \leq [\theta(p(T^{n-1}x_1, T^{n+1}x_1))]^{\lambda} \leq [\theta(p(T^{n-2}x_1, T^{n}x_1))]^{\lambda^2}$$
$$\leq \cdots \leq [\theta(p(x_1, T^2x_1))]^{\lambda^n},$$

for large n. Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1. \tag{2.10}$$

Then in all cases, we obtain that (2.10) holds. By using (2.10) and (Θ2), we get that

$$\lim_{n\to\infty} p(T^n x_1, T^{n+2}x_1) = 0.$$

As an analogous proof as above, from (Θ3), there exists $n_2 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+2}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{2.11}$$

Let $M = \max\{n_1, n_2\}$. We consider the following two cases.

Case 1: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. Using (2.8), for all $n \geq M$, we get that

$$\begin{aligned}
p(T^n x_1, T^{n+m}x_1) &\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) - \\
&\quad p(T^{n+1}x_1, T^{n+1}x_1) - p(T^{n+2}x_1, T^{n+2}x_1) \\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) \\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \\
&\quad p(T^{n+3}x_1, T^{n+4}x_1) + p(T^{n+4}x_1, T^{n+2L+1}x_1) \\
&\;\;\vdots \\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + \cdots + p(T^{n+2L}x_1, T^{n+2L+1}x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \qquad (2.12) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

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Case 2: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. Using (2.8) and (2.11), for all $n \geq M$, we get that

$$\begin{aligned}
p(T^n x_1, T^{n+m} x_1) &\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+2L} x_1) - \\
&\quad p(T^{n+2} x_1, T^{n+2} x_1) - p(T^{n+3} x_1, T^{n+3} x_1) \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+2L} x_1) \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+4} x_1) + \\
&\quad p(T^{n+4} x_1, T^{n+5} x_1) + p(T^{n+5} x_1, T^{2L} x_1) \\
&\vdots \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + \cdots + p(T^{n+2L-1} x_1, T^{n+2L} x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L-1)^{1/r}} \qquad (2.13) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

From Case 1 and Case 2, we have

$$p(T^n x_1, T^{n+m} x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq M. \tag{2.14}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (2.14), we have

$$\lim_{n,m\to\infty} p(T^n x_1, T^{n+m} x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in (X, p). By Lemma 1.9, we have $\{T^n x_1\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, then (X, d_p) is complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_p(T^n x_1, z) = 0$. Using Lemma 1.8, we have

$$\lim_{n\to\infty} p(T^n x_1, z) = \lim_{n\to\infty} p(T^n x_1, T^n x_1) = p(z, z).$$

By applying Proposition 1.6, we obtain that

$$\begin{aligned}
2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\
&\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1) + p(T^n x_1, z).
\end{aligned}$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n\to\infty} p(T^n x_1, z) = 0$. We now suppose that $p(z, Tz) > 0$. By condition (4), there exists a subsequence $\{T^{n(k)} x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)} x_1, z) \geq \eta(T^{n(k)} x_1, z)$ for all $k \in \mathbb{N}$. Since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $m \neq n$, without loss of generality, we can assume that $T^{n(k)+1} x_1 \neq Tz$. And applying the condition (2.1), we obtain that

$$\begin{aligned}
\theta(p(T^{n(k)+1} x_1, Tz)) &= \theta(p(T(T^{n(k)} x_1), Tz)) \\
&\leq [\theta(R(T^{n(k)} x_1, z))]^{\lambda},
\end{aligned}$$

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where

$$\begin{aligned} R(T^{n(k)}x_1, z) &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T(T^{n(k)}x_1)), p(z, Tz), \\ &\qquad \frac{p(T^{n(k)}x_1, T(T^{n(k)}x_1))p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\} \\ &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \\ &\qquad \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}. \end{aligned}$$

Thus we have

$$\begin{aligned} \theta(p(T^{n(k)+1}x_1, Tz)) \leq \Big[\theta\Big(&\max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \\ &\frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}. \end{aligned} \tag{2.15}$$

Taking the limit as $k \to \infty$ in (2.15), using the continuity of θ and Lemma 2.1, we obtain that

$$\theta(p(z, Tz)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. Thus we obtain that $p(z, Tz) = 0$. By Remark 1.3, we have $Tz = z$, which contradicts to the assumption that T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty. Then we have $q > 1$ and $p(z, Tz) > 0$. By using (2.1) and condition (6), we get that

$$\theta(p(z, Tz)) = \theta(p(T^q z, T^{q+1}z)) \leq [\theta(p(z, Tz))]^{\lambda^q} < \theta(p(z, Tz)),$$

which is a contradiction. This implies that the set of fixed points of T is non-empty. Hence T has at least one fixed point. ■

Example 2.3 Let $X = \{0, 1, 2, 3, 4, 5\}$ and define $p : X \times X \to [0, +\infty)$ such that

$$p(x, y) = \begin{cases} x & \text{if } x = y; \\ \frac{2x+y}{2} & \text{if } x, y \in \{0, 1, 2\},\ x \neq y; \\ \frac{2+x+2y}{2} & \text{otherwise.} \end{cases}$$

Then (X, p) is a complete partial rectangular metric space. Since, for all $x \in X$ and $x > 0$, then we have $p(x, x) = x > 0$. Therefore (X, p) is not a rectangular metric space. Define a mapping $T : X \to X$ by

$$T0 = T1 = T4 = 0, T2 = T3 = 2, \text{ and } T5 = 4.$$

We can see that 0 and 2 are periodic points of T. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be functions defined by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in \{0, 1, 2, 3\}; \\ 0 \text{ otherwise.} \end{cases}$$

$$\eta(x,y) = \begin{cases} \frac{1}{2} \text{ if } x,y \in \{0,1,2,3\}; \\ 1 \text{ otherwise.} \end{cases}$$

Also define $\theta : (0,\infty) \to (1,\infty)$ by $\theta(t) = e^{\sqrt{t}}$. We next illustrate that all conditions in Theorem 2.1 hold. Taking $x_1 = 1$, we have $\alpha(1,T1) = \alpha(1,0) = 1 \geq \frac{1}{2} = \eta(1,0) = \eta(1,T1)$. Next, we prove that T is α-orbital admissible with respect to η. Let $\alpha(x,Tx) \geq \eta(x,Tx)$. Thus $x, Tx \in \{0,1,2,3\}$. By the definitions of a, η, we obtain that $\alpha(Tx,T^2x) \geq \eta(Tx,T^2x)$ for all $x \in \{0,1,2,3\}$. It follows that T is α-orbital admissible with respect to η. Let $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$. By definitions of α, η, we have $x, y, Ty \in \{0,1,2,3\}$. This yields $\alpha(x,Ty) \geq \eta(x,Ty)$ for all $x,y \in \{0,1,2,3\}$. This implies that T is triangular α-orbital admissible with respect to η. Let $x,y \in X$ be such that $p(Tx,Ty) > 0$. We could observe that if $x,y \in \{0,1,4\}$, then $Tx = Ty = 0$. This implies that $p(Tx,Ty) = 0$. So we consider the following cases:

- $x \in \{0,1,4\}$ and $y \in \{2,3\}$ or
- $x \in \{0,1,4\}$ and $y = 5$ or
- $x = \{2,3\}$ and $y = 5$.

If $x = 4$ and $y \in \{2,3\}$ or $x \in \{0,1,4\}$ and $y = 5$ or $x = \{2,3\}$ and $y = 5$, then we have $\alpha(x,y) \ngeq \eta(x,y)$. We divide the proof into four cases as follows:
(1) If $(x,y) \in \{(0,2),(2,0)\}$, then

$$R(0,2) = \max\Big\{p(0,2), p(0,0), p(2,2), \frac{p(0,0)p(2,2)}{1+p(0,2)}\Big\} = \max\Big\{1,0,2,0\Big\} = 2.$$

This implies that

$$\psi(p(T0,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(0,2))]^{0.71}.$$

(2) If $(x,y) \in \{(1,2),(2,1)\}$, then

$$R(1,2) = \max\Big\{p(1,2), p(1,0), p(2,2), \frac{p(1,0)p(2,2)}{1+p(1,2)}\Big\} = \max\Big\{2,1,2,\frac{2}{3}\Big\} = 2.$$

This implies that

$$\psi(p(T1,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(1,2))]^{0.71}.$$

(3) If $(x,y) \in \{(0,3),(3,0)\}$, then

$$R(0,3) = \max\Big\{p(0,3), p(0,0), p(3,2), \frac{p(0,0)p(3,2)}{1+p(0,3)}\Big\} = \max\Big\{4,0,\frac{9}{2},0\Big\} = \frac{9}{2}.$$

This implies that

$$\psi(p(T0,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \leq [\psi(R(0,3))]^{0.5}.$$

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(4) If $(x, y) \in \{(1, 3), (3, 1)\}$, then

$$R(1,3) = \max\Big\{p(1,3), p(1,0), p(3,2), \frac{p(1,0)p(3,2)}{1+p(1,3)}\Big\} = \max\Big\{\frac{9}{2}, 1, \frac{9}{2}, \frac{9}{11}\Big\} = \frac{9}{2}.$$

This implies that

$$\psi(p(T1, T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \le [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \le [\psi(R(1,3))]^{0.5}.$$

It follows that $\psi(p(Tx, Ty)) \le [\psi(R(x, y))]^{\lambda}$. Hence all assumptions in Theorem 2.1 are satisfied and thus T has a fixed point which are $x = 0$ and $x = 2$.

We now prove the existence of the fixed point theorem by replacing triangular mappings and condition (4) in Theorem 2.2 by α-orbital attractive mappings.

Theorem 2.4 *Let (X, p) be a complete partial rectangular metric space, $T : X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Suppose that the following conditions hold :*
(1) there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$p(Tx, Ty) > 0 \ \text{ and } \ \alpha(x, y) \ge \eta(x, y) \ \text{ imply } \ \theta(p(Tx, Ty)) \le [\theta(R(x, y))]^{\lambda}, \tag{2.16}$$

where

$$R(x, y) = \max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Tx)p(y, Ty)}{1 + p(x, y)}\Big\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge \eta(x_1, Tx_1)$ and $\alpha(x_1, T^2x_1) \ge \eta(x_1, T^2x_1)$;
(3) T is an α-orbital admissible mapping with respect to η;
(4) T is an α-orbital attractive mapping with respect to η;
(5) θ is continuous;
(6) if z is a periodic point of T, then $\alpha(z, Tz) \ge \eta(z, Tz)$.
Then T has a fixed point.

Proof. By (2), there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge \eta(x_1, Tx_1)$ and $\alpha(x_1, T^2x_1) \ge \eta(x_1, T^2x_1)$. Define the iterative sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_1$ for all $n \in \mathbb{N}$. Since T is an α-orbital admissible mapping with respect to η, we obtain that

$\alpha(x_1, Tx_1) \ge \eta(x_1, Tx_1)$ implies $\alpha(Tx_1, T^2x_1) \ge \eta(Tx_1, T^2x_1)$

and

$\alpha(x_1, T^2x_1) \ge \eta(x_1, T^2x_1)$ implies $\alpha(Tx_1, T^3x_1) \ge \eta(Tx_1, T^3x_1)$.

By continuing this process, we get that

$$\alpha(T^n x_1, T^{n+1}x_1) \ge \eta(T^n x_1, T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N} \tag{2.17}$$

and

$$\alpha(T^n x_1, T^{n+2}x_1) \ge \eta(T^n x_1, T^{n+2}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{2.18}$$

If $T^n x_1 = T^{n+1} x_1$ for some $n \in \mathbb{N}$, then $T^n x_1$ is a fixed point of T. Thus we suppose that $T^n x_1 \neq T^{n+1} x_1$ for all $n \in \mathbb{N}$. That is $p(T^n x_1, T^{n+1} x_1) > 0$. Applying (2.16) and (2.17), for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+1} x_1)) &= \theta(p(T(T^{n-1} x_1), T(T^n x_1))) \\ &\leq [\theta(R(T^{n-1} x_1, T^n x_1))]^{\lambda}, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned}R(T^{n-1} x_1, T^n x_1) &= \max \Big\{ p(T^{n-1} x_1, T^n x_1), p(T^{n-1} x_1, T^n x_1), p(T^n x_1, T^{n+1} x_1), \\ &\qquad \frac{p(T^{n-1} x_1, T^n x_1) p(T^n x_1, T^{n+1} x_1)}{1 + p(T^{n-1} x_1, T^n x_1)} \Big\} \\ &= \max\{p(T^{n-1} x_1, T^n x_1), p(T^n x_1, T^{n+1} x_1)\}.\end{aligned}$$

If $R(T^{n-1} x_1, T^n x_1) = p(T^n x_1, T^{n+1} x_1)$. By using (2.19), we get that

$$\theta(p(T^n x_1, T^{n+1} x_1)) \leq [\theta(p(T^n x_1, T^{n+1} x_1))]^{\lambda}.$$

This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1} x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1} x_1))],$$

which is a contradiction with $\lambda \in (0, 1)$. It follows that $R(T^{n-1} x_1, T^n x_1) = p(T^{n-1} x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (2.19), we obtain that

$$\theta(p(T^n x_1, T^{n+1} x_1)) \leq [\theta(p(T^{n-1} x_1, T^n x_1))]^{\lambda} \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1} x_1)) \leq \cdots \leq [\theta(p(x_1, T x_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \tag{2.20}$$

Taking the limit as $n \to \infty$, we obtain that

$$\lim_{n \to \infty} \theta(p(T^n x_1, T^{n+1} x_1)) = 1. \tag{2.21}$$

By using condition ($\Theta 2$), we have

$$\lim_{n \to \infty} p(T^n x_1, T^{n+1} x_1) = 0.$$

As in the proof of Theorem 2.2, we can prove that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1} x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{2.22}$$

We now prove that T has a periodic point. Suppose that T does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using (2.16) and (2.18), we get that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+2} x_1)) &= \theta(p(T(T^{n-1} x_1), T(T^{n+1} x_1))) \\ &\leq [\theta(R(T^{n-1} x_1, T^{n+1} x_1))]^{\lambda},\end{aligned}$$

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where

$$\begin{aligned}R(T^{n-1}x_1, T^{n+1}x_1) &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1),\\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^{n+1}x_1, T^{n+2}x_1)}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\}\\ &= \max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\}.\end{aligned}$$

By the analogous proof in Theorem 2.2, we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+2}x_1) = 0$$

and there exists $n_2 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+2}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{2.23}$$

Let $h = \max\{n_1, n_2\}$. We consider the following two cases.

Case 1: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. By using (2.22), for all $n \geq h$, we obtain that

$$\begin{aligned}p(T^n x_1, T^{n+m}x_1) &\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) -\\ &\quad p(T^{n+1}x_1, T^{n+1}x_1) - p(T^{n+2}x_1, T^{n+2}x_1)\\ &\vdots\\ &\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + \cdots + p(T^{n+2L}x_1, T^{n+2L+1}x_1)\\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}}\\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.\end{aligned}$$

Case 2: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. By using (2.22) and (2.23), for all $n \geq h$, we get that

$$\begin{aligned}p(T^n x_1, T^{n+m}x_1) &\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+2L}x_1) -\\ &\quad p(T^{n+2}x_1, T^{n+2}x_1) - p(T^{n+3}x_1, T^{n+3}x_1)\\ &\vdots\\ &\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \cdots + p(T^{n+2L-1}x_1, T^{n+2L}x_1)\\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}}\\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.\end{aligned}$$

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From Case 1 and Case 2, we obtain that

$$p(T^n x_1, T^{n+m} x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq h. \tag{2.24}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (2.24), we have

$$\lim_{n,m\to\infty} p(T^n x_1, T^{n+m} x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in (X, p). By Lemma 1.9, we have $\{T^n x_1\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, then (X, d_p) is complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_p(T^n x_1, z) = 0$. Using Lemma 1.8, we have

$$\lim_{n\to\infty} p(T^n x_1, z) = \lim_{n\to\infty} p(T^n x_1, T^n x_1) = p(z, z).$$

By applying Proposition 1.6, we obtain that

$$\begin{aligned} 2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\ &\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1) + p(T^n x_1, z). \end{aligned}$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n\to\infty} p(T^n x_1, z) = 0$. We now prove that $z = Tz$. Suppose that $z \neq Tz$. Since T is α-orbital attractive with respect to η, we obtain that for all $n \in \mathbb{N}$,

$\alpha(T^n x_1, z) \geq \eta(T^n x_1, z)$ or $\alpha(z, T^{n+1} x_1) \geq \eta(z, T^{n+1} x_1)$.

We divide the proof in two cases as follows.

(1) There exists an infinite subset J of $\mathbb{N}$ such that $\alpha(T^{n(k)} x_1, z) \geq \eta(T^{n(k)} x_1, z)$ for every $k \in J$.

(2) There exists an infinite subset L of $\mathbb{N}$ such that $\alpha(z, T^{n(k)+1} x_1) \geq \eta(z, T^{n(k)+1} x_1)$ for every $k \in L$.

For the case (1), since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $n \neq m$, without loss of the generality, we can assume that $T^{n(k)+1} x_1 \neq z$ for all $k \in J$. Applying the condition (2.16), we get that

$$\begin{aligned} \theta(p(T^{n(k)+1} x_1, Tz)) &= \theta(p(T(T^{n(k)} x_1), Tz)) \\ &\leq [\theta(R(T^{n(k)} x_1, z))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} R(T^{n(k)} x_1, z) = \max\Big\{& p(T^{n(k)} x_1, z), p(T^{n(k)} x_1, T(T^{n(k)} x_1)), p(z, Tz), \\ & \frac{p(T^{n(k)} x_1, T(T^{n(k)} x_1)) p(z, Tz)}{1 + p(T^{n(k)} x_1, z)}\Big\} \\ = \max\Big\{& p(T^{n(k)} x_1, z), p(T^{n(k)} x_1, T^{n(k)+1} x_1), p(z, Tz), \\ & \frac{p(T^{n(k)} x_1, T^{n(k)+1} x_1) p(z, Tz)}{1 + p(T^{n(k)} x_1, z)}\Big\}. \end{aligned}$$

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Then we have

$$\theta(p(T^{n(k)+1}x_1, Tz)) \leq \Big[\theta\Big(\max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}.$$

Taking the limit as $k \to \infty$ in the above equality, using the continuity of θ and Lemma 2.1, we obtain that

$$\theta(p(z, Tz)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. For the case (2), the proof is similar. Therefore $z = Tz$, which is a contradiction with the assumption that T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty, Then we have $q > 1$ and $p(z, Tz) > 0$. Applying (2.16) and condition (6), we get that

$$\theta(p(z, Tz)) = \theta(p(T^q z, T^{q+1}z)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. Thus the set of fixed points of T is non-empty. Hence T has at least one fixed point. ■

Since a rectangular metric space is a partial rectangular metric space, we immediately obtain Theorem 17 and Theorem 19 in [1] by applying Theorem 2.2 and Theorem 2.4, respectively.

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On stability problems of a functional equation deriving from a quintic function

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Abstract. In this paper, we obtain a solution of new type quintic functional equations and prove the Hyers-Ulam-Rassias stability for a quintic functional equation by the directed method and a subaddtive function approach and also, present a counterexample. Finally, we investigate the Hyers-Ulam-Rassias stability for a quintic functional equation with an involution by the fixed point method.

1. Introduction and preliminaries

The concept of stability problem of a functional equation was first posed by Ulam [18] concerning the stability of group homomorphisms. In 1941, Hyers [6] solved the problem of Ulam. This result was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [13] has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. Since then, several stability problems for various functional equations have been investigated by numerous mathematicians; c.f e.g. [5], [20], [14], [2], [21] and [11].

In [4], Cho and et al. introduced the following quintic functional equation

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)=20\{f(x+y)+f(x-y)\}+90f(x)\,. \tag{1.1}$$

Since $f(x)=x^5$ is a solution of the equation (1.1), the equation (1.1) is called a quintic functional equation.

Stetkær [17] introduced the following quadratic functional equation with an involution

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(\sigma(y))$$

and solved the general solution, Belaid and et al. [3] established generalized Hyers-Ulam stability in Banach space for this functional equation.

In this paper we consider the following another type quintic functional equation

$$f(5x+y)+f(5x-y)+3[f(x+y)+f(x-y)]=2[f(4x+y)+f(4x-y)]+2f(5x)-4090f(x) \tag{1.2}$$

for all $x,y\in\mathcal{X}$. Here our purpose is to find out a solution and to prove the generalized Hyers-Ulam-Rassias stability problem and give a counterexample for the equation (1.2). Also, we introduce a quintic functional equation with an involution σ as follows;

$$f(3x+y)+f(3x+\sigma(y))+5[f(x+y)+f(x+\sigma(y))]=4[f(2x+y)+f(2x+\sigma(y))]+2f(3x)-246f(x) \tag{1.3}$$

for all $x,y\in\mathcal{X}$. We will investigate the generalized Hyers-Ulam-Rassias stability for this functional equation by using a fixed point method.

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2. Solutions of Equations (1.2) and (1.3)

In this section let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces and we will obtain the result that the functional equations (1.2) and (1.3) are solutions of a quintic functional equation by using 5-additive symmetric mapping. Before we proceed, we will introduce some basic concepts concerning 5-additive symmetric mappings. A mapping $A_5 : \mathcal{X}^5 \to \mathcal{Y}$ is called 5-additive if it is additive in each variable. A mapping A_5 is said to be symmetric if $A_5(x_1, x_2, x_3, x_4, x_5) = A_5(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)})$ for every permutation $\{\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)\}$ of $\{1, 2, 3, 4, 5\}$. If $A_5(x_1 , x_2 , x_3 , x_4 , x_5)$ is a 5-additive symmetric mapping, then $A^5(x)$ will denote the diagonal $A_5(x , x , x , x , x)$ and $A^5(qx) = q^5 A^5(x)$ for all $x \in \mathcal{X}$ and all $q \in \mathbb{Q}$. A mapping $A^5(x)$ is called a monomial function of degree 5 (assuming $A^5 \not\equiv 0$). On taking $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1} = x_{s+2} = \cdots = x_5 = y$ in $A_5(x_1 , x_2 , x_3 , x_4 , x_5)$, it is denoted by $A^{s,5-s}(x , y)$. We note that the generalized concepts of n-additive symmetric mappings are found in [16] and [19].

Theorem 2.1. *A function $f : \mathcal{X} \to \mathcal{Y}$ is a solution of the functional equation (1.2) if and only if f is of the form $f(x) = A^5(x)$ for all $x \in \mathcal{X}$, where $A^5(x)$ is the diagonal of the 5-additive symmetric map $A_5 : \mathcal{X}^5 \to \mathcal{Y}$.*

Proof. Suppose f satisfies the functional equation (1.2). Letting $x = 0$ and replacing y by x in the equation (1.2), we have $f(x) = -f(-x)$, for all $x \in \mathcal{X}$. Hence f is an odd mapping and also we have $f(0) = 0$. Putting $y = 0$ in the equation (1.2), we get $f(4x) = 4^5 f(x)$, for all $x \in \mathcal{X}$. Hence we have

$$f(4^n x) = 4^{5n} f(x)\,, \tag{2.1}$$

for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Note that $f(x) = \frac{1}{4^{5n}} f(4^n x)$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$.

On the other hand, we can rewrite the functional equation (1.2) in the following form

$$\begin{aligned} f(x) + \frac{1}{4090} f(5x + y) + \frac{1}{4090} f(5x - y) - \frac{1}{2045} f(4x + y) - \frac{1}{2045} f(4x - y) \\ + \frac{3}{4090} f(x + y) + \frac{3}{4090} f(x - y) - \frac{1}{2045} f(5x) = 0\,, \end{aligned}$$

for all $x \in \mathcal{X}$. By [19, Theorem 3.5 and Theorem 3.6] f is a general polynomial function of degree at most 6, that is, f is of the following form

$$f(x) = A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x)$$

for all $x \in \mathcal{X}$. Note that $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : \mathcal{X}^i \to \mathcal{Y}$ for $i = 1, 2, 3, 4, 5$. Since $f(0) = 0$ and f is odd, we have $A^0(x) = A^0 = 0$ and $A^4(x) = A^2(x) = 0$. It follows that $f(x) = A^5(x) + A^3(x) + A^1(x)$, for all $x \in \mathcal{X}$. By (2.1) and $A^n(rx) = r^n A^n(x)$ whenever $x \in \mathcal{X}$ and $r \in \mathbb{Q}$, we obtain

$$4^5 A^5(x) + 4^3 A^3(x) + 4A^1(x) = f(4x) = 4^5 f(x) = 4^5 A^5(x) + 4^5 A^3(x) + 4^5 A^1(x)\,,$$

for all $x \in \mathcal{X}$. Then $A^1(x) = -\frac{16}{17} A^3(x)$, for all $x \in \mathcal{X}$. Hence $A^3(x) = A^1(x) = 0$, for all $x \in \mathcal{X}$. Thus $f(x) = A^5(x)$. Conversely, suppose $f(x) = A^5(x)$ for all $x \in \mathcal{X}$, where $A^5(x)$ is the diagonal of the 5-additive symmetric map $A_5 : X^5 \to Y$. We note that

$$\begin{aligned} A^5(ax + by) &= a^5 A^5(x) + b^5 A^5(y) + 5a^4 b A^{4,1}(x, y) + 10a^3 b^2 A^{3,2}(x, y) \\ &+ 10a^2 b^3 A^{2,3}(x, y) + 5ab^4 A^{1,4}(x, y)\,, \end{aligned}$$

for all $x , y \in \mathcal{X}$ and $a , b \in \mathbb{Q}$. The remains of the proof can be easily checked. □

Theorem 2.2. *Let $\sigma(x) = -x$, for all $x \in \mathcal{X}$. A function $f : \mathcal{X} \to \mathcal{Y}$ is a solution of the functional equation (1.3) if and only if f is of the form $f(x) = A^5(x)$ for all $x \in \mathcal{X}$, where $A^5(x)$ is the diagonal of the 5-additive symmetric map $A_5 : \mathcal{X}^5 \to \mathcal{Y}$.*

Proof. Suppose f satisfies the functional equation (1.3). Letting $x = y = 0$ in the equation (1.3), we have $f(0) = 0$. Putting $y = 0$ in the equation (1.3), we get $f(2x) = 2^5 f(x)$, for all $x \in \mathcal{X}$. The remains are similar to the proof of Theorem 2.1. □

3. Hyers-Ulam-Rassias stability of (1.2) in Banach spaces

In this section, we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.2). Throughout this section, we assume that $\mathcal{X}$ is a normed real linear space with norm $\|\cdot\|_{\mathcal{X}}$ and $\mathcal{Y}$ is a real Banach space with norm $\|\cdot\|_{\mathcal{Y}}$.

We use the abbreviation for the given mapping $f : \mathcal{X} \to \mathcal{Y}$ as follows:

$$\mathcal{D}f(x,y) := f(5x+y)+f(5x-y)+3[f(x+y)+f(x-y)]-2[f(4x+y)+f(4x-y)]-2f(5x)+4090f(x)$$

for all $x, y \in \mathcal{X}$.

Theorem 3.1. *Suppose that there exists a mapping $\phi : \mathcal{X}^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies $f(0) = 0$,*

$$||\mathcal{D}f(x,y)||_{\mathcal{Y}} \le \phi(x,y) \tag{3.1}$$

and the series $\sum_{j=0}^{\infty} \frac{1}{4^{5j}}\phi(4^j x, 4^j y)$ converges for all $x, y \in \mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (1.2) and the inequality

$$||f(x) - \mathcal{Q}(x)||_{\mathcal{Y}} \le \frac{1}{4^6}\sum_{j=0}^{\infty} \frac{1}{4^{5j}}\phi(4^j x, 0), \tag{3.2}$$

for all $x \in \mathcal{X}$.

Proof. By letting $y = 0$ in the inequality (3.1), we have

$$||\mathcal{D}f(x,0)||_{\mathcal{Y}} = 4^6||f(x) - \frac{1}{4^5}f(4x)||_{\mathcal{Y}} \le \phi(x,0),$$

that is,

$$||f(x) - \frac{1}{4^5}f(4x)||_{\mathcal{Y}} \le \frac{1}{4^6}\phi(x,0), \tag{3.3}$$

for all $x \in \mathcal{X}$. For any positive integer k replacing x by $4^k x$ and multiplying $\frac{1}{4^{5k}}$ in the inequality (3.3),

$$||\frac{1}{4^{5k}}f(4^k x) - \frac{1}{4^{5(k+1)}}f(4^{k+1}x)||_{\mathcal{Y}} \le \frac{1}{4^6}\frac{1}{4^{5k}}\phi(4^k x, 0), \tag{3.4}$$

for all $x \in \mathcal{X}$. For any positive integers n and m with $n > m$,

$$||\frac{1}{4^{5m}}f(4^m x) - \frac{1}{4^{5n}}f(4^n x)||_{\mathcal{Y}} \le \frac{1}{4^6}\sum_{j=m}^{n-1} \frac{1}{4^{5j}}\phi(4^j x, 0), \tag{3.5}$$

for all $x \in \mathcal{X}$. As $n \to \infty$, the right-hand side in the inequality (3.5) close to 0. Hence $\{\frac{1}{4^{5n}}f(4^n x)\}$ is a Cauchy sequence in the Banach space $\mathcal{Y}$. Thus we can define a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{4^{5n}}f(4^n x),$$

for all $x \in \mathcal{X}$.

