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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
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H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
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Chattanooga, TN 37304 USA
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Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
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Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
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Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
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Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of
Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

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Journal of Computational Analysis and Applications

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Fixed point theorems for F-contractions on closed ball in partial metric spaces

Muhammad Nazam[1], Choonkil Park[2], Aftab Hussain[3], Muhammad Arshad[1] and Jung-Rye Lee[4*]

[1]Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
e-mail: nazim254.butt@gmail.com, marshadzia@iiu.edu.pk

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr

[3]Depatment of Mathematics, Khawaja Farid University, Rahim Yar Khan, Punjab, Pakistan
e-mail: aftabshh@gmail.com

[4]Depatment of Mathematics, Daejin University,Kyunggi 11159
e-mail: jrlee@daejin.ac.kr

Abstract. In this paper, we present new fixed point theorems for Kannan type F_p-contraction and Kannan type (α, η, GF_p)-contraction on a closed ball contained in a complete partial metric space. Some comparative examples are constructed to illustrate the significance of these results. Our results provide substantial generalizations and improvements of several well known results existing in the comparable literature.

1. Introduction and preliminaries

The recent study in Fixed Point Theory is due to a Polish mathematician Stefan Banach who, in 1922, presented a revolutionary contraction principle known as Banach's Contraction Principle. He proved that every contraction T in a complete metric space X has a unique fixed point ($T(x) = x$; $x \in X$). After the appearance of this remarkable result many generalizations of this result have appeared in literature (see for example [1–3, 6–11, 13, 14, 16, 19, 20, 22, 24, 25, 29]). One of these generalizations is known as F-contraction presented by Wardowski [30]. Wardowski [30] evinced that every F-contraction defined on a complete metric space has a unique fixed point. The concept of F-contraction proved another milestone in fixed point theory and numerous research papers on F-contraction have been published (see [21, 23, 28, 31]). Hussain *et al.* [12] introduced an α-GF-contraction with respect to a general family of functions G and established Wardowski type fixed point results in ordered metric spaces. Batra *et al.* [4, 5] extended the concept of F-contraction on graphs and altered distances and proved some fixed point and coincidence point results.

Motivated by Kannan [15], Wardowski [30], Matthews [18] and Kryeyszig [17], in this paper, we introduce Kannan type F-contraction and Kannan type (α, η, GF)-contraction on a closed ball contained in a complete partial metric space and present related fixed point theorems. We construct examples to illustrate these results. F-contraction on partial metric spaces is more general than F-contraction defined on metric spaces.

The notion of a partial metric space (PMS) was introduced in 1992 by Matthews [18] to model computation over a metric space. The PMS is a generalization of the usual metric space in which the self-distance is no longer necessarily zero.

Definition 1. [18] Let X be a nonempty set and $p : X \times X \to \mathbb{R}_0^+$ satisfy the following properties: for all $x, y, z \in X$,

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$.

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*Corresponding author.

Then (X,p) is called PMS. We present some new nontrivial examples of PMS.

Example 1. Let the set of rational numbers be $\mathbb{Q}=\{x_1,x_2,\cdots\}$. We define $p:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ by

$$p(x,y)=\begin{cases} 1 & \text{if } x=y\in\mathbb{R}-\mathbb{Q};\\ \frac{3}{2} & \text{if } x\neq y\in\mathbb{R}-\mathbb{Q};\\ \frac{1}{3} & \text{if } x=y\in\mathbb{Q};\\ 1+\frac{1}{m}+\frac{1}{n} & \text{if } x=x_m, y=x_n \text{ and } m\neq n;\\ 1+\frac{1}{n} & \text{if } \{x,y\}\cap\mathbb{Q}=\{x_n\} \text{ and } \{x,y\}-\mathbb{Q}\neq\phi. \end{cases}$$

Clearly p satisfies $(p_1)-(p_3)$. To prove p_4, let $x,y,z\in\mathbb{R}-\mathbb{Q}$ and $m\neq n$. Then

$$\begin{aligned} p(x,y)+p(z,z) &\leq p(x,z)+p(y,z);\\ p(x_n,y)+p(z,z) &= 2+\frac{1}{n}\leq p(x_n,z)+p(y,z);\\ p(x_n,x_n)+p(z,z) &= \frac{4}{3}<p(x_n,z)+p(x_n,z);\\ p(x_m,x_n)+p(z,z) &= 2+\frac{1}{m}+\frac{1}{n}=p(x_m,z)+p(x_n,z);\\ p(x,y)+p(x_k,x_k) &< 2<p(x,x_k)+p(y,x_k);\\ p(x_n,y)+p(x_k,x_k) &= \frac{4}{3}+\frac{1}{n}<2+\frac{1}{n}+\frac{2}{k}=p(x_n,x_k)+p(y,x_k);\\ p(x_n,x_n)+p(x_k,x_k) &= \frac{2}{3}\leq p(x_n,x_k)+p(x_n,x_k);\\ p(x_m,x_n)+p(x_k,x_k) &= \frac{4}{3}+\frac{1}{m}+\frac{1}{n}\leq p(x_m,x_k)+p(x_n,x_k). \end{aligned}$$

Example 2. Let X be uncountable, $a\in X$ and $\mathcal{T}=\{A\subset X:\ a\in A\}$ be a topology on X. It is easy to show that $(X,\mathcal{T})$ is a PMS with the partial metric p defined by

$$\begin{cases} p(a,a)=0, & \\ p(a,x)=p(x,x)=1 & \text{if } x\neq a,\\ p(x,y)=2 & \text{if } x=y \text{ and } x,y\in X-\{a\}. \end{cases}$$

Example 3. Let $X=\{x_i:i\in\mathbb{N}\}$ be a countably infinite set. Define $p:X\times X\to[0,\infty)$ by

$$p(x,y)=\begin{cases} 0 & \text{if } x=y=x_0,\\ \sum_{k=1}^{n}\frac{1}{2^k} & \text{if } (x,y)\in\{(x_m,x_n);0\leq m\leq n \text{ and } n\geq 1. \end{cases}$$

Then (X,p) is a PMS.

Example 4. Let $A=\{a_i:i\in\mathbb{N}\}$ and $B=\{b_i:i\in\mathbb{N}\}$ be two disjoint infinitely countable sets, and let $X=A\cup B$. Define $p:X\times X\to[0,\infty)$ by

$$p(x,y)=\begin{cases} 1 & \text{if } x=y\in A,\\ 0 & \text{if } x=y\in B,\\ 1+\frac{1}{i}+\frac{1}{j} & \text{if } x=y \text{ and } \{x,y\}\in\{\{a_i,a_j\},\{a_i,b_j\},\{b_i,b_j\}\}. \end{cases}$$

Then (X,p) is a PMS.

In [18], Matthews proved that every partial metric p on M induces a metric $d_p:M\times M\to\mathbb{R}_0^+$ defined by

$$d_p(r_1,r_2)=2p(r_1,r_2)-p(r_1,r_1)-p(r_2,r_2)$$

for all $r_1, r_2 \in M$. Matthews described that if $p(r_1, r_2) = \max\{r_1, r_2\}$, then $d_p(r_1, r_2) = |r_1 - r_2|$ a usual metric on M. Notice that every metric d on a set M is a partial metric p such that $p(r, r) = 0$ for all $r \in M$ and $p(r_1, r_2) = 0$ implies $r_1 = r_2$ (using (p_1) and (p_2)) but not conversely. The notions such as convergence, completeness, Cauchy sequence in the setting of partial metric spaces, can be found in [18] and references there in.

Definition 2. [18] Let (M, p) be a partial metric space.

(1) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in (M, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(r_n, r_m)$ exists and is finite.
(2) A partial metric space (M, p) is said to be complete if every Cauchy sequence $\{r_n\}_{n\in\mathbb{N}}$ in M converges, with respect to $\tau(p)$, to a point $r \in X$ such that $p(r, r) = \lim_{n,m\to\infty} p(r_n, r_m)$.

The following lemma will be helpful in the sequel.

Lemma 1. [18]

(1) A sequence r_n is a Cauchy sequence in a partial metric space (M, p) if and only if it is a Cauchy sequence in metric space (M, d_p)
(2) A partial metric space (M, p) is complete if and only if the metric space (M, d_p) is complete.
(3) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in M converges to a point $r \in M$, with respect to $\tau(d_p)$ if and only if $\lim_{n\to\infty} p(r, r_n) = p(r, r) = \lim_{n,m\to\infty} p(r_n, r_m)$.
(4) If $\lim_{n\to\infty} r_n = \upsilon$ such that $p(\upsilon, \upsilon) = 0$ then $\lim_{n\to\infty} p(r_n, r) = p(\upsilon, r)$ for all $r \in M$.

Remark 1. *Since* $\left(\overline{B_p(x_0, r)}, p\right) \subseteq (X, p)$*, Lemma 1 holds for* $\left(\overline{B_p(x_0, r)}, p\right)$.

Let F_d denote F-contraction on metric spaces and F_p denote F-contraction on partial metric spaces. Wardowski [30] investigated a nonlinear function $F : \mathbb{R}^+ \to \mathbb{R}$ complying with the following axioms:

(F_1) F is strictly increasing;
(F_2) For each sequence $\{r_n\}$ of positive numbers $\lim_{n\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} F(r_n) = -\infty$;
(F_3) There exists $\theta \in (0, 1)$ such that $\lim_{\xi\to 0^+} (\xi)^\theta F(\xi) = 0$.

We denote by Δ_F the set of all functions satisfying the conditions $(F_1) - (F_3)$.

Example 5. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by

(a) $F(r) = \ln(r)$,
(b) $F(r) = r + \ln(r)$,
(c) $F(r) = \ln(r^2 + r)$,
(d) $F(r) = -\dfrac{1}{\sqrt{r}}$.

It is easy to check that (a),(b),(c) and (d) are members of Δ_F.

Wardowski utilized function F in an excellent manner and gave the following remarkable result.

Theorem 1. [30] Let (M, d) be a complete metric space and $T : M \to M$ be a mapping satisfying

$$(d(T(r_1), T(r_2)) > 0 \Rightarrow \tau + F(d(T(r_1), T(r_2)) \leq F(d(r_1, r_2)))$$

for all $r_1, r_2 \in M$ and some $\tau > 0$. Then T has a unique fixed point $\upsilon \in M$ and for every $r_0 \in M$ the sequence $\{T^n(r_0)\}$ for all $n \in \mathbb{N}$ is convergent to υ.

Remark 2. [30, Remark 2.1] In metric spaces a mapping giving fulfillment to F-contraction, is always a Banach contraction and hence a continuous map.

Example 6 explains that F_p-contraction is more general than F_d-contraction.

Example 6. Let $M = [0,1]$ and define partial metric by $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \in M$. The metric d induced by partial metric p is given by $d(r_1, r_2) = |r_1 - r_2|$ for all $r_1, r_2 \in M$. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ and T by

$$T(r) = \begin{cases} \frac{r}{5} & \text{if } r \in [0,1), \\ 0 & \text{if } r = 1. \end{cases}$$

Note that for all $r_1, r_2 \in M$ with $r_1 \leq r_2$ or $r_2 \leq r_1$

$$\begin{aligned} \tau + F(p(T(r_1), T(r_2))) &\leq F(p(r_1, r_2)) \text{ implies} \\ \tau + F\left(\frac{r_1}{5}\right) &\leq F(r_1) \text{ or } \tau + F\left(\frac{r_2}{5}\right) \leq F(r_2). \end{aligned}$$

But T is neither continuous and nor satisfies F-contraction in a metric space (M, d). Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, $d(T(r_1), T(r_2)) > 0$ and we have

$$\begin{aligned} \tau + F(d(T(r_1), T(r_2))) &\leq F(d(r_1, r_2)), \\ \tau + F\left(d(T(1), T(\frac{5}{6}))\right) &\leq F\left(d(1, \frac{5}{6})\right), \\ \tau + F\left(d(0, \frac{1}{6})\right) &\leq F\left(\frac{1}{6}\right), \\ \frac{1}{6} &< \frac{1}{6}, \end{aligned}$$

which is a contradiction for all possible values of τ.

The following result plays a vital role regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball.

Theorem 2. [17, Theorem 5.1.4] Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0,1)$ with

$$d(T(x), T(y)) \leq kd(x, y), \text{ for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, T(x_0)) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = T(x^*)$.

Definition 3. [15] Let (X, p) be a partial metric space. A mapping $T : X \to X$ is said to be a Kannan contraction if it satisfies the following condition:

$$p(T(x), T(y)) \leq \frac{k}{2}[p(x, T(x)) + p(y, T(y))]$$

for all $x, y \in X$ and some $k \in [0, 1[$.

2. Kannan type F_p-contraction on closed ball

Definition 4. Let (X, p) be a partial metric space, $r > 0$ and x_0 be an arbitrary point in X. The mapping $T : X \to X$ is called Kannan type F_p-contraction on closed ball if for all $x, y \in \overline{B_p(x_0, r)} \subseteq X$ we have

$$\tau + F(p(T(x), T(y))) \leq F\left(\frac{k}{2}[p(x, T(x)) + p(y, T(y))]\right), \tag{2.1}$$

where $0 \leq k < 1$, $F \in \Delta_F$ and $\tau > 0$.

Remark 3. *(1) F_p-contraction and Kannan type F_p-contraction are independent.*
(2) Let F be a Kannan type F_p-contraction. From (2.1), for all $x, y \in \overline{B_p(x_0, r)}$ with $T(x) \neq T(y)$, we have

$$F(p(T(x), T(y))) \leq \tau + F(p(T(x), T(y))) \leq F\left(\frac{k}{2}[p(x, T(x)) + p(y, T(y))]\right).$$

Due to (F_1), *we obtain*

$$p(T(x),T(y)) < \frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right] \text{ for all } x,y\in X, T(x)\neq T(y).$$

Theorem 3. *Let* (X,p) *be a complete partial metric space,* $r>0$ *and* x_0 *be an arbitrary point in* X. *Assume that* $T: X\to X$ *is a Kannan type* F_p*-contraction on closed ball* $\overline{B_p(x_0,r)}\subseteq X$ *with*

$$p(x_0,T(x_0)) \leq (1-\lambda)[r+p(x_0,x_0)], \lambda = \frac{k}{2-k}. \tag{2.2}$$

If T *or* F *is continuous, then there exists a point* x^* *in* $\overline{B_p(x_0,r)}$ *such that* $T(x^*)=x^*$ *with* $p(x^*,x^*)=0$.

Proof. Let x_0 be an initial point in X such that $x_1=T(x_0)$, $x_2=T(x_1)=T^2(x_0)$. Continuing in this way we can construct an iterative sequence $\{x_n\}$ such that $x_{n+1}=T(x_n)=T^n(x_0)$, for all $n\geq 0$. We show that $x_n\in\overline{B_p(x_0,r)}$ for all $n\in\mathbb{N}$. From (2.2), we have

$$p(x_0,x_1)=p(x_0,T(x_0))\leq(1-\lambda)[r+p(x_0,x_0)]<r+p(x_0,x_0),$$

which shows that $x_1\in\overline{B_p(x_0,r)}$. From (2.1) and (F_1), we get

$$F\left(p(x_1,x_2)\right)=F(p(T(x_0),T(x_1)))\leq F\left(\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]\right)-\tau,$$

which implies

$$p(x_1,x_2)<\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]<\lambda p(x_0,x_1)\leq\lambda[r+p(x_0,x_0)]$$

$$p(x_0,x_2)\leq p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1)<(1-\lambda)[r+p(x_0,x_0)]+\lambda[r+p(x_0,x_0)]=r+p(x_0,x_0).$$

This shows that $x_2\in\overline{B_p(x_0,r)}$. Inductively, we obtain that $x_n\in\overline{B_p(x_0,r)}$, for all $n\in\mathbb{N}$ and hence from the contractive condition (2.1), we have

$$\begin{aligned}
F\left(p(x_n,x_{n+1})\right) &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+p(x_n,x_{n+1})\right]\right)-\tau\\
&\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+\frac{k}{2-k}p(x_{n-1},x_n)\right]\right)-\tau\\
&\leq F\left(\frac{k}{2-k}p(x_{n-1},x_n)\right)-\tau
\end{aligned} \tag{2.3}$$

and also

$$F\left(p(x_{n-1},x_n)\right)\leq F\left(\lambda p(x_{n-2},x_{n-1})\right)-\tau.$$

From (2.3), we obtain

$$F\left(p(x_n,x_{n+1})\right)\leq F\left(\lambda p(x_{n-2},x_{n-1})\right)-2\tau.$$

Repeating these steps, we get

$$F\left(p(x_n,x_{n+1})\right)\leq F\left(p(x_0,x_1)\right)-n\tau. \tag{2.4}$$

From (2.4), we obtain $\lim_{n\to\infty}F\left(p(x_n,x_{n+1})\right)=-\infty$. Since $F\in\Delta_F$,

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0. \tag{2.5}$$

From the property (F_3) of F-contraction, there exists $\kappa\in(0,1)$ such that

$$\lim_{n\to\infty}\left((p(x_n,x_{n+1}))^{\kappa}F\left(p(x_n,x_{n+1})\right)\right)=0. \tag{2.6}$$

Following (2.4), for all $n\in\mathbb{N}$, we obtain

$$\left(p(x_n,x_{n+1})\right)^{\kappa}\left(F\left(p(x_n,x_{n+1})\right)-F\left(p(x_0,x_1)\right)\right)\leq-\left(p(x_n,x_{n+1})\right)^{\kappa}n\tau\leq 0. \tag{2.7}$$

Considering (2.5), (2.6) and letting $n\to\infty$ in (2.7), we have

$$\lim_{n\to\infty}\left(n\left(p(x_n,x_{n+1})\right)^{\kappa}\right)=0. \tag{2.8}$$

Since (2.8) holds, there exists $n_1 \in \mathbb{N}$ such that $n\left(p(x_n, x_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$ or

$$p(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{2.9}$$

Using (2.9), we get for $m > n \geq n_1$,

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) + \cdots + p(x_{m-1}, x_m) - \sum_{j=n+1}^{m-1} p(x_j, x_j) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) + \cdots + p(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} p(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B_p(x_0, r)}, p\right)$. By Lemma 1, $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B(x_0, r)}, d_p\right)$. Moreover, since $\left(\overline{B_p(x_0, r)}, p\right)$ is a complete partial metric space, by Lemma 1, $\left(\overline{B(x_0, r)}, d_p\right)$ is also a complete metric space. Thus there exists $x^* \in (\overline{B(x_0, r)}, d_p)$ such that $x_n \to x^*$ as $n \to \infty$ and using Lemma 1, we have

$$\lim_{n\to\infty} p(x^*, x_n) = p(x^*, x^*) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.10}$$

Due to $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, we infer from (2.10) that $p(x^*, x^*) = 0$ and $\{x_n\}$ converges to x^* with respect to $\mathcal{T}_p$. In order to show that x^* is a fixed point of T, we have two cases.

Case (1). T is *continuous*. We have

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^{n+1}(x_0) = T(\lim_{n\to\infty} T^n(x_0)) = T(x^*).$$

Hence $x^* = T(x^*)$, that is, x^* is a fixed point of T.

Case (2). F is *continuous*. We complete this case in two steps. First, if for each $n \in \mathbb{N}$ there exists $b_n \in \mathbb{N}$ such that $x_{b_n+1} = T(x^*)$ and $b_n > b_{n-1}$ with $b_0 = 1$. Then we have

$$x^* = \lim_{n\to\infty} x_{b_n+1} = \lim_{n\to\infty} T(x^*) = T(x^*).$$

This shows that x^* is a fixed point of T. Second, there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq T(x^*)$ for all $n \geq n_0$. Using contractive condition (2.1), we obtain

$$F\left(p(T(x_n), T(x^*))\right) \leq F\left(\frac{k}{2}\left[p\left(x_n, x_{n+1}\right) + p\left(x^*, T(x^*)\right)\right]\right) - \tau.$$

On taking limit as $n \to \infty$ and using the continuity of F and the fact that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we have

$$F(p(x^*, T(x^*))) < F(\frac{k}{2} p\left(x^*, T(x^*)\right)).$$

Since F is strictly increasing, the above inequality leads us to conclude that $p(x^*, T(x^*)) = 0$. Thus, by using the properties (p_1) and (p_2), we obtain $x^* = T(x^*)$, which completes the proof.

To prove the uniqueness of x^*, assume on contrary, that $y^* \in \overline{B_p(x_0, r)}$ is another fixed point of T, that is, $y^* = T(y^*)$. From (2.1), we have

$$\tau + F\left(p(T(x^*), T(y^*))\right) \leq F\left(\frac{k}{2}\left[p(x^*, T(x^*)) + p(y^*, T(y^*))\right]\right) \leq F\left(\frac{k}{2} \times 2p(x^*, y^*)\right). \tag{2.11}$$

The inequality (2.11) leads to a contradiction. Hence $p(x^*, y^*) = 0$. Thus, due to (p_1) and (p_2), we obtain $x^* = y^*$. □

The following example explains the significance of Theorem 3.

Example 7. Let $X = \mathbb{R}^+$. Define $p : X^2 \to [0, \infty)$ by $p(x, y) = \max\{x, y\}$ for all $(x, y) \in X^2$. Then (X, p) is a complete partial metric space. Define the mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{14} & \text{if } x \in [0, 1], \\ x - \frac{1}{2} & \text{if } x \in (1, \infty). \end{cases}$$

Set $k=\frac{2}{5}$, $x_0=\frac{1}{2}$, $r=\frac{1}{2}$ and $p(x_0,x_0)=\frac{1}{2}$. Then $\overline{B_p(x_0,r)}=[0,1]$ and

$$p(x_0,T(x_0))=\max\left\{\frac{1}{2},\frac{1}{28}\right\}=\frac{1}{2}<(1-\lambda)[r+p(x_0,x_0)].$$

For all $x,y\in\overline{B_p(x_0,r)}$, we note that

$$\begin{aligned} p(T(x),T(y)) &= \max\left\{\frac{x}{14},\frac{y}{14}\right\}=\frac{1}{14}\max\{x,y\} \\ &< \frac{1}{5}[x+y]=\frac{1}{5}\left[\max\left\{x,\frac{x}{14}\right\}+\max\left\{y,\frac{y}{14}\right\}\right] \\ &= \frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right] \end{aligned}$$

Thus

$$\tau+\ln\left(p(T(x),T(y))\right)\leq\ln\left(\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\right).$$

If $F(\alpha)=\ln(\alpha)$, $\alpha>0$ and $\tau>0$, then

$$\tau+F\left(p(T(x),T(y))\right)\leq F\left(\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\right).$$

However, for $x=100, y=10\in(1,\infty)$,

$$\begin{aligned} p(T(x),T(y)) &= \max\left\{x-\frac{1}{2},y-\frac{1}{2}\right\} \\ &\geq \frac{1}{5}[x+y]=\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]. \end{aligned}$$

Consequently, the contractive condition (2.1) does not hold on X. Hence, all the hypotheses of Theorem 3 are satisfied on closed ball and so $x=0$ is a fixed point of T.

3. Kannan type (α,η,GF_p)-contraction on closed ball

Definition 5. [27]. Let $T:X\to X$ and $\alpha:X\times X\to[0,+\infty)$ be two functions. We say that T is an α-admissible if for all $x,y\in X$, $\alpha(x,y)\geq 1$ implies $\alpha(T(x),T(y))\geq 1$.

Example 8. Let $X=\mathbb{R}$. Define $\alpha:X\times X\to[0,\infty)$ and $f:X\to X$ by

$$\alpha(x,y)=\begin{cases} e^{x+y} & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise }.\end{cases}\quad f(x)=\begin{cases}\frac{x^2}{7} & \text{if } x\in[0,1],\\ \ln(x) & \text{if } x\in(1,\infty).\end{cases}$$

Apparently, $\alpha(x,y)\geq 1$ implies $\alpha(fx,fy)\geq 1$.

Definition 6. [26]. Let $T:X\to X$ and $\alpha,\eta:X\times X\to[0,+\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if for all $x,y\in X$, $\alpha(x,y)\geq\eta(x,y)$ implies $\alpha(T(x),T(y))\geq\eta(T(x),T(y))$.

Example 9. Let $X=\mathbb{R}$. Define $\alpha,\eta:X\times X\to[0,\infty)$ and $f:X\to X$ by

$$\alpha(x,y)=\begin{cases}\pi^{x+y} & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise },\end{cases}\quad \eta(x,y)=\begin{cases} e^{x+y} & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise }.\end{cases}$$

$$f(x)=\begin{cases}\frac{x^2}{7} & \text{if } x\in[0,1],\\ \ln(x) & \text{if } x\in(1,\infty).\end{cases}$$

Apparently, $\alpha(x,y)\geq\eta(x,y)$ implies $\alpha(fx,fy)\geq\eta(fx,fy)$.

If $\eta(x,y)=1$, then the above definition reduces to Definition 5.

We begin by introducing the following family of new functions.

Let Δ_G denote the set of all functions $G:(\mathbb{R}^+)^4\to\mathbb{R}^+$ which satisfy the property

(G): for all $p_1,p_2,p_3,p_4\in\mathbb{R}^+$, if

$$\frac{p_1+p_2+p_3+p_4}{4}\leq\frac{p_i+p_{i+1}}{2},\ i=1,2,3,4,$$

then there exists $\tau>0$ such that $G(p_1,p_2,p_3,p_4)=\tau$.

Definition 7. Let (X, p) be a partial metric space and T be a self mapping on X. Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ are two functions. The mapping T is said to be an (α, η, GF_p)-contraction if for all $x, y \in X$, with $\eta(x, y) \leq \alpha(x, y)$ and $d(T(x), T(y)) > 0$, we have

$$G(p(x, T(x)), p(y, T(y)), p(x, T(y)), p(y, T(x))) + F\left(p(T(x), T(y))\right) \leq F\left(p(x, y)\right),$$

where $G \in \Delta_G$ and $F \in \Delta_F$.

Definition 8. Let (X, p) be a partial metric space and $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. T is said to be an (α, η)-continuous mapping on (X, p) if for a given $x \in X$, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ implies } T(x_n) \to T(x).$$

Example 10. Let $X = [0, \infty)$ and $p : X \times X \to [0, \infty)$ be defined by $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \in X$. Define

$$T(r) = \begin{cases} \sin(\pi r) & \text{if } r \in [0, 1], \\ \cos(\pi r) + 2 & \text{if } r \in (1, \infty), \end{cases} \quad \alpha(r_1, r_2) = \begin{cases} r_1^3 + r_2^3 + 1 & \text{if } r_1, r_2 \in [0, 1], \\ 0 & \text{otherwise}, \end{cases}$$

$$\eta(r_1, r_2) = \begin{cases} r_1^3 + r_2^3 & \text{if } r_1, r_2 \in [0, 1], \\ 0 & \text{otherwise}. \end{cases}$$

Then apparently, T is not continuous on X, however T is an (α, η)-continuous.

Definition 9. Let (X, p) be a partial metric space and $\alpha, \eta : X \times X \to [0, +\infty)$ are two functions, $r > 0$ and x_0 be an arbitrary point in X. The mapping $T : X \to X$ is said to be a Kannan type (α, η, GF_p)-contraction on closed ball if for all $x, y \in \overline{B_p(x_0, r)} \subseteq X$ with $\eta(x, y) \leq \alpha(x, y)$ and $p(T(x), T(y)) > 0$, we have

$$\tau(G) + F\left(p(T(x), T(y))\right) \leq F\left(\frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right]\right), \tag{3.1}$$

where $\tau(G) = G(p(x, T(x)), p(y, T(y)), p(x, T(y)), p(y, T(x)))$, $0 \leq k < 1, G \in \Delta_G$ and $F \in \Delta_F$.

Theorem 4. *Let (X, p) be a complete metric space and $T : X \to X$ be a Kannan type (α, η, GF_p)-contraction mapping on a closed ball $\overline{B_p(x_0, r)}$ satisfying the following assertions*

(1) *T is an α-admissible mapping with respect to η,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq \eta(x_0, T(x_0))$,*
(3) *there exist $r > 0$ and $x_0 \in X$ such that $p(x_0, T(x_0)) \leq (1 - \lambda)[r + p(x_0, x_0)]$, where $\lambda = \frac{k}{2-k}$.*

Then there exists a point x^* in $\overline{B_p(x_0, r)}$ such that $T(x^*) = x^*$ with $p(x^*, x^*) = 0$.

Proof. Suppose that x_0 is an initial point of X, we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{n+1} = T(x_n) = T^{n+1}(x_0)$ for all $n \in \mathbb{N}$. By assumption (2) there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq \eta(x_0, T(x_0))$. Since T is an α-admissible mapping with respect to η,

$$\alpha(x_0, T(x_0)) \geq \eta(x_0, T(x_0)) \text{ implies } \alpha(x_1, x_2) \geq \eta(x_1, x_2), \text{ which implies } \alpha(x_2, x_3) \geq \eta(x_2, x_3).$$

In general, we have

$$\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0}, T(x_{n_0})) = 0$, then x_{n_0} is a fixed point of T. We assume that $p(x_n, T(x_n)) > 0$ for all $n \in \mathbb{N}$. We show that $x_n \in \overline{B_p(x_0, r)}$ for all $n \in N$. Assumption (3) implies

$$p(x_0, x_1) = p(x_0, T(x_0)) \leq (1 - \lambda)[r + p(x_0, x_0)] < [r + p(x_0, x_0)]$$

and thus $x_1 \in \overline{B_p(x_0, r)}$. Note that $\tau(G) = \tau$. Indeed, $\tau(G) = G(p(x_0, x_1), p(x_1, x_2), p(x_0, x_2), p(x_1, x_1))$ satisfies

$$\frac{p(x_0, x_1) + p(x_1, x_2) + p(x_0, x_2) + p(x_1, x_1)}{4} \leq \frac{p(x_0, x_1) + p(x_1, x_2)}{2}.$$

By the property (G), there exists $\tau > 0$ such that

$$G(p(x_0, x_1), p(x_1, x_2), p(x_0, x_2), p(x_1, x_1)) = \tau.$$

Due to (3.1) and (F_1), we have

$$\begin{aligned} F\left(p(x_1,x_2)\right) &= F(p(T(x_0),T(x_1))) \leq F\left(\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]\right)-\tau(G) \\ &\leq F\left(\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]\right)-\tau. \\ \text{This implies } p(x_1,x_2) &< \frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right] < \lambda p(x_0,x_1) \leq \lambda[r+p(x_0,x_0)], \end{aligned}$$

$$\begin{aligned} p(x_0,x_2) &\leq p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1) \\ &< (1-\lambda)[r+p(x_0,x_0)]+\lambda[r+p(x_0,x_0)] = r+p(x_0,x_0). \end{aligned}$$

This shows that $x_2 \in \overline{B_p(x_0,r)}$. Inductively, we obtain that $x_n \in \overline{B_p(x_0,r)}$ for all $n \in \mathbb{N}$ and hence from the contractive condition (3.1), we have

$$F\left(p(x_n,x_{n+1})\right) \leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+p(x_n,x_{n+1})\right]\right)-\tau(G). \tag{3.2}$$

Note that $\tau(G) = G(p(x_{n-1},x_n),p(x_n,x_{n+1}),p(x_{n-1},x_{n+1}),p(x_n,x_n))$ satisfies

$$\frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})+p(x_{n-1},x_{n+1})+p(x_n,x_n)}{4} \leq \frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})}{2},$$

and so by the property (G), there exists $\tau > 0$ such that

$$G(p(x_{n-1},x_n),p(x_n,x_{n+1}),p(x_{n-1},x_{n+1}),p(x_n,x_n)) = \tau.$$

Thus, from (3.2), we get

$$\begin{aligned} F\left(p(x_n,x_{n+1})\right) &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+\frac{k}{2-k}p(x_{n-1},x_n)\right]\right)-\tau \\ &\leq F\left(\frac{k}{2-k}p(x_{n-1},x_n)\right)-\tau = F(\lambda p(x_{n-1},x_n))-\tau \end{aligned} \tag{3.3}$$

but

$$F\left(p(x_{n-1},x_n)\right) \leq F\left(\lambda p(x_{n-2},x_{n-1})\right)-\tau.$$

From (3.3), we obtain

$$F\left(p(x_n,x_{n+1})\right) \leq F\left(\lambda p(x_{n-2},x_{n-1})\right)-2\tau.$$

Continuing in the same way we obtain

$$F\left(p(x_n,x_{n+1})\right) \leq F\left(p(x_0,x_1)\right)-n\tau.$$

By the same reasoning as in the proof of Theorem 3, there exists $x^* \in \overline{B_p(x_0,r)}$ such that $p(x^*,x^*)=0$ and $\{x_n\}$ converges to x^* with respect to $\mathcal{T}_p$. We show that x^* is a fixed point of T. We have two cases.

Case (1). T is an (α,η)-continuous.
Since $x_n \to x^*$ as $n \to \infty$ and $\eta(x_{n-1},x_n) \leq \alpha(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, (α,η)-continuity of T implies $x_{n+1} = T(x_n) \to T(x^*)$ as $n \to \infty$. That is, $x^* = T(x^*)$. Hence x^* is a fixed point of T.

Case (2). F is *continuous*. We complete this case in two steps. First, if for each $n \in \mathbb{N}$ there exists $b_n \in \mathbb{N}$ such that $x_{b_n+1} = T(x^*)$ and $b_n > b_{n-1}$ with $b_0 = 1$, then we have

$$x^* = \lim_{n\to\infty} x_{b_n+1} = \lim_{n\to\infty} T(x^*) = T(x^*).$$

This shows that x^* is a fixed point of T. Second, there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq T(x^*)$ for all $n \geq n_0$. Using the contractive condition (3.1), we obtain

$$F\left(p(x_n,T(x^*))\right) \leq F\left(\frac{k}{2}\left[p\left(x_{n-1},x_n\right)+p\left(x^*,T(x^*)\right)\right]\right)-\tau(G),$$

where $\tau(G)=G(p(x_{n-1},x_n),p(x^*,T(x^*)),p(x_{n-1},T(x^*)),p(x^*,x_n))$. Using the continuity of F and the property (G), we have

$$F\left(\lim_{n\to\infty}p(x_n,T(x^*))\right)\leq F\left(\frac{k}{2}\left[\lim_{n\to\infty}p\left(x_{n-1},x_n\right)+\lim_{n\to\infty}p\left(x^*,T(x^*)\right)\right]\right)-\tau.$$

Since F is strictly increasing, the above inequality leads us to conclude that $p(x^*,T(x^*))=0$. Thus, by using properties (p_1) and (p_2), we obtain $x^*=T(x^*)$, which completes the proof. □

Example 11. Let $X=\mathbb{R}^+$. Define $p:X^2\to[0,\infty)$ by $p(x,y)=\max\{x,y\}$. Then (X,p) is a complete partial metric space. Define $T:X\to X$, $\alpha:X\times X\to[0,+\infty)$, $\eta:X\times X\to\mathbb{R}^+$, $G:(\mathbb{R}^+)^4\to\mathbb{R}^+$ and $F:\mathbb{R}^+\to\mathbb{R}$ by

$$T(x)=\begin{cases}\frac{5x}{19} & \text{if } x\in[0,1],\\ x-\frac{1}{3} & \text{if } x\in(1,\infty),\end{cases}\quad \alpha(x,y)=\begin{cases}e^{x+y} & \text{if } x\in[0,1],\\ \frac{1}{3} & \text{otherwise},\end{cases}$$

$\eta(x,y)=\frac{1}{2}$ for all $x,y\in X$, $G(t_1,t_2,t_3,t_4)=\tau>0$ and $F(t)=\ln(t)$ with $t>0$. Set $k=\frac{4}{5}$ $x_0=\frac{1}{2}$, $r=\frac{1}{2}$ and $p(x_0,x_0)=\frac{1}{2}$. Then $\overline{B(x_0,r)}=[0,1]$, $\alpha(0,T(0))\geq\eta(0,T(0))$ and

$$p\left(\frac{1}{2},T\left(\frac{1}{2}\right)\right)=\max\left\{\frac{1}{2},\frac{5}{38}\right\}<(1-\lambda)[r+p(x_0,x_0)].$$

For if $x,y\in\overline{B(x_0,r)}$, then $\alpha(x,y)=e^{x+y}\geq\frac{1}{2}=\eta(x,y)$. On the other hand, $T(x)\in[0,1]$ for all $x\in[0,1]$ and so $\alpha(T(x),T(y))\geq\eta(T(x),T(y))$ for $x\neq y$, $p(T(x),T(y))=\left\{\frac{5x}{19},\frac{5y}{19}\right\}>0$. For all $x,y\in\overline{B_p(x_0,r)}$, we have

$$p(T(x),T(y))=\left\{\frac{5x}{19},\frac{5y}{19}\right\}=\frac{5}{19}\max\{x,y\},$$

$$\frac{5}{19}\max\{x,y\}<\frac{k}{2}\left[\max\left\{x,\frac{5x}{19}\right\}+\max\left\{y,\frac{5y}{19}\right\}\right]=\frac{14k}{38}[x+y].$$

Thus

$$p(T(x),T(y))<\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right].$$

Consequently,

$$\tau+\ln\left(p(T(x),T(y))\right)\leq\ln\left(\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\right)$$

leads to

$$\tau+F\left(p(T(x),T(y))\right)\leq F\left(\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\right).$$

If $x\notin\overline{B_p(x_0,r)}$ or $y\notin\overline{B_p(x_0,r)}$, then $\alpha(x,y)=\frac{1}{3}\ngeq\frac{1}{2}=\eta(x,y)$. Moreover, if $x=100, y=10\in(1,\infty)$, then

$$p(T(x),T(y))=\max\left\{x-\frac{1}{3},y-\frac{1}{3}\right\}\geq\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right].$$

Hence, all the hypotheses of Theorem 4 are satisfied on closed ball and $x=0$ is a unique fixed point of T.

4. Conclusion

In this paper, the main aim of our paper is to present new concepts of F-contraction on closed ball which are different from those given in [12, 23, 30]. Existence and uniqueness of a fixed point of such type of F-contractions on closed ball in complete partial metric space are discussed. The study of such results is very useful in the sense that it requires the F-contraction mapping defined only on the closed ball instead the whole space. These new concepts shall lead the readers for further investigations and applications. It will also be interesting to apply these concepts in different metric spaces.

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Lacunary sequence spaces defined by Euler transform and Orlicz functions

Abdullah Alotaibi[1], **Kuldip Raj**[2], **Ali H. Alkhaldi**[3] and **S. A. Mohiuddine**[4]

[1,4]Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2]School of Mathematics, Shri Mata Vaishno Devi University, Katra 182320, J&K India
[3]Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia
Emails: [1]mathker11@hotmail.com; [2]kuldipraj68@gmail.com; [3]ahalkhaldi@kku.edu.sa; [4]mohiuddine@gmail.com

Abstract

We introduce lacunary sequence spaces over n-normed space defined by Euler transform and Musielak-Orlicz functions with the help of an infinite matrix. We also make an effort to study some topological properties and prove some inclusion relations. Finally, we study the notion of statistical convergence over mentioned sequence space.

Keywords and phrases: Musielak-Orlicz function; matrix transformation; Euler transformation; statistical convergence.
AMS subject classification (2010): 40A05; 40C05; 46A45.

1 Introduction and preliminaries

Euler transform is used for improving the convergence of certain series which is widely used in numerical analysis. These techniques are useful in computer science especially in making graphics. We may find the application of the results to physical chemistry and crystallography. Further, we may use these results in the accelerated convergence techniques to find eigenvalues and eigenvectors of the dynamical systems.

Let $\sum_{k=0}^{\infty} a_k$ be an infinite series with sequence of partial sums (s_k) and $q > 0$ be any real number. The Euler transform (E, q) of the sequence $S = (s_n)$ is defined by $E_n^q(S) = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v$. A series $\sum_{n=0}^{\infty} a_n$ is said to be summable (E, q) to the number s if $E_n^q(S) = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v \to s$ as $n \to \infty$, and is said to be absolutely summable (E, q) or summable $|E, q|$, if $\sum_k |E_k^q(S) - E_{k-1}^q(S)| < \infty$. Let $x = (x_n)$ be a sequence of scalars, for $k \geq 1$ we will denote by $N_n(x)$ the difference $E_n^q(x) - E_{n-1}^q(x)$, where E_n^q is defined as above. Using Abel's transform we have

$$N_n(x) = -\frac{1}{(1+q)^{n-1}} \sum_{k=0}^{n-2} x_{k+1} A_k + \frac{s_{n-1} A_{n-1}}{(1+q)^{n-1}} + \frac{s_n}{(1+q)^n} - \frac{q^{n-1}}{(1+q)^n} s_0,$$

where

$$A_k = \sum_{i=0}^{k} \Big[\frac{q}{1+q} \begin{pmatrix} n \\ i \end{pmatrix} - \begin{pmatrix} n-1 \\ i \end{pmatrix} \Big] q^{n-i-1}.$$

Clearly, for any sequence $x = (x_n)$, $y = (y_n)$ and scalar λ, we have: $N_n(x+y) = N_n(x) + N_n(y)$ and $N_n(\lambda x) = \lambda N_n(x)$.

Let

$$\mathfrak{M} = [m_{ij}] = \begin{bmatrix} p_1 & w_1^{(1)} & w_1^{(2)} & \cdots \\ w_1^{(-1)} & p_2 & w_2^{(1)} & \cdots \\ w_1^{(-2)} & w_2^{(-1)} & p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where $p = (p_i)$ and $w^{(t)} = (w_i)^{(t)}$ are some fixed numerical sequences $t \in \mathbb{Z}\backslash\{0\}$. For a fixed $k_f \in \mathbb{N}$, we define a finite sequence t_n with k_f terms as $t_n = \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ \frac{-n}{2}, & n \text{ is even} \end{cases}$. We construct a matrix $\mathfrak{M}_{(p,w^t,k_f)} = \mathfrak{M}$, $w^{t_i} = 0 \;\forall\; i > k_f$ and for $i = 1, 2, \ldots, k_f$ we have some fixed sequences w^{t_i} and p.

Example 1.1. *For $k_f = 2$ we have $t_1 = 1, t_2 = -1$, we define $p_i = -1 \;\forall\; i$ and*

$$w_i^{(t)} = \begin{cases} 1, & for\, t = 1, -1 \\ 0, & \forall\; t \in \mathbb{Z}\backslash\{0, 1, -1\} \end{cases},$$

then we have

$$\mathfrak{M}_{(p,w^t,2)}x = \left\langle \sum_{j=1}^{\infty} m_{ij}\xi_j \right\rangle_n =< -\xi_1 + \xi_2, \xi_1 - \xi_2 + \xi_3, \xi_2 - \xi_3 + \xi_4, \xi_3 - \xi_4 + \xi_5 \cdots > .$$

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ $(x \to \infty)$. Clerly, if M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$ for all $\lambda \in (0,1)$. Using the idea of Orlicz function, Lindenstrauss and Tzafriri [15] constructed the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

is called Orlicz sequence space and showed that ℓ_M is a Banach space with the following norm:

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \le p < \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be a Musielak-Orlicz function [22]. A sequence $\mathcal{V} = (V_k)$ is defined by $V_k(v) = \sup\{|v|u - (M_k) : u \ge 0\}$ $(k = 1, 2, \cdots)$ is said to complementary function of $\mathcal{M}$. For a given $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined by

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ denotes the convex modular and is defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} (M_k)(x_k) \quad (x = (x_k) \in t_{\mathcal{M}}).$$

It is noted that $t_{\mathcal{M}}$ equipped with the Luxemburg norm or equipped with the Orlicz norm, where Luxemburg and Orlicz norms are given by

$$||x|| = \inf \Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\} \ \text{ and } \ ||x||^0 = \inf \Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\},$$

respectively.

Kızmaz [14] was the first who introduced the idea of difference sequence spaces and studied $Z(\Delta) = \{x = (x_k) \in w : \Delta x \in Z\}$ $(Z = l_\infty, c, c_0)$, where $\Delta x = x_k - x_{k+1}$ for all $k \in \mathbb{N}$ ($\mathbb{N}$ and w denote the set of natural numbers and the set of all real and complex sequences) and the standard notations l_∞, c and c_0 denote bounded, convergent and null sequences respectively. Et and Çolak [7] presented a generalization of these difference sequence spaces and introduced the space $Z(\Delta^n)$ $(n \in \mathbb{N})$, in this case, $\Delta^n x$ is given by $\Delta^n x = \Delta(\Delta^{n-1}x) = \Delta^{n-1}x_k - \Delta^{n-1}x_{k+1}$ for $n \geq 2$, which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}.$$

We remark that if we take $n = 1$, then difference sequence space $Z(\Delta^n)$ is reduced to $Z(\Delta)$.

Gähler [12] extended the usual notion of normed spaces into 2-normed spaces while the notion was again extended to n-normed spaces by Misiak [16]. Assume that X is a linear space over the field $\mathbb{K}$ of real or complex numbers of dimension $d \geq n \geq 2$, $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of natural numbers). A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the conditions:

(N1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent in X;

(N2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

(N3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \, ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$;

(N4) $||x_1 + x_1', x_2, \cdots, x_n|| \leq ||x_1, x_2, \cdots, x_n|| + ||x_1', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over $\mathbb{K}$.

For more details about these notions we refer to [3–5, 13, 18, 19, 21, 23] and references therein.

We used the standard notation $\theta = (k_r)$ to denotes the lacunary sequence, where θ is a sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ $(r \to \infty)$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_r - 1}$ by q_r (see [9]).

2 Main results

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ a sequence of strictly positive numbers. We define the following sequence space in the present paper:

$$E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) = \Big\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s}$$
$$\times \Big[M_k\Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k} < \infty, \;\; s \geq 0, \text{ for some } \;\rho > 0 \Big\}.$$

We will use the following inequality to prove our results. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the space $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$ is linear over the field $\mathbb{R}$ of real numbers.*

Proof. Let $x = (x_k), y = (y_k) \in E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive integers ρ_1 and ρ_2 such that

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho_1}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k} < \infty$$

and

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r y)}{\rho_2}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k} < \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is nondecreasing and convex function so we have

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} (\alpha\Delta^r x + \beta\Delta^r y))}{\rho_3}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k}$$
$$\leq \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \alpha\Delta^r x)}{\rho_3}, z_1, \cdots, z_{n-1} \Big\| \Big)$$
$$+ \Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \beta\Delta^r y)}{\rho_3}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k}$$
$$\leq K \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho_1}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k}$$
$$+ K \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r y)}{\rho_2}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k}$$

$$< \infty.$$

Therefore, $\alpha x + \beta y \in E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, ||\cdot, \cdots, \cdot||)$. This proves that $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, ||\cdot, \cdots, \cdot||)$ is a linear space. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then, the space $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$ is paranormed space with the paranorm defined by*

$$g(x) = \inf\left\{\rho^{p_n/H} : \left(\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) = g(-x)$ and $g(x+y) \leq g(x)+g(y)$. Since $M_k(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = 0$. Finally, we prove that multiplication is continuous. Let λ be any number then,

$$g(\lambda x) = \inf\left\{\rho^{p_n/H} : \lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{\lambda u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1\right\}$$

implies that

$$g(\lambda x) = \inf\left\{(\lambda s)^{p_n/H} : \lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{s}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1\right\},$$

where $s = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$, then $|\lambda|^{p_k/H} \leq \left(\max\left(1, |\lambda|^H\right)\right)^{\frac{1}{H}}$. Hence

$$g(\lambda x) \leq \left(\max\left(1, |\lambda|^H\right)\right)^{\frac{1}{H}} \inf\left\{(s)^{p_n/H} : \left(\lim_r \frac{1}{h_r}\right.\right.$$
$$\left.\left.\times \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}$$

and therefore, $g(x)$ converges to zero when $g(x)$ converges to zero in $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$. Now suppose that $\lambda_n \to 0$ as $n \to \infty$ and $x \in E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$. For arbitrary $\epsilon > 0$, let n_0 be a positive integer such that

$$\lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} < \frac{\epsilon}{2}$$

for some $\rho > 0$. This implies that

$$\left(\lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq \frac{\epsilon}{2}.$$

Let $0 < |\lambda| < 1$, then using convexity of (M_k), we get

$$\lim_r \frac{1}{h_r} \sum_{k=n_0+1}^{\infty} k^{-s} \Big[M_k \Big(\Big\| \frac{\lambda u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k}$$

$$< |\lambda| \lim_r \frac{1}{h_r} \sum_{k=n_0+1}^{\infty} k^{-s} \Big[M_k \Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k} < \Big(\frac{\epsilon}{2}\Big)^H.$$

Since (M_k) is continuous everywhere in $[0, \infty)$, so

$$h(t) = \lim_r \frac{1}{h_r} \sum_{k=1}^{n_0} k^{-s} \Big[M_k \Big(\Big\| \frac{t u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k}$$

is continuous at 0. Hence, there is $0 < \delta < 1$ such that $|h(t)| < \epsilon/2$ for $0 < t < \delta$. Let K be such that $|\lambda_n| < \delta$ for $n > K$ we have

$$\Big(\lim_r \frac{1}{h_r} \sum_{k=1}^{n_0} k^{-s} \Big[M_k \Big(\Big\| \frac{\lambda_n u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k} \Big)^{\frac{1}{H}} < \frac{\epsilon}{2}.$$

Thus, for $n > K$,

$$\Big(\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k \Big(\Big\| \frac{\lambda_n u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_k} \Big)^{\frac{1}{H}} < \epsilon.$$

Hence, $g(\lambda x) \to 0$ as $\lambda \to 0$. This completes the proof of the theorem. □

Theorem 2.3. *If $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ are two Musielak-Orlicz functions and s, s_1, s_2 be non-negative real numbers, then we have*

(i) $E_n^q(\mathcal{M}', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \cap E_n^q(\mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E_n^q(\mathcal{M}' + \mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.

(ii) If the inequality $s_1 \leq s_2$ holds, then $E_n^q(\mathcal{M}', u, p, s_1, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E_n^q(\mathcal{M}', u, p, s_2, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.

(iii) If $\mathcal{M}'$ and $\mathcal{M}''$ are equivalent, then $E_n^q(\mathcal{M}', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) = E_n^q(\mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.

Proof. It is obvious so we omit the details. □

Theorem 2.4. *Suppose that $0 < r_k \leq p_k < \infty$ for each k. Then*

$$E_n^q(\mathcal{M}, u, r, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|).$$

Proof. Let $x \in E_n^q(\mathcal{M}, u, r, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$. Then there exists some $\rho > 0$ such that

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k \Big(\Big\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{r_k} < \infty.$$

This implies that

$$M_k\Big(\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\leq 1$$

for sufficiently large value of k, say $k\geq k_0$ for some fixed $k_0\in\mathbb{N}$. Since (M_k) is nondecreasing, we have

$$\lim_r\frac{1}{h_r}\sum_{k\geq k_0}^{\infty}k^{-s}\Big[M_k\Big(\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho},z_1,\cdots,z_{n-1}\Big|\Big)\Big]^{p_k}$$
$$\leq\lim_r\frac{1}{h_r}\sum_{k\geq k_0}^{\infty}k^{-s}\Big[M_k\Big(\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{r_k}<\infty.$$

Hence, $x\in E_n^q(\mathcal{M},u,p,s,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|)$. □

Theorem 2.5. *(i) If $0<p_k\leq 1$ for each k, then $E_n^q(\mathcal{M},u,p,s,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|)\subseteq E_n^q(\mathcal{M},u,s,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|)$.*
(ii) If $p_k\geq 1$ for all k, then $E_n^q(\mathcal{M},u,s,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|)\subseteq E_n^q(\mathcal{M},u,p,s,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|)$.

Proof. It is easy to prove by using above so we omit the details. □

3 Applications to statistical convergence

Fast [8] extended the notion of usual convergence of a sequence of real or complex numbers and called it statistical convergence. This notion turned out to be one of the most active areas of research in summability theory after the works of Fridy [10] and Šalát [24]. Fridy and Orhan [11] defined and studied the notion of lacunary statistical convergence. Some recent related work and applications we refer to [1,2,6,17,20]. We are now ready to define following notions:

Definition 3.1. *Let $\theta=(k_r)$ be a lacunary sequence. Then, the sequence $x=(x_k)$ is $N_k(u)$-lacunary statistically convergent to the number l provided that for every $\epsilon>0$,*

$$\lim_r\frac{1}{h_r}\Big|\Big\{k\in I_r:\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1\cdots,z_{n-1}\Big\|\geq\epsilon\Big\}\Big|=0.$$

In symbols, we shall write $[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta$-$\lim x=l$ or $x_k\to l([N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta)$. If we take $\theta=(2^r)$, then we shall write $[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]$ instead of $[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta$.

Definition 3.2. *Let $\theta=(k_r)$ be a lacunary sequence, $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $u=(u_k)$ be a sequence of strictly positive real numbers and $p=(p_k)$ be a bounded sequence of positive real numbers. We say that $x=(x_k)$ is strongly $N_k(u,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r)$-lacunary convergent to l with respect to $\mathcal{M}$ provided that*

$$\lim_r\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}=0.$$

The set of all strongly $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary convergent sequences to l with respect to $\mathcal{M}$ is denoted by $[N_k, u, \mathcal{M}, p, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$. In symbols, we shall write $x_k \to l([N_k, u, \mathcal{M}, p, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta)$.

Note that, in the special case, $\mathcal{M}(x) = x$, $p_k = p_0$ for all $k \in \mathbb{N}$, we shall write $[N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$ instead of $[N_k, u, \mathcal{M}, p, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$.

Theorem 3.3. *Let $\theta = (k_r)$ be a lacunary sequence.*

(i) If a sequence $x = (x_k)$ is strongly $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary convergent to l, then it is $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary statistically convergent to l.

(ii) If a bounded sequence $x = (x_k)$ is $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary statistically convergent to l, then it is strongly $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary convergent to l.

Proof. (i) Let $\epsilon > 0$ and $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta)$. Then, we have

$$\sum_{k\in I_r} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$\geq \sum_{\substack{k\in I_r \\ \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon}} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$\geq \epsilon^{p_0} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon \right\} \right|.$$

Hence, $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta)$.

(ii) Suppose that $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta)$ and let $x \in l_\infty$. Let $\epsilon > 0$ be given and take N_ϵ such that

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \left(\frac{\epsilon}{2}\right)^{\frac{1}{p_0}} \right\} \right| \leq \frac{\epsilon}{2K^{p_0}}$$

for all $r > N_\epsilon$ and set

$$T_r = \left\{ k \in I_r : \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \left(\frac{\epsilon}{2}\right)^{\frac{1}{p_0}} \right\},$$

where $K = \sup_k |x_k| < \infty$. Now for all $r > N_\epsilon$ we have

$$\lim_r \frac{1}{h_r} \sum_{k\in I_r} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$= \lim_r \frac{1}{h_r} \sum_{\substack{k\in I_r \\ k\in T_r}} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$+\lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r\\ k\notin T_r}} \Big\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho}, z_1,\cdots,z_{n-1}\Big\|^{p_0}$$

$$\leq \lim_r \frac{1}{h_r}\Big(\frac{h_r\epsilon}{2K^{p_0}}\Big)K^{p_0}+\frac{\epsilon}{2h_r}h_r=\epsilon.$$

Thus, $(x_k)\in[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|]_\theta$. □

Theorem 3.4. *For any lacunary sequence θ, if $\lim_{r\to\infty}\inf q_r>1$ then*

$$[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]\subset[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta.$$

Proof. If $\lim_{r\to\infty}\inf q_r>1$, then there exists a $\delta>0$ such that $1+\delta\leq q_r$ for sufficiently large r. Since $h_r=k_r-k_{r-1}$, we have $\frac{k_r}{h_r}\leq\frac{1+\delta}{\delta}$. Let $x_k\to l([N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r])$. Then for every $\epsilon>0$, we have

$$\frac{1}{k_r}\Big|\Big\{k\leq k_r:\Big\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|\geq\epsilon\Big\}\Big|$$

$$\geq\frac{1}{k_r}\Big|\Big\{k\in I_r:\Big\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|\geq\epsilon\Big\}\Big|$$

$$\geq\Big(\frac{\delta}{1+\delta}\Big)\frac{1}{h_r}\Big|\Big\{k\in I_r:\Big\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|\geq\epsilon\Big\}\Big|.$$

Hence, $[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]\subset[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta$. □

In the next results we denote the quantity $\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho}$ by $x_k^{l,\rho}$.

