Volume 27, Number 3 September 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.’s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board

Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He
Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive
Linear Operators)

Xing-Biao Hu
Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Modified Halpern's iteration without assumptions on fixed point set in metric space

Kanyarat Cheawchan, Atid Kangtunyakarn*
Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand
E-mail addresses: Kkanyarat.cheaw@gmail.com; beawrock@hotmail.com

Abstract

By improving Halpern's iteration and studing convergence theorem of [1] and [2] in a complete uniformly convex metric space, we prove convergence theorem of a finite family of nonexpansive mappings without the assumption that " the set of common fixed points of nonexpansive mappings is nonempty". We also introduce a mapping in metric space using a concept of the S-mapping defined by [3] for proving our main results.
Keywords: Convex metric space; Nonexpansive mapping; S-mapping.
Mathematics Subject Classification (2000): 31E05, 54E40, 54E50, 47H09.

1 Introduction

Many researchers have theorized for finding a solution of fixed point problems by taking advantage of iteration process, see for instance [4], [5], [6]. Halpern's iteration is a method which has been very popular for finding a solution to fixed point problem. It was introduced for the first time by Halpern [7] and defined by the vector u, x_0 belonging to a closed convex C subset of Hilbert (Banach) space and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n,$$

for all $n \geq 1$, where $T : C \to C$ is a mapping and parameter $\{\alpha_n\} \subseteq [0, 1]$.

It has been developed and improved to fixed point theorem to increase efficiency by several researchers, see example [4], [5], [6]. Although the proof of the theorem has been well developed, but the proof is still under critical conditions below;

*Corresponding author

$i)^*$ $F(T) \neq \emptyset$;

$ii)^*$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Can we prove a convergence theorem by developing Halpern iteration and without conditions $i)^*$ and $ii)^*$ in space which is more general than Hilbert and Banach spaces?

Throughout this paper, we assume that (X,d) is a complete metric space and C is a nonempty closed convex subset of (X,d). A point x is called a fixed point of T if $Tx = x$. We use $F(T)$ to denote the set of fixed point of T. Recall the following definitions;

Definition 1.1. *The mapping $T : C \to C$ is said to be nonexpansive if*

$$d(Tx,Ty) \leq d(x,y), \quad \forall x,y \in C.$$

In 1970, Takahashi [8] introduced the following definition:

Definition 1.2. *Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and for all $u \in X$,*

$$d\big(u, W(x,y,\lambda)\big) \leq \lambda d(u,x) + (1-\lambda)d(u,y).$$

If the mapping W is defined by $W(x,y,\lambda) = \lambda x + (1-\lambda)y$, then it is a convex structure on a normed linear space. A metric space (X,d) together with a convex structure W is called a *convex metric space* denoted by (X,d,W). A nonempty subset C of X is said to be *convex* if $W(x,y,\lambda) \in C$ for all $x,y \in C$ and $\lambda \in [0,1]$.

Definition 1.3. *(See [9]) A convex metric space (X,d,W) is said to be uniformly convex if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $r > 0$ and $x,y,z \in X$ with $d(z,x) < r, d(z,y) < r$ and $d(x,y) \geq r\epsilon$,*

$$d\big(z, W(x,y,\frac{1}{2})\big) \leq (1-\delta)r.$$

It is well known that Hilbert space is uniformly convex metric space.

Very recently, Hafiz Fukhar-ud-din [1] proved convergence theorem in uniformly convex metric spaces (X,d,W) with convex structure but he still assumed the fixed point set is nonempty as follows;

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a uniformly convex complete metric space X with continuous convex structure W and $S,T : C \to C$ be nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Then the sequence $\{x_n\}$, defined by $x_{n+1} = W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right)$, Δ-converges to an element of $F(S) \cap F(T)$, where $0 < a \leq \alpha_n, \beta_n \leq b < 1$ with $\alpha_n + \beta_n < 1$.*

In 2013, Phuengrattana and Suantai [2] proved convergence theorem in uniformly convex metric space for infinite family of nonexpansive mapping by leveraging the map K_n, see [2] for more details, but still assume that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ as follows;

Theorem 1.2. *Let C be a nonempty compact convex subset of a complete uniformly convex metric space (X, d, W) with the property (H). Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Assume that $x_1 \in C$ and the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = W(x_n, K_n x_n, \alpha_n),$$

for all $n \geq 1$ where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$. Then sequence $\{x_n\}$ converges to an element of $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$.

Inspired by Theorem 1.1 and 1.2 and improved process of Halpern's iteration, we prove convergence theorem in uniformly convex metric space for a finite family of nonexpansive mappings without using the conditions $i)^*$ and $ii)^*$.

2 Preliminaries

In this section, in order to prove our main theorem, we provide definitions, lemma and also prove the importance lemma to be used as a tool to prove the main theorem:

Lemma 2.1. *(See [8], [10]) Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, we have the following.*
(i) $W(x, x, \lambda) = x$, $W(x, y, 0) = y$ and $W(x, y, 1) = x$.
(ii) $d\big(x, W(x, y, \lambda)\big) = (1 - \lambda)d(x, y)$ and $d\big(y, W(x, y, \lambda)\big) = \lambda d(x, y)$.
(iii) $d(x, y) = d\big(x, W(x, y, \lambda)\big) + d\big(W(x, y, \lambda), y\big)$.
(iv) $|\lambda_1 - \lambda_2| d(x, y) \leq d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big)$.

We say that a convex metric space (X, d, W) has the following properties:
(C) if $W(x, y, \lambda) = W(y, x, 1 - \lambda)$ for all $x, y \in X$ and $\lambda \in [0, 1]$,
(I) if $d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big) \leq |\lambda_1 - \lambda_2| d(x, y)$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$,
(H) if $d\big(W(x, y, \lambda), W(x, z, \lambda)\big) \leq (1 - \lambda)d(y, z)$ for all $x, y, z \in X$ and $\lambda \in [0, 1]$,
(S) if $d\big(W(x, y, \lambda), W(z, w, \lambda)\big) \leq \lambda d(x, z) + (1 - \lambda)d(y, w)$ for all $x, y, z, w \in X$ and $\lambda \in [0, 1]$.

Remark 2.2. It is easy to see that the property (C) and (H) imply continuity of a convex structure $W : X \times X \times [0, 1] \to X$ and the property (S) implies the property (H). In 2005, Aoyama et al. [10] proved that a convex metric space with property (C) and (H) has the property (S).

In 2011, Phuengrattana and Suantai [2] proved the following lemma as follows;

Lemma 2.3. *(See [2]) Property (C) holds in uniformly convex metric space.*

Remark 2.4. (See [2]) From Lemma 2.3, a uniformly convex metric space (X, d, W) with the property (H) has the property S and the convex structure W is also continuous.

Lemma 2.5. *(See [11]) Let (X, d, W) be a uniformly convex metric space with continuous convex structure. Then for arbitrary positive number ϵ, there exists $\eta = \eta(\epsilon) > 0$ such that*

$$d\big(z, W(x, y, \lambda)\big) \leq (1 - 2\min\{\lambda, 1-\lambda\}\eta)r,$$

for all $r > 0$ and $x, y, z \in X, d(z, x) \leq r, d(z, y) \leq r, d(x, y) \geq r\varepsilon$ and $\lambda \in [0, 1]$.

We introduce the following definition to use in the next section.

Definition 2.1. *Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of (X, d, W). Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into C. For each $j = 1, 2, \cdots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $x \in C$, we define the mapping $S : C \times C \times [0, 1] \to C$ as follows;*

$$\begin{aligned}
U_0x &= x,\\
U_1x &= W\big(T_1U_0x, W(U_0x, x, \frac{\alpha_2^1}{1-\alpha_1^1}), \alpha_1^1\big),\\
U_2x &= W\big(T_2U_1x, W(U_1x, x, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big),\\
&\vdots\\
U_{N-1}x &= W\big(T_{N-1}U_{N-2}x, W(U_{N-2}x, x, \frac{\alpha_2^{N-1}}{1-\alpha_1^{N-1}}), \alpha_1^{N-1}\big),\\
Sx &= U_Nx = W\big(T_NU_{N-1}x, W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big).
\end{aligned}$$

This mapping is called $S-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.6. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0, 1]$ α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Proof. From Lemma 2.1 and definition of $S-$mapping, it is easy to see that $\bigcap_{i=1}^{N} F(T_i) \subseteq F(S)$. Next, we show that $F(S) \subseteq \bigcap_{i=1}^{N} F(T_i)$. To show this let $x_0 \in F(S)$ and $q \in \bigcap_{i=1}^{N} F(T_i)$, we have

$$
\begin{aligned}
d(q, Sx_0) &= d\Big(q, W\big(T_N U_{N-1}x_0, W(U_{N-1}x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\Big) \\
&\leq \alpha_1^N d(q, T_N U_{N-1}x_0) + (1-\alpha_1^N) d\Big(q, W(U_{N-1}x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N})\Big) \\
&\leq \alpha_1^N d(q, T_N U_{N-1}x_0) + (1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N} d(q, U_{N-1}x_0) \\
&+ (1-\frac{\alpha_2^N}{1-\alpha_1^N}) d(q, x_0)\Big) \\
&= \alpha_1^N d(q, T_N U_{N-1}x_0) + \alpha_2^N d(q, U_{N-1}x_0) + \alpha_3^N d(q, x_0) \\
&\leq (1-\alpha_3^N) d(q, U_{N-1}x_0) + \alpha_3^N d(q, x_0) \\
&\leq (1-\alpha_3^N)\Big((1-\alpha_3^{N-1}) d(q, U_{N-2}x_0) + \alpha_3^{N-1} d(q, x_0)\Big) \\
&\quad + \alpha_3^N d(q, x_0) \\
&= (1-\alpha_3^N)(1-\alpha_3^{N-1}) d(q, U_{N-2}x_0) + \alpha_3^{N-1}(1-\alpha_3^N) d(q, x_0) \\
&\quad + \alpha_3^N d(q, x_0) \\
&= \Pi_{j=N-1}^{N}(1-\alpha_3^j) d(q, U_{N-2}x_0) + \big(1-\Pi_{j=N-1}^{N}(1-\alpha_3^j)\big) d(q, x_0) \\
&\vdots \\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j) d(q, U_2 x_0) + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0) \\
&= \Pi_{j=3}^{N}(1-\alpha_3^j) d\big(q, W\big(T_2 U_1 x_0, W(U_1 x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big)\big) \\
&\quad + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0) \\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2 U_1 x_0) + (1-\alpha_1^2) d\big(q, W(U_1 x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2})\big)\Big) \\
&\quad + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0) \\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2 U_1 x_0) + (1-\alpha_1^2)\big(\frac{\alpha_2^2}{1-\alpha_1^2} d(q, U_1 x_0) \\
&\quad + (1-\frac{\alpha_2^2}{1-\alpha_1^2}) d(q, x_0)\big)\Big) + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0) \\
&= \Pi_{j=3}^{N}(1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2 U_1 x_0) + \alpha_2^2 d(q, U_1 x_0) + \alpha_3^2 d(q, x_0)\Big) \\
&\quad + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0) \\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j)\Big((1-\alpha_3^2) d(q, U_1 x_0) + \alpha_3^2 d(q, x_0)\big)\Big) \\
&\quad + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0)
\end{aligned}
$$

5

$$
\begin{aligned}
&= \Pi_{j=2}^{N}(1-\alpha_3^j)d(q,U_1x_0) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\
&= \Pi_{j=2}^{N}(1-\alpha_3^j)d\big(q,W\big(T_1U_0x_0,W(U_0x_0,x_0,\frac{\alpha_2^1}{1-\alpha_1^1}),\alpha_1^1\big)\big)\\
&\quad +\big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\
&= \Pi_{j=2}^{N}(1-\alpha_3^j)d\big(q,W\big(T_1x_0,x_0,\alpha_1^1\big)\big) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\
&\le \Pi_{j=2}^{N}(1-\alpha_3^j)\big(\alpha_1^1d(q,T_1x_0)+(1-\alpha_1^1)d(q,x_0)\big) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\
&\le \Pi_{j=2}^{N}(1-\alpha_3^j)d(q,x_0) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0)\\
&= d(q,x_0). \qquad (2.1)
\end{aligned}
$$

From (2.1), we have

$$d(q,U_1x_0) = d\big(q,W(T_1x_0,x_0,\alpha_1^1)\big) = d(q,x_0) \text{ and } d(q,T_1x_0) = d(q,x_0).$$

Suppose $x_0 \neq T_1x_0$, we have $d(x_0,T_1x_0) > 0$. Choose $r = d(q,x_0) > 0$ and $\epsilon = \dfrac{d(x_0,T_1x_0)}{r}$, we have $d(q,T_1x_0) \le d(q,x_0) = r$, $d(q,x_0) \le r$ and $d(x_0,T_1x_0) \ge r\epsilon$. From Lemma 2.5, we have

$$d\big(q,W(T_1x_0,x_0,\alpha_1^1)\big) < d(q,x_0) \text{ for } \alpha_1^1 \in (0,1).$$

This is a contradiction, we have $x_0 = T_1x_0$ that is $x_0 \in F(T_1)$. Since $x_0 = T_1x_0$ definition of U_1 and Lemma 2.1, we have $U_1x_0 = x_0$ that is $x_0 \in F(U_1)$. From (2.1) and $x_0 = U_1x_0$, we have

$$d(q,U_2x_0) = d\big(q,W\big(T_2x_0,x_0,\alpha_1^2\big)\big) = d(q,x_0) \text{ and } d(q,T_2x_0) = d(q,x_0).$$

Suppose $x_0 \neq T_2x_0$, we have $d(x_0,T_2x_0) > 0$. Choose $r_1 = d(q,x_0) > 0$ and $\epsilon = \dfrac{d(x_0,T_2x_0)}{r_1}$, we have $d(q,T_2x_0) \le d(q,x_0) = r_1$, $d(q,x_0) \le r_1$ and $d(x_0,T_2x_0) \ge r_1\epsilon$. From Lemma 2.5, we have

$$d\big(q,W(T_2x_0,x_0,\alpha_1^2)\big) < d(q,x_0) \text{ for } \alpha_1^2 \in (0,1).$$

This is a contradiction, we have $x_0 = T_2x_0$ that is $x_0 \in F(T_2)$. Since $x_0 = T_2x_0$ definition of U_2 and Lemma 2.1, we have $U_2x_0 = x_0$ that is $x_0 \in F(U_2)$.
By continuing on this way, we can conclude that $x_0 \in F(T_i)$ and $x_0 \in F(U_i)$ for all $i = 1,2,\ldots,N-1$.
Finally, we show that $x_0 \in F(T_N)$. From definition of S and Lemma 2.1, we have

$$Sx_0 = W\big(T_NU_{N-1}x_0,W(U_{N-1}x_0,x_0,\frac{\alpha_2^N}{1-\alpha_1^N}),\alpha_1^N\big) = W(T_Nx_0,x_0,\alpha_1^N).$$

Since

$$0 = d(x_0,Sx_0) = d(x_0,W(T_Nx_0,x_0,\alpha_1^N)) = \alpha_1^N d(T_Nx_0,x_0),$$

we have $x_0 = T_Nx_0$, that is, $x_0 \in F(T_N)$. Hence $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$. □

Remark 2.7. From Theorem 2.6, we have the mapping S is nonexpansive. To show this, let $x, y \in C$. By Remark 2.4, we have

$$\begin{aligned}
d(Sx,Sy) &= d\Big(W\big(T_NU_{N-1}x, W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big),\\
&\quad W\big(T_NU_{N-1}y, W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\Big)\\
&\leq \alpha_1^N d(T_NU_{N-1}x, T_NU_{N-1}y)\\
&\quad +(1-\alpha_1^N)d\big(W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N})\big)\\
&\leq \alpha_1^N d(T_NU_{N-1}x, T_NU_{N-1}y)\\
&\quad +(1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N}d(U_{N-1}x, U_{N-1}y) + \big(1-\frac{\alpha_2^N}{1-\alpha_1^N}\big)d(x,y)\Big)\\
&\leq \alpha_1^N d(U_{N-1}x, U_{N-1}y) + \alpha_2^N d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y)\\
&= (1-\alpha_3^N)d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y)\\
&\leq (1-\alpha_3^N)\big((1-\alpha_3^{N-1})d(U_{N-2}x, U_{N-2}y) + \alpha_3^{N-1}d(x,y)\big) + \alpha_3^N d(x,y)\\
&= \Pi_{j=N-1}^N(1-\alpha_3^j)d(U_{N-2}x, U_{N-2}y) + \big(1-\Pi_{j=N-1}^N(1-\alpha_3^j)\big)d(x,y)\\
&\leq\\
&\vdots\\
&= \Pi_{j=1}^N(1-\alpha_3^j)d(U_0x, U_0y) + \big(1-\Pi_{j=1}^N(1-\alpha_3^j)\big)d(x,y)\\
&= d(x,y).
\end{aligned}$$

Example 2.8. Let the metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Let the mapping $W : \mathbb{R}^2 \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ be defined by

$$W(x,y,\lambda) = \lambda x + (1-\lambda)y = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2),$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

For every $i = 1, 2, \ldots, N$, let the mapping $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_i x = \frac{ix}{i+1},$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = \left(\alpha_1^j, \alpha_2^j, \alpha_3^j\right) = \left(\frac{1}{2^j}, \frac{2^j-1}{2^i(2+j)}, \frac{2^j-1}{2^j}\cdot\left(\frac{j+1}{j+2}\right)\right)$ for all $j = 1, 2, \ldots, N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.

Solution. From the properties of $d, W, \mathbb{R}^2$, $\left(\mathbb{R}^2, d, W\right)$ is a complete uniformly

7

convex metric space. Next, we show that $\left(\mathbb{R}^2, d, W\right)$ has a property (H). Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ and $a \in [0,1]$, then

$$d\left(W(x,y,a), W(x,z,a)\right) = \max\left\{(1-a)\left|y_1 - z_1\right|, (1-a)\left|y_2 - z_2\right|\right\}.$$

Since $d(y,z) = \max\left\{\left|y_1 - z_1\right|, \left|y_2 - z_2\right|\right\}$, we get

$$d\left(W(x,y,a), W(x,z,a)\right) \leq (1-a)\, d(y,z).$$

Then $\left(\mathbb{R}^2, d, W\right)$ has a property (H).

It is clear that T_i is a nonexpansive mapping for all $i = 1, 2, \ldots, N$ and $\bigcap_{i=1}^{N} F(T_i) = \{0\}$, due to the properties of T_i. From Lemma 2.6, we have $F(S) = \bigcap_{i=1}^{N} F(T_i)$.

Remark 2.9. Lemma 2.8 in [3] is a spacial case of Lemma 2.6.

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0,1]$ $, \alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0,1]$ α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = W(u, Sx_n, \alpha) \tag{3.1}$$

for all $n \geq 1$ and $\alpha \in [0,1]$. Then the following statements are equivalent:

i) The sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^{N} F(T_i)$,

ii) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, N$.

Proof. $i) \Rightarrow ii)$. Since $\{x_n\}$ converges to $z \in \bigcap_{i=1}^{N} F(T_i)$ and

$$d(x_n, T_i x_n) \leq d(x_n, z) + d(T_i x_n, z) \leq 2d(x_n, z)$$

for all $i = 1, 2, \ldots, N$, so we can prove that ii) is true.

For the next result, we prove $ii) \Rightarrow i)$. For every $n \in \mathbb{N}$ and remark (S property), we have

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq d\left(W(u, Sx_n, \alpha), W(u, Sx_{n-1}, \alpha)\right) \\
&\leq (1-\alpha)\, d(x_n, x_{n-1}) \\
&\leq (1-\alpha)^2\, d(x_{n-1}, x_{n-2}) \\
&\vdots \\
&\leq (1-\alpha)^n\, d(x_1, x_0).
\end{aligned}$$

Using the benefits from the inequality above, we have

$$\begin{aligned} d\left(x_{n+k}, x_n\right) &\leq \sum_{j=n}^{n+k-1} d\left(x_{j+1}, x_j\right) \\ &\leq \sum_{j=n}^{n+k-1} (1-\alpha)^j d\left(x_1, x_0\right) \\ &\leq \frac{(1-\alpha)^n}{\alpha} \cdot d\left(x_1, x_0\right), \end{aligned}$$

for all $k \in \mathbb{N}$. Since $\lim_{n\to\infty} (1-\alpha)^n = 0$, we get the sequence $\{x_n\}$ is Cauchy. Then there exists $z \in C$ such that $\lim_{n\to\infty} x_n = z$.

From the condition $ii)$ and

$$d\left(z, T_i z\right) \leq d\left(x_n, z\right) + d\left(x_n, T_i x_n\right) + d\left(T_i x_n, T_i z\right) \leq 2d\left(x_n, z\right) + d\left(x_n, T_i x_n\right),$$

for all $i = 1, 2, \ldots, N$, we have $d\left(z, T_i z\right) = 0$. We can conclude that $z \in \bigcap_{i=1}^N F\left(T_i\right)$. Hence the sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F\left(T_i\right)$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for all $i = 1, 2, \ldots, N$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0, 1]$, α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = W\left(u, Sx_n, \alpha\right) \tag{3.2}$$

for all $n \geq 1$ and $\alpha \in [0, 1]$. Then the sequence $\{x_n\}$ converges to $z \in F(S)$.

Proof. The sequence $\{x_n\}$ is a Cauchy by using the same method of Theorem 3.1. Then there exists $z \in C$ such that $\lim_{n\to\infty} x_n = z$.

Since $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for all $i = 1, 2, \ldots, N$ and

$$d\left(z, T_i z\right) \leq 2d\left(x_n, z\right) + d\left(x_n, T_i x_n\right),$$

for all $i = 1, 2, \ldots, N$, we have $z \in \bigcap_{i=1}^N F\left(T_i\right)$. From Lemma 2.6, we have $z \in F(S)$. Hence the sequence $\{x_n\}$ converges to $z \in F(S)$. □

If the condition ii) in Theorem 3.1 and 3.2 are replaced by "$\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N F\left(T_i\right)\right) = 0$" where $d\left(x_n, \bigcap_{i=1}^N F\left(T_i\right)\right) = \inf_{v\in\bigcap_{i=1}^N F(T_i)} d\left(x_n, v\right)$". Then, the following theorems are still true.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in$*

$I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0, 1]$, α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and

$$x_{n+1} = W(u, Sx_n, \alpha) \tag{3.3}$$

for all $n \geq 1$ and $\alpha \in [0, 1]$. Then the following statements are equivalent:
i) The sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F(T_i)$.
ii) $\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N F(T_i)\right) = 0$ where $d\left(x_n, \bigcap_{i=1}^N F(T_i)\right)$
$= \inf_{v \in \bigcap_{i=1}^N F(T_i)} d(x_n, v)$.

Proof. It is very clear that case $i) \Rightarrow ii)$. Next, we show that case $ii) \Rightarrow i)$. Using the same method in Theorem 3.1, we obtain that the sequence $\{x_n\}$ is a Cauchy sequence. Then, there exists $z \in C$ such that $\lim_{n\to\infty} x_n = z$.

For every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$d\left(x_n, \cap_{j=1}^N F(T_i)\right) < \frac{\varepsilon}{2}$$

and

$$d(x_n, z) < \frac{\varepsilon}{2},$$

for all $n \geq N_0$.

From the above inequality, there exists $p \in \bigcap_{i=1}^N F(T_i)$ such that $d(x_n, p) < \frac{\varepsilon}{2}$.

Since

$$d(p, z) \leq d(x_n, p) + d(x_n, z) < \varepsilon$$

and ε is arbitrary, we have $d(p, z) = 0$. Hence $z = p$. Therefore, the sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F(T_i)$. □

Theorem 3.4. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N F(T_i)\right) = 0$ where $d\left(x_n, \bigcap_{i=1}^N F(T_i)\right) = \inf_{v \in \bigcap_{i=1}^N F(T_i)} d(x_n, v)$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0, 1]$, α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = W(u, Sx_n, \alpha) \tag{3.4}$$

for all $n \geq 1$ and $\alpha \in [0, 1]$. Then the sequence $\{x_n\}$ converges $z \in F(S)$.

Proof. Applying the method of Theorem 3.2 and 3.3, we can obtain the desired result. □

Acknowledgements This paper was supported by the Royal Golden Jubilee (RGJ) Ph.D. Programme, the Thailand Research Fund (TRF), under Grant No. PHD/0082/2558 and the Research and Innovation Services of King Mongkuts Institute of Technology Ladkrabang.

References

[1] H. Fukhar-ud-din, Convergence of Ishikawa type iteration process for three quasi-nonexpansive mappings in a convex metric space 23(2), 83-92 2015.5

[2] W. Phuengrattana and S. Suantai, Strong Convergence Theorems for a Countable Family of Nonexpansive Mappings in Convex Metric Spaces, Abstract and Applied Analysis 2011, Article ID 929037, 18 pages (2011).9

[3] A. Kangtunyakarn and S. Suantai, Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions, Comput. Math. Appl. 60, 680-694 (2010).11

[4] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis 69, 1025-1033 (2008).2

[5] G. Cai and S. Bu, Strong convergence theorems for variational inequality problems and fixed point problems in uniformly smooth and uniformly convex Banach spaces, J Glob Optim 56, 1529-1542 (2013).3

[6] C. Tian and Y. Song, Strong convergence of a regularization method for Rockafellars proximal point algorithm, J Glob Optim 55, 831-837 (2013).4

[7] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73, 957-961 (1967).1

[8] W. Takahashi, A convexity in metric space and nonexpansive mappings. Kodai Mathematical Seminar Reports 22, 142-149 (1970).10

[9] T. Shimizu and W. Takahashi, Fixed points of multivalued mappings in certain convex metric spaces, Topological Methods in Nonlinear Analysis 8, 197-203 (1996).8

[10] K. Aoyama, K. Eshita, and W. Takahashi, Iteration processes for nonexpansive mappings in convex metric spaces, Proceedings of the 4th International Conference on Nonlinear Analysis and Convex Analysis, Yokohama Publishers, 31-39 (2007).6

[11] T. Shimizu, A convergence theorem to common fixed points of families of nonexpansive mappings in convex metric spaces, Proceedings of the 4th International Conference on Nonlinear Analysis and Convex Analysis, Yokohama Publishers, 575-585 (2007).7

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

CONVERGENCE OF DOUBLE ACTING ITERATIVE SCHEME FOR A FAMILY OF GENERALIZED φ-WEAK CONTRACTION MAPPINGS IN $CAT(0)$ SPACES

Kyung Soo Kim

Graduate School of Education, Mathematics Education
Kyungnam University, Changwon, Gyeongnam, 51767, Republic of Korea
e-mail: kksmj@kyungnam.ac.kr

Abstract. The purpose of this paper, we discuss the convergence theorems for the double acting iterative scheme to a common fixed point for a family of generalized φ-weak contraction mappings in $CAT(0)$ spaces.

1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \le \alpha \cdot d(x, y), \quad \forall\, x, y \in X,$$

holds. A mapping $T : X \to X$ is a *φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)), \quad \forall\, x, y \in X, \tag{1.1}$$

holds.

The concept of the φ-weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later Rhoades [15] in 2001, who extended the results of [1] to metric spaces.

Theorem 1.1. ([15]) *Let (X, d) be a complete metric space, $T : X \to X$ be a φ-weak contractive self-map on X. The T has a unique fixed point p in X.*

Remark 1.1. Theorem 1.1 is one of generalizations of the Banach contraction principle because it takes $\varphi(t) = (1 - \alpha)t$ for $\alpha \in (0, 1)$, then φ-weak contraction contains contraction as special cases.

In 2016, Xue [18] introduced a new contraction type mapping as follows.

[0]2010 Mathematics Subject Classification: 47H09, 47H10, 47J25, 41A65.
[0]Keywords: generalized φ-weak contraction mapping, common fixed point, double acting iterative scheme, CAT(0) space.

Definition 1.1. ([18]) A mapping $T : X \to X$ is a *generalized φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X \tag{1.2}$$

holds.

We notice immediately that if $T : X \to X$ is φ-weak contraction, then T satisfies the following inequality

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X.$$

However, the converse is not true in general.

Example 1.1. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x-y|$ and let $Tx = \frac{2}{5}x$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{4}{3}t$. Then T satisfies (1.2), but T does not satisfy inequality (1.1). Indeed,

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\leq |x-y| - \frac{4}{3} \cdot \frac{2}{5}|x-y| \\ &= d(x, y) - \varphi(d(Tx, Ty)) \end{aligned}$$

and

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\geq |x-y| - \frac{4}{3}|x-y| \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

for all $x, y \in X$.

Example 1.2. ([18]) Let $X = (-1, +\infty)$ be endowed by $d(x, y) = |x - y|$ and let $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t^2}{1+t}$. Then

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \frac{|x-y|}{(1+x)(1+y)} \\ &\leq \frac{|x-y|}{1+|x-y|} = |x-y| - \frac{|x-y|^2}{1+|x-y|} \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

holds for all $x, y \in X$. So T is a φ-weak contraction. However T is not a contraction.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Remark 1.2. The above examples show that the class of generalized φ-weak contractions properly includes the class of φ-weak contractions and the class of φ-weak contractions properly includes the class of contractions. In fact, let $T : X \to X$ be a contraction, there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \ \ \forall\, x, y \in X.$$

Then

$$\begin{aligned} d(Tx, Ty) &\leq \alpha \cdot d(x, y) = d(x, y) - (1 - \alpha)d(x, y) \\ &= d(x, y) - \varphi(d(x, y)), \end{aligned}$$

where, $\varphi(d(x, y)) = (1 - \alpha)d(x, y)$. So, T is a φ-weak contraction. Moreover, let T be a φ-weak contraction, from property of φ, we have $d(Tx, Ty) \leq d(x, y)$ and

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)).$$

From (1.1),

$$\begin{aligned} d(Tx, Ty) &\leq d(x, y) - \varphi(d(x, y)) \\ &\leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X. \end{aligned}$$

Therefore, T is a generalized φ-weak contraction.

In the meantime, if T is a φ-weak contractive self mapping for one mapping φ so we do not expect that the φ-weak contractivity should be satisfied with the same function φ. Let us suppose that T is a φ-weak contractive self mapping and consider

$$\tilde{\varphi}(x) = \min\{\varphi(x/2); x/2\}.$$

Then, if $d(Tx, Ty) > \frac{1}{2}d(x, y)$, we have

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)) \leq d(x, y) - \varphi\left(\frac{1}{2}d(x, y)\right)$$

on account of monotonocity of φ and finally

$$d(Tx, Ty) \leq d(x, y) - \tilde{\varphi}(d(x, y)).$$

On the other hand, if $d(Tx, Ty) < \frac{1}{2}d(x, y)$, we get

$$d(Tx, Ty) < d(x, y) - \frac{1}{2}d(x, y) \leq d(x, y) - \tilde{\varphi}(d(x, y)).$$

So T is just thr $\tilde{\varphi}$-weak contractive mapping. The continuity and monotonocity of $\tilde{\varphi}$ follows directly from properties of min function, φ and the metric.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

One of the most interesting aspects of metric fixed point theory is to extend a linear version of known result to the nonlinear case in metric spaces. To achieve this, Takahashi [16] introduced a convex structure in a metric space (X, d). A mapping $W : X \times X \times [0, 1] \to X$ is a *convex structure* in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. A metric space with a convex structure W is known as a convex metric space which denoted by (X, d, W). A nonempty subset K of a convex metric space is said to be *convex* if

$$W(x, y, \lambda) \in K$$

for all $x, y \in K$ and $\lambda \in [0, 1]$. In fact, every normed linear space and its convex subsets are convex metric spaces but the converse is not true, in general (see, [16]).

Example 1.3. ([9]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $\lambda \in [0, 1]$. We define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1 - \lambda)y_1, \frac{\lambda x_1 x_2 + (1 - \lambda)y_1 y_2}{\lambda x_1 + (1 - \lambda)y_1}\right)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (X, d, W) is a convex metric space but not a normed linear space.

A metric space X is a $CAT(0)$ space. This term is due to M. Gromov [6] and it is an acronym for E. Cartan, A.D. Aleksandrov and V.A. Toponogov. If X is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane(see, *e.g.*, [2, p.159]). It is well known that any complete, simply connected Riemannian manifold nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [2] or Burago *et al.* [4].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or, *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely* *geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists(see, [2]).

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,
>
> $$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces*(see, [11]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if satisfies the (CN) inequality (*cf.* [2, p.163]). The above inequality has been extended by [5] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha)y) \\ &\leq \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \alpha(1-\alpha)d^2(x, y), \end{aligned} \tag{CN*}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality(see, [2, p.163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha)d(z, y) \tag{1.3}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$. In view of the above inequality, $CAT(0)$ space have Takahashi's convex structure

$$W(x, y, \alpha) = \alpha x \oplus (1-\alpha)y.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

It is easy to see that for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(x, (1-\lambda)x \oplus \lambda y) = \lambda d(x, y),$$
$$d(y, (1-\lambda)x \oplus \lambda y) = (1-\lambda) d(x, y).$$

As a consequence,

$$1 \cdot x \oplus 0 \cdot y = x,$$
$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Moreover, a subset K of $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$(see, [1, 10, 13]).

The purpose of this paper, we discuss the convergence theorems for the double acting iterative scheme to a common fixed point for a family of generalized φ-weak contraction mappings in $CAT(0)$ spaces.

2. Convergence theorems of double acting iterative schemes

Xue [18] proved the following very intersting fixed point theorem in complete metric space.

Theorem 2.1. ([18]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a generalized φ-weak contraction. Then the Picard iterative scheme* ([14])

$$x_{n+1} = Tx_n$$

converges to the unique fixed point.

Theorem 2.2. *Let T be a generalized φ-weak contractive self mapping of a closed convex subset K of a Banach space X. Then the Picard iterative scheme*

$$x_{n+1} = Tx_n$$

converges strongly to the fixed point p with the following error estimate:

$$\|x_{n+1} - p\| \le \Phi^{-1}(\Phi(\|x_1 - p\| - n)),$$

where Φ is defined by the antiderivative

$$\Phi(t) = \int \frac{1}{\varphi(t)}\, dt, \quad \Phi(0) = 0$$

and Φ^{-1} is the inverse of Φ.

Proof. The proof is similar as Rhoades ([15], Theorem 2). However, for completeness, we give a sketch of the proof. We can obtain convergence follows from Theorem 2.1. To establish the error estimete, from (1.2) with $\lambda_n = \|x_n - p\|$,

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

$$\begin{aligned}\lambda_{n+1} &= \|x_{n+1} - p\| = \|Tx_n - p\| \\ &\leq \|x_n - p\| - \varphi(\|x_{n+1} - p\|) \\ &= \lambda_n - \varphi(\lambda_{n+1}),\end{aligned}$$

so, we have

$$\varphi(\lambda_{n+1}) \leq \lambda_n - \lambda_{n+1}. \tag{2.1}$$

Thus

$$\Phi(\lambda_n) - \Phi(\lambda_{n+1}) = \int_{\lambda_{n+1}}^{\lambda_n} \frac{1}{\varphi(t)}\, dt = \frac{\lambda_n - \lambda_{n+1}}{\varphi(\mu_n)},$$

for some $\lambda_{n+1} < \mu_n < \lambda_n$. Since φ is nondecreasing, from (2.1),

$$\Phi(\lambda_n) - \Phi(\lambda_{n+1}) = \frac{\lambda_n - \lambda_{n+1}}{\varphi(\mu_n)} \geq \frac{\lambda_n - \lambda_{n+1}}{\varphi(\lambda_n)} \geq 1.$$

Thus

$$\Phi(\lambda_{n+1}) \leq \Phi(\lambda_n) - 1 \leq \cdots \leq \Phi(\lambda_1) - n.$$

This completes the proof of Theorem 2.2. □

In this section, we will use $I = \{1, 2, \cdots, r\}$, where $r \geq 1$. Let $\{T_i : i \in I\}$ be a family of generalized φ-weak contraction self mappings on K. The scheme introduced in [8] is

$$x_1 \in K, \quad x_{n+1} = U_{n(r)}x_n, \quad n \geq 1, \tag{2.2}$$

where

$$\begin{aligned}U_{n(0)} &= I_d \ \ (: \text{the identity mapping}), \\ U_{n(1)}x &= \alpha_{n(1)}x \oplus (1 - \alpha_{n(1)})T_1^n U_{n(0)}x, \\ U_{n(2)}x &= \alpha_{n(2)}x \oplus (1 - \alpha_{n(2)})T_2^n U_{n(1)}x, \\ &\vdots \\ U_{n(r-1)}x &= \alpha_{n(r-1)}x \oplus (1 - \alpha_{n(r-1)})T_{r-1}^n U_{n(r-2)}x, \\ U_{n(r)}x &= \alpha_{n(r)}x \oplus (1 - \alpha_{n(r)})T_r^n U_{n(r-1)}x,\end{aligned}$$

where $\alpha_{n(i)} \in [0, 1]$ for each $i \in I$.

After this, the we called the iterative scheme (2.2) is *double acting iterative scheme.*

The existence of fixed (or common fixed) points of one mapping (or two mappings or a family of mappigs) is not known in many situations. So the

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

approximation of fixed (or common fixed) points of one or more mappings by various iterations have been extensively studied in many other spaces.

In the sequel, it is assumed that

$$\mathcal{F} = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset,$$

where $F(T_i) = \{x \in K : T_i x = x,\ i \in I\}$.

Now, we shall investigate the convergence of double acting iterative scheme applied to $\{T_i : i \in I\}$.

Theorem 2.3. *Let (X, d) be a complete $CAT(0)$ space, K be a closed convex subset of X, $\{T_i : i \in I\}$ be a family of generalized φ-weak contraction self mappings of K. Then the double acting iterative scheme* (2.2) *satisfies*

(i) $0 \le \alpha_{n(i)} \le 1,\ i \in I,$
(ii) $\sum_{n=1}^{\infty}(1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)}) = \infty$

converges to commom fixed point $p \in \mathcal{F}$.

Proof. For $p \in \mathcal{F}$, using (2.2) and (1.3),

$$\begin{aligned}
&d(U_{n(1)}x_n, p)\\
&= d(\alpha_{n(1)}x_n \oplus (1-\alpha_{n(1)})T_1^n U_{n(0)}x_n, p)\\
&\le \alpha_{n(1)}d(x_n, p) + (1-\alpha_{n(1)})d(T_1^n x_n, p)\\
&\le \alpha_{n(1)}d(x_n, p) + (1-\alpha_{n(1)})[d(x_n, p) - \varphi(d(T_1^n x_n, p))]\\
&\le d(x_n, p) - (1-\alpha_{n(1)})\varphi(d(T_1^n x_n, p)). \qquad (2.3)
\end{aligned}$$

Using (2.3), we get

$$\begin{aligned}
&d(U_{n(2)}x_n, p)\\
&= d(\alpha_{n(2)}x_n \oplus (1-\alpha_{n(2)})T_2^n U_{n(1)}x_n, p)\\
&\le \alpha_{n(2)}d(x_n, p) + (1-\alpha_{n(2)})d(T_2^n U_{n(1)}x_n, p)\\
&\le \alpha_{n(2)}d(x_n, p) + (1-\alpha_{n(2)})[d(U_{n(1)}x_n, p) - \varphi(d(T_2^n U_{n(1)}x_n, p))]\\
&\le \alpha_{n(2)}d(x_n, p) + (1-\alpha_{n(2)})[d(x_n, p) - (1-\alpha_{n(1)})\varphi(d(T_1^n x_n, p))]\\
&\quad - (1-\alpha_{n(2)})\varphi(d(T_2^n U_{n(1)}x_n, p))\\
&\le d(x_n, p) - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\varphi(d(T_1^n x_n, p))\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\varphi(d(T_2^n U_{n(1)}x_n, p))
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

and

$$\begin{aligned}
&d(U_{n(3)}x_n, p)\\
&= d(\alpha_{n(3)}x_n \oplus (1-\alpha_{n(3)})T_2^n U_{n(2)}x_n, p)\\
&\le d(x_n, p) - (1-\alpha_{n(1)})(1-\alpha_{n(2)})(1-\alpha_{n(3)})\varphi(d(T_1^n x_n, p))\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})(1-\alpha_{n(3)})\varphi(d(T_2^n U_{n(1)}x_n, p))\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})(1-\alpha_{n(3)})\varphi(d(T_3^n U_{n(2)}x_n, p)).
\end{aligned}$$

Continue this processing, we obtain

$$\begin{aligned}
&d(U_{n(r)}x_n, p)\\
&= d(\alpha_{n(r)}x_n \oplus (1-\alpha_{n(r)})T_r^n U_{n(r-1)}x_n, p)\\
&\le d(x_n, p) - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_1^n x_n, p))\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_2^n U_{n(1)}x_n, p))\\
&\quad \vdots\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_r^n U_{n(r-1)}x_n, p))\\
&\le d(x_n, p) - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_i^n U_{n(i-1)}x_n, p)), \quad (2.4)
\end{aligned}$$

for each $i \in I$. From property of φ, we conclude

$$d(U_{n(r)}x_n, p) \le d(x_n, p),$$

that is

$$d(x_{n+1}, p) \le d(x_n, p).$$

Therefore, $\{d(x_n, p)\}$ is a nonnegative nonincreasing sequence, which converges to a limit $L \ge 0$.

