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Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

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Common fixed point theorems in G_b-metric space

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Abstract In this paper, we introduce a new type of common fixed point for three mappings in G_b-complete G_b-metric space. On the other hand, we prove that the theory is also established in G-metric space and several corollaries and examples are listed.

Keywords: G_b-metric space; common fixed point; G-metric space

1 Preliminaries

Mustafa and Sims [1] generalized the concept of metric space and Mustafa [2,3,7] obtained some fixed point theorems in his papers. After that, many authors established fixed point and common fixed point theorems for different contractive-type condition in G-metric space. In 1998, Czerwik [10] introduced the notion of b-metric space, and then Aghajani [12] based on the notion gave the concept of G_b-metric space and some authors obtained the existence and uniqueness fixed point in G_b-metric space [7,11].

Fixed point theory has a large number of applications in many branches of nonlinear analysis and has been extended in many different directions. Let A, B and C are self mappings of a nonempty set X, if there exists a $p \in X$, such that $Ap = Bp = Cp = p$, then we call p is a common fixed point of A, B and C. For a mapping T on nonempty set X to itself, we have $Tx = x$, and x is unique then we call x is a Picard operator.

In this paper, we mainly obtain a unique common fixed point for three mappings in G_b-metric space. First, we recall some basic properties of G_b-metric space.

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ be the set of all natural numbers. Denote $\mathbb{N}^+$ the set of all positive integers.

Definition 1.1 ([12]) Let X be a nonempty set and $s \geq 1$ be a given real number, and let the function $G : X \times X \times X \to [0, \infty)$ satisfy the following properties:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$;

(G_b2) $0 \leq G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z;

(G_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$ for all $x, y, z \in X$.

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Then G is called a G_b-metric on X, and (X, G) is called a G_b-metric space.

Definition 1.2 ([12]) A G_b-metric space G is said to be symmetric if $G(x, x, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.3 ([12]) Let X be a G_b-metric space, then for each $x, y, z, a \in X$ it follows that:

(1) if $G(x, y, z) = 0$ then $x = y = z$;

(2) $G(x, y, z) \leq sG(G(x, y, y) + G(x, x, z))$;

(3) $G(x, y, y) \leq 2s(G(y, x, x))$;

(4) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$.

Definition 1.4 ([12]) Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) G_b-Cauchy if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(2) G_b-convergent to a point $x \in X$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, \geq n_0$, $G(x_n, x_m, x) < \varepsilon$;

Definition 1.5 ([12]) A G_b-metric space X is called complete if every G_b-Cauchy sequence is G_b-convergent in X.

lemma 1.6 ([11]) Let $(X, , G)$ be a G_b-metric space with $s > 1$.

(1) Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_b-convergent to x, y and z, respectively. Then we have

$$\frac{1}{s^3}G(x, y, z) \leq \liminf_{n\to\infty} G(x_n, y_n, z_n) \leq \limsup_{n\to\infty} G(x_n, y_n, z_n) \leq s^3 G(x, y, z).$$

(2) If $\{z_n\} = c$ is constant, then

$$\frac{1}{s^2}G(x, y, c) \leq \liminf_{n\to\infty} G(x_n, y_n, c) \leq \limsup_{n\to\infty} G(x_n, y_n, c) \leq s^2 G(x, y, c).$$

(3) If $\{y_n\} = b$ and $\{z_n\} = c$ are constant, then

$$\frac{1}{s}G(x, b, c) \leq \liminf_{n\to\infty} G(x_n, b, c) \leq \limsup_{n\to\infty} G(x_n, b, c) \leq sG(x, b, c).$$

2 Common fixed point theorems in G_b-metric space

Theorem 2.1 Let (X, G) be a G_b-complete G_b-metric space and A, B and C are mappings from X to itself. Suppose that A, B and C satisfy the following condition:

$$G(Ax, By, Cz) \leq \frac{G(x, Ax, Ax) + G(x, By, By) + G(z, Cz, Cz)}{G(x, Ax, By) + G(y, By, Cz) + G(z, Cz, Ax) + 1})G(x, y, z) \tag{2.1}$$

for all $x, y, z \in X$. Then either one of A, B and C has a fixed point, or, A, B and C have a unique common fixed point.

Proof. Define the sequence $\{x_n\}$ as $x_{3n+1} = Ax_{3n}, x_{3n+2} = Bx_{3n+1}, x_{3n+3} = Bx_{3n+2}$ for all $n = 0, 1, 2, \cdots$.

If $x_{3n} = x_{3n+1}$, then x_{3n} is a fixed point of A.

If $x_{3n+1} = x_{3n+2}$, then x_{3n+1} is a fixed point of B.

If $x_{3n+2} = x_{3n+3}$, then x_{3n+2} is a fixed point of C.

If the above conclusions are not true, then we assume that $x_n \neq x_{n+1}$ for all n. Let $d_n = G(x_n, x_{n+1}, x_{n+2})$, then for (2.1) we have

$$\begin{aligned}
&G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2})\\
\leq&\frac{G(x_{3n}, Ax_{3n}, Ax_{3n}) + G(x_{3n+1}, Bx_{3n+1}, Bx_{3n+1}) + G(x_{3n+2}, Cx_{3n+2}, Cx_{3n+2})}{G(x_{3n}, Ax_{3n}, Bx_{3n+1}) + G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(x_{3n+2}, Cx_{3n+2}, Ax_{3n}) + 1})G(x_{3n}, x_{3n+1}, x_{3n+2})\\
=&\frac{G(x_{3n}, x_{3n+1}, x_{3n+1}) + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + G(x_{3n+2}, x_{3n+3}, , x_{3n+3})}{G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + G(x_{3n+2}, x_{3n+3}, x_{3n+1}) + 1})G(x_{3n}, x_{3n+1}, x_{3n+2})\\
\leq&\frac{G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + G(x_{3n+1}, x_{3n+2}, , x_{3n+3})}{G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + G(x_{3n+2}, x_{3n+3}, x_{3n+1}) + 1})G(x_{3n}, x_{3n+1}, x_{3n+2})
\end{aligned}$$

so we have

$$d_{3n+1} \leq \frac{d_{3n} + 2d_{3n+1}}{d_{3n} + 2d_{3n+1} + 1} d_{3n}$$

Let

$$\alpha_{3n} = \frac{d_{3n} + 2d_{3n+1}}{d_{3n} + 2d_{3n+1} + 1}$$

so we have

$$d_{3n+1} \leq \alpha_{3n} d_{3n}$$

by introduction, we have

$$d_{3n+1} \leq \alpha_{3n}\alpha_{3n-1} \cdots \alpha_1 d_1$$

It is obvious that for any natural number $n \in \mathbb{N}$, we have $0 < \alpha_n < 1$,and so

$$d_n \leq d_{n-1}$$

then we have

$$\begin{aligned}
d_n \leq d_{n-1} &\Rightarrow d_n + d_{n+1} \leq d_{n-1} + d_n\\
&\Rightarrow 1 + \frac{1}{d_{n-1} + 2d_n} \leq 1 + \frac{1}{d_n + 2d_{n+1}}\\
&\Rightarrow \frac{1}{\alpha_{n-1}} \leq \frac{1}{\alpha_n}
\end{aligned}$$

Hence, we can get

$$\alpha_{n-1} \geq \alpha_n$$

so we can obtain

$$\alpha_{3n}\alpha_{3n-1} \cdots \alpha_1 \leq {\alpha_1}^{3n}$$

taking the limit as $n \to \infty$, so we have

$$\lim_{n\to\infty} d_{3n+1} \leq \lim_{n\to\infty} \alpha_{3n}\alpha_{3n-1}\cdots\alpha_1 d_1 \leq \lim_{n\to\infty} {\alpha_1}^{3n} d_1 = 0$$

so

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0.$$

Next,we will show that $\{x_n\}$ is a G_b-Cauchy sequence. on the other hand, according to (G_b3) we have

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) \leq \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0 \tag{2.2}$$

for any $n, m \in \mathbb{N}$, $m > n$, using (G_b5), so we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq sG(x_n, x_{n+1}, x_{n+1}) + sG(x_{n+1}, x_m, x_m) \\
&\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2G(x_{n+1}, x_{n+2}, x_{n+2}) + s^2G(x_{n+2}, x_m, x_m) \\
&\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + s^{m-n}G(x_{m-1}, x_{m-1}, x_m) \\
&\leq sG(x_n, x_{n+1}, x_{n+2}) + s^2G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + s^{m-n}G(x_{m-1}, x_m, x_{m+1}) \\
&= d_1(s{\alpha_1}^n + s^2{\alpha_1}^{n+1} + \cdots + s^{m-n}{\alpha_1}^{m-1}) \\
&= d_1\frac{s{\alpha_1}^n(1-(s\alpha)^{m-n-1})}{1-s{\alpha_1}^n}
\end{aligned}$$

taking the limit as $n \to \infty$, then we have

$$\lim_{n\to\infty} G(x_n, x_m, x_m) \leq \lim_{n\to\infty} d_1\frac{s{\alpha_1}^n(1-(s\alpha)^{m-n-1})}{1-s{\alpha_1}^n} = 0$$

so $\{x_n\}$ is a G_b-Cauchy sequence.

Since X is complete, so there exists a $p \in X$, such that $\{x_n\}$ is a G_b-Cauchy sequence and G_b-converges to p such that

$$\begin{aligned}
\lim_{n\to\infty} x_{3n+1} &= \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} x_{3n+2} = \lim_{n\to\infty} Bx_{3n+1} \\
&= \lim_{n\to\infty} x_{3n+3} = \lim_{n\to\infty} Cx_{3n+2} = p.
\end{aligned}$$

Now we prove that p is a common fixed point of A, B and C.

Using Lemma 1.6 and (2.1), taking the upper limit as $n \to \infty$, we get

$$\begin{aligned}
&G(Ap, p, p) \\
&\leq s^2 \lim_{n\to\infty} \sup G(Ap, Bx_{3n+1}, Cx_{3n+2}) \\
&\leq s^2 \lim_{n\to\infty} \sup \frac{G(p, Ap, Ap) + G(x_{3n+1}, Bx_{3n+1}, Bx_{3n+1}) + G(x_{3n+2}, Cx_{3n+2}, Cx_{3n+2})}{G(p, Ap, Bx_{3n+1}) + G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(x_{3n+2}, Cx_{3n+2}, Ap) + 1} G(p, x_{3n+1}, x_{3n+1}) \\
&\leq s^4 \lim_{n\to\infty} \sup \frac{G(p, Ap, Ap) + G(x_{3n+1}, Bx_{3n+1}, Bx_{3n+1}) + G(x_{3n+2}, Cx_{3n+2}, Cx_{3n+2})}{G(p, Ap, Bx_{3n+1}) + G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(x_{3n+2}, Cx_{3n+2}, Ap) + 1} G(p, p, p) \\
&= 0
\end{aligned}$$

then we get $G(Ap, p, p) = 0$. Hence by (1) of Proposition 1.1, we can get $Ap = p$. Similarly, letting $x = x_{3n}$, $y = p$, $z = x_{3n+2}$ and $x = x_{3n}$, $y = x_{3n+1}$, $z = p$ we can get $Bp = p$ and $Cp = p$ respectively, so we have $Ap = Bp = Cp = p$.

Now, we show that the common fixed point of A, B and C is unique. Assume on contrary that q is another fixed point, i.e. $Aq = Bq = Cq = q$ such that $p \neq q$. Then, by our assumption, we apply (2.1) to obtain

$$\begin{aligned} G(p,p,q) &= G(Ap,Bp,Cq) \\ &\leq \frac{G(p,Ap,Ap)+G(p,Bp,Bp)+G(q,Cq,Cq)}{G(p,Ap,Bp)+G(p,Bp,Cq)+G(q,Cq,Ap)+1}G(p,p,q) \\ &= \frac{G(p,p,p)+G(p,p,p)+G(q,q,q)}{G(p,p,p)+G(p,p,q)+G(q,q,p)+1}G(p,p,q) \\ &= 0 \end{aligned}$$

so by the Proposition 1.1, we have $G(p,p,q) = 0$, then $p = q$.

Corollary 2.2 Let (X, G) be a G_b-complete G_b-metric space and T be a mapping from X to itself. Suppose that T satisfy the following condition:

$$G(Tx,Ty,Tz) \leq \frac{G(x,Tx,Tx)+G(x,Tx,Tx)+G(x,Tx,Tx)}{G(x,Tx,Ty)+G(y,Ty,Tz)+G(z,Tz,Tx)+1})G(x,y,z)$$

for all $x, y, z \in X$. Then T has a unique fixed point.

Proof. Taking $A = B = C = T$, the result follow from Theorem 2.1.

Theorem 2.3 Let (X, G) be a G_b-complete G_b-metric space and A, B and C are mappings from X to itself. Suppose that A, B and C satisfy the following condition:

$$G(Ax,By,Cz) \leq \alpha\frac{\min\{G(y,By,By),G(z,Cz,Cz)\}}{G(z,Cz,Ax)+1}G(x,Ax,Ax)+\beta G(x,y,z) \tag{2.3}$$

for all $x, y, z \in X$, where $\alpha + \beta \leq 1$.

Then either one of A, B and C has a fixed point, or, A, B and C have a unique common fixed point.

Proof. Let $d_n = G(x_n, x_{n+1}, x_{n+2})$, then for (2.3) we have

$$\begin{aligned} G(Ax_{3n},Bx_{3n+1},Cx_{3n+2}) &\leq \alpha\frac{\min\{G(x_{3n+1},Bx_{3n+1},Bx_{3n+1}),G(x_{3n+2},Cx_{3n+2},Cx_{3n+2})\}}{G(x_{3n+2},Cx_{3n+2},Ax_{3n})+1}) \\ &\quad G(x_{3n},Ax_{3n},Ax_{3n})+\beta G(x_{3n},x_{3n+1},x_{3n+2}) \\ &= \alpha\frac{\min\{G(x_{3n+1},x_{3n+2},x_{3n+2}),G(x_{3n+2},x_{3n+3},x_{3n+3})\}}{G(x_{3n+2},x_{3n+3},x_{3n+1})+1}) \\ &\quad G(x_{3n},x_{3n+1},x_{3n+1})+\beta G(x_{3n},x_{3n+1},x_{3n+2}) \\ &\leq \alpha\frac{d_{3n+1}}{d_{3n+1}+1}d_{3n}+\beta d_{3n} \\ &= (\alpha\frac{d_{3n+1}}{d_{3n+1}+1}+\beta)d_{3n} \end{aligned}$$

Since $\alpha\frac{d_{3n+1}}{d_{3n+1}+1} + \beta \leq 1$, the following proof is similar to Theorem 2.1.

Corollary 2.4 Let (X, G) be a G_b-complete G_b-metric space and T be mapping from X to itself. Suppose that T satisfy the following condition:

$$G(Tx,Ty,Tz) \leq \alpha(\frac{\min\{G(y,Ty,Ty),G(z,Tz,Tz)\}}{G(z,Tz,Tx)+1})G(x,Tx,Tx)+\beta G(x,y,z)$$

for all $x, y, z \in X$, where $\alpha + \beta \leq 1$.

Then T has a unique fixed point.

Proof. Taking $A = B = C = T$, the result follow from Theorem 2.4.

Theorem 2.5 Let (X, G) be a G-complete G-metric space and A, B and C are mappings from X to itself. Suppose that A, B and C satisfy the following condition:

$$G(Ax, By, Cz) \leq \frac{G(x, Ax, Ax) + G(y, By, By) + G(z, Cz, Cz)}{G(x, Ax, By) + G(y, By, Cz) + G(z, Cz, Ax) + 1})G(x, y, z)$$

for all $x, y, z \in X$. Then either one of A, B and C has a fixed point, or, A, B and C have a unique common fixed point.

Proof. The proof is similar to Theorem 2.1. There is a little difference between them.

First, when we prove that $\{x_n\}$ is a G_b-Cauchy sequence, we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m) \\
&\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m) \\
&\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_{m-1}, x_m) \\
&\leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+1})
\end{aligned}$$

so we can get

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+1}) \\
&\leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+2}) \\
&= \sum_{i=0}^{m-n} d_{n+i}
\end{aligned}$$

taking the limit as $n \to \infty$, then we have

$$\lim_{n\to\infty} G(x_n, x_m, x_m) \leq \lim_{n\to\infty} \sum_{i=0}^{m-n} {\alpha_1}^{n+i} d_1 = 0$$

so $\{x_n\}$ is a G-Cauchy sequence.

Secondly, since that G-metric space is continuous so when we prove that p is a common fixed point in G-metric space we have

$$G(Ap, p, p) \leq \frac{G(p, Ap, Ap) + G(p, Bp, Bp) + G(p, Cp, Cp)}{G(p, Ap, Bp) + G(p, Bp, Cp) + G(p, Cp, Ap) + 1} G(p, p, p) = 0$$

Corollary 2.6 Let (X, G) be a G-complete G-metric space and T be a mapping from X to itself. Suppose that T satisfy the following condition:

$$G(Tx, Ty, Tz) \leq \frac{G(x, Tx, Tx) + G(x, Tx, Tx) + G(x, Tx, Tx)}{G(x, Tx, By) + G(y, Ty, Tz) + G(z, Tz, Tx) + 1})G(x, y, z)$$

for all $x, y, z \in X$. Then T has a unique fixed point.

Proof. Taking $A = B = C = T$, the result follow from Theorem 2.6.

3 An example

Example 3.1 Let $G(x,y,z) = (\max\{|x-y|,|y-x|,|z-x|\})^2$ for all $x,y,z \in X$ then G is a G_b-metric on X where $s=2$. Define self-mappings A,B and C on x by

$$A(x) = 1, B(x) = 1, C(x) = \frac{7+x}{8}$$

Then we have

$$\begin{aligned} G(Ax,By,Cz) &= (\max\{|1-1|,|1-\frac{7+z}{8}|,|1-\frac{7+z}{8}|\})^2 \\ &= (\frac{1}{8}-\frac{z}{8})^2 \end{aligned}$$

and we also have

$$G(x,Ax,Ax) = (1-x)^2, G(y,By,By) = (1-y)^2$$
$$G(z,Cz,Cz) = (\frac{7-7z}{8})^2, G(x,Ax,By) = (1-x)^2$$
$$G(y,By,Cz) = \max\{(1-y)^2,(\frac{1-z}{8})^2,(\frac{7+z}{8}-y)^2\}$$
$$G(z,Cz,Ax) = (1-z)^2$$
$$G(x,y,z) = \max\{|x-y|^2,|y-z|^2,|z-x|^2\}$$

Case1: when $z \leq x, z \leq y$ $\alpha \geq \frac{1}{64}, \beta = 0$ and $y \leq \frac{1+7z}{8}$, we have the (2.3) established. Then $x=y=z=1$ is a common fixed point.

Case2: when $y \geq \frac{1+7z}{8}$ $\alpha = 1, \beta = 0$ and $z \geq -6+7x$, we have the (2.3) established. Then $x=y=z=1$ is a common fixed point.

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A modified collocation method for weakly singular Fredholm integral equations of second kind*

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Abstract

In this paper, a collocation method with high precision by using the polynomial basis functions is proposed to solve the Fredholm integral equation of second kind with weakly singular kernel. We introduce the polynomial basis functions and use it to reduce the given equation to a system of linear algebraic equation. Thus, we can simplify the solving of the equation. The error analysis are given. Numerical examples are given to illustrate the efficiency of our method.

Keyword: Weakly Singular · Fredholm Integral Equation · Polynomial basis function Method

AMS subject classification: 65D10 · 65D32

1 Introduction

This paper is concerned with collocation method for weakly singular Fredholm integral equations of the second kind as follows

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$$\phi(x) + \lambda \int_a^b \kappa(x,t)\phi(t)dt = f(x), 0 \leq x \leq 1, \tag{1.1}$$

where $\kappa(x,t) = \frac{H(x,t)}{|x-t|^\alpha}, 0 < \alpha < 1, H(x,t), f(x)$ are continue and bounded functions and $\phi(x)$ is the function to be determined.

Numerical methods for weakly singular Fredholm integral equations of the second kind have been developed by many scholars in recent years because of their important applications in science and engineering. These methods can be classified into two types. One type is through making approximations to the analytical solutions directly. For instance, Tricomi used successive approximations method to solve the integral equations in his book [1]. Variational iteration method and Adomian decomposition method were introduced in [2] and [3] respectively. Also, The homotopy analysis method was proposed by Liao [4] and has applied it in [5] et. Another type is through shifting the equations into a form which easier to solve than the original equations. For example, Taylor expansion collocation methods are presented to solve integral equations in [6-8]. In [9], the orthogonal triangular basis functions were used by Babolian et al. to solve some integral equations systems. And Legendre wavelets method was proposed by Jafari et al. in [10] to find the numerical solutions of linear integral equations systems. Moreover, in [12] architecture artificial neural networks was suggested to approximate the solutions of linear integral equations systems. Furthermore, Jafarian et al. [13] using the Bernstein polynomials to obtain the numerical solutions of linear Fredholm and Volterra integral equations systems of the second kind. And application of Bernstein polynomial have been made by scholar for solving both differential equations and integral equations, see [11]. And piecewise polynomial collocation method were applied to solve the Volterra integro-differential equations with weakly singular kernel in [14] respectively. And the stability of piecewise polynomial collocation methods for solving weakly singular integral equations of the second kind has been discussed by Kangro et al. in [15]. Besides, Baratella et al. [16] had proposed an approach with product integration to solve the weakly singular Volterra integral equations. Kolk et al. And Pallaw et al. [17] used the quadratic spline collocation to solve the smoothed weakly singular Fredholm integral equations. However, these methods introduced above do not provide a good accuracy in the solution near the singular points.

In this paper, we are going to use polynomial basis functions collocation method to approximate the solution of singular Fredholm integral equations of the second kind. The proposed approach converted the given equation with unique solution into a system of linear algebraic equations in general case. To do this, first the polynomial basis functions of certain degree n of unknown functions are substituted in the given integral equations. So that the solution of the unknown function of given equations have converted into the solutions of the coefficients of the unknown polynomial basis functions, such that we can solve the integral equations in a convenient way.

The layout of this paper is as follows: In section 2 we presented the procedure of the polynomial basis functions collocation method to obtain the approximate solution of the weakly singular Fredholm integral equation. In section 3, we had demonstrated that the proposed method is convergent to all the weakly singular Fredholm integral equations of second kind. In section 4, we give numerical example to test the effectiveness and efficiency of the method. Finally, Numerical examples are given to illustrate the efficiency of our method.

2 The Polynomial Basis Function Method

We are going to use the polynomial basis functions to solve the eq.(1.1). The form of the functions are as follows:

$$U = \sum_{k=0}^{m-1} x^k,$$

where the polynomial basis functions $1, x, x^2, \cdots, x^{m-1}$ are linear independent.

Since eq.(1.1) is a weakly singular integral equation, the singularity of the e-quation must be removed such that the procedure of solving the problem can be move on. But since the proposed method of this paper is belong to the colloca-tion method, which can smooth the singular points of the discretion, so that we can use the method directly. Then we provided the procedure of using polynomial basis functions to solve the kind of the integral equations proposed in this paper concretely as follows:

Step 1. Choosing the basis functions $u = [1, x, x^2, ..., x^k], (k = 0, 1, 2, ..., m-1)$ the unknown function $\phi(x)$ is substituted by the following polynomials

$$\phi(x) \approx \phi_m(x) = \sum_{k=0}^{m-1} a_k x^k, \tag{2.1}$$

Step 2. Substituting (2.1) into (1.1) we have

$$\sum_{k=0}^{m-1} a_k x^k + \lambda \sum_{k=0}^{m-1} a_k x^k \int_a^b \kappa(x,t) t^k dt = f(x), \tag{2.2}$$

Step 3. Discrete the interval [a,b] into n sections uniformly, we obtained the systems of the coefficient a_k as follows

$$\sum_{k=0}^{m-1} a_k x_j^k + \lambda \sum_{k=0}^{m-1} a_k x_j^k \int_a^b \kappa(x,t) t^k dt = f(x_j), \tag{2.3}$$

where $j = 1, 2, ..., n, x_j = a + j(b - a)/n$. We transformed the equations into the form of linear matrix as follows

$$(U + KU)A = f, \tag{2.4}$$

where

$$U = \begin{pmatrix} 1 & x_1 & \cdots & x_1^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{m-1} \end{pmatrix}, A = \begin{pmatrix} a_0 \\ \vdots \\ a_{m-1} \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \tag{2.5}$$

and $K = \int_a^b \kappa(x,t)dt$ which is the integral operator.

Step 4. Solve the system we obtained the solutions of the coefficients of a_k as follows

$$a_0, a_1, ..., a_{m-1}.$$

Substituting them into eq.(2.1) we obtained the approximate solution $\phi_m(x)$.

3 Convergence and Error Analysis

In this section, we are going to prove that the approximate method we proposed in this paper is convergent to the analytic solution of eq.(1.1).

Firstly, we rewrite the form of the weakly singular kernel as follows

$$K(x,t) = \frac{H(x,t)}{|x-t|^\alpha}.$$

Let $0 < \alpha \leq \frac{1}{2}$, and $H(x,t)$ is continuously bounded. Then the eigenvalue integral equation with weakly singular kernel is as follows

$$\lambda\phi(x) = \int_a^b K(x,t)\phi(t)dt, 0 \leq x \leq 1, \tag{3.1}$$

where $K(x,t)$ is the weakly singular kernel, λ is the eigenvalue of the $K(x,t)$, $\phi(x)$ is the eigenfunction of λ.

Lemma 3.1 [14]. If $x_1, x_2 \in C^{m,v}(0,T], m \in N, v < 1$, then $x_1x_2 \in C^{m,v}(0,T]$, and

$$\|x_1x_2\|_{m,v} \leq c\|x_1\|_{m,v}\|x_2\|_{m,v}$$

with a constant c which is independent of x_1 and x_2.

Proof. See [14].

Lemma 3.2 [18] Suppose that the function $\phi_m(x)$ obtained by the polynomial basis function is the approximation of eq.(1) and eq.(1) is with bounded first derivative, then eq.(1) can be expanded as an infinite sum of the polynomial basis functions, that is, $\phi(x) = \sum_{k=0}^{m-1} c_k x^k$, and the coefficients c_k are bounded as

$$c_k < \frac{K}{(m+1)2^{\frac{3k}{2}}}$$

where K is a constant.

Proof. See [18].

Let the linear operator $\mathbf{K} : L^1_{[0,1]} \to L^1_{[0,1]}$

$$(\mathbf{K}\phi)(x) = \int_0^1 K(x,t)\phi(t)dt, 0 \leq x \leq 1$$

then (3.1) can be written as

$$\mathbf{K}\phi = \lambda\phi, \tag{3.2}$$

using $\mathbf{K}$ operating two sides of (3.2) we yield

$$\mathbf{K}^2\phi = \lambda^2\phi$$

where

$$\mathbf{K}^2\phi(x) = \int_0^1 K_2(x,t)\phi(t)dt$$

and $K_2(x,t)$ is the iterative kernel of $K(x,t)$

$$K^2(x,t) = \int_0^1 K_1(x,r)K_1(r,t)dr$$

$$K_1(x,r) = K(x,r).$$

Theorem 3.3. Let $\phi_m(x)$ be the polynomial basis function of degree $m-1$ and whose coefficients has been obtained by solving linear system (2.4), the given polynomial basis function is converge to the analytical solution of the weak singular Fredholm integral equations of the second kind (1.1), when $m \to \infty$.

Proof. Since

$$\phi(x) = \lim_{m\to\infty} \phi_m(x),$$

substitute $\phi_m(x)$ into eq.(1.1), we have

$$\phi_m(x) + \lambda \int_a^b K(x,t)\phi_m(t)dt = f(x), 0 \leq x \leq 1. \tag{3.3}$$

We defined the error function $\|e_m\|$ by subtracting (2.1) and (2.5) as follows

$$\|e_m\| = \|\phi_m(x) - \phi(x)\| + |\lambda| \| \int_0^1 \|K(x,t)\| \cdot \|\phi_m(t) - \phi(t)\| dt\|,$$

According to lemma 3.1 and 3.2, since the subinterval of integral equation is compact and the coefficients obtained by the polynomial basis functions are bounded and the kernel $K(x,t)$ can be continuous and bounded through iteration, therefore, whether

$$\|e_m\| \to 0$$

depends on

$$\|\phi_m(x) - \phi(x)\| \to 0,$$

since

$$\phi(x) = \lim_{m \to \infty} \phi_m(x),$$

that is,

$$\|e_m\| \to 0$$

when $m \to \infty$.

Thus, the proof is completed. □

Remark 3.4. When we use this method we can find that it is similar to the piecewise linear spline function interpolation method which is convergent and numerical stable. The speed of the convergency is accelerated with the increasing of the degree m of the polynomial basis function.

4 Numerical Experiments

Example 1: Consider the following Fredholm integral equations of the second kind with weakly singular kernel

$$\phi(x) - \frac{1}{10} \int_0^1 K(x,t)\phi(t)dt = f(x), 0 \leq x \leq 1, \tag{4.1}$$

where $K(x,t) = |x-t|^{-\frac{1}{3}}$,

$$f(x) = x^2(1-x)^2 - \frac{27}{30800}[x^{\frac{8}{3}}(54x^2 - 126x + 77) + (1-x)^{\frac{8}{3}}(54x^2 + 18x + 5)].$$

the exact solution of eq.(4.1) is $\phi(x) = x^2(1-x)^2$.

Using the method we proposed in section 2 and the successive approximation method and using MATLAB writing the program codes we obtained the figures and tables so that we can make a comparison for the accuracy of the two methods.

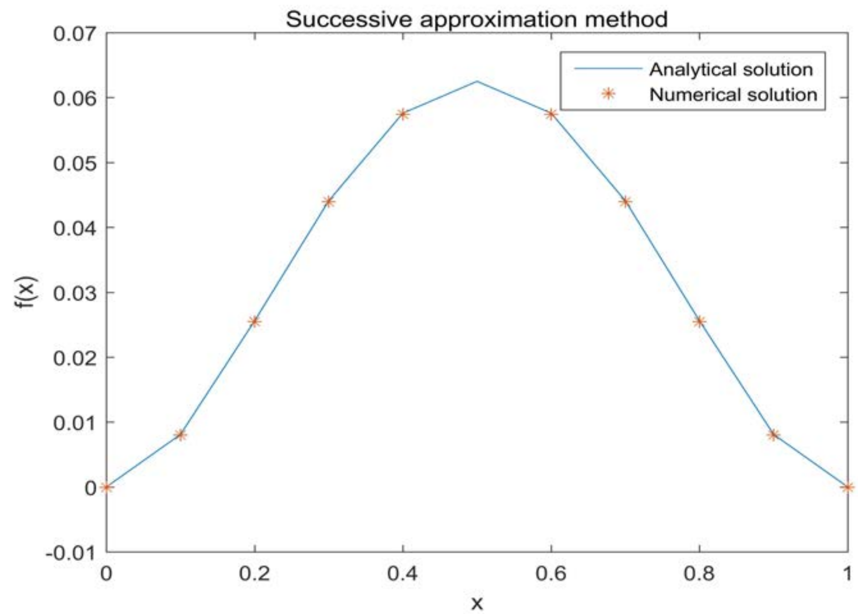

Figure 1: The nodes are 11, iterations are 6, The result of Successive approximation method and the analytical solutions.

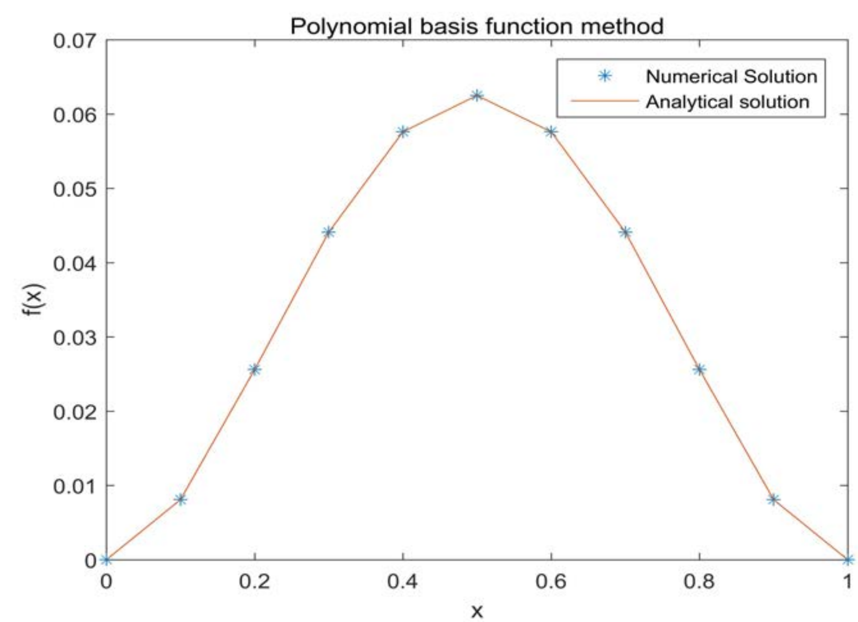

Figure 2: The nodes are 11, k=4, The result of Polynomial basis function method and the analytical solutions.