By letting $m=0$ in the inequality (3.5), we have

$$||f(x)-\frac{1}{4^{5n}}f(4^nx)||_{\mathcal{Y}} \le \frac{1}{4^6}\sum_{j=0}^{n-1}\frac{1}{4^{5j}}\phi(4^jx,0)\,, \tag{3.6}$$

for all $x\in\mathcal{X}$, $n\in\mathbb{N}$. As $n\to\infty$ in the inequality (3.6),

$$||f(x)-\mathcal{Q}(x)||_{\mathcal{Y}} \le \frac{1}{4^6}\sum_{j=0}^{\infty}\frac{1}{4^{5j}}\phi(4^jx,0)\,, \tag{3.7}$$

for all $x\in\mathcal{X}$. It satisfies the inequality (3.2). Now, replacing x and y by 4^nx and 4^ny respectively and dividing by 4^{5n} in the inequality (3.1), we have

$$||\mathcal{D}\mathcal{Q}(x,y)||_{\mathcal{Y}} = \frac{1}{4^{5n}}||\mathcal{D}f(4^nx,4^ny)||_{\mathcal{Y}} \le \frac{1}{4^{5n}}\phi(4^nx,4^ny)\,,$$

for all $x,\,y\in\mathcal{X}$. By taking $n\to\infty$, the definition of $\mathcal{Q}$ implies that $\mathcal{Q}$ satisfies (1.2) for all $x,\,y\in\mathcal{X}$, that is, $\mathcal{Q}$ is the quintic mapping. Next, it remains to show the uniqueness. Assume that there exists $\mathcal{T}:\mathcal{X}\to\mathcal{Y}$ satisfying (1.2) and (3.2). The Theorem 2.1 implies that $\mathcal{T}(4^nx)=4^{5n}\mathcal{T}(x)$ and $\mathcal{Q}(4^nx)=4^{5n}\mathcal{Q}(x)$, for all $x\in\mathcal{X}$. Then

$$\begin{aligned}||\mathcal{T}(x)-\mathcal{Q}(x)||_{\mathcal{Y}} &= \frac{1}{4^{5n}}||\mathcal{T}(4^nx)-\mathcal{Q}(4^nx)||_{\mathcal{Y}}\\ &\le \frac{1}{4^{5n}}\Big(||\mathcal{T}(4^nx)-f(4^nx)||_{\mathcal{Y}}+||f(4^nx)-\mathcal{Q}(4^nx)||_{\mathcal{Y}}\Big)\\ &\le \frac{2}{4^6}\sum_{j=0}^{\infty}\frac{1}{4^{5(n+j)}}\phi(4^{n+j}x,0)\,,\end{aligned}$$

for all $x\in\mathcal{X}$. By letting $n\to\infty$, we immediately have the uniqueness of $\mathcal{Q}$. □

Corollary 3.2. *Let θ, r be positive real numbers with $r<5$ and let $f:\mathcal{X}\to\mathcal{Y}$ be a mapping with $f(0)=0$ such that*

$$||\mathcal{D}f(x,y)||_{\mathcal{Y}} \le \theta(||x||_{\mathcal{Y}}^r+||y||_{\mathcal{Y}}^r) \tag{3.8}$$

for all $x,y\in\mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q}:\mathcal{X}\to\mathcal{Y}$ satisfying

$$||f(x)-\mathcal{Q}(x)||_{\mathcal{Y}} \le \frac{\theta||x||_{\mathcal{Y}}^r}{4(4^5-4^r)}$$

for all $x\in\mathcal{X}$.

Proof. On taking $\phi(x,y)=\theta(||x||_{\mathcal{Y}}^r+||y||_{\mathcal{Y}}^r)$ for all $x,y\in\mathcal{X}$, it is easy to show that the inequality (3.8) holds. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}||f(x)-\mathcal{Q}(x)||_{\mathcal{Y}} &\le \frac{1}{4^6}\sum_{j=0}^{\infty}\frac{1}{4^{5j}}\phi(4^jx,0)\\ &= \frac{\theta}{4^6}\sum_{j=0}^{\infty}\frac{4^r}{4^{5j}}||x||_{\mathcal{Y}}^r\\ &= \frac{\theta\,||x||_{\mathcal{Y}}^r}{4}\frac{1}{4^5-4^r}\end{aligned}$$

for all $x\in\mathcal{X}$ and $r<5$. □

Now, we will investigate the stability of the given quintic functional equation (1.2) using the subadditive function method under the condition that the space $\mathcal{Y}$ is a p-Banach space. Before proceeding the proof, we will state the the basic definition of subadditive function. It follows from the reference [12].

A function $\phi : A \to B$ having a domain A and a codomain $(B, \leq)$ that are both closed under addition is called

(1) *a subadditive function* if $\phi(x+y) \leq \phi(x) + \phi(y)$, for all $x, y \in A$.

(2) *a contractively subadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x+y) \leq L(\phi(x) + \phi(y))$, for all $x, y \in A$.
We note that ϕ satisfies the following properties $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2nx) \leq (2L)n\phi(x)$. It follows by the contractively subadditive condition of ϕ that

$$\phi(\lambda x) \leq \lambda L \phi(x)\,, \text{ and so } \phi(\lambda^j x) \leq (\lambda L)^j \phi(x),\ i \in \mathbb{N}\,,$$

for all $x \in A$ and all positive integer $\lambda \geq 2$.

(3) *a expansively superadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x+y) \geq \frac{1}{L}(\phi(x) + \phi(y))$, for all $x, y \in A$.
We note that ϕ satisfies the following properties $\phi(x) \leq \frac{L}{2}\phi(2x)$ and so $\phi(\frac{x}{2^n}) \leq \frac{L}{2^n}\phi(x)$. We observe that an expansively superadditive mapping ϕ satisfies the following properties

$$\phi(\lambda x) \geq \frac{\lambda}{L}\phi(x) \text{ and so } \phi(\frac{x}{\lambda^j}) \leq (\frac{L}{\lambda})^j \phi(x), j \in \mathbb{N},$$

for all $x \in A$ and all positive integer $\lambda \geq 2$.

Theorem 3.3. *Suppose that there exists a mapping $\phi : \mathcal{X}^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies $f(0) = 0$ and*

$$||\mathcal{D}f(x,y)||_{\mathcal{Y}} \leq \phi(x,y) \tag{3.9}$$

for all x, $y \in \mathcal{X}$ and the map ϕ is contractively subadditive with a constant L such that $\frac{4L}{4^5} < 1$. Then there exists a unique quintic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ which satisfies the equation (1.2) and the inequality

$$||f(x) - \mathcal{Q}(x)||_{\mathcal{Y}} \leq \frac{\phi(x,0)}{4\sqrt[p]{4^{5p} - (4L)^p}}\,, \tag{3.10}$$

for all $x \in \mathcal{X}$.

Proof. By the inequalities (3.3) and (3.5) of the proof of Theorem 3.1, we have

$$\begin{aligned}
&||\frac{1}{4^{5m}}f(4^m x) - \frac{1}{4^{5n}}f(4^n x)||_{\mathcal{Y}}^p \\
&\leq \frac{1}{4^{6p}}\sum_{j=m}^{n-1}\frac{1}{4^{5jp}}||f(4^j x) - \frac{1}{4^5}f(4^{j+1}x)||_{\mathcal{Y}}^p \\
&\leq \frac{1}{4^{6p}}\sum_{j=m}^{n-1}\frac{1}{4^{5jp}}\phi(4^j x, 0)^p \\
&\leq \frac{1}{4^{6p}}\sum_{j=m}^{n-1}\frac{1}{4^{5jp}}(4L)^{jp}\phi(x,0)^p \\
&= \frac{\phi(x,0)^p}{4^{6p}}\sum_{j=m}^{n-1}\Big(\frac{4L}{4^5}\Big)^{jp},
\end{aligned}$$

that is,

$$||\frac{1}{4^{5m}}f(4^m x) - \frac{1}{4^{5n}}f(4^n x)||_{\mathcal{Y}}^p \leq \frac{\phi(x,0)^p}{4^{6p}}\sum_{j=m}^{n-1}\Big(\frac{4L}{4^5}\Big)^{jp}, \tag{3.11}$$

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for all $x \in \mathcal{X}$, and for all m and n with $m < n$. Hence $\{\frac{1}{4^{5n}} f(4^n x)\}$ is a Cauchy sequence in the space $\mathcal{Y}$. Thus we may define

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{4^{5n}} f(4^n x),$$

for all $x \in \mathcal{X}$. Now, we will show that the map $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ is a generalized quintic mapping. Then

$$\begin{aligned} ||\mathcal{D}\mathcal{Q}(x,y)||_{\mathcal{Y}}^p &= \lim_{n\to\infty} \frac{||\mathcal{D}f(4^n x, 4^n y)||_{\mathcal{Y}}^p}{4^{5pn}} \\ &\le \lim_{n\to\infty} \frac{\phi(4^n x, 4^n y)^p}{4^{5pn}} \\ &\le \lim_{n\to\infty} \phi(x,y)^p (\frac{4L}{4^5})^{pn} = 0, \end{aligned}$$

for all $x \in \mathcal{X}$. Hence the mapping $\mathcal{Q}$ is a quintic mapping. Note that the inequality (3.11) implies the inequality (3.10) by letting $m = 0$ and taking $n \to \infty$. Assume that there exists $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) and (3.10). We know that $\mathcal{T}(4^n x) = 4^{5n}\mathcal{T}(x)$, for all $x \in \mathcal{X}$. Then

$$\begin{aligned} ||\mathcal{T}(x) - \frac{1}{4^{5n}} f(4^n x)||_{\mathcal{Y}}^p &= \frac{1}{4^{5pn}} ||\mathcal{T}(4^n x) - f(4^n x)||_{\mathcal{Y}}^p \\ &\le \frac{1}{4^{5pn}} \frac{\phi(4^n x, 0)^p}{4^p(4^{5p} - (4L)^p)} \\ &\le \Big(\frac{4L}{4^5}\Big)^{pn} \frac{\phi(x,0)^p}{4^p(4^{5p} - (4L)^p)}, \end{aligned}$$

that is,

$$||\mathcal{T}(x) - \frac{1}{4^{5n}} f(4^n x)||_{\mathcal{Y}} \le \Big(\frac{4L}{4^5}\Big)^n \frac{\phi(x,0)}{4\sqrt[p]{4^{5p} - (4L)^p}},$$

for all $x \in \mathcal{X}$. By letting $n \to \infty$, we immediately have the uniqueness of $\mathcal{Q}$. □

4. Counterexample

In this section, we will present a counterexample to show that the functional equation (1.2) is not stable for $r = 5$ in Corollary 3.2.

Example 4.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a mapping defined by*

$$\phi(x) = \begin{cases} \theta x^5 & \text{for } |x| < 1 \\ \theta & \text{otherwise} \end{cases}$$

where $\theta > 0$ is a constant and a mapping $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^{5i}} \tag{4.1}$$

for all $x \in \mathbb{R}$. Then the mapping f satisfies the inequality

$$|\mathcal{D}f(x,y)| \le 4092 \frac{4^{15}\theta}{4^5 - 1}(|x|^5 + |y|^5) \tag{4.2}$$

for all $x \in \mathbb{R}$. Then there does not exist a quintic mapping $\mathcal{Q} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - \mathcal{Q}(x)| \le \beta |x|^5 \tag{4.3}$$

for all $x \in \mathbb{R}$.

Proof. The definitions of ϕ and f imply that

$$|f(x)| = \Big|\sum_{i=0}^{\infty}\frac{\phi(4^ix)}{4^{5i}}\Big| \le \sum_{i=0}^{\infty}\frac{\theta}{4^{5i}} = \frac{\theta 4^5}{4^5-1}$$

for all $x \in \mathbb{R}$. Hence f is bounded by $\frac{\theta 4^5}{4^5-1}$. If $|x|^5+|y|^5 \ge 1$, then the inequality (4.2) holds. Now, we suppose that $0<|x|^5+|y|^5<1$. Then there exists a positive integer t such that

$$\frac{1}{4^{5(t+2)}} \le |x|^5+|y|^5 < \frac{1}{4^{5(t+1)}}\,. \tag{4.4}$$

Since $|x|^5+|y|^5 < \frac{1}{4^{5(t+1)}}$ we have

$$4^{5t}x^5 < \frac{1}{4^5} \text{ and } 4^{5t}y^5 < \frac{1}{4^5}\,.$$

That is,

$$4^tx < \frac{1}{4} \text{ and } 4^ty < \frac{1}{4}\,.$$

These imply that $4^{t-1}x, 4^{t-1}y, 4^{t-1}5x, 4^{t-1}(x+y), 4^{t-1}(x-y), 4^{t-1}(4x+y), 4^{t-1}(4x-y), 4^{t-1}(5x+y), 4^{t-1}(5x-y) \in (-1,1)$. Hence we obtain that $4^jx, 4^jy, 4^j5x, 4^j(x+y), 4^j(x-y), 4^j(4x+y), 4^j(4x-y), 4^j(5x+y), 4^j(5x-y) \in (-1,1)$ for each $j=0,1,\cdots,t-1$. Also, for each $j=0,1,\cdots,t-1$,

$$\begin{aligned}&\phi(4^j(5x+y))+\phi(4^j(5x-y))+3[\phi(4^j(x+y))+\phi(4^j(x-y))]\\&-2[\phi(4^j(4x+y))+\phi(4^j(4x-y))]-2\phi(4^j5x)+4090\phi(4^jx)=0\,.\end{aligned}$$

From the definition of f and the inequality (4.4), we have

$$\begin{aligned}|\mathcal{D}f(x,y)| \quad &\le \sum_{j=0}^{\infty}\Big\{\phi(4^j(5x+y))+\phi(4^j(5x-y))\\&+3[\phi(4^j(x+y))+\phi(4^j(x-y))]\\&-2[\phi(4^j(4x+y))+\phi(4^j(4x-y))]\\&-2\phi(4^j5x)+\phi(4^jx)\Big\}\\&\le \sum_{j=t}^{\infty}\frac{4092\theta}{4^{5j}}\\&\le 4092\theta\frac{4^5\,4^{5\cdot 2}}{4^5-1}\frac{1}{4^{5(t+2)}}\\&\le \frac{4092\cdot 4^{15}\theta}{4^5-1}(|x|^5+|y|^5)\,,\end{aligned}$$

for all $x,y\in\mathbb{R}$. We claim that the quintic functional equation (1.2) is not stable in Corollary 3.2. Assume that there exists a quintic mapping $\mathcal{Q}:\mathbb{R}\to\mathbb{R}$ and a constant $\beta>0$ satisfying the inequality (4.3). Since f is bounded and continuous for all $x\in\mathbb{R}$, $\mathcal{Q}$ is bounded on any open interval containing the origin and continuous at the origin. In the view of Corollary 3.2, $\mathcal{Q}(x)$ must have the form $\mathcal{Q}(x)=\gamma x^5$ for all $x\in\mathbb{R}$. Hence we have that

$$|f(x)| \le (\beta+|\gamma|)|x|^5\,. \tag{4.5}$$

But we can choose a positive integer m with $m\theta>\beta+|\gamma|$. If $x\in(0\,,\frac{1}{4^{5(m-1)}})$, then $4^{5t}\in(0\,,1)$ for all $t=0,1,\cdots,m-1$. For this x, we have

$$f(x)=\sum_{i=0}^{\infty}\frac{\phi(4^ix)}{4^{5i}} \ge \sum_{i=0}^{m-1}\frac{\theta(4^ix)^5}{4^{5i}} = m\theta x^5 > (\beta+|\gamma|)x^5$$

This implies that it is a contradiction to the inequality (4.5). Therefore the quintic functional equation (1.2) is not stable. □

5. Hyers-Ulam-Rassias stability with an involution via a fixed point method

In this section, we will investigate the Hyers-ulam-Rassias stability of a quintic functional equation with a involution over a non-Archimedean normed space $\mathcal{X}$.

A *non-Archimedean field* is a field $\mathcal{K}$ equipped with a (valuation) function from $\mathcal{K}$ into $[0, \infty)$ satisfying the following properties: (1) $|a| \geq 0$ and equality holds if and only if $a = 0$, (2) $|ab| = |a|\,|b|$, (3) $|a+b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. An example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$; see [10]. Also, the most important examples of non-Archimedean spaces are p-adic numbers; see [8]. We will reproduce the following definitions due to Moslehian and Sadeghi [9] and Mirmostafaee and Moslehian [8].

Definition 5.1. [9] *Let $\mathcal{X}$ be a linear space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $||\cdot|| : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following properties:*

(1) $||x|| = 0$ *if and only if* $x = 0$
(2) $||rx|| = |r| \cdot ||x||$ $(r \in \mathcal{K})$
(3) $||x+y|| \leq$ *max* $\{||x||, ||y||\}$,

for all $x, y \in \mathcal{X}$ and $r \in \mathcal{K}$. Then $(\mathcal{X}, ||\cdot||)$ is called a non-Archimedean normed space.

Before proceed the proof, we will introduce a notion of an involution. For an additive mapping $\sigma : \mathcal{X} \to \mathcal{X}$ with $\sigma(\sigma(x)) = x$ for all $x \in \mathcal{X}$, then σ is called an involution of $\mathcal{X}$; see [3] and [17]. Let $(\mathcal{Y}, ||\cdot||)$ be a non-Archimedean normed space. We use the abbreviation for the given mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ as follows:

$$\begin{aligned}\mathcal{D}_\sigma f(x,y) \quad := \quad & f(3x+y) + f(3x+\sigma(y)) + 5[f(x+y) + f(x+\sigma(y))] \\ & -4[f(2x+y) + f(2x+\sigma(y))] - 2f(3x) + 246f(x)\end{aligned}$$

for all $x, y \in \mathcal{X}$.

The following statements are relative to the alternative of fixed point; see [7] and [15]. By using this method, we will prove the Hyers-Ulam-Rassias stability.

Theorem 5.2 (The alternative of fixed point [7], [15]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant l. Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

(1) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\};$$

(4) $d(y, y^*) \leq \frac{1}{1-l}\, d(y, Ty)$ *for all* $y \in \triangle$.

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Theorem 5.3. *Suppose that $\phi : \mathcal{X}^2 \to [0, 1)$ is a mapping and there exists a real number l with $0 < l < 1$ such that*

$$\phi(2x\,, 2y) \le |2| l \phi(x\,, y)\,, \;\; \phi(x + \sigma(x)\,, y + \sigma(y)) \le |2| l \phi(x\,, y) \tag{5.1}$$

for all $x\,, y \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping such that $f(0) = 0$ and

$$||\mathcal{D}_\sigma f(x, y)|| \le \phi(x\,, y) \tag{5.2}$$

for all $x\,, y \in \mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q} : X \to Y$ with an involution such that

$$||f(x) - \mathcal{Q}(x)|| \le \frac{1 + |2|^3\, l}{|2|^8(1 - l)} \Phi(x) \tag{5.3}$$

where $\Phi(x) = max\{\phi(x\,, 0)\,, \phi(0\,, x)\}$ for all $x \in \mathcal{X}$.

Proof. We will consider the following set

$$\Omega = \{g \,|\, g : \mathcal{X} \to \mathcal{X}\,, g(0) = 0\}\,.$$

Then there is the generalized metric on Ω,

$$d(g,\, h) = \inf\{\lambda \in (0, \infty) \,|\, \parallel g(x) - h(x) \parallel \le \lambda \Phi(x)\,, \text{for all } x \in \mathcal{X}\}\,.$$

It is not hard to prove that (Ω, d) is a complete space. A function $T : \Omega \to \Omega$ is defined by

$$T(g)(x) = \frac{1}{2^5}\{g(2x) + g(x + \sigma(x))\} \tag{5.4}$$

for all $x \in \mathcal{X}$. We know that there is an arbitrary positive constant λ with $d(g, h) \le \lambda$, for all $g, h \in \Omega$. Then

$$||g(2x) - h(2x)|| \le |2|\lambda\, l \Phi(x) \;\; \text{and} \;\; ||g(x + \sigma(x)) - h(x + \sigma(x))|| \le |2|\lambda\, l \Phi(x) \tag{5.5}$$

for all $x \in \mathcal{X}$. Hence we have

$$\begin{aligned} ||T(g)(x) - T(h)(x)|| &= \frac{1}{|2|^5}||g(2x) - h(2x) + g(x + \sigma(x)) - h(x + \sigma(x))|| \\ &\le \frac{1}{|2|^5} \max\{||g(2x) - h(2x)||\,, ||g(x + \sigma(x)) - h(x + \sigma(x))||\} \\ &\le \frac{l}{|2|^4} \lambda \Phi(x) \le l\, \lambda \Phi(x)\,, \end{aligned}$$

for all $x \in \mathcal{X}$. This implies that $d(T(g)\,, T(h)) \le l\, d(g\,, h)$ for all $g\,, h \in \Omega$ and hence T is a strictly contractive mapping with Lipschitz constant $0 < l < 1$. Now, letting $y = 0$ and $x = 0$ in the inequality (5.2), respectively we have

$$||f(2x) - 2^5 f(x)|| \le \frac{1}{|2|^3} \phi(x, 0) \tag{5.6}$$

and

$$||2f(y) + 2f(\sigma(y))|| \le \phi(0, y) \tag{5.7}$$

for all $x\,, y \in \mathcal{X}$. Replacing y by $x + \sigma(x)$ in the inequality (5.7), we get

$$||f(x + \sigma(x))|| \le \frac{1}{|2|}\, \phi(0, x + \sigma(x)) \le l\, \phi(0, x) \tag{5.8}$$

for all $x \in \mathcal{X}$. The inequalities (5.6) and (5.7) imply that

$$||T(f)(x) - f(x)|| = \frac{1}{|2|^5}||f(2x) - 2^5 f(x) + f(x + \sigma(x))|| \le \frac{1 + |2|^3\, l}{|2|^8}\, \Phi(x) \tag{5.9}$$

for all $x \in \mathcal{X}$. Hence we have $d(T(f), f) \leq \frac{1+|2|^3 l}{|2|^8} < \infty$. By Theorem 5.2, there exits a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that $\lim_{n\to\infty} d(T^n(f), \mathcal{Q}) = 0$. Using mathematical induction, we may define

$$T^n(f)(x) = \lim_{n\to\infty} \frac{1}{2^{5n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\}$$

for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} d(T^n(f), \mathcal{Q}) = 0$, there exists a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that $\lambda_n \to 0$ as $n \to \infty$ and $d(T^n f, \mathcal{Q}) \leq \lambda_n$ for $n \in \mathbb{N}$. The definition of d implies that

$$||T^n(f)(x) - \mathcal{Q}(x)|| \leq \lambda_n \Phi(x)$$

for all $x \in \mathcal{X}$. For each fixed $x \in \mathcal{X}$, we have

$$\lim_{n\to\infty} ||T^n(f)(x) - \mathcal{Q}(x)|| = 0 .$$

Thus we may conclude that

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{2^{5n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\} \tag{5.10}$$

for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} ||\mathcal{D}_\sigma \mathcal{Q}(x, y)|| &\leq \lim_{n\to\infty} \frac{1}{|2|^{5n}} \max\{\phi(2^n x, 2^n y), |2^n - 1|\phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y))\} \\ &\leq \lim_{n\to\infty} \frac{l^n}{|2|^{4n}} \max\{\phi(x, y), |2^n - 1|\phi(x, y)\} \\ &\leq \lim_{n\to\infty} l^n \phi(x, y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{X}$. The mapping $\mathcal{Q}$ satisfies the Theorem 2.2. This means that $\mathcal{Q}$ is a quintic mapping. By Theorem 5.2, we have

$$d(f, \mathcal{Q}) \leq \frac{1}{1-l} d(T(f), f) \leq \frac{1 + |2|^3 l}{|2|^8(1-l)} .$$

This implies that the inequality (5.3) holds for all $x \in \mathcal{X}$. The uniqueness of the quintic mapping follows from (3) in Theorem 5.2. □

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Comparisons of isolation numbers and semiring ranks of fuzzy matrices

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Abstract. Let $\mathbb{F}$ be the fuzzy semiring and A be an $m \times n$ fuzzy matrix over $\mathbb{F}$. The semiring rank of a fuzzy matrix A is the smallest k such that A can be factored as an $m \times k$ fuzzy matrix times a $k \times n$ fuzzy matrix. The isolation number of A is the maximum number of nonzero entries in A such that no two are in any row or any column, and no two are in a 2×2 submatrix of all nonzero entries. We have that the isolation number of A is a lower bound on the semiring rank of A. We also compare the isolation number with the Boolean rank of the support of A, and determine the equal cases of them.

1. **Introduction**

There are many papers on the study of rank of matrices over several semirings containing binary Boolean algebra, fuzzy semiring, semiring of nonegative integers, and so on ([2], [3], [6], and [7]). But there are few papers on isolation numbers of matrices. Gregory et al ([7]) introduced set of isolated entries and compared Boolean rank with biclique covering number. Recently Beasley ([2]) introduced isolation number of Boolean matrix and compare it with Boolean rank.

In this paper, we investigate the possible isolation number of fuzzy matrix and compare it with semiring rank of fuzzy matrix and the Boolean rank of the support of the fuzzy matrix.

2. **Preliminaries**

A *semiring* is a binary system $(\mathbb{S}, +, \cdot)$ such that $(\mathbb{S}, +)$ is an abelian monoid with identity 0, $(\mathbb{S}, \cdot)$ is a monoid with identity 1, $\cdot$ distributes over $+$ from both sides and $0 \cdot s = s \cdot 0 = 0$ for all $s \in \mathbb{S}$ and $1 \neq 0$. We use juxtaposition for $\cdot$ for convenience. If $(\mathbb{S}, \cdot)$ is abelian then we say $\mathbb{S}$ is commutative. If 0 is the only element of $\mathbb{S}$ that has an additive inverse then $\mathbb{S}$ is *antinegative.* Note that all rings with unity are semirings, but none are antinegative. The set, Z_+, of nonnegative integers with usual addition and multiplication is an example of combinatorially interesting antinegative semiring.

Let $\mathcal{M}_{m,n}(\mathbb{S})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{S}$ with matrix addition and multiplication following the usual rules. Let $\mathcal{M}_n(\mathbb{S}) = \mathcal{M}_{m,n}(\mathbb{S})$ if $m = n$, let I_m denote the $m \times m$ identity matrix, $O_{m,n}$ denote the zero matrix in $\mathcal{M}_{m,n}(\mathbb{S})$, $J_{m,n}$ denote the matrix of all ones in $\mathcal{M}_{m,n}(\mathbb{S})$. The subscripts are usually omitted if the order is obvious, and we write I, O, J.

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The matrix $A \in \mathcal{M}_{m,n}(\mathbb{S})$ is said to be of *semiring rank* k if there exist matrices $B \in \mathbf{M}_{m,k}(\mathbb{S})$ and $C \in \mathbf{M}_{k,n}(\mathbb{S})$ such that $A = BC$ and k is the smallest positive integer that such a factorization exists. We denote $r_{\mathbb{S}}(A) = k$.

We say that a matrix A *dominates* a matrix B if $a_{i,j} = 0$ implies $b_{i,j} = 0$.

Given a matrix X, we let $\mathbf{x}^{(j)}$ denote the j^{th} column of X and $\mathbf{x}_{(i)}$ denote the i^{th} row. Now if $r_{\mathbb{S}}(A) = k$ and $A = BC$ is a factorization of $A \in \mathcal{M}_{m,n}(\mathbb{S})$, then $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(k)}\mathbf{c}_{(k)}$. Since each of the terms $\mathbf{b}^{(i)}\mathbf{c}_{(i)}$ is a semiring rank one matrix, the semiring rank of A is also the minimum number of semiring rank one matrices whose sum is A.

Let $\mathbb{S}$ be any set of two or more elements. If $\mathbb{S}$ is totally ordered by $<$, that is, $\mathbb{S}$ is a chain under $<$(i.e. $x < y$ or $y < x$ for all distinct x, y in $\mathbb{S}$), then define $x + y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all x, y in $\mathbb{S}$. If $\mathbb{S}$ has a universal lower bound and a universal upper bound then $\mathbb{S}$ becomes a semiring: a *chain semiring*.

Let $\mathbb{H}$ be any nonempty family of sets nested by inclusion, $0 = \bigcap_{x\in\mathbb{H}} x$ and $1 = \bigcup_{x\in\mathbb{H}} x$. Then $\mathbb{S} = \mathbb{H}\bigcup\{0, 1\}$ is a chain semiring.