(I) Most of all, we want to show that

$$d(T_i^n U_{n(i-1)}x_n, p) \ge L, \quad \forall\, n \ge 1,\ i \in I. \tag{2.5}$$

To show (2.5), it is sufficient to show that there exists $k \in \mathbb{N}$ such that

$$d(x_k, p) \le d(T_i^n U_{n(i-1)}x_n, p), \quad n \ge 1,\ i \in I.$$

To verify (2.5), suppose that $d(T_i^n U_{n(i-1)}x_n, p) < L$. Then

$$d(x_k, p) > d(T_i^n U_{n(i-1)}x_n, p), \quad \forall\, k \in \mathbb{N}, \tag{2.6}$$

for $n \ge 1$, $i \in I$. Since $\{d(x_n, p)\}$ is a nonincreasing sequence, we have

$$d(x_n, p) \ge d(x_{n+1}, p) \ge \cdots \ge L, \quad \forall\, n \ge 1. \tag{2.7}$$

Let

$$\frac{\varepsilon}{2n} = L - d(T_i^n U_{n(i-1)}x_n, p) > 0. \tag{2.8}$$

Since $\lim_{n\to\infty} d(x_n, p) = L$ and (2.6), there exists $N \in \mathbb{N}$ with

$$d(x_N, p) < d(T_i^n U_{n(i-1)} x_n, p) + \frac{\varepsilon}{4n} \tag{2.9}$$

such that

$$\begin{aligned}
|d(x_n, p) - L| &\le |L - d(T_i^n U_{n(i-1)} x_n, p)| + |d(T_i^n U_{n(i-1)} x_n, p) - d(x_n, p)| \\
&= L - d(T_i^n U_{n(i-1)} x_n, p) + d(x_n, p) - d(T_i^n U_{n(i-1)} x_n, p) \\
&\le \frac{\varepsilon}{2n} + d(x_N, p) - d(T_i^n U_{n(i-1)} x_n, p) \qquad \text{(from (2.7))} \\
&< \frac{\varepsilon}{2n} + \frac{\varepsilon}{4n} < \varepsilon, \quad \forall\, n \ge N.
\end{aligned}$$

On the other hand, from (2.9), (2.8) and (2.6), we obtain

$$\begin{aligned}
d(x_N, p) &< d(T_i^n U_{n(i-1)} x_n, p) + \frac{\varepsilon}{4n} \\
&= d(T_i^n U_{n(i-1)} x_n, p) + \frac{1}{2}(L - d(T_i^n U_{n(i-1)} x_n, p)) \\
&= \frac{1}{2}(L + d(T_i^n U_{n(i-1)} x_n, p)) \\
&< \frac{1}{2}(L + d(x_N, p)),
\end{aligned}$$

i.e.,

$$d(x_N, p) < L.$$

This is a contradiction to (2.7). Therefore, (2.5) holds. That is

$$d(T_i^n U_{n(i-1)} x_n, p) \ge L, \quad \forall\, n \ge 1,\ i \in I.$$

(II) We claim that $L = 0$. Suppose that $L > 0$. It follows that, from (2.4) and (2.5), for any fixed integer $N \in \mathbb{N}$ and $i \in I$

$$\begin{aligned}
&\sum_{n=N}^{\infty} (1 - \alpha_{n(1)})(1 - \alpha_{n(2)}) \cdots (1 - \alpha_{n(r)}) \varphi(L) \\
&\le \sum_{n=N}^{\infty} (1 - \alpha_{n(1)})(1 - \alpha_{n(2)}) \cdots (1 - \alpha_{n(r)}) \varphi(d(T_i^n U_{n(i-1)} x_n, p)) \\
&\le \sum_{n=N}^{\infty} (d(x_n, p) - d(x_{n+1}, p)) \\
&\le d(x_N, p).
\end{aligned}$$

This is a contradiction to the condition (ii). Therefore, $L \le 0$. Thus

$$\lim_{n\to\infty} d(x_n, p) = L = 0.$$

This completes the proof of Theorem 2.3. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Remark 2.1. The author does not apply the real $CAT(0)$ properties of a space such as for example (CN*) inequality,

$$\begin{aligned} &d^2(\alpha x \oplus (1-\alpha)y, z) \\ &\leq \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y), \end{aligned} \tag{CN*}$$

but only the fact that

$$d(\alpha x \oplus (1-\alpha)y, z) \leq \alpha d(x,z) + (1-\alpha)d(y,z),$$

i.e., the convexity of the metric.

Corollary 2.1. *Let (X,d) be a complete $CAT(0)$ space, K be a closed convex subset of X, T be a generalized φ-weak contraction self mapping of K. Then the Noor iterative scheme* ([17])

$$\begin{aligned} x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n)Ty_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tz_n, \\ z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tx_n \end{aligned}$$

satisfies

(i) $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$,
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n)(1-\gamma_n) = \infty$

converges to fixed point $p \in F(T)$.

Proof. In the double acting iterative scheme (2.2), if $r = 3$ and $T_1 = T_2 = T_3 = T$, then it reduces to the Noor iterative scheme. So the proof is similar to that of Theorem 2.3, and will be omitted. □

Corollary 2.2. *Let (X,d) be a complete $CAT(0)$ space, K be a closed convex subset of X, T be a generalized φ-weak contraction self mapping of K. Then the Ishikawa iterative scheme* ([7])

$$\begin{aligned} x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n)Ty_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tx_n \end{aligned}$$

satisfies

(i) $0 \leq \alpha_n, \beta_n \leq 1$,
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = \infty$

converges to fixed point $p \in F(T)$.

Proof. In the double acting iterative scheme (2.2), if $r = 2$ and $T_1 = T_2 = T$, then it reduces to the Ishikawa iterative scheme. So the proof is similar to that of Theorem 2.3, and will be omitted. □

Corollary 2.3. *Let (X, d) be a complete $CAT(0)$ space, K be a closed convex subset of X, T be a generalized φ-weak contraction self mapping of K. Then the Mann iterative scheme* ([12])

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)Tx_n,$$

satisfies

(i) $0 \le \alpha_n \le 1$,
(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$

converges to fixed point $p \in F(T)$.

Proof. In the double acting iterative scheme (2.2), if $r =$ and $T_1 = T$, then it reduces to the Mann iterative scheme. So the proof is similar to that of Theorem 2.3, and will be omitted. □

Competing interests

The authors declares that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This work was supported by Kyungnam University Research Fund, 2017.

References

[1] Y.I. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*, in: I. Gohberg, Yu. Lyubich(Eds.), New Results in Operator Theory, in: Advances and Appl., vol. 98, Birkhäuser, Basel, 1997, 7–22.

[2] M. Bridson and A. Haefliger, Metric spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, 1999.

[3] F. Bruhat and J. Tits, *Groups réductifss sur un corps local. I. Données radicielles valuées*, Publ. Math. Inst. Hautes Études Sci., **41** (1972), 5–251.

[4] D. Burago, Y. Burago and S. Ivanov, A course in metric Geometry, in:Graduate studies in Math., 33, Amer. Math. Soc., Providence, Rhode Island, 2001.

[5] S. Dhompongsa and B. Panyanak, *On triangle-convergence theorems in $CAT(0)$ spaces*, Comput. Math. Anal., **56** (2008), 2572–2579.

[6] M. Gromov, Hyperbolic groups, Essays in group theory, Math. Sci. Res. Inst. Publ. **8**. Springer, New York, 1987.

[7] S. Ishikawa, *Fixed point by a new iteration*, Proc. Amer. Math. Soc., **44** (1974), 147–150.

[8] A.R. Khan, M.A. Khamsi and H. Fukhar-ud-din, *Strong convergence of a general iteration scheme in $CAT(0)$ spaces*, Nonlinear Anal., **74**(3) (2011), 783-791.

[9] J.K. Kim, K.S. Kim and S.M. Kim, *Convergence theorems of implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces*, Proc. of RIMS Kokyuroku, Kyoto Univ., **1484** (2006), 40–51.

[10] K.S. Kim, *Some convergence theorems for contractive type mappings in $CAT(0)$ spaces*, Abstract and Applied Analysis, 2013, Article ID 381715, 9 pages, http://dx.doi.org/10.1155/2013/381715

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

[11] W.A. Kirk, *A fixed point theorem in $CAT(0)$ spaces and $\mathbb{R}$-trees*, Fixed Point Theory Appl., **2004(4)** (2004), 309–316.
[12] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.
[13] A. Nicolae, *Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces*, Nonlinear Analysis, 2013, **87**, 102–115
[14] E. Picard, *Sur les groupes de transformation des équations différentielles linéaires*, Comptes Rendus Acad. Sci. Paris, **96** (1883), 1131–1134.
[15] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001), 2683–2693.
[16] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep., **22** (1970), 142–149.
[17] B.L. Xu and M.A. Noor, *Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **267** (2002), 444–453.
[18] Z. Xue, *The convergence of fixed point for a kind of weak contraction*, Nonlinear Func. Anal. Appl., **21**(3) (2016), 497–500.

On solution of a system of differential equations via fixed point theorem

Muhammad Nazam[1], Muhammad Arshad[1], Choonkil Park[2*], Özlem Acar[3], Sungsik Yun[4*], George A. Anastassiou[5]

[1]Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
e-mail: nazim254.butt@gmail.com; marshadzia@iiu.edu.pk

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr

[3]Department of Mathematics, Faculty of Science and Arts, Mersin University, 33343, Yenişehir, Mersin, Turkey
e-mail: ozlemacar@mersin.edu.tr

[4]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
e-mail: ssyun@hs.ac.kr

[5]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
e-mail: ganastss@@memphis.edu

Abstract. The purpose of the present paper is to study the existence of solution of a system of differential equations using fixed point technique. In this regard, in the first part of this article, along with some properties of partial b-metric topology, we prove a common fixed point theorem for generalized Geraghty type contraction mappings in a complete partial b-metric spaces. Then in second part we apply this result to show the existence of the solution of a system of ordinary differential equations.

1. Introduction and preliminaries

One of the most important results in fixed point theory is the Banach contraction principle introduced by Banach [4]. There were many authors who have studied and proved the results for fixed point theory by generalizing the Banach contraction principle in several directions (see [1, 5–7, 18, 22, 24]).

Czerwik [9] introduced the notion of b-metric to generalize the concept of a distance. The analog of the famous Banach fixed point theorem was proved by Czerwik in the frame of complete b-metric spaces. Following these initial papers, the existence and the uniqueness of (common) fixed points for the classes of both singlevalued and multivalued operators in the setting of (generalized) b-metric spaces have been investigated extensively (see [2, 3, 10, 13, 15, 16, 20, 23, 26–28] and related references therein).

Shukla [29] introduced the concept of partial b-metric space and established some fixed point theorems. Shukla, in fact, generalized Matthews partial metric to partial b-metric. Recently, Mustafa *et al.* [20], Latif *et al.* [19] and Piri *et al.* [21] have established some fixed point results in complete partial b-metric spaces.

In this paper, we introduce the notion of generalized Geraghty type contraction mappings and develop new common fixed point theorems for such mappings in complete partial b-metric spaces and properties of partial b-metric topology. Examples are given to support the usability of our results. In the last section of this paper, we utilize our results to present an application on existence of a solution of a pair of ordinary differential equations. We also study well-posedness of common fixed point problem for generalized Geraghty type contraction mappings.

First of all, we recall some definitions and properties of partial b-metric spaces.

Definition 1. [29] Let X be a nonempty set and $s \geq 1$ be a real number. A function $p_b : X \times X \to [0, \infty)$ is said to be a partial b-metric if for all $x, y, z \in X$, we have

[0]2010 Mathematics Subject Classification: 47H10; 54H25
[0]Keywords: complete partial b-metric space; generalized Geraghty type contraction mapping; differential equation; well posed.
*Corresponding authors.

(p_b1) $x = y$ if and only if $p_b(x,y) = p_b(x,x) = p_b(y,y)$,
(p_b2) $p_b(x,x) \leq p_b(x,y)$,
(p_b3) $p_b(x,y) = p_b(y,x)$,
(p_b4) $p_b(x,y) \leq s\,[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

In this case, the pair (X, p_b) is called a partial b-metric space (with constant s).

It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not to hold. The self distance $p_b(x,x)$, referred to as the size or weight of x, is a feature used to describe the amount of information contained in x.

Definition 2. Let (X, p_b) be a partial b-metric space. The distance function $d_{p_b} : X \times X \to \mathbb{R}_0^+$, defined by

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y), \text{ for all } x, y \in X,$$

defines a metric on X called an induced metric.

Example 1. [29] Let $X = \mathbb{R}^+$ and $l > 1$. Then the functional $p_b : X \times X \to \mathbb{R}^+$, defined by

$$p_b(x,y) = \left\{(\max\{x,y\})^l + |x-y|^l\right\}, \text{ for all } x, y \in X,$$

is a partial b-metric.

Example 2. [29] Let X be a nonempty set such that p is a partial metric and d is a b-metric with coefficient $s > 1$ on X. Then the function $p_b : X \times X \to \mathbb{R}^+$, defined by $p_b(x,y) = p(x,y) + d(x,y)$ for all $x, y \in X$, is a partial b-metric on X and (X, p_b) is a partial b-metric space.

Example 3. [29] Let X be a nonempty set and p be a partial metric defined on X. The functional $p_b : X \times X \to \mathbb{R}^+$, defined by

$$p_b(x,y) = [p(x,y)]^q \text{ for all } x, y \in X \text{ and } q > 1,$$

defines a partial b-metric.

For a partial b-metric space (X, p_b), we immediately have a natural definition for the open balls:

$$B_\epsilon(x;p_b) = \{y \in X | p_b(x,y) < p_b(x,x) + \epsilon\} \text{ for each } x \in X \text{ and } \epsilon > 0.$$

Proposition 1. *The set $\{B_\epsilon(x;p_b) | x \in X,\ \epsilon > 0\}$ of open balls forms the basis for partial b-metric topology denoted by $\mathcal{T}[p_b]$.*

Proof. It is obvious that

$$X = \cup_{x \in X} B_\epsilon(x;p_b)$$

and for any two open balls $B_\epsilon(x;p_b)$, $B_\delta(y;p_b)$ we note that

$$B_\epsilon(x;p_b) \cap B_\delta(y;p_b) = \cup\{B_\kappa(c;p_b) |\, c \in B_\epsilon(x;p_b) \cap B_\delta(y;p_b)\}$$
$$\text{where, } \kappa = p_b(c,c) + \min\{\epsilon - p_b(x,c), \delta - p_b(y,c)\},$$

as desired. □

Proposition 2. *Each partial b-metric topology is T_0 topology but not T_1.*

Proof. Suppose $p_b : X \times X \to \mathbb{R}_0^+$ is a partial b-metric and $x \neq y$. Then without loss of generality, we have $p_b(x,x) < p_b(x,y)$ for all $x, y \in X$. Choose $\epsilon = p_b(x,y) - p_b(x,x)$. Since

$$p_b(x,x) < p_b(x,x) + \epsilon = p_b(x,y),$$

$x \in B_\epsilon(x;p_b)$ and $y \notin B_\epsilon(x;p_b)$. Otherwise we obtain an absurdity $(p_b(x,y) < p_b(x,y))$. It is obvious that for $x \neq v$,

$$x \in B_\delta(x;p_b) \subseteq B_\epsilon(v;p_b),$$

which contradicts T_1 axiom. □

The following definition and lemma describe the convergence criteria established by Shukla in [29].

Definition 3. [29] Let (X, p_b) be a partial b-metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p_b) is called a Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists and is finite.

(2) A partial b-metric space (X, p_b) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}[p_b]$, to a point $\upsilon \in X$ such that
$$p_b(x, x) = \lim_{n,m\to\infty} p_b(x_n, x_m).$$

Lemma 1. [29] Let (X, p_b) be a partial b-metric space.

(1) Every Cauchy sequence in (X, d_{p_b}) is also a Cauchy sequence in (X, p_b).

(2) A partial b-metric (X, p_b) is complete if and only if the metric space (X, d_{p_b}) is complete.

(3) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $\upsilon \in X$ with respect to $\mathcal{T}[(d_{p_b})]$ if and only if
$$\lim_{n\to\infty} p_b(\upsilon, x_n) = p_b(\upsilon, \upsilon) = \lim_{n\to\infty} p_b(x_n, x_m).$$

(4) If $\lim_{n\to\infty} x_n = \upsilon$ such that $p_b(\upsilon, \upsilon) = 0$, then $\lim_{n\to\infty} p_b(x_n, k) = p_b(\upsilon, k)$ for every $k \in X$.

The following important lemma is useful in the sequel.

Lemma 2. [20] Let (X, p_b) be a partial b-metric space with coefficient $s > 1$. Suppose that the sequences $\{x_n\}$,$\{y_n\}$ converge to x, y, respectively. Then we have
$$\begin{aligned}\frac{1}{s^2}p_b(x,y) - \frac{1}{s}p_b(x,x) - p_b(y,y) &\leq \lim_{n\to\infty}\inf p_b(x_n,y_n) \leq \lim_{n\to\infty}\sup p_b(x_n,y_n)\\ &\leq sp_b(x,x) + s^2p_b(y,y) + s^2p_b(x,y).\end{aligned}$$
If $p_b(x, y) = 0$ then we have $\lim_{n\to\infty} p_b(x_n, y_n) = 0$. Moreover, for each $x^* \in X$ we obtain
$$\begin{aligned}\frac{1}{s}p_b(x,x^*) - p_b(x,x) &\leq \lim_{n\to\infty}\inf p_b(x_n,x^*) \leq \lim_{n\to\infty}\sup p_b(x_n,x^*)\\ &\leq sp_b(x,x^*) + sp_b(x,x).\end{aligned}$$
If $p_b(x, x) = 0$, then we have
$$\frac{1}{s}p_b(x,x^*) \leq \lim_{n\to\infty}\inf p_b(x_n,x^*) \leq \lim_{n\to\infty}\sup p_b(x_n,x^*) \leq sp_b(x,x^*).$$

Let Ω denote to the class of all functions $\beta : [0, +\infty) \to [0, 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$. Geraghty [11] presented a very important generalization of Banach Contraction Principle as follows:

Theorem 1. [11] Let (X, d) be a metric space. Let $S : X \to X$ be a self-mapping. Suppose that there exists $\beta \in \Omega$ such that for all $x, y \in X$,
$$d(Sx, Sy) \leq \beta(d(x,y))\, d(x,y).$$
Then S has a unique fixed point $x^* \in X$ and $\{S^n x\}$ converges to x^* for each $x \in X$.

Following [8], we let Ψ denote to the class of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(1) ψ is nondecreasing,
(2) ψ is continuous,
(3) $\psi(t) = 0$ if and only if $t = 0$.

Definition 4. Let $S, T : X \to X$ be two self-mappings and $F(S)$ and $F(T)$ denote the set of fixed points of S and T, respectively. Then a fixed point problem for S and T is well posed if for any sequence $\{x_n\}$ in X and $x^* \in F(S) \cap F(T)$, $\lim_{n\to\infty} p_b(x_n, S(x_n)) = 0$ or $\lim_{n\to\infty} p_b(x_n, T(x_n)) = 0$ implies $\lim_{n\to\infty} p_b(x_n, x^*) = p_b(x^*, x^*)$.

2. Fixed point results

We begin with the introduction of the concept of generalized Geraghty type contraction mappings as follows:

Definition 5. Let (X, p_b) be a partial b-metric space. The pair $S, T : X \to X$ of self-mappings is called a generalized Geraghty type contraction mapping if there exist $\beta \in \Omega$ and $\psi \in \Psi$ such that for $x, y \in X$, the pair (S, T) satisfies the following inequality:

$$\psi\left(s^3 p_b(Sx, Ty)\right) \leq \beta(\psi(\mathcal{M}(x, y))) \cdot \psi(\mathcal{M}(x, y)) \tag{2.1}$$

where

$$\mathcal{M}(x, y) = \max\left\{p_b(x, y), p_b(x, Sx), p_b(y, Ty), \frac{p_b(x, Ty) + p_b(y, Sx)}{2s}\right\}.$$

The main result of this section is the following.

Theorem 2. *Let (X, p_b) be a complete partial b-metric space and $S, T : X \to X$ be two self-mappings satisfying the following conditions:*

1. *(S, T) is a pair of generalized Geraghty type contraction mappings;*
2. *S or T is a continuous mapping.*

Then S and T have a common fixed point $x^* \in X$.

Proof. First, we suppose that $s > 1$. Let $x_0 \in X$ and choose $x_1 = S(x_0)$, $x_2 = T(x_1)$. Continuing in the same way we construct a sequence $\{x_n\}$ in X such that $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$, $i = 0, 1, 2, \ldots$. Without loss of generality, we can assume that $\mathcal{M}(x, y) > 0$ for $x \neq y$. Now, for $i \in \mathbb{N}$, we have

$$\begin{aligned} 0 < \psi(p_b(x_{2i+1}, x_{2i+2})) &\leq \psi\left(s^3 p_b(Sx_{2i}, Tx_{2i+1})\right) \\ &\leq \beta(\psi(\mathcal{M}(x_{2i}, x_{2i+1}))) . \psi(\mathcal{M}(x_{2i}, x_{2i+1})), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{M}(x_{2i}, x_{2i+1}) &= \max\left\{ \begin{array}{c} p_b(x_{2i}, x_{2i+1}), p_b(x_{2i}, Sx_{2i}), p_b(x_{2i+1}, Tx_{2i+1}), \\ \frac{p_b(x_{2i}, Tx_{2i+1}) + p_b(x_{2i+1}, Sx_{2i})}{2s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} p_b(x_{2i}, x_{2i+1}), p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2}), \\ \frac{p_b(x_{2i}, x_{2i+2}) + p_b(x_{2i+1}, x_{2i+1})}{2s} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} p_b(x_{2i}, x_{2i+1}), p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2}), \\ \frac{p_b(x_{2i}, x_{2i+1}) + p_b(x_{2i+1}, x_{2i+2})}{2s} \end{array} \right\} \\ &= \max\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2})\}. \end{aligned}$$

If $\max\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2})\} = p_b(x_{2i+1}, x_{2i+2})$, then from (2.2) we have

$$\begin{aligned} \psi(p_b(x_{2i+1}, x_{2i+2})) &\leq \beta(\psi(p_b(x_{2i+1}, x_{2i+2}))) . \psi(p_b(x_{2i+1}, x_{2i+2})) \\ &< \psi(p_b(x_{2i+1}, x_{2i+2})), \end{aligned}$$

which is a contradiction. Thus we conclude that

$$\max\{p_b(x_{2i}, x_{2i+1}), p_b(x_{2i+1}, x_{2i+2})\} = p_b(x_{2i}, x_{2i+1}).$$

By (2.2), we get that $\psi(p_b(x_{2i+1}, x_{2i+2})) < \psi(p_b(x_{2i}, x_{2i+1}))$. Since ψ is nondecreasing, we have

$$p_b(x_{2i+1}, x_{2i+2}) < p_b(x_{2i}, x_{2i+1}).$$

This implies that

$$p_b(x_{n+1}, x_{n+2}) < p_b(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N}.$$

Hence we deduce that the sequence $\{p_b(x_n, x_{n+1})\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that $\lim_{n\to\infty} p_b(x_n, x_{n+1}) = r$.

Now we shall prove that $r = 0$. Suppose that $r > 0$. From (2.1), we have

$$\begin{aligned}\psi\left(p_b\left(x_{n+1}, x_{n+2}\right)\right) &\leq \psi\left(s^3 p_b\left(Sx_n, Tx_{n+1}\right)\right) \\ &\leq \beta\left(\psi\left(\mathcal{M}\left(x_n, x_{n+1}\right)\right)\right).\psi\left(\mathcal{M}\left(x_n, x_{n+1}\right)\right),\end{aligned}$$

which implies

$$\psi\left(p_b\left(x_{n+1}, x_{n+2}\right)\right) \leq \beta\left(\psi\left(p_b\left(x_n, x_{n+1}\right)\right)\right).\psi\left(p_b\left(x_n, x_{n+1}\right)\right).$$

Hence

$$\frac{\psi\left(p_b\left(x_{n+1}, x_{n+2}\right)\right)}{\psi\left(p_b\left(x_n, x_{n+1}\right)\right)} \leq \beta\left(\psi\left(p_b\left(x_n, x_{n+1}\right)\right)\right) < 1.$$

This implies that $\lim_{n\to\infty} \beta\left(\psi\left(p_b\left(x_n, x_{n+1}\right)\right)\right) = 1$. Since $\beta \in \Omega$, we have

$$\lim_{n\to\infty} \psi\left(p_b\left(x_n, x_{n+1}\right)\right) = 0,$$

which yields

$$r = \lim_{n\to\infty} p_b\left(x_n, x_{n+1}\right) = 0, \tag{2.3}$$

which is a contradiction.

Now we will show that $\{x_n\}$ is a Cauchy sequence. For this purpose, we use Lemma 1. Suppose that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $m(k) > n(k) > k$ with $d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) \geq \varepsilon$. Let $m(k)$ be the smallest number satisfying the condition above. Then we have $d_{p_b}\left(x_{n(k)}, x_{m(k)-1}\right) < \varepsilon$. Therefore,

$$\begin{aligned}\varepsilon \leq d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) &\leq s\left[d_{p_b}\left(x_{n(k)}, x_{m(k)-1}\right) + dp_b\left(x_{m(k)-1}, x_{m(k)}\right)\right] \\ &< s\left[\varepsilon + d_{p_b}\left(x_{m(k)-1}, x_{m(k)}\right)\right].\end{aligned} \tag{2.4}$$

By taking the upper limit as $k \to \infty$ in (2.4) and using (2.3), we get

$$\varepsilon \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) < s\varepsilon. \tag{2.5}$$

From the triangular inequality, we have

$$d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) \leq s[d_{p_b}\left(x_{n(k)}, x_{n(k)+1}\right) + d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right)] \tag{2.6}$$

and

$$d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s[d_{p_b}\left(x_{n(k)+1}, x_{n(k)}\right) + d_{p_b}\left(x_{n(k)}, x_{m(k)}\right)]. \tag{2.7}$$

By taking upper limit as $k \to \infty$ in (2.6) and applying (2.3) and (2.5),

$$\varepsilon \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) \leq s\left(\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right)\right).$$

Again, by taking the upper limit as $k \to \infty$ in (2.7), we get

$$\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s\left(\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)}\right)\right) \leq s.s\varepsilon = s^2\varepsilon.$$

Thus

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s^2\varepsilon. \tag{2.8}$$

Similarly

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)+1}\right) = \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+2}\right) \leq s^2\varepsilon. \tag{2.9}$$

By the triangular inequality, we have

$$d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s[d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right) + d_{p_b}\left(x_{m(k)+1}, x_{m(k)}\right)]. \tag{2.10}$$

Letting $k \to \infty$ in (2.10) and using (2.3) and (2.8), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right). \tag{2.11}$$

Following the above process, we find

$$\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right) \leq s^3\varepsilon. \tag{2.12}$$

From (2.11) and (2.12), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right) \leq s^3\varepsilon.$$

Since $x_{n(k)} \neq x_{m(k)+1}$, we get

$$\begin{aligned} & \psi\left(d_{p_b}\left(x_{n(k)+1}, x_{m(k)+2}\right)\right) \\ \leq\ & \psi\left(s^3 d_{p_b}\left(Sx_{n(k)}, Tx_{m(k)+1}\right)\right) \\ \leq\ & \beta\left(\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right)\right)\cdot\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right) \\ \leq\ & \beta\left(\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right)\right)\cdot\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right), \end{aligned}$$

where

$$\begin{aligned} M\left(x_{n(k)}, x_{m(k)+1}\right) &= \max\left\{\begin{array}{l} d_{p_b}\left(x_{n(k)}, x_{m(k)+1}\right), d_{p_b}\left(x_{n(k)}, Sx_{n(k)}\right), \\ d_{p_b}\left(x_{m(k)+1}, Tx_{m(k)+1}\right), \\ \frac{d_{p_b}\left(x_{n(k)}, Tx_{m(k)+1}\right)+d_{p_b}\left(x_{m(k)+1}, Sx_{n(k)}\right)}{2s} \end{array}\right\} \\ &= \max\left\{\begin{array}{l} d_{p_b}\left(x_{n(k)}, x_{m(k)-1}\right), d_{p_b}\left(x_{n(k)}, x_{n(k)+1}\right), \\ d_{p_b}\left(x_{m(k)+1}, x_{m(k)+2}\right), \\ \frac{d_{p_b}\left(x_{n(k)}, x_{m(k)+2}\right)+d_{p_b}\left(x_{m(k)+1}, x_{n(k)+1}\right)}{2s} \end{array}\right\}. \end{aligned}$$

Taking the limit as $k \to \infty$ and using $(2.3), (2.5), (2.8)$ and (2.9), we get

$$\frac{\varepsilon}{s} = \max\left\{\frac{\varepsilon}{s}, \frac{s\varepsilon}{4}\right\} \leq \lim_{k\to\infty} \sup \mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right) \leq \max\left\{s^2\varepsilon, \frac{s^2\varepsilon}{4}\right\} = s^2\varepsilon.$$

Similarly, we can show that

$$\frac{\varepsilon}{s} = \max\left\{\frac{\varepsilon}{s}, \frac{s\varepsilon}{4}\right\} \leq \lim_{k\to\infty} \inf \mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right) \leq \max\left\{s^2\varepsilon, \frac{s^2\varepsilon}{4}\right\} = s^2\varepsilon.$$

From (2.9), we have

$$\begin{aligned} \psi\left(s^2\varepsilon\right) &= \psi\left(s^3\left(\frac{\varepsilon}{s}\right)\right) \leq \psi\left(s^3 \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+2}\right)\right) \\ &\leq \beta\left(\psi\left(\lim_{k\to\infty} \sup \mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right)\right)\cdot\psi\left(\lim_{k\to\infty} \sup \mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right) + 0 \\ &\leq \beta\left(\psi\left(s^2\varepsilon\right)\right)\psi\left(s^2\varepsilon\right) \\ &< \psi\left(s^2\varepsilon\right), \end{aligned}$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in (X, d_{p_b}). Since (X, p_b) is a complete partial b-metric space, from Lemma 1, (X, d_{p_b}) is a complete b-metric space. Therefore, the sequence $\{x_n\}$ converges to some $x^* \in (X, d_{p_b})$. From Lemma 1, there exists $x^* \in X$ such that $\lim_{n\to\infty} d_{p_b}(x_n, x^*) = 0$ if and only if

$$\lim_{n\to\infty} p_b(x^*, x_n) = p_b(x^*, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m). \tag{2.13}$$

Since $d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y)$, considering (2.3) and the axiom (p_b2) with $\lim_{n\to\infty} d_{p_b}(x_n, x^*) = 0$, we conclude that

$$\lim_{n\to\infty} p_b(x_n, x_m) = 0. \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$\lim_{n\to\infty} p_b(x^*, x_n) = p_b(x^*, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m) = 0.$$

Now $\lim_{n\to\infty} p_b(x^*, x_n) = 0$ implies that $\lim_{i\to\infty} p_b(x_{2i+1}, x^*) = 0$ and $\lim_{i\to\infty} p_b(x_{2i+2}, x^*) = 0$. As one of S and T is a continuous mapping, so we $\lim_{i\to\infty} p_b(Sx_{2i+1}, Sx^*) = 0$. Thus

$$p_b(x^*, Sx^*) = \lim_{i\to\infty} p_b(x_{2i+2}, Sx^*) \leq \lim_{i\to\infty} p_b(Sx_{2i+1}, Sx^*) = 0,$$

and so $x^* = Sx^*$. By (2.1), we have

$$\begin{aligned}\psi\left(s^3 p_b\left(x^*, T(x^*)\right)\right) &= \psi\left(s^3 p_b\left(S(x^*), T(x^*)\right)\right) \\ &\leq \beta\left(\psi\left(\mathcal{M}\left(x^*, x^*\right)\right)\right).\psi\left(\mathcal{M}\left(x^*, x^*\right)\right) \\ &\leq \beta\left(\psi\left(p_b\left(x^*, T(x^*)\right)\right)\right).\psi\left(p_b\left(x^*, T(x^*)\right)\right).\end{aligned}$$

Due to the definitions of β and ψ, we deduce that $x^* = Tx^*$. Therefore, S and T have a common fixed point $x^* \in X$. It is easy to check that x^* is unique. □

Remark 1. *We note that Theorem 2 is more general than the results established in [11, 12, 14, 25, 26, 29].*

Example 4. Let $X = [0, 1]$. Define a function $p_b : X \times X \to [0, +\infty)$ by $p_b(x, y) = (\max\{x, y\})^2 + (x - y)^2$. Clearly, (X, p_b) is a complete partial b-metric space with the constant $s = 2$. Let β be a function on $[0, +\infty)$ defined by $\beta(t) = \frac{1}{1+t}$ for all $t > 0$ and $\beta(0) = 0$. Then $\beta \in \Omega$. Let ψ be a function on $[0, +\infty)$ defined by $\psi(t) = t$. Then $\psi \in \Psi$. Define the mappings $S, T : X \to X$ by

$$T(x) = \begin{cases} \frac{2}{245}x, & \text{if } x \in \left[0, \frac{1}{2}\right) \\ 1, & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and } S(x) = 0.$$

If $\{x_n\}$ is a Cauchy sequence such that $\{x_n\} \subseteq \left[0, \frac{1}{2}\right)$. Since $\left(\left[0, \frac{1}{2}\right), p_b\right)$ is a complete partial b-metric space, the sequence $\{x_n\}$ converges in $\left[0, \frac{1}{2}\right) \subseteq X$. Thus (X, p_b) is a complete partial b-metric space. We note that $x, y, Sy, Ty \in \left[0, \frac{1}{2}\right)$ and S is continuous. It is easy to check that for all $x, y \in \left[0, \frac{1}{2}\right)$, the following inequality is true

$$\psi\left(s^3 p_b\left(Sx, Ty\right)\right) \leq \beta\left(\psi\left(\mathcal{M}\left(x, y\right)\right)\right) \cdot \psi\left(\mathcal{M}\left(x, y\right)\right),$$

Thus all the conditions of Theorem 2 are satisfied. Hence S and T have a common fixed point $(x = 0)$.

3. Derived results

In Theorem 2, if we set $S = T$ and

$$\mathcal{M}\left(x, y\right) = \max\left\{p_b\left(x, y\right), p_b\left(x, Sx\right), p_b\left(y, Sy\right), \frac{p_b\left(x, Sy\right) + p_b\left(y, Sx\right)}{2s}\right\},$$

then we obtain the following result.

Corollary 1. Let (X, p_b) be a complete partial b-metric space. Suppose that $S : X \to X$ is a self-mapping satisfying the following conditions:

(1) S is a generalized Geraghty type contraction mapping;
(2) S is a continuous mapping.

Then S has a fixed point $x^* \in X$.

In Theorem 2, if $\psi(t) = t$, then we obtain the following corollary.

Corollary 2. Let (X, p_b) be a complete partial b-metric space. Suppose that $S, T : X \to X$ are two self-mappings such that

(1) there exists $\beta \in \Omega$ such that for $x, y \in X$, the pair (S, T) satisfies the following inequality

$$s^3 p_b\left(Sx, Ty\right) \leq \beta\left(\left(\mathcal{M}\left(x, y\right)\right)\right).\left(\mathcal{M}\left(x, y\right)\right),$$

where

$$\mathcal{M}\left(x, y\right) = \max\left\{p_b\left(x, y\right), p_b\left(x, Sx\right), p_b\left(y, Ty\right), \frac{p_b\left(x, Ty\right) + p_b\left(y, Sx\right)}{2s}\right\}.$$

(2) S or T is a continuous mapping

Then S and T have a common fixed point $x^* \in X$.

In particular, if $p_b(x,x)=0$ for all $x\in X$, then the following result can easily be obtained from Theorem 2.

Corollary 3. Let (X,d) be a b-metric space. Suppose that $S,T:X\to X$ are two self-mappings satisfying the following conditions:

(1) (S,T) is a pair of Geraghty type contraction mappings;
(2) S or T is a continuous mapping.

Then S and T have a common fixed point $x^*\in X$.

In the following, we see that the problem stated in Theorem 2 is well posed.

Theorem 3. *Let (X,p_b) be a complete partial b-metric space. Let $S,T\colon X\to X$ be two self-mappings as in Theorem 2 with $\psi(t)=t$. Then the fixed point problem for S and T is well posed.*

Proof. Let $\{x_n\}$ be a sequence in X and $x^*\in F(S)\cap F(T)$. Suppose that $\lim_{n\to\infty}p_b(x_n,S(x_n))=0$. If $\lim_{n\to\infty}p_b(x_n,x^*)=0$, then we are done. Assume that $\lim_{n\to\infty}p_b(x_n,x^*)=r>0$. Using (p_b3), we have

$$\begin{aligned}
s^3p_b(x_n,x^*) &\le s^4[p_b(x_n,S(x_n))+p_b(S(x_n),x^*)-p_b(S(x_n),S(x_n))],\\
s^2p_b(x_n,x^*) &\le s^3p_b(x_n,S(x_n))+s^3p_b(S(x_n),T(x^*))\\
&\le s^3p_b(x_n,S(x_n))+\beta\left(\mathcal{M}\left(x_n,x^*\right)\right)\cdot\mathcal{M}\left(x_n,x^*\right),\\
\frac{1}{s}\lim_{n\to\infty}p_b(x_n,x^*) &\le s^3\lim_{n\to\infty}p_b(x_n,S(x_n))+\lim_{n\to\infty}\beta(p_b(x_n,x^*))\cdot p_b(x_n,x^*),\\
\frac{r}{s} &\le 0+\frac{r}{s^3}\beta(r),\ \text{a contradiction due to the definition of } \beta.
\end{aligned}$$

Similarly, we obtain $\lim_{n\to\infty}x_n=x^*$ if we assume $\lim_{n\to\infty}d(x_n,T(x_n))=0$. □

4. Application

In this section, we present an application on existence of a solution of a pair of ordinary differential equations. In particular, inspired from [17] and using Theorem 2, we consider the following pair of differential equations:

$$\begin{cases} -\frac{d^2x}{dt^2}=f\left(t,x\left(t\right)\right), & t\in[0,1]\\ x(0)=x\left(1\right)=0\end{cases} \quad\text{and}\quad \begin{cases} -\frac{d^2y}{dt^2}=K\left(t,y\left(t\right)\right), & t\in[0,1]\\ y(0)=y\left(1\right)=0\end{cases} \tag{4.1}$$

where $f,K:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions. The Green function associated to (4.1) is defined by

$$G\left(t,s\right)=\begin{cases} t\left(1-s\right), & 0\le t\le s\le 1\\ s\left(1-t\right), & 0\le t\le s\le 1.\end{cases}$$

Let $C\left(I\right)$ be the space of all continuous functions defined on I, where $I=[0,1]$. Suppose that

$$p_b(x,y)=\left(\sup_{t\in I}|x(t)-y(t)|\right)^2+(\max\{x(t),y(t)\})^2.$$

It is known that $(C\left(I\right),p_b)$ is a complete partial b-metric space with constant $s=2$. Now, define the operators $S,T:C\left(I\right)\to C\left(I\right)$ by

$$Sx(t)=\int_0^1 G(t,s)f(s,x(s))ds \text{ and } Tx(t)=\int_0^1 G(t,s)K(s,y(s))ds$$

for all $t\in I$. Note that (4.1) has a solution if and only if the operators S and T have a common fixed point.

The main result is the following.

Theorem 4. *Assume that*

(1) *there exist continuous functions $f,K:[0,1]\times\mathbb{R}\to\mathbb{R}$ such that for all $a,b,\rho\in\mathbb{R}$, we have*

$$|f(t,a)-K(t,b)|^2\le 64\ln\left(\frac{\mathcal{M}(a,b)+1}{\rho}\right)\ \textit{for all } t\in I,$$

where

$$\mathcal{M}(a,b) = \max\left\{p_b(a,b), p_b(a,S(a)), p_b(b,T(b)), \frac{p_b(a,S(b)) + p_b(b,T(a))}{2s}\right\} > \rho;$$

(2) *the operators S, T are such that*

$$(\max\{Sx(t), Ty(t)\})^2 \leq \ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2.$$

Then the system of ordinary differential equations (4.1) has a solution.

Proof. It is well known that $x^* \in C^2(I)$ is a solution of (4.1) if and only if $x^* \in C(I)$ is a solution of the integral equation (see [17]). Define the mappings $S, T : C(I) \to C(I)$ by

$$Sx(t) = \int_0^1 G(t,s) f(s, x(s))ds \text{ and } Tx(t) = \int_0^1 G(t,s) K(s, y(s))ds.$$

Hence the solution of (4.1) is equivalent to find $x^* \in C(I)$, that is, a fixed point of T. By (1), we get

$$\begin{aligned}
p_b(Sx,Ty) &= \left(\sup_{t\in I}|Sx(t) - Ty(t)|\right)^2 + (\max\{Sx(t), Ty(t)\})^2 \\
&\leq \left[\sup_{t\in I}\left|\int_0^1 G(t,s)\,[f(s,x(s)) - K(s,y(s))]\,ds\right|\right]^2 + \ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2 \\
&\leq \left[\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2 |f(s,x(s)) - K(s,y(s))|^2\right] + \ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2 \\
&\leq \left[64\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2 \ln\left(\frac{\mathcal{M}(a,b)+1}{\rho}\right)\right] + \ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2 \\
&= \left[8^2\ln\left(\frac{\mathcal{M}(a,b)+1}{\rho}\right) + \ln(\rho)\right]\left(\sup_{t\in I}\left[\int_0^1 G(t,s)ds\right]^2\right).
\end{aligned}$$

Since $\int G(t,s)ds = -\frac{t^2}{2} + \frac{t}{2}$ for all $t \in I$, we have $\left(\sup_{t\in I}\left[\int_0^1 G(t,s)ds\right]^2\right) = \frac{1}{8^2}$. Therefore,

$$p_b(Sx,Ty) \leq \ln(\mathcal{M}(a,b) + 1),$$

which implies that

$$\begin{aligned}
\ln(p_b(Sx,Ty) + 1) &\leq \ln(\ln(\mathcal{M}(x,y) + 1) + 1) \\
&= \frac{\ln(\ln(\mathcal{M}(x,y) + 1) + 1)}{\ln(\mathcal{M}(x,y) + 1)}\ln(\mathcal{M}(x,y) + 1).
\end{aligned}$$

Define the functions $\psi : [0,\infty) \to [0,\infty)$ and $\beta : [0,\infty) \to [0,1)$ by

$$\psi(x) = \ln(x+1) \text{ and } \beta(x) = \begin{cases} \frac{\psi(x)}{x}, & if\ x \neq 0 \\ 0, & \text{otherwise.}\end{cases}$$

Note that $\psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, positive in $(0,\infty)$, $\psi(0) = 0$ and $\psi(x) < x$. Hence $\beta \in \Omega$, $\psi \in \Psi$ and

$$\psi\left(s^3 p_b(Sx,Ty)\right) \leq \beta(\psi(\mathcal{M}(x,y))).\psi(\mathcal{M}(x,y))$$

for all $x, y \in C(I)$. Therefore, all the assumptions of Theorem 2 are satisfied. Hence S and T have a common fixed point $x^* \in C(I)$, that is, $Sx^* = Tx^* = x^*$, which is a solution of (4.1). □

References

[1] G. A. Anastassiou, I. K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[2] H. Aydi, M. F. Bota, E. Karapinar, S. Moradi, *A common fixed point for weak ϕ-contractions on b-metric spaces*, Fixed Point Theory **13** (2012), 337–346.

[3] H. Aydi, A. Felhi, S. Sahmim, *Common fixed points in rectangular b-metric spaces using $(E : A)$ property*, J. Adv. Math. Studies **8** (2015), 159–169.