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Firstly, we obtained the figures of the results of the Polynomial basis function method and the successive approximation method

From the figures we can find that both of the curves of successive approximation method and the polynomial basis functions method are simulated very well, but there is a defection of the numerical solution of successive approximation method that the singular point in the function can not be removed. But the polynomial basis functions method almost accordant with the analytical solutions. Namely, the accuracy of polynomial basis functions method is better than the successive approximation methods.

Table 1: The comparison of the solutions of the two kinds of methods.

Node	Exact Solution	Successive Approximation	Polynomial Basis Function Method
0	0	-2.8235e-04	-1.136e-016
0.1	8.1000e-003	7.7634e-03	8.1000e-003
0.2	2.5600e-002	2.5228e-02	2.5600e-002
0.3	4.4100e-002	4.3685e-02	4.4100e-002
0.4	5.7600e-002	5.7144e-02	5.7600e-002
0.5	6.2500e-002	NaN	6.2500e-002
0.6	5.7600e-002	5.7144e-02	5.7600e-002
0.7	4.4100e-002	4.3685e-02	4.4100e-002
0.8	2.5600e-002	2.5228e-02	2.5600e-002
0.9	8.1000e-003	7.7634e-03	8.1000e-003
1	0	-2.8235e-04	0

The Table 1 shows the results of the solutions of the example 1 using successive approximation method with the iterations k=8 and the polynomial basis function method with the orders m=5 of the polynomial basis function, respectively. From the table we can find that the results of the polynomial basis function method is more approximate to the exact solutions than the successive approximation method.

From the Table 2 we can easily find that with the increasing of the iterations of k, there is little increasing of the error accuracy of the successive approximation method. And it is obvious that there is a singular point of the discrete interval.

The Table 3 shows the errors accuracy results of the polynomial basis function method when the orders of the polynomials are n=3,4,5,6, respectively. We can find that the results is much superior than the successive approximation method. The best error effectiveness of successive approximation is $O(10^{-4})$, but we obtained the high accuracy of the polynomial basis function method when the orders of the polynomial basis functions is $n = 5$ and the effective errors have reached $O(10^{-16})$, which is much better than the successive approximation methods.

Table 2: The error comparison of Successive Approximation methods.

Node	Exact solution	k=2	k=4	k=6	k=8
0	0	9.2298e-03	6.6592e-05	2.7807e-04	2.8225e-04
0.1	8.1000e-03	1.0586e-02	7.7066e-05	3.3105e-04	3.3646e-04
0.2	2.5600e-02	1.1068e-02	1.1413e-04	3.6606e-04	3.7179e-04
0.3	4.4100e-02	1.1342e-02	1.5033e-04	4.0921e-04	4.1510e-04
0.4	5.7600e-02	1.1480e-02	1.8759e-04	4.4996e-04	4.5593e-04
0.5	6.2500e-02	NaN	NaN	NaN	NaN
0.6	5.7600e-02	1.1480e-02	1.8759e-04	4.4996e-04	4.5593e-04
0.7	4.4100e-02	1.1342e-02	1.5033e-04	4.0921e-04	4.1510e-04
0.8	2.5600e-02	1.1068e-02	1.1413e-04	3.6606e-04	3.7179e-04
0.9	8.1000e-03	1.0586e-02	7.7066e-05	3.3105e-04	3.3646e-04
1	0	9.2298e-03	6.6592e-05	2.7807e-04	2.8225e-04

Table 3: The error comparison of Polynomial basis function methods.

Node	Exact solution	n=3	n=4	n=5	n=6
0	0	6.7365e-03	6.7365e-03	2.0322e-16	3.6580e-16
0.1	8.1000e-03	7.5301e-03	7.5301e-03	8.3267e-17	2.9490e-16
0.2	2.5600e-02	7.4263e-03	7.4263e-03	4.1633e-17	2.1164e-16
0.3	4.4100e-02	1.3522e-03	1.3522e-03	6.2450e-17	1.8041e-16
0.4	5.7600e-02	4.6922e-03	4.6922e-03	4.8572e-17	1.3184e-16
0.5	6.2500e-02	7.1071e-03	7.1071e-03	1.3878e-17	9.7145e-17
0.6	5.7600e-02	4.6922e-03	4.6922e-03	2.7756e-17	8.3267e-17
0.7	4.4100e-02	1.3522e-03	1.3522e-03	3.4694e-17	1.3878e-17
0.8	2.5600e-02	7.4263e-03	7.4263e-03	1.4572e-16	1.5613e-16
0.9	8.1000e-03	7.5301e-03	7.5301e-03	2.2204e-16	2.7756e-16
1	0	6.7365e-03	6.7365e-03	0	1.5260e-16

Example 2: Consider the following Fredholm integral equations of the second kind with weakly singular kernel

$$\phi(x) - \frac{1}{10}\int_0^1 K(x,t)\phi(t)dt = f(x), 0 \leq x \leq 1, \tag{4.2}$$

where $K(x,t) = |x-t|^{-\frac{1}{2}}$,

$$f(x) = x^2(1-x)^2 - \frac{27}{30800}[x^{\frac{8}{3}}(54x^2 - 126x + 77) + (1-x)^{\frac{8}{3}}(54x^2 + 18x + 5)].$$

We have not the exact solutions of the example 2, but we compared the accuracy of the two methods through the error accuracy when the iterations increased of

Table 4: The error comparison of succcessive approximation methods.

Node	n=2	n=4	n=6	errors of (c3-c2)	errors of (c4-c3)
0.0139	5.5529e-04	7.8461e-04	7.9539e-04	2.2932e-04	1.0780e-05
0.0556	3.3207e-03	3.5784e-03	3.5903e-03	2.5772e-04	1.1917e-05
0.1250	1.2872e-02	1.3386e-02	1.3399e-02	5.1414e-04	1.2785e-05
0.2222	3.2102e-02	3.2388e-02	3.2402e-02	2.8611e-04	1.3450e-05
0.3472	5.4893e-02	5.5188e-02	5.5202e-02	2.9552e-04	1.3437e-05
0.5000	NaN	NaN	NaN	NaN	NaN
0.6528	5.4893e-02	5.5188e-02	5.5202e-02	2.9552e-04	1.3437e-05
0.7778	3.2102e-02	3.2388e-02	3.2402e-02	2.8611e-04	1.3450e-05
0.8750	1.2872e-02	1.3386e-02	1.3399e-02	5.1414e-04	1.2785e-05
0.9444	3.3207e-03	3.5784e-03	3.5903e-03	2.5772e-04	1.1917e-05
0.9861	5.5529e-04	7.8461e-04	7.9539e-04	2.2932e-04	1.0780e-05

the successive approximation method and when the orders of the polynomial basis function increased, respectively. The column 2 to column 4 of Table 4 shows the solutions of the method when the iterations k=2,4,6, respectively, and it shows the error accuracy of the solutions of column 3 minus column 2 and column 4 minus column 3 and we get column 5 and column 6, respectively. From the Table 4 we can easily find that, with the increasing of the iterations of the successive approximation method, the error accuracy increased accordingly.

Table 5: The error comparison of polynomial basis function methods.

Node	n=4	n=5	n=6	errors of (c3-c2)	errors of (c4-c3)
0.0139	-1.2677e-04	6.3355e-03	6.3355e-03	6.4623e-03	1.8388e-16
0.0556	1.1224e-02	9.5719e-03	9.5719e-03	-1.6520e-03	3.1225e-17
0.1250	2.7883e-02	2.0391e-02	2.0391e-02	-7.4917e-03	-1.3878e-17
0.2222	4.6462e-02	4.0972e-02	4.0972e-02	-5.4890e-03	9.7145e-17
0.3472	6.2217e-02	6.5462e-02	6.5462e-02	3.2455e-03	1.8041e-16
0.5000	6.9050e-02	7.8091e-02	7.8091e-02	9.0411e-03	6.9389e-17
0.6528	6.2217e-02	6.5462e-02	6.5462e-02	3.2455e-03	-1.8041e-16
0.7778	4.6462e-02	4.0972e-02	4.0972e-02	-5.4890e-03	-1.0408e-16
0.8750	2.7883e-02	2.0391e-02	2.0391e-02	-7.4917e-03	2.0470e-16
0.9444	1.1224e-02	9.5719e-03	9.5719e-03	-1.6520e-03	4.5103e-17
0.9861	-1.2677e-04	6.3355e-03	6.3355e-03	6.4623e-03	-8.5001e-17

Table 5 shows the results of the solutions of the method we proposed in this paper. It shows the results of the solutions of the proposed method from the column 2 to column 4, and the error accuracy results obtained by column 3 minus column

2 and column 4 minus column 3 and we get column 5 and column 6, respectively. From the data of the table 5 we can easily find that the polynomial basis function method is much superior than the successive approximation method. The best error effectiveness of successive approximation we finally obtained is $O(10^{-5})$, but we obtained the high accuracy of the polynomial basis function method when we let $n = 5$ and the effective errors have reached $O(10^{-16})$, which is much nearly to the exact solutions.

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SHARP COEFFICIENT ESTIMATES FOR NON-BAZILEVIČ FUNCTIONS

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ABSTRACT. The class $\bar{\mathcal{B}}(\alpha)$ of non-Bazilevič functions was introduced by Obradović. Later, estimates on the second coefficient and Fekete–Szegö functional for normalized analytic functions in the class $\bar{\mathcal{B}}(\alpha)$ were investigated by Tuneski and Darus. In the present work, sharp estimate on third to eighth coefficients for normalized analytic functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \bar{\mathcal{B}}(\alpha)$ are investigated. Further sharp estimate on the functional $|a_2a_3 - a_4|$ is also obtained.

1. Introduction

The class of analytic functions defined in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and having the Taylor series expansion of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{1.1}$$

is denoted by $\mathcal{A}$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. De Branges, in 1984, proved that if $f \in \mathcal{S}$, then $|a_n| \leq n$. This result was put before by Bieberbach in 1916 and is popularly known as the Bieberbach conjecture. Among the many subclasses of $\mathcal{S}$, the class of starlike and convex functions are the most investigated. The class of starlike and convex functions are defined, respectively, by $\mathcal{S}^* := \{f \in \mathcal{S} : \mathrm{Re}(zf'(z)/f(z)) > 0\}$ and $\mathcal{K} := \{f \in \mathcal{S} : \mathrm{Re}(1 + zf''(z)/f'(z)) > 0\}$. Thomas [15], in 1967, introduced a general form of the class of starlike functions. Thomas [15], for a starlike functions g, defined the class $\mathcal{B}_\alpha := \{f \in \mathcal{S} : \mathrm{Re}(zf'(z)f(z)^{\alpha-1}/g(z)^\alpha) > 0\}$. This class is popularly known as the class of Bazilevič functions of type α. In 1973, Singh [12] investigated a special case of $\mathcal{B}_\alpha$. For $\alpha \geq 0$ and setting $g(z) = z$, he considered a subclass of $\mathcal{B}_\alpha$ defined by

$$\mathcal{B}_1(\alpha) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} \right) > 0 \right\}.$$

In his paper, he obtained the sharp radius estimates for certain integral operator to be a member of the class $\mathcal{B}_1(\alpha)$ and he also obtained the sharp upper bound on the first four initial coefficients. He also investigated the sharp bound on the Fekete-Szegö functional for functions in this class. It should be noted that the class $\mathcal{B}_1(1)$ is a subclass of close-to-convex

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functions and hence univalent in $\mathbb{D}$. Moreover, $\mathcal{B}_1(0) = \mathcal{S}^*$. In 2015, Thomas [13] proved the sharp bound $|a_2a_4 - a_3^2| \leq 4/(2+\alpha)^2$ for functions in the class $\mathcal{B}_1(\alpha)$ for $\alpha \in [0,1]$. In 2017, Marjono *et al.* [6] investigated the sharp upper bound on fifth and sixth coefficients. They also conjectured that if $f \in \mathcal{B}_1(\alpha)$, then

$$|a_n| \leq \frac{2}{n-1+\alpha} \quad (n = 2, 3, 4, \cdots)$$

holds for all $\alpha \geq 1$. This conjecture for the fifth coefficient, for certain range of α, was recently settled by Cho and Kumar [1]. For many results related to the Bazilevič functions we refer the reader to the papers [11, 14, 15, 17] and the references cited therein. A class $\mathcal{B}(\alpha,\beta)$ with stronger conditions was considered by Ponnusamy [8]. For $\alpha > 0$ and $0 < \beta < 1$, he defined

$$\mathcal{B}(\alpha,\beta) := \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} - 1 \right| < \beta \right\}.$$

For the negative value of $\alpha \in (-1,0)$, the class $\mathcal{B}(\alpha,\beta)$ can be rewritten as

$$\bar{\mathcal{B}}(\alpha,\beta) := \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{z}{f(z)} \right)^{\alpha+1} - 1 \right| < \beta \right\}.$$

This class was introduced and investigated by Obradović, in 1998. He obtained the conditions on the parameter β that embeds this class into the class of starlike functions. Later in 2002, Tuneski and Darus [16], for $0 < \alpha < 1$, considered the class

$$\bar{\mathcal{B}}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(f'(z) \left(\frac{z}{f(z)} \right)^{\alpha+1} \right) > 0 \right\}.$$

This class, as mentioned by Obradović in the conference "Computational Methods and Function Theory 2001" is called to be class of functions of non-Bazilevič type, see [16]. Tuneski and Darus investigated the sharp bounds on $|a_2|$ and the Fekete-Szegö functional $|a_3 - \mu a_2^2|$. Some typographical errors in the result [16, Theorem 1, p. 64] were reported by Kumar and Kumar [5]. For a more general result and the correct version of their result one can refer to [5]. Starlikeness of multivalent non-Bazilevič functions were investigated by Guo *et al.* [2]. Estimate on the second Hankel determinant for the class of functions $f \in \mathcal{A}$ satisfying $\operatorname{Re}\left(f'(z)\left(z/f(z)\right)^{\alpha}\right) > 0$ for $\alpha \in (0, 1/3]$ was obtained by Krishna and Reddy [4].

Motivated by the above works, in this paper, sharp bound on the third to eighth coefficients of functions in the class $\bar{\mathcal{B}}(\alpha)$ are investigated. Moreover, sharp bound on the functional $|a_2a_3 - a_4|$ for functions in the class $\bar{\mathcal{B}}(\alpha)$ is also obtained.

Let $\mathcal{P}$ be the class of analytic functions having the Taylor series of the form $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ and mapping the unit disk $\mathbb{D}$ onto the right-half of the complex plane i.e. satisfying the condition $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. Let $\mathbf{B}$ be the class of Schwarz functions consisting of analytic functions of the form $w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$ $(z \in \mathbb{D})$

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and satisfying the condition $|w(z)| < 1$ for $z \in \mathbb{D}$. The following correspondence between the classes $\mathbf{B}$ and $\mathcal{P}$ holds:

$$p \in \mathcal{P} \text{ if and only if } w(z) = \frac{p(z)-1}{p(z)+1} \in \mathbf{B}. \tag{1.2}$$

Comparing coefficients in (1.2), we have

$$c_1 = \frac{p_1}{2},\ c_2 = \frac{2p_2 - p_1^2}{4},\ c_3 = \frac{4p_3 - 4p_1p_2 + p_1^3}{8},\ c_4 = \frac{8p_4 - 8p_1p_3 - 4p_2^2 + 6p_1^2p_2 - p_1^4}{16}. \tag{1.3}$$

Lemma 1.1. [3](see also [10]) *If $p \in \mathcal{P}$, then, for any complex number ν,*

$$|p_2 - \nu p_1^2| \leq 2\max\{1; |2\nu - 1|\}$$

and the equality holds for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Consider the functional $\Psi(\mu,\nu) = |c_3 + \mu c_1 c_2 + \nu c_1^3|$ for $w \in \mathbf{B}$ and $\mu, \nu \in \mathbb{R}$. Let us assume that the symbols Ω_k's are defined as follows:

$$\Omega_1 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \leq 1/2,\ \ |\nu| \leq 1\right\},$$

$$\Omega_2 := \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \leq |\mu| \leq 2,\ \ \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leq \nu \leq 1\right\},$$

$$\Omega_3 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \leq \frac{1}{2},\ \ \nu \leq -1\right\}, \Omega_4 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \geq 1/2,\ \ \nu \leq -\frac{2}{3}(|\mu|+1)\right\},$$

$$\Omega_5 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \leq 2,\ \ \nu \geq 1\right\}, \Omega_6 := \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \leq |\mu| \leq 4,\ \ \nu \geq \frac{1}{12}(\mu^2+8)\right\},$$

$$\Omega_7 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \geq 4,\ \nu \geq \frac{2}{3}(|\mu|-1)\right\},$$

$$\Omega_8 := \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \leq |\mu| \leq 2,\ -\frac{2}{3}(|\mu|+1) \leq \nu \leq \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1)\right\},$$

$$\Omega_9 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \geq 2,\ -\frac{2}{3}(|\mu|+1) \leq \nu \leq \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\right\},$$

$$\Omega_{10} := \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \leq |\mu| \leq 4,\ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \leq \nu \leq \frac{1}{12}(\mu^2+8)\right\},$$

$$\Omega_{11} := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \geq 4,\ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \leq \nu \leq \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4}\right\},$$

$$\Omega_{12} := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \geq 4,\ \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4} \leq \nu \leq \frac{2}{3}(|\mu|-1)\right\}.$$

The following result is due to Prokhorov and Szynal [9] which we need in our investigation.

Lemma 1.2. [9, Lemma 2, p. 128] *If $w \in \mathbf{B}$, then for any real numbers μ and ν, we have*

$$|\Psi(\mu,\nu)| \leq \begin{cases} 1, & (\mu,\nu) \in \Omega_1 \cup \Omega_2 \cup \{(2,1)\}; \\ |\nu|, & (\mu,\nu) \in \bigcup_{k=3}^{7} \Omega_k; \\ \frac{2}{3}(|\mu|+1)\left(\frac{|\mu|+1}{3(|\mu|+\nu+1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_8 \cup \Omega_9; \\ \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\left(\frac{\mu^2-4}{3(\nu-1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_{10} \cup \Omega_{11} \setminus \{(2,1)\}; \\ \frac{2}{3}(|\mu|-1)\left(\frac{|\mu|-1}{3(|\mu|-\nu-1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w_1(z) = z^3, \quad w_2(z) = z, \quad w_3(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \quad w_4(z) = \frac{z(t_2 + z)}{1 + t_2 z}$$

and $w_5(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, where the parameters t_1, t_2 and the coefficients c_i are given by

$$t_1 = \left(\frac{|\mu|+1}{3(|\mu|+\nu+1)}\right)^{1/2}, \quad t_2 = \left(\frac{|\mu|-1}{3(|\mu|-\nu-1)}\right)^{1/2}, \quad c_1 = \left(\frac{2\nu(\mu^2+2)-3\mu^2}{3(\nu-1)(\mu^2-4\nu)}\right)^{1/2},$$

$$c_2 = (1-c_1^2)e^{i\theta_0}, \quad c_3 = -c_1 c_2 e^{i\theta_0}, \quad \theta_0 = \pm \arccos\left[\frac{\mu}{2}\left(\frac{\nu(\mu^2+8)-2(\mu^2+2)}{2\nu(\mu^2+2)-3\mu^2}\right)^{1/2}\right].$$

2. Coefficient Estimates

The following theorem gives the sharp estimates on $|a_3|$, $|a_4|$ and on the functional $|a_2a_3 - a_4|$ for functions in the class $\bar{\mathcal{B}}(\alpha)$.

Theorem 2.1. *Let $\alpha_0 \approx 2.36, \alpha_1 \approx 2.68$ and $\alpha_2 \approx 2.71$ are the smallest positive roots of the equations $3\alpha^4 - 11\alpha^3 + \alpha^2 + 11\alpha + 20 = 0$, $\alpha^6 - 11\alpha^5 + 56\alpha^4 - 138\alpha^3 + 151\alpha^2 - 7\alpha - 148 = 0$ and $\alpha^3 - 5\alpha^2 + 11\alpha - 13 = 0$, respectively. Let $f \in \bar{\mathcal{B}}(\alpha)$ has the from (1.1). Then, the following sharp inequalities hold:*

$$|a_3| \leq \begin{cases} \frac{2}{\alpha-2}, & \text{if } \alpha \in (0,3] \setminus \{1,2\}; \\ \frac{2(\alpha-3)}{(\alpha-2)(\alpha-1)^2}, & \text{if } \alpha > 3, \end{cases} \tag{2.1}$$

$$|a_4| \leq \begin{cases} \frac{2\left(\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36\right)}{3(\alpha-1)(\alpha-2)(\alpha-3)}, & \text{if } \alpha \in (0,\alpha_0] \setminus \{1,2\} \text{ or } \alpha_2 \leq \alpha < 3; \\ \frac{4(|a|-1)^{3/2}}{3(\alpha-3)(|a|-b-1)^{1/2}}, & \text{if } \alpha_0 \leq \alpha \leq \alpha_1; \\ \frac{2(\alpha-1)^2(a^2-4)^{3/2}}{(\alpha-3)(a^2-4b)(3(b-1))^{1/2}}, & \text{if } \alpha_1 \leq \alpha \leq \alpha_2; \\ \frac{2}{\alpha-3}, & \text{if } 3 < \alpha, \end{cases} \tag{2.2}$$

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where a and b are given by

$$a := -\frac{2(\alpha-5)}{(\alpha-1)(\alpha-2)} \quad \text{and} \quad b := \frac{\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36}{3(\alpha-1)^3(\alpha-2)}.$$

Proof. Since $f \in \bar{\mathcal{B}}(\alpha)$, it follows that there exists $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots \in \mathcal{P}$ such that

$$f'(z)\left(\frac{z}{f(z)}\right)^{\alpha+1} = p(z). \tag{2.3}$$

Comparing coefficients of like-power terms in (2.3), we get

$$a_2 = -\frac{p_1}{\alpha-1} \quad \text{and} \quad a_3 = \frac{(\alpha-2)(\alpha+1)p_1^2-2(\alpha-1)^2p_2}{2(\alpha-2)(\alpha-1)^2}. \tag{2.4}$$

Now consider

$$\begin{aligned} a_3 &= \frac{(\alpha-2)(\alpha+1)p_1^2-2(\alpha-1)^2p_2}{2(\alpha-2)(\alpha-1)^2} \\ &= -\frac{1}{\alpha-2}\left[p_2-\frac{(\alpha-2)(\alpha+1)}{2(\alpha-1)^2}p_1^2\right]. \end{aligned} \tag{2.5}$$

An application of Lemma 1.1 on (2.5), gives

$$|a_3| \le \frac{2}{\alpha-2}\max\left\{1; \frac{|\alpha-3|}{(\alpha-1)^2}\right\}$$

which equivalently can be written as

$$|a_3| \le \begin{cases} \frac{2}{\alpha-2}, & \alpha \in (0,3] \setminus \{1,2\}; \\ \frac{2(\alpha-3)}{(\alpha-2)(\alpha-1)^2}, & \alpha > 3. \end{cases}$$

This is the required bound on third coefficient as stated in the theorem. In the first case of (2.1), equality occurs for the function $f_0 \in \bar{\mathcal{B}}(\alpha)$ defined by

$$f_0'(z)\left(\frac{z}{f_0(z)}\right)^{\alpha+1} = \frac{1+z^2}{1-z^2}, \tag{2.6}$$

whereas in the second case of (2.2), equality holds for the function $\tilde{f}_0 \in \bar{\mathcal{B}}(\alpha)$ defined by

$$\tilde{f}_0{}'(z)\left(\frac{z}{\tilde{f}_0(z)}\right)^{\alpha+1} = \frac{1+z}{1-z}. \tag{2.7}$$

Next we shall find the estimate on $|a_4|$. From (2.3), we have

$$a_4 = \frac{-(\alpha-3)(\alpha-2)(\alpha+1)(2\alpha+1)p_1^3+6(\alpha-1)^2(\alpha-3)(\alpha+1)p_1p_2-6(\alpha-2)(\alpha-1)^3p_3}{6(\alpha-1)^3(\alpha-2)(\alpha-3)}. \tag{2.8}$$

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In view of the interconnections in (1.2) and (1.3), Eqn. (2.8) can be rewritten as:

$$a_4 = -\frac{2\left[(\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36)\,c_1^3-6(\alpha-5)(\alpha-1)^2c_1c_2+3(\alpha-2)(\alpha-1)^3c_3\right]}{3(\alpha-1)^3(\alpha-2)(\alpha-3)} \tag{2.9}$$

or equivalently

$$a_4 = -\frac{2}{\alpha-3}\left[c_3+ac_1c_2+bc_1^3\right],$$

where the parameters a and b are given by

$$a := -\frac{2(\alpha-5)}{(\alpha-1)(\alpha-2)} \text{ and } b := \frac{\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36}{3(\alpha-1)^3(\alpha-2)}. \tag{2.10}$$

Assume that Ω_i's are defined as in Lemma 1.2 with the settings $\mu = a$ and $\nu = b$. We now proceed further in the proof with the following steps:

(1) Assume that $\alpha \geq (\sqrt{73}-1)/2 \approx 3.772$. In this case, we see that $-1/2 \leq a \leq 1/2$ holds. Moreover, $b \leq 1$ holds if and only if $\alpha^4-5\alpha^3+8\alpha^2-\alpha-15 \geq 0$, which holds for all $\alpha \geq 3$. Thus for all $\alpha \geq (\sqrt{73}-1)/2$, we conclude that $(a,b) \in \Omega_1$.

(2) Next assume that $3 < \alpha \leq (\sqrt{73}-1)/2$. Then, we see that the condition $-1/2 \leq a \leq 2$ holds for all such α and $(4/27)(a+1)^3-(a+1) \leq b \leq 1$ all $\alpha > 3$. Therefore, for $3 < \alpha \leq (\sqrt{73}-1)/2$, we must have $(a,b) \in \Omega_2$.

(3) Let

$$\alpha_2 := \frac{1}{3}\left(\sqrt[3]{53+9\sqrt{41}}-\frac{8}{\sqrt[3]{53+9\sqrt{41}}}+5\right) \approx 2.71$$

and

$$\alpha_0 := \frac{11}{12}+\frac{1}{12}\sqrt{4\ (2^{2/3})\sqrt[3]{8989+9\sqrt{14717}}+4\sqrt[3]{35956-36\sqrt{14717}}+113}-\frac{1}{2}\sqrt{\hat{C}+\hat{D}} \approx 2.36$$

with

$$\hat{C} := \frac{113}{18}-\frac{1}{9}(2^{2/3})\sqrt[3]{8989+9\sqrt{14717}}-\frac{1}{9}\sqrt[3]{35956-36\sqrt{14717}},$$

and

$$\hat{D} := \frac{407}{18\sqrt{4\ (2^{2/3})\sqrt[3]{8989+9\sqrt{14717}}+4\sqrt[3]{35956-36\sqrt{14717}}+113}}$$

are the smallest positive roots of the equations $\alpha^3-5\alpha^2+11\alpha-13=0$ and $3\alpha^4-11\alpha^3+\alpha^2+11\alpha+20=0$, respectively. Now assume that $0<\alpha<1$ or $2<\alpha\leq\alpha_0$. Then $a \geq 4$ and $b \geq 2(a-1)/3$ hold and hence $(a,b) \in \Omega_7$. Moreover, $a \leq -1/2$ and $b \leq -2(-a+1)/3$ holds whenever $1<\alpha<2$. Therefore, $(a,b) \in \Omega_4$ whence $1<\alpha<2$. Also it can be easily seen that $2 \leq a \leq 4$ and $b \geq (a^2+8)/12$ hold for $\alpha_2 \leq \alpha < 3$.

(4) Let $2.69 \approx (5+\sqrt{33})/4 \leq \alpha \leq \alpha_0$. Then a and b satisfy $2 \leq a \leq 4$ and
$$\frac{2a(a+1)}{a^2+2a+4} \leq b \leq \frac{a^2+8}{12}.$$
Therefore, for this range of α, we see that $(a,b) \in \Omega_{10}$. Let $\alpha_1 \approx 2.68$ is the smallest positive root of $\alpha^6 - 11\alpha^5 + 56\alpha^4 - 138\alpha^3 + 151\alpha^2 - 7\alpha - 148 = 0$. Further, when $\alpha_1 \leq \alpha \leq (\sqrt{33}+5)/4$, the parameters a and b satisfy $a \geq 4$ and
$$\frac{2a(a+1)}{a^2+2a+4} \leq b \leq \frac{2a(a-1)}{a^2-2a+4}.$$
Hence, in view of Lemma 1.2, we have $(a,b) \in \Omega_{11}$.

(5) Assume that $\alpha_0 \leq \alpha \leq \alpha_1$. In this case, it is a simple matter to check that $a \geq 4$ and
$$\frac{2a(a-1)}{a^2-2a+4} \leq b \leq \frac{2(a-1)}{3}.$$
Therefore, Lemma 1.2 gives $(a,b) \in \Omega_{12}$.

In the light of the above discussions, an application of Lemma 1.2 gives the desired estimates on $|a_4|$. In the first case of (2.2), the equality holds for the function f_0 defined in (2.6), whereas in the forth case of (2.2), the equality holds for the function function $\tilde{f}_0$ defined in (2.7). In the case third of (2.2), the extremal function f_1 is given by
$$f_1'(z)\left(\frac{z}{f_1(z)}\right)^{\alpha+1} = \frac{1+w(z)}{1-w(z)} \tag{2.11}$$
with choice of the Schwarz function (up to rotation) $w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots \in \mathbf{B}$, where the coefficients c_i are given by
$$c_1 = \left(\frac{2b(a^2+2)-3a^2}{3(b-1)(a^2-4b)}\right)^{1/2},\ c_2 = (1-c_1^2)e^{i\theta_0},\ c_3 = -c_1c_2e^{i\theta_0},$$
with
$$\theta_0 = \pm \arccos\left[\frac{a}{2}\left(\frac{b(a^2+8)-2(a^2+2)}{2b(a^2+2)-3a^2}\right)^{1/2}\right],$$
where a and b are given by (2.10). Finally, in the second case of (2.2), the equality holds for the function $\tilde{f}_1$ defined by
$$\tilde{f}_1'(z)\left(\frac{z}{\tilde{f}_1(z)}\right)^{\alpha+1} = \frac{1+w(z)}{1-w(z)} \tag{2.12}$$
with the Schwarz function given by $w(z) = z(\kappa+z)/(1+\kappa z)$, where
$$\kappa := \left(\frac{|a|-1}{3\,(|a|-b-1)}\right)^{1/2}.$$
This completes the proof. ∎

The following theorem provides sharp bound on the fifth, sixth, seventh and eighth coefficients for functions in the class $\bar{\mathcal{B}}(\alpha)$.