Let α, ω be real numbers with $\alpha < \omega$. Define $\mathbb{S}_R = \{\beta \in \mathbb{R} : \alpha \leq \beta \leq \omega\}$. Then $\mathbb{S}_R$ is a chain semiring with $\alpha =$ "0" and $\omega =$ "1". It is isomorphic to the chain semiring $\mathbb{H} = \{[\alpha, \beta] : \alpha \leq \beta \leq \omega\}$.

If in particular we choose the real numbers 0 and 1 as α and ω in the previous example $\mathbb{S}_R$, then the chain semiring $\mathbb{F} = \{\beta \in \mathbb{R} : 0 \leq \beta \leq 1\}$ is called *fuzzy semiring* and the $m \times n$ matrices over $\mathbb{F}$ is called the *fuzzy matrices.*

Now let $\mathcal{M}_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ fuzzy matrices with entries in $\mathbb{F}$. The *fuzzy rank* of $A \in \mathcal{M}_{m,n}(\mathbb{F})$ is the semiring rank over $\mathbb{F}$ and denoted $r_{\mathbb{F}}(A)$.

If we take $\mathbb{H}$ to be a sington, say $\{a\}$, and denote empty subset by 0 and $\{a\}$ by 1, the resulting chain semiring is called a *Boolean algebra* $\mathbb{B} = \{0, 1\}$, and the $m \times n$ matrices over $\mathbb{B}$ is called *Boolean matrices.* This Boolean algebra is a subsemiring of every chain semiring.

Now let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ Boolean matrices with entries in $\mathbb{B}$. The *Boolean rank* of $D \in \mathcal{M}_{m,n}(\mathbb{B})$ is the semiring rank over $\mathbb{B}$ and denoted $b(D)$ or $r_{\mathbb{B}}(D)$. Also, $r_{\mathbb{S}}(O) = 0$, and O is the only matrix of semiring rank 0 over any semiring $\mathbb{S}$.

The Boolean rank has many applications in combinatorics, especially graph theory, for example, if $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is the adjacency matrix of the bipartite graph G with bipartition (X, Y), the Boolean rank of A is the minimum number of bicliques that cover the edges of G, called the *biclique covering number.*

Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{S})$, a set of *isolated entries* ([7]) is a set of entries, usually written as $E = \{a_{i,j}\}$ such that $a_{i,j} \neq 0$, no two entries in E are in the same row, no two entries in E are in the same column, and, if $a_{i,j}, a_{k,l} \in E$ then, $a_{i,l} = 0$ or $a_{k,j} = 0$. That is, isolated entries are independent entries and any two isolated entries $a_{i,j}$ and $a_{k,l}$ do not lie in a submatrix of A of the form $\begin{bmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{bmatrix}$ with all entries nonzero. The *isolation number of* A, $\iota(A)$, is the maximum size of a set of isolated entries. Note that $\iota(A) = 0$ if and only if $A = O$.

Example 2.1. Let

$$A = \begin{bmatrix} 1 & 1 & 0.2 & 0 & 0 \\ 0.2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0.2 \\ 0 & 0.2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0.2 & 1 \end{bmatrix}$$

be a fuzzy matrix and $E_1 = \{a_{1,3}, a_{2,1}, a_{3,5}, a_{4,2}, a_{5,4}\}$. The entries (0.2's) of A in E_1 are isolated entries and hence $\iota(A) = 5$. But the entries of A in the position in $E_2 = \{a_{1,1}, a_{2,2}, a_{3,5}, a_{4,4}, a_{5,3}\}$ are not isolated.

Suppose that $A \in \mathcal{M}_{m,n}(\mathbb{S})$ and $r_{\mathbb{S}}(A) = k$. Then there are k semiring rank one matrices A_i such that

$$A = A_1 + A_2 + \cdots + A_k. \tag{2.1}$$

Because each semiring rank one matrix can be permuted to a matrix of the form $\begin{bmatrix} N & O \\ O & O \end{bmatrix}$ with $\overline{N} = J$, it is easily seen that the matrix consisting of two isolated entries of A cannot be dominated by any one A_i among the semiring rank one summand of A in (1.1). Thus

$$i(A) \leq r_{\mathbb{S}}(A). \tag{2.2}$$

Many functions, sets and relations concerning matrices do not depend upon the magnitude or nature of the individual entries of a matrix, but rather only on whether the entry is zero or nonzero. These combinatorially significant matrices have become increasingly important in recent years. Of primary interest is the Boolean rank. Finding the Boolean rank of a $(0,1)$-matrix is an NP-Complete problem ([8]), and consequently finding bounds on the Boolean rank of a matrix is of interest to those researchers that would use Boolean rank in their work. If the $(0,1)$-matrix is the reduced adjacency matrix of a bipartite graph, the isolation number of the matrix is the maximum size of a non-competitive matching in the bipartite graph. This is related to the study of such combinatorial problems as the patient hospital problem, the stable marriage problem, etc. An additional reason for studying the isolation number is that it is a lower bound on the rank of a matrix over $\mathbb{S}$. While finding the isolation number as well as finding the semiring rank of a matrix is an NP-Complete problem ([1]), for some matrices finding the isolation number can be easier than finding the semiring rank especially if the matrix is sparse:

Example 2.2. Let

$$F = \begin{bmatrix} 1 & 1 & 1 & 0.2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0.2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0.2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a fuzzy matrix.

Then we can easily see $r_{\mathbb{F}}(F) \leqq 6$ from first 3 rows and columns, however to find that fuzzy rank is not 5, requires much calculation if the isolation number is not considered. However, the isolation number is easily seen to be 6, both computationally and visually, the 0.2's in the matrix represent a set of isolated entries. Thus we have $r_{\mathbb{F}}(F) = 6$ by (2.2).

Note that if any of the 1's in F are replaced by zeros, the resulting matrix still has fuzzy rank 6 as well as isolation number 6.

Terms not specifically defined here can be found in Brualdi and Ryser [5] for matrix terms, or Bondy and Murty [4] for graph theoretic terms.

For our use in the next section, we define the support matrix of a fuzzy matrix. If $A \in \mathcal{M}_{m,n}(\mathbb{F})$, then the *support* of A is the Boolean matrix $\overline{A} = (\overline{a_{i,j}}) \in \mathcal{M}_{m,n}(\mathbb{B})$ such that $\overline{a_{i,j}} = 1$ if $a_{i,j} \neq 0$ and $\overline{a_{i,j}} = 0$ if $a_{i,j} = 0$.

3. Comparisons between isolation numbers and semiring ranks of fuzzy matrices

In this section, we compare the isolation number with semiring rank of fuzzy matrix, and also we compare the isolation number with Boolean rank of the support of fuzzy matrix.

Lemma 3.1. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{F})$, $r_{\mathbb{F}}(A+B) \leq r_{\mathbb{F}}(A) + r_{\mathbb{F}}(B)$. And for $A, B \in \mathcal{M}_{m,n}(\mathbb{B})$, $b(A+B) \leq b(A)+b(B)$.*

Proof. It follows from the definition of fuzzy (and Boolean) rank and equation (2.1). ■

Lemma 3.2. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{F})$, $\overline{A+B} = \overline{A} + \overline{B}$ in $\mathcal{M}_{m,n}(\mathbb{B})$.*

Proof. It follows from the facts that all the entries of $A, B \in \mathcal{M}_{m,n}(\mathbb{B})$ are nonnegative and $1 + 1 = 1$ in $\mathbb{B}$. ■

Lemma 3.3. *For $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $b(\overline{A}) \leq r_{\mathbb{F}}(A)$.*

Proof. If $r_{\mathbb{F}}(A) = k$, then A has a fuzzy rank one factorization such that $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(k)}\mathbf{c}_{(k)}$ with $B = [\mathbf{b}^{(1)}\mathbf{b}^{(2)} \cdots \mathbf{b}^{(k)}] \in \mathbf{M}_{m,k}(\mathbb{F})$ and $C = [\mathbf{c}_{(1)}\mathbf{c}_{(2)} \cdots \mathbf{c}_{(k)}]^t \in \mathbf{M}_{k,n}(\mathbb{F})$ from (2.1). Therefore
$b(\overline{A}) = b(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(k)}\mathbf{c}_{(k)}}) = b(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)}} + \overline{\mathbf{b}^{(2)}\mathbf{c}_{(2)}} + \cdots + \overline{\mathbf{b}^{(k)}\mathbf{c}_{(k)}}) \leq k$, from Lemma 3.2.
Hence $b(\overline{A}) \leq r_{\mathbb{F}}(A)$. ■

We may have strict inequality in Lemma 3.3 as we see in the following example.

Example 3.4. Consider $A = \begin{bmatrix} 1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0.2 \\ 0.6 & 0.6 \end{bmatrix}$ in $\mathcal{M}_{m,n}(\mathbb{F})$. Then $r_{\mathbb{F}}(A) = 2$ but $b(\overline{A}) = b(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = 1$. Hence $b(\overline{A}) < r_{\mathbb{F}}(A)$. But $r_{\mathbb{F}}(B) = b(\overline{B}) = 1$ since $B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \begin{bmatrix} 1 & 0.2 \end{bmatrix}$ over $\mathbb{F}$.

Lemma 3.5. *For $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{F})$, $\iota(A) = \iota(\overline{A})$.*

Proof. If $a_{i,j}$ and $a_{k,l}$ are any isolated entries in A, then $i \neq k$ and $j \neq l$, and that $a_{i,l} = 0$ or $a_{k,j} = 0$. Hence $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are isolated entries in $\overline{A}$, so we have $\iota(A) \leq \iota(\overline{A})$.

Conversely, if $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are any isolated entries in $\overline{A}$, then $a_{i,j} \neq 0$ and $a_{k,l} \neq 0$ and that $a_{i,l} = \overline{a_{i,l}} = 0$ or $a_{k,j} = \overline{a_{k,j}} = 0$. Hence $a_{i,j}$ and $a_{k,l}$ are isolated entries in A, so we have $\iota(\overline{A}) \leq \iota(A)$. ■

Theorem 3.6. *If $A \in \mathcal{M}_{m,n}(\mathbb{F})$, then $\iota(A) = 1$ if and only if $b(\overline{A}) = 1$.*

Proof. Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$. If $b(\overline{A}) = 1$ then $A \neq O$ so that $\iota(A) \neq 0$ and since $\iota(A) = \iota(\overline{A}) \leq b(\overline{A})$ by (2.2), we have $\iota(A) = 1$.

Conversely, suppose that $\iota(A) = 1$ and that $b(\overline{A}) \geq 2$. Then, for some P and Q, permutation matrices of the appropriate orders, $P\overline{A}Q = \begin{bmatrix} J_{r,s} & O \\ O & O \end{bmatrix} + \overline{A}_2$ for some r, s with either $r < m$ or $s < n$. Partition $\overline{A}_2$ as $\overline{A}_2 = \begin{bmatrix} \overline{A}_{2,1} & \overline{A}_{2,2} \\ \overline{A}_{2,3} & \overline{A}_{2,4} \end{bmatrix}$, where $\overline{A}_{2,1}$ is $r \times s$. Since $b(P\overline{A}Q) = b(\overline{A}) \geq 2$, we have $\overline{A} \neq J$, and hence, one of

$\overline{A}_{2,2}, \overline{A}_{2,3}, \overline{A}_{2,4}$ has a zero entry. Further, one of $\overline{A}_{2,2}, \overline{A}_{2,3}, \overline{A}_{2,4}$ has an entry of 1 since $P\overline{A}Q \neq \begin{bmatrix} J_{r,s} & O \\ O & O \end{bmatrix}$. Thus, in $P\overline{A}Q$, there is some zero entry, say $\overline{a_{i,j}} = 0$, which lies in a nonzero column j and a nonzero row i. Then, any of the ones in that column j together with a one in the nonzero row i form a set of two isolated entries, a contradiction, thus $b(\overline{A}) = 1$. ■

It follows that the subset of $\mathcal{M}_{m,n}(\mathbb{F})$ of matrices with isolation number one is the same as the set of matrices whose support has Boolean rank one.

For $A = A_1 + A_2 + \cdots + A_k$ with $r_{\mathbb{F}}(A) = k$, let $\mathcal{R}_i$ denote the indices of the nonzero rows of A_i and $\mathcal{C}_j$ denote the indices of the nonzero columns of A_j, $i, j = 1.\cdots, k$. Let $r_i = |\mathcal{R}_i|$, the number of nonzero rows of A_i and $c_j = |\mathcal{C}_j|$, the number of nonzero columns of A_j.

Lemma 3.7. *Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$. Then if $r_{\mathbb{F}}(A) \geq b(\overline{A}) = 2$ then $\iota(A) = 2$, and if $\iota(A) = 2$ then $b(\overline{A}) \neq 3$.*

Proof. If $b(\overline{A}) = 2$, then $\iota(A) > 1$ by Theorem 3.6. Since $\iota(A) = \iota(\overline{A}) \leq b(\overline{A})$ from Lemma 3.5 and (2.2), we have that $\iota(A) = \iota(\overline{A}) = 2$.

Now, suppose that $\iota(A) = 2$ and that $b(\overline{A}) = 3$. Let $\overline{A} = \overline{A_1} + \overline{A_2} + \overline{A_3}$ where $b(\overline{A_i}) = 1$.

Permute the rows of $\overline{A}$ so that $\mathcal{R}_1 = \{1, 2, \cdots, r_1\}$ and permute the columns of $\overline{A}$ so that $\mathcal{C}_2 = \{1, 2, \cdots, c_2\}$ and $\mathcal{C}_3 = \{k+1, k+2, \cdots, k+c_3\}$ where $k \leq c_2$.

Note that $\mathcal{R}_i \neq \mathcal{R}_j$ and $\mathcal{C}_i \neq \mathcal{C}_j$ unless $i = j$ otherwise $\overline{A_i} + \overline{A_j}$ would be Boolean rank 1.

Suppose that $\mathcal{R}_1 \subset \mathcal{R}_2$. Permute the remaining rows so that $\mathcal{R}_2 = \{1, 2, \cdots, r_2\}$, and $\mathcal{R}_3 = \{a+1, a+2, \cdots, a+b+c, r_2+1, r_2+2\cdots, r_2+e\}$ where $a+b \leq r_1$, $r_1 \leq a+b+c \leq a+b+c+d \leq r_2$ and $r_2 \leq a+b+c+d+e$.

Thus, we have that

$$\overline{A} = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & J_{b,v} & O_{b,w} \\ J_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v} & O_{c,w} \\ J_{d,k} & J_{d,g} & O_{d,h} & O_{d,u} & O_{d,v} & O_{d,w} \\ O_{e,k} & J_{e,g} & J_{e,h} & J_{e,u} & O_{e,v} & O_{e,w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v} & O_{f,w} \end{bmatrix},$$

for some $a, b, c, d, e, f, g, h, k, u, v$ and w. Thus, with this notation,

$$\overline{A_1} = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O \\ J_{b,k} & J_{b,g} & J_{b,h} & O_{b,u} & J_{b,v} & O \\ O & O & O & O & O & O \end{bmatrix},$$

$$\overline{A_2} = \begin{bmatrix} J_{a,k} & J_{a,g} & O \\ J_{b,k} & J_{b,g} & O \\ J_{c,k} & J_{c,g} & O \\ J_{d,k} & J_{d,g} & O \\ O & O & O \end{bmatrix}, \text{ and } \overline{A_3} = \begin{bmatrix} O_{a,k} & O_{a,g} & O_{a,h} & O_{a,u} & O_{a,v+w} \\ O_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & O_{b,v+w} \\ O_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v+w} \\ O_{d,k} & O_{d,g} & O_{d,h} & O_{d,u} & O_{d,v+w} \\ O_{e,k} & J_{e,g} & J_{e,h} & J_{e,u} & O_{e,v+w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v+w} \end{bmatrix}.$$

Now, if $\overline{A}[r_1+1,\cdots,m|1,\cdots,n] = \overline{A_2}[r_1+1,\cdots,m|1,\cdots,n] + \overline{A_3}[r_1+1,\cdots,m|1,\cdots,n]$ has Boolean rank 1 then $d=e=0$ and hence $\overline{A}$ has the form

$$\overline{A} = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & J_{b,v} & O_{b,w} \\ J_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v} & O_{c,w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v} & O_{f,w} \end{bmatrix},$$

$$= \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O \\ J_{b,k} & J_{b,g} & J_{b,h} & O_{b,u} & J_{b,v} & O \\ O & O & O & O & O & O \end{bmatrix} + \begin{bmatrix} O & O & O & O & O & O \\ J_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & O_{b,v} & O \\ J_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v} & O \\ O & O & O & O & O & O \end{bmatrix},$$

so that $b(\overline{A})=2$, a contradiction to the assumption $b(\overline{A})=3$. Thus, $\overline{A}[r_1+1,\cdots,m|1,\cdots,n]$ must have Boolean rank 2, and hence it has two isolated entries, say i_2 and i_3. If $\mathcal{C}_1 \not\subseteq \mathcal{C}_2 \cup \mathcal{C}_3$ then without loss of generality we have that $\overline{a_{1,x}} \neq 0$ for $x = k+g+h+u+1$, but then, $\{\overline{a_{1,x}}, i_2, i_3\}$ is a set of three isolated entries, a contradiction to $\iota(\overline{A}) = \iota(A) = 2$. Thus, $v=0$ and hence, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cup \mathcal{C}_3$. Further, $\mathcal{C}_1 \neq \mathcal{C}_2 \cup \mathcal{C}_3$, otherwise, $\overline{A}$ can be expressed as

$$\overline{A} = \begin{bmatrix} J_{a,k} & J_{a,g} & O \\ J_{b,k} & J_{b,g} & O \\ J_{c,k} & J_{c,g} & O \\ J_{d,k} & J_{d,g} & O \\ O & O & O \end{bmatrix} + \begin{bmatrix} O_{a,k} & J_{a,g} & J_{a,h} & O \\ O_{b,k} & J_{b,g} & J_{b,h} & O \\ O_{c,k} & J_{c,g} & J_{c,h} & O \\ O_{d,k} & O_{d,g} & O_{d,h} & O \\ O_{e,k} & J_{e,g} & J_{e,h} & O \\ O_{f,k} & O_{f,g} & O_{f,h} & O \end{bmatrix},$$

so that $b(\overline{A})=2$, contradiction to the assumption $b(\overline{A})=3$.

Note that $a, u, d \neq 0$, for otherwise $b(\overline{A})=2$. If $e=0$ then $b+c\neq 0$ so that $\{\overline{a_{1,c_1}}, \overline{a_{a+1,k+c_3}}, \overline{a_{r_2,1}}\}$ is a set of three isolated entries, a contradiction to $\iota(\overline{A}) = \iota(A) = 2$. If $e \neq 0$, then $\{\overline{a_{1,c_1}}, \overline{a_{r_2,1}}, \overline{a_{r_2+e,k+c_3}}\}$ is a set of three isolated entries, contradicting that $\iota(\overline{A}) = \iota(A) = 2$. Thus, $\mathcal{R}_1 \not\subset \mathcal{R}_2$.

By renumbering and/or transposing we have proved that $\mathcal{R}_i \not\subset \mathcal{R}_j$ and $\mathcal{C}_i \not\subset \mathcal{C}_j$ for any pair i and j.

Now, permute the rows and columns of $\overline{A}$ so that $\mathcal{R}_1 = \{1,2,\cdots,r_1\}$, $\mathcal{R}_2 = \{a+1, a+2, \cdots, a+b, a+b+c+1, a+b+c+2, \cdots, a+b+c+d+e+f\}$, and $\mathcal{R}_3 = \{a+b+1, a+b+2, \cdots, a+b+c+d+e, a+b+c+e+f+1, a+b+c+e+f+2\cdots, a+b+c+e+f+g\}$ for some a,b,c,d,e,f,g where $a+b+c+d = r_1$, so that $\overline{A}$ has the form:

$$\overline{A} = \begin{bmatrix} J_{a,k} & O_{a,l} & J_{a,p} & O_{a,q} & J_{a,r} & O_{a,s} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,l} & J_{b,p} & J_{b,q} & J_{b,r} & O_{b,s} & J_{b,v} & O_{b,w} \\ J_{c,k} & O_{c,l} & J_{c,p} & J_{c,q} & J_{c,r} & J_{c,s} & J_{c,v} & O_{c,w} \\ J_{d,k} & J_{d,l} & J_{d,p} & J_{d,q} & J_{d,r} & J_{d,s} & J_{d,v} & O_{d,w} \\ J_{e,k} & J_{e,l} & J_{e,p} & J_{e,q} & J_{e,r} & J_{e,s} & O_{e,v} & O_{e,w} \\ J_{f,k} & J_{f,l} & J_{f,p} & J_{f,q} & O_{f,r} & O_{f,s} & O_{f,v} & O_{f,w} \\ O_{g,k} & O_{g,l} & J_{g,p} & J_{g,q} & J_{g,r} & J_{g,s} & O_{g,v} & O_{g,w} \\ O_{h,k} & O_{h,l} & O_{h,p} & O_{h,q} & O_{h,r} & O_{h,s} & O_{h,v} & O_{h,w} \end{bmatrix}, \tag{3.1}$$

for some $a, b, c, d, e, f, g, h, k, l, p, q, r, s, v$, and w, so that

$$\overline{A_1} = \begin{bmatrix} J_{a,k} & O_{a,l} & J_{a,p} & O_{a,q} & J_{a,r} & O_{a,s} & J_{a,v} & O_{a,w} \\ J_{b,k} & O_{b,l} & J_{b,p} & O_{b,q} & J_{b,r} & O_{b,s} & J_{b,v} & O_{b,w} \\ J_{c,k} & O_{c,l} & J_{c,p} & O_{c,q} & J_{c,r} & O_{c,s} & J_{c,v} & O_{c,w} \\ J_{d,k} & O_{d,l} & J_{d,p} & O_{d,q} & J_{d,r} & O_{d,s} & J_{d,v} & O_{d,w} \\ O & O & O & O & O & O & O & O \end{bmatrix},$$

$$\overline{A_2} = \begin{bmatrix} O_{a,k} & O_{a,l} & O_{a,p} & O_{a,q} & O \\ J_{b,k} & J_{b,l} & J_{b,p} & J_{b,q} & O \\ O_{c,k} & O_{c,l} & O_{c,p} & O_{c,q} & O \\ J_{d,k} & J_{d,l} & J_{d,p} & J_{d,q} & O \\ J_{e,k} & J_{e,l} & J_{e,p} & J_{e,q} & O \\ J_{f,k} & J_{f,l} & J_{f,p} & J_{f,q} & O \\ O & O & O & O & O \end{bmatrix}, \text{and}$$

$$\overline{A_3} = \begin{bmatrix} O_{a,k} & O_{a,l} & O_{a,p} & O_{a,q} & O_{a,r} & O_{a,s} & O \\ O_{b,k} & O_{b,l} & O_{b,p} & O_{b,q} & O_{b,r} & O_{b,s} & O \\ O_{c,k} & O_{c,l} & J_{c,p} & J_{c,q} & J_{c,r} & J_{c,s} & O \\ O_{d,k} & O_{d,l} & J_{d,p} & J_{d,q} & J_{d,r} & J_{d,s} & O \\ O_{e,k} & O_{e,l} & J_{e,p} & J_{e,q} & J_{e,r} & J_{e,s} & O \\ O_{f,k} & O_{f,l} & O_{f,p} & O_{f,q} & O_{f,r} & O_{f,s} & O \\ O_{g,k} & O_{g,l} & J_{g,p} & J_{g,q} & J_{g,r} & J_{g,s} & O \\ O_{h,k} & O_{h,l} & O_{h,p} & O_{h,q} & O_{h,r} & O_{h,s} & O \end{bmatrix}.$$

Suppose that $v \neq 0$ and $\overline{A}[r_1+1, \cdots, m|1, \cdots, n] = \overline{A_2}[r_1+1, \cdots, m|1, \cdots, n] + \overline{A_3}[r_1+1, \cdots, m|1, \cdots, n]$ has Boolean rank 1. Then, $f = g = 0$ and we must have $l, s \neq 0$, for otherwise $b(\overline{A}) = 2$, a contradiction. Further, if $b = c = 0$ then $b(\overline{A}) = 2$, again a contradiction. Thus, using a 1 from each of the blocks subscripted $(a, v), (b, l)$ and (e, s) of $\overline{A}$ or a 1 from each of the blocks subscripted $(a, v), (e, l)$ and (c, s) of $\overline{A}$ we have three isolated entries, a contradiction since $\iota(A) = \iota(\overline{A}) = 2$. Thus the Boolean rank of $\overline{A}[r_1 + 1, \cdots, m|1, \cdots, n]$ must be 2, and hence has two isolated entries, say i_2 and i_3. If $\mathcal{C}_1 \not\subseteq \mathcal{C}_2 \cup \mathcal{C}_3$ then $\overline{a_{1,x}} \neq 0$ for $x = k + l + p + q + r + s + 1$ then, $\{\overline{a_{1,x}}, i_2, i_3\}$ is a set of three isolated entries, a contradiction to $\iota(A) = \iota(\overline{A}) = 2$. Thus, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cup \mathcal{C}_3$. Further, $\mathcal{C}_1 \neq \mathcal{C}_2 \cup \mathcal{C}_3$, otherwise, $\overline{A}$ would have Boolean rank 2. Thus, $v = 0$, and hence, $\mathcal{C}_1 \subset \mathcal{C}_2 \cup \mathcal{C}_3$.

Since the choice of rows versus columns can be changed by transposition and the index of $\mathcal{R}_i$ and $\mathcal{C}_j$ by renumbering, we have shown that if $\{i, j, k\} = \{1, 2, 3\}$ then $\mathcal{R}_i \subset \mathcal{R}_j \cup \mathcal{R}_k$ and $\mathcal{C}_i \subset \mathcal{C}_j \cup \mathcal{C}_k$.

Consider the matrix (3.1). Since $\mathcal{R}_1 \subset \mathcal{R}_2 \cup \mathcal{R}_3$ we have that $a = 0$; since $\mathcal{R}_2 \subset \mathcal{R}_1 \cup \mathcal{R}_3$ we have that $f = 0$; since $\mathcal{C}_2 \subset \mathcal{C}_1 \cup \mathcal{C}_3$ we have that $l = 0$; and since $\mathcal{C}_3 \subset \mathcal{C}_1 \cup \mathcal{C}_2$ we have that $s = 0$. That is, since $a = f = l = s = v = 0$, $\overline{A}$ has the form

$$\overline{A} = \begin{bmatrix} J & J & J & J & O \\ J & J & J & J & O \\ J & J & J & J & O \\ J & J & J & J & O \\ O & J & J & J & O \\ O & O & O & O & O \end{bmatrix},$$

where the indices have been omitted. Thus $b(\overline{A}) = 2$, a contradiction. Thus, if $\iota(A) = 2$ then $b(\overline{A}) \neq 3$. ■

Theorem 3.8. *Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$. Then, $\iota(A) = 2$ if and only if $b(\overline{A}) = 2$.*

Proof. From Lemma 3.7, we have the sufficiency. So we only need show the necessity.

Suppose there exists $A \in \mathcal{M}_{m,n}(\mathbb{F})$ with $\iota(A) = \iota(\overline{A}) = 2$ and $b(\overline{A}) > 2$. By Lemma 3.7, $b(\overline{A}) \neq 3$, and hence $b(\overline{A}) \geq 4$. Thus we choose A such that if $b(\overline{A}) > r_{\mathbb{B}}(\overline{C}) > 2$ then $\iota(C) > 2$. Suppose that $\overline{A} = \overline{A_1} + \overline{A_2} + \cdots + \overline{A_k}$ for $k = b(\overline{A})$ where each $\overline{A_i}$ is Boolean rank one, i.e., k is the minimum k such that $b(\overline{A}) = k$ and $\iota(A) = 2$. Suppose that $\overline{A_1}$ has the fewest number of nonzero rows of the $\overline{A_i}$'s. As in the proof of the above lemma 3.7, permute the rows of $\overline{A}$ so that $\overline{A_1}$ has nonzero rows $1, 2, \cdots, r_1$. For $j = 1, \cdots, r_1$, let $\overline{B_j}$ be the matrix whose first j rows are the first j rows of $\overline{A}$ and whose last $m - j$ rows are all zero. Let $\overline{C_j}$ be the matrix whose first j rows are all zero and whose last $m - j$ rows are the last $m - j$ rows of $\overline{A}$. Then $\overline{A} = \overline{B_j} + \overline{C_j}$. Further any set of isolated entries of $\overline{C_j}$ is a set of isolated entries for $\overline{A}$. Now, from $b(\overline{A}) \leq b(\overline{B_j}) + b(\overline{C_j})$, and the fact that $b(\overline{C_j}) = b(\overline{C_{j-1}})$ or $b(\overline{C_j}) = b(\overline{C_{j-1}}) - 1$, there is some t such that $b(\overline{C_t}) = b(\overline{A}) - 1$. Since $b(\overline{C_t}) < k$ by the choice of $\overline{A}$, for this t, we have that $\iota(\overline{C_t}) > 2$ since $b(\overline{C_t}) \geq 3$. That is, $\iota(A) = \iota(\overline{A}) > 2$, a contradiction. ■

Now, as we can see in the following example, there is a matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$ such that $\iota(\overline{A}) = 3$ and $b(\overline{A})$ is relative large, depending on m and n.