Theorem 3.5. *Let $\theta=(k_r)$ be a lacunary sequence, $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H$. Then $[N_k,\mathcal{M},u,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|]_\theta\subset[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta$.*

Proof. Let $x\in[N_k,\mathcal{M},u,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|]_\theta$. Then there exists a number $\rho>0$ such that $\lim_r\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\|x_k^{l,\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\to 0$, as $r\to\infty$. Then given $\epsilon>0$, we have

$$\lim_r\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\Big\|x_k^{l,\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}$$

$$\geq\lim_r\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_k^{l,\rho},z_1,\cdots,z_{n-1}\|\geq\epsilon}}\Big[M_k\Big(\Big\|x_k^{l,\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}$$

$$\geq\lim_r\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_k^{l,\rho},z_1,\cdots,z_{n-1}\|\geq\epsilon}}[M_k(\epsilon_1)]^{p_k},\quad\text{where }\ \epsilon/\rho=\epsilon_1$$

$$\geq\lim_r\frac{1}{h_r}\sum_{k\in I_r}\min\Big\{[M_k(\epsilon_1)]^h,[M_k(\epsilon_1)]^H\Big\}$$

$$\geq \lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\right\| \geq \epsilon\right\}\right|. \min\left\{[M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H\right\}.$$

Hence, $x \in [N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta$. This completes the proof of the theorem. □

Theorem 3.6. *Let* $\theta = (k_r)$ *be a lacunary sequence,* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function and* $p = (p_k)$ *be a bounded sequence, then* $[N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta \subset [N_k, \mathcal{M}, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$.

Proof. Let $x \in l_\infty$ and $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta)$. Since $x \in l_\infty$, there is a constant $T > 0$ such that $\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\| \leq T$ and given $\epsilon > 0$ we have

$$\begin{aligned}
&\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M_k\Big(\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\Big)\right]^{p_k} \\
&= \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\geq\epsilon}}\left[M_k\Big(\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\Big)\right]^{p_k} \\
&\quad + \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|<\epsilon}}\left[M_k\Big(\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\Big)\right]^{p_k} \\
&\leq \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\geq\epsilon}}\max\left\{\left[M_k\left(\frac{T}{\rho}\right)\right]^h, \left[M_k\left(\frac{T}{\rho}\right)\right]^H\right\} \\
&\quad + \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|<\epsilon}}\left[M_k\left(\frac{\epsilon}{\rho}\right)\right]^{p_k} \\
&\leq \max\left\{[M_k(K)]^h, [M_k(K)]^H\right\}\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\| \geq \epsilon\right\}\right| \\
&\quad + \max\left\{[M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H\right\}, \quad \left(\frac{T}{\rho} = K,\ \frac{\epsilon}{\rho} = \epsilon_1\right).
\end{aligned}$$

Hence, $x \in [N_k, \mathcal{M}, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$. □

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OSCILLATION ANALYSIS FOR HIGHER ORDER DIFFERENCE EQUATION WITH NON-MONOTONE ARGUMENTS

ÖZKAN ÖCALAN AND UMUT MUTLU ÖZKAN

ABSTRACT. The aim of this paper is to obtain the some new oscillatory conditions for all solutions of higher order difference equation with general argument

$$(\star)\qquad \Delta^m x(n) + p(n)x(\tau(n)) = 0,\ n = 0, 1, \cdots,$$

where $(p(n))$ is a sequence of nonnegative real numbers and $(\tau(n))$ is a sequence of integers and non-monotone.

1. Introduction

Oscillation theory of difference equations has attracted many researchers. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of delay difference equations. For these oscillatory and nonoscillatory results, we refer, for instance, $[1-22]$.

Consider the higher order difference equation with general argument

$$\Delta^m x(n) + p(n)x(\tau(n)) = 0,\ n = 0, 1, \cdots, \tag{1.1}$$

where $(p(n))_{n\geq 0}$ is a sequence of nonnegative real numbers and $(\tau(n))_{n\geq 0}$ is a sequence of integers such that

$$\tau(n) \leq n-1 \text{ for all } n \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = \infty. \tag{1.2}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$.

Define

$$r = -\min_{n\geq 0} \tau(n). \quad \text{(Clearly, } k \text{ is a positive integer.)}$$

By a solution of the difference equation (1.1), we mean a sequence of real numbers $(x(n))_{n\geq -r}$ which satisfies (1.1) for all $n \geq 0$.

A solution $(x(n))_{n\geq -r}$ of the difference equation (1.1) is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.

If $m = 1$, then Eq.(1.1) take the form

$$\Delta x(n) + p(n)x(\tau(n)) = 0,\ n = 0, 1, \cdots. \tag{1.3}$$

In particular, if we take $\tau(n) = n - l$ where $l > 0$, then Eq.(1.3) reduces to

$$\Delta x(n) + p(n)x(n-l) = 0. \tag{1.4}$$

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Corresponding author : Özkan ÖCALAN; email address: ozkanocalan@akdeniz.edu.tr.

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In 1989, Erbe and Zhang [8] proved that each one of the conditions

$$\liminf_{n\to\infty} p(n) > \frac{l^l}{(l+1)^{l+1}} \tag{1.5}$$

and

$$\limsup_{n\to\infty} \sum_{j=n-l}^{n} p(j) > 1 \tag{1.6}$$

is sufficient for all solutions of (1.4) to be oscillatory.

In the same year, 1989, Ladas, Philos and Sficas [11] established that all solutions of (1.4) are oscillatory if

$$\liminf_{n\to\infty} [\frac{1}{l} \sum_{j=n-l}^{n-1} p(j)] > \frac{l^l}{(l+1)^{l+1}}. \tag{1.7}$$

Clearly, condition (1.6) improves to (1.4).

In 1991, Philos [14] extended the oscillation criterion (1.7) to the general case of the Eq.(1.3), by establishing that, if the sequence $(\tau(n))_{n\geq 0}$ is increasing, then the condition

$$\liminf_{n\to\infty} \left[\frac{1}{n-\tau(n)} \sum_{j=\tau(n)}^{n-1} p(j) \right] > \limsup_{n\to\infty} \frac{(n-\tau(n))^{n-\tau(n)}}{(n-\tau(n)+1)^{n-\tau(n)+1}} \tag{1.8}$$

suffices for the oscillation of all solutions of Eq.(1.3).

In 1998, Zhang and Tian [19] obtained that if $(\tau(n))$ is non-decreasing,

$$\lim_{n\to\infty} (n-\tau(n)) = \infty \tag{1.9}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.10}$$

then all solutions of Eq.(1.3) are oscillatory.

Later, in 1998, Zhang and Tian [20] obtained that if $(\tau(n))$ is non-decreasing or non-monotone,

$$\limsup_{n\to\infty} p(n) > 0 \tag{1.11}$$

and (1.10) holds, then all solutions of Eq.(1.3) are oscillatory.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [3] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n) = \max_{0\leq s\leq n} \tau(s)$,

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1, \tag{1.12}$$

then all solutions of Eq.(1.3) are oscillatory.

In the same year, Chatzarakis, Koplatadze and Stavroulakis [4] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n) = \max_{0\leq s\leq n} \tau(s)$,

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) < \infty \tag{1.13}$$

and (1.10) holds, then all solutions of Eq.(1.3) are oscillatory.

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In 2006, Yan, Meng and Yan [17] obtained that if $(\tau(n))$ is non-decreasing,

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j)>0 \tag{1.14}$$

and

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j)\left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1}>1, \tag{1.15}$$

then all solutions of Eq.(1.3) are oscillatory.

Finally, in 2016, Öcalan [16] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n)=\max_{0\le s\le n}\tau(s)$ and (1.15) holds, then all solutions of Eq.(1.3) are oscillatory.

Set

$$k(n)=\left(\frac{n-\tau(n)+1}{n-\tau(n)}\right)^{n-\tau(n)+1},\quad n\ge 1. \tag{1.16}$$

Clearly

$$e\le k(n)\le 4,\quad n\ge 1. \tag{1.17}$$

Observe that, it is easy to see that

$$\sum_{j=\tau(n)}^{n-1}p(j)k(j)\ge e\sum_{j=\tau(n)}^{n-1}p(j)$$

and therefore condition (1.15) is better than condition (1.10).

In 2006, Zhou [22] studied the following delay difference equation with constant delays

$$\Delta^m x(n)+\sum_{i=1}^{l}p_i(n)x(n-k_i)=0,\ n=0,1,\cdots, \tag{1.18}$$

where $(p_i(n))_{n\ge 0}$ are sequences of nonnegative real numbers and k_i is a positive integer for $i=1,2,\cdots,l$. He obtained some new criteria for all solutions of Eq.(1.18) to be oscillatory.

2. Main Results

In this section we investigated the oscillatory behavior of all solutions of Eq.(1.1).

Further, we need the following lemmas proved in [1, 2].

Lemma 2.1. *(Discrete Kneser's Theorem) Let $x(n)$ be defined for $n\ge n_0$, and $x(n)>0$ with $\Delta^m x(n)$ of constant sing for $n\ge n_0$ and not identically zero. Then, there exists an integer j, $0\le j\le m$ with $(m+j)$ odd for $\Delta^m x(n)\le 0$ or $(m+j)$ even for $\Delta^m x(n)\ge 0$ and such that*

$$j\le m-1 \text{ implies } (-1)^{j+i}\Delta^i x(n)>0,\quad \text{for all } n\ge n_0,\ j\le i\le m-1,$$

and

$$j\ge 1 \text{ implies } \Delta^i x(n)>0,\quad \text{for all large } n\ge n_0,\ 1\le i\le j-1.$$

Specially, if $\Delta^m x(n)\le 0$ for $n\ge n_0$, and $(x(n))$ is bounded, then

$$(-1)^{i+1}\Delta^{m-i}x(n)\ge 0,\ \text{for all } n\ge n_0,\quad 1\le i\le m-1,$$

and

$$\lim_{n\to\infty} \Delta^i x(n) = 0,\ 1 \le i \le m-1.$$

Lemma 2.2. *Let $x(n)$ be defined for $n \ge n_0$, and $x(n) > 0$ with $\Delta^m x(n) \le 0$ for $n \ge n_0$ and not identically zero. Then, there exists a large $n_1 \ge n_0$ such that*

$$x(n) \ge \frac{(n-n_1)^{m-1}}{(m-1)!}\Delta^{m-1} x(2^{m-j-1}n)\ ;\ \ n \ge n_1,$$

where j is defined in Lemma 2.1*. Further, if $x(n)$ is increasing, then*

$$x(n) \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1} \Delta^{m-1} x(n)\ ;\ \ n \ge 2^{m-1}n_1.$$

Set

$$h(n) = \max_{0\le s\le n} \tau(s) \tag{2.1}$$

Clearly, $h(n)$ is nondecreasing, and $\tau(n) \le h(n)$ for all $n \ge 0$. We note that if $\tau(n)$ is nondecreasing, then we have $\tau(n) = h(n)$ for all $n \ge 0$.

Theorem 2.3. *Assume that* (1.2) *holds. If $(\tau(n))$ is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)k(j) > (m-1)!, \tag{2.2}$$

where $k(n)$ is defined by (1.16)*, then every solution of Eq.*(1.1) *either oscillates or* $\lim_{n\to\infty} x(n) = 0$.

Proof. Assume, for the sake of contradiction, that $(x(n))$ is an eventually positive solution of (1.1) and $\lim_{n\to\infty} x(n) > 0$. Then there exists $n_1 \ge n_0$ such that $x(n),\ x(\tau(n)),\ x(h(n)) > 0$, for all $n \ge n_1$. Thus, from Eq.(1.1) we have

$$\Delta^m x(n) = -p(n)x(\tau(n)) \le 0, \quad \text{for all } n \ge n_1. \tag{2.3}$$

By Lemma 2.1, $\Delta^i x(n)$ are eventually of one sign for every $i \in \{1, 2, \dots, m-1\}$, and $\Delta^{m-1}x(n) > 0$ holds for large n, and there exist two cases to consider: (A) $\Delta x(n) > 0$ and (B) $\Delta x(n) < 0$.

Case A: This says that $(x(n))$ is increasing. By Lemma 2.2, there exists an integer $n_2 \ge n_1$ such that

$$x(n) \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1} \Delta^{m-1} x(n)\ ,\ \ n \ge n_2$$

and

$$x(\tau(n)) \ge \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} \Delta^{m-1} x(\tau(n))\ ,\ \ n \ge n_2 \tag{2.4}$$

Letting $y(n) = \Delta^{m-1}x(n)$. So, we have

$$y(n) > 0,\ y(\tau(n)) > 0 \text{ for } n \ge n_2,$$

which implies that

$$\Delta y(n) + p(n)x(\tau(n)) = 0,\ n \ge n_2. \tag{2.5}$$

On the other hand, by (2.4) and since $\lim_{n\to\infty}\tau(n)=\infty$, there exists an integer $n_3 \geq n_2$ such that

$$x(\tau(n)) \geq \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} y(\tau(n)) \geq \frac{1}{(m-1)!} y(\tau(n)), \quad n \geq n_3. \tag{2.6}$$

In view of (2.6), Eq.(2.5) gives

$$\Delta y(n) + \frac{1}{(m-1)!} p(n) y(\tau(n)) \leq 0, \quad n \geq n_3. \tag{2.7}$$

Taking into account that $y(n)$ is nonincreasing and $h(n)$ is nondecreasing, $\tau(n) \leq h(n)$ for all $n \geq 0$, from (2.7) we get

$$\Delta y(n) + \frac{1}{(m-1)!} p(n) y(h(n)) \leq 0, \quad n \geq n_3. \tag{2.8}$$

It follows that

$$\Delta y(n) + \widetilde{p}(n) y(h(n)) \leq 0, \quad n \geq n_3, \tag{2.9}$$

where $\widetilde{p}(n) = \frac{p(n)}{(m-1)!}$, which means that inequality (2.9) has an eventually positive solution.

On the other hand, we know from Lemma 2.3 in [16] that

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)k(j) = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j)k(j), \tag{2.10}$$

where $h(n)$ is defined by (2.1).

Therefore, condition (2.2) and (2.10) imply that

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \widetilde{p}(j)k(j) = \frac{1}{(m-1)!} \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j)k(j) > 1 \tag{2.11}$$

Thus, by Theorem 1 in [16], Eq.(2.9) has no eventually positive solution. This is a contradiction.

Case B: Note that, by Lemma 2.1, it is impossible that the case that m is even. In what follows, we only consider the case that m is odd. Case B says that $x(n)$ is decreasing and bounded, and so, $(x(n))$ converges a constant a. By Lemma 2.1, we get

$$(-1)^{i+1}\Delta^{m-i}x(n) > 0, \text{ for all large } n \geq n_1, \quad 1 \leq i \leq m-1, \tag{2.12}$$

and

$$\lim_{n\to\infty} \Delta^{m-1}x(n) = 0. \tag{2.13}$$

By (2.13), there exists an integer $n_4 \geq n_1$ such that

$$0 \leq \Delta^{m-1}x(n) \leq \varepsilon, \text{ for any } \varepsilon > 0, \ n \geq n_4. \tag{2.14}$$

It is obvious that $a > 0$. So, there exists an integer $n_5 \geq n_4$ such that

$$x(n) > \frac{1}{2}a, \ x(\tau(n)) > \frac{1}{2}a, \ n \geq n_5. \tag{2.15}$$

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Thus, Eq.(1.1) implies that

$$\Delta^m x(n) + \frac{a}{2}p(n) \leq 0, \ n \geq n_5. \tag{2.16}$$

Summing both sides of (2.16) from n_5 to n, we obtain

$$\Delta^{m-1}x(n+1) - \Delta^{m-1}x(n_5) + \frac{a}{2}\sum_{s=n_5}^{n} p(s) \leq 0, \ n \geq n_5. \tag{2.17}$$

Letting $n \to \infty$, we have

$$\frac{a}{2}\sum_{s=n_5}^{n} p(s) \leq \varepsilon, \text{ for large } n. \tag{2.18}$$

On the other hand, condition (2.2) says that there exist an integer $n_6 \geq n_5$ such that

$$\sum_{s=\tau(n)}^{n-1} p(s)k(s) > \frac{(m-1)!}{2}, \ n \geq n_6. \tag{2.19}$$

Since $k(n) \leq 4$ for $n \geq 1$, by (2.19) we get

$$\frac{a}{2}\sum_{s=\tau(n)}^{n-1} p(s) \geq \frac{a(m-1)!}{8}, \text{ for large } n, \tag{2.20}$$

which contradicts (2.18) and (2.20). The proof is completed. □

Theorem 2.4. *Assume that m is even and* (1.2) *holds. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \tau^{m-1}(j)p(j)k(j) > 2^{(m-1)^2}(m-1)!, \tag{2.21}$$

where $k(n)$ is defined by (1.16)*, then every solution of Eq.*(1.1) *oscillates.*

Proof. Assume, for the sake of contradiction, that $(x(n))$ is an eventually positive solution of (1.1). Then there exists $n_1 \geq n_0$ such that $x(n), \ x(\tau(n)), \ x(h(n)) > 0$, for all $n \geq n_1$. According to the proof of Theorem 2.3, there exists a positive integer n_1 such that (2.3) holds. By Lemma 2.1, we have

$$\Delta x(n) > 0$$

which implies $x(n)$ is increasing. In view of proof of Theorem 2.3, we have

$$x(\tau(n)) \geq \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} y(\tau(n)), \tag{2.22}$$

where $y(n) = \Delta^{m-1}x(n)$. Therefore, from Eq.(2.5) and (2.22), we obtain

$$\Delta y(n) + \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} p(n)y(\tau(n)) \leq 0, \ n \geq n_2. \tag{2.23}$$

Taking into account that $y(n)$ is nonincreasing and $h(n)$ is nondecreasing, $\tau(n) \leq h(n)$ for all $n \geq 0$, from (2.23) we get,

$$\Delta y(n) + \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} p(n)y(h(n)) \leq 0, \quad n \geq n_3. \tag{2.24}$$

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It follows that

$$\Delta y(n) + \widetilde{p}(n) y\left(h(n)\right) \leq 0, \quad n \geq n_3, \tag{2.25}$$

where $\widetilde{p}(n) = \left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} \frac{p(n)}{(m-1)!}$, which means that inequality (2.25) has an eventually positive solution.

On the other hand, we know from Lemma 2.3 in [16] that

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)k(j) = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j)k(j), \tag{2.26}$$

where $h(n)$ is defined by (2.1).

Therefore, condition (2.21) and (2.26) imply that

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \widetilde{p}(j)k(j) = \frac{1}{(m-1)!} \frac{1}{2^{(m-1)^2}} \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \tau^{m-1}(j)p(j)k(j) > 1 \tag{2.27}$$

Thus, by Theorem 1 in [16], Eq.(2.25) has no eventually positive solution. This contradiction completes the proof. □

Now, using (1.16), (1.17), Theorem 2.3 and Theorem 1 in [16], we have the following results immediately.

Corollary 2.5. *Assume that* (1.2) *holds. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}(m-1)!, \tag{2.28}$$

then every solution of Eq.(1.1) *either oscillates or* $\lim_{n\to\infty} x(n) = 0$.

Corollary 2.6. *Assume that* m *is even and* (1.2) *holds. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \tau^{m-1}(j)p(j) > \frac{2^{(m-1)^2}}{e}(m-1)!, \tag{2.29}$$

then every solution of Eq.(1.1) *oscillates.*

Finally, using the proofs of Theorem 2.3 and Theorem 2.4, and from the Theorem 2.1 in [3], we obtain the following results by removing the proofs.

Theorem 2.7. *Assume that* (1.2) *and* (1.14) *hold. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) > (m-1)!, \tag{2.30}$$

where $h(n)$ *is defined by* (2.1), *then every solution of Eq.*(1.1) *either oscillates or* $\lim_{n\to\infty} x(n) = 0$.

Theorem 2.8. *Assume that m is even and (1.2) holds. If $(\tau(n))$ is non-decreasing or non-monotone,*

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} \tau^{m-1}(j)p(j) > 2^{(m-1)^2}(m-1)!, \tag{2.31}$$

where $h(n)$ is defined by (2.1), then every solution of Eq.(1.1) oscillates.

we present an example to show the significance of our new result.

Example 2.1. *Consider the retarded difference equation*

$$\Delta^3 x(n) + \frac{3}{e}x(\tau(n)) = 0, \qquad n \geq 0, \tag{2.32}$$

with

$$\tau(n) = \begin{cases} n-3, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}.$$

Here, it is clear that (1.2) is satisfied. By (2.1), we see that

$$h(n) = \max_{0\leq s\leq n} \tau(s) = \begin{cases} n-2, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}.$$

Computing, we get

$$\sum_{j=\tau(n)}^{n-1} p(j) = \begin{cases} 6/e, & \text{if } n \text{ is even} \\ 3/e, & \text{if } n \text{ is odd} \end{cases}.$$

Thus

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) = \frac{3}{e} > \frac{1}{e}(m-1)! = \frac{2}{e},$$

that is, condition (2.28) of Corollary 2.5 is satisfied and therefore every solution of Eq.(2.32) either oscillates or $\lim_{n\to\infty} x(n) = 0$.

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Akdeniz University, Faculty of Science, Department of Mathematics, 07058, Antalya, Turkey

E-mail address: ozkanocalan@akdeniz.edu.tr

Afyon Kocatepe University, Faculty of Science and Arts, Department of Mathematics, ANS Campus, 03200, Afyon, Turkey

E-mail address: umut_ozkan@aku.edu.tr

On Orthonormal Wavelet Bases

Richard A. Zalik *

Abstract

Given a multiresolution analysis with one generator in $L^2(\mathbb{R}^d)$, we give a characterization in closed form and in the frequency domain, of all orthonormal wavelets associated to this MRA. Examples are given. This theorem corrects a previous result of the author.

1 Introduction

In what follows $\mathbb{Z}$ will denote the set of integers, and $\mathbb{R}$ the set of real numbers. We will always assume that $\mathbf{A}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$; that is, $\mathbf{A}\mathbb{Z}^d \subset \mathbb{Z}^d$ and all its eigenvalues have modulus greater than 1; $\mathbf{A}^*$ will denote the transpose of $\mathbf{A}$ and $\mathbf{B} := (A^*)^{-1}$. The underlying space will be $L^2(\mathbb{R}^d)$, where $d \geq 1$ is an integer and $\mathbf{I}$ will stand for the identity matrix. Boldface lowcase letters will denote elements of $\mathbb{R}^d$, which will be represented as column vectors; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$.

Let $\mathbf{A} \in \mathbb{R}^{d \times d}$ and $a := |\det \mathbf{A}|$. For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D^{\mathbf{A}} f(\mathbf{t}) := a^{1/2} f(\mathbf{A}\mathbf{t}) \qquad \text{and} \qquad T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t} + \mathbf{k})$$

respectively.

Let $\mathbf{u} = \{u_1, \ldots, u_m\} \subset L^2(\mathbb{R}^d)$; then $T(u_1, \ldots, u_m) = T(\mathbf{u})$, $S(u_1, \ldots, u_m) = S(\mathbf{u})$ and $S(\mathbf{A}; u_1, \ldots, u_m) = S(\mathbf{A}; \mathbf{u})$ are respectively defined by

$$T(\mathbf{u}) := \{T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}, \qquad S(\mathbf{u}) := \overline{\text{span}}\, T(\mathbf{u}),$$

and

$$S(\mathbf{A}, \mathbf{u}) := \overline{\text{span}}\, \{D^{\mathbf{A}} T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}.$$

In [5] we formulated a representation theorem for multiresolution analyses having an arbitrary set $u_1, \ldots, u_n$ of scaling functions, i.e., the set of translates of all these functions constitutes an orthonormal basis of V_0. However the proof was based on the implicit (and incorrect) assumption that any such function u_ℓ is contained in $S(\mathbf{A}, u_\ell)$, and it is therefore not valid. The purpose of this paper is to apply the method of proof

*Department of Mathematics and Statistics, Auburn University, AL 36849-5310, zalik@auburn.edu

employed in [5] to prove a representation theorem for MRA's having a single scaling function, and to provide some examples.

A function f will be called $\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}}f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$.

The Fourier transform of a function f will be denoted by $\widehat{f}$ or $\mathfrak{F}(f)$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi\mathbf{x}\cdot\mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

The Fourier transform is extended to $L^2(\mathbb{R}^d)$ in the usual way.

Our starting point and motivation is the following well known characterization in Fourier space of affine MRA orthonormal wavelets in $L^2(\mathbb{R})$ (see e.g. Hernández and Weiss [2], Wojtaszczyk [4]) which, with the definition of Fourier transform we have adopted, may be stated as follows.

Theorem A. *Let φ be a scaling function for a multiresolution analysis M with associated low pass filter p. The following propositions are equivalent:*
(a) ψ is an MRA orthonormal wavelet associated with M.
(b) There is a measurable unimodular $\mathbb{Z}$–periodic function $\mu(x)$ such that

$$\widehat{\psi}(2x) = e^{i2\pi x}\mu(2x)\overline{p(x+1/2)}\widehat{\varphi}(x) \qquad a.e.$$

Recall that a *multiresolution analysis* (MRA) in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}, f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{At}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \emptyset$.

(v) There is a function u (called the *scaling function* of the MRA) such that $T(u)$ is an orthonormal basis of V_0.

A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ is called an orthonormal wavelet system if the affine sequence

$$\{D_j^{\mathbf{A}} T_{\mathbf{k}}\psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be an orthonormal wavelet system in $L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$; for $j \in \mathbb{Z}$ we define

$$V_j = \sum_{r<j} S(\mathbf{A}^r; \boldsymbol{\psi}).$$

We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\boldsymbol{\psi}$ is associated with M. Let W_j denote the orthogonal complement of V_j in V_{j+1}. Then it is esily seen that $\boldsymbol{\psi}$ is an orthonormal wavelet associated with M if and only if $T(\boldsymbol{\psi})$ is an orthonormal basis of W_0.

Let $\mathbf{e} := (1, 0, \cdots, 0)^T \in \mathbb{R}^m$ and let $\mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{m}}$ denote the $m \times m$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries. The following proposition was implicitly established by Jia and Shen in the discussion that follows the proof of [3, Lemma 3.3] (we adopt the convention that $\mathrm{Arg}\ 0 = 0$).

Theorem B. *Let* $\mathbf{b} = (b_1, \cdots, b_m)^T \in \mathbb{C}^m$ *be unimodular,* $\omega := Arg\, b_1$ *and* $\mathbf{q} := \mathbf{b} + e^{i\omega}\mathbf{e}$. *Then the matrix*

$$\mathbf{Q} = (q_{r,k})_{r,k=1}^m := \mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{m}} \left[\overline{\mathbf{I} - 2\mathbf{q}\mathbf{q}^*/\mathbf{q}^*\mathbf{q}}\right]$$

is unitary. Moreover

$$q_{r,k} = \begin{cases} b_k & \text{if } r = 1, 1 \le k \le m \\ -\overline{b_r} e^{i\omega} & \text{if } 1 < r \le m, k = 1 \\ \delta_{r,k} - \dfrac{\overline{b_r} b_k}{1 + |b_1|} & \text{if } 1 < r \le m, 1 < k \le m, \end{cases}$$

where $\delta_{r,k}$ *is Krönecker's delta.*

The following proposition is a particular case of [5, Theorem 3].

Lemma C. *Let* $u \in L^2(\mathbb{R}^d)$ and assume that $T(u)$ is an orthonormal sequence. Let $\mathbf{A}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\{j_1, \ldots, j_a\}$ be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$, and let

$$v_k(\mathbf{t}) := a^{1/2} u(\mathbf{A}t + j_k), \quad k = 1, \ldots a. \tag{1}$$

Then $T(v_1, \cdots, v_a)$ is an orthonormal basis of $S(\mathbf{A}; u)$.

Since $\widehat{v}_k(\mathbf{x}) = e^{i2\pi \mathbf{Bx} \cdot j_k} \widehat{u}(\mathbf{Bx})$, a straightforward consequence of Lemma C and [5, Lemma E] is the following

Corollary 1. *Let* $u \in L^2(\mathbb{R}^d)$ *and assume that* $T(u)$ *is an orthonormal sequence. Let* $\mathbf{A}$ *be a dilation matrix preserving the lattice* $\mathbb{Z}^d$, $B := (A^*)^{-1}$, *let* $\{j_1, \ldots, j_a\}$ *be a full collection of representatives of* $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$, *and let* $v_k(\mathbf{t})$ *be defined by* (1). *If* $u \in S(A, u)$, *then there are* $\mathbb{Z}^d$*–periodic functions* $q_k \in L^2(\mathbb{T}^d)$ *such that*

$$\sum_{k=1}^{a} |q_k(\mathbf{x})|^2 = 1 \quad a.e., \tag{2}$$

and

$$\widehat{u}(\mathbf{x}) = \sum_{k=1}^{a} q_k(\mathbf{x}) \widehat{v}_k(\mathbf{x}) = \sum_{k=1}^{a} q_k(\mathbf{x}) e^{i2\pi \mathbf{Bx} \cdot j_k} \widehat{u}(\mathbf{Bx}) = p(B\mathbf{x}) \widehat{u}(B\mathbf{x}), \tag{3}$$

where

$$p(\mathbf{x}) := a^{-1/2} \sum_{k=1}^{a} q_k(A^*\mathbf{x}) e^{i2\pi \mathbf{x}\cdot j_k}.$$

We can now prove

Theorem 1. *Let M be a multiresolution analysis generated by $\mathbf{A}$ with scaling function u, let $v_k(\mathbf{t})$ be defined by (1), $B := (A^*)^{-1}$, and let the functions $q_k(\mathbf{x})$ be $\mathbb{Z}^d$–periodic, in $L^2(\mathbb{T}^d)$, and satisfy (2) and (3). Let*

$$\alpha(\mathbf{x}) := Arg\, q_1(\mathbf{x}), \tag{4}$$

$$w_{r,k}(\mathbf{x}) := \begin{cases} q_k(\mathbf{x}) & \text{if } r = 1, 1 \le k \le a \\ -\overline{q_r(\mathbf{x})} e^{i\alpha(\mathbf{x})} & \text{if } 1 < r \le a, k = 1 \\ \delta_{r,k} - \dfrac{\overline{q_r(\mathbf{x})} q_k(\mathbf{x})}{1 + |q_1(\mathbf{x})|} & \text{if } 1 < r \le a, 1 < k \le a \end{cases} \tag{5}$$

and

$$\widehat{z}_r(\mathbf{x}) := \sum_{k=1}^{a} w_{r,k}(\mathbf{x}) \widehat{v}_k(\mathbf{x}),$$

and let

$$\mathbf{Z}(\mathbf{x}) := \left(\widehat{z}_2(\mathbf{x}), \ldots, \widehat{z}_a(\mathbf{x})\right)^T.$$

Then

$$\{\psi_1, \ldots, \psi_{(a-1)}\}$$

is an orthonormal wavelet system associated with M if and only if there exists an $(a-1) \times (a-1)$ unitary matrix function $\mathbf{U}(x)$ such that

$$(\widehat{\psi}_1(\mathbf{x}), \ldots, \widehat{\psi}_{(a-1)}(\mathbf{x}))^T = \mathbf{U}(\mathbf{x})\mathbf{Z}(\mathbf{x}).$$

Proof. The existence of functions $q_k(\mathbf{x})$ satisfying (2) and (3) is a consequence of Corollary 1. Setting

$$\widehat{\mathbf{v}}(\mathbf{x}) := (\widehat{v}_1(\mathbf{x}), \cdots, \widehat{v}_a(\mathbf{x}))^T$$

and applying Theorem B, we see that

$$(\widehat{z}_1(\mathbf{x}), \cdots \widehat{z}_a(\mathbf{x}))^T = \mathbf{Q}(\mathbf{x})\widehat{\mathbf{v}}(\mathbf{x}),$$

and that $\mathbf{Q}(\mathbf{x})$ has $(q_1(\mathbf{x}), \cdots, q_a(\mathbf{x}))$ as its first row. Therefore [5, Theorem 8] implies that $\{z_2, \ldots z_a\}$ is an orthonormal wavelet system associated with M, which is equivalent to saying that $S(z_2, \ldots z_a)$ is an orthonormal basis generator of W_0. Applying now [5, Theorem 5], the assertion follows. □

Example 1. Let us verify that Theorem A is a particular case of Theorem 1.

For $d = 1$ and $A = 2$ we have $j_1 = 0$ and $j_2 = 1$, and Corollary 1 implies that

$$p(x) = 2^{-1/2}[q_1(2x) + e^{i2\pi x}q_2(2x)],$$

whence the periodicity of $q_1(x)$ and $q_2(x)$ implies that

$$p(x + 1/2) = 2^{-1/2}[q_1(2x) - e^{i2\pi x}q_2(2x)].$$

On the other hand, since $|q_1(x)|^2+|q_2(x)|^2 = 1$ a.e., (5) implies that $w_{2,1}(x) = -\overline{q_2(x)}e^{i\alpha(x)}$ and

$$w_{2,2}(x) = 1 - \frac{|q_2(x)|^2}{1 + |q_1(x)|} = 1 - \frac{|q_2(x)|^2(1 - |q_1(x)|)}{|q_2(x)|^2} = |q_1(x)|.$$

Since $\mathbf{B} = 1/2$, it follows that $\widehat{v}_1(x) = 2^{-1/2}\widehat{u}(x/2)$ and $\widehat{v}_2(x) = 2^{-1/2}e^{-i\pi x}\widehat{u}(x/2)$, and Theorem 1 implies that

$$\begin{aligned}\widehat{z}_2(x) = 2^{-1/2}[-e^{i\alpha(x)}\overline{q_2(x)} + e^{i\pi x}|q_1(x)|]\widehat{u}(x/2) = \\ 2^{-1/2}e^{i\pi x}e^{i\alpha(x)}[-\overline{q_2(x)}e^{-i\pi x} + e^{-i\alpha(x)}|q_1(x)|]\widehat{u}(x/2) = \\ 2^{-1/2}e^{-i\pi x}e^{i\alpha(x)}[\overline{q_1(x)} - e^{i\pi x}\overline{q_2(x)}]\widehat{u}(x/2),\end{aligned}$$

and therefore

$$\widehat{z}_2(2x) = 2^{-1/2}e^{-i2\pi x}e^{i\alpha(2x)}[\overline{q_1(2x)} - e^{i2\pi x}\overline{q_2(2x)}]\widehat{u}(x) = e^{-i2\pi x}\mu(2x)\overline{p(x + 1/2)}\widehat{u}(x),$$

where $\mu(x) := e^{i\alpha(x)}$ is unimodular and $\mathbb{Z}$-periodic.

Example 2. Let

$$\mathbf{A} := \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

and let $\phi(\mathbf{t})$ be the characteristic function of $[0, 1] \times [0, 1]$. Gröchenig and Madych [1] have shown that ϕ is a scaling function of an MRA generated by the dilation matrix $\mathbf{A}$ and that the function ψ defined by

$$\psi(\mathbf{t}) := \begin{cases} 1 & \text{if } \mathbf{t} \in [0, 1] \times [0, 1/2] \\ -1 & \text{if } \mathbf{t} \in [0, 1] \times [1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

is a wavelet associated with this MRA. Let us see how this assertion follows from Theorem 1.

Since $\{(0, 0)^T, (1, 0)^T\}$ is a a full collection of representatives of $A/A\mathbb{Z}^2$, from Lemma C we deduce that if $v_1(\mathbf{t}) := 2^{-1/2}\phi(A\mathbf{t})$ and $v_2(\mathbf{t}) := 2^{-1/2}\phi(A\mathbf{t}+(1, 0)^T)$, then $T(v_1, v_2)$ is an orthonormal basis of $S(A, \phi)$, and a straightforward computation shows that

$$\phi(\mathbf{t}) = 2^{-1/2}\left(v_1(\mathbf{t} - (1, 0)^T) + v_2(\mathbf{t} - (1, 1)^T)\right),$$

which implies that if $\mathbf{x} = (x_1, x_2)^T$, then

$$\widehat{\phi}(\mathbf{x}) = 2^{-1/2}\left(e^{-i2\pi x_1}\widehat{v}_1(\mathbf{x}) + e^{-i2\pi(x_1+x_2)}\widehat{v}_2(\mathbf{x})\right).$$

Thus $q_1(\mathbf{x}) = 2^{-1/2}e^{-i2\pi x_1}$, $q_2(\mathbf{x}) = 2^{-1/2}e^{-i2\pi(x_1+x_2)}$ and $\alpha(\mathbf{x}) = i2\pi x_1$, and proceeding as in Example 1 we see that

$$w_{2,1}(\mathbf{x}) = -\overline{q_2(x)}e^{i\alpha(x)} = 2^{-1/2}e^{-i2\pi x_2} \quad \text{and} \quad w_{2,2}(\mathbf{x}) = |q_1(\mathbf{x})| = 2^{-1/2}.$$

Thus,

$$\widehat{z_2}(\mathbf{x}) = w_{2,1}(\mathbf{x})\widehat{v}_1(\mathbf{x}) + w_{2,2}(\mathbf{x})\widehat{v}_2(\mathbf{x}) = 2^{-1/2}\left(\widehat{v}_2(\mathbf{x}) - e^{-i2\pi x_2}\widehat{v}_1(\mathbf{x})\right),$$

which by Theorem 1 implies that $\sigma(\mathbf{t})$ is a wavelet associated with A if and only if there is a measurable unimodular $\mathbb{Z}^2$–periodic function $\mu(\mathbf{x})$ such that

$$\widehat{\sigma}(\mathbf{x}) = \mu(\mathbf{x})\widehat{z_2}(\mathbf{x}).$$

In particular, $\widehat{\psi}(\mathbf{x}) = e^{-i2\pi x_1}\widehat{z_2}(\mathbf{x})$.

Example 3. Gröchenig and Madych have also shown in [1] that the characteristic function ϕ of $[0,1]\times[0,1]$ which we considered in the previous example is also a scaling function of an MRA generated by the dilation matrix

$$\mathbf{A} := 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $a = 4$, from e.g. [5, Theorem H] we know that any orthonormal wavelet associated with this MRA has exactly three generators.. Let us construct an orthonormal wavelet basis using Theorem 1. The vectors $j_1 := (0,0)^T, j_2 := (1,0)^T, j_3 := (0,1)^T$ and $j_4 := (1,1)^T$ are a full collection of representatives of $A/A\mathbb{Z}^2$. Let

$$v_k(\mathbf{t}) := 2\phi(A\mathbf{t} + j_k) = 2\phi(2\mathbf{t} + j_k).$$

Lemma C implies that $T(v_1, v_2, v_3, v_4)$ is an orthonormal basis of $S(A,\phi)$. Moreover, it is easily verified that

$$\phi(\mathbf{t}) = \sum_{k=1}^{4}\phi(2\mathbf{t} - j_k) = (1/2)\sum_{k=1}^{4} v_k(\mathbf{t} - j_k).$$

Since

$$\mathfrak{F}\{v_k(\cdot - j_k)\}(\mathbf{x}) = e^{-i2\pi\mathbf{x}\cdot j_k}\widehat{v_k}(\mathbf{x})$$

we see that

$$\widehat{\phi}(\mathbf{x}) = (1/2)\sum_{k=1}^{4} e^{-i2\pi\mathbf{x}\cdot j_k}\widehat{v}_k(\mathbf{x}),$$

and therefore

$$q_k(\mathbf{x}) = (1/2)e^{-i2\pi\mathbf{x}\cdot j_k}, k = 1, \ldots 4.$$

Since $\alpha(\mathbf{x}) = 0$, (5) implies that

$$w_{r,k}(\mathbf{x}) := \begin{cases} \frac{1}{2}e^{-i2\pi\mathbf{x}\cdot j_k} & \text{if } r = 1, 1 \leq k \leq 4 \\ -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_r} & \text{if } 1 < r \leq 4, k = 1 \\ -\frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_k - j_r)} & \text{if } 1 < r \leq 4, 1 < k \leq 4, k \neq r. \\ \frac{5}{6} & \text{if } 1 < r \leq 4, 1 < k \leq 4, k = r. \end{cases}$$

Thus,

$$\widehat{z}_2(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_2}\widehat{v}_1(\mathbf{x}) + \frac{5}{6}\widehat{v}_2(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_3-j_2)}\widehat{v}_3(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_4-j_2)}\widehat{v}_4(\mathbf{x})$$

$$\widehat{z}_3(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_3}\widehat{v}_1(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_2-j_3)}\widehat{v}_2(\mathbf{x}) + \frac{5}{6}\widehat{v}_3(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_4-j_3)}\widehat{v}_4(\mathbf{x})$$

and

$$\widehat{z}_4(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_4}\widehat{v}_1(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_2-j_4)}\widehat{v}_2(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_3-j_4)}\widehat{v}_3(\mathbf{x}) + \frac{5}{6}\widehat{v}_4(\mathbf{x}).$$

i.e.,

$$z_2(t) = -\frac{1}{2}v_1(\mathbf{t} + j_2) + \frac{5}{6}v_2(\mathbf{t}) - \frac{1}{6}v_3(\mathbf{t} + (j_3 - j_2)) - \frac{1}{6}v_4(\mathbf{t} + (j_4 - j_2)),$$

$$z_3(\mathbf{t}) = -\frac{1}{2}v_1(\mathbf{t} + j_3) - \frac{1}{6}v_2(\mathbf{t} + (j_2 - j_3)) + \frac{5}{6}v_3(\mathbf{t}) - \frac{1}{6}v_4(\mathbf{t} + (j_4 - j_3)),$$

and

$$z_4(\mathbf{t}) = -\frac{1}{2}v_1(\mathbf{t} + j_4) - \frac{1}{6}v_2(\mathbf{t} + (j_2 - j_4)) - \frac{1}{6}v_3(\mathbf{t} + (j_3 - j_4)) + \frac{5}{6}v_4(\mathbf{t}).$$

Applying Theorem 1 we conclude that $\{z_2, z_3, z_4\}$ is an orthonormal wavelet system associated with the dilation matrix $\mathbf{A}$, and that $\{\psi_1, \psi_2, \psi_3\}$ is an orthonormal wavelet system associated with $\mathbf{A}$ if and only if there exists a 3×3 unitary matrix function $\mathbf{U}(x)$ such that

$$(\widehat{\psi}_1(\mathbf{x}), \widehat{\psi}_2(\mathbf{x}), \widehat{\psi}_3(\mathbf{x}))^T = \mathbf{U}(\mathbf{x})(\widehat{z}_2(\mathbf{x}), \widehat{z}_3(\mathbf{x}), \widehat{z}_4(\mathbf{x}))^T.$$

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Neutrosophic sets applied to mighty filters in BE-algebras

Jung Mi Ko[1] **and Sun Shin Ahn**[2,*]

[1] *Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea*

[2] *Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. The notion of a neutrosophic subalgebra of a BE-algebra is introduced and consider characterizations of a neutrosophic subalgebra and a neutrosophic filter. We defined the notion of a neutrosophic mighty filter of a BE-algebra, and investigated some properties of it. We provide conditions for a neutrosophic filter to be a neutrosophic mighty filter.

1. Introduction

In 2007, Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Y. B. Jun et. al [4] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy filters of BE-algebras and investigated their relations and properties. J. S. Han et. al [3] defined the notion of hesitant fuzzy implicative filter of a BE-algebra, and considered some properties of it.

Zadeh [11] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). In 2015, neutrosophic set theory is applied to BE-algebra, and the notion of neutrosophic filter is introduced [9]. A new definition of neutrosopic filter is established and some basic properties are presented [12].

In this paper, we introduce the notion of a neutrosophic subalgebra of a BE-algebra and consider characterizations of a neutrosophic subalgebra and a neutrosophic filter. We defined the notion of a neutrosophic mighty filter of a BE-algebra, and investigated some properties of it. We provide conditions for a neutrosophic filter to be a neutrosophic mighty filter.

2. Preliminaries

By a *BE-algebra* ([6]) we mean a system $(X; *, 1)$ of type $(2, 0)$ which the following axioms hold:

(BE1) $(\forall x \in X)\,(x * x = 1)$,

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* Correspondence: Tel: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).

[0] **E-mail**: `jmko@gwnu.ac.kr (J. M. Ko); sunshine@dongguk.edu (S. S. Ahn).`

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(BE2) $(\forall x \in X)\,(x * 1 = 1)$,

(BE3) $(\forall x \in X)\,(1 * x = x)$,

(BE4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general ([6]).

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

(2.1) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,

(2.2) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,

(2.3) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X ([6]) if

(F1) $1 \in F$;

(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is a *mighty filter* ([8]) of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(z * (y * x), z \in F \Rightarrow ((x * y) * y) * x \in F)$.

Theorem 2.2. ([8]) *A filter F of a BE-algebra X is mighty if and only if*

(2.4) $(\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F)$.

Definition 2.3. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosopic set A can be denoted by

$$A := \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.4. ([10]) A neutrosophic set A is contained in the other neutrosophic B, denoted by $A \subseteq B$, if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$, and $F_A(x) \geq F_B(x)$ for any $x \in X$. Two neutrosophic sets A and B are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 2.5. ([12]) Let A be a neutrosophic set in a BE-algebra X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$ and an (α, β, γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}.$$

3. Neutrosophic subalgebras in BE-algebras

Definition 3.1. A neutrosophic set A in a BE-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $\min\{T_A(x), T_A(y)\} \leq T_A(x*y), \max\{I_A(x), I_A(y)\} \geq I_A(x*y),$ and $\max\{F_A(x), F_A(y)\} \geq F_A(x*y)$, for any $x, y \in X$.

Example 3.2. Let $X := \{1, a, b, c\}$ be a BE-algebra ([4]) with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.83, & \text{if } x \in \{1, a\} \\ 0.13, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.15, & \text{if } x \in \{1, a\} \\ 0.82, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.15, & \text{if } x \in \{1, a\} \\ 0.82, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Definition 3.3. ([12]) A neutrosophic set A in a BE-algebra X is called a *neutrosophic filter* of X if it satisfies:

(NSF1) $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1),$ and $F_A(x) \geq F_A(1)$, for any $x \in X$;

(NSF2) $\min\{T_A(x), T_A(x*y)\} \leq T_A(y), \max\{I_A(x), I_A(x*y)\} \geq I_A(y),$ and $\max\{F_A(x), F_A(x*y)\} \geq F_A(y)$, for any $x, y \in X$.

Proposition 3.4. *Every neutrosophic filter of a BE-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be a neutrosophic filter of X. For any $x, y \in X$, we have $\min\{T_A(x), T_A(y)\} \leq \min\{T_A(1), T_A(y)\} = \min\{T_A(y*(x*y)), T_A(y)\} \leq T_A(x*y)$, $\max\{I_A(x), I_A(y)\} \geq \max\{I_A(1), I_A(y)\} = \max\{I_A(y*(x*y)), I_A(y)\} \geq I_A(x*y)$, and $\max\{F_A(x), F_A(y)\} \geq \max\{F_A(1), F_A(y)\} = \max\{F_A(y*(x*y)), F_A(y)\} \geq F_A(x*y)$. Hence A is a neutrosophic subalgebra of X. □

The converse of Proposition 3.4 may not be true in general (see Example 3.5).

Example 3.5. Let $X := \{1, a, b\}$ be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	1	1

Define a neutrosophic set A in X as follows: $T_A = \{(1, 0.83), (a, 0.13), (b, 0.16)\}, I_A = \{(1, 0.15), (a, 0.15), (b, 0.82)\}$, and $F_A = \{(1, 0.15), (a, 0.15), (b, 0.82)\}$. It is easy to check that A is a neutrosophic subalgebra of X. But it is not a neutrosophic filter of X, since $\min\{T_A(b*a), T_A(b)\} = \min\{T_A(1), T_A(b)\} = 0.16 \nleq 0.13 = T_A(a)$.

Theorem 3.6. *Let A be a neutrosophic set in a BE-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x) \geq \alpha, T_A(y) \geq \alpha, I_A(x) \leq \beta, I_A(y) \leq \beta$ and $F_A(x) \leq \gamma, F_A(y) \leq \gamma$. Using (NSS), we have $\alpha \leq \min\{T_A(x), T_A(y)\} \leq T_A(x * y)$, $\beta \geq \max\{I_A(x), I_A(y)\} \geq I_A(x * y)$, and $\gamma \geq \max\{F_A(x), F_A(y)\} \geq F_A(x * y)$. Hence $x * y \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Assume that there exist $a_t, b_t, a_i, b_i \in X$ and $a_f, b_f \in X$ such that $\min\{T_A(a_t), T_A(b_t)\} > T_A(a_t * b_t), \max\{I_A(a_i), I_A(b_i)\} < I_A(a_i * b_i)$, and $\max\{F_A(a_f), F_A(b_f)\} < F_A(a_f * b_f)$. Then $\min\{T_A(a_t), T_A(b_t)\} \geq t_{\alpha_1} > T_A(a_t * b_t), \max\{I_A(a_i), I_A(b_i)\} \leq t_{\alpha_2} < I_A(a_i * b_i)$, and $\max\{F_A(a_f), F_A(b_f)\} \leq t_{\alpha_3} < F_A(a_f * b_f)$ for some $t_{\alpha_1} \in (0, 1]$, and $t_{\alpha_2}, t_{\alpha_3} \in [0, 1)$. Hence $a_t, b_t, a_i, b_i, a_f, b_f \in A^{(t_{\alpha_1},t_{\alpha_2},t_{\alpha_3})}$, but $a_t * b_t, a_i * b_i, a_f * b_f \notin A^{(t_{\alpha_1},t_{\alpha_2},t_{\alpha_3})}$, which is a contradiction. Hence $\min\{T_A(x), T_A(y)\} \leq T_A(x * y), \max\{I_A(x), I_A(y)\} \geq I_A(x * y)$, and $\max\{F_A(x), F_A(y)\} \geq F_A(x * y)$ for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.7. *If $\{A_i|i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a BE-algebra X, then $(\{A_i|i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Proposition 3.8. *If A is a neutrosopic subalgebra of a BE-algebra X, then $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$, and $F_A(x) \geq F_A(1)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 3.9. *Let A be a neutrosophic subalgebra of a BE-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 1, \lim_{n\to\infty} I_A(a_n) = 0$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(1) = 1, I_A(1) = 0$, and $F_A(1) = 0$.*

Proof. By Proposition 3.8, we have $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$, and $F_A(x) \geq F_A(1)$ for all $x \in X$. Hence we have $T_A(a_n) \leq T_A(1), I_A(a_n) \geq I_A(1)$, and $F_A(a_n) \geq F_A(1)$ for every positive integer n. Therefore $1 = \lim_{n\to\infty} T_A(a_n) \leq T_A(1) \leq 1, 0 = \lim_{n\to\infty} I_A(a_n) \geq I_A(1) \geq 0$, and $0 = \lim_{n\to\infty} F_A(a_n) \geq F_A(1) \geq 0$. Thus we have $T_A(1) = 1, T_A(1) = 0$, and $F_A(1) = 0$. □

Proposition 3.10. *If every neutrosophic subalgebra A of a BE-algebra X satisfies the condition*

(3.1) $T_A(x * y) \geq T_A(x), I_A(x * y) \leq I_A(x), F_A(x * y) \leq F_A(x)$, *for any $x, y \in X$,*

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.1) that $T_A(x) = T_A(1 * x) \geq T_A(1), I_A(x) = I_A(1 * x) \leq I_A(1)$, and $F_A(x) = F_A(1 * x) \leq F_A(1)$ for any $x \in X$. By Proposition 3.8, we have $T_A(x) = T_A(1), I_A(x) = I_A(1)$, and $F_A(x) = F_A(1)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Proposition 3.11. *Let A be a neutrosophic filter of a BE-algebra X. Then*

(i) $\min\{T_A(x * (y * z)), T_A(y)\} \leq T_A(x * z), \max\{I_A(x * (y * z)), I_A(y)\} \geq I_A(x * z)$, *and* $\max\{F_A(x * (y * z)), F_A(y)\} \geq F_A(x * z)$ *for any* $x, y \in X$.

(ii) $T_A(a) \leq T_A((a * x) * x), I_A(a) \geq I_A((a * x) * x)$, *and* $F_A(a) \geq F_A((a * x) * x)$ *for any* $a, x \in X$.

Proof. (i) Using (BE4) and (NSF2), we have $T_A(x * z) \geq \min\{T_A(y * (x * z)), T_A(y)\} = \min\{T_A(x * (y * z)), T_A(y)\}, I_A(x * z) \leq \max\{I_A(y * (x * z)), I_A(y)\} = \max\{I_A(x * (y * z)), I_A(y)\}$, and $F_A(x * z) \leq \max\{F_A(y * (x * z)), F_A(y)\} = \max\{F_A(x * (y * z)), F_A(y)\}$ for any $x, y \in X$.
(ii) Taking $y := (a * x) * x$ and $x := a$ in (NSF2), we have $T_A((a * x) * x) \geq \min\{T_A(a * ((a * x) * x)), T_A(a)\} = \min\{T_A((a * x) * (a * x)), T_A(a)\} = \min\{T_A(1), T_A(a)\} = T_A(a), I_A((a * x) * x) \leq \max\{I_A(a * ((a * x) * x)), I_A(a)\} = \max\{I_A((a * x) * (a * x)), I_A(a)\} = \max\{I_A(1), I_A(a)\} = I_A(a)$, and $F_A((a * x) * x) \leq \max\{F_A(a * ((a * x) * x)), F_A(a)\} = \max\{F_A((a * x) * (a * x)), F_A(a)\} = \max\{F_A(1), F_A(a)\} = F_A(a)$ for any $a, x \in X$.

□

Theorem 3.12. ([12]) *Let A be a neutrosophic set in a BE-algebra. Then A is a neutrosophic filter of X if and only if it satisfies (NSF1) and*

(3.2) *if* $x \leq y * z$ *for any* $x, y \in X$, *then* $\min\{T_A(x), T_A(y)\} \leq T_A(z), \max\{I_A(x), I_A(y)\} \geq I_A(z)$, *and* $\max\{F_A(x), F_A(y)\} \geq F_A(z)$.

Theorem 3.13. *If every neutrosophic set of a BE-algebra X satisfies (NSF1) and Proposition 3.11(i), then it is a neutrosophic filter of X.*

Proof. Taking $x := 1$ in Proposition 3.11(i) and using (BE3), we get $T_A(z) = T_A(1 * z) \geq \min\{T_A(1 * (y * z)), T_A(y)\} = \min\{T_A(y * z), T_A(y)\}, I_A(z) = I_A(1 * z) \leq \max\{I_A(1 * (y * z)), T_A(y)\} = \max\{I_A(y * z), I_A(y)\}$, and $F_A(z) = F_A(1 * z) \leq \max\{F_A(1 * (y * z)), F_A(y)\} = \max\{F_A(y * z), F_A(y)\}$ for any $y, z \in X$. Hence A is a neutrosophic filter of X. □

Corollary 3.14. *Let A be a neutrosophic set of a BE-algebra X. Then A is a neutrosophic filter of X if and only if it satisfies (NSF1) and Proposition 3.11(i).*

Theorem 3.15. *Let A be a neutrosophic set of a BE-algebra X. Then A is a neutrosophic filter of X if and only if it satisfies the following conditions:*

(i) $T_A(y * x) \geq T_A(x), I_A(y * x) \leq I_A(x)$, *and* $F_A(y * x) \leq F_A(x)$;

(ii) $T_A((a*(b*x))*x) \geq \min\{T_A(a), T_A(b)\}, I_A((a*(b*x))*x) \leq \max\{I_A(a), I_A(b)\}$, *and* $F_A((a*(b*x))*x) \leq \max\{F_A(a), F_A(b)\}$ *for any* $a, b, x \in X$.