Theorem 2.2. *Let us denote*

$$\Psi := 2\alpha^6 - 28\alpha^5 + 137\alpha^4 - 331\alpha^3 + 437\alpha^2 - 433\alpha + 360,$$

$$\hat{\Psi} := -6\alpha^9+96\alpha^8-674\alpha^7+2836\alpha^6-8942\alpha^5+22504\alpha^4-40886\alpha^3+45124\alpha^2-30132\alpha+21600,$$

$$\begin{aligned}\chi := 23\alpha^{12} - 756\alpha^{11} + 10218\alpha^{10} - 77686\alpha^9 + 376014\alpha^8 - 1243398\alpha^7 + 2969824\alpha^6 \\ - 5401638\alpha^5 + 7729083\alpha^4 - 8432486\alpha^3 + 6389238\alpha^2 - 3333636\alpha + 1360800,\end{aligned}$$

and

$$\begin{aligned}\hat{\chi} := -(45\alpha^{15} - 1530\alpha^{14} + 23641\alpha^{13} - 221500\alpha^{12} + 1438032\alpha^{11} - 7061480\alpha^{10} + 27696314\alpha^9 \\ - 88000680\alpha^8 + 222370901\alpha^7 - 435300650\alpha^6 + 653299149\alpha^5 - 763502860\alpha^4 \\ + 703545502\alpha^3 - 473136900\alpha^2 + 206026416\alpha - 76204800).\end{aligned}$$

If $f \in \bar{\mathcal{B}}(\alpha)$ has the from (1.1)*, then for $0 < \alpha < 1$, the following sharp inequalities hold:*

$$|a_5| \leq \frac{2\Psi}{3(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^4},$$

$$|a_6| \leq \frac{\hat{\Psi}}{15(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^5},$$

$$|a_7| \leq \frac{2\chi}{45(\alpha-6)(\alpha-5)(\alpha-4)(\alpha-3)^2(\alpha-2)^3(\alpha-1)^6}$$

and

$$|a_8| \leq \frac{2\hat{\chi}}{315(\alpha-7)(\alpha-6)(\alpha-5)(\alpha-4)(\alpha-3)^2(\alpha-2)^3(\alpha-1)^7}.$$

Proof. From (2.3), on comparing the coefficients, we have

$$a_5 = \frac{\tau_1 p_4 + \tau_2 p_1^2 p_2 + \tau_3 p_2^2 + \tau_4 p_1 p_3 + \tau_5 p_1^4}{24(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^4}, \tag{2.13}$$

where τ_i's are given by

$$\begin{aligned}\tau_1 := -24(\alpha-3)(\alpha-2)^2(\alpha-1)^4, \tau_2 := -12(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)^2(\alpha+1)(2\alpha+1), \\ \tau_3 := 12(\alpha-3)(\alpha-4)(\alpha-1)^4(\alpha+1), \tau_4 := 24(\alpha-4)(\alpha-2)^2(\alpha-1)^3(\alpha+1), \\ \tau_5 := (\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha+1)(2\alpha+1)(3\alpha+1).\end{aligned}$$

Similarly, the sixth coefficient is given by

$$a_6 = -\frac{\hat{\tau}_1 p_5 + \hat{\tau}_2 p_2^2 p_1 + \hat{\tau}_3 p_2 p_3 + \hat{\tau}_4 p_1^3 p_2 + \hat{\tau}_5 p_1^2 p_3 + \hat{\tau}_6 p_1 p_4 + \hat{\tau}_7 p_1^5}{120(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^5}, \tag{2.14}$$

where $\hat{\tau}_i$'s are defined by

$$\hat{\tau}_1 := 120(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^5, \hat{\tau}_2 := 60(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-1)^4(\alpha+1)(2\alpha+1),$$

$$\hat{\tau}_3 := -120(\alpha-5)(\alpha-4)(\alpha-2)(\alpha-1)^5(\alpha+1),$$

$$\hat{\tau}_4 := -20(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)^2(\alpha+1)(2\alpha+1)(3\alpha+1),$$

$$\hat{\tau}_5 := 60(\alpha-5)(\alpha-4)(\alpha-2)^2(\alpha-1)^3(\alpha+1)(2\alpha+1),$$

$$\hat{\tau}_6 := -120(\alpha-5)(\alpha-3)(\alpha-2)^2(\alpha-1)^4(\alpha+1),$$

$$\hat{\tau}_7 := (\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha+1)(2\alpha+1)(3\alpha+1)(4\alpha+1).$$

To find the estimate on $|a_5|$, we observe from (2.13) that the coefficients τ_i $(i = 1, 2, 3, 4, 5, 6, 7)$ of $p_4, p_1^2 p_2, p_2^2, p_1 p_3$ and p_1^4 are positive. Hence applying triangle inequality in (2.13) and using the fact that $|p_j| \leq 2$, we get the required estimate on $|a_5|$. A similar argument can be used to obtained the estimates on $|a_6|$, $|a_7|$ and $|a_8|$. In all the cases, equality hold for the function $\tilde{f}_0$ given by (2.7). This completes the proof. ∎

The following theorem gives the sharp bound on the functional $|a_2a_3 - a_4|$ for the functions in the class $\bar{\mathcal{B}}(\alpha)$.

Theorem 2.3. *Let $f \in \bar{\mathcal{B}}(\alpha)$ has the form* (1.1)*. Then, the following sharp result holds:*

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{2(\alpha^3-4\alpha^2+\alpha+18)}{3(\alpha-1)^2(\alpha-2)(\alpha-3)}, & \text{if } \alpha \in (0,2) \setminus \{1\}; \\ \frac{2(\alpha^3-4\alpha^2+\alpha+18)}{3(\alpha-1)^2(\alpha-2)(3-\alpha)}, & \text{if } 2 < \alpha < 3; \\ \frac{2}{\alpha-3}, & \text{if } \alpha > 3. \end{cases} \tag{2.15}$$

Proof. Proceeding as in the proof of previous theorem and using (2.4) and (2.9), we can write

$$a_2a_3 - a_4 = \frac{2\left[(\alpha^3 - 4\alpha^2 + \alpha + 18)\, c_1^3 + 12(\alpha-1)c_1c_2 + 3(\alpha-2)(\alpha-1)^2 c_3\right]}{3(\alpha-3)(\alpha-2)(\alpha-1)^2}. \tag{2.16}$$

By setting

$$s := \frac{4}{(\alpha-1)(\alpha-2)} \quad \text{and} \quad t := \frac{\alpha^3 - 4\alpha^2 + \alpha + 18}{3(\alpha-2)(\alpha-1)^2}$$

the expression in (2.16) can be written as

$$a_2a_3 - a_4 = \frac{2}{\alpha-3}\left[c_3 + sc_1c_2 + tc_1^3\right].$$

Assume that the symbols Ω_i's are as defined in Lemma 1.2 with the settings $\mu = s$ and $\nu = t$. Now the proof is accomplished in the following steps:

(1) Let $(3+\sqrt{33})/2 \leq \alpha$. Then it can be easily verified that

$$-\frac{1}{2} \leq s \leq \frac{1}{2} \text{ and } -1 \leq t \leq 1.$$

Therefore, for the range $(3+\sqrt{33})/2 \leq \alpha$, we have $(s,t) \in \Omega_1$. Further, when $3 < \alpha \leq (3+\sqrt{33})/2$, wee see that $(s,t) \in \Omega_2$.

(2) Let $0 < \alpha < 1$ or $1 < \alpha < 2$. Then in a similar way we have $(s,t) \in \Omega_4$. Further if $(3+\sqrt{5})/2 \leq \alpha < 3$, then $(s,t) \in \Omega_6$ and when $2 < \alpha \leq (3+\sqrt{5})/2$, then $(s,t) \in \Omega_7$.

In the light of the above discussions, an application of Lemma 1.2, establish the required estimate on $|a_2a_3 - a_4|$. In the first two cases of (2.15), the equality hold for the function $\tilde{f}_0 \in \bar{\mathcal{B}}(\alpha)$ defined by (2.7). In the third case of (2.15), the equality holds for the function f_2 defined by

$$f_2'(z)\left(\frac{z}{f_2(z)}\right)^{\alpha+1} = \frac{1+z^3}{1-z^3}. \tag{2.17}$$

This completes the proof. ∎

Remark 2.4. It would be interesting to find out the sharp bound on $|a_i|$ $(i = 5,6,7,8)$ for the functions $\tilde{f} \in \bar{\mathcal{B}}(\alpha)$ in the case when $\alpha > 1$.

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A new extragradient method for the split feasibility and fixed point problems *

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Abstract: In this paper, we propose a new extragradient method with regularization for finding a common element of the solution set Γ of the split feasibility problem and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in infinite-dimensional Hilbert spaces, combining the regularization method and the technique of averaged operator, we prove the sequences generated by the proposed algorithm converge weakly to an element of $\mathrm{Fix}(S)\bigcap\Gamma$ under mild conditions.

Keywords: split feasibility problem , extragradient, regularization.

1. Introduction

Throughout this paper, let H be a Hilbert space, $\langle\cdot,\cdot\rangle$ denotes the inner product, and $\|\cdot\|$ denotes for the corresponding norm. The split feasibility problem (SFP) which was first introduced by Censor and Elfving [1] in 1994 for modeling inverse problems arising from phase retrievals and in medical image reconstruction. Let C and Q be closed convex sets in the infinite-dimensional real Hilbert spaces H_1 and H_2, respectively. The SFP is to find a vector x^* satisfying

$$x^* \in C \ \text{ such that } \ Ax^* \in Q, \tag{1.1}$$

where $A \in B(H_1, H_2)$ which denotes the family of all bounded linear operators from H_1 to H_2. Some related work in the infinite-dimensional setting can be found in [2, 3, 4, 5, 9, 10, 12] and the references therein.

Many methods have been developed to solve the SFP, The basic algorithm have CQ algorithm proposed by Byrne [2], the relaxed CQ algorithm proposed by Yang [9], the half-space relaxation projection method proposed by Qu and Xiu [11], the variable Krasnosel skii-Mann algorithm proposed by Xu [12]. The projections of a point onto C and Q are difficult to compute when C and Q fail to have closed-form expressions, though theoretically we can prove the (weak) convergence of the algorithm.

Very recently, Xu [6] gave a continuation of the study on the CQ algorithm and its convergence. He applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm which was proved to be weakly convergent to a solution of the SFP. On the other hand, Korpelevich

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[7] introduced the so-called extragradient method for finding a solution of a saddle point problem. He proved that the sequences generated by the proposed iterative algorithm converge to a solution of a saddle.

Motivated by the idea of an extragradient method, Nadezhina and Takahashi [8] introduced an iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of a variational inequality problem [13] for a monotone, Lipschitz continuous mapping in a real Hilbert space. They obtained a weak convergence theorem for two sequence generated by the proposed algorithm.

In our paper, we introduce and analyze a new extragradient iterative algorithm to find a common element of the solution set Γ of the split feasibility problem and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in infinite-dimensional Hilbert spaces, furthermore, we prove its convergence. The results of this paper represent the improvement of the corresponding results in [6] and [14].

2. Preliminaries

Throughout this paper, we use $x_n \to x$ and $x_n \rightharpoonup x$ to denote strong and weak convergence to x of the sequence x_n, respectively. Let K be a nonempty closed convex subset of H. Recall that the projection (nearest point or metric) from H onto K, denoted by P_K, is defined in such a way that, for each $x \in H$, $P_K x$ is the unique point in K with the property

$$\| x - P_K x \| = \inf_{y \in K} \| x - y \| =: d(x, K),$$

i.e.

$$P_K(x) = \operatorname{argmin}\{\| x - y \| \mid y \in K\}.$$

Some important properties of projections are gathered in the following Lemma.

Lemma 2.1 *For given $x \in H$ and $z \in K$, the following properties hold:*
(1) $x \in K \Leftrightarrow P_K(x) = x$;
(2) $\langle x - P_K(x), z - P_K(x)\rangle \leq 0, \ \forall x \in H$ *and* $\forall z \in K$;
(3) $\langle x - y, P_K(x) - P_K(y)\rangle \geq \|P_K(x) - P_K(y)\|^2, \ \forall x, \ y \in H$;
(4) $\|P_K(x) - z\|^2 \leq \|x - z\|^2 - \|P_K(x) - x\|^2, \ \forall x \in H$ *and* $\forall z \in K$;
(5) $\|P_K(x) - P_K(y)\| \leq \|x - y\|, \ \forall x, \ y \in H$.

Proof. See Facchinei and Pang [15].

Definition 2.1 Let T be a mapping from $K \subseteq H$ into H, then
(a) T is called monotone on K if

$$\langle T(x) - T(y), x - y\rangle \geq 0, \ \forall \ x, \ y \in K.$$

(b) T is called strongly monotone on K if there is a $\mu > 0$, such that

$$\langle T(x) - T(y), x - y\rangle \geq \mu\|x - y\|^2, \ \forall \ x, \ y \in K.$$

(c) F is called co-coercive (or ν-inverse strongly monotone) on K if there is a $\nu > 0$, such that

$$\langle T(x) - T(y), x - y\rangle \geq \nu\|T(x) - T(y)\|^2, \ \forall \ x, \ y \in K.$$

(d) F is called pseudo-monotone on K if

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq 0, \ \forall \ x, \ y \in K.$$

(e) T is called Lipschitz continuous on K if there exists a constant $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\|, \ \forall \ x, \ y \in K.$$

Definition 2.2 A mapping $T : H \to H$ is said to be:
(a) nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|, \ \forall x, \ y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \geq \| Tx - Ty \|, \ \forall x, \ y \in H,$$

or alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S)$$

where $S : H \to H$ is nonexpansive.
Remark 2.1 From Lemma 2.1 and Definition 2.1-2.2, we can infer that if S is nonexpansive, then I-S is monotone; A monotone mapping is pseudo-monotone mapping; An inverse strongly monotone mapping is monotone and Lipschitz continuous; A Lipschitz continuous and strongly monotone mapping is an inverse strongly monotone mapping; The projection operator is 1-ism and nonexpansive.
Lemma 2.2 *A mapping T is 1-ism if and only if the mapping I-T is 1-ism, where I is the identity operator.*
Proof. See [16, Lemma 2.3].
Remark 2.2 If T is an inverse strongly monotone mapping, then T is a nonexpansive mapping.

Definition 2.3 A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping S, that is,

$$T = (1 - \alpha)I + \alpha S \tag{2.1}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (for example, projections) are $\frac{1}{2}$-averaged mappings.
Proposition 2.1 *([16]). Let $T : H \to H$ be a given mapping:*
(1) T is nonexpansive if and only if the complement I-T is $\frac{1}{2}$-ism.
(2) If T is μ-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.
(3) T is averaged if and only if the complement I-T is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if I-T is $\frac{1}{2\alpha}$-ism.
Proposition 2.2 *([16, 17]). Let $S, \ T, \ V : H \to H$ be given operators.*
(1) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.
(2) T is firmly nonexpansive if and only if the complement I-T is firmly nonexpansive.
(3) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.
(4) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \ \alpha_2 \in (0, 1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.
(5) If the mapping $\{T_i\}_i^N$ are averaged and have a common fixed point, then

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \circ \cdots \circ T_N).$$

The notation $\mathrm{Fix}(T)$ *denotes the set of all fixed points of the mapping* T, *that is* $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

The so-called demiclosedness principle plays an important role in our argument.

Definition 2.4 Let $T : H \to H$ be an operator. We say that I-T is demiclosed (at zero), if for any sequence x_n in H, there holds the following implication:

$$x_n \rightharpoonup x \text{ and } (I\text{-}T)x_n \to 0 \Rightarrow (I\text{-}T)x = 0.$$

Lemma 2.3 *([18]). Let* H *be a Hilbert space. Then for all* $x,\ y \in H$ *and* $\lambda \in [0,1]$,

$$\| \lambda x + (1-\lambda)y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2 .$$

Lemma 2.4 *([19]). Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n,\ \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

Corollary 2.1 *([20]). Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n,\ \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

Recall that a Banach space X is said to satisfy the Opial condition [22] if for any sequence $\{x_n\}$ in X the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that the inequality

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \|$$

holds for every $y \in X$ with $y \ne x$.

It is well-known that every Hilbert space satisfies the Opial condition.

3. Main results

Throughout this paper, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty.

It is easy to see that SFP is equivalent to the following minimization problem

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2, \tag{3.1}$$

where $f : H_1 \to R$ is a continuous differentiable function, however it is ill-posed. Therefore, Xu [6] considered the following Tikhonov regularized problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{3.2}$$

where $\alpha > 0$ is the regularization parameter.

We observe that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I \tag{3.3}$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone.

Proposition 3.1 *([21]) Given* $x^* \in H_1$, *the following statements are equivalent:*

(1) x^* *solves the SFP;*

(2) x^* *solves the fixed point equation*

$$P_C(I - \lambda\nabla f) = P_C[I - \lambda A^*(I - P_Q)A]x^* = x^* \qquad (3.4).$$

(3) *x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C, \tag{3.5}$$

where $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A.

Remark 3.1. It is clear from Proposition 3.1 that

$$\Gamma = \mathrm{Fix}(P_C(I - \lambda \nabla f)) = VI(C, \nabla f)$$

for any $\lambda > 0$, where $\mathrm{Fix}(P_C(I - \lambda \nabla f))$ and $VI(C, \nabla f)$ denote the set of fixed points of $P_C(I - \lambda \nabla f)$ and the solution set of VIP(3.5).

Next, we will present our method for solving the SFP and prove its convergence.

Theorem 3.1 *Let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$ in Hilbert space. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences in C generated by the following extragradient algorithm:*

$$\begin{cases} x_0 \in C \ \ \text{chosen arbitrarily}, \\ z_n = (1 - \gamma_n)x_n + \gamma_n P_C(I - \lambda_n \nabla f_{\alpha_n})x_n, \\ y_n = (1 - \beta_n)z_n + \beta_n S P_C(I - \lambda_n f_{\alpha_n})z_n, \\ x_{n+1} = (1 - \mu_n)y_n + \mu_n S P_C(I - \lambda_n \nabla f_{\alpha_n})y_n, \ \forall n > 0, \end{cases} \tag{3.6}$$

where the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ satisfy the following conditions:
(a) $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(b) $\{\lambda_n\} \subset \left(0, \frac{1}{\|A\|^2}\right)$ *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|A\|^2}$;
(c) $\{\gamma_n\} \subset (0, 1)$, *and* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;
(d) $\{\beta_n\} \subset (0, 1)$, *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(e) $\{\mu_n\} \subset (0, 1)$, *and* $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$.
Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all converge weakly to an element $\bar{x} \in \mathrm{Fix}(S) \bigcap \Gamma$.

Proof. It [21] has been proved $P_C(I - \lambda \nabla f_\alpha)$ is ζ-averaged for each $\lambda \in \left(0, \frac{2}{\alpha + \|A\|^2}\right)$, where $\zeta = \frac{2 + \lambda\left(\alpha + \|A\|^2\right)}{4}$, so $P_C(I - \lambda \nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \subset \left(0, \frac{1}{\|A\|^2}\right)$, we have

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|A\|^2} = \lim_{0\to\infty} \frac{1}{\alpha_n + \|A\|^2}.$$

Without loss of generality, we may assume that

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\alpha_n + \|A\|^2}, \quad \forall n \geq 0.$$

Consequently, $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is ζ_n-averaged for each integer $n \geq 0$, where

$$\zeta_n = \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \in (0, 1).$$

This implies that $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is nonexpansive for all $n \geq 0$.

Next, we show the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated in Theorem 3.1 are bounded. Indeed, take a fixed $p \in \mathrm{Fix}(S) \bigcap \Gamma$ arbitrarily. Then, we get $Sp = p$ and $P_C(I - \lambda \nabla f)p = p$ for

$\lambda \in \left(0, \frac{1}{\|A\|^2}\right)$. From (3.6), it follows that

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)(x_n - p) + \gamma_n[P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - p]\| \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - p\| \\
&= (1-\gamma_n)\|x_n - p\| + \gamma_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f)p\| \\
&= (1-\gamma_n)\|x_n - p\| + \gamma_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\
&\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n(\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\
&\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\|) \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n(\|x_n - p\| + \|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\|) \\
&= \|x_n - p\| + \lambda_n \alpha_n \gamma_n \|p\|,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)(z_n - p) + \beta_n[SP_C(I - \lambda_n \nabla f_{\alpha_n})z_n - p]\| \\
&\le (1-\beta_n)\|z_n - p\| + \beta_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - p\| \\
&= (1-\beta_n)\|z_n - p\| + \beta_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f)p\| \\
&= (1-\beta_n)\|z_n - p\| + \beta_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\
&\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \\
&\le (1-\beta_n)\|z_n - p\| + \beta_n(\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\
&\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\|) \\
&\le (1-\beta_n)\|z_n - p\| + \beta_n(\|z_n - p\| + \|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\|) \\
&= \|z_n - p\| + \lambda_n \alpha_n \beta_n \|p\|,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\mu_n)(y_n - p) + \mu_n[SP_C(I - \lambda_n \nabla f_{\alpha_n})y_n - p]\| \\
&\le (1-\mu_n)\|y_n - p\| + \mu_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})y_n - p\| \\
&= (1-\mu_n)\|y_n - p\| + \mu_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})y_n - P_C(I - \lambda_n \nabla f)p\| \\
&= (1-\mu_n)\|y_n - p\| + \mu_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\
&\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \\
&\le (1-\mu_n)\|y_n - p\| + \mu_n(\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\
&\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\|) \\
&\le (1-\mu_n)\|y_n - p\| + \mu_n(\|y_n - p\| + \|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\|) \\
&= \|y_n - p\| + \lambda_n \alpha_n \mu_n \|p\| \\
&\le \|x_n - p\| + \lambda_n \alpha_n (\gamma_n + \beta_n + \mu_n)\|p\|,
\end{aligned} \tag{3.9}$$

where the last inequality follows from (3.7) and (3.8).

Since $\Sigma_{n=1}^{\infty} \alpha_n < \infty$, and $\{\lambda_n\}$, $\{\gamma_n\}$, $\{\beta_n\}$, $\{\mu_n\}$ are bounded, then from Corollary 2.1, we conclude that

$$\lim_{n \to \infty} \|x_n - p\| \text{ exists for each } p \in \mathrm{Fix}(S) \bigcap \Gamma. \tag{3.10}$$

Hence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$.

In the following, we will show

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|y_n - z_n\| = \lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|y_n - Sw_n\| = \lim_{n\to\infty} \|z_n - Sv_n\| = 0,$$

where $u_n = P_C(I - \lambda_n \nabla f_{\alpha_n})x_n$, $v_n = P_C(I - \lambda_n \nabla f_{\alpha_n})z_n$, $w_n = P_C(I - \lambda_n \nabla f_{\alpha_n})y_n$.

Note that

$$\begin{aligned}
\|u_n - p\| &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - p\| \\
&= \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\
&\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p + P_C(I - \lambda_n \nabla f)p\| \\
&\le \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\
&\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p + P_C(I - \lambda_n \nabla f)p\| \\
&\le \|x_n - p\| + \lambda_n \alpha_n \|p\|.
\end{aligned} \tag{3.11}$$

Similarly, we can obtain that

$$\|v_n - p\| \le \|z_n - p\| + \lambda_n\alpha_n\|p\| \tag{3.12}$$

and

$$\|w_n - p\| \le \|y_n - p\| + \lambda_n\alpha_n\|p\|. \tag{3.13}$$

Indeed, observe that

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-\gamma_n)(x_n - p) + \gamma_n(u_n - p)\|^2 \\
&= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p)\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n(\|x_n - p\| + \lambda_n\alpha_n\|p\|)^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|x_n - p\| + \lambda_n^2\alpha_n^2\|p\|^2) \\
&\quad -\gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&= \|x_n - p\|^2 + \alpha_n\gamma_n(2\lambda_n\|p\|\|x_n - p\| + \alpha_n\lambda_n^2\|p\|^2) - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n M_1 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2,
\end{aligned} \tag{3.14}$$

where $M_1 = \sup_{n\ge 0}\{\gamma_n(2\lambda_n\|p\|\|x_n - p\| + \alpha_n\lambda_n^2\|p\|^2)\} < \infty$ and the first inequality follows from (3.11).

Also, observe that

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(z_n - p) + \beta_n(Sv_n - p)\|^2 \\
&= (1-\beta_n)\|z_n - p\|^2 + \beta_n\|Sv_n - p)\|^2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&\le (1-\beta_n)\|z_n - p\|^2 + \beta_n\|v_n - p)\|^2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&\le (1-\beta_n)\|z_n - p\|^2 + \beta_n(\|z_n - p\| + \lambda_n\alpha_n\|p\|)^2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&= (1-\beta_n)\|z_n - p\|^2 + \beta_n(\|z_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|z_n - p\| + \lambda_n^2\alpha_n^2\|p\|^2) \\
&\quad -\beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&= \|z_n - p\|^2 + \alpha_n\beta_n(2\lambda_n\|p\|\|z_n - p\| + \alpha_n\lambda_n^2\|p\|^2) - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&\le \|z_n - p\|^2 + \alpha_n M_2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2,
\end{aligned} \tag{3.15}$$

where $M_2 = \sup_{n\ge 0}\{\beta_n(2\lambda_n\|p\|\|z_n - p\| + \alpha_n\lambda_n^2\|p\|^2)\} < \infty$ and the second inequality follows from (3.12).

And

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\mu_n)(y_n - p) + \mu_n(Sw_n - p)\|^2 \\
&= (1-\mu_n)\|y_n - p\|^2 + \mu_n\|Sw_n - p)\|^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&\le (1-\mu_n)\|y_n - p\|^2 + \mu_n\|w_n - p)\|^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&\le (1-\mu_n)\|y_n - p\|^2 + \mu_n(\|y_n - p\| + \lambda_n\alpha_n\|p\|)^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&= (1-\mu_n)\|y_n - p\|^2 + \mu_n(\|y_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\| + \lambda_n^2\alpha_n^2\|p\|^2) \\
&\quad -\mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&= \|y_n - p\|^2 + \alpha_n\mu_n(2\lambda_n\|p\|\|y_n - p\| + \alpha_n\lambda_n^2\|p\|^2) - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&\le \|y_n - p\|^2 + \alpha_n M_3 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2,
\end{aligned} \tag{3.16}$$

where $M_3 = \sup_{n\ge 0}\{\mu_n(2\lambda_n\|p\|\|y_n - p\| + \alpha_n\lambda_n^2\|p\|^2)\} < \infty$ and the second inequality follows from (3.13).

Substitute (3.14) and (3.15) into (3.16), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|x_n - p\|^2 + \alpha_n(M_1 + M_2 + M_3) - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|z_n - Sv_n\|^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2.
\end{aligned} \tag{3.17}$$

Hence, it follows that

$$\begin{aligned} &\gamma_n(1-\gamma_n)\|x_n-u_n\|^2+\beta_n(1-\beta_n)\|z_n-Sv_n\|^2+\mu_n(1-\mu_n)\|y_n-Sw_n\|^2 \\ &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(M_1+M_2+M_3). \end{aligned} \tag{3.18}$$

Since $\Sigma_{n=1}^{\infty}\alpha_n < \infty$, $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, and $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < 1$, we deuce from the existence of $\lim_{n\to\infty}\|x_n-p\|$ that

$$\lim_{n\to\infty}\|x_n-u_n\| = \lim_{n\to\infty}\|y_n-Sw_n\| = \lim_{n\to\infty}\|z_n-Sv_n\| = 0. \tag{3.19}$$

Then, utilizing (3.6) we get

$$\lim_{n\to\infty}\|z_n-x_n\| = \lim_{n\to\infty}\gamma_n\|u_n-x_n\| = 0, \tag{3.20}$$

$$\lim_{n\to\infty}\|y_n-z_n\| = \lim_{n\to\infty}\beta_n\|Sv_n-z_n\| = 0, \tag{3.21}$$

and

$$\lim_{n\to\infty}\|x_n-y_n\| = \lim_{n\to\infty}\mu_n\|Sw_n-y_n\| = 0. \tag{3.22}$$

This implies that

$$\lim_{n\to\infty}\|x_n-y_n\| = \lim_{n\to\infty}\|y_n-z_n\| = \lim_{n\to\infty}\|x_n-u_n\| = \lim_{n\to\infty}\|y_n-Sw_n\| = \lim_{n\to\infty}\|z_n-Sv_n\| = 0.$$

Furthermore, note that

$$\begin{aligned} \|Sv_n-v_n\| &\leq \|Sv_n-z_n\|+\|z_n-x_n\|+\|x_n-u_n\|+\|u_n-v_n\| \\ &= \|Sv_n-z_n\|+\|z_n-x_n\|+\|x_n-u_n\| \\ &\quad +\|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n-P_C(I-\lambda_n\nabla f_{\alpha_n})z_n\| \\ &\leq \|Sv_n-z_n\|+\|z_n-x_n\|+\|x_n-u_n\|+\|x_n-z_n\|. \end{aligned}$$

From (3.20-3.22), we can get that

$$\lim_{n\to\infty}\|u_n-v_n\| = \lim_{n\to\infty}\|Sv_n-v_n\| = 0. \tag{3.23}$$

Similarly, we can prove

$$\lim_{n\to\infty}\|u_n-w_n\| = \lim_{n\to\infty}\|Sw_n-w_n\| = 0. \tag{3.24}$$

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $\bar{x}$. Next, we will show $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$. We first show $\bar{x}\in\Gamma$, let $T=P_C(I-\lambda_n\nabla f)$, then

$$\begin{aligned} \|x_n-Tx_n\| &\leq \|x_n-u_n\|+\|u_n-Tx_n\| \\ &= \|x_n-u_n\|+\|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n-P_C(I-\lambda_n\nabla f)x_n\| \\ &\leq \|x_n-u_n\|+\|(I-\lambda_n\nabla f_{\alpha_n})x_n-(I-\lambda_n\nabla f)x_n\| \\ &= \|x_n-u_n\|+\lambda_n\alpha_n\|x_n\|. \end{aligned} \tag{3.25}$$

From $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|\alpha_n\|=0$ and $\{\lambda_n\}$, $\{x_n\}$ are bounded, we can get that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Taking into account $x_{n_i}\rightharpoonup\bar{x}$ and Definition 2.4, we obtain $\bar{x}\in\mathrm{Fix}(T)$. Thus, utilizing Remark 3.1, we have $\bar{x}\in\Gamma$. On the other hand, since

$$\lim_{n\to\infty}\|x_n-u_n\| = \lim_{n\to\infty}\|u_n-v_n\| = \lim_{n\to\infty}\|Sv_n-v_n\| = 0,$$

there is subsequence v_{n_j} of v_n that converges weakly to $\bar{x}$ and $\lim_{n\to\infty}\|Sv_{n_j}-v_{n_j}\|=0$. Then from Definition 2.4, we have $\bar{x}\in \mathrm{Fix}(S)$. Therefore, we get $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup\tilde{x}$. Then, $\tilde{x}\in \mathrm{Fix}(S)\bigcap\Gamma$. Next, we prove $\tilde{x}=\bar{x}$. Assume that $\tilde{x}\neq\bar{x}$. From the Opial condition [22], we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-\tilde{x}\| &= \liminf_{i\to\infty}\|x_{n_i}-\tilde{x}\| < \liminf_{i\to\infty}\|x_{n_i}-\bar{x}\|\\ &= \lim_{n\to\infty}\|x_n-\bar{x}\| = \liminf_{j\to\infty}\|x_{n_j}-\bar{x}\|\\ &< \liminf_{j\to\infty}\|x_{n_j}-\tilde{x}\| = \lim_{n\to\infty}\|x_n-\tilde{x}\|,\end{aligned}$$

which is a contradiction. Thus, we have $\tilde{x}=\bar{x}$. This implies $x_n\rightharpoonup\bar{x}\in\mathrm{Fix}(S)\bigcap\Gamma$. Furthermore, from $\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|z_n-x_n\|=0$, we can get $y_n\rightharpoonup\bar{x}$ and $z_n\rightharpoonup\bar{x}$. This shows that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all converge weakly to an element $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$.

Theorem 3.2 *Let $S: C\to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S)\cap\Gamma\neq\emptyset$ *in Hilbert space. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences in C generated by the following extragradient algorithm:*

$$\begin{cases} x_0\in C \quad \textit{chosen arbitrarily},\\ z_n=(1-\gamma_n)x_n+\gamma_n P_C(I-\lambda_n\nabla f)x_n,\\ y_n=(1-\beta_n)z_n+\beta_n SP_C(I-\lambda_n\nabla f)z_n,\\ x_{n+1}=(1-\mu_n)y_n+\mu_n SP_C(I-\lambda_n\nabla f)y_n,\ \forall n>0,\end{cases} \tag{3.26}$$

where the sequences of parameters $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ satisfy the following condition:
(a) $\{\lambda_n\}\subset\left(0,\frac{1}{\|A\|^2}\right)$ *and* $0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<\frac{1}{\|A\|^2}$;
(b) $\{\gamma_n\}\subset(0,1)$, *and* $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$;
(c) $\{\beta_n\}\subset(0,1)$, *and* $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;
(d) $\{\mu_n\}\subset(0,1)$, *and* $0<\liminf_{n\to\infty}\mu_n\le\limsup_{n\to\infty}\mu_n<1$.
Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all converge weakly to an element $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$.

Proof. Let α_n=0 in Theorem 3.1, then we can obtain the desired result.

Remark 3.2. Our iteration method improves the corresponding results of [6], [8] and [14].

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Behavior of Meromorphic Solutions of Composite Functional-Difference Equations *†

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Abstract In view of Nevanlinna value distribution theory, we will investigate the behavior of meromorphic solutions of four types of composite functional-difference equations, and a type of system of composite functional-difference equations, some results are obtained. Moreover, we also give some examples to show that the conditions of our theorems are accurate.

Key words: meromorphic solutions; composite functional-difference equations; behavior; growth order
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1.Introduction

Recently, with the establishment of the difference analogues of Nevanlinna value distribution theory, researchers obtained many interesting theorems about the existence and growth of solutions of difference equations, functional equations and so on([3-6]). To state the results, a number of basic definition and standard notations should be introduced. We shall assume that the reader is familiar with the standard notations and results of Nevanlinna value distribution theory such as $m(r, f(z)), n(r, f(z)), N(r, f(z))$ and $T(r, f(z))$([15,18,22]) denote the proximity function, the non-integrated counting function, the counting function and the characteristic function of $f(z)$, respectively. For the integrated counting function for distinct poles of $f(z)$ we use the notations $\overline{N}(r, f(z))$, and $N_1(r, f) = N(r, f) - \overline{N}(r, f)$.

In this article, a meromorphic function means meromorphic in the whole complex plane. Given a meromorphic function $f(z)$, recall that a meromorphic function $h(z)$ is said to be a small function of $f(z)$, if $T(r, h(z)) = S(r, f)$,where $S(r, f)$ is used to denote

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any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside of a set of r of finite logarithmic measure.