Example 3.9. For $n \geq 3$, let $\overline{D_n} = J \setminus I \in \mathcal{M}_n(\mathbb{B})$. Then, it is easily shown that $\iota(\overline{D_n}) = 3$ while $b(\overline{D_n}) = k$ where $k = min\left\{k : n \leq \begin{pmatrix} k \\ \frac{k}{2} \end{pmatrix}\right\}$, see [6](Corollary 2). So, $\iota(\overline{D_{20}}) = 3$ while $b(\overline{D_{20}}) = 6$.

A *tournament matrix* $[T] \in \mathcal{M}_n(\mathbb{B})$ is the adjacency matrix of a directed graph called a tournament, T. It is characterized by $[T] \circ [T]^t = O$ and $[T] + [T]^t = J - I$.

Now, for each $k = 1, 2, \cdots, \min\{m, n\}$, can we characterize the matrices in $\mathcal{M}_{m,n}(\mathbb{F})$ for which $\iota(A) = b(\overline{A})$? Of course it is done if $k = 1$ or $k = 2$ in the above theorems, but only in those cases. For $k = m$ we can also find a characterization:

Theorem 3.10. *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{F})$. Then, $\iota(A) = b(\overline{A}) = m$ if and only if there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B})$ and $Q \in \mathcal{M}_n(\mathbb{B})$ such that $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T} \in \mathcal{M}_m(\mathbb{B})$ where $\overline{T} \in \mathcal{M}_m(\mathbb{B})$ is dominated by a tournament matrix. (There are no restrictions on C.)*

Proof. Suppose that $\iota(A) = m$. Then we permute A by permutation matrices P and Q so that the set of isolated entries are in the (d, d) positions, $d = 1, \cdots, m$. That is, if $X = PAQ$ then $I = \{x_{1,1}, x_{2,2}, \cdots, x_{m,m}\}$ is the set of isolated entries in X. Therefore $X = [B|C]$, with $\overline{b_{i,i}} = \overline{x_{i,i}} = 1$ and $\overline{b_{i,j}} \cdot \overline{b_{j,i}} = 0$ for every i and $j \neq i$ from the definition of the isolated entries. Thus, $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix. Thus, $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T}$ and clearly there are no conditions on C.

Conversely, if $PAQ = [B|C]$ and $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix, then the diagonal entries of B form a set of isolated entries for PAQ and hence A has a set of m isolated entries. Thus $\iota(A) = b(\overline{A}) = m$. ■

Corollary 3.11. *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{F})$. If there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B})$ and $Q \in \mathcal{M}_n(\mathbb{B})$ such that $PAQ = [B|C]$ where $B \in \mathcal{M}_m(\mathbb{F})$ is a diagonal matrix or a triangular matrix with nonzero diagonal entries, then $\iota(A) = b(\overline{A}) = m$.*

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Some properties of certain difference polynomials

Yong Liu, Yuqing Zhang and Xiaoguang Qi

Abstract

This research is a continuation of a recent paper [16]. In this paper, we utilize Nevanlinna value distribution theory to study some properties of difference polynomial $Y_n(z) = \sum_{j=1}^{k} v_j y(z+\eta_j) - ay^n(z)$.

Keywords: Meromorphic functions; Difference; Fixed point; Finite order.

1 Introduction and main results

In this article, we assume familiarity with the basics of Nevanlinna theory (see, e.g., [12, 17]). In addition, we will use the notation $\sigma(y)$ to denote the order of the meromorphic function $y(z)$, and $\lambda(f)$ and $\lambda(\frac{1}{y})$ to denote, respectively, the exponent of convergence of zeros and poles of $y(z)$.

In 1959, Hayman [11] obtained the following famous theorem.

Theorem A [11]. Let $y(z)$ be a transcendental meromorphic function and $a \neq 0, b$ be finite complex constants. Then $y^n(z) + ay'(z) - b$ has infinitely many zeros for $n \geq 5$. If $y(z)$ is transcendental entire, this holds for $n \geq 3$, resp. $n \geq 2$, if $b = 0$.

Recently, several articles (see, e.g., [1-3, 5-10, 13-15]) have focused on complex difference equations and difference analogues of Nevanlinna's theory.

In 2013, the first author and Yi [16] established partial difference polynomial counterparts of Theorem A, and obtained the following result:

Theorem B [16]. Let $y(z)$ be a transcendental entire function of finite order $\rho(y)$, let $a, b, a_j, c_j (j = 1, 2, \cdots, k)$ be complex constants. Set $Y_n(z) = \sum_{j=1}^{k} a_j y(z+c_j) - ay^n(z)$,

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where $n \geq 3$ is an integer. Then $Y_n(z)$ have infinitely many zeros and $\lambda(F_n(z) - b) = \rho(f)$ provided that $\sum_{j=1}^{k} a_j(z)y(z + c_j) \not\equiv b$.

Theorem C [16]. Suppose that $y(z)$ be a finite order transcendental entire function with a Borel exceptional value d. Let $a(z)(\not\equiv 0), b(z), a_j(z)(j = 1, 2, \cdots, k)$ be polynomials, and let $c_j(j = 1, 2, \cdots, k)$ be complex constants. If either $d = 0$ and $\sum_{j=1}^{k} a_j(z)y(z + c_j) \not\equiv 0$, or, $d \neq 0$ and $\sum_{j=1}^{k} da_j(z) - d^2a(z) - b(z) \not\equiv 0$, then $F_2(z) - b(z) = \sum_{j=1}^{k} a_j(z)f(z + c_j) - a(z)y^2(z) - b(z)$ has infinitely many zeros and $\lambda(Y_2(z) - b(z)) = \rho(y)$.

In this paper, we will improve the above results from entire functions to meromorphic functions.

Theorem 1.1. *Suppose $y(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{y}) < \rho(y) < \infty$, suppose $\eta_j(j = 1, 2, \cdots, k)$ are complex constants, and $a(z), v_j(j = 1, 2, \cdots, k)$ be polynomials, and $\varphi(z)$ be a meromorphic function, small compared to $y(z)$. Suppose $Y_n(z) = \sum_{j=1}^{k} v_j y(z + \eta_j) - ay^n(z)$, where $n \geq 3$ is an integer, and $\sum_{j=1}^{k} v_j(z)y(z + \eta_j) \not\equiv \varphi(z)$. Then $\lambda(Y_n(z) - \varphi(z)) = \rho(y)$.*

In Theorem 1.1, we consider difference polynomial $Y_n(z)$ with $n \geq 3$. The following result is about the case $n = 2$:

Theorem 1.2. *Suppose that $y(z)$ is a finite order transcendental meromorphic function with two Borel exceptional value d, ∞. Suppose $a(z)(\not\equiv 0), v_j(z)(j = 1, 2, \cdots, k)$ are polynomials, $\varphi(z)$ is a meromorphic function, small compared to $y(z)$, and suppose $\eta_j(j = 1, 2, \cdots, k)$ are complex constants. If either $d = 0$ and $\sum_{j=1}^{k} v_j(z)y(z + \eta_j) \not\equiv 0$, or, $d \neq 0$ and $\sum_{j=1}^{k} dv_j(z) - d^2a(z) - \varphi(z) \not\equiv 0$, then $\lambda(Y_2(z) - \varphi(z)) = \rho(y)$, where $Y_2(z) - \varphi(z) = \sum_{j=1}^{k} v_j(z)y(z + \eta_j) - a(z)y^2(z) - \varphi(z)$.*

Example 1.3. *Let $y(z) = \frac{\exp\{z\}-1}{\exp\{z\}+1}, a(z) = -1, \eta_1 = 3\pi i, \eta_2 = \pi i, \eta_3 = 0, \eta_4 = 5\pi i, \eta_5 = 7\pi i, v_1(z) = 1, v_2(z) = -3, v_3(z) = -1, v_4(z) = 2, v_5(z) = 1, v_6(z) = \cdots = v_k(z) = 0, \varphi(z) = -1$. Then we have*

$$Y_2(z) - \varphi(z) = \sum_{j=1}^{k} v_j(z)y(z + \eta_j) - a(z)y^2(z) - \varphi(z) = \frac{8\exp\{z\}}{(\exp\{z\} + 1)^2(\exp\{z\} - 1)}.$$

Here $y(z)$ has no two Borel exceptional values, but $Y_2(z) - \varphi(z)$ has no zeros. Hence the condition that $y(z)$ has two Borel exceptional value cannot be omitted in Theorem 1.2.

2 Preliminary lemmas

In order to prove Theorem 1.1 and Theorem 1.2, we need the following lemmas.The following lemma is a generalisation of Borel's Theorem on linear combinations of entire functions.

Lemma 2.1 $[17, pp.79-80]$ *Let $f_j(z)(j = 1, 2, \cdots, n)(n \geq 2)$ be meromorphic function, $g_j(z)(j = 1, 2, \cdots, n)$ be entire functions, and let them satisfy*
(i) $f_1(z)e^{g_1(z)} + \cdots + f_k(z)e^{g_k(z)} \equiv 0$;

(ii) when $1 \le j < k \le n$, then $g_j(z) - g_k(z)$ is not a constant.
(iii) when $1 \le j \le n$, $1 \le h < k \le n$, then

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \ \ (r \to \infty, \ \ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j \equiv 0 (j = 1, \cdots, n)$.

Let $c_j, (j = 1, \cdots, n)$ be a finite collection of complex numbers. Then a difference polynomial in $f(z)$ is a function which is polynomial in $f(z + c_j)$ with meromorphic coefficients $a_\lambda(z)$ such that $T(r, a_\lambda) = S(r, f)$ for all λ. As for difference counterparts of the Clunie lemma, see [4; Corollary 3.3]. The following lemma due to Laine and Yang [14] is a more general version.

Lemma 2.2 [14] *Let $f(z)$ be a transcendental meromorphic solution of finite order of a difference equation of the form*

$$U(z, f)P(z, f) = Q(z, f),$$

where $U(z, f), P(z, f)$, and $Q(z, f)$ are difference polynomials such that the total degree $\deg U(z, f) = n$ in $f(z)$ and its shifts, and $\deg Q(z, f) \le n$. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then

$$m(r, P(z, f)) = o\Big(\frac{T(r + |c|, f)}{r^\delta}\Big) + o(T(r, f)).$$

The following lemma is a difference analogue of the logarithmic derivative lemma.

Lemma 2.3 [8, 10] *Let $f(z)$ be a meromorphic function of finite order and let c be a non-zero complex number. Then we have*

$$m\Big(r, \frac{f(z + c)}{f(z)}\Big) = S(r, f).$$

Lemma 2.4 [8, 10] *If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \infty$, and let c be a non-zero complex number. Then for each $\varepsilon > 0$, we have*

$$N(r, f(z + c)) = N(r, f) + O(r^{\lambda - 1 + \varepsilon}) + O(\log r).$$

3 Proof of Theorem 1.1

Combining Lemma 2.3 and $Y_n(z)-\varphi(z)=\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)$, we have

$$\begin{aligned}
nm(r,y(z)) &= m(r,ay^n(z))+O(\log r)\\
&= m\Big(r,\sum_{j=1}^{k}v_jy(z+\eta_j)-\varphi(z)-(Y_n(z)-\varphi(z))\Big)+O(\log r)\\
&\le m\Big(r,y(z)\frac{\sum_{j=1}^{k}v_jy(z+\eta_j)}{y(z)}\Big)\\
&+m(r,Y_n(z)-\varphi(z))+m(r,\varphi(z))+O(\log r)\\
&\le m(r,y(z))+\sum_{j=1}^{k}m\Big(r,\frac{y(z+\eta_j)}{y(z)}\Big)+\sum_{j=1}^{k}m(r,v_j(z))\\
&+m(r,Y_n(z)-\varphi(z))+O(\log r)\\
&= m(r,y(z))+m(r,Y_n(z)-\varphi(z))+S(r,y).
\end{aligned}\tag{1}$$

By $\lambda(\frac{1}{y})<\rho(y)$, we obtain

$$N(r,y)=O(r^{\rho-1+\varepsilon}).\tag{2}$$

Hence, by (1) and (2), we have

$$(n-1)T(r,y)\le m(r,Y_n(z)-\varphi(z))+O(r^{\rho-1+\varepsilon})+S(r,y).$$

On the other hand, Lemma 2.3 and $Y_n(z)-\varphi(z)=\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)$ imply that

$$\begin{aligned}
T(r,Y_n(z)-\varphi(z)) &= m(r,Y_n(z)-\varphi(z))+N(r,Y_n(z)-\varphi(z))\\
&= m\Big(r,\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)\Big)\\
&+N\Big(r,\sum_{j=1}^{k}v_jy(z+\eta_j)-ay^n(z)-\varphi(z)\Big)\\
&\le m(r,y(z))+\sum_{j=1}^{k}m\Big(r,\frac{y(z+\eta_j)}{y(z)}\Big)+\sum_{j=1}^{k}T(r,v_j)\\
&+m(r,ay^n(z))+(k+n)N(r,y)+T(r,\varphi(z))\\
&\le (k+n)T(r,y(z))+S(r,y).
\end{aligned}\tag{3}$$

Together (1) with (3), we can obtain $\rho(y)=\rho(Y_n-\varphi(z))$. We next break the rest of the proof into two parts.

Case 1. If $\rho(y)=0$, then by $0\le\lambda(Y_n-\varphi(z))\le\rho(Y_n-\varphi(z))=\rho(y)=0$, we have $\lambda(Y_n-\varphi(z))=\rho(y)$, we have proved Theorem 1.1.

Case 2. If $\rho(y) > 0$, then we assume $\lambda(Y_n - \varphi(z)) < \rho(y)$. By this and $\rho(Y_n - \varphi(z)) = \rho(y)$, $Y_n(z) - \varphi(z)$ can be written as

$$\begin{aligned} Y_n(z) - \varphi(z) &= \sum_{j=1}^{k} v_j y(z+\eta_j) - a y^n(z) - \varphi(z) \\ &= \frac{r_1(z)}{r_2(z)} \exp\{q(z)\} = p(z)\exp\{q(z)\}, \end{aligned} \tag{4}$$

where $q(z)$ is a nonzero polynomial, $r_1(z)$ is an entire function with $\rho(r_1) < \rho(y)$, and $r_2(z)$ is the canonical product formed with the poles $Y_n(z) - \varphi(z)$. So $\rho(r_2) = \lambda(r_2) = \lambda(\frac{1}{p}) \leq \lambda(\frac{1}{y}) < \rho(y)$, and $\rho(p) \leq \max\{\rho(r_1), \rho(r_2)\} < \rho(y)$. Differentiating (3) and eliminating $\exp\{q(z)\}$, we get

$$\begin{aligned} & y^{(n-1)}(z)\Big(anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z)\Big) \\ &= p(z)[\sum_{j=1}^{k} v_j y'(z+\eta_j) - \varphi'(z)] - \{p'(z) + p(z)q'(z)\}[\sum_{j=1}^{k} v_j y(z+\eta_j) - \varphi(z)]. \end{aligned} \tag{5}$$

We assume that

$$anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z) \equiv 0. \tag{6}$$

Integrating (6)

$$y^n(z) = dp(z)\exp\{q(z)\}, \tag{7}$$

where $d \in \mathbb{C} \setminus \{0\}$ is a constant. Therefore, by (4) and (7), we obtain that

$$Y_n(z) - \varphi(z) = \sum_{j=1}^{k} v_j y(z+\eta_j) - a y^n(z) - \varphi(z) = \frac{1}{d} y^n(z), \tag{8}$$

by computing (8), we have

$$d\Big(\sum_{j=1}^{k} v_j y(z+\eta_j) - \varphi(z)\Big) = (ad+1)y^n(z). \tag{9}$$

By the condition of theorem 1.1, we know $\sum_{j=1}^{k} v_j y(z+\eta_j) \not\equiv \varphi(z)$, hence we have $ad \neq -1$. Differentiating (9) and then dividing by $y'(z)$, we obtain

$$d\Big(\sum_{j=1}^{k} \frac{v_j y'(z+\eta_j)}{y'(z)}\Big) - d\frac{\varphi'(z)}{y'(z)} = n(ad+1)y^{n-1}(z). \tag{10}$$

We have from (10) and Lemma 2.3 that

$$\begin{aligned}
&(n-1)m(r,y) = m(r,(ad+1)y^{n-1}(z)) + O(1)\\
&= m(r, d\Big(\sum_{j=1}^{k} \frac{v_j y'(z+\eta_j)}{y'(z)}\Big) - d\frac{\varphi'(z)}{y'(z)}) + O(1)\\
&\leq \sum_{j=1}^{k} m(r, \frac{v_j y'(z+\eta_j)}{y'(z)}) + m(r,\varphi'(z)) + m(r,\frac{1}{y'}) + O(1)\\
&= S(r,y') + m(r,\varphi') + m(r,\frac{1}{y'}) \leq S(r,y') + T(r,y') = S(r,y) + T(r,y),
\end{aligned}$$

On the other hand, by (7), we know that the poles of $y(z)$ comes from the poles of $p(z)$, hence we obtain

$$(n-1)N(r,y) \leq O(N(r,p))$$

so

$$(n-2)T(r,y) \leq O(T(r,p)) + S(r,y)$$

we can obtain $\rho(y) \leq \rho(p)$, a contradiction, since $n \geq 3$. Hence $p(z,y) \not\equiv 0$. Since $n \geq 3$, Lemma 2.2 and (5) imply that

$$\begin{aligned}
&m(r, anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z))\\
&= o\big(\frac{T(r+|c|,y)}{r^{\delta}}\big) + o(T(r,y)) + O(m(r,p(z))),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&m(r, y(z)(anp(z)y'(z) - a(p'(z) + q'(z)p(z))y(z)))\\
&= o\big(\frac{T(r+|c|,y)}{r^{\delta}}\big) + o(T(r,y)) + O(m(r,p(z))),
\end{aligned} \tag{12}$$

for all r outside of an exceptional set of finite logarithmic measure. From (11) and (12), we obtain

$$m(r,y) = o\big(\frac{T(r+|c|,y)}{r^{\delta}}\big) + o(T(r,y)) + O(m(r,p(z))) \tag{13}$$

for all r outside of an exceptional set of finite logarithmic measure. (13) and $N(r,y) \leq O(N(r,p))$ yield that $\rho(y) \leq \rho(p)$. A contradiction. So $\lambda(Y_n(z) - \varphi(z)) = \rho(y)$. The proof of Theorem 1.1 is complete.

4 Proof of Theorem 1.2

Since $y(z)$ has a Borel exceptional value d, we see that $y(z)$ takes the form

$$y(z) = d + \frac{x(z)}{q(z)}\exp\{\mu z^k\}, \tag{14}$$

where $\mu \in \mathbb{C}\setminus\{0\}$, $k \in \mathbb{N}\setminus\{0\}$, and $x(z)$ is an entire function such that $x(z)(\not\equiv 0), \rho(x) < k$, and $q(z)$ is the canonical product formed with the poles of $y(z)$ satisfying $\rho(q) = \lambda(q) = \lambda(\frac{1}{y}) < \rho(y)$. (14) implies that

$$y(z+\eta_j) = d + \frac{x(z+\eta_j)}{q(z+\eta_j)} x_j(z)\exp\{\mu z^k\}, (j = 1, 2, \cdots, k) \tag{15}$$

where $x_j(z)$ are entire functions, and $\rho(x_j) = k-1$. If $Y_2(z) - \varphi(z)$ is a rational function, then

$$\sum_{j=1}^{k} v_j(z)y(z+\eta_j) - a(z)y^2(z) - \varphi(z) = p(z), \tag{16}$$

where $p(z)$ is a rational function, we deduce from Lemma 2.3 and (16)

$$\begin{aligned} m(r, a(z)y^2(z)) &= m\Big(r, \sum_{j=1}^{k} v_j(z)y(z+\eta_j) - \varphi(z) - p(z)\Big) \\ &\le m(r, y(z)) + \sum_{j=1}^{k} m\Big(r, \frac{y(z+\eta_j)}{y(z)}\Big) + m(r, \varphi(z)) \\ &+ m(r, p(z)) + \sum_{j=1}^{k} m(r, v_j(z)) + S(r, y) \\ &= m(r, y(z)) + S(r, y), \end{aligned} \tag{17}$$

We obtain form Lemma 2.4

$$\begin{aligned} N(r, a(z)y^2(z)) &= N\Big(r, \sum_{j=1}^{k} v_j(z)y(z+\eta_j) - \varphi(z) - p(z)\Big) \\ &= kN(r, y) + O(r^{\lambda-1+\varepsilon}) + S(r, y). \end{aligned} \tag{18}$$

Together (17) and (18), we have

$$\begin{aligned} T(r, a(z)y^2(z)) &= T\Big(r, \sum_{j=1}^{k} v_j(z)y(z+\eta_j) - \varphi(z) - p(z)\Big) \\ &\le T(r, y) + (k-1)N(r, y) + O(r^{\lambda-1+\varepsilon}) + S(r, y). \end{aligned} \tag{19}$$

(16), (19) and $T(r, ay^2) = 2T(r, y(z)) + S(r, y)$ imply that

$$T(r, y) \le (k-1)N(r, y) + O(r^{\lambda-1+\varepsilon}) + S(r, y).$$

A contradiction, since $\lambda(\frac{1}{y}) < \rho(y)$. Hence $Y_2(z) - \varphi(z)$ is transcendental. (14) and (15) imply that

$$\begin{aligned} Y_2(z) - \varphi(z) &= \Big(\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)} x_j(z) - 2da(z)\frac{x(z)}{q(z)}\Big)\exp\{\mu z^k\} \\ &- a(z)\frac{x^2(z)}{q^2(z)}\exp\{2\mu z^k\} + \sum_{j=1}^{k} dv_j(z) - d^2a(z) - \varphi(z). \end{aligned} \tag{20}$$

By $\frac{x(z)}{q(z)} \not\equiv 0$, we obtain $\rho(Y_2(z)-\varphi(z)) = \rho(y) = k$. Suppose $\lambda(Y_2(z)-\varphi(z)) < \rho(y)$. Then

$$Y_2(z)-\varphi(z) = \frac{l(z)}{m(z)}\exp\{\beta z^k\} = l*(z)\exp\{\beta z^k\}, \tag{21}$$

where $\beta \in \mathbb{C}\setminus\{0\}$, $l(z)$ is an entire function satisfying $\rho(l) < k$, and $\rho(m) = \lambda(m) = \lambda(\frac{1}{y}) < \rho(y) = k$. We obtain from (14), (15) and (21)

$$\begin{aligned}&(\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)}x_j(z) - 2da(z)\frac{x(z)}{q(z)})\exp\{\mu z^k\} - a(z)\frac{x^2(z)}{q^2(z)}\exp\{2\mu z^k\}\\ &= l*(z)\exp\{\beta z^k\} + \sum_{j=1}^{k} dv_j(z) - d^2a(z) + \varphi(z).\end{aligned} \tag{22}$$

We divided the discussion into the following three cases.

Case I. $\beta \neq \mu$ and $\beta \neq 2\mu$, Lemma 2.1 and (22) imply that $\frac{x^2(z)}{q^2(z)} \equiv 0$, by (14) and this, we have $y(z) \equiv d$. A contradiction.

Case II. $\beta = \mu$ and $\beta \neq 2\mu$. By Lemma 2.1 and (22), we can obtain $\frac{x^2(z)}{q^2(z)} \equiv 0$, we use the similar method as case I, we also get a contradiction.

Case III. $\beta = 2\mu$ and $\beta \neq \mu$, we divided this into the following two subcases.

Subcase I. If $d = 0$, then we obtain from (14), (15) and (20)

$$\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)}x_j(z)\exp\{\mu z^k\} - a(z)\frac{x^2(z)}{q^2(z)}\exp\{2\mu z^k\} - \varphi(z) = l*(z)\exp\{\beta z^k\}. \tag{23}$$

Since $\frac{x(z)}{q(z)} \not\equiv 0$, (23) implies that $\beta = 2\mu$. Hence we can write (22) as follows

$$\sum_{j=1}^{k} v_j(z)\frac{x(z+\eta_j)}{q(z+\eta_j)}x_j(z)\exp\{\mu z^k\} - (a(z)\frac{x^2(z)}{q^2(z)} + l*(z))\exp\{2\mu z^k\} - \varphi(z) = 0. \tag{24}$$

Combing Lemma 2.1 and (24), we have $\sum_{j=1}^{k} v_j(z)x(z+\eta_j)x_j(z) \equiv 0$. This is impossible, sicne $\sum_{j=1}^{k} v_j(z)y(z+\eta_j) \not\equiv 0$.

Subcase II. Suppose that $d \neq 0$. Using the similar method as above, we also obtain $\sum_{j=1}^{k} dv_j(z) - d^2a(z) - \varphi(z) \equiv 0$, a contradiction. So $\lambda(Y_2(z)-\varphi(z)) = k$.

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Stability of ternary Jordan bi-derivations on C^*-ternary algebras for bi-Jensen functional equation

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Abstract. In this paper, we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on C^*-ternary algebras for bi-Jensen functional equation.

1. Introduction and preliminaries

The stability problem of functional equations had been first raised by Ulam [15]. In 1941, Hyers [8] gave a first affirmative answer to the question of Ulam for Banach spaces. The generalizations of this result have been published by Aoki [1] and Rassias [14] for additive mappings and linear mappings, respectively. Several stability problems for various functional equations have been investigated in [3, 4, 6, 7, 11, 12, 13].

Let A be a C^*-ternary algebra (see [16]). An additive mapping $D : A \to A$ is called a ternary ring derivation if

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]$$

for all $x, y, z \in A$. An additive mapping $D : A \to A$ is called a ternary Jordan ring derivation if

$$D([x, x, x]) = [D(x), x, x] + [x, D(x), x] + [x, x, D(x)]$$

for all $x \in A$.

The following definition was defined by Eshaghi Gordji et al. [5].

Definition 1.1. ([5]) Let A be a C^*-ternary algebra. A bi-additive mapping $D : A \times A \to A$ is called a *ternary bi-derivation* if it satisfies

$$\begin{aligned} D([x, y, z], w) &= [D(x, w), y, z] + [x, D(y, w^*), z] + [x, y, D(z, w)], \\ D(x, [y, z, w]) &= [D(x, y), z, w] + [y, D(x^*, z), w] + [y, z, D(x, w)] \end{aligned}$$

for all $x, y, z, w \in A$.

A bi-additive mapping $D : A \times A \to A$ is called a ternary Jordan bi-derivation if it satisfies

$$\begin{aligned} D([x, x, x], w) &= [D(x, w), x, x] + [x, D(x, w^*), x] + [x, x, D(x, w)], \\ D(x, [w, w, w]) &= [D(x, w), w, w] + [w, D(x^*, w), w] + [w, w, D(x, w)] \end{aligned}$$

for all $x, w \in A$.

Let A and B be C^*-ternary algebras. A mapping $J : A \to A$ is called a Jensen mapping if J satisfies the functional equation $2J\left(\frac{x+y}{2}\right) = J(x) + J(y)$. For a given mapping $f : A \times A \to B$, we define

$$Jf(x, y, z, w) = 4f\left(\frac{x + y}{2}, \frac{z + w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w)$$

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for all $x, y, z, w \in A$. A mapping $f : A \times A \to B$ is called a bi-Jensen mapping if f satisfies the equation $Jf(x, y, z, w) = 0$ and the functional equation $Jf = 0$ is called a bi-Jensen functional equation. For more details about the result concerning such problems, see ([2, 9]).

In this paper, we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on C^*-ternary algebras for the bi-Jensen functional equation.

2. Stability of ternary Jordan bi-derivations on C^*-ternary algebras for the bi-Jensen functional equation

Throughout this section, assume that A is a ternary C^*-algebra. We need the following lemmas to prove the main theorems.

The following lemma was proved in [7].

Lemma 2.1. ([7]) *Let $f : A \to A$ be an additive mapping. Then*

$$\begin{aligned} f([a,a,a],w) &= [f(a,w),a,a] + [a,f(a,w^*),a] + [a,a,f(a,w)], \\ f(a,[w,w,w]) &= [f(a,w),a,a] + [a,f(a,w^*),a] + [a,a,f(a,w)] \end{aligned}$$

hold for all $a, w \in A$ if and only if

$$\begin{aligned} f([a,b,c]+[b,c,a]+[c,a,b],[w,w,w]) &= [f(a,w),b,c] + [a,f(b,w^*),c] + [a,b,f(c,w)] + [f(b,w),c,a] \\ +[b,f(c,w^*),a] &+ [b,c,f(a,w)] + [f(c,w),a,b] + [c,f(a,w^*),b] + [c,a,f(b,w)], \\ f([a,a,a],[b,c,w]+[c,w,b]+[w,b,c]) &= [f(a,b),c,w] + [b,f(a^*,c),w] + [b,c,f(a,w)] + [f(a,c),w,b] \\ +[c,f(a^*,w),b] &+ [c,w,f(a,b)] + [f(a,w),b,c] + [w,f(a^*,b),c] + [w,b,f(a,w)] \end{aligned}$$

hold for all $a, b, c, w \in A$.