Proof. Assume that A is a neutrosophic filter of X. Using (NSF2), we have $T_A(y * x) \geq \min\{T_A(x * (y * x)), T_A(x)\} = \min\{T_A(1), T_A(x)\} = T_A(x), I_A(y * x) \leq \max\{I_A(x * (y * x)), I_A(x)\} = \max\{I_A(1), I_A(x)\} = I_A(x)$, and $F_A(y * x) \leq \max\{F_A(x * (y * x)), F_A(x)\} = \max\{F_A(1), F_A(x)\} = F_A(x)$, for any $x, y \in X$. It follows from Proposition 3.11 that $T_A((a*(b*x))*x) \geq \min\{T_A((a*(b*x))*(b*x)), T_A(b)\} \geq \min\{T_A(a), T_A(b)\}$, $I_A((a*(b*x))*x) \leq \max\{I_A((a*(b*x))*(b*x)), I_A(b)\} \leq \max\{I_A(a), I_A(b)\}$, and $F_A((a*(b*x))*x) \leq \max\{F_A((a*(b*x))*(b*x)), F_A(b)\} \leq \max\{F_A(a), F_A(b)\}$ for any $x, a, b \in X$.

Conversely, assume that A is a neutrosophic set of X satisfying conditions (i) and (ii). Taking $y := x$ in (i), we have $T_A(1) = T_A(x*x) \geq T_A(x), I_A(1) = I_A(x*x) \leq I_A(x)$ and $F_A(1) = F_A(x*x) \leq F_A(x)$ for any $x \in X$. Using (ii), we get $T_A(y) = T_A(1*y) = T_A(((x*y)*(x*y))*y) \geq \min\{T_A(x*y), T_A(x)\}, I_A(y) = I_A(1*y) = I_A(((x*y)*(x*y))*y) \leq \max\{I_A(x*y), I_A(x)\}, F_A(y) = F_A(1*y) = F_A(((x*y)*(x*y))*y) \leq \max\{F_A(x*y), F_A(x)\}$ for any $x, y \in X$. Hence A is a neutrosophic filter of X. ☐

4. Neutrosophic mighty filters in BE-algebras

Definition 4.1. A neutrosophic set A in a BE-algebra X is called a *neutrosophic mighty filter* of X if it satisfies (NSF1) and

(NSF3) $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(((x*y)*y)*x)), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(((x*y)*y)*x)$ for any $x, y, z \in X$.

Example 4.2. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra ([8]) with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.83, & \text{if } x \in \{1, b, c\} \\ 0.12, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.14, & \text{if } x \in \{1, b, c\} \\ 0.81, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.14, & \text{if } x \in \{1, b, c\} \\ 0.81, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic mighty filter of X.

Proposition 4.3. *Every neutrosophic mighty filter of a BE-algebra X is a neutrosophic filter of X.*

Proof. Let A be a neutrosophic mighty filter of X. Putting $y := 1$ in (NSF3), we obtain $\min\{T_A(z*(1*x)), T_A(z)\} = \min\{T_A(z*x), T_A(z)\} \leq T_A(((x*1)*1)*x) = T_A(x), \max\{I_A(z*(1*x)), I_A(z)\} = \max\{I_A(z*x), I_A(z)\} \geq I_A(((x*1)*1)*x) = I_A(x)$, and $\max\{F_A(z*(1*x)), F_A(z)\} = \max\{F_A(z*x), F_A(z)\} \geq F_A(((x*1)*1)*x) = F_A(x)$ for any $x, y, z \in X$. Hence A is a neutrosophic filter of X. ☐

The converse of Proposition 4.3 may be not true in general (see Example 4.4).

Example 4.4. Let $X := \{1, a, b, c, d\}$ be a BE-algebra ([5]) with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.84, & \text{if } x = 1 \\ 0.11, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.13, & \text{if } x = 1 \\ 0.81, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.13, & \text{if } x = 1 \\ 0.81, & \text{otherwise.} \end{cases}$$

Then A is a neutrosophic filter of X, but not a neutrosophic mighty filter of X, since $\min\{T_A(1*(c*a)), T_A(1)\} = T_A(1) = 0.84 \nleq T_A(((a*c)*c)*a) = T_A(a) = 0.11$.

Theorem 4.5. *Any neutrosophic filter A of a BE-algebra X is mighty if and only if it satisfies the following conditions:*

(4.1) $T_A(y*x) \leq T_A(((x*y)*y)*x), I_A(y*x) \geq I_A(((x*y)*y)*x)$, *and* $F_A(y*x) \geq F_A(((x*y)*y)*x)$ *for any* $x, y \in X$.

Proof. Suppose that a neutrosophic filter A of a BE-algebra X satisfies the condition (4.1). Using (NSF2) and (4.1), we have $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(y*x) \leq T_A(((x*y)*y)*x), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(y*x) \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(y*x) \geq F_A(((x*y)*y)*x)$ for any $x, y \in X$. Hence A is a neutrosophic mighty filter of X.

Conversely, assume that the neutrosophic filter A of X is mighty. Setting $z := 1$ in (NSF3), we have $\min\{T_A(1*(y*x)), T_A(1)\} = T_A(y*x) \leq T_A(((x*y)*y)*x), \max\{I_A(1*(y*x)), I_A(1)\} = I_A(y*x) \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(1*(y*x)), F_A(1)\} = F_A(y*x) \geq F_A(((x*y)*y)*x)$ for any $x, y \in X$. Hence (4.1) holds. □

Proposition 4.6. *Let A be a neutrosophic mighty filter of a BE-algebra X. Denote that $X_T := \{x \in X | T_A(x) = T_A(1)\}, X_I := \{x \in X | I_A(x) = I_A(1)\}$, and $X_F := \{x \in X | F_A(x) = F_A(1)\}$. Then X_T, X_I, and X_F are mighty filters of X.*

Proof. Clearly, $1 \in X_T, X_I, X_F$. Let $z*(y*x), z \in X_T$. Then $T_A(z*(y*x)) = T_A(1), T_A(z) = T_A(1)$. Hence $\min\{T_A(z*(y*x)), T_A(z)\} = T_A(1) \leq T_A(((x*y)*y)*x)$ and so $T_A((x*y)*y)*x) = T_A(1)$. Therefore $((x*y)*y)*x \in X_T$. Thus X_T is a mighty filter of X. Similarly, X_I, X_F are mighty filters of X. □

Theorem 4.7. *Let A, B be neutrosophic filters of a transitive BE-algebra X such that $A \subseteq B$ and $T_A(1) = T_B(1), I_A(1) = I_B(1), F_A(1) = F_B(1)$. If A is mighty, then B is mighty.*

Proof. Let $x, y \in X$. Since A is a neutrosophic mighty filter of a BE-algebra X, by (4.1) and $A \subseteq B$ we have $T_A(1) = T_A(y*((y*x)*x)) \leq T_A(((((y*x)*x)*y)*y)*((y*x)*x)) \leq T_B(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $T_A(1) = T_B(1)$, we get $T_B((y*x)*((((y*x)*x)*y)*y)*x)) = T_B(((((y*x)*x)*y)*y)*((y*x)*x)) = T_B(1)$. It follows from (NSF1) and (NSF2) that

$$\begin{aligned} T_B(y*x) =& \min\{T_B(1), T_B(y*x)\} \\ =& \min\{T_B((y*x)*(((((y*x)*x)*y)*y)*x)), T_B(y*x)\} \\ \leq& T_B(((((y*x)*x)*y)*y)*x). \end{aligned} \tag{4.2}$$

Since X is transitive, we get

$$\begin{aligned} [((((y*x)*x)*y)*y)*x]*&[((x*y)*y)*x] \\ &\geq ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\geq (((y*x)*x)*y)*(x*y) \\ &\geq x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1. \end{aligned}$$

It follows from Theorem 3.12 that $\min\{T_B(((((y*x)*x)*y)*y)*x), T_B(1)\} = T_B(((((y*x)*x)*y)*y)*x) \leq T_B(((x*y)*y)*x)$. Using (4.2), we have $T_B(y*x) \leq T_B(((((y*x)*x)*y)*y)*x) \leq T_B(((x*y)*y)*x)$. Therefore $T_B(y*x) \leq T_B(((x*y)*y)*x)$. Similarly, we have $I_B(y*x) \geq T_B(((x*y)*y)*x)$ and $F_B(y*x) \geq F_B(((x*y)*y)*x)$. By Theorem 4.5, B is a neutrosophic mighty filter of X. □

Theorem 4.8. *Let A be a neutrosophic set in a BE-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Then A is a neutrosophic mighty filter of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are mighty filters of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic mighty filter of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $z*(y*x), z \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(z*(y*x)) \geq \alpha, T_A(z) \geq \alpha, I_A(z*(y*x)) \leq \beta, I_A(z) \leq \beta$, and $F_A(z*(y*x)) \leq \gamma, F_A(z) \leq \gamma$. By Definition 4.1, we have $T_A(1) \geq T_A(((x*y)*y)*x) \geq \min\{T_A(z*(y*x)), T_A(z)\} \geq \alpha, I_A(1) \leq I_A(((x*y)*y)*x) \leq \max\{I_A(z*(y*x)), I_A(z)\} \leq \beta$, and $F_A(1) \leq F_A(((x*y)*y)*x) \leq \max\{F_A(z*(y*x)), F_A(z)\} \leq \gamma$. Hence $1, ((x*y)*y)*x \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ are mighty filters of X.

Conversely, suppose that there exist $a, b, c \in X$ such that $T_A(a) > T_A(1), I_A(b) < I_A(1)$, and $F_A(c) < F_A(1)$. Then there exist $a_t \in (0, 1]$ and $b_t, c_t \in [0, 1)$ such that $T_A(a) \geq a_t > T_A(1), I_A(b) \leq b_t < I_A(1)$ and $F_A(c) \leq c_t < F_A(1)$. Hence $1 \notin A^{(a_t,b_t,c_t)}$, which is a contradiction. Therefore $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$ for all $x \in X$. Assume that there exist $a_t, b_t, c_t, a_i, b_i, c_i \in X$ and $a_f, b_f, c_f \in X$ such that $T_A(((a_t*b_t)*b_t)*a_t) < \min\{T_A(c_t*(b_t*a_t)), T_A(c_t)\}, I_A(((a_i*b_i)*b_i)*a_i) > \max\{I_A(c_i*(b_i*a_i)), I_A(c_i)\}$, and $F_A(((a_f*b_f)*b_f)*a_f) > \max\{F_A(c_f*(b_f*a_f)), F_A(c_f)\}$. Then there exist $s_t \in (0, 1]$ and $s_i, s_f \in [0, 1)$ such that $T_A(((a_t*b_t)*b_t)*a_t) < s_t \leq \min\{T_A(c_t*(b_t*a_t)), T_A(c_t)\}, I_A(((a_i*b_i)*b_i)*a_i) > s_i \geq \max\{I_A(c_i*(b_i*a_i)), I_A(c_i)\}$, and $F_A(((a_f*b_f)*b_f)*a_f) > s_f \geq \max\{F_A(c_f*(b_f*a_f)), F_A(c_f)\}$. Hence $c_t*(b_t*a_t), c_t, c_i*(b_i*a_i), c_i \in A^{(s_t,s_i,s_f)}$ and $c_f*(b_f*a_f), c_f \in A^{(s_t,s_i,s_f)}$ but $((a_t*b_t)*b_t)*a_t, ((a_i*b_i)*b_i)*a_i \notin A^{(s_t,s_i,s_f)}$, and $((a_f*b_f)*b_f)*a_f \notin A^{(s_t,s_i,s_f)}$, which is a contradiction. Therefore $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(((x*y)*y)*x)), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(((x*y)*y)*x))$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(((x*y)*y)*x))$ for any $x, y, z \in X$. Thus A is a neutrosophic mighty filter of X □

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Coupled fixed point of firmly nonexpansive mappings by Mann's iterative processes in Hilbert spaces

TAMER NABIL[a,b1]

[a]King Khalid University, College of Science, Department of Mathematics, 61413, Abha, Saudi Arabia

[b]Suez Canal University, Faculty of Computers and Informatics, Department of Basic Science, Ismailia, Egypt

Abstract

We study the weak convergence of Mann's explicit iteration processes to common coupled fixed point of firmly nonexpansive coupled mappings in Hilbert spaces.Our results extend and generalized the results due to Nabil and Soliman for coupled fixed point approach (T. Nabil and A. H. Soliman, weak convergence theorems of explicit iteration process with errors and applications in optimization, J. Ana. Num. Theor., 5(2017) 81: 89).

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1 Introduction

The study of finding the fixed point of iterative processes has attracted the interest of many researchers due to its applications in physics, optimization, image processing and economics can be recast in terms of a fixed point problem of nonlinear mappings in Hilbert space [[1],[2], [3], [4], [5], [6]]. A lot of this studies consider this mappings as nonexpansive which is defined as: let H be a real Hilbert space and K be a nonempty closed convex subset of H. Then, a mapping R of K into H is called nonexpansive if $\|Rx - Ry\| \leq \|x - y\|$ for all $x, y \in K$. R is called firmly nonexpansive if

$$||Rx - Ry||^2 + ||(Id - R)x - (Id - R)y||^2 \leq ||x - y||^2 \quad (1)$$

[1] $t_3$bdelsadek@yahoo.com

for all $x, y \in K$, where $Id : K \to K$ denote the identity operator. We have known that every firmly nonexpansive mapping is a nonexpansive mapping. The finding of common fixed point for iteration process have been investigated since the early 1953 by Mann [7] which consider the following iteration scheme

$$\begin{cases} x_1 \in C \text{ is chosen arbitrarily} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) R x_n, \forall\, n \in N \end{cases}$$

where $\{\alpha_n\}$ is a sequence in [0,1]. Several authors studied another types of iteration process such as: Halpern [8] , Bauschke [9] and Xu and Ori [10] . In 2005, Kimura et al. [11], studied the convergence of an iterative scheme to a common fixed point of a finite family of nonexpansive mappings in Banach space.

The problem of finding a common fixed point of families of nonlinear mappings has been investigated by many researchers; see, for instance, ([12]-[17]).

Recently, Chuang and Takahashi [18] defined the new Mann's type iteration process by metric projection from H to K and gave weak convergence theorems for finding a common fixed point of a sequence of firmly nonexpansive mappings in a Hilbert space. More recently, in 2017 Nabil and Soliman [19] studied the weak convergen theorem f a new Mann iterative proesses with errors.

The idea of coupled fixed point was started in 1987 by Guo and Lakshmikantham [20]. Several authors studied the coupled fixed point Theorem See [[21],[22], [23], [24], [25]]

In this work, we prove the weak convergence theorem for finding the coupled fixed points of iteration processes for the families of nonlinear coupled mappings in Hilbert spaces.

2 Firmly nonexpansive coupled mappings

Throughout this paper we denote by N the set of positive integers and strongly (respectively weak) convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ (respectively $x_n \rightharpoonup x$). Let H be a Hilbert space . The inner product and the induced norm on H are denoted by $< .,. >$ and $\| . \|$ respectively. Consider $F(T)$ be the set of fixed points of T (i.e., $F(T) = \{x \in C : Tx = x\}$).

Let $C \neq \emptyset$ be a closed and convex subset of a real Hilbert space H, and consider the coupled mapping

$T : C \times C \to H$. Then $(w_1, w_2) \in C \times C$ is said to be coupled fixed point of T if $T(w_1, w_2) = w_1$ and $T(w_2, w_1) = w_2$, thus we can define the set of all coupled fixed points of T (denoted by $CF(T)$) as : $CF(T) = \{(x, y) \in C \times C : T(x, y) = x, T(y, x) = y\}$.

$T : C \times C \to C$ is said to be nonexpansive coupled mapping (denoted by NCM) if for every (x, y) and $(u, v) \in C \times C$,

$$\|T(x, y) - T(u, v)\| \leq \frac{1}{2}[\|x - u\| + \|y - v\|]$$

;T is said to be firmly nonexpansive coupled mapping (denoted ny $FNCM$) if,

$$\|T(x, y) - T(u, v)\|^2 \leq \frac{1}{2}[\langle x - u, T(x, y) - T(u, v)\rangle + \langle y - v, T(x, y) - T(u, v)\rangle],$$

equivalent;

$$\|T(x, y) - T(u, v)\|^2 \leq \frac{1}{2}\langle x - u + y - v, T(x, y) - T(u, v)\rangle,$$

for all $(x, y), (u, v) \in C \times C$. The following lemma give the relation between NCM and $FNCM$.

Lemma 2.1 Let $C \neq \emptyset$ be subset of real Hilbert space H. If $T : C \times C \to H$ be $FNCM$. Then T is NCM

Proof. Since T is $FNCM$, for all $(x, u), (u, v) \in C \times C$ we get that,

$$\|T(x, y) - T(u, v)\|^2 \leq \frac{1}{2}[\langle x - u, T(x, y) - T(u, v)\rangle + \langle y - v, T(x, y) - T(u, v)\rangle]$$
$$\leq \frac{1}{2}[\|x - u\|\|T(x, y) - T(u, v)\| + \|y - v\|\|T(x, y) - T(u, v)\|].$$

Therefore, we get that;

$$\|T(x, y) - T(u, v)\| \leq \frac{1}{2}[\|x - u\| + \|y - v\|].$$

Thus , T is NCM.

The following example show that the converse of lemma 2.1 is may not be true.

Example 2.1 Let $H = \Re$, and consider $T : \Re \times \Re \to \Re$ such as:for all $(x, y) \in \Re \times \Re$, define $T(x, y) = \frac{1}{2}x$. Let, $(x, y), (u, v) \in \Re \times \Re$, then we have that:

$$\|T(x, y) - T(u, v)\| = \|\frac{1}{2}(x - u)\| \leq \frac{1}{2}[\|x - u\| + \|y - v\|].$$

Thus, T is NCM. However,

$$\langle 1 - 0 - 2 - 0, T(1, -2) - T(0, 0)\rangle = \frac{-1}{2} < 2\|T(1, -2) - T(0, 0)\|^2.$$

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Hence, T is not $FNCM$.

Let $C \neq \emptyset$ be closed convex subset of H. Let us recall that : the metric projection of H onto C (denoted by P_C) is defined as the mapping $P_C : H \to C$ such that: for each $x \in H$, there exist a unique $y \in C$ such that: $P_C x = y$ if and only if $\|x - y\| \leq \|x - z\|$ for every $z \in C$. A mapping P_C satisfied some important properties such as:$\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$. Also $\|P_C x - P_C y\|^2 \leq \frac{1}{2}\langle x - y, P_C x - P_C y\rangle$, for all $x, y \in H$. The following lemma give one of useful properties of metric projection mapping.

Lemma 2.2 [18]. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H, and let P_C be the metric projection from H onto C. Then $\langle x - P_C x, P_C x - y\rangle \geq 0, \forall x \in H, y \in C$.

Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{T_n : C \times C \to H\}$ be a $FNCM$. Then we say that $\{T_n\}$ satisfies then resolvent coupled property (denoted by RCP) if there exist a NCM, $T : C \times C \to H$ and two natural numbers n_0 and k such that: $||x - T(x, y)|| \leq k||x - T_n(x, y)||$ and $||y - T(y, x)|| \leq k||y - T_n(y, x)||$ for all $x, y \in C$ and $n \in N$ with $n \geq n_0$ and $CF(T) = \cap_{n=1}^{\infty} CF(T_n)$. The next example give sequence of mapping which satisfy $FRCP$.

Example 2.2. Let $H = \Re$ and $C = [0, 2.]$ Define $T_1 : C \times C \to \Re$ and $T_2 : C \times C \to \Re$ such as:

$$T_1(x, y) = \begin{cases} 0 & \text{if } x \in [0, \frac{3}{2}], y \in [0, 2], \\ 0 & \text{if } x \in [0, 2], y \in [0, \frac{3}{2}], \\ \frac{1}{2}(x + y) - \frac{3}{2} & \text{if } x \in (\frac{3}{2}, 2], y \in (\frac{3}{2}, 2], \end{cases}$$

and

$$T_2(x, y) = \begin{cases} 0 & \text{if } x \in [0, 1], y \in [0, 2], \\ 0 & \text{if } x \in [0, 2], y \in [0, 1], \\ \frac{1}{2}(x + y) - 1 & \text{if } x \in (1, 2], y \in (1, 2], \end{cases}$$

let $T_{2n-1}(x, y) = T_1(x, y)$ and $T_{2n}(x, y) = T_2(x, y)$ for all $n \in N$. Therefore,it is clear that: $CF(T_1) = CF(T_2) = \{(0, 0)\}$. Now , If $(x, y), (u, v) \in [\frac{3}{2}, 2] \times [\frac{3}{2}, 2]$, we get that,

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$$\|T_1(x,y)-T_1(u,v)\|^2 = \|\frac{1}{2}(x+y)-\frac{1}{2}(u-v)\|^2$$
$$= \frac{1}{4}\|x-u+y-v\|^2 = \frac{1}{4}\langle x-u+y-v, x-u+y-v\rangle$$
$$= \frac{1}{2}\langle x-u+y-v, \frac{1}{2}(x-u)+\frac{1}{2}(y-v)\rangle$$
$$= \frac{1}{2}\langle x-u+y-v, T_1(x,y)-T_1(u,v)\rangle. \tag{2}$$

In the other hand, If $(x,y),(u,v)$ in other region , we get that the same result. Also, if $(x,y),(u,v) \in [1,2]\times[1,2]$, we have that:

$$\|T_2(x,y)-T_2(u,v)\|^2 = \|\frac{1}{2}(x+y)-\frac{1}{2}(u+v)\|^2$$
$$= \frac{1}{4}\|(x-u)+(y-v)\|^2 = \frac{1}{2}\langle x-u+y-v, \frac{1}{2}(x-u)+\frac{1}{2}(y-v)\rangle$$
$$= \frac{1}{2}\langle x-u+y-v, \frac{1}{2}(x+y-1)-\frac{1}{2}(u+v-1)\rangle.$$
$$= \frac{1}{2}\langle x-u+y-v, T_2(x,y)-T_2(u,v)\rangle. \tag{3}$$

By the same method, we can prove that: if $(x,y),(u,v)$ in other regions of the mapping T_2 we get the same above results. Thus, T_1 and T_2 are $FNCM$. Let $T(x,y) = T_1(x,y)$. Thus, T is NCM , $CFT = \{(0,0)\}$ and:

$$\|x - T(x,y)\| \le 2\|x - T_n(x,y)\|$$

Also, we get that:

$$\|y - T(y,x)\| \le 2\|y - T_n(y,x)\|$$

Then $\{T_n\}$ satisfies a RCP.

Lemma 2.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T : C\times C \to C$ be even mapping in the second variable (i.e. $T(x,-y) = T(x,y)$, for all $(x,y) \in C\times C$) and $FNCM$ with $CF(T) \neq \emptyset$. Then $\langle x - T(x,y), T(x,y) - w_1\rangle \ge 0$ and $\langle y - T(y,x), T(y,x) - w_2\rangle \ge 0$ for all $(x, y \in C\times C$ and $(w_1,w_2) \in CF(T))$.

Proof. Since $(w_1,w_2) \in CF(T)$,we get that: for all $(x,y) \in C\times C$,

$$\|T(x,y)-T(w_1,w_2)\|^2 = \|T(x,y)-w_1\|^2 \le \frac{1}{2}\langle x-w_1+y-w_2, T(x,y)-w_1\rangle.$$

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Therefore,

$$\|T(x,-y)-T(w_1,-w_2)\|^2 = \|T(x,y)-w_1\|^2 \leq \frac{1}{2}\langle x-w_1-y+w_2, T(x,y)-w_1\rangle.$$

Thus, we have that:

$$\langle x-T(x,y), T(x,y)-w_1\rangle = \langle x-w_1+y-w_2-T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle$$
$$= \langle x-w_1+y-w_2, T(x,y)-w_1\rangle + \langle -T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle$$
$$\geq 2\|T(x,y)-w_1\|^2 + \langle -T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle$$
$$= \langle 2T(x,y)-2w_1, T(x,y)-w_1\rangle + \langle -T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle$$
$$= \langle T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle$$
$$= \langle T(x,y)-x, T(x,y)-w_1\rangle + \langle x-w_1-y+w_2, T(x,y)-w_1\rangle \tag{4}$$

Hence , we get that:

$$2\langle x-T(x,y), T(x,y)-w_1\rangle \geq \langle x-w_1+y-w_2, T(x,y)-w_1\rangle \geq 2\|T(x,y)-w_1\|^2 \geq 0. \tag{5}$$

Similarly, we can prove that: $\langle y-T(y,x), T(y,x)-w_2\rangle \geq 0$.

Definition 2.1 [26]. A space X is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in X which $x_n \rightharpoonup x$, we have $\forall\, y \in X, y \neq x$ the following:

(i) $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$,

(ii) $\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$.

We recall that: every Hilbert space has Opial's property [26] .

3 Main results

In this section, we prove the main weak convergence theorems for families of $FNCM$ in Hilbert spaces. To prove it, we use the following Lemma.

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Lemma 3.1.([27]) Let T be a closed convex subset of a real Hilbert space H. Let T be a nonexpansive non self-mapping of K into H such that $F(T) \neq \emptyset$. Then $F(T) = F(P_K T)$.

Now, we prove the main theorem in this paper.

Theorem 3.1. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $\{T_n\} : C \times C \to H$ be a sequence of $FNCM$ and be even mappings in the second variable (i.e. $T_n(x, -y) = T_n(x, y)$, for all $(x, y) \in C \times C$) with $S := \bigcap_{n=1}^{\infty} CF(T_n) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2). Let $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T_n(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\{T_n\}$ satisfies RCP and $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

Proof. Let $(w_1.w_2) \in S$, now we will prove that : $P_C(w_1) = w_1$, and $P_C(w_2) = w_2$. Consider the mapping : $f_{w_2} : C \to H$ such that: $f_{w_2} = T_1(x, w_2)$. Thus, we have that: $f_{w_2}(w_1) = T_1(w_1, w_2) = w_1$. Then, w_1 is fixed point of f_{w_2}. Therefore, let $x, y \in C$. Then, we get that:

$$\begin{aligned} &\|f_{w_2}(x) - f_{w_2}(y)\|^2 = \|T_1(x, w_2) - T_2(y, w_2)\|^2 \\ &= \frac{1}{2}\langle x - y, f_{w_2}(x) - f_{w_2}(y)\rangle \\ &\leq \frac{1}{2}\|x - y\|\|f_{w_2}(x) - f_{w_2}(y)\| \end{aligned} \tag{6}$$

Hence, we have that:

$$\|f_{w_2}(x) - f_{w_2}(y)\| \leq \frac{1}{2}\|x - y\| \leq \|x - y\|.$$

Then, f_{w_2} is nonexpansive mapping . By applying lemma 3.1 , we get that: $P_C(w_1) = w_1$. By the same method, let $f_{w_1} : C \to H$, which defined as: $f_{w_1}(x) = T_1(x, w_1)$. It is clear that: $f_{w_1}(w_2) = T_1(w_2, w_1) = w_2$. Thus, w_2 is fixed point of the mapping f_{w_1} and therefore f_{w_1} is nonexpansive. Then , we get that: $P_C(w_2) = w_2$.

Also, by applying lemma 2.3, we get that: $\langle x_n - T_n(x_n, y_n), T_n(x_n, y_n) - w_1\rangle \geq 0$ and

$\langle y_n - T_n(y_n, x_n), T_n(y_n, x_n) - w_2\rangle \geq 0$, for all $n \in N$. Then, we get that:

$$\|x_{n+1} - w_1\|^2 = \|P_C((1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n)) - P_C(w_1)\|^2 \leq \|(1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n) - w_1\|^2$$

$$= \|(x_n - w_1) + \alpha_n(T_n(x_n, y_n) - x_n)\|^2$$

$$= \langle (x_n - w_1) + \alpha_n(T_n(x_n, y_n) - x_n), (x_n - w_1) + \alpha_n(T_n(x_n, y_n) - x_n)\rangle$$

$$= \langle x_n - w_1, x_n - w_1\rangle + 2\alpha_n\langle x_n - w_1, T_n(x_n, y_n) - x_n\rangle + \alpha_n^2\langle T_n(x_n, y_n) - x_n, T_n(x_n, y_n) - x_n\rangle$$

$$= \|x_n - w_1\|^2 + \alpha_n^2\|x_n - T_n(x_n, y_n)\|^2 - 2\alpha_n\langle x_n - w_1, T_n(x_n, y_n) - x_n\rangle$$

$$= \|x_n - w_1\|^2 + \alpha_n^2\|x_n - T_n(x_n, y_n)\|^2 + 2\alpha_n\langle x_n - T_n(x_n, y_n), T_n(x_n, y_n) - x_n\rangle$$

$$-2\alpha_n\langle T_n(x_n, y_n) - w_1, T_n(x_n, y_n) - x_n\rangle$$

$$\leq \|x_n - w_1\|^2 - \alpha_n(2-\alpha_n)\|x_n - T_n(x_n, y_n)\|^2$$

for all $n \in N$. By doing the same steps , we get also:

$$\|y_{n+1} - w_2\|^2 \leq \|y_n - w_2\|^2 - \alpha_n(2-\alpha_n)\|y_n - T_n(y_n, x_n)\|^2;$$

for all $n \in N$. Then we have that, $\{x_n\}$ and $\{y_n\}$ are bounded sequence in C, , therefore, $\lim_{n\to\infty}\|x_n - w_1\|$ exist and $\lim_{n\to\infty}\|y_n - w_2\|$ exist. Therefore, we get that:

$$\lim_{n\to\infty} \alpha_n(2-\alpha_n)\|x_n - T_n(x_n, y_n)\| = 0.$$

Also, we have that:

$$\lim_{n\to\infty} \alpha_n(2-\alpha_n)\|y_n - T_n(y_n, x_n)\| = 0,$$

and since $\lim_{n\to\infty}\alpha_n(2-\alpha_n) > 0$, then, we have that:

$$\lim_{n\to\infty}\|x_n - T_n(x_n, y_n)\| = 0, \lim_{n\to\infty}\|y_n - T_n(y_n, x_n)\| = 0.$$

Since $\{T_n\}$ satisfies the RCP , then there exist NCM $T : C \times C \to C$ and $n_0, k \in N$ such that:

$$\|x - T(x, y)\| \leq k\|x - T_n(x, y)\|, \|y - T(y, x)\| \leq k\|y - T_n(y, x)\|,$$

therefore , we get that:

$$\|x_n - T(x_n, y_n)\| \leq k\|x_n - T_n(x_n, y_n)\|, \|y_n - T(y_n, x_n)\| \leq k\|y_n - T_n(y_n, x_n)\|,$$

for every $n \geq 0$. Then we have that :

$$\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0, \lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exist subsequences $\{x_{n_{k_1}}\}$ of $\{x_n\}$, $\{y_{n_{k_2}}\}$ of $\{y_n\}$ and $(u_1, u_2) \in C \times C$ such that : $x_{n_{k_1}} \rightharpoonup u_1$ and $y_{n_{k_2}} \rightharpoonup u_2$. By applying one of the useful property of the Hilbert space , we get:

$$\|T(u_1, u_2) - x_n\|^2 = \|T(u_1, u_2) - u_1\|^2 + 2\langle T(u_1, u_2) - u_1, u_1 - x_n\rangle + \|u_1 - x_n\|^2$$

Since, $\{x_n\}$ convergent weakly to u_1, then we obtain that:

$$\lim_{n\to\infty} \langle T(u_1, u_2) - u_1, u_1 - x_n\rangle = 0.$$

Hence, we find that:

$$\lim_{n\to\infty} \sup \|T(u_1, u_2) - x_n\|^2 = \|T(u_1, u_2) - u_1\|^2 + \lim_{n\to\infty} \sup \|u_1 - x_n\|^2.$$

Also , using the condition of coupled firmly non-expansive , we get that:

$$\lim_{n\to\infty} \|T(u_1, u_2) - x_n\| \leq \lim_{n\to\infty} \|T(u_1, u_2) - T(x_n, y_n)\| + \lim_{n\to\infty} \|x_n - T(x_n, y_n)\| \leq \lim_{n\to\infty} \| u_1 - x_n \| .$$

Thus, we have that:

$$\| T(u_1, u_2) - u_1 \|^2 + \lim_{n\to\infty} \sup \|u_1 - x_n\|^2 \leq \lim_{n\to\infty} sup \| u_1 - x_n \|^2 .$$

Then, we have that:

$$\| T(u_1, u_2) - u_1 \|^2 = 0.$$

Therefore, we get that: $T(u_1, u_2) = u_1$ and similarly we can prove that: $T(u_2, u_1) = u_2$. Thus , it clear that : $(u_1, u_2) \in S$. Now we prove that $\{(x_n, y_n)\} \rightharpoonup (\overline{x}, \overline{y}) \in S$. Let, $\{x_{n_l}\}$ and $\{x_{n_m}\}$ be subsequences of $\{x_n\}$ which converge weakly to $u, v \in C$ respectively. If $u \neq v$, from the the Opial property,

$$\begin{aligned} &\lim_{l\to\infty} \|x_{n_l} - u\| < \lim_{l\to\infty} \|x_{n_l} - v\| = \lim_{m\to\infty} \|x_{n_m} - v\| \\ &< \lim_{m\to\infty} \|x_{n_m} - u\| = \lim_{l\to\infty} \|x_{n_l} - u\||. \end{aligned} \tag{7}$$

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This is contradiction. Therefore, $x_n \rightharpoonup \overline{x}$. By the same method, we can prove $y_n \rightharpoonup \overline{y}$. Thus $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y}) \in S$.

Corollary 3.1. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $T : C \times C \to H$ be $FNCM$ and even mapping in the second variable (i.e. $T(x, -y) = T(x, y)$, for all $(x, y) \in C \times C$) with $CF(T) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2). Let $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in CF(T)$.

Lemma 3.2. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $\{T_n\} : C \times C \to H$ be a sequence of $FNCM$ and be even mappings in the second variable (i.e. $T_n(x, -y) = T_n(x, y)$, for all $(x, y) \in C \times C$). Suppose that : $\sum_{n=1}^{n=\infty} \sup\{\|T_{n+1}(x, y) - T_n(x, y)\| < \infty : (x, y) \in C \times C\}$. Then $\{T_n(x, y)\}$ converges strongly to some point of $C \times C$. In the other hand, if $T : C \times C \to C$ defined by: $T(x, y) = \lim_{n\to\infty} T_n(x, y)$, for all $(x, y) \in C \times C$. Then $\lim_{n\to\infty} \sup\{\|T(x, y) - T_n(x, y)\| : (x, y) \in C \times C\} = 0$.

Proof. First, we will prove that $\{T_n(x, y)\}$ is Cauchy sequence for all $(x, y) \in C \times C$. Let $i, j \in N$ and $i > j$. we get that:

$$\|T_i(x,y) - T_j(x,y)\| \le \|\sup\{\|T_i(x,y) - T_j(x,y)\| : (x,y) \in C \times C\}$$

$$\le \sup\{\|T_i(x,y) - T_{i-1}(x,y)\| : (x,y) \in C \times C\} + \sup\{\|T_{i-1}(x,y) - T_j(x,y)\| : (x,y) \in C \times C\} \le ....$$

$$\le \sum_{i=1}^{\infty} \sup\{\|T_{n+i}(x,y) - T_n(x,y)\| : (x,y) \in C \times C\}$$

Let $i \to \infty$, Then we get that $\{T_n(x, y)\}$ is a Cauchy sequence. Thus $\{T_n(x, y)\}$ converges strongly to some point of $C \times C$. Also, we have :

$$\|T(x,y) - T_j(x,y)\| \le \sum_{i=1}^{\infty} \sup\{\|T_{n+i}(x,y) - T_n(x,y)\| : (x,y) \in C \times C\}$$

Thus, we have that: $\lim_{j\to\infty} \sup\{\|T(x, y) - T_j(x, y)\| : (x, y) \in C \times C\} = 0$.

Theorem 3.2. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $\{T_n\} : C \times C \to H$ be a sequence of $FNCM$ and be even mappings in the second variable (i.e. $T_n(x, -y) = T_n(x, y)$, for all

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$(x, y) \in C \times C)$ with $S := \bigcap_{n=1}^{\infty} CF(T_n) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2). Let $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T_n(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\{T_n\}$ satisfies the property: $\sum_{n-1}^{n=\infty} \sup\{\|T_{n+1}(x, y) - T_n(x, y)\| < \infty : (x, y) \in C \times C\}$ and $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

Proof. First, we will apply lemma 3.2. Define $T : C \times C \to H$ by $T(x, y) = \lim_{n\to\infty} T_n(x, y)$

$$\begin{aligned} &\|T(x, y) - T(u, v)\| = \|\lim_{n\to\infty} T_n(x, y) - \lim_{n\to\infty} T_n(u, v)\| \\ &= \lim_{n\to\infty} \|T_n(x, y) - T_n(u, v)\| \leq \lim_{n\to\infty} \frac{1}{2}(\|x - u\| + \|y - v\|). \end{aligned} \tag{8}$$

For all $(x, y), (u, v) \in C \times C$. Hence T is a NCM . Therefore, we get that:

$$\lim_{n\to\infty} \sup\{\|T(x, y) - T_n(x, y)\| : (x, y) \in B\} = 0, \tag{9}$$

for each bounded subset B of $C \times C$. Then by doing the same steps as in Theorem 3.1, we get that

$$\|x_n - w_1\|^2 \leq \|x_n - w_1\|^2 - \alpha_n(2 - \alpha_n)\|x_n - T_n(x_n, y_n)\|^2. \tag{10}$$

therefore,

$$\|y_n - w_2\|^2 \leq \|y_n - w_2\|^2 - \alpha_n(2 - \alpha_n)\|y_n - T_n(y_n, x_n)\|^2$$

Thus , we have that:

$$\lim_{n\to\infty} \|T(x_n, y_n) - T_n(x_n, y_n)\| = 0. \tag{11}$$

Then, we get the following:

$$\|x_n - T(x_n, y_n)\| \leq \|x_n - T_n(x_n, y_n)\| + \|T(x_n, y_n) - T_n(x_n, y_n)\|.$$

therefore, we get that:

$$\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0. \tag{12}$$

By doing the same step we can prove that:

$$\lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0.$$

Agian , by doing the same steps as the proof of Theorem 3.1, we get the proof of Theorem 3.2.

Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ and Γ be two families of NCM mappings of $C \times C$ into C and even in the second variable, such that: $\emptyset \neq CF(\Gamma) = \bigcap_{n=1}^{\infty} CF(T_n)$, where $CF(T_n)$ is the set of all coupled fixed points of $\{T_n\}$ and $CF(\Gamma)$ is the set of all common coupled fixed points of Γ. We gave the following Condition.

Condition 3.1. For each bounded sequence $\{(x_n, y_n)\}$ of $C \times C$, if we have that: $\lim_{n\to\infty} \|x_n - T_n(x_n, y_n)\| = 0$ and $\lim_{n\to\infty} \|y_n - T_n(y_n, x_n)\| = 0$, then $\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0$ and $\lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0$ for all $T \in \Gamma$.

Theorem 3.3. Let H be a Hilbert space, C be a nonempty, closed and convex subset of H. Consider $\{T_n\}$: $C \times C \to C$ be a sequence of $FNCM$ mappings. Let Γ be a family of NCM of $C \times C$ into C , which satisfies $\emptyset \neq CF(\Gamma) \subseteq \bigcap_{n=1}^{\infty} CF(T_n)$ and condition (3.1). Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2), and $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T_n(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

Proof. By doing the same steps as in the proof of Theorem 3.1, we get $\{(x_n, y_n)\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - T_n(x_n, y_n)\| = 0,$$

also,

$$\lim_{n\to\infty} \|y_n - T_n(y_n, x_n)\| = 0.$$

By condition (3.1),

$$\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0, \lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0,$$

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for all $T \in \Gamma$. Since $\{(x_n, y_n)\}$ is bounded , there exist a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ and $(u_1, u_2) \in C \times C$ such that: $x_{n_k} \to u_1$ and $y_{n_k} \to u_2$. By lemma 2.6, we have that $(u_1, u_2) \in CF(T)$ for all $T \in \Gamma$. Thus we have that: $(u_1, u_2) \in CF(\Gamma) \subseteq \bigcap_{n=1}^{\infty} CF(T_n)$. Then the same steps as in the proof of Theorem 3.1 lead to $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$, where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

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Dynamics of the zeros of analytic continued the second kind q-Euler polynomial

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper we study that the second kind q-Euler numbers $E_{n,q}$ and q-Euler Euler polynomials $E_{n,q}(x)$ are analytic continued to $E_q(s)$ and $E_q(s,w)$. We investigate the new concept of dynamics of the zeros of analytic continued polynomials. Finally, we observe an interesting phenomenon of 'scattering' of the zeros of $E_q(s,w)$.

Key words : Second kind Euler polynomial, Euler Zeta function, Analytic Continuation, complex zeros, dynamics.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Several mathematicians have studied the Bernoulli numbers and polynomials, Euler numbers and polynomials, q-Bernoulli numbers and polynomials, q-Euler numbers and polynomials, the second kind Euler numbers and polynomials(see [1-11]). These numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. Throughout this paper, we always make use of the following notations:$\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. We introduced the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties(see [6]). Let q be a complex number with $|q| < 1$. We define the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows:

$$F_q(t) = \frac{2e^t}{qe^{2t}+1} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}, \tag{1}$$

$$F_q(x,t) = \left(\frac{2e^t}{qe^{2t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{2}$$

By the above definition (2) and Cauchy product, we have

$$\begin{aligned}\sum_{l=0}^{\infty} E_{l,q}(x)\frac{t^l}{l!} &= \left(\frac{2e^t}{e^{2t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m\frac{t^m}{m!}\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,q}\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right) = \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}E_{n,q}x^{l-n}\right)\frac{t^l}{l!}.\end{aligned}$$

By using comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

Theorem 1. For $n \in \mathbb{N}_0$, one has

$$E_{n,q}(x) = \sum_{k=0}^{n}\binom{n}{k}E_{k,q}x^{n-k}.$$

By Theorem 1 and some calculations, we have

$$\int_a^b E_{n,q}(x)dx = \sum_{l=0}^{n}\binom{n}{l}E_{l,q}\int_a^b x^{n-l}dx = \sum_{l=0}^{n}\binom{n}{l}E_{l,q}\ \frac{x^{n-l+1}}{n-l+1}\Big|_a^b$$
$$= \frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}E_{l,q}\ x^{n-l+1}\big|_a^b.$$

By Theorem 1, we get

$$\int_a^b E_{n,q}(x)dx = \frac{E_{n+1,q}(b)-E_{n+1,q}(a)}{n+1}. \tag{3}$$

Since $E_{n,q}(0) = E_{n,q}$, by (3), we have the following theorem.

Theorem 2. For $n \in \mathbb{N}$, one has

$$E_{n,q}(x) = E_{n,q} + n\int_0^x E_{n-1,q}(t)dt.$$

By using computer, the second kind q-Euler polynomials $E_{n,q}(x)$ can be determined explicitly. A few of them are

$$E_{0,q}(x) = \frac{2}{1+q},$$
$$E_{1,q}(x) = \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2} + \frac{2x}{(1+q)},$$
$$E_{2,q}(x) = \frac{4}{(1+q)^3} - \frac{8q}{(1+q)^3} + \frac{4q^2}{(1+q)^3} - \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2} + \frac{4x}{(1+q)^2} - \frac{4qx}{(1+q)^2} + \frac{2x^2}{(1+q)}.$$

2. Analytic Continuation of the second kind q-Euler numbers and the q-Euler Zeta function

By using the second kind q-Euler numbers and polynomials, the second kind q-Euler zeta function and Hurwitz q-Euler zeta functions are defined. From (1), we note that

$$\frac{d^k}{dt^k}F_q(t)\Big|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n q^n(2n+1)^k = E_{k,q}, (k \in \mathbb{N}).$$

By using the above equation, we are now ready to define the second kind q-Euler zeta functions.

Definition 3. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, define the second kind q-Euler zeta function by

$$\zeta_E(s) = 2\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(2n+1)^s}.$$

Notice that the Euler zeta function can be analytically continued to the whole complex plane, and these q-zeta function have the values of the q-Euler numbers at negative integers. That is, the second kind q-Euler numbers are related to the second kind q-Euler zeta function as

$$\zeta_{E,q}(-k) = E_{k,q}.$$

By using (2), we note that

$$\frac{d^k}{dt^k} F_q(x,t)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n q^n (2n+x+1)^k, (k \in \mathbb{N}), \tag{4}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}\right)\bigg|_{t=0} = E_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{5}$$

By (4) and (5), we are now ready to define the Hurwitz q-Euler zeta functions.

Definition 4. We define the Hurwitz q-zeta function $\zeta_{E,q}(s,x)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ by

$$\zeta_{E,q}(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(2n+x+1)^s}.$$

Note that $\zeta_{E,q}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_{E,q}(s,x)$ and $E_{k,q}(x)$ is given by the following theorem.

Theorem 5. For $k \in \mathbb{N}$, we have

$$\zeta_{E,q}(-k,x) = E_{k,q}(x). \tag{6}$$

We now consider the function $E_q(s)$ as the analytic continuation of the second kind q-Euler numbers. From the above analytic continuation of the second kind q-Euler numbers, we consider

$$\begin{aligned} &E_{n,q} \mapsto E_q(s), \\ &\zeta_{E,q}(-n) = E_{n,q} \mapsto \zeta_{E,q}(-s) = E_q(s). \end{aligned} \tag{7}$$

All the second kind q-Euler number $E_{n,q}$ agree with $E_q(n)$, the analytic continuation of the second

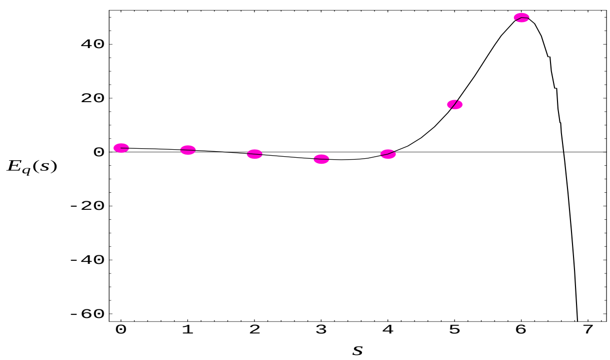

Figure 1: The curve $E_q(s)$ runs through the points of all $E_{n,q}$

kind q-Euler numbers evaluated at n(see Figure 1). Consider

$$E_{n,q} = E_q(n) \text{ for } n \geq 0 \tag{8}$$

In Figure 1, we choose $q = 1/3$. In fact, we can express $E'_q(s)$ in terms of $\zeta'_{E,q}(s)$, the derivative of $\zeta_{E,q}(s)$. Consider

$$\begin{aligned} &E_q(s) = \zeta_{E,q}(-s),\\ &E'_q(s) = -\zeta'_{E,q}(-s)\\ &E'_q(2n+1) = -\zeta'_{E,q}(-2n-1) \text{ for, } n \in \mathbb{N}_0. \end{aligned} \tag{9}$$

From the relation (9), we can define the other analytic continued half of the second kind q-Euler numbers

$$\begin{aligned} &E_q(s) = \zeta_{E,q}(-s), \quad E_q(-s) = \zeta_{E,q}(s)\\ &\Rightarrow E_q(-n) = \zeta_{E,q}(n), n \in \mathbb{N}. \end{aligned} \tag{10}$$

By (10), we have

$$\lim_{n\to\infty} E_q(-n) = \zeta_{E,q}(n) = 2.$$

The curve $E_q(s)$ runs through the points $E_{-n,q} = E_q(-n)$ and grows ~ 2 asymptotically as $-n \to \infty$(see Figure 2).

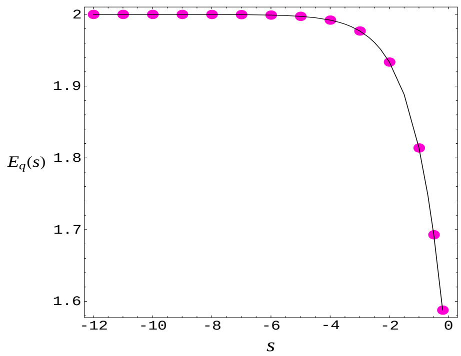

Figure 2: The curve $E_q(s)$ runs through the points $E_{-n,q}$ for $q = \frac{1}{3}$

3. Dynamics of the zeros of analytic continued polynomials

Our main purpose in this section is to investigate the new concept of dynamics of the zeros of analytic continued polynomials. Let $\Gamma(s)$ be the gamma function. The analytic continuation can be then obtained as

$$\begin{aligned} &n \mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C},\\ &E_{k,q} \mapsto E_q(k+s-[s]) = \zeta_{E,q}(-(k+(s-[s]))),\\ &\binom{n}{k} \mapsto \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\ &\Rightarrow E_{n,q}(w) \mapsto E_q(s,w) = \sum_{k=-1}^{[s]} \frac{\Gamma(1+s)E_q(k+s-[s])w^{[s]-k}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\ &= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s)E_q((k-1)+s-[s])w^{[s]+1-k}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)}, \end{aligned} \tag{11}$$

where $[s]$ gives the integer part of s, and so $s-[s]$ gives the fractional part. By (11), we obtain analytic continuation of the second kind q-Euler polynomials for $q=1/3$. Consider

$$\begin{aligned}
&E_{0,q}(w) \approx 1.5,\\
&E_q(1,w) \approx 0.75+1.5w,\\
&E_q(2,w) \approx -0.75+1.5w+1.5w^2,\\
&E_q(2.2,w) \approx -1.14137+1.13863w+1.84171w^2+0.15595w^3,\\
&E_q(2.4,w) \approx -1.54674+0.60395w+2.13491w^2+0.38568w^3,\\
&E_q(2.6,w) \approx -1.94844-0.12719w+2.33741w^2+0.69096w^3,\\
&E_q(2.8,w) \approx -2.32024-1.07449w+2.39690w^2+1.06697w^3,\\
&E_q(3,w) \approx -2.625-2.25w+2.25w^2+1.5w^3.
\end{aligned} \tag{12}$$

By using (12), we plot the deformation of the curve $E_q(2,w)$ into the curve of $E_q(3,w)$ via the real analytic continuation $E_q(s,w), 2 \le s \le 3, w \in \mathbb{R}$(see Figure 3). In [6], we observe that $E_q(n,w), w \in$

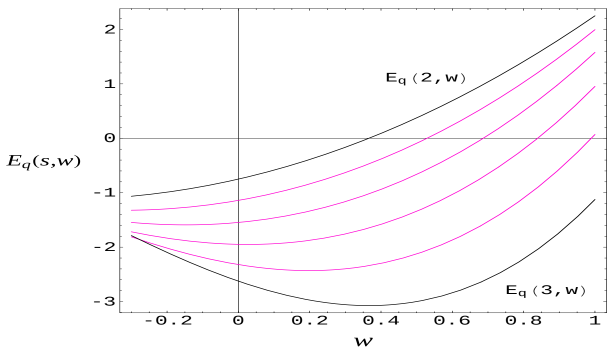

Figure 3: The curve of $E_q(s,w), 2 \le s \le 3, -0.3 \le w \le 1$

$\mathbb{C}$, has $Im(w)=0$ reflection symmetry analytic complex functions(see Figure 4). The zeros of $E_q(n,w)$ will also inherit these symmetries.

$$\text{If } E_q(n,w_0)=0, \text{ then } E_q(n,w_0^*)=0,$$

where $*$ denotes complex conjugation.

For $n \in \mathbb{N}_0$, it is easy to deduce that the second kind q-Euler polynomials $E_{n,q}(x)$ satisfy

$$\sum_{n=0}^{\infty} E_{n,q^{-1}}(-x)\frac{(-t)^n}{n!} = \frac{2e^{-t}}{q^{-1}e^{-2t}+1}e^{(-x)(-t)} = \frac{2qe^t}{e^{2t}+1}e^{xt} = q\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}.$$

By using comparing coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 6 (Theorem of complement). For any positive integer n, we have

$$E_{n,q}(x) = (-1)^n q^{-1} E_{n,q^{-1}}(-x). \tag{13}$$

The question is as follows: what happens with the reflexive symmetry (13), when one considers the second kind q-Euler polynomials? Prove that $E_q(n,w), w \in \mathbb{C}$, has not $Re(w)=0$ reflection symmetry analytic complex functions(see Figure 4). Next, we investigate the beautiful zeros of the $E_q(s,w)$ by using a computer. We plot the zeros of $E_q(s,w)$ for $s = 9, 9.3, 9.7, 10, q = 1/3$, and $w \in \mathbb{C}$(Figure 4). In Figure 4(top-left), we choose $s = 9$. In Figure 4(top-right), we choose $s = 9.3$.

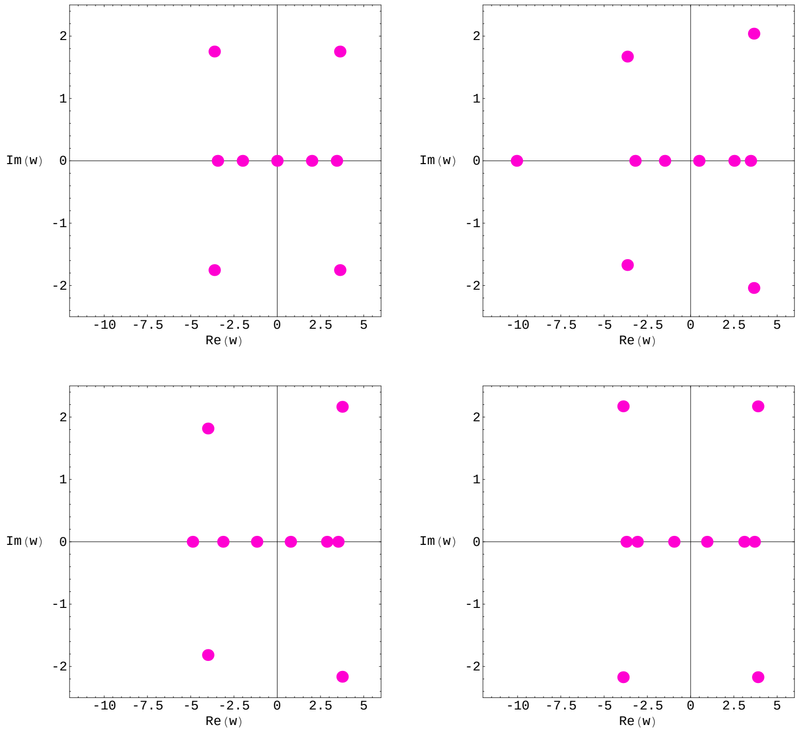

Figure 4: Zeros of $E_q(s,w)$ for $s = 9, 9.3, 9.7, 10$

In Figure 4(bottom-left), we choose $s = 9.7$. In Figure 4(bottom-right), we choose $s = 10$.

Stacks of zeros of $E_q(s,w)$ for $s = n + 1/3, 1 \leq n \leq 50$, forming a 3D structure are presented(Figure 5).