Let c be a fixed, non-zero complex number, $\Delta_c f(z) = f(z + c) - f(z)$, and $\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) = \Delta_c^{n-1} f(z + c) - \Delta_c^{n-1} f(z)$ for each integer $n \geq 2$.Equations written with the above difference operators $\Delta_c^n f(z)$ are difference equations. Let E be a subset on the positive real axis. We define the logarithmic measure of E to be

$$\log(E) = \int_{E\cap(1,+\infty)} \frac{dr}{r}.$$

A set $E \in (1, +\infty)$ is said to have finite logarithmic measure if $\log(E) < \infty$.

Difference equations have been studied in many aspects see e.g.,[1],[5-6],[17]. Some expositions consider (system of) difference equations in real domains, or discrete domain. So far, the previous researches are only on complex differential equations (systems) or difference equations (systems)[5,6], but not on composite functional-difference equations (systems). Therefore, it is very important and meaningful to study the cases of composite functional-difference equations (systems). That will be an innovative contribution of this paper.

The remainder of the paper is organised as follows. In section 2, we will study the existence of meromorphic solutions or the form on some type of composite functional-difference equations, and obtain three theorems, some examples are give to show that our results hold. In section 3, we will discuss the growth order of meromorphic solutions on some types of composite functional-difference equations or system of composite functional-difference equations, which extend the result of Theorem B.

2. Existence of meromorphic solutions of difference equations and form of difference equations

In 2003, H.Silvennoinen[21] was devoted to considering many types of composite functional equations, he got some good results, for example, the following theorem A is one of his results.

Theorem A([21]) The composite functional equation

$$f(p(z)) = \frac{a_0(z) + a_1(z)f(z)}{b_0(z) + b_1(z)f(z)}$$

where the coefficients a_i, b_j are of growth $S(r, f)$ such that $a_0(z)b_1(z) - a_1(z)b_0(z) \neq 0$ and $p(z)$ is a polynomial of $\deg p(z) = k \geq 2$, does not have meromorphic solutions.

A question is,whether or not the assertion of Theorem A remains valid, if we replace the equation

$$f(p(z)) = \frac{a_0(z) + a_1(z)f(z)}{b_0(z) + b_1(z)f(z)}$$

with the following form

$$\sum_{(i)} a_{(i)}(z)(f(z))^{i_0}(\Delta_c f(z))^{i_1} \cdots (\Delta_c^n f(z))^{i_n} = \frac{a_0(z) + a_1(z)f(p(z))}{b_0(z) + b_1(z)f(p(z))}.$$

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In this section, the authors will pay attention to considering the properties of meromorphic solutions on three types of composite functional difference equations in complex domain, and extend the results obtained by H.Silvennoinen [21] to types of composite functional-difference equations (1)-(3) of the following forms, which are different from the complex differential equations or systems of complex difference equations.

At this point we pause briefly to introduce the notation used in this paper. Let I be a finite set of multi-indexes $i=(i_0,...,i_n)$,J be a finite set of multi-indexes $j=(j_0,...,j_n)$. Difference polynomials $\Omega_1(z,f),\Omega_2(z,f)$ of a meromorphic function $f(z)$ are defined as

$$\Omega_1(z,f)=\sum_{(i)\in I}a_{(i)}(f(z))^{i_0}(\Delta_c f(z))^{i_1}\cdots(\Delta_c^n f(z))^{i_n},$$

$$\Omega_2(z,f)=\sum_{(j)\in J}b_{(j)}(f(z))^{j_0}(\Delta_c f(z))^{j_1}\cdots(\Delta_c^n f(z))^{j_n},$$

where each $\{a_{(i)}(z)\},\{b_{(j)}(z)\}$ is a small meromorphic function with respect to f.

We denote that

$$u_1=\max\{\sum_{l=0}^{n}(l+1)i_l\},u_2=\max\{\sum_{l=0}^{n}(l+1)j_l\}.$$

First, we will investigate the existence of meromorphic solutions of a type of composite functional-difference equations of the form

$$\sum_{(i)}a_{(i)}(z)(f(z))^{i_0}(\Delta_c f(z))^{i_1}\cdots(\Delta_c^n f(z))^{i_n}=\frac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))}, \tag{1}$$

where the coefficients $\{a_i(z)\}$,$\{b_j(z)\}(i,j=0,1)$ and $\{a_{(i)}(z)\}$ are of growth $S(r,f)$ such that $a_0(z)b_1(z)-a_1(z)b_0(z)\not\equiv 0$, $p(z)=c_kz^k+\cdots+c_0$,$\deg p(z)\geq 2$.

For the composite functional-difference equations (1), the main theorem can be stated as follows.

Theorem 2.1 Let $u_1<k$. The composite function-difference equation (1) does not have meromorphic solutions.

Remark 1 The example 1 shows that Theorem 2.1 does not hold if at least $a_i(z),b_j(z)$ and $a_{(i)}(z)$ are not of growth $S(r,f)$,there may exist a rational solution.

Example 1 Let $p(z)=z^2,c=1$.Then function $f(z)=\frac{1}{z-1}$ is a solution of the following equation

$$\frac{1-z^2f(p(z))}{(1+(z-1)f(p(z))}=\frac{z(z-1)}{2+z}f\Delta_c f.$$

Second, we will study the properties of $p(z)$ of composite functional-difference equations of the following

$$\sum_{i=0}^{l}a_i(z)f(p(z))^i=\frac{\Omega_1(z,f)}{\Omega_2(z,f)}, \tag{2}$$

where $p(z)$ is an entire function,$\{a_i(z)\},\{a_{(i)}(z)\},\{b_{(j)}(z)\}$ are small functions.

We obtain the following result

Theorem 2.2 Let f be a non-constant meromorphic solution of the composite functional-difference equations (2).Then $p(z)$ is a polynomial.

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Third, we shall consider the growth and characteristic estimate of meromorphic solutions of the following composite functional-difference equation

$$\sum_{(i)\in I} a_{(i)}(z)f^{i_0}(\Delta_c f)^{i_1}\cdots(\Delta_c^n f)^{i_n} = \sum_{i=0}^{m} a_i(z)(f(p(z)))^i, \tag{3}$$

where $\{a_i(z)\}$ are meromorphic functions, $a_{(i)} \not\equiv 0, a_m(z) \neq 0$, $p(z)$ is a polynomial of degree $k \geq 2$.

We get the main result below.

Theorem 2.3 Let $f(z)$ be a finite order transcendental meromorphic solution of (3),$\{a_{(i)}(z)\}$ be polynomials,

$$T(r, a_i) < KT(r^s, f), i = 0, 1, 2, \cdots, m,$$

where K and s are positive constants, r is large enough. If $s < k$, then for given $\varepsilon > 0$,

$$T(r, f) = O((\log r)^{\alpha+\varepsilon}),$$

where

$$\alpha = \frac{\log((m+1)K + \frac{u_1}{ms})}{\log \frac{k}{s}}, if \quad 1 \leq s < k,$$

and

$$\alpha = \frac{\log \frac{u_1 + m(m+1)Ks}{m}}{\log k}, if \quad s < 1 < k,$$

where $u_1 = \max\{\sum_{l=0}^{n}(l+1)i_l\}$.

Remark 2 The example 2 shows that the condition $s < k$ in Theorem 2.3 is best possible.

Example 2 Let $p(z) = c_k z^k + \cdots + c_0$,$\deg p(z) \geq 2$,

$$a_i(z) = C_m^i \frac{e^{2z}}{(1 + e^{p(z)})^m}, i = 0, 1, 2, ..., m.$$

Then

$$\sum_{i=0}^{m} a_i(z) f(p(z))^i = \frac{e^{2z}}{(1 + e^{p(z)})^m} \sum_{i=0}^{m} C_m^i f(p(z))^i,$$

$f = e^z$ is a transcendental meromorphic solution of the composite functional-difference equation of the form

$$\frac{1+z(e^c-1)^2}{(e^c-1)^3}(\Delta_c f)(\Delta_c^2 f)^2 - f(\Delta_c f)^2 - z(\Delta_c f)^2(\Delta_c^2 f) + (e^c-1)^3 f^2(\Delta_c f)$$
$$-f(\Delta_c^2 f)^2 + f^2 = \sum_{i=0}^{m} a_i(z)(f(p(z)))^i.$$

In this case, $f(z)$ satisfirs

$$T(r, f(z)) = \frac{r}{\pi} + O(1).$$

However, by $k \geq 2$,we have

$$T(r, a_i(z)) = (1 + o(1))\frac{m|c_k|r^k}{\pi},$$

it shows that Theorem 2.3 does not hold if $s = k$.

To prove Theorem 2.1-2.3, we need some lemmas as follows.

Lemma 2.1([13]) Let f be a transcendental meromorphic function and $p(z) = a_k z^k + a_{k-1}z^{k-1} + \ldots + a_1(z) + a_0, a_k \neq 0, k \geq 1$,be a polynomial of degree k.Given $0 < \delta < |a_k|$,let $\lambda = |a_k| + \delta, \mu = |a_k| - \delta$.Then, given $\varepsilon > 0$,for any $a \in \mathbf{C} \cup \{\infty\}$ and for r large enough,we have

$$kn(\mu r^k, \frac{1}{f-a}) \leq n(r, \frac{1}{f(p)-a}) \leq kn(\lambda r^k, \frac{1}{f-a}),$$

$$N(\mu r^k, \frac{1}{f-a}) + O(\log r) \leq N(r, \frac{1}{f(p)-a}) \leq N(\lambda r^k, \frac{1}{f-a}) + O(\log r),$$

$$(1-\varepsilon)T(\mu r^k, f) \leq T(r, f(p)) \leq (1+\varepsilon)T(\lambda r^k, f).$$

Lemma 2.2([12]) Let ψ:$[r_0, +\infty) \to (0, +\infty)$ be positive and bounded in every finite interval. Suppose that

$$\psi(\mu r^m) \leq A\psi(r) + B, (r \geq r_0),$$

where $\mu > 0, m > 1, A > 1$ and B are real constants.Then

$$\psi(r) = O((\log r)^\alpha),$$

where

$$\alpha = \frac{\log A}{\log m}.$$

Lemma 2.3([18]) Let $R(z,f) = \frac{\sum_{i=0}^{p} a_i(z)f^i}{\sum_{j=0}^{q} b_j(z)f^j}$ be an irreducible rational function in $f(z)$ with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$.If $f(z)$ is a meromorphic function,then

$$T(r, R(z,f)) = \max\{p,q\}T(r,f) + O\{\sum T(r,a_i) + \sum T(r,b_j)\}.$$

Lemma 2.4([3]) Let f be a non-constant meromorphic function and let g be a transcendental entire function.Then there exists an increasing sequence,$r_n \to \infty$,such that

$$T(r, f(g(z))) \geq T((M(\frac{r}{4}, g))^{\frac{1}{30}}, f)$$

holds for $r = r_n$.

Lemma 2.5([18]) Let g:$(0, +\infty) \to \mathbf{R}$, h:$(0, +\infty) \to \mathbf{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure.Then,for any $\alpha > 1$,there exists r_0 such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

Lemma 2.6([17]) Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function,let $\delta \in (0,1)$,and let $s \in (0, \infty)$.If T is of finite order,i.e.,

$$\lim_{r\to\infty} \frac{\log T(r)}{\log r} < \infty,$$

then

$$T(r+s) = T(r) + o(\frac{T(r)}{r^{\delta}}),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Lemma 2.7 Let f be a meromorphic function of finite order,

$$\Omega_1(z,f) = \sum_{(i)\in I} a_{(i)}(z) f^{i_0}(\Delta_c f)^{i_1}\cdots(\Delta_c^n f)^{i_n},$$

$$\Omega_2(z,f) = \sum_{(j)\in I} b_{(j)}(z) f^{j_0}(\Delta_c f)^{j_1}\cdots(\Delta_c^n f)^{j_n}.$$

Then

$$T(r,\Omega_1(z,f)) \le u_1 T(r,f) + S_1(r,f) + \sum_{(i)\in I} T(r,a_{(i)}),$$

and

$$T(r,\frac{\Omega_1(z,f)}{\Omega_2(z,f)}) \le (u_1+u_2)T(r,f) + S_1(r,f) + \sum_{(i)\in I} T(r,a_{(i)}) + \sum_{(j)\in J} T(r,b_{(j)}),$$

where $u_1 = \max\{\sum\limits_{l=0}^{n}(l+1)i_l\}, u_2 = \max\{\sum\limits_{l=0}^{n}(l+1)j_l\}$, the exceptional set E associated to $S(r,f)$ is of finite logarithmic measure $\int_E \frac{dr}{r} < +\infty$.

Proof It follows from

$$\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) = \Delta_c^{n-1} f(z+c) - \Delta_c^{n-1} f(z)$$

that

$$\Delta_c^m f(z) = \sum_{i=0}^{m} C_m^i (-1)^{m-i} f(z+ci).$$

Similar to the proof of Lemma 4.2 in [16](pp. 181-182), we have

$$m(r,\Omega(z,f)) = \lambda m(r,f) + S(r,f),$$

where $\lambda = \sum\limits_{l=0}^{n} i_l$.

In order to estimate the poles of $\Omega(z,f)$, we consider the term of

$$\Omega_{(i)}(z,f) = a_{(i)}(z) f^{i_0}(\Delta_c f)^{i_1}\cdots(\Delta_c^n f)^{i_n}.$$

Noting that

$$n(r,f(z+c)) \le n(r+C,f) + S(r,f) = n(r,f) + S(r,f), C = |lc|,$$

it is easy to get that

$$n(r,\Omega_{(i)}(z,f)) \le \sum_{l=0}^{n} i_l(l+1) n(r,f(z+lc)) + n(r,a_{(i)}(z)).$$

Hence, we get

$$n(r,\Omega(z,f)) \leq \max(\sum_{l=0}^{n} i_l(l+1))n(r,f(z)) + S(r,f) + \sum_{(i)} n(r,a_{(i)}(z)).$$

By the above equality, we get

$$T(r,\Omega_1(z,f)) \leq uT(r,f) + S(r,f) + \sum_{(i)} T(r,a_{(i)}(z)),$$

where $u_1 = \max\{\sum\limits_{l=0}^{n}(l+1)i_l\}$,$r$ runs to infinity outside of a set of finite logarithmic measure.

Further,we have

$$T(r,\Omega_2(z,f)) \leq u_2T(r,f) + S(r,f) + \sum_{(j)\in J} T(r,b_{(j)}),$$

where $u_2 = \max\{\sum\limits_{l=0}^{n}(l+1)j_l\}$.

Hence,we obtain

$$\begin{aligned} T(r,\tfrac{\Omega_1(z,f)}{\Omega_2(z,f)}) \leq & \ T(r,\Omega_1(z,f)) + T(r,\tfrac{1}{\Omega_2(z,f)}) \\ \leq & \ (u_1+u_2)T(r,f) + S(r,f) + \sum_{(i)\in I} T(r,a_{(i)}) + \sum_{(j)\in I} T(r,b_{(j)}). \end{aligned}$$

Lemma 2.8([21]) Let $P(z,f) = \sum\limits_{i=0}^{p} a_i(z)f^i$ be polynomial in $f(z)$ with the meromorphic coefficients $\{a_i(z)\}$.If $f(z)$ is a meromorphic function,then

$$T(r,P(z,f)) \leq pT(r,f) + \sum_{i=0}^{p} T(r,a_i) + O(1),$$

$$T(r,P(z,f)) \geq p(T(r,f) - \sum_{i=0}^{p} T(r,a_i)) + O(1).$$

Lemma 2.9([21]) Let f be a meromorphic function.Then $T(r,f)$ is an increasing function of $\log r$ and convex function of $\log r$,$\frac{T(r,f)}{\log r}$ is an incresing function of r.

Proof of Theorem 2.1 First, we suppose that there is a transcendental meromorphic solution $f(z)$ of composite functional-difference equation (1) .

For a sufficiently small $\varepsilon > 0$, by Lemma 2.1, Lemma 2.3 and Lemma 2.7, we get

$$(1-\varepsilon)T(\mu r^k, f) \leq T(r, f(p(z))) \leq (u_1+\varepsilon)T(r,f),$$

where $u_1 = \max\{\sum\limits_{l=0}^{n}(l+1)i_l\}$, $\mu = |c_k|(1-\varepsilon)$, outside a possible exceptional set of finite logarithmic measure.

Hence, for $\alpha > 1$ and for r large enough

$$(1-\varepsilon)T(\mu r^k, f) \leq (u_1+\varepsilon)T(\alpha r, f).$$

Set $t=\alpha r$. Then
$$T(\frac{\mu}{\alpha^k}t^k,f)\le\frac{u_1+\varepsilon}{(1-\varepsilon)}T(t,f).$$
By Lemma 2.2 we obtain
$$T(t,f)=O((\log t)^{\alpha_1}),$$
where
$$\alpha_1=\frac{\log\frac{u_1+\varepsilon}{(1-\varepsilon)}}{\log k}<1,$$
there is a contradiction.

Second, we suppose that $f(z)$ is a rational solution of (1). Then the coefficients $a_{(i)}(z),a_0(z),\ a_1(z),b_0(z),b_1(z)$ must be constants.

Set
$$f(z)=\frac{P(z)}{Q(z)}=\frac{\alpha_pz^p+\alpha_{p-1}z^{p-1}+\cdots+\alpha_0}{\beta_qz^q+\beta_{q-1}z^{q-1}+\cdots+\beta_0},$$
where $\alpha_p\ne0,\beta_q\ne0,\deg w(z)=\max\{p,q\}=l$.

If $p\ne q$, we immediately have $\deg(\frac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))})=kl$.

If $p=q$, we have

$$\begin{aligned}\frac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))}=\frac{a_0+a_1f(p(z))}{b_0+b_1f(p(z))}&=\frac{a_0+a_1\frac{\alpha_p(p(z))^p+\alpha_{p-1}(p(z))^{p-1}+\cdots+\alpha_0}{\beta_q(p(z))^q+\beta_{q-1}(p(z))^{q-1}+\cdots+\beta_0}}{b_0+b_1\frac{\alpha_p(p(z))^p+\alpha_{p-1}(p(z))^{p-1}+\cdots+\alpha_0}{\beta_q(p(z))^q+\beta_{q-1}(p(z))^{q-1}+\cdots+\beta_0}}\\&=\frac{(a_0\beta_q+a_1\alpha_p)(p(z))^p+(a_0\beta_{q-1}+a_1\alpha_{p-1})(p(z))^{p-1}+\cdots+(a_0\beta_0+a_1\alpha_0)}{(b_0\beta_q+b_1\alpha_p)(p(z))^p+(b_0\beta_{q-1}+b_1\alpha_{p-1})(p(z))^{p-1}+\cdots+(b_0\beta_0+b_1\alpha_0)}.\end{aligned}$$

It follows from the equation above that $a_0\beta_q+a_1\alpha_p=0$ and $b_0\beta_q+b_1\alpha_p=0$ can not hold at the same time. Otherwise $\frac{a_0(z)+a_1(z)w(p(z))}{b_0(z)+b_1(z)w(p(z))}=c$, c is a constant.

Hence, we get

$$\begin{aligned}kl=&\ \deg(\tfrac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))})\\=&\ \deg(\textstyle\sum_{(i)}a_{(i)}(z)(f(z))^{i_0}(\Delta_cf(z))^{i_1}\cdots(\Delta_c^nf(z))^{i_n})\\\le&\ \max\{i_0+2i_1+\cdots+(n+1)i_n\}l=u_1l.\end{aligned}$$

So, $u_1\ge k$, there is also a contradiction. Thus, $f(z)$ is not a rational solution of (1).

Combined with the first and second steps above, the assertion follows.

Proof of Theorem 2.2 Suppose that $p(z)$ is transcendental entire function, we have
$$\liminf_{r\to\infty}\frac{\log M(r,p(z))}{\log r}=\infty.$$
Hence, for any given $K>30$ and for r large enough
$$M(r,p)>r^K.$$
There exists an increasing sequence $r_n\to\infty$, as in Lemma 2.4, for any n such that
$$M(\frac{r_n}{4},p)>(\frac{r_n}{4})^K.$$

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Applying Lemma 2.3 and Lemma 2.7 to equation (2), we have

$$lT(r, f(p(z))) \le (u_1 + u_2)T(r, f) + S(r, f),$$

outside a possible exceptional set of finite linear measure. According to Lemma 2.5, for $\forall\alpha > 1, r \ge r_\alpha$, we obtain

$$T(r, f(p(z))) \le \frac{(u_1 + u_2)(1 + o(1))}{l} T(\alpha r, f). \tag{4}$$

It follows from Lemma 2.4 that

$$T(r_n, f(p(z))) \ge T((\frac{r_n}{4})^{\frac{K}{30}}, f). \tag{5}$$

Note that $\frac{T(r,f)}{\log r}$ is an increasing function of r. As

$$(\frac{r_n}{4})^{\frac{K}{30}} > \alpha r_n,$$

for sufficiently large n, we have

$$T((\frac{r_n}{4})^{\frac{K}{30}}, f) > \frac{K/30(\log r_n - \log 4)}{\log r_n + \log\alpha} T(\alpha r_n, f) > \frac{K}{40} T(\alpha r_n, f), \tag{6}$$

as $n \to \infty$. By (4),(5) and (6), we get

$$\frac{(u_1 + u_2)(1 + o(1))}{l} T(\alpha r_n, f) \ge T((\frac{r_n}{4})^{\frac{K}{30}}, f) > \frac{K}{40} T(\alpha r_n, f), \tag{7}$$

as $n \to \infty$.

Because K can be arbitrarily large, this is a contradiction in (7). This shows that $p(z)$ is a polynomial.

Proof of Theorem 2.3 By the equation (3), Lemma 2.7 and Lemma 2.8, we have

$$mT(r, f(p(z))) - m\sum_{i=0}^{m} T(r, a_i(z)) \le (u_1 + \varepsilon)T(r, f),$$

i.e.,

$$mT(r, f(p(z))) \le (u_1 + \varepsilon)T(r, f) + m\sum_{i=0}^{m} T(r, a_i(z)). \tag{8}$$

Combining (8) and

$$T(r, a_i(z)) < KT(r^s, f), i = 0, 1, 2, \cdots, m,$$

we obtain

$$T(r, f(p(z))) \le \frac{u_1 + \varepsilon}{m} T(r, f) + (m + 1)KT(r^s, f), \tag{9}$$

where K is a positive constant.

Case (1): If $s \geq 1$, by Lemma 2.9, we have $\frac{T(r,f)}{\log r}$ is increasing functions of r, we can obtain for any positive constant C and any $t \geq 1$

$$\frac{T(Cr^t, f)}{T(r,f)} \geq \frac{\log C + t\log r}{\log r} > (1-\varepsilon)t.$$

Hence, for r sufficiently large,

$$T(r,f) < \frac{1}{(1-\varepsilon)t} T(Cr^t, f).$$

Let $s = t, C = 1$. Then

$$T(r,f) < \frac{1}{(1-\varepsilon)s} T(r^s, f). \tag{10}$$

It follows from (9)and (10) that

$$\begin{aligned} T(r, f(p)) \leq & \ (m+1)KT(r^s, f) + \tfrac{u_1+\varepsilon}{(1-\varepsilon)ms} T(r^s, f) \\ \leq & \ ((m+1)K + \tfrac{u_1}{ms} + \varepsilon_1) T(r^s, f). \end{aligned}$$

By Lemma 2.1

$$(1-\varepsilon)T(\mu r^k, f) \leq ((m+1)K + \frac{u_1}{ms} + \varepsilon_1) T(r^s, f).$$

From the above inequality we further get

$$(1-\varepsilon)T(\mu r^{\frac{k}{s}}, f) \leq ((m+1)K + \frac{u_1}{ms} + \varepsilon_2) T(r, f). \tag{11}$$

Since $k > s$, then by (11) and Lemma 2.2, we obtain

$$T(r, f(z)) = O((\log r)^{\alpha_1 + \varepsilon}),$$

where

$$\alpha_1 = \frac{\log((m+1)K + \frac{u_1}{ms})}{\log \frac{k}{s}}.$$

Case (2):If $s < 1$, by Lemma 2.9, since $\frac{T(r,f)}{\log r}$ is increasing function of r, we obtain

$$\frac{T(r,f)}{\log r} \geq \frac{T(r^s, f)}{\log r^s},$$

i.e.

$$\frac{T(r,f)}{T(r^s, f)} \geq \frac{1}{s}. \tag{12}$$

From (9) and (12) we get

$$T(r, f(p(z))) \leq (\frac{u_1 + m(m+1)Ks + \varepsilon_3}{m}) T(r, f).$$

According to Lemma 2.1, we obtain

$$T(\mu r^k, f) \leq (\frac{u_1 + m(m+1)Ks + \varepsilon_4}{m}) T(r, f).$$

We obtain from Lemma 2.2

$$T(r, f(z)) = O((\log r)^{\alpha_2 + \varepsilon}),$$

where

$$\alpha_2 = \frac{\log \frac{u_1 + m(m+1)Ks}{m}}{\log k}.$$

Combining case (1) and case (2), we get the proof of Theorem 2.3.

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3. Growth of meromorphic solutions

Since the 1970's, R.Goldstein[10-13], W.Bergweiler[2-4], J.Heittokangas[16] et al had investigated the existence and growth of meromorphic solutions on composite functional equations in the whole complex plane and a number of important results were obtained. Particularly, J.Rieppo [20] discussed the growth on meromorphic solutions of many types of functional equations, he also obtained some interesting results, for example, the following theorem B is one of his some results.

For the following functional equations

$$Q(z, f(az+b)) = R(z, f(z)), \tag{$*$}$$

where $Q(z, f), R(z, f)$ are rational functions in f with small meromorphic coefficients relative to f such that $0 < q = \deg_f^Q \leq d = \deg_f^R$ and $a, b \in \mathbf{C}, a \neq 0$ and $|a| \neq 1$.

He obtained

Theorem B([20]) Suppose that f is a transcendental meromorphic solution of the equation $(*)$. Then

$$\mu(f) = \rho(f) = \frac{\log d - \log q}{\log |a|}.$$

It is known that when treating the meromorphic solutions of difference equations, the basic task is to estimate their growth order, while in the case of complex composite functional difference equations, considering the growth order of them is also an interesting task. Hence, this section is devoted to investigating the growth order of meromorphic solutions on two types of composite functional-difference equations (3), (13) and systems of difference equations (14) in complex domain.

As regards the growth order of meromorphic solutions of complex composite functional-difference equations (3), we obtain Theorem 3.1.

Theorem 3.1 Let $\{a_i(z)\}, \{a_{(i)}(z)\}$ be of growth order of $S(r, f)$, $u_1 \geq km$. Then the lower order and the order of meromorphic solution f of the equation (3) satisfy

$$\rho(f) = \mu(f) = 0.$$

In the following, we will also investigate the growth of meromorphic solutions about a type of composite functional-difference equations of the form

$$\frac{\sum\limits_{i=0}^{l} d_i f(a_{1i}z + b_{1i})^i}{\sum\limits_{j=0}^{t} e_j f(a_{2j}z + b_{2j})^j} = \frac{\Omega_1(z, f)}{\Omega_2(z, f)}, \tag{13}$$

where $\{a_{1i}\}, \{a_{2i}\}, \{b_{1j}\}, \{b_{2j}\}, \{d_i\}, \{e_j\}$ are constants ,$\{a_{(i)}(z)\}, \{b_{(j)}(z)\}$ are small functions and $a_{(i)}(z) \not\equiv 0, b_{(j)}(z) \not\equiv 0$.

For complex composite functional-difference equations (13), we obtain the following main result.

Theorem 3.2 Suppose that f is a transcendental meromorphic solution of composite functional-difference equations (13), $a_{1i}, a_{2j}, b_{1i}, b_{2j} \in \mathbf{C}$, $|a_{1i}| > 1, |a_{2j}| > 1$, and the coefficients $a_{(i)}(z)$ are of growth $S(r, f)$.

(i). If $l > t$, then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{l}}{\log |a_{1l}|};$$

(ii). If $l < t$, then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{t}}{\log |a_{2t}|};$$

(iii). If $l = t$, then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{l}}{\log |a|},$$

where $|a| = \max\{|a_{1l}|, |a_{2t}|\}$.

Remark 3 The example 3 shows that the upper bound in Theorem 3.2 can be reached.

Example 3 $f(z) = e^z$ is a meromorphic solution of the following equation

$$\frac{(e^c-1)^2 f(6z+c)}{e^c f(5z+c)} = \frac{f\Delta_c^2 f}{\Delta_c f + f}.$$

We see that $u_1 = 4, u_2 = 2, \rho(f) = 1 = \frac{\log \frac{u_1+u_2}{\max\{l,t\}}}{\log |a_{12}|} = \frac{\log \frac{6}{1}}{\log 6} = \frac{\log 6}{\log 6}$.

By using the Nevanlinna value distribution theory of meromorphic functions, difference equation theory, a large number of papers also have considered the properties of meromorphic solutions of some types of system of functional equations, and obtained some results([7-9]). Now, we consider the problem of the growth order on a class of system of composite functional equations as follows

$$\begin{cases} \sum\limits_{i=0}^{l} d_i f_1(c_{1i}z + d_{1i})^i = \dfrac{\sum\limits_{\mu=0}^{m_1} a_{1\mu}(z) f_2(z)^\mu}{\sum\limits_{\nu=0}^{n_1} a_{2\nu}(z) f_2(z)^\nu}, \\ \sum\limits_{j=0}^{t} e_j f_2(c_{2j}z + d_{2j})^j = \dfrac{\sum\limits_{s=0}^{m_2} b_{1s}(z) f_1(z)^s}{\sum\limits_{k=0}^{n_2} b_{2k}(z) f_1(z)^k}, \end{cases} \tag{14}$$

where $\{c_{1i}\}, \{c_{2j}\}, \{d_{1i}\}, \{d_{2j}\}, d_i, e_j$ are constants,$\{a_{1\mu}(z)\}, \{a_{2\nu}(z)\}, \{b_{1s}(z)\}, \{b_{2k}(z)\}$ are small functions,$|c_{1l}| > 1, |c_{2t}| > 1$.

The growth order of meromorphic solutions (f_1, f_2) of (14) is defined by

$$\rho(f_1, f_2) = \max\{\rho(f_1), \rho(f_2)\},$$

$$\rho(f_k) = \limsup_{r\to\infty} \frac{\log^+ T(r, f_k)}{\log r}, k = 1, 2.$$

The lower order of meromorphic function $f_i, i = 1, 2$ are defined by

$$\mu(f_k) = \liminf_{r\to\infty} \frac{\log^+ T(r, f_k)}{\log r}, k = 1, 2.$$

As regards the complex composite functional-difference equation (14), we obtain Theorem 3.3 and Theorem 3.4 as follows.

Theorem 3.3 Suppose that f is a transcendental meromorphic solution of the system (14), $c_{ij}, d_{ij} \in \mathbf{C}$, $|c_{1l}| > 1, |c_{2t}| > 1$, and the coefficients $a_{ij}(z)$ and $b_{ij}(z)$ are of growth $S(r, f_i)$.Then

$$\rho(f_1, f_2) \leq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{lt}}{\log |c_{1l}||c_{2t}|}.$$

Example 4 Let $b \in \mathbf{C}$ be a constant such that $b \neq \frac{m\pi}{2}$,where $m \in \mathbf{Z}$. We see that $(f_1(z), f_2(z)) = (\tan z, -\tan z)$ is a meromorphic solution of the following system of composite functional equations of the form

$$\begin{cases} f_1(2z+b) = \frac{-2f_2(z)-C(1-f_2^2)}{1-f_2^2-2Cf_2}, \\ f_2(2z+b) = \frac{2f_1(z)-C(1-f_1^2)}{1-f_1^2+2Cf_1}, \end{cases}$$

where $C = -\tan b \neq 0, \infty$.

In this case, $|a_{1l}||a_{2t}| = 4, \max\{m_1, n_1\}\max\{m_2, n_2\}\} = 4, lt = 1$, thus,

$$\rho(f_1, f_2) = 1 = \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{lt}}{\log |a_{1l}||a_{2t}|} = \frac{\log 4}{\log 4}.$$

It shows that the upper bound in Theorem 3.3 can be reached.

Theorem 3.4 Let (f_1, f_2) be a transcendental meromorphic solution of the system (14), and $\mu(f_1), \mu(f_2)$ be the lower order of f_1, f_2, respectively. Then

$$\mu(f_1) + \mu(f_2) \geq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{lt}}{\log |c_{1l}||c_{2t}|},$$

where $\{a_{1\mu}(z)\}, \{a_{2\nu}(z)\}, \{b_{1s}(z)\}, \{b_{2k}(z)\}$ are small functions are small functions.

In order to prove Theorems 3.1-3.4, we need the following Lemmas.