The following lemma was proved in [10].

Lemma 2.2. ([10]) *Let $f : A \times A \to A$ be a bi-Jensen mapping and let n be a positive integer. Then the following are equivalent:*

$$(1) \qquad f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (\frac{1}{2^n} - \frac{1}{4^n})(f(2^n x, 0) + f(0, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

holds for all $x, y \in A$.

$$(2) \qquad f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (2^n - 1)(f(2^n x, 0) + f(0, 2^n y)) + (2^{n+1} - 3 + \frac{1}{4^n})^2 f(0,0)$$

holds for all $x, y \in A$.

$$(3) \qquad f(x,y) = 4^n f(\frac{1}{2^n} x, \frac{1}{2^n} y) + (2^n - 4^n)(f(\frac{1}{2^n} x, 0) + f(0, \frac{1}{2^n} y)) + (2^n - 1)^2 f(0,0)$$

holds for all $x, y \in A$.

$$(4) \qquad f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n}(1 - \frac{1}{2^n}) f(0, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

holds for all $x, y \in A$.

$$(5) \qquad f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^{n+1}}(1 - \frac{1}{2^n})(f(x, 2^n y) + f(-x, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

holds for all $x, y \in A$.

Theorem 2.3. *Let $p \in (0,1)$ and $\theta > 0$. Let $f : A \times A \to A$ be a mapping such that*

$$\|Jf(x,y,z,w)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \tag{2.1}$$

$$\begin{aligned}&\|f([x,y,z]+[y,z,x]+[z,x,y],w) - [f(x,w),y,z] + [x,f(y,w^*),z] - [x,y,f(z,w)] - [f(y,w),z,x]\\ &-[y,f(z,w^*),x] - [y,z,f(x,w)] - [f(z,w),x,y] - [z,f(x,w^*),y] - [z,x,f(y,w)]\|\\ &+\|f(x,[y,z,w]+[z,w,y]+[w,y,z]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x^*,w)]\\ &-[f(x,z),w,y] - [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\|\\ &\le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)\end{aligned} \tag{2.2}$$

for all $x,y,z,w \in A$. Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that

$$\|f(x,y) - D(x,y)\| \le \Big(\frac{2^p}{2(2-2^p)} + \frac{2\cdot 2^p}{4-2^p}\Big)\theta(\|x\|^p + \|y\|^p) \tag{2.3}$$

for all $x,y,z,w \in A$ with $D(0,0) = f(0,0)$. The mapping $D : A \times A \to A$ is given by

$$D(x,y) := \lim_{j\to\infty} \frac{1}{4^j} f(2^j x, 2^j y) + \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 0) + \lim_{j\to\infty} \frac{1}{2^j} f(0, 2^j y) + f(0,0)$$

for all $x,y \in A$

Proof. By the same reasoning as in the proof of [10, Theorem 2], there exists a unique bi-Jensen mapping $D : A \times A \to A$ satisfying (2.3). The mapping $D : A \times A \to A$ is given by

$$D(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x, 0) = \lim_{n\to\infty} \frac{1}{2^n} f(0, 2^n y) = 0$$

for all $x,y \in A$. It follows from (2.2) that

$$\begin{aligned}&\Big\|D([x,y,z]+[y,z,x]+[z,x,y],w) - [D(x,w),y,z] - [x,D(y,w^*),z] - [x,y,D(z,w)]\\ &-[D(y,w),z,x] - [y,D(z,w^*),x] - [y,z,D(x,w)] - [D(z,w),x,y] - [z,D(x,w^*),y] - [z,x,D(y,w)]\Big\|\\ &+\Big\|D(x,[y,z,w]+[z,w,y]+[w,y,z]) - [D(x,y),z,w] - [y,D(x^*,z),w] - [y,z,D(x^*,w)]\\ &-[D(x,z),w,y] - [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\Big\|\\ &= \lim_{n\to\infty}\Big(\Big\|\frac{1}{16^n} f(2^{3n}[x,y,z] + 2^{3n}[y,z,x] + 2^{3n}[z,x,y], 2^n w)\\ &-[\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [x, \frac{1}{4^n} f(2^n y, 2^n w^*), z] - [x, y, \frac{1}{4^n} f(2^n z, 2^n w)]\\ &-[\frac{1}{4^n} f(2^n y, 2^n w), z, x] - [y, \frac{1}{4^n} f(2^n z, 2^n w^*), x] - [y, z, \frac{1}{4^n} f(2^n x, 2^n w)]\\ &-[\frac{1}{4^n} f(2^n z, 2^n w), x, y] - [z, \frac{1}{4^n} f(2^n x, 2^n w^*), y] - [z, x, \frac{1}{4^n} f(2^n y, 2^n w)]\Big\|\Big)\end{aligned}$$

$$+\lim_{n\to\infty}\Big(\Big\|\frac{1}{16^n}f(2^nx,2^{3n}[y,z,w]+2^{3n}[z,w,y]+2^{3n}[z,w,y])$$
$$-[\frac{1}{4^n}f(2^nx,2^ny),z,w]-[y,\frac{1}{4^n}f(2^nx^*,2^nz),w]-[y,z,\frac{1}{4^n}f(2^nx,2^nw)]$$
$$-[\frac{1}{4^n}f(2^nx,2^nz),w,y]-[z,\frac{1}{4^n}f(2^nx^*,2^nw),y]-[z,w,\frac{1}{4^n}f(2^nx,2^ny)]$$
$$-[\frac{1}{4^n}f(2^nx,2^nw),y,z]-[w,\frac{1}{4^n}f(2^nx^*,2^ny),z]-[w,y,\frac{1}{4^n}f(2^nx,2^nz)]\Big\|\Big)$$
$$\le\lim_{n\to\infty}\frac{2^{np}}{16^n}\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)=0$$

for all $x,y,z,w\in A$. So

$$D([x,y,z]+[y,z,x]+[z,x,y],w)=[D(x,w),y,z]+[x,D(y,w^*),z]+[x,y,D(z,w)]+[D(y,w),z,x]$$
$$+[y,D(z,w^*),x]+[y,z,D(x,w)]+[D(z,w),x,y]+[z,D(x,w^*),y]+[z,x,D(y,w)]$$

and

$$D(x,[y,z,w]+[z,w,y]+[w,y,z])=[D(x,y),z,w]+[y,D(x^*,z),w]+[y,z,D(x^*,w)]+[D(x,z),w,y]$$
$$+[z,f(x^*,w),y][z,w,f(x,y)]+[f(x,w),y,z]+[w,f(x^*,y),z]+[w,y,f(x,z)]$$

for all $x,y,z,w\in A$. Therefore, the mapping D is a unique ternary Jordan bi-derivation satisfying (2.3). □

Now we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on C^*-ternary algebras for the bi-Jensen mapping for the case $p>2$ in the following theorem.

Theorem 2.4. *Let $p>2$ and $\theta>0$. Let $f:A\times A\to A$ be a mapping satisfying (2.1) and (2.2). Then there exists a unique ternary Jordan bi-derivation $D:A\times A\to A$ such that*

$$\|f(x,y)-D(x,y)\|\le\Big(\frac{2^p}{2(2^p-2)}+\frac{2\cdot2^p}{2^p-4}\Big)\theta(\|x\|^p+\|y\|^p)\tag{2.4}$$

for all $x,y\in A$.

Proof. By the same reasoning as in the proof of [10, Theorem 2], there exists a unique bi-Jensen mapping $D:A\times A\to A$ satisfying (2.4). By Lemma 2.2, the mapping $D:A\times A\to A$ is given by

$$D(x,y):=\lim_{j\to\infty}4^j\Big(f(\frac{x}{2^j},\frac{y}{2^j})-f(\frac{x}{2^j},0)-f(0,\frac{y}{2^j})+f(0,0)\Big)+\lim_{j\to\infty}2^j\Big(f(\frac{x}{2^j},0)+f(0,0)\Big)$$
$$+\lim_{j\to\infty}2^j\Big(f(0,\frac{y}{2^j})+f(0,0)\Big)+f(0,0)$$

for all $x,y\in A$. It follows from (2.2) that

$$\Big\|D([x,y,z]+[y,z,x]+[z,x,y],w)-[D(x,w),y,z]-[x,D(y,w^*),z]-[x,y,D(z,w)]$$
$$-[D(y,w),z,x]-[y,D(z,w^*),x]-[y,z,D(x,w)]-[D(z,w),x,y]-[z,D(x,w^*),y]-[z,x,D(y,w)]\Big\|$$
$$+\Big\|D(x,[y,z,w]+[z,w,y]+[w,y,z])-[D(x,y),z,w]-[y,D(x^*,z),w]-[y,z,D(x^*,w)]$$
$$-[D(x,z),w,y]-[z,f(x^*,w),y]-[z,w,f(x,y)]-[f(x,w),y,z]-[w,f(x^*,y),z]-[w,y,f(x,z)]\Big\|$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big(\Big\| \frac{1}{16^n} f(2^{3n}[x,y,z] + 2^{3n}[y,z,x] + 2^{3n}[z,x,y], 2^n w) \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [x, \frac{1}{4^n} f(2^n y, 2^n w^*), z] - [x, y, \frac{1}{4^n} f(2^n z, 2^n w)] \\
&\quad - [\frac{1}{4^n} f(2^n y, 2^n w), z, x] - [y, \frac{1}{4^n} f(2^n z, 2^n w^*), x] - [y, z, \frac{1}{4^n} f(2^n x, 2^n w)] \\
&\quad - [\frac{1}{4^n} f(2^n z, 2^n w), x, y] - [z, \frac{1}{4^n} f(2^n x, 2^n w^*), y] - [z, x, \frac{1}{4^n} f(2^n y, 2^n w)] \Big\| \Big) \\
&\quad + \lim_{n\to\infty} \Big(\Big\| \frac{1}{16^n} f(2^n x, 2^{3n}[y,z,w] + 2^{3n}[z,w,y] + 2^{3n}[z,w,y]) \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n y), z, w] - [y, \frac{1}{4^n} f(2^n x^*, 2^n z), w] - [y, z, \frac{1}{4^n} f(2^n x, 2^n w)] \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n z), w, y] - [z, \frac{1}{4^n} f(2^n x^*, 2^n w), y] - [z, w, \frac{1}{4^n} f(2^n x, 2^n y)] \\
&\quad - [\frac{1}{4^n} f(2^n x, 2^n w), y, z] - [w, \frac{1}{4^n} f(2^n x^*, 2^n y), z] - [w, y, \frac{1}{4^n} f(2^n x, 2^n z)] \Big\| \Big) \\
&\quad \le \lim_{n\to\infty} \frac{2^{np}}{16^n} \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) = 0
\end{aligned}
$$

for all $x, y, z, w \in A$. So

$$
\begin{aligned}
&D([x,y,z] + [y,z,x] + [z,x,y], w) = [D(x,w), y, z] + [x, D(y,w^*), z] + [x, y, D(z,w)] \\
&+ [D(y,w), z, x] + [y, D(z,w^*), x] + [y, z, D(x,w)] + [D(z,w), x, y] + [z, D(x,w^*), y] + [z, x, D(y,w)]
\end{aligned}
$$

and

$$
\begin{aligned}
&D(x, [y,z,w] + [z,w,y] + [w,y,z]) = [D(x,y), z, w] + [y, D(x^*,z), w] + [y, z, D(x^*,w)] \\
&+ [D(x,z), w, y] + [z, f(x^*,w), y][z, w, f(x,y)] + [f(x,w), y, z] + [w, f(x^*,y), z] + [w, y, f(x,z)]
\end{aligned}
$$

for all $x, y, z, w \in A$.

Now, let $\delta : A \times A \to A$ be another bi-Jensen mapping satisfying (2.4). By Lemma 2.2 and $D(0,0) = f(0,0) = \delta(0,0)$, we have

$$
\begin{aligned}
\Big\| D(x,y) - \delta(x,y) \Big\| &= 4^n \Big\| D(\frac{x}{2^j}, \frac{y}{2^j}) - \delta(\frac{x}{2^j}, \frac{y}{2^j}) \Big\| \\
&\le 4^n \Big\| D(\frac{x}{2^j}, \frac{y}{2^j}) - f(\frac{x}{2^j}, \frac{y}{2^j}) \Big\| + \Big\| f(\frac{x}{2^j}, \frac{y}{2^j}) - \delta(\frac{x}{2^j}, \frac{y}{2^j}) \Big\| \\
&\le \frac{4^n \theta}{2^{(n-1)p}} \Big(\frac{2}{2^p - 2} + \frac{8}{2^p - 4} \Big) (\|x\|^p + \|y\|^p),
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x, y \in A$. So we can conclude that $D(x,y) = \delta(x,y)$ for all $x, y \in A$. Thus the bi-Jensen mapping $D : A \times A \to A$ is unique. □

Now we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on C^*-ternary algebras for the bi-Jensen mapping for the case $p \in (1,2)$ in the following theorem.

Theorem 2.5. *Let $p \in (1,2)$ and $\theta > 0$. Let $f : A \times A \to A$ be a mapping satisfying* (2.1) *and* (2.2)*. Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that*

$$
\|f(x,y) - D(x,y)\| \le \Big(\frac{2^p}{2^p - 2} + \frac{4 \cdot 2^p}{4 - 2^p} \Big) \theta (\|x\|^p + \|y\|^p)
$$

for all $x, y \in A$.

Proof. The rest of the proof is similar to the proof of Theorem 2.3. □

Finally, we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on C^*-ternary algebras for the bi-Jensen mapping for the case $p \in (0,1)$ in the following theorem.

Theorem 2.6. *Let $p \in (0,1)$, $\theta > 0$ and $\delta > 0$. Let $f : A \times A \to A$ be a mapping satisfying* (2.1), (2.2) *and* $D(0,0) = f(0,0)$. *Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that*

$$\|f(x,y) - D(x,y)\| \le \frac{2^p\theta}{2(2-2^p)}\|x\|^p + (\frac{2^p\theta}{2(2-2^p)} + \theta)\|y\|^p + \delta$$

for all $x, y \in A$ with $D(0,0) = f(0,0)$. The mapping $D : A \times A \to A$ is given by

$$D(x,y) := \lim_{j\to\infty} \frac{1}{2^j}(f(2^j x, y) + f(0, 2^j y)) + f(0,0)$$

for all $x, y \in A$.

Proof. The rest of the proof is similar to the proof of Theorem 2.3. □

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A nonmonotone smoothing Newton algorithm for circular cone complementarity problems

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Abstract The circular cone complementarity problem (CCCP) is a particular nonsymmetric cone optimization problem, which is widely used in real engineering problems. In this paper, we first reformulate the CCCP as a nonlinear system of equations by a one-parametric class of smoothing functions, and then propose a nonmonotone smoothing Newton method for solving the CCCP. A new nonmonotone line search scheme is used in the proposed algorithm, which can help to improve the convergence speed of the algorithm and find the optimal solution more rapidly. Under suitable assumptions, the global convergence and local quadratic convergence are achieved. Finally, numerical results of the force optimization problem for a quadruped robot and random generated CCCPs illustrate the effectiveness of our new algorithm.

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1 Introduction

The circular cone (CC) [1] is a pointed closed convex cone having hyper-spherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. The n_i-dimensional circular cone $C^{n_i}_{\theta_i}(i = 1, \ldots, m)$ is given by

$$C^{n_i}_{\theta_i} := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} | \cos\theta_i \|x^i\| \leq x^{i0}\} \tag{1}$$

with the rotation angle $\theta_i \in (0, \frac{\pi}{2})$, where $\|\cdot\|$ represents the Euclidean norm. And $(C^{n_i}_{\theta_i})^*(i = 1, \ldots, m)$ is the dual cone of $C^{n_i}_{\theta_i}(i = 1, \ldots, m)$ defined by

$$(C^{n_i}_{\theta_i})^* := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} | \sin\theta_i \|x^i\| \leq x^{i0}\}.$$

When $\theta_i = \frac{\pi}{4}$, the circular cone $C^{n_i}_{\theta_i}$ becomes the second-order cone (SOC) $K^{n_i}(i = 1, \ldots, m)$ [2] given by

$$K^{n_i} := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} \mid \|x^{i1}\| \leq x^{i0}\}, \tag{2}$$

and the interior of the SOC K^{n_i} is expressed as

$$(K^{n_i})^\circ := \{x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1} | \|x^{i1}\| < x^{i0}\}.$$

In this paper, we consider the circular cone complementarity problem (CCCP), that is to find a pair of vectors $(x, y) \in R^n \times R^n$ satisfying

$$x \in C^n_\theta,\ y = f(x) \in (C^n_\theta)^*,\ \langle x, y\rangle = 0, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ refers to the Euclidean inner product, $f : R^n \to R^n$ is a continuously differentiable function, and $C^n_\theta \subset R^n$ is the Cartesian product of circular cones, i.e.,

$$C^n_\theta = C^{n_1}_{\theta_1} \times C^{n_2}_{\theta_2} \times \cdots \times C^{n_m}_{\theta_m}$$

with $n = n_1+n_2+\cdots+n_m$. Thus, the second-order cone complementarity problem (SOCCP) is a special class of the CCCP.

Recently, the CCCP is widely used in real engineering problems. For example, it is easy to find that circular cone constraints are involved in force optimization problems for legged robots, the optimal grasping force manipulation for the multifingered hand-arm robot, and the control for quadruped robots [3, 4]. Furthermore, the nonsymmetric cone optimization plays an important role in combinatorial NP-hard problems and nonconvex quadratic problems [5]. Therefore, it is meaningful to study theories and algorithms for the CCCP. Zhou and Chen [6] studied the properties and spectral decomposition of the CC. In order to solve convex quadratic circular cone optimization problem, Wang et al. [7] proposed a primal-dual interior-point algorithm, and proved polynomial convergence of the proposed algorithm. Bai et al. [8] proposed interior-point methods for circular cone programming by kernel functions. Miao et al. [9] constructed some complementarity functions for the CCCP and proposed some merit functions for the CCCP. However, the algorithms for the CCCP are still rare at the moment.

In contrast to nonsymmetric cone complementarity problems, there are many numerical methods [10-14] for solving symmetric cone complementarity problems, such as interior-point methods [11], merit functions methods [12] and smoothing Newton methods [13, 14]. Among them, people pay more attention to smoothing Newton methods. Since C_θ^n and $(C_\theta^n)^*$ in (3) are usually not the same cone with $\theta \neq 45°$, we can not directly adopt smoothing Newton methods for the SOCCP to solve the CCCP (3).

Note that in [6], for any $x^i = (x^{i0}, x^{i1}) \in R \times R^{n_i-1}$ $(i = 1, \ldots, m)$ and $y^i = (y^{i0}, y^{i1}) \in R \times R^{n_i-1}$, the algebraic relationship between the CC and the SOC is as follows:

$$x^i \in K^{n_i} \Leftrightarrow H_i^{-1} x^i \in C_{\theta_i}^{n_i},\ y^i \in K^{n_i} \Leftrightarrow H_i y^i \in (C_{\theta_i}^{n_i})^*, \tag{4}$$

where $H_i = \begin{bmatrix} \tan\theta_i & 0^T \\ 0 & I_{n_i-1} \end{bmatrix}$, and H_i^{-1} denotes the inverse matrix of H_i.

Based on the algebraic relationship (4), the CCCP (3) can be rewritten as the SOCCP: find vectors $(x, y) \in R^n \times R^n$ satisfying

$$x \in K^n,\ y = H^{-1} f(H^{-1}x) \in K^n,\ \langle x, y\rangle = 0, \tag{5}$$

where $K^n = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m}$ with $n = n_1 + n_2 + \cdots + n_m$ is the Cartesian product of SOCs, and $H = H_1 \oplus H_2 \oplus \cdots H_m$. Thus a smoothing Newton method can be used to solve the SOCCP (5). Recently, in order to find the optimal solution more rapidly and improve the convergence speed of the algorithm, the nonmonotone line search has been adopted to solve symmetric cone complementarity problems [13, 14]. Therefore, we ask whether we can use a nonmonotone smoothing Newton method to solve the CCCP.

We propose a nonmonotone smoothing Newton algorithm for solving the CCCP in this paper. Without restrictions regarding its starting point, the proposed algorithm performs one line search and solves one linear system of equations approximately at each iteration. The global convergence and local quadratic convergence are achieved without strict complementarity. Moreover, numerical results about the force optimization problem for a quadruped robot and random generated CCCPs illustrate the effectiveness of our new algorithm.

For simplicity, in the following analysis, we assume that $m = 1$, i.e., $C_\theta^n = C_{\theta_1}^{n_1}$. This does not lose any generality, because we can easily extended our analysis to the general case.

The organization of this paper is as follows. We briefly review the Euclidean Jordan algebra and some basic concepts in the next section. In Section 3, a smoothing function and its properties are given. In Section 4, we present a nonmonotone smoothing Newton method for solving the CCCP, and show its well-definedness under suitable assumptions. In Section 5, the global convergence and local quadratic convergence of the proposed algorithm are investigated. Some preliminary numerical results are reported in Section 6. Finally, we close this paper with some conclusions in Section 7.

We use the following notations. R^n and R denote the set of n-dimensional real column vectors and real numbers, respectively. $\|x\| := \sqrt{x^T x}$ is the Euclidean norm for any $x \in R^n$. For convenience, we use $x = (x^0, x^1)$ instead of $x = (x^0, (x^1)^T)^T \in R \times R^{n-1}$. Given two matrices C and D, we define

$$C \oplus D = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}.$$

When $\varrho \to 0$, we write $\nu = o(\varrho)$ (respectively, $\nu = O(\varrho)$) to mean that ν/ϱ tends to zero (respectively, is uniformly bounded) for any $\nu, \varrho > 0$.

2 Preliminaries

The Euclidean Jordan algebra associated with the SOC K^n [2] plays an important role in this paper. For any $x = (x^0, x^1) \in R \times R^{n-1}$ and $y = (y^0, y^1) \in R \times R^{n-1}$, we have the following Jordan algebra associated with the SOC K^n

$$x \circ y = (x^T y, x^0 y^1 + y^0 x^1).$$

The unit element of this algebra is $\boldsymbol{e} = (1, 0, \cdots, 0) \in R^n$. For any $x = (x^0, x^1) \in R \times R^{n-1}$, the symmetric matrix is defined by

$$W(x) = \begin{pmatrix} x^0 & (x^1)^T \\ x^1 & x^0 I_{n-1} \end{pmatrix}.$$

It is easy to verify that

$$x \circ y = W(x)y = W(y)x, \quad \forall x, y \in R^n.$$

Furthermore, $W(x)$ is invertible if and only if $x \in (K^n)^\circ$.

Given $x = (x^0, x^1) \in R \times R^{n-1}$, the spectral factorization of vectors in R^n associated with the SOC K^n can be decomposed as

$$x = \lambda_1(x) u^{(1)}(x) + \lambda_2(x) u^{(2)}(x),$$

where

$$\lambda_i(x) = x_0 + (-1)^i \|x_1\|, \ i = 1, 2,$$

and

$$u^{(i)}(x) = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{x_1}{\|x_1\|}), & \text{if } x_1 \neq 0, \\ \frac{1}{2}(1, (-1)^i \varpi), & \text{otherwise}, \end{cases} \qquad i = 1, 2,$$

with any $\varpi \in R^{n-1}$ satisfying $\|\varpi\| = 1$.

Lemma 1 [11] Let $a,\ b,\ r,\ g \in R^n$ and $a \succ_{K^n} 0$, $b \succ_{K^n} 0$, $a \circ b \succ_{K^n} 0$. If $\langle r, g \rangle \geq 0$ and $a \circ r + b \circ g = 0$, then $r = g = 0$.

The concept of semismoothness is closely related to the local convergence of the proposed algorithm. Mifflin [15] originally introduced the concept of semismoothness for functionals. Then Qi and Sun [16] extended it to vector-valued functions.

Definition 1 A locally Lipschitz function $H : R^n \to R^m$, if H is directionally differentiable at x and for any $V \in \partial H(x + \Delta x)$,

$$H(x + \Delta x) - H(x) - V(\Delta x) = o(\|\Delta x\|),$$

where ∂H stands for the generalized Jacobian of H [17], then it is said to be semismooth at x. If H is semismooth at x and

$$H(x+\Delta x)-H(x)-V(\Delta x)=O(\|\Delta x\|^2),$$

then H is said to be strongly semismooth at x. Suppose a function $H: R^n \to R^m$ is (strongly) semismooth everywhere in R^n, then it is a (strongly) semismooth function.

Next, we introduce the concept of a monotone function, which will be used in our subsequent analysis.

Definition 2 [18] If a nonlinear mapping $f: R^n \to R^n$ for any $x, y \in R^n$ with $x \neq y$ satisfies

$$\langle x-y, f(x)-f(y)\rangle \geq 0,$$

then it is said to be a monotone function. Moreover, if there exists $\xi > 0$ such that

$$\langle x-y, f(x)-f(y)\rangle \geq \xi\|x-y\|^2,$$

we say f is a strongly monotone function. When f is continuously differentiable, we have that f is monotone (respectively, strongly monotone) if and only if ∇f is positive-semidefinite (respectively, positive definite) for all $x \in R^n$.

3 A smoothing function and its properties

Given any $(x, y) \in R^n \times R^n$, we know that a one-parametric class of functions [12]

$$\vartheta_\tau(x,y) := x+y-\sqrt{(x-y)^2+4\tau(x\circ y)} \tag{6}$$

with $\tau \in (0,1)$ is an SOC complementarity function, i.e.,

$$\vartheta_\tau(x,y)=0 \Leftrightarrow x\in K^n,\ y\in K^n,\ x^Ty=0. \tag{7}$$

However, $\vartheta_\tau(x,y)$ is not continuously differentiable at $(0,0) \in R^n \times R^n$, and thus it is nonsmooth.

In this paper, we introduce the following smoothing function [19] of the SOC complementarity function (6)

$$\vartheta_\tau(\mu,x,y) := x+y-\sqrt{(x-y)^2+4\tau(x\circ y)+4\mu^2\boldsymbol{e}}, \tag{8}$$

where $\tau \in [0,1)$ is a given constant. It is easy to see that (8) is continuously differentiable at any $(\mu,x,y) \in R_{++}\times R^n\times R^n$. When $\tau=0$, $\vartheta_\tau(\mu,x,y)$ reduces to the well-known smoothing Chen-Harker-Kanzow-Smale function [20]

$$\vartheta_0(\mu,x,y) := x+y-\sqrt{(x-y)^2+4\mu^2\boldsymbol{e}}.$$

When $\tau=\frac{1}{2}$, $\vartheta_\tau(\mu,x,y)$ becomes the smoothing Fischer-Burmeister function [21]

$$\vartheta_{\frac{1}{2}}(\mu,x,y) := x+y-\sqrt{x^2+y^2+4\mu^2\boldsymbol{e}}. \tag{9}$$

Define $\Phi_\tau(\omega)$ by

$$\Phi_\tau(\omega) := \begin{pmatrix} \mu \\ y-H^{-1}f(H^{-1}x) \\ \vartheta_\tau(\mu,x,y) \end{pmatrix} \tag{10}$$

with $\omega := (\mu, x, y) \in R_+ \times R^n \times R^n$, where $\vartheta_\tau(\mu, x, y)$ is defined by (8). It follows from (3),(4),(5),(7) and (10) that

$$\Phi_\tau(\omega) = 0 \Leftrightarrow (x, y) \text{ solves the SOCCP (5) } \Leftrightarrow (H^{-1}x, Hy) \text{ solves the CCCP (3).}$$

Therefore, when $\mu > 0$, we can use the Newton's method to solve the nonlinear system of equations $\Phi_\tau(\omega) = 0$ approximately at each iteration. By driving $\|\Phi_\tau(\omega)\| \to 0$, we can find a solution of the SOCCP (5). Thus by the algebraic relationship (4), a solution of the CCCP (3) can be obtained.

Theorem 1 Let the function $\Phi_\tau(\omega)$ be given as in (10). Then we have the following results.

(i) $\Phi_\tau(\omega)$ is continuously differentiable at any $\omega = (\mu, x, y) \in R_{++} \times R^n \times R^n$ with its Jacobian

$$\Phi'_\tau(\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -H^{-1}f'(H^{-1}x)H^{-1} & I \\ C_\tau(\omega) & D_\tau(\omega) & E_\tau(\omega) \end{pmatrix}, \tag{11}$$

where

$$C_\tau(\omega) = (\vartheta_\tau)'_\mu(\omega) = -4\mu W^{-1}(\psi_\tau)\boldsymbol{e},$$

$$D_\tau(\omega) = (\vartheta_\tau)'_x(\omega) = I - W^{-1}(\psi_\tau)W[x + (2\tau - 1)y], \tag{12}$$

$$E_\tau(\omega) = (\vartheta_\tau)'_y(\omega) = I - W^{-1}(\psi_\tau)W[y + (2\tau - 1)x], \tag{13}$$

$$\psi_\tau := \sqrt{(x-y)^2 + 4\tau(x \circ y) + 4\mu^2\boldsymbol{e}}.$$

(ii) Suppose a function f is continuously differentiable and monotone, then $\Phi'_\tau(\omega)$ is invertible for any $\omega = (\mu, x, y) \in R_{++} \times R^n \times R^n$.