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux equations itegrales*, Fund. Math. **3** (1922), 133–181.

[5] A. Batool, T. Kamran, S. Jang, C. Park, *Generalized φ-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[6] M. Berzig, E. Karapınar, *On modified α-ψ-contractive mappings with application,* Thai J. Math. **13** (2015), 147–152.

[7] S. H. Cho, J. S. Bae, E. Karapinar, *Fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2013**, 2013:329.

[8] P. Chuadchawna, A. Kaewcharoen, S. Plubtieng, *Fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings in α-η-complete metric spaces*, J. Nonlinear Sci. App. **9** (2016), 471–485.

[9] S. Czerwik, *Contraction mappings in b-metric spaces.* Acta Math. Inf. Univ. Ostrav. **1** (1993), 5–11.

[10] A. Felhi, S. Sahmim, H. Aydi, *Ulam-Hyers stability and well-posedness of fixed point problems for α-λ-contractions on quasi b-metric spaces*, Fixed Point Theory Appl. **2016**, 2016:1.

[11] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40** (1973), 604–608.

[12] V. Gupta, W. Shatanawi, N. Mani, *Fixed point theorems for (Ψ, β)-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations*, J. Fixed Point Theory Appl. **19** (2017). 1251–1267.

[13] H. Huang, S. Xu, *Fixed point theorems of contractive mappings in cone b-metric spaces and applications*, Fixed Point Theory Appl. **2013**, 2013:112.

[14] N. Hussain, M. A. Kutbi, P. Salimi, *Fixed point theory in α-complete metric space with applications*, Abstr. Appl. Anal. **2014**, Art. ID 280817 (2014).

[15] N. Hussain, M. H. Shah, *KKM mappings in cone b-metric spaces*, Comput. Math. Appl. **62** (2011), 1677–1684.

[16] M. Jovanovic, Z. Kadelburg, S. Radenovic, *Common fixed point results in metric type spaces*, Fixed Point Theory Appl. **2010** , Art. ID 978121 (2010).

[17] E. Karapinar, *α-ψ-Geraghty contraction type mappings and some related fixed point results*, Filomat **28** (2014), 37–48.

[18] E. Karapınar, P. Kumam, P. Salimi, *On α-ψ-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:94.

[19] A. Latif, J. R. Roshan, V. Parvaneh, N. Hussain, *Fixed point results via α-admissible mappings and cyclic contractive mappings in partial b-metric spaces*, J. Inequal. Appl. **2014**, 2014:345.

[20] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Some common fixed point results in ordered partial b-metric spaces*, J. Inequal. Appl. **2013**, 2013:562.

[21] H. Piri, H. Afshari, *Some fixed point theorems in complete partial b-metric spaces*, Adv. Fixed Point Theory **4** (2014), 444–461.

[22] O. Popescu, *Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2014**, 2014:190.

[23] J. R. Roshan, V. Parvaneh, Sh. Sedghi, N. Shobkolaei, W. Shatanawi, *Common fixed points of almost generalized $(\psi$-$\varphi)_s$-contraction mappings in ordered b-metric spaces.* Fixed Point Theory Appl. **2013**, 2013:159.

[24] P. Salimi, A. Latif, N. Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory Appl. **2013**, 2013:151.

[25] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

[26] R. J. Shahkoohi, A. Razani, *Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces*, J. Inequal. Appl. **2014**, 2014:373.

[27] W. Shatanawi, M. B. Hani, *A fixed point theorem in b-metric spaces with nonlinear contractive condition*, Far East J. Math. Sci. **100** (2016), 1901–1908.

[28] L. Shi, S. Xu, *Common fixed point theorems for two weakly compatible self-mappings in cone b-metric spaces*, Fixed Point Theory Appl. **2013**, 2013:120.

[29] S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math. **11** (2014), 703–711.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

SOME EQUALITIES AND INEQUALITIES FOR K-G-FRAMES

ZHONG-QI XIANG† AND YIN-SUO JIA

Abstract. In this paper we establish some equalities and inequalities for K-g-frames. Our results generalize the remarkable results obtained by Balan et al. and Găvruţa. We also give several new inequalities for K-g-frames by using operator theory methods, which differ in structure from those for frames.

1. Introduction

Throughout this paper, $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces, $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ is a sequence of closed subspaces of $\mathscr{K}$, where $\mathbb{J}$ is a finite or countable index set. For any $\mathbb{I}\subset\mathbb{J}$, we denote $\mathbb{I}^c=\mathbb{J}\backslash\mathbb{I}$. The notation $B(\mathscr{H},\mathscr{K})$ is reserved for the set of all linear bounded operators from $\mathscr{H}$ to $\mathscr{K}$, and $B(\mathscr{H},\mathscr{H})$ is abbreviated to $B(\mathscr{H})$; $K\in B(\mathscr{H})$.

Frames for Hilbert spaces, appeared first in the early 1950's, have now been applied in a variety of fields because of their redundancy and flexibility. For more information on frame theory and its applications, the interested reader can consult [4–8,16,19]. G-frames, proposed by Sun in [17], generalize the concept of frames extensively and possess some distinct properties though they share many similar properties with frames, see [15,18].

A K-frame is a generalization of a frame, which was put forward by Găvruţa in [10] to investigate the atomic systems associated with a linear bounded operator K. When K is an orthogonal projection, a K-frame is just an atom system for subspace which was introduced by Feichtinger and Werther in [9]. It should be remarked that the properties of K-frames are quite different from those of frames as shown in [1,12,20,22], though the definition of a K-frame is similar to a frame in form. Recently, Xiao et al. [23] applied Găvruţa's idea to the case of g-frames, thereby leading to the notion of K-g-frames, which have attracted much attention, see [2,13].

Balan et al. [3] found a surprising identity for Parseval frames when they devoted to the study of efficient algorithms for signal reconstruction, given below.

Theorem 1.1. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval frame for $\mathscr{H}$, then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$, we have*

$$\sum_{j\in\mathbb{I}}|\langle f,f_j\rangle|^2-\Big\|\sum_{j\in\mathbb{I}}\langle f,f_j\rangle f_j\Big\|^2=\sum_{j\in\mathbb{I}^c}|\langle f,f_j\rangle|^2-\Big\|\sum_{j\in\mathbb{I}^c}\langle f,f_j\rangle f_j\Big\|^2. \tag{1.1}$$

In [3], the following inequality was also obtained.

Theorem 1.2. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval frame for $\mathscr{H}$, then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$, we have*

$$\sum_{j\in\mathbb{I}}|\langle f,f_j\rangle|^2+\Big\|\sum_{j\in\mathbb{I}^c}\langle f,f_j\rangle f_j\Big\|^2\geq\frac{3}{4}\|f\|^2. \tag{1.2}$$

---

† Corresponding author.

2010 *Mathematics Subject Classification*. Primary 42C15; Secondary 42C40.
Key words and phrases. Parseval K-g-frame; K-dual g-frame; operator; pseudo-inverse.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Later on, Găvruţa [11] extended Theorems 1.1 and 1.2 to alternate dual frames:

Theorem 1.3. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a frame for $\mathscr{H}$ and $\{g_j\}_{j\in\mathbb{J}}$ be an alternate dual frames of $\{f_j\}_{j\in\mathbb{J}}$. Then for all $\mathbb{I}\subset\mathbb{J}$ and all $f\in\mathscr{H}$, we have*

$$\begin{aligned}&\operatorname{Re}\sum_{j\in\mathbb{I}}\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{I}^c}\langle f,g_j\rangle f_j\Big\|^2\\ &=\operatorname{Re}\sum_{j\in\mathbb{I}^c}\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{I}}\langle f,g_j\rangle f_j\Big\|^2\geq\frac{3}{4}\|f\|^2. \qquad (1.3)\end{aligned}$$

In fact, Theorem 1.3 is a particular case of the following result, given in [11].

Theorem 1.4. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a frame for $\mathscr{H}$ and $\{g_j\}_{j\in\mathbb{J}}$ be an alternate dual frame of $\{f_j\}_{j\in\mathbb{J}}$. Then for all bounded sequence $\{\omega_j\}_{j\in\mathbb{J}}$ and all $f\in\mathscr{H}$, we have*

$$\begin{aligned}&\operatorname{Re}\sum_{j\in\mathbb{J}}\omega_j\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{J}}(1-\omega_j)\langle f,g_j\rangle f_j\Big\|^2\\ &=\operatorname{Re}\sum_{j\in\mathbb{J}}(1-\omega_j)\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{J}}\omega_j\langle f,g_j\rangle f_j\Big\|^2\geq\frac{3}{4}\|f\|^2. \qquad (1.4)\end{aligned}$$

In this paper we generalize the equalities and inequalities (1.1), (1.2) and (1.4) to K-g-frames. Since g-frames can be considered as a class of K-g-frames, Theorem 2.2 in [21] and Theorem 4.1 in [24] which are a generalization of Theorems 1.1 and 1.2, and Theorem 1.4 respectively, can be obtained as a special case of the results we establish on K-g-frames. We also present some new inequalities for K-g-frames by using operator theory methods, which are different in structure from those in (1.2)–(1.4).

2. Preliminaries

In the following we briefly recall some definitions and basic properties of operators.

Definition 2.1. We call a sequence $\{\Lambda_j\in B(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, if there exist $0<C\leq D<\infty$ such that

$$C\|K^*f\|^2\leq\sum_{j\in\mathbb{J}}\|\Lambda_jf\|^2\leq D\|f\|^2,\quad\forall f\in\mathscr{H}. \qquad (2.1)$$

If we only require the right-hand inequality of (2.1), then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said to be a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with g-Bessel bound D.

Remark 2.2. If $K=I_{\mathscr{H}}$, the identity operator on $\mathscr{H}$, then a K-g-frame is just a g-frame.

A K-g-frame $\{\Lambda_j\in B(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ is said to be Parseval if

$$\|K^*f\|^2=\sum_{j\in\mathbb{J}}\|\Lambda_jf\|^2,\quad\forall f\in\mathscr{H}. \qquad (2.2)$$

Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then it is easy to check that

$$KK^*f=\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_jf,\quad\forall f\in\mathscr{H}. \qquad (2.3)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Definition 2.3. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. A g-Bessel sequence $\{\Gamma_j\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ is called a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$, if

$$Kf = \sum_{j\in\mathbb{J}} \Lambda_j^*\Gamma_j f, \quad \forall f \in \mathscr{H}. \tag{2.4}$$

To prove the main results, we need the following lemmas.

Lemma 2.4. *(see [6]) Suppose that $\mathscr{T} \in B(\mathscr{H})$ has closed range, then there exists a unique operator $\mathscr{T}^\dagger \in B(\mathscr{H})$, called the pseudo-inverse of $\mathscr{T}$, satisfying*

$$\mathscr{T}\mathscr{T}^\dagger\mathscr{T} = \mathscr{T}, \quad \mathscr{T}^\dagger\mathscr{T}\mathscr{T}^\dagger = \mathscr{T}^\dagger, \quad (\mathscr{T}^\dagger\mathscr{T})^* = \mathscr{T}^\dagger\mathscr{T}, \quad (\mathscr{T}^\dagger)^* = (\mathscr{T}^*)^\dagger.$$

In the following, the notation $\Theta^\dagger$ is reserved to denote the pseudo-inverse of the linear bounded operator Θ (if it exists).

Lemma 2.5. *(see [14]) Suppose that $U, V, \mathscr{T} \in B(\mathscr{H})$, that $U + V = \mathscr{T}$, and that $\mathscr{T}$ has closed range. Then we have*

$$\mathscr{T}^*\mathscr{T}^\dagger U + V^*\mathscr{T}^\dagger V = V^*\mathscr{T}^\dagger\mathscr{T} + U^*\mathscr{T}^\dagger U.$$

Lemma 2.6. *If $U, V, K \in B(\mathscr{H})$ satisfy $U + V = K$, then*

$$U^*U + \frac{1}{2}(V^*K + K^*V) \geq \frac{3}{4}K^*K.$$

Proof. We have

$$\begin{aligned}
U^*U + \frac{1}{2}(V^*K + K^*V) &= (K^* - V^*)(K - V) + \frac{1}{2}(V^*K + K^*V)\\
&= V^*V - (K^*V + V^*K) + K^*K + \frac{1}{2}(V^*K + K^*V)\\
&= V^*V - \frac{1}{2}(V^*K + K^*V) + K^*K\\
&= \left(V - \frac{1}{2}K\right)^*\left(V - \frac{1}{2}K\right) + \frac{3}{4}K^*K\\
&\geq \frac{3}{4}K^*K.
\end{aligned}$$

□

3. Main results and their proofs

We begin with several equalities and inequalities for Parseval K-g-frames and K-dual g-frames.

Theorem 3.1. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$ we have*

$$\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\|^2 = \overline{\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j KK^* f\rangle} - \left\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\right\|^2. \tag{3.1}$$

Proof. For every $\mathbb{I} \subset \mathbb{J}$, one can easily check that the operators $S_\mathbb{I}$ and $S_{\mathbb{I}^c}$ defined by

$$S_\mathbb{I} f = \sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f, \quad S_{\mathbb{I}^c} f = \sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

are positive, linear bounded and self-adjoint. Moreover, the definition of a Parseval K-g-frame gives $S_{\mathbb{I}} + S_{\mathbb{I}^c} = KK^*$. Hence for each $f \in \mathscr{H}$,

$$\begin{aligned}\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2 &= \langle S_{\mathbb{I}} f, KK^* f\rangle - \|S_{\mathbb{I}} f\|^2\\ &= \langle (KK^* - S_{\mathbb{I}})S_{\mathbb{I}} f, f\rangle\\ &= \langle S_{\mathbb{I}^c}(KK^* - S_{\mathbb{I}^c})f, f\rangle\\ &= \langle S_{\mathbb{I}^c}KK^* f, f\rangle - \langle S_{\mathbb{I}^c}^2 f, f\rangle\\ &= \overline{\langle S_{\mathbb{I}^c} f, KK^* f\rangle} - \|S_{\mathbb{I}^c} f\|^2\\ &= \overline{\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j KK^* f\rangle} - \Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2.\end{aligned}$$

□

A version of the equality obtained in Theorem 3.1 for overlapping divisions is derived in the following result.

Theorem 3.2. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$, every $\mathbb{E} \subset \mathbb{I}^c$, and every $f \in \mathscr{H}$, we have*

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{I}\cup\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c\setminus\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2\\ &\quad= \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2 + 2\mathrm{Re}\sum_{j\in\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle.\end{aligned}$$

Proof. Applying Theorem 3.1 twice yields

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{I}\cup\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c\setminus\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2\\ &\quad= \sum_{j\in\mathbb{I}\cup\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \overline{\sum_{j\in\mathbb{I}^c\setminus\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle}\\ &\quad= \sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \overline{\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j KK^* f\rangle} + 2\mathrm{Re}\sum_{j\in\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle\\ &\quad= \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2 + 2\mathrm{Re}\sum_{j\in\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle.\end{aligned}$$

□

Theorem 3.3. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\{\alpha_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$ and every $f \in \mathscr{H}$ we have*

$$\begin{aligned}&\sum_{j\in\mathbb{J}}(1-\alpha_j)\langle\Gamma_j f, \Lambda_j K f\rangle + \Big\|\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f\Big\|^2\\ &\quad= \overline{\sum_{j\in\mathbb{J}}\alpha_j\langle\Gamma_j f, \Lambda_j K f\rangle} + \Big\|\sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f\Big\|^2. \qquad (3.2)\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Proof. For any $\{\alpha_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$ and any $f \in \mathscr{H}$, we let

$$S_\alpha f = \sum_{j\in\mathbb{J}} \alpha_j \Lambda_j^* \Gamma_j f, \quad S_{1-\alpha} f = \sum_{j\in\mathbb{J}} (1-\alpha_j)\Lambda_j^*\Gamma_j f.$$

Denote by D_1 and D_2 the g-Bessel bounds of $\{\Lambda_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ respectively. Then

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f\Big\|^2 &= \sup_{g\in\mathscr{H},\|g\|=1}\Big|\Big\langle\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f, g\Big\rangle\Big|^2\\ &= \sup_{g\in\mathscr{H},\|g\|=1}\Big|\sum_{j\in\mathbb{J}}\alpha_j\langle\Gamma_j f, \Lambda_j g\rangle\Big|^2\\ &\le \sup_{g\in\mathscr{H},\|g\|=1}\Big(\sum_{j\in\mathbb{J}}|\alpha_j|\,|\langle\Gamma_j f,\Lambda_j g\rangle|\Big)^2\\ &\le \sup_{g\in\mathscr{H},\|g\|=1}\|\{\alpha_j\}_{j\in\mathbb{J}}\|^2\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2\cdot\sum_{j\in\mathbb{J}}\|\Lambda_j g\|^2\\ &\le D_1D_2\|\{\alpha_j\}_{j\in\mathbb{J}}\|^2\|f\|^2.\end{aligned}$$

It follows that S_α is well defined and bounded. By the same way we can show that $S_{1-\alpha}$ is also well defined and bounded. Since $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$, we have

$$S_\alpha f + S_{1-\alpha} f = \sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f + \sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f = \sum_{j\in\mathbb{J}}\Lambda_j^*\Gamma_j f = Kf$$

for each $f \in \mathscr{H}$. It follows that

$$\begin{aligned}\sum_{j\in\mathbb{J}}(1-\alpha_j)\langle\Gamma_j f,\Lambda_j Kf\rangle + \Big\|\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f\Big\|^2 &= \langle S_{1-\alpha}f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle\\ &= \langle (K-S_\alpha)f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle\\ &= \langle Kf, Kf\rangle - \langle S_\alpha f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle, \quad (3.3)\end{aligned}$$

and

$$\begin{aligned}&\overline{\sum_{j\in\mathbb{J}}\alpha_j\langle\Gamma_j f,\Lambda_j Kf\rangle} + \Big\|\sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f\Big\|^2\\ &= \overline{\langle S_\alpha f, Kf\rangle} + \langle S_{1-\alpha}f, S_{1-\alpha}f\rangle\\ &= \langle Kf, S_\alpha f\rangle + \langle (K-S_\alpha)f, (K-S_\alpha)f\rangle\\ &= \langle Kf, S_\alpha f\rangle + \langle Kf, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle - \langle S_\alpha f, Kf\rangle - \langle Kf, S_\alpha f\rangle\\ &= \langle Kf, Kf\rangle - \langle S_\alpha f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle. \quad (3.4)\end{aligned}$$

Combination of (3.3) and (3.4) yields (3.2). □

Corollary 3.4. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I}\subset\mathbb{J}$ and every $f \in \mathscr{H}$ we have

$$\sum_{j\in\mathbb{I}^c}\langle\Gamma_j f,\Lambda_j Kf\rangle + \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Gamma_j f\Big\|^2 = \overline{\sum_{j\in\mathbb{I}}\langle\Gamma_j f,\Lambda_j Kf\rangle} + \Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Gamma_j f\Big\|^2.$$

Proof. The result follows directly from Theorem 3.3 if we take $\mathbb{I}\subset\mathbb{J}$ and

$$\alpha_j = \begin{cases} 1, & j \in \mathbb{I},\\ 0, & j\in\mathbb{I}^c.\end{cases}$$

□

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Theorem 3.5. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\{\alpha_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$ and every $f \in \mathscr{H}$ we have*

$$\Big\|\sum_{j\in\mathbb{J}} \alpha_j\Lambda_j^*\Gamma_j f\Big\|^2 + \mathrm{Re}\Big(\sum_{j\in\mathbb{J}}(1-\alpha_j)\langle\Gamma_j f, \Lambda_j K f\rangle\Big)$$

$$= \Big\|\sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f\Big\|^2 + \mathrm{Re}\Big(\sum_{j\in\mathbb{J}} \alpha_j\langle\Gamma_j f, \Lambda_j K f\rangle\Big) \geq \frac{3}{4}\|Kf\|^2. \tag{3.5}$$

Proof. The equality is obtained immediately if we take the real part on both sides of (3.2). For the inequality, taking

$$Uf = \sum_{j\in\mathbb{J}} \alpha_j\Lambda_j^*\Gamma_j f \quad \text{and} \quad Vf = \sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f$$

for each $f \in \mathscr{H}$ in Lemma 2.6, then we have

$$\begin{aligned}
&\Big\|\sum_{j\in\mathbb{J}} \alpha_j\Lambda_j^*\Gamma_j f\Big\|^2 + \mathrm{Re}\Big(\sum_{j\in\mathbb{J}}(1-\alpha_j)\langle\Gamma_j f, \Lambda_j K f\rangle\Big)\\
&= \|Uf\|^2 + \mathrm{Re}\langle Vf, Kf\rangle = \langle Uf, Uf\rangle + \frac{1}{2}(\langle Vf, Kf\rangle + \langle Kf, Vf\rangle)\\
&= \Big\langle\Big(U^*U + \frac{1}{2}(V^*K + K^*V)\Big)f, f\Big\rangle\\
&\geq \frac{3}{4}\langle K^*Kf, f\rangle = \frac{3}{4}\|Kf\|^2.
\end{aligned}$$

□

Theorem 3.6. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$ we have*

$$\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Lambda_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j KK^* f\rangle$$

$$= \Big\|\sum_{j\in\mathbb{I}^c} \Lambda_j^*\Lambda_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle \geq \frac{3}{4}\|KK^* f\|^2. \tag{3.6}$$

Proof. The equality follows if we take the real part on both sides of (3.1). It remains to prove the inequality. Since $S_{\mathbb{I}} + S_{\mathbb{I}^c} = KK^*$, it follows that

$$\begin{aligned}
S_{\mathbb{I}}^2 + S_{\mathbb{I}^c}^2 &= S_{\mathbb{I}}^2 + (KK^* - S_{\mathbb{I}})^2\\
&= 2S_{\mathbb{I}}^2 + (KK^*)^2 - KK^*S_{\mathbb{I}} - S_{\mathbb{I}}KK^*\\
&= 2\Big(\frac{KK^*}{2} - S_{\mathbb{I}}\Big)^2 + \frac{(KK^*)^2}{2}.
\end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned}
KK^*S_{\mathbb{I}^c} + S_{\mathbb{I}^c}KK^* + 2S_{\mathbb{I}}^2 &= KK^*(S_{\mathbb{I}} + S_{\mathbb{I}^c}) - KK^*S_{\mathbb{I}} + S_{\mathbb{I}^c}KK^* + 2S_{\mathbb{I}}^2\\
&= (KK^*)^2 - (S_{\mathbb{I}} + S_{\mathbb{I}^c})S_{\mathbb{I}} + S_{\mathbb{I}^c}(S_{\mathbb{I}} + S_{\mathbb{I}^c}) + 2S_{\mathbb{I}}^2\\
&= (KK^*)^2 - S_{\mathbb{I}}^2 - S_{\mathbb{I}^c}S_{\mathbb{I}} + S_{\mathbb{I}^c}S_{\mathbb{I}} + S_{\mathbb{I}^c}^2 + 2S_{\mathbb{I}}^2\\
&= (KK^*)^2 + S_{\mathbb{I}}^2 + S_{\mathbb{I}^c}^2 \geq \frac{3}{2}(KK^*)^2.
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Thus for every $f \in \mathscr{H}$ we have

$$\begin{aligned}
&\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Lambda_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Lambda_j f, \Lambda_j KK^* f\rangle \\
&= \|S_{\mathbb{I}} f\|^2 + \frac{1}{2}(\langle S_{\mathbb{I}^c} f, KK^* f\rangle + \langle KK^* f, S_{\mathbb{I}^c} f\rangle) \\
&= \frac{1}{2}(2\langle S_{\mathbb{I}}^2 f, f\rangle + \langle S_{\mathbb{I}^c} f, KK^* f\rangle + \langle KK^* f, S_{\mathbb{I}^c} f\rangle) \\
&= \frac{1}{2}\langle (KK^* S_{\mathbb{I}^c} + S_{\mathbb{I}^c} KK^* + 2S_{\mathbb{I}}^2) f, f\rangle \geq \frac{3}{4}\langle (KK^*)^2 f, f\rangle = \frac{3}{4}\|KK^* f\|^2.
\end{aligned}$$

□

We give an upper bound condition for the left-hand-side of the equality in (3.6) under the condition that K has closed range.

Theorem 3.7. *Suppose that K has closed range and that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$ we obtain*

$$\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Lambda_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Lambda_j f, \Lambda_j KK^* f\rangle \leq \|K\|\,\|K^\dagger\|(1+\|K\|\,\|K^\dagger\|)\|KK^* f\|^2.$$

Proof. For each $f\in\mathscr{H}$, by Lemma 2.4 we have

$$\begin{aligned}
\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Lambda_j f\Big\|^2 &\leq \|S_{\mathbb{I}}\| \sum_{j\in\mathbb{I}} \|\Lambda_j f\|^2 \leq \|S_{\mathbb{I}}\| \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \\
&\leq \|K\|^2\|K^* f\|^2 = \|K\|^2\|K^*(K^*)^\dagger K^* f\|^2 \\
&= \|K\|^2\|K^\dagger KK^* f\|^2 \leq \|K\|^2\|K^\dagger\|^2\|KK^* f\|^2, \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Lambda_j f, \Lambda_j KK^* f\rangle &\leq \Big(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\Big)^{\frac{1}{2}}\Big(\sum_{j\in\mathbb{J}} \|\Lambda_j KK^* f\|^2\Big)^{\frac{1}{2}} \\
&= \|K^* f\|\,\|K^* KK^* f\| \\
&= \|K^\dagger KK^* f\|\,\|K^* KK^* f\| \\
&\leq \|K\|\,\|K^\dagger\|\,\|KK^* f\|^2. \qquad (3.9)
\end{aligned}$$

Now, the result follows by combining (3.8) and (3.9). □

In the following we give some new inequalities for K-g-frames, which possess different structure comparing with the inequalities for frames shown in Theorems 1.2, 1.3 and 1.4.

Theorem 3.8. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$ we have*

$$\frac{3}{4}\|Kf\|^2 \leq \Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Gamma_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j Kf\rangle \leq \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4}\|f\|^2, \qquad (3.10)$$

where the operators $F_{\mathbb{I}}$ is defined by $F_{\mathbb{I}} f = \sum_{j\in\mathbb{I}} \Lambda_j^*\Gamma_j f$.

Proof. The left-hand inequality follows from Theorem 3.5 if we consider $\mathbb{I}\subset\mathbb{J}$ and

$$\alpha_j = \begin{cases} 1, & j\in\mathbb{I}, \\ 0, & j\in\mathbb{I}^c. \end{cases}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

We now prove the right-hand inequality of (3.10). For any $f \in \mathscr{H}$ we get

$$\begin{aligned}
&\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Gamma_j f\Big\|^2 + \operatorname{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j K f\rangle \\
&= \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle + \operatorname{Re}\langle Kf, F_{\mathbb{I}^c} f\rangle \\
&= \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle + \operatorname{Re}\langle Kf, (K - F_{\mathbb{I}}) f\rangle \\
&= \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle + \langle Kf, Kf\rangle - \operatorname{Re}\langle Kf, F_{\mathbb{I}} f\rangle \\
&= \langle Kf, Kf\rangle - \operatorname{Re}(\langle (K - F_{\mathbb{I}}) f, F_{\mathbb{I}} f\rangle) \\
&= \langle Kf, Kf\rangle - \operatorname{Re}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \langle Kf, Kf\rangle - \frac{1}{2}\langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \frac{3}{4}\|Kf\|^2 + \frac{1}{4}\langle F_{\mathbb{I}} f + F_{\mathbb{I}^c} f, F_{\mathbb{I}} f + F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \frac{3}{4}\|Kf\|^2 + \frac{1}{4}\langle (F_{\mathbb{I}} - F_{\mathbb{I}^c}) f, (F_{\mathbb{I}} - F_{\mathbb{I}^c}) f\rangle \\
&\le \frac{3}{4}\|K\|^2\|f\|^2 + \frac{1}{4}\|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2\|f\|^2 \\
&= \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4}\|f\|^2.
\end{aligned}$$

This completes the proof. □

Theorem 3.9. *Suppose that K is positive and that it has closed range. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$ we have*

$$\begin{aligned}
&\operatorname{Re}\sum_{j\in\mathbb{I}}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}^c} K^\dagger \Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}^c}\Lambda_j^*\Gamma_j f\Big\rangle \\
&\qquad = \operatorname{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}} K^\dagger \Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}}\Lambda_j^*\Gamma_j f\Big\rangle \ge \frac{3}{4}\|K^{\frac{1}{2}} f\|^2.
\end{aligned}$$

Proof. Since K is positive, it is self-adjoint and thus by Lemma 2.4, $(K^\dagger)^* = (K^*)^\dagger = K^\dagger$. Hence, $\langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle$ and $\langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f\rangle$ are real numbers for every $f \in \mathscr{H}$. From Lemma 2.5 it follows that

$$\begin{aligned}
&\operatorname{Re}\sum_{j\in\mathbb{I}}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}^c} K^\dagger \Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}^c}\Lambda_j^*\Gamma_j f\Big\rangle \\
&= \operatorname{Re}\langle F_{\mathbb{I}} f, K^\dagger K f\rangle + \langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f\rangle \\
&= \operatorname{Re}\langle K K^\dagger F_{\mathbb{I}} f, f\rangle + \langle F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c} f, f\rangle \\
&= \operatorname{Re}\langle (K K^\dagger F_{\mathbb{I}} + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f\rangle \\
&= \operatorname{Re}\langle (F_{\mathbb{I}^c}^* K^\dagger K + F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}}) f, f\rangle \\
&= \operatorname{Re}(\langle F_{\mathbb{I}^c}^* K^\dagger K f, f\rangle + \langle F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}} f, f\rangle) \\
&= \operatorname{Re}(\langle K^\dagger K f, F_{\mathbb{I}^c} f\rangle + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle) \\
&= \operatorname{Re}\langle F_{\mathbb{I}^c} f, K^\dagger K f\rangle + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle \\
&= \operatorname{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}} K^\dagger \Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}}\Lambda_j^*\Gamma_j f\Big\rangle.
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Again by Lemma 2.4 we have

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle\Gamma_j f,\Lambda_j K^\dagger K f\rangle+\Big\langle\sum_{j\in\mathbb{I}}K^\dagger\Lambda_j^*\Gamma_j f,\sum_{j\in\mathbb{I}}\Lambda_j^*\Gamma_j f\Big\rangle\\
&\quad=\mathrm{Re}\langle(KK^\dagger F_{\mathbb{I}}+F_{\mathbb{I}^c}^*K^\dagger F_{\mathbb{I}^c})f,f\rangle\\
&\quad=\mathrm{Re}\langle(KK^\dagger(K-F_{\mathbb{I}^c})+F_{\mathbb{I}^c}^*K^\dagger F_{\mathbb{I}^c})f,f\rangle\\
&\quad=\langle Kf,f\rangle-\mathrm{Re}\langle KK^\dagger F_{\mathbb{I}^c}f,f\rangle+\langle F_{\mathbb{I}^c}^*K^\dagger F_{\mathbb{I}^c}f,f\rangle\\
&\quad=\langle K^{\frac12}f,K^{\frac12}f\rangle-\mathrm{Re}\langle K^{\frac12}K^{\frac12}K^\dagger F_{\mathbb{I}^c}f,f\rangle+\langle(K^{\frac12}K^\dagger F_{\mathbb{I}^c})^*(K^{\frac12}K^\dagger F_{\mathbb{I}^c})f,f\rangle\\
&\quad=\frac34\|K^{\frac12}f\|^2+\Big\langle\frac12K^{\frac12}f-K^{\frac12}K^\dagger F_{\mathbb{I}^c}f,\frac12K^{\frac12}f-K^{\frac12}K^\dagger F_{\mathbb{I}^c}f\Big\rangle\geq\frac34\|K^{\frac12}f\|^2
\end{aligned}$$

for each $f\in\mathscr{H}$ and the proof is finished. □

Theorem 3.10. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. If $S_{\mathbb{I}}$ commutes with $S_{\mathbb{I}^c}$ for every $\mathbb{I}\subset\mathbb{J}$, then for every $f\in\mathscr{H}$ we have*

$$\frac12\|KK^*f\|^2\leq\Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2+\Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2\leq\|KK^*f\|^2. \tag{3.11}$$

$$0\leq\sum_{j\in\mathbb{I}}\langle\Lambda_j KK^*f,\Lambda_j f\rangle-\Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2\leq\frac14\|KK^*f\|^2. \tag{3.12}$$

Proof. From (3.7) it follows that

$$\begin{aligned}
\Big\|\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j f\Big\|^2+\Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2&=\|S_{\mathbb{I}}f\|^2+\|S_{\mathbb{I}^c}f\|^2=\langle S_{\mathbb{I}}f,S_{\mathbb{I}}f\rangle+\langle S_{\mathbb{I}^c}f,S_{\mathbb{I}^c}f\rangle\\
&=\langle(S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2)f,f\rangle\geq\frac12\langle(KK^*)^2f,f\rangle=\frac12\|KK^*f\|^2
\end{aligned}\tag{3.13}$$

for every $f\in\mathscr{H}$. Since $S_{\mathbb{I}}$ commutes with $S_{\mathbb{I}^c}$, $S_{\mathbb{I}^c}S_{\mathbb{I}}\geq 0$ and

$$0\leq S_{\mathbb{I}}S_{\mathbb{I}^c}=S_{\mathbb{I}}(KK^*-S_{\mathbb{I}})=S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2. \tag{3.14}$$

It follows that

$$\begin{aligned}
S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2&=S_{\mathbb{I}}^2+(KK^*)^2-KK^*S_{\mathbb{I}}-S_{\mathbb{I}}KK^*+S_{\mathbb{I}}^2\\
&=(KK^*)^2+(S_{\mathbb{I}}^2-S_{\mathbb{I}}KK^*)+(S_{\mathbb{I}}^2-KK^*S_{\mathbb{I}})\\
&=(KK^*)^2-(S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2)-S_{\mathbb{I}^c}S_{\mathbb{I}}\leq(KK^*)^2.
\end{aligned}$$

Hence for every $f\in\mathscr{H}$ we have

$$\Big\|\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j f\Big\|^2+\Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2=\langle(S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2)f,f\rangle\leq\langle(KK^*)^2f,f\rangle=\|KK^*f\|^2.$$

This together with (3.13) gives (3.11). We next prove (3.12). Using formula (3.14) we get

$$\begin{aligned}
\sum_{j\in\mathbb{I}}\langle\Lambda_j KK^*f,\Lambda_j f\rangle-\Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2&=\langle S_{\mathbb{I}}KK^*f,f\rangle-\|S_{\mathbb{I}}f\|^2=\langle(S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2)f,f\rangle\\
&=\langle S_{\mathbb{I}}(KK^*-S_{\mathbb{I}})f,f\rangle=\langle S_{\mathbb{I}}S_{\mathbb{I}^c}f,f\rangle\geq 0
\end{aligned}$$

for every $f \in \mathscr{H}$. On the other hand we obtain

$$\begin{aligned}\sum_{j\in\mathbb{I}}\langle\Lambda_j KK^* f, \Lambda_j f\rangle - \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2 &= \langle(S_{\mathbb{I}}KK^* - S_{\mathbb{I}}^2)f, f\rangle \\ &= \left\langle -\left(S_{\mathbb{I}} - \frac{KK^*}{2}\right)^2 f + \frac{(KK^*)^2}{4} f, f\right\rangle \\ &\le \left\langle \frac{(KK^*)^2}{4} f, f\right\rangle \\ &= \frac{1}{4}\|KK^* f\|^2.\end{aligned}$$

This completes the proof. □

Acknowledgements

The research was partially supported by the National Natural Science Foundation of China (Grant Nos. 11761057 and 11561057), the Natural Science Foundation of Jiangxi Province (Grant No. 20151BAB201007), and the Science Foundation of Jiangxi Education Department (Grant No. GJJ151061).

References

[1] F. Arabyani Neyshaburi and A. Arefijamaal, Some constructions of K-frames and their duals, to appear in *Rocky Mountain J. Math*.

[2] M.S. Asgari and H. Rahimi, Generalized frames for operators in Hilbert spaces, *Infin. Dimens. Anal. Quantum. Probab. Relat. Top*. 17, 1450013, 20 pp (2014).

[3] R. Balan, P.G. Casazza, D. Edidin, and G. Kutyniok, A new identity for Parseval frames, *Proc. Amer. Math. Soc*. 135, 1007–1015 (2007).

[4] J.J. Benedetto, A.M. Powell, and O. Yilmaz, Sigma-Delta (ΣΔ) quantization and finite frames, *IEEE Trans. Inform. Theory* 52, 1990–2005 (2006).

[5] P.G. Casazza, The art of frame theory, *Taiwanese J. Math*. 4, 129–201 (2000).

[6] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[7] I. Daubechies, A. Grossmann, and Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys*. 27, 1271–1283 (1986).

[8] R.J. Duffin and A.C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc*. 72, 341–366 (1952).

[9] H.G. Feichtinger and T. Werther, Atomic systems for subspaces, in *Proceedings SampTA 2001* (L. Zayed, eds.), Orlando, FL, 2001, pp. 163–165.

[10] L. Găvruţa, Frames for operators, *Appl. Comput. Harmon. Anal*. 32, 139–144 (2012).

[11] P. Găvruţa, On some identities and inequalities for frames in Hilbert spaces, *J. Math. Anal. Appl*. 321, 469–478 (2006).

[12] X.X. Guo, Canonical dual K-Bessel sequences and dual K-Bessel generators for unitary systems of Hilbert spaces, *J. Math. Anal. Appl*. 444, 598–609 (2016).

[13] D.L. Hua and Y.D. Huang, K-g-frames and stability of K-g-frames in Hilbert spaces, *J. Korean Math. Soc*. 53, 1331-1345 (2016).

[14] J.Z. Li and Y.C. Zhu, Some equalities and inequalities for g-Bessel sequences in Hilbert spaces, *Appl. Math. Lett*. 25, 1601–1607 (2012).

[15] J.Z. Li and Y.C. Zhu, Exact g-frames in Hilbert spaces, *J. Math. Anal. Appl*. 374, 201–209 (2011).

[16] T. Strohmer and R. Heath, Grassmannian frames with applications to coding and communication, *Appl. Comput. Harmon. Anal*. 14, 257–275 (2003).

[17] W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl*. 322, 437–452 (2006).

[18] W. Sun, Stability of g-frames, *J. Math. Anal. Appl*. 326, 858–868 (2007).

[19] W. Sun, Asymptotic properties of Gabor frame operators as sampling density tends to infinity, *J. Funct. Anal*. 258, 913–932 (2010).

[20] Z.Q. Xiang and Y.M. Li, Frame sequences and dual frames for operators, *ScienceAsia* 42, 222–230 (2016).

[21] X.C. Xiao, Y.C. Zhu, and X.M. Zeng, Some properties of g-Parseval frames in Hilbert spaces, *Acta Math. Sin. (Chin. Ser.)* 51, 1143–1150 (2008).

[22] X.C. Xiao, Y.C. Zhu, and L. Găvruţa, Some properties of *K*-frames in Hilbert spaces, *Results Math*. 63, 1243–1255 (2013).

[23] X.C. Xiao, Y.C. Zhu, Z.B. Shu, and M.L. Ding, G-frames with bounded linear operators, *Rocky Mountain J. Math*. 45, 675–693 (2015).

[24] X.H. Yang and D.F. Li, Some new equalities and inequalities for g-frames and their dual frames, *Acta Math. Sin. (Chin. Ser.)* 52, 1033–1040 (2009).

College of Mathematics and Computer Science, Shangrao Normal University, Shangrao, Jiangxi 334001, P.R. China

E-mail address: lxsy20110927@163.com; jiayinsuo2002@sohu.com.

AQ-functional equation in matrix non-Archimedean fuzzy normed spaces

Jung-Rye Lee[1], George A. Anastassiou[2], Choonkil Park[3*], Murali Ramdoss[4*] and Vithya Veeramani[5]

[1]Department of Mathematics, Daejin University, Kyunggi 11159, Korea
[2]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[3]Research Institute for Natural Sciences Hanyang University, Seoul 04763, Korea
[4,5]Department of Mathematics, Sacred Heart College, Tirupattur - 635 601, TamilNadu, India
e-mail: rlee@daejin.ac.kr; ganastss@memphis.edu; baak@hanyang.ac.kr; shcrmurali@yahoo.co.in; viprutha26@gmail.com

Abstract. Using the fixed point method, we establish some stability results concerning the following new mixed type AQ-functional equation

$$f(-x+2y)+2[f(3x-2y)+f(2x+y)-f(y)-f(y-x)]=3[f(x+y)+f(x-y)+f(-x)]+4f(2x-y)$$

in matrix non-Archimedean fuzzy normed spaces.

1. Introduction and preliminaries

A basic question in the theory of functional equations is as follows: "When is it true that a function, which approximately satisfies a functional equation must be close to an act solution of the equation? If the problem accepts a solution, we say the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [29] in 1940 and affirmatively solved by Hyers [7]. The result of Hyers was generalized by Aoki [1] for approximate additive mappings and by Rassias [24] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [6] who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\chi(x,y)$. In addition, Rassias [20]–[23] generalized the Hyers-Ulam stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product sum of powers of norms, respectively. applied to the cases of other functional equations in various spaces [2, 5, 13, 15, 16, 26, 27]. In particular Mirmostafafe and Moslehian [14] introduced a notation of non-Archimedean fuzzy normed spaces. They presented the interdisciplinary relation between the theory of fuzzy spaces, the theory of non-Archimedean spaces and the theory of functional equations. Many authors [8, 11, 12, 14, 19, 25, 32] investigated the Hyers-Ulam stability in non-Archimedean fuzzy normed spaces.

Definition 1. [8, 32] Let X be a linear space over a non-Archimedean field $\mathbb{K}$. A function $N : X \times R \to [0,1]$ is said to be a non-Archimedean fuzzy norm on X if for all $x, y \in X$ and all $s, t \in R$
(N1) $N(x,c) = 0$ for $c \le 0$;
(N2) $x = 0 \Leftrightarrow N(x,c) = 1$ for all $c > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, \max\{s+t\}) \ge \min\{N(x,s), N(y,t)\}$;
(N5) $\lim_{t\to\infty} N(x,t) = 1$.
The pair (X,N) is called a non-Archimedean fuzzy normed space. Clearly, if $(N4)$ holds then so does
(N6) $N(x+y, s+t) \ge \min\{N(x,s), N(y,t)\}$. A classical vector space over a complex or real field satisfying $(N1)$ and $(N5)$ is called fuzzy normed space. It is easy to see that $(N4)$ is equivalent to the following condition
(N7) $N(x+y,t) \ge \min\{N(x,t), N(y,t)\}$ $\quad (x, y \in X; t \in R)$.