Lemma 3.1([14]) Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function,and let f be a nonconstant meromorphic function.If for some real constant $\alpha \in (0, 1)$,there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that

$$T(r, f) \leq K_1\Phi(\alpha r) + K_2T(\alpha r, f) + S(\alpha r, f),$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r}.$$

Lemma 3.2([3]) Suppose that a meromorphic function f has finite lower order λ.Then for every constant $c > 1$ and a given ε there exists a sequence $r_n = r_n(c, \varepsilon) \to \infty$ such that

$$T(cr_n, f) \leq c^{\lambda+\varepsilon}T(r_n, f).$$

Proof of Theorem 3.1 For a sufficiently small $\varepsilon > 0$, by Lemma 2.1 and Lemma 2.3, we get

$$m(1-\varepsilon)T(\mu r^k, f) \leq mT(r, f(p(z))) \leq (u_1 + \varepsilon)T(r, f),$$

where $\mu = |c_k|(1-\varepsilon), u_1 = \max\{\sum_{l=0}^{n}(l+1)i_l\}$, outside a possible exceptional set of finite logarithmic measure of r.

Hence, for $\alpha > 1$ and for r large enough

$$m(1-\varepsilon)T(\mu r^k, f) \leq (u_1+\varepsilon)T(\alpha r, f).$$

Set $t = \alpha r$. Then

$$T(\frac{\mu}{\alpha^k}t^k, f) \leq \frac{u_1+\varepsilon}{m(1-\varepsilon)}T(t, f).$$

By Lemma 2.2 we obtain

$$T(t, w) = O((\log t)^{\alpha_1}),$$

where

$$\alpha_1 = \frac{\log \frac{u_1}{m}}{\log k} + \varepsilon_1.$$

From the above equation, we can obtain that

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} = 0,$$

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} = 0.$$

Thus, we have completed the proof of Theorem 3.1.

Proof of Theorem 3.2 Applying Lemma 2.3 and Lemma 2.7 to equation (13),we get

$$\max\{l, t\}T(r, f(a_{sk}z + b_{sk})) = T(r, \frac{\sum_{i=0}^{l} d_i f(a_{1i}z + b_{1i})^i}{\sum_{j=0}^{t} e_j f(a_{2j}z + b_{2j})^j}) \leq (u_1+u_2)T(r, f) + S(r, f),$$

where $s = 1$ or 2, $k = \max\{l, t\}$.

Applying Lemma 2.1 to equation (13), we get

$$(1-\varepsilon)\max\{l, t\}T(\mu r, f) \leq (u_1+u_2)T(r, f) + S(r, f),$$

that is

$$T(\mu r, f) \leq \frac{u_1+u_2}{(1-\varepsilon)\max\{l, t\}}T(r, f) + S(r, f),$$

where $\mu = |a| - \delta > 1, |a| = \max\{|a_{1k}|, |a_{2k}|\}, \delta > 0$. Denoting $\alpha = \frac{1}{\mu}$, we have $0 < \alpha < 1$, and we deduce that

$$T(r, f) \leq \frac{u_1+u_2}{(1-\varepsilon)\max\{l, t\}}T(\alpha r, f) + S(\alpha r, f).$$

By Lemma 3.1, we obtain

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{(1-\varepsilon)\max\{l,t\}}}{-\log \alpha}.$$

Let $\varepsilon \to 0$ and $\delta \to 0$. Then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{\max\{l,t\}}}{\log |a|}.$$

Proof of Theorem 3.3 Applying Lemma 2.3 to system (14), we get

$$lT(r, f_1(c_{1l}z + d_{1l})) = \max\{m_1, n_1\}T(r, f_2) + S(r, f_2). \tag{15}$$

$$tT(r, f_2(c_{2t}z + d_{2t})) = \max\{m_2, n_2\}T(r, f_1) + S(r, f_1). \tag{16}$$

Applying Lemma 2.1 to equations (15) and (16), we get

$$(1-\varepsilon)lT(\mu_1 r, f_1) \leq \max\{m_1, n_1\}T(r, f_2) + S(r, f_2),$$

$$(1-\varepsilon)tT(\mu_2 r, f_2) \leq \max\{m_2, n_2\}T(r, f_1) + S(r, f_1),$$

that is

$$T(\mu_1 r, f_1) \leq \frac{\max\{m_1, n_1\}}{(1-\varepsilon)l}T(r, f_2) + S(r, f_2),$$

$$T(\mu_2 r, f_2) \leq \frac{\max\{m_2, n_2\}}{(1-\varepsilon)t}T(r, f_1) + S(r, f_1),$$

where $\mu_1 = |c_{1l}| - \delta_1 > 1, \delta_1 > 0, \mu_2 = |c_{2t}| - \delta_2 > 1, \delta_2 > 0$.

Denoting $\alpha_1 = \frac{1}{\mu_1}, \alpha_2 = \frac{1}{\mu_2}$, we have $0 < \alpha_1 < 1, 0 < \alpha_2 < 1$, and we deduce that

$$T(r, f_1) \leq \frac{\max\{m_1, n_1\}}{(1-\varepsilon)l}T(\alpha_1 r, f_2) + S(\alpha_1 r, f_2), \tag{17}$$

$$T(r, f_2) \leq \frac{\max\{m_2, n_2\}}{(1-\varepsilon)t}T(\alpha_2 r, f_1) + S(\alpha_2 r, f_1), \tag{18}$$

outside a possible exceptional set of finite logarithmic measure of r.

Combining (17) and (18), it yields

$$T(r, f_1) \leq \frac{(1+o(1))\max\{m_1, n_1\}\max\{m_2, n_2\}}{(1-\varepsilon)^2 lt}T(\alpha_1\alpha_2 r, f_1) + S(\alpha_1\alpha_2 r, f_1),$$

outside a possible exceptional set of finite logarithmic measure of r.

By Lemma 3.1, we obtain

$$\rho(f_1) \leq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{(1-\varepsilon)^2 lt}}{-\log \alpha_1\alpha_2}.$$

By a similar reasoning as to above, we also can get

$$\rho(f_2) \leq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{(1-\varepsilon)^2 lt}}{-\log \alpha_1\alpha_2}.$$

Let $\varepsilon \to 0$ and $\delta_i \to 0, i = 1, 2$. Then Theorem 3.3 is proved.

Proof of Theorem 3.4 We assume conversely that f_1, f_2 are transcendental meromorphic functions.

By Lemma 2.3 and $T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f) + M([17])$, where M is a constant, we have

$$\begin{cases} \max\{m_1, n_1\}T(r, f_2) & \leq lT(r, f_1(c_{1l}(z+\frac{d_{1l}}{c_{1l}}))) + S(r, f_2) \\ & \leq (1+o(1))lT(|c_{1l}|r + |\frac{d_{1l}}{c_{1l}}|, f_1) + S(r, f_2), \\ \max\{m_2, n_2\}T(r, f_1) & \leq tT(r, f_2(c_{2t}(z+\frac{d_{2t}}{c_{2t}}))) + S(r, f_1) \\ & \leq (1+o(1))tT(|c_{2t}|r + |\frac{d_{2t}}{c_{2t}}|, f_2) + S(r, f_1). \end{cases} \tag{19}$$

There are two constants $c_1 = |c_{1l}| + \varepsilon_1, c_2 = |c_{2t}| + \varepsilon_2, \varepsilon_i > 0, i = 1, 2$, such that

$$T(|c_{1l}|r + |\frac{d_{1l}}{c_{1l}}|, f_1) \leq T(c_1 r, f_1), T(|c_{2t}|r + |\frac{d_{2t}}{c_{2t}}|, f_2) \leq T(c_2 r, f_2). \tag{20}$$

When r is large enough, we can obtain from (19) and (20)

$$\begin{cases} \max\{m_1, n_1\}T(r, f_2) \leq (1+o(1))lT(c_1 r, f_1) + S(r, f_2), \\ \max\{m_2, n_2\}T(r, f_1) \leq (1+o(1))tT(c_2 r, f_2) + S(r, f_1), \end{cases}$$

outside a possible exceptional set of finite linear measure of r.

According to Lemma 2.5, for given $\sigma_1 > 1, \sigma_2 > 1$,

$$\begin{cases} \max\{m_1, n_1\}T(r, f_2) \leq (1+o(1))lT(\sigma_1 c_1 r, f_1) + S(r, f_2), \\ \max\{m_2, n_2\}T(r, f_1) \leq (1+o(1))tT(\sigma_2 c_2 r, f_2) + S(r, f_1). \end{cases} \tag{21}$$

Let $\mu(f_1), \mu(f_2)$ be the finite lower order in f_1, f_2,respectively. By Lemma 3.2, for any given $\varepsilon_i > 0, i = 1, 2$, there exists a sequence $r_n \to \infty$ such that for $r_n > r_0$

$$T(c_1 r_n, f_1) \leq c_1^{\mu(f_1)+\varepsilon_1} T(r_n, f_1), T(c_2 r_n, f_2) \leq c_2^{\mu(f_2)+\varepsilon_2} T(r_n, f_2).$$

By (21)

$$\begin{cases} \max\{m_1, n_1\}T(r_n, f_2) \leq (1+o(1))l(\sigma_1 c_1)^{\mu(f_1)+\varepsilon_1} T(r_n, f_1) + S(r_n, f_2), \\ \max\{m_2, n_2\}T(r_n, f_1) \leq (1+o(1))t(\sigma_2 c_2)^{\mu(f_2)+\varepsilon_2} T(r_n, f_2) + S(r_n, f_1). \end{cases} \tag{22}$$

From (22), we get

$$\begin{cases} \max\{m_1, n_1\} \leq (1+o(1))l(\sigma_1 c_1)^{\mu(f_1)+\varepsilon_1} \frac{T(r_n, f_1)}{T(r_n, f_2)} + \frac{S(r_n, f_2)}{T(r_n, f_2)}, \\ \max\{m_2, n_2\} \leq (1+o(1))t(\sigma_2 c_2)^{\mu(f_2)+\varepsilon_2} \frac{T(r_n, f_2)}{T(r_n, f_1)} + \frac{S(r_n, f_1)}{T(r_n, f_1)}. \end{cases} \tag{23}$$

Taking lower limit as $n \to \infty$, and $\lim_{n\to\infty} \inf \frac{S(r_n, f_i)}{T(r_n, f_i)} = 0, i = 1, 2$. Then (23) becomes

$$\max\{m_1, n_1\} \max\{m_2, n_2\} \leq lt(\sigma_1 c_1)^{\mu(f_1)+\varepsilon_3} (\sigma_2 c_2)^{\mu(f_2)+\varepsilon_3},$$

where $\varepsilon_3 = \max\{\varepsilon, \varepsilon_1, \varepsilon_2\}, \varepsilon_3 \to 0, \sigma_1 \to 1, \sigma_2 \to 1$. Hence

$$\mu(f_1) + \mu(f_2) \geq \frac{\log \frac{\max\{m_1, n_1\} \max\{m_2, n_2\}}{lt}}{\log |c_{1l}||c_{2t}|}.$$

Thus, we have completed the proof of Theorem 3.4.

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Locally and globally small Riemann sums and Henstock-Stieltjes integral for n-dimensional fuzzy-number-valued functions

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Abstract: In this paper, we study locally and globally small Riemann sums with respect to α for n-dimensional fuzzy-number-valued functions. And we prove that a fuzzy-number-valued functions in n-dimensional is Henstock-Stieltjes (HS) integrable on $[a, b]$ if and only if it has $(LSRS)$ with respect to α on $[a, b]$. Also we shall prove that a fuzzy-number-valued functions in n-dimensional is Henstock-Stieltjes (HS) integrable on $[a, b]$ if and only if it has $(GSRS)$ with respect to α on $[a, b]$.
Keywords: Fuzzy-number-valued functions in E^n; Henstock-Stieltjes integral (HS); locally small Riemann sums $(LSRS)$; globally small Riemann sums $(GSRS)$.

1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [13], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on.

The locally and globally small Riemann sums have been introduced by many authors from different points of views including [3, 4, 5, 7, 8, 10, 11]. In 1986, Schurle characterized the Lebesgue integral in $(LSRS)$ (locally small Riemann sums) property [10]. The $(LSRS)$ property has been used to characterized the Perron (P) integral on $[a, b]$ [11]. By considering the equivalency between the (P) integral and the Henstock-Kurzweil (HK) integral, the $(LSRS)$ property has been used to characterized the (HK) integral on $[a, b]$ [8]. In 2015, Indrati [7] introduced a countably Lipschitz condition of a function which is simpler than the ACG*, and proved that the (HK) integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the (HK) integral in the $(GSRS)$ property, it showed that the relationship between $(GSRS)$ property and countably Lipschitz condition of an (HK) integrable function on $[a, b]$. In 2018, Hamid et al. [5] introduced locally and globally small Riemann sums for fuzzy-number-valued functions and established two main theorems: (i) A fuzzy-number-valued functions $\tilde{f}(x)$ is (HS) integrable on $[a, b]$ iff $\tilde{f}(x)$ has $(LSRS)$. (ii) A fuzzy-number-valued functions $\tilde{f}(x)$ is (HS) integrable on $[a, b]$ iff $\tilde{f}(x)$ has $(GSRS)$.

In this paper, the concept of locally small Riemann sums for n-dimensional fuzzy-number-valued functions with respect to α is introduced and discussed. Furthermore, we provide a characterizations of globally small Riemann sums in n-dimensional fuzzy-number-valued functions with respect to α.

The rest of this paper is organized as follows. To make our analysis possible, in Section 2 we shall review the relevant concepts and properties of fuzzy-number-valued functions in E^n and the definition of Henstock-Stieltjes (HS) integral for fuzzy-number-valued functions in E^n. In Section 3, we introduce the support function characterizations of locally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n. In section 4, we shall discuss the support function characterizations of globally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n. The last section provides the Conclusions.

2 Preliminaries

In this paper the close interval $[a, b]$ denotes a compact interval on R. The set of intervals-point $\{([a_1, b_1], \xi_1), ([a_2, b_2], \xi_2), \cdots, ([a_k, b_k], \xi_k)\}$ is called a division of $[a, b]$ that is $\xi_1, \xi_2, \cdots, \xi_k \in [a, b]$, intervals $[a_1, b_1], [a_2, b_2], \cdots, [a_k, b_k]$ are non-intersect and $\bigcup_{i=1}^{k} [a_i, b_i] = [a, b]$. Marking the division of $[a, b]$ as $P = \{([a_1, b_1], \xi_1), ([a_2, b_2], \xi_2), \cdots, ([a_k, b_k], \xi_k)\}$, shortening as $P = \{[u, v]; \xi\}$ [9].

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Definition 2.1 [6, 8] Let $\delta : [a,b] \to \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a δ-fine division, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

For brevity, we write $P = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in P and ξ is the associated point of $[u, v]$.

Definition 2.2 [12] E^n is said to be a fuzzy number space if $E^n = \{u : R^n \to [0,1] : u$ satisfies (1)-(4) below$\}$:

(1) u is normal, i.e., there exists a $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is a convex fuzzy set, i.e., $u(rx + (1-r)y) \geqslant \min(u(x), u(y)),\ x, y \in R^n,\ r \in [0,1]$;

(3) u is upper semi-continuous;

(4) $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$ is compact, for $0 < r \leq 1$, denote $[u]^r = \{x : x \in R^n\ and\ u(x) \geqslant r\}$, $[u]^0 = \overline{\bigcup_{r\in(0,1]}[u]^r}$.

From (1)-(4), it follows that for any $u \in E^n$ and $r \in [0,1]$ the $r-$level set $[u]^r$ is a compact convex set. For any $u, v \in E^n$

$$D(u,v) = \sup_{r\in[0,1]} d([u]^r, [v]^r), \tag{1}$$

where d is Hausdorff metric. It is well known that (E^n, d) is an metric space [12]. The norm of fuzzy number $u \in E^n$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{\alpha\in[u]^0} |\alpha|, \tag{2}$$

where the $\|\cdot\|$ is norm on E^n, $\tilde{0}$ is fuzzy number on E^n and $\tilde{0} = \chi_{\{0\}}$.

Definition 2.3 [12] For $A \in P_k(R^n),\ x \in S^{n-1}$, define the support function of A as $\sigma(x, A) = \sup\limits_{y\in A}\langle y, x\rangle$, where S^{n-1} is the unit sphere of R^n, i.e., $S^{n-1} = \{x \in R^n : \|x\| = 1\}$, $\langle\cdot,\cdot\rangle$ is the inner product in R^n.

Definition 2.4 [2] Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ is said to be fuzzy Henstock-Stieltjes (FHS) integrable with respect to α on $[a,b]$, if there exists $\tilde{A} \in E^n$, for every $\varepsilon > 0$, there is a function $\delta(\xi) > 0$, such that for any δ-fine division $P = \{[u,v], \xi\}$ of $[a,b]$, we have

$$D\Big(\sum_{(P)} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{A}\ \Big) < \varepsilon. \tag{3}$$

We write $(FHS)\int\limits_a^b \tilde{f}(x)\mathrm{d}\alpha = \widetilde{A}$.

Lemma 2.1 [12] If $u, v \in E^n,\ k \in R,$ for any $r \in [0,1]$, we have

$$[u+v]^r = [u]^r + [v]^r,\ [ku]^r = k[u]^r. \tag{4}$$

Lemma 2.2 [12] Suppose $u \in E^n$, then

(1) $u^*(r, x+y) \leq u^*(r,x) + u^*(r,y)$,

(2) if $u, v \in E^n$, $r \in [0,1]$, then

$$d([u]^r, [v]^r) = \sup_{x\in S^{n-1}} |u^*(r,x) - v^*(r,x)|, \tag{5}$$

(3) $(u+v)^*(r,x) = u^*(r,x) + v^*(r,x)$,

(4) $(ku)^*(r,x) = ku^*(r,x),\ k \geq 0$.

Lemma 2.3 [1, 12] Given $u, v \in E^n$ the distance $D : E^n \times E^n \to [0, +\infty)$ between u and v is defined by the equation $D(u,v) = \sup\limits_{r\in[0,1]} d([u]^r, [v]^r)$, then

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$,

(3) $D(u+v, w+e) \leqslant D(u,w) + D(v,e)$,

(4) $D(ku, kv) = |k|D(u,v), k \in R$,

(5) $D(u+v, \widetilde{0}) \leqslant D(u, \widetilde{0}) + D(v, \widetilde{0})$,

(6) $D(u+v, w) \leqslant D(u,w) + D(v, \widetilde{0})$.

Where $u, v, w, e, \widetilde{0} \in E^n,\ \widetilde{0} = \mathcal{X}_{(\{0\})}$.

Lemma 2.4 [2] Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\widetilde{F} : [a,b] \to E^n$ is (FHS) integrable with respect to α on $[a,b]$ if and only if $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, we have

$$\left((FHS)\int_a^b \widetilde{F}(t)\mathrm{d}\alpha\right)^*(r,x) = (RHS)\int_a^b F^*(t)(r,x)\mathrm{d}\alpha. \tag{6}$$

Uniformly for any $r \in [0,1]$.

3 Support function characterizations of locally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n

In this section, we shall define locally small Riemann sums or in short $(LSRS)$ with respect to α on $[a,b]$ by using support function $f^*(\xi)(r,x)$ and show that it is the necessary and sufficient condition for $\tilde{f}$ to be (HS) integrable with respect to α on $[a,b]$.

Definition 3.1 Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}:[a,b]\to E^n$ is said to be have locally small Riemann sums or $(LSRS)$ with respect to α on $[a,b]$ if for every $\varepsilon>0$ there is a $\delta(\xi)>0$ such that for every $t\in[a,b]$, we have

$$\Big\| \sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big\|_{E^n} < \varepsilon, \tag{7}$$

whenever $P=\{[u,v];\xi\}$ is a δ-fine division of an interval $C\subset(t-\delta(t),t+\delta(t))$, $t\in C$ and Σ sums over P. (Where $C=[y,z]$).

The following Theorem 3.1 shows that $\tilde{f}$ has $(LSRS)$ with respect to α on $[a,b]$ is equal to the type of it's support functions.

Theorem 3.1 Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function and let $\tilde{f}:[a,b]\to E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ has locally small Riemann sums or $(LSRS)$ with respect to α on $[a,b]$ if and only if for every $\varepsilon>0$, there is a $\delta(\xi)>0$ such that for every $t\in[a,b]$, we have

$$\Big| \sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\Big| < \varepsilon, \tag{8}$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, whenever $P=\{[u,v];\xi\}$ is a δ-fine division of an interval $C\subset(t-\delta(t),t+\delta(t))$, $t\in C$ and Σ sums over P.

Proof Let $\tilde{0}\in E^n$ denote the (FHS) integral of $\tilde{f}$ with respect to α on $[a,b]$. Given $\varepsilon>0$ there is a $\delta(\xi)>0$ such that for any δ-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$D\Big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)],\tilde{0}\Big)<\varepsilon. \tag{9}$$

That is

$$\sup_{r\in[0,1]} d\Big(\big[\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\big]^r,[\tilde{0}]^r\Big)<\varepsilon. \tag{10}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\Big|\big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\big)^*(r,x)-\sigma(x,0)\Big|<\varepsilon. \tag{11}$$

Furthermore, by $\sigma(x,A)=\sup\limits_{y\in A}\langle y,x\rangle$, we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\Big|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-\sigma(x,0)\Big|<\varepsilon. \tag{12}$$

Hence, for any $r\in[0,1]$, $x\in S^{n-1}$ and for any δ-fine division P we have

$$\Big|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\Big|<\varepsilon. \tag{13}$$

Where $\sigma(x,0)=0$.
This completes the proof. □

Lemma 3.1 (Henstock Lemma). Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function and let $\tilde{f}:[a,b]\to E^n$ be a fuzzy-number-valued function and (HS) integrable to $\tilde{A}$ with respect to α on $[a,b]$. Then, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ on $[a,b]$ is (HS) integrable to $A^*(r,x)$ with respect to α on $[a,b]$ uniformly for any $r\in[0,1]$, $x\in S^{n-1}$ and $\tilde{A}\in E^n$, i.e., for every $\varepsilon>0$ there is a positive function $\delta(\xi)>0$, for δ-fine division $P=\{[u,v];\xi\}$ of $[a,b]$ and for any $x\in S^{n-1}$, we have

$$\Big|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-A^*(r,x)\Big|<\varepsilon. \tag{14}$$

Furthermore, for any sum of parts $\sum\limits_1$ from $\sum$ we have

$$\Big|\sum_1 f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-A^*(r,x)\Big|<\varepsilon. \tag{15}$$

Proof Let $\tilde{A} \in E^n$ denote the (FHS) integral of $\tilde{f}$ with respect to α on $[a,b]$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\bigg(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{A}\bigg) < \varepsilon. \tag{16}$$

That is

$$\sup_{r\in[0,1]} d\bigg(\big[\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\big]^r, [\tilde{A}]^r\bigg) < \varepsilon. \tag{17}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}} \bigg|\big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\big)^*(r,x) - A^*(r,x)\bigg| < \varepsilon. \tag{18}$$

Furthermore, by $A^*(r,x) = \sup_{y\in[A]^r}\langle y,x\rangle$, we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}} \bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - A^*(r,x)\bigg| < \varepsilon. \tag{19}$$

Hence, for any $r \in [0,1]$, $x \in S^{n-1}$ and for any δ-fine division P we have

$$\bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - A^*(r,x)\bigg| < \varepsilon. \tag{20}$$

For proof

$$\bigg|\sum_1 f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - A^*(r,x)\bigg| < \varepsilon, \tag{21}$$

the proof is similar to the Theorem 3.7 in [8].
This completes the proof. □

Theorem 3.2 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ is (HS) integrable to $\tilde{F}([a,b])$ with respect to α on $[a,b]$, then $\tilde{f}$ has $LSRS$ with respect to α on $[a,b]$.

Proof Since $\tilde{f}$ is (HS) integrable to $\tilde{F}([a,b])$ with respect to α on $[a,b]$, by Theorem 3.1 the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ on $[a,b]$ is (HS) integrable to $F^*([a,b])(r,x)$ with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$, $x \in S^{n-1}$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, for δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$ and for any $x \in S^{n-1}$, we have

$$\bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - F^*([a,b])(r,x)\bigg| < \frac{\varepsilon}{2}. \tag{22}$$

For each $t \in [a,b]$, there is a closed interval $C = [y,z] \subset (t-\delta(t), t+\delta(t))$ such that

$$\bigg|F^*([y,z])(r,x)\bigg| < \frac{\varepsilon}{2}. \tag{23}$$

According to Henstock Lemma, for each $t \in [a,b]$ and δ-fine division $P = \{[u,v];\xi\}$ of $C \subset (t-\delta(t), t+\delta(t))$, we have

$$\begin{aligned}\bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\bigg| &\leq \bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - F^*([a,b])(r,x)\bigg| + \bigg|F^*([y,z])(r,x)\bigg| \\ &< \varepsilon.\end{aligned}$$

Applies Theorem 3.1 again $\tilde{f}$ has $LSRS$ with respect to α on $[a,b]$.
This completes the proof. □

Lemma 3.2 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ is (FHS) integrable with the $\tilde{F}$ as primitive then for each number $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, such that for any $[u,v] \subset [a,b]$ with $\big(\alpha(v)-\alpha(u)\big) < \delta(\xi)$, we have

$$\bigg\|\tilde{F}([u,v])\bigg\|_{E^n} = \bigg\|(FHS)\int_{[u,v]} \tilde{f}\,\mathrm{d}\alpha\bigg\|_{E^n} < \varepsilon. \tag{24}$$

Proof The continuity follows from Lemma 3.1 and the following inequality:

$$\begin{aligned}\bigg\|\tilde{F}([u,v])\bigg\|_{E^n} &= D\bigg(\tilde{F}(u), \tilde{F}(v)\bigg) \\ &\leq D\bigg(\tilde{F}([u,v]), \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\bigg) + \bigg\|\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\bigg\|_{E^n} \\ &< \varepsilon.\end{aligned}$$

We only need set $\delta(\xi) < \frac{\varepsilon}{2(\|\tilde{f}(\xi)\|_{E^n}+1)}$.
This completes the proof. □

Theorem 3.3 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ has $LSRS$ with respect to α on $[a,b]$, then $\tilde{f}$ is (FHS) integrable with respect to α on $[a,b]$.

Proof Given any $\varepsilon > 0$ and $P = \{([a,b],\xi)\} = \{([a_1,b_1],\xi_1), ([a_2,b_2],\xi_2), \cdots, ([a_n,b_n],\xi_n)\}$ is a δ-fine partition of $[a,b]$. For each $i (i = 1,2,\cdots,n)$ there is a positive function δ_i with $P_i = \{([u_i,v_i],\xi_i)\}$ is a δ_i-fine partition of $[a_i,b_i]$. Since $\tilde{f}$ has $LSRS$ with respect to α on $[a_i,b_i]$, then we have

$$\Big\| \sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^n} < \frac{\varepsilon}{2n}. \tag{25}$$

Taken $\eta = \max\{\delta(\xi),\ \xi \in [a,b]\}$, according to the Lemma 3.2 we have

$$\Big\| \tilde{F}([a_i,b_i]) \Big\|_{E^n} = \Big\| (FHS) \int_{[a_i,b_i]} \tilde{f} \mathrm{d}\alpha \Big\|_{E^n} < \frac{\varepsilon}{2n}. \tag{26}$$

Therefore, for any δ_i-fine partition $P_i = \{([u_i,v_i],\xi_i)\}$ of $[a_i,b_i]$, we have

$$\begin{aligned} \Big(\sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}([a_i,b_i]) \Big) &\leq \Big\| \sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^n} + \Big\| \tilde{F}([a_i,b_i]) \Big\|_{E^n} \\ &< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n}, \end{aligned}$$

for each i.

Subsequently taken $\delta^*(\xi) = \min\{\delta(\xi), \delta_i(\xi)\}$, then $P = \bigcup_{i=1}^{n} P_i$ denote δ^*-fine partition of $[a,b]$.

Therefore we have

$$\begin{aligned} \Big(\sum_{P} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}([a,b]) \Big) &= \sum_{i=1}^{n} D\Big(\sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}([a_i,b_i]) \Big) \\ &< n \cdot \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

Then $\tilde{f}$ is (FHS) integral with respect to α on $[a,b]$.
This completes the proof. □

4 Support function characterizations of globally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n

In this section, we shall define globally small Riemann sums or in short $(GSRS)$ integral with respect to α on $[a,b]$ by using support function $f^*(\xi)(r,x)$ and show that it is the necessary and sufficient condition for $\tilde{f}$ to be (HS) integrable on $[a,b]$.

Definition 4.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ is said to be have globally small Riemann sums or $(GSRS)$ with respect to α on $[a,b]$ if for every $\varepsilon > 0$ there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\Big\| \sum_{\|\tilde{f}(\xi)\|_{E^n} > n} \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^n} < \varepsilon, \tag{27}$$

where the $\sum$ is taken over P and for which $\big\|\tilde{f}(\xi)\big\|_{E^n} >_n$.

The following Theorem 4.1 shows that $\tilde{f}$ has $(GSRS)$ with respect to α on $[a,b]$ is equal to the type of it's support functions.

Theorem 4.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ has globally small Riemann sums or $(GSRS)$ with respect to α on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\Big| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \Big| < \varepsilon, \tag{28}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, where the $\sum$ is taken over P and for which $\big|f^*(\xi)(r,x)\big| > n$.

Proof First, we can prove the following statements are equivalent:
(1) $\|\tilde{f}(\xi)\|_{E^n} > n$.
(2) $|f^*(\xi)(r,x)| > n$.
In fact

$$\begin{aligned}\|\tilde{f}(\xi)\|_{E^n} > n &= \sup_{r\in[0,1]} d\big([\tilde{f}(\xi)]^r, [\tilde{0}]^r\big)\\ &= \sup_{r\in[0,1]}\sup_{x\in S^{n-1}} |f^*(\xi)(r,x)|.\end{aligned}$$

Second, let $\tilde{0}\in E^n$ denote the (FHS) integral of $\tilde{f}$ with respect to α on $[a,b]$. Given $\varepsilon>0$ there exists a positive integer N such that for every $n\geqslant N$ there is a $\delta_n(\xi)>0$ and for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$D\bigg(\sum_{\|\tilde{f}(\xi)\|_{E^n}>n}\tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{0}\bigg)<\varepsilon. \tag{29}$$

That is

$$\sup_{r\in[0,1]} d\bigg([\sum_{\|\tilde{f}_r(\xi)\|_{E^n}>n}\tilde{f}(\xi)[\alpha(v)-\alpha(u)]]^r, [\tilde{0}]^r\bigg)<\varepsilon. \tag{30}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\bigg|\big(\sum_{|f^*(\xi)(r,x)|>n} f(\xi)[\alpha(v)-\alpha(u)]\big)^*(r,x)-\sigma(x,0)\bigg|<\varepsilon. \tag{31}$$

Furthermore, by $\sigma(x,A)=\sup_{y\in A}\langle y,x\rangle$, we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}}\bigg|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-\sigma(x,0)\bigg|<\varepsilon. \tag{32}$$

Hence, for any $r\in[0,1]$, $x\in S^{n-1}$ and for any δ-fine division P we have

$$\bigg|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\bigg|<\varepsilon. \tag{33}$$

Where $\sigma(x,0)=0$.
This completes the proof. □

Theorem 4.2 Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function and let $\tilde{f}:[a,b]\to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$ then $\tilde{f}$ is (HS) integrable with respect to α on $[a,b]$.

Proof Because $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$, then by Theorem 4.1 for every $\varepsilon>0$, there exists a positive integer N such that for every $n\geqslant N$ there is a $\delta_n(\xi)>0$ and for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$\bigg|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\bigg|<\varepsilon. \tag{34}$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, where the $\sum$ is taken over P and for which $|f^*(\xi)(r,x)|>n$.
For each two δ-fine divisions $P_1=\{[u_1,v_1];\xi_1\}$, $P_2=\{[u_2,v_2];\xi_2\}$ of $[a,b]$, we have

$$\begin{aligned}&\Big|\sum f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]-\sum f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\Big|\\ \le\ &\Big|\sum f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]\Big|+\Big|\sum f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\Big|\\ \le\ &\bigg|\sum_{|f^*(\xi_1)(r,x)|>n} f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]\bigg|+\bigg|\sum_{|f^*(\xi_1)(r,x)|\le n} f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]\bigg|\\ +\ &\bigg|\sum_{|f^*(\xi_2)(r,x)|>n} f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\bigg|+\bigg|\sum_{|f^*(\xi_2)(r,x)|\le n} f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\bigg|\\ <\ &4\varepsilon.\end{aligned}$$

According to the properties of Cauchy, $\tilde{f}$ is (HS) integrable on $[a,b]$.
This completes the proof. □

Theorem 4.3 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing. Given a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$, for each $r \in [0,1]$ and $x \in S^{n-1}$ defined the support function $f_n^*(\xi)(r,x)$ of $\tilde{f}_n$ by the formula:

$$f_n^*(\xi)(r,x) = \begin{cases} f^*(\xi)(r,x), \xi \in [a,b] & \text{if } |f^*(\xi)(r,x)| \le n, \\ 0, & \text{others.} \end{cases}$$

A fuzzy-number-valued function $\tilde{f}$ is (HS) integrable with respect to α on $[a,b]$ if and only if $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$ and $\tilde{F}_n([a,b]) \to \tilde{F}([a,b])$ as $n \to \infty$. (Where $\tilde{F}([a,b])$ and $\tilde{F}_n([a,b])$ the integral of $\tilde{f}$ and $\tilde{f}_n$ with respect to α on $[a,b]$ respectively).