Proof (i) According to the proof of Proposition 2.1 [19], it is not difficult to see that (i) holds.

(ii) Let an arbitrary vector $\Delta\omega := (\Delta\mu, \Delta x, \Delta y) \in R \times R^n \times R^n$ satisfy $\Phi'_\tau(\omega)\Delta\omega = 0$. It is sufficient to show $\Delta\omega = 0$. By (11), $\Phi'_\tau(\omega)\Delta\omega = 0$ gives

$$\Delta\mu = 0, \tag{14}$$

$$-H^{-1}f'(H^{-1}x)H^{-1}\Delta x + \Delta y = 0, \tag{15}$$

$$D_\tau(\omega)\Delta x + E_\tau(\omega)\Delta y = 0. \tag{16}$$

Since f is a continuously differentiable and monotone function, we have by (15)

$$\langle \Delta x, \Delta y\rangle = \langle \Delta x, H^{-1}f'(H^{-1}x)H^{-1}\Delta x\rangle = \langle H^{-1}\Delta x, f'(H^{-1}x)H^{-1}\Delta x\rangle \geq 0. \tag{17}$$

By (12), (13) and (16), we obtain

$$\{I - W^{-1}(\psi_\tau)W[x + (2\tau - 1)y]\}\Delta x + \{I - W^{-1}(\psi_\tau)W[y + (2\tau - 1)x]\}\Delta y = 0. \tag{18}$$

Applying $W(\psi_\tau)$ to both sides of (18) and using $W(x)y = x \circ y$ for any $x, y \in R^n$ yield

$$\{\psi_\tau - [x + (2\tau - 1)y]\} \circ \Delta x + \{\psi_\tau - [y + (2\tau - 1)x]\} \circ \Delta y = 0. \tag{19}$$

On the other hand, from the definition of ψ_τ, we have

$$\psi_\tau^2 - [x + (2\tau - 1)y]^2 = 4\tau(1-\tau)y^2 + 4\mu^2\mathbf{e} \succ_{K^n} 0,$$

$$\psi_\tau^2 - [y + (2\tau - 1)x]^2 = 4\tau(1-\tau)x^2 + 4\mu^2 \mathbf{e} \succ_{K^n} 0.$$

Thus it follows from Proposition 3.4 [21] that

$$\psi_\tau - [x + (2\tau - 1)y] \succ_{K^n} 0, \ \{\psi_\tau - [y + (2\tau - 1)x]\} \succ_{K^n} 0. \tag{20}$$

Furthermore, note that

$$\begin{aligned} &\{\psi_\tau - [x + (2\tau - 1)y]\} \circ \{\psi_\tau - [y + (2\tau - 1)x]\} \\ &= \tau(\psi_\tau - x - y)^2 + 4(1-\tau)\mu^2 \mathbf{e} \succ_{K^n} 0. \end{aligned} \tag{21}$$

Therefore, from (17), (19)-(21) and Lemma 1, we have $\Delta x = \Delta y = 0$. The proof is completed.

4 A nonmonotone smoothing Newton algorithm for CCCP

Let Φ_τ be defined by (10). We define

$$\Psi_\tau(\omega) := \|\Phi_\tau(\omega)\|^2 = \mu^2 + \left\|y - H^{-1} f(H^{-1}x)\right\|^2 + \|\vartheta_\tau(\mu, x, y)\|^2. \tag{22}$$

Algorithm 1 (A nonmonotone smoothing Newton algorithm for CCCP)

Step 0 Choose $\theta \in (0, \frac{\pi}{2})$, $\delta \in (0,1)$, $\tau \in [0,1)$, $\sigma \in (0, \frac{1}{2})$ and $\mu_0 > 0$. And choose $\gamma \in (0,1)$ such that $\gamma\mu_0 < 1$. Let $\overline{u} := (\mu_0, 0, 0) \in R_{++} \times R^n \times R^n$ and $(x^0, y^0) \in R^n \times R^n$ be an arbitrary point. Let $\omega^0 := (\mu_0, x^0, y^0)$, $\Upsilon_0 := \Psi_\tau(\omega^0)$ and $\phi_\tau(\omega^0) := \gamma \min\{1, \Psi_\tau(\omega^0)\}$. Choose an integer $P \geq 0$. Set $k := 0$, $m(0) = 0$.

Step 1 If $\left\|\Phi_\tau(\omega^k)\right\| = 0$, stop. Otherwise, let

$$\phi_\tau(\omega^k) := \min \gamma\{1, \Psi_\tau(\omega^0), ..., \Psi_\tau(\omega^k)\}. \tag{23}$$

Step 2 Compute $\Delta\omega^k := (\Delta\mu_k, \Delta x^k, \Delta y^k) \in R \times R^n \times R^n$ by

$$\Phi_\tau(\omega^k) + \Phi'_\tau(\omega^k)\Delta\omega^k = \phi_\tau(\omega^k)\overline{u}. \tag{24}$$

Step 3 Let $\lambda_k = \max\{\delta^l \,|\, l = 0, 1, 2, \ldots\}$ such that

$$\Psi_\tau(\omega^k + \lambda_k \Delta\omega^k) \leq [1 - 2\sigma(1 - \gamma\mu_0)\lambda_k]\Upsilon_k. \tag{25}$$

Step 4 Set $\omega^{k+1} := \omega^k + \lambda_k \Delta\omega^k$, $k := k+1$.

Step 5 Set $m(k) = \min\{m(k-1) + 1, P\}$ and

$$\Psi_\tau(\omega^{l(k)}) := \max_{0 \leq j \leq m(k)} \{\Psi_\tau(\omega^{k-j})\}, \ \Upsilon_k := \frac{(k - m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k - m(k) + 1}. \tag{26}$$

Go to Step 1.

Remark 1

(i) In Algorithm 1, we employ a new nonmonotone line search, which can be used to find the optimal solution more rapidly and improve the convergence speed of the algorithm. If we choose $P = 0$ or P to be sufficiently large, then (25) is the monotone line search.

(ii) If P is a given positive integer, there are the following two cases in the iteration process:

(a) if $k < P$, then $m(k) = k$ and $\Upsilon_k = \Psi_\tau(\omega^k)$, i.e., we use a monotone line search in Algorithm 1. In fact, smoothing Newton algorithms with a monotone line search possess local fast convergence when $\left\|\Phi_\tau(\omega^k)\right\|$ is small enough [22]. So now it is not necessary to use the nonmonotone line search in Algorithm 1;

(b) if $k \geq P$, then $m(k) = P$ and

$$\Upsilon_k := \frac{(k-P)\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k-P+1} = \frac{(k-P)\Psi_\tau(\omega^{l(k)})}{k-P+1} + \frac{\Psi_\tau(\omega^k)}{k-P+1}, \tag{27}$$

i.e., we use a nonmonotone line search in Algorithm 1.

Let $\phi_\tau(\omega)$ be given by (23), and denote

$$\Gamma = \{\omega = (\mu, x, y) \in R_{++} \times R^n \times R^n : \mu \geq \phi_\tau(\omega)\mu_0\}. \tag{28}$$

Lemma 2 Suppose that a function f is continuously differentiable and monotone, and consider the sequence $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then
(i) $\{\phi_\tau(\omega^k)\}$ is monotonically decreasing.
(ii) For any $k \geq 0$, we have $\mu_k > 0$ and $\omega^k \in \Gamma$.
(iii) $\{\mu_k\}$ is monotonically decreasing.
Proof The proof is similar to Lemma 4.1 [14]. We omit the details for brevity.

Lemma 3 Suppose that a function f is continuously differentiable and monotone, and consider the sequence $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then we have $\Psi_\tau(\omega^k) \leq \Upsilon_k \leq \Psi_\tau(\omega^{l(k)})$.
Proof We obtain from (26)

$$\Upsilon_k = \frac{(k-m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k-m(k)+1} \leq \frac{(k-m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^{l(k)})}{k-m(k)+1} = \Psi_\tau(\omega^{l(k)}),$$

and

$$\Upsilon_k = \frac{(k-m(k))\Psi_\tau(\omega^{l(k)}) + \Psi_\tau(\omega^k)}{k-m(k)+1} \geq \frac{(k-m(k))\Psi_\tau(\omega^k) + \Psi_\tau(\omega^k)}{k-m(k)+1} = \Psi_\tau(\omega^k).$$

This completes the proof.

Theorem 2 Assume that a function f is continuously differentiable and monotone, and consider the sequence $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then Algorithm 1 is well defined.

Proof Since $\Phi'_\tau(\omega)$ is invertible for any $\mu > 0$ by Theorem 1, then Step 2 is well defined. Next we show that Step 3 is well defined. From the definition of $\phi_\tau(\omega^k)$ in (23), we have $\phi_\tau(\omega^k) \leq \gamma \min\{1, \Psi_\tau(\omega^k)\}$ for any $k \geq 0$. If $\Psi_\tau(\omega^k) \geq 1$, then $\phi_\tau(\omega^k) \leq \gamma \leq \gamma\sqrt{\Psi_\tau(\omega^k)}$; If $\Psi_\tau(\omega^k) < 1$, then $\phi_\tau(\omega^k) \leq \gamma\Psi_\tau(\omega^k) \leq \gamma\sqrt{\Psi_\tau(\omega^k)}$. Therefore, we obtain for any $k \geq 0$,

$$\phi_\tau(\omega^k) \leq \gamma\sqrt{\Psi_\tau(\omega^k)} = \gamma \left\|\Phi_\tau(\omega^k)\right\|. \tag{29}$$

For any $\lambda \in (0, 1]$, denote

$$r_\tau^k(\lambda) := \Psi_\tau(\omega^k + \lambda\Delta\omega^k) - \Psi_\tau(\omega^k) - \lambda\Psi'_\tau(\omega^k)\Delta\omega^k. \tag{30}$$

Since $\Psi_\tau(\cdot)$ is continuously differentiable at any $\omega^k \in R^{1+2n}$, we have

$$|r_\tau^k(\lambda)| = o(\lambda). \tag{31}$$

It follows from (22), (24), (29)-(31) and Lemma 3 that

$$\begin{aligned}
\Psi_\tau(\omega^k+\lambda\Delta\omega^k) &= \Psi_\tau(\omega^k)+\lambda\Psi_\tau'(\omega^k)\Delta\omega^k+r_\tau^k(\lambda)\\
&= \Psi_\tau(\omega^k)+2\lambda{\Phi_\tau}^T(\omega^k){\Phi_\tau}'(\omega^k)\Delta\omega^k+o(\lambda)\\
&= \Psi_\tau(\omega^k)+2\lambda{\Phi_\tau}^T(\omega^k)\phi_\tau(\omega^k)\overline{u}-2\lambda\big\|\Phi_\tau(\omega^k)\big\|^2+o(\lambda) \qquad (32)\\
&\le (1-2\lambda)\Psi_\tau(\omega^k)+2\lambda\gamma\mu_0\Psi_\tau(\omega^k)+o(\lambda)\\
&\le [1-2(1-\gamma\mu_0)\lambda]\Upsilon_k+o(\lambda).
\end{aligned}$$

Since $\gamma\mu_0<1$, there exists $\bar{\lambda}\in(0,1)$ such that for any $\lambda\in(0,\bar{\lambda}]$ and $\sigma\in(0,\frac{1}{2})$,

$$\Psi_\tau(\omega^k+\lambda\Delta\omega^k)\le[1-2\sigma(1-\gamma\mu_0)\lambda]\Upsilon_k.$$

This demonstrates that Step 3 is well defined. We complete the proof.

Lemma 4 Assume that a function f is continuously differentiable and monotone, and consider the sequence $\{\omega^k=(\mu_k,x^k,y^k)\}$ generated by Algorithm 1. Then $\{\Psi_\tau(\omega^{l(k)})\}$ is monotonically decreasing.

Proof We have $\Upsilon_k\le\Psi_\tau(\omega^{l(k)})$ for any $k\ge0$ by Lemma 3. Thus, it follows from (25) that

$$\Psi_\tau(\omega^k+\lambda_k\Delta\omega^k)\le[1-2\sigma(1-\gamma\mu_0)\lambda_k]\Upsilon_k\le[1-2\sigma(1-\gamma\mu_0)\lambda_k]\Psi_\tau(\omega^{l(k)}). \qquad (33)$$

Since $\gamma\mu_0<1$, it follows from (33) that $\Psi_\tau(\omega^{k+1})\le\Psi_\tau(\omega^{l(k)})$. We obtain from (26)

$$\begin{aligned}
\Psi_\tau(\omega^{l(k+1)}) &= \max_{0\le j\le m(k+1)}\{\Psi_\tau(\omega^{k+1-j})\}\\
&\le \max_{0\le j\le m(k)+1}\{\Psi_\tau(\omega^{k+1-j})\}=\max\{\Psi_\tau(\omega^{l(k)}),\Psi_\tau(\omega^{k+1})\},
\end{aligned}$$

Therefore, we have $\Psi_\tau(\omega^{l(k+1)})\le\Psi_\tau(\omega^{l(k)})$ for any $k\ge0$. We complete the proof.

5 Convergence Analysis

The global convergence and local quadratic convergence of Algorithm 1 will be analyzed in this section. In order to establish the global convergence of Algorithm 1, we first give the coerciveness of the function $\Psi_\tau(\omega)$ given by (22).

From the proof of Theorem 4.1 [22], we have the result as follows.

Lemma 5 Let $\vartheta_\tau(\mu,x,y)$ be given by (8), and $s,t\in R_{++}$ with $s<t$. Suppose that $\{\omega^k=(\mu_k,x^k,y^k)\}$ is a sequence satisfying
(a) $\mu_k\in[s,t]$, and $\{(x^k,y^k)\}$ is unbounded; and
(b) there is a bounded sequence $\{(u^k,v^k)\}$ such that $\{\langle x^k-u^k,y^k-v^k\rangle\}$ is bounded below.
Then $\{\vartheta_\tau(\mu_k,x^k,y^k)\}$ is unbounded.

By Lemma 5, it is not difficult to obtain the coerciveness of the function $\Psi_\tau(\omega)$ given in (22).

Lemma 6 Assume that a function f is continuously differentiable and monotone, and consider the sequence $\Psi_\tau(\omega)$ given by (22). Then $\Psi_\tau(\mu, x, y)$ is coercive in (x, y) for each $\mu > 0$, that is, $\lim_{\|(x,y)\|\to\infty} \Psi_\tau(\mu, x, y) = +\infty$.

Proof The proof is similar to Lemma 5.3 [22]. We omit it here for brevity.

Theorem 3 Suppose that a function f is continuously differentiable and monotone, and consider $\{\omega^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm 1. Then $\{\mu_k\}$ and $\{\| \Phi_\tau(\omega^k) \|\}$ converge to zero as $k \to \infty$, and any accumulation point $(H^{-1}x^*, Hy^*)$ is a solution of the CCCP (3).

Proof From Lemma 2, we know that $\{\phi_\tau(\omega^k)\}$ is convergent, i.e., there exists a scalar $\bar{\beta} \geq 0$ such that $\lim\limits_{k\to\infty} \phi_\tau(\omega^k) = \bar{\beta}$. Suppose that $\bar{\beta} > 0$. Then it follows from Lemma 2 (ii) that $0 < \mu_0\bar{\beta} \leq \mu_* = \lim\limits_{k\to\infty} \mu_k$. By (22), Lemma 3 and Lemma 4,

$$\mu_k{}^2 \leq \Psi_\tau(\omega^k) \leq \Upsilon_k \leq \Psi_\tau(\omega^{l(k)}) \leq \Psi_\tau(\omega^{l(k-1)}) \leq \cdots \leq \Psi_\tau(\omega^0). \tag{34}$$

Therefore we obtain from Lemma 6 that $\{\omega^k\}$ is bounded, and hence there exists a convergent sequence $\{\omega^k\}_{k\in J}$, where $J \subseteq \{0, 1, ..., k, ...\}$. Let $\omega^* := (\mu_*, x^*, y^*) = \lim\limits_{J\ni k\to\infty} (\mu_k, x^k, y^k)$ such that $\Psi_\tau(\omega^*) = \lim\limits_{J\ni k\to\infty} \Psi_\tau(\omega^k) = \limsup_{k\to\infty} \Psi_\tau(\omega^k)$ and $\phi_\tau(\omega^*) = \lim\limits_{J\ni k\to\infty} \phi_\tau(\omega^k) = \bar{\beta}$. It follows from (34) and $\bar{\beta} > 0$ that $\Psi_\tau(\omega^*) > 0$. We now prove that Theorem 3 holds by considering the following two cases.

(1) Assume that there is a constant ρ such that $\lambda_k \geq \rho > 0$ for any $k \in J$. Then we obtain from (25)

$$\Psi_\tau(\omega^k + \lambda_k\Delta\omega^k) \leq [1 - 2\sigma(1 - \gamma\mu_0)\lambda_k]\Upsilon_k \leq [1 - 2\sigma(1 - \gamma\mu_0)\rho]\Upsilon_k. \tag{35}$$

By letting $J \ni k \to \infty$ in (35), we have

$$\Psi_\tau(\omega^*) \leq [1 - 2\sigma(1 - \gamma\mu_0)\rho]\Upsilon_*. \tag{36}$$

It is not difficult to verify that $\Upsilon_* := \limsup_{J\ni k\to\infty} \Upsilon_k = \Psi_\tau(z^*) > 0$ by (26). Thus we get $1 \leq 1 - 2\sigma(1 - \gamma\mu_0)\rho$, which contradicts the fact that $\gamma\mu_0 < 1$.

(2) Suppose that $\lim\limits_{J\ni k\to\infty} \lambda_k = 0$. Then the stepsize $\hat{\lambda}_k := \lambda_k/\delta$ does not satisfy (25) for any sufficiently large $k \in J$, i.e.

$$\Psi_\tau(\omega^k + \hat{\lambda}_k\Delta\omega^k) > [1 - 2\sigma(1 - \gamma\mu_0)\hat{\lambda}_k]\Upsilon_k \geq [1 - 2\sigma(1 - \gamma\mu_0)\hat{\lambda}_k]\Psi_\tau(\omega^k),$$

which implies

$$\frac{\Psi_\tau(\omega^k + \hat{\lambda}_k\Delta\omega^k) - \Psi_\tau(\omega^k)}{\hat{\lambda}_k} \geq -2\sigma(1 - \gamma\mu_0)\Psi_\tau(\omega^k). \tag{37}$$

Since $0 < \mu_0\phi_\tau(\omega^*) \leq \mu_*$, we have that $\Psi_\tau(\omega)$ is continuously differentiable at $\omega^* \in R^{1+2n}$. By taking the limit on both sides of (37), we obtain

$$\begin{aligned}
-2\sigma(1 - \gamma\mu_0)\Psi_\tau(\omega^*) &\leq 2\Phi_\tau{}^T(\omega^*)\Phi_\tau'(\omega^*)\Delta\omega^* \\
&= 2\Phi_\tau{}^T(\omega^*)[-\Phi_\tau(\omega^*) + \phi_\tau(\omega^*)\overline{u}] \\
&= -2\Phi_\tau{}^T(\omega^*)\Phi_\tau(\omega^*) + 2\phi_\tau(\omega^*)\Phi_\tau{}^T(\omega^*)\overline{u} \\
&\leq -2(1 - \gamma\mu_0)\Psi_\tau(\omega^*).
\end{aligned}$$

Since $\Psi_\tau(\omega^*) > 0$ and $\gamma\mu_0 < 1$, we have $\sigma \geq 1$, which contradicts the fact that $0 < \sigma < \frac{1}{2}$. Thus we have $\bar{\beta} = 0$. It follows from (23) that there is a sequence $\{\omega^{k_n}\}$ such that $\lim_{k_n\to\infty} \Psi_\tau(\omega^{k_n}) = 0$ holds. By (26) and Lemma 4, we have $\lim_{k_n\to\infty} \Psi_\tau(\omega^{l(k_n)}) = \lim_{k\to\infty} \Psi_\tau(\omega^{l(k)}) = \Psi_\tau(\omega^{l(*)}) = 0$. Then, we obtain from (34) that $\lim_{k\to\infty} \Psi_\tau(\omega^k) = \Psi_\tau(\omega^*) = 0$ and hence $\|\Phi_\tau(\omega^*)\| = 0$. Thus $(H^{-1}x^*, Hy^*)$ is a solution of the CCCP (3). This completes the proof.

Next the local convergence of Algorithm 1 will be analyzed. It is easy to see that $\Phi_\tau(\omega)$ is strongly semismooth at any $\omega \in R^{1+2n}$ by Theorem 1. Then by the proof of Theorem 8 [23], we obtain the local quadratic convergence of Algorithm 1 for the CCCP.

Lemma 7 Suppose that a function f is continuously differentiable and monotone, and the solution set of the CCCP is nonempty and bounded. Let the sequence $\{\omega^k\}$ be generated by Algorithm 1 and $\omega^* := (\mu^*, x^*, y^*)$ be an accumulation point of $\{\omega^k\}$. If all $V \in \partial\Phi_\tau(\omega^*)$ are nonsingular, then the sequence $\{\omega^k\}$ converges to ω^* quadratically, i.e.,

$$\|\omega^{k+1} - \omega^*\| = O(\|\omega^k - \omega^*\|^2) \text{ and } \mu_{k+1} = O((\mu_k)^2).$$

6 Numerical examples

In this section, we have conducted some numerical experiments of Algorithm 1 for solving the CCCP. All the experiments were done on a PC with Intel(R) Celeron(R) CPU N2930 1.83 GHz×2 and 4.0 GB memory. Algorithm 1 was implemented in MATLAB 8.1.0.604 (R2013a). We chose the following parameters in all the numerical experiments:

$$\mu_0 = 0.1, \delta = 0.75, \sigma = 0.3, \gamma = 0.45, \tau = 0.4.$$

We used $\Psi_\tau(\omega^k) \leq 10^{-8}$ as the stopping criterion.

In the following tables, n denote the size of problems; ACPU and AIter denote the CPU time in seconds and the number of iterations, respectively.

Firstly, we use Algorithm 1 to solve the force optimization problem for a quadruped robot [4, 7], which can be expressed as the circular cone programming:

$$(P)\ \min\ \left\{c^T x : Ax = b,\ x \in C_\theta^{12}\right\}, \tag{38}$$

where $c = (c_1, c_2, c_3, c_4) \in R^{12}$, and $C_\theta^{12} = C_\theta^3 \times C_\theta^3 \times C_\theta^3 \times C_\theta^3$. The dual problem of (38) is defined by

$$(D)\ \max\ \left\{b^T s : A^T s + y = c,\ y \in (C_\theta^{12})^*\right\}.$$

If $F^\circ(P) \times F^\circ(D) \neq \emptyset$, then (x^*, s^*, y^*) is the solution of (P) and (D) if and only if it is the solution of

$$Ax = b,\ x \in C_\theta^{12},\ y = c - A^T s \in (C_\theta^{12})^*,\ x^T y = 0. \tag{39}$$

According to the algebraic relationship between the CC and the SOC (4), we reformulate (39) as

$$AH^{-1}x = b,\ x \in K,\ A^T s + Hy = c,\ y \in K,\ x^T y = 0 \tag{40}$$

with $K = K^3 \times K^3 \times K^3 \times K^3$. Let

$$\Phi_\tau(\mu, x, s, y) := \begin{pmatrix} \mu \\ AH^{-1}x - b \\ A^T s + Hy - c \\ \vartheta_\tau(\mu, x, y) \end{pmatrix}. \tag{41}$$

We adopt Algorithm 1 to solve $\Phi_\tau(\mu, x, s, y) = 0$, where $\vartheta_\tau(\mu, x, y)$ is defined by (8). We use parameters:
$A_1 = [5\ 1\ 1; 1\ 1\ 1; 4\ 6\ 3; 1\ 4\ 3; 3\ 3\ 5; 3\ 3\ 3]$; $\quad A_2 = [3\ 6\ 6; 1\ 6\ 2; 6\ 2\ 1; 5\ 4\ 1; 6\ 5\ 1; 4\ 3\ 4]$;
$A_3 = [4\ 3\ 6; 3\ 2\ 6; 2\ 5\ 1; 1\ 5\ 2; 5\ 6\ 5; 4\ 3\ 3]$; $\quad A_4 = [3\ 3\ 1; 6\ 1\ 2; 6\ 2\ 6; 5\ 2\ 5; 4\ 4\ 5; 6\ 1\ 6]$;
$b = (43, 32, 51, 39, 54, 44)^T$; $\quad c_i = (2, 1, 0)^T,\ i = 1, 2, 3, 4$;

The initial points are $x^0 = (1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0)^T$ and $s^0 = (0,\ 0,\ 0,\ 0,\ 0,\ 0)^T$. Let $\theta = \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{8}$, or $\frac{\pi}{12}$, respectively. Table 1 shows the value x^* and the objective function value $Z^* = (c^*)^T x^*$ of the force optimization problem for a quadruped robot.

Moreover, we solve the randomly generated linear CCCP with different problem sizes n and $m = 1$ by Algorithm 1. In details, let a random vector $q = rand(n, 1)$ and a random matrix $A = rand(n, n)$ be generated, and $M := A^T A$. Since the matrix M is semidefinite positive, the generated problem (3) with $f(x) = Mx + q$ is the monotone CCCP, i.e, the generated problem (5) with $H^{-1} f(H^{-1}x) = H^{-1}[MH^{-1}x + q]$ is the monotone SOCCP. The random problems of each size are generated 10 times. Choose initial points $x^0 = \boldsymbol{e} \in R^n$, $y^0 = 0 \in R^n$, and $\boldsymbol{e}$ denotes the unit element in K^n.

Table 2 reveals that the AIter and ACPU for the CCCP with different rotation angles and problem sizes. It shows that Algorithm 1 can be used efficiently to solve the CCCP with different rotational angles.

Table 3 reveals that the AIter and ACPU of Algorithm 1 with a monotone line search or a nonmonotone line search for the SOCCP with different problem sizes. It shows that our algorithm usually works worse with the monotone line search than the nonmonotone line search.

From the numerical results in Tables 1-3, we see that the nonmonotone smoothing Newton algorithm is successful for solving the CCCP. Moreover, we can use Algorithm 1 to solve the force optimization problem for a quadruped robot. Furthermore, we also find that our algorithm usually works worse with the monotone line search than the nonmonotone line search, in the sense that the former tends to require more AIter and more ACPU than the latter in most cases.

7 Conclusions

In this paper, a smoothing Newton method for the CCCP with a new nonmonotone line is proposed. Under suitable assumptions, the global convergence and local quadratic convergence are achieved. From the numerical experiments, we can see that Algorithm 1 can effectively solve the CCCP with different problem sizes and different rotation angles, and also can be applied to real-world problems, such as the force optimization problem for a quadruped robot. And the nonmonotone smoothing Newton method is better than the

Table 1 Numerical results of the force optimization problem for a quadruped robot.

θ	$\theta = \frac{\pi}{4}$	$\theta = \frac{\pi}{5}$	$\theta = \frac{\pi}{8}$	$\theta = \frac{\pi}{12}$
x^*	2.42056	2.41022	2.40052	2.10235
	2.27904	1.64542	0.76492	0.50586
	0.81553	0.59920	0.63521	0.24776
	1.34655	1.70622	2.52821	3.50136
	1.34503	1.23334	1.03455	0.91945
	−0.06425	−0.12509	−0.16275	−0.18712
	1.21045	1.21569	1.22029	1.59579
	0.40717	0.28628	0.17841	0.14573
	1.13991	0.83560	0.47297	0.40209
	0.72928	1.22722	1.99630	2.22975
	0.08791	0.62921	0.82427	0.43971
	0.72395	0.63169	0.06585	−0.40467
Z^*	15.53282	16.91290	19.09283	20.86925

Table 2 Results for the CCCP with different θ and problem sizes.