[0]2010 Mathematics Subject Classification: 46S40, 46S50, 47L25, 47H10, 54C30, 54E70.
[0]Keywords: Hyers-Ulam stability, fixed point, mixed type additive-quadratic functional equation, matrix non-Archemedian fuzzy normed space.
*Corresponding authors.

Definition 2. Let (X, N) be a non-Archimedean fuzzy normed space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$

Definition 3. Let (X, N) be a non-Archimedean fuzzy normed space. A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$. Due to $N(x_{n+p} - x_n, t) \geq \min \{N(x_{n+p} - x_{n+p-1}, t), ..., N(x_{n+1} - x_n, t)\}$ the sequence $\{x_n\}$ is Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $N(x_{n+1} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a (non-Archimedean) fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the (non-Archimedean) fuzzy normed space is called a (non-Archimedean) fuzzy Banach space.

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of matricially normed spaces [4] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory [18]. Recently, Lee *et al.* [9] researched the Hyers-Ulam stability of the Cauchy functional equation and the quadratic functional equation in matrix normed spaces. This terminology may also be applied to the cases of other functional equations [3, 10, 28, 30, 31].

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that jth component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero. For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|.\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|.\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\| \|B\| \|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $B \in M_{n,k}(\mathbb{C})$ and $x = (x_{ij}) \in M_n(X)$, and that $(X, \{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|.\|_n\})$ is a matrix normed space. A matrix normed space $(X, \{\|.\|_n\})$ is called an L^∞ -matrix normed space if $\|x \otimes y\|_{n+k} = max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$. Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by $h_n([x_{ij}]) = [h(x_{ij})]$ for all $[x_{ij}] \in M_n(E)$.

We introduce the concept of matrix non-Archimedean fuzzy normed space.

Definition 4. Let (X, N) be a non-Archimedean fuzzy normed space.
(i) $(X, \{N_n\})$ is called a matrix non-Archimedean fuzzy normed space if for each positive integer n, $(M_n(X), N_n)$ is a non-Archimedean fuzzy normed space and $N_k(AxB, t) \geq N_n\left(x, \frac{t}{\|A\|\cdot\|B\|}\right)$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $B \in M_{n,k}(\mathbb{R})$ and $x = [x_{ij}] \in M_n(X)$ with $\|A\| \cdot \|B\| \neq 0$.
(ii) $(X, \{N_n\})$ is called a complete matrix non-Archimedean fuzzy normed space if (X, N) is a non-Archimedean fuzzy Banach space and $(X, \{N_n\})$ is a matrix non-Archimedean fuzzy normed space.

Example 5. Let $(X, \{\|.\|_n\})$ is a matrix normed space. Let $N_n(x, t) = \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then

$$N_k(AxB, t) = \frac{t}{t+\|AxB\|_k} \geq \frac{t}{t+\|A\|\cdot\|x\|_n\cdot\|B\|} = \frac{\frac{t}{\|A\|\cdot\|B\|}}{\frac{t}{\|A\|\|B\|}+\|x\|_n}$$

for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $B \in M_{n,k}(\mathbb{R})$ and $x = [x_{ij}] \in M_n(X)$ with $\|A\| \cdot \|B\| \neq 0$. So $(X, \{N_n\})$ is a matrix non-Archimeaden fuzzy normed space.

Definition 6. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ forall $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ forall $x, y, z \in X$.

Theorem 7. [17] Let (x, d) be a complete generalized metric space and let $J : X \to Y$ be a strictly contractive mapping with a Lipschitz constant $\alpha < 1$.Then, for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$. for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$converges to a fixed point $y*$ of J;
(3) $y*$ is the unique fixed pointof J;
(4) $d(y, y*) \leq \frac{1}{1-\alpha} d(y, Jy)$ forall $y \in Y$.

In this paper, we establish some stability results concerning the following new mixed type AQ-functional equation

$$\begin{aligned} f(-x+2y) + 2[f(3x-2y) + f(2x+y) - f(y) - f(y-x)] \\ = 3[f(x+y) + f(x-y) + f(-x)] + 4f(2x-y) \end{aligned} \tag{1.1}$$

in matrix non-Archimedean fuzzy normed spaces by using the fixed point method.

Theorem 8. Let $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces. If an odd mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies (1.1), then f is additive.

Proof. Suppose that f is an odd mapping. Then (1.1) is equivalent to

$$-f(x-2y) + 2[f(3x-2y) + f(2x+y) - f(y)] = 3[f(x+y) - f(x)] + f(x-y) + 4f(2x-y) \tag{1.2}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x + y$ in (1.2), we obtain

$$-f(x-y) + 2[f(3x+y) + f(2x+3y) - f(y)] = 3[f(x+2y) - f(x+y)] + f(x) + 4f(2x+y) \tag{1.3}$$

for all $x, y \in \mathcal{A}$. Replacing (x, y) by $(x + y, -y)$ in (1.3), we obtain

$$-f(x+2y) + 2[f(3x+2y) + f(2x-y) + f(y)] = 3[f(x-y) - f(x)] + f(x+y) + 4f(2x+y) \tag{1.4}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.3) from (1.4) and then dividing the resulting equation by 2, we get

$$\begin{aligned} -f(x+2y) + f(x-y) + f(2x+3y) - f(3x+2y) + f(3x+y) - f(2x-y) \\ = -2f(x+y) + 2f(x) + 2f(y) \end{aligned} \tag{1.5}$$

for all $x, y \in \mathcal{A}$. Interchanging x and y in (1.5) and then adding the resulting equation to (1.5), we get

$$\begin{aligned} -f(x+2y) - f(2x+y) + f(3x+y) + f(x+3y) - f(2x-y) + f(x-2y) \\ = -4f(x+y) + 4f(x) + 4f(y) \end{aligned} \tag{1.6}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x - y$ in (1.6), we obtain

$$\begin{aligned} -f(x+y) - f(2x-y) + f(3x-2y) + f(x+2y) - f(2x-3y) + f(x-3y) \\ = -4f(x) + 4f(x-y) + 4f(y) \end{aligned} \tag{1.7}$$

for all $x, y \in \mathcal{A}$. Replacing y by $-y$ in (1.7), we obtain

$$\begin{aligned} -f(x-y) - f(2x+y) + f(3x+2y) + f(x-2y) - f(2x+3y) + f(x+3y) \\ = -4f(x) + 4f(x+y) - 4f(y) \end{aligned} \tag{1.8}$$

for all $x, y \in \mathcal{A}$. Adding (1.7) to (1.8), we get

$$\begin{aligned} -f(x+2y) - f(2x+y) + f(3x+y) + f(x+3y) - f(2x-y) + f(x-2y) \\ = -2f(x) + 2f(x+y) - 2f(y) \end{aligned} \tag{1.9}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.9) from (1.6) and then dividing the resulting equation by 6, we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathcal{A}$, as desired. □

Theorem 9. Let $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces. If an even mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies (1.1), then f is quadratic.

Proof. Suppose that f is an even mapping. Then (1.1) is equivalent to

$$f(x-2y)+2[f(3x-2y)+f(2x+y)-f(y)] \\ =3[f(x+y)+f(x)]+5f(x-y)+4f(2x-y) \tag{1.10}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x+y$ and y by $x+y$ in (1.10) and then comparing the two resulting equations, we get

$$2f(x)+4f(x+2y)-2f(2x+3y) \\ =-f(x+y)+5f(x-y)+3f(y)-f(2x+y)-2f(x-2y) \tag{1.11}$$

for all $x, y \in \mathcal{A}$. Interchanging x and y in (1.11), we obtain

$$2f(y)+4f(2x+y)-2f(3x+2y) \\ =-f(x+y)+5f(x-y)+3f(x)-f(x+2y)-2f(2x-y) \tag{1.12}$$

for all $x, y \in \mathcal{A}$. Replacing y by $-y$ in (1.12), we get

$$2f(y)+4f(2x-y)-2f(3x-2y) \\ =-f(x-y)+5f(x+y)+3f(x)-f(x-2y)-2f(2x+y) \tag{1.13}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.13) from (1.10) and then dividing the resulting equation by 2, we get

$$2f(y)+4f(2x-y)-2f(3x-2y)=-3f(x-y)+f(x+y) \tag{1.14}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x+y$ in (1.14), we get

$$f(x+2y)+2[f(3x+y)-f(y)]=3f(x)+4f(2x+y) \tag{1.15}$$

for all $x, y \in \mathcal{A}$. Replacing y by $y-x$ in (1.15), we get

$$f(-x+2y)+2[f(2x+y)-f(y-x)]=3f(x)+4f(x+y) \tag{1.16}$$

for all $x, y \in \mathcal{A}$. Replacing y by $-y$ in (1.16), we obtain

$$f(x+2y)+2[f(2x-y)-f(x+y)]=3f(x)+4f(x-y) \tag{1.17}$$

for all $x, y \in \mathcal{A}$. Replacing x by y and y by x in (1.16), we obtain that

$$f(2x-y)+2[f(x+2y)-f(x-y)]=3f(y)+4f(x+y) \tag{1.18}$$

for all $x, y \in \mathcal{A}$. Adding (1.17) to (1.18) and then dividing the resulting equation by 3, we get

$$f(x+2y)+f(2x-y)=f(x)+f(y)+2f(x+y)+2f(x-y) \tag{1.19}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.17) from (1.18) and then adding the resulting equation to (1.19), we get

$$f(x+2y)+f(x)=2f(y)+2f(x+y) \tag{1.20}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x-y$ in (1.20), we obtain $f(x+y)+f(x-y)=2f(x)+2f(y)$ for all $x, y \in \mathcal{A}$. This completes the proof. □

2. Hyers-Ulam stability of the additive-quadratic functional equation (1.1)

Throughout this paper, we assume that $\mathbb{K}$ is a non-Archimedean field, X is a vector space over $\mathbb{K}$ and $(Y, \mathcal{N}_n)$ is a complete matrix non-Archimedean fuzzy normed space over $\mathbb{K}$, and $(Z, \mathcal{N}')$ is (an Archimedean or a non-Archimedean fuzzy) normed space.
For a mapping $f: X \to Y$, define $\mathcal{G}\, f: X^2 \to Y$ and $\mathcal{G}\, f_n: M_n(X^2) \to M_n(Y)$ by

$$\begin{aligned}\mathcal{G}f(a,b) &= f(-a+2b)+2[f(3a-2b)+f(2a+b)-f(b)-f(b-a)] \\ &\quad -3[f(a+b)+f(a-b)+f(-a)]-4f(2a-b), \\ \mathcal{G}f_n([x_{ij}],[y_{ij}]) &= f_n([-x_{ij}+2y_{ij}])+2[f_n([3x_{ij}-2y_{ij}])+f_n([2x_{ij}+y_{ij}])-f_n([y_{ij}])-f_n([y_{ij}-x_{ij}])] \\ &\quad -3[f_n([x_{ij}+y_{ij}])+f_n([x_{ij}-y_{ij}])+f_n([-x_{ij}])]-4f_n([2x_{ij}-y_{ij}])\end{aligned}$$

for all $a, b \in X$ and all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$.

In this section, we investigate the Hyers-Ulam stability for the functional equation (1.1) in matrix non-Archimedean fuzzy normed spaces by using the fixed point method.

Theorem 10. Let $q = \pm 1$ be fixed and let $\psi : X \times X \to Z$ be a mapping such that for some $\eta \neq 2$ with $\left(\frac{\eta}{2}\right)^q < 1$

$$\mathcal{N}'(\psi(2^q a, 2^q b) \geq \mathcal{N}'\left(\psi(a,b), \eta^{-q} t\right) \tag{2.1}$$

for all $a, b \in X$ and $t > 0$, and

$$\lim_{k\to\infty} \mathcal{N}(2^{-kq}\mathcal{G}f(2^{kq}a, 2^{kq}b), t) = 1$$

for all $a, b \in X$ and $t > 0$. Suppose that an odd mapping $f : X \to Y$ satisfies the inequality

$$\mathcal{N}(\mathcal{G}f_n([x_{ij}], [y_{ij}]), t) \geq \mathcal{N}'\left(\sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}), t\right) \quad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X), \text{ and } t > 0. \tag{2.2}$$

Then there exists a unique additive mapping $\mathcal{A} : X \to Y$ such that

$$\mathcal{N}_n(f_n([x_{ij}]) - \mathcal{A}_n([x_{ij}]), t) \geq min\left\{\mathcal{N}'(\psi(x_{ij}, 0), |\eta - 2|\, n^{-2} t) : i, j = 1, 2, \cdots, n\right\} \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$.

Proof. For the cases $q = 1$ and $q = -1$, we consider $\eta < 2$ and $\eta > 2$, respectively. Letting $n = 1$ in (2.2), we obtain

$$\mathcal{N}(\mathcal{G}f(a,b), t) \geq \mathcal{N}'\left(\psi(a,b), t\right) \tag{2.4}$$

for all $a, b \in X$ and $t > 0$. Replacing (a, b) by $(0, a)$ in (2.4), we get

$$\mathcal{N}\left(f(2a) - 2f(a), t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right)$$

for all $a \in X$ and $t > 0$. Thus

$$\mathcal{N}\left(f(a) - \frac{1}{2^q} f(2^q a), \frac{\eta^{\frac{q-1}{2}}}{|2|^{\left(\frac{1+q}{2}\right)}} t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right) \qquad \forall\ a \in X \text{ and } t > 0. \tag{2.5}$$

Consider the set $\mathcal{M} = \{f : X \to Y\}$ and introduce the generalized metric ρ on $\mathcal{M}$ as follows:

$$\rho(f,g) = \in \rho(f,g) = f\left\{\mu \in \mathbb{R}_+ : \mathcal{N}\left(f(a) - g(a), \mu t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right), \forall a \in X, t > 0\right\}$$

We will prove that $(\mathcal{M}, \rho)$ is a complete generalized metric, First we will prove that ρ is a generalized metric on $\mathcal{M}$. Let $\rho(f,g) = \mu_1$ and $\rho(g,h) = \mu_2$. Then $\mathcal{N}\left(f(a) - g(a), \mu_1 t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right)$ and $\mathcal{N}\left(g(a) - h(a), \mu_2 t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right)$ for all $a \in X$ and $t > 0$. Therefore, $\mathcal{N}\left(f(a) - h(a), (\mu_1 + \mu_2)t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right)$. By definition of ρ, $\rho(f,h) \leq \mu_1 + \mu_2 = \rho(f,g) + \rho(g,h)$. which means that ρ satisfies the triangle inequality. One can show that other properties are satisfied. So ρ is a generalized metric on $\mathcal{M}$.

Next we will prove that $(\mathcal{M}, \rho)$ is a complete generalized metric.
Suppose that $\{f_n\}$ is ρ-Cauchy, i.e., for any $\tau > 0$, there exist $n_0, n > m \geq n_0$, such that $\rho(f_n, f_m) < \tau$.
By definition of ρ, there exists $0 < \mu_0 < \tau$, which satisfies

$$\mathcal{N}\left(f_n(a) - f_m(a), \tau t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right)$$

for all $a \in X$ and $t > 0, n > m \geq n_0$, i.e., $\{f_n(a)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $\{f_0(a)\} \subseteq Y$ and $\{f_n(a)\} \to \{f_0(a)\}$. Taking the limit as $m \to \infty$, we obtain

$$\mathcal{N}(f_n(a) - f_0(a), \tau t) \geq \mathcal{N}'\left(\psi(0,a), t\right)$$

for all $a \in X$ and $t > 0, n \geq n_0$. Therefore,

$$\rho(f_n, f_0) = inf\left\{\mu \in \mathbb{R}_+ : \mathcal{N}\left(f_n(a) - f_0(a), \mu t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right)\right\} < \tau.$$

for all $n \geq n_0$, so that $\{f_n\}$ is ρ-convergent, i.e., $(\mathcal{M}, \rho)$ is a complete generalized metric.

Now consider the mapping $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{P}f(a) = \frac{1}{2^q} f(2^q a) \qquad \forall\ f \in \mathcal{M} \text{ and } a \in X.$$

Let $f, g \in \mathcal{M}$ and ν be an arbitrary constant with $\rho(f,g) \leq \nu$. Then

$$\mathcal{N}\left(f(a) - g(a), \nu t\right) \geq \mathcal{N}'\left(\psi(0,a), t\right) \qquad \text{for all } a \in X \text{ and } t > 0.$$

Therefore, using (2.1), we get

$$\mathcal{N}\left(\mathcal{P}f(a) - \mathcal{P}g(a), 2^{-q}\nu t\right) = \mathcal{N}\left(f(2^q a) - g(2^q a), \nu t\right), \geq \mathcal{N}'\left(\psi(0,a), \eta^{-q} t\right)$$

for all $a \in X$ and $t > 0$. Hence by definition $\rho(\mathcal{P}f, \mathcal{P}g) \le \left(\frac{\eta}{2}\right)^q \nu$, that is, $\rho(\mathcal{P}f, \mathcal{P}g) \le L\rho(f, g)$ for all $f, g \in \mathcal{M}$. This means that $\mathcal{P}$ is a contractive mapping with Lipschitz constant $L = \left(\frac{\eta}{2}\right)^q < 1$.

It follows from (2.5) that $\rho(f, \mathcal{P}f) \le \frac{\eta^{\left(\frac{q-1}{2}\right)}}{|2|^{\left(\frac{1+q}{2}\right)}}$. Therefore according to Theorem 7, there exists a mapping $\mathcal{A} : X \to Y$ which satisfies

(1) $\mathcal{A}$ is a unique fixed point of $\mathcal{P}$ in the set $\mathcal{S} = \{g \in \mathcal{M} : \rho(f, g) < \infty\}$, which satisfies

$$\mathcal{A}(2^q a) = 2^q \mathcal{A}(a) \qquad \forall\ a \in X.$$

In other words, there exists a $\mu > 0$ satisfying

$$\mathcal{N}\left(f(a) - g(a), \mu t\right) \ge \mathcal{N}'\left(\psi(0, a), t\right) \qquad \forall\ a \in X \text{ and } t > 0.$$

(2) $\rho(\mathcal{P}^k f, \mathcal{V}_{\mathcal{U}}) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} \frac{1}{2^{kq}} f(2^{kq} a) = \mathcal{A}(a) \qquad \forall\ a \in X.$$

(3) $\rho(f, \mathcal{A}) \le \frac{1}{1-\eta}\rho(f, \mathcal{P}f)$, which implies the inequality $\rho(f, \mathcal{A}) \le \frac{1}{|2-\eta|}$. So

$$\mathcal{N}\left(f(a) - \mathcal{A}(a), \frac{1}{|2-\eta|} t\right) \ge \mathcal{N}'\left(\psi(0, a), t\right) \qquad \forall\ a \in X \text{ and } t > 0. \tag{2.6}$$

By (2.4),

$$\mathcal{N}(\mathcal{G}\mathcal{A}(a, b), t) = \lim_{k\to\infty} \mathcal{N}(2^{-kq}\mathcal{G}f(2^{kq}a, 2^{kq}b), t) \ge \lim_{k\to\infty} \mathcal{N}'(2^{-kq}\psi(2^{kq}a, 2^{kq}b), t) = 1.$$

Hence by (N2), $\mathcal{G}\mathcal{A}(a, b) = 0$. Thus $\mathcal{A}$ is additive.

We note that $e_j \in M_{1,n}(\mathbb{R})$ means that the j-th component is 1 and the others are zero, $E_{ij} \in M_n(X)$ means that (i, j)-component is 1 and the others are zero, and $E_{ij} \otimes x \in M_n(X)$ means that (i, j)-component is x and the others are zero. Since $N(E_{kl} \otimes x, t) = N(x, t)$, we have

$$\begin{aligned} N_n([x_{ij}], t) = N_n\left(\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}, t\right) &\ge min\left\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i, j = 1, 2, ..., n\right\} \\ &= min\left\{N(x_{ij}, t_{ij}) : i, j = 1, 2, ..., n\right\}, \end{aligned}$$

where $t = \sum_{i,j=1}^{n} t_{ij}$. So $N_n([x_{ij}], t) \ge \mathcal{T}\left\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \cdots, n\right\}$.

By (2.6),

$$\begin{aligned} \mathcal{N}\left(f_n([x_{ij}]) - \mathcal{A}_n([x_{ij}]), t\right) &\ge min\left\{\mathcal{N}\left(f(x_{ij}) - \mathcal{A}(x_{ij}), \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n.\right\} \\ &\ge min\left\{\mathcal{N}'\left(\psi(0, x_{ij}), |2-\eta|\, n^{-2} t\right) : i, j = 1, 2, ..., n\right\} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$. Thus $\mathcal{A} : X \to Y$ is a unique additive mapping satisfying (2.3). □

Corollary 1. Let $q = \pm 1$ be fixed and let p be a nonnegative real number with $p \ne 1$ and $\Upsilon \in Z$. Let $f : X \to Y$ be an odd mapping such that

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}], [y_{ij}]), t) \ge \sum_{i,j=1}^{n} \mathcal{N}'(\Upsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), t)$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and $t > 0$. Then there exists a unique additive mapping $\mathcal{A} : X \to Y$ such that

$$\mathcal{N}\left(f_n([x_{ij}]) - \mathcal{A}_n([x_{ij}]), t\right) \ge min\left\{\mathcal{N}'(\|x\|^p\, \Upsilon, |2 - 2^p|\, n^{-2} t) : i, j = 1, 2, \cdots, n\right\} \tag{2.7}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$.

Proof. The proof follows from Theorem 10 by taking $\psi(a, b) = \Upsilon(\|a\|^p + \|b\|^p)$ for all $a, b \in X$. Then we can choose $\eta = 2^{q(p-1)}$, and we can obtain the required result. □

The following corollary gives the Hyers-Ulam stability for the additive functional equation (1.1).

Corollary 2. Let $q = \pm 1$ be fixed and let p be a nonnegative real number with $p = v + w \neq 1$ and $\Upsilon \in Z$. Let $f : X \to Y$ be an odd mapping such that

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}], [y_{ij}]), t) \geq \sum_{i,j=1}^{n} \mathcal{N}'(\Upsilon(\|x_{ij}\|^v \cdot \|y_{ij}\|^w + \|x_{ij}\|^{v+w} + \|y_{ij}\|^{v+w}), t)$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and $t > 0$. Then there exists a unique additive mapping $\mathcal{A} : X \to Y$ satisfying (2.7).

Proof. The proof follows from Theorem 10 by taking $\psi(a, b) = \Upsilon(\|a\|^v \cdot \|b\|^w + \|a\|^p + \|b\|^p)$ for all $a, b \in X$. Then we can choose $\eta = 2^{q(p-1)}$, and we can obtain the required result. □

Theorem 11. Let $q = \pm 1$ be fixed and let $\psi : X \times X \to Z$ be a mapping such that for some $\eta \neq 4$ with $\left(\frac{\eta}{4}\right)^q < 1$

$$\mathcal{N}'(\psi(2^q a, 2^q b) \geq \mathcal{N}'\left(\psi(a, b), \eta^{-q} t\right) \tag{2.8}$$

for all $a, b \in X$ and $t > 0$, and $\lim_{k\to\infty} \mathcal{N}(4^{-kq}\mathcal{G}f(2^{kq}a, 2^{kq}b), t) = 1$ for all $a, b \in X$ and $t > 0$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\mathcal{N}(\mathcal{G}f_n([x_{ij}], [y_{ij}]), t) \geq \mathcal{N}'\left(\sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}), t\right) \qquad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X), \text{ and } t > 0. \tag{2.9}$$

Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\mathcal{N}_n(f_n([x_{ij}]) - \mathcal{Q}_n([x_{ij}]), t) \geq min\left\{\mathcal{N}'(\psi(0, x_{ij}), |\eta - 4|\, n^{-2} t) : i, j = 1, 2, \cdots, n\right\} \tag{2.10}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$.

Proof. For the cases $q = 1$ and $q = -1$, we consider $\eta < 4$ and $\eta > 4$, respectively. Letting $n = 1$ in (2.9), we obtain

$$\mathcal{N}(\mathcal{G}f(a, b), t) \geq \mathcal{N}'\left(\psi(a, b), t\right) \tag{2.11}$$

for all $a, b \in X$ and $t > 0$. Replacing (a, b) by $(0, a)$ in (2.11), we get

$$\mathcal{N}\left(f(2a) - 4f(a), t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right) \tag{2.12}$$

for all $a \in X$ and $t > 0$. Thus

$$\mathcal{N}\left(f(a) - \frac{1}{4^q} f(2^q a), \frac{\eta^{\frac{q-1}{2}}}{|4|^{\left(\frac{1+q}{2}\right)}} t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right) \qquad \forall\ a \in X \text{ and } t > 0. \tag{2.13}$$

We consider the set $\mathcal{M} = \{f : X \to Y\}$ and introduce the generalized metric ρ on $\mathcal{M}$ as follows:

$$\rho(f, g) = \inf\left\{\mu \in \mathbb{R}_+ : \mathcal{N}\left(f(a) - g(a), \mu t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right), \forall a \in X, t > 0\right\}.$$

It is easy to check that $(\mathcal{M}, \rho)$ is a complete generalized metric (see also Theorem 10).

Define the mapping $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ by $\mathcal{P}f(a) = \dfrac{1}{4^q} f(2^q a)$ for all $f \in \mathcal{M}$ and $a \in X$.

Let $f, g \in \mathcal{M}$ and ν be an arbitrary constant with $\rho(f, g) \leq \nu$. Then

$$\mathcal{N}\left(f(a) - g(a), \nu t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right)$$

for all $a \in X$ and $t > 0$. Therefore, using (2.8), we get

$$\mathcal{N}\left(\mathcal{P}f(a) - \mathcal{P}g(a), 4^{-q}\nu t\right) = \mathcal{N}\left(f(2^q a) - g(2^q a), \nu t\right), \geq \mathcal{N}'\left(\psi(0, a), \eta^{-q} t\right)$$

for all $a \in X$ and $t > 0$. Hence by definition $\rho(\mathcal{P}f, \mathcal{P}g) \leq \left(\dfrac{\eta}{4}\right)^q \nu$, that is, $\rho(\mathcal{P}f, \mathcal{P}g) \leq L\rho(f, g)$ for all $f, g \in \mathcal{M}$. This means that $\mathcal{P}$ is a contractive mapping with Lipschitz constant $L = \left(\dfrac{\eta}{4}\right)^q < 1$.

It follows from (2.13) that $\rho(f, \mathcal{P}f) \leq \dfrac{\eta^{\left(\frac{q-1}{2}\right)}}{|4|^{\left(\frac{1+q}{2}\right)}}$. Therefore according to Theorem 7, there exists a mapping $\mathcal{Q} : X \to Y$ which satisfies

(1) $\mathcal{Q}$ is a unique fixed point of $\mathcal{P}$, which satisfies $\mathcal{Q}(2^q a) = 4^q \mathcal{Q}(a)$ for all $a \in X$.

(2) $\rho(f,\mathcal{Q}) \leq \frac{1}{1-\eta}\rho(f,\mathcal{P}f)$, which implies the inequality $\rho(f,\mathcal{Q}) \leq \frac{1}{|4-\eta|}$. So

$$\mathcal{N}\left(f(a)-\mathcal{Q}(a),\frac{1}{|4-\eta|}t\right) \geq \mathcal{N}'\left(\psi(0,a),t\right) \qquad \forall\ a \in X \text{ and } t>0. \tag{2.14}$$

By (2.11), $\mathcal{N}(\mathcal{G}\mathcal{Q}(a,b),t) = \lim_{k\to\infty}\mathcal{N}(4^{-kq}\mathcal{G}f(2^{kq}a,2^{kq}b),t) \geq \lim_{k\to\infty}\mathcal{N}'(4^{-kq}\psi(2^{kq}a,2^{kq}b),t) = 1$. Hence by (N2), $\mathcal{G}\mathcal{Q}(a,b)=0$. Thus $\mathcal{Q}$ is quadratic.

Since $N(E_{kl}\otimes x,t) = N(x,t)$, we have

$$\begin{aligned} N_n([x_{ij}],t) &= N_n\left(\sum_{i,j=1}^{n} E_{ij}\otimes x_{ij},t\right) \geq min\left\{N_n(E_{ij}\otimes x_{ij},t_{ij}) : i,j=1,2,...,n\right\} \\ &= min\left\{N(x_{ij},t_{ij}) : i,j=1,2,...,n\right\}, \end{aligned}$$

where $t=\sum_{i,j=1}^{n} t_{ij}$. So $N_n([x_{ij}],t) \geq min\left\{N(x_{ij},\frac{t}{n^2}) : i,j=1,2,...,n\right\}$.

By (2.14),

$$\begin{aligned} \mathcal{N}\left(f_n([x_{ij}])-\mathcal{Q}_n([x_{ij}]),t\right) &\geq min\left\{\mathcal{N}\left(f(x_{ij})-\mathcal{Q}(x_{ij}),\frac{t}{n^2}\right) : i,j=1,2,\cdots,n.\right\} \\ &\geq min\left\{\mathcal{N}'\left(\psi(0,x_{ij}),|4-\eta|\,n^{-2}t\right) : i,j=1,2,\cdots,n\right\} \end{aligned}$$

for all $x=[x_{ij}]\in M_n(X)$ and $t>0$. Thus $\mathcal{Q}:\mathcal{X}\to\mathcal{Y}$ is a unique quadratic mapping satisfying (2.10). □

Corollary 3. Let $q=\pm1$ be fixed and let p be a nonnegative real number with $p\neq 2$ and $\Upsilon\in Z$. Let $f: X\to Y$ be an even mapping satisfying $f(0)=0$ and

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}],[y_{ij}]),t) \geq \mathcal{N}'(\sum_{i,j=1}^{n}\Upsilon(\|x_{ij}\|^p+\|y_{ij}\|^p),t)$$

for all $x=[x_{ij}], y=[y_{ij}]\in M_n(X)$ and $t>0$. Then there exists a unique quadratic mapping $\mathcal{Q}: X\to Y$ such that

$$\mathcal{N}\left(f_n([x_{ij}])-\mathcal{Q}_n([x_{ij}]),t\right) \geq min\left\{\mathcal{N}'(\|x\|^p\,\Upsilon,|4-2^p|\,n^{-2}t) : i,j=1,2,\cdots,n\right\} \tag{2.15}$$

for all $x=[x_{ij}]\in M_n(X)$ and $t>0$.

Proof. The proof follows from Theorem 11 by taking $\psi(a,b)=\Upsilon(\|a\|^p+\|b\|^p)$ for all $a,b\in X$. Then we can choose $\eta=2^{q(p-2)}$, and we can obtain the required result. □

The following corollary gives the Hyers-Ulam stability for the quadratic functional equation (1.1).

Corollary 4. Let $q=\pm1$ be fixed and let p be a nonnegative real number with $p=v+w\neq 2$ and $\Upsilon\in Z$. Let $f: X\to Y$ be an even mapping satisfying $f(0)=0$ and

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}],[y_{ij}]),t) \geq \mathcal{N}'(\sum_{i,j=1}^{n}\Upsilon(\|x_{ij}\|^v\cdot\|y_{ij}\|^w+\|x_{ij}\|^{v+w}+\|y_{ij}\|^{v+w}),t)$$

for all $x=[x_{ij}], y=[y_{ij}]\in M_n(X)$ and $t>0$. Then there exists a unique quadratic mapping $\mathcal{Q}: X\to Y$ satisfying (2.15).

Proof. The proof follows from Theorem 11 by taking $\psi(a,b)=\Upsilon(\|a\|^v\cdot\|b\|^w+\|a\|^p+\|b\|^p)$ for all $a,b\in X$. Then we can choose $\eta=2^{q(p-2)}$, and we can obtain the required result. □

References

[1] T. Aoki , On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan* **2** (1950), 64–66.

[2] A. Bodaghi, C. Park, J. M. Rassias, Fundamental stabilities of the nonic functional equation In intuitionistic fuzzy normed spaces , *Commun. Korean Math. Soc.* **31** (2016), 729–743.

[3] A. Ebadian, S. Zolfaghari, S. Ostadbashi, C. Park, Approximation on the reciprocal functional equation in several variables in matrix non-Archimedean random normed spaces, *Adv. Difference Equ.* **2015**, 2015:314.

[4] E. Effros, Z. J. Ruan, On approximation properties for operator spaces, *Int. J. Math.* **1** (1990), 163–187.

[5] Iz. EL-Fassi, S. Kabbaj, Non-Archimedean random stability of σ-quadratic functional equation, *Thai J. Math.* **14)** (2016), 151–165.

[6] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431–436.
[7] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222–224.
[8] A. Kumar, S. Kumar, On stability of cubic functional equation in non-Archimedean fuzzy normed spaces , *Int. J. Math. Archive* **7** (2016), No. 10, 167–174.
[9] J. Lee, C. Park, D. Shin, Functional equations in matrix normed spaces, *Proc. Indian Acad. Sci.* **125** (2015), 399–412.
[10] J. Lee, D. Shin, C. Park, Fuzzy stabilty of functional inequalities in matrix fuzzy normed spaces, *J. Inequal. Appl.* **2013**, 2013:224.
[11] D. Mihet, Fuzzy φ-contractive mapping in non-Archimedean fuzzy metric spaces, *Fuzzy Sets Syst.* **159** (2008), 739–744.
[12] D. Mihet, The stability of the additive Cauchy functional equation in non-Archimedean fuzzy normed spaces, *Fuzzy Sets Syst.* **161** (2010), 2206–2212.
[13] D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J. Math. Anal. Appl.* **343** (2008), 567–572.
[14] A. K. Mirmostafaee, M. S. Moslehian, Stability of additive mappings in non- Archimedean fuzzy normed spaces, *Fuzzy Sets Syst.* **160** (2009), 1643–1652.
[15] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, Approximate n-Jordan $*$-homomorphisms in C^*-algebras, *J. Comput. Anal. Appl.* **15** (2013), 365–368.
[16] C. Park, A. Najati, S. Jang, Fixed points and fuzzy stability of an additive-quadratic functional equation, *J. Comput. Anal. Appl.* **15** (2013), 452–462.
[17] V. Radu, The fixed point alternative and the stability of functional equations, *Fixed Point Theory* **4** (2003), 91–96.
[18] Z. J. Ruan, Subspaces of $C*$-algebras, *J. Funct. Anal.* **76** (1988), 217–230.
[19] C. Renu, Sushma, A fixed point approach to Ulam stability problem for cubic and quartic mapping in non-Archimedean fuzzy normed spaces, *Proceeding of the World Congress on Engineering 2010 vol VIII WCE*, (2010).
[20] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.* **46** (1982), 126–130.
[21] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *Bull. Sci. Math.* **108** (1984), 445–446.
[22] J. M. Rassias, On a new approximation of approximately linear mappings by linear mappings, *Discuss. Math.* **7** (1985), 193–196.
[23] J. M. Rassias, Solution of a problem of Ulam, *J. Approx. Theory* **57** (1989), 268–273.
[24] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Am. Math. Soc.* **72** (1978), 297–300.
[25] B. V. Senthil Kumar, A. Kumar, P. Narasimman, Estimation of approximate nonic functional equation in non-Archimedean fuzzy normed spaces, *Int. J. Pure Appl. Math. Tech.* **1** (2016), No. 2, 18–29.
[26] D. Shin, C. Park, Sh. Farhadabadi, On the superstability of ternary Jordan C^*-homomorphisms, *J. Comput. Anal. Appl.* **16** (2014), 964–973.
[27] D. Shin, C. Park, Sh. Farhadabadi, Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation, *J. Comput. Anal. Appl.* **17** (2014), 125–134.
[28] A. Song, The Ulam stability of matrix intuitionistic fuzzy normed spaces, *J. Intelligent Fuzzy Syst.* **32** (2017), 629–641.
[29] S. M. Ulam, Problems in Modern Mathematics, *Science Editions*, Wiley, NewYork, 1964.
[30] Z. Wang, P. K. Sahoo, Stability of an ACQ-functional equation in various matrix normed spaces, *J. Nonlinear Sci. Appl.* **8** (2015), 64–85.
[31] Z. Wang, P. K. Sahoo, Stability of the generalized quadratic and quartic type functional equation in non-Archimedean fuzzy normed spaces, *J. Appl. Anal. Comput.* **6** (2016), 917–938.
[32] T. Z. Xu, J. M. Rassias, W. X. Xu, Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces, *J. Math. Phys.* **51** (2010), 1–19.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Existence of continuous selection for some special kind of multivalued mappings

G. Poonguzali[a], Muthiah Marudai[b], George A. Anastassiou[c] and Choonkil Park[d*]

[a]Department of Mathematics, Bharathidasan University, Tiruchirappalli 620 024, Tamil Nadu, India
[b]Department of Mathematics, Bharathidasan University, Tiruchirappalli 620 024, Tamil Nadu, India
[c]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[d]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail: poomath09@gmail.com; mmarudai@yahoo.co.in; ganastss@memphis.edu; baak@hanyang.ac.kr

Abstract

This paper deals with the existence of continuous selection of a multivalued mapping in product space. Many authors provided existence of continuous map for lower semicontinuous. We provide continuous selection for weakly lower semicontinuous. Rhybinski [9] proved the existence for contraction type mapping. We prove the existence for some general type of mapping different from contraction mapping.

1 Introduction and preliminaries

Let X be a normed linear space. Then $B = \{x \in X : \|x\| \leq 1\}$ represents the closed unit ball and $B^0 = \{x \in X : \|x\| < 1\}$ represents the open unit ball in X. First we quote some notations and basic facts that are used in the sequal

$$\mathcal{P}(X) = \{A \subset X : A \neq \emptyset\},$$
$$\mathcal{P}_{cl}(X) = \{A \in \mathcal{P}(X) : A \textit{ is closed}\},$$
$$\mathcal{P}_{cv}(X) = \{A \in \mathcal{P}(X) : A \textit{ is convex}\},$$
$$\mathcal{P}_{cl,cv}(X) = \{A \in \mathcal{P}(X) : A \textit{ is closed, convex}\}.$$

For $x \in X,\ A, B \in \mathcal{P}(X)$,

$$\delta(A, B) = \sup\{d(x, B) : x \in A\}.$$

$\mathcal{H}(A, B) = \max\{\delta(A, B), \delta(B, A)\}$.

Let us consider the mapping $T : X \times Y \to \mathcal{P}_{cl,cv}(Y)$. Then the fixed point set is defined as $P_T(x) := \{y \in Y : y \in T(x, y)\}$. See [1, 2] for more information on fixed point theory.

Definition 1.1. *A multivalued mapping $F : X \to \mathcal{P}(Y)$ is called lower semicontinuous (l.s.c.) at $x_0 \in X$ if and only if for every $\epsilon > 0$ and $z \in F(x_0)$ there exists a neighborhood U_z containing x_0 with the property that*

$$z \in \cap\{F(x) + \epsilon B^0 : x \in U_z\}.$$

2010 Mathematics Subject Classification: 47H04, 47H10.

Keywords: weakly lower semicontinuous map, continuous selection, paracompact space, perfectly normal space.

*Corresponding author: Choonkil Park (email: baak@hanyang.ac.kr, office: +82-2-2220-0892).

Definition 1.2. *A multivalued mapping F is said to be weakly lower semicontinuous (w.l.s.c.) at $x_0 \in X$ if ond only if for every $\epsilon > 0$ and for every neighborhood V containing x_0, there exists a point $x_1 \in V$ so that for every $z \in F(x_1)$ there is a neighborhood U_z containing x_0 satifying the condition that*

$$z \in \cap\{F(x) + \epsilon B^0 : x \in U_z\}.$$

It is well known that F is l.s.c. (w.l.s.c.) if and only if F is l.s.c. (w.l.s.c.) at every $x \in X$. Also, it is easy to see that f is l.s.c. implies that F is w.l.s.c., but the converse is not true [8].

A topological space X is said to be paracompact if every open cover of X has a locally finite refinement. A cover $\{U_\beta\}_{\beta\in J}$ is called a refinement of $\{W_\alpha\}_{\alpha\in I}$ if for all $\beta \in J$, there exists $\alpha \in I$ such that $U_\beta \subset W_\alpha$. Also, a collection $\{A_i : i \in I\}$ of subsets of X is locally finite if and only if for each $x \in X$ there is an open $U \ni x$ with $|\{i \in I : A_i \cap U \neq \emptyset\}| < \infty$. A topological space X is said to be perfectly normal if it is normal and every closed subset is a G_δ subset. A multivalued mapping $T : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ is said to satisfy condition $\mathcal{C}$ if there exists $K < 1$ such that

$$\mathcal{H}(T(x, y_1), T(x, y_2)) \leq K\|y_1 - y_2\| \ for\ x \in X,\ y_1, y_2 \in Y.$$

In a similar way, a multivalued mapping $H : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ is said to satisfy condition $\mathcal{N}$ if it satisfies

$$\mathcal{H}(T(x, y_1), T(x, y_2)) \leq \|y_1 - y_2\| \ for\ x \in X,\ y_1, y_2 \in Y.$$

In 1956, Michael [6] was the first person to study about continuous selection for a given multivalued mapping under some suitable conditions. The following theorem is due to Michael.

Theorem 1.3. [6] *In a paracompact space X, the lower semi-continuous multivalued mapping $F : X \to \mathcal{P}_{cl,cv}(Y)$ has a continuous selection, where Y is a Banach space.*

The importance of the above theorem was first noticed by Browder [4], who used the theorem to prove Fan Browder theorem. Later, many researchers established results on continuous selections with applications (see [3, 5, 7, 10, 11]). Further, in [8], Przeslawski and Rybinski has generalized Michael selection theorem for weakly lower semicontinuous mapping. They proved the existence of continuous selection for w.l.s.c. which is weaker than l.s.c. Rybinski [9] proved the following theorem.

Theorem 1.4. *Let X be a paracompact and perfectly normal topological space and Y be a closed subset of a Banach space $(Z, \|\cdot\|)$. Assume that $T : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ satisfies condition $\mathcal{C}$ and also, satisfies the condition that for every $y \in Y$ the multivalued mapping $T(\cdot, y)$ is w.l.s.c. Then there exists a continuous mapping $h : X \times Y \to Y$ such that $h(x, y) \in P_T(x)$ for every $(x, y) \in X \times Y$.*

In this direction, we study the existence of continuous selection for multivalued mapping with certain conditions. For that, we need the following lemma and theorem.