Proof First we shall prove the necessity. Because a fuzzy-number-valued function $\tilde{f}$ is (HS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, i.e., for every $\varepsilon > 0$ there is a positive function δ^*, for δ^*-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F^*([a,b])(r,x) \right| < \frac{\varepsilon}{3}. \tag{35}$$

For each $n \in \mathbb{N}$, there is a positive function δ_n, for δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum f_n^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F_n^*([a,b])(r,x) \right| < \frac{\varepsilon}{3}, \tag{36}$$

for each $r \in [0,1]$ and $x \in S^{n-1}$.

Because $\{F_n^*([a,b])(r,x)\}$ converge to $F^*([a,b])(r,x)$ of $[a,b]$ then there is a positive number N so if $n \ge N$ we have

$$\left| F_n^*([a,b])(r,x) - F^*([a,b])(r,x) \right| < \frac{\varepsilon}{3}. \tag{37}$$

For $n \ge N$, defined a positive function δ on $[a,b]$ by the formula:

$$\delta(\xi) = \min\{\delta^*(\xi), \delta_n(\xi)\}. \tag{38}$$

Therefor, for each δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\begin{aligned} & \left| \sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ = & \left| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - \sum f_n^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ \le & \left| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F^*([a,b])(r,x) \right| + \left| F_n^*([a,b])(r,x) - F^*([a,b])(r,x) \right| \\ + & \left| F^*([a,b])(r,x) - \sum f_n^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$.

Second we shall prove the sufficiency. Because $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$, then by Theorem 4.1 for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| < \varepsilon, \tag{39}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, where the $\sum$ is taken over P and for which $\left| f^*(\xi)(r,x) \right| > n$.

Note that $\tilde{f}_n$, is Henstock-Stieltjes integrable with respect to α on $[a,b]$ for all n. Choose N so that whenever $n, m \geqslant N$ we have

$$\left| F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x) \right| < \varepsilon. \tag{40}$$

Then for $n, m \geqslant N$ and a suitably chosen δ-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned} & \left| F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x) \right| \\ \le & \left| F_n^*([a,b])(r,x) - \sum_{|f^*(\xi)(r,x)|\le n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| + \left| \sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ + & \left| \sum_{|f^*(\xi)(r,x)|\le m} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F_m^*([a,b])(r,x) \right| + \left| \sum_{|f^*(\xi)(r,x)|>m} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ < & 4\varepsilon. \end{aligned}$$

That is, $\{F_n^*([a,b])(r,x)\}$ converge to $F^*([a,b])(r,x)$, as $n\to\infty$. Again, for suitably chosen N and $\delta(\xi)$ and for every δ-fine division $P=\{[u,v];\xi\}$, we have

$$\begin{aligned}
& \left|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-F^*([a,b])(r,x)\right| \\
\leq\ & \left|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-F_N^*([a,b])(r,x)\right|+\left|F_N^*([a,b])(r,x)-F^*([a,b])(r,x)\right| \\
\leq\ & \left|\sum_{|f^*(\xi)(r,x)|\leq N} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]-F_N^*([a,b])(r,x)\right|+\left|\sum_{|f^*(\xi)(r,x)|>N} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\right| \\
+\ & \left|F_N^*([a,b])(r,x)-F^*([a,b])(r,x)\right| \\
<\ & 3\varepsilon.
\end{aligned}$$

That is, $\tilde{f}$ is (HS) integrable on $[a,b]$.
This completes the proof. □

5 conclusions

In this paper, the notions of locally and globally small Riemann sums modifications with respect to fuzzy-number-valued functions in E^n are introduced and studied. The basic properties and characterizations are presented. In particular, it is proved that a fuzzy-number-valued functions in E^n is (HS) integrable on $[a,b]$ iff it has $(LSRS)$, and also it is proved that a fuzzy-number-valued functions in E^n is (HS) integrable on $[a,b]$ iff it has $(GSRS)$.

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Solving Systems of Nonhomogeneous Coupled Linear Matrix Differential Equations in Terms of Mittag-Leffler Matrix Functions

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Abstract

In this paper, we investigate systems of nonhomogeneous coupled linear matrix differential equations. Applying Kronecker products, the vector operator, and matrix convolution product, we obtain explicit formula of the general solution to this system in terms of matrix series concerning exponentials and Mittag-Leffler functions.

Keywords: linear matrix differential equation, Kronecker product, vector operator, matrix convolution product, Mittag-Leffler function.
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1 Introduction

Theory of linear matrix differential equations can be applied in a broad range of scientific fields, e.g. statistics [2, 6, 8], game theory [4], ecometrics and Leondief model [6, 8, 11], control and system theory [3, 7]. The simplest first-order homogeneous linear matrix differential equation with time-invariant coefficient is given by

$$X'(t) = AX(t). \tag{1.1}$$

Here, A is a given square matrix and $X(t)$ is an unknown matrix-valued function to be solved. The system (1.1) has been widely studied, and the solution relies on the computation of e^{tA}; see more information in [12, 13]. The nonhomogeneous case appears in the form

$$X'(t) = AX(t) + U(t), \tag{1.2}$$

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here $U(t)$ is a given matrix-valued function. In fact, the equation (1.2) has a general solution given by a one-parameter matrix-valued function

$$X(t) \;=\; e^{(t-t_0)A}X(t_0) + e^{tA} * U(t), \tag{1.3}$$

where $*$ denotes the matrix convolution product. See related works on nonhomogeneous case in [10, 15] and references therein.

Coupled matrix differential equations have numerous applications in pure and applied mathematics. For example, to obtain the solution of an optimal control problem with performance index we need to solve the system [7]

$$\begin{aligned} X'(t) &= AX(t) + BY(t), \\ Y'(t) &= CX(t) - A^T Y(t). \end{aligned}$$

A general system of nonhomogeneous coupled linear matrix differential equations with time-invariant coefficient takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned} \tag{1.4}$$

In [5], a homogeneous case of (1.4) when $E = C$, $F = D$, $G = A$, $H = B$ was investigated under the assumption that $AC = CA$ and $BD = DB$. In this case, the solution is given in terms of Kronecker products, the vector operator, and matrix series concerning exponentials and hyperbolic functions. A nonhomogeneous case of (1.4) was discussed in [1].

In this work, we investigate the system (1.4) under the assumption that $AC = CG$, $GE = EA$, $DB = HD$, $FH = BF$. We apply Kronecker products and the vector operator to reduce our complex system to the simplest form. Thus, an explicit formula of the general solution to this system is obtained in terms of Mittag-Leffler matrix functions. In particular, we obtain general solution of several special cases of the main system. When initial conditions are imposed to these problems, its solution is uniquely determined. Our results also include the previous works [1, 5].

This paper is structured as follows. In Section 2, we supply useful facts for solving linear matrix differential equations, including matrix functions defined by power series, Kronecker product, vector operator, and matrix convolution product. The main part of the paper, Section 3, deals with solving the system (1.4) and its interesting special cases. In Sections 4, we treat an initial value problem related to (1.4) and illustrate it with a numerical example.

2 Preliminaries

In this section, we provide adequate tools for solving system of linear matrix differential equations. We shall denote the set of all m-by-n complex matrices by $M_{m,n}$, and we set $M_n = M_{n,n}$.

2.1 Functions of a matrix defined by power series

Consider $A \in M_n$ and a holomorphic function f defined on a region in the complex plane containing the origin and the spectrum of A. Let $R > 0$ be such that f admits the Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for} \quad |z| < R,$$

where $a_0 = f(0)$ and $a_k = f^{(k)}(0)/k!$ for any $k \in \mathbb{N}$. If the spectral radius of A is less than R, then the matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges, denoted by $f(A)$. Hence if f is an entire function then $f(A)$ is a well-defined matrix for any $A \in M_n$. In particular, the following matrix series converge for any $A \in M_n$:

$$\sinh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1}, \quad \cosh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k}.$$

Recall that the two-parameter Mittag-Leffler functions (e.g. [14]) is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.1}$$

where Γ is the Gamma function. The power series (2.1) converges for all complex numbers z.

The Mittag-Leffler function of a matrix $A \in M_n$ with parameters $\alpha > 0$ and $\beta > 0$ is defined by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} A^k = I_n + \frac{1}{\Gamma(\alpha + \beta)} A + \frac{1}{\Gamma(2\alpha + \beta)} A^2 + \cdots .$$

The class of matrix Mittag-Leffler functions include the following functions:

$$E_{1,1}(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = e^A, \quad E_{2,1}(A^2) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} = \cosh(A).$$

An expansion shows that $\big(E_{2,2}(A^2)\big)A = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} = \sinh(A)$.

Lemma 2.1 (see e.g. [9])**.** *If (A, B) is a pair of commuting complex matrices, then $e^{A+B} = e^A e^B$.*

The next lemma is useful for deriving explicit formulas of solutions for system of linear matrix differential equations in Section 3.

Lemma 2.2. *For any $A \in M_n(\mathbb{C})$ and $B \in M_n(\mathbb{C})$, we have*

$$e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} E_{2,1}(AB) & \big(E_{2,2}(AB)\big)A \\ \big(E_{2,2}(BA)\big)B & E_{2,1}(BA) \end{bmatrix}.$$

Proof. A computation using matrix analysis reveals that

$$
\begin{aligned}
e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \begin{bmatrix} 0 & (AB)^k A \\ (BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!}(AB)^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{(2k)!}(BA)^k \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)}(AB)^k & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)}(AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)}(BA)^k B & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)}(BA)^k \end{bmatrix} \\
&= \begin{bmatrix} E_{2,1}(AB) & \big(E_{2,2}(AB)\big)A \\ \big(E_{2,2}(BA)\big)B & E_{2,1}(BA) \end{bmatrix}.
\end{aligned}
$$

□

2.2 Kronecker product and vector operator

Given two matrices $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{p,q}$ the Kronecker product of A and B is defined by

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}.$$

The the vector operator Vec : $M_{m,n} \to \mathbb{C}^{mn}$ is defined for each $A = [a_{ij}]$ by

$$\operatorname{Vec} A = [a_{11} \dots a_{m1} \dots a_{12} \dots a_{m2} \dots a_{1m} \dots a_{mn}]^T.$$

It is clear that Vec is a linear isomorphism. Algebraic properties of the Kronecker product and the vector operator used in this paper are as follows:

Lemma 2.3 (see e.g. [9])**.** *The map $(A, B) \mapsto A \otimes B$ is bilinear. The following properties hold for matrices of appropriate sizes:*

1. $I_m \otimes I_n = I_{mn}$,
2. $(A \otimes B)(C \otimes D) = AC \otimes BD$,
3. $\operatorname{Vec}(AXB) = (B^T \otimes A) \operatorname{Vec} X$.

The Kronecker product is compatible with holomorphic functions in the following sense.

Lemma 2.4 (see e.g.[9]). *Let f be a holomorphic function defined on a region including the origin and the spectrum of $A \in M_n$. Then $f(I \otimes A) = I \otimes f(A)$ and $f(A \otimes I) = f(A) \otimes I$. In particular, the following relations hold for any $A \in M_n$:*

$$\begin{array}{lll} E_{\alpha,\beta}(A \otimes I) = E_{\alpha,\beta}(A) \otimes I & \text{and} & E_{\alpha,\beta}(I \otimes A) = I \otimes E_{\alpha,\beta}(A), \\ \sinh(A \otimes I) = \sinh(A) \otimes I & \text{and} & \sinh(I \otimes A) = I \otimes \sinh(A), \\ \cosh(A \otimes I) = \cosh(A) \otimes I & \text{and} & \cosh(I \otimes A) = I \otimes \cosh(A). \end{array}$$

2.3 Matrix convolution product

Let $\Omega = [0, \infty)$ or $\Omega = [0, b]$ for some $b > 0$. The convolution is a binary operation assigned to each pair of integrable function f and g defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau, \quad t \in \Omega.$$

The convolution is bilinear and commutative. Given two integrable matrix-valued functions $A : \Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ and $B : \Omega \to M_{n,p}(\mathbb{R})$, $B(t) = [b_{ij}(t)]$, we define the matrix convolution product of A and B by

$$(A * B)(t) \ = \ \left[\sum_{k=1}^{n} a_{ik}(t) * b_{kj}(t)\right] \in M_{m,p}(\mathbb{R}), \quad t \in \Omega.$$

We may write $A(t) * B(t)$ for $(A * B)(t)$. The matrix convolution product is bilinear, but not commutative in general.

3 General solutions of systems of nonhomogeneous coupled linear matrix differential equations

From now on, let $A, B, C, D, E, F, G, H, J, K \in M_n(\mathbb{C})$ be given constant matrices and let $U, V : \Omega \to M_n(\mathbb{C})$ be given matrix-valued functions. We wish to solve certain systems of linear matrix differential equations in unknown matrix-valued functions $X, Y : \Omega \to M_n(\mathbb{C})$.

Theorem 3.1. *Assume that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. Then the general solution of the system of nonhomogeneous coupled linear matrix differential equations:*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned} \tag{3.1}$$

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is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\big(E_{2,1}((t-t_0)^2M)\big)\operatorname{Vec} X(t_0)\\
&+\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big)\operatorname{Vec} Y(t_0)\\
&+\big(E_{2,1}((t-t_0)^2M)\big)*\operatorname{Vec} U(t)\\
&+\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big)*\operatorname{Vec} V(t)\big\},\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T\otimes G)}\big\{(t-t_0)\big(E_{2,2}((t-t_0)^2N\big)\big(F^T\otimes E\big)\operatorname{Vec} X(t_0)\\
&+\big(E_{2,1}((t-t_0)^2N)\big)\operatorname{Vec} Y(t_0)\\
&+\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2N)\big)\big(F^T\otimes E\big)*\operatorname{Vec} U(t)\\
&+\big(E_{2,1}((t-t_0)^2N)\big)*\operatorname{Vec} V(t)\big\},
\end{aligned} \tag{3.2}$$

where $M=(FD)^T\otimes CE$ *and* $N=(DF)^T\otimes EC$.

Proof. Using Lemma 2.3, we can transform the system (3.1) into the vector form:

$$\begin{bmatrix}\operatorname{Vec} X'(t)\\ \operatorname{Vec} Y'(t)\end{bmatrix}=\begin{bmatrix}B^T\otimes A & D^T\otimes C\\ F^T\otimes E & H^T\otimes G\end{bmatrix}\begin{bmatrix}\operatorname{Vec} X(t)\\ \operatorname{Vec} Y(t)\end{bmatrix}+\begin{bmatrix}\operatorname{Vec} U(t)\\ \operatorname{Vec} V(t)\end{bmatrix}.$$

Let us denote $P=\begin{bmatrix}B^T\otimes A & 0\\ 0 & H^T\otimes G\end{bmatrix}$ and $Q=\begin{bmatrix}0 & D^T\otimes C\\ F^T\otimes E & 0\end{bmatrix}$. From (1.3), this system has the following solution:

$$\begin{bmatrix}\operatorname{Vec} X(t)\\ \operatorname{Vec} Y(t)\end{bmatrix}=e^{(t-t_0)S}\begin{bmatrix}\operatorname{Vec} X(t_0)\\ \operatorname{Vec} Y(t_0)\end{bmatrix}+e^{(t-t_0)S}*\begin{bmatrix}\operatorname{Vec} U(t)\\ \operatorname{Vec} V(t)\end{bmatrix},$$

where $S=P+Q$. Now, we will compute e^S. Since $DB=HD$, $AC=CG$, $FH=BF$ and $GE=EA$, by Lemma 2.3 we have $PQ=QP$. From which it follows from Lemma 2.1 that $e^S=e^{P+Q}=e^Pe^Q$. By expanding the power series of matrix exponential, we have

$$e^P=\begin{bmatrix}e^{B^T\otimes A} & 0\\ 0 & e^{H^T\otimes G}\end{bmatrix}.$$

By Lemma 2.2, we have

$$e^Q=\begin{bmatrix}E_{2,1}(M) & \big(E_{2,2}(M)\big)\big(D^T\otimes C\big)\\ \big(E_{2,2}(N)\big)\big(F^T\otimes E\big) & E_{2,1}(N)\end{bmatrix}.$$

Thus

$$\begin{aligned}
e^S &= \begin{bmatrix}e^{B^T\otimes A} & 0\\ 0 & e^{H^T\otimes G}\end{bmatrix}\begin{bmatrix}E_{2,1}(M) & \big(E_{2,2}(M)\big)\big(D^T\otimes C\big)\\ \big(E_{2,2}(N)\big)\big(F^T\otimes E\big) & E_{2,1}(N)\end{bmatrix}\\
&= \begin{bmatrix}e^{B^T\otimes A}E_{2,1}(M) & e^{B^T\otimes A}\big(E_{2,2}(M)\big)\big(D^T\otimes C\big)\\ e^{H^T\otimes G}\big(E_{2,2}(N)\big)\big(F^T\otimes E\big) & e^{H^T\otimes G}E_{2,1}(N)\end{bmatrix}.
\end{aligned}$$

Denoting

$$\begin{aligned}
R_1 &= e^{(t-t_0)(B^T\otimes A)}E_{2,1}((t-t_0)^2M),\\
R_2 &= e^{(t-t_0)(B^T\otimes A)}\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big),\\
R_3 &= e^{(t-t_0)(H^T\otimes G)}\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2N)\big)\big(F^T\otimes E\big),\\
R_4 &= e^{(t-t_0)(H^T\otimes G)}E_{2,1}((t-t_0)^2N),
\end{aligned}$$

we obtain

$$e^{(t-t_0)S}\begin{bmatrix}\operatorname{Vec}X(t_0)\\ \operatorname{Vec}Y(t_0)\end{bmatrix}=\begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}\begin{bmatrix}\operatorname{Vec}X(t_0)\\ \operatorname{Vec}Y(t_0)\end{bmatrix}=\begin{bmatrix}R_1\operatorname{Vec}X(t_0)+R_2\operatorname{Vec}Y(t_0)\\ R_3\operatorname{Vec}X(t_0)+R_4\operatorname{Vec}Y(t_0)\end{bmatrix}.$$

We also have

$$e^{(t-t_0)S}*\begin{bmatrix}\operatorname{Vec}U(t)\\ \operatorname{Vec}V(t)\end{bmatrix}=\begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}*\begin{bmatrix}\operatorname{Vec}U(t)\\ \operatorname{Vec}V(t)\end{bmatrix}=\begin{bmatrix}R_1*\operatorname{Vec}U(t)+R_2*\operatorname{Vec}V(t)\\ R_3*\operatorname{Vec}U(t)+R_4*\operatorname{Vec}V(t)\end{bmatrix}.$$

Therefore, the general solution of (3.1) is given by (3.2). □

Corollary 3.2. *Assume that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. Then the general solution of the system*

$$\begin{aligned}
X'(t) &= AX(t)B + CY(t)D,\\
Y'(t) &= EX(t)F + GY(t)H
\end{aligned}$$

is given by

$$\begin{aligned}
\operatorname{Vec}X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\big(E_{2,1}((t-t_0)^2M)\big)\operatorname{Vec}X(t_0)\\
&\quad+\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big)\operatorname{Vec}Y(t_0),\\
\operatorname{Vec}Y(t) &= e^{(t-t_0)(H^T\otimes G)}\big\{\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2N)\big)\big(F^T\otimes E\big)\operatorname{Vec}X(t_0)\\
&\quad+\big(E_{2,1}((t-t_0)^2N)\big)\operatorname{Vec}Y(t_0)\big\}
\end{aligned}\tag{3.3}$$

where $M = (FD)^T\otimes CE$ and $N = (DF)^T\otimes EC$.

Proof. Put $U(t) = V(t) = 0$ in (3.2) and then use Lemma 2.3. □

The next result was firstly established in [1].

Corollary 3.3. *The general solution of the system*

$$\begin{aligned}
X'(t) &= AX(t)B + CY(t)D + U(t),\\
Y'(t) &= CX(t)D + AY(t)B + V(t)
\end{aligned}\tag{3.4}$$

under the assumption that $AC = CA$ and $BD = DB$, is given by

$$\begin{aligned}
\operatorname{Vec}X(t) =& e^{(t-t_0)(B^T\otimes A)}\big\{\cosh L\operatorname{Vec}X(t_0)+\sinh L\operatorname{Vec}Y(t_0)\\
&+\cosh L*\operatorname{Vec}U(t)+\sinh L*\operatorname{Vec}V(t)\big\},\\
\operatorname{Vec}Y(t) =& e^{(t-t_0)(B^T\otimes A)}\big\{\sinh L\operatorname{Vec}X(t_0)+\cosh L\operatorname{Vec}Y(t_0)\\
&+\sinh L*\operatorname{Vec}U(t)+\cosh L*\operatorname{Vec}V(t)\big\},
\end{aligned}\tag{3.5}$$

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where $L=(t-t_0)(D^T\otimes C)$.

Proof. Put $E=C$, $F=D$, $G=A$ and $H=B$ in (3.2), and use Lemma 2.3. □

The corresponding homogeneous system of (3.4) is given by

$$\begin{aligned} X'(t) &= AX(t)B+CY(t)D, \\ Y'(t) &= CX(t)D+AY(t)B. \end{aligned} \tag{3.6}$$

If $AC=CA$ and $BD=DB$, then the general solution of (3.6) is reduced to

$$\begin{aligned} \operatorname{Vec}X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\cosh L\operatorname{Vec}X(t_0)+\sinh L\operatorname{Vec}Y(t_0)\big\}, \\ \operatorname{Vec}Y(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\sinh L\operatorname{Vec}X(t_0)+\cosh L\operatorname{Vec}Y(t_0)\big\}. \end{aligned}$$

This result was firstly obtained in [5].

Corollary 3.4. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B+CY(t) \quad +U(t), \\ Y'(t) &= EX(t) \quad +GY(t)B+V(t) \end{aligned}$$

under the condition $AC=CG, GE=EA$, *is given by*

$$\begin{aligned} \operatorname{Vec}X(t) &= e^{(t-t_0)(B^T\otimes A)}\operatorname{Vec}\Big\{\big(E_{2,1}(K_1)\big)X(t_0)+\big(t-t_0\big)\big(E_{2,2}(K_1)\big)CY(t_0)\Big\} \\ &\quad + e^{(t-t_0)(B^T\otimes A)}\Big\{\big(I_n\otimes E_{2,1}(K_1)\big)*\operatorname{Vec}U(t) \\ &\quad + \big(I_n\otimes \big(t-t_0\big)\big(E_{2,2}(K_1)\big)C\big)*\operatorname{Vec}V(t)\Big\}, \\ \operatorname{Vec}Y(t) &= e^{(t-t_0)(B^T\otimes G)}\operatorname{Vec}\Big\{\big(t-t_0\big)\big(E_{2,2}(K_2)\big)EX(t_0)+\big(E_{2,1}(K_2)\big)Y(t_0)\Big\} \\ &\quad + e^{(t-t_0)(B^T\otimes G)}\Big\{\big(I_n\otimes \big(t-t_0\big)\big(E_{2,2}(K_2)\big)E\big)*\operatorname{Vec}U(t) \\ &\quad + \big(I_n\otimes E_{2,1}(K_2)\big)*\operatorname{Vec}V(t)\Big\}, \end{aligned}$$

where $K_1=(t-t_0)^2CE$ *and* $K_2=(t-t_0)^2EC$.

Proof. Put $H=B$, $D=F=I_n$ in (3.2) and then use Lemmas 2.3 and 2.4. □

Corollary 3.5. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B+ \quad Y(t) \quad +U(t), \\ Y'(t) &= \quad X(t) \quad +AY(t)B+V(t) \end{aligned}$$

is given by

$$\begin{aligned} \operatorname{Vec}X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\cosh(t-t_0)\operatorname{Vec}X(t_0)+\sinh(t-t_0)\operatorname{Vec}Y(t_0) \\ &\quad +\cosh(t-t_0)\big(I_{n^2}*\operatorname{Vec}U(t)\big)+\sinh(t-t_0)\big(I_{n^2}*\operatorname{Vec}V(t)\big)\big\}, \\ \operatorname{Vec}Y(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\sinh(t-t_0)\operatorname{Vec}X(t_0)+\cosh(t-t_0)\operatorname{Vec}Y(t_0) \\ &\quad +\sinh(t-t_0)\big(I_{n^2}*\operatorname{Vec}U(t)\big)+\cosh(t-t_0)\big(I_{n^2}*\operatorname{Vec}V(t)\big)\big\}. \end{aligned}$$

Proof. Put $C = D = I_n$ in (3.5) and then use Lemma 2.3. □

Corollary 3.6. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned}$$

under the condition $FH = BF$ and $GE = EA$, is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\big\{ \operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t)\big\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T \otimes G)} \operatorname{Vec}\big\{(t-t_0)EX(t_0)F + Y(t_0)\big\} \\ &\quad + e^{(t-t_0)(H^T \otimes G)}\big\{(t-t_0)(F^T \otimes E) * \operatorname{Vec} U(t) + I * \operatorname{Vec} V(t)\big\}. \end{aligned}$$

Proof. Put $C = D = 0$ in (3.2) and then use Lemma 2.3. □

Corollary 3.7. *The general solution of equation $X'(t) = AX(t)B + U(t)$ is given by* $\operatorname{Vec} X(t) = e^{(t-t_0)(B^T \otimes A)}\big\{ \operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t)\big\}$.

Proof. Put $E = F = 0$ in Corollary 3.6. □

4 Unique solution of initial value problem and a numerical example

Consider the following initial value problem associated with the system (3.1):

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned}$$

subject to initial conditions $X(0) = J$ and $Y(0) = K$. Suppose $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. In this case, the solution of this problem is unique and given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{t(B^T \otimes A)}\big\{\big(E_{2,1}(t^2M)\big) \operatorname{Vec} J + t\big(E_{2,2}(t^2M)\big)\big(D^T \otimes C\big) \operatorname{Vec} K \\ &\quad + \big(E_{2,1}(t^2M)\big) * \operatorname{Vec} U(t) + t\big(E_{2,2}(t^2M)\big)\big(D^T \otimes C\big) * \operatorname{Vec} V(t)\big\}, \\ \operatorname{Vec} Y(t) &= e^{t(H^T \otimes G)}\big\{t\big(E_{2,2}(t^2N)\big(F^T \otimes E\big) \operatorname{Vec} J + \big(E_{2,1}(t^2N)\big) \operatorname{Vec} K \\ &\quad + t\big(E_{2,2}(t^2N)\big)\big(F^T \otimes E\big) * \operatorname{Vec} U(t) + \big(E_{2,1}(t^2N)\big) * \operatorname{Vec} V(t)\big\}, \end{aligned}$$

where $M = (FD)^T \otimes CE$ and $N = (DF)^T \otimes EC$.

Let us see a numerical example.

Example 4.1. *The initial value problem*

$$\begin{aligned} X'(t) &= AX(t)B + \quad Y(t) \quad + U(t), \\ Y'(t) &= \quad X(t) \quad + AY(t)B + V(t) \\ X(0) &= J \quad \textit{and} \quad Y(0) = K \end{aligned}$$

with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, J = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, K = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix},$
$U(t) = \begin{bmatrix} -e^{2t} & 1 \\ 1 & \sin t \end{bmatrix}, V(t) = \begin{bmatrix} 1 & e^{2t} \\ \cos t & \sin 2t \end{bmatrix}$ *has a unique solution given by*

$$\operatorname{Vec} X(t) = e^{tW} \operatorname{Vec} \begin{bmatrix} w_1(t)\cosh t + w_2(t)\sinh t & w_3(t)\cosh t + w_4(t)\sinh t \\ w_5(t)\cosh t + w_6(t)\sinh t & w_7(t)\cosh t + w_8(t)\sinh t \end{bmatrix},$$

$$\operatorname{Vec} Y(t) = e^{tW} \operatorname{Vec} \begin{bmatrix} w_2(t)\cosh t + w_1(t)\sinh t & w_4(t)\cosh t + w_3(t)\sinh t \\ w_6(t)\cosh t + w_5(t)\sinh t & w_8(t)\cosh t + w_7(t)\sinh t \end{bmatrix}.$$

Here, $W = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ -1 & -2 & 1 & 2 \\ -3 & -4 & 3 & 4 \end{bmatrix},$

$w_1(t) = \frac{1}{2}(5 - e^{2t}),\ w_2(t) = 3 + t,\ w_3(t) = -1 + t,\ w_4(t) = \frac{1}{2}(1 + e^{2t}),$ $w_5(t) = 1 + t,\ w_6(t) = 1 + \sin t,\ w_7(t) = 1 - \cos t,\ w_8(t) = -\frac{1}{2}(1 + \cos 2t).$

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Expressions of the solutions of some systems of difference equations

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ABSTRACT

In this paper, we deal with the form of the solutions and the periodicity character of the following systems of nonlinear difference equations of order two

$$z_{n+1}=\frac{z_nt_{n-1}}{\pm t_n\pm t_{n-1}},\ t_{n+1}=\frac{t_nz_{n-1}}{\pm z_n\pm z_{n-1}},$$

where the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are nonzero real numbers.

Keywords: recursive sequences, difference equations, periodic solution, solution of difference equation, system of difference equations.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Through this paper, we will obtain the form of the solutions of some nonlinear difference equations systems of order two of the following form

$$z_{n+1}=\frac{t_{n-1}z_n}{\pm t_n\pm t_{n-1}},\ t_{n+1}=\frac{z_{n-1}t_n}{\pm z_n\pm z_{n-1}},$$

where the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are nonzero real numbers. We will then investigate the periodicity character of the solutions of the systems under study. Finally we will present some numerical examples and some figures will be given to explain the behavior of the obtained solutions.

The study of difference equations is a very rich research field, and difference equations have been applied in several mathematical models in biology, population dynamics, genetics, economics, medicine, and so forth. Solving difference equations and studying the asymptotic behavior of their solutions has attracted the attention of many authors, see for example [1-39].

El-Dessoky et al. [6] studied the periodic nature and the form of the solutions of nonlinear difference equations systems of order four

$$x_{n+1}=\frac{x_ny_{n-3}}{y_{n-2}(\pm1\pm x_ny_{n-3})},\ y_{n+1}=\frac{y_nx_{n-3}}{x_{n-2}(\pm1\pm y_nx_{n-3})},$$

Grove et al. [7] obtained the existence and behavior of solutions of the rational system

$$x_{n+1}=\frac{a}{x_n}+\frac{b}{y_n},\ \ y_{n+1}=\frac{c}{x_n}+\frac{d}{y_n}.$$

Mansour et al. [8] investigated the periodic nature and get the form of the solutions of the following systems of rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{\pm x_{n-1}y_n-g},\ y_{n+1}=\frac{y_{n-1}}{\pm y_{n-1}x_n-f}.$$

El-Dessoky [9] studied the solutions of the rational equation systems

$$x_{n+1}=\frac{y_{n-1}y_{n-2}}{x_n(\pm1\pm y_{n-1}y_{n-2})},\ y_{n+1}=\frac{x_{n-1}x_{n-2}}{y_n(\pm1\pm x_{n-1}x_{n-2})}.$$

Touafek et al. [10] investigated the periodic nature and gave the form of the solutions of the following systems of rational second order difference equations

$$x_{n+1}=\frac{y_n}{x_{n-1}(\pm1\pm y_n)},\ y_{n+1}=\frac{x_n}{y_{n-1}(\pm1\pm x_n)}.$$

Yang et al. [11] studied global behavior of the system of the two nonlinear difference equations

$$x_{n+1}=\frac{Ax_n}{1+y_n^p},\quad y_{n+1}=\frac{By_n}{1+x_n^p}.$$

Din et al. [6] studied the behavior of the solutions of the following system of difference equations

$$x_{n+1}=\frac{\alpha x_{n-3}}{\beta+\gamma y_n y_{n-1}y_{n-2}y_{n-3}},\ y_{n+1}=\frac{\alpha_1 y_{n-3}}{\beta_1+\gamma_1 x_n x_{n-1}x_{n-2}x_{n-3}}.$$

Definition 1. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.

Definition 2. (Fibonacci Sequence)

The sequence $\{f_m\}_{m=1}^{\infty}=\{1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ ...\}$ i.e. $f_{m+1}=f_m+f_{m-1},\ m\geq 0,\ f_{-1}=0,\ f_0=1$ is called Fibonacci Sequence.