In Figure 5(top-right), we draw y and z axes but no x axis in three dimensions. In Figure 5(bottom-left), we draw x and y axes but no z axis in three dimensions. In Figure 5(bottom-right), we draw x and z axes but no y axis in three dimensions. However, we observe that $E_q(n,w), w \in \mathbb{C}$, has $Im(w) = 0$ reflection symmetry analytic complex functions(see Figure 4 and Figure 5).

Our numerical results for approximate solutions of real zeros of $E_q(s,w), q = 1/3$, are displayed. We observe a remarkably regular structure of the complex roots of the second kind q-Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of the second kind q-Euler polynomials(Table 1).

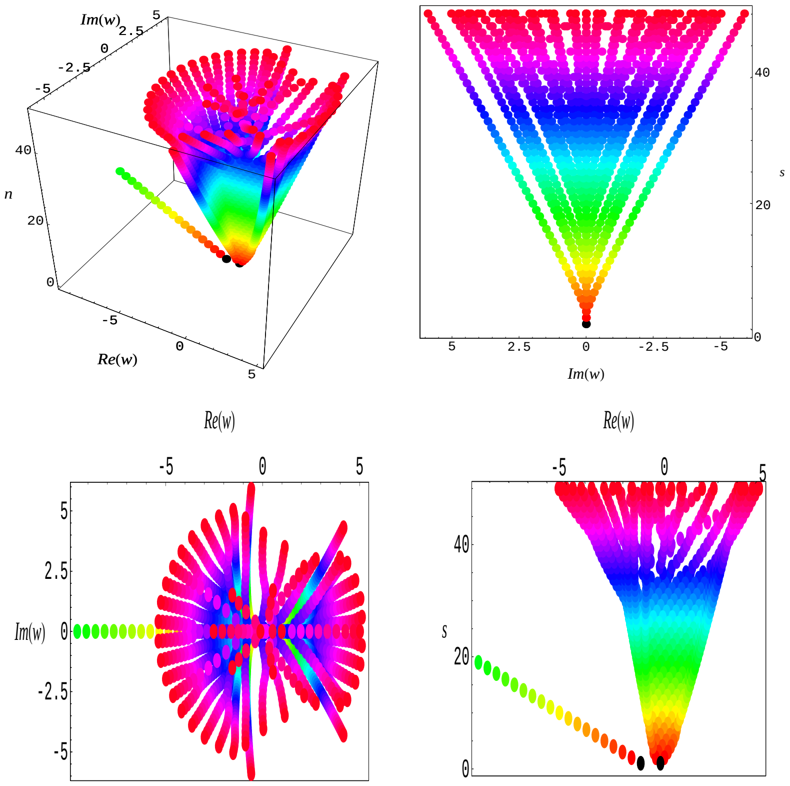

Figure 5: Stacks of zeros of $E_q(s,w)$ for $1 \leq n \leq 50$

Table 1. Numbers of real and complex zeros of $E_q(s,w)$

s	real zeros	complex zeros
1.5	2	0
2.5	3	0
3.5	4	0
4.5	3	2
5.5	4	2
6.5	5	2
7.5	6	2
8.5	3	6
9	3	6
9.3	4	6
9.5	4	6
9.8	4	6
10	4	6

Next, we calculated an approximate solution satisfying $E_q(s,w), q = 1/3, w \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $E_q(s,w)=0, w\in\mathbb{R}$

s	w
6	$-2.25291,\quad -0.499167,\quad 1.50121,\quad 2.89899$
6.5	$-8.19021,\quad -1.97235,\quad -0.106447,\quad 1.90361,\quad 3.03711$
7	$-2.65744,\quad -1.71446,\quad 0.286584,\quad 2.31062,\quad 3.2536$
7.5	$-9.25827,\quad -2.51685,\quad -1.32105,\quad 0.679634,\quad 2.83991,\quad 3.19538$
8	$-0.927418,\quad 1.07258$
8.5	$-10.3265,\quad -0.534533,\quad 1.46541$
9	$-2.1399,\quad -0.141641,\quad 1.85831$
9.2	$-35.7141,\quad -1.98173,\quad 0.0155236,\quad 2.01523$
9.5	$-11.3949,\quad -1.74785,\quad 0.251276,\quad 2.2499$
9.7	$-6.68645,\quad -1.59132,\quad 0.408446,\quad 2.40587$
10	$-3.09896,\quad -1.3558,\quad 0.644202,\quad 2.64146$

In Figure 6, we plot the real zeros of the the second kind q-Euler polynomials $E_q(s,w)$ for $s=n+\frac{1}{3}, 1\le n\le 30, q=1/3$, and $w\in\mathbb{C}$ (Figure 7). In Figure 6(right), we choose $E_q(s,w)$ for

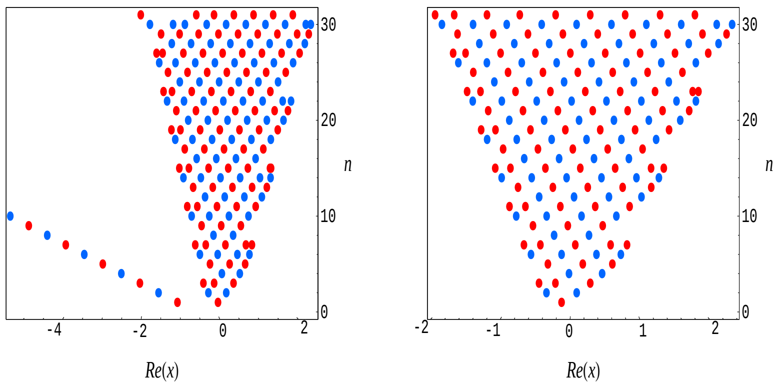

Figure 6: Real zeros of $E_q(s,w)$

$s=n+\frac{1}{3}, 1\le n\le 30$. In Figure 6(left), we choose $E_q(n,w)$ for $1\le n\le 30$.

The second kind q-Euler polynomials $E_{n,q}(w)$ is a polynomials of degree n. Thus, $E_{n,q}(w)$ has n zeros and $E_{n+1,q}(w)$ has $n+1$ zeros. When discrete n is analytic continued to continuous parameter s, it naturally leads to the question: How does $E_q(s,w)$, the analytic continuation of $E_{n,q}(w)$, pick up an additional zero as s increases continuously by one? This introduces the exciting concept of the dynamics of the zeros of analytic continued polynomials-the idea of looking at how the zeros move about in the w complex plane as we vary the parameter s. To have a physical picture of the motion of the zeros in the complex w plane, imagine that each time, as s increases gradually and continuously by one, an additional real zero flies in from positive infinity along the real positive axis, gradually slowing down as if " it is flying through a viscous medium ". More studies and results in this subject we may see references [5], [6], [7], [10].

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Remarks on the blow-up for damped Klein-Gordon equations with a gradient nonlinearity *

Hongwei Zhang, Jian Dang, Qingying Hu
(Department of Mathematics, Henan University of Technology, Zhengzhou 450001, China)

Abstract We consider initial boundary value problem for a class of damped Klein-Gordon type wave equations with a gradient nonlinearity and derive sufficient conditions for finite time blow-up of its solutions. To prove blow-up of the solution, we use eigenfunction method combining with a modification of Glassey's inequality. This extend the early results.

Keywords Klein-Gordon equations; blow-up; initial-boundary value problem; gradient nonlinearity

AMS Classification (2010): 35L20,35B44.

1 Introduction

The aim of this paper is to give some sufficient conditions for blow-up of solutions to the following damped Klein-Gordon type wave equations with a gradient nonlinearity

$$u_{tt} - \Delta u + cu_t = f(u, \nabla u),\ in\ \Omega \times (0, T), \tag{1.1}$$

$$u(x, t) = 0,\ x \in \partial\Omega, t \in (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),\ x \in \Omega; \tag{1.3}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$, $f(u, \nabla u) = a|u|^{p-1}u + b|\nabla u|^q$,$p, q > 1, a, b \in R, ab < 0$, and $c > 0$.

Nonlinear wave equations of the form (1.1) arise in differential geometry, controllability theory of partial differential equations, and in various areas of physics(see [1] and its references). The derivative Klein-Gordon type wave problem (1.1)-(1.3) can be viewed as a simplification of the Boussinesq equation [2, 3, 4, 1] with higher order spatial derivative terms appearing neither in the linear part nor in the nonlinearity. It belongs to the family of nonlinear wave equations of the form $u_{tt} + Au = \rho(u)\nabla u + g(u)$. This family of wave equations have as an important subclass the Yang-Mills-type equations with $\rho(u) = u$ and $g(u) = u^3$. Yang-Mills-type wave equations have the same scaling as the cubic nonlinear wave equation, but are more difficult technically because of the derivative term $u\nabla u$. Other important examples of the type equations include the Maxwell-Klein-Gordon and Yang-Mills-Higgs equations in the Lorenz gauge at least, as well as the simplified model equations of these (see [1]). If $b = 0$ and $a \neq 0$, then equation (1.1) is the standard Klein-Gordon wave problem. The standard Klein-Gordon wave problem in the critical exponent has been studied by many authors. In this case, the blowup behavior of solutions is by

*Corresponding author:Zhang H.W., Email: whz661@163.com

now fairly well understood, and various sufficient conditions for blowup have been provided and qualitative properties have also been investigated (see for example [5, 6, 7, 8, 9, 10, 11, 12, 13, 14], to cite just a few). However, very little is known in literature concerning the asymptotic dynamics exhibited by the derivative Klein-Gordon type wave equations of the form (1.1) and in space dimensions greater than one or two (see [1]). Recently, D'Abbicco [15] proved global existence of small data solutions to the following Cauchy problem for the doubly dissipative wave equation with power nonlinearity $|\nabla u|^p$:

$$u_{tt} - \Delta u + u_t - \Delta u_t = |\nabla u|^p,$$

for $p > 1 + \frac{1}{n+1}$, in any space dimension $n \geq 1$, and he also derive optimal energy estimates and $L^1 - L^1$ estimates for the solution to the semilinear problems. Willie[1] studied a nonlinear wave problem of the form

$$u_{tt} - \Delta u + du_t = -\rho|\nabla u|^2 + \gamma|u|^{p-1}u, \rho \geq 0, \gamma > 0,$$

its linear problem well-posedness, behaviour of the spectrum of the wave differential operator in varied damping and diffusion constants, as well as the asymptotic dynamics defined by the derivative Klein-Gordon type wave problem.

We mention also some related mathematical work involving the derivative nonlinearity term in the literature. Ebihara [16, 17, 18] established global existence of classical solutions and asymptotic behavior of solutions of the following nonlinear wave equation

$$u_{tt} - \Delta u = f(u, u_t, \nabla u), \tag{1.4}$$

where $f(u, u_t, \nabla u) = -u^p - |\nabla u|^{2r} - u_t^q$ (or $f(u, u_t, \nabla u) = -u^p|\nabla u|^q u_t^r$, here $p, q, r > 0$). When $f(u, u_t, \nabla u) = -a(x)\beta(u_t, \nabla u)$ in (1.4), where $\beta(\lambda_1, \lambda_2, ..., \lambda_n)\lambda_1 \geq 0$, Slemrod [19], Vancostenoble [20] and Haraux [21] proved the weak asymptotic stabilization of solutions. Quite recently, Nakao [22, 23, 24, 25, 26] considered the nonlinear wave equations of the form

$$u_{tt} - \Delta u + \rho(x, u_t) = f(u, u_t, \nabla u), \tag{1.5}$$

and he proved the global existence and decay of solutions.

On the other hand, relatively little is known on the blowup for nonlinearities with a dependence on spatial derivatives of u. As far as we know, the previous studies of blow-up of solutions of (1.1) were performed in [27, 28, 29, 30]. In [27], Sideris gives blow-up of small data solutions in finite time for the Cauchy problem in three dimensions when the nonlinear gradient term $a|u|^{p-1}u + b|\nabla u|^q$ in (1.1) is replaced with term $f(u, u_t, \nabla u) = a^2|\nabla u|^2 + b^2|\Delta u|^2$. To our knowledge, this is the first blow-up result for nonlinear wave equation when the nonlinear perturbation term depends on the derivatives of u. Then the result was extended by Schaeffer [28] and Rammaha [29, 30]. However, very little is known in the literature concerning the blow-up of solutions for initial boundary problem of equation (1.1) and such a method in [27, 28, 29, 30] cannot applied this case. Levine [7] has pointed that the eigenfunction method can easily be modified to include nonlinear terms of the form $f(u, \nabla u)$ provided that for all $s \in R^1, p \in R^n$,$f(s, p) \geq G(s)$, where $G(s)$ is a convex function and the function $G(s)$ satisfy the following conditions:(1)$G(s)-(\lambda+1)s$ is nonnegative and nondecreasing on (s_0,∞) for some $s_0>0$; (2)$\int_0^s G(\rho)d\rho-\frac{1}{2}s^2$ is nondecreasing on (s_0,∞);(3)$[\int_0^s G(\rho)d\rho-\frac{1}{2}s^2]^{\frac{-1}{2}}$ is integrable at $+\infty$ for s. However, when $f(u,\nabla u)=a|u|^{p-1}u+b|\nabla u|^q, a,b\in R$, we can't find any function $G(s)$ such that $f(s,p)\geq G(s)$.

Motivated by the eigenfunction method in [5, 7], the main purpose of this paper is to give sufficient conditions for finite time blow-up of solutions for the initial boundary value problem of equation (1.1) under certain conditions. We will generalize Glassey's inequality (Lemma 1.1 in [5],and see also [7]), and get sufficient conditions for blow-up of solutions to problem (1.1)-(1.3) for various $a,b\in R$ and $ab<0$ by eigenfunction method. In this sense, we extend the result [5, 7]. This method applies also to the case of the equation (1.1) with Neumann boundary condition and it remains valid for more general equation

$$u_{tt}-\Delta u+cu_t=|u|^{p-1}u+f(u,|\nabla u|), \tag{1.6}$$

where f is locally Lipschitz continuous and satisfies certain growth condition (see remark 2.4).

2 Main results

Throughout this paper we assume all function spaces are considered over real field and their notations and definitions are same as those [31]. By the usual Galerkin method and similar to the proof in [16], we can obtain regular solution in the local sense. Now we extend Lemma 1.1 in [5](see also [7]) to the following lemma, which play an essential role in this paper.

Lemma 1 Let $\phi(t)\in C^2$ satisfy

$$\phi_{tt}+k_1\phi_t\geq h(\phi),\quad t\geq 0 \tag{2.1}$$

with $\phi(0)=\alpha>0, \phi_t(0)=\beta>0$, where $k_1>0$. Suppose that $h(s)\geq 0$ for all $s\geq\alpha$. If $\delta_0=k_1\int_\alpha^{+\infty}[\beta^2+2\int_\alpha^s h(\rho)d\rho]^{-\frac{1}{2}}ds<1$, then $\phi_t(t)>0$ where $\phi_t(t)$ exists and $\lim\limits_{t\to T^-}\phi(t)=+\infty$ where $T\leq T^*=-\frac{1}{k_1}ln(1-\delta_0)$.

Proof Because $\phi(0)=\alpha>0$ and $\phi_t(0)=\beta>0$ then there exist an interval $[0,T_0)$ such that $\phi_t(t)>0$ and $\phi(t)>\alpha$ for $t\in[0,T_0)$. If it is false, let

$$t_1=\inf\{t:\phi(t)=\alpha\}, t_2=\inf\{t:\phi_t(t)=0\}.$$

If $t_2<t_1$, taking into account the condition (2.1) and the fact that $h(s)\geq 0$ for all $s\geq\alpha$, we have

$$\frac{d}{dt}(e^{k_1t}\phi_t)=e^{k_1t}(\phi_{tt}+k_1\phi_t)\geq e^{k_1t}h(\phi)>0.$$

Thus $\phi_t(t_2)>e^{-k_1t_2}\phi_t(0)>0$, which contradicts $\phi_t(t_2)=0$, and so we have $t_2\geq t_1$. Furthermore, we have $\phi_t(t)>0$ for $t\in[0,t_1)$. In this case, we get that $\phi(t_1)=\phi(0)+\int_0^{t_1}\phi_t(s)ds>\phi(0)=\alpha>0$, this is a contradiction of the fact $\phi(t_1)=\alpha$. Thus, there exist an interval $[0,T_0)$ such that $\phi_t(t)>0$ and $\phi(t)>\alpha$ for $t\in[0,T_0)$.

A multiplication of (2.1) by $2e^{2k_1t}\phi_t(t)$ gives

$$2e^{2k_1t}\phi_t\phi_{tt}+2k_1e^{2k_1t}(\phi_t)^2\geq 2e^{2k_1t}h(\phi)\phi_t,$$

that is,

$$\frac{d}{dt}[e^{2k_1t}(\phi_t)^2] \geq 2e^{2k_1t}h(\phi)\phi_t \geq 2h(\phi)\phi_t = 2\frac{d}{dt}\int_\alpha^\phi h(s)ds. \tag{2.2}$$

Integrating (2.2) from 0 to t yields

$$e^{2k_1t}(\phi_t)^2 - (\phi_t(0))^2 \geq 2\int_\alpha^\phi h(s)ds,$$

since $\phi_t > 0$, hence

$$\phi_t \geq e^{-k_1t}(\beta^2 + 2\int_\alpha^\phi h(s)ds)^{-\frac{1}{2}}. \tag{2.3}$$

We may separate variables and integrate over $(0,t)$ to obtain

$$1 - e^{-k_1t} \leq k_1\int_\alpha^{+\infty}(\beta^2 + 2\int_\alpha^y h(s)ds)^{-\frac{1}{2}}dy = \delta_0.$$

Therefore we get the result.

We consider the following spectral problem

$$\Delta w + \lambda w = 0 \ in \ \Omega, \tag{2.4}$$

$$w = 0, \ on \ \partial\Omega. \tag{2.5}$$

It is well known that problem (2.4)-(2.5) has the smallest eigenvalue $\lambda_1 > 0$ and the corresponding normalized eigenfunction $w_1 > 0$ in Ω, $\int_\Omega w_1(x)dx = 1$. Then we denote

$$k_0 = (\int_\Omega \frac{|\nabla w_1|^{\frac{q}{q-1}}}{w_1^{1/(q-1)}}dx)^{\frac{q-1}{q}}. \tag{2.6}$$

Theorem 2 Suppose $q > 1, a = 0$ and $b > 0$. Let $u(x,t)$ be a regular solution of problem (1.1)-(1.3). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x)w_1(x)dx = \alpha, \int_\Omega u_1(x)\psi_1(x)dx = \beta,$$

where $\alpha > \frac{k_0^{q/(q-1)}}{\lambda_1} > 0, \beta > 0$, and that $(\frac{\lambda_1}{k_0})^q s^q - \lambda_1 s$ is a nongeative, nondecreasing function for $s \geq \alpha$. If $\delta_1 = c\int_\alpha^{+\infty}[\beta^2 + 2\int_\alpha^s[(\frac{\lambda_1}{k_0})^q\rho^q - \lambda_1\rho]d\rho]^{-\frac{1}{2}}ds < 1$, then the solution of problem (1.1)-(1.3) blows up in a finite time.

Proof Let

$$U(t) = \int_\Omega u(x,t)w_1(x)dx.$$

Then $U(0) = \alpha > 0, U_t(0) = \beta > 0$ and as it follows from (1.1)-(1.3), $U(t)$ satisfies

$$U_{tt} + cU_t + \lambda_1 U = \int_\Omega |\nabla u|^q w_1 dx. \tag{2.7}$$

By (2.4) and Holder inequality, we get

$$\begin{aligned}\lambda_1 U &\leq |\int_\Omega u(x,t)\lambda_1 w_1(x)dx| = |\int_\Omega u(x,t)\Delta w_1(x)dx| \\ &= |\int_\Omega \nabla u\nabla w_1 dx| \leq \int_\Omega |\nabla u||\nabla w_1|dx = \int_\Omega(|\nabla u|w_1^{1/q})\frac{|\nabla w_1|}{w_1^{1/q}}dx \\ &\leq (\int_\Omega \frac{|\nabla w_1|^{\frac{q}{q-1}}}{w_1^{1/(q-1)}}dx)^{\frac{q-1}{q}}(\int_\Omega |\nabla u|^q w_1 dx)^{\frac{1}{q}} = k_0(\int_\Omega |\nabla u|^q w_1 dx)^{\frac{1}{q}},\end{aligned}$$

that is to say

$$\int_\Omega |\nabla u|^q w_1 dx \geq (\frac{\lambda_1}{k_0})^q U^q. \tag{2.8}$$

Therefore, from (2.7) and inequality(2.2), we obtain the ordinary differential inequality

$$U_{tt} + cU_t \geq (\frac{\lambda_1}{k_0})^q U^q - \lambda_1 U, \tag{2.9}$$

with $U(0) = \alpha > 0, U_t(0) = \beta > 0$. Denote $h(s) = (\frac{\lambda_1}{k_0})^q s^q - \lambda_1 s$, since $h(s) > 0$ for $s \geq \alpha$, it follows from Lemma 6 that $\lim\limits_{t\to T_0^-} U(t) = \infty$, for some $T_0 \leq T^* = -\frac{1}{c}ln(1-\delta_1)$. Furthermore, since $U(t) > 0$, we have $U(t) = |U(t)| \leq sup_\Omega |u(x,t)| \int_\Omega w_1 dx \leq sup_\Omega |u(x,t)|$, and we get $\lim\limits_{t\to T_0^-} ||u||_p^p = \infty, \forall 1 \leq p \leq \infty$, for some $T_0 \leq T^* = -\frac{1}{c}ln(1-\delta_1)$, which proves the theorem.

Theorem 3 Suppose $q \geq 2, 0 < p < 2, a < 0$ and $b > 0$. Let $u(x,t)$ be a regular solution of problem (1.1)-(1.3). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x) w_1(x) dx = \alpha_0, \int_\Omega u_1(x)\psi_1(x) dx = \beta_0,$$

where $\beta_0 > 0$ and α_0 is the positive root of the equation $b(\frac{\lambda_1}{k_0})^q s^q - |a| s^p - \lambda_1 s = 0$. If $\delta_2 = c\int_\alpha^{+\infty}[\beta^2 + 2\int_\alpha^s [(\frac{\lambda_1}{k_0})^q \rho^q - |a|\rho^p - \lambda_1 \rho] d\rho]^{-\frac{1}{2}} ds < 1$, then the solution of problem (1.1)-(1.3) blows up in a finite time.

Proof Let

$$U(t) = \int_\Omega u(x,t) w_1(x) dx.$$

Then $U(0) = \alpha_0 > 0, U_t(0) = \beta_0 > 0$ and as it follows from (1.1)-(1.3), $U(t)$ satisfies

$$U_{tt} + cU_t + \lambda_1 U = a\int_\Omega |u|^p w_1 dx + b\int_\Omega |\nabla u|^q w_1 dx. \tag{2.10}$$

Then (2.8) and the inequality $\int_\Omega |u|^p w_1 dx \geq U^p$ yield the ordinary differential inequality

$$U_{tt} + cU_t \geq b(\frac{\lambda_1}{k_0})^q U^q - |a| U^p - \lambda_1 U = h_2(U), \tag{2.11}$$

with $U(0) = \alpha_0 > 0, U_t(0) = \beta_0 > 0$. Since $h_2(s) > 0$ for $s \geq \alpha_0$, then the rest of the proof is similar to the proof of Theorem 2 and the proof is complete.

Theorem 4 Suppose $p \geq 2, 0 < q < 2, b < 0$ and $a > 0$. Let $u(x,t)$ be a regular solution of problem (1.1)-(1.3). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x) w_1(x) dx = \alpha_1, \int_\Omega u_1(x)\psi_1(x) dx = \beta_1,$$

where $\beta_1 > 0$ and α_1 is the positive root of the equation $as^p - |b|(\frac{\lambda_1}{k_0})^q s^q - \lambda_1 s = 0$. If $\delta_3 = c\int_\alpha^{+\infty}[\beta^2 + 2\int_\alpha^s [a\rho^p - |b|(\frac{\lambda_1}{k_0})^q \rho^q - \lambda_1 \rho] d\rho]^{-\frac{1}{2}} ds < 1$, then the solution of problem (1.1)-(1.3) blows up in a finite time.

Proof Similar to the proof Theorem 3, $U(t)$ satisfies

$$U_{tt} + cU_t + \lambda_1 U = a\int_\Omega |u|^p w_1 dx + b\int_\Omega |\nabla u|^q w_1 dx, \tag{2.12}$$

with $U(0) = \alpha_1 > 0, U_t(0) = \beta_1 > 0$, and then we have

$$U_{tt} + cU_t \geq aU^p - |b|(\frac{\lambda_1}{k_0})^q U^q - \lambda_1 U = h_3(U), \tag{2.13}$$

with $U(0) = \alpha_1 > 0, U_t(0) = \beta_1 > 0$. Since $h_3(s) > 0$ for $s \geq \alpha_0$, then the rest of the proof is similar to the proof of Theorem 3 and the proof is complete.

Remark 1 By Theorem 2-Theorem 4, we can also prove that the blowup result holds under the similar initial conditions for the case $a > 0, p > 2, p > q$ or $b > 0, q > 2, q > p$.

Remark 2 The same results hold if the boundary condition is of the form $a\frac{\partial u}{\partial n} + bu = 0$.

Remark 3 The results remain true when $\triangle u$ is replaced by p-Laplace operator $div(|\nabla u|^p \nabla u)$.

Remark 4 The method remains valid for more general equation (1.6), where f is locally Lipschitz continuous and satisfies the growth condition $f(u, |\nabla u|) \leq C(1 + |u|^k + |\nabla u|^q)$.

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The γ-fuzzy topological semigroups and γ-fuzzy topological ideals

Cheng-Fu Yang

(School of Mathematics and Statistics of Hexi University, Zhangye Gansu,734000, P. R. China)

Abstract: Based on the concepts of semigroup and Chang's fuzzy topological space, this paper gives the defines of the γ-fuzzy topological semigroups, γ-fuzzy topological left ideals (γ-fuzzy topological right ideals, γ-fuzzy topological intrinsic ideals and γ-fuzzy topological double ideals) and discusses the fuzzy continuous homomorphic image and the fuzzy continuous homomorphic inverse image of them.

Keywords: Fuzzy topological space; γ-fuzzy topological semigroup; γ-fuzzy topological ideal; F-continuous; homomorphic image and homomorphic inverse image

1.Introduction

Since Zadeh [15] introduced fuzzy sets and fuzzy set operations in 1965. The concept of fuzzy sets has been widely used in various fields. For example, in 1968, Chang [2] applied the fuzzy set to topological space to give fuzzy topological space. After that, Pu and Liu [9,10] introduced neighborhood structure of a fuzzy point, moore-smith convergence and product and quotient spaces in fuzzy topological space. Afterwards Rosenfeld [12] formulated the elements of the theory of fuzzy groups and Foster [4] introduced the fuzzy topological groups. In 2011, Tanay et al. [13] gave the notion of fuzzy soft topological spaces and studied neighborhood and interior of a fuzzy soft set and then used these to characterize fuzzy soft open sets. Then Nazmul and Samanta [8] introduced the fuzzy soft topological groups. Subsequently, Coker [3] used the notion of intuitionistic fuzzy sets gave by Atanassov in [1] to introduce the notion of intuitionistic fuzzy topological spaces and obtained several preservation properties and some characterizations concerning fuzzy compactness and fuzzy connectedness. After that, Kul [6] introduced the intuitionistic fuzzy topological groups.

Recently, Rajesh gave the notion of γ-fuzzy topological group in [11] and discussed the connection between fuzzy topological group and γ-fuzzy topological group. Based on this idea, in this paper, we give the concepts of the γ-fuzzy topological semigroups, γ-fuzzy topological left ideals (right ideals, intrinsic ideals and double ideals) and then discuss the homomorphic image and inverse image of them.

2.Preliminary

Definition 2.1.[15] A fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \tilde{A}(x)) : x \in X\}$$

Where $\tilde{A}(x): X \to I = [0,1]$ is a mapping and $\tilde{A}(x)$ states the grade of belongness of x in $\tilde{A}$. The family of all fuzzy sets in *X* is denoted by I^X.

Particularly, the fuzzy set

$$x_\lambda(y) = \begin{cases} \lambda, if\ x = y \\ 0, otherwise \end{cases}, \forall y \in X$$

is called a fuzzy point in *X*, denoted by x_λ.

Deflnition 2.2.[15] Let $\tilde{A}$, $\tilde{B}$ be two fuzzy sets of I^X

1) $\tilde{A}$ is contained in $\tilde{B}$ if and only if $\tilde{A}(x) \le \tilde{B}(x)$ for every $x \in X$.

2)The union of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$, denoted by $\tilde{A} \bigcup \tilde{B} = \tilde{C}$, whose membership function $\tilde{C}(x) = \tilde{A}(x) \vee \tilde{B}(x)$ for every $x \in X$.

3)The intersection of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$, denoted by $\tilde{A} \bigcap \tilde{B} = \tilde{C}$, whose membership function $\tilde{C}(x) = \tilde{A}(x) \wedge \tilde{B}(x)$ for every $x \in X$.

4)The complement of $\tilde{A}$ is a fuzzy set, denoted by $\tilde{A}^c$, whose membership function $\tilde{A}^c(x) = 1 - \tilde{A}(x)$ for every $x \in X$.

Definition 2.3.[2] Let *X*, *Y* be two nonempty sets, *f* a function from *X* to *Y* and $\tilde{B}$ a fuzzy set in *Y* with membership function $\tilde{B}(y)$. Then the inverse of $\tilde{B}$, written as $f^{-1}(\tilde{B})$, is a fuzzy set in *X* whose membership function is defined by $f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$ for all *x* in *X*.

Conversely, let $\tilde{A}$ be a fuzzy set in *X* with membership function $\tilde{A}$(x). The image of $\tilde{A}$, written as f ($\tilde{A}$), is a fuzzy set in *Y* whose membership function is given by

$$f(\tilde{A})(y) = \begin{cases} \vee_{x \in f^{-1}(y)} \tilde{A}(x), if\ f^{-1}(y) \ne \phi \\ 0, \qquad otherwise, \end{cases} \quad \text{for } \forall y \in Y$$

where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Proposition 2.4.[2] Let *f* be a function from *X* to *Y*. Then:

(1) $f^{-1}[\tilde{B}^c] = [f^{-1}\tilde{B}]^c$ for any fuzzy set $\tilde{B}$ in *Y*.

(2) $f[A^c] \supset [f(A)]^c$ for any fuzzy set $\tilde{A}$ in *X*.

(3) $\tilde{B}_1 \subset \tilde{B}_2 \Rightarrow f^{-1}[\tilde{B}_1] \subset f^{-1}[\tilde{B}_2]$, where $\tilde{B}_1$, $\tilde{B}_2$ are fuzzy sets in *Y*.

(4) $\tilde{A}_1 \subset \tilde{A}_2 \Rightarrow f[\tilde{A}_1] \subset f[\tilde{A}_2]$, where $\tilde{A}_1$, $\tilde{A}_2$ are fuzzy sets in *X*.

(5) $\tilde{B} \supset f[f^{-1}[\tilde{B}]]$ for any fuzzy set $\tilde{B}$ in Y.

(6) $\tilde{A} \subset f[f^{-1}[\tilde{A}]]$ for any fuzzy set $\tilde{A}$ in X.

Proposition 2.5. Let f be a function from X to Y. Then:

(1) $f(\tilde{A} \cap \tilde{B}) \subseteq f(\tilde{A}) \cap f(\tilde{B})$ and $f(\tilde{A} \cup \tilde{B}) = f(\tilde{A}) \cup f(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$.

(2) $f^{-1}(\tilde{A} \cap \tilde{B}) = f^{-1}(\tilde{A}) \cap f^{-1}(\tilde{B})$ and $f^{-1}(\tilde{A} \cup \tilde{B}) = f^{-1}(\tilde{A}) \cup f^{-1}(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$.

Proof. This proposition can be directly verified by the Definition 2.3.

Next example shows that $f(\tilde{A} \cap \tilde{B}) \supseteq f(\tilde{A}) \cap f(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$ has not hold.

Example. Let $X = \{a,b,c,d\}$, $Y = \{x,y\}$, $\tilde{A} = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0.6}{c} + \frac{0.1}{d}$, $\tilde{B} = \frac{0.5}{a} + \frac{0.8}{b} + \frac{0.1}{c} + \frac{0.3}{d}$. Define $f: X \to Y$ as $f(a) = f(b) = f(c) = x, f(d) = y$. Then $\tilde{A} \cap \tilde{B} = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0.1}{c} + \frac{0.1}{d}$, $f(\tilde{A} \cap \tilde{B}) = \frac{0.3}{x} + \frac{0.1}{y}$, $f(\tilde{A}) = \frac{0.6}{x} + \frac{0.1}{y}$,

$f(\tilde{B}) = \frac{0.8}{x} + \frac{0.3}{y}$, $f(\tilde{A}) \cap f(\tilde{B}) = \frac{0.6}{x} + \frac{0.1}{y}$. Thus $f(\tilde{A} \cap \tilde{B}) \supseteq f(\tilde{A}) \cap f(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$ has not hold.

3.Fuzzy topological space

Definition 3.1.[2] A fuzzy topology is a family τ of fuzzy sets in X which satisfies the following conditions:

(1) $\tilde{0}, \tilde{1} \in \tau$;

(2)If $\tilde{A}, \tilde{B} \in \tau$,then $\tilde{A} \cap \tilde{B} \in \tau$;

(3) If $\tilde{A}_i \in \tau, \forall i \in \Gamma$,then $\bigcup_{i \in \Gamma} \tilde{A}_i \in \tau$;

τ is called a fuzzy topology for X, and the pair (X, τ) is called a fuzzy topological space, or fts for short. Every member of τ is called a τ-open fuzzy set. A fuzzy set is τ-closed if and only if its complement is τ-open. In the sequel, when no confusion is likely to arise, we shall call a τ-open (τ-closed) fuzzy set simply an open (closed) set.

Proposition 3.2. Let X be a nonempty set. If τ and J are two fuzzy topologicals for X, then $\tau \cap J$ is a fuzzy topology for X, where $\tau \cap J = \{A \cap B \mid A \in \tau, B \in J\}$.

Proof. Straightforward.

Proposition 3.3. Let X, Y be two nonempty sets, f be an one-to-one mapping from X to Y. If τ is a fuzzy topological for X, then $f(\tau)$ is a fuzzy topology for Y, where $f(\tau)=\{f(A)\mid A\in\tau\}$.

Proof. (1) Obviously, $f(\tilde{0})=\tilde{0}$, $f(\tilde{1})=\tilde{1}$;

(2) If $\tilde{A},\tilde{B}\in f(\tau)$, then there exist $\tilde{A}_1,\tilde{B}_1\in\tau$ such that $\tilde{A}=f(\tilde{A}_1)$ and $\tilde{B}=f(\tilde{B}_1)$ respectively. $\forall y\in Y$,

$$(\tilde{A}\cap\tilde{B})(y)=(f(\tilde{A}_1)\cap f(\tilde{B}_1))(y)=f(\tilde{A}_1)(y)\wedge f(\tilde{B}_1)(y)=(\vee_{x\in f^{-1}(y)}\tilde{A}_1(x))\wedge(\vee_{x\in f^{-1}(y)}\tilde{B}_1(x))$$

$$=\vee_{x\in f^{-1}(y)}(\tilde{A}_1(x)\wedge\tilde{B}_1(x))=\vee_{x\in f^{-1}(y)}(\tilde{A}_1\cap\tilde{B}_1)(x)=f(\tilde{A}_1\cap\tilde{B}_1)(y).$$

This means $\tilde{A}\cap\tilde{B}=f(\tilde{A}_1\cap\tilde{B}_1)$. Since $\tilde{A}_1\cap\tilde{B}_1\in\tau$, thus $f(\tilde{A}_1\cap\tilde{B}_1)\in f(\tau)$

(3) If $\tilde{A}_i\in f(\tau),\forall i\in\Gamma$, then for any $i\in\Gamma$ there exists a $\tilde{A}_1'\in\tau$ such that $\tilde{A}_i=f(\tilde{A}_i')$. And then

$$(\bigcup_{i\in\Gamma}\tilde{A}_i)(y)=\vee_{i\in\Gamma}(\tilde{A}_i(y))=\vee_{i\in\Gamma}(f(\tilde{A}_i')(y))=\vee_{i\in\Gamma}(\vee_{x\in f^{-1}(y)}\tilde{A}_1'(x))$$

$$=\vee_{x\in f^{-1}(y)}(\vee_{i\in\Gamma}\tilde{A}_1'(x))=\vee_{x\in f^{-1}(y)}((\bigcup_{i\in\Gamma}\tilde{A}_1')(x))=f(\bigcup_{i\in\Gamma}\tilde{A}')(y).$$

This means $\bigcup_{i\in\Gamma}\tilde{A}_i=f(\bigcup_{i\in\Gamma}\tilde{A}')$. Since $\bigcup_{i\in\Gamma}\tilde{A}_i\in\tau$, thus $f(\bigcup_{i\in\Gamma}\tilde{A}')\in f(\tau)$. This completes the proof.

Proposition 3.4. Let X, Y be two nonempty sets and f a mapping from X to Y. If τ is a fuzzy topological for Y, then $f^{-1}(\tau)$ is a fuzzy topology for X, where $f^{-1}(\tau)=\{f^{-1}(\tilde{A})\mid\tilde{A}\in\tau\}$.

Proof. (1) Obviously, $f^{-1}(\tilde{0})=\tilde{0}$, $f^{-1}(\tilde{1})=\tilde{1}$;

(2)If $\tilde{A},\tilde{B}\in f^{-1}(\tau)$,then there exist $\tilde{A}_1,\tilde{B}_1\in\tau$, such that $\tilde{A}=f^{-1}(\tilde{A}_1)$ and $\tilde{B}=f^{-1}(\tilde{B}_1)$ respectively. $\forall x\in X$,

$$(\tilde{A}\cap\tilde{B})(x)=(f^{-1}(\tilde{A}_1)\cap f^{-1}(\tilde{B}_1))(x)=f^{-1}(\tilde{A}_1)(x)\wedge f^{-1}(\tilde{B}_1)(x)=\tilde{A}_1(f(x))\wedge\tilde{B}_1(f(x))$$

$$=(\tilde{A}_1\cap\tilde{B}_1)(f(x))=f^{-1}(\tilde{A}_1\cap\tilde{B}_1)(x).$$

This means $\tilde{A}\bigcap\tilde{B}=f^{-1}(\tilde{A}_1\bigcap\tilde{B}_1)$. Since $\tilde{A}_1\bigcap\tilde{B}_1\in\tau$, thus $\tilde{A}\bigcap\tilde{B}=f^{-1}(\tilde{A}_1\bigcap\tilde{B}_1)\in f^{-1}(\tau)$.

(3) If $\tilde{A}_i\in f^{-1}(\tau),\forall i\in\Gamma$, then for any $i\in\Gamma$ there exists a $\tilde{A}_1'\in\tau$ such that $\tilde{A}_i=f^{-1}(\tilde{A}_i')$. And then

$$(\bigcup_{i\in\Gamma}\tilde{A}_i)(x)=\vee_{i\in\Gamma}(\tilde{A}_i(x))=\vee_{i\in\Gamma}(f^{-1}(\tilde{A}_i')(x))=\vee_{i\in\Gamma}(\tilde{A}_1'(f(x)))$$

$$=\vee_{i\in\Gamma}(\tilde{A}_1'(f(x)))=(\bigcup_{i\in\Gamma}\tilde{A}_1')(f(x))=f^{-1}(\bigcup_{i\in\Gamma}\tilde{A}_1')(x).$$

This means $\bigcup_{i\in\Gamma}\tilde{A}_i=f^{-1}(\bigcup_{i\in\Gamma}\tilde{A}_1')$. Since $\bigcup_{i\in\Gamma}\tilde{A}_1'\in\tau$, thus $\bigcup_{i\in\Gamma}\tilde{A}_i=f^{-1}(\bigcup_{i\in\Gamma}\tilde{A}_1')\in f^{-1}(\tau)$.

This complete the proof.

Definition 3.5.[2] A function *f* from a fts (X,τ) to a fts (Y,U) is *F*-continuous iff the inverse of each open set in *Y* is open set in *X*.

Definition 3.6.[9] Let $\tilde{A}$ be a fuzzy set in (X, τ) and the union of all the open sets contained in $\tilde{A}$ is called the interior of $\tilde{A}$, denoted by $\tilde{A}^O$. Evidently $\tilde{A}^O$ is the largest open set contained in $\tilde{A}$ and $\tilde{A}^{OO} = \tilde{A}^O$.

Proposition 3.7.[2] Let $\tilde{A}$ be a fuzzy set in a fts (X, τ). Then $\tilde{A}$ is open iff $\tilde{A} = \tilde{A}^O$.

Definition 3.8.[9] The intersection of all the closed sets containing $\tilde{A}$ is called the closure of $\tilde{A}$, denoted by $\overline{\tilde{A}}$. Obviously $\overline{\tilde{A}}$ is the smallest closed set containing $\tilde{A}$ and $\overline{\overline{\tilde{A}}} = \overline{\tilde{A}}$.

By the definitions of the interior and closure, obviously $\tilde{A}^O \subset \tilde{A} \subset \overline{\tilde{A}}$.

Proposition 3.9.[9] Let $\tilde{A}$ be a fuzzy set in a fts (X, τ). Then $\tilde{A}$ is closed iff $\tilde{A} = \overline{\tilde{A}}$.

Proposition 3.10. Let $\tilde{A}$ be a fuzzy set in a fts (X, τ).

(1) If $\tilde{A} \subset \tilde{B}$, then $\tilde{A}^O \subset \tilde{B}^O$.

(2) If $\tilde{A} \subset \tilde{B}$, then $\overline{\tilde{A}} \subset \overline{\tilde{B}}$.

Proof. According to the definition can be directly proved.

Proposition 3.11.[10] Let f : $(X, \tau)\rightarrow (Y, U)$ be a function, then the ollowing are equivalent:

(1) *f* is *F*-continuous.

(2) For every closed set $\tilde{A}$ in Y, $f^{-1}(\tilde{A})$ is closed set in X.

(3) For any fuzzy set $\tilde{A}$ in X, $f(\overline{\tilde{A}}) \subset \overline{f(\tilde{A})}$.

(4) For any fuzzy set $\tilde{B}$ in Y, $\overline{f^{-1}(\tilde{B})} \subset f^{-1}(\overline{\tilde{B}})$.

Proposition 3.12. Let f : $(X, \tau) \to (Y, U)$ be a function; then the following are equivalent:

(1) f is F-continuous.

(2) For any fuzzy set $\tilde{B}$ in Y, $f^{-1}(\tilde{B}^O) \subset (f^{-1}(\tilde{B}))^o$.

Proof. (1)$\Rightarrow$ (2). For any fuzzy set $\tilde{B}$ in Y, by the definition of the interior and f is F-continuous, this means $f^{-1}(\tilde{B}^O)$ is an open set in X. On the other hand, since $\tilde{B}^O \subset \tilde{B}$, by (3) of Proposition 2.4, $f^{-1}(\tilde{B}^O) \subset f^{-1}(\tilde{B})$. Considering $(f^{-1}(\tilde{B}))^o$ is the union of all the open sets contained in $f^{-1}(\tilde{B})$, thus $f^{-1}(\tilde{B}^O) \subset (f^{-1}(\tilde{B}))^o$.

(2) $\Rightarrow$ (1). Let $\tilde{B}$ be any open fuzzy set in Y, then $\tilde{B}^O = \tilde{B}$. By condition, $f^{-1}(\tilde{B}) = f^{-1}(\tilde{B}^O) \subset (f^{-1}(\tilde{B}))^o$. On the other hand, since $f^{-1}(\tilde{B}) \supset (f^{-1}(\tilde{B}))^O$, thus $f^{-1}(\tilde{B}) = (f^{-1}(\tilde{B}))^o$. This means $f^{-1}(\tilde{B})$ is an open set in X, thus f is F-continuous.

Proposition 3.13. Let (X, τ) be a fts and f : $X \to Y$ be an one-to-one F-continuous mapping, then the following are hold:

(1) For any fuzzy set $\tilde{A}$ in X, $f(\tilde{A}^O) = (f(\tilde{A}))^o$.

(2) For any fuzzy set $\tilde{A}$ in X, $f((\overline{\tilde{A}})^O) \subset (\overline{f(\tilde{A})})^o$

Proof. According to the previous conclusion, $(Y, f(\tau))$ is a fts.

(1) For any fuzzy set $\tilde{A}$ in X, since $\tilde{A}^O = \bigcup\left\{\tilde{B} \mid \tilde{B} \subset \tilde{A}, \tilde{B} \in \tau\right\}$, by (1) of Proposition 2.5, $f(\tilde{A}^O) = \bigcup\left\{f(\tilde{B}) \mid f(\tilde{B}) \subset f(\tilde{A}), f(\tilde{B}) \in f(\tau)\right\}$. On the other hand, by the definition of the interior, $(f(\tilde{A}))^O = \bigcup\left\{f(\tilde{B}) \mid f(\tilde{B}) \subset f(\tilde{A}), f(\tilde{B}) \in f(\tau)\right\}$. Thus $f(\tilde{A}^O) = (f(\tilde{A}))^o$.

(2) For any fuzzy set $\tilde{A}$ in X, since $\overline{\tilde{A}}^O = \bigcup\left\{\tilde{B} \mid \tilde{B} \subset \overline{\tilde{A}}, \tilde{B} \in \tau\right\}$, by (1) of Proposition 2.5, thus $f((\overline{\tilde{A}})^O) = \bigcup\left\{f(\tilde{B}) \mid f(\tilde{B}) \subset f(\overline{\tilde{A}}), f(\tilde{B}) \in f(\tau)\right\}$. Considering f is a F-continuous, thus for any $f(\tilde{B}) \subset f(\overline{\tilde{A}})$ and $f(\tilde{B}) \in f(\tau)$, by (3) of

Proposition 3.11, $f(\tilde{B}) \subset f(\overline{\tilde{A}}) \subset \overline{f(\tilde{A})}$ holds. By the definition of the interior of $(\overline{f(\tilde{A})})^o$ and (1) of Proposition 3.10, and then $f(\underline{\tilde{B}}) = (f(\underline{\tilde{B}}))^O \subset (\overline{f(\tilde{A})})^o$, thus $f((\overline{\tilde{A}})^O) \subset (\overline{f(\tilde{A})})^o$.

Definition 3.14.[5, 11] A fuzzy set $\tilde{A}$ in fts (X, τ) is said to be fuzzy γ-open if $\tilde{A} \subset (\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}$. The complement of a fuzzy γ-open set is called a fuzzy γ-closed set. The family of all fuzzy γ-open sets of X is denoted by *γO(X)*.

Proposition 3.15. Let (X, τ) be a fts and $f : X \rightarrow Y$ be an one-to-one F-continuous mapping. If $\tilde{A}$ is a γ-open set in X, then $f(\tilde{A})$ is a γ-open set in Y.

Proof. Since $\tilde{A}$ is a γ-open set in X, then $\tilde{A} \subset (\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}$. And then $f(\tilde{A}) \subset f((\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}) = f((\overline{\tilde{A}})^o) \cup f(\overline{(\tilde{A})^O})$ .By Proposition 3.11 and Proposition 3.13, $f(\overline{(\tilde{A})^O}) \subset \overline{f(\tilde{A}^O)} = \overline{(f(\tilde{A}))^O}$ and $f((\overline{\tilde{A}})^o) \subset (\overline{f(\tilde{A})})^O$. This means $f(\tilde{A}) \subset f((\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}) = f((\overline{\tilde{A}})^o) \cup f(\overline{(\tilde{A})^O}) \subset (\overline{f(\tilde{A})})^o \cup \overline{(f(\tilde{A}))^o}$. By the Definition of the fuzzy γ-open set, $f(\tilde{A})$ is a γ-open set in Y.

Proposition 3.16. Let (Y, J) be a fts and $f^{-1} : Y \rightarrow X$ be an one-to-one F-continuous. If $\tilde{B}$ is a γ-open set in Y, then $f^{-1}(\tilde{B})$ is a γ-open set in X.

Proof. Let f^{-1} as f in proposition 3.15, the proof is similar to proposition 3.15.

Definition 3.17. A fuzzy set $\tilde{A}$ in a fts (X, τ) is called a γ-neighborhood of fuzzy point x_λ, if there exists a γ-open set $\tilde{B} \in \tau$ such that $x_\lambda \in \tilde{B} \subset \tilde{A}$. The family consisting of all γ-neighborhoods of x_λ is called the system of γ-neighborhoods of x_λ.

Definition 3.18. A fuzzy point x_λ is said to be quasi-coincident with a fuzzy set $\tilde{A}$, denoted by $x_\lambda \, q \, \tilde{A}$, if $\lambda + \tilde{A}(x) > 1$. A fuzzy set $\tilde{A}$ is said to be a Q_γ-neighborhood x_λ of if there exists a γ-open set $\tilde{B} \in \tau$ such that $x_\lambda \, q \, \tilde{B} \subset \tilde{A}$. The family consisting of all the Q_γ-neighborhoods of x_λ is called the system of Q_γ-neighborhoods of x_λ.

Proposition 3.19. Let (X, τ) be a fts and f an one-to-one F-continuous mapping from X to Y. If U is a Q_γ-neighborhood of a in X, then $f(U)$ is a Q_γ-neighborhood of $f(a_\lambda)=[f(a)]_\lambda$ in fts $(Y, f(\tau))$.

Proof. In order to avoid confusion, here record $y=f(a)$. Without losing generality, let $U\in\gamma O(X)$, Since

$$f(U)(f(a))+\lambda=f(U)(y)+\lambda=\vee_{x\in f^{-1}(y)}U(x)+\lambda>U(a)>1,$$

this means $[f(a)_\lambda qf(U)]$ .Considering $f(U)\in\gamma O(Y)$, thus $f(U)$ is a Q$_\gamma$-neighborhood of $[f(a)]_\lambda$.

Proposition 3.20. Let (Y, J) be a fts and f^{-1} an one-to-one F-continuous mapping from Y to X. If V is a Q$_\gamma$-neighborhood of $(f(a))_\lambda$ in Y, then $f^{-1}(V)$ is a Q$_\gamma$-neighborhood of a_λ in fts $(X, f^{-1}(J))$.

Proof. Without losing generality, let $V\in\gamma O(Y)$, Since

$$f^{-1}(V)(a)+\lambda=V(f(a))+\lambda>1,$$

this means $a_\lambda q f^{-1}(V)$. Considering $f^{-1}(V)\in\gamma O(X)$, thus $f^{-1}(V)$ is a Q$_\gamma$-neighborhood of a_λ.

4 γ-Fuzzy topological semigroup

Definition 4.1.[7] Let X be a semigroup and $\tilde{A}$, $\tilde{B}$ two fuzzy sets in X. $\tilde{A}\tilde{B}$ is defined as a fuzzy set in X, which membership function is as follows:

$$\tilde{A}\tilde{B}(x)=\vee_{x_1x_2=x}(\tilde{A}(x_1)\wedge\tilde{B}(x_2))\quad for\quad x\in X.$$

Proposition 4.2. Let X, Y be two semigroups and f an epimorphism from X to Y. If $\tilde{A}$, $\tilde{B}$ are any two the fuzzy sets in X, then $f(\tilde{A}\tilde{B})=f(\tilde{A})f(\tilde{B})$.

Proof. For any $y\in Y$, since

$$f(\tilde{A}\tilde{B})(y)=\vee_{x\in f^{-1}(y)}(\tilde{A}\tilde{B})_{(x)}=\vee_{x\in f^{-1}(y)}(\vee_{x_1x_2=x}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\vee_{x_1x_2=x}(\vee_{x\in f^{-1}(y)}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\vee_{x_1x_2=x}(\vee_{x_1x_2\in f^{-1}(y)}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\vee_{f(x_1x_2)=f(x)=y}(\vee_{x_1x_2\in f^{-1}(y)}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\vee_{f(x_1)f(x_2)=f(x)=y}(\vee_{x_1x_2\in f^{-1}(y)}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\vee_{y_1y_2=y}(\vee_{x_1x_2\in f^{-1}(y)}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\vee_{y_1y_2=y}(\vee_{x_1x_2\in f^{-1}(y_1y_2)}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\vee_{y_1y_2=y}(\vee_{x_1x_2\in f^{-1}(y_1)f^{-1}(y_2)}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\vee_{y_1y_2=y}((\vee_{x_1x_2\in f^{-1}(y_1)f^{-1}(y_2)}(\tilde{A}(x_1))\wedge(\vee_{x_1x_2\in f^{-1}(y_1)f^{-1}(y_2)}\tilde{B}(x_2)))$$

$$=\vee_{y_1y_2=y}((\vee_{x_1\in f^{-1}(y_1)}(\tilde{A}(x_1))\wedge(\vee_{x_2\in f^{-1}(y_2)}\tilde{B}(x_2)))$$

$$= \vee_{y_1 y_2 = y} (f(\tilde{A})(y_1) \wedge f(\tilde{B})(y_2)) = f(\tilde{A}\tilde{B})(y),$$

thus $f(\tilde{A}\tilde{B}) = f(\tilde{A}) f(\tilde{B})$.

Proposition 4.3. Let *X*, *Y* be two semigroups and *f* a monomorphism from *X* to *Y*. If $\tilde{C}, \tilde{D}$ are any two the fuzzy sets in *Y*, then $f^{-1}(\tilde{C}\tilde{D}) = f^{-1}(\tilde{C}) f^{-1}(\tilde{D})$.

Proof. For any $x \in X$, since

$$f^{-1}(\tilde{C}\tilde{D})(x) = \tilde{C}\tilde{D}(f(x)) = \vee_{y_1 y_2 = f(x)} (\tilde{C}(y_1) \wedge \tilde{D}(y_2)) = \vee_{f^{-1}(y_1) f^{-1}(y_2) = x} (\tilde{C}(y_1) \wedge \tilde{D}(y_2))$$

$$= \vee_{x_1 x_2 = x} (\tilde{C}(f(x_1)) \wedge \tilde{D}(f(x_2))) = \vee_{x_1 x_2 = x} (f^{-1}(\tilde{C}(x_1)) \wedge f^{-1}(\tilde{D}(x_2))) = (f^{-1}(\tilde{C}) f^{-1}(\tilde{D}))(x),$$

thus $f^{-1}(\tilde{C}\tilde{D}) = f^{-1}(\tilde{C}) f^{-1}(\tilde{D})$.

Definition 4.4. Let X be a semigroup and (X, τ) a fts. Then (X, τ) is called a γ-fuzzy topological semigroup, or γ-ftsg for short, if for all $a, b \in X$ and any Q_γ-neighborhood W of fuzzy point $(ab)_\lambda$ there exist Q_γ-neighborhoods U of a_λ and V of b_λ such that $UV \subset W$.

Proposition 4.5. Let *X, Y* be two semigroups and (X, τ) a γ-ftsg. If *f* is an one-to-one *F*-continuous homomorphic mapping from *X* to *Y*, then $(Y, f(\tau))$ is a γ-ftsg.

Proof. By Proposition 3.3, $(Y, f(\tau))$ is a fts. For any Q_γ-neighborhood *W* of fuzzy point $(ab)_\lambda$ in *Y*, according to Proposition 3.20, $f^{-1}(W)$ is a Q_γ-neighborhood of fuzzy point $f^{-1}((ab)_\lambda)$ in *X*. Since (X, τ) is a γ-ftsg, thus there exist Q_γ-neighborhoods U of $f^{-1}(a_\lambda)$ and V of $f^{-1}(b_\lambda)$ such that $UV \subset f^{-1}(W)$, and then $f(UV) \subset W$. By proposition 3.19, $f(U)$ and $f(V)$ is the Q_γ-neighborhoods of a_λ and b_λ respectively. Again by Proposition 4.2, $f(U) f(V) = f(UV) \subset W$. Thus $(Y, f(\tau))$ is a γ-ftsg.