	$\theta = \frac{\pi}{3}$		$\theta = \frac{\pi}{4}$		$\theta = \frac{\pi}{5}$		$\theta = \frac{\pi}{6}$	
n	ACPU	AIter	ACPU	AIter	ACPU	AIter	ACPU	AIter
100	0.0700	5.0	0.0863	6.0	0.0900	6.2	0.0951	6.9
200	0.2783	5.0	0.3343	6.0	0.3804	7.0	0.4107	7.5
300	0.7516	5.9	0.9180	6.9	1.0528	8.0	1.0323	7.9
400	1.6756	6.0	1.9657	7.0	2.2717	8.1	2.5356	9.0
500	2.9128	6.0	3.4351	7.6	3.8557	8.2	4.3882	8.9
600	4.7698	6.0	5.7232	7.7	6.6746	9.0	7.2381	9.2
700	6.6911	6.0	8.9254	8.0	10.0183	9.0	10.5241	9.5
800	9.5330	6.0	12.6333	8.0	14.3293	9.3	16.4956	10.5
900	13.1500	6.2	16.7760	8.0	19.4508	9.3	22.5813	10.8
1000	18.4252	6.6	22.2793	8.0	26.5724	9.5	30.5843	11.0
1100	28.179	7.0	36.036	8.8	40.461	10.0	53.848	11.4
1200	37.528	7.0	48.298	9.0	63.445	10.2	67.497	11.2
1300	45.091	7.0	57.184	9.0	80.317	10.2	82.146	11.2
1400	54.362	7.0	71.051	9.0	89.219	10.0	103.418	11.0
1500	66.070	7.0	86.972	9.0	100.303	10.6	132.516	11.4

Table 3 Numerical results for SOCCP with a nonmonotone or monotone line search.

	P=3		P=0	
n	ACPU	AIter	ACPU	AIter
100	0.0872	6.0	0.0882	6.0
200	0.3844	6.0	0.3961	6.1
300	1.0185	6.9	1.0305	7.0
400	2.2106	7.0	2.3637	7.4
500	4.0018	7.4	4.3376	8.0
600	6.4876	7.6	6.9378	8.1
700	8.9622	8.0	10.3067	8.8
800	12.7327	8.0	14.7962	9.0
900	16.8958	8.0	20.1800	9.2
1000	22.4432	8.0	29.7001	9.7
1100	34.8014	8.5	44.9750	9.9
1200	47.9557	9.0	57.4256	10.0
1300	57.1843	9.0	70.5862	10.0

monotone smoothing Newton method for solving the CCCP. Therefore, the smoothing Newton method with a nonmonotone line search is promising for solving the CCCP.

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Duality in nondifferentiable multiobjective fractional programming problems involving second order $(F, b, \phi, \rho, \theta)-$ univex functions

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Abstract

In the present paper a nondifferentiable multiobjective fractional programming problem is considered in which every component of objective functions includes a term involving the support function of a compact convex set. Finally a second order Mond-weir type dual is formulated and weak, strong and converse duality results are proved under $(F, b, \phi, \rho, \theta)-$ univexity types assumptions.

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1 Introduction

In recent years, the concept of convexity and generalized convexity is well recognized in optimization theory and play an imperative role in mathematical economics, management science and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important tool in mathematical programming. The differential convex function $f : R^n \to R$ is characterized by the following inequality

$$f(x) - f(y) \geq \nabla f(y)^t (x - y)$$

for all $x, y \in R^n$, where ∇ denotes the gradient of f. In general a function $f(x)$ is said to be convex on a convex set $X \subseteq R^n$ if for any $x, y \in X, \lambda \in [0, 1]$, $f(x)$ satisfies the following inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

In 1981, Hanson [15] generalized convex functions to introduce the concept of invex functions, which was a significant landmark in the optimization theory. Normally, a differentiable function $f : R^n \to R$ is said to be invex function if there exits a vector valued function $\eta : R^n \times R^n \to R^n$ such that the following inequality

$$f(x) - f(y) \geq \nabla f(y)\eta^t(x, y)$$

holds, for all $x, y \in R^n$.

Consequently, several classes of generalized convexity and invexity have been introduced. More specifically, Preda [28] introduced the concept of $(F, \rho)-$ convexity as an extension of $F-$ convexity [14] and $\rho-$ convexity [13] and he used this concept to investigate some duality for Wolfe vector dual, Mond-weir dual and general Mond-weir dual for multiobjective programming problem. Gulati and Islam [26] and Ahmad [9] deliberate optimality and duality results for multiobjective programming problems involving $F-$convexity and $(F, \rho)-$ convexity assumptions respectively.

Mangasarian [19] first formulated the second order dual for a nonlinear programming problem and obtained duality results under generalized convex type assumptions. Mond [3] reproved the second order duality results under some easier assumptions than those used by Mangasarian [19].

The class of $(F, \rho)-$convex functions was extended to the second order $(F, \rho)-$convex functions by [12] and they obtained the duality results for Mangasarian type, Mond-weir type and general Mond-weir type multiobjective programming problem. Motivated by different concepts of generalized convexity, Liang et al. [30, 31] formulated the $(F, \alpha, \rho, d)-$convexity and acquired some optimality conditions and duality results for the multiobjective problems.

Further, stimulated by Liang et al. [30] and Aghezzaf [4], I. Ahmad and Z. Husain [10] introduced the notion of second order $(F, \alpha, \rho, d)-$convex functions and their generalization and they developed weak, strong and strict converse duality theorems for the second order Mond-weir type multiobjective dual. Moreover, Bector et al. [4] introduced the concept of univex functions and considered optimality and duality for multiobjective optimization problem. Rueda et al. [18] studied optimality and duality results for several mathematical programming problems by combining the concepts of type I and univex functions. A step ahead Zalmai [7] introduced the notion of second order $(F, b, \phi, \rho, \theta)-$univex functions and obtained optimality and duality results for multiobjective programming problems.

On the other hand, the optimization problems in which the objective function is a ratio of two functions usually identified as fractional programming problems. Basically, these types of problems occur in design of electronic circuits, engineering design, portfolio selection problems [1, 6, 11, 20]. Due to the fact that minimax fractional problems has wide varieties of applications in real life problems, so it becomes a fascinating and interesting topic for research. Necessary and sufficient optimality conditions for minimax fractional programming problems first developed by Schmittendorf [29]. Tonimoto [25] used the necessary conditions formulated in [29] and construct a dual problem for minimax fractional programming problems. Recently, Ramu Dubey et al. [21] and S. K. Mishra et al. [22] taken up the nondifferentiable multi objective fractional problem and obtained the optimality and duality results under higher order $(C, \alpha, \gamma, \rho, d)-$ convexity and $(C, \alpha, \rho, d)-$ convexity type assumptions. More recently, many articles in this direction have been appeared in the literature [see 17, 23, 24, 27, 32].

In this paper, a class of nondifferentiable multiobjective fractional programming problem is considered in which the numerator as well as denominator of every component of objective function contains a term concerning the support functions. Further, we prove sufficient optimality conditions and duality theorems for nondifferentiable minimax fractional programming problems with support functions under the second order $(F, b, \rho, \alpha, \theta)-$ univex functions.

2 Notations and Preliminaries

In this paper following generalized nondifferentiable multiobjective minimax fractional problem is considered

$$\text{(GMFP)} \qquad \min_{x \in R^n} \sup_{y \in Y} \frac{F(x,y)}{G(x,y)} = \min_{x \in R^n} \sup_{y \in Y} \frac{f(x,y) + s(x|C)}{g(x,y) - s(x|D)}$$

$$\text{Subject to } \; h_j(x) + s(x|E_j) \leq 0, \quad j = 1, 2, \ldots, m,$$

where Y is a compact subset of R^m, $f, g : R^n \times R^m \to R$ and $h_j : R^n \to R^m$ $(j = 1, 2, \ldots, m)$ are continuously differentiable functions of $R^n \times R^m$. C, D and $E_j(j = 1, \ldots, m)$ are compact convex sets of R^m and $s(x|C), s(x|D)$ and $s(x|E_j), (j = 1, \ldots, m)$ represent the support functions of the compact sets and $f(x, y) + s(x|C) \geq 0$ and $g(x, y) - s(x|D) > 0$ for all feasible x. Let S be the set of all feasible solutions of (GMFP). We define the following sets for every $x \in S$.

$$J(x) = \{j \in J : h_j(x) + s(x|E_j) = 0\},$$

$$Y(x) = \{y \in Y : \frac{f(x, y) + s(x|C)}{g(x, y) - s(x|D)} = \sup_{z \in Y} \frac{f(x, z) + s(x|C)}{g(x, z) - s(x|D)}\}.$$

$K(x) = \{(s, t, \bar{y}) \in N \times R^s_+ \times R^m : 1 \leq s \leq n + 1, \; t = (t_1, t_2, \ldots, t_s\} \in R_+$ with $\sum_{i=1}^s t_i = 1, \; \bar{y} = (\bar{y}_1, \ldots \bar{y}_s)$ and $\bar{y}_i \in Y(x), i = 1, 2, \ldots s\}$.

Since f and g are continuously differentiable functions and Y is compact subset of R^m, it follows that for each $x^* \in S \; Y(x^*) \neq \phi$. Thus for any $\bar{y}_i \in Y(\bar{x})$, we have a positive constant

$$\lambda_0 = \frac{f(x^*, \bar{y}_i) + s(x^*|C)}{g(x^*, \bar{y}_i) - s(x^*|D)}.$$

Definition 2.1 [2]. Let K be a compact convex set in R^n. The support function $s(x|K)$ is defined as

$$s(x|K) = \max\{x^t y : y \in K\}.$$

The support function $s(.|K)$ has a subdifferential. The subdifferential of $s(.|K)$ at x is defined as

$$\partial s(x|K) = \{z \in K | z^t x = s(x|K)\}.$$

Consistently, we can write

$$z^t x = s(x|K).$$

Now we describe the generalized $(F, b, \phi, \rho, \theta)-$ univex function in the following steps

Definition 2.3. A function $F : X \times X \times R^n \to R$, where $X \subseteq R^n$ is said to be a sublinear in its third argument if for all $x, \bar{x} \in X$, the following conditions are satisfied

(i) $\mathcal{F}(x, \bar{x}, a_1 + a_2) \leq \mathcal{F}(x, \bar{x}, a_1) + \mathcal{F}(x, \bar{x}, a_2)$

(ii) $\mathcal{F}(x, \bar{x}, \alpha a) = \alpha \mathcal{F}(x, \bar{x}, a)$,

$\forall \; a_1, a_2, a \in R^n, \alpha \in R_+$.

Definition 2.4 [7]. The function $f(x)$ is said to be second order $(F, b, \phi, \rho, \theta)-$ (strict) univex at z if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) :$ $R^n \times R^n \times R^n \to R$ such that for each $x \in X (x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2} p^t \nabla^2 f(z) p)(>) \geqq F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z) p]) + \rho(x, z) \|\theta(x, z)\|^2,$$

where $\|.\|^2$ is a norm on R^n.

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ univex at $x = z$, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$ univex at z. Now we define generalized second order $(F, b, \phi, \rho, \theta)-$ univex functions.

Definition 2.5. A twice differentiable function f, over X is said to be second order $(F, b, \phi, \rho, \theta)-$ pseudo univex at z if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2}p^T \nabla^2 f(z)p) < 0$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) < -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order $(F, b, \phi, \rho, \theta)-$ pseudo univex at $x = z$, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$ pseudo univex at z.

Definition 2.6. A twice differentiable function f, over X is said to be second order $(F, b, \phi, \rho, \theta)-$ strictly pseudo univex at z if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) \geqq -\rho(x, z)\|\theta(x, z)\|^2.$$

$$\Rightarrow \phi(f(x) - f(z) + \frac{1}{2}p^t \nabla^2 f(z)p) > 0,$$

or equivalently

$$\phi(f(x) - f(z) + \frac{1}{2}p^t \nabla^2 f(z)p) \geqq 0,$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) < -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order $(F, b, \phi, \rho, \theta)-$ strictly pseudo univex at $x = z$, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$ strictly pseudo univex at z.

Definition 2.7. A twice differentiable function f over X is said to be second order $(F, b, \phi, \rho, \theta)-$ quasi univex at z if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2}p^t \nabla^2 f(z)p) \geqq 0$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) \geqq -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order $(F, b, \phi, \rho, \theta)-$ quasi univex at $x = z$, if each of its components f_i is

$(F, b, \phi, \rho, \theta)-$ quasi univex at z.

Definition 2.8. A twice differentiable function f, over X is said to be second order strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at z if there exist functions $b : X \times X \to (0, \infty), \phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $\mathcal{F}(x, z; .) : R^n \times R^n \times R^n \to R$ such that for each $x \in X(x \neq z)$ and $p \in R^n$,

$$\phi(f(x) - f(z) + \frac{1}{2}p^T \nabla^2 f(z)p) \leq 0$$

$$\Rightarrow F(x, z; b(x, z)[\nabla f(z) + \nabla^2 f(z)p]) \leq -\rho(x, z)\|\theta(x, z)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at $x = z$, if each of its components f_i is strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at z.

Note 2.1. Now we have the following special cases

(i) If $\phi(x) = x$ and $\theta(.,.) = d(.,.) : X \times X \to R$, then the second order $(F, b, \phi, \rho, \theta)-$univexity becomes the second order $(F, \alpha, \rho, d)-$convexity defined by I. Ahmad and Z. Husain [10]

(ii) If $\phi(x) = x$, $b(x, z) = 1$ and $\theta(.,.) : X \times X \to R$, then second order $(F, b, \phi, \rho, \theta)-$ univexity becomes the second order $(F, \rho)-$ convexity introduced by Zhang and Mond [12]. Moreover, if second order terms become zero i.e., $p = 0$, then it reduces to $(F, \rho)-$convexity defined in [9, 28].

Now we have the following necessary condition

Theorem 2.1 (Necessary optimal condition). Let x^* be an optimal solution for (GMFP) satisfying $\langle w, x\rangle > 0, \langle v, x\rangle > 0$ and if $\nabla(h_j(x^*) + \langle u_j, x^*\rangle), j \in J(x^*)$ are linearly independent. Then there exists $(s, t^*, \bar{y}) \in K(x^*)$, $\lambda_0 \in R_+, w, v \in R^n, u_j \in R^m$ and $\mu_j^* \in R_+^m$ such that

$$\sum_{i=1}^{s} t_i^*(\nabla(f(x^*, \bar{y}_i) + \langle w, x^*\rangle) - \lambda_0(\nabla(g(x^*, \bar{y}_i) - \langle v, x^*\rangle)))$$

$$+\sum_{j=1}^{m} \mu_j^* \nabla(h_j(x^*) + \langle u_j, x^*\rangle) = 0, \tag{2.1}$$

$$f(x^*, \bar{y}_i) + \langle w, x^*\rangle - \lambda_0(\nabla(g(x^*, \bar{y}_i) - \langle v, x^*\rangle)) = 0, \tag{2.2}$$

$$\sum_{j=1}^{m} \mu_j^* \nabla(h_j(x^*) + \langle u_j, x^*\rangle) = 0, \tag{2.3}$$

$$\langle w, x^*\rangle = s(x^*|C) \tag{2.4}$$

$$\langle v, x^*\rangle = s(x^*|D) \tag{2.5}$$

$$\langle u_j, x^*\rangle = s(x^*|E_j) \tag{2.6}$$

$$t_i^* \geq 0, \quad i = 1, \ldots s, \quad \sum_{i=1}^{s} t_i = 1.$$

3 Duality Model

In this section, we consider the following Mond-weir type dual to (GMFP)

$$\max_{(s,t,\bar{y})\in K(z)} \sup_{(z,\mu,\lambda,u,v,w,p)\in H_1(s,t,\bar{y})} \lambda, \tag{DI}$$

$$\nabla \sum_{i=1}^{s} t_i(f(z,\bar{y}_i) + \langle w,z\rangle - \lambda(g(z,\bar{y}_i) - \langle v,z\rangle)) + \nabla^2 \sum_{i=1}^{s} t_i(f(z,\bar{y}_i) + \langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p+\nabla \sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j,z\rangle)+\nabla^2 \sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j,z\rangle)p = 0, \tag{3.1}$$

$$\sum_{i=1}^{s} t_i(f(z,\bar{y}_i) + \langle w,z\rangle - \lambda(g(z,\bar{y}_i) - \langle v,z\rangle)) - \frac{1}{2}p^t\nabla^2 \sum_{i=1}^{s} t_i(f(z,\bar{y}_i) + \langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i) - \langle v,z\rangle))p \geq 0. \tag{3.2}$$

$$\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j,z\rangle) - \frac{1}{2}p^t\nabla^2 \sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j,z\rangle)p \geq 0. \tag{3.3}$$

Theorem 3.1(*Weak duality Theorem*). Suppose that x and (z,μ,λ,v,w,u,p) are feasible solutions of (GMFP) and (DI) respectively. Let

(i) $h_j(.) + \langle u_j,.\rangle$ is second order $(F,b,\phi,\rho,\theta)-$quasi univex at z,

(ii) $f(.,\bar{y}_i) + \langle w,.\rangle$ and $-g(.,\bar{y}_i) + \langle v,.\rangle$ for $i = 1,\dots,s$ are respectively strong $(F,b,\phi,\rho,\theta)-$ pseudo univex at z with $\frac{\rho}{b} + \frac{\rho_1}{b_1} \geqq 0$,

(iii) $u \leq 0 \Rightarrow \phi(u) \leq 0$ and $v \leqq 0 \Rightarrow \phi(v) \leqq 0$, for all $u,v \in R^n$.

Then

$$\sup_{y\in Y} \frac{f(x,y) + \langle w,x\rangle}{g(x,y) - \langle v,x\rangle} \geq \lambda. \tag{3.4}$$

Proof. Suppose contrary to the result

$$\sup_{y\in Y} \frac{f(x,y) + \langle w,x\rangle}{g(x,y) - \langle v,x\rangle} < \lambda.$$

Then, we find

$$f(x,\bar{y}_i) + \langle w,x\rangle - \lambda(g(x,\bar{y}_i) - \langle v,x\rangle) < 0,$$

for all $\bar{y}_i \in Y$.

It follows $t_i \geq 0,\ i = 1,\dots,s$ with $\sum_{i=1}^{s} t_i = 1$, that

$$t_i(f(x,\bar{y}_i) + \langle w,x\rangle - \lambda(g(x,\bar{y}_i) - \langle v,x\rangle)) \leq 0,$$

since $t = (t_1,\dots,t_s) \neq 0$, then there is at least one strict inequality. Now we have the following

$$\sum_{i=1}^{s} t_i(f(x,\bar{y}_i) + \langle w,x\rangle - \lambda(g(x,\bar{y}_i) - \langle v,x\rangle)) < 0 \leq \sum_{i=1}^{s} t_i(f(z,\bar{y}_i) + \langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle)-\frac{1}{2}p^t\nabla^2(f(z,y_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p),$$

or

$$\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)+\langle w,x\rangle-\lambda(g(x,\bar{y}_i)-\langle v,x\rangle)-(f(z,\bar{y}_i)+\langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))+\frac{1}{2}p^t\nabla^2(f(z,y_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p)\leq 0.$$

From the condition (iii), we get

$$\phi(\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)+\langle w,x\rangle-\lambda(g(x,\bar{y}_i)-\langle v,x\rangle)-(f(z,\bar{y}_i)+\langle w,z\rangle$$

$$-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))+\frac{1}{2}p^t\nabla^2(f(z,y_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p))\leq 0.$$

By the second order strong $(F,b,\phi,\rho,\theta)-$ pseudo univexity of $f(.,\bar{y}_i)+\langle w,.\rangle$ and $-g(.\bar{y}_i)+\langle v,.\rangle$, we have

$$F(x,z,b_1(x,z)(\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))$$

$$+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p))\leq-\rho_1(x,z)\|\theta(x,z)\|^2,$$

or

$$F(x,z,\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))$$

$$+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p)\leq-\frac{\rho_1}{b_1}\|\theta(x,z)\|^2. \quad (3.5)$$

By use of the sublinearity on dual constraints (3.1), we get

$$F(x,z;\nabla\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)+\nabla^2\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)p$$

$$\geqq -F(x,z;\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))$$

$$+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)+\langle w,z\rangle-\lambda(g(z,\bar{y}_i)-\langle v,z\rangle))p).$$

Applying (3.5) in above inequality, we have

$$F(x,z;\nabla\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)+\nabla^2\sum_{j=1}^{m}\mu_j(h_j(z)+\langle u_j,z\rangle)p)>\frac{\rho_1}{b_1}\|\theta(x,z)\|^2 \quad (3.6)$$

Let x and $(z, \mu, \lambda, u, v, w, p)$ are any feasible solutions of (GMFP) and (DI)

$$\sum_{j=1}^{m} \mu_j(h_j(x)+\langle u_j, x\rangle) \leq 0 \leq \sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)-\frac{1}{2}p^t\nabla^2\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)p. \tag{3.7}$$

By using assumption (iii), equation (3.7) yields

$$\phi(\sum_{j=1}^{m} \mu_j(h_j(x)+\langle u_j, x\rangle)-\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)+\frac{1}{2}p^t\nabla^2\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)p) \leq 0.$$

Using the second order $(F, b, \phi, \rho, \theta)-$ quasi univexity of $\sum_{j=1}^{m} \mu_j(h_j(.)+\langle u_j, .\rangle)$, we get

$$F(x, z; b(x, z)(\nabla\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)+\nabla^2\sum_{j=1}^{m} \mu_j(h_j(z)+\langle u_j, z\rangle)p)) \leqq -\rho\|\theta(x, z)\|^2. \tag{3.8}$$

Since $b(x, z) > 0$, the above inequality with the sublinearity of F give

$$F(x, z; \nabla\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle) + \nabla^2\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle)p) \leqq -\frac{\rho}{b}\|\theta(x, z)\|^2. \tag{3.9}$$

Now utilizing the assumption $-\frac{\rho}{b} \leq \frac{\rho_1}{b_1}$, the equation (3.9) provides

$$F(x, z; \nabla\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle) + \nabla^2\sum_{j=1}^{m} \mu_j(h_j(z) + \langle u_j, z\rangle)p) \leqq \frac{\rho_1}{b_1}\|\theta(x, z)\|^2, \tag{3.10}$$

which contradict (3.6), hence (3.4) hold.

Theorem 3.2 (*Strong duality*). Assume that x^* is an efficient solution of (GMFP) and $\nabla h_j(x^*)$ $j \in J(x^*)$ are linearly independent. Then there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0) \in H_1(s^*, t^*, u^*)$ such that $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ is a feasible solution of (DI) and the two objectives have the same values. If in addition, the assumptions of weak duality (Theorem 3.1) hold for all feasible solutions of (GMFP) and (DI), then $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ is an optimal solution of (DI).

Proof. Since x^* is an optimal solution of (GMFP) and $\nabla h_j(x^*), j \in J(x^*)$ are linearly independent, by Theorem 2.1, there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0) \in H_1(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ is a feasible solution of (DI) and the two objectives have the same value. Optimality of $(x^*, \mu^*, \lambda^*, u^*, v^*, w^*, p^* = 0)$ for DI follows from weak duality theorem (Theorem 3.1).

Theorem 3.3 (*Strict converse duality*). Let $\bar{x}$ and $(\bar{z}, \bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}, \bar{y}, \bar{p})$ be the efficient solutions of (GMFP) and (DI), respectively such that

$$\sup_{y\in Y} \frac{f(\bar{x}, \bar{y}) + \langle w, \bar{x}\rangle}{g(\bar{x}, \bar{y}) - \langle v, \bar{x}\rangle} = \bar{\lambda}. \tag{3.11}$$

Suppose

(i) $h_j(.) + \langle u_j, .\rangle$ is second order $(F, b, \phi, \rho, \theta)-$quasi univex at z

(ii) $f(., \bar{y}_i) + \langle w, .\rangle$ and $-g(., \bar{y}_i) + \langle v, .\rangle$ for $i = 1, \ldots, s$, are respectively strong $(F, b, \phi, \rho, \theta)-$ pseudo univex at z with $\frac{\rho}{b} + \frac{\rho_1}{b_1} \geqq 0$,

(iii) $u \leq 0 \Rightarrow \phi(u) \leq 0$ and $v \leqq 0 \Rightarrow \phi(v) \leqq 0$, for all $u, v \in R^n$.

Then

$$\bar{x} = \bar{z}.$$

Proof. We assume that $\bar{x} \neq \bar{z}$ and reach a contradiction, since $\bar{x}$ and $(\bar{z}, \bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{w}, \bar{y}, \bar{p})$ are the feasible solutions of (GMFP) and (DI) respectively, then we have

$$\sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{x})+\langle \bar{u}_j, \bar{x}\rangle) \leq 0 \leq \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)-\frac{1}{2}\bar{p}\nabla^2 \sum_{j=1}^{m} \mu_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)\bar{p}, \tag{3.12}$$

by assumption (iii) equation (3.12) yields

$$\phi(\sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{x}) + \langle \bar{u}_j, \bar{x}\rangle - (h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle)) + \frac{1}{2}p\nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle)p) \leq 0.$$

Utilizing second order $(F, b, \phi, \rho, \theta)-$ quasi univexity of $\sum_{j=1}^{m} \bar{\mu}_j h_j(.) + \langle u_j, .\rangle$, we get

$$F(\bar{x}, \bar{z}; b(\bar{x}, \bar{z})(\nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)+\nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z})+\langle \bar{u}_j, \bar{z}\rangle)\bar{p})) \leqq -\rho\|\theta(\bar{x}, \bar{z})\|^2. \tag{3.13}$$

Since $b(\bar{x}, \bar{z}) > 0$, the above inequality with the sublinearity of F gives

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle u_j, \bar{z}\rangle)\bar{p}) \leqq -\frac{\rho}{b}\|\theta(\bar{x}, \bar{z})\|^2. \tag{3.14}$$

Now utilizing the assumption $-\frac{\rho}{b} \leq \frac{\rho_1}{b_1}$, the inequality (3.14) yields

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle u_j, \bar{z}\rangle)\bar{p}) \leqq \frac{\rho_1}{b_1}\|\theta(\bar{x}, \bar{z})\|^2. \tag{3.15}$$

Suppose (3.11) does not hold, then we have

$$\sup_{y \in Y} \frac{f(\bar{x}, \bar{y}) + \langle \bar{w}, \bar{x}\rangle}{g(\bar{x}, \bar{y}) - \langle \bar{v}, \bar{x}\rangle} < \bar{\lambda}.$$

It is straightforward to see that

$$f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x}\rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x}\rangle) < 0,$$

for all $\bar{y}_i \in Y$.

It follows $t_i \geq 0,\ i = 1, \ldots, s$ with $\sum_{i=1}^{s} t_i = 1$, that

$$t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x}\rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x}\rangle)) \leq 0,$$

with at least one strict inequality, since $t = (t_1, \dots, t_s) \neq 0$. Now we have

$$\sum_{i=1}^{s} t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x}\rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x}\rangle)) < 0 \leq \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle$$
$$-\bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle) - \frac{1}{2}\bar{p}^t\nabla^2(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, z\rangle))\bar{p}),$$

or

$$\sum_{i=1}^{s} t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x}\rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x}\rangle) - (f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle$$
$$-\bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle)) + \frac{1}{2}\bar{p}^t\nabla^2(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))\bar{p}) \leq 0.$$

From the condition (iii), we get

$$\phi(\sum_{i=1}^{s} t_i(f(\bar{x}, \bar{y}_i) + \langle \bar{w}, \bar{x}\rangle - \bar{\lambda}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x}\rangle) - (f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle$$
$$-\bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle)) + \frac{1}{2}\bar{p}^t\nabla^2(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))\bar{p})) \leq 0.$$

By the second order strong $(F, b, \phi, \rho, \theta)-$ pseudo univexity of $f(., \bar{y}_i) + \langle \bar{w}, .\rangle$ and $-g(., \bar{y}_i) + \langle \bar{v}, .\rangle$, we have

$$F(\bar{x}, \bar{z}, b_1(\bar{x}, \bar{z})(\nabla \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))$$
$$+\nabla^2 \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))\bar{p})) \leq -\rho_1(\bar{x}, \bar{z})\|\theta(\bar{x}, \bar{z})\|^2,$$

or

$$F(\bar{x}, \bar{z}, \nabla \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))$$
$$+\nabla^2 \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))\bar{p}) \leq -\frac{\rho_1}{b_1}\|\theta(\bar{x}, \bar{z})\|^2. \quad (3.16)$$

Using sublinearity on dual constraints (3.1), we get

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle)\bar{p})$$
$$\geqq -F(\bar{x}, \bar{z}; \nabla \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))$$
$$+\nabla^2 \sum_{i=1}^{s} t_i(f(\bar{z}, \bar{y}_i) + \langle \bar{w}, \bar{z}\rangle - \bar{\lambda}(g(\bar{z}, \bar{y}_i) - \langle \bar{v}, \bar{z}\rangle))\bar{p}).$$

Applying (3.16) in above inequality, we have

$$F(\bar{x}, \bar{z}; \nabla \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle) + \nabla^2 \sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{z}) + \langle \bar{u}_j, \bar{z}\rangle)\bar{p}) > \frac{\rho_1}{b_1}\|\theta(\bar{x}, \bar{z})\|^2, \quad (3.17)$$

which is a contradiction of (3.15). Hence the result follows immediately.

4 Conclusion

On the basis of application point of view second order duality is very practical and competent as it provides tighter lower bounds. So it is very significant to generalize the existing results to second order environment. In the present study the notion of second order $(F, b, \rho, \alpha, \theta)-$ univexity and its generalizations is considered. Many generalized convexity, invexity and univexity concepts are special cases of second order $(F, b, \rho, \alpha, \theta)-$univexity. This notion is appropriate to study the weak, strong and converse duality theorems for second order dual (DI) of a nondifferentiable fractional problem with support function (GMFP).