Lemma 1.5. [8] *Let X and Y be any topological spaces. If $T : X \to \mathcal{P}_{cl,cv}(Y)$ is a w.l.s.c. multivalued mapping and $f : X \to Y$ is a continuous and open mapping, then $T \circ f$ is w.l.s.c.*

Theorem 1.6. [8] *If X is a paracompact topological space, Y is a normed linear space and $F : X \to \mathcal{P}_{cl,cv}(Y)$ is w.l.s.c., then F has a continuous selection.*

2 Existence of continuous selections

In this section, we provide continuous selection for some general type of mapping.

Theorem 2.1. *Let $F : X_1 \times X_2 \to X_2$ be any multivalued mapping with the property that*

1. *$F(.,x_2)$ is w.l.s.c for every $x_2 \in X_2$,*
2. *F satisfies property $(\mathcal{N})$.*

Then, for a given continuous mapping $\alpha : X_1 \times X_2 \to X_2$, the new mapping $(x_1,x_2) \to F(x_1,\alpha(x_1,x_2))$ is w.l.s.c.

Proof. Let us define $S := X_1 \times X_2$ and define $G : S \times X_2 \to \mathcal{P}_{cl,cv}(X_2)$ by $G(s,u) = F(P_X(s),u) = F(x_1,u)$ for $s \in S$ and $u \in X_2$. Now our aim is to show that the mapping $s \to G(s,\alpha(s))$ is w.l.s.c. By Lemma 1.5, it is clear that $G(.,u)$ is w.l.s.c. for every $u \in X_2$.

Step 1: For an $s_0 \in S$ and an $\epsilon > 0$ and a neighborhood $O \ni s_0$, by continuity of α, we can choose a neighborhood $V \subset O$ of s_0 with the property that

$$\|\alpha(s) - \alpha(s_0)\| < \frac{\epsilon}{3}$$

for each $s \in V$.

Step 2: Since $G(.,u)$ is w.l.s.c., by applying the definition of weakly lower semicontinuity for this V, we can find a point s_1 so that for any $v \in G(s_1,\alpha(s_0))$, there exists a neighborhood U_v of s_0 with

$$v \in \cap\{G(s,\alpha(s_0)) + \frac{\epsilon}{3}B^0 : s \in U_v\}. \tag{1}$$

Step 3: Let $v_1 \in G(s_1,\alpha(s_1))$. Since F satisfies property $(\mathcal{N})$, we have

$$H(G(s,u_1),G(s,u_2)) \le \|u_1 - u_2\|.$$

Using the above, we can find $v \in G(s_1,\alpha(s_0))$ such that

$$\|v - v_1\| \le \|\alpha(s_1) - \alpha(s_0)\| < \frac{\epsilon}{3}.$$

For such v, applying Step 2, we get U_v which satisfies (1). Observe that $G(s,\alpha(s_0)) \subseteq G(s,\alpha(s)) + \frac{\epsilon}{3}B^0$. Hence $G(s,\alpha(s_0)) + \frac{\epsilon}{3} \subseteq G(s,\alpha(s)) + 2\frac{\epsilon}{3}B^0$, and so $v \in \cap\{G(s,\alpha(s)) + 2\frac{\epsilon}{3}B^0 : s \in U_v \cap V\}$. Thus $v_1 \in \cap\{G(s,\alpha(s) + \epsilon B^0 : s \in U_v \cap V)\}$, which gives our claim. □

Lemma 2.2. *Let $H : X \to \mathcal{P}_{cl}(Y)$ be w.l.s.c. and $h : X \to Y$ is continuous. Then for every continuous function $d : X \to [0,\infty)$ such that $H(x) \cap (h(x) + d(x)B) \ne \emptyset$, the multivalued mapping $S : X \to \mathcal{P}_{cl}(Y)$ defined by $S(x) = H(x) \cap (h(x) + d(x)B)$ is w.l.s.c.*

Proof. Fix any $x_0 \in X$. If $d(x_0) = 0$, then nothing to prove. Suppose $d(x_0) > 0$. Fix $\epsilon > 0$ and any neighborhood V of x_0. Then choose $\delta > 0$ such that $\delta < min\{d(x_0),\epsilon\}$. Now, choose a neighborhood W of x_0, $W \subseteq V$, such that for $x_1,x_2 \in W$,

$$\begin{aligned} |d(x_1) - d(x_2)| &< \frac{\delta}{2}, \\ \|f(x_1) - f(x_2)\| &< \frac{\delta}{2}. \end{aligned}$$

Choose a point x' in W such that for every $z \in H(x')$, there exists U_z of x_0 such that

$$z \in \cap\{H(x) + \delta B^0 : x \in U_z\}. \tag{2}$$

Now our claim is that this x' is the required point. To see this, take any arbitrary

$$z' \in H(x') \cap (h(x') + d(x')B).$$

Since $z' \in H(x')$, by (2), there exists $U_{z'} \ni x_0$ such that $z' \in \cap\{H(x) + \delta B^0 : x \in U_{z'}\}$. Let us define $W_{z'} := W \cap U_{z'}$. Then this is our required neighborhood for each z'.

$$\begin{aligned} \|z' - h(x)\| &\leq \|z' - h(x')\| + \|f(x') - h(x)\| \\ &\leq d(x') + \frac{\delta}{2} \\ &< d(x) + \delta. \end{aligned}$$

It follows that

$$\begin{aligned} &z' \in \cap\{(H(x) + \delta B^0) \cap (h(x) + d(x)B + \delta B^0) : x \in W_{z'}\}, \\ &z' \in \cap\{(H(x) \cap (h(x) + d(x)B) + \delta B^0 : x \in W_{z'}\}, \\ &z' \in \cap\{G(x) + \delta B^0 : x \in W_{z'}\}. \end{aligned}$$

Hence $z' \in \cap\{G(x) + \epsilon B^0 : x \in W_{z'}\}$. □

Theorem 2.3. *Let α_1, $\alpha_2 : X_1 \times X_2 \to X_2$ be mappings such that α_2 is a selection of the multivalued mapping $(x_1, x_2) \to F(x_1, \alpha(x_1, x_2))$. Then there exists a continuous selection α_3 of the multivalued mapping $(x_1, x_2) \to F(x_1, \alpha_2(x_1, x_2))$ such that*

$$\begin{aligned} \|\alpha_1(x_1, x_2) - \alpha_2(x_1, x_2)\| &\leq \frac{\lambda}{1-\lambda}\|\alpha_2(x_1, x_2) - \alpha_1(x_1, x_2)\|, \\ d(\alpha_3(x_1, x_2), F(x_1, \alpha_3(x_1, x_2))) &\leq \frac{\lambda}{1-\lambda}\|\alpha_2(x_1, x_2) - \alpha_1(x_1, x_2)\| \end{aligned}$$

for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. By hypothesis, we have $\alpha_2(x_1, x_2) \in F(x_1, \alpha_1(x_1, x_2))$. Then

$$\begin{aligned} d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2))) &\leq H(F(x_1, \alpha_1(x_1, x_2)), F(x_1, \alpha_2(x_1, x_2))) \\ &\leq \lambda[d(\alpha_1(x_1, x_2), F(x_1, \alpha_1(x_1, x_2))) \\ &\quad + d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2)))], \end{aligned}$$

$$(1-\lambda)d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2))) \leq \lambda d(\alpha_1(x_1, x_2), F(x_1, \alpha_1(x_1, x_2))),$$

$$\begin{aligned} d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2))) &\leq \frac{\lambda}{1-\lambda} d(\alpha_1(x_1, x_2), F(x_1, \alpha_1(x_1, x_2))) \\ &\leq \frac{\lambda}{1-\lambda}\|\alpha_1(x_1, x_2) - \alpha_2(x_1, x_2)\|. \end{aligned}$$

Now, define a new mapping $G : X_1 \times X_2 \to \mathcal{P}_{cl,cv}(X_2)$ by $G(x_1, x_2) := F(x_1, \alpha_2(x_1, x_2)) \cap (\alpha_2(x_1, x_2)) + \frac{\lambda}{1-\lambda}\|\alpha_1(x_1, x_2) - \alpha_2(x_1, x_2)\|$. Then, clearly, G is well defined and by Lemma 2.2 G is w.l.s.c. By Theorem 1.6, G has a continuous selection $\alpha_3 : X_1 \times X_2 \to X_,$. which is our required mapping. □

Theorem 2.4. *Let X_1 be a paracompact and perfectly normal topological space and X_2 be a Banach space. Assume that*

1. *$F : X_1 \times X_2 \to \mathcal{P}_{cl,cv}(X_2)$ satisfies property $(\mathcal{N})$,*
2. *for a given $x \in X$, the mapping F satifies $\mathcal{H}(F(x, v_1), F(x, v_2)) \leq \lambda[d(v_1, F(x, v_1)) + d(v_2, F(x, v_2))]$, where $\lambda < \frac{1}{2}$,*

3. for each $x_2 \in X_2$, the mapping $F(., x_2)$ is w.l.s.c.

Then there exists a continuous mapping $f : X_1 \times X_2 \to X_2$ such that $f(x_1, x_2) \in P_H(x_1)$ for every $(x_1, x_2) \in X_1 \times X_2$.

Proof. Choose $\alpha_0 : X_1 \times X_2 \to X_2$ by $\alpha_0(x_1, x_2) = x_2$. Then α_0 is continuous. Now using Theorem 2.1, we get $(x_1, x_2) \to F(x_1, \alpha_0(x_1, x_2))$ is w.l.s.c. Applying Theorem 1.6, we get a continuous selection $\alpha_1 : X_1 \times X_2 \to X_2$ for the mapping $(x_1, x_2) \to F(x_1, \alpha_0(x_1, x_2))$.

By Theorem 2.3, there exists a continuous selection $\alpha_2 : X_1 \times X_2 \to X_2$ for $(x_1, x_2) \to F(x_1, \alpha_1(x_1, x_2))$ satisfying the following two conditions

$$\|\alpha_2(x_1, x_2) - \alpha_1(x_1, x_2)\| \le \frac{\lambda}{1-\lambda}\|\alpha_1(x_1, x_2) - \alpha_0(x_1, x_2)\|,$$
$$d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2))) \le \frac{\lambda}{1-\lambda}\|\alpha_1(x_1, x_2) - \alpha_0(x_1, x_2)\|$$

for every $(x_1, x_2) \in X_1 \times X_2$.

By proceeding the above process, we get a sequence of continuous functions $\alpha_n : X_1 \times X_2 \to X_2$ with the following properties:

$$\|\alpha_n(x_1, x_2) - \alpha_{n-1}(x_1, x_2)\| \le \frac{\lambda}{1-\lambda}\|\alpha_{n-1}(x_1, x_2) - \alpha_{n-2}(x_1, x_2)\|,$$
$$d(\alpha_n(x_1, x_2), F(x_1, \alpha_n(x_1, x_2))) \le \frac{\lambda}{1-\lambda}\|\alpha_n(x_1, x_2) - \alpha_{n-1}(x_1, x_2)\|$$

for $n = 1, 2, \ldots,$ $(x_1, x_2) \in X_1 \times X_2$. For a fixed pair (x_1, x_2), the sequence $(\alpha_n(x_1, x_2))$ is a Cauchy sequence. To see this, using the following inequality

$$\|\alpha_n(x_1, x_2) - \alpha_{n-1}(x_1, x_2)\| \le (\frac{\lambda}{1-\lambda})^{n-1}\|\alpha_1(x_1, x_2) - \alpha_0(x_1, x_2)\|.,$$

we show that $(\alpha_n(x_1, x_2))$ is a Cauchy sequence. Since X_2 is complete, this Cauchy sequence converges.

Now, define $f : X_1 \times X_2 \to X_2$ by $f(x_1, x_2) = \lim_{n\to\infty} \alpha_n(x_1, x_2)$. It is clear that f is well-defined.

Next, our aim is to claim that f is continuous. For that, fix any $(x_1', x_2') \in X_1 \times X_2$. Then, consider

$$\begin{aligned}\|f(x_1, x_2) - f(x_1', x_2')\| \le &\|f(x_1, x_2) - \alpha_n(x_1, x_2)\| \\ &+ \|\alpha_n(x_1, x_2) - \alpha_n(x_1', x_2')\| \\ &+ \|\alpha_n(x_1', x_2') - f(x_1, x_2)\|.\end{aligned}$$

Since $\alpha_n' s$ are continuous and $\alpha_n(x_1, x_2)$ is convergent for every $(x_1, x_2) \in X_1 \times X_2$, applying all these in the above inequality, we can conclude f is continuous.

Next, consider

$$\begin{aligned}d(f(x_1, x_2), F(x_1, f(x_1, x_2))) &\le \|f(x_1, x_2) - \alpha_n(x_1, x_2)\| \\ &+ d(\alpha_n(x_1, x_2), F(x_1, f(x_1, x_2))) \\ &\le \sum_{m=n}^{\infty}(\frac{\lambda}{1-\lambda})^m\|\alpha_1(x_1, x_2) - \alpha_0(x_1, x_2)\| \\ &+ \frac{\lambda}{1-\lambda}\|\alpha_m(x_1, x_2) - \alpha_{m-1}(x_1, x_2)\| \\ &\le \sum_{m=n}^{\infty}(\frac{\lambda}{1-\lambda})^m\|\alpha_1(x_1, x_2) - \alpha_0(x_1, x_2)\| \\ &+ \sum_{m=n}^{\infty}(\frac{\lambda}{1-\lambda})^n\|\alpha_1(x_1, x_2) - \alpha_0(x_1, x_2)\|.\end{aligned}$$

Hence $f(x_1, x_2) \in F(x_1, f(x_1, x_2))$. □

Acknowledgments

G. Poonguzali was funded by Human Resource Development Group and Council of Scientific and Industrial Research, Sanction No. 09/475(0198)/2016-EMR-I dated 15.11.2016. C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

References

[1] G.A. Anastassiou, I.K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[2] A. Batool, T. Kamran, S. Jang, C. Park, *Generalized φ-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[3] F. E. Browder, *A new generation of the Schauder fixed point theorem*, Math. Ann. **174** (1967), 285–290.

[4] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces,* Math. Ann. **177** (1968), 283–301.

[5] C. D. Horvath,it Extension and selection theorems in topological vector spaces with a generalized convexity structure, Ann. Fac. Sci. **2** (1993), 253–269.

[6] E. Michael, *Continuous selections I*. Ann. Math. **63** (1956), 361–382.

[7] S. Park, *Continuous selection theorems in generalized convex spaces,* Numer. Funct. Anal. Optim. **25** (1999), 567–583.

[8] K. Przeslawski, L. E. Rybinski, *Michael selection theorem under weak lower semicontinuity assumption*. Proc. Amer. Math. Soc. **109** (1990), 537–543.

[9] L. Rybinski, *An application of the continuous selection theorem to the study of the fixed points of multivalued mappings*, J. Math. Anal. Appl. **153** (1990), 391–396.

[10] N. C. Yannelis, N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Economics **12** (1983), 233–245.

[11] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-point theorems*, Springer-Verlag, New York, 1986.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

REFINED STABILITY OF SET-VALUED FUNCTIONAL EQUATIONS

HONG-MEI LIANG, HARK-MAHN KIM, AND HWAN-YONG SHIN

Abstract. Recently, stability results of set-valued functional equations on domain of cones in Banach spaces are obtained by several authors. In this paper, we present the refined stability results of set-valued functional equations which is stable in the sense of Aoki, Rassias and Găvrută on domain of cones.

1. Introduction

The Hyers–Ulam stability problem was originated by S. M. Ulam [16] in 1940 as follows: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) \le \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$.*

Ulam's question was partially solved by D. H. Hyers [6] in the case of approximately additive functions and when the groups in the question are Banach spaces. In fact, Hyers proved that each solution of the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all x and y can be approximated by an exact solution, say an additive function. In this case, it is said that the Cauchy additive functional equation $f(x+y) = f(x)+f(y)$ satisfies Hyers–Ulam stability or that the equation is stable in the sense of Hyers–Ulam.

Many mathematicians attempted to moderate the condition for the bound of the norm of the Cauchy difference. First, T. Aoki [1] proved the stability of Cauchy functional equations by changing the bound of Cauchy difference as follows

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p),$$

where $p \in (0, 1)$, and Rassias [14] obtained additional linear properties of this results. Furthermore, the control function of Cauchy difference with some regularity conditions has been employed by Găvrută [5] as follows

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y).$$

1991 *Mathematics Subject Classification.* 39B52, 39B82, 54C65.

Key words and phrases. set-valued functional equation; generalized Hyers–Ulam stability; Cantor intersection theorem; cone subset in Banach spaces.

† Corresponding author. hyshin31@cnu.ac.kr.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Recently, as the development of non-convex analysis, cone sets were investigated by many authors and were applied for various regions of optimization theory and mathematical physics [10, 11]. Let X be a real Banach space and P a subset of X. P is called a *cone* [15] if

(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

Set-valued functions in Banach spaces also have received a lot of attention in the literature [2]. Functional inclusion is a tool for defining many notions of set-valued analysis, e.g., linear, affine, convex, concave, subadditive, superadditive set-valued maps. Finding a selection of such set-valued maps, with some special properties, is one of the main problems of set-valued analysis (see [2]). The stability theory of functional equations leads in some cases to such problems and solving them provides Hyers–Ulam stability results [3, 4, 7]. In setting domain of set-valued functions as a cone, some stability results of set-valued functional equations were obtained by several authors [9, 13].

In this sequel, we introduce a result concerning with stability of set-valued functional equations under cone domain. Let Y be a Banach space and P be a cone. We define the following families of sets :

$$\mathcal{P}_0(Y) := \{A \subseteq Y : \text{A is nonempty set}\}$$
$$cl(Y) := \{A \in \mathcal{P}_0(A) : \text{A is closed set}\}$$
$$cz(Y) := \{A \in \mathcal{P}_0(A) : \text{A is closed set containing zero}\}.$$

Theorem 1.1. (*C. Park, D. O'Regan, R. Saadati,* [13]) *If $F : P \to cz(Y)$ is a set-valued mapping satisfying $F(0) = \{0\}$,*

$$F(x) + F(y) \subseteq 2F\Big(\frac{x+y}{2}\Big) \tag{1.1}$$

and

$$sup\{diam(F(x)) : x \in P\} < +\infty$$

for all $x, y \in P$, then there exists a unique additive mapping $g : P \to Y$ such that $g(x) \in F(x)$ for all $x \in P$.

In view of Theorem 1.1, if $diam(F(x)) = \varepsilon$, then $\sup\{diam(F(x)) : x \in P\} < +\infty$. So if, in addition, F satisfies (1.1), we may understand Theorem 1.1 works good in the sense of Hyers–Ulam. On the other hand, if $diam(F(x)) = \|x\|^p, p \neq 0$, we confirm that $sup\{diam(F(x)) : x \in P\} = \infty$, and so Theorem 1.1 cannot be favorably applied in this case.

Thus, in this paper, we are devoted to investigate refined stability results of Theorem 1.1, and also we present alternative new stability theorems and examples to provide refined stability theorems of Theorem 1.1.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$A + B = \{x \in X : x = a + b,\ a \in A,\ b \in B\}$$
$$\lambda A = \{x \in X : x = \lambda a,\ a \in A\}.$$

Lemma 1.2. [12] *Let λ and μ be real numbers. If A and B are empty subsets of a real vector space X, then*

$$\lambda(A + B) = \lambda A + \lambda B$$
$$(\lambda + \mu)A \subseteq \lambda A + \mu A.$$

Moreover, if A is convex in X and $\lambda\mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$

Lemma 1.3. *If A_n and B_n are non-empty subsets of a real vector space X for all nonnegative positive integer n, then*

$$\bigcap_{n=0}^{l} A_n + \bigcap_{n=0}^{l} B_n = \bigcap_{n=0}^{l} (A_n + B_n)$$

for any given $l \in \mathbb{N}$.

The following famous theorem is a crucial tool to prove our main theorems.

Theorem 1.4. (*Cantor Intersection Theorem,* [8]) *Suppose (X, d) is a non-empty complete metric space, and $\{C_n\}_{n\geq 0}$ closed subsets of X which satisfies*

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots .$$

If $\lim_{n\to\infty} diam(C_n) = 0$, where $diam(C_n)$ is defined by $diam(C_n) = \sup\{d(x, y)|x, y \in C_n\}$, then $\bigcap_{n=1}^{\infty} C_n$ consists of a single point.

From now on, let P be a cone for a Banach space Y. We present a main theorem, which is an extended Hyers–Ulam stability of a set-valued functional equations on the domain of cones.

Theorem 1.5. *If $F : P \to cl(Y)$ is a set-valued mapping satisfying*

$$\sum_{i=1}^{m} F(x_i) \subseteq mF\Big(\frac{\sum_{j=1}^{m} x_j}{m}\Big) \tag{1.2}$$

and

$$\lim_{n\to\infty} \frac{diam(F(m^n x))}{m^n} = 0 \tag{1.3}$$

for all $x_1, \cdots, x_m, x \in P$, where $m > 1$ is a positive integer, then there exists a unique additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Proof. Since $F(0) \in cl(Y)$, $F(0)$ has at least an element, say $p \in F(0)$.

Letting $x_1 = x$ and $x_k = 0$ for all $k \neq 1$, in (1.2), we have

$$F(x) + (m-1)\{p\} \subseteq F(x) + (m-1)F(0) \subseteq mF(\frac{x}{m}) \tag{1.4}$$

and so

$$F(x) + (-1)\{p\} \subseteq m\big(F(\frac{x}{m}) + (-1)\{p\}\big) \tag{1.5}$$

for all $x \in P$. Replacing x by $m^{n+1}x$ in (1.5), then we obtain

$$F(m^{n+1}x) + (-1)\{p\} \subseteq m\big(F(m^n x) + (-1)\{p\}\big)$$

and hence

$$\frac{F(m^{n+1}x) + (-1)\{p\}}{m^{n+1}} \subseteq \frac{F(m^n x) + (-1)\{p\}}{m^n}$$

for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$. Denoting $F_n(x) := \dfrac{F(m^n x) + (-1)\{p\}}{m^n}$ for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$, it results that $\{F_n(x)\}_{\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$diam(F_n(x)) = \frac{1}{m^n} diam(F(m^n x) + (-1)\{p\}) = \frac{1}{m^n} diam(F(m^n x)).$$

By (1.3), we get $\lim_{n\to\infty} diam(F_n(x)) = 0$ for all $x \in P$. Using the Cantor Intersection Theorem for the sequence $\{F_n(x)\}_{n\geq 0}$, the intersection $\bigcap_{n\geq 0} F_n(x)$ is a singleton and we denote this intersection by $g(x)$ for all $x \in P$. Thus we obtain a mapping $g : P \to cl(Y)$, defined as $g(x) := \bigcap_{n\geq 0} F_n(x)$, which is a singleton from F because $g(x) \subseteq F_0(x) = F(x) + (-1)\{p\} \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Now, we show that g is additive. It follows from the definition of g and Lemma 1.3 that

$$\sum_{i=1}^{m} g(x_i) \subseteq \sum_{i=1}^{m} \bigcap_{n=0}^{l} F_n(x_i) = \bigcap_{n=0}^{l} \sum_{i=1}^{m} F_n(x_i) \subseteq \bigcap_{n=0}^{l} \Big(mF_n\Big(\frac{\sum_{j=1}^{m} x_j}{m}\Big)\Big)$$

for any $l \in \mathbb{N} \cup \{0\}$, thus

$$\sum_{i=1}^{m} g(x_i) \subseteq \bigcap_{n=0}^{\infty} \Big(mF_n\Big(\frac{\sum_{j=1}^{m} x_j}{m}\Big)\Big)$$

for all $x_1, \cdots, x_m \in P$. On the other hand, one obtains vacuously

$$mg\Big(\frac{\sum_{j=1}^{m} x_j}{m}\Big) \subseteq \bigcap_{n=0}^{\infty} \Big(mF_n\Big(\frac{\sum_{j=1}^{m} x_j}{m}\Big)\Big)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x_1, \cdots, x_m \in P$. Thus, since $\bigcap_{n=0}^{\infty}\Big(mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big)\Big)$ is a singleton, we arrive at

$$\sum_{i=1}^m g(x_i) = mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big)$$

for all $x_1, \cdots, x_m \in P$. Thus g is additive since $g(0) = \{0\}$. Therefore, we conclude that there exists an additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F_0(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Next, we will finalize the proof by proving the uniqueness of g. Suppose that $g' : P \to cl(Y)$ is another additive mapping such that $g'(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$. Then we have

$$m^n g(x) = g(m^n x) \subseteq F(m^n x) + (-1)F(0)$$
$$m^n g'(x) = g'(m^n x) \subseteq F(m^n x) + (-1)F(0)$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in P$. Thus, we get

$$\begin{aligned} m^n diam(g(x) - g'(x)) &= diam(m^n g(x) - m^n g'(x)) \\ &= diam(g(m^n x) - g'(m^n x)) \\ &\leq diam(F(m^n x) + (-1)F(0)) \\ &= diam(F(m^n x)) + diam((-1)F(0)) \end{aligned}$$

which implies

$$diam(g(x) - g'(x)) \leq \frac{1}{m^n}\big[diam(F(m^n x)) + diam((-1)F(0))\big]$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in P$. Therefore, it follows from $\lim_{n\to\infty} \frac{diam(F(m^n x))}{m^n} = 0$ that $g(x) = g'(x)$ for all $x \in P$, as desired. □

The following corollary is a refined stability result of Theorem 1.1, if we take $m = 2$.

Corollary 1.6. *If $F : P \to cl(Y)$ is a set-valued mapping satisfying*

$$\sum_{i=1}^m F(x_i) \subseteq mF\big(\frac{\sum_{j=1}^m x_j}{m}\big)$$

and

$$\sup\{diam(F(x)) : x \in X\} < +\infty$$

for all $x_1, \cdots, x_m, x \in P$, then there exists a unique additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Proof. Since $\sup\{diam(F(x)) : x \in P\} < +\infty$, $\lim_{n\to\infty} \dfrac{diamF(m^n x)}{m^n} = 0$ for all $x \in P$. Applying Theorem 1.5, we complete the proof. □

Now, let us consider the following example with nontrivial set-valued function at zero.

Example 1.7. Let $F : [0,\infty) \to cl(\mathbb{R})$ be defined by

$$F(x) = \begin{cases} [ax, ax + bx^p], & \text{if} \quad x \neq 0, \\ [0, c], & \text{if} \quad x = 0, \end{cases}$$

where a, b are positive real numbers, $c \geq 0$ and $p \in (-\infty, 0) \cup (0, 1)$. It is easy to see that

$$F(x) + F(y) \subseteq 2F\big(\frac{x+y}{2}\big), \quad \lim_{n\to\infty} \frac{diam(F(2^n x))}{2^n} = 0$$

for all $x, y \in [0,\infty)$. Also, we can check that

$$\bigcap_{n=0}^{\infty} \frac{F(2^n x) + (-1)F(0)}{2^n} = \{ax\}$$

for all $x \in [0,\infty)$. Therefore, there exists additive mapping $g : [0,\infty) \to cl(\mathbb{R})$ defined by $g(x) = \{ax\}$ such that $g(x) \subseteq F(x) + (-1)F(0) = [ax - c, ax + bx^p]$ for all $x \in [0,\infty)$. This result can be found by applying Theorem 1.5.

However, it is noted that we cannot apply Theorem 1.1 to this example because

$$\sup\{diam(F(x)) : x \in [0,\infty)\} = +\infty.$$

Next, we provide an alternative main theorem of Theorem 1.5.

Theorem 1.8. *If $F : P \to cl(Y)$ is a set-valued mapping satisfying*

$$mF\big(\frac{\sum_{j=1}^m x_j}{m}\big) \subseteq \sum_{i=1}^m F(x_i) \tag{1.6}$$

and

$$\lim_{n\to\infty} m^n diam(F(\frac{x}{m^n})) = 0 \tag{1.7}$$

for all $x_1, \cdots, x_m, x \in P$, then there exists a unique additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Proof. By assumption (1.7), one has

$$\lim_{n\to\infty} m^n diam(F(0)) = 0$$

and so $F(0)$ is a singleton, say $F(0) = \{p\}$. Taking $x_1 = x$ and $x_k = 0$ for all $k \neq 0$ in (1.6), we obtain

$$m\big(F\big(\frac{x}{m}\big) + (-1)\{p\}\big) \subseteq F(x) + (-1)\{p\}. \tag{1.8}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x \in P$. And if we replace x by $\frac{x}{m^n}$ in (1.8), then we obtain

$$m\big(F(\frac{x}{m^{n+1}}) + (-1)\{p\}\big) \subseteq F\big(\frac{x}{m^n}\big) + (-1)\{p\}$$

and so

$$m^{n+1}\big(F\big(\frac{x}{m^{n+1}}\big) + (-1)\{p\}\big) \subseteq m^n\big(F\big(\frac{x}{m^n}\big) + (-1)\{p\}\big)$$

for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$. Defining $F_n(x) = m^n\big(F(\frac{x}{m^n}) + (-1)\{p\}\big)$ for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\{F_n(x)\}_{n\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. It is noted that

$$diam(F_n(x)) = diam\big(m^n\big(F\big(\frac{x}{m^n}\big) + (-1)\{p\}\big)\big) = m^n diam\big(F\big(\frac{x}{m^n}\big)\big),$$

which implies $\lim_{n\to\infty} diam(F_n(x)) = 0$ for all $x \in P$ by (1.7).

Employing the Cantor Intersection Theorem to the sequence $\{F_n(x)\}_{n\geq 0}$, $\bigcap_{n\geq 0} F_n(x)$ is a singleton set and so we may define a mapping $g : P \to cl(Y)$ by $g(x) := \bigcap_{n\geq 0} F_n(x), x \in P$, which satisfies $g(x) \subseteq F_0(x) = F(x) + (-1)\{p\} \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Now, we show that g is additive. It follows from Lemma 1.2 that

$$\begin{aligned} mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) &= m \cdot m^n\Big(F\big(\sum_{j=1}^m \frac{x_j}{m^n \cdot m}\big) + (-1)\{p\}\Big) \\ &\subseteq m^n \sum_{i=1}^m \big(F\big(\frac{x_i}{m^n}\big) + (-1)\{p\}\big) = \sum_{i=1}^m F_n(x_i) \end{aligned}$$

for all $x_1, \cdots, x_m \in P$. By the definition of g, we can get

$$mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) \subseteq \bigcap_{n=0}^{l} mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) \subseteq \bigcap_{n=0}^{l}\sum_{i=1}^m F_n(x_i)$$

for any $l \in \mathbb{N} \cup \{0\}$ and all $x_1, \cdots, x_m \in P$, which yields

$$mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) \subseteq \bigcap_{n=0}^{\infty}\sum_{i=1}^m F_n(x_i). \tag{1.9}$$

Moreover, it is easy to show that, for all $x_1, \cdots, x_m \in P$,

$$\sum_{i=1}^m g(x_i) \subseteq \sum_{i=1}^m F_n(x_i), \forall n \in \mathbb{N} \cup \{0\}$$

and so

$$\sum_{i=1}^m g(x_i) \subseteq \bigcap_{n=0}^{\infty}\sum_{i=1}^m F_n(x_i). \tag{1.10}$$

Therefore, it follows from (1.9) and (1.10) that

$$mg\Big(\frac{\sum_{j=1}^{m} x_j}{m}\Big) = \sum_{i=1}^{m} g(x_i),$$

that is, g is additive because $g(0) = \{0\}$.

Finally, let us prove the uniqueness of g. Suppose that $g' : P \to cl(Y)$ is an additive mapping such that $g'(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$. Then we have

$$\frac{1}{m^n} g(x) = g\big(\frac{x}{m^n}\big) \subseteq F\big(\frac{x}{m^n}\big) + (-1)F(0),$$
$$\frac{1}{m^n} g'(x) = g'\big(\frac{x}{m^n}\big) \subseteq F\big(\frac{x}{m^n}\big) + (-1)F(0)$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in P$. Thus, noting singleton $F(0)$, we get

$$\begin{aligned} \frac{1}{m^n} diam(g(x) - g'(x)) &= diam(g(\frac{x}{m^n}) - g'(\frac{x}{m^n})) \\ &\leq diam\Big(F(\frac{x}{m^n}) + (-1)F(0)\Big) = diam\Big(F(\frac{x}{m^n})\Big) \end{aligned}$$

for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$. It follows from (1.7) that $g(x) = g'(x)$ for all $x \in P$, as desired. □

Corollary 1.9. *If $F : P \to cl(Y)$ is a set-valued mapping satisfying $F(0) = \{0\}$,*

$$mF\big(\frac{\sum_{j=1}^{m} x_j}{m}\big) \subseteq \sum_{i=1}^{m} F(x_i)$$

and

$$\lim_{n\to\infty} m^n diam(F(\frac{x}{m^n})) = 0$$

for all $x_1, \cdots, x_m, x \in P$, then there exists a unique additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F(x)$ for all $x \in P$.

Example 1.10. Let $F : [0, \infty) \to cl(\mathbb{R})$ be defined by $F(x) = [ax, ax + bx^p]$, where a, b are positive real numbers and $p > 1$. Then, since the function x^p is convex, it is easily checked that $2F\big(\frac{x+y}{2}\big) \subseteq F(x) + F(y)$ and $\lim_{n\to\infty} 2^n diam(F(\frac{x}{2^n})) = 0$ for all $x, y \in [0, \infty)$. Thus, there exists an additive mapping $g : [0, \infty) \to cl(\mathbb{R})$ such that $g(x) = \{ax\} \subseteq F(x)$ for all $x \in [0, \infty)$ by Corollary 1.9.

Example 1.11. Finally, let $H : [0, \infty) \to cl(\mathbb{R})$ be defined by $H(x) = [ax, ax + bx]$, where a, b are positive real numbers. Then, it follows easily that $2H\big(\frac{x+y}{2}\big) = H(x) + H(y)$ for all $x, y \in [0, \infty)$. However, there are two different additive mappings $g_1(x) := \{ax\}, g_2(x) := \{(a + b)x\}$ such that $g_1(x), g_2(x) \subseteq H(x)$ for all $x \in [0, \infty)$. In fact, one notes that either

(1.3) or (1.7) is not satisfied for the function H, and so one cannot apply Theorems 1.5 and 1.8 to this example.

Thus, we remark that the set-valued function $F(x) = [ax, ax + bx]$ has no Hyers–Ulam stability property for the set-valued Cauchy–Jensen additive functional equation.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2016R1D1A3B03930971). Correspondence should be addressed to Hark-Mahn Kim(hmkim@cnu.ac.kr) and Hwan-Yong Shin(hyshin31@cnu.ac.kr).

References

[1] T. Aoki : On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan **2**, 64–66 (1950)

[2] J.P. Aubin, H. Frankowska, : Set-valued analysis, in Modern Birkhäuser Classics. Birkhäuser, Boston (2008)

[3] H.-Y. Chu, S. K. Yoo, : On the Stability of the Generalized Quadratic Set-Valued Functional Equation. Journal of Computational Analysis and Applicartions. **20**, 1007-1020 (2016)

[4] J. Brzdęk, D. Popa, B. Xu,: Selections of set-valued maps satisfying a linear inclusion in a single variable. Nonlinear Analysis: Theory, Methods and Applications **74**, 324–330 (2011)

[5] P. Găvruta,: A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431–436 (1994)

[6] D.H. Hyers,: On the stability of the linear functional equation. Proc. Nat. Acad. Sci. **27**, 222–224 (1941)

[7] D. Kang,: Stabiltiy of generalized cubic set-valued functional equations Journal of Computational Analysis and Applicartions. **20**, 296-306 (2016)

[8] J. Lewin,: An Interactive Introduction to Mathematical Analysis. Cambridge University Press (2003)

[9] G. Lu, C. Park,: Hyers–Ulam stability of additive set-valued functional equations. Appl. Math. Lett. **24**, 1312–1316 (2011)

[10] H. Mohebi,: Topical functions and their properties in a class of ordered Banach spaces, in Continuous Optimization. Applied Optimization, Springer **99**, 343–361 (2005)

[11] H. Mohebi, H. Sadeghi, A.M. Rubinov, : Best approximation in a class of normed spaces with star-shaped cone. Current Numer. Funct. Anal. Optim. **27**, 411–436 (2006)

[12] K. Nikodem,: K-convex and K-concave set-valued functions. Z. K. Nr. 559 (1989)

[13] C. Park, D. O'Regon, R. Saadati,: Stability of some set-valued functional equations. Appl. Math. Lett. **24**, 1910–1914 (2011)

[14] T.M. Rassias,: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc. **72**, 297–300 (1978)

[15] S. Rezapour, R. Hamlbarani,: Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings". J. Math. Anal. Appl. **345**, 719–724 (2008)

[16] S.M. Ulam,: Problems in Modern Mathematics. Chapter 6 Wiley Interscience, New York (1964)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Hong-Mei Liang, Department of Mathematics, Qiqihar University, Qiqihar, 161006, China; Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hmliang124@126.com

Hark-Mahn Kim, Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hmkim@cnu.ac.kr

Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hyshin31@cnu.ac.kr

APPROXIMATE CAUCHY-JENSEN AND BI-QUADRATIC MAPPINGS IN 2-BANACH SPACES

WON-GIL PARK AND JAE-HYEONG BAE

ABSTRACT. In this paper, we obtain the stability of the Cauchy-Jensen and bi-quadratic functional equation

$$2f\left(x+y, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w),$$

$$\begin{aligned} f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) \\ = 4[f(x,z) + f(x,w) + f(y,z) + f(y,w)], \end{aligned}$$

respectively, in 2-Banach spaces.

1. Introduction

In 1940, Ulam [7] suggested the stability problem of functional equations concerning the stability of group homomorphisms: Let a group G and a metric group H with the metric ρ be given. For each $\varepsilon > 0$, the question is whether or not there is a $\delta > 0$ such that if $f : G \to H$ satisfies $\rho\big(f(xy), f(x)f(y)\big) < \delta$ for all $x, y \in G$, then there exists a group homomorphism $h : G \to H$ satisfying $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$.

We introduce some definitions on 2-Banach spaces [2], [3].

Definition 1. Let X be a real linear space with $\dim X \geq 2$ and $\|\cdot,\cdot\| : X^2 \to \mathbb{R}$ be a function. Then $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space* if the following conditions hold:
(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$,
(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$
for all $\alpha \in \mathbb{R}$ and $x, y, z \in X$. In this case, the function $\|\cdot,\cdot\|$ is called a *2-norm* on X.

Definition 2. Let $\{x_n\}$ be a sequence in a linear 2-normed space X. The sequence $\{x_n\}$ is said to *convergent* in X if there exits an element $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

1991 *Mathematics Subject Classification.* 39B52, 39B72.

Key words and phrases. linear 2-normed space, Cauchy-Jensen mapping, bi-quadratic mapping.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(grant number 2017028238).

Competing interests. The authors declare that they have no competing interests.

1

for all $y \in X$. In this case, we say that a sequence $\{x_n\}$ converges to the limit x, simply dented by $\lim_{n\to\infty} x_n = x$.

Definition 3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$ for all $y \in X$. For convenience, we will write $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for a Cauchy sequence $\{x_n\}$. A 2-*Banach space* is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.

In the following lemma, we obtain some basic properties in a linear 2-normed space which will be used to prove the stability results.

Lemma 4. ([1]) Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $x \in X$.
(a) If $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.
(b) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ for all $x, y, z \in X$.
(c) If a sequence $\{x_n\}$ is convergent in X, then $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for all $y \in X$.

Throughout this paper, let X be a normed space and Y a 2-Banach space. We introduce the definitions of Cauchy-Jensen and bi-quadratic mappings.

Definition 5. A mapping $f : X \times X \to Y$ is called a *Cauchy-Jensen mapping* if f satisfies the system of equations

$$\begin{aligned} f(x+y,z) &= f(x,z)+f(y,z), \\ 2f\big(x,\tfrac{y+z}{2}\big) &= f(x,y)+f(x,z). \end{aligned} \tag{1}$$

Definition 6. A mapping $f : X \times X \to Y$ is called *bi-quadratic* if f satisfies the system of equations

$$\begin{aligned} f(x+y,z)+f(x-y,z) &= 2f(x,z)+2f(y,z), \\ f(x,y+z)+f(x,y-z) &= 2f(x,y)+2f(x,z). \end{aligned} \tag{2}$$

For a mapping $f : X \times X \to Y$, consider the functional equations:

$$2f\left(x+y, \frac{z+w}{2}\right) = f(x,z)+f(x,w)+f(y,z)+f(y,w) \tag{3}$$

and

$$\begin{aligned} f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)& \\ = 4[f(x,z)+f(x,w)+f(y,z)+f(y,w)]&. \end{aligned} \tag{4}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) := axy + bx$ and $f(x, y) := ax^2y^2$ are solutions of (3) and (4), respectively.

In 2011, W.-G. Park [4] investigate approximate additive, Jensen and quadratic mappings in 2-Banach spaces. In this papaer, we also investigate Cauchy-Jensen and bi-quadratic mappings in 2-Banach spaces with different assumptions from [4].

2. Approximate Cauchy-Jensen mappings

Let $\varphi : X^5 \to [0,\infty)$ be a function satisfying

$$\begin{aligned}\tilde{\varphi}(x,y,z,w,s) := \sum_{j=0}^{\infty} \frac{1}{6^{j+1}} \Big[&\varphi(2^j x, 2^j y, 3^j z, 3^j w, s) + 2\varphi(2^j x, 2^j y, -3^j z, 3^j w, s) \\ &+ \varphi(2^j x, 2^j y, -3^j z, 3^{j+1} w, s) + \frac{1}{2}\varphi(2^j x, 2^j y, 3^{j+1} z, 3^{j+1} w, s) \\ &+ 3\varphi(2^j x, 2^j y, 3^j z, -3^j w, s) + 2\|f(2^{j+1}x, 0), t\| + 5\|f(x,0), t\| \Big] < \infty \end{aligned} \tag{5}$$

for all $x, y, z, w, s \in X$, where $t = f(s)$.

Theorem 7. *Suppose that $f : X \times X \to Y$ is a surjective mapping such that*

$$\left\| 2f\left(x+y, \frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w), t \right\| \le \varphi(x,y,z,w,s) \tag{6}$$

for all $x, y, z, w, s \in X$, where $t = f(s)$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - f(x,0) - F(x,y), t\| \le \tilde{\varphi}(x,x,y,y,s) \tag{7}$$

for all $x, y, s \in X$, where $t = f(s)$.