Proposition 4.6. Let *X, Y* be two semigroups and *(Y, J)* a γ-ftsg. If f^{-1} is an one-to-one *F*-continuous homomorphic mapping from *X* to *Y,* then $(X, f^{-1}(J))$ is a γ-ftsg.

Proof. By Proposition 3.4, $f^{-1}(J)$ is a fts. For any Q$_\gamma$-neighborhood *W* of fuzzy point $(ab)_\lambda$ in *X*, according to Proposition 3.19, $f(W)$ is a Q$_\gamma$-neighborhood *W* of fuzzy point $f((ab)_\lambda)$ in *Y*. Since (*Y, J*) is a γ-ftsg, thus there exist Q$_\gamma$-neighborhoods *U* of $f(a_\lambda)$ and *V* of $f(b_\lambda)$ such that $UV \subset f(W)$, and then $f^{-1}(UV) \subset W$. By proposition 3.20, $f^{-1}(U)$ and $f^{-1}(V)$ is the Q$_\gamma$-neighborhoods of a and b respectively.Again by Proposition 4.3, $f^{-1}(U) f^{-1}(V) = f^{-1}(UV) \subset W$. Thus $(X, f^{-1}(J))$ is a γ-ftsg.

5. γ-Fuzzy topological ideal

Definition 5.1. Let *X* be a semigroup and (X, τ) a fts. Then (X, τ) is called a -fuzzy topological left ideal (right ideal), if for all $a, b \in X$ and any Q$_\gamma$-neighborhood W of fuzzy point $(ab)_\lambda$ there exists Q$_\gamma$-neighborhood *U* of b_λ (Q$_\gamma$-neighborhood V of a_λ) such that $U \subset W$ ($V \subset W$).

Definition 5.2. Let *X* be a semigroup and (X, τ) a fts. Then (X, τ) is called a γ-fuzzy topological intrinsic ideal (double ideal), if for all $a, b, c \in X$ and any Q$_\gamma$-neighborhood *W* of fuzzy point $(abc)_\lambda$ there exists Q$_\gamma$-neighborhood *U* of b_λ (Q$_\gamma$-neighborhood *U* of a_λ and Q -neighborhood V of c_λ) such that $U \subset W$ (such that $UV \subset W$).

Proposition 5.3. Let *X, Y* be two semigroups and (X, τ) a -fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal). If *f* is an one-to-one *F*-continuous homomorphic mapping from *X* to *Y*, then $(Y, f(\tau))$ is a γ-fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal).

Proof. Similar to the proof of Proposition 4.5.

Proposition 5.4. Let *X, Y* be two semigroups and (*Y, J*) a γ-fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal). If f^{-1} is an one-to-one *F*-continuous homomorphic mapping from *X* to *Y*, then $(X, f^{-1}(J))$ is a γ-fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal).

Proof. Similar to the proof of Proposition 4.6.

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The Behavior and Closed Form of the Solutions of Some Difference Equations

E. M. Elsayed[1,2], and Hanan S. Gafel[1]

[1]Mathematics Department, Faculty of Science, King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.

[2]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

E-mail: emelsayed@yahoo.com, h-s-g2006@hotmail.com.

ABSTRACT:

In this paper, we will investigate the local stability of the equilibrium points, global attractor, boundedness and the form of the solutions for the following equations

$$x_{n+1}=\frac{Ax_nx_{n-3}}{Bx_n+Cx_{n-2}} \text{and } x_{n+1}=\frac{Ax_nx_{n-3}}{Bx_n-Cx_{n-2}},$$

where the coefficients A, B and C are real positive numbers, and the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary non zero real numbers.

Keywords: difference equations, global attractor, local stability, equilibrium point, boundedness.

Mathematics Subject Classification: 39A10.

1. Introduction

The study of difference equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory ,quelling theory, statistical problems, stochastic time series. combinatorial analysis number theory, geometry, electrical network, quanta in radiation, genetics in biology ,economics, psychology. sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole see [1]-[20]. The purpose of this article is to investigate the global attractivity of the equilibrium point, and the asymptotic behavior of the solutions of the following difference equations

$$x_{n+1}=\frac{Ax_nx_{n-3}}{Bx_n+Cx_{n-2}}. \tag{1}$$

$$x_{n+1}=\frac{Ax_nx_{n-3}}{Bx_n-Cx_{n-2}}. \tag{2}$$

Where the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers, and A, B, C are positive constants. We obtain the form of the solution of some special cases of equation (1) and (2) and some numerical simulation to the equation are given to illustrate our results.

Lemma 1.1. Let I be some interval of real numbers and let $f: I^{k+1} \rightarrow I$, be a continuously differentiable function. Then for every initial conditions $x_{-k}, \dots, x_{-1}, x_0 \in I, k \in \{1, 2, 3, \dots\}$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, 2, \dots, \tag{3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Difinition 1.1. A point $\bar{x} \in I$ is called an **equilibrium point** of equation (3) if $\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x})$. That is, $x_n = \bar{x}\ for\ n \geq 0$, is a solution of equation (3), or equivalently, $\bar{x}$ is a fixed point of f.

Difinition 1.2. The equilibrium point $\bar{x}$ of equation (3) is called **locally stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x_{-k}, $x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \varepsilon, \forall n \geq -k$.

Difinition 1.3. The equilibrium point $\bar{x}$ of equation (3) is called **locally asymptotically stable** if it is locally stable and if there exists $\gamma > 0$ such that for all x_{-k}, $x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \gamma,$$

we have $\lim_{n\to\infty} x_n = \bar{x}$.

Difinition 1.4. The equilibrium point $\bar{x}$ of equation (3) is called **global attractor** if for all x_{-k}, $x_{-k+1}, \dots, x_{-1}, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

Difinition 1.5. The equilibrium point $\bar{x}$ of equation (3) is called **global asymptotically stable** if it is locally stable and a global attractor of equation (3).

The equilibrium point $\bar{x}$ of equation (3) is **unstable** if it is not locally stable.

The linearized equation of equation (3) about the equilibrium point $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i} \tag{4}$$

Now suppose that the characteristic equation associated with equation (4) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + \dots + p_{k-1}\lambda + p_k = 0,$$

where $p_i = \frac{\partial f(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}}$.

Theorem A. [32] Suppose that $p_i \in R$, $i = 1,2,3, \dots k$ and $\mathrm{k} \in \{0,1,2, \cdots\}$. Then $\sum_{i=1}^{k} |p_i| < 1$, is a sufficient condition for the **asymptotic stability** of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \qquad n = 0,1,2, \dots \ .$$

Theorem B. [33] Let $[a, b]$ be an interval of real numbers and assume that $g: [a, b]^3 \to [a, b]$ is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ is non-decreasing in x and z in $[a, b]$ for each $y \in [a, b]$, and is non-increasing in $y \in [a, b]$ for each x and z in $[a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(M, m, M) \quad and \quad m = g(m, M, m).$$

Then $m = M$. Then equation $x_{n+1} = g(x_n, x_{n-1}, x_{n-2})$, has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of this equation converges to $\bar{x}$.

Theorem C. [33] Let $[a, b]$ be an interval of real numbers and assume that $g: [a, b]^3 \to [a, b]$ is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ is non-decreasing in y and z in $[a, b]$ for each $x \in [a, b]$, and is non-increasing in $x \in [a, b]$ for each y and z in $[a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(m, M, M) \quad and \quad m = g(M, m, m).$$

Then $m = M$. Then equation $x_{n+1} = g(x_n, x_{n-1}, x_{n-2})$, has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of this equation converges to $\bar{x}$.

Theorem D. [33] Let $[a, b]$ be an interval of real numbers and assume that $g: [a, b]^3 \to [a, b]$ is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ is non-decreasing in x in $[a, b]$ for each y and $z \in [a, b]$, and is non-increasing in x and $z \in [a, b]$ for each y in $[a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(m, M, m) \quad and \quad m = g(M, m, M).$$

Then $m = M$. Then equation $x_{n+1} = g(x_n, x_{n-1}, x_{n-2})$, has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of this equation converges to $\bar{x}$.

Many researchers have investigated the behavior of the solution of difference equation for example: Cinar [5-7] has got the solutions of the following difference

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{1 + bx_n x_{n-1}}.$$

Elabbasy et al. [10] studied the behavior of the difference equation

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

El-Metwally et al. [12] investigated the asymptotic behavior of the population model

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}.$$

Karatas et al. [28] got the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Zayed and El-Moneam [45] deal with the behavior of the following rational recursive sequence

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}.$$

Wang and Zhang [38] considered the sufficient and necessary condition for the existence and uniqueness of the initial value problem of difference equations of higher order. In addition, they investigated the local stability, asymptotic behavior, periodicity and oscillation of solutions for the same equation. See also [21]-[47].

2. The Behavior of Equation (1)

This section will examine the behavior of solutions of equation (1). The constants A, B and C within the equation are real positive numbers.

2.1. Local Stability of Equation (1)

In this subsection, we explore the local stability character of the solution of equation (1).

Equation (1) make sure has a unique equilibrium point is set as follows:

$$\overline{x} = \frac{A\overline{x}\overline{x}}{B\overline{x} + C\overline{x}} \quad \Rightarrow \quad \overline{x} = 0.$$

Then the unique equilibrium point is $\bar{x} = 0$ if $B + C \neq A$.

Theorem 2.1. Assume that $A(B + 3C) < (B + C)^2$, then the equilibrium point of equation (1) is locally asymptotically stable.

Proof: Let $f: (0, \infty)^3 \to (0, \infty)$ be a function define by

$$f(u, v, w) = \frac{Auw}{Bu + Cv}. \tag{5}$$

Thus, it follows that

$$\frac{\partial f}{\partial u} = \frac{ACvw}{(Bu + Cv)^2}, \quad \frac{\partial f}{\partial v} = \frac{-ACuw}{(Bu + Cv)^2}, \quad \frac{\partial f}{\partial w} = \frac{Au}{Bu + Cv}.$$

As it can be seen

$$\frac{\partial f}{\partial u}\Big|_{\bar{x}=0} = \frac{AC}{(B + C)^2}, \quad \frac{\partial f}{\partial v}\Big|_{\bar{x}=0} = \frac{-AC}{(B + C)^2}, \quad \frac{\partial f}{\partial w}\Big|_{\bar{x}=0} = \frac{A}{B + C}.$$

Then the linearized equation associated with equation (1) about $\bar{x} = 0$ is

$$y_{n+1} - \frac{AC}{(B + C)^2} y_n + \frac{AC}{(B + C)^2} y_{n-2} - \frac{A}{B + C} y_{n-3} = 0,$$

and it associated characteristic equation is

$$\lambda^4 - \frac{AC}{(B + C)^2}\lambda^3 + \frac{AC}{(B + C)^2}\lambda - \frac{A}{B + C} = 0.$$

It follows by theorem A that equation (1) is asymptotically stable if

$$\left| \frac{AC}{(B + C)^2} \right| + \left| \frac{AC}{(B + C)^2} \right| + \left| \frac{A}{B + C} \right| < 1,$$

thus,

$$A(B + 3C) < (B + C)^2.$$

Therefore, the proof is complete.

2.2. Global Attractivity of the Equilibrium Point of Equation (1)

The global attractivity character of solutions of equation (1) will be investigated in this section.

Theorem 2.2. The equilibrium point of equation (1) is global attractor if $B \neq A$.

Proof. Let p, q are real numbers and suppose that $f: [p, q]^3 \to [p, q]$ be a function define by equation $f(u, v, w) = \frac{Auw}{Bu+Cv}$, then we can easily see that the function increasing in u, w and decreasing in v. Assume that (m, M) is a solution of the system

$$M = f(M, m, M) \quad \text{and } m = f(m, M, m).$$

Then from equation (1), we see that

$$M = \frac{AM^2}{BM + Cm}, \qquad m = \frac{Am^2}{Bm + CM}$$

$$\Rightarrow BM^2 + CMm = AM^2, \quad Bm^2 + CMm = Am^2.$$

Formerly

$$(B - A)(M - m)(M + m) = 0.$$

Then $M = m$ if $B \neq A$. Therefore, it can be concluded from Theorem B that $\bar{x}$ is a global attractor.

2.3. Boundedness of the Solutions of Equation (1)

The boundedness of the solutions of Equation (1) will be discussed in this section.

Theorem 2.3. Every solution of equation (1) is bounded if $\frac{A}{B} < 1$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (1). It follows from equation (1) that

$$x_{n+1} = \frac{Ax_n x_{n-3}}{Bx_n + Cx_{n-2}} \leq \frac{Ax_n x_{n-3}}{Bx_n} = \frac{A}{B} x_{n-3}, \qquad \text{for all } n \geq 1.$$

By using a comparison, we can write the right hand side as follows $y_{n+1} = \frac{A}{B} y_{n-3}$.

So $y_n = \left(\frac{A}{B}\right)^n K$, K is constant, and this equation is locally asymptotically stable if $\frac{A}{B} < 1$, and converges to the equilibrium point $\bar{y} = 0$. Thus the solution of equation (1) is bounded.

2.4. Special Case of Equation (1)

In this subsection, the solution of the fourth order difference equation will be presented here

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n + x_{n-2}}. \tag{6}$$

Such that the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary non zero real numbers.

Theorem 2.4. The solution of equation (6) is given by the following formulas for $n = 0,1,2, \ldots$

$$x_{10n-3} = \frac{(abc)^n d^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n}, \qquad x_{10n-2} = \frac{(abd)^n c^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n},$$

$$x_{10n-1}=\frac{(acd)^{n}b^{n+1}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n}},\qquad x_{10n}=\frac{(bcd)^{n}a^{n+1}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n}},$$

$$x_{10n+1}=\frac{(ad)^{n+1}(bc)^{n}}{(a+c)^{n+1}(b+d)^{n}(ab+ad+bc)^{n}},\qquad x_{10n+2}=\frac{(acd)^{n+1}b^{n}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n+1}},$$

$$x_{10n+3}=\frac{(cbd)^{n+1}a^{n}}{(a+c)^{n+1}(b+d)^{n+1}(ab+ad+bc)^{n}},x_{10n+4}=\frac{(bca)^{n+1}d^{n}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n+1}},$$

$$x_{10n+5}=\frac{(abd)^{n+1}c^{n}}{(a+c)^{n+1}(b+d)^{n+1}(ab+ad+bc)^{n}},x_{10n+6}=\frac{(cd)^{n+1}b^{n}a^{n+2}}{(a+c)^{n+1}(b+d)^{n}(ab+ad+bc)^{n+1}},$$

where $x_{-3}=d, x_{-2}=c, x_{-1}=b, x_0=a$.

Proof. By using mathematical induction, we can prove as follow. For $n=0$ the result holds. Assume that the result holds for $n-1$, as follows

$$x_{10n-7}=\frac{(cbd)^{n}a^{n-1}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n-1}},x_{10n-6}=\frac{(bca)^{n}d^{n-1}}{(a+c)^{n-1}(b+d)^{n-1}(ab+ad+bc)^{n}}$$

$$x_{10n-5}=\frac{(abd)^{n}c^{n-1}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n-1}},x_{10n-4}=\frac{(cd)^{n}b^{n-1}a^{n+1}}{(a+c)^{n}(b+d)^{n-1}(ab+ad+bc)^{n}}.$$

We see from equation (6) that

$$x_{10n-3}=\frac{x_{10n-4}x_{10n-7}}{x_{10n-4}+x_{10n-6}}$$

$$=\frac{(cd)^{n}b^{n-1}a(acbd)^{n}a^{n-1}}{(a+c)^{2n}(b+d)^{2n-1}(ab+ad+bc)^{2n-1}}\div[\frac{(acbd)^{n}b^{-1}a(a+c)^{-1}+(bcad)^{n}d^{-1}}{(a+c)^{n-1}(b+d)^{n-1}(ab+ad+bc)^{n}}]$$

$$=\frac{(cd)^{n}b^{n-1}a^{n}\div[b^{-1}d^{-1}(a+c)^{-1}(da+b(a+c)]}{(a+c)^{n+1}(b+d)^{n}(ab+ad+bc)^{n-1}}=\frac{(acb)^{n}d^{n+1}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n}}.$$

$$x_{10n-2}=\frac{x_{10n-3}x_{10n-6}}{x_{10n-3}+x_{10n-5}}$$

$$=\frac{(acbd)^{2n}}{(a+c)^{2n-1}(b+d)^{2n-1}(ab+ad+bc)^{2n}}\div[\frac{(acbd)^{n}d(ab+ad+bc)^{-1}+(abcd)^{n}c^{-1}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n-1}}]$$

$$=\frac{(acbd)^{n}\div(ab+ad+bc)^{-1}c^{-1}[dc+(ab+ad+bc)]}{(a+c)^{n-1}(b+d)^{n-1}(ab+ad+bc)^{n+1}}=\frac{(abd)^{n}c^{n+1}}{(a+c)^{n}(b+d)^{n}(ab+ad+bc)^{n}}.$$

Also, the other relations can be proved similarly. The proof is completed.

2.5. Numerical Examples

In this subsection, numerical examples which represent different types of solutions to equation (1). Are considered to confirm the results.

Example 5.1. We assume the initial condition as follows: $x_{-3} = 14,\ x_{-2} = 2,\ x_{-1} = 7,\ x_0 = 5$ and the constants $A = 2,\ B = 4,\ C = 1$. See Fig. 1.

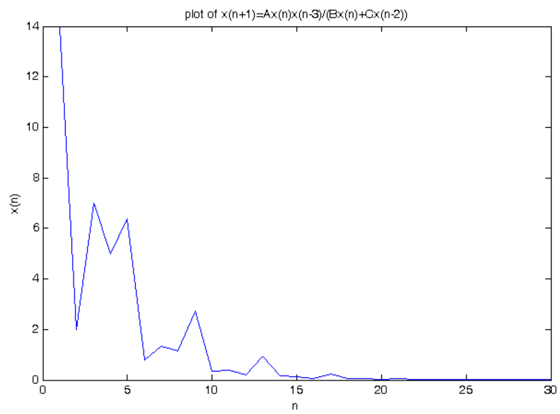

Figure 1.

Example 5.2. See Fig. 2 since we put $x_{-3} = 4,\ x_{-2} = 2,\ x_{-1} = 7,\ x_0 = 5$ and the constants $A = 12,\ B = 4,\ C = 3$.

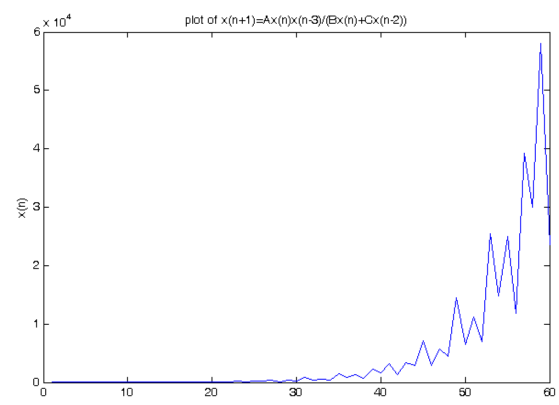

Figure 2.

3. The Behavior of Equation (2)

This section will examine the behavior of solutions of equation (2). The constants A, B and C within the equation are real positive numbers.

3.1. Local Stability of Equation (2)

In this subsection, we explore the local stability character of the solution of equation (2). Equation (2) make sure a unique equilibrium point is set as follows:

$$\bar{x} = \frac{A\overline{xx}}{B\bar{x} - C\bar{x}} \quad \Rightarrow \quad \bar{x} = 0.$$

Then the unique equilibrium point is $\bar{x} = 0$ if $A + C \neq B$.

Theorem 3.1. Assume that $A(B - 3C) < (B - C)^2$, then the equilibrium point of equation (2) is locally asymptotically stable.

Proof: Let $f: (0,\infty)^3 \to (0,\infty)$ be a function define by

$$f(u, v, w) = \frac{Auw}{Bu - Cv}. \tag{7}$$

Thus, it follows that

$$\frac{\partial f}{\partial u} = \frac{-ACvw}{(Bu - Cv)^2}, \qquad \frac{\partial f}{\partial v} = \frac{ACuw}{(Bu - Cv)^2}, \qquad \frac{\partial f}{\partial w} = \frac{Au}{Bu - Cv}.$$

As it can be seen

$$\frac{\partial f}{\partial u}\Big|_{\bar{x}=0} = \frac{-AC}{(B - C)^2}, \qquad \frac{\partial f}{\partial v}\Big|_{\bar{x}=0} = \frac{AC}{(B - C)^2}, \qquad \frac{\partial f}{\partial w}\Big|_{\bar{x}=0} = \frac{A}{B - C}.$$

Then the linearized equation associated with equation (2) about $\bar{x} = 0$ is

$$y_{n+1} + \frac{AC}{(B - C)^2} y_n - \frac{AC}{(B - C)^2} y_{n-2} - \frac{A}{B - C} y_{n-3} = 0,$$

and it associated characteristic equation is

$$\lambda^4 + \frac{AC}{(B - C)^2}\lambda^3 - \frac{AC}{(B - C)^2}\lambda - \frac{A}{B - C} = 0.$$

It follows by theorem A that equation (2) is asymptotically stable if

$$\left|\frac{AC}{(B-C)^2}\right| + \left|\frac{AC}{(B-C)^2}\right| + \left|\frac{A}{B-C}\right| < 1,$$

or

$$A(B-3C) < (B-C)^2.$$

Therefore, the proof is complete.

3.2. Global Attractivity of the Equilibrium Point of Equation (2)

The global attractivity character of solutions of equation (2) will be investigated in this section.

Theorem 3.2. The equilibrium point of equation (2) is global attractor if $C \neq A$.

Proof. Let p, q are real numbers and suppose that $f: [p, q]^3 \to [p, q]$ be a function define by $f(u, v, w) = \frac{Auw}{Bu-Cv}$. Then we can easily see that the function $f(u, v, w)$ decreasing in u and increasing in v. So we have two cases we prove case (1) and case (2) is similar and so will be omitted.

Case (1):- If $Bu - Cv > 0$, then we can easily see that the function $f(u, v, w)$ increasing in w. Assume that (m, M) is a solution of the system

$$M = f(m, M, M) \quad \text{and } m = f(M, m, m).$$

Then from equation (2), we see that

$$M = \frac{AMm}{Bm - CM}, \qquad m = \frac{AMm}{BM - Cm}$$

$$\Rightarrow BMm - CM^2 = AMm, \quad BMm - Cm^2 = AMm.$$

Formerly $C(M - m)(M + m) = 0$. Then $M = m$. Therefore, it can be concluded from Theorem C that $\bar{x}$ is a global attractor.

3.3. Special Case of Equation (2)

In this section, we study the following special cases of equation (2) where the constants A, B and C are integers numbers. The solution of the fourth order difference equation will be presented here

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n - x_{n-2}}. \tag{8}$$

Such that the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ arearbitrary nonzero realnumbers.

Theorem 3.3. The solution of equation (8) is given by the following formulas for $n = 0,1,2, ...$

$$x_{10n-3} = \frac{(abc)^n d^{n+1}}{(a-c)^n (b-d)^n (bc-ab+ad)^n}, \qquad x_{10n-2} = \frac{(abd)^n c^{n+1}}{(a-c)^n (b-d)^n (bc-ab+ad)^n},$$

$$x_{10n-1}=\frac{(acd)^{n}b^{n+1}}{(a-c)^{n}(b-d)^{n}(bc-ab+ad)^{n}},\qquad x_{10n}=\frac{(bcd)^{n}a^{n+1}}{(a-c)^{n}(b-d)^{n}(bc-ab+ad)^{n}},$$

$$x_{10n+1}=\frac{(ad)^{n+1}(bc)^{n}}{(a-c)^{n+1}(b-d)^{n}(bc-ab+ad)^{n}},\qquad x_{10n+2}=\frac{(acd)^{n+1}b^{n}}{(a-c)^{n}(b-d)^{n}(bc-ab+ad)^{n+1}},$$

$$x_{10n+3}=\frac{(cbd)^{n+1}a^{n}}{(a-c)^{n+1}(b-d)^{n+1}(bc-ab+ad)^{n}},x_{10n+4}=\frac{(bca)^{n+1}d^{n}}{(a-c)^{n}(b-d)^{n}(bc-ab+ad)^{n+1}},$$

$$x_{10n+5}=\frac{(abd)^{n+1}c^{n}}{(a-c)^{n+1}(b-d)^{n+1}(bc-ab+ad)^{n}},x_{10n+6}=\frac{(cd)^{n+1}b^{n}a^{n+2}}{(a-c)^{n+1}(b-d)^{n}(bc-ab+ad)^{n+1}}.$$

Proof. As the proof of Theorem 2.4 and will be omitted.

3.4. Numerical Examples

Example 3.4. We suppose that initial condition are taken as follows: $x_{-3}=14,\ x_{-2}=32,\ x_{-1}=-7,\ x_0=5$ and the constants $A=12,\ B=3,\ C=8$. See Fig. 3.

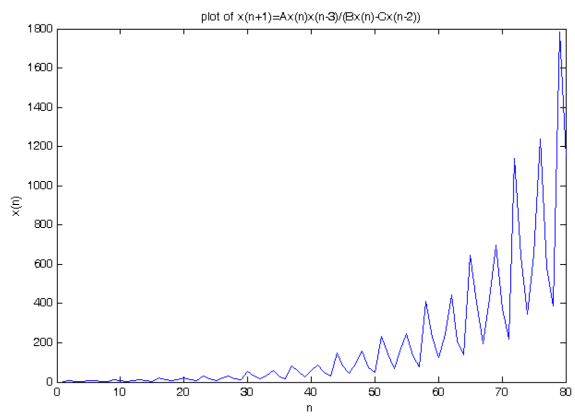

Figure 3.

Example 3.5. The Figure 4 shows the behavior of the solutions of equation (2) when $x_{-3}=1.55,\ x_{-2}=2.20,\ x_{-1}=5.45,\ x_0=7$ and the constants $A=4,\ B=2,\ C=3$. See Fig. 4.

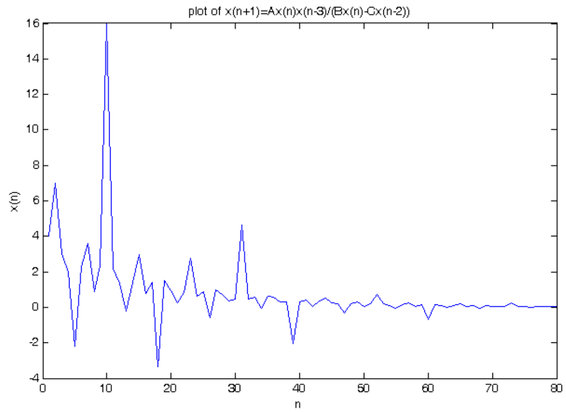

Figure 4.

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Convexity and Monotonicity of Certain Maps Involving Hadamard Products and Bochner Integrals for Continuous Fields of Operators

Pattrawut Chansangiam*

Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.

Abstract

We investigate the convexity and the monotonicity of certain maps involving Hadamard products and Bochner integrals for continuous fields of Hilbert space operators. Their special cases and consequences are then discussed. In particular, we obtain certain arithmetic mean-harmonic mean, Jensen, and Fiedler type inequalities.

Keywords: Hadamard product, tensor product, continuous field of operators, Bochner integral
Mathematics Subject Classifications 2010: 26D15, 474A63, 46G10, 47A80.

1 Introduction

Throughout, let $\mathfrak{B}(\mathbb{H})$ be the algebra of bounded linear operators on a complex separable Hilbert space $\mathbb{H}$. The positive cone $\mathfrak{B}(\mathbb{H})^+$ of $\mathfrak{B}(\mathbb{H})$ consists of all positive operators on $\mathbb{H}$. The identity operator is denoted by I, where the underlying space should be clear from contexts. The spectrum of $A \in \mathfrak{B}(\mathbb{H})$ is written as $\mathrm{sp}(A)$. For self-adjoint operators A and B, the situation $A \geqslant B$ means that $A - B \in \mathfrak{B}(\mathbb{H})^+$. If A is an invertible positive operator, we write $A > 0$. The operator norm of $A \in \mathfrak{B}(\mathbb{H})$ is denoted by $\|A\|$. The notation $\|\cdot\|_{\infty,X}$ is used for the supremum norm on the set X.

The *Hadamard product* of A and B in $\mathfrak{B}(\mathbb{H})$ is defined to be the bounded linear operator $A \circ B$ satisfying

$$\langle (A \circ B)e, e\rangle = \langle Ae, e\rangle \langle Be, e\rangle \quad \text{for all } e \in \mathcal{E}. \tag{1.1}$$

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

Here, $\mathcal{E}$ is a fixed countable orthonormal basis for $\mathbb{H}$. This definition is independent on a choice of the orthonormal basis. In [7], it was shown that there is a positive linear map Φ taking the tensor product $A \otimes B$ to the Hadamard product $A \circ B$ for any $A, B \in \mathfrak{B}(\mathbb{H})$. Indeed, the map Φ is given by $\Phi(X) = Z^* X Z$ where $Z : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined on the basis $\mathcal{E}$ by

$$Ze = e \otimes e \quad \text{for all } e \in \mathcal{E}. \tag{1.2}$$

From the condition (1.1), the Hadamard product is commutative, bilinear, and positivity preserving. When $\mathbb{H}$ is the finite-dimensional space $\mathbb{C}^n$, the Hadamard product for square complex matrices is just a principal submatrix of their Kronecker product, and it can be computed easily as the entrywise product.

In the literature, there are many results concerning Hadamard products for matrices/operators, see e.g. [5, 8, 10]. A well known result is Fiedler's inequality:

Theorem 1.1 ([6]). *For any positive definite matrix A, we have*

$$A \circ A^{-1} \geqslant I.$$

Theorem 1.1 can be extended in the following way:

Theorem 1.2 ([11]). *For each $i = 1, 2, \ldots, n$, let A_i be a positive definite matrix and X_i a positive semidefinite matrix of the same size. Then the map*

$$\alpha \mapsto \sum_{i=1}^{n} X_i^{1/2} A_i^{\alpha} X_i^{1/2} \circ \sum_{i=1}^{n} X_i^{1/2} A_i^{-\alpha} X_i^{1/2}$$

is increasing on $[0, \infty)$.

In this paper, we shall investigate the convexity and the monotonicity of an integral map

$$\alpha \mapsto \int_{\Omega} X_t^* A_t^{\alpha} X_t \, d\mu(t) \circ \int_{\Omega} X_t^* A_t^{-\alpha} X_t \, d\mu(t) \tag{1.3}$$

where α is a real constant. Here, $(A_t)_{t \in \Omega}$ and $(X_t)_{t \in \Omega}$ are two operator-valued maps parametrized by a locally compact Hausdorff space Ω. Some interesting special cases of this map are discussed. Moreover, we obtain certain arithmetic mean - harmonic mean (AM-HM), Jensen, and Fiedler type inequalities as consequences. When we set Ω to be a finite space endowed with the counting measure, our results are reduced to the corresponding discrete inequalities. In particular, these include Theorems 1.1 and 1.2.

The paper is structured as follows. In Section 2, we set up basic notations, and discuss Bochner integrability of continuous field of operators on a locally compact Hausdorff space. The main part of the paper, Section 3, discusses convexity and monotonicity of the map (1.3) and its interesting special cases. As consequences, we obtain certain AM-GM, Jensen, and Fielder type inequalities in the last section.

2 Continuous field of operators on a locally compact Hausdorff space

In this section, we provide fundamental background on continuous fields of operators and their integrability. See, e.g., [1, 3, 12] for more information.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A family $(A_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ is said to be a *continuous field of operators* if the parametrization $t \mapsto A_t$ is norm-continuous on Ω. If, in addition, the norm function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω, then we can form the Bochner integral $\int_\Omega A_t \, d\mu(t)$ which is the unique operator in $\mathfrak{B}(\mathbb{H})$ such that

$$\phi\left(\int_\Omega A_t \, d\mu(t)\right) \;=\; \int_\Omega \phi(A_t)\, d\mu(t)$$

for every bounded linear functional ϕ on $\mathfrak{B}(\mathbb{H})$ (see e.g. [14, pp. 75-78]).

In what follows, suppose further that the measure μ on Ω is finite. Next, we shall prove the Bochner integrability of an operator-valued map involving a continuous field of operators (Proposition 2.3). To do this we need some auxiliary results about functional calculus and vector-valued integration.

Lemma 2.1. *Let Δ be a nonempty compact subset of $\mathbb{C}$ and $f : \Delta \to \mathbb{C}$ a continuous function. Let $\mathcal{A}$ be the subset of $\mathfrak{B}(\mathbb{H})$ consisting of all normal operators whose spectra are contained in Δ. Then the map $\Psi : \mathcal{A} \to \mathfrak{B}(\mathbb{H})$, $A \mapsto f(A)$ is continuous. Here, $f(A)$ is the continuous functional calculus of f on the spectrum of A.*

Proof. See [4, Lemma 2.1]. □

Lemma 2.2. *Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space, and let (Γ, ν) be a finite measure space. Suppose that $f : \Gamma \to \mathbb{X}$ is a measurable function. Then f is Bochner integrable if and only if its norm function $\|f\|$ is Lebesgue integrable, i.e.,*

$$\int_\Gamma \|f\|\, d\nu \;<\; \infty.$$

Here, $\|f\|$ is defined by $\|f\|(x) = \|f(x)\|_{\mathbb{X}}$ for any $x \in \mathbb{X}$.

Proof. See e.g. [1, Theorem 11.44]. □

Now we are in a position to prove the Bochner integrability of a map related to the map (1.3).

Proposition 2.3. *Let $(A_t)_{t\in\Omega}$ be a continuous field of normal operators in $\mathfrak{B}(\mathbb{H})$ such that $\mathrm{sp}(A_t) \subseteq [m, M]$ for all $t \in \Omega$. Let $(X_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. For any continuous function $f : [m, M] \to \mathbb{C}$, we can form the Bochner integral*

$$\int_\Omega X_t^* f(A_t) X_t \, d\mu(t).$$

In addition, if $f([m, M]) \subseteq [0, \infty)$, then this operator is positive.

Proof. Let $K > 0$ be such that $\|X_t\| \leqslant K$ for all $t \in \Omega$. By Lemma 2.2, it suffices to prove the Lebesgue integrability of the norm function $t \mapsto \|X_t^* f(A_t) X_t\|$. We shall show that the map $t \mapsto X_t^* f(A_t) X_t$ is continuous and bounded. Since $t \mapsto A_t$ is continuous, the map $t \mapsto f(A_t)$ is continuous on Ω by Lemma 2.1, and hence so is the map $t \mapsto X_t^* f(A_t) X_t$. For boundedness, we have that for each $t \in \Omega$,

$$\begin{aligned} \|X_t^* f(A_t) X_t\| &\leqslant \|X_t^*\| \cdot \|f(A_t)\| \cdot \|X_t\| \\ &\leqslant \|X_t\|^2 \cdot \|f\|_{\infty,[m,M]} \\ &\leqslant K^2 \|f\|_{\infty,[m,M]}. \end{aligned}$$

Now, suppose that f is positive on $[m, M]$. Then $f(A_t)$ is a positive operator for all $t \in \Omega$. Hence the resulting integral is positive since the integrand is positive. □

Remark 2.4. For convenience to all results in this paper, we may assume that Ω is a compact Hausdorff space. In this case, any Radon measure on Ω is always finite. It follows that every continuous field of operators on Ω is automatically bounded, and hence is Bochner integrable.

Lemma 2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let (Γ, ν) be a measure space. Suppose that a function $f : \Gamma \to \mathbb{X}$ is Bochner integrable. If $T : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator, then the composition $T \circ f$ is Bochner integrable and*

$$\int_\Gamma (T \circ f)\, d\nu \;=\; T\left(\int_\Gamma f\, d\nu\right).$$

Proof. See e.g. [1, Lemma 11.45]. □

The next property will be used to prove the main result of the paper.

Proposition 2.6. *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. For any $X \in \mathfrak{B}(\mathbb{H})$, we have*

$$\int_\Omega A_t\, d\mu(t) \circ X \;=\; \int_\Omega (A_t \circ X)\, d\mu(t). \tag{2.1}$$

Proof. By Lemma 2.2, the map $t \mapsto A_t$ is Bochner integrable on Ω since it is continuous and bounded. Note that the map $T \mapsto T \circ X$ is a bounded linear operator from $\mathfrak{B}(\mathbb{H})$ to itself. It follows from Lemma 2.5 that the map $t \mapsto A_t \circ X$ is Bochner integrable on Ω and the property (2.1) holds. □

3 Convexity and Monotonicity of certain maps for Hadamard products of operators

In this section, we consider convexity and monotonicity of the map

$$\alpha \mapsto \int_\Omega X_t^* A_t^\alpha X_t\, d\mu(t) \circ \int_\Omega X_t^* A_t^{-\alpha} X_t\, d\mu(t)$$

where α is a real constant. We start with an auxiliary result.

4

Lemma 3.1. *For each $A > 0$, the map $\alpha \mapsto A^\alpha + A^{-\alpha}$ is convex on $\mathbb{R}$, increasing on $[0,\infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$.*

Proof. The convexity of the map $F(\alpha) = A^\alpha + A^{-\alpha}$ means that for each $\alpha, \beta \in \mathbb{R}$ and $\omega \in (0,1)$, we have $F((1-\omega)\alpha + \omega\beta) \leqslant (1-\omega)F(\alpha) + \omega F(\beta)$ or equivalently,

$$A^{(1-\omega)\alpha+\omega\beta} + A^{-((1-\omega)\alpha+\omega\beta)} \leqslant (1-\omega)(A^\alpha + A^{-\alpha}) + \omega(A^\beta + A^{-\beta}). \tag{3.1}$$

Indeed, for each fixed $x > 0$, consider the function $f(\alpha) = x^\alpha + x^{-\alpha}$ in a real variable α. Then

$$f''(\alpha) = (\ln x)^2 (x^\alpha + x^{-\alpha}) > 0, \quad \alpha \in \mathbb{R}.$$

It follows that f is convex on $\mathbb{R}$, i.e., for each $\alpha, \beta \in \mathbb{R}$ and $\omega \in (0,1)$ we have

$$x^{(1-\omega)\alpha+\omega\beta} + x^{-((1-\omega)\alpha+\omega\beta)} \leqslant (1-\omega)(x^\alpha + x^{-\alpha}) + \omega(x^\beta + x^{-\beta}). \tag{3.2}$$

Applying the functional calculus on the spectrum of A yields the desired inequality (3.1). Note also that $f'(\alpha) = \alpha(x^{\alpha-1} - x^{-\alpha-1})$ for each $\alpha \in \mathbb{R}$. Hence, f is increasing on $[0,\infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$. Similarly, applying the functional calculus yields the desired results. □

A proof of a part of this fact in matrix context was given in [13], using diagonalization.

Theorem 3.2. *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathfrak{B}(\mathbb{H})$ such that $\mathrm{sp}(A_t) \subseteq [m, M] \subseteq (0,\infty)$ for all $t \in \Omega$. Let $(X_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. Then the map*

$$\alpha \mapsto \int_\Omega X_t^* A_t^\alpha X_t \, d\mu(t) \circ \int_\Omega X_t^* A_t^{-\alpha} X_t \, d\mu(t) \tag{3.3}$$

is convex on $\mathbb{R}$, increasing on $[0,\infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$.

Proof. Denote this map by F. Proposition 2.3 asserts the Bochner integrability of the map $t \mapsto X_t^* A_t^\alpha X_t$ for each $\alpha \in \mathbb{R}$. For each $\alpha \in \mathbb{R}$, we have by Proposition 2.6 and Fubini's theorem for Bochner integrals [2] that

$$\begin{aligned}
F(\alpha) &= \int_\Omega \left(X_t^* A_t^\alpha X_t \circ \int_\Omega X_s^* A_s^{-\alpha} X_s \, d\mu(s) \right) d\mu(t) \\
&= \iint_{\Omega^2} (X_t^* A_t^\alpha X_t \circ X_s^* A_s^{-\alpha} X_s) \, d\mu(s) \, d\mu(t) \qquad (3.4) \\
&= \frac{1}{2} \iint_{\Omega^2} (X_t^* A_t^\alpha X_t \circ X_s^* A_s^{-\alpha} X_s) + (X_t^* A_t^{-\alpha} X_t \circ X_s^* A_s^\alpha X_s) \, d\mu(s) \, d\mu(t).
\end{aligned}$$

Then, appealing the isometry Z defined by (1.2), we have

$$\begin{aligned} F(\alpha) &= \frac{1}{2}\iint_{\Omega^2} Z^*\left[(X_t^* A_t^{\alpha} X_t \otimes X_s^* A_s^{-\alpha} X_s) + (X_t^* A_t^{-\alpha} X_t \otimes X_s^* A_s^{\alpha} X_s)\right] Z \\ &\quad d\mu(s)\, d\mu(t) \\ &= \frac{1}{2}\iint_{\Omega^2} Z^*(X_t \otimes X_s)^*\left[(A_t \otimes A_s^{-1})^{\alpha} + (A_t \otimes A_s^{-1})^{-\alpha}\right](X_t \otimes X_s) Z \\ &\quad d\mu(s)\, d\mu(t). \end{aligned} \tag{3.5}$$

Now, for each $\alpha, \beta \in \mathbb{R}$ and $\omega \in (0,1)$, we have from Lemma 3.1 and (3.5) that

$$\begin{aligned} &F((1-\omega)\alpha + \omega\beta) \\ &\leqslant \frac{1}{2}\iint_{\Omega^2} Z^*(X_t \otimes X_s)^*\big[(1-\omega)\{(A_t \otimes A_s^{-1})^{\alpha} + (A_t \otimes A_s^{-1})^{-\alpha}\} \\ &\qquad + \omega\{(A_t \otimes A_s^{-1})^{\beta} + (A_t \otimes A_s^{-1})^{-\beta}\}\big](X_t \otimes X_s) Z d\mu(s)\, d\mu(t) \\ &= \frac{1}{2}\iint_{\Omega^2} Z^*\big[(1-\omega)(X_t^* A_t^{\alpha} X_t \otimes X_s^* A_s^{-\alpha} X_s + X_t^* A_t^{-\alpha} X_t \otimes X_s^* A_s^{\alpha} X_s) \\ &\qquad + \omega(X_t^* A_t^{\beta} X_t \otimes X_s^* A_s^{-\beta} X_s + X_t^* A_t^{-\beta} X_t \otimes X_s^* A_s^{\beta} X_s)\big] Z d\mu(s)\, d\mu(t) \\ &= \frac{1-\omega}{2}\iint_{\Omega^2} (X_t^* A_t^{\alpha} X_t \circ X_s^* A_s^{-\alpha} X_s + X_t^* A_t^{-\alpha} X_t \circ X_s^* A_s^{\alpha} X_s) d\mu(s)\, d\mu(t) \\ &\qquad + \frac{\omega}{2}\iint_{\Omega^2} (X_t^* A_t^{\beta} X_t \circ X_s^* A_s^{-\beta} X_s + X_t^* A_t^{-\beta} X_t \circ X_s^* A_s^{\beta} X_s) d\mu(s)\, d\mu(t) \\ &= (1-\omega)F(\alpha) + \omega F(\beta). \end{aligned}$$

Therefore, F is convex. In the rest, it suffices to show that F is increasing on $[0,\infty)$ since the Hadamard product is commutative. It follows from (3.5) and Lemma 3.1 that for $0 \leqslant \alpha \leqslant \beta$,

$$\begin{aligned} F(\alpha) &\leqslant \frac{1}{2}\iint_{\Omega^2} Z^*(X_t \otimes X_s)^*\left[(A_t \otimes A_s^{-1})^{\beta} + (A_t \otimes A_s^{-1})^{-\beta}\right](X_t \otimes X_s) Z\, d\mu(s)\, d\mu(t) \\ &= \frac{1}{2}\iint_{\Omega^2} Z^*\left[(X_t^* A_t^{\beta} X_t \otimes X_s^* A_s^{-\beta} X_s) + (X_t^* A_t^{-\beta} X_t \otimes X_s^* A_s^{\beta} X_s)\right] Z\, d\mu(s)\, d\mu(t) \\ &= \iint_{\Omega^2} (X_t^* A_t^{\beta} X_t \circ X_s^* A_s^{-\beta} X_s)\, d\mu(s)\, d\mu(t). \end{aligned}$$

From (3.4), the right-hand side is equal to $F(\beta)$. Thus, F is increasing on $[0,\infty)$. □

In the rest of section, we discuss certain special cases of Theorem 3.2.

Corollary 3.3. *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be two bounded continuous field of positive operators in $\mathfrak{B}(\mathbb{H})$ such that $\mathrm{sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ and $A_tB_t = B_tA_t$ for all $t \in \Omega$. Then the map*

$$\alpha \mapsto \int_{\Omega} A_t^{\alpha} B_t\, d\mu(t) \circ \int_{\Omega} A_t^{-\alpha} B_t\, d\mu(t) \tag{3.6}$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha = 0$*.*

Proof. Set $X_t = B_t^{1/2}$ for each $t \in \Omega$. Then $(X_t)_{t\in\Omega}$ is a continuous field by Lemma 2.1. The family $(X_t)_{t\in\Omega}$ is bounded due to the boundedness of $(B_t)_{t\in\Omega}$. The result now follows from Theorem 3.2. □

An interesting special case of Corollary 3.3 is when $B_t = f(A_t)$ where f is a complex-valued continuous function on $[m, M]$. In this case, the field $(B_t)_{t\in\Omega}$ is bounded since

$$\|f(A_t)\| \leqslant \|f\|_{\infty,[m,M]}$$

for all $t \in \Omega$. Hence we obtain monotonicity information of the map

$$\alpha \mapsto \int_\Omega A_t^\alpha f(A_t)\, d\mu(t) \circ \int_\Omega A_t^{-\alpha} f(A_t)\, d\mu(t).$$

In particular, when $f(x) = x^\lambda$, we get the following result.

Corollary 3.4. *For any* $\lambda \in \mathbb{R}$*, the map*

$$\alpha \mapsto \int_\Omega A_t^{\lambda+\alpha}\, d\mu(t) \circ \int_\Omega A_t^{\lambda-\alpha}\, d\mu(t)$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha = 0$*.*

The next result is also a special case of Theorem 3.2 in which the weights are scalars.

Corollary 3.5. *Let* $(A_t)_{t\in\Omega}$ *be a continuous field of positive operators in* $\mathfrak{B}(\mathbb{H})$ *such that* $\mathrm{sp}(A_t) \subseteq [m, M] \subseteq (0,\infty)$ *for all* $t \in \Omega$*. For any bounded continuous function* $w : \Omega \to [0,\infty)$*, the map*

$$\alpha \mapsto \int_\Omega w(t) A_t^\alpha\, d\mu(t) \circ \int_\Omega w(t) A_t^{-\alpha}\, d\mu(t) \tag{3.7}$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha = 0$*.*

Proof. Set $X_t = \sqrt{w(t)}I$ for all $t \in \Omega$ in Theorem 3.2. We see that $(X_t)_{t\in\Omega}$ is a bounded continuous field of operators. □

Corollary 3.6. *Let* $f : \Omega \to \mathbb{C}$ *and* $g : \Omega \to [0,\infty)$ *be bounded continuous functions such that* $\mathrm{Range}(f) \subseteq [m, M] \subseteq (0,\infty)$*. Then the map*

$$\alpha \mapsto \int_\Omega g f^\alpha\, d\mu \int_\Omega g f^{-\alpha}\, d\mu$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha = 0$*.*

Proof. Set $\mathbb{H} = \mathbb{C}$ in Corollary 3.3. $\square$

A discrete version of Theorem 3.2 is obtained in the next corollary, which is an operator extension of Theorem 1.2.

Corollary 3.7. *For each $i = 1, 2, \ldots, n$, let $A_i, X_i \in \mathfrak{B}(\mathbb{H})$ be such that A_i is positive and invertible. Then the map*

$$\alpha \mapsto \sum_{i=1}^{n} X_i^* A_i^{\alpha} X_i \circ \sum_{i=1}^{n} X_i^* A_i^{-\alpha} X_i$$

is convex on $\mathbb{R}$, increasing on $[0, \infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$.

Proof. In Theorem 3.2, set Ω to be the finite space $\{1, 2, \ldots, n\}$ equipped with the counting measure. $\square$

4 AM-GM, Jensen, and Fiedler type inequalities

From the main result (Theorem 3.2), we get three interesting inequalities. The first consequence is an integral version of the weighted arithmetic-harmonic mean inequality for bounded continuous function defined on a locally compact Hausdorff space:

Corollary 4.1. *Let f be a bounded continuous function defined on Ω such that* $\operatorname{Range}(f) \subseteq [m, M] \subseteq (0, \infty)$. *Let $w : \Omega \to [0, \infty)$ be a weight function, i.e., $\int_\Omega w \, d\mu = 1$. We obtain the following bound for the weight integral of f:*

$$\|wf\|_1 \geqslant \frac{1}{\|w/f\|_1}. \tag{4.1}$$

Here, $\|\cdot\|_1$ denotes the L^1-norm on Ω.

Proof. Setting $\mathbb{H} = \mathbb{C}$ in Corollary 3.5 yields that the function

$$\alpha \mapsto \int_\Omega w f^{\alpha} \, d\mu \int_\Omega \frac{w}{f^{\alpha}} \, d\mu$$

is increasing on $[0, \infty)$. In particular, this implies that

$$\int_\Omega wf \, d\mu \int_\Omega \frac{w}{f} \, d\mu \geqslant \left(\int_\Omega w \, d\mu \right)^2 = 1.$$

Now, since $\int \frac{w}{f} \, d\mu \geqslant \int w M^{-1} \, d\mu = M^{-1} > 0$, the inequality (4.1) follows. $\square$

The second consequence is a Jensen type inequality for a continuous field of strictly positive operators.

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Corollary 4.2. *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathfrak{B}(\mathbb{H})$ such that $\operatorname{sp}(A_t) \subseteq [m, M] \subseteq (0, \infty)$ for all $t \in \Omega$. Suppose that $\mu(\Omega) = 1$. Then*

$$\int_\Omega A_t^2 \, d\mu(t) \circ I \;\geqslant\; \int_\Omega A_t \, d\mu(t) \circ \int_\Omega A_t \, d\mu(t).$$

Proof. Form Corollary 3.4, the map

$$G(\alpha) \;\equiv\; \int_\Omega {A_t}^{1+\alpha} \, d\mu(t) \circ \int_\Omega {A_t}^{1-\alpha} \, d\mu(t)$$

is increasing on $[0, \infty)$. In particular, we have $G(1) \geqslant G(0)$, which is the desired inequality. □

Corollary 4.2 may be regarded as a Jensen type inequality for continuous field of operators (cf. [9]). Indeed, the case $\mathbb{H} = \mathbb{C}$ of this corollary can be described as follows. Suppose that (Ω, μ) is a probability space. For any continuous function $f : \Omega \to (0, \infty)$, we have

$$\int_\Omega f^2 \, d\mu \;\geqslant\; \left(\int_\Omega f \, d\mu \right)^2,$$

which is Jensen's inequality for the convex function $\phi(x) = x^2$.

The final result is an operator extension of Fiedler's inequality (Theorem 1.1).

Corollary 4.3. *For each invertible positive operator A, we have*

$$A \circ A^{-1} \;\geqslant\; I. \tag{4.2}$$

Proof. The case $n = 1$ in Corollary 3.7 says that the map $\alpha \mapsto A^\alpha \circ A^{-\alpha}$ has a minimum at $\alpha = 0$. It follows that $A^\alpha \circ A^{-\alpha} \geqslant I$ for any $\alpha \in \mathbb{R}$. Replacing A with $A^{1/\alpha}$ yields the inequality (4.2). □

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Fibonacci periodicity and Fibonacci frequency

Hee Sik Kim[1], **J. Neggers**[2] **and Keum Sook So**[3,*]

[1]**Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul, 04763, Korea**

[2]**Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A**

[3,*]**Department of Statistics and Financial Informatics, Hallym University, Chuncheon, 24252, Korea**

Abstract. In this paper we introduce the notion of Fibonacci periodicity modulo n, denoting this period by the function $\widehat{F}(n)$. We note that $\widehat{F}(n)$ is an integral multiple of a fundamental frequency $\widehat{f}(n)$, where the ratio $\widehat{F}(n)/\widehat{f}(n)$ is a power of 2 for a collection of observed values of n. It is demonstrated that if a, b are natural numbers with $gcd(a, b) = 1$, then $\widehat{F}(n) = lcm\{\widehat{F}(a), \widehat{F}(b)\}$ and thus that $\widehat{F}$ is a non-trivial example of a function which we refer to as radical. From observations it also seems clear that $\frac{\widehat{F}(p^{s+1})}{\widehat{F}(p^s)} = p$ for primes p.

1. Introduction and Preliminaries

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. One of the amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio (section) rather quickly as they go to infinity probably has a good deal to do with the observation made in the previous sentence. Surveys and connections of the type just mentioned are provided in [1] and [2] for a very minimal set of examples of such texts, while in [3] an application (observation) concerns itself with a theory of a particular class of means which has apparently not been studied in the fashion done there by two of the authors the present paper. Surprisingly novel perspectives are still available.

Kim and Neggers [6] showed that there is a mapping $D : M \to DM$ on means such that if M is a Fibonacci mean so is DM, that if M is the harmonic mean, then DM is the arithmetic mean, and if M is a Fibonacci mean, then $\lim_{n\to\infty} D^n M$ is the golden section mean.

In [5] Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x+2) = f(x+1)+f(x)$, and developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. Moreover, they showed that if f is a Fibonacci function then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. The present authors [8] discussed Fibonacci functions using the (ultimately) periodicity and we also discuss the exponential Fibonacci functions. Especially, given a non-negative real-valued function, we obtain several exponential Fibonacci functions.

The present authors [9] introduced the notions of Fibonacci (co-)derivative of real-valued functions, and found general solutions of the equations $\triangle(f(x)) = g(x)$ and $(\triangle + I)(f(x)) = g(x)$. Moreover, they [10] defined and studied a function $F : [0, \infty) \to \mathbf{R}$ and extensions $F : \mathbf{R} \to \mathbf{C}$, $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ which are continuous and such that if $n \in \mathbf{Z}$, the set of all integers, then $F(n) = F_n$, the n^{th} Fibonacci number based on $F_0 = F_1 = 1$. If x is not an integer and $x < 0$, then $F(x)$ may be a complex number, e.g., $F(-1.5) = \frac{1}{2} + i$. If $z = a + bi$,

[0*] Correspondence: Tel.: +82 33 248 2011, Fax: +82 33 256 2011 (K. S. So).

then $\widetilde{F}(z) = F(a) + iF(b-1)$ defines complex Fibonacci numbers. In connection with this function (and in general) they defined a Fibonacci derivative of $f : \mathbf{R} \to \mathbf{R}$ as $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$ so that if $(\triangle f)(x) \equiv 0$ for all $x \in \mathbf{R}$, then f is a (real) Fibonacci function. A complex Fibonacci derivative $\widetilde{\triangle}$ is given as $\widetilde{\triangle} f(a+bi) = \triangle f(a) + i \triangle f(b-1)$ and its properties are discussed in same detail.

The notion of Fibonacci means was introduced by

$$M(x,y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$$

where $M(x,x) = \dfrac{2ax+2bx^2}{2a+2bx} = x$ provided $2a+2bx \neq 0$ ([6]).