2. THE FIRST SYSTEM: $Z_{N+1}=\frac{Z_N T_{N-1}}{T_N-T_{N-1}},\ T_{N+1}=\frac{T_N Z_{N-1}}{Z_N-Z_{N-1}}$

In this section, we investigate the solutions of the two difference equations system

$$z_{n+1}=\frac{z_n t_{n-1}}{t_n-t_{n-1}},\ t_{n+1}=\frac{t_n z_{n-1}}{z_n-z_{n-1}}, \tag{1}$$

where $n\in\mathbb{N}_0$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary nonzero real numbers

Theorem 2.1. *Assume that $\{z_n,\ t_n\}$ are solutions of system (1). Then for $n=0,\ 1,\ 2,\ ...$, we see that all solutions of system (1) are given by the following formulae*

$$z_{2n-1}=z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},\ z_{2n}=z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},$$

and

$$t_{2n-1}=t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})},\ t_{2n}=t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})},$$

where $\{f_m\}_{m=-2}^{\infty}=\{1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ ...\}$.

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned} z_{2n-3} &= z_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},\ z_{2n-2}=z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},\\ t_{2n-3} &= t_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})},\ t_{2n-2}=t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}, \end{aligned}$$

Now we find from system (1) that

$$\begin{aligned}
z_{2n-1} &= \frac{z_{2n-2}t_{2n-3}}{t_{2n-2}-t_{2n-3}} \\
&= \frac{\left(z_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)\left(t_{-1}\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)}{\left(t_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)-\left(t_{-1}\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)} \\
&= \frac{z_0t_{-1}\prod\limits_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{\left(t_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}t_0-f_{2i}t_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i+1}t_0-f_{2i+2}t_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}\right)-t_{-1}} = \frac{z_0t_{-1}\prod\limits_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{\left(t_0\frac{(f_{-1}t_0-f_0t_{-1})(f_{2n-4}t_0-f_{2n-3}t_{-1})}{(f_{2n-3}t_0-f_{2n-2}t_{-1})(f_{-2}t_0-f_{-1}t_{-1})}\right)-t_{-1}} \\
&= \frac{z_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{-\frac{(f_{2n-4}t_0-f_{2n-3}t_{-1})}{(f_{2n-3}t_0-f_{2n-2}t_{-1})}-1}\left(\frac{f_{2n-3}t_0-f_{2n-2}t_{-1}}{f_{2n-3}t_0-f_{2n-2}t_{-1}}\right) \\
&= \frac{z_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}(f_{2n-3}t_0-f_{2n-2}t_{-1})}{-f_{2n-4}t_0+f_{2n-3}t_{-1}-f_{2n-3}t_0+f_{2n-2}t_{-1}} \\
&= z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\frac{(f_{2n-3}t_0-f_{2n-2}t_{-1})}{(-f_{2n-2}t_0+f_{2n-1}t_{-1})} \\
&= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},
\end{aligned}$$

$$\begin{aligned}
t_{2n-1} &= \frac{t_{2n-2}z_{2n-3}}{z_{2n-2}-z_{2n-3}} \\
&= \frac{\left(t_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)\left(z_{-1}\prod\limits_{i=0}^{n-2}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)}{\left(z_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)-\left(z_{-1}\prod\limits_{i=0}^{n-2}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)} \\
&= \frac{z_{-1}t_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}}{z_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}z_0-f_{2i+1}z_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i-2}z_0-f_{2i-1}z_{-1})}-z_{-1}} \\
&= \frac{z_{-1}t_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}}{z_0\frac{(f_{-1}z_0-f_0z_{-1})(f_{2n-4}z_0-f_{2n-3}z_{-1})}{(f_{2n-3}z_0-f_{2n-2}z_{-1})(f_{-2}z_0-f_{-1}z_{-1})}-z_{-1}} \\
&= \frac{t_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}}{-\frac{(f_{2n-4}z_0-f_{2n-3}z_{-1})}{(f_{2n-3}z_0-f_{2n-2}z_{-1})}-1}\left(\frac{f_{2n-3}z_0-f_{2n-2}z_{-1}}{f_{2n-3}z_0-f_{2n-2}z_{-1}}\right) \\
&= \frac{t_0\prod\limits_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}(f_{2n-3}z_0-f_{2n-2}z_{-1})}{-f_{2n-4}z_0+f_{2n-3}z_{-1}-f_{2n-3}z_0+f_{2n-2}z_{-1}} \\
&= t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\frac{(f_{2n-3}z_0-f_{2n-2}z_{-1})}{(-f_{2n-2}z_0+f_{2n-1}z_{-1})} \\
&= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}.
\end{aligned}$$

Also, we infer from system (1) that

$$\begin{aligned}
z_{2n} &= \frac{z_{2n-1}t_{2n-2}}{t_{2n-1}-t_{2n-2}} \\
&= \frac{\left(z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)\left(t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)}{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)-\left(t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)} \\
&= \frac{z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{\left(-\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})}\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})}\right)-1} \\
&= \frac{z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{-\frac{(f_{2n-3}z_0-f_{2n-2}z_{-1})}{(f_{2n-2}z_0-f_{2n-1}z_{-1})}-1}\left(\frac{(f_{2n-2}z_0-f_{2n-1}z_{-1})}{(f_{2n-2}z_0-f_{2n-1}z_{-1})}\right) \\
&= \frac{z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}(f_{2n-2}z_0-f_{2n-1}z_{-1})}{-f_{2n-3}z_0+f_{2n-2}z_{-1}-f_{2n-2}z_0+f_{2n-1}z_{-1}} \\
&= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\frac{(f_{2n-2}z_0-f_{2n-1}z_{-1})}{(-f_{2n-1}z_0+f_{2n}z_{-1})} \\
&= z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},
\end{aligned}$$

and so,

$$\begin{aligned}
t_{2n} &= \frac{t_{2n-1}z_{2n-2}}{z_{2n-1}-z_{2n-2}} \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)\left(z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)}{\left(z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)-\left(z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)} \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)}{\left(-\prod_{i=0}^{n-1}\frac{(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i}t_0-f_{2i+1}t_{-1})}\prod_{i=0}^{n-2}\frac{(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i-1}t_0-f_{2i}t_{-1})}\right)-1} \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)}{-\frac{(f_{2n-3}t_0-f_{2n-2}t_{-1})}{(f_{2n-2}t_0-f_{2n-1}t_{-1})}-1}\left(\frac{(f_{2n-2}t_0-f_{2n-1}t_{-1})}{(f_{2n-2}t_0-f_{2n-1}t_{-1})}\right) \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)(f_{2n-2}t_0-f_{2n-1}t_{-1})}{-f_{2n-3}t_0+f_{2n-2}t_{-1}-f_{2n-2}t_0+f_{2n-1}t_{-1}} \\
&= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\frac{(f_{2n-2}t_0-f_{2n-1}t_{-1})}{(-f_{2n-1}t_0+f_{2n}t_{-1})} \\
&= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}t_0-f_{2i-1}t_{-1})}{f_{2i-1}t_0-f_{2i}t_{-1})}\frac{(f_{2n-2}t_0-f_{2n-1}t_{-1})}{(-f_{2n-1}t_0+f_{2n}t_{-1})} \\
&= t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}.
\end{aligned}$$

The proof is complete.

Example 1. For confirming the results of this section, we consider numerical example for the difference system (1) with the initial conditions $z_{-1} = 0.3,\ z_0 = 0.4,\ t_{-1} = 0.15$ and $t_0 = -0.1$. (See Fig. 1).

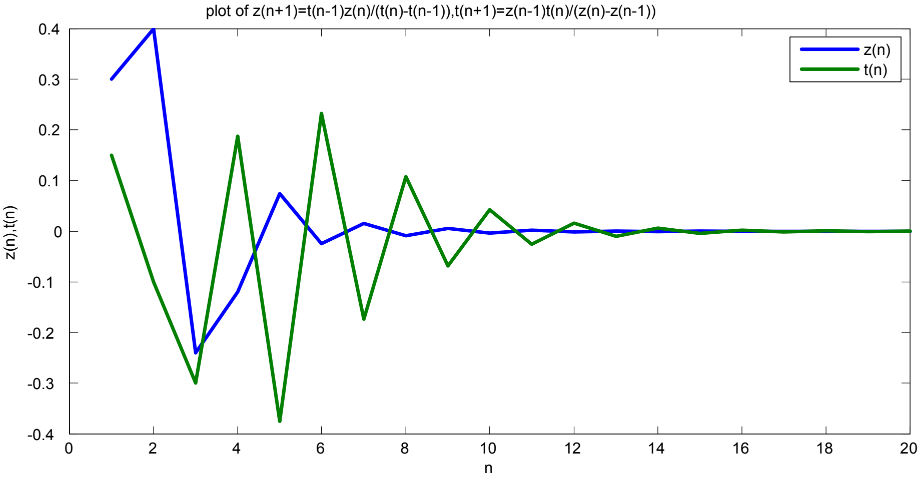

Figure 1. Plot the behavior of the solution of the system (1).

3. THE SECOND SYSTEM: $Z_{N+1} = \frac{Z_N T_{N-1}}{T_N - T_{N-1}},\ T_{N+1} = \frac{T_N Z_{N-1}}{-Z_N - Z_{N-1}}$

We obtain, in this section, the form of the solutions of the difference equations system

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n - z_{n-1}}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 are arbitrary non zero real numbers with $z_{-1} \neq -z_0$.

THEOREM 3.1. *Let* $\{z_n,\ t_n\}_{n=-1}^{+\infty}$ *be solutions of system (2).Then* $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the formulae for* $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned}
z_{4n} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})},\\
z_{4n+1} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\
z_{4n+2} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\
z_{4n+3} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})(f_{2n+2}t_0-f_{2n+1}t_{-1})},
\end{aligned}$$

and

$$\begin{aligned}
t_{4n} &= \frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})},\ t_{4n+1} = \frac{-(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})},\\
t_{4n+2} &= \frac{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0+z_{-1})},\ t_{4n+3} = \frac{-(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0+z_{-1})}.
\end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned}
z_{4n-4} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-4}z_0+f_{2n-2}z_{-1})(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-3}t_0-f_{2n-4}t_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})},\\
z_{4n-3} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-4}z_0+f_{2n-2}z_{-1})(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})},\\
z_{4n-2} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})},\\
z_{4n-1} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})},
\end{aligned}$$

$$
\begin{aligned}
t_{4n-4} &= \frac{(f_{2n-4}z_0+f_{2n-2}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})}{(z_0+z_{-1})},\ t_{4n-3}=\frac{-(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})}{(z_0+z_{-1})},\\
t_{4n-2} &= \frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})},\ t_{4n-1}=\frac{-(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})}.
\end{aligned}
$$

Now, we obtain from system (2) that

$$
\begin{aligned}
z_{4n+1} &= \frac{z_{4n}t_{4n-1}}{t_{4n}-t_{4n-1}}=\frac{\left(\frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)\left(\frac{-(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})}\right)}{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)-\left(\frac{-(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})}\right)}\\
&= \frac{\left(\frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)}{(f_{2n-1}t_0-f_{2n-2}t_{-1})+(f_{2n}t_0-f_{2n-1}t_{-1})}\\
&= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+11}t_0-f_{2n}t_{-1})},
\end{aligned}
$$

$$
\begin{aligned}
t_{4n+1} &= \frac{t_{4n}z_{4n-1}}{-z_{4n}-z_{4n-1}}=\frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)\left(\frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)}{\left[-\left(\frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)-\left(\frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)\right]}\\
&= \frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{-1+\frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}}=-\frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{1-\frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}}\\
&= -\frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{1-\frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}}=-\frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{\frac{(f_{2n-2}z_0+f_{2n}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}}\\
&= \frac{-(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}.
\end{aligned}
$$

Also, we can prove the other relations. This completes the proof.

Example 2. We assume that the initial conditions for the difference system (2) are $z_{-1}=0.38$, $z_0=-17$, $t_{-1}=0.85$ and $t_0=1.26$. (See Fig. 2).

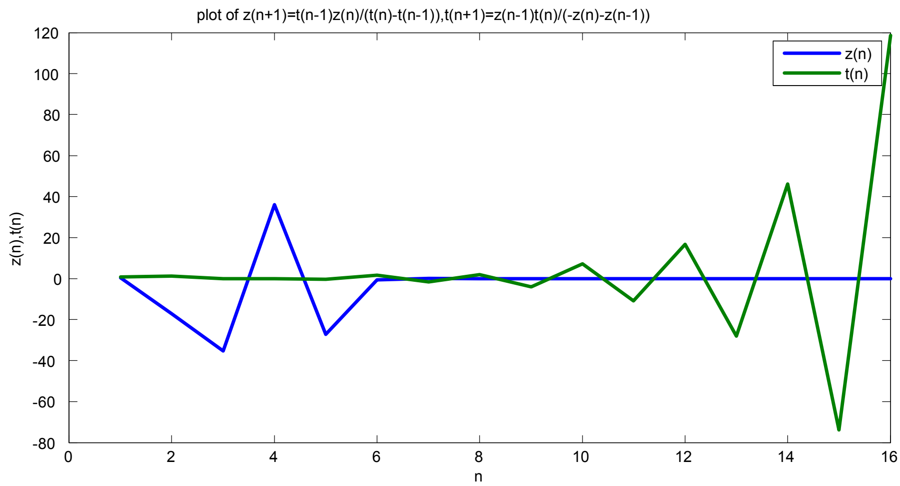

Figure 2. Sketch the behavior of the solution of the system (2).

4. PERIODICITY OF THE SYSTEMS

In this section, we study the periodicity nature of the solutions of the following systems of the difference equations

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{z_n + z_{n-1}}. \tag{3}$$

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n + t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{z_n - z_{n-1}}. \tag{4}$$

$$z_{n+1} = \frac{z_n t_{n-1}}{-t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n - z_{n-1}}. \tag{5}$$

Where $n = 0,\ 1,\ 2,\ ...$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary nonzero real numbers.

Theorem 4.1. *Suppose that $\{z_n,\ t_n\}$ are solutions of difference equation system (3) with $z_0 \neq -z_{-1},\ t_0 \neq t_{-1}$. Then all solutions of system (3) are periodic with period six and for $n = 0,\ 1,\ 2,\ ...$,*

$$z_{6n-1} = z_{-1},\ z_{6n} = z_0,\ z_{6n+1} = \frac{z_0 t_{-1}}{t_0 - t_{-1}},\ z_{6n+2} = \frac{t_{-1}(z_0 + z_{-1})}{(t_{-1} - t_0)},\ z_{6n+3} = \frac{t_0(z_0 + z_{-1})}{(t_0 - t_{-1})},\ z_{6n+4} = \frac{z_{-1} t_0}{(t_{-1} - t_0)},$$

and

$$t_{6n-1} = t_{-1},\ t_{6n} = t_0,\ t_{6n+1} = \frac{z_{-1} t_0}{z_0 + z_{-1}},\ t_{6n+2} = \frac{z_{-1}(t_0 - t_{-1})}{(z_0 + z_{-1})},\ t_{6n+3} = \frac{z_0(t_{-1} - t_0)}{(z_0 + z_{-1})},\ t_{6n+4} = \frac{z_0 t_{-1}}{(z_0 + z_{-1})}.$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$z_{6n-7} = z_{-1},\ z_{6n-6} = z_0,\ z_{6n-5} = \frac{z_0 t_{-1}}{t_0 - t_{-1}},\ z_{6n-4} = \frac{t_{-1}(z_0 + z_{-1})}{(t_{-1} - t_0)},\ z_{6n-3} = \frac{t_0(z_0 + z_{-1})}{(t_0 - t_{-1})},\ z_{6n-2} = \frac{z_{-1} t_0}{(t_{-1} - t_0)},$$

and

$$t_{6n-7} = t_{-1},\ t_{6n-6} = t_0,\ t_{6n-5} = \frac{z_{-1} t_0}{z_0 + z_{-1}},\ t_{6n-4} = \frac{z_{-1}(t_0 - t_{-1})}{(z_0 + z_{-1})},\ t_{6n-3} = \frac{z_0(t_{-1} - t_0)}{(z_0 + z_{-1})},\ t_{6n-2} = \frac{z_0 t_{-1}}{(z_0 + z_{-1})}.$$

Now, we obtain from system (3) that

$$\begin{aligned}
z_{6n-1} &= \frac{z_{6n-2} t_{6n-3}}{t_{6n-2} - t_{6n-3}} = \frac{\left(\frac{z_{-1} t_0}{(t_{-1} - t_0)}\right)\left(\frac{z_0(t_{-1} - t_0)}{(z_0 + z_{-1})}\right)}{\left(\frac{z_0 t_{-1}}{(z_0 + z_{-1})}\right) - \left(\frac{z_0(t_{-1} - t_0)}{(z_0 + z_{-1})}\right)} = \frac{z_{-1} t_0 z_0}{z_0 t_{-1} - z_0(t_{-1} - t_0)} = \frac{z_{-1} t_0 z_0}{z_0 t_0} = z_{-1}, \\
t_{6n-1} &= \frac{t_{6n-2} z_{6n-3}}{z_{6n-2} + z_{6n-3}} = \frac{\left(\frac{z_0 t_{-1}}{(z_0 + z_{-1})}\right)\left(\frac{t_0(z_0 + z_{-1})}{(t_0 - t_{-1})}\right)}{\left(\frac{z_{-1} t_0}{(t_{-1} - t_0)}\right) + \left(\frac{t_0(z_0 + z_{-1})}{(t_0 - t_{-1})}\right)} = \frac{z_0 t_{-1} t_0}{-z_{-1} t_0 + t_0(z_0 + z_{-1})} = \frac{z_0 t_{-1} t_0}{t_0 z_0} = t_{-1}, \\
z_{6n} &= \frac{z_{6n-1} t_{6n-2}}{t_{6n-1} - t_{6n-2}} = \frac{z_{-1} \frac{z_0 t_{-1}}{(z_0 + z_{-1})}}{t_{-1} - \frac{z_0 t_{-1}}{(z_0 + z_{-1})}} = \frac{z_{-1} z_0 t_{-1}}{t_{-1}(z_0 + z_{-1}) - z_0 t_{-1}} = z_0, \\
t_{6n} &= \frac{t_{6n-1} z_{6n-2}}{z_{6n-1} + z_{6n-2}} = \frac{t_{-1} \frac{z_{-1} t_0}{(t_{-1} - t_0)}}{z_{-1} + \frac{z_{-1} t_0}{(t_{-1} - t_0)}} = \frac{t_{-1} z_{-1} t_0}{z_{-1}(t_{-1} - t_0) + z_{-1} t_0} = t_0.
\end{aligned}$$

We can prove the other relations similarly. The proof is completed.

Theorem 4.2. *If $\{z_n,\ t_n\}$ are solutions of system (4) with $z_0 \neq z_{-1},\ t_0 \neq -t_{-1}$. Then all solutions of system (4) are periodic with period six and given by the formulae*

$$\begin{aligned}
z_{6n-1} &= z_{-1},\ z_{6n} = z_0,\ z_{6n+1} = \frac{z_0 t_{-1}}{t_0 + t_{-1}},\ z_{6n+2} = \frac{t_{-1}(z_0 - z_{-1})}{(t_0 + t_{-1})},\ z_{6n+3} = \frac{t_0(z_{-1} - z_0)}{(t_0 + t_{-1})},\ z_{6n+4} = \frac{z_{-1} t_0}{t_0 + t_{-1}}, \\
t_{6n-1} &= t_{-1},\ t_{6n} = t_0,\ t_{6n+1} = \frac{z_{-1} t_0}{z_0 - z_{-1}},\ t_{6n+2} = \frac{z_{-1}(t_0 + t_{-1})}{(z_{-1} - z_0)},\ t_{6n+3} = \frac{z_0(t_0 + t_{-1})}{(z_0 - z_{-1})},\ t_{6n+4} = \frac{z_0 t_{-1}}{z_{-1} - z_0}.
\end{aligned}$$

THEOREM 4.3. *Assume that* $\{z_n,\ t_n\}$ *are solutions of difference equation system (5) with* $z_0 \neq -z_{-1},\ t_0 \neq -t_{-1}$. *Then all solutions of system (5) are periodic with period six and for* $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{6n-1} &= z_{-1},\ z_{6n} = z_0,\ z_{6n+1} = -\frac{z_0 t_{-1}}{t_0+t_{-1}},\ z_{6n+2} = \frac{t_{-1}(z_0+z_{-1})}{(t_0+t_{-1})},\ z_{6n+3} = \frac{t_0(z_0+z_{-1})}{(t_0+t_{-1})},\ z_{6n+4} = -\frac{z_{-1}t_0}{t_0+t_{-1}}, \\ t_{6n-1} &= t_{-1},\ t_{6n} = t_0,\ t_{6n+1} = -\frac{z_{-1}t_0}{z_0+z_{-1}},\ t_{6n+2} = \frac{z_{-1}(t_0+t_{-1})}{(z_0+z_{-1})},\ t_{6n+3} = \frac{z_0(t_{-1}+t_0)}{(z_0+z_{-1})},\ t_{6n+4} = -\frac{z_0 t_{-1}}{z_0+z_{-1}}. \end{aligned}$$

Example 3. See Figure (3) where we take system (3) with the initial conditions $z_{-1} = 0.18$, $z_0 = 0.17$, $t_{-1} = 0.5$ and $t_0 = 0.86$.

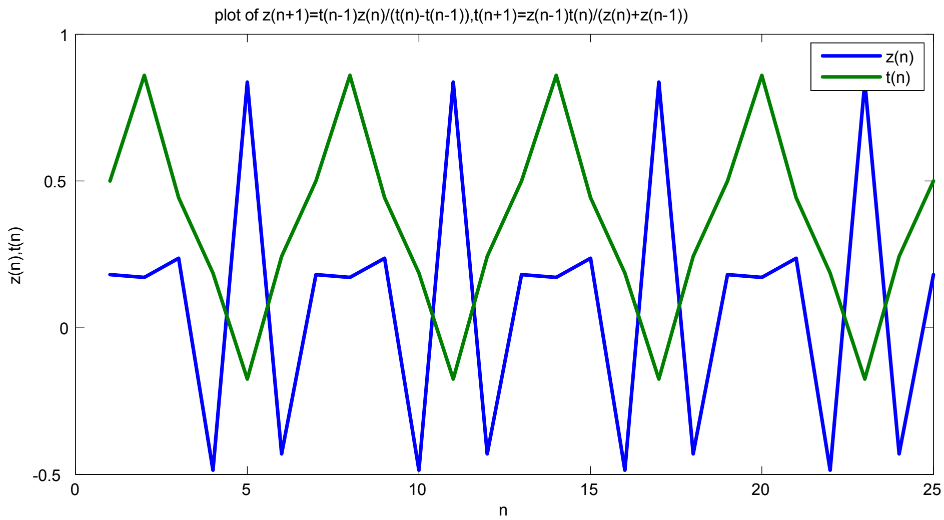

Figure 3. Draw the behavior of the solution of the system (3).

5. OTHER SYSTEMS

In this section, we obtain the form of the solutions of the follwing systems of the difference equations.

THEOREM 5.1. *If* $\{z_n,\ t_n\}$ *are solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{-t_n+t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n-z_{n-1}}, \tag{6}$$

where $n \in \mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers, then for* $n = 0, 1, 2, ...,$

$$\begin{aligned} z_{2n-1} &= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0+f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0+f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},\ z_{2n} = z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0+f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0+f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}, \\ t_{2n-1} &= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0+f_{2i}z_{-1})(f_{2i-1}t_{-1}-f_{2i-2}t_0)}{(f_{2i}z_0+f_{2i+1}z_{-1})(f_{2i}t_{-1}-f_{2i-1}t_0)},\ t_{2n} = t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0+f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0+f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}. \end{aligned}$$

such that $\prod_{i=0}^{-1} A_i = 1.$

THEOREM 5.2. *The solutions of system*

$$z_{n+1}=\frac{z_nt_{n-1}}{-t_n-t_{n-1}},\ t_{n+1}=\frac{t_nz_{n-1}}{-z_n+z_{n-1}}, \tag{7}$$

are given by the relations

$$\begin{aligned} z_{2n-1} &= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_{-1}-f_{2i-2}z_0)(f_{2i-1}t_0+f_{2i}t_{-1})}{(f_{2i}z_{-1}-f_{2i-1}z_0)(f_{2i}t_0+f_{2i+1}t_{-1})},\ z_{2n}=z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0+f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0+f_{2i+1}t_{-1})},\\ t_{2n-1} &= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0+f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0+f_{2i}t_{-1})},\ t_{2n}=t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0+f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0+f_{2i+2}t_{-1})}, \end{aligned}$$

where $n\in\mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers and* $\prod_{i=0}^{-1}A_i=1$.

THEOREM 5.3. *Suppose that* $\{z_n,\ t_n\}_{n=-1}^{+\infty}$ *are solutions of system*

$$z_{n+1}=\frac{z_nt_{n-1}}{-t_n-t_{n-1}},\ t_{n+1}=\frac{t_nz_{n-1}}{z_n+z_{n-1}}. \tag{8}$$

where $n\in\mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers with* $z_{-1}\neq -z_0$. *Then* $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the formula for* $n=0,\ 1,\ 2,\ \ldots,$

$$\begin{aligned} z_{4n} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_{-1}+f_{2n-2}t_0)(f_{2n}t_{-1}+f_{2n-1}t_0)},\\ z_{4n+1} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_{-1}+f_{2n-1}t_0)(f_{2n+1}t_{-1}+f_{2n}t_0)},\\ z_{4n+2} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n}t_{-1}+f_{2n-1}t_0)(f_{2n+1}t_{-1}+f_{2n}t_0)},\\ z_{4n+3} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n+1}t_{-1}+f_{2n}t_0)(f_{2n+2}t_{-1}+f_{2n+1}t_0)}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0+f_{2n-1}t_{-1})}{(z_0+z_{-1})},\ t_{4n+1}=\frac{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0+f_{2n-1}t_{-1})}{(z_0+z_{-1})},\\ t_{4n+2} &= \frac{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n+1}t_0+f_{2n}t_{-1})}{(z_0+z_{-1})},\ t_{4n+3}=\frac{(f_{2n}z_0+f_{2n+2}b)(f_{2n+1}t_0+f_{2n}t_{-1})}{(z_0+z_{-1})}. \end{aligned}$$

THEOREM 5.4. *Let* $\{z_n,t_n\}_{n=-1}^{+\infty}$ *be solutions of system*

$$z_{n+1}=\frac{z_nt_{n-1}}{-t_n+t_{n-1}},\ t_{n+1}=\frac{t_nz_{n-1}}{z_n-z_{n-1}}. \tag{9}$$

Then $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the following expressions for* $n=0,1,2,\ldots,$

$$\begin{aligned} z_{4n} &= \frac{z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-2}z_0-f_{2n}z_{-1})(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})},\\ z_{4n+1} &= \frac{z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-2}z_0-f_{2n}z_{-1})(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\ z_{4n+2} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}z_0-f_{2n+2}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\ z_{4n+3} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}z_0-f_{2n+2}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})(f_{2n+2}t_0-f_{2n+1}t_{-1})}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{(f_{2n-2}z_0-f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0-z_{-1})},\ t_{4n+1}=\frac{-(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0-z_{-1})},\\ t_{4n+2} &= \frac{(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0-z_{-1})},\ t_{4n+3}=\frac{-(f_{2n}z_0-f_{2n+2}b)(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0-z_{-1})}. \end{aligned}$$

where $n \in \mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers with* $z_{-1} \neq z_0$.

THEOREM 5.5. *Let* $\{z_n,\ t_n\}_{n=-1}^{+\infty}$ *be solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{-t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{z_n - z_{n-1}}, \tag{10}$$

where $n \in \mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers with* $t_{-1} \neq -t_0$. *Then* $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the following relations for* $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{4n} &= \frac{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})}{t_0 + t_{-1}},\ z_{4n+1} = \frac{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}}, \\ z_{4n+2} &= \frac{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}},\ z_{4n+3} = \frac{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}{t_0 + t_{-1}}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n-1}z_0 - f_{2n-2}z_{-1})(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}, \\ t_{4n+1} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}, \\ t_{4n+2} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}, \\ t_{4n+3} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n+2}z_0 - f_{2n+1}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}. \end{aligned}$$

THEOREM 5.6. *Suppose that* $\{z_n,\ t_n\}_{n=-1}^{+\infty}$ *be solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n + z_{n-1}}. \tag{11}$$

Then $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the following relations for* $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{4n} &= \frac{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 - f_{2n}t_{-1})}{t_0 - t_{-1}},\ z_{4n+1} = \frac{-(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}{t_0 - t_{-1}}, \\ z_{4n+2} &= \frac{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}{t_0 - t_{-1}},\ z_{4n+3} = \frac{-(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n}t_0 - f_{2n+2}t_{-1})}{t_0 - t_{-1}}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n-1}z_0 - f_{2n-2}z_{-1})(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 - f_{2n}t_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}, \\ t_{4n+1} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-2}t_0 - f_{2n}t_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}, \\ t_{4n+2} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})(f_{2n}t_0 - f_{2n+2}t_{-1})}, \\ t_{4n+3} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n+2}z_0 - f_{2n+1}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})(f_{2n}t_0 - f_{2n+2}t_{-1})}. \end{aligned}$$

where $n \in \mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers with* $t_{-1} \neq t_0$.

THEOREM 5.7. *Let* $\{z_n, t_n\}_{n=-1}^{+\infty}$ *be solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n + t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n - z_{n-1}}. \tag{12}$$

Then $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the following relations for* $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{4n} &= \frac{(f_{2n}z_0 + f_{2n-1}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})}{t_0 + t_{-1}},\ z_{4n+1} = \frac{(f_{2n}z_0 + f_{2n-1}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}}, \\ z_{4n+2} &= \frac{(f_{2n+1}z_0 + f_{2n}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}},\ z_{4n+3} = \frac{(f_{2n+1}z_0 + f_{2n}z_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}{t_0 + t_{-1}}, \end{aligned}$$

and

$$\begin{aligned}
t_{4n} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n-1}z_0+f_{2n-2}z_{-1})(f_{2n}z_0+f_{2n-1}z_{-1})(f_{2n-2}t_0+f_{2n}t_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})},\\
t_{4n+1} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n}z_0+f_{2n-1}z_{-1})(f_{2n+1}z_0+f_{2n}z_{-1})(f_{2n-2}t_0+f_{2n}t_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})},\\
t_{4n+2} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n}z_0+f_{2n-1}z_{-1})(f_{2n+1}z_0+f_{2n}z_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})(f_{2n}t_0+f_{2n+2}t_{-1})},\\
t_{4n+3} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n+1}z_0+f_{2n}z_{-1})(f_{2n+2}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})(f_{2n}t_0+f_{2n+2}t_{-1})}.
\end{aligned}$$

where $n \in \mathbb{N}_0$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary non zero real numbers with $t_{-1} \neq -t_0$.

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Hardy type inequalities for Choquet integrals

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Abstract

Here we present Hardy type integral inequalities for Choquet integrals. These are very general inequalities involving convex and increasing functions. Initially we collect a rich machinery of results about Choquet integrals needed next, and we prove also results of their own merit such as, Choquet-Hölder's inequalities for more than two functions and a multivariate Choquet-Fubini's theorem. The main proving tool here is the property of comonotonicity of functions. We finish with independent estimates on left and right Riemann-Liouville-Choquet fractional integrals.

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1 Introduction

To motivate the work in this article we mention the Riemann-Liouville fractional integrals, see [9]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left and right Riemann-Liouville fractional integrals $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ (respectively) of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a),$$
$$\left(I_{b-}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad (x < b),$$

where Γ is the Gamma function.

We mention a basic property of the operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha > 0$, see also [11]: It holds that the fractional integral operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\left\| I_{a+}^{\alpha} f \right\|_p \leq K \|f\|_p, \quad \left\| I_{b-}^{\alpha} f \right\|_p \leq K \|f\|_p,$$

where

$$K = \frac{(b-a)^{\alpha}}{\alpha \Gamma(\alpha)}.$$

The first inequality that is the result involving the left-sided fractional integral, was proved by H.G. Hardy in one of his first papers, see [7]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

General Hardy inequalities of the above type were derived also in [8] and [1]. We continue this kind of research for Choquet integrals based on the comonotonicity property of functions and convexity. We derive a wide range of Choquet integral inequalities of Hardy type.

2 Background

In this section we give some definitions and basic properties of Choquet integral essential for this work.

Definition 1 *([15]) Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X and $\mu : \mathcal{F} \to [0,\infty]$ be a nonnegative real-valued set function, μ is said to be a fuzzy measure iff:*

(1) $\mu(\varnothing) = 0$,

(2) for any $A, B \in \mathcal{F}$, $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity),

(3) for $\{A_n\} \subseteq \mathcal{F}$, $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$, implies $\lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n)$ (continuity from below)

(4) for $\{A_n\} \subseteq \mathcal{F}$, $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$, $\mu(A_1) < \infty$, implies $\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n)$ (continuity from above).