Proof. Let $t = f(s)$. Letting $y = x$ in (6), we gain

$$\left\| 2f\left(2x, \frac{z+w}{2}\right) - 2f(x,z) - 2f(x,w), t \right\| \le \varphi(x,x,z,w,s) \tag{8}$$

for all $x, z, w, s \in X$. Putting $w = -z$ in (8), we get

$$\| -2f(2x,0) + 2f(x,z) + 2f(x,-z), t\| \le \varphi(x,x,z,-z,s) \tag{9}$$

for all $x, z, s \in X$. Replacing z by $-z$ and w by $-z$ in (8), we have

$$\|f(2x,-z) - 2f(x,-z), t\| \le \frac{1}{2}\varphi(x,x,-z,-z,s) \tag{10}$$

for all $x, z, s \in X$. By (9) and (10),

$$\|f(2x,-z) + 2f(x,z) - 2f(2x,0), t\| \le \frac{1}{2}\varphi(x,x,-z,-z,s) + \varphi(x,x,z,-z,s) \tag{11}$$

for all $x, z, s \in X$. Setting $w = -3z$ in (8),

$$\|2f(2x,-z) - 2f(x,z) - 2f(x,-3z), t\| \le \varphi(x,x,z,-3z,s)$$

for all $x, z, s \in X$. By (11) and the above inequality,

$$\|6f(x,z) + 2f(x,-3z) - 4f(2x,0), t\| \le \varphi(x,x,-z,-z,s) + 2\varphi(x,x,z,-z,s) + \varphi(x,x,z,-3z,s) \tag{12}$$

for all $x, z, s \in X$. Replacing z by $3z$ in (10),

$$\|f(2x,-3z) - 2f(x,-3z), t\| \le \frac{1}{2}\varphi(x,x,-3z,-3z,s)$$

for all $x, z, s \in X$. By (12) and the above inequality,

$$\begin{aligned}&\|6f(x,z)+f(2x,-3z)-4f(2x,0),t\|\\&\quad\le\varphi(x,x,-z,-z,s)+2\varphi(x,x,z,-z,s)+\varphi(x,x,z,-3z,s)+\frac{1}{2}\varphi(x,x,-3z,-3z,s)\end{aligned}$$

for all $x, z, s \in X$. Replacing z by $-z$ in the above inequality,

$$\begin{aligned}&\|6f(x,-z)+f(2x,3z)-4f(2x,0),t\|\\&\quad\le\varphi(x,x,z,z,s)+2\varphi(x,x,-z,z,s)+\varphi(x,x,-z,3z,s)+\frac{1}{2}\varphi(x,x,3z,3z,s)\end{aligned}$$

for all $x, z, s \in X$. By (9) and the above inequality,

$$\begin{aligned}&\|6f(x,z)-f(2x,3z)-2f(2x,0),t\|\\&\quad\le\varphi(x,x,z,z,s)+2\varphi(x,x,-z,z,s)+\varphi(x,x,-z,3z,s)+\frac{1}{2}\varphi(x,x,3z,3z,s)+3\varphi(x,x,z,-z,s)\end{aligned}$$

for all $x, z, s \in X$. Replacing x by 2^jx and z by 3^jy in the above inequality and dividing 6^{j+1},

$$\begin{aligned}&\left\|\frac{1}{6^j}f(2^jx,3^jy)-\frac{1}{6^{j+1}}f(2^{j+1}x,3^{j+1}y)-\frac{2}{6^{j+1}}f(2^{j+1}x,0),t\right\|\\&\quad\le\frac{1}{6^{j+1}}\Big[\varphi(2^jx,2^jx,3^jy,3^jy,s)+2\varphi(2^jx,2^jx,-3^jy,3^jy,s)\\&\qquad+\varphi(2^jx,2^jx,-3^jy,3^{j+1}y,s)+\frac{1}{2}\varphi(2^jx,2^jx,3^{j+1}y,3^{j+1}y,s)+3\varphi(2^jx,2^jx,3^jy,-3^jy,s)\Big]\end{aligned}$$

for all $x, y, s \in X$. For given integers $l, m (0 \le l < m)$,

$$\begin{aligned}(13)\quad&\left\|\frac{1}{6^l}f(2^lx,3^ly)-\frac{1}{6^m}f(2^mx,3^my)-\sum_{j=l}^{m-1}\frac{2}{6^{j+1}}f(2^{j+1}x,0),t\right\|\\&\quad\le\sum_{j=l}^{m-1}\frac{1}{6^{j+1}}\Big[\varphi(2^jx,2^jx,3^jy,3^jy,s)+2\varphi(2^jx,2^jx,-3^jy,3^jy,s)\\&\qquad+\varphi(2^jx,2^jx,-3^jy,3^{j+1}y,s)+\frac{1}{2}\varphi(2^jx,2^jx,3^{j+1}y,3^{j+1}y,s)+3\varphi(2^jx,2^jx,3^jy,-3^jy,s)\Big]\end{aligned}$$

for all $x, y, s \in X$. By (14) and (13), the sequence $\{\frac{1}{6^j}f(2^jx,3^jy)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{6^j}f(2^jx,3^jy)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x,y):=\lim_{j\to\infty}\frac{1}{6^j}f(2^jx,3^jy)$$

for all $x, y \in X$.

By (6),

$$\begin{aligned}&\left\|\frac{1}{6^j}f\left(2^j(x+y),\frac{3^j(z+w)}{2}\right)-\frac{1}{6^j}f(2^jx,3^jz)-\frac{1}{6^j}f(2^jx,3^jw)\right.\\&\left.-\frac{1}{6^j}f(2^jy,3^jz)-\frac{1}{6^j}f(2^jy,3^jw),t\right\|\le\frac{1}{6^j}\varphi(2^jx,2^jy,3^jz,3^jw,s)\end{aligned}$$

for all $x,y,z,w,s\in X$. Letting $j\to\infty$ and using (14), F satisfies (3). By Theorem 4 in [6], F is a Cauchy-Jensen mapping. Setting $l=0$ and taking $m\to\infty$ in (13), one can obtain the inequality (7). If $G:X\times X\to Y$ is another Cauchy-Jensen mapping satisfying (7),

$$\begin{aligned}\|F(x,y)-G(x,y),t\| &= \frac{1}{6^n}\|F(2^nx,3^ny)-G(2^nx,3^ny),t\|\\ &\le \frac{1}{6^n}\|F(2^nx,3^ny)-f(2^nx,0)-f(2^nx,3^ny),t\|\\ &\quad+\frac{1}{6^n}\|f(2^nx,2^ny)+f(2^nx,0)-G(2^nx,3^ny),t\|\\ &\le \frac{2}{6^n}\tilde{\varphi}(2^nx,2^nx,3^ny,3^ny,s)\to 0 \text{ as } n\to\infty\end{aligned}$$

for all $x,y,s\in X$. Hence the mapping F is the unique Cauchy-Jensen mapping, as desired. □

Corollary 8. *Let $\varepsilon>0$. Suppose that $f:X\times X\to Y$ is a surjective mapping satisfying*

$$\left\|2f\left(x+y,\frac{z+w}{2}\right)-f(x,z)-f(x,w)-f(y,z)-f(y,w),t\right\|\le\varepsilon,$$

for all $x,y,z,w,s\in X$, where $t=f(s)$ and $\varphi_\varepsilon(x,s):=\frac{3}{2}\varepsilon+\|f(x,0),t\|+\sum_{j=0}^\infty\frac{2}{6^{j+1}}\|f(2^{j+1}x,0),t\|<\infty$ for all $x,s\in X$, where $t=f(s)$. Then there exists a unique Cauchy-Jensen mapping $F:X\times X\to Y$ such that

$$\|f(x,y)-f(x,0)-F(x,y),t\|\le\varphi_\varepsilon(x,s)$$

for all $x,y,s\in X$, where $t=f(s)$.

Proof. Taking $\varphi(x,y,z,w,s):=\varepsilon$ in Theorem 7, we have

$$\tilde{\varphi}(x,x,y,y,s)=\frac{3}{2}\varepsilon+\|f(x,0),t\|+\sum_{j=0}^{\infty}\frac{2}{6^{j+1}}\|f(2^{j+1}x,0),t\|=\varphi_\varepsilon(x,s)$$

for all $x,y,z,w,s\in X$, where $t=f(s)$. □

3. Approximate bi-quadratic mappings

From now on, let $\varphi:X^5\to[0,\infty)$ be a function satisfying

$$\tilde{\varphi}(x,y,z,w,s):=\sum_{j=0}^{\infty}\frac{1}{16^{j+1}}\varphi(2^jx,2^jy,2^jz,2^jw,s)<\infty \qquad (14)$$

for all $x,y,z,w,s\in X$.

Theorem 9. *Let $f : X \times X \to Y$ be a surjective mapping such that*

$$\begin{aligned}&\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)\\ &-4[f(x,z)-f(x,w)-f(y,z)-f(y,w)],t\| \le \varphi(x,y,z,w,s)\end{aligned} \tag{15}$$

and let $f(x,0)=0$ and $f(0,y)=0$ for all $x,y,z,w,s \in X$, where $t=f(s)$. Then there exists a unique bi-quadratic mapping $F : X \times X \to Y$ such that

$$\|f(x,y)-F(x,y),t\| \le \tilde{\varphi}(x,x,y,y,s) \tag{16}$$

for all $x,y,s \in X$, where $t=f(s)$.

Proof. Let $t=f(s)$. Putting $y=x$ and $w=z$ in (15), we have

$$\left\|f(x,z)-\frac{1}{16}f(2x,2z),t\right\| \le \frac{1}{16}\varphi(x,x,z,z,s)$$

for all $x,z,s \in X$. Thus we obtain

$$\left\|\frac{1}{16^j}f(2^jx,2^jz)-\frac{1}{16^{j+1}}f(2^{j+1}x,2^{j+1}z),t\right\| \le \frac{1}{16^{j+1}}\varphi(2^jx,2^jx,2^jz,2^jz,s)$$

for all $x,z,s \in X$ and all j. Replacing z by y in the above inequality, we see that

$$\left\|\frac{1}{16^j}f(2^jx,2^jy)-\frac{1}{16^{j+1}}f(2^{j+1}x,2^{j+1}y),t\right\| \le \frac{1}{16^{j+1}}\varphi(2^jx,2^jx,2^jy,2^jy,s)$$

for all $x,y,s \in X$ and all j. For given integers $l,m (0 \le l < m)$, we get

$$\left\|\frac{1}{16^l}f(2^lx,2^ly)-\frac{1}{16^m}f(2^mx,2^my),t\right\| \le \sum_{j=l}^{m-1}\frac{1}{16^{j+1}}\varphi(2^jx,2^jx,2^jy,2^jy,s) \tag{17}$$

for all $x,y,s \in X$. By (17), the sequence $\{\frac{1}{16^j}f(2^jx,2^jy)\}$ is a Cauchy sequence for all $x,y \in X$. Since Y is complete, the sequence $\{\frac{1}{16^j}f(2^jx,2^jy)\}$ converges for all $x,y \in X$. Define $F : X \times X \to Y$ by

$$F(x,y) := \lim_{j\to\infty}\frac{1}{16^j}f(2^jx,2^jy)$$

for all $x,y \in X$. By (15), we have

$$\begin{aligned}&\bigg\|\frac{1}{16^j}f\big(2^j(x+y),2^j(z+w)\big)+\frac{1}{16^j}f\big(2^j(x+y),2^j(z-w)\big)\\ &+\frac{1}{16^j}f\big(2^j(x-y),2^j(z+w)\big)+\frac{1}{16^j}f\big(2^j(x-y),2^j(z-w)\big)\\ &-\frac{4}{16^j}f(2^jx,2^jz)-\frac{4}{16^j}f(2^jx,2^jw)-\frac{4}{16^j}f(2^jy,2^jz)-\frac{4}{16^j}f(2^jy,2^jw),t\bigg\|\\ &\le \frac{1}{16^j}\varphi(2^jx,2^jy,2^jz,2^jw,s)\end{aligned}$$

for all $x,y,z,w,s \in X$ and all j. Letting $j \to \infty$ and using (14), we see that F satisfies (4). By Theorem 4 in [5], we obtain that F is bi-quadratic. Setting $l=0$ and taking $m \to \infty$ in (17), one

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

can obtain the inequality (16). If $G : X \times X \to Y$ is another bi-quadratic mapping satisfying (16), we obtain

$$\begin{aligned}
&\|F(x,y) - G(x,y), t\| \\
&= \frac{1}{16^n}\|F(2^n x, 2^n y) - G(2^n x, 2^n y), t\| \\
&\le \frac{1}{16^n}\|F(2^n x, 2^n y) - f(2^n x, 2^n y), t\| + \frac{1}{16^n}\|f(2^n x, 2^n y) - G(2^n x, 2^n y), t\| \\
&\le \frac{2}{16^n}\tilde{\varphi}(2^n x, 2^n x, 2^n y, 2^n y, s) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

for all $x, y, s \in X$. Hence the mapping F is the unique bi-quadratic mapping, as desired. □

Corollary 10. *Let $\varepsilon > 0$. Suppose that $f : X \times X \to Y$ is a surjective mapping satisfying*

$$\begin{aligned}
\|f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) \\
-4[f(x,z) - f(x,w) - f(y,z) - f(y,w)], t\| \le \varepsilon,
\end{aligned}$$

for all $x, y, z, w, s \in X$, where $t = f(s)$. Then there exists a unique bi-quadratic mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y), t\| \le \frac{1}{15}\varepsilon$$

for all $x, y, s \in X$, where $t = f(s)$.

Proof. Taking $\varphi(x, y, z, w, s) := \varepsilon$ in Theorem 9, we have $\tilde{\varphi}(x, x, y, y, s) = \frac{1}{15}\varepsilon$ for all $x, y, z, w, s \in X$, where $t = f(s)$. □

References

[1] H.-Y. Chu, A. Kim and J. Park, *On the Hyers-Ulam stabilities of functional equations on n-Banach spaces*, Math. Nachr. **289** (2016), 1177–1188.

[2] S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–148.

[3] S. Gähler, *Lineare 2-normierte Räumen*, Math. Nachr. **28** (1964), 1–43.

[4] W.-G. Park, *Approximate additive mappings in 2-Banach spaces and related topics*, J. Math. Anal. Appl. **376** (2011), 193–202.

[5] W.-G. Park and J.-H. Bae, *On a bi-quadratic functional equation and its stability*, Nonlinear Anal. **62** (2005), 643–654.

[6] W.-G. Park and J.-H. Bae, *On a Cauchy-Jensen functional equation and its stability*, J. Math. Anal. Appl. **323** (2006), 634–643.

[7] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1960.

Won-Gil Park, Department of Mathematics Education, College of Education, Mokwon University, Daejeon 35349, Republic of Korea

E-mail address: wgpark@mokwon.ac.kr

Jae-Hyeong Bae, Humanitas College, Kyung Hee University, Yongin 17104, Republic of Korea

E-mail address: jhbae@khu.ac.kr

Birkhoff Normal Forms, KAM theory and continua of periodic points for certain planar system

M. R. S. Kulenović[†1,2] E. Pilav[‡] and N. Mujić[‡]

[†]Department of Mathematics
University of Rhode Island, Kingston, Rhode Island 02881-0816, USA

[‡]Department of Mathematics
University of Sarajevo, Sarajevo, Bosnia and Herzegovina

Abstract. By using the KAM theory and time reversal symmetries we investigate the stability of the equilibrium solutions of the system:

$$\begin{cases} x_{n+1} &= \frac{a}{x_n+y_n} \\ y_{n+1} &= \frac{x_n}{y_n} \end{cases}, \quad n = 0, 1, 2, \ldots,$$

where the parameter $a > 0$, and initial conditions x_0 and y_0 are positive numbers. We obtain the Birkhoff normal form for this system and prove the existence of periodic points with arbitrarily large periods in every neighborhood of the unique positive equilibrium. We also use the time reversal symmetry method to find effectively some feasible periods and the corresponding periodic orbits. Finally, we give computational procedure for finding an infinite number of periodic solutions with the given period. The second order difference equation obtained by eliminating x_n from this system is an equation of the type $y_{n+1} = f(y_n, y_{n-1})$, where f is decreasing in both variables. Such equation can be embedded into fifth order difference equation which is increasing in all its arguments and it exhibits chaotic behavior.

Keywords. area preserving map, Birkhoff normal form, difference equation, KAM theory, periodic solutions, symmetry, time reversal, competitive map, global stable manifold, monotonicity, period-two solution.

AMS 2010 Mathematics Subject Classification: 37E40, 37J40, 37N25, 39A28, 39A30

1 Introduction

In this paper we consider the following rational system of difference equations

$$\begin{cases} x_{n+1} &= \frac{a}{x_n+y_n} \\ y_{n+1} &= \frac{x_n}{y_n} \end{cases}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

and the corresponding equation

$$y_{n+1} = \frac{a}{y_n y_{n-1}(1+y_n)}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where the parameter $a > 0$, and initial conditions x_0 and y_0 are positive numbers. System (1) was first considered in [6], where boundedness of all its solutions was proved using the invariant. We will use this invariant in Section 3 to prove the stability of the unique equilibrium. Equation (2) gives an example of second order difference equation where transition function decreases in both variables and yet equation exhibits complicated dynamics. First such example was given in [5]. We will use similar techniques as in [5] with the addition of the new computational procedure from [7], which uses an invariant of the system to find effectively continua of periodic solutions of certain feasible periods.

We will show that the corresponding map can be transformed into an area preserving map and using Birkhoff Normal form we will apply the KAM theorem to prove stability of the unique positive equilibrium and the existence of periodic points with arbitrarily large period in every neighborhood of the unique positive equilibrium. In addition, we will prove that the corresponding map is conjugate to its inverse map through the involution map and then use this conjugacy to find some feasible periods of this map. The method of invariants for proving stability of the equilibrium solution for all values of parameter a will be used along with Morse's lemma to prove that the level sets of the invariants are diffeomorphic with circles. This method was used successfully in [11, 12] and the KAM theory was used for the same objective in [8, 10, 13, 14].

Let T be the map associated to the system (1), i.e.,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a}{x+y} \\ \frac{x}{y} \end{pmatrix}. \tag{3}$$

[1]Corresponding author, *e-mail: mkulenovic@uri.edu*

[2]Partially supported by Maitland P. Simmons Foundation

The map (3) has the unique fixed point $(\bar{y}^2, \bar{y})$ in the positive quadrant, where

$$\bar{y}^3(\bar{y}+1) = a.$$

An invertible mapping T is area preserving if the area of $T(A)$ coincides with the area of A for all measurable subsets A [9, 12, 18]. We claim that in logarithmic coordinates, i.e., $u = \ln(x/\bar{y}^2)$, $v = \ln(y/\bar{y})$ the map (3) is area preserving.

Lemma 1 *The map* (3) *is area preserving in the logarithmic coordinates.*

Proof. The Jacobian matrix of the corresponding transformation T is

$$J_T(x,y) = \begin{pmatrix} -\frac{a}{(x+y)^2} & -\frac{a}{(x+y)^2} \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix} \tag{4}$$

with

$$det J_T(x,y) = \frac{a}{y^2(x+y)}.$$

We substitute $u = \ln(x/\bar{y}^2)$, $v = \ln(y/\bar{y})$ and rewrite the map in (u,v) coordinates to obtain the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \ln a - 3\ln\bar{y} - \ln(\bar{y}e^u + e^v) \\ u - v \end{pmatrix} \tag{5}$$

The Jacobian of this transformation is

$$J(u,v) = \begin{pmatrix} -\frac{e^u\bar{y}}{e^u\bar{y}+e^v} & -\frac{e^v}{e^u\bar{y}+e^v} \\ 1 & -1 \end{pmatrix} \tag{6}$$

It is easy to see that $det J(u,v) = 1$. □

A point $(\bar{x}, \bar{y})$ is a fixed point of T if $T(\bar{x},\bar{y}) = (\bar{x},\bar{y})$. A fixed point is elliptic if the eigenvalues of $J_T(\bar{x},\bar{y})$ form a complex conjugate pair $\lambda, \bar{\lambda}$ on the unit circle and is hyperbolic if the the modulus of the eigenvalues is different from 1, see [9, 12].

Lemma 2 *The map T in the (x,y) coordinates has elliptic fixed point $(\bar{y}^2, \bar{y})$. In the logarithmic coordinates, the corresponding fixed points is* $(0,0)$.

Proof. For the fixed points in (x, y) coordinates, solving $a/(\bar{x}+\bar{y}) = x$ and $\bar{x}/\bar{y} = \bar{y}$ yields the fixed points $(\bar{y}^2, \bar{y})$ where $\bar{y}^3(\bar{y}+1) = a$. Evaluating the Jacobian matrix (4) of T at $(\bar{y}^2, \bar{y})$ gives

$$J_T(\bar{y}^2, \bar{y}) = \begin{pmatrix} -\frac{a}{(\bar{y}^2+\bar{y})^2} & -\frac{a}{(\bar{y}^2+\bar{y})^2} \\ \frac{1}{\bar{y}} & -1 \end{pmatrix} \tag{7}$$

By using $a = \bar{y}^3(1+\bar{y})$ we obtain that the eigenvalues of $J_T(\bar{y}^2,\bar{y})$ are λ and $\bar{\lambda}$ where

$$\lambda = \frac{-1 - 2\bar{y} + i\sqrt{4\bar{y}+3}}{2\bar{y}+2}. \tag{8}$$

It is easy to see that $|\lambda| = 1$ and so $(\bar{y}^2, \bar{y})$ is an elliptic fixed point.

Under the logarithmic coordinate change $(x,y) \to (u,v)$, the fixed point $(\bar{y}^2, \bar{y})$ becomes $(0,0)$. Evaluating the Jacobian matrix (6) of T at $(0,0)$ gives

$$J(0,0) = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix} \tag{9}$$

with eigenvalues which are given by (8). □

This paper is organized as follows. In section 2 the KAM theorem is explained in some detail and Birkhoff normal form for map T is derived. By using the KAM theory stability of the unique equilibrium and existence of infinite number of periodic solution is proven except for a single value of the parameter a. Section 3 uses the invariant of the equation (2) in proving stability for all values of a. In section 4 by using symmetries it is shown that the map T is conjugate to its inverse through an involution. Then by using time reversal symmetry method some feasible periods and corresponding orbits of the map T are found. Finally in Section 5 we use the recent method of Gasull and *al.* [7] to find continua of p-periodic points lying on the level sets of the invariant I. The method is based on use of resultants and is implemented by *Mathematica*. The special attention is given to period-seven solution.

2 The KAM theory and Birkhoff normal form

The KAM Theorem asserts that in any sufficiently small neighborhood of a non degenerate elliptic fixed point of a smooth area-preserving map there exists many invariant closed curves.We explain this theorem in some detail. Consider a smooth, area-preserving mapping $(x, y) \to T(x, y)$ of the plane that has $(0, 0)$ as an elliptic fixed point. After a linear transformation one can put the map in the form

$$z \to \lambda z + g(z, \bar{z})$$

where λ is the eigenvalue of the elliptic fixed point, $z = x + iy$ and $\bar{z} = x - iy$ are complex variables, and g vanishes with its derivative at $z = 0$. Assume that the eigenvalue λ of the elliptic fixed point satisfies the non-resonance condition $\lambda^k \neq 1$ for $k = 1, \ldots, q$, for some $q \geq 4$. Then Birkhoff showed that there exist new, canonical complex coordinates $(\zeta, \bar{\zeta})$ relative to which the mapping takes the normal form

$$\zeta \to \lambda \zeta e^{i\tau(\zeta\bar{\zeta})} + h(\zeta, \bar{\zeta})$$

in a neighborhood of the elliptic fixed point, where $\tau(\zeta\bar{\zeta}) = \tau_1|\zeta|^2 + \ldots + \tau_s|\zeta|^{2s}$ is a real polynomial, $s = [(q-2)/2]$, and h vanishes with its derivatives up to order $q-1$. The numbers $\tau_1, \ldots, \tau_s$ are called twist coefficients. Consider an invariant annulus $\epsilon < |\zeta| < 2\epsilon$ in a neighborhood of the elliptic fixed point, for ϵ a very small positive number. Note that under the neglect of the remainder h, the normal form approximation $\zeta \to \lambda \zeta e^{i\tau(\zeta\bar{\zeta})}$ leaves invariant all circles $|\zeta|^2 = const$. The motion restricted to each of these circles is a rotation by some angle. Also note that if at least one of the twist coefficients τ_j is nonzero, the angle of rotation will vary from circle to circle. A radial line through the fixed point will undergo twisting under the mapping. The KAM theorem (Moser's twist theorem) says that, under the addition of the remainder term, most of these invariant circles will survive as invariant closed curves under the full map.

Theorem 1 *Assuming that $\tau(\zeta\bar{\zeta})$ is not identically zero and ϵ is sufficiently small, then the map T has a set of invariant closed curves of positive Lebesque measure close to the original invariant circles. Moreover the relative measure of the set of surviving invariant curves approaches full measure as ϵ approaches 0. The surviving invariant closed curves are filled with dense irrational orbits.*

The KAM theorem requires that the elliptic fixed point be non-resonant and non degenerate. Note that for $q = 4$ the non-resonance condition $\lambda^k \neq 1$ requires that $\lambda \neq \pm 1$ or $\lambda \neq \pm i$. The above normal form yields the approximation

$$\zeta \to \lambda\zeta + c_1\zeta^2\bar{\zeta} + O(|\zeta|^4)$$

with $c_1 = i\lambda\tau_1$ and τ_1 being the first twist coefficient. We will call an elliptic fixed point non-degenerate if $\tau_1 \neq 0$.

Consider a general map T that has a fixed point at the origin with complex eigenvalues λ and $\bar{\lambda}$ satisfying $|\lambda| = 1$ and $Im(\lambda) \neq 0$. By putting the linear part of such a map into Jordan Canonical form, we may assume T to have the following form near the origin

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix} \tag{10}$$

One can now switch to the complex coordinates $z = x_1 + ix_2$ to obtain the complex form of the system

$$z \to \lambda z + \xi_{20}z^2 + \xi_{11}z\bar{z} + \xi_{02}\bar{z}^2 + \xi_{30}z^3 + \xi_{21}z^2\bar{z} + \xi_{12}z\bar{z}^2 + \xi_{03}\bar{z}^3 + O(|z|^4)$$

The coefficient c_1 can be computed directly using the formula below derived by Wan in the context of Hopf bifurcation theory [19]. In [16] it is shown that when one uses area-preserving coordinate changes this formula by Wan yields the twist coefficient τ_1 that is used to verify the non-degeneracy condition necessary to apply the KAM theorem. We use the formula:

$$c_1 = \frac{\xi_{20}\xi_{11}(\bar{\lambda} + 2\lambda - 3)}{(\lambda^2 - \lambda)(\bar{\lambda} - 1)} + \frac{|\xi_{11}|^2}{1 - \bar{\lambda}} + \frac{2|\xi_{02}|^2}{\lambda^2 - \bar{\lambda}} + \xi_{21} \tag{11}$$

where

$$\xi_{20} = \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} + 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} - 2(g_1)_{x_1x_2}\right]\right\},$$

$$\xi_{11} = \frac{1}{4}\left\{(g_1)_{x_1x_1} + (g_1)_{x_2x_2} + i\left[(g_2)_{x_1x_1} + (g_2)_{x_2x_2}\right]\right\},$$

$$\xi_{02} = \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} - 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} + 2(g_1)_{x_1x_2}\right]\right\},$$

$$\xi_{21} = \frac{1}{16}\left\{(g_1)_{x_1x_1x_1} + (g_1)_{x_1x_2x_2} + (g_2)_{x_1x_1x_2} + (g_2)_{x_2x_2x_2} + i\left[(g_2)_{x_1x_1x_1} + (g_2)_{x_1x_2x_2} - (g_1)_{x_1x_1x_2} - (g_1)_{x_2x_2x_2}\right]\right\}.$$

Theorem 2 *The elliptic fixed point $(0,0)$, in the (u,v) coordinates, is non-degenerate for $a \neq \frac{3}{16}$ and non-resonant for $a > 0$.*

Proof.
Let F be the function defined by

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \ln a - 3\ln\bar{y} - \ln(\bar{y}e^u + e^v) \\ u - v \end{pmatrix}. \tag{12}$$

Then F has the unique elliptic fixed point $(0,0)$. The Jacobian matrix of F is given by

$$J_F(u,v) = \begin{pmatrix} -\frac{e^u\bar{y}}{e^u\bar{y}+e^v} & -\frac{e^v}{e^u\bar{y}+e^v} \\ 1 & -1 \end{pmatrix}. \tag{13}$$

At $(0,0)$, $J_F(u,v)$ has the form

$$J_0 = J_F(0,0) = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix}. \tag{14}$$

The eigenvalues of (14) are λ and $\bar{\lambda}$ where

$$\lambda = \frac{-1-2y+i\sqrt{4y+3}}{2y+2}. \tag{15}$$

One can prove that

$$\begin{aligned} &|\lambda| = 1, \\ \lambda^2 =& \frac{2\bar{y}^2-1}{2(\bar{y}+1)^2} - \frac{i(2\bar{y}+1)\sqrt{4\bar{y}+3}}{2(\bar{y}+1)^2}, \\ \lambda^3 =& \frac{\bar{y}((3-2\bar{y})\bar{y}+6)+2}{2(\bar{y}+1)^3} + \frac{i\bar{y}\sqrt{4\bar{y}+3}\,(3\bar{y}+2)}{2(\bar{y}+1)^3}, \\ \lambda^4 =& \frac{2\bar{y}(\bar{y}((\bar{y}-4)\bar{y}-8)-4)-1}{2(\bar{y}+1)^4} - \frac{i(2\bar{y}+1)\sqrt{4\bar{y}+3}\,(2\bar{y}^2-1)}{2(\bar{y}+1)^4}, \end{aligned} \tag{16}$$

from which follows that $\lambda^k \neq 1$ for $k = 1,2,3,4$ and $a > 0$.
Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(\delta,u,v) \\ f_2(\delta,u,v) \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} f_1(\delta,u,v) =& -\ln(e^u\bar{y}+e^v) + \frac{u\bar{y}}{\bar{y}+1} + \frac{v}{\bar{y}+1} - 3\ln\bar{y} + \ln a \\ f_2(\delta,u,v) =& 0. \end{aligned} \tag{18}$$

The system $(u_{n+1}, v_{n+1}) = F(u_n, v_n)$ takes the form

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n,v_n) \\ f_2(u_n,v_n) \end{pmatrix}, \tag{19}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P\begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix},$$

where

$$P = \frac{1}{\sqrt{D}}\begin{pmatrix} \frac{1}{2\bar{y}+2} & -\frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2} \\ 1 & 0 \end{pmatrix}$$

and

$$P^{-1} = \sqrt{D}\begin{pmatrix} 0 & 1 \\ -\frac{2\bar{y}+2}{\sqrt{4\bar{y}+3}} & \frac{1}{\sqrt{4\bar{y}+3}} \end{pmatrix},$$

with

$$D = \frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2}.$$

Then the system $(u_{n+1}, v_{n+1}) = F(u_n, v_n)$ becomes

$$\begin{pmatrix} \tilde{u}_{n+1} \\ \tilde{v}_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{-2\bar{y}-1}{2\bar{y}+2} & -\frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2} \\ \frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2} & \frac{-2\bar{y}-1}{2\bar{y}+2} \end{pmatrix}\begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix} + P^{-1}H\left(P\begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix}\right), \tag{20}$$

where

$$H\begin{pmatrix}u\\v\end{pmatrix} := \begin{pmatrix}f_1(u,v)\\f_2(u,v)\end{pmatrix}.$$

Let

$$G\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}g_1(u,v)\\g_2(u,v)\end{pmatrix} = P^{-1}H\left(P\begin{pmatrix}u\\v\end{pmatrix}\right).$$

The straightforward calculation yields

$$\begin{aligned} g_1(u,v) =& 0\\ g_2(u,v) =& -\frac{1}{\sqrt{D}}\ln\left(\frac{a}{\bar{y}^3}\right) - \frac{\bar{y}\left(u-2Dv\left(\bar{y}+1\right)\right)}{2D\left(\bar{y}+1\right)^2} + \frac{1}{\sqrt{D}}\ln\left(\bar{y}e^{\frac{u-2Dv(\bar{y}+1)}{2\sqrt{D}(\bar{y}+1)}} + e^{\frac{u}{\sqrt{D}}}\right) - \frac{u}{D\left(\bar{y}+1\right)}. \end{aligned} \tag{21}$$

By straightforward calculation we obtain that

$$\begin{aligned} \xi_{20}|_{u=v=0} =& \frac{\bar{y}\left(2\bar{y}\left(\sqrt{4\bar{y}+3}+i\bar{y}\right)+\sqrt{4\bar{y}+3}-i\right)}{8\left(\bar{y}+1\right)^3\sqrt{D\left(4\bar{y}+3\right)}},\\ \xi_{11}|_{u=v=0} =& \frac{i\bar{y}}{2\left(\bar{y}+1\right)\sqrt{D\left(4\bar{y}+3\right)}},\\ \xi_{02}|_{u=v=0} =& \frac{i\bar{y}\left(2\bar{y}\left(\bar{y}+i\sqrt{4\bar{y}+3}\right)+i\sqrt{4\bar{y}+3}-1\right)}{8\left(\bar{y}+1\right)^3\sqrt{D\left(4\bar{y}+3\right)}},\\ \xi_{21}|_{u=v=0} =& \frac{\left(\bar{y}-1\right)\bar{y}\left(2i\bar{y}+\sqrt{4\bar{y}+3}+i\right)}{16D\left(\bar{y}+1\right)^3\sqrt{4\bar{y}+3}}. \end{aligned} \tag{22}$$

Since

$$\begin{aligned} \xi_{21}\xi_{11} =& \frac{i\bar{y}^2\left(2\bar{y}\left(\sqrt{4\bar{y}+3}+i\bar{y}\right)+\sqrt{4\bar{y}+3}-i\right)}{16D\left(\bar{y}+1\right)^4\left(4\bar{y}+3\right)},\\ \xi_{11}\overline{\xi_{11}} =& \frac{\bar{y}^2}{4D\left(\bar{y}+1\right)^2\left(4\bar{y}+3\right)},\\ \xi_{02}\overline{\xi_{02}} =& \frac{\bar{y}^2}{16D\left(\bar{y}+1\right)^2\left(4\bar{y}+3\right)}. \end{aligned} \tag{23}$$

the simplification of the expression for c_1 yields

$$\begin{aligned} c_1 =& \frac{\xi_{20}\xi_{11}(\bar{\lambda}+2\lambda-3)}{(\lambda^2-\lambda)(\bar{\lambda}-1)} + \frac{|\xi_{11}|^2}{1-\bar{\lambda}} + \frac{2|\xi_{02}|^2}{\lambda^2-\bar{\lambda}} + \xi_{21}\\ =& \frac{\bar{y}\left(2\bar{y}-1\right)\left(2\bar{y}+2\right)\left(2i\bar{y}+\sqrt{4\bar{y}+3}+i\right)}{8\left(\bar{y}+1\right)^2\left(4\bar{y}+3\right)^2} \end{aligned} \tag{24}$$

One can prove that

$$\tau_1 = -i\bar{\lambda}c_1 = -\frac{\bar{y}\left(2\bar{y}-1\right)}{2\left(4\bar{y}+3\right)^2},$$

which implies that $\tau_1 \neq 0$ for $a \neq \frac{3}{16}$ since $\bar{y}^2(1+\bar{y}) = a$.

□

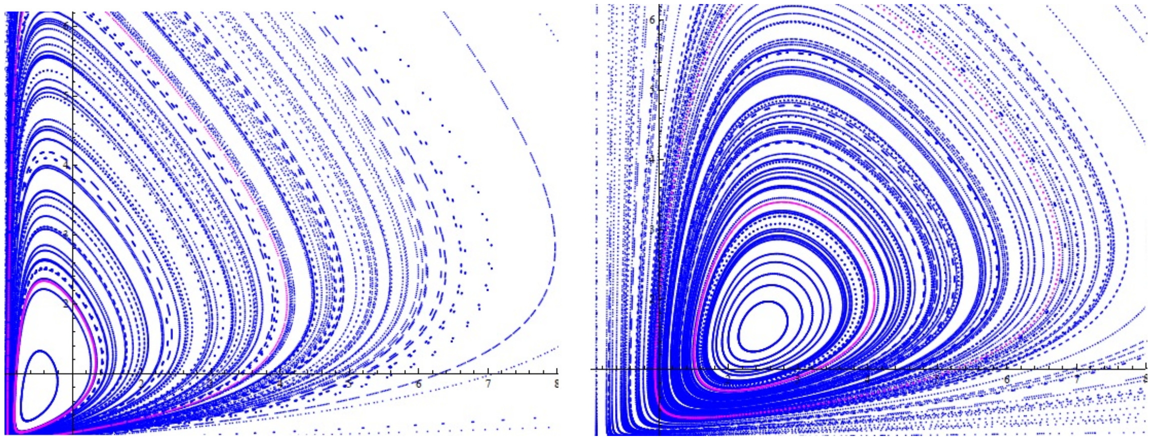
The following result is a consequence of Moser's twist map theorem [8, 15, 17, 18].

Theorem 3 *Let T be a map (3) associated to the system (1), and $(\bar{x}, \bar{y})$ a non-degenerate elliptic fixed point. If $a \neq \frac{3}{16}$ then there exist periodic points with arbitrarily large period in every neighbourhood of $(\bar{x}, \bar{y})$. In adition, $(\bar{x}, \bar{y})$ is a stable fixed point.*

3 Invariant

In this section we prove that the restriction $a \neq \frac{3}{16}$ is not necessary for stability of the equilibrium solution.

The system (1) possesses the invariant given by



(a) (b)

Figure 1: Some orbits of the map T for (a) $a = 0.5$ and (b) $a = 10.0$

$$I(x_n, y_n) = x_n + y_n + \frac{a}{x_n} + \frac{x_n}{y_n}. \tag{25}$$

Indeed, it is easy to see that I is continuous and that $I(x_{n+1}, y_{n+1}) = I(x_n, y_n)$. In this section we use the invariant I to find a Lyapunov function and prove stability of the equilibrium point for all values of parameter $a > 0$, see[11, 12].

The partial derivatives of the function $I(x, y)$ are given with

$$\begin{aligned} \frac{\partial I}{\partial x} &= -\frac{a}{x^2} + \frac{1}{y} + 1, \\ \frac{\partial I}{\partial y} &= 1 - \frac{x}{y^2}. \end{aligned} \tag{26}$$

The unique positive equilibrium of (1) satisfies that $\bar{x} = \bar{y}^2$ and $\bar{y}^3(\bar{y} + 1) = a$. Equation (26) implies that any critical point (x, y) of (25) satisfies the system

$$\begin{aligned} x &= y^2 \\ y^4 + y^3 &= a. \end{aligned}$$

Hance, $(\bar{y}^2, \bar{y})$ of (1) is the unique positive solution of this system and $(\bar{y}^2, \bar{y})$ is critical point of the invariant (25). Thus the unique equilibrium $(\bar{y}^2, \bar{y})$ is critical point of the invariant (25).

Lemma 3 *The graph of the function $I(x, y)$ associated with* (25) *is a simple closed curve in a neighborhood of the equilibrium point of* (1). *The equilibrium point $(\bar{y}^2, \bar{y})$ is stabile.*

Proof. The Hessian matrix assosiated with $I(x, y)$ is

$$H(x, y) = \begin{pmatrix} \frac{2a}{x^3} & -\frac{1}{y^2} \\ -\frac{1}{y^2} & \frac{2x}{y^3} \end{pmatrix}$$

with determinant

$$\det(H(x, y)) = \frac{4ay - x^2}{x^2 y^4}.$$

For the equilibrium $(\bar{y}^2, \bar{y})$ we have

$$\det(H(\bar{y}^2, \bar{y})) = \frac{4a\bar{y} - \bar{x}^2}{\bar{x}^2 \bar{y}^4} = \frac{4\bar{y}(\bar{y}^4 + \bar{y}^3) - \bar{y}^4}{\bar{y}^8} = \frac{4\bar{y}^5 + 3\bar{y}^4}{\bar{y}^8} = \frac{4\bar{y} + 3}{\bar{y}^4} > 0$$

Thus, in view of Morse's lema, [9], the level sets of the function $I(x, y)$ are diffeomorphic to circles in the neighborhood of $(\bar{x}, \bar{y})$. In adition, the function

$$V(x, y) = I(x, y) - I(\bar{x}, \bar{y})$$

is Lyapunov function, and so the equilibrium point $(\bar{x}, \bar{y})$ is stable, see [11]. □

4 Symmetries

In this section we will show that mat T is conjugate to its inverse map and use this conjugacy to find some feasible periods of T and corresponding periodic orbits. A transformation R of the plane is said to be a time reversal symmetry for T if

$$R^{-1} \circ T \circ R = T^{-1}.$$

If the time reversal symmetry R is an involution, i.e. $R^2 = I$, where I is identity map then the time reversal symmetry condition is equivalent to

$$R \circ T \circ R = T^{-1},$$

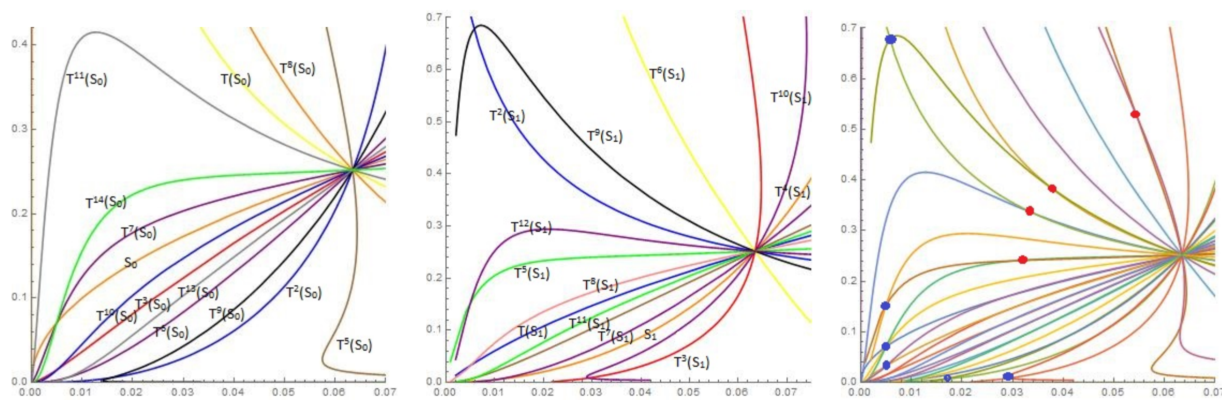
and T can be written as the composition of two involutions $T = I_1 \circ I_0$ where $I_0 = R$ and $I_1 = T \circ R$. Let us note here that if $I_0 = R$ is reversor then so is $I_1 = T \circ R$. Also, the jth involution defined as $I_j = T^j \circ R$ is also a reversor. The invariant sets of the involution maps

$$S_{0,1} = \{(x, y) | I_{0,1}(x, y) = (x, y)\}$$

are one-dimensional sets called the symmetry lines of the map. When the sets $S_{0,1}$ are known the search for periodic orbits can be reduced to one-dimensional root finding problem using the following result, see [2, 8]

Theorem 4 *If $(x, y) \in S_{0,1}$ then $T^n(x, y) = (x, y)$ if and only if*

$$\begin{cases} T^{n/2}(x, y) \in S_{0,1}, & \text{for } n \text{ even} \\ T^{(n\pm1)/2}(x, y) \in S_{1,0}, & \text{for } n \text{ odd.} \end{cases}$$



(a) (b) (c)

Figure 2: a) The first fourteen iterations of symmetry line S_0 of the map T for $a = 0.02$ (b) The first twelve iterations of symmetry line S_1 of the map T for $a = 0.02$ (c) The periodic orbits of period 14 (blue) and 17 (red)

The inverse map of the map T is

$$T^{-1}(x, y) = \left(\frac{ay}{x(y+1)}, \frac{a}{x(y+1)}\right).$$

The involution $R = \left(x, \frac{x}{y}\right)$ is reversor for T. Indeed,

$$(R \circ T \circ R)(x, y) = (R \circ T)\left(x, \frac{x}{y}\right) = R\left(\frac{ay}{x(y+1)}, y\right) = \left(\frac{ay}{x(y+1)}, \frac{a}{x(y+1)}\right) = T^{-1}(x, y).$$

Thus $T = I_1 \circ I_0$ where $I_0(x, y) = R(x, y)$ and

$$I_1(x, y) = T \circ R = \left(\frac{ay}{x(y+1)}, y\right).$$

The symmetry lines corresponding to I_0 and I_1 are

$$S_0 = \{(x, y) : x = y^2\}, \quad S_1 = \{(x, y) : ay = x^2(y+1)\}.$$

Periodic orbits of different orders can be found at the intersection of the symmetry lines S_j, $j = 1, 2, ...$ associated to the jth involution. So if $(x, y) \in S_j \cap S_k$ then $T^{j-k}(x, y) = (x, y)$. The symmetry lines are also related to each other by the relation

$$S_{2j+i} = T^j(S_i), \quad S_{2j-i} = I_j(S_i), \quad \forall i, j.$$

Now we start with the point $(x_0, \sqrt{x_0}) \in S_0$ in search for periodic orbits on the symmetry line S_0 with even period n and impose that $(x_{n/2}, y_{n/2}) \in S_0$, where

$$(x_{n/2}, y_{n/2}) = T^{n/2}(x_0, \sqrt{x_0}).$$

This reduces to one-dimensional root finding for the equation $x_{n/2} = y_{n/2}^2$, where the unknown is x_0.