Particular cases are $a > 0, b = 0, M(x,y) = \frac{x+y}{2}$, the average (arithmetic mean), $a = 0, b > 0, M(x,y) = \frac{2xy}{x+y}$, the harmonic mean, and if $q = \frac{1+\sqrt{5}}{2}$, $M_q(x,y) = \frac{q(x+y)+2xy}{2q+(x+y)}$, the golden section mean. Hence both the harmonic mean, the arithmetic mean and golden section mean are special cases of the Fibonacci mean.

The golden section mean $M_q(x,y)$ is defined by $M_q(x,y) = \frac{q(x+y)+2xy}{2q+(x+y)}$ where $q = \frac{1+\sqrt{5}}{2}$, and we define $M_{q^*}(x,y)$ by $\frac{q^*(x+y)+2xy}{2q^*+(x+y)}$ where $q^* = \frac{1-\sqrt{5}}{2}$, which is called a *conjugate golden section mean.*

It was shown that: if $M(x,y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$ is a Fibonacci mean and if $M(x,y) = DM(x,y)$, then either $M(x,y) = M_q(x,y)$ or $M(x,y) = M_{q^*}(x,y)$.

2. Fibonacci frequency

Given a positive integer $n \geq 2$, let $\widehat{F}(n) = m$ provided $F_k \equiv F_{k+m} \pmod{n}$ for all positive integers k, where m is the smallest positive integer with this property and F_k is the k^{th} Fibonacci number relative to arbitrary inputs $F_1 = a, F_2 = b$, non-negative integers.

For example, for $n = 2$ we have with inputs $a = 1, b = 1$:

$$1,\ 1,\ 0,\ 1, 1, 0, \cdots$$

whence $\widehat{F}(2) \geq 3$. Also, $a, b, a+b, a, b, a+b, \cdots$ shows that $\widehat{F}(2) \leq 3$. Thus, combining these observations we establish that:

Proposition 2.1. $\widehat{F}(2) = 3$.

For $n = 3$, $a = 1, b = 1$ yields a lower bound computation is:

$$1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, \cdots$$

and $\widehat{F}(3) \geq 8$. Also, $a, b, a+b, a+2b, 2a, 2b, 2a+2b, 2a+b, a, b, \cdots$ and $\widehat{F}(3) \leq 8$. Hence, it follows that:

Proposition 2.2. $\widehat{F}(3) = 8$.

When we consider the method of proof of the above two propositions, we note that the pattern $0, 1, 1, 2, \cdots$ corresponds to a pattern $\alpha a + \beta b, a, b, a+b, \cdots$ in the second sequence where $\alpha + \beta \equiv 0 \pmod{n}$, and that if $\cdots, s, \cdots$ is any term in the $a = 1, b = 1$ sequence, then $\cdots, s, \cdots$ corresponds to $\cdots, \lambda a + \mu b, \cdots$ where $\lambda + \mu \equiv s \pmod{n}$. Hence if $s = 1$, then we have $\lambda \equiv 1 \pmod{n}, \mu \equiv 0 \pmod{n}$ or $\lambda \equiv 0 \pmod{n}, \mu \equiv 1 \pmod{n}$, i.e., in the first case we find the input a, whereas in the second case we find the input b. Notice that $\alpha a + \beta b, a, b, a+b, \cdots$ with $\alpha + \beta \equiv 0 \pmod{n}$ means $(\alpha+1)a + \beta b \equiv b \pmod{n}$, $\beta \equiv -\alpha \pmod{n}$, $(\alpha+1)(a-b) \equiv 0$

(mod n), and $a-b$ arbitrary means $\alpha+1 \equiv 0 \pmod{n}$, i.e., $\alpha \equiv n-1 \pmod{n}$, $\beta \equiv 1 \pmod{n}$. Hence, the sequence looks like $(n-1)a+b, a, b, \cdots$. If we continue the construction by including one more term, then $\lambda a+\mu b, (n-1)a+b, a, b, \cdots$ yields $(\lambda+(n-1))a+(\mu+1)b \equiv a \pmod{n}$ and $(\lambda+n-2)a+(\mu+1)b \equiv 0 \pmod{n}$. Hence $a=0$ yields $\mu \equiv n-1 \pmod{n}$ and $\lambda+n-2 \equiv \lambda-2 \equiv 0 \pmod{n}$, i.e., $\lambda \equiv 2 \pmod{n}$, i.e., the sequence is $\cdots, 2a+(n-1)b, (n-1)a+b, a, b, \cdots$. Letting $a=b=1$, the corresponding pattern is $\cdots, 2+(n-1) \equiv 1, (n-1)+1 \equiv 0, 1, 1, \cdots$. Thus, if this occurs at the m^{th} location, then $\widehat{F}(n) \geq m$ from $a=1, b=1$ and $\widehat{F}(n) \leq m$ from a, b unspecified, whence $\widehat{F}(n)=m$. We thus obtain:

Theorem 2.3. *To determine $\widehat{F}(n)$ it suffices to take $a=1, b=1$, and construct the Fibonacci sequence modulo n until the pattern $\cdots, 1, 0$ is obtained. If the sequence has m terms, then $\widehat{F}(n)=m$.*

Suppose for example that we wish to determine $\widehat{F}(4)$. Using Theorem 2.3 we let $a=1, b=1$, whence the sequence is

$$1, 1, 2, 3, 1, 0,$$

and $\widehat{F}(4)=6$. Note that $\widehat{F}(4)/\widehat{F}(2)=6/3=2$.

As another example consider the computation of $\widehat{F}(9)$. Again, using Theorem 3.3 and $a=1, b=1$, we obtain the following sequence:

$$1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 0$$

and $\widehat{F}(9)=24$. Note that $\widehat{F}(9)/\widehat{F}(3)=24/8=3$.

In this case we also note that the first 0 shows up after 12 steps. Accordingly we take $\widehat{f}(9)=12$ and $\widehat{F}(9)=\widehat{f}(9)\widehat{m}(9), \widehat{F}(9)=2$. We consider $\widehat{f}(9)$ to be the *fundamental frequency* and $\widehat{m}(9)$ to be the *multiplicity*, with $\widehat{F}(9)$ the *Fibonacci frequency* of the integer 9 (≥ 2).

Fibonacci numbers have been studied in great detail over many years and the literature on the subject is quite substantial with entire books on the subject dedicated to their study and the study of these numbers also meriting chapters in books on number theory ([1, 2]). Recently, two of the authors of this paper were able to make a different but not entirely surprising connection between Fibonacci numbers and the Golden Section than the usual one ([3]).

If $a=b=1$, then it is well-known that $F_m \mid F_n$ if and only if $m \mid n$ for example. Another known result is the following: For any prime p, there are infinitely many Fibonacci numbers which are divisible by p and these are equally spaced in the Fibonacci sequence. The case $\widehat{f}(3)=4$ is an instance of this observation. Our point of view allows us to consider a more general situation and obtain some results on relationships connecting various values of $\widehat{F}(n)$ and to make some conjectures on these relationships which appear to be interesting.

3. Radical functions

In the number-theoretical setting, a function f on the natural numbers is multiplicative if $\gcd(a, b)=1$ implies $f(ab)=f(a)f(b)$. Certainly any function for which $f(xy)=f(x)f(y)$ is multiplicative. The Euler-phi-function is multiplicative in the number-theoretical sense without being multiplicative in the strict sense.

Given a natural number $m = p_1^{r_1} \cdots p_n^{r_n}$, where the p_i's are distinct primes in the factorization of m, we let $rad(m) = p_1 \cdots p_n$, according to conventional ring-theoretical practice. Thus, for natural numbers m_1, m_2, we find that $rad(m_1 m_2) = lcm\{rad(m_1), rad(m_2)\}$. This function is an example of functions on the natural numbers satisfying the following "multiplicative" condition: a function f on the natural numbers is a *radical function* if $\gcd(a, b) = 1$ implies $f(ab) = lcm\{f(a), f(b)\}$.

When we check the table in the previous section, we observe that in the available examples it is true that $\gcd(a, b) = 1$ implies $\widehat{F}(ab) = lcm\{\widehat{F}(a), \widehat{F}(b)\}$. For example, $\widehat{F}(4) = 6, \widehat{F}(5) = 20, \widehat{F}(20) = 60, \gcd(4, 5) = 1$ and $\widehat{F}(20) = lcm\{6, 20\} \neq 120$, i.e., $\widehat{F}$ is not a multiplicative function in the number-theoretical sense. Thus, it is our goal in this section to prove that $\widehat{F}$ is a radical function in the sense described above.

Lemma 3.1. *If $d|n$, then $\widehat{F}(d) \leq \widehat{F}(n)$.*

Proof. Since $d|n$, there exists an integer q such that $n = dq$. If we let $\widehat{F}(n) = m$, then $F_k \equiv F_{k+m} \pmod n$ for any integer k, so that $F_{k+m} - F_k = nu = dqu$ for some $u \in Z$, i.e., $d|F_{k+m} - F_k$. This means that $\widehat{F}(d) \leq m = \widehat{F}(n)$. □

Lemma 3.2. *If $d|n$, then $\widehat{F}(d)|\widehat{F}(n)$.*

Proof. Using Division Algorithm, we have $\widehat{F}(n) = \widehat{F}(d)q + r$ for some $q, r \in Z$, where $0 \leq r < \widehat{F}(d)$. Let $\widehat{F}(n) := m$ and $\widehat{F}(d) := t$. Then $m = qt + r$ and hence $F_k \equiv F_{k+m} \pmod n$, so that $n|F_{k+m} - F_k$. Since $d|n$, we have $d|\, F_{k+m} - F_k$. We claim that $F_{k+r} \equiv F_{k+qt+r} \pmod d$. Since $\widehat{F}(d) := t$,

$$F_k \equiv F_{k+t} \pmod d \tag{1}$$

for any natural number $k \in N$. If we take $k := k + t$ in (1), then $F_{k+t} \equiv F_{(k+t)+t} \equiv F_{k+2t} \pmod d$. Similarly, we obtain

$$F_k \equiv F_{k+(q-1)t} \pmod d \tag{2}$$

for any natural number $k \in N$ and natural number $q > 1$. If we replace k by $k+r$ in (2), then $F_{k+r} \equiv F_{k+r+(q-1)t} \equiv F_{k+r+qt} \pmod d$. Hence $F_k \equiv F_{k+m} \equiv F_{k+qt+r} \equiv F_{k+r} \pmod d$. Since $\widehat{F}(d) = t$ is the smallest positive integer with this property and $0 \leq t < t$, we have $r = 0$, i.e., $m = qt$, proving the assertion. □

Theorem 3.3. *If $n = ab$, where a, b are natural numbers with $gcd(a, b) = 1$, then $\widehat{F}(n) = lcm\{\widehat{F}(a), \widehat{F}(b)\}$.*

Proof. Suppose that $n = ab$, where a, b are natural numbers with $gcd(a, b) = 1$. Then $\widehat{F}(a)|\widehat{F}(n)$, $\widehat{F}(b)|\widehat{F}(n)$ by Lemma 4. Hence $lcm\{\widehat{F}(a), \widehat{F}(b)\} \leq \widehat{F}(n)$. If we let $m := lcm\{\widehat{F}(a), \widehat{F}(b)\}$, then there exists natural numbers r, s such that $m = r\widehat{F}(a) = s\widehat{F}(b)$ where $gcd(r, s) = 1$. Let $\alpha := \frac{m}{r}, \beta := \frac{m}{s}$. Then $F_k \equiv F_{k+\alpha} \pmod a$, $F_k \equiv F_{k+\beta} \pmod b$ for any positive integer k. Hence $F_k \equiv F_{k+r\alpha} \equiv F_{k+r\frac{m}{r}} \equiv F_{k+m} \pmod a$. Similarly, $F_k \equiv F_{k+s\beta} \equiv F_{k+s\frac{m}{s}} \equiv F_{k+m} \pmod b$ for any positive integer k. This means that $a|F_k - F_{k+m}, b|F_k - F_{k+m}$. Since $gcd(a, b) = 1$, it follows that $F_k \equiv F_{k+m} \pmod{ab}$, i.e., $F_{k+m} \equiv F_k \pmod n$, so that $\widehat{F}(n) \leq m$ by the minimality property of $\widehat{F}$, proving the theorem. □

Corollary 3.4. *Let a, b, c are natural numbers which are relatively prime. Then $\widehat{F}(abc) = lcm\{\widehat{F}(a), \widehat{F}(b), \widehat{F}(c)\}$.*

Corollary 3.5. *Let a, b are natural numbers which are relatively prime. Then*

$$gcd\{\widehat{F}(a), \widehat{F}(b)\} = \frac{\widehat{F}(a)\widehat{F}(b)}{\widehat{F}(ab)}$$

Example 3.6. $\widehat{F}(1147) = \widehat{F}(1517) = 760$ using the table above along with Theorem 3.3. Indeed, $1147 = 31 \cdot 37$ and $1517 = 41 \cdot 37$, $\widehat{F}(31) = \widehat{F}(41) = 40$, $\widehat{F}(37) = 76$ and $lcm\{40, 76\} = 760$. It is of course true that F_{760} is not a small integer.

4. Powers of primes

From the table given above, it is not immediately clear that there is any pattern to the values of $\widehat{F}(p)$, where p is a prime. However, in all cases we have seen, the following properties holds:

Conjecture 4.1. For any prime p,

$$\frac{\widehat{F}(p^{s+1})}{\widehat{F}(p^s)} = p$$

Thus, for example $\widehat{F}(27)/\widehat{F}(9) = 72/24 = 3$ and $\widehat{F}(25)/\widehat{F}(5) = 5$. Accepting Conjecture 4.1 as true, we note that if $\eta(n) = \widehat{F}(n)/n$, then $\eta(p^{s+1}) = \widehat{F}(p^{s+1})/p^{s+1} = p\widehat{F}(p^s)/p^{s+1} = \widehat{F}(p^s)/p^s = \eta(p^s) = \cdots = \eta(p)$. For example, $\eta(27) = \eta(3) = \widehat{F}(3)/3 = 8/3 = 72/27$. If $n = p^{r+1}q^{s+1}$, then

$$\begin{aligned}\widehat{F}(n) &= lcm\{\widehat{F}(p^{r+1}, \widehat{F}(q^{s+1}\} \\ &= lcm\{p^r\widehat{F}(p), q^s\widehat{F}(q)\} \\ &= \frac{p^r q^s \widehat{F}(p)\widehat{F}(q)}{\gcd\{p^r\widehat{F}(p), q^s\widehat{F}(q)\}} \\ &= \frac{n\widehat{F}(p)\widehat{F}(q)}{\gcd\{\widehat{F}(p^{r+1}), \widehat{F}(q^{s+1})\}pq}\end{aligned}$$

and thus

$$\eta(n) = \frac{\widehat{F}(p)\widehat{F}(q)}{pq\gcd\{\widehat{F}(p^{r+1}), \widehat{F}(q^{s+1})\}}$$

Now, $pq = rad(n)$. Continuing in the same fashion, if $m = p_1^{r_1} \cdots p_n^{r_n}$, then we find that

$$\eta(m) = \frac{\widehat{F}(p_1)\cdots\widehat{F}(p_n)}{rad(m)\gcd\{\widehat{F}(p_1^{r_1}), \cdots, \widehat{F}(p_n^{r_n})\}}$$

Global properties of the function $\widehat{F}(n)$ may then be gathered in the function:

$$Z_\eta(s) = \sum_{n=2}^{\infty} \frac{\widehat{F}(n)}{n^s},$$

so that $Z_\eta(1) = \sum_{n=2}^{\infty} \frac{\widehat{F}(n)}{n} = \sum_{n=2}^{\infty} \eta(n)$, which does not appear to converge for $s = 1$, but may well converge for complex variables with $Re(s)$ sufficiently large. Other generating functions may also be constructed such as $\sum_{n=2}^{\infty} \widehat{F}(n)z^n$, $\sum_{n=2}^{\infty} \eta(n)z^n$, $\sum_{n=2}^{\infty} \widehat{F}(n)\frac{z^n}{n!}$, etc..

Given $\widehat{F}(n)$ for $n \geq 2$, define $\widehat{F}(1) = 1$ and for positive integers a, b with $\gcd(a, b) = 1$, let $\widehat{F}(\frac{a}{b})$ satisfying the following equation:

$$\widehat{F}(\frac{a}{b}) = \frac{lcm\{\widehat{F}(a), \widehat{F}(b)\}}{lcm\{\widehat{F}(b), \widehat{F}(b^2)\}}$$

Thus, if $b = 1$, then $\widehat{F}(\frac{a}{1}) = \widehat{F}(a)/1 = \widehat{F}(a)$. Also, $\widehat{F}(\frac{1}{b}) = \widehat{F}(b)/lcm\{\widehat{F}(b), \widehat{F}(b^2)\}$. Thus, if $b = p$ is a prime and if Conjecture 4.1 is accepted, then $\widehat{F}(\frac{1}{p}) = \widehat{F}(p)/p\widehat{F}(p) = 1/p$. The meaning or interpretation of the values of $\widehat{F}$ on fractions is not quite clear. It does however demonstrate that the function $\widehat{F}$ defined on integers $n \geq 2$ has extensions to the positive rationals, the one described here being one of them. Since $mn = (-m)(-n)$, it makes sense to define $\widehat{F}(q) = \widehat{F}(-q)$ for rationals $q > 0$. Also, since we expect $\widehat{F}(\frac{a}{b})$ to be "near zero" if $\frac{a}{b}$ is "near zero", $\widehat{F}(0) = 0$ appears to be a sensible decision also.

For irrational values α, the definition of $\widehat{F}(\alpha)$ could be as follows: if we define $S(n, \alpha) := \sup\{F(q) \mid q \in Q \cap [\alpha - \frac{1}{n}, \alpha + \frac{1}{n}]\}$, then $0 \leq S(n+1, \alpha) \leq S(n, \alpha)$ and hence $\lim_{n\to\infty} S(n, \alpha) = \inf_{n\in\omega} S(n, \alpha)$. Since $\cap_{n\in\omega} S(n, \alpha) = \{\alpha\}$, it follows that this permits us to define $\widehat{F}(\alpha)$ for α an irrational number. If $\widehat{F}(\alpha) = \infty$, then $S(n, \alpha) = \infty$ for all integers n. Thus, if this is the case, there is a sequence of rational numbers $\{q_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} q_i = \alpha$ and at the same time $\lim_{i\to\infty} \widehat{F}(q_i) = \alpha$ and at the same time $\lim_{i\to\infty} \widehat{F}(q_i) = \infty$. We conjecture the following:

Conjecture 4.2. *There is no sequence* $\{q_i\}_{i=1}^{\infty}$ *of rational numbers such that* $\lim_{i\to\infty} q_i = \alpha$ *and* $\lim_{i\to\infty} \widehat{F}(q_i) = \infty$.

Given that the conjecture holds, $\widehat{F}(\alpha)$ is defined for irrational values of α as well, i.e., the domain of $\widehat{F}$ is the real numbers.

5. Comments

In this paper we have considered several aspects of the sequence of Fibonacci numbers with inputs a, b arbitrary related to the periodicity of such a sequence modulo n. Because of the plenitude of relations known to exist among various Fibonacci numbers it was not surprising that patterns would be observed. We were pleased to discover that there were numerous relationships to be found, even if not all of them are explainable. The most mysterious values are those for $\widehat{F}(p)$ where p is an arbitrary prime. Thus, $\widehat{F}(29) = 14, \widehat{F}(31) = 40$, which insists on announcing that from the "Fibonacci point of view" there is a "big difference' between these two primes in the twin-prime couple. Also, given an integer n, then fact that $\widehat{F}(n^2)/\widehat{F}(n) \neq n$, suffices to identify it as a composite number without knowing anything about any factorization of n. Thus, e.g., $\widehat{F}(36)/\widehat{F}(6) = 24/24 = 1$. Since Fibonacci numbers grow rather quickly, this observation may prove useful in the exercise of primality testing. Also, if $\widehat{F}(n^2)/\widehat{F}(n) = n$, then, although this does not guarantee (maybe) that n is a prime, it seems that it ought to greatly improve the probability that it is.

6. Appendix

n	$\widehat{F}(n)$	$\widehat{f}(n)$	n	$\widehat{F}(n)$	$\widehat{f}(n)$	n	$\widehat{F}(n)$	$\widehat{f}(n)$	n	$\widehat{F}(n)$	$\widehat{f}(n)$
2	3	3	3	8	4	4	6	6	5	20	5
6	24	12	7	16	8	8	12	6	9	24	12
10	60	15	11	10	10	12	24	12	13	28	7
14	48	24	15	40	20	16	24	12	17	36	9
18	24	12	19	18	18	20	60	30	21	16	8
22	30	30	23	48	24	24	24	12	25	100	25
26	84	21	27	72	36	28	48	24	29	14	14
30	120	60	31	30	30	32	48	24	33	40	20
34	36	9	35	80	40	36	24	12	37	76	19
38	18	18	39	56	28	40	60	30	41	40	20
42	48	24	43	88	44	44	30	30	45	120	60
46	48	24	47	32	16	48	24	12	49	112	56
50	300	75	51	72	36	52	84	42	53	108	27
54	72	36	55	20	10	56	48	24	57	72	36
58	42	42	59	58	58	60	120	60	61	60	15
62	30	30	63	48	24	64	96	48	65	140	35
66	120	60	67	136	68	68	36	18	69	48	24
70	240	120	71	70	70	72	24	12	73	148	37
74	228	57	75	200	100	76	18	18	77	80	40
78	168	84	79	78	78	80	120	60	81	216	108
82	120	60	83	168	84	84	48	24	85	180	45
86	264	132	87	56	28	88	60	30	89	44	11
90	120	60	91	112	56	92	48	24	93	120	60
94	96	48	95	180	90	96	48	24	97	196	49
98	336	168	99	120	60	100	300	150			

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The weighted moving averages for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules†

Zeng-Tai Gong[a,*], Wen-Jing Lei[a,b]

[a] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China*

[b] *School of Economics and Management, Tongji University, Shanghai, 200092, China*

Abstract Non-additive measure theory is an important mathematical tool to deal with inter-dependent or interactive information. The concept of fuzzy number provides an effective means of describing vague and uncertain system. The aim of this study is to integrate moving average with non-additive measures with $\sigma - \lambda$ rules under fuzzy environment. That is, the moving average for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules is proposed. Further, its specific calculation is invested and some properties are discussed. In particular, triangular fuzzy numbers about this method are also discussed. Finally, an example is given to illustrate our results.

Keywords: Fuzzy number; Fuzzy measure; Moving average.

1. Introduction

Non-additive measure theory, as an extension of classical measure theory for the study of inter-dependent or interactive information, was proposed by Sugeon [18] by replacing additivity with monotonicity. Many studies have focused on theoretical aspects and applications of non-additive measures. Asahina [1] studied implication relationship among six continuity conditions and two null-additivity conditions with respect to non-additive measures. Li [8] discussed four versions of Egoroff's theorem in non-additive measure theory by using special condition. In particular, the Choquet integral with respect to non-additive measures has lbeen successfully applied in decision-making [23, 19], information fusion [6], economic theory [17] and so on.

Considering the inherent uncertain and imprecise of information in practical life, another key mathematical structure is introduced to model uncertain and incomplete systems, which is called fuzzy number, proposed by Zadeh [25], on the basis of fuzzy sets [24]. Fuzzy number has been investigated intensively by researches from various aspects. Gong [5] generalized convexity from vector-valued maps to n-dimensional fuzzy number-valued functions. Saeidifar [16] introduced the concepts of nearest weighted interval and point approximations of a fuzzy number. And Wang [22] applied triangular fuzzy number to study management knowledge performance evaluation.

Moving average is that, given a series of numbers and fixed subset size, the first element of the moving average is obtained by taking the average of the initial fixed subset of the number series [2]. The moving average has been widely applied in time series analysis [20], cloud computing [14] signal processing and financial mathematics, etc. However, when we use moving average to make forecasting, it is not reasonable to assume that all data are real data before we elicit them from practical data, fuzzy data may exit, such as in financial and sociological application. So we need to take the vagueness of the universe of importance. Furthermore, there is interaction among data in real application. The aim of this paper is to propose the moving average for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules. In particular, triangular fuzzy numbers about this method are also discussed.

The structure of this paper is as follows. In Section 2, we review some basic cpconcepts and properties about non-additive measure with $\sigma - \lambda$ rules and fuzzy numbers. And the definition of conduct between a non-negative matrix and fuzzy number vector is given to make our analysis possible. In Section 3, we propose the moving average for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules.

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*Corresponding author. Tel.: +8613993196400.

Email addresses: zt-gong@163.com(Zeng-Tai Gong), leiwenjingbz@163.com(Wen-Jing Lei).

In Section 4, the calculation of the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma-\lambda$ rules is invested and some properties are discussed. The paper ends with conclusion in In Section 5.

2. Preliminaries

Throughout this study, R^m denotes the m-dimension real Euclidean space and $R^+=(0,\infty)$.

Definition 2.1 [18, 10, 3]. Let X denote a nonempty set and $\mathscr{A}$ a $\sigma-$ algebra on the X. A set function μ is referred to as a regular fuzzy measure if

(1) $\mu(\emptyset)=0$;

(2) $\mu(X)=1$;

(3) for every A and $B\in\mathscr{A}$ such that $A\subseteq B$, $\mu(A)\le\mu(B)$.

Definition 2.2 [18, 10, 3]. g_λ is called a fuzzy measure based on $\sigma-\lambda$ rules if it satisfies

$$g_\lambda\left(\bigcup_{i=1}^{\infty}A_i\right)=\begin{cases}\dfrac{1}{\lambda}\left\{\prod_{i=1}^{\infty}[1+\lambda g_\lambda(A_i)]-1\right\}, & \lambda\neq 0,\\ \sum_{i=1}^{\infty}g_\lambda(A_i), & \lambda=0,\end{cases}$$

where $\lambda\in(-\frac{1}{\sup\mu},\infty)\bigcup\{0\}$, $\{A_i\}\subset\mathscr{A}$, $A_i\cap A_j=\emptyset$ for all $i,j=1,2,\cdots$ and $i\neq j$.

Particularly, if $\lambda=0$, then g_λ is a classic probability measure.

A regular fuzzy measure μ is called Sugeno measure based on $\sigma-\lambda$ rules if μ satisfies $\sigma-\lambda$ rules, briefly denoted as g_λ. The fuzzy measure denoted in this paper is Sugeno measure.

Remark 2.1. In the Definition, if $n=2$, then

$$\mu\left(A\cup B\right)=\begin{cases}\mu(A)+\mu(B)+\lambda\mu(A)\mu(B), & \lambda\neq 0,\\ \mu(A)+\mu(B), & \lambda=0.\end{cases}$$

Remark 2.2. If X be a finite set, for any subset A of X, then

$$g_\lambda\left(A\right)=\begin{cases}\dfrac{1}{\lambda}\left\{\prod_{x\in A}[1+\lambda g_\lambda(\{x\})]-1\right\}, & \lambda\neq 0,\\ \sum_{i=1}^{\infty}g_\lambda(\{x\}), & \lambda=0.\end{cases}$$

Remark 2.3 [3]. If X be a finite set, then the parameter λ of a regular Sugeno measure based on $\sigma-\lambda$ rules is determined by the equation

$$\prod_{i=1}^{n}(1+\lambda g_{\lambda i})=1+\lambda.$$

Let $g_\lambda(\{x_i\})=g_i, i=1,2,...,m$, then $g_\lambda(A_i)$ is obtained from the following recurrence relation

$$g_\lambda(A_m)=g_\lambda(\{x_m\})=g_m,\quad g_\lambda(A_i)=g_\lambda(A_{i+1})+\lambda g_i g_\lambda(A_{i+1}), 1\le i<m.$$

Let $\tilde{A}(x)\in\tilde{E}, r\in(0,1]$ and $[\tilde{A}]^r=\{x\in R: u_{\tilde{A}}(X)\ge r\}$. If $\tilde{A}$ satisfies

(1) $\tilde{A}$ is a normal fuzzy set, i.e., an $x_0\in R$ exists such that $u_{\tilde{A}}(x_0)=1$;

(2) $\tilde{A}$ is a convex fuzzy set, i.e., $u_{\tilde{A}}(\lambda x+(1-\lambda)y)\ge\min\{u_{\tilde{A}}(x),u_{\tilde{A}}(y)\}$ for any $x,y\in R$, and $\lambda\in(0,1]$;

(3) $\tilde{A}$ is a upper semicontinuous fuzzy set;

(4) $[A]^0=\overline{X\in R: u_{\tilde{A}}(x)>0}=\overline{\bigcup_{r\in(0,1]}[A]^r}$ is compact, where $\bar{A}$ denotes the closure of A.

Then, $\tilde{A}$ is called a fuzzy number. We use $\tilde{E}$ to denote the fuzzy number space [9].

It is clear that each $x \in R$ can be consider as a fuzzy number $\tilde{A}$ defined by

$$u_{\tilde{A}}(x) = \begin{cases} 1, & x = A, \\ 0, & otherwise. \end{cases}$$

Given any two fuzzy numbers $\tilde{A}_1, \tilde{A}_2, k, k_1k_2 \geq 0$, the operational rules are as follows:
(1) $k(\tilde{A}_1 + \tilde{A}_2) = k\tilde{A}_1 + k\tilde{A}_2$,
(2) $k_1(k_2\tilde{A}_1) = (k_1k_2)\tilde{A}_1$,
(3) $(k_1 + k_2)\tilde{A}_1) = k_1\tilde{A}_1 + k_2\tilde{A}_1$.
Lemma 2.1 [11, 12, 9]. For a fuzzy set $\tilde{A}$, it satisfy the following equation

$$\tilde{A} = \bigcup_{r\in[0,1]} (r^* \bigcap [\tilde{A}]^r),$$

where r^* denotes the fuzzy set whose membership function is constant function r.
Let $\tilde{A}, \tilde{B} \in \tilde{E}, k \in \mathbb{R}$, the addition and scalar conduct are defined by

$$[\tilde{A} + \tilde{B}]^r = [\tilde{A}]^r + [\tilde{B}]^r, \qquad [k\tilde{A}]^r = k[\tilde{A}]^r,$$

respectively, where $[\tilde{A}]^r = \{x : u_{\tilde{A}}(x) \geqslant r\} = [A^-(r), A^+(r)]$, for any $r \in (0, 1]$.
Lemma 2.2 [11, 12, 9]. If $\tilde{A} \in \tilde{E}$, then
(1) $[\tilde{A}]^r$ is a nonempty bounded closed interval for any $r \in (0, 1]$;
(2) $[\tilde{A}]^{r_1} \supset [\tilde{A}]^{r_2}$ where $0 \leqslant r_1 \leqslant r_2 \leqslant 1$;
(3) if $r_n > 0$ and $\{r_n\}$ converging increasingly to $r \in (0, 1]$, then

$$\bigcap_{n=1}^{\infty} [\tilde{A}]^{r_n} = [\tilde{A}]^r.$$

Conversely, if for any $r \in [0, 1]$, there exists $B_r \subset \mathbb{R}$ satisfying $(1) - (3)$, then there exists a unique $\tilde{A} \in \tilde{E}$ such that $[\tilde{A}]^r = A^r, r \in (0, 1]$, and $[\tilde{A}]^0 = \overline{\bigcup_{r\in(0,1]} [\tilde{A}]^r} \subset B_0$.
Definition 2.3 [21]. A triangle fuzzy number $\tilde{A}$ is a fuzzy number with piecewise linear membership function $\tilde{A}$ defined by

$$u_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_l}{a_m - a_l}, & a_l \leq x \leq a_m, \\ \dfrac{a_n - x}{a_n - a_m}, & a_m < x \leq a_n, \\ 0, & otherwise, \end{cases}$$

which can be indicated as a triplet (a_l, a_m, a_n).
Given any two triangle fuzzy numbers $\tilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,1}), \tilde{x}_j = (x_j - \delta_{j,1}, x_j, x_j + \delta_{j,1}))$, and $k \geq 0$, the operational rules are as follows:
(1) $\tilde{x}_i + \tilde{x}_j = (x_i - \delta_{i,1} + x_j - \delta_{j,1}, x_i + x_j, x_i + \delta_{i,1} + x_j + \delta_{j,1})$,
(2) $k \cdot \tilde{x}_i = (kx_i - k\delta_{i,1}, kx_i, kx_i + k\delta_{i,2})$.
Definition 2.4. Given a nonnegative matrix $P = [p_{ij}]$ and a fuzzy-number vector $\tilde{X}$, if $P \in R_+^{m\times m}$ and $\tilde{X} = [\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m]^\intercal \in \tilde{E}^m$(The $^\intercal$ denotes the conjugate transpose of a vector or a matrix.), then the product of P and X is defined as follows:

$$P\tilde{X}_{n-1} = \begin{bmatrix} \sum\limits_{j=1}^{m} p_{ij}\tilde{x}_j \\ \vdots \\ \sum\limits_{j=1}^{m} p_{mj}\tilde{x}_j \end{bmatrix}.$$

3. The weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma-\lambda$ rules

Definition 3.1. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \cdots, t_m) \in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Denote $A_i = \{t_i, t_{i+1}, \cdots, t_m\}$, $i = 1, 2, ..., m$, $A_{m+1} = \emptyset$. Then the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma-\lambda$ rules is defined as follows:

$$\tilde{x}_n = (g_\lambda(A_1) - g_\lambda(A_2))\tilde{x}_{n-m} + (g_\lambda(A_2) - g_\lambda(A_3))\tilde{x}_{n-m+1} + \cdots + (g_\lambda(A_m) - g_\lambda(A_{m+1}))\tilde{x}_{n-1},$$

where $n > m$.

When we use moving average to make forecasting, it is not reasonable to assume that all data are real data before we elicit them from practical data, fuzzy data may exit, such as in financial and sociological application. So we need to take the vagueness of the universe of importance. Furthermore, there is interaction among data in real application.

Remark 3.1. If $\lambda = 0$, and $\tilde{x}_i$ is a special fuzzy number, namely, real number, $i = 1, 2, \cdots$, the weighted moving averages for a series of fuzzy numbers based on non-additive measures with $\sigma-\lambda$ rules degenerates to the classic weighted moving average in Ref. [2].

Theorem 3.1. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \cdots, t_m) \in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Let $A_i = \{t_i, t_{i+1}, \cdots, t_m\}$, $i = 1, 2, ..., m$, $A_{m+1} = \emptyset$, $\tilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m-1}]^\intercal$, then

$$\tilde{X}_n = \boldsymbol{P}\tilde{X}_{n-1} = \boldsymbol{P}^2\tilde{X}_{n-2} = \cdots = \boldsymbol{P}^{n-1}\tilde{X}_1,$$

$n = 1, 2, 3, ...$, where

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ g_\lambda(A_1) - g_\lambda(A_2) & g_\lambda(A_2) - g_\lambda(A_3) & g_\lambda(A_3) - g_\lambda(A_4) & \cdots & g_\lambda(A_m) - g_\lambda(A_{m+1}) \end{bmatrix}.$$

Proof. Based on Definition 3.1 and the operational rules of fuzzy numbers, we have

$$\boldsymbol{P}\tilde{X}_{n-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ g_\lambda(A_1) - g_\lambda(A_2) & g_\lambda(A_2) - g_\lambda(A_3) & \cdots & g_\lambda(A_m) - g_\lambda(A_{m+1}) \end{bmatrix} \begin{bmatrix} \tilde{x}_{n-1} \\ \tilde{x}_n \\ \vdots \\ \tilde{x}_{n+m-3} \\ \tilde{x}_{n+m-2} \end{bmatrix} =$$

$[\tilde{x}_n, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m-3}, (g_\lambda(A_1)-g_\lambda(A_2))\tilde{x}_{n-1}+(g_\lambda(A_2)-g_\lambda(A_3))\tilde{x}_n+\cdots+(g_\lambda(A_m)-g_\lambda(A_{m+1}))\tilde{x}_{n+m-2}]^\intercal$.

And we know that

$$(g_\lambda(A_1) - g_\lambda(A_2))\tilde{x}_{n-1} + (g_\lambda(A_2) - g_\lambda(A_3))\tilde{x}_n + \cdots + (g_\lambda(A_m) - g_\lambda(A_{m+1}))\tilde{x}_{n+m-2} = \tilde{x}_{n+m-1},$$

This follows that

$$\boldsymbol{P}\tilde{X}_{n-1} = \tilde{X}_n.$$

The proof is complete. □

Theorem 3.1. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \cdots, t_m) \in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Let $A_i = \{t_i, t_{i+1}, \cdots, t_m\}$, $i = 1, 2, ..., m$, $A_{m+1} = \emptyset$, $\tilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m-1}]^\intercal$, and $\tilde{X}_n^-(r)$ and $\tilde{X}_n^+(r)$ as follows

$$\tilde{X}_n^-(r) = [\tilde{x}_n^-(r), \tilde{x}_{n+1}^-(r), \cdots, \tilde{x}_{n+m-1}^-(r)]^\intercal,$$

$$\tilde{X}_n^+(r) = [\tilde{x}_n^+(r), \tilde{x}_{n+1}^+(r), \cdots, \tilde{x}_{n+m-1}^+(r)]^\intercal,$$

where $[\tilde{x}_i]_r = [x_i^-(r), x_i^+(r)]$. Then

$$\tilde{X}_n^-(r) = \boldsymbol{P}\tilde{X}_{n-1}^-(r) = \boldsymbol{P}^2\tilde{X}_{n-2}^-(r) = \cdots = \boldsymbol{P}^{n-1}\tilde{X}_1^-(r),$$

$$\tilde{X}_n^+(r) = \boldsymbol{P}\tilde{X}_{n-1}^+(r) = \boldsymbol{P}^2\tilde{X}_{n-2}^+(r) = \cdots = \boldsymbol{P}^{n-1}\tilde{X}_1^+(r),$$

where $n = 1, 2, 3, ...$, and $\boldsymbol{P}$ is the same matrix in Theorem 3.1.
Proof. Based on Theorem 3.1, we have

$$\boldsymbol{P}\tilde{X}_{n-1}^-(r) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ g_\lambda(A_1) - g_\lambda(A_2) & g_\lambda(A_2) - g_\lambda(A_3) & \cdots & g_\lambda(A_m) - g_\lambda(A_{m+1}) \end{bmatrix} \begin{bmatrix} \tilde{x}_{n-1}^-(r) \\ \tilde{x}_n^-(r) \\ \vdots \\ \tilde{x}_{n+m-3}^-(r) \\ \tilde{x}_{n+m-2}^-(r) \end{bmatrix} =$$

$$[\tilde{x}_n^-(r), \cdots, \tilde{x}_{n+m-3}^-(r), (g_\lambda(A_1) - g_\lambda(A_2))\tilde{x}_{n-1}^-(r) + \cdots + (g_\lambda(A_m) - g_\lambda(A_{m+1}))\tilde{x}_{n+m-2}^-(r)]^\intercal.$$

By Definition 3.1, we get

$$(g_\lambda(A_1) - g_\lambda(A_2))\tilde{x}_{n-1} + (g_\lambda(A_2) - g_\lambda(A_3))\tilde{x}_n^-(r) + \cdots + (g_\lambda(A_m) - g_\lambda(A_{m+1}))\tilde{x}_{n+m-2}^-(r) = \tilde{x}_{n+m-1}^-(r).$$

This follows that

$$\boldsymbol{P}\tilde{X}_{n-1}^-(r) = \tilde{X}_n^-(r).$$

Similarly, we can prove that

$$\boldsymbol{P}\tilde{X}_{n-1}^+(r) = \tilde{X}_n^+(r).$$

The proof is complete. □

Theorem 3.2. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \cdots, t_m) \in R^m$, and g_λ be fuzzy measure satisfying $\delta - \lambda$ rules. Let $A_i = \{t_i, t_{i+1}, \cdots, t_m\}$, $i = 1, 2, ..., m$, $A_{m+1} = \emptyset$, $\tilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m-1}]^\intercal$,

If $\tilde{x}_i$ is a triangle fuzzy number, and $\tilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2})$, $i = 1, 2, \cdots$, then

$$\begin{aligned}\tilde{X}_n^-(r) &= [\tilde{x}_n^-(r), \tilde{x}_{n+1}^-(r), \cdots, \tilde{x}_{n+m-1}^-(r)]^\intercal \\ &= [\delta_{n,1}r + x_n - \delta_{n,1}, \delta_{n+1,1}r + x_n - \delta_{n+1,1}, \cdots, \delta_{n+m-1,,1}r + x_{n+m-1} - \delta_{n+m-1,1}]^\intercal,\end{aligned}$$

$$\begin{aligned}\tilde{X}_n^+(r) &= [\tilde{x}_n^+(r), \tilde{x}_{n+1}^+(r), \cdots, \tilde{x}_{n+m-1}^+(r)]^\intercal \\ &= [-\delta_{n,2}r + x_n + \delta_{n,2}, -\delta_{n+1,2}r + x_{n+1} + \delta_{n+1,2}, \cdots, -\delta_{n+m-1,2}r + x_{n+m-1} + \delta_{n+m-1,2}]^\intercal.\end{aligned}$$

Proof. Based on the operational rules we have

$$\begin{aligned}\tilde{X}_n^-(r) &= [\tilde{x}_n^-(r), \tilde{x}_{n+1}^-(r), \cdots, \tilde{x}_{n+m-1}^-(r)]^\intercal \\ &= [\delta_{n,1}r + x_n - \delta_{n,1}, \delta_{n,1}r + x_n - \delta_{n,1}, \cdots, \delta_{n+m-1,1}r + x_{n+m-1} - \delta_{n+m-1,1}]^\intercal, \\ \tilde{X}_n^+(r) &= [\tilde{x}_n^+(r), \tilde{x}_{n+1}^+(r), \cdots, \tilde{x}_{n+m-1}^+(r)]^\intercal \\ &= [-\delta_{n,2}r + x_n + \delta_{n,2}, -\delta_{n,2}r + x_n + \delta_{n,2}, \cdots, -\delta_{n+m-1,2}r + x_{n+m-1} + \delta_{n+m-1,2}]^\intercal.\end{aligned}$$

The proof is complete. □

4. The calculation of the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma - \lambda$ rules

Lemma 4.1 [15]. Let$(d_1, d_2, \cdots, d_m) \in R_+^m$ and set

$$q(x) = x^m - d_1x^{m-1} - \cdots - d_m.$$

Suppose that $\gcd\{k \in \{1, 2, \cdots, m\} : d_k > 0\} = 1$, where the greatest common division of a set S is denoted by $\gcd(S)$. Then q has a unique positive root$r$. Moreover, the algebraic multiplicity of r is 1, coinciding with the geometric multiplicity of r, and the modulus of every other root of q is strictly less than r.

Lemma 4.2 [13]. Let$B \in C^{m\times m}$, where C denotes plural numbers. Then the following holds

(1) $\{B^n\}$ converges to nonzero matrix if and only if 1 is a eigenvalue of B, whose algebraic multiplicities and geometric multiplicities coincide, and every other eigenvalues of B has modulus strictly less than 1;

(2) If $\rho(B) = \max\limits_{\lambda\in\sigma(B)} |\lambda| = 1$ is a eigenvalue of B whose algebraic multiplicity and geometric multiplicity of 1 coincide, equal to 1, with right-hand and left-hand eigenvalue x and $y^\intercal$ respectively, then

$$\lim_{n\to\infty} B^n = \frac{xy^\intercal}{y^\intercal x},$$

where $\sigma(B)$ is the set of eigenvalues of B.

Theorem 4.1. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \cdots, t_m) \in R^m$, and g_λ be fuzzy measure satisfying $\delta - \lambda$ rules. Let $A_i = \{t_i, t_{i+1}, \cdots, t_m\}$, $i = 1, 2, ..., m$, $A_{m+1} = \emptyset$, $\tilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m-1}]^\intercal$, For the matrix $\mathbf{P}$ satisfying the following recurrence relation in Theorem 3.1

$$\tilde{X}_n = \boldsymbol{P}\tilde{X}_{n-1} = \boldsymbol{P}^2\tilde{X}_{n-2} = \cdots = \boldsymbol{P}^{n-1}\tilde{X}_1,$$

if gcd $\{i \in \{1, 2, \cdots, m\} | g_\lambda(A_i) - g_\lambda(A_{i+1}) > 0\} = 1$, then $\lim\limits_{n\to\infty} \boldsymbol{P}^n$ exists, and

$$\lim_{n\to\infty} \boldsymbol{P}^n = \frac{ea^\intercal}{a^\intercal e} = eb^\intercal,$$

where $e = \sum\limits_{i=1}^{m} e_k = [1, 1, \cdots, 1]^\intercal \in R^m$, e_k is the ith standard unit column vector,

$a = [a_1, a_2, \cdots, a_m]^\intercal$, $b = [b_1, b_2, \cdots, b_m]^\intercal$, $a_k = \sum\limits_{i=1}^{k} (g_\lambda(A_i) - g_\lambda(A_{i+1}))$,

$b_k = \frac{a^\intercal e_k}{a^\intercal e} = \frac{a_k}{\sum\limits_{i=1}^{m} a_i} = \frac{g_\lambda(A_1) - g_\lambda(A_{k+1})}{mg_\lambda(A_1) - \sum\limits_{i=2}^{m} g_\lambda(A_i)}, k = 1, 2, 3, ..., m.$

Proof. For matrix $\boldsymbol{P}$, its characteristic polynomial is $p(t) = \det(t\boldsymbol{Id} - \boldsymbol{P})$, where $\boldsymbol{Id}$ is the unit matrix of order m . It is easy to obtain

$$p(t) = t^m - (g_\lambda(A_m) - g_\lambda(A_{m+1}))t^{m-1} - \cdots - (g_\lambda(A_2) - g_\lambda(A_3))t - (g_\lambda(A_1) - g_\lambda(A_2)).$$

Since $\sum\limits_{i=1}^{m} (g_\lambda(A_i) - g_\lambda(A_{i+1})) = g_\lambda(A_1) = 1$, $t = 1$ is a positive root of $p(t)$. Note that

$$\gcd\{k \in \{1, 2, \cdots, m\} : g_\lambda(A_i) - g_\lambda(A_{i+1}) > 0\} = 1.$$

According to Lemma 4.1, we can obtain $t = 1$ is the unique root of $p(t)$, whose algebraic multiplicity and geometric multiplicity of 1 are both equal to 1, and the modulus of every other root of q is strictly less than r.

Let x be the right-hand eigenvector of matrix $\boldsymbol{P}$ with respect to eigenvalue 1, then $\boldsymbol{P}x = x$. By using the elementary line transformation and the first elementary column transformation to matrix P, we can obtain

$$\boldsymbol{Id} - \boldsymbol{P}$$
$$= \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \\ g_\lambda(A_2) - g_\lambda(A_1) & g_\lambda(A_3) - g_\lambda(A_2) & g_\lambda(A_4) - g_\lambda(A_3) & \cdots & 1 - (g_\lambda(A_{m+1}) - g_\lambda(A_m)) \end{bmatrix}$$
$$\to \cdots \to \begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, a basic systeme of solutions for homogeneous linear equation set $(\boldsymbol{Id}-\boldsymbol{P})x=[0,0,\cdots,0]^{\intercal}$ is determined. It follows that the right-hand eigenvalue of $\boldsymbol{P}$ with respect to 1 is $x=[1,1,\cdots,1]^{\intercal}=e$.

Let $y^{\intercal}$ be the left-hand eigenvector of matrix $\boldsymbol{P}$ with respect to eigenvalue 1, then $y^{\intercal}\boldsymbol{P}=y^{\intercal}$. By using the elementary line transformation and the first elementary column transformation to matrix $\boldsymbol{Id}-\boldsymbol{P}^{\intercal}$, we can obtain

$$\boldsymbol{Id}-\boldsymbol{P}^{\intercal}=\begin{bmatrix}1&0&0&\cdots&g_\lambda(A_2)-g_\lambda(A_1)\\0&1&0&\cdots&g_\lambda(A_3)-g_\lambda(A_2)\\\vdots&\cdots&\cdots&\cdots&\vdots\\0&0&0&\cdots&g_\lambda(A_m)-g_\lambda(A_{m-1})\\0&0&0&\cdots&1-(g_\lambda(A_m)-g_\lambda(A_{m+1}))\end{bmatrix}$$

$$\to\cdots\to\begin{bmatrix}1&0&0&\cdots&g_\lambda(A_2)-g_\lambda(A_1)\\0&1&0&\cdots&g_\lambda(A_3)-g_\lambda(A_1)\\\vdots&\cdots&\cdots&\cdots&\vdots\\0&0&0&\cdots&g_\lambda(A_m)-g_\lambda(A_1)\\0&0&0&\cdots&0\end{bmatrix}.$$

Thus, a basic system of solutions for homogeneous linear equation set $(\boldsymbol{Id}-\boldsymbol{P}^{\intercal})y=[0,0,\cdots,0]$ is determined as follows:

$$[g_\lambda(A_1)-g_\lambda(A_2),\ g_\lambda(A_1)-g_\lambda(A_3),\ \cdots,\ g_\lambda(A_1)-g_\lambda(A_m)]^{\intercal},$$

It follows that the left-hand eigenvalue of $\boldsymbol{P}$ with respect to 1 is $a^{\intercal}=[a_1,a_2,\cdots,a_m], a_k=\sum\limits_{i=1}^{k}(g_\lambda(A_i)-g_\lambda(A_{i+1})),\ k=1,2,3,...,m$. According to Lemma 4.2(1), we know that $\{\boldsymbol{P}^n\}$ converges to a nonzero matrix. Combing Lemma 4.2(2), we can get

$$\lim_{n\to\infty}\boldsymbol{P}^n=\frac{ea^{\intercal}}{a^{\intercal}e}=eb^{\intercal}.$$

The proof is complete. □

Theorem 4.2. Let $(\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_m)\in\tilde{E}^m$, $(t_1,t_2,\cdots,t_m)\in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Let $A_i=\{t_i,t_{i+1},\cdots,t_m\}$, $i=1,2,...,m$, $A_{m+1}=\emptyset$, $\tilde{X}_n=[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}$. For the matrix P satisfying the recurrence relation in Theorem 3.1

$$\tilde{X}_n=\boldsymbol{P}\tilde{X}_{n-1}=\boldsymbol{P}^2\tilde{X}_{n-2}=\cdots=\boldsymbol{P}^{n-1}\tilde{X}_1,$$

if gcd $\{i\in\{1,2,\cdots,m\}:g_\lambda(A_i)-g_\lambda(A_{i+1})>0\}=1$, then $\lim\limits_{n\to\infty}\tilde{x}_n$ exists, and

$$\lim_{n\to\infty}\tilde{x}_n=\sum_{i=1}^{m}b_i\tilde{x}_i,$$

where $e=\sum\limits_{i=1}^{m}e_k=[1,1,\cdots,1]^{\intercal}\in\mathbb{R}^{m\times1}$, e_k is the ith standard unit column vector,

$a=[a_1,a_2,\cdots,a_m]^{\intercal}$, $b=[b_1,b_2,\cdots,b_m]^{\intercal}$, $a_k=\sum\limits_{i=1}^{k}(g_\lambda(A_i)-g_\lambda(A_{i+1}))$,

$b_k=\frac{a^{\intercal}e_k}{a^{\intercal}e}=\frac{a_k}{\sum\limits_{i=1}^{m}a_i}=\frac{g_\lambda(A_1)-g_\lambda(A_{k+1})}{mg_\lambda(A_1)-\sum\limits_{i=2}^{m}g_\lambda(A_i)}, k=1,2,3,...,m.$

Proof. Since

$$\tilde{X}_n=\boldsymbol{P}\tilde{X}_{n-1}=\boldsymbol{P}^2\tilde{X}_{n-2}=\cdots=\boldsymbol{P}^{n-1}\tilde{X}_1,$$

we have

$$\lim_{n\to\infty}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}=\lim_{n\to\infty}\boldsymbol{P}^{n-1}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}.$$

then, by Theorem 4.2, we can get

$$\lim_{n\to\infty}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}=er^{\intercal}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal},$$

i.e. $\lim_{n\to\infty} \tilde{x}_n$ is determined by the operation of the first row of $\lim_{n\to\infty} \boldsymbol{P}^{n-1}$ and $\tilde{X}_1 = [\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m]^\intercal$. It follows that

$$\lim_{n\to\infty} \tilde{x}_n = \frac{a^\intercal x}{a^\intercal e} = \frac{\sum_{i=1}^{m} a_i \tilde{x}_i}{\sum_{i=1}^{m} a_i} = \sum_{i=1}^{m} b_i \tilde{x}_i.$$

The proof is complete. □

In moving weighted average, the weight of the information contained in the data is not the same, and is independent of each other, so to identify the data of each phase is not reasonable. And introducing the non-additive measure into the moving weighted average is of practical significance.

Example 4.1. Given a closing stock prices system over 5 days. The closing prices of each day is denoted as $\tilde{x}_i$, $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_5) \in \tilde{E}^5$, and every $\tilde{x}_i$ is a triangle fuzzy number, $\tilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2})$, $i = 1, 2, \cdots, 5$. Suppose $(t_1, t_2, \cdots, t_5) \in R^5$, $A_i = \{t_i, t_{i+1}, \cdots, t_5\}$, $i = 1, 2, ..., 5$, $A_6 = \emptyset$. The value and the weight of each $\tilde{x}_i$ is shown in Table 1, $i = 1, 2, \cdots, 5$, then we can get the closing stock price over 10 days and some relevant results.

Day	Closing stock price	g_λ
1	(19,20,21)	0.1
2	(21,22,23)	0.2
3	(23,24,25)	0.3
4	(24,25,26)	0.15
5	(22,23,24)	0.175

Table 1: The closing stock prices over 5 days.

According to Remark 2.3 again, we know that $\prod_{i=1}^{5}(1 + \lambda g_{\lambda i}) = 1 + \lambda$, hence we can gain $\lambda = 0.218$.

Then, by Remark 2.3, we have

$g_\lambda(A_1) = 1, \quad g_\lambda(A_2) = \frac{1}{\lambda}\left\{\prod_{i=2}^{5}[1 + \lambda g_\lambda(\{x\})] - 1\right\} = 0.88,$

$g_\lambda(A_3) = \frac{1}{\lambda}\left\{\prod_{i=3}^{5}[1 + \lambda g_\lambda(\{x\})] - 1\right\} = 0.65,\ g_\lambda(A_4) = \frac{1}{\lambda}\left\{\prod_{i=4}^{5}[1 + \lambda g_\lambda(\{x\})] - 1\right\} = 0.33,$

$g_\lambda(A_5) = g_\lambda(\{x_5\}) = 0.175, \quad g_\lambda(A_6) = 0.$

By Definition 3.1, we have

$$\begin{aligned}\tilde{x}_6 = (&\sum_{i=1}^{5}((x_i - \delta_{i,1})(g_\lambda(A_i) - g_\lambda(A_{i+1})),\\ &\sum_{i=1}^{5} x_i(g_\lambda(A_i) - g_\lambda(A_{i+1})), \sum_{i=1}^{5}(x_i + \delta_{i,2})(g_\lambda(A_i) - g_\lambda(A_{i+1}))),\\ &= (22.04, 23.04, 24.04).\end{aligned}$$

Similarly, we can also calculate $\tilde{x}_n, n = 7, 8, 9, 10$, with respect to fuzzy measure g_λ on A, shown in Table 2. And by Theorem 4.1 and Theorem 4.3, we have

$$\begin{aligned}\lim_{n\to\infty} \boldsymbol{P}^n &= \frac{ea^\intercal}{a^\intercal e}\\ &= \frac{1}{0.12 + 0.35 + 0.67 + 0.825 + 1}\begin{bmatrix} 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1\end{bmatrix}\end{aligned}$$

Day	Closing stock price	g_λ
1	(19,20,21)	0.1
2	(21,22,23)	0.2
3	(23,24,25)	0.3
4	(24,25,26)	0.15
5	(22,23,24)	0.175
6	(22.04,23.04,24.04)	
7	(22.76,23.76,24.76)	
8	(22.72,23,72,24.72)	
9	(22.5,23.5,24.5)	
10	(22.45,23.45,24.45)	

Table 2: The closing stock prices over 10 days.

$$= \begin{bmatrix} 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \end{bmatrix},$$

$$\lim_{n\to\infty} \tilde{x}_n = \frac{a^\intercal \tilde{X}_1}{a^\intercal e} = \frac{1}{0.12+0.35+0.67+0.825+1}(66.84, 69.865, 72.77) = (22.54, 23.56, 24.54),$$

where $e = \sum\limits_{i=1}^{5} e_k = [1, 1, \cdots, 1]^\intercal \in R^{5\times 1}$, e_k is the ith standard unit column vector,

$a_1 = g_\lambda(A_1) - g_\lambda(A_2) = 0.12$, $a_2 = \sum\limits_{i=1}^{2}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.35$, $a_3 = \sum\limits_{i=1}^{3}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.67$, $a_4 = \sum\limits_{i=1}^{4}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.825$, $a_5 = \sum\limits_{i=1}^{5}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.1$.