If μ is a fuzzy measure from $\mathcal{F}$ to $[0,1]$ with $\mu(X) = 1$, μ is called a regular fuzzy measure. If μ is a fuzzy measure, $(X, \mathcal{F}, \mu)$ is called a fuzzy measure space and $(X, \mathcal{F})$ is a fuzzy measurable space. Clearly μ is not necessarily an additive measure. Let F be the set of all real-valued nonnegative measurable functions defined on X.

Definition 2 *([10]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, μ is said to be submodular (supermodular) if*

$$\mu(A \cap B) + \mu(A \cup B) \leq (\geq) \mu(A) + \mu(B), \quad \forall\ A, B \subseteq \mathcal{F}. \tag{1}$$

Definition 3 *([4]) Let $f, g \in F$, f and g are said to be comonotonic iff $f(x) < f(x')$ implies $g(x) \leq g(x')$, $\forall\ x, x' \in X$.*

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Definition 4 *([5], [16]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, $f \in F$ and $A \in \mathcal{F}$. The Choquet integral of f with respect to μ on A is defined by*

$$(C) \int_A f d\mu = \int_0^\infty \mu(A \cap \{x | f(x) \geq \alpha\}) \, d\alpha. \tag{2}$$

If $(C) \int_X f d\mu < \infty$, we call f (C)-integrable, $L_1(\mu)$ is the set of all (C)-integrable function.

Clearly $(C) \int_X f d\mu < \infty$, implies $(C) \int_A f d\mu < \infty$.

Theorem 5 *([14]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measurable space, $\{f_1, f_2, f\} \subset F$, $A, B \in \mathcal{F}$ and $c \geq 0$ constant. Then,*

(1) if $\mu(A) = 0$, then $(C) \int_A f d\mu = 0$,

(2) $(C) \int_A c d\mu = c\mu(A)$,

(3) if $f_1 \leq f_2$, then

$$(C) \int_A f_1 d\mu \leq (C) \int_A f_2 d\mu, \tag{3}$$

(4) if $A \subset B$, then $(C) \int_A f d\mu \leq (C) \int_B f d\mu$,

(5) $(C) \int_A (f + c) \, d\mu = (C) \int_A f d\mu + c\mu(A)$,

(6) $(C) \int_A c f d\mu = c\left((C) \int_A f d\mu\right)$.

Theorem 6 *([5]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g \in F$. Then*

(1) if f, g are comonotonic, then for any $A \in \mathcal{F}$,

$$(C) \int_A (f + g) \, d\mu = (C) \int_A f d\mu + (C) \int_A g d\mu, \tag{4}$$

(2) if μ is submodular, then for any $A \in \mathcal{F}$,

$$(C) \int_A (f + g) \, d\mu \leq (C) \int_A f d\mu + (C) \int_A g d\mu. \tag{5}$$

The Jensen's inequality for Choquet integrals follows:

Theorem 7 *([13]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f \in L_1(\mu)$. If μ is a regular fuzzy measure and $\Phi : [0, \infty) \to [0, \infty)$ is a convex function, then*

$$\Phi\left((C) \int_X f d\mu\right) \leq (C) \int_X \Phi(f) \, d\mu. \tag{6}$$

Corollary 8 *([13]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f \in L_1(\mu)$. If μ is a regular fuzzy measure, then*

$$\left((C) \int_X f d\mu\right)^p \leq (C) \int_X f^p d\mu, \tag{7}$$

for any $1 < p < \infty$.

Theorem 9 *([13]) (Hölder's inequality) Let* $(X, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $f, g \in F$. *If* μ *is a submodular fuzzy measure and* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$(C) \int_X fg d\mu \leq \left((C) \int_X f^p d\mu \right)^{\frac{1}{p}} \left((C) \int_X g^q d\mu \right)^{\frac{1}{q}}. \tag{8}$$

Theorem 10 *([13]) (Minkowski inequality) Let* $(X, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $f, g \in F$. *If* μ *is a submodular fuzzy measure and* $1 \leq p < \infty$, *then*

$$\left((C) \int_X (f+g)^p \, d\mu \right)^{\frac{1}{p}} \leq \left((C) \int_X f^p d\mu \right)^{\frac{1}{p}} + \left((C) \int_X g^p d\mu \right)^{\frac{1}{p}}. \tag{9}$$

We give

Theorem 11 *(Hölder's inequality for three functions) Let* $(X, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $f_1, f_2, f_3 \in F$. *If* μ *is a submodular fuzzy measure and* $1 < p_1 \leq p_2 \leq p_3 < \infty$ *with* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, *then*

$$(C) \int_X f_1 f_2 f_3 d\mu \leq \left((C) \int_X f_1^{p_1} d\mu \right)^{\frac{1}{p_1}} \left((C) \int_X f_2^{p_2} d\mu \right)^{\frac{1}{p_2}} \left((C) \int_X f_3^{p_3} d\mu \right)^{\frac{1}{p_3}}. \tag{10}$$

Proof. Let $p = \frac{p_3}{p_3 - 1} > 1$ and $q = p_3$. Notice that $\frac{1}{p} + \frac{1}{q} = 1$.

We apply (8) as follows

$$(C) \int_X f_1 f_2 f_3 d\mu \leq \left((C) \int_X (f_1 f_2)^p \, d\mu \right)^{\frac{1}{p}} \left((C) \int_X f_3^{p_3} d\mu \right)^{\frac{1}{p_3}}. \tag{11}$$

We see that

$$\frac{p}{p_1} + \frac{p}{p_2} = p \left(\frac{1}{p_1} + \frac{1}{p_2} \right) = p \left(1 - \frac{1}{p_3} \right) = p \left(\frac{p_3 - 1}{p_3} \right) = 1. \tag{12}$$

Clearly it holds $\frac{p_1}{p}, \frac{p_2}{p} > 1$.

Therefore we get

$$(C) \int_X f_1^p f_2^p d\mu \overset{(8)}{\leq} \left((C) \int_X f_1^{p \frac{p_1}{p}} d\mu \right)^{\frac{p}{p_1}} \left((C) \int_X f_2^{p \frac{p_2}{p}} d\mu \right)^{\frac{p}{p_2}} = \tag{13}$$

$$\left((C) \int_X f_1^{p_1} d\mu \right)^{\frac{p}{p_1}} \left((C) \int_X f_2^{p_2} d\mu \right)^{\frac{p}{p_2}}.$$

That is

$$\left((C) \int_X (f_1 f_2)^p \, d\mu \right)^{\frac{1}{p}} \leq \left((C) \int_X f_1^{p_1} d\mu \right)^{\frac{1}{p_1}} \left((C) \int_X f_2^{p_2} d\mu \right)^{\frac{1}{p_2}}. \tag{14}$$

Combining (11) and (14), we produce (10). ■

In general we have

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Theorem 12 *(Hölder's inequality for n functions) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f_i \in F$, $i = 1, ..., n \in \mathbb{N}$. If μ is a submodular fuzzy measure and $1 < p_1 \leq p_2 \leq ... \leq p_n < \infty$ with $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, then*

$$(C) \int_X \prod_{i=1}^{n} f_i d\mu \leq \prod_{i=1}^{n} \left((C) \int_X f_i^{p_i} d\mu \right)^{\frac{1}{p_i}}. \tag{15}$$

Proof. By induction. ■

Remark 13 *Let $\mathcal{A}$ be a σ-algebra, and let $\{A_k\}_{k \in \mathbb{N}} \subseteq \mathcal{A}$ be a family of pairwise disjoint sets. Here $\mathcal{P}$ is a probability measure on $(X, \mathcal{A})$ with only the finite additivity property valid: i.e.,*

$$P\left(\cup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P\left(A_k\right), \quad \forall\ n \in \mathbb{N}.$$

We observe that

$$P\left(\cup_{k=1}^{\infty} A_k\right) = \lim_{n \to \infty} P\left(\cup_{k=1}^{n} A_k\right) = \lim_{n \to \infty} \sum_{k=1}^{n} P\left(A_k\right) = \sum_{k=1}^{\infty} P\left(A_k\right). \tag{16}$$

That is, the countable additivity property holds, hence P is a usual probability measure.

Notice that a σ-algebra on X is also an algebra of subsets of X.

Definition 14 *([3], [6]) For every space Ω and algebra $\mathcal{A}$ of subsets of Ω a set-function $\sigma : \mathcal{A} \to \mathbb{R}$ is called a (normalized) capacity if it satisfies the following:*

(i)

$$\sigma(\varnothing) = 0, \quad \sigma(\Omega) = 1, \tag{17}$$

(ii) $\forall\ A, B \in \mathcal{A} : A \subseteq B \Rightarrow \sigma(A) \leq \sigma(B)$.

From (i) and (ii) we get that the range of σ is contained in $[0, 1]$.

In general the Choquet integral is defined as follows:

Definition 15 *([3], [12]) Let $(\Omega, \mathcal{A})$ be an algebra and $f : \Omega \to \mathbb{R}$ is a bounded $\mathcal{A}$-measurable function and σ is any (normalized) capacity on Ω we define the Choquet integral of f with respect to σ to be the number*

$$(C) \int_\Omega f(\omega)\, d\sigma(\omega) = \int_0^\infty \sigma\left(\{\omega \in \Omega : f(\omega) \geq \alpha\}\right) d\alpha + \tag{18}$$

$$\int_{-\infty}^{0} \left[\sigma\left(\omega \in \Omega : f(\omega) \geq \alpha\}\right) - 1\right] d\alpha,$$

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where the integrals are taken in the sense of Riemann.

A (normalized) capacity σ is called probability ([6]) iff

$$\forall\ A, B \in \mathcal{A} : \sigma(A \cup B) + \sigma(A \cap B) = \sigma(A) + \sigma(B). \tag{19}$$

Notice that since the integrands are monotone, the Choquet integral always exists, and if σ is a probability it collapses to a usual Lebesgue integral.

Definition 16 *([6]) Let $f, g : \Omega \to \mathbb{R}$ be two bounded $\mathcal{A}$-measurable functions. We say that f and g are comonotonic, if for every $\omega, \omega' \in \Omega$,*

$$(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0. \tag{20}$$

A class of functions $\mathcal{F}^*$ is said to be comonotonic if for every $f, g \in \mathcal{F}^*$, f and g are comonotonic.

Proposition 17 *([6]) If σ and λ are (normalized) capacities on the algebra $(\Omega, \mathcal{A})$, and $f, g : \Omega \to \mathbb{R}$ are bounded $\mathcal{A}$-measurable functions then:*

(i)

$$(C)\int_{\Omega} 1_A d\sigma = \sigma(A), \ \ \forall\ A \in \mathcal{A}, \tag{21}$$

where 1_A is the characteristic function on A,

(ii) (positive homogeneity)

$$(C)\int_{\Omega} pf d\sigma = p\left((C)\int_{\Omega} f d\sigma\right), \ \ \text{for every } p \geq 0, \tag{22}$$

(iii) (monotonicity) $f \geq g$ implies

$$(C)\int_{\Omega} f d\sigma \geq (C)\int_{\Omega} g d\sigma, \tag{23}$$

(iv)

$$(C)\int_{\Omega} (f + p) d\sigma = (C)\int_{\Omega} f d\sigma + p, \ \ \forall\ p \in \mathbb{R}, \tag{24}$$

(v) (comonotonic additivity) If f, g are comonotonic then

$$(C)\int_{\Omega} (f + g) d\sigma = (C)\int_{\Omega} f d\sigma + (C)\int_{\Omega} g d\sigma. \tag{25}$$

We need the very important

Lemma 18 *([6]) Let $(\Omega, \mathcal{A})$ be an algebra. Suppose that $\mathcal{F}^*$ is a comonotonic class of bounded and $\mathcal{A}$-measurable functions from Ω into $\mathbb{R}$ and σ is a (normalized) capacity on $(\Omega, \mathcal{A})$. Then there exists a probability measure P on $(\Omega, \mathcal{A})$ such that for every $f \in \mathcal{F}^*$*

$$\int_{\Omega} f d\sigma = \int_{\Omega} f dP. \tag{26}$$

Here $\int_{\Omega} f dP$ is a standard integral of Lebesgue type.

Based on Remark 13, Lemma 18 is still valid in case that $(\Omega, \mathcal{A})$ is a σ-algebra.

Definition 19 *([6]) Let X, Y be two sets and $Z = X \times Y$. Let $f : Z \to \mathbb{R}$. We say that f has comonotonic x-sections if for every $x, x' \in X$, $f(x, \cdot) : Y \to \mathbb{R}$, and $f(x', \cdot) : Y \to \mathbb{R}$ are comonotonic functions. Comonotonicity of y-sections is similarly defined. We call f slice-comonotonic if it has both comonotonic x-sections and y-sections.*

Remark 20 *Notice that Definitions 14-16 and Proposition 17, are still valid when $(\Omega, \mathcal{A})$ is a σ-algebra.*

Next we mention Fubini's theorem for Choquet integrals.

Theorem 21 *([2]) Let (Ω_1, Σ_1), (Ω_2, Σ_2) be σ-algebras. Let u_i, $i = 1, 2$ be submodular (or supermodular) regular fuzzy measures on Ω_i, respectively. Let $\Omega = \Omega_1 \times \Omega_2$ be endowed with the product σ-algebra $\Sigma = \Sigma_1 \otimes \Sigma_2$. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a slice-comonotonic bounded Σ-measurable mapping, then:*

1) $f(\cdot, \omega_2)$ is Σ_1-measurable and $\omega_2 \in \Omega_2 \to (C)\int_{\Omega_1} f(s, \omega_2)\, du_1(s)$ is bounded and Σ_2-measurable,

$f(\omega_1, \cdot)$ is Σ_2-measurable and $\omega_1 \in \Omega_1 \to (C)\int_{\Omega_2} f(\omega_1, t)\, du_2(t)$ is bounded and Σ_1-measurable,

2) the iterated integrals $(C)\int_{\Omega_2}\int_{\Omega_1} f du_1 du_2$, $(C)\int_{\Omega_1}\int_{\Omega_2} f du_2 du_1$ exist and are equal:

$$(C)\int_{\Omega_2}\left((C)\int_{\Omega_1} f(\omega_1, \omega_2)\, du_1\right) du_2 = (C)\int_{\Omega_1}\left((C)\int_{\Omega_2} f(\omega_1, \omega_2)\, du_2\right) du_1. \tag{27}$$

We give

Definition 22 *Let $f : \prod_{i=1}^{n} \Omega_i \to \mathbb{R}$, $n \in \mathbb{N}$. If the i-sections $f(x_1, ..., x_{i-1}, \cdot, x_{i+1}, ..., x_n)$ and $f\left(x_1', ..., x_{i-1}', \cdot, x_{i+1}', ..., x_n'\right)$ are comonotonic functions, for all $i = 1, ..., n$; where the vectors $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, $\left(x_1', ..., x_{i-1}', x_{i+1}', ..., x_n'\right) \in \prod_{\substack{j=1 \\ j \neq i}}^{n-1} \Omega_j$ are different, for all $i = 1, 2, ..., n$, we call f slice-n-comonotonic function.*

We denote by θ a permutation of the set $\{1, 2, ..., n\}$ into itself, $n \in \mathbb{N}$. There are $n!$ permutations.

In [2] is mentioned that Theorem 21 can be generalized for n spaces. Next we state in brief Fubini's theorem for n Choquet iterated integrals.

Theorem 23 *Let (Ω_i, Σ_i) be σ-algebras, $i = 1, 2, ..., n \in \mathbb{N}$. Let u_i, $i = 1, 2, ..., n$ be submodular (or supermodular) regular fuzzy measures on Ω_i, respectively. Let $\Omega = \prod_{i=1}^{n} \Omega_i$ be endowed with the product σ-algebra $\Sigma = \otimes_{i=1}^{n} \Sigma_i$. Let $f : \prod_{i=1}^{n} \Omega_i \to \mathbb{R}$ be a slice-comonotonic bounded Σ-measurable mapping, then*

$$(C) \int_{\Omega_n} \int_{\Omega_{n-1}} ... \int_{\Omega_1} f du_1 du_2 ... du_n =$$

$$(C) \int_{\Omega_{\theta(n)}} \int_{\Omega_{\theta(n-1)}} ... \int_{\Omega_{\theta(1)}} f du_{\theta(1)} du_{\theta(2)} ... du_{\theta(n)}, \tag{28}$$

for any permutation θ on the set $\{1, ..., n\}$. All the iterated Choquet integrals in (28) exist and are equal.

Proof. By induction, (23) and using Theorem 21. ■

Remark 24 *If μ is a countably additive bounded measure, then the Choquet integral $(C) \int_A f d\mu$ reduces to the usual Lebesgue type integral (see, e.g. [5], p. 62, or [17], p. 226), above it is $A \subseteq \Omega$.*

3 Main Results

This section is motivated by [8].

Let the fuzzy measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, where μ_1, μ_2 are regular fuzzy measures, furthermore μ_1, μ_2 are assumed to be submodular.

Let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ which is a bounded measurable function and $k(x, y)$ is $slice-$comonotonic and belongs to a comonotonic class F_1^* $as\ a\ function\ of\ y$.

Consider the function

$$K(x) = (C) \int_{\Omega_2} k(x, y) d\mu_2(y), \quad x \in \Omega_1, \tag{29}$$

and assume that $K(x) > 0$.

Notice that K is bounded.

Denote by $W(k)$ the class of functions $g : \Omega_1 \to \mathbb{R}_+$, such that

$$g(x) = (C) \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \tag{30}$$

where $f : \Omega_2 \to \mathbb{R}_+$ is a bounded measurable function, such that $k(x, y) f(y)$ is $slice-$comonotonic and belongs to a comonotonic class F_2^* $as\ a\ function\ of\ y$.

Notice that g is also bounded.

We give

Theorem 25 *Let u be a nonnegative measurable function on Ω_1. Assume that $\frac{u(x)}{K(x)}$ is bounded on Ω_1. Define υ on Ω_2 by*

$$\upsilon(y) = (C)\int_{\Omega_1} \frac{u(x)}{K(x)} k(x,y)\, d\mu_1(x), \tag{31}$$

which is bounded. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex and increasing function, such that $k(x,y)\,\Phi(f(y))$ is x-section comonotonic with comonotonic class F_3^*. Assume here that $(F_1^* \cup F_2^* \cup F_3^*) \subseteq F^*$, where F^* is one comonotonic class of functions on Ω_2. Assume further that $u(x)(K(x))^{-1} k(x,y)\,\Phi(f(y))$ is slice-comonotonic. Then

$$(C)\int_{\Omega_1} u(x)\,\Phi\left(\frac{g(x)}{K(x)}\right) d\mu_1(x) \le (C)\int_{\Omega_2} \upsilon(y)\,\Phi(f(y))\, d\mu_2(y), \tag{32}$$

holds for all $g \in W(k)$, with f as in (30).

Proof. We observe that

$$(C)\int_{\Omega_1} u(x)\,\Phi\left(\frac{g(x)}{K(x)}\right) d\mu_1(x) =$$

$$(C)\int_{\Omega_1} u(x)\,\Phi\left(\frac{1}{K(x)}(C)\int_{\Omega_2} k(x,y)\, f(y)\, d\mu_2(y)\right) d\mu_1(x) = \tag{33}$$

(next we use Lemma 18, where P is a probability measure on Ω_2)

$$(C)\int_{\Omega_1} u(x)\,\Phi\left(\frac{1}{K(x)}(C)\int_{\Omega_2} k(x,y)\, f(y)\, dP(y)\right) d\mu_1(x) \le$$

(we can also write $K(x) = \int_{\Omega_2} k(x,y)\, dP(y)$, hence by classic Jensen's inequality)

$$(C)\int_{\Omega_1} u(x)(K(x))^{-1}\left((C)\int_{\Omega_2} k(x,y)\,\Phi(f(y))\, dP(y)\right) d\mu_1(x) = \tag{34}$$

(again by Lemma 18)

$$(C)\int_{\Omega_1} u(x)(K(x))^{-1}\left((C)\int_{\Omega_2} k(x,y)\,\Phi(f(y))\, d\mu_2(y)\right) d\mu_1(x) =$$

$$(C)\int_{\Omega_1}\left((C)\int_{\Omega_2} u(x)(K(x))^{-1} k(x,y)\,\Phi(f(y))\, d\mu_2(y)\right) d\mu_1(x) =$$

(since the functions $\Phi(f(y))$ and $u(x)(K(x))^{-1} k(x,y)\,\Phi(f(y))$ are bounded and the second one is slice-comonotonic, we can apply Fubini's Theorem 21)

$$(C)\int_{\Omega_2}\left((C)\int_{\Omega_1} u(x)(K(x))^{-1} k(x,y)\,\Phi(f(y))\, d\mu_1(x)\right) d\mu_2(y) =$$

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$$(C)\int_{\Omega_2}\Phi(f(y))\left((C)\int_{\Omega_1}u(x)(K(x))^{-1}k(x,y)\,d\mu_1(x)\right)d\mu_2(y)\overset{(31)}{=} \quad (35)$$

$$(C)\int_{\Omega_2}\Phi(f(y))\,\upsilon(y)\,d\mu_2(y),$$

proving the claim. ■

We also give

Corollary 26 *All as in Theorem 25, with $\Phi =$ identity mapping. Then*

$$(C)\int_{\Omega_1}\frac{u(x)}{K(x)}g(x)\,d\mu_1(x)\le(C)\int_{\Omega_2}\upsilon(y)\,f(y)\,d\mu_2(y), \quad (36)$$

holds for all $g\in W(k)$, with f as in (30).

Corollary 27 *All as in Theorem 25, with $\Phi(x)=x^p$, $\forall\ x\in\mathbb{R}_+$, $p>1$. Then*

$$(C)\int_{\Omega_1}\frac{u(x)}{K^p(x)}g^p(x)\,d\mu_1(x)\le(C)\int_{\Omega_2}\upsilon(y)\,f^p(y)\,d\mu_2(y), \quad (37)$$

holds for all $g\in W(k)$, with f as in (30).

Corollary 28 *All as in Theorem 25, with $\Phi(x)=e^x$, $\forall\ x\in\mathbb{R}_+$. Then*

$$(C)\int_{\Omega_1}u(x)\,e^{\frac{g(x)}{K(x)}}d\mu_1(x)\le(C)\int_{\Omega_2}\upsilon(y)\,e^{f(y)}d\mu_2(y), \quad (38)$$

holds for all $g\in W(k)$, with f as in (30).

Corollary 29 *All as in Theorem 25, with $\Phi =$ identity mapping and $u(x)=K(x)$. Then*

$$(C)\int_{\Omega_1}g(x)\,d\mu_1(x)\le(C)\int_{\Omega_2}\upsilon(y)\,f(y)\,d\mu_2(y), \quad (39)$$

holds for all $g\in W(k)$, with f as in (30). Here $\upsilon(y)=(C)\int_{\Omega_1}k(x,y)\,d\mu_1(x)$ is bounded.

Corollary 30 *All as in Theorem 25, with $\Phi(x)=x^p$, $\forall\ x\in\mathbb{R}_+$, $p>1$, and $u(x)=K^p(x)$. Then*

$$(C)\int_{\Omega_1}g^p(x)\,d\mu_1(x)\le(C)\int_{\Omega_2}\upsilon(y)\,f^p(y)\,d\mu_2(y), \quad (40)$$

holds for all $g\in W(k)$, with f as in (30). Here

$$\upsilon(y)=(C)\int_{\Omega_1}K^{p-1}(x)\,k(x,y)\,d\mu_1(x)\ \text{ is bounded.} \quad (41)$$

Remark 31 *(on Corollary 30) Let us assume that* $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$, *then* $K(x) \leq M$. *And from (41),* $\upsilon(y) \leq M^p$.

Consequently, from (40), it holds

$$(C)\int_{\Omega_1} g^p(x)\,d\mu_1(x) \leq M^p\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right), \tag{42}$$

and even better written

$$\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)^{\frac{1}{p}} \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)^{\frac{1}{p}}. \tag{43}$$

Next we rewrite the result of (43) in detail.

Theorem 32 *Assume that* $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$, *and let* $p > 1$. *Define*

$$\upsilon(y) = (C)\int_{\Omega_1} K^{p-1}(x)\,k(x,y)\,d\mu_1(x), \tag{44}$$

which is bounded. Here $k(x,y)(f(y))^p$ *is* x*-section comonotonic with comonotonic class* F_3^*. *Assume that* $(F_1^* \cup F_2^* \cup F_3^*) \subseteq F^*$, *where* F^* *one comonotonic class on* Ω_2. *Assume further that* $(K(x))^{p-1}k(x,y)(f(y))^p$ *is slice-comonotonic. Then*

$$\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)^{\frac{1}{p}} \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)^{\frac{1}{p}}, \tag{45}$$

holds for all $g \in W(k)$, *with* f *as in (30).*

Remark 33 *Assume that* $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$. *Hence directly by (30) we get*

$$g(x) \leq M\left((C)\int_{\Omega_2} f(y)\,d\mu_2(y)\right), \quad \forall\ x \in \Omega_1.$$

Therefore

$$\int_{\Omega_1} g(x)\,d\mu_1(x) \leq M\left((C)\int_{\Omega_2} f(y)\,d\mu_2(y)\right), \tag{46}$$

holds for all $g \in W(k)$, *with* f *as in (30).*

Theorem 34 *Define* υ *on* Ω_2 *by* $\upsilon(y) = (C)\int_{\Omega_1} k(x,y)\,d\mu_1(x)$, *which is bounded. Let* $p > 1$. *Here* $k(x,y)(f(y))^p$ *is slice−comonotonic and belongs to a comonotonic class* F_3^* *as a function of* y. *Assume that* $(F_1^* \cup F_2^* \cup F_3^*) \subseteq F^*$, *where* F^* *one comonotonic class on* Ω_2. *Then*

$$(C)\int_{\Omega_1} (K(x))^{1-p}\,g^p(x)\,d\mu_1(x) \leq (C)\int_{\Omega_2} \upsilon(y)\,f^p(y)\,d\mu_2(y), \tag{47}$$

holds for all $g \in W(k)$, *with* f *as in (30).*

Proof. By Theorem 25, take $f(x) = x^p$, $x \geq 0$, $p > 1$, and $u(x) = K(x)$. ■

Corollary 35 *All as in Theorem 34. Then*

$$\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)^{\frac{1}{p}} \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)^{\frac{1}{p}}. \tag{48}$$

holds for all $g \in W(k)$, *with* f *as in (30). Here* $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$.

Proof. Since $p > 1$, $1 - p < 0$. Hence the left hand side of (47) is greater equal to $M^{1-p}\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)$, by $K(x) \leq M$ and $(K(x))^{1-p} \geq M^{1-p}$. And the right hand side of (47) is less equal to $M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)$, by $v(y) \leq M$. Therefore

$$M^{1-p}\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right) \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right), \tag{49}$$

proving the claim. ■

4 Appendix

Here $\mathcal{B}$ stands for the Borel σ-algebra on $[a,b]$.

Let the fuzzy measure spaces $([a,b], \mathcal{B}, \mu_1)$ and $([a,b], \mathcal{B}, \mu_2)$, where $[a,b] \subset \mathbb{R}$ and μ_1, μ_2 are bounded fuzzy measures with μ_2 submodular. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \to \mathbb{R}_+$ which is bounded and $\mathcal{B}$-measurable.

We define the left and right Riemann-Liouville-Choquet fractional integrals of order $\alpha > 1$ (respectively):

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}(C)\int_a^x (x-t)^{\alpha-1} f(t)\,d\mu_2(t), \tag{50}$$

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}(C)\int_x^b (t-x)^{\alpha-1} f(t)\,d\mu_2(t), \tag{51}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function.

We assume that $\left(I_{a+}^{\alpha} f\right)$ and $\left(I_{b-}^{\alpha} f\right)$ are $\mathcal{B}$-measurable functions. Clearly $I_{a+}^{\alpha} f, I_{b-}^{\alpha} f$ are nonnegative and bounded over $[a,b]$.

Remark 36 *By Theorem 9 we obtain*

$$\left(I_{a+}^{\alpha} f\right)(x) \leq \frac{1}{\Gamma(\alpha)}\left((C)\int_a^x (x-t)^{p(\alpha-1)}\,d\mu_2(t)\right)^{\frac{1}{p}}\left((C)\int_a^x f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}} \leq \tag{52}$$

$$\frac{1}{\Gamma(\alpha)}\left((b-a)^{p(\alpha-1)}\mu_2([a,b])\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}.$$

Hence it holds

$$\left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p \leq \frac{1}{(\Gamma(\alpha))^p}(b-a)^{p(\alpha-1)}\mu_2([a,b])\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{p}{q}}, \tag{53}$$

$\forall\ x\in[a,b]$.

Therefore

$$(C)\int_a^b \left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p d\mu_1(x) \leq$$

$$\frac{\mu_1([a,b])}{(\Gamma(\alpha))^p}(b-a)^{p(\alpha-1)}\mu_2([a,b])\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{p}{q}}. \tag{54}$$

We have proved that

$$\left((C)\int_a^b \left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \leq$$

$$\frac{(\mu_1([a,b])\,\mu_2([a,b]))^{\frac{1}{p}}(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}. \tag{55}$$

Similarly, we have

$$\left(I_{b-}^{\alpha}f\right)(x) \overset{(8)}{\leq} \frac{1}{\Gamma(\alpha)}\left((C)\int_x^b (t-x)^{p(\alpha-1)}d\mu_2(t)\right)^{\frac{1}{p}}\left((C)\int_x^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}} \tag{56}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left((b-a)^{p(\alpha-1)}\mu_2([a,b])\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}.$$

As before we obtain

$$\left((C)\int_a^b \left(\left(I_{b-}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \leq$$

$$\frac{(\mu_1([a,b])\,\mu_2([a,b]))^{\frac{1}{p}}(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}. \tag{57}$$

We have proved

Theorem 37 *Here $\alpha > 1$ and the rest are as in this section. It holds*

$$\max\left\{\left((C)\int_a^b \left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}}, \left((C)\int_a^b \left(\left(I_{b-}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}}\right\}$$

$$\leq \frac{\left(\mu_1([a,b])\,\mu_2([a,b])\right)^{\frac{1}{p}}(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}. \quad (58)$$

Remark 38 *From (52) we get*

$$\left(I_{a+}^{\alpha}f\right)(x) \leq \frac{1}{\Gamma(\alpha)}\left((x-a)^{p(\alpha-1)}\mu_2([a,x])\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}, \quad (59)$$

and from (56) we derive (by exchanging the roles of p and q)

$$\left(I_{b-}^{\alpha}f\right)(x) \leq \frac{1}{\Gamma(\alpha)}\left((b-x)^{q(\alpha-1)}\mu_2([x,b])\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}. \quad (60)$$

Therefore by multiplying (59), (60) we get

$$\left(I_{a+}^{\alpha}f\right)(x)\left(I_{b-}^{\alpha}f\right)(x) \leq \frac{1}{(\Gamma(\alpha))^2}\left((x-a)^{p(\alpha-1)}\mu_2([a,x])\right)^{\frac{1}{p}} \cdot \quad (61)$$

$$\left((b-x)^{q(\alpha-1)}\mu_2([x,b])\right)^{\frac{1}{q}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p}+\frac{b}{q}$)

$$\leq \frac{1}{(\Gamma(\alpha))^2}\left(\frac{(x-a)^{p(\alpha-1)}\mu_2([a,x])}{p}+\frac{(b-x)^{q(\alpha-1)}\mu_2([x,b])}{q}\right)\cdot$$

$$\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}. \quad (62)$$

We have that

$$\frac{\left(I_{a+}^{\alpha}f\right)(x)\left(I_{b-}^{\alpha}f\right)(x)}{\left[\frac{(x-a)^{p(\alpha-1)}\mu_2([a,x])}{p}+\frac{(b-x)^{q(\alpha-1)}\mu_2([x,b])}{q}\right]} \leq$$

$$\frac{1}{(\Gamma(\alpha))^2}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}. \quad (63)$$

Notice that the denominator of left hand side of (63) is never zero.

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Integrating (63) with respect to x we obtain:

Theorem 39 *Here $\alpha > 1$ and the rest are as in this section. It holds*

$$(C)\int_a^b \frac{\left(I_{a+}^{\alpha} f\right)(x)\left(I_{b-}^{\alpha} f\right)(x)\, d\mu_1(x)}{\left[\frac{(x-a)^{p(\alpha-1)}\mu_2([a,x])}{p}+\frac{(b-x)^{q(\alpha-1)}\mu_2([x,b])}{q}\right]} \leq$$

$$\frac{\mu_1([a,b])}{(\Gamma(\alpha))^2}\left((C)\int_a^b f^p(t)\, d\mu_2(t)\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\, d\mu_2(t)\right)^{\frac{1}{q}}. \qquad (64)$$

Inequality (64) is a Hilbert-Pachpatte type inequality for Choquet fractional integrals.

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