Periodic orbits on S_0 with odd period n are obtained by solving for x_0 the equation

$$ay_{(n+1)/2} = x_{(n+1)/2}^2(y_{(n+1)/2} + 1),$$

where

$$(x_{(n+1)/2}, y_{(n+1)/2}) = T^{(n+1)/2}(x_0, \sqrt{x_0}).$$

For example, for $a = 0.02$ in Figure 2 we have an intersection between the symmetry lines S_0 and $S_{14} = T^7(S_0)$, $S_4 = T^2(S_0)$ and $S_{18} = T^9(S_0)$, $S_5 = T^2(S_1)$ and $S_{19} = T^9(S_1)$, and $S_{11} = T^5(S_1)$ and $S_{25} = T^{12}(S_1)$ of the map T. The intersection points of these lines correspond to the periodic orbits of period 14.

5 Continua of periodic points for map T

In this section we use resultants and technique from [7] for finding continua of p-periodic points lying on the level sets of the invariant I.

Let

$$T^p(x, y) = (T_1^p(x, y), T_2^p(x, y)).$$

The idea is to find the values of h for which the system

$$\begin{aligned} T_1^p(x, y) &= x \\ I(x, y) &= h \end{aligned} \tag{27}$$

has continua of solutions. Let

$$F(y, h) := Res(\text{numerator } (T_1^p(x, y) - x), \text{ numerator } (I(x, y) - h)), \tag{28}$$

where Res denote the resultant of corresponding expressions. The values of h have to be such that $F(y, h)$ vanishes identically. We need to collect the factors of the above resultant that only depend on h. Denote by $D_p(a, h)$ the product of these factors. We introduce the functions $d_p(a, h)$ as those factors of $D_p(a, h)$ that remain after removing from this polynomial all the factors that already appear in some $D_k(a, h)$ where k is either 1 or a proper divisor of p. We call the conditions $d_p(a, h) = 0$ the resultant p-periodicity conditions associated to the invariant I (RPC from now on), see [7]. The main fact is that the energy levels filled with periodic points must satisfy the RPC what gives us the necessary condition for periodic point because the resultant (28) can contain some spurious factors. We will prove in our examples that the RPC we obtain actually give continua of p-periodic points.

Theorem 5 *The RPC of the map T associated to the invariant I for $p \leq 10$ are given by $d_p(a, h) = 0$, where:*

$$\begin{aligned}
d_2(a, h) &= a \\
d_3(a, h) &= 1 \\
d_4(a, h) &= 1 + h \\
d_5(a, h) &= a - h - 1 \\
d_6(a, h) &= a - h^2 - 3h - 2 \\
d_7(a, h) &= a^2 - ah - a - h^3 - 3h^2 - 3h - 1 \\
d_8(a, h) &= 2a^2 - ah^2 - 5ah - 4a + h^2 + 2h + 1 \\
d_9(a, h) &= -3 + 4a - 3a^2 + a^3 - 12h + 9ah - 3a^2h \\
&\quad - 19h^2 + 6ah^2 - 15h^3 + ah^3 - 6h^4 - h^5 \\
d_{10}(a, h) &= 1 + 5a - 5a^2 + a^3 + 5h + 15ah - 8a^2h + 10h^2 \\
&\quad + 16ah^2 - 3a^2h^2 + 10h^3 + 7ah^3 + 5h^4 + ah^4 + h^5
\end{aligned}$$

Proof. For $p=2$ we obtain

$$numerator(T_1^2(x,y)-x)=-x^3-x^2y+ay^2,$$

and

$$Res(numerator(T_1^2(x,y)-x),numerator(I(x,y)-h),x)=$$
$$ay^3(a^2+4ay+4ahy+4ay^2+3ahy^2-h^2y^2-h^3y^2+2ay^3+2hy^3+2h^2y^3-y^4+ay^4-hy^4).$$

So there is no factor of resultant without dependence on the variable y that can be equal to zero since $a>0$ so we can ensure there are no energy levels formed by continua of period-two points. We come to the same conclusion for $p=3$, so we continue with $p=4$ where we have

$$numerator(T_1^4(x,y)-x)=-x^7+ax^4y-ax^5y-4x^6y-x^7y+2a^2x^2y^2+2ax^3y^2-2ax^4y^2-6x^5y^2-4x^6y^2+$$
$$a^3y^3+2a^2xy^3+ax^2y^3+a^2x^2y^3-ax^3y^3-4x^4y^3-6x^5y^3+a^2xy^4-x^3y^4-4x^4y^4-x^3y^5,$$

and

$$Res(numerator(T_1^4(x,y)-x),numerator(I(x,y)-h),x)=a^3(1+h)^2y^8(1+y)$$
$$(a-2a^2+a^3+2ah-2a^2h+ah^2+8ay-4a^2y+20ahy-4a^2hy+16ah^2y+4ah^3y+8ay^2-4a^2y^2-hy^2+22ahy^2-$$
$$5a^2hy^2-4h^2y^2+19ah^2y^2-6h^3y^2+5ah^3y^2-4h^4y^2-h^5y^2+2y^3-4ay^3+2a^2y^3+8hy^3-6ahy^3+12h^2y^3-2ah^2y^3+$$
$$8h^3y^3+2h^4y^3-y^4-ay^4+a^2y^4-3hy^4-ahy^4-3h^2y^4-h^3y^4).$$

The only factor independent of y is $1+h$ which gives $d_4(a,h)=1+h$. In an analogous way we compute $d_5(a,h)$ and $d_6(a,h)$. For $p=7$ we consider the equation

$$T^4(x,y)=T^{-3}(x,y)$$

so we obtain

$$Res(numerator(T_1^4(x,y)-T_1^{-3}(x,y)),numerator(I(x,y)-h),x)=$$
$$a^4(-1-a+a^2-3h-ah-3h^2-h^3)^2y^{11}(1+y)^3(a+4ay+4ay^2-hy^2-h^2y^2+2y^3+2hy^3),$$

and $d_7(a,h)=-1-a+a^2-3h-ah-3h^2-h^3$. The computation of $d_8(a,h),d_9(a,h)$ and $d_{10}(a,h)$ is analogous to the previous computation. □

Let us now determine the feasibility region $\mathcal{R}$ of a map T, that is those pairs $(a,h)\in\mathbb{R}^2$ that satisfy the condition

$$\{I(x,y)=h\}\cap\mathbb{R}^2=\{x^2+ay-hxy+x^2y+xy^2=0\}\cap\mathbb{R}^2\neq 0.$$

From the property of the invariant I in Lemma 3 we obtain that the equilibrium point $(\bar{y}^2,\bar{y})$ is the absolute minimum of the invariant (25). Let us denote the value of I in the absolute minimum with

$$h_c(\bar{y})=I(\bar{y}^2,\bar{y})=\bar{y}(2\bar{y}+3)$$

and therefore the region

$$\mathcal{R}=\{(a,h),a>0 \text{ and } h\geq h_c(\bar{y}) \text{ and } a=\bar{y}^3(\bar{y}+1)\}$$

is a feasibility region for the map T.

5.1 Analysis of the 7-periodic RPC

In this section we will determine the number of the level curves associated to the 7-periodic RPC. We will use the following Lemma from [7],

Lemma 4 *Let*

$$G_a(h)=g_n(a)h^n+g_{n-1}(a)h^{n-1}+...+g_1(a)h+g_0(a),$$

be a family of real polynomials depending also polynomially on a real parameter a. Set $I_a=(\phi(a),+\infty)$ where $\phi(a)$ is a continuous function. Suppose that there exists an open interval $\Lambda\subset\mathbb{R}$ such that

i) There exists $a_0\in\Lambda$ such that $G_{a_0}(h)$ has exactly $r\geq 0$ simple roots in I_{a_0}.

ii) For all $a\in\Lambda$, $G_a(\phi(a))\cdot g_n(a)\neq 0$.

iii) For all $a\in\Lambda$, $\Delta_h(G_a)\neq 0$, where $\Delta_h(G_a)$ is disriminant of the polynomial $G_a(h)$.

Then for all $a\in\Lambda$, $G_a(h)$ has exactly $r\geq 0$ simple roots in I_a.

The disriminant $\Delta_h(G_a)$ of the polynomial $G_a(h)$ is given as

$$\Delta_h(G_a) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} Res(G_a(h), G_a'(h), h).$$

Let us now for the sake of the convenience rewrite $d_7(a,h)$ as one-parametric family of polynomials in h depending on the parameter $\bar{y}$ where $a = \bar{y}^3(1+\bar{y})$:

$$G_{\bar{y}}(h) := d_7(a,h) = g_3(\bar{y})h^3 + g_2(\bar{y})h^2 + g_1(\bar{y})h + g_0(\bar{y}),$$

where $g_3(\bar{y}) = -1$, $g_2(\bar{y}) = -3$, $g_1(\bar{y}) = -3 - \bar{y}^3(1+\bar{y})$ and $g_0(\bar{y}) = y^3(y+1)\left(y^4+y^3-1\right) - 1$. Since $(a,h) \in \mathcal{R}$ if and only if $h \in [h_c(\bar{y}), +\infty) = [\bar{y}(2\bar{y}+3), +\infty)$ and $a = \bar{y}^3(1+\bar{y})$, we have to study the number of real roots of $G_{\bar{y}}(h) = 0$ in the feasibility region. Let us note that

$$\Delta_h(G_{\bar{y}}(h)) = \bar{y}^9(\bar{y}+1)^3\left(27\bar{y}^4 + 27\bar{y}^3 + 4\right) > 0,$$

so hypotheses (iii) of the Lemma 4 is satisfied.

Further, according to Lemma 4, since $g_3(\bar{y}) \neq 0$, the number of real simple roots in $G_{\bar{y}}(h)$ is constant on any open interval where $G_a(h_c(\bar{y})) \neq 0$. We have

$$G_{\bar{y}}(\bar{y}(2\bar{y}+3)) = (\bar{y}+1)^5\left(\bar{y}^3 - 3\bar{y}^2 - 4\bar{y} - 1\right),$$

and it vanishes in $\bar{y}_0 \approx 4.04892$, which is the only positive root of $\bar{y}^3 - 3\bar{y}^2 - 4\bar{y} - 1 = 0$. Hence, the map T has a constant number of real roots in $I_{\bar{y}} = [h_c(\bar{y}), +\infty) = [\bar{y}(2\bar{y}+3), +\infty)$ for $\bar{y}$ in each of the intervals $(0, \bar{y}_0)$ and $(\bar{y}_0, \infty)$. So we have to determine the number of roots of $G_{\bar{y}}$ in $I_{\bar{y}} = [\bar{y}(2\bar{y}+3), +\infty)$ for the intervals $(0, \bar{y}_0)$ and $(\bar{y}_0, +\infty)$. We can reduce the problem to study one concrete value of $\bar{y}$ in each of the intervals mentioned above. Let us consider the value $\bar{y} = 2 \in (0, \bar{y}_0)$. We can compute

$$G_2(h) = -h^3 - 3h^2 - 27h + 551,$$

and it is easy to see that $G_2(h)$ has no simple roots in $I_{\bar{y}}$. By Lemma 4 we have that $G_{\bar{y}}(h)$ has no simple roots in $I_{\bar{y}} = [\bar{y}(2\bar{y}+3), +\infty)$ for $\bar{y} \in (0, \bar{y}_0)$, i.e. for $a \in (0, \bar{y}_0^3(\bar{y}_0+1))$. Similarly, one can see that $G_{\bar{y}}(h)$ has one simple root in $I_{\bar{y}} = [\bar{y}(2\bar{y}+3), +\infty)$ for $\bar{y} \in (\bar{y}_0, +\infty)$, i.e. $a \in (\bar{y}_0^3(\bar{y}_0+1), +\infty)$.

From the previous discussion we obtain the following theorem:

Theorem 6 *Consider the map T given by (3) with positive parameter a and value $a_0 = \bar{y}_0^3(\bar{y}_0+1) \approx 335.13213$. The set of real 7-periodic points is empty set for $a \in (0, a_0)$ and it is given by smooth non-empty level sets $I_a(x,y) = h$ for the values of h satisfying $d_7(a,h) = 0$ for $a > a_0$, with d_7 given in Theorem 5 and it is formed by one closed curve diffeomorphic to $\mathbb{S}^1$.*

References

[1] A. M. Amleh, E. Camouzis, and G. Ladas, On the Dynamics of a Rational Difference Equation, Part I, *Int. J. Difference Equ.* 3(2008), 1–35.

[2] D. del-Castillo-Negrete, J. M. Greene, E. J. Morrison, Area preserving nontwist maps: periodic orbits and transition to chaos, *Physica D*, 91(1996), 1–23.

[3] A. Cima, A. Gasull, V. Mañosa, Studying discrete dynamical systems through differential equations, *J. Differential Equations* 244 (2008), 630-648.

[4] A. Cima, A. Gasull, V. Mañosa, Non-autonomus 2-periodic Gumovski-Mira difference equations, Internat. *J. Bifur. Chaos Appl. Sci. Engrg.* 22, (2012), 1250264, 14 pp.

[5] E. Denette, M. R. S. Kulenović and E. Pilav, Birkhoff normal forms, KAM theory and time reversal symmetry for certain rational map, *Mathematics, MDPI*, 2016; 4(1):20.

[6] E. Drymmonis, E. Camouzis, G. Ladas, G. and W. Tikjha, Patterns of boundedness of the rational system $x_{n+1} = \frac{\alpha_1}{A_1+B_1x_n+C_1y_n}$ and $y_{n+1} = \frac{\alpha_2+\beta_2x_n+\gamma_2y_n}{A_2+B_2x_n+C_2y_n}$. *J. Difference Equ. Appl.* 18 (2012), 89–110.

[7] A. Gasull, M. Liorens, V. Mañosa, Continua of periodic points for planar integrable rational maps, *Int. J. Difference Equ.*, 11(2016), 37–63.

[8] M. Gidea, J. D. Meiss, I. Ugarcovici, H. Weiss, Applications of KAM Theory to Population Dynamics, *J. Biological Dynamics* 5:1,44-63 (2011).

[9] J. K. Hale and H. Kocak, *Dynamics and Bifurcation*, Springer-Verlag, New York, (1991).

[10] V. L. Kocic, G. Ladas, G. Tzanetopoulos, and E. Thomas, On the stability of Lyness' equation, *Dynam. Contin. Discrete Impuls. Systems*, 1(1995), 245–254.

[11] M. R. S. Kulenović, Invariants and related Liapunov functions for difference equations, *Appl. Math. Lett.* 13(2000), 1-8.

[12] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall/CRC, Boca Raton, London, 2002.

[13] M. R. S. Kulenović and Z. Nurkanović, Stability of Lyness' Equation with Period-Two Coefficient via KAM Theory, *J. Concr. Appl. Math.*, 6(2008), 229-245.

[14] G. Ladas, G. Tzanetopoulos, and A. Tovbis, On May's host parasitoid model, *J. Difference Equ. Appl.* 2 (1996), 195–204.

[15] R. S. MacKay, *Renormalization in Area-Preserving Maps*, World Scientific, River Edge, NJ, 1993.

[16] R. Moeckel, Generic bifurcations of the twist coefficient, *Ergodic Theory Dyn. Syst.* 10(1) (1990), pp. 185–195.

[17] C. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer-Varlag, New York, 1971.

[18] M. Tabor, *Chaos and integrability in nonlinear dynamics. An introduction.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989.

[19] Y. H. Wan, Computation of the stability condition for the Hopf bifurcation of diffeomorphisms on $\mathcal{R}^2$, *SIAM J. Appl. Math.* 34(1) (1978), pp. 167–175.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Durrmeyer type $(p,\, q)$-Baskakov operators for functions of one and two variables

Qing-Bo Cai[a] and Guorong Zhou[b,*]

[a]School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China

[b]School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China

E-mail: qbcai@126.com, goonchow@xmut.edu.cn.

Abstract. In this paper, we construct a generalization of Durrmeyer type Baskakov operators based on the concept of $(p,\, q)$-integers and bivariate tensor product form. For the univariate case, we obtain the estimates of moments and central moments of these operators, establish a local approximation theorem, obtain the estimates on the rate of convergence and weighted approximation of those operators. For the bivariate case, we give the rate of convergence by using the weighted modulus of continuity, give some graphs and numerical examples to illustrate the convergent properties of these operators to certain functions. We also compare these operators $D_{n,p,q}$ with another forms.

2000 Mathematics Subject Classification: 41A10, 41A25, 41A36.
Key words and phrases: $(p,\, q)$-integers, Baskakov operators, modulus of continuity, rate of convergence, bivariate tensor product.

1 Introduction

In recent years, $(p,\, q)$-integers have been introduced to linear positive operators to construct new approximation processes. A sequence of $(p,\, q)$-analogue of Bernstein operators was first introduced by Mursaleen [1, 2]. Besides, $(p,\, q)$-analogues of Szász-Mirakyan operators [3] , $(p,\, q)$-Baskakov Kantorovich operators [4, 5], $(p,\, q)$-Baskakov-Beta operators [6] and Kantorovich-type Bernstein-Stancu-Schurer operators [7] were also considered. For further developments, one can also refer to [8, 9, 28]. These operators are double parameters corresponding to p and q versus single parameter q-type operators [11, 12, 13]. The aim of these generalizations is to provide appropriate and powerful tools to these application areas such as numerical analysis, CAGD and solutions of differential equations (see, e. g., [14]).

*Corresponding author.

In 2010, Aral and Gupta [15], Gupta [16] introduced certain Durrmeyer type q-Baskakov operators and got some important approximation properties, motivated by them, in 2012, Cai and Zeng [17] introduced a new modification of Durrmeyer type one. Recently, Acar et al. [18] introduced a generalization of Durrmeyer type $(p,\ q)$-Baskakov operators which having Baskakov and Szász basis functions defined by

$$B_n^{p,q}(f;x) = [n]_{p,q} \sum_{k=0}^{\infty} b_{n,k}(p,q;x) \int_0^{\infty} p^{\frac{k(k-1)}{2}} \frac{([n]_{p,q}t)^k\, E_{p,q}\left(-q[n]_{p,q}t\right)}{[k]_{p,q}!} f\left(\frac{p^{k+n-1}}{q^{k-1}}t\right) d_{p,q}t, \quad (1)$$

where

$$b_{n,k}(p,q;x) = \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} p^{k+\frac{n(n-1)}{2}} q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_{p,q}^{n+k}}. \quad (2)$$

From [5], we know $\sum_{k=0}^{\infty} b_{n,k}(p,q;x) = 1$. In 2016, Mishra and Pandey [19] introduced the Stancu type base on operators (1).

Inspired by these results, in this paper, we introduce a generalization of Durrmeyer type $(p,\ q)$-Baskakov operators $D_{n,p,q}(f;x)$ as

$$D_{n,p,q}(f;x) = [n-1]_{p,q} \sum_{k=0}^{\infty} \widetilde{b_{n,k}}(p,q;\mu(x)) \int_0^{\infty} \widetilde{b_{n,k}}(p,q;pu) f(p^k u) d_{p,q}u, \quad (3)$$

where $\mu(x) = \frac{p^{n-2}\left(p^2 q[n-2]_{p,q}x-1\right)}{[n]_{p,q}}$, $x \in \left[\frac{1}{p^2 q[n-2]_{p,q}}, \infty\right)$, $0 < q < p \leq 1$ and

$$\widetilde{b_{n,k}}(p,q;x) = \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} p^{\frac{n(n-1)+(k+1)(k+2)}{4}} q^{\frac{k^2-1}{2}} \frac{x^k}{(1+x)_{p,q}^{n+k}}. \quad (4)$$

The paper is organized as follows: In section 2, we give some basic definitions regarding $(p,\ q)$-integers and $(p,\ q)$-calculus. In section 3, we estimate the moments and central moments of these operators (3). In section 4, we establish a local approximation theorem, obtain the estimates on the rate of convergence and weighted approximation. In section 5, we give some graphs and numerical examples to illustrate the convergent properties for one variable functions. In section 6-7, we propose the bivariate case, give the rate of convergence by using the weighted modulus of continuity and give some graphs and numerical analysis for two variables functions. In the last section, we compare the operators $D_{n,p,q}$ with $\widetilde{D_{n,p,q}}$, and show the former operators give better approximation to f than the latter ones by graphs.

2 Some notations

We mention some definitions based on $(p,\ q)$-integers, details can be found in [20, 21, 22, 23, 24]. For any fixed real number $0 < q < p \leq 1$ and each nonnegative integer k, we denote $(p,\ q)$-integers by $[k]_{p,q}$, where

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}.$$

Also (p, q)-factorial and (p, q)-binomial coefficients are defined as follows:

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q}...[1]_{p,q}, & k = 1, 2, ..., \\ 1, & k = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad (n \geq k \geq 0).$$

Let n be a non-negative integer, the (p, q)-Gamma function is defined as

$$\Gamma_{p,q}(n+1) = \frac{(p-q)^n_{p,q}}{(p-q)^n} = [n]_{p,q}!,$$

where $(p-q)^n_{p,q} = (p-q)(p^2-q^2)...(p^n-q^n)$.

For $m, n \in \mathbb{N}$, the (p, q)-Beta function of second kind is given by

$$B_{p,q}(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+pt)^{m+n}_{p,q}} d_{p,q}t,$$

where the (p, q)-power basis is given by

$$(1+pt)^{m+n}_{p,q} = (1+pt)(p+pqt)(p^2+pq^2t)...(p^{m+n-1}+pq^{m+n-1}t).$$

The relationship by the (p, q)-Beta and Gamma functions is shown as follows

$$B_{p,q}(m, n) = \frac{q\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{(p^{m+1}q^{m-1})^{m/2}\,\Gamma_{p,q}(m+n)},$$

if $p = 1, q \to 1^-$, it reduces to the classic type $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

The improper (p, q)-integral of $f(x)$ on $[0, \infty)$ is defined to be

$$\int_0^\infty f(x)d_{p,q}x = \sum_{j=-\infty}^{\infty} \int_{\frac{q^{j+1}}{p^{j+1}}}^{\frac{q^j}{p^j}} f(x)d_{p,q}x = (p-q)\sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right).$$

When $p = 1$, all the definitions of (p, q)-calculus above are reduced to q-calculus.

3 Auxiliary results

Lemma 3.1. *For $x \in [0, \infty)$ and sufficiently large n, the following equalities hold*

$$D_{n,p,q}(1; x) = 1, \tag{5}$$

$$D_{n,p,q}(t; x) = x, \tag{6}$$

$$\begin{aligned} D_{n,p,q}(t^2; x) = {} & \frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}} x^2 + \frac{\left(p^2+q^2\right)}{p^3q^3[n-3]_{p,q}} x - \frac{2p^{n-2}}{q^3[n-3]_{p,q}[n]_{p,q}} x \\ & + \frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}, \end{aligned} \tag{7}$$

$$\begin{aligned}
&D_{n,p,q}(t^3;x)\\
=\ & \frac{[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}^2}{[n-3]_{p,q}[n-4]_{p,q}[n]_{p,q}^2}x^3+\frac{\left([5]_{p,q}+[2]_{p,q}^2pq\right)p^{n-2}[2]_{p,q}[n+1]_{p,q}[n+2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2\\
&+\frac{\left([5]_{p,q}q^2+[2]_{p,q}^2pq^3-3p^2\right)[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2+\frac{q^5+3p^3q^2-p^5}{p^7q^7[n-3]_{p,q}[n-4]_{p,q}}x\\
&+\frac{3p^{2n-4}[2]_{p,q}}{q^8[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x-\frac{p^{n-6}\left(2p^4+2q^4+4pq^3+p^3q\right)}{q^8[n]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}x\\
&-\frac{[2]_{p,q}^2}{p^7q^7[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}+\frac{p^{n-8}\left([5]_{p,q}+[2]_{p,q}pq^2\right)}{q^9[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}\\
&-\frac{p^{2n+3}[2]_{p,q}}{p^9q^9[n]_{p,q}^2[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}, \qquad (8)\\
&D_{n,p,q}(t^4;x)=\frac{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}[n-2]_{p,q}^3}{q^{12}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}[n]_{p,q}^3}x^4+O\left(\frac{1}{[n]_{p,q}}\right)\phi(x), \qquad (9)
\end{aligned}$$

where $\phi(x)$ is depend on x.

Proof. Since

$$\begin{aligned}
\int_0^\infty \frac{u^k}{(1+pu)_{p,q}^{n+k}}d_{p,q}u &= B_{p,q}(k+1,n-1)=\frac{q\Gamma_{p,q}(k+1)\Gamma_{p,q}(n-1)}{\left(p^{k+2}q^k\right)^{\frac{k+1}{2}}\Gamma_{p,q}(n+k)}\\
&= \frac{q[k]_{p,q}![n-2]_{p,q}!}{p^{\frac{(k+1)(k+2)}{2}}q^{\frac{k(k+1)}{2}}[n+k-1]_{p,q}!},
\end{aligned}$$

we have

$$\begin{aligned}
D_{n,p,q}(1;x) &= [n-1]_{p,q}\sum_{k=0}^\infty \widetilde{b_{n,k}}(p,q;\mu(x))\int_0^\infty \widetilde{b_{n,k}}(p,q;pu)d_{p,q}u\\
&= [n-1]_{p,q}\sum_{k=0}^\infty \widetilde{b_{n,k}}(p,q;\mu(x))\frac{[n+k-1]_{p,q}!}{[k]_{p,q}![n-1]_{p,q}!}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\\
&\quad\times\frac{p^kq[k]_{p,q}![n-2]_{p,q}!}{p^{\frac{(k+1)(k+2)}{2}}q^{\frac{k(k+1)}{2}}[n+k-1]_{p,q}!}\\
&= \sum_{k=0}^\infty b_{n,k}(p,q;\mu(x))=1.
\end{aligned}$$

Similarly, we get

$$\int_0^\infty \frac{u^{k+1}}{(1+pu)_{p,q}^{n+k}}d_{p,q}u=\frac{q[k+1]_{p,q}![n-3]_{p,q}!}{p^{\frac{(k+2)(k+3)}{2}}q^{\frac{(k+1)(k+2)}{2}}[n+k-1]_{p,q}!},$$

thus,

$$D_{n,p,q}(t;x) = [n-1]_{p,q}\sum_{k=0}^\infty \widetilde{b_{n,k}}(p,q;\mu(x))\int_0^\infty \widetilde{b_{n,k}}(p,q;pu)p^kud_{p,q}u$$

$$\begin{aligned}
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\frac{[n+k-1]_{p,q}!}{[k]_{p,q}![n-1]_{p,q}!}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\\
&\quad\times\frac{p^{2k}q[k+1]_{p,q}![n-3]_{p,q}!}{p^{\frac{(k+2)(k+3)}{2}}q^{\frac{(k+1)(k+2)}{2}}[n+k-1]_{p,q}!}\\
&= \sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\frac{p^{2k}q[k+1]_{p,q}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}}{p^{\frac{(k+2)(k+3)}{2}}q^{\frac{(k+1)(k+2)}{2}}[n-2]_{p,q}}.
\end{aligned}$$

Since $[k+1]_{p,q}=q^k+p[k]_{p,q}$, by simple computations, we have

$$\begin{aligned}
D_{n,p,q}(t;x) &= \frac{[n]_{p,q}\mu(x)}{p^nq^2[n-2]_{p,q}}\sum_{k=0}^{\infty}\left[\begin{array}{c}n+k\\k\end{array}\right]_{p,q}\frac{p^{k+\frac{n(n+1)}{2}}q^{\frac{k(k-1)}{2}}(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k+1}}\\
&\quad+\frac{1}{p^2q[n-2]_{p,q}}\sum_{k=0}^{\infty}b_{n,k}(p,q;\mu(x))\\
&= \frac{[n]_{p,q}\mu(x)}{p^nq^2[n-2]_{p,q}}+\frac{1}{p^2q[n-2]_{p,q}}=x.
\end{aligned}$$

Next,

$$\int_0^{\infty}\frac{u^{k+2}}{(1+pu)_{p,q}^{n+k}}d_{p,q}u=\frac{q[k+2]_{p,q}![n-4]_{p,q}!}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}[n+k-1]_{p,q}!},$$

we get

$$\begin{aligned}
&D_{n,p,q}(t^2;x)\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)p^{2k}u^2d_{p,q}u\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\left[\begin{array}{c}n+k-1\\k\end{array}\right]_{p,q}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\\
&\quad\times\int_0^{\infty}\frac{p^{3k}u^{k+2}}{(1+pu)_{p,q}^{n+k}}d_{p,q}u\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\left[\begin{array}{c}n+k-1\\k\end{array}\right]_{p,q}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\\
&\quad\times\frac{p^{3k}q[k+2]_{p,q}![n-4]_{p,q}!}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}[n+k-1]_{p,q}!}\\
&= \sum_{k=0}^{\infty}\left[\begin{array}{c}n+k-1\\k\end{array}\right]_{p,q}\frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}}\frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}}p^{3k}q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}}\frac{[k+1]_{p,q}[k+2]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} \quad (10)
\end{aligned}$$

Using $[k+1]_{p,q}=q^k+p[k]_{p,q}$ and some computations, we obtain

$$[k+1]_{p,q}[k+2]_{p,q}=[2]_{p,q}q^{2k}+p[2]_{p,q}^2q^{k-1}[k]_{p,q}+p^4[k]_{p,q}[k-1]_{p,q}. \quad (11)$$

Since

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} \frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}}p^{3k}q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}} \frac{p^4[k]_{p,q}[k-1]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} \\
= & \frac{[n]_{p,q}[n+1]_{p,q}x^2}{p^{2n}q^6[n-2]_{p,q}[n-3]_{p,q}} \sum_{k=0}^{\infty}\left[\begin{array}{c} n+k+1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k+2}} p^{k+\frac{(n+1)(n+2)}{2}} q^{\frac{k(k-1)}{2}} \\
= & \frac{[n]_{p,q}[n+1]_{p,q}\,(\mu(x))^2}{p^{2n}q^6[n-2]_{p,q}[n-3]_{p,q}}, \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} \frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}}p^{3k}q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}} \frac{q^{k-1}[k]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} \\
= & \frac{[n]_{p,q}\mu(x)}{p^{n+4}q^5[n-2]_{p,q}[n-3]_{p,q}} \sum_{k=0}^{\infty}\left[\begin{array}{c} n+k \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k+1}} p^{k+\frac{n(n+1)}{2}} q^{\frac{k(k-1)}{2}} \\
= & \frac{[n]_{p,q}\mu(x)}{p^{n+4}q^5[n-2]_{p,q}[n-3]_{p,q}}, \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} \frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}}p^{3k}q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}} \frac{[2]_{p,q}q^{2k}}{[n-2]_{p,q}[n-3]_{p,q}} \\
= & \frac{[2]_{p,q}}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}} \sum_{k=0}^{\infty} b_{n,k}(p,q;\mu(x)) \\
= & \frac{[2]_{p,q}}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}}, \qquad (14)
\end{aligned}$$

combining (10), (11), (12), (13) and (14), we have

$$\begin{aligned}
&D_{n,p,q}(t^2;x) \\
= & \frac{[n]_{p,q}[n+1]_{p,q}\,(\mu(x))^2}{p^{2n}q^6[n-2]_{p,q}[n-3]_{p,q}} + \frac{[2]_{p,q}^2[n]_{p,q}\mu(x)}{p^{n+3}q^5[n-2]_{p,q}[n-3]_{p,q}} + \frac{[2]_{p,q}}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}} \\
= & \frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}}x^2 + \frac{\left(p^2+q^2\right)}{p^3q^3[n-3]_{p,q}}x - \frac{2p^{n-2}}{q^3[n-3]_{p,q}[n]_{p,q}}x \\
& + \frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}.
\end{aligned}$$

Using the same methods, we have

$$\begin{aligned}
&D_{n,p,q}(t^3;x) \\
= & [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)p^{3k}u^3d_{p,q}u
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} p^{\frac{n(n-1)+2k-18}{2}} q^{\frac{k^2-7k-12}{2}} \frac{[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}},$$

since

$$\begin{aligned} & [k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q} \\ = \; & p^9[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q} + p^4 q^{k-2}\left([5]_{p,q}+[2]_{p,q}^2 pq\right)[k]_{p,q}[k-1]_{p,q} \\ & + pq^{2k-2}[2]_{p,q}\left([5]_{p,q}+[2]_{p,q}^2 pq\right)[k]_{p,q} + q^{3k}[2]_{p,q}[3]_{p,q}, \end{aligned}$$

by some computations, we have

$$\begin{aligned} & D_{n,p,q}(t^3;x) \\ = \; & \frac{[n]_{p,q}[n+1]_{p,q}[n+2]_{p,q}\,(\mu(x))^3}{p^{3n}q^{12}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} + \frac{\left([5]_{p,q}+[2]_{p,q}^2 pq\right)[n]_{p,q}[n+1]_{p,q}\,(\mu(x))^2}{p^{2n+4}q^{11}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} \\ & + \frac{[2]_{p,q}\left([5]_{p,q}+[2]_{p,q}^2 pq\right)[n]_{p,q}\mu(x)}{p^{n+7}q^9[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} + \frac{[2]_{p,q}[3]_{p,q}}{p^9q^6[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} \\ = \; & \frac{[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}^2}{[n-3]_{p,q}[n-4]_{p,q}[n]_{p,q}^2}x^3 + \frac{\left([5]_{p,q}+[2]_{p,q}^2 pq\right)p^{n-2}[2]_{p,q}[n+1]_{p,q}[n+2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2 \\ & + \frac{\left([5]_{p,q}q^2+[2]_{p,q}^2 pq^3-3p^2\right)[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2 + \frac{q^5+3p^3q^2-p^5}{p^7q^7[n-3]_{p,q}[n-4]_{p,q}}x \\ & + \frac{3p^{2n-4}[2]_{p,q}}{q^8[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x - \frac{p^{n-6}\left(2p^4+2q^4+4pq^3+p^3q\right)}{q^8[n]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}x \\ & - \frac{[2]_{p,q}^2}{p^7q^7[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} + \frac{p^{n-8}\left([5]_{p,q}+[2]_{p,q}pq^2\right)}{q^9[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} \\ & - \frac{p^{2n+3}[2]_{p,q}}{p^9q^9[n]_{p,q}^2[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}. \end{aligned}$$

Finally,

$$\begin{aligned} & D_{n,p,q}(t^4;x) \\ = \; & [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)p^{4k}u^4 d_{p,q}u \\ = \; & \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} p^{\frac{n(n-1)+2k-28}{2}} q^{\frac{k^2-9k-20}{2}} \frac{[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}} \\ & \times \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}}, \end{aligned}$$

since

$$\begin{aligned} & [k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q} \\ = \; & p^{16}[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q}[k-3]_{p,q} + \left([7]_{p,q}+pq[5]_{p,q}+[2]_{p,q}^2p^2q^2\right)p^9q^{k-3}[k]_{p,q} \end{aligned}$$

$$\times[k-1]_{p,q}[k-2]_{p,q}+p^4q^{2k-4}\left([5]_{p,q}+[2]_{p,q}^2pq\right)\left([6]_{p,q}+p^2q^2[2]_{p,q}\right)[k]_{p,q}[k-1]_{p,q}$$
$$+pq^{3k-3}[2]_{p,q}\left([5]_{p,q}^2+[2]_{p,q}^2[5]_{p,q}pq+p^3q^3[3]_{p,q}\right)[k]_{p,q}+q^{4k}[2]_{p,q}[3]_{p,q}[4]_{p,q},$$

we have

$$\begin{aligned}
&D_{n,p,q}(t^4;x)\\
=\;&\frac{[n]_{p,q}[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}{p^{4n}q^{20}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}(\mu(x))^4\\
&+\frac{\left([7]_{p,q}+pq[5]_{p,q}+[2]_{p,q}^2p^2q^2\right)[n]_{p,q}[n+1]_{p,q}[n+2]_{p,q}}{p^{3n+5}q^{19}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\left(\mu(x)^3\right)\\
&+\frac{\left([5]_{p,q}+[2]_{p,q}^2pq\right)\left([6]_{p,q}+p^2q^2[2]_{p,q}\right)}{p^{2n+9}q^{17}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\left(\mu(x)^2\right)\\
&+\frac{[2]_{p,q}\left([5]_{p,q}^2+[2]_{p,q}^2[5]_{p,q}pq+p^3q^3[3]_{p,q}\right)[n]_{p,q}}{p^{n+12}q^{14}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\mu(x)\\
&+\frac{[2]_{p,q}[3]_{p,q}[4]_{p,q}}{p^{14}q^{10}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\\
=\;&\frac{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}[n-2]_{p,q}^3}{q^{12}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}[n]_{p,q}^3}x^4+O\left(\frac{1}{[n]_{p,q}}\right)\phi(x).
\end{aligned}$$

Lemma 3.1 is proved. □

Lemma 3.2. *For sufficiently large n, we have*

$$D_{n,p,q}(t-x;x)=0, \tag{15}$$

$$\begin{aligned}
&D_{n,p,q}((t-x)^2;x)\\
=\;&\frac{p^nx^2}{q[n]_{p,q}}+\frac{p^{n-3}x^2}{q[n-3]_{p,q}}+\frac{p^{2n-3}x^2}{q^2[n-3]_{p,q}[n]_{p,q}}+\frac{\left(p^2+q^2\right)x}{p^3q^3[n-3]_{p,q}}-\frac{2p^{n-2}x}{q^3[n-3]_{p,q}[n]_{p,q}}\\
&+\frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}}-\frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}=B_{n,p,q}(x) \qquad (16)\\
=\;&O\left(\frac{1}{[n]_{p,q}}\right)\left(x^2+x+1\right), \qquad (17)
\end{aligned}$$

$$D_{n,p,q}((t-x)^4;x)=O\left(\frac{1}{[n]_{p,q}}\right)\left(x^4+x^3+x^2+x+1\right). \tag{18}$$

Proof. (15) is obtained by (5) and (6). Since

$$\begin{aligned}
\frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}} &= \frac{\left(p^{n-3}+q[n-3]_{p,q}\right)\left(p^n+q[n]_{p,q}\right)}{q^2[n-3]_{p,q}[n]_{p,q}}\\
&= 1+\frac{p^n}{q[n]_{p,q}}+\frac{p^{n-3}}{q[n-3]_{p,q}}+\frac{p^{2n-3}}{q^2[n-3]_{p,q}[n]_{p,q}},
\end{aligned}$$

using lemma 3.1, we have

$$D_{n,p,q}((t-x)^2;x)$$

$$\begin{aligned}
&= D_{n,p,q}(t^2;x) - 2xM_{n,p,q}(t;x) + x^2 \\
&= D_{n,p,q}(t^2;x) - x^2 \\
&= \left[\frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}} - 1\right]x^2 + \frac{\left(p^2+q^2\right)}{p^3q^3[n-3]_{p,q}}x - \frac{2p^{n-2}}{q^3[n-3]_{p,q}[n]_{p,q}}x \\
&\quad + \frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}} \\
&= \frac{p^n x^2}{q[n]_{p,q}} + \frac{p^{n-3}x^2}{q[n-3]_{p,q}} + \frac{p^{2n-3}x^2}{q^2[n-3]_{p,q}[n]_{p,q}} + \frac{\left(p^2+q^2\right)x}{p^3q^3[n-3]_{p,q}} - \frac{2p^{n-2}x}{q^3[n-3]_{p,q}[n]_{p,q}} \\
&\quad + \frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}.
\end{aligned}$$

Similarly, by some computations, we can obtain (18). □

Lemma 3.3. *(See theorem 2.1 of [25]). For $0 < q_n < p_n \leq 1$, set $q_n := 1-\alpha_n$, $p_n := 1-\beta_n$ such that $0 \leq \beta_n < \alpha_n < 1$, $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$. The following statements are true*

(A) *If* $\lim\limits_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ *and* $e^{n\beta_n}/n \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

(B) *If* $\overline{\lim\limits_{n\to\infty}} e^{n(\beta_n-\alpha_n)} < 1$ *and* $e^{n\beta_n}(\alpha_n-\beta_n) \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

(C) *If* $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim\limits_{n\to\infty}} e^{n(\beta_n-\alpha_n)} = 1$ *and* $\max\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n-\beta_n)\} \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

4 Approximation properties

In this section, we establish a local approximation theorem. We give the following definitions at first, the space of all real valued continuous bounded functions f defined on the interval $[0,\infty)$ is denoted by $C_B[0,\infty)$. The norm on $C_B[0,\infty)$ is defined by $||f|| = \sup\{|f(x)| : x \in [0,\infty)\}$. The Peetre's K-functional is given by

$$K_2(f;\delta) = \inf_{g\in W^2}\{||f-g|| + \delta||g''||\}, \tag{19}$$

where $\delta > 0$, $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. For $f \in C_B[0,\infty)$, the second order modulus of smoothness is defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)}|f(x+2h) - 2f(x+h) + f(x)|. \tag{20}$$

By [27], there exists a constant $C > 0$, such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}). \tag{21}$$

In order to obtain the convergence of operators defined in (3), in the sequel, let $p = \{p_n\}$ and $q = \{q_n\}$ be sequences satisfying $p_n^n \to 1 (n\to\infty)$ and the conditions of lemma 3.3 (A), (B) or (C).