Here when n is infinite, the forecasting value of x_n will become a stable value (22.54,23.56,24.54) by the weighted moving averages for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules.

5. Conclusion

In this paper, the moving average for a series of fuzzy numbers was proposed by means of non-additive measures with $\sigma - \lambda$ rules and fuzzy number. Meanwhile, the special case, i,e. the moving average for a series of triangular fuzzy numbers based on non-additive measures with $\sigma-\lambda$ were also discussed. Further, the calculation of the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma - \lambda$ rules was invested and some properties were discussed. Finally, an example was given to illustrate the practical importance of the main results.

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A Periodic Observer Based Stabilization Synthesis Approach for LDP Systems based on iteration *

Lingling Lv †, Wei He ‡, Zhe Zhang §, Lei Zhang ¶, Xianxing Liu ‖

Abstract

The stabilization problem of state observer based for linear discrete-time periodic (LDP) system and its robust consideration are discussed in this paper. It is proved that the periodic controller and the full-dimensional periodic state observer can be designed separately. Based on the well-known CG-algorithm for matrix equation Ax = b as well as applying the lifting technique and algebraic operations, an iterative algorithm for both periodic observer gains and periodic state feedback gains can be generated simultaneously. By optimizing the free parameter matrix in the proposed algorithm, a robust stabilization algorithm based on periodic observer for LDP systems is presented. One numerical example is worked out to illustrate the effect of the proposed approaches.

Keywords: Linear discrete-time periodic (LDP) systems; periodic state observers; stabilization; iterative method.

1 Introduction

The controller design requires us to master the state characteristics of the system. However, it is impractical to direct measure all state variables precisely in practical applications. So it requires us to make reliable estimates of the states that cannot be measured directly. The state observer is also called state reconstruction. The basic design idea is to design a state equivalent to the original system and use the designed state equivalent to the original state (see [1]-[2] and references therein). Especially, full-dimensional state observer in the construction idea is based on the original observed coefficient matrix in accordance with the same structure to establish a copy system. The difference between the observed system y and the copy system output $\widehat{y}$ is taken as a fixed variable and fed back to the input of the integrator group in the copy system to form a closed-loop system (see [3]-[5] and references therein). The design of observer has always been a research hot topic in control theory and control engineering, one can see [6, 7, 8] and references therein for instance.

Because of its extensive applications in cyclostationary process, multirate digital control, economics and management, biology, etc., and advantages of improving control performance by using periodic controllers, linear discrete periodic systems have been paid renewed attentions in the control theory community(see [9]-[11] and the references therein). The stabilization problem of dynamic systems has a fundamental importance in engineering, and hence it is among the most studied problems in modern control theory. Particularly, the

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†1. College of Environment and Planning, Henan University, Kaifeng, 475004, P. R. China. 2. Institute of Electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: lingling_lv@163.com (Lingling Lv).

‡Institute of electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: 13298369256@163.com (Wei He).

§Institute of electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: zhe_Zhang5218@163.com (Zhe Zhang)

¶Institute of Data and Knowledge Engineering, School of Computer and Information Engineering, Henan University, Kaifeng 475004, P.R. China. Email: zhanglei@henu.edu.cn (Lei Zhang).

‖Computer and Information Engineering College, Henan University, Kaifeng 475004, P. R. China. Email: Liuxianxing@henu.edu.cn (Xianxing Liu). Corresponding author.

stabilization of periodic motions of dynamic systems has drawn much attention over the past years (see [12]-[16] and references therein). In [14], LMI based conditions for stabilization via static periodic state feedback as well as via static periodic output feedback are presented, and the problem of quadratic stabilization in the presence of either norm-bounded or polytopic parameter uncertainty is also treated. The output stabilization problem for discrete-time linear periodic systems is solved in [15], where both the state-feedback control law and the state-predictor are based on a suitable time-invariant state-sampled reformulation associated with a periodic system. In addition, utilizing parametric poles assignment algorithm and robust performance index, an algorithm of robust stabilization based on periodic observers is proposed in [16].

In this paper, the problem of stabilization of discrete-time periodic systems based on state observer is transformed into the solution of the corresponding matrix equations, and a neat iterative algorithm is given based on the well-known conjugate gradient algorithm. Initially, we consider the stabilization problem for linear discrete-lime periodic systems without disturbances and give the expected algorithm. On this basis, in case that uncertain disturbances exist in the system parameters, a robust control algorithm for purpose of stabilization is also derived.

Notation 1 *The superscripts "T" and "−1" stand for matrix transposition and matrix inverse, respectively;* $\mathbb{R}^n$ *denotes the n-dimensional Euclidean space;* $\overline{i,j}$ *represents the integer set* $\{i, i+1, \ldots, j-1, j\}$, $\mathrm{tr}(A)$ *means the trace of matrix A. Norm* $\|A\|$ *is a Frobenius norm of matrix A.* $\Lambda(A)$ *means the eigenvalue set of matrix A and* Ψ_A *denotes the monodromy matrix* $A_{T-1}A_{T-2}\cdots A_0$ *with period T.*

2 Preliminaries

Consider the completely observable and completely reachable LDP systems with the following state space representation

$$\begin{cases} x_{t+1} = A_t x_t + B_t u_t \\ y_t = C_t x_t \end{cases} \tag{1}$$

where $t \in \mathbb{Z}$, the set of integers, $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^r$ and $y_t \in \mathbb{R}^m$ are respectively the state vector, the input vector and the output vector, A_t, B_t, C_t are matrices of compatible dimensions satisfying

$$A_{t+T} = A_t,\ B_{t+T} = B_t,\ C_{t+T} = C_t.$$

In case that the state of system (1) can be measured, by periodic feedback control law

$$u_t = -K_t x_t + v(t), \quad K_{t+T} = K_t, \quad K_t \in \mathbb{R}^{r\times n} \tag{2}$$

where v_t is the reference input, we can obtain the following combined system with period T

$$\begin{cases} x_{t+1} = (A_t - B_t K_t) x_t + B_t v_t \\ y_t = C_t x_t \end{cases} \tag{3}$$

When there exists some restrictions in practice, the state of system (1) can not be gotten by hardware, but the input u_t and the output y_t can be measured. In this case, we need build another periodic system which can give an asymptotic estimation of system states. The system with the following form can be adopted:

$$\hat{x}_{t+1} = A_t \hat{x}_t + B_t u_t - L_t(C_t \hat{x} - y_t) \tag{4}$$

where $\hat{x} \in \mathbb{R}^n$ and $L(t) \in \mathbb{R}^{n\times m}$, $t \in \mathbb{Z}$ are real matrices of period T. Obviously, equation 4 has the following equivalent presentation:

$$\hat{x}_{t+1} = (A_t - L_t C_t)\hat{x}_t + B_t u_t + L_t y_t \tag{5}$$

Integrating (4) and (3) gives the following augmented system:

$$\begin{cases} \begin{bmatrix} x_{t+1} \\ \hat{x}_{t+1} \end{bmatrix} = \begin{bmatrix} A_t & B_t K_t \\ L_t C_t & \widetilde{A}_t - B_t K_t \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} B_t \\ B_t \end{bmatrix} v_t \\ y_t = \begin{bmatrix} C_t & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} \end{cases} \tag{6}$$

where $\widetilde{A}_t = A_t - L_t C_t$.

Then the problem of stabilization based on periodic observer for LDP system (1) can be represented as

Problem 1 *Given a completely reachable and completely observable LDP system (1), find periodic matrix* $K(t) \in \mathbb{R}^{r\times n}, t \in \overline{0, T-1}$ *and* $L(t) \in \mathbb{R}^{n\times m}, t \in \overline{0, T-1}$, *such that the augmented system (6) is asymptotically stable.*

When the system is disturbed by external environment, the closed loop system matrix will deviate from the nominal matrix $\widetilde{A}_t$, which can be generally expressed as

$$A_t - B_t K_t \mapsto A_t + \Delta_{a,t} - (B_t + \Delta_{b,t}) K_t, \ t \in \overline{0, T-1},$$

$$A_t + L_t C_t \mapsto A_t + \Delta_{a,t} + L_t (C_t + \Delta_{c,t}), \ t \in \overline{0, T-1},$$

in which $\Delta_{a,t} \in \mathbb{R}^{n\times n}$, $\Delta_{b,t} \in \mathbb{R}^{n\times r}$, $\Delta_{c,t} \in \mathbb{R}^{m\times n}, t \in \overline{0, T-1}$ are random small perturbations. Thus, the problem of robust observer design for linear discrete-time periodic system (1) can be portrayed as

Problem 2 *Consider the completely observable and completely reachable linear discrete-time periodic system (1), seek the periodic matrix* $K(t) \in \mathbb{R}^{r\times n}, t \in \overline{0, T-1}$ *and* $L_t \in \mathbb{R}^{n\times m}, t \in \overline{0, T-1}$, *such that the following conditions are met:*

1. *The augmented system (6) is asymptotically stable;*
2. *Eigenvalues of the augmented system (6) are as insensitive as possible to small perturbations on systems matrices.*

3 Main result

The first thing to consider is the existence condition for a periodic state observer and a periodic state feedback controller. To do this, we would like to give the following theorem firstly.

Theorem 1 *For a given completely observable and completely reachable LDP system (1), the transfer function of the closed-loop system (6) is equal to the transfer function of the closed-loop system (3).*

Proof. It is easy to calculate that the transfer function of the closed-loop system (3) is:

$$G(s) = C_t(sI - A_t - B_t K_t)^{-1} B_t \tag{7}$$

Let

$$P_t = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}.$$

It is easily computed that

$$P_t^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}.$$

Noticing the coefficient matrices of system (6), we can obtain that

$$P_t \begin{bmatrix} A_t & B_t K_t \\ -L_t C_t & \widetilde{A}_t + B_t K_t \end{bmatrix} P_t^{-1} = \begin{bmatrix} A_t + B_t K_t & B_t K_t \\ 0 & \widetilde{A}_t \end{bmatrix},$$

$$P \begin{bmatrix} B_t \\ B_t \end{bmatrix} = \begin{bmatrix} B_t \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} C_t & 0 \end{bmatrix} P_t^{-1} = \begin{bmatrix} C_t & 0 \end{bmatrix}.$$

Obviously, system (6) is algebra equivalent to the following system:

$$\left(\begin{bmatrix} A_t + B_t K_t & B_t K_t \\ 0 & \widetilde{A}_t \end{bmatrix}, \begin{bmatrix} B_t \\ 0 \end{bmatrix}, \begin{bmatrix} C_t & 0 \end{bmatrix} \right) \tag{8}$$

Since the systems which are algebra equivalent to each other have the same transfer function, we only need to prove that the transfer function of system (8) is as shown in (7). By noticing

$$\begin{bmatrix} sI - A_t - B_tK_t & B_tM_t \\ 0 & sI - \widetilde{A}_t \end{bmatrix}^{-1} = \begin{bmatrix} (sI - A_t - B_tK_t)^{-1} & * \\ 0 & (sI - \widetilde{A}_t)^{-1} \end{bmatrix} \tag{9}$$

the transfer function corresponding to (8) can be calculated as

$$\begin{aligned} \bar{G}(s) &= \begin{bmatrix} C_t & 0 \end{bmatrix} \begin{bmatrix} sI - A_t - B_tK_t & B_tK_t \\ 0 & SI - \widetilde{A}_t \end{bmatrix}^{-1} \begin{bmatrix} B_t \\ 0 \end{bmatrix} \\ &= C_t(sI - A_t - B_tK_t)^{-1}B_t \end{aligned}$$

which is exactly equal to the transfer function of system (6). Thus the proof is accomplished. ■

According to theorem 1, the introduction of periodic state observer has no influence on the desired poles of the closed-loop systems via periodic state feedback. Similarly, the introduction of periodic state feedback has no influence on the designed poles of observer. In this point, the LDP systems keep pace with the linear time invariant systems. Therefore, for the stabilization problem of LDP systems based on periodic observer, the periodic state feedback controller and periodic observer can be designed separately. In the following, poles assignment techniques are adopt to realize the desired purpose.

Let Γ_1 and Γ_2 be the predetermined set of poles of the close-loop system (3) and (5) respectively, which are both symmetric with respect to the real axis. Let $\bar{F}_j^{\mathrm{K}}, \bar{F}_j^{\mathrm{L}} \in \mathbb{R}^{n\times n}$ be the T-periodic matrix satisfying $\Lambda(\Psi_{\bar{F}^{\mathrm{K}}}) = \Gamma_1$ and $\Lambda(\Psi_{\bar{F}^{\mathrm{L}}}) = \Gamma_2$, respectively. Clearly, to make system (3) and (5) possess the pole set Γ_1 and Γ_2 if and only if there exists a T-periodic invertible matrix X_j and Y_j such that

$$X_{j+1}^{-1}(A_j - B_jK_j)X_j = -F_j^{\mathrm{K}}. \tag{10}$$

and

$$Y_{j+1}^{-1}(A_j^{\mathrm{T}} - C_j^{\mathrm{T}}L_j^{\mathrm{T}})Y_j = -F_j^{\mathrm{L}}. \tag{11}$$

where $F_j^{\mathrm{K}} = -\bar{F}_j^{\mathrm{K}}, F_j^{\mathrm{L}} = -\bar{F}_j^{\mathrm{L}}, j \in \overline{0, T-1}$. Obviously, equations (10) and (11) can be rewritten as the following periodic Sylvester matrices:

$$A_jX_j - B_jK_jX_j = -X_{j+1}F_j^{\mathrm{K}}, \tag{12}$$

and

$$A_j^{\mathrm{T}}Y_j - C_j^{\mathrm{T}}L_j^{\mathrm{T}}Y_j = -Y_{j+1}F_j^{\mathrm{L}}, \tag{13}$$

Next, an iterative algorithm of stabilization problem based on periodic observer via periodic state feedback is presented firstly, and its correctness will be strictly verified in the subsequence.

Algorithm 1 *(Periodic CG-based Algorithm of problem 1)*

1. *Let* $F_j^{\mathrm{K}} \in \mathbb{R}^{n\times n}, F_j^{\mathrm{L}} \in \mathbb{R}^{n\times n}, j \in \overline{0, T-1}$ *be a real periodic matrix, which satisfies* $\Lambda(\Psi_{F_j^{\mathrm{K}}}) = \Gamma_1$ *and* $\Lambda(\Psi_{F_j^{\mathrm{K}}}) \bigcap \Lambda(\Psi_{A_j}) = 0$*;* $\Lambda(\Psi_{F_j^{\mathrm{L}}}) = \Gamma_2$ *and* $\Lambda(\Psi_{F_j^{\mathrm{L}}}) \bigcap \Lambda(\Psi_{A_j^{\mathrm{T}}}) = 0$*. Further, let* $G_j = K_jX_j \in \mathbb{R}^{r\times n}, D_j = L_j^{\mathrm{T}}Y_j \in \mathbb{R}^{m\times n}$ *are real parametric matrix such that periodic matrix pair* (F_j^{K}, G_j) *and* (F_j^{L}, D_j) *is completely observable.*

2. *Set tolerance* ε*; Choose arbitrary initial periodic matrix* $X_j(0) \in \mathbb{R}^{n\times n}, Y_j(0) \in \mathbb{R}^{n\times n}$*,* $j \in \overline{0, T-1}$*; Calculated as follows:*

$$\begin{aligned} Q_j(0) &= B_jG_j - A_jX_j(0) - X_{j+1}(0)F_j^{\mathrm{K}}, \\ W_j(0) &= C_j^{\mathrm{T}}D_j - A_j^{\mathrm{T}}Y_j(0) - Y_{j+1}(0)F_j^{\mathrm{L}}; \\ R_j(0) &= A_j^{\mathrm{T}}Q_j(0) + Q_{j-1}(0)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}}; \\ N_j(0) &= A_jW_j(0) + W_{j-1}(0)(F_{j-1}^{\mathrm{L}})^{\mathrm{T}}; \\ P_j(0) &= -R_j(0); \\ H_j(0) &= -N_j(0); \\ t &:= 0. \end{aligned}$$

3. *If* $\sum_{j=0}^{T-1}\|R_j(t)\|\leq\varepsilon$ *and* $\sum_{i=0}^{T-1}\|N_j(t)\|\leq\varepsilon$*, stop; else, go to next step.*

4. *While* $\sum_{j=0}^{T-1}\|R_j(t)\|\geq\varepsilon$ *and* $\sum_{i=0}^{T-1}\|N_j(t)\|\geq\varepsilon$*, calculate*

$$\begin{aligned}
\alpha_j(t) &= \frac{\sum_{j=0}^{T-1}\operatorname{tr}\left[P_j^{\mathrm{T}}(t)R_j(t)\right]}{\sum_{j=0}^{T-1}\|A_jP_j(t)+P_{j+1}(t)B_j\|^2};\\
\beta_j(t) &= \frac{\sum_{j=0}^{T-1}\operatorname{tr}\left[H_j^{\mathrm{T}}(t)N_j(t)\right]}{\sum_{j=0}^{T-1}\left\|A_j^{T}H_j(t)+H_{j+1}(t)C_j^{\mathrm{T}}\right\|^2};\\
X_j(t+1) &= X_j(t)+\alpha_j(t)P_j(t);\\
Y_j(t+1) &= Y_j(t)+\beta_j(t)H_j(t);\\
Q_j(t+1) &= B_jG_j-A_jX_j(t+1)-X_{j+1}(t+1)F_j^{\mathrm{K}};\\
W_j(t+1) &= C_j^{\mathrm{T}}D_j-A_j^{T}Y_j(t+1)-Y_{j+1}(t+1)F_j^{\mathrm{L}};\\
R_j(t+1) &= A_j^{\mathrm{T}}Q_j(t+1)+Q_{j-1}(t+1)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}},\\
N_j(t+1) &= A_jW_j(t+1)+W_{j-1}(t+1)(F_j^{\mathrm{L}})^{\mathrm{T}};\\
P_j(t+1) &= -R_j(t+1)+\frac{\sum_{j=0}^{T-1}\|R_j(t+1)\|^2}{\sum_{j=0}^{T-1}\|R_j(t)\|^2}P_j(t);\\
H_j(t+1) &= -N_j(t+1)+\frac{\sum_{j=0}^{T-1}\|N_j(t+1)\|^2}{\sum_{j=0}^{T-1}\|N_j(t)\|^2}H_j(t);\\
t &= t+1;
\end{aligned}$$

5. *Let* $X_j=X_j(t)$*,*$Y_j=Y_j(t)$*. The real periodic matrix* K_j *and* L_j *can be obtained as*

$$\begin{aligned}
K_j &= G_jX_j^{-1}, j\in\overline{0,T-1},\\
L_j &= (D_jY_j^{-1})^{\mathrm{T}}, j\in\overline{0,T-1}.
\end{aligned}$$

Remark 1 *The main part of the algorithm does not contain nested loops, so the computational complexity of the algorithm is* $O(n)$*.*

Next, the convergence and correctness of the algorithm are proved.

Lemma 1 *For sequences* $\{R_j(k)\}$*,* $\{P_j\}(k)$*,*$\{N_j(k)\}$*,* $\{H_j(k)\}$*,* $j\in\overline{0,T-1}$*, the following relations hold for* $k\geq 0$*:*

$$\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k+1)P_j(k)\right]=0,\quad \sum_{j=0}^{T-1}\operatorname{tr}\left[N_j^{\mathrm{T}}(k+1)H_j(k)\right]=0, \tag{14}$$

$$\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k)P_j(k)\right]+\sum_{j=0}^{T-1}\|R_j(k)\|^2=0,\quad \sum_{j=0}^{T-1}\operatorname{tr}\left[N_j^{\mathrm{T}}(k)H_j(k)\right]+\sum_{j=0}^{T-1}\|N_j(k)\|^2=0 \tag{15}$$

$$\sum_{k>0}\frac{\left(\sum_{j=0}^{T-1}\|R_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|P_j(k)\|^2}<\infty,\quad \sum_{k>0}\frac{\left(\sum_{j=0}^{T-1}\|N_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|H_j(k)\|^2}<\infty \tag{16}$$

Proof. By the expression of $R_j(k+1)$ in Algorithm 1, the following deduction is established.

$$\begin{aligned}
R_j(k+1) &= A_j^{\mathrm{T}}Q_j(k+1) + Q_{j-1}(k+1)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&= A_j^{\mathrm{T}}\left(C_jG_j - A_jX_j(k+1) - X_{j+1}(k+1)F_j^{\mathrm{K}}\right) \\
&\quad + \left(C_{j-1}G_{j-1} - A_{j-1}X_{j-1}(k) - X_j(k)F_{j-1}^{\mathrm{K}}\right)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&= A_j^{\mathrm{T}}\left(C_jG_j - A_jX_j(k) - X_jF_j^{\mathrm{K}}\right) \\
&\quad + \left(C_{j-1}G_{j-1} - A_{j-1}X_{j-1} - X_j(k)F_{j-1}^{\mathrm{K}}\right)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&\quad -\alpha(k)A_j^{\mathrm{T}}\left(A_jP_j(k) + P_{j+1}(k)F_j^{\mathrm{K}}\right) \\
&\quad -\alpha(k)\left(A_{j-1}P_{j-1}(k) + P_j(k)F_{j-1}^{\mathrm{K}}\right)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&= R_j(k) - \alpha(k)\left[A_j^{\mathrm{T}}\left(A_jP_j + P_{j+1}(k)F_j^{\mathrm{K}}\right) + \left(A_{j-1}P_{j-1}(k) + P_j(k)F_{j-1}^{\mathrm{K}}\right)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}}\right]
\end{aligned}$$

Noticing the formula of $\alpha(k)$ in step 3 of Algorithm 1, we can obtain that

$$\begin{aligned}
\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k+1)P_j(k)\right] &= \sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{T}(k)P_j(k)\right] - \alpha(k)\sum_{j=0}^{T-1}\left[\left(A_jP_j(k) + P_{j+1}(k)F_j^{\mathrm{K}}\right)^{\mathrm{T}}A_jP_j(k)\right] \\
&\quad +\alpha(k)\sum_{j=0}^{T-1}\left[\left(A_{j-1}P_{j-1}(k) + P_j(k)F_{j-1}^{\mathrm{K}}\right)^{\mathrm{T}}P_j(k)F_{j-1}^{\mathrm{K}}\right] \\
&= \sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{T}(k)P_j(k)\right] \\
&\quad -\alpha(k)\sum_{j=0}^{T-1}\left[\left(A_jP_j(k) + P_{j+1}(k)F_j^{\mathrm{K}}\right)^{\mathrm{T}}\left(A_jP_j(k) + P_{j+1}(k)F_j^{\mathrm{K}}\right)\right] \\
&= \sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{T}(k)P_j(k)\right] - \alpha(k)\sum_{j=0}^{T-1}\left\|A_jP_j(k) + P_{j+1}(k)F_j^{\mathrm{K}}\right\| \\
&= 0
\end{aligned}$$

The second equation in (14) can be verified by similar deduction.

It is easily to check that equation (15) holds for $k = 0$. Then, according to the expression of $P_j(k+1)$ and Equation (14), the following deduction holds.

$$\begin{aligned}
\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k+1)P_j(k+1)\right] &= -\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k+1)R_j(k+1)\right] + \frac{\sum_{j=0}^{T-1}\|R_j(k+1)\|^2}{\sum_{j=0}^{T-1}\|R_j(k)\|^2}\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k+1)P_j(k)\right] \\
&= -\sum_{j=0}^{T-1}\|R_j(k+1)\|^2
\end{aligned}$$

That's to say Equation (15) holds. Applying Kronecker product, we get

$$\begin{aligned}
&\sum_{j=0}^{T-1}\left\|A_jP_j(k)+P_{j+1}(k)F_j^{\mathrm{K}}\right\|^2=\sum_{j=0}^{T-1}\left\|(E\otimes A_j)\operatorname{vec}(P_j(k))+\left((F_j^{\mathrm{K}})^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_{j+1}(k))\right\|^2\\
=&\left\|\begin{array}{c}(E\otimes A_0)\operatorname{vec}(P_0(k))+\left((F_0^K)^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_1(k))\\(E\otimes A_1)\operatorname{vec}(P_1(k))+\left((F_1^K)^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_2(k))\\\vdots\\(E\otimes A_{T-1})\operatorname{vec}(P_{T-1}(k))+\left((F_{T-1}^K)^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_0(k))\end{array}\right\|^2\\
=&\left\|\begin{bmatrix}E\otimes A_0&(F_0^K)^{\mathrm{T}}\otimes E&&&\\&E\otimes A_1&(F_1^K)^{\mathrm{T}}\otimes E&&\\&&E\otimes A_2&&\\&&&\ddots&(F_{T-2}^K)^{\mathrm{T}}\otimes E\\(F_{T-1}^K)^{\mathrm{T}}\otimes E&&&&E\otimes A_{T-1}\end{bmatrix}\begin{bmatrix}\operatorname{vec}(P_0(k))\\\operatorname{vec}(P_1(k))\\\operatorname{vec}(P_2(k))\\\vdots\\\operatorname{vec}(P_{T-1}(k))\end{bmatrix}\right\|^2\\
\leq&\ \Pi\sum_{j=0}^{T-1}\|P_j(k)\|^2, \qquad (17)
\end{aligned}$$

where,

$$\Pi=\left\|\begin{bmatrix}E\otimes A_0&(F_0^K)^{\mathrm{T}}\otimes E&&&\\&E\otimes A_1&(F_1^K)^{\mathrm{T}}\otimes E&&\\&&E\otimes A_2&&\\&&&\ddots&(F_{T-2}^K)^{\mathrm{T}}\otimes E\\(F_{T-1}^K)^{\mathrm{T}}\otimes E&&&&E\otimes A_{T-1}\end{bmatrix}\right\|^2.$$

Define the following function:

$$J_1(k)=\frac{1}{2}\sum_{j=0}^{T-1}\left\|B_jG_j-A_jX_j(t+1)-X_{j+1}(t+1)F_j^{\mathrm{K}}\right\|^2, \qquad (18)$$

$$J_2(k)=\frac{1}{2}\sum_{j=0}^{T-1}\left\|C_j^{\mathrm{T}}D_j-A_j^TY_j(t+1)-Y_{j+1}(t+1)F_j^{\mathrm{L}}\right\|^2, \qquad (19)$$

By using the expression of $\alpha(k)$, $\beta(k)$, the following relations hold for $k\geq 0$:

$$J_1(k+1)=J_1(k)-\frac{1}{2}\alpha(k)\sum_{j=0}^{T-1}\operatorname{tr}\left[P_j^{\mathrm{T}}(k)R_j(k)\right]\ J_2(k+1)=J_2(k)-\frac{1}{2}\alpha(k)\sum_{j=0}^{T-1}\operatorname{tr}\left[H_j^{\mathrm{T}}(k)N_j(k)\right]$$

Then, one has

$$\begin{aligned}
&J_1(k+1)-J(k)\\
=&-\frac{1}{2}\frac{(\sum_{j=0}^{T-1}\operatorname{tr}\left[P_j^{\mathrm{T}}(k)R_j(k)\right])^2}{\sum_{j=0}^{T-1}\left\|A_j^{\mathrm{T}}P_j(k)+P_{j+1}(k)F_j^K\right\|^2}\\
\leq&\ 0, \qquad (20)
\end{aligned}$$

which means that $\{J(k)\}$ is a descent sequence, so that

$$J_1(k+1)\leq J(0)$$

holds for all $k\geq 0$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty}[J_1(k)-J_1(k+1)]=J_1(0)-\lim_{k\to\infty}J(k) \qquad (21)\\
<&\ \infty.
\end{aligned}$$

In view of Equation (15), (17) and (21), the following deduction holds:

$$\begin{aligned}
\sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\|R_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|P_j(k)\|^2} &= \sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k)P_j(k)\right]\right)^2}{\sum_{j=0}^{T-1}\|P_j(k)\|^2} \\
\leq\ & \pi\sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k)P_j(k)\right]\right)^2}{\sum_{j=0}^{T-1}\left\|A_jP_j(k)+P_{j+1}(k)F_j^K\right\|^2} = 2\pi(J(0)-\lim_{k\to\infty}J(k)) \\
<\ & \infty.
\end{aligned}$$

Similar argument on $J_2(k)$ gives the conclusion

$$\sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\|N_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|H_j(k)\|^2} < \infty.$$

To summarize, the Lemma 1 has been proved. ■

Based on the above lemma, the following conclusion could be drawn as:

Theorem 2 *Consider the completely observable and completely reachable periodic discrete-time linear system (1), the T-periodic matrix $L_j, j\in\overline{0,T-1}$, $K_j, j\in\overline{0,T-1}$, derived from Algorithm 1 is a solution of Problem 1.*

Proof. Let us first prove the convergence of matrix sequence $\{R_j(k)\}, j\in\overline{0,T-1}$ generated from Algorithm 1.

By Lemma 1 and the expressions of $P_j(k+1)$ in Algorithm 1, we have

$$\begin{aligned}
\sum_{j=0}^{T-1}\|P_j(k+1)\|^2 &= \sum_{j=0}^{T-1}\left\|-R_j(k+1)+\frac{\sum_{j=0}^{T-1}\|R_j(k+1)\|^2}{\sum_{j=0}^{T-1}\|R_j(k)\|^2}P_j(k)\right\|^2 \\
&= \left(\frac{\sum_{j=0}^{T-1}\|R_j(k+1)\|^2}{\sum_{j=0}^{T-1}\|R_j(k)\|^2}\right)^2\sum_{j=0}^{T-1}\|P_j(k)\|^2+\sum_{j=0}^{T-1}\|R_j(k+1)\|^2. \qquad (22)
\end{aligned}$$

Equation (22) can be written as

$$t(k+1)=t(k)+\frac{1}{\sum_{j=0}^{T-1}\|R_j(k+1)\|^2} \qquad (23)$$

equivalently, where

$$t(k)=\frac{\sum_{j=0}^{T-1}\|P_j(k)\|^2}{\left(\sum_{j=0}^{T-1}\|R_j(k)\|^2\right)^2}.$$

Assume that

$$\lim_{k\to\infty}\sum_{j=0}^{T-1}\|R_j(k)\|^2\neq 0, \qquad (24)$$

which implies that there exists a constant $\delta>0$ such that

$$\sum_{j=0}^{T-1}\|R_j(k)\|^2\geq\delta$$

for all $k\geq 0$. It follows from (23) and (24) that

$$t(k+1)\leq t(k)+\frac{1}{\delta}\leq\cdots\leq t(0)+\frac{k+1}{\delta},$$

which means

$$\frac{1}{t(k+1)} \geq \frac{\delta}{\delta t(0)+k+1}.$$

So we have

$$\sum_{k=1}^{\infty} \frac{1}{t(k)} \geq \sum_{k=1}^{\infty} \frac{\delta}{\delta t(0)+k+1} = \infty.$$

However, according to Equation (16) that

$$\sum_{j=1}^{\infty} \frac{1}{t(j)} < \infty.$$

This gives a contradiction. Thus, there holds

$$\lim_{k\to\infty} \sum_{j=0}^{T-1} \|R_j(k)\|^2 = 0,$$

Similarity, we have

$$\lim_{k\to\infty} \sum_{j=0}^{T-1} \|R_j(k)\|^2 = 0,$$

which indicates that the matrix sequence $\{X_j(k)\}$, $\{Y_j(k)\}$, $j \in \overline{0, T-1}$, generated by Algorithm 1 are convergent to matrices $\{X_j\}$, $\{Y_j\}$, $j \in \overline{0, T-1}$, which are respectively the solutions to the two periodic Sylvester equations (12) and (13). According to the poles assignment theory as previously mentioned, matrix L_j, K_j derived from Algorithm 1 are solutions to Problem 1. ■

3.1 Minimum norm and robust consideration

In this section, we will consider robust poles assignment problem raised in problem 2. In previous work, we have discussed the sensitivity of the closed-loop LDP systems with respect to parameter uncertainties. Here, we revisit it in the following lemma.

Lemma 2 *[17] Let $\Psi = A(T-1)A(T-2)\cdots A(0) \in \mathbb{R}^{n\times n}$ be diagonalizable and $Q \in \mathbb{C}^{n\times n}$ be a nonsingular matrix such that $\Psi = Q^{-1}\Lambda Q \in \mathbb{R}^{n\times n}$, where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is the Jordan canonical form of matrix Ψ. For a real scalar $\varepsilon > 0$, $\Delta_i(\varepsilon) \in \mathbb{R}^{n\times n}, i \in \overline{0, T-1}$, are matrix functions of ε satisfying*

$$\lim_{\varepsilon\to 0^+} \frac{\Delta_i(\varepsilon)}{\varepsilon} = \Delta_i,$$

where $\Delta_i \in \mathbb{R}^{n\times n}, i \in \overline{0, T-1}$ are constant matrices. Then for any eigenvalue λ of matrix

$$\Psi(\varepsilon) = \left(A(T-1) + \Delta_{T-1}(\varepsilon)\right)\left(A(T-2) + \Delta_{T-2}(\varepsilon)\right)\cdots\left(A(0) + \Delta_0(\varepsilon)\right),$$

the following relation holds:

$$\min_i\{|\lambda_i - \lambda|\} \leq \varepsilon n \kappa_{\text{F}}(Q) \left(\sum_{i=0}^{T-1} \|A(i)\|_{\text{F}}^{T-1}\right) \max_i\{\|\Delta_i\|_{\text{F}}\} + \text{O}(\varepsilon^2). \tag{25}$$

According to Lemma 2, combining the Algorithm 1, one could take the robust performance index of problem 2 as

$$J(G_j, D_j) = \kappa_{\text{F}}(X_0) \sum_{j=0}^{T-1} \|A_j + B_j K_j\|_{\text{F}}^{T-1} + \kappa_{\text{F}}(Y_0) \sum_{j=0}^{T-1} \left\|A_j^{\text{T}} + C_j^{\text{T}} L_j^{\text{T}}\right\|_{\text{F}}^{T-1} \tag{26}$$

Based on the above discussion, the algorithm for robust stabilization based on observer design for LDP systems can be presented as follows.

Algorithm 2 *(Robust stabilization based on periodic observer)*

1. *Perform the operations of step 1-4 of Algorithm 1.*
2. *Based on gradient-based search methods and the index (26), solve the optimization problem*

$$\text{Minimize } J(G_j, D_j),$$

and denote the optimal decision matrix by $G_j^{\rm opt}, D_j^{\rm opt}, j \in \overline{0, T-1}$.

3. *Substituting* $G_j^{\rm opt}, D_j^{\rm opt}$ *into steps 2-4 of algorithm 1 gives optimization matrices* $X_j^{\rm opt}, Y_j^{\rm opt}$.
4. *The robust controller and observer gains can be obtained as*

$$K_j^{\rm opt} = G_j^{\rm opt}(X_j^{\rm opt})^{-1},\ L_j^{\rm opt} = \left(D_j^{\rm opt}(Y_j^{\rm opt})^{-1}\right)^{\rm T}, j \in \overline{0, T-1}.$$

4 A Numerical Example

Consider LDP system (1) with parameters as follows:

$$A(0) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix},\ A(1) = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix},\ A(2) = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$$
$$B(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ B(1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ B(2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$C(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ C(1) = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ C(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This is a diverging system and it is easy to prove that the system is completely reachable and completely observable. Hence, we can claim that it can be stabilized by a periodic state feedback law based on a full-dimensional state observer. Without loss of generality, let the pole set the system (3) and (5) be $\Gamma_1 = \{-0.3, 0.3\}$ and $\Gamma_2 = \{-0.4, 0.4\}$, respectively.

According to algorithm 1, by choosing parameter matrices G and D randomly, we obtain a group of solutions as follows:

$$\begin{cases} K_0^{\rm rand} = \begin{bmatrix} 1.7699 & -1.8268 \end{bmatrix} \\ K_1^{\rm rand} = \begin{bmatrix} -1.9615 & -2.3782 \end{bmatrix} \\ K_2^{\rm rand} = \begin{bmatrix} -1.1669 & -0.8084 \end{bmatrix} \end{cases},\ \begin{cases} L_0^{\rm rand} = \begin{bmatrix} -2.5762 & -1.4737 \end{bmatrix}^{\rm T} \\ L_1^{\rm rand} = \begin{bmatrix} 0.1217 & 2.0509 \end{bmatrix}^{\rm T} \\ L_2^{\rm rand} = \begin{bmatrix} -1.1765 & -1.6305 \end{bmatrix}^{\rm T} \end{cases}$$

Furthermore, employing the robust stabilization algorithm 2, we obtain a group of solution as follows:

$$\begin{cases} K_0^{\rm robu} = \begin{bmatrix} 1.8432 & -3.5251 \end{bmatrix} \\ K_1^{\rm robu} = \begin{bmatrix} -3.1085 & 1.4631 \end{bmatrix} \\ K_2^{\rm robu} = \begin{bmatrix} -1.1128 & -2.4826 \end{bmatrix} \end{cases},\ \begin{cases} L_0^{\rm robu} = \begin{bmatrix} -0.6456 & 0.9869 \end{bmatrix}^{\rm T} \\ L_1^{\rm robu} = \begin{bmatrix} 0.3933 & 1.3929 \end{bmatrix}^{\rm T} \\ L_2^{\rm robu} = \begin{bmatrix} -1.0894 & -1.7176 \end{bmatrix}^{\rm T} \end{cases}$$

Let discrete reference input $v(t) = 0.1\sin(\frac{\pi}{2} + t)$ and the initial values of state and the observer state be $x_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^{\rm T}$, $\hat{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\rm T}$. We depict the trajectory of state variable x for the original system (1), state variable x and its estimated state of system (6) under $(K_{\rm rand}, L_{\rm rand})$, state variable x and its estimated state of system (6) under $(K_{\rm robu}, L_{\rm robu})$ in Fig.1 respectively, where the red line denote the histories of xL and the green line denote the histories the observed state $\hat{x}$. From the simulation results, we can see the good performance of the controller and observer generated by the proposed algorithm.

5 Conclusion

A stabilizing controller design method for LDP systems based on periodic full-dimensional state observer is introduced in this paper. As similar with linear time variant systems, the periodic state feedback controller

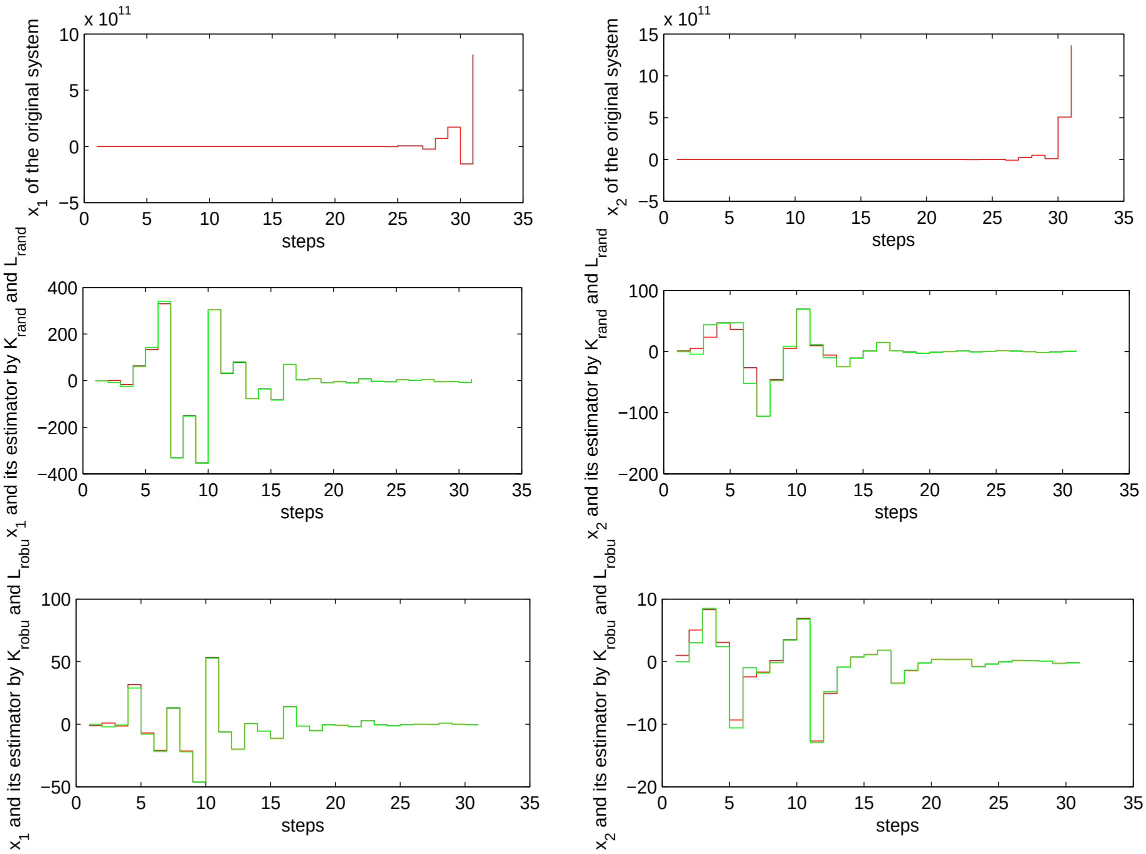

Figure 1: Comparison of state and the observed state under different cases

and periodic observer are designed separately, based on the periodic poles assignment technique. An iterative algorithm is presented to generate periodic observer gains and periodic controller gains simultaneously. In addition, robust stabilization problem is also discussed in this paper, and the corresponding algorithm is derived. The effectiveness of the proposed algorithms are shown by simulation results on an illustrate example.

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SUBORDINATION AND SUPERORDINATION PROPERTIES FOR CERTAIN FAMILY OF INTEGRAL OPERATORS ASSOCIATED WITH MULTIVALENT FUNCTIONS

M. K. AOUF, H. M. ZAYED, AND N. E. CHO

Abstract. The object of the present paper is to obtain subordination, superordination and sandwich-type results related to a certain family of integral operators defined on the space of multivalent functions in the open unit disk. Also we point out relevant connections of the results presented here with those obtained in earlier.

Keywords and phrases: $p-$valent function, differential subordination, superordination, subordination chain, integral operator.

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1. Introduction

Let $\mathcal{H}=\mathcal{H}(\mathbb{U})$ be the class of functions analytic in $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$ and denote $\mathcal{H}_0:=\mathcal{H}[0,1]$ and $\mathcal{H}:=\mathcal{H}[1,1]$.

Let $\mathcal{P}$ denote the class of functions

$$\mathcal{P}=\{h\in\mathcal{H}[0,1]:h(z)h'(z)\neq0,\ z\in\mathbb{U}^*:=\mathbb{U}\setminus\{0\}\}\,, \tag{1}$$

and $\mathcal{A}(p)$ be the class of all functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p}\ (p\in\mathbb{N}=\{1,2,...\}), \tag{2}$$

which are analytic in $\mathbb{U}$. We note that $\mathcal{A}(1)=\mathcal{A}$.

For $f,g\in\mathcal{H}(\mathbb{U})$, the function $f(z)$ is said to be subordinate to $g(z)$ or $g(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U})$, such that $f(z)=g(\omega(z))$. In such a case we write $f(z)\prec g(z)$. If g is univalent, then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$ (see [14, 15]).

Let $\phi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\phi\left(\mathfrak{p}\left(z\right),z\mathfrak{p}'\left(z\right);z\right)\prec h\left(z\right), \tag{3}$$

then $\mathfrak{p}(z)$ is a solution of the differential subordination (3). The univalent function $\mathfrak{q}(z)$ is called a dominant of the solutions of the differential subordination (3) if $\mathfrak{p}(z)\prec\mathfrak{q}(z)$

for all $\mathfrak{p}(z)$ satisfying (3). A univalent dominant $\tilde{\mathfrak{q}}$ that satisfies $\tilde{\mathfrak{q}} \prec \mathfrak{q}$ for all dominants of (3) is called the best dominant. If $\mathfrak{p}(z)$ and $\phi(\mathfrak{p}(z), z\mathfrak{p}'(z); z)$ are univalent in $\mathbb{U}$ and if $\mathfrak{p}(z)$ satisfies the first order differential superordination:

$$h(z) \prec \phi(\mathfrak{p}(z), z\mathfrak{p}'(z); z), \tag{4}$$

then $\mathfrak{p}(z)$ is a solution of the differential superordination (4). An analytic function $\mathfrak{q}(z)$ is called a subordinant of the solutions of the differential superordination (4) if $\mathfrak{q}(z) \prec \mathfrak{p}(z)$ for all $\mathfrak{p}(z)$ satisfying (4). A univalent subordinant $\tilde{\mathfrak{q}}$ that satisfies $\mathfrak{q} \prec \tilde{\mathfrak{q}}$ for all subordinants of (4) is called the best subordinant (see [14, 15]).

For the functions $f_i(z) \in \mathcal{A}(p)$ $(p \in \mathbb{N},\ i = 2, 3, ..., m)$, $h(z) \in \mathcal{P}$ and the parameters $\beta,\ \alpha_1,\ \alpha_2, ..., \alpha_m \in \mathbb{C}$ with $\beta \neq 0$, we introduce the integral operator $I^{p,m}_{h;\alpha_1,\alpha_i,\beta} : \mathcal{A}(p) \to \mathcal{A}(p)$ as follows:

$$I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z) = \left[\frac{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}{z^{\alpha_1 - p\beta + p\sum\limits_{i=2}^{m}\alpha_i}} \int_0^z \left(\prod_{i=2}^{m} f_i^{\alpha_i}(t) \right) h^{\alpha_1 - 1}(t) h'(t) dt \right]^{\frac{1}{\beta}}. \tag{5}$$

(All powers are principal ones).

We note the next special cases of the above defined integral operator:

(i) For $p = 1,\ m = 2,\ \alpha_1 = \gamma,\ \alpha_2 = \beta$ and $f_2(t) = f(t)$, we obtain

$$I_{h;\beta,\gamma}(f)(z) = \left(\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) h^{\gamma - 1}(t) h'(t) dt \right)^{\frac{1}{\beta}},$$

where the operator $I_{h;\beta,\gamma}$ was introduced and studied by Cho and Bulboacă [6].

(ii) For $p = 1,\ m = 2,\ \alpha_1 = \gamma,\ \alpha_2 = \beta,\ f_2(t) = f(t)$ and $h(t) = t$, we obtain

$$I_{\beta,\gamma}(f)(z) = \left(\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma - 1}(t) dt \right)^{\frac{1}{\beta}},$$

where the operator $I_{\beta,\gamma}$ was introduced by Miller *et al.* [16] and studied by Bulboacă [3–5].

To prove our results, we need the following definitions and lemmas.

Definition 1. *[14] Denote by $\mathcal{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$ where*

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(q)$. Further, denote by $\mathcal{Q}(a)$ the subclass of $\mathcal{Q}$ for which $q(0) = a$.

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Definition 2. *[14] A function $L(z,t)$ $(z \in \mathbb{U},\ t \geq 0)$ is said to be a subordination chain (or Löwner chain) if $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for all $0 \leq s \leq t$.*

Lemma 1. *[17] The function $L(z,t) : \mathbb{U} \times [0,\infty) \longrightarrow \mathbb{C}$ of the form*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + ... \quad (a_1(t) \neq 0;\ t \geq 0)$$

and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\operatorname{Re}\left\{ \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} \right\} > 0 \quad (z \in \mathbb{U};\ t \geq 0),$$

and

$$|L(z,t)| \leq K_0 |a_1(t)| \quad (|z| < r_0 < 1;\ t \geq 0),$$

for some positive constants K_0 and r_0.

Lemma 2. *[10] Suppose that the function $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\operatorname{Re}\{\mathbf{H}(\mathrm{is};\mathrm{t})\} \leq 0$$

for all real s and for all $t \leq -n(1+s^2)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + ...$ is analytic in $\mathbb{U}$ and

$$\operatorname{Re}\{\mathbf{H}(p(z); zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then $\operatorname{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$.

Lemma 3. *[11] Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the following differential equation:*

$$\mathfrak{q}(z) + \frac{z\mathfrak{q}'(z)}{\kappa\mathfrak{q}(z) + \gamma} = h(z) \quad (z \in \mathbb{U};\ \mathfrak{q}(0) = c)$$

is analytic in $\mathbb{U}$ and satisfies $\operatorname{Re}\{\kappa\mathfrak{q}(z) + \gamma\} > 0$ for $z \in \mathbb{U}$.

Lemma 4. *[14] Let $\mathfrak{p} \in \mathcal{Q}(a)$ and let $q(z) = a + a_n z^n + a_{n+1}z^{n+1} + ...$ be analytic in $\mathbb{U}$ with $q(z) \neq a$ and $n \geq 1$. If q is not subordinate to $\mathfrak{p}$, then there exists two points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(q)$ such that*

$$q(\mathbb{U}_{r_0}) \subset \mathfrak{p}(\mathbb{U}),\ q(z_0) = \mathfrak{p}(\zeta_0) \text{ and } z_0\mathfrak{p}'(z_0) = m\zeta_0 q'(\zeta_0) \quad (m \geq n).$$

Lemma 5. *[15] Let $q \in \mathcal{H}[a;1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi(q(z), zq'(z)) = h(z)$. If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $q \in H[a,1] \cap \mathcal{Q}(a)$, then*

$$h(z) \prec \varphi(q(z), zq'(z)),$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}(a)$, then q is the best subordinant.

Let $c \in \mathbb{C}$ with $\mathrm{Re}(c) > 0$ and

$$N = N(c) = \frac{|c| \sqrt{1 + 2\mathrm{Re}(c)} + Im(c)}{\mathrm{Re}(c)}.$$

If $R = R(z) = \frac{2Nz}{1-z^2}$ is a univalent function and $b = R^{-1}(c)$, then the open door function $R_c(z)$ is defined by

$$R_c(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) \ (z \in \mathbb{U}).$$

The function R_c is univalent in $\mathbb{U}$, $R_c(0) = c$ and $R_c(\mathbb{U}) = R(\mathbb{U})$ is the complex plane slit along the half lines $\mathrm{Re}(w) = 0$, $\mathrm{Im}(\mathrm{w}) \geq \mathrm{N}$ and $\mathrm{Re}(w) = 0$, $\mathrm{Im}(\mathrm{w}) \leq -\mathrm{N}$.

Lemma 6. *(Integral Existence Theorem [12–14]) Let $\phi, \Phi \in H$ with $\phi(z) \neq 0$, $\Phi(z) \neq 0$ for $z \in \mathbb{U}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\mathrm{Re}(\alpha + \delta) > 0$. If the function $g(z) \in \mathcal{A}$ and*

$$\alpha \frac{zg'(z)}{g(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta}(z),$$

then

$$G(z) = \left(\frac{\beta + \gamma}{z^{\gamma}\Phi(z)} \int_0^z g^{\alpha}(t)\phi(t)t^{\delta-1}(t)dt \right)^{\frac{1}{\beta}} \in \mathcal{A},$$

$\frac{G(z)}{z} \neq 0 \ (z \in \mathbb{U})$ *and*

$$\mathrm{Re}\left(\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right) > 0 \ (z \in \mathbb{U}).$$

(All powers are principal ones).

Indeed, Lemma 6 is extended for p-valent functions as follows:

Lemma 7. *[18] (see also [1]) Let $p \in \mathbb{N}$, $\phi, \Phi \in H$ with $\phi(z) \neq 0$, $\Phi(z) \neq 0$ for $z \in \mathbb{U}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $p\alpha + \delta = p\beta + \gamma$ and $\mathrm{Re}(p\alpha + \delta) > 0$. If the function $f(z) \in \mathcal{A}(p)$ and*

$$\mathcal{A}_{p,\alpha,\delta} = \left\{ f(z) \in \mathcal{A}(p) : \alpha \frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{p\alpha+\delta}(z) \right\},$$

then

$$F(z) = \left(\frac{p\beta + \gamma}{z^{\gamma}\Phi(z)} \int_0^z f^{\alpha}(t)\phi(t)t^{\delta-1}dt \right)^{\frac{1}{\beta}} = z^p + ... \in \mathcal{A}(p),$$

$\frac{F(z)}{z^p} \neq 0 \ \ (z \in \mathbb{U})$ *and*

$$\operatorname{Re}\left(\beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right) > 0 \ \ (z \in \mathbb{U}).$$

(All powers are principal ones).

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $h \in \mathcal{P}$, $\beta,\ \alpha_1,\ \alpha_2, ..., \alpha_m \in \mathbb{C}$ with $\beta \neq 0$ such that $\operatorname{Re}\left(\alpha_1 + p\sum_{i=2}^{m} \alpha_i\right) > 0$ and all powers are principal ones.

Using similar arguments to Lemma 7, we obtain the following lemma.