The results proved in this paper can be further generalized for the following non-differentiable minimax fractional programming problem with square root terms i.e.,

$$\min \sup_{y \in Y} \frac{f(x,y) + (x^t B x)^{1/2}}{g(x,y) - (x^t C x)^{1/2}},$$

$$\text{subject to} \;\; h_j(x) \leq 0, \quad j = 1, 2, \ldots, p,$$

where Y is a compact subset of R^m, $f(.,.), g(.,.) : R^n \times R^m \to R$ and $h(.) : R^n \to R^p$ are twice differentiable functions. B and C are $n \times n$ positive semi definite symmetric matrices.

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COUPLED COINCIDENCE POINT THEOREMS AND CONE b-METRIC SPACES OVER BANACH ALGEBRAS

YOUNG-OH YANG* AND HONG JOON CHOI

ABSTRACT. In this paper, we obtain some coupled coincidence point results for two nonlinear contractive mappings in cone b-metric spaces over Banach algebras without assumption of normality by virtue of the properties of spectral radius. Also we give two examples as applications of the main results.

1. INTRODUCTION

In 2007 the concept of cone metric space was introduced by Huang and Zhang in [4], where they generalized metric space by replacing the set of real numbers with an ordering Banach space, and proved some fixed point theorems for contractive mappings on these spaces. Recently, in ([1],[3], [4], [5], [6], [7], [9], [10]) some common fixed point theorems have been proved for contractive maps on cone metric spaces. Gnana Bhaskar and Lakshmikantham([2]) introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is extended and used in various directions([2]).

In 2013, in order to generalize the Banach contraction principle to more general form, Liu and Xu([7]) introduced the concept of cone metric spaces over Banach algebras, by replacing Banach spaces with Banach algebras as the underlying spaces of cone metric spaces, and proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constants by means of spectral radius. Furthermore, they gave an example to explain that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces.

Motivated by the above works, in this paper, we obtain some coupled coincidence point results for two nonlinear contractive mappings in cone b-metric spaces over Banach algebras without assumption of normality by virtue of the properties of spectral radius. Our main results extends the corresponding similar results in cone metric spaces. Also we give two examples as applications of the main results.

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Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that A is a real Banach algebra with a unit (i.e., a multiplicative identity) e. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Let A be a real Banach algebra with a unit e and θ the zero element of A. A nonempty closed subset P of Banach algebra A is called a *cone* if

(i) $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β ;
(iii) $P^2 = PP \subset P$;
(iv) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \neq \emptyset$ then P is called a *solid cone.* A cone P is called *normal* if there exists a number K such that for all $x, y \in A$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number K satisfying condition (1.1) is called the *normal constant* of P.

In the following we always assume that P is a solid cone of A and $\preceq$ is the partial ordering with respect to P.

Definition 1.1. Let X be a nonempty set, $s \geq 1$ be a constant and A be a real Banach algebra. Suppose the mapping $d : X \times X \to A$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a *cone b-metric* on X, and (X, d) is called a *cone b-metric space* over the Banach algebra A.

If $s = 1$,then every cone b-metric is a cone metric.

Example 1.2. Let $A = C[a, b]$ be the set of continuous functions on $[a, b]$ with the supremum. Define multiplication in the usual way. Then A is a Banach algebraa with a unit 1. Set $P = \{x \in A : x(t) \geq 0,\ t \in [a, b]\}$ and $X = \mathbb{R}$. We define a mapping $d : X \times X \to A$ by $d(x, y)(t) = |x - y|^p e^t$ for all $x, y \in X$ and for each $t \in [a, b]$, where $p > 1$ is a constant. This makes (X, d) into a cone b-metric space over Banach algebra with the coefficient $s = 2^{p-1}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Definition 1.3. Let (X, d) be a cone b-metric space over the Banach algebra A. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in A$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x, and the point x is the *limit* of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

(2) If for all $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

(3) A cone b-metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Definition 1.4. Let E be a real Banach space with a solid cone P. A sequence $\{x_n\} \subset P$ is called a $c-$*sequence* if for any $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $x_n \ll c$ for all $n \geq N$.

Lemma 1.5. *([5], [7]) Let E be a real Banach space with a cone P. Then*

(p_1) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(p_2) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*
(p_3) *If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.*
(p_4) *If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*
(p_5) *If $\{x_n\}, \{y_n\}$ are sequences in E such that $x_n \to x$, $y_n \to y$ and $x_n \preceq y_n$ for all $n \geq 1$, then $x \preceq y$.*

Lemma 1.6. *([7]) Let A ba a real Banach algebra with a unit e and P be a solid cone in A. We define the spectral radius $\rho(x)$ of $x \in A$ by*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

(1) *If* $0 \le r(x) < 1$, *then then* $e - x$ *is invertible,*

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i \quad and \quad r((e-x)^{-1}) \le \frac{1}{1-r(k)}.$$

(2) *If* $r(x) < 1$ *then* $\|x^n\| \to 0$ *as* $n \to \infty$.
(3) *If* $x \in P$ *and* $r(x) < 1$, *then* $(e-x)^{-1} \in P$.
(4) *If* $k, u \in P$, $r(k) < 1$ *and* $u \preceq ku$, *then* $u = \theta$.
(5) $r(x) \le \|x\|$ *for all* $x \in A$.
(6) *If* $x, y \in A$ *and* x, y *commute, then the following holds:*
  (a) $r(xy) \le r(x)r(y)$
  (b) $r(x+y) \le r(x) + r(y)$ *and*
  (c) $|r(x) - r(y)| \le r(x-y)$.

Lemma 1.7. *(*[5], [7]*) Let* (X, d) *be a complete cone b-metric space over a Banach algebra* A *and let* P *be a solid cone in* A. *Let* $\{x_n\}$ *be a sequence in* X. *Then*

(1) *If* L $\|x_n\| \to 0$ *as* $n \to \infty$, *then* $\{x_n\}$ *is a* $c-$*sequence.*
(2) *If* $k \in P$ *is any vector and* $\{x_n\}$ *is* $c-$*sequence in* P, *then* $\{kx_n\}$ *is a* $c-$*sequence.*
(3) *If* $x, y \in A$, $a \in P$ *and* $x \preceq y$, *then* $ax \preceq ay$.
(4) *If* $\{x_n\}$ *converges to* $x \in X$, *then* $\{d(x_n, x)\}$, $\{d(x_n, x_{n+p})\}$ *are c-sequences for any* $p \in \mathbb{N}$.

2. Main results

Gnana Bhaskar and Lakshmikantham([2]) introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is extended and used in various directions.

In this section, we establish some coupled coincidence point results for a mapping $F : X \times X \to X$ satisfying certain contractive condition on cone metric spaces over Banach algebras without assumption of normality.

Definition 2.1. ([2], [8]) Let (X, d) be a cone b-metric space over the Banach algebra A.

(1) An element $(x, y) \in X \times X$ is called a *coupled fixed point* of $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.
(2) An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of mappings $F : X \times X \to X$and $g : X \times X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called coupled point of coincidence;
(3) An element $(x, y) \in X \times X$ is called a *common coupled fixed point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

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(4) The mappings $F : X \times X \to X$ and $g : X \times X$ are called *w-compatible* if $g(F(x,y)) = F(gx, gy)$ whenever $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Note that if (x,y) is a coupled fixed point of F, then (y,x) is also a coupled fixed point of F.

Theorem 2.2. *Let (X,d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and let P be a solid cone in A. Let $F : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(F(x,y), F(u,v)) &\preceq a_1 d(gx, gu) + a_2 d(F(x,y), gx) + a_3 d(gy, gv) \\ &+ a_4 d(F(u,v), gu) + a_5 d(F(x,y), gu) \\ &+ a_6 d(F(u,v), gx) \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$, where $a_i \in P$, $a_i a_j = a_j a_i$ $(i = 1, 2, \cdots, 6)$ and

$$2s(r(a_1) + r(a_3)) + (s+1)(r(a_2) + r(a_4)) + (s^2 + s)(r(a_5) + r(a_6)) < 2.$$

If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X, then F and g have a coupled coincidence point in X.

Proof. Let x_0, y_0 be any two arbitrary points in X. Set $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. This can be done because $F(X \times X) \subseteq g(X)$. Continuing this process we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$. From (2.1), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq a_1 d(gx_{n-1}, gx_n) + a_2 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &+ a_4 d(F(x_n, y_n), gx_n) + a_5 d(F(x_{n-1}, y_{n-1}), gx_n) \\ &+ a_6 d(F(x_n, y_n), gx_{n-1}) \\ &= a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &+ a_4 d(gx_{n+1}, gx_n) + a_5 d(gx_n, gx_n) + a_6 d(gx_{n+1}, gx_{n-1}) \\ &\preceq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) \\ &+ a_4 d(gx_{n+1}, gx_n) + sa_6 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})] \\ &= (a_1 + a_2 + sa_6) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) \\ &+ (a_4 + sa_6) d(gx_n, gx_{n+1}), \end{aligned}$$

and so we get

$$\begin{aligned} (e - a_4 - sa_6) d(gx_n, gx_{n+1}) &\preceq (a_1 + a_2 + sa_6) d(gx_{n-1}, gx_n) \\ &+ a_3 d(gy_{n-1}, gy_n) \end{aligned} \tag{2.2}$$

Similarly, we have

$$(e - a_4 - sa_6)d(gy_n, gy_{n+1}) \preceq (a_1 + a_2 + sa_6)d(gy_{n-1}, gy_n) + a_3d(gx_{n-1}, gx_n) \quad (2.3)$$

Because of the symmetry in (2.1),

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\preceq a_1d(gx_n, gx_{n-1}) + a_2d(F(x_n, y_n), gx_n) + a_3d(gy_n, gy_{n-1}) \\ &+ a_4d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + a_5d(F(x_n, y_n), gx_{n-1}) \\ &+ a_6d(F(x_{n-1}, y_{n-1}), gx_n) \\ &= a_1d(gx_n, gx_{n-1}) + a_2d(gx_{n+1}, gx_n) + a_3d(gy_n, gy_{n-1}) \\ &+ a_4d(gx_n, gx_{n-1}) + a_5d(gx_{n+1}, gx_{n-1}) + a_6d(gx_n, gx_n) \\ &\preceq a_1d(gx_n, gx_{n-1}) + a_2d(gx_{n+1}, gx_n) + a_3d(gy_n, gy_{n-1}) \\ &+ a_4d(gx_n, gx_{n-1}) + sa_5[d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})] \end{aligned}$$

that is,

$$(e - a_2 - sa_5)d(gx_{n+1}, gx_n) \preceq (a_1 + a_4 + sa_5)d(gx_{n-1}, gx_n) + a_3d(gy_n, gy_{n-1}) \quad (2.4)$$

Similarly,

$$(e - a_2 - sa_5)d(gy_{n+1}, gy_n) \preceq (a_1 + a_4 + sa_5)d(gy_{n-1}, gy_n) + a_3d(gx_n, gx_{n-1}) \quad (2.5)$$

Let $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$. Now, by (2.2), (2.3),(2.4) and (2.5), we obtain that

$$(e - a_4 - sa_6)\delta_n \preceq (a_1 + a_2 + a_3 + sa_6)\delta_{n-1} \quad (2.6)$$

$$(e - a_2 - sa_5)\delta_n \preceq (a_1 + a_3 + a_4 + sa_5)\delta_{n-1} \quad (2.7)$$

Finally, from (2.6) and (2.7) we have

$$(2e - a_2 - a_4 - sa_5 - sa_6)\delta_n \preceq (2a_1 + 2a_3 + a_2 + a_4 + sa_5 + sa_6)\delta_{n-1}$$

By hypothesis and Lemma 1.6,

$$r(a_2 + a_4 + sa_5 + sa_6) \leq r(a_2) + r(a_4) + sr(a_5) + sr(a_6) < 1$$

and so $2e - (a_2 + a_4 + sa_5 + sa_6)$ is invertible by Lemma **??**. Putting

$$\eta = (2e - a_2 - a_4 - sa_5 - sa_6)^{-1}(2a_1 + 2a_3 + a_2 + a_4 + sa_5 + sa_6),$$

we have, by hypothesis,

$$r(\eta) = \frac{2r(a_1) + 2r(a_3) + r(a_2) + r(a_4) + sr(a_5) + sr(a_6)}{2 - r(a_2) - r(a_4) - sr(a_5) - sr(a_6)} < \frac{1}{s},$$

and so

$$\delta_n \preceq \eta\delta_{n-1}, \quad r(\eta) < 1 \quad (2.8)$$

Consequently, we have

$$\theta \preceq \delta_n \preceq \eta\delta_{n-1} \preceq \cdots \preceq \eta^n\delta_0 \tag{2.9}$$

If $\delta_0 = \theta$ then (x_0, y_0) is a coupled coincidence point of F and g. So let $\theta \prec \delta_0$. If $m > n$, we have

$$\begin{aligned} d(gx_m, gx_n) &\preceq s[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\ &\preceq sd(gx_n, gx_{n+1}) + s^2[d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, gx_m)] \\ &\vdots \\ &\preceq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\ &\quad + s^{m-n-1}d(gx_{m-2}, gx_{m-1}) + s^{m-n}d(gx_{m-1}, gx_m) \end{aligned} \tag{2.10}$$

and similarly

$$\begin{aligned} d(gy_m, gy_n) &\preceq sd(gy_n, gy_{n+1}) + s^2 d(gy_{n+1}, gy_{n+2}) + \cdots \\ &\quad + s^{m-n-1}d(gy_{m-2}, gy_{m-1}) + s^{m-n}d(gy_{m-1}, gy_m) \end{aligned} \tag{2.11}$$

Adding both the above inequalities, we get

$$\begin{aligned} d(gx_m, gx_n) + d(gy_m, gy_n) &\preceq s^{m-n}\delta_{m-1} + s^{m-n-1}\delta_{m-2} + \cdots s\delta_n \\ &\preceq (s^{m-n}\eta^{m-1} + s^{m-n-1}\eta^{m-2} + \cdots + s\eta^n)\delta_0 \\ &\preceq s\eta^n(\sum_{i=0}^{\infty}(s\eta)^i)\delta_0 = s\eta^n(e - s\eta)^{-1}\delta_0 \ \rightarrow \ \theta \end{aligned}$$

as $n \to \infty$. From Lemma 1.7, it follows that for $\theta \ll c$ and large n, $\eta^n(1-\eta)^{-1}\delta_0 \ll c$. Thus, according to (p_2), $d(gx_n, gx_m) + d(gy_n, gy_m) \ll c$. Hence, by Definition, $\{d(gx_n, gx_m) + d(gy_n, gy_m)\}$ is a Cauchy sequence. Since, $d(gx_n, gx_m) \preceq d(gx_n, gx_m) + d(gy_n, gy_m)$ and $d(gy_n, gy_m) \preceq d(gx_n, gx_m) + d(gy_n, gy_m)$, then again by (p_2), $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is a complete subset of X, there exist x and y in X such that $gx_n \to gx$ and $gy_n \to gy$.

Now, we prove that $F(x,y) = gx$ and $F(y,x) = gy$. For that we have

$$\begin{aligned} d(F(x,y),gx) &\preceq s[d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx)] \\ &= s[d(F(x,y),F(x_n,y_n)) + d(gx_{n+1},gx)] \\ &\preceq sa_1 d(gx,gx_n) + sa_2 d(F(x,y),gx) + sa_3 d(gy,gy_n) \\ &+ sa_4 d(F(x_n,y_n),gx_n) + sa_5 d(F(x,y),gx_n) \\ &+ sa_6 d(F(x_n,y_n),gx) + sd(gx_{n+1},gx) \\ &= sa_1 d(gx,gx_n) + sa_2 d(F(x,y),gx) + sa_3 d(gy,gy_n) + sa_4 d(gx_{n+1},gx_n) \\ &+ sa_5 d(F(x,y),gx_n) + sa_6 d(gx_{n+1},gx) + sd(gx_{n+1},gx) \\ &\preceq sa_1 d(gx,gx_n) + sa_2 d(F(x,y),gx) + sa_3 d(gy,gy_n) + s^2 a_4 d(gx_{n+1},gx) \\ &+ s^2 a_4 d(gx,gx_n) + s^2 a_5 [d(F(x,y),gx) + d(gx,gx_n)] \\ &+ sa_6 d(gx_{n+1},gx) + sd(gx_{n+1},gx) \end{aligned}$$

which further implies that

$$\begin{aligned} d(F(x,y),gx) &\preceq (e - sa_2 - s^2 a_5)^{-1}(sa_1 + s^2(a_4 + a_5))d(gx_n,gx) \\ &+ (e - sa_2 - s^2 a_5)^{-1}(e + s^2 a_4 + sa_6)d(gx_{n+1},gx) \\ &+ (e - sa_2 - s^2 a_5)^{-1} sa_3 d(gy_n,gy). \end{aligned} \tag{2.12}$$

since $e - sa_2 - s^2 a_5$ is invertible. Since $gx_n \to gx$ and $gy_n \to gy$, then for any $\theta \ll c$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(gx_n,gx) \ll \frac{(1 - sr(a_2) - s^2 r(a_5))c}{3(sr(a_1) + s^2 r(a_4) + s^2 r(a_5))}, \quad d(gx_{n+1},gx) \ll \frac{(1 - sr(a_2) - s^2 r(a_5))c}{3(s + s^2 r(a_4) + sr(a_6))}$$

and

$$d(gy_n,gy) \ll \frac{(1 - sr(a_2) - s^2 r(a_5))c}{3sr(a_3)}.$$

Thus, for all $n \geq N$,

$$d(F(x,y),gx) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c. \tag{2.13}$$

Now, according to (p_4), it follows that $d(F(x,y),gx) = \theta$ and so $F(x,y) = gx$. Similarly, $F(y,x) = gy$. Hence (x,y) is a coupled coincidence point of the mappings F and g. □

Corollary 2.3. *Let (X,d) be a complete cone metric space over Banach algebra A and let P be a solid cone in A. Let $F : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(F(x,y),F(u,v)) &\preceq a_1 d(gx,gu) + a_2 d(F(x,y),gx) + a_3 d(gy,gv) \\ &+ a_4 d(F(u,v),gu) + a_5 d(F(x,y),gu) \\ &+ a_6 d(F(u,v),gx) \end{aligned}$$

for all $x, y, u, v \in X$*, where* $a_i \in P$, $a_i a_j = a_j a_i$ $(i = 1, 2, \cdots, 6)$ *and* $\sum_{i=1}^{6} r(a_i) < 1$*. If* $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X*, then* F *and* g *have a coupled coincidence point in* X*.*

Proof. Taking $s = 1$ in Theorem 2.2, we get the required result. □

Corollary 2.4. *Let* (X, d) *be a complete cone b-metric space over Banach algebra* A *with the coefficient* $s \geq 1$ *and let* P *be a solid cone in* A*. Let* $F : X \times X \to X$ *be mappings satisfying*

$$\begin{aligned} d(F(x,y), F(u,v)) &\preceq a_1 d(x,u) + a_2 d(F(x,y),x) + a_3 d(y,v) \\ &+ a_4 d(F(u,v),u) + a_5 d(F(x,y),u) \\ &+ a_6 d(F(u,v),x) \end{aligned}$$

for all $x, y, u, v \in X$*, where* $a_i \in P$ *and* $a_i a_j = a_j a_i$ $(i = 1, 2, \cdots, 6)$ *If*

$$2s(r(a_1) + r(a_3)) + (s+1)(r(a_2) + r(a_4)) + (s^2 + s)(r(a_5) + r(a_6)) < 2,$$

then F *has a coupled fixed point in* X*.*

Proof. Taking $g = I_X$, identity mapping of X in Theorem 2.2, we get the required result. □

Corollary 2.5. *Let* (X, d) *be cone b-metric space over Banach algebra* A *with the coefficient* $s \geq 1$ *and let* P *be a solid cone in* A*. Suppose that two mappings* $F : X \times X \to X$ *and* $g : X \to X$ *satisfy*

$$\begin{aligned} d(F(x,y), F(u,v)) &\preceq a[d(gx,gu) + d(F(x,y),gx)] + b[d(gy,gv) + d(F(u,v),gu)] \\ &+ c[d(F(x,y),gu) + d(F(u,v),gx)] \end{aligned}$$

for all $x, y, u, v \in X$*, where* $a, b, c \in P$ *commute and*

$$(3s+1)[r(a) + r(b)] + 2(s^2 + s)r(c) < 2.$$

If $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is complete subset of* X*, then* F *and* g *have a coupled coincidence point in X.*

Proof. Taking $a_1 = a_2 = a, a_3 = a_4 = b, a_5 = a_6 = c$ in Theorem 2.2, we get the required result. □

Corollary 2.6. *Let* (X, d) *be a complete cone b-metric space over Banach algebra* A *and let* P *be a solid cone in* A*. Suppose that* $F : X \times X \to X$ *satisfies the following contractive condition for all* $x, y, u, v \in X$ *:*

$$d(F(x,y), F(u,v)) \preceq kd(x,u) + ld(y,v)$$

where $k, l \in P$ commute and $s[r(k) + r(l)] < 1$. Then F has a unique coupled fixed point.

Proof. Taking $a_1 = k, a_3 = l, a_2 = a_4 = a_5 = a_6 = \theta$ and $g = I_X$ in Theorem 2.2, we get the required result. □

Corollary 2.7. *Let (X, d) be a complete cone b-metric space over Banach algebra A and let P be a solid cone in A. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \preceq kd(F(x, y), x) + ld(F(u, v), u)$$

where $k, l \in P$ commute and $(s + 1)[r(k) + r(l)] < 2$. Then F has a unique coupled fixed point.

Proof. Taking $a_2 = k, a_4 = l, a_1 = a_3 = a_5 = a_6 = \theta$ and $g = I_X$ in Theorem 2.2, we get the required result. □

Corollary 2.8. *Let (X, d) be a complete cone b-metric space over Banach algebra A and let P be a solid cone in A. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \preceq kd(F(x, y), u) + ld(F(u, v), x)$$

where $k, l \in P$ commute and $(s^2 + s)[r(k) + r(l)] < 2$. Then F has a unique coupled fixed point.

Proof. Taking $a_5 = k, a_6 = l, a_1 = a_2 = a_3 = a_5 = \theta$ and $g = I_X$ in Theorem 2.2, we get the required result. □

Now we present two examples showing that Theorem 2.2 is a proper extension of known results. In this example, the conditions of Theorem 2.2 are fulfilled.

Example 2.9. (The case of non-normal cone) Let $A = C^1_{\mathbb{R}}[0, 1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in A$. Define multiplication in A as just pointwise multiplication. Then A is a real Banach algebra with unit $e = 1(e(t) = 1$ for all $t \in [0, 1])$. The set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal.

Let $X = \{1, 2, 3\}$. Define $d : X \times X \to A$ by $d(1, 2)(t) = d(2, 1)(t) = d(2, 3)(t) = d(3, 2)(t) = e^t, d(1, 3)(t) = d(3, 1)(t) = 3e^t, d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and for each $x \in X$. Then (X, d) is a solid cone b-metric space over Banach algebra with the coefficient $s = \frac{3}{2}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Define two mappings $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} 3, & (x,y) = (3,1) \\ 2, & \text{otherwise} \end{cases}$$

and $g : X \to X$ by $g1 = 3, g2 = 2, g3 = 1$. Then $F(X \times X) = \{2,3\} \subset \{1,2,3\} = g(X)$. Let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with

$$a_1(t) = a_3(t) = 0.03, a_2(t) = 0.02, a_4(t) = 0.25, a_5(t) = a_6(t) = 0.154$$

for all $t \in [0,1]$. Then, by definition of spectral radius, $r(a_1) = r(a_3) = 0.03, r(a_2) = 0.02, r(a_4) = 0.25, r(a_5) = r(a_6) = 0.15$ and so

$$2s(r(a_1) + r(a_3)) + (s+1)(r(a_2) + r(a_4)) + (s^2 + s)(r(a_5) + r(a_6)) = 1.89 < 2.$$

Since $d(F(x,y), F(3,1))(t) = d(2,3)(t) = e^t$ for any $x, y \in X$, by careful calculations, we can get that for any $x, y, u, v \in X$, F and g satisfy the contractive condition (2.1) of Theorem 2.2. Hence the hypotheses are satisfied and so by Theorem 2.2, F and g have a coupled coincidence point in a complete cone b-metric space X over Banach algebra. Since $F(2,2) = 2 = g2$, F and g are w-compatible and $(2,2)$ is the unique coupled coincidence point of F and g.

Example 2.10. (The case of normal cone) Let $A = \mathbb{R}^2$ and define a norm on A by $\|(x_1, x_2)\| = |x_1| + |x_2|$ for $x = (x_1, x_2) \in A$. Define the multiplication in A by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Put $P = \{x = (x_1, x_2) \in A : x_1, x_2 \geq 0\}$. Then P is a normal cone and A is a real Banach algebra with unit $e = (1,1)$.

Let $X = [0, \infty)$. Define a mapping $d : X \times X \to A$ by $d(x,y) = (|x-y|^2, |x-y|^2)$ for each $x, y \in X$. Then (X, d) is a complete cone b-metric space over Banach algebra with the coefficient $s = 2$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Consider the mappings $F : X \times X \to X$ and $g : X \to X$ defined by

$$F(x,y) = x + \frac{|\sin y|}{2}$$

and

$$g(x) = 3x.$$

Then $F(X \times X) \subseteq g(X) = X$. Let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with

$$a_1 = (\frac{2}{9}, 0),\ a_3 = (\frac{1}{18}, 0),\ a_2 = a_4 = (0,0),\ a_5 = a_6 = (0.07, 0).$$

Then, by definition of spectral radius, $r(a_1) = \frac{2}{9}$, $r(a_3) = \frac{1}{18}$, $r(a_2) = r(a_4) = 0$, $r(a_5) = r(a_6) = 0.07$, and so

$$4(r(a_1) + r(a_3)) + 3(r(a_2) + r(a_4)) + 6(r(a_5) + r(a_6)) < 2.$$

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By careful calculations, it is easy to verify that for any $x, y, u, v \in X$, F and g satisfy the contractive condition (2.1) of Theorem 2.2. Thus by Theorem 2.2, F and g have a coupled coincidence point in a complete cone b-metric space X over Banach algebra. Since $F(0,0) = g0 = 0$, $(0,0)$ is the common coupled coincidence point of F and g.

Theorem 2.11. *Let $F : X \times X \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Theorem 2.2. If F and g are w-compatible, then F and g have unique common coupled fixed point. Moreover, common fixed point of F and g is of the form (u, u) for some $u \in X$.*

Proof. First we claim that coupled point of coincidence is unique. Suppose that $(x, y), (x^*, y^*) \in X \times X$ with $g(x) = F(x, y)$, $g(y) = F(y, x)$ and $g(x^*) = F(x^*, y^*)$, $g(y^*) = F(y^*, x^*)$. Using (2.1), we get

$$\begin{aligned} d(gx, gx^*) &\preceq d(F(x,y), F(x^*,y^*)) \\ &\preceq a_1 d(gx, gx^*) + a_2 d(F(x,y), gx) + a_3 d(gy, gy^*) \\ &+ a_4 d(F(x^*,y^*), gx^*) + a_5 d(F(x,y), gx^*) + a_6 d(F(x^*,y^*), gx) \\ &= (a_1 + a_5 + a_6) d(gx, gx^*) + a_3 d(gy, gy^*) \end{aligned}$$

and so

$$d(gx, gx^*) \preceq (a_1 + a_5 + a_6) d(gx, gx^*) + a_3 d(gy, gy^*). \tag{2.14}$$

Similarly

$$d(gy, gy^*) \preceq (a_1 + a_5 + a_6) d(gy, gy^*) + a_3 d(gx, gx^*). \tag{2.15}$$

Thus

$$d(gx, gx^*) + d(gy, gy^*) \preceq (a_1 + a_3 + a_5 + a_6)(d(gx, gx^*) + d(gy, gy^*)).$$

Since $s \geq 1$ and $r(a_1) + r(a_3) + r(a_5) + r(a_6) < 1$, therefore by Lemma 1.6(4), we have $d(gx, gx^*) + d(gy, gy^*) = \theta$, which implies that $gx = gx^*$ and $gy = gy^*$. Similarly we prove that $gx = gy^*$ and $gy = gx^*$. Thus $gx = gy$. Therefore (gx, gx) is unique coupled point of coincidence of F and g.

Now, let $g(x) = u$. Then we have $u = g(x) = F(x, x)$. By w- compatibility of F and g, we have

$$g(u) = g(g(x)) = g(F(x,x)) = F(gx, gx) = F(u, u). \tag{2.16}$$

Then (gu, gu) is a coupled point of coincidence of F and g. Consequently $gu = gx$. Therefore $u = gu = F(u, u)$. Hence (u, u) is unique common coupled fixed point of F and g. □

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