Lemma 8. *If* $f_i \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$ $(i = 2, 3, \cdots, m)$, *where*

$$\mathcal{A}_{p,h;\alpha_1,\alpha_i} = \left\{ f_i(z) \in \mathcal{A}(p) : \sum_{i=2}^{m} \alpha_i \frac{zf_i'(z)}{f_i(z)} + 1 + \frac{zh''(z)}{h'(z)} \right. \\ \left. + (\alpha_1 - 1)\frac{zh'(z)}{h(z)} \prec R_{\alpha_1 + p\sum_{i=2}^{m}\alpha_i}(z) \right\}, \tag{6}$$

then $I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z) \in \mathcal{A}(p)$, $\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p} \neq 0$ *and*

$$\operatorname{Re}\left[\beta \frac{z\left(I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)\right)'}{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)} + \alpha_1 + p\sum_{i=2}^{m}\alpha_i - p\beta\right] > 0 \ \ (z \in \mathbb{U}),$$

where $I^{p,m}_{h;\alpha_1,\alpha_i,\beta}$ *is the integral operator defined by (5).*

Theorem 1. *Let* $f_i, g_i \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$ $(i = 2, 3, \cdots, m)$ *and*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \tag{7}$$

$$\left(\phi(z) = z\prod_{i=2}^{m}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1 - 1} h'(z);\ z \in \mathbb{U}\right),$$

where δ *is given by*

$$\delta = \frac{1 + |a|^2 - |1 - a^2|}{4\operatorname{Re}\{a\}} \quad \left(a = \alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1,\ \operatorname{Re}\{a\} > 0\right). \tag{8}$$

Then the subordination condition:

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1 - 1} h'(z) \prec \phi(z) \tag{9}$$

implies that

$$z\left(\frac{I_{h;\alpha_1,\alpha_i,\beta}^{p,m}[f_i](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I_{h;\alpha_1,\alpha_i,\beta}^{p,m}[g_i](z)}{z^p}\right)^{\beta} \tag{10}$$

and the function $z\left(\frac{I_{h;\alpha_1,\alpha_i,\beta}^{p,m}[g_i](z)}{z^p}\right)^{\beta}$ *is the best dominant.*

Proof. Define the functions $\Psi(z)$ and $\Phi(z)$ in $\mathbb{U}$ by

$$\Psi(z) = z\left(\frac{I_{h;\alpha_1,\alpha_i,\beta}^{p,m}[f_i](z)}{z^p}\right)^{\beta} \text{ and } \Phi(z) = z\left(\frac{I_{h;\alpha_1,\alpha_i,\beta}^{p,m}[g_i](z)}{z^p}\right)^{\beta} \quad (z \in \mathbb{U}). \tag{11}$$

From Lemma 8, it follows that these two functions are well defined. We first show that, if

$$q(z) = 1 + \frac{z\Phi''(z)}{\Phi'(z)} \quad (z \in \mathbb{U}), \tag{12}$$

then

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

From (5) and the definitions of the functions $\phi(z)$ and $\Phi(z)$, we obtain

$$\left(\alpha_1 + p\sum_{i=2}^{m}\alpha_i\right)\phi(z) = z\Phi'(z) + \left(\alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1\right)\Phi(z). \tag{13}$$

Hence, it follows that

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i - 1} = h(z) \quad (z \in \mathbb{U}). \tag{14}$$

It follows from (7) and (14) that

$$\operatorname{Re}\left\{h(z) + \alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1\right\} > 0 \quad (z \in \mathbb{U}). \tag{15}$$

Moreover, by using Lemma 3, we conclude that the differential equation (14) has a solution $q(z) \in H(\mathbb{U})$ with $h(0) = q(0) = 1$. Let

$$\mathbf{H}(u,v) = u + \frac{v}{u + \alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i - 1} + \delta,$$

where δ is given by (8). From (14) and (15), we obtain $\operatorname{Re}\{\mathbf{H}(q(z); zq'(z))\} > 0 \ (z \in \mathbb{U})$. To verify the condition

$$\operatorname{Re}\{\mathbf{H}(is;t)\} \leq 0 \quad \left(s \in \mathbb{R};\ t \leq -\frac{1+s^2}{2}\right), \tag{16}$$

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we proceed as follows:

$$\mathrm{Re}\left\{\mathbf{H}\left(is;t\right)\right\}=\mathrm{Re}\left\{is+\frac{t}{is+a}+\delta\right\}=\delta+\frac{t\mathrm{Re}\{a\}}{\left|is+a\right|^{2}}\leq-\frac{E_{\delta}\left(s\right)}{2\left|a+is\right|^{2}},$$

where

$$E_{\delta}\left(s\right)=\left(\mathrm{Re}\{a\}-2\delta\right)s^{2}-4\delta\left(\mathrm{Im}a\right)s+\left(\mathrm{Re}\{a\}-2\delta\left|a\right|^{2}\right). \tag{17}$$

For δ given by (8), the coefficient of s^2 in the quadratic expression $E_{\delta}(s)$ given by (17) is positive or equal to zero and $E_{\delta}(s)\geq 0$. Thus, we see that $\mathrm{Re}\{\mathbf{H}(is;t)\}\leq 0$ for all $s\in\mathbb{R}$ and $t\leq-\frac{1+s^2}{2}$. Thus, by using Lemma 2, we conclude that

$$\mathrm{Re}\left\{q\left(z\right)\right\}>0\ \ \left(z\in\mathbb{U}\right),$$

that is, that $\Phi(z)$ defined by (11) is convex (univalent) in $\mathbb{U}$. Next, we prove that the subordination condition (9) implies that

$$\Psi\left(z\right)\prec\Phi\left(z\right),$$

for $\Psi(z)$ and $\Phi(z)$ defined by (11). Without loss of generality, we assume that $\Phi(z)$ is analytic, univalent on $\overline{\mathbb{U}}$ and

$$\Phi'(\zeta)\neq 0\ \ (|\zeta|=1).$$

If not, then we replace $\Psi(z)$ and $\Phi(z)$ by $\Psi(\rho z)$ and $\Phi(\rho z)$, respectively, with $0<\rho<1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, so we can use them in the proof of our result and the result would follow by letting $\rho\to 1$. Consider the function $L(z,t)$ given by

$$L\left(z,t\right)=\left(1-\frac{1}{\alpha_{1}+p\sum\limits_{i=2}^{m}\alpha_{i}}\right)\Phi\left(z\right)+\frac{\left(1+t\right)}{\alpha_{1}+p\sum\limits_{i=2}^{m}\alpha_{i}}z\Phi'\left(z\right)\ \ \left(0\leq t<\infty;\ z\in\mathbb{U}\right). \tag{18}$$

We note that

$$\left.\frac{\partial L\left(z,t\right)}{\partial z}\right|_{z=0}=\left(1+\frac{t}{\alpha_{1}+p\sum\limits_{i=2}^{m}\alpha_{i}}\right)\Phi'\left(0\right)\neq 0\ \ \left(0\leq t<\infty;\ z\in\mathbb{U}\right).$$

This show that the function

$$L\left(z,t\right)=a_{1}\left(t\right)z+a_{2}(t)z^{2}+...,$$

satisfy the conditions $\lim\limits_{t\to\infty}\left|a_{1}\left(t\right)\right|=\infty$ and $a_{1}\left(t\right)\neq 0\ \ \left(0\leq t<\infty\right)$. Further, we have

$$\mathrm{Re}\left\{\frac{z\dfrac{\partial L\left(z,t\right)}{\partial z}}{\dfrac{\partial L\left(z,t\right)}{\partial t}}\right\}=\mathrm{Re}\left\{\alpha_{1}+p\sum_{i=2}^{m}\alpha_{i}-1+\left(1+t\right)\left(1+\frac{z\Phi''\left(z\right)}{\Phi'\left(z\right)}\right)\right\}>0$$

$$\left(0\leq t<\infty;\ z\in\mathbb{U}\right),$$

since $\Phi(z)$ is convex and $\mathrm{Re}\left\{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i - 1\right\} > 0$, by using the well-known growth and distortion sharp inequalities for convex functions (see [8]), the second inequality of Lemma 1 is satisfied and so $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = \left(1 - \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi(z) + \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i} z\Phi'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \ \ (0 \le t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \ \ (0 \le t < \infty;\ \zeta \in \partial\mathbb{U}). \tag{19}$$

If $\Psi(z)$ is not subordinate to $\Phi(z)$, by using Lemma 4, we know that there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$\Psi(z_0) = \Phi(\zeta_0) \ \text{and}\ z_0\Psi'(z_0) = (1+t)\zeta_0\Phi'(\zeta_0) \ \ (0 \le t < \infty). \tag{20}$$

Hence, by using (10), (18), (20) and (8), we have

$$\begin{aligned} L(\zeta_0,t) &= \left(1 - \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi(\zeta_0) + \frac{(1+t)}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\zeta_0\Phi'(\zeta_0) \\ &= \left(1 - \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\right)\Psi(z_0) + \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}z_0\Psi'(z_0) \\ &= z_0\prod_{i=2}^{m}\left(\frac{f_i(z_0)}{z_0^p}\right)^{\alpha_i}\left(\frac{h(z_0)}{z_0}\right)^{\alpha_1-1}h'(z_0) \in \phi(\mathbb{U}). \end{aligned}$$

This contradicts (19). Thus, we deduce that $\Psi \prec \Phi$. Considering $\Psi = \Phi$, we see that the function Φ is the best dominant. This completes the proof of Theorem 1. ∎

We now derive the following superordination result.

Theorem 2. *Let $f_i, g_i \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$ $(i = 2, 3, \cdots, m)$ and*

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$

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$$\left(\phi\left(z\right)=z\prod_{i=2}^{m}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\ z\in\mathbb{U}\right),$$

where δ *is given by (8). If the function*

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z)$$

is univalent in $\mathbb{U}$ *and* $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}\in\mathcal{H}[0,1]\cap\mathcal{Q}$. *Then the superordination condition*

$$\phi\left(z\right)\prec z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \tag{21}$$

implies that

$$z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta}\prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \tag{22}$$

and the function $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta}$ *is the best subordinant.*

Proof. Suppose that the functions $\Psi(z)$, $\Phi(z)$ and $q(z)$ are defined by (11) and (12), respectively. We will use similar method as in the proof of Theorem 1. As in Theorem 1, we have

$$\phi\left(z\right)=\left(1-\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi\left(z\right)+\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}z\Phi'\left(z\right)=\varphi\left(G\left(z\right),zG'\left(z\right)\right)$$

and we obtain

$$\operatorname{Re}\left\{q\left(z\right)\right\}>0\ \ \left(z\in\mathbb{U}\right).$$

Next, to obtain the desired result, we show that $\Phi(z)\prec\Psi(z)$. For this, we suppose that the function

$$L\left(z,t\right)=\left(1-\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi\left(z\right)+\frac{t}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}z\Phi'\left(z\right)\ \ \left(0\leq t<\infty;\ z\in\mathbb{U}\right).$$

We note that

$$\left.\frac{\partial L\left(z,t\right)}{\partial z}\right|_{z=0}=\left(1-\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi'\left(0\right)\neq0\ \ \left(0\leq t<\infty;\ z\in\mathbb{U}\right).$$

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This show that the function

$$L(z,t) = a_1(t)\,z + a_2(t)z^2 + ...$$

satisfy the conditions $\lim_{t\to\infty}|a_1(t)| = \infty$ and $a_1(t) \neq 0 \ (0 \leq t < \infty)$. Further, we have

$$\mathrm{Re}\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} = \mathrm{Re}\left\{\alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1 + t\left(1+\frac{z\Phi''(z)}{\Phi'(z)}\right)\right\} > 0$$

$$(0 \leq t < \infty;\ z \in \mathbb{U}),$$

since $\Phi(z)$ is convex and $\mathrm{Re}\left\{\alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1\right\} > 0$. By using the well-known growth and distortion sharp inequalities for convex functions (see [8]), the second inequality of Lemma 1 is satisfied and so $L(z,t)$ is a subordination chain. Therefore, by using Lemma 5, we conclude that the superordination condition (21) must imply the superordination given by (22). Moreover, since the differential equation has a univalent solution Φ, it is the best subordinant. This completes the proof of Theorem 2. ∎

Combining Theorems 1 and 2, the following sandwich-type results are derived.

Theorem 3. *Let* $f, g_j \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$, $(i = 2, 3, \cdots, m;\ j = 1,\ 2)$ *and*

$$Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta$$

$$\left(\phi_j(z) = z\prod_{i=2}^{m}\left(\frac{g_{i,j}(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\ z \in \mathbb{U}\right),$$

where δ *is given by (8). If the function*

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z)$$

is univalent in $\mathbb{U}$ *and* $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \in \mathcal{H}[0,1] \cap \mathcal{Q}$. *Then*

$$\phi_1(z) \prec z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \prec \phi_2(z)$$

implies that

$$z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^{\beta}.$$

Moreover, the functions $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^{\beta}$ *and* $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^{\beta}$ *are, respectively, the best subordinant and the best dominant.*

We note that the assumption of Theorem 3 that the functions

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \text{ and } z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}$$

need to be univalent in $\mathbb{U}$, may be replaced as in the following corollary.

Corollary 1. *Let* $f, g_j \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$, $(i = 2, 3, \cdot, m;\ j = 1,\ 2)$ *and*

$$Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta$$

$$\left(\phi_j(z) = z\prod_{i=2}^{m}\left(\frac{g_{i,j}(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\ z\in\mathbb{U}\right)$$

and

$$\text{Re}\left\{1+\frac{z\Theta''(z)}{\Theta'(z)}\right\} > -\delta \tag{23}$$

$$\left(\Theta(z) = z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\ z\in\mathbb{U}\right),$$

where δ *is given by (8). Then*

$$\phi_1(z) \prec z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \prec \phi_2(z)$$

implies that

$$z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^{\beta}.$$

Proof. To prove Corollary 1, we have to show that condition (23) implies the univalence of $\Theta(z)$ and $\Psi(z) = z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}$. Since $0 \le \delta < \frac{1}{2}$, it follows that $\Theta(z)$ is close to convex function in $\mathbb{U}$ (see [9]) and hence $\Theta(z)$ is univalent in $\mathbb{U}$. Also, by using the same techniques as in the proof of Theorem 1, we can prove that Ψ is convex (univalent) in $\mathbb{U}$, and so the details may be omitted. Therefore, by applying Theorem 3, we obtain the desired result. ∎

Remark 1. *(i) Putting $p = 1,\ m = 2,\ \alpha_1 = \gamma,\ \alpha_2 = \beta$ and $f_2(t) = f(t)$ in Theorem $1, 2,\ 3$ and Corollary 1, we obtain the results by Cho and Bulboacă [6] and the results by Al-Kharsani et al. [2];*

(ii) If we take $\alpha_1 = 0$ in the results mentioned above, then we have those by Aouf et. al [1]. Moreover, putting $p = 1,\ m = 2,\ \alpha_1 = 0,\ \alpha_2 = \beta$ and $f_2(t) = f(t)$ in Theorem $1, 2,\ 3$ and Corollary 1, we obtain the results by Cho and Kim [7].

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(M. K. Aouf) Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

E-mail address: mkaouf127@yahoo.com

(H. M. Zayed) Department of Mathematics, Faculty of Science, Menofia University, Shebin Elkom 32511, Egypt

E-mail address: hanaa_zayed42@yahoo.com

(N. E. Cho) Corresponding Author, Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

E-mail address: necho@pknu.ac.kr

ADDITIVE s-FUNCTIONAL INEQUALITIES AND DERIVATIONS ON BANACH ALGEBRAS

TAEKSEUNG KIM, YOUNGHUN JO*, JUNHA PARK, JAEMIN KIM, CHOONKIL PARK*, AND JUNG RYE LEE

ABSTRACT. In this paper, we introduce the following new additive s-functional inequalities

$$\|f(x-y)+f(y)+f(-x)\| \le \|s\left(f(x+y)-f(x)-f(y)\right)\|, \tag{0.1}$$

$$\|f(x+y)-f(x)-f(y)\| \le \|s\left(f(x-y)+f(y)+f(-x)\right)\|, \tag{0.2}$$

where s is a fixed complex number with $|s|<1$, and prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequalities (0.1) and (0.2).

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [20] concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 9, 13, 14, 17, 18, 19]).

Gilányi [4] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

See also [16]. Fechner [2] and Gilányi [5] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [12] investigated the Cauchy additive functional inequality

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\| \tag{1.3}$$

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*Corresponding authors.

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [10, 11] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.1).

In Section 3, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.2).

Throughout this paper, assume that s is a fixed complex number with $|s| < 1$.

2. Stability of linear derivations on Banach algebras associated to the functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.1).

Theorem 2.1. *Let $\theta \geq 0$ and p be real numbers with $p > 2$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying*

$$\|f(\lambda(x-y)) + \lambda f(y) + \lambda f(-x)\| \leq \|s\left(f(x+y) - f(x) - f(y)\right)\| + \theta\left(\|x\|^p + \|y\|^p\right), \tag{2.1}$$

$$\|f(xy) - xf(y) - yf(x)\| \leq \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.2}$$

for all $\lambda \in \mathbb{S}^1 := \{\mu \in \mathbb{C} | |\mu| = 1\}$ and all $x, y \in \mathcal{B}$. Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that

$$\|f(x) - D(x)\| \leq \frac{2\theta}{(2^p - 2)(1 - |s|)}\|x\|^p \tag{2.3}$$

for all $x \in \mathcal{B}$.

Proof. Letting $x = y = 0$ and $\lambda = -1 \in \mathbb{S}^1$ in (2.1), we get $\|f(0)\| \leq \|sf(0)\|$ and so we get $f(0) = 0$.

Replacing y by x and letting $\lambda = 1$ in (2.1), we get

$$\|f(x) + f(-x)\| \leq \|s\left(f(2x) - 2f(x)\right)\| + 2\theta\|x\|^p \tag{2.4}$$

for all $x \in \mathcal{B}$.

Replacing x by $-x$ and y by x and letting $\lambda = -1$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \leq \|s\left(f(x) + f(-x)\right)\| + 2\theta\|x\|^p \tag{2.5}$$

for all $x \in \mathcal{B}$.

From (2.4) and (2.5), we get

$$\|f(2x) - 2f(x)\| \leq |s|^2\|f(2x) - 2f(x)\| + 2(1 + |s|)\theta\|x\|^p$$

and so

$$\|f(2x) - 2f(x)\| \leq \frac{2\theta}{1 - |s|}\|x\|^p \tag{2.6}$$

for all $x \in \mathcal{B}$. So one can obtain that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{2\theta}{2^p(1 - |s|)}\|x\|^p$$

and hence

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\| \le \frac{2\cdot 2^{n(1-p)}\theta}{2^p(1-|s|)}\|x\|^p$$

for all $x \in \mathcal{B}$. So we get

$$\left\|f(x) - 2^n f\left(\frac{x}{2^n}\right)\right\| \le \sum_{l=0}^{n-1} \frac{2\cdot 2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p \tag{2.7}$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{l=m}^{n-1} \frac{2\cdot 2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p,$$

which tends to zero as $m \to \infty$. So $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by

$$D(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.8}$$

for all $x \in \mathcal{B}$.

Letting $x = 0$ in (2.1), we get

$$\|f(\lambda x) + \lambda f(-x)\| \le \theta\|x\|^p$$

for all $\lambda \in \mathbb{S}^1$ and all $x \in \mathcal{B}$.

By (2.8), we get

$$\|D(\lambda x) + \lambda D(-x)\| = \lim_{n\to\infty}\left\|2^n f\left(\frac{\lambda x}{2^n}\right) + 2^n \lambda f\left(-\frac{x}{2^n}\right)\right\| \le \lim_{n\to\infty}\frac{2^n}{2^{pn}}\theta\|x\|^p = 0$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{S}^1$. Hence

$$D(\lambda x) + \lambda D(-x) = 0 \tag{2.9}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{S}^1$.

Letting $\lambda = 1$ in (2.9), we get

$$D(x) + D(-x) = 0 \tag{2.10}$$

for all $x \in \mathcal{B}$. Hence

$$D(\lambda x) = \lambda D(x) \tag{2.11}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{S}^1$.

Let $\lambda = 1$ in (2.1). By (2.1), (2.8) and (2.10), we get

$$\begin{aligned}
&\|D(x-y) - D(x) + D(y)\| = \lim_{n\to\infty}\left\|2^n f\left(\frac{x-y}{2^n}\right) + 2^n f\left(\frac{y}{2^n}\right) + 2^n f\left(-\frac{x}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty}\left(\left\|s\left(2^n f\left(\frac{x+y}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right)\right\| + 2^{n(1-p)}\theta(\|x\|^p + \|y\|^p)\right) \\
&= \|s\left(D(x+y) - D(x) - D(y)\right)\|
\end{aligned}$$

for all $x, y \in \mathcal{B}$. Hence

$$\|D(x-y) - D(x) + D(y)\| \le \|s(D(x+y) - D(x) - D(y))\| \tag{2.12}$$

for all $x, y \in \mathcal{B}$.

Replacing y by $-y$ in (2.12), we get

$$\|D(x+y) - D(x) - D(y)\| \le \|s(D(x-y) - D(x) + D(y))\| \tag{2.13}$$

for all $x, y \in \mathcal{B}$.

It follows from (2.12) and (2.13) that

$$\|D(x+y) - D(x) - D(y)\| \le \|s^2(D(x+y) - D(x) - D(y))\|$$

for all $x, y \in \mathcal{B}$. Since $|s| < 1$, we get

$$\|D(x+y) - D(x) - D(y)\| = 0$$

for all $x, y \in \mathcal{B}$. So one can obtain that D is additive. Moreover, by passing to the limit in (2.7) as $n \to \infty$, we get the inequality (2.3).

Now let $S : \mathcal{B} \to \mathcal{B}$ be another additive mapping satisfying

$$\|f(x) - S(x)\| \le \frac{2^p}{2^p - 2}\theta\|x\|^p$$

for all $x \in \mathcal{B}$.

$$\begin{aligned}
\|D(x) - S(x)\| &= 2^l \left\|D\left(\frac{x}{2^l}\right) - S\left(\frac{x}{2^l}\right)\right\| \\
&\le 2^l \left\|D\left(\frac{x}{2^l}\right) - f\left(\frac{x}{2^l}\right)\right\| + 2^l \left\|f\left(\frac{x}{2^l}\right) - S\left(\frac{x}{2^l}\right)\right\| \\
&\le \frac{2^{l+1}}{2^{lp}} \times \frac{2^p}{2^p - 2}\theta\|x\|^p,
\end{aligned}$$

which tends to zero as $l \to \infty$. Thus $D(x) = S(x)$ for all $x \in \mathcal{B}$. This proves the uniqueness of D.

Now let $\mu \in \mathbb{C}$ ($\mu \ne 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [7, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{S}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. By (2.11),

$$\begin{aligned}
D(\lambda x) &= D\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) = M \cdot D\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) = \frac{M}{3}D\left(3\frac{\lambda}{M}x\right) \\
&= \frac{M}{3}D(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(D(\mu_1 x) + D(\mu_2 x) + D(\mu_3 x)) \\
&= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)D(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}D(x) \\
&= \lambda D(x)
\end{aligned}$$

for all $x \in \mathcal{B}$. Hence

$$D(\alpha x + \beta y) = D(\alpha x) + D(\beta y) = \alpha D(x) + \beta D(y)$$

for all $\alpha, \beta \in \mathbb{C}(\alpha, \beta \ne 0)$ and all $x, y \in \mathcal{B}$. And $D(0x) = 0 = 0D(x)$ for all $x \in \mathcal{B}$. So the unique additive mapping $D : \mathcal{B} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

It follows from (2.2) and (2.8) that

$$\begin{aligned}
\|D(xy) - xD(y) - yD(x)\| &= \lim_{n\to\infty} \left\|2^{2n} f\left(\frac{x}{2^n}\frac{y}{2^n}\right) - 2^n x f\left(\frac{y}{2^n}\right) - 2^n y f\left(\frac{x}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 2^{n(2-p)}\theta(\|x\|^p + \|y\|^p) = 0
\end{aligned}$$

for all $x, y \in \mathcal{B}$. Hence

$$D(xy) = xD(y) + yD(x)$$

for all $x, y \in \mathcal{B}$. Hence the mapping $D : \mathcal{B} \to \mathcal{B}$ is a $\mathbb{C}$-linear derivation satisfying (2.3). □

Theorem 2.2. *Let $\theta \geq 0$ and p be real numbers with $0 < p < 1$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying (2.1) and (2.2). Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that*

$$\|f(x) - D(x)\| \leq \frac{2\theta}{(2 - 2^p)(1 - |s|)} \|x\|^p \tag{2.14}$$

for all $x \in \mathcal{B}$.

Proof. It follows from (2.6) that $\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{\theta}{1-|s|}\|x\|^p$ and hence

$$\left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x)\right\| \leq \frac{2^{pn}\theta}{2^n(1 - |s|)}\|x\|^p$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\right\| \leq \sum_{l=m}^{n-1} \frac{2^{pl}\theta}{2^l(1 - |s|)}\|x\|^p, \tag{2.15}$$

which tends to zero as $m \to \infty$. So $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by $D(x) = \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$ for all $x \in \mathcal{B}$.

Moreover, by letting $m = 0$ and passing to the limit in (2.15) as $n \to \infty$, we get (2.14).

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Stability of linear derivations on Banach algebras associated to the functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.2).

Theorem 3.1. *Let $\theta \geq 0$ and p be real numbers with $p > 2$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying (2.2) and*

$$\|f(\lambda(x + y)) - \lambda f(x) - \lambda f(y)\| \leq \|s(f(x - y) + f(y) + f(-x))\| + \theta(\|x\|^p + \|y\|^p) \tag{3.1}$$

for all $\lambda \in \mathbb{S}^1$ and all $x, y \in \mathcal{B}$. Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that

$$\|f(x) - D(x)\| \leq \frac{(2 - |s|)\theta}{(2^p - 2)(1 - |s|)}\|x\|^p \tag{3.2}$$

for all $x \in \mathcal{B}$.

Proof. Letting $x = y = 0$ and $\lambda = -1 \in \mathbb{S}^1$ in (3.1), we get

$$\|3f(0)\| \leq \|3sf(0)\|$$

and so we get $f(0) = 0$.

Letting $y = 0$ and $\lambda = -1$ in (3.1), we get

$$\|f(-x) + f(x)\| \leq \|s(f(x) + f(-x))\| + \theta\|x\|^p$$

and so

$$\|f(-x)+f(x)\| \le \frac{1}{1-|s|}\theta\|x\|^p$$

for all $x \in \mathcal{B}$.

Letting $y = x$ and $\lambda = 1$ in (3.1), we get

$$\begin{aligned} \|f(2x)-2f(x)\| &\le \|s\left(f(x)+f(-x)\right)\| + 2\theta\|x\|^p \\ &\le \frac{|s|}{1-|s|}\theta\|x\|^p + 2\theta\|x\|^p = \frac{2-|s|}{1-|s|}\theta\|x\|^p \end{aligned} \tag{3.3}$$

for all $x \in \mathcal{B}$. So one can obtain that

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \le \frac{(2-|s|)\theta}{2^p(1-|s|)}\|x\|^p$$

and hence

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\| \le \frac{(2-|s|)2^{n(1-p)}\theta}{2^p(1-|s|)}\|x\|^p$$

for all $x \in \mathcal{B}$. So we get

$$\left\|f(x) - 2^n f\left(\frac{x}{2^n}\right)\right\| \le \sum_{l=0}^{n-1} \frac{(2-|s|)2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p \tag{3.4}$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{l=m}^{n-1} \frac{(2-|s|)2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p,$$

which tends to zero as $m \to \infty$. So $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by

$$D(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{3.5}$$

for all $x \in \mathcal{B}$.

It follows from (3.1) and (3.5) that

$$\begin{aligned} &\|D(\lambda(x+y)) - \lambda D(x) - \lambda D(y)\| = \lim_{n\to\infty} \left\|2^n f\left(\lambda\frac{x+y}{2^n}\right) - 2^n\lambda f\left(\frac{x}{2^n}\right) - 2^n\lambda f\left(\frac{y}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty}\left(\left\|s\left(2^n f\left(\frac{x-y}{2^n}\right) + 2^n f\left(\frac{y}{2^n}\right) + 2^n f\left(\frac{-x}{2^n}\right)\right)\right\| + 2^{n(1-p)}\theta(\|x\|^p + \|y\|^p)\right) \\ &= \|s\left(D(x-y)+D(y)+D(-x)\right)\| \end{aligned}$$

for all $\lambda \in \mathbb{S}^1$ and all $x, y \in \mathcal{B}$. Hence

$$\|D(\lambda(x+y)) - \lambda D(x) - \lambda D(y)\| = \|s\left(D(x-y)+D(y)+D(-x)\right)\| \tag{3.6}$$

for all $\lambda \in \mathbb{S}^1$ and all $x, y \in \mathcal{B}$.

Letting $\lambda = -1$ and $x = y = 0$ in (3.6), we get $\|3D(0)\| \le \|3sD(0)\|$ and so $D(0) = 0$.

Replacing x by $-x$ and letting $y = -x$ and $\lambda = -1$ in (3.6), we get

$$\|D(2x) + 2D(-x)\| \le \|s(D(-x)+D(x))\|$$

for all $x \in \mathcal{B}$.

Letting $y = -x$ and $\lambda = 1$ in (3.6), we get

$$\|D(x) + D(-x)\| \le \|s(D(2x) + 2D(-x))\| \le |s|^2 \|D(x) + D(-x)\|$$

and so $D(-x) = -D(x)$ for all $x \in \mathcal{B}$.

Replacing y by $-y$ and letting $\lambda = 1$ in (3.6), we get

$$\|D(x-y) - D(x) + D(y)\| \le \|s\left(D(x+y) - D(y) - D(x)\right)\|$$

for all $x, y \in \mathcal{B}$.

Letting $\lambda = 1$ in (3.6), we get

$$\begin{aligned} \|D(x+y) - D(x) - D(y)\| &\le \|s\left(D(x-y) + D(y) + D(-x)\right)\| \\ &\le |s|^2 \|D(x+y) - D(x) - D(y)\| \end{aligned}$$

for all $x, y \in \mathcal{B}$. Thus $D(x+y) = D(x) + D(y)$ for all $x, y \in \mathcal{B}$.

Letting $y = 0$ in (3.6), we get

$$\|D(\lambda x) - \lambda D(x)\| \le 0$$

and so $D(\lambda x) = \lambda D(x)$ for all $\lambda \in \mathbb{S}^1$ and $x \in \mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.2. *Let $\theta \ge 0$ and p be real numbers with $0 < p < 1$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying* (3.1) *and* (2.2). *Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that*

$$\|f(x) - D(x)\| \le \frac{(2-|s|)\theta}{(2-2^p)(1-|s|)} \|x\|^p \tag{3.7}$$

for all $x \in \mathcal{B}$.

Proof. It follows from (3.3) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{(2-|s|)\theta}{2(1-|s|)} \|x\|^p$$

and hence

$$\left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^{n+1}} f(2^{n+1} x) \right\| \le \frac{(2-|s|)2^{pn}\theta}{2^{n+1}(1-|s|)} \|x\|^p$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x) \right\| \le \sum_{l=m}^{n-1} \frac{(2-|s|)2^{pl}\theta}{2^{l+1}(1-|s|)} \|x\|^p, \tag{3.8}$$

which tends to zero as $m \to \infty$. So $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by $D(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in \mathcal{B}$.

Moreover, by letting $m = 0$ and passing to the limit in (3.8) as $n \to \infty$, we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

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Taekseung Kim, Younghun Jo, Junha Park, Jaemin Kim
Mathematics Branch, Seoul Science High School, Seoul 03066, Korea
E-mail address: andyk0202@naver.com; stop0422@naver.com; mathmario@naver.com; jm1997a@naver.com

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

Jung Rye Lee
Depatment of Mathematics, Daejin University, Kyunggi 11159, Korea
E-mail address: jrlee@daejin.ac.kr

On modulus of convexity of quasi-Banach spaces

Shin Min Kang[1,2], Hussain Minhaj Uddin Ahmad Qadri[3],
Waqas Nazeer[4,*] and Absar Ul Haq[5]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan

[3]Aitchison College, Lahore 54000, Pakistan
e-mail: minhaj_h@hotmail.com

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[5]Department of Mathematics, University of Management and Technology, Sialkot Campus, Lahore 51410, Pakistan
e-mail: absarulhaq@hotmail.com

Abstract

The aim of this report is to study modulus of convexity $\delta_{\mathcal{B}}$ of a quasi-Banach space $\mathcal{B}$. We prove that $\delta_{\mathcal{B}}$ is convex, continuous, nondecreasing and for arbitrary uniformly convex quasi-Banach space $\mathcal{B}$, $\delta_{\mathcal{B}}(\epsilon) = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}$. We also prove that a quasi-Banach space $\mathcal{B}$ is uniformly convex if and only if $\delta_{\mathcal{B}}(\epsilon) \geq 0$. Moreover we prove that a non-trivial quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $\delta_{\mathcal{B}}(\epsilon) > 0$.

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Key words and phrases: modulus of convexity, uniformly convex, uniformly non-square, quasi Banach space.

1 Introduction

Many of the geometric constants for Banach spaces have been investigated so far. These constants play an important role in the description of various geometric structures of Banach spaces. In 1899 Jung [10] was the first who introduced a geometric constant for Banach spaces. In 1936 and 1937, Clarkson [4,5] introduced classical modulus of convexity to define a uniformly convex space. A great number of such moduli have been defined and introduced since then. The theory of the geometry of a Banach space has evolved very rapidly over the past fifty years. By contrast the study of a quasi-Banach space has lagged far behind, even though the first research papers in the subject appeared in the early 1940's [2,4–6]. There are very sound reasons to develop the understanding of these space, but the absence of one of the fundamental tools of functional analysis, the Hahn-Banach

* Corresponding author

theorem, has proved a very significant stumbling block. However, there has been some progress in the non-convex theory and arguably it has contributed to our appreciation of Banach space theory. A systematic study of a quasi-Banach space only really started in the late 1950's and early 1960's with the work of several authors. The efforts of these researchers tended to go in rather separate directions. The subject was given great impetus by the paper of Duren et al. [7] in 1969 which demonstrated both the possibilities for using quasi-Banach spaces in classical function theory and also high-lighted some key problems related to the Hahn-Banach theorem. This opened up many new directions of research. The 1970's and 1980's saw a significant increase in activity with a number of authors contributing to the development of a coherent theory. An important breakthrough was the work of Roberts [13, 14] who showed that the Krein-Milman Theorem fails in general quasi-Banach spaces by developing powerful new techniques. Quasi-Banach spaces (H^p-space when $p < 1$) were also used significantly in Alexandrov's solution of the inner function problem in 1982 [1]. During this period three books on the subject appeared by Turpin [16], Rolewicz [15] (actually an expanded version of a book first published in 1972 and the author, Roberts [14]. In the 1990's it seems to the author that while more and more analysts find that quasi-Banach spaces have uses in their research, paradoxically the interest in developing a general theory has subsided somewhat. The strictly convex Banach spaces were introduced in 1936 by Clarkson, [4], who also studied the concept of uniform convexity. The uniform convexity of L^p spaces, $1 < p < \infty$, was established by Clarkson [4]. The concept of duality map was introduced in 1962 by Beurling and Livingston [3] and was further developed by many others and, De Figueiredo [8]. General properties of the duality map can be found in De Figueiredo [8].

In this paper we aim study modulus of convexity in the setting of quasi Banach spaces.

2 Preliminaries

Throughout this paper $S_{\mathcal{B}}$ is a closed unit ball in a quasi Banach space.

Definition 2.1. A uniformly convex space is a normed vector space so that, for every $0 < \epsilon \leq 2$ there is some $\delta > 0$ so that for any two vectors with $\|x\| = 1$ and $\|y\| = 1$, the condition $\|x - y\| \geq \varepsilon$ implies that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. Intuitively, the center of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short.

Definition 2.2. A quasi-Banach space $\mathcal{B}$ is said to be uniformly non-square if there exists a positive number $\delta < 2$ such that for any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\min\left(\left\|\frac{x_1 + x_2}{C}\right\|, \left\|\frac{x_1 + x_2}{C}\right\|\right) \leq \delta.$$

Definition 2.3. Let $\epsilon \in [0, 2]$ and $C \geq 1$. For a quasi-Banach space $\mathcal{B}$, the modulus of convexity is a function $\delta_{\mathcal{B}} : (0, 2] \longrightarrow [0, 1]$ defined as

$$\delta_{\mathcal{B}}(\epsilon) = \inf\left\{1 - \frac{\|x_1 + x_2\|}{2C} : x_1, x_2 \in S_{\mathcal{B}};\ \frac{\|x_1 - x_2\|}{C} \geq \epsilon\right\}. \tag{2.1}$$

A characteristic or related coefficient of this modulus is

$$\delta_0(\mathcal{B}) = \sup\left\{\epsilon \in [0, 2] : \delta_{\mathcal{B}}(\epsilon) = 0\right\}. \tag{2.2}$$

3 Main results

Lemma 3.1. ([9]) *Every convex function f with convex domain in $\mathbb{R}$ is continuous.*

Lemma 3.2. *Let $\mathcal{B}$ be a quasi-Banach space, and $x_1, x_2 \in S_{\mathcal{B}}$. Then*

$$\frac{\|x_1 + x_2\|}{2C} \leq 1 - \delta_{\mathcal{B}}\left(\frac{\|x_1 - x_2\|}{C}\right). \tag{3.1}$$

Proof. Let $\dim(\mathcal{B}) < \infty$. Let $\epsilon \in [0, 2]$ and choose $u, v \in S_{\mathcal{B}}$ such that $\frac{\|u-v\|}{C}$ is maximal subject to $\frac{\|u-v\|}{C} = \epsilon$. So, here this is enough to prove that $\|u\| = \|v\| = 1$.

The case $\epsilon = 0$ is trivial.

Assume that $\epsilon \neq 0$. Let $x^* \in X^*$ satisfying $\|x^*\| = 1$ and $x^*(u + v) = \frac{\|u+v\|}{2C}$. It would be suffices to prove that if, say, $\|v\| < 1$, then $x^*(v - u) = \epsilon$ and $\|u\| < 1$. Indeed an analogous reasoning would then yields, $x^*(u - v) = \epsilon$ and hence $\epsilon = -\epsilon$, which is a contradiction.

To this end, let $\mathcal{A} = \{w \in \mathcal{B} : \frac{\|w-u\|}{C} = \epsilon\}$. If $w \in \mathcal{A} \cap S_{\mathcal{B}}$, then by maximality of $\frac{\|u+v\|}{2C}$ we get

$$x^*(u + w) \leq \frac{\|u + w\|}{2C} \leq \frac{\|u + v\|}{2C} \leq x^*(u + v).$$

Hence, if we had $\|v\| < 1$, then x^* would attain at v local maximum on $\mathcal{A}$. Consequently, x^* would norm the vector $v - u$, that is, $x^*(v - u) = \frac{\|v-u\|}{C} = \epsilon$. And also

$$\begin{aligned}\|u\| &< \frac{1}{2C}\Big(\|u + v\| + \|u - v\|\Big) \\ &< \frac{1}{2C}\Big[x^*(u + v) + x^*(u - v)\Big] \\ &= x^*(v) < 1\end{aligned}$$

as permitted. This completes the proof. □

Lemma 3.3. *Let $\mathcal{B}$ be a quasi-Banach space. and $\epsilon \in (0, 2]$. Then the following statements holds:*

(a) $\delta_{\mathcal{B}}(\epsilon)$ is convex and continuous function.

(b) $\delta_{\mathcal{B}}(\epsilon)$ is a non-decreasing function.

(c) $\delta_{\mathcal{B}}(\epsilon)/\epsilon$ is a non-decreasing function.

Proof. (a) Consider any two vectors $u, v \in \mathcal{B}$, we denote by $N(u, v)$ the set of all pairs $x, y \in \mathcal{B}$ with $x, y \in S_{\mathcal{B}}(0)$ such that for some real scalars α_1, β_1 we have $x - y = \alpha u$ and $x + y = \beta v$, that is, $N(u, v) = \big\{(x, y) : x - y = \alpha u, x + y = \beta v\big\}$. For $r \in (0, 2)$ define

$$\delta(u, v, r) = \inf\left\{1 - \frac{\|x + y\|}{2C} : \ x, y \in N(u, v), \ \frac{\|x - y\|}{C} \geq r\right\}. \tag{3.2}$$

It is easy to check $\delta(u, v, r) = 0$ for (3.2) as $\|x\| = 1, \forall x \in N(u, v)$. Moreover, for r, for any given $\lambda_1, \lambda_2 \in (0, 2)$ and $\epsilon > 0$ we can choose $x_k, y_k \in N(u, v)$ such that (for $k = 1, 2$)

$$x_k + y_k \geq \lambda_k \quad \text{and} \quad \delta(u, v, \lambda_k) + \frac{\epsilon}{2} \geq 1 - \frac{\|x_k + y_k\|}{2C}.$$

The choice of (x_k, y_k) is possible because of the definition of $\delta(u, v, r)$ in (3.2) as infimum. Now, for $\lambda \in (0, 1)$ we assume

$$x_3 = \lambda x_1 + (1 - \lambda)x_2, \quad y_3 = \lambda y_1 + (1 - \lambda)y_2. \tag{3.3}$$

$\|x_3\| = \lambda\|x_1\| + (1 - \lambda)\|x_2\|$ because $x_1, x_2 \in \overline{S_{\mathcal{B}}(0)}$. Similarly, $(x_k, y_k) \in N(u, v)$ implies that there exist constants such that (for $k = 1, 2$)

$$x_k - y_k = \alpha_k u, x_k - y_k = \beta_k v. \tag{3.4}$$

From equation (3.3) we have

$$\begin{aligned}
x_3 - y_3 &= \lambda x_1 + (1 - \lambda)x_2 - \lambda y_1 - (1 - \lambda)y_2 \\
&= \lambda[x_1 - y_1] + (1 - \lambda)[x_2 - y_2] \\
&= \lambda[\alpha_1 u] + (1 - \lambda)[\alpha_2 u] \\
&= [\lambda\alpha_1 + (1 - \lambda)\alpha_2]u.
\end{aligned}$$

Similarly,

$$\begin{aligned}
x_3 - y_3 &= \lambda x_1 + (1 - \lambda)x_2 - \lambda y_1 + (1 - \lambda)y_2 \\
&= \lambda[x_1 - y_1] + (1 - \lambda)[x_2 - y_2] \\
&= \lambda[\beta_1 v] + (1 - \lambda)[\beta_2 v] \\
&= [\lambda\beta_1 + (1 - \lambda)\beta_2]v.
\end{aligned}$$

Now we have

$$\|x_3 - y_3\| = [\lambda\alpha_1 + (1 - \lambda)\alpha_2]\|u\|. \tag{3.5}$$

Similarly,

$$\|x_3 - y_3\| = [\lambda\beta_1 + (1 - \lambda)\beta_2]\|v\|. \tag{3.6}$$

Therefore, using (3.5) and (3.6), generally, we get,

$$\|x_3 - y_3\| = \lambda\epsilon_1 + (1 - \lambda)\epsilon_2. \tag{3.7}$$

Now we have

$$\begin{aligned}
\delta(u, v, [\lambda(\epsilon_1) + (1 - \lambda)\epsilon_2]) &\le 1 - \frac{\|x_3 + y_3\|}{2C} \\
&= 1 - \frac{\lambda\|x_1 + y_1\| + (1 - \lambda)\|x_2 + y_2\|}{2C} \\
&\le \lambda\left[1 - \frac{\|x_1 + y_1\|}{2C}\right] + (1 - \lambda)\left[1 - \frac{\|x_2 + y_2\|}{2C}\right] \\
&= \lambda[\delta(u, v, \epsilon_1] + (1 - \lambda)[\delta(u, v, \epsilon_2].
\end{aligned}$$

Belonging to some $N(u, v)$ since $\delta_{\mathcal{B}}(u, v, \epsilon)$ is convex, which shows that $\delta_{\mathcal{B}}(\epsilon)$ is convex. Since $\delta_{\mathcal{B}}(\epsilon)$ is convex, so is continuous by Lemma 3.1.

(b) Let $0 \le \epsilon_1 \le \epsilon_2 \le 2$ and $x_1, x_2 \in S_{\mathcal{B}}$ satisfying $\frac{\|x_1 - x_2\|}{C} \le \epsilon_2$ and $\frac{\|x_1 + x_2\|}{2C} \le 1 - \delta_{\mathcal{B}}(\epsilon_2)$. Then letting $\mathcal{E} = \frac{\epsilon_2 - \epsilon_1}{2\epsilon_2}$ and $x = x_1 + \mathcal{E}(x_2 - x_1)$ and $y = x_2 - \mathcal{E}(x_2 - x_1)$ we have $x, y \in S_{\mathcal{B}}$ and $\frac{\|x - y\|}{C} \le \epsilon_1$.

Now applying Lemma 3.2 we get

$$\delta_{\mathcal{B}}(\epsilon_1) \leq 1 - \frac{x+y}{2C} \leq 1 - \frac{x_1+x_2}{2C} \leq \delta_{\mathcal{B}}(\epsilon_2),$$

which shows that $\delta_{\mathcal{B}}(\epsilon)$ is a non-decreasing function.

(c) Fix $\eta \in (0,2]$ with $\eta < \epsilon$. Let $x_1, x_2 \in \mathcal{B}$ such that $\|x_1\| = \|x_2\| = 1$ and $\frac{\|x_1-x_2\|}{C} = \epsilon$. Here, it will be suffices to show that

$$\frac{\delta_{\mathcal{B}}(\eta)}{\eta} \leq \frac{\delta_{\mathcal{B}}(\epsilon)}{\epsilon}.$$

Consider

$$u_1 = \frac{\eta}{\epsilon}x_1 + \left(1 - \frac{\eta}{\epsilon}\right)\left[\frac{x_1+x_2}{\|x_1+x_2\|}\right],$$

$$u_2 = \frac{\eta}{\epsilon}x_2 + \left(1 - \frac{\eta}{\epsilon}\right)\left[\frac{x_1+x_2}{\|x_1+x_2\|}\right],$$

then

$$u_1 - u_2 = \frac{\eta}{\epsilon}(x_1 - x_2) \quad \Longrightarrow \quad \frac{\|u_1-u_2\|}{\eta} = C.$$

And

$$\frac{u_1+u_2}{2} = \left[\frac{x_1+x_2}{\|x_1+x_2\|}\right]\left(1 - \frac{\eta}{\epsilon} + \frac{\eta\|x_1+x_2\|}{2\epsilon}\right),$$

thus

$$\frac{u_1+u_2}{2C} = \left[\frac{x_1+x_2}{\|x_1+x_2\|}\right]\left(\frac{1}{C} - \frac{\eta}{\epsilon C} + \frac{\eta\|x_1+x_2\|}{2\epsilon C}\right).$$

This implies that

$$\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{u_1+u_2}{2C}\right\| = 1 - \left(\frac{1}{C} - \frac{\eta}{\epsilon C} + \frac{\eta\|x_1+x_2\|}{2\epsilon C}\right)$$
$$= 1 - \frac{\|u_1+u_2\|}{2C}.$$

Here note that

$$\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{x_1+x_2}{2C}\right\| = \|x_1+x_2\|\left(\frac{1}{\|x_1+x_2\|} - \frac{1}{2C}\right)$$
$$= 1 - \frac{\|x_1+x_2\|}{2C}.$$

Now we have

$$\frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{u_1+u_2}{2C}\right\|}{\|u_1-u_2\|} = \frac{C}{\eta}\left(\frac{\eta}{\epsilon C} - \frac{\eta\|x_1+x_2\|}{2\epsilon C}\right)$$
$$= \frac{1}{\epsilon}\left(1 - \frac{\|x_1+x_2\|}{2C}\right)$$
$$= \frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{x_1+x_2}{2C}\right\|}{\|x_1-x_2\|},$$

then

$$\begin{aligned}\frac{\delta(\eta)}{\eta} = \frac{1-\frac{\|u_1+u_2\|}{2C}}{\|u_1-u_2\|} &= \frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{u_1+u_2}{2c}\right\|}{\|u_1-u_2\|} \\ &= \frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{x_1+x_2}{2C}\right\|}{\|x_1-x_2\|} \\ &= \frac{1-\frac{\|x_1+x_2\|}{2C}}{\|x_1-x_2\|} \leq \frac{1}{\epsilon}\left[1-\frac{\|x_1+x_2\|}{2C}\right] \\ &= \frac{1}{\epsilon}\big(\delta_{\mathcal{B}}(\epsilon)\big).\end{aligned}$$

This completes the proof. □

Proposition 3.4. *Let $\mathcal{B}$ be a quasi-Banach space. Then $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq 2$ we have*

$$\frac{\delta_{\mathcal{B}}(\epsilon_2)-\delta_{\mathcal{B}}(\epsilon_1)}{\epsilon_2-\epsilon_1} \leq \frac{1-\delta_{\mathcal{B}}(\epsilon_1)}{2-\epsilon_1}. \tag{3.8}$$

Proof. Let

$$\epsilon_2 = 2\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) + \epsilon_1\left(1-\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right).$$

Then we have

$$\begin{aligned}\delta_{\mathcal{B}}(\epsilon_2) &= \delta_{\mathcal{B}}\left[2\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) + \epsilon_1\left(1-\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)\right] \\ &\leq \delta_{\mathcal{B}}(2)\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) + \delta_{\mathcal{B}}(\epsilon_1)\left(1-\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) \\ &\leq \delta_{\mathcal{B}}(2)\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) + \delta_{\mathcal{B}}(\epsilon_1) - \delta_{\mathcal{B}}(\epsilon_1)\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) \\ &= \left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)[\delta_{\mathcal{B}}(2)-\delta_{\mathcal{B}}(\epsilon_1)] + \delta_{\mathcal{B}}(\epsilon_1).\end{aligned}$$

Now

$$\delta_{\mathcal{B}}(\epsilon_2)-\delta_{\mathcal{B}}(\epsilon_1) \leq \left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)[1-\delta_{\mathcal{B}}(\epsilon_1)].$$

Hence

$$\frac{\delta_{\mathcal{B}}(\epsilon_2)-\delta_{\mathcal{B}}(\epsilon_1)}{\epsilon_2-\epsilon_1} \leq \frac{1-\delta_{\mathcal{B}}(\epsilon_1)}{2-\epsilon_1}.$$

This completes the proof. □

Theorem 3.5. *Let $\mathcal{B}$ be a uniformly convex space. Then for every $d > 0$, $\varepsilon > 0$, and for arbitrary vectors, $x_1, x_2 \in \mathcal{B}$ with $\|x_1\| \leq d$, $\|x_2\| \leq d$ and $\frac{\|x_1-x_2\|}{C} \geq \varepsilon$, there exists $\delta > 0$ such that*

$$\frac{\|x_1+x_2\|}{2C} \leq \left[1-\delta\left(\frac{\varepsilon}{d}\right)\right]d.$$

Proof. For any arbitrary $x_1, x_2 \in \mathcal{B}$ we assume that

$$z_1 = \frac{x_1}{d}, \quad z_2 = \frac{x_2}{d}, \quad \text{and set } \epsilon = \frac{\varepsilon}{d}.$$

Obviously $\epsilon > 0$. Moreover, with $\|x_1\| \leq 1$ and $\|x_2\| \leq 1$ we have

$$\|z_1 - z_2\| = \frac{1}{d}\|x_1 - x_2\| \geq \frac{\varepsilon}{d} = \epsilon.$$

Now, for uniform convexity, we have

$$\delta = \delta\left(\frac{\varepsilon}{d}\right) > 0 \quad \text{and} \quad \frac{\|z_1 + z_2\|}{2C} \leq 1 - \delta(\epsilon),$$

which implies that

$$\frac{\|x_1 + x_2\|}{2dC} \leq 1 - \delta\left(\frac{\varepsilon}{d}\right),$$

thus

$$\frac{\|x_1 + x_2\|}{2C} \leq \left[1 - \delta\left(\frac{\varepsilon}{d}\right)\right] d.$$

This completes the proof. □

Theorem 3.6. *A quasi-Banach space $\mathcal{B}$ is uniformly convex iff $\delta_{\mathcal{B}}(\epsilon) \geq 0$.*

Proof. If X is uniformly convex, then, for given $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x_1, x_2 \in \mathcal{B}$ with $\|x_1\| = 1$, $\|x_2\| = 1$ and $\frac{\|x_1 - x_2\|}{C} \geq \epsilon$

$$1 - \frac{\|x_1 + x_2\|}{2C} \geq \delta \quad \Longrightarrow \quad \delta_{\mathcal{B}}(\epsilon) > 0.$$

Conversely, assume that $\delta_{\mathcal{B}}(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let fix $\epsilon \in (0, 2]$ and then take $x_1, x_2 \in \mathcal{B}$ with $\|x_1\| = 1$, $\|x_2\| = 1$ and $\frac{\|x_1 - x_2\|}{C} \geq \epsilon$. Then

$$0 < \delta_{\mathcal{B}}(\epsilon) \leq 1 - \frac{\|x_1 + x_2\|}{2C}.$$

This implies that $1 - \frac{\|x_1 + x_2\|}{2C} \leq 1 - \delta$ with $\delta = \delta_{\mathcal{B}}(\epsilon)$, which does not depends upon either x_1 or x_2. This completes the proof. □

Theorem 3.7. *For arbitrary uniformly convex quasi-Banach space $\mathcal{B}$,*

$$\delta_{\mathcal{B}}(\epsilon) = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}.$$

Proof. Let $x_1, x_2 \in \mathcal{B}$ with $\|x_1\| = 1$, $\|x_2\| = 1$ and $\frac{\|x_1 - x_2\|}{C} = \epsilon$. Then using the parallelogram identity,

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2\big(\|x_1\|^2 + \|x_2\|^2\big),$$

thus

$$\begin{aligned}
\|x_1 + x_2\|^2 &= 2\big(\|x_1\|^2 + \|x_2\|^2\big) - \|x_1 - x_2\|^2 \\
&= 2(1^2 + 1^2) - \|x_1 - x_2\|^2 \\
&= 2(2) - (\epsilon C)^2 \\
&= 4 - \epsilon^2 C^2,
\end{aligned}$$

hence

$$\|x_1 + x_2\| = \sqrt{4 - \epsilon^2 C^2},$$

thus we have

$$1 - \frac{\|x_1 + x_2\|}{2C} = 1 - \frac{\sqrt{4 - \epsilon^2 C^2}}{2C},$$

which implies that

$$\inf \left\{ 1 - \frac{\|x_1 + x_2\|}{2C} \right\} = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}.$$

Hence, we get

$$\delta_{\mathcal{B}}(\epsilon) = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}.$$

This completes the proof. □

Theorem 3.8. *A non-trivial quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $\delta_{\mathcal{B}}(\epsilon) > 0$.*

Proof. Let $\mathcal{B}$ be uniformly non-square. Set $\epsilon = 2 - 2\delta$, $\epsilon \in (0, 2)$. Then

$$\delta_{\mathcal{B}}(\epsilon) \geq 1 - \frac{\epsilon}{2} > 0.$$

Conversely, let there is $\epsilon_0 \in (0, 2)$ such that $\delta_{\mathcal{B}}(\epsilon) > 0$, that is,

$$\delta_{\mathcal{B}}(\epsilon) \geq \eta_0 > 0 \quad \text{for some } \eta_0 \in (0, 1).$$

Let $2 - 2\delta = \epsilon \in [\epsilon_0, 2)$. Then

$$\delta \in (0, 1 - \epsilon_0/2] \quad \text{and} \quad \delta_{\mathcal{B}}(2 - 2\delta) = \delta_{\mathcal{B}}(\epsilon) \geq \eta_0 > 0.$$

This indicates that for any $x, y \in S_{\mathcal{B}}$, if

$$\frac{\|x - y\|}{C} \geq 2 - 2\delta,$$

then

$$1 - \frac{\|x + y\|}{2C} \geq \eta_0.$$

Let $\delta' = \min\{\delta, \eta_0\}$. Then of course $\delta' \in (0, 1)$.

Now we just need to show that either $\frac{\|x-y\|}{2C} \leq 1 - \delta'$ or $\frac{\|x+y\|}{2C} \leq 1 - \delta'$. If

$$\frac{\|x + y\|}{2C} \leq 1 - \delta',$$

then we are done:

Let we consider

$$\frac{\|x + y\|}{2C} > 1 - \delta'.$$

Then

$$\|x - y\| > 2C(1 - \delta') \geq 2C(1 - \delta).$$

By this assumption we get

$$1-\frac{\|x+y\|}{2C}\geq \eta_0,$$

which implies that

$$\frac{\|x+y\|}{2C}\leq 1-\eta_0\leq (1-\delta'),$$

which shows that $\mathcal{B}$ is uniformly non-square. This completes the proof. □

Proposition 3.9. *Let $\mathcal{B}$ be a quasi-Banach space and H be a Hilbert space. Then*

$$\delta_{\mathcal{B}}(\epsilon)\leq \delta_H(\epsilon),\quad \forall\epsilon\in[0,2]. \tag{3.9}$$

Proof. From Theorem 3.7 we can easily prove the result. □

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