Theorem 4.1. *For $f \in C_B[0,\infty)$ and $n \geq 6$, we have*

$$|D_{n,p,q}(f;x) - f(x)| \leq C\omega_2\left(f;\sqrt{B_{n,p,q}(x)/2}\right), \tag{22}$$

where C is a positive constant, $B_{n,p,q}(x)$ is defined in (16).

Proof. Let $g \in W^2$, by Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \tag{23}$$

applying $D_{n,p,q}$ to (23), using (15), we get

$$D_{n,p,q}(g;x) - g(x) = D_{n,p,q}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Thus, we have,

$$\begin{aligned}|D_{n,p,q}(g;x) - g(x)| &= \left|D_{n,p,q}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\leq D_{n,p,q}\left(\left|\int_x^t |t-u||g''(u)|du\right|;x\right) \\ &\leq D_{n,p,q}\left((t-x)^2;x\right)||g''|| \\ &= B_{n,p,q}(x)||g''||. \end{aligned} \tag{24}$$

On the other hand, by (3) and (5), we have

$$D_{n,p,q}(f;x) = [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;x)\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)|f(p^k u)|d_{p,q}u \leq ||f||. \tag{25}$$

Now (24) and (25) imply

$$\begin{aligned}|D_{n,p,q}(f;x) - f(x)| &\leq |D_{n,p,q}(f-g;x) - (f-g)(x)| + |D_{n,p,q}(g;x) - g(x)| \\ &\leq 2||f-g|| + B_{n,p,q}(x)||g''||,\end{aligned}$$

from (19), taking infimum on the right hand side over all $g \in W^2$, we obtain

$$|D_{n,p,q}(f;x) - f(x)| \leq 2K_2\left(f;B_{n,p,q}(x)/2\right).$$

Finally, using (21), we get

$$|D_{n,p,q}(f;x) - f(x)| \leq C\omega_2\left(f;\sqrt{B_{n,p,q}(x)/2}\right).$$

Theorem 4.1 is proved. □

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \leq M_f(1+x^2)$, where M_f is the constant depending only on f. We denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is

$$||f||_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

We denote the usual modulus of continuity of f on the closed interval $[0,a]$ $(a>0)$ by

$$\omega_a(f;\delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

Obviously, for a function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 4.2. *Let $f \in C_{x^2}[0,\infty)$, $\omega_{a+1}(f;\delta)$ be the modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a>0$. Then for $n \geq 6$, we have*

$$||D_{n,p,q}(f)-f||_{C_{x^2}[0,a]} \leq 4M_f(1+a^2)B_{n,p,q}(a) + 2\omega_{a+1}\left(f;\sqrt{B_{n,p,q}(a)}\right),$$

where $B_{n,p,q}(a)$ is defined in (16).

Proof. For $x\in[0,a]$ and $t>a+1$, we have

$$|f(t)-f(x)| \leq M_f\left(2+x^2+t^2\right) \leq M_f\left(2+3x^2+2(t-x)^2\right).$$

Since $t-x \geq t-a > 1$, then $(t-x)^2>1$. Thus $2+3x^2+2(t-x)^2 \leq (2+3x^2)(t-x)^2 + 2(t-x)^2 = (4+3x^2)(t-x)^2 \leq (4+3a^2)(t-x)^2 \leq 4(1+a^2)(t-x)^2$. Thus, we obtain

$$|f(t)-f(x)| \leq 4M_f(1+a^2)(t-x)^2. \tag{26}$$

For $x\in[0,a]$ and $t \leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f;|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta),\ (\delta>0) \tag{27}$$

From (26) and (27), we get

$$|f(t)-f(x)| \leq 4M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{28}$$

For $x\in[0,a]$ and $t\geq 0$, by Schwarz's inequality and lemma 3.2, we have

$$\begin{aligned}
& |D_{n,p,q}(f;x)-f(x)| \\
\leq\ & D_{n,p,q}(|f(t)-f(x)|;x) \\
\leq\ & 4M_f(1+a^2)D_{n,p,q}\left((t-x)^2;x\right) + \omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{D_{n,p,q}\left((t-x)^2;x\right)}\right) \\
\leq\ & 4M_f(1+a^2)B_{n,p,q}(x) + \omega_{a+1}(f;\delta)\left(1+\frac{\sqrt{B_{n,p,q}(x)}}{\delta}\right).
\end{aligned}$$

By taking $\delta=\sqrt{B_{n,p,q}(x)}$, we get the assertion of theorem 4.2. □

Now we discuss the weighted approximation theorem.

Theorem 4.3. *For $f \in C^*_{x^2}[0,\infty)$ and $n \geq 6$, we have*

$$\lim_{n\to\infty} ||D_{n,p_n,q_n}(f) - f||_{x^2} = 0. \tag{29}$$

Proof. By using the Korovkin theorem, we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} ||D_{n,p_n,q_n}(t^i;x) - x^i||_{x^2} = 0, \quad i = 0,1,2. \tag{30}$$

Since $D_{n,p_n,q_n}(1;x) = 1$ and $D_{n,p_n,q_n}(t;x) = x$, equality (30) holds true for $i = 0$ and $i = 1$. Finally, for $i = 2$, from lemma 3.2, we have

$$\begin{aligned}
&||D_{n,p_n,q_n}(t^2;x) - x^2||_{x^2} \\
= & \sup_{x\in[0,\infty)} \frac{|D_{n,p_n,q_n}(t^2;x) - x^2|}{1+x^2} \\
\leq & \left(\frac{p_n^n}{q_n[n]_{p_n,q_n}} + \frac{p_n^{n-3}}{q_n[n-3]_{p_n,q_n}} + \frac{p_n^{2n-3}}{q_n^2[n-3]_{p_n,q_n}[n]_{p_n,q_n}} \right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
& + \left(\frac{p_n^2+q_n^2}{p_n^3 q_n^3[n-3]_{p_n,q_n}} + \frac{2p_n^{n-2}}{q_n^3[n-3]_{p_n,q_n}[n]_{p_n,q_n}} \right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
& + \left(\frac{p_n^{n-4}}{q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{p_n^3 q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} \right) \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\
\leq & \frac{p_n^n}{q_n[n]_{p_n,q_n}} + \frac{p_n^2+q_n^2+p_n^n q_n^2}{p_n^3 q_n^3[n-3]_{p_n,q_n}} + \frac{p_n^{2n-3}q_n + 2p_n^{n-2}}{q_n^3[n-3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{p_n^3 q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} \\
& + \frac{p_n^{n-4}}{q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}[n]_{p_n,q_n}}.
\end{aligned}$$

We can obtain $\lim_{n\to\infty} ||D_{n,p_n,q_n}(t^2;x) - x^2|| = 0$ by using lemma 3.3 and $\lim_{n\to\infty} p_n^n = 1$, theorem 4.3 is proved. □

Table 1: The errors of the approximation of $D_{n,p_n,q_n}(t^2;x)$ with $p_n = 0.999999$ and different values of q_n and n .

q_n	$\|f(x) - D_{n,p_n,q_n}(f;x)\|_\infty$			
	$n=10$	$n=20$	$n=30$	$n=50$
0.95	0.756459	0.471385	0.396404	0.348978
0.99	0.545694	0.264869	0.185654	0.126539
0.999	0.502874	0.224749	0.146152	0.087113
0.9999	0.498679	0.220856	0.142352	0.083372
0.99999	0.498261	0.220468	0.141973	0.083000

Table 2: The errors of the approximation of $D_{n,p_n,q_n}(t^2;x)$ with $q_n = 0.99$ and different values of p_n and n .

$p_n = 1 - 1/10^m$	$\|f(x) - D_{n,p_n,q_n}(f;x)\|_\infty$			
	$n = 10$	$n = 20$	$n = 30$	$n = 50$
$m = 3$	0.545703746	0.264908767	0.185736472	0.126723749
$m = 4$	0.545694253	0.264872541	0.185660670	0.126555374
$m = 5$	0.545693781	0.264869729	0.185654271	0.126540622
$m = 6$	0.545693738	0.264869456	0.185653644	0.126539167
$m = 7$	0.545693733	0.264869429	0.185653581	0.126539022

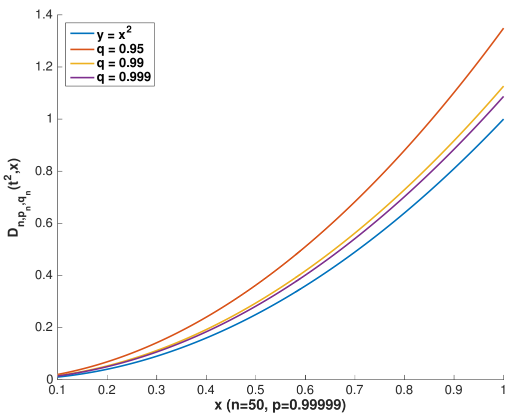


Figure 1: The figures of $D_{n,p_n,q_n}(t^2;x)$ for $n = 50$, $p_n = 0.99999$ and different values of q_n.

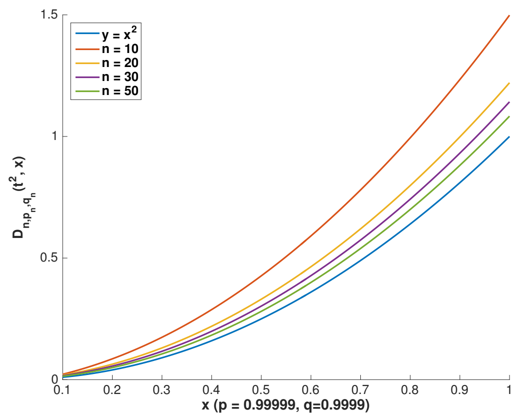


Figure 2: The figures of $D_{n,p_n,q_n}(t^2;x)$ for $p_n = 0.99999$, $q_n = 0.9999$ and different values of n.

5 Graphical and numerical analysis for one variable functions

In this section, we give several graphs and numerical examples to show the convergence of $D_{n,p_n,q_n}(f;x)$ to $f(x)$ with different values of parameters which satisfy the conclusions of lemma 3.3.

Let $f(x) = x^2$, the graphs of $D_{n,p_n,q_n}(f;x)$ with $n = 50$, $p_n = 0.99999$ and different values of q_n is shown in Figure 1. The graphs of $D_{n,p_n,q_n}(f;x)$ with $p_n = 0.99999$, $q_n = 0.9999$ and different values of n is shown in Figure 2. The graphs of $D_{n,p_n,q_n}(f;x)$ with $n = 50$, $q_n = 0.95$ and different values of p_n is shown in Figure 3. Moreover, we give the errors of the approximation of $D_{n,p_n,q_n}(f;x)$ to $f(x)$ with different parameters in Table 1 and Table 2.

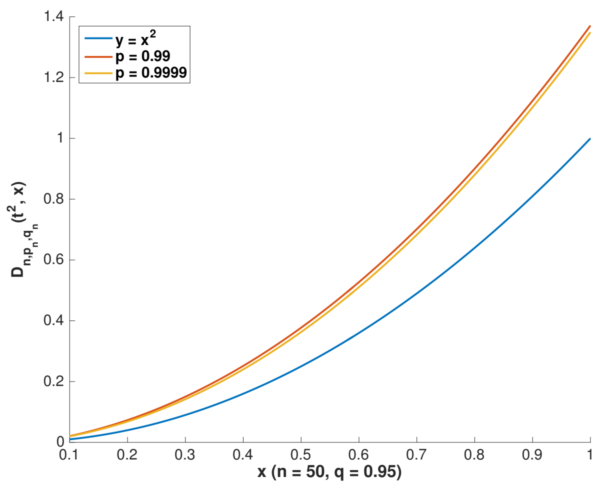


Figure 3: The figures of $D_{n,p_n,q_n}(t^2;x)$ for $n=50,\ q_n=0.95$ and different values of p_n.

6 Construction of bivariate operators and approximation properties

We introduce the bivariate tensor product $(p,\,q)$-analogue of Durrmeyer type Baskakov operators as follows

$$\begin{aligned} & D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y) \\ = & [n_1-1]_{p_{n_1},q_{n_1}}[n_2-1]_{p_{n_2},q_{n_2}}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\widetilde{b_{n_1,k_1}}(p_{n_1},q_{n_1};\mu(x))\widetilde{b_{n_2,k_2}}(p_{n_2},q_{n_2};\nu(y))\int_0^\infty\int_0^\infty \\ & \widetilde{b_{n_1,k_1}}(p_{n_1},q_{n_1};p_{n_1}u)\widetilde{b_{n_2,k_2}}(p_{n_2},q_{n_2};p_{n_2}v)f\left(p_{n_1}^{k_1}u,p_{n_2}^{k_2}v\right)d_{p_{n_1},q_{n_1}}u\,d_{p_{n_2},q_{n_2}}v, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mu(x) &= \frac{p_{n_1}^{n_1-2}q_{n_1}\left(p_{n_1}^2q_{n_1}[n_1-2]_{p_{n_1},q_{n_1}}x-1\right)}{[n_1]_{p_{n_1},q_{n_1}}},\ \left(x\geq\frac{1}{p_{n_1}^2q_{n_1}[n_1-2]_{p_{n_1},q_{n_1}}}\right), \\ \nu(y) &= \frac{p_{n_2}^{n_2-2}q_{n_2}\left(p_{n_2}^2q_{n_2}[n_2-2]_{p_{n_2},q_{n_2}}y-1\right)}{[n_2]_{p_{n_2},q_{n_2}}},\ \left(y\geq\frac{1}{p_{n_2}^2q_{n_2}[n_2-2]_{p_{n_2},q_{n_2}}}\right), \end{aligned}$$

$0<q_{n_1},q_{n_2}<p_{n_1},p_{n_2}\leq 1$ and $\widetilde{b_{n,k}}(p,q;x)$ is defined in (4).

Lemma 6.1. *Let $e_{i,j}(x,y)=x^iy^j, i,j\in\mathbb{N},\ (x,y)\in([0,\infty)\times[0,\infty))$ be the two dimensional test functions and $n_1,n_2\geq 6$, using lemma 3.1, we easily obtain the following equalities*

$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{0,0};x,y)=1, \tag{32}$$
$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{1,0};x,y)=x, \tag{33}$$

$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{0,1};x,y) = y, \tag{34}$$
$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{1,1};x,y) = xy, \tag{35}$$
$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{2,0};x,y) = \frac{[n_1-2]_{p_{n_1},q_{n_1}}[n_1+1]_{p_{n_1},q_{n_1}}}{q^2_{n_1}[n_1-3]_{p_{n_1},q_{n_1}}[n_1]_{p_{n_1},q_{n_1}}}x^2 + \frac{\left(p^2_{n_1}+q^2_{n_1}\right)}{p^3_{n_1}q^3_{n_1}[n_1-3]_{p_{n_1},q_{n_1}}}x$$
$$-\frac{2p^{n_1-2}_{n_1}}{q^3_{n_1}[n_1-3]_{p_{n_1},q_{n_1}}[n_1]_{p_{n_1},q_{n_1}}}x + \frac{p^{n_1-4}_{n_1}}{q^4_{n_1}[n_1-2]_{p_{n_1},q_{n_1}}[n_1-3]_{p_{n_1},q_{n_1}}[n_1]_{p_{n_1},q_{n_1}}}$$
$$-\frac{1}{p^3_{n_1}q^4_{n_1}[n_1-2]_{p_{n_1},q_{n_1}}[n_1-3]_{p_{n_1},q_{n_1}}}, \tag{36}$$
$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{0,2};x,y) = \frac{[n_2-2]_{p_{n_2},q_{n_2}}[n_2+1]_{p_{n_2},q_{n_2}}}{q^2_{n_2}[n_2-3]_{p_{n_2},q_{n_2}}[n_2]_{p_{n_2},q_{n_2}}}y^2 + \frac{\left(p^2_{n_2}+q^2_{n_2}\right)}{p^3_{n_2}q^3_{n_2}[n_2-3]_{p_{n_2},q_{n_2}}}y$$
$$-\frac{2p^{n_2-2}_{n_2}}{q^3_{n_2}[n_2-3]_{p_{n_2},q_{n_2}}[n_2]_{p_{n_2},q_{n_2}}}y + \frac{p^{n_2-4}_{n_2}}{q^4_{n_2}[n_2-2]_{p_{n_2},q_{n_2}}[n_2-3]_{p_{n_2},q_{n_2}}[n_2]_{p_{n_2},q_{n_2}}}$$
$$-\frac{1}{p^3_{n_2}q^4_{n_2}[n_2-2]_{p_{n_2},q_{n_2}}[n_2-3]_{p_{n_2},q_{n_2}}}. \tag{37}$$

Lemma 6.2. *For sufficiently large n_1 and n_2, using lemma 6.1 and lemma 3.2, we have the following statements*

$$\begin{aligned}
D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(t-x;x,y) &= 0,\\
D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(s-y;x,y) &= 0,\\
D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}((t-x)^2;x,y) &= O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)\left(x^2+x+1\right) = O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2,\\
D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}((s-y)^2;x,y) &= O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)\left(y^2+y+1\right) = O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2,\\
D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}((t-x)^4;x,y) &= O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)\left(x^4+x^3+x^2+x+1\right)\\
&= O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^4,\\
D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}((s-y)^4;x,y) &= O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)\left(y^4+y^3+y^2+y+1\right)\\
&= O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^4.
\end{aligned}$$

Let B_ρ be the space of all functions f defined on $[0,\infty)\times[0,\infty)$ satisfying the condition $|f(x)| \le M_f\rho(x,y)$, where M_f is a positive constant depending only on f and $\rho(x,y) = 1+x^2+y^2$ is a weighted function. We denote the subspace of all continuous functions belong to B_ρ by C_ρ. Let C^*_ρ be the subspace of all functions $f \in C_\rho$, for which $\lim_{\sqrt{x^2+y^2}\to\infty} \frac{f(x,y)}{\rho(x,y)}$ is finite. The norm on C^*_ρ is $||f||_\rho = \sup_{x,y\in[0,\infty)} \frac{|f(x,y)|}{\rho(x,y)}$. For the infinite interval $[0,\infty)$,

$f \in C^*_\rho$ and $\delta_1, \delta_2 > 0$, İspir and Atakut [28] introduced the weighted modulus of continuity as

$$\Omega_\rho(f;\delta_1,\delta_2) = \sup_{x,y\in[0,\infty)} \sup_{0\le|k_1|\le\delta_1, 0\le|k_2|\le\delta_2} \frac{|f(x+k_1,y+k_2)-f(x,y)|}{\rho(x,y)\rho(k_1,k_2)},$$

which satisfy the following inequality

$$\Omega_\rho(f;d_1\delta_1,d_2\delta_2) \le 4(1+d_1)(1+d_2)(1+\delta_1^2)(1+\delta_2^2)\Omega_\rho(f;\delta_1,\delta_2), \quad d_1,d_2>0. \tag{38}$$

From the definition of Ω_ρ, we have

$$\begin{aligned} & |f(t,s)-f(x,y)| \\ \le\ & \rho(x,y)\rho(|t-x|,|s-y|)\Omega_\rho(f;|t-x|,|s-y|) \\ \le\ & \left(1+x^2+y^2\right)\left(1+(t-x)^2\right)\left(1+(s-y)^2\right)\Omega_\rho(f;|t-x|,|s-y|) \end{aligned} \tag{39}$$

Now, we establish the degree approximation of operators $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}$ in the weighted space C^*_ρ by the weighted modulus of continuity Ω_ρ.

Theorem 6.3. *For $f \in C^*_\rho$, then for sufficiently large n_1, n_2, we have the following inequality*

$$\sup_{x,y\in[0,\infty)} \frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)}{(\rho(x,y))^3} \le C\Omega_\rho\left(f;\frac{1}{\sqrt{[n_1]_{p_{n_1},q_{n_1}}}},\frac{1}{\sqrt{[n_2]_{p_{n_2},q_{n_2}}}}\right),$$

where C is a positive constant.

Proof. From (38) and (39), for $\delta_{n_1}, \delta_{n_2} > 0$, we get

$$\begin{aligned} & |f(t,s)-f(x,y)| \\ =\ & 4\left(1+x^2+y^2\right)\left(1+(t-x)^2\right)\left(1+(s-y)^2\right)\left(1+\frac{|t-x|}{\delta_{n_1}}\right)\left(1+\frac{|s-y|}{\delta_{n_2}}\right)\left(1+\delta_{n_1}^2\right) \\ & \times\left(1+\delta_{n_2}^2\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}) \\ =\ & 4\left(1+x^2+y^2\right)\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)\left(1+\frac{|t-x|}{\delta_{n_1}}+(t-x)^2+\frac{|t-x|}{\delta_{n_1}}(t-x)^2\right) \\ & \times\left(1+\frac{|s-y|}{\delta_{n_2}}+(s-y)^2+\frac{|s-y|}{\delta_{n_2}}(s-y)^2\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}), \end{aligned}$$

applying the operators $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}$ on the above inequality, we have

$$\begin{aligned} & |D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)| \\ \le\ & D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(|f(t,s)-f(x,y)|;x,y) \\ \le\ & 4(1+x^2+y^2)\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(1+\frac{|t-x|}{\delta_{n_1}}+(t-x)^2\right. \\ & \left.+\frac{|t-x|}{\delta_{n_1}}(t-x)^2;x,y\right)D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(1+\frac{|s-y|}{\delta_{n_2}}+(s-y)^2+\frac{|s-y|}{\delta_{n_2}}(s-y)^2;x,y\right) \end{aligned}$$

$$
\begin{aligned}
&\times\Omega_\rho(f;\delta_{n_1},\delta_{n_2})\\
= \;& 4\left(1+x^2+y^2\right)\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)\left(1+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(|t-x|;x,y)}{\delta_{n_1}}\right.\\
&\left.+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(|t-x|(t-x)^2;x,y\right)}{\delta_{n_1}}\right)\\
&+\left(1+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(|s-y|;x,y)}{\delta_{n_2}}+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)\right.\\
&\left.+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(|s-y|(s-y)^2;x,y\right)}{\delta_{n_2}}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}).
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&|D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)|\\
\leq\;& 4(1+x^2+y^2)\left(1+\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)}}{\delta_{n_1}}+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)\right.\\
&\left.+\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)}\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^4;x,y\right)}}{\delta_{n_1}}\right)\\
&\times\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)\left(1+\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)}}{\delta_{n_2}}\right.\\
&+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)+\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)}\\
&\left.\times\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^4;x,y\right)}}{\delta_{n_2}}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}).
\end{aligned}
$$

Using lemma 6.2, we have

$$
\begin{aligned}
&|D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)|\\
\leq\;& 4(1+x^2+y^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\left(1+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2}\right.\\
&\left.+O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2}\sqrt{O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^4}\right)\\
&\times\left(1+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2}+O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2\right.\\
&\left.+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2}\sqrt{O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^4}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}).
\end{aligned}
$$

Then, we have

$$\begin{aligned} & |D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)| \\ \leq\ & 4(1+x^2+y^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\left(1+\frac{1}{\delta_{n_1}}\sqrt{\frac{C_1}{[n_1]_{p_{n_1},q_{n_1}}}(x+1)+\frac{C_1}{[n_1]_{p_{n_1},q_{n_1}}}(x+1)^2}\right. \\ & \left.+\frac{1}{\delta_{n_1}}\sqrt{\frac{C_1^2}{[n_1]^2_{p_{n_1},q_{n_1}}}(x+1)^3}\right)\left(1+\delta_{n_2}\sqrt{\frac{C_2}{[n_2]_{p_{n_2},q_{n_2}}}(y+1)+\frac{C_2}{[n_2]_{p_{n_2},q_{n_2}}}(y+1)^2}\right. \\ & \left.+\frac{1}{\delta_{n_2}}\sqrt{\frac{C_2^2}{[n_2]^2_{p_{n_2},q_{n_2}}}(y+1)^3}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}). \end{aligned}$$

Let $\delta_{n_1}=\frac{1}{\sqrt{[n_1]_{p_{n_1},q_{n_1}}}}$ and $\delta_{n_2}=\frac{1}{\sqrt{[n_2]_{p_{n_2},q_{n_2}}}}$, we have

$$\begin{aligned} & |D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)| \\ \leq\ & 4(1+x^2+y^2)\left(1+\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)\left(1+\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)C(1+x^2+y^2)^2\Omega_\rho(f;\delta_{n_1},\delta_{n_2}), \end{aligned}$$

where C is a positive constant. Theorem 6.3 is proved. □

7 Graphical and numerical analysis for two variables functions

In this section, we give several graphs and numerical examples to show the convergence of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ to $f(x,y)$ with different values of parameters which satisfy the conclusions of lemma 3.3.

Let $f(x,y)=x^2y^2$, the graphs of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $n_1=n_2=30$, $p_{n_1}=p_{n_2}=0.9999$, $q_{n_1}=q_{n_2}=0.999$ and $f(x,y)=x^2y^2$ are shown in Figure 4. The graphs of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $n_1=n_2=50$, $p_{n_1}=p_{n_2}=0.99999$, $q_{n_1}=q_{n_2}=0.9999$ and $f(x,y)=x^2y^2$ are shown in Figure 5. Moreover, we give the errors of the approximation of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ to $f(x,y)$ with different parameters in Table 3 and Table 4.

Table 3: The errors of the approximation of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $p_{n_1}=p_{n_2}=0.99999$ and different values of $q_{n_1}=q_{n_2}=q$ and $n_1=n_2=n$.

q	$\|f(x)-D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)\|_\infty$			
	$n=10$	$n=20$	$n=30$	$n=50$
0.95	2.085144	1.164978	0.949957	0.819780
0.99	1.389169	0.599895	0.405776	0.269094
0.999	1.258631	0.500011	0.313666	0.181816
0.9999	1.246041	0.490490	0.3049678	0.173697

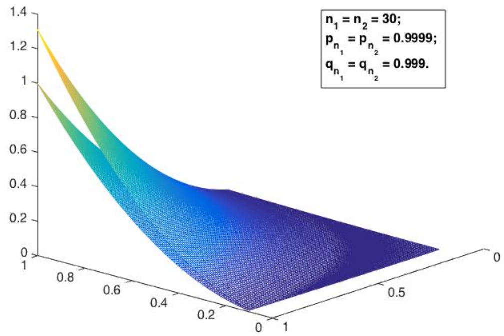


Figure 4: The figures of (the upper one) $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ for $n_1 = n_2 = 30$, $p_{n_1} = p_{n_2} = 0.9999$, $q_{n_1} = q_{n_2} = 0.999$, and (the below one) $f(x,y) = x^2y^2$.

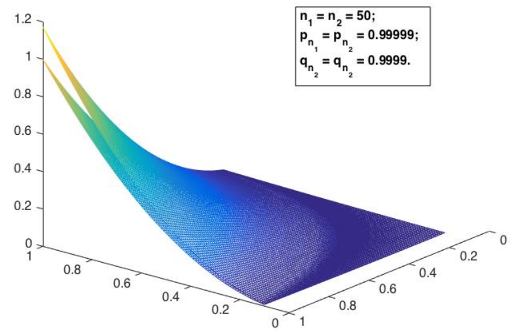


Figure 5: The figures of (the upper one) $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ for $n_1 = n_2 = 50$, $p_{n_1} = p_{n_2} = 0.99999$, $q_{n_1} = q_{n_2} = 0.9999$, and (the below one) $f(x,y) = x^2y^2$.

Table 4: The errors of the approximation of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $q_{n_1} = q_{n_2} = 0.999$ and different values of $p_{n_1} = p_{n_2} = p$ and $n_1 = n_2 = n$.

$p = 1 - 1/10^m$	$\Vert f(x) - D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)\Vert_\infty$			
	$n = 10$	$n = 20$	$n = 30$	$n = 50$
$m = 4$	1.25863727	0.50001245	0.31366763	0.18181861
$m = 5$	1.25863086	0.50001052	0.31366566	0.18181572
$m = 6$	1.25863023	0.50001034	0.31366548	0.18181546
$m = 7$	1.25863016	0.50001033	0.31366546	0.18181544

8 Further discussion

If we consider the following modified forms $\widetilde{D_{n,p,q}}$,

$$\widetilde{D_{n,p,q}}(f;x) = [n-1]_{p,q} \sum_{k=0}^{\infty} \widetilde{b_{n,k}}(p,q;x) \int_0^\infty \widetilde{b_{n,k}}(p,q;pu) f(p^k u) d_{p,q}u, \tag{40}$$

where $x \in [0,\infty)$, $\widetilde{b_{n,k}}(p,q;x)$ is defined in (4). Here we omit the bivariate forms of operators (40). By similar computations in section 3, we know these operators (40) reproduce only constant functions, but not linear functions. We also provide two graphs to show that the operators $D_{n,p,q}$ give a better approximation to f than $\widetilde{D_{n,p,q}}$ and so is the bivariate case (See Figure 6 and Figure 7), hence it is more appropriate to consider the operators $D_{n,p,q}$ and the bivariate ones defined in (31).

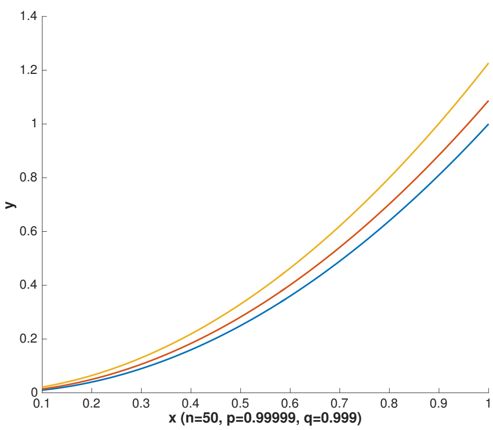


Figure 6: The figures of $D_{n,p_n,q_n}(f;x)$ (the red one), $\widetilde{D_{n,p_n,q_n}}(f;x)$ (the yellow one)for $n = 50$, $p_n = 0.99999$, $q_n = 0.999$, and $f(x) = x^2$ (the blue one).

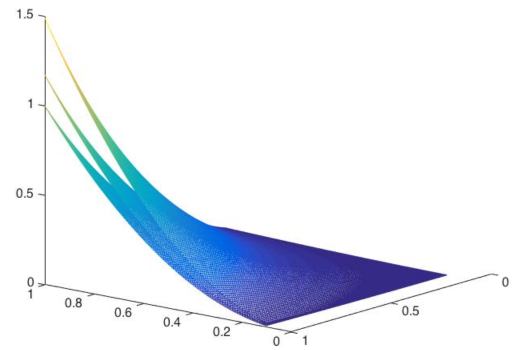


Figure 7: The figures of (the middle one) $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ and $\widetilde{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}}(f;x,y)$ (the upper one) for $n_1 = n_2 = 50$, $p_{n_1} = p_{n_2} = 0.99999$, $q_{n_1} = q_{n_2} = 0.9999$, and $f(x,y) = x^2y^2$ (the below one).

Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11601266 and 11626201), the Natural Science Foundation of Fujian Province of China (Grant No. 2016J05017) and the Program for New Century Excellent Talents in Fujian Province University. We also thank Fujian Provincial Key Laboratory of Data Intensive Computing and Key Laboratory of Intelligent Computing and Information Processing of Fujian Province University.

References

[1] M. Mursaleen, K. J. Ansari, A. Khan, On (p, q)-analogue of Bernstein operators. Appl. Math. Comput., **266** (2015), 874-882.

[2] M. Mursaleen, K. J. Ansari, A. Khan, Erratum to "On (p, q)-analogue of Bernstein operators [Appl. Math. Comput. 266 (2015) 874-882]", Appl. Math. Comput., **278** (2016), 70-71.

[3] T. Acar, (p, q)-Generalization of Szász-Mirakyan operators, Math. Methods Appl. Sci., **39(10)** (2016), 2685-2695.

[4] T. Acar, A. Aral, S. A. Mohiuddine, On Kantorovich modification of (p, q)-Baskakov operators, J. Ineq. App., (2016), Doi: 10.1186/s13660-016-1045-9.

[5] V. Gupta, (p, q)-Baskakov-Kantorovich operators, Appl. Math. Inf. Sci., **10(4)** (2016), 1551-1556.

[6] N. Malik, V. Gupta, Approximation by (p, q)-Baskakov-Beta operators, Appl. Math. Comput., **293** (2017), 49-56.

[7] Q. -B. Cai, G. Zhou, On (p, q)-analogue of Kantorovich type Bernstein-Stancu-Schurer operators, Appl. Math. Comput., **276** (2016), 12-20.

[8] T. Acar, On pointwise convergence of q-Bernstein operators and their q-derivatives, Nurmer. Funct. Anal. Optim., **36(3)** (2015), 287-304.

[9] T. Acar, P. Agrawal, A. Kumar, On a modification of (p, q)-Szász-Mirakyan operators, Complex Anal. Oper. Theory, (2016), Doi: 10.1007/s11785-016-0613-9.

[10] H. Ilarslan, T. Acar, Approximation by bivariate (p, q)-Baskakov-Kantorovich operators, Georgian Math. J., (2016), Doi: 10.1515/gmj-2016-0057.

[11] G. M. Phillips, Bernstein polynomials based on the q-integers, Ann. Number. Math., **4** (1997), 511-518.

[12] V. Gupta, T. Kim, On the rate of approximation by q modified Beta operators, J. Math. Anal. Appl., **377** (2011), 471-480.

[13] V. Gupta, A. Aral, Convergence of the q analogue of Szász-Beta operators, Appl. Math. Comput., **216** (2010), 374-380.

[14] K. Khan, D. K. Lobiyal, Bézier curves based on Lupas (p, q)-analogue of Bernstein functions in CAGD, J. Comput. Appl. Math., **317** (2017), 458-477.

[15] A. Aral, V. Gupta, On the Durrmeyer type modification of the q-Baskakov type operators, Nonlinear Anal., **72** (2010), 1171-1180.

[16] V. Gupta, On certain Durrmeyer type q Baskakov operators, Ann. Univ. Ferrara, **56** (2010), 295-303.

[17] Q. -B. Cai, X. -M. Zeng, Convergence of modification of the Durrmeyer type q-Baskakov operators, Georgian Math. J., **19** (2012), 49-61.

[18] T. Acar, A. Aral, M. Mursaleen, Approximation by Baskakov-Durrmeyer operators based on (p, q)-integers, arXiv: submit/1450876 [math. CA].

[19] V. N. Mishra, S. Pandey, On (p, q)-Baskakov-Durrmeyer-Stancu operators, (2016), arXiv: 1602. 06719.

[20] M. N. Hounkonnou, J. Désiré, B. Kyemba, $R(p, q)$-calculus: differentiation and integration, SUT Journal of Mathematics, **49** (2013), 145-167.

[21] R. Jagannathan, K. S. Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, Proceedings of the International Conference on Number Theory and Mathematical Physics, (2005) 20-21.

[22] J. Katriel, M. Kibler, Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers, J. Phys. A: Math. Gen. (1992) 24, 2683-2691, printed in the UK.

[23] P. N. Sadjang, On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas, (2015) arXiv: 1309.3934v1.

[24] V. Sahai, S. Yadav, Representations of two parameter quantum algebras and p, q-special functions, J. Math. Anal. Appl., **335**(2007), 268-279.

[25] Q. -B. Cai, X. -W. Xu, A basic problem of (p, q)-Bernstein operators, J. Ineq. Appl., **140** (2017), Doi: 10. 1186/s13660-017-1413-0.

[26] G. A. Anastassiou, S. G. Gal, Approximation theory: moduli of continuity and global smoothness preservation, Birkhauser, Boston, 2000.

[27] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.

[28] N. İspir, C. Atakut, Approximation by modified Szász-Mirakjan operators on weighted spaces, Proc. Indian Acad. Sci. Math. Sci., **112(4)** (2002), 571-578.

A subclass of analytic functions defined by a fractional integral operator

Alb Lupaş Alina
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro, alblupas@gmail.com

Abstract

Making use the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative, we introduce a new class of analytic functions $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ defined on the open unit disc, and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$.

Keywords: Analytic functions, univalent functions, radii of starlikeness and convexity, neighborhood property, Salagean operator, Ruscheweyh operator.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}: |z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}(p,t)=\{f\in\mathcal{H}(U): f(z)=z^p+\sum_{j=p+t}^{\infty}a_jz^j, z\in U\}$, with $\mathcal{A}(1,t)=\mathcal{A}_t$ and $\mathcal{H}[a,t]=\{f\in\mathcal{H}(U): f(z)=a+a_tz^t+a_{t+1}z^{t+1}+\ldots, z\in U\}$, where $p,t\in\mathbb{N}$, $a\in\mathbb{C}$.

Definition 1.1 *(Sălăgean [4]) For $f\in\mathcal{A}_t$, and $n\in\mathbb{N}$, the operator S^n is defined by $S^n:\mathcal{A}_t\to\mathcal{A}_t$,*

$$S^0f(z)=f(z),\ S^1f(z)=zf'(z),\ldots,\ S^{n+1}f(z)=z\left(S^nf(z)\right)',\ z\in U.$$

Remark 1.1 *If $f\in\mathcal{A}_t$, $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j$, then $S^nf(z)=z+\sum_{j=t+1}^{\infty}j^na_jz^j$, $z\in U$.*

Definition 1.2 *(Ruscheweyh [3]) For $f\in\mathcal{A}_t$ and $n\in\mathbb{N}$, the operator R^n is defined by $R^n:\mathcal{A}_t\to\mathcal{A}_t$,*

$$R^0f(z)=f(z),\ R^1f(z)=zf'(z),\ldots,\ (n+1)R^{n+1}f(z)=z\left(R^nf(z)\right)'+nR^nf(z),\ z\in U.$$

Remark 1.2 *If $f\in\mathcal{A}_t$, $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j$, then $R^nf(z)=z+\sum_{j=t+1}^{\infty}\frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)}a_jz^j$ for $z\in U$.*

Definition 1.3 *Let $n,m\in\mathbb{N}$. Denote by $SR_{\lambda}^{m,n}:\mathcal{A}_t\to\mathcal{A}_t$ the operator given by the Hadamard product of the Sălăgean operator S^m and the Ruscheweyh derivative R^n, $SR^{m,n}f(z)=(S^m*R^n)f(z)$, for any $z\in U$ and each nonnegative integers m,n.*

Remark 1.3 *If $f\in\mathcal{A}_t$ and $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j$, then $SR^{m,n}f(z)=z+\sum_{j=t+1}^{\infty}j^m\frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)}a_j^2z^j$, $z\in U$.*

Definition 1.4 *([2]) The fractional integral of order λ ($\lambda>0$) is defined for a function f by $D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(t)}{(z-t)^{1-\lambda}}dt$, where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t)>0$.*

From Definition 1.3 and Definition 1.4, we get the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative, which has the following form

$$D_z^{-\lambda}SR^{m,n}f(z)=\frac{1}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=t+1}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda},$$

for a function $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j\in\mathcal{A}_t$.

Following the work from [1] we can define the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ as follows.

Definition 1.5 *For $\mu \geq 0$, $\lambda \in \mathbb{N}$, $\alpha \in \mathbb{C} - \{0\}$ and $0 < \beta \leq 1$, let $\mathcal{D}(\mu, \lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}_t$ consisting of functions that satisfying the inequality*

$$\left| \frac{\lambda (1-\mu) \frac{D_z^{-\lambda} SR^{m,n} f(z)}{z} + \mu \left(D_z^{-\lambda} SR^{m,n} f(z)\right)'}{\lambda (1-\mu) \frac{D_z^{-\lambda} SR^{m,n} f(z)}{z} + \mu \left(D_z^{-\lambda} SR^{m,n} f(z)\right)' - \alpha} \right| < \beta \tag{1.1}$$

2 Coefficient bounds

In this section we obtain coefficient bounds and extreme points for functions is $\mathcal{D}(\mu, \lambda, \alpha, \beta)$.

Theorem 2.1 *Let the function $f \in \mathcal{A}_t$. Then $f \in \mathcal{D}(\mu, \lambda, \alpha, \beta)$ if and only if*

$$\sum_{j=t+1}^{\infty} \frac{(\beta+1)(\lambda+\mu j) j^{m+1} \Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < \beta |\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}. \tag{2.1}$$

The result is sharp for the function $F(z)$ defined by $F(z) = z + \sqrt{\frac{\left(\beta|\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}} z^j$, $j \geq t+1$.

Proof. Suppose f satisfies (2.1). Then for $|z| < 1$, we have
$\left|\lambda (1-\mu) \frac{D_z^{-\lambda} SR^{m,n} f(z)}{z} + \mu \left(D_z^{-\lambda} SR^{m,n} f(z)\right)'\right| - \beta \left|\lambda (1-\mu) \frac{D_z^{-\lambda} SR^{m,n} f(z)}{z} + \mu \left(D_z^{-\lambda} SR^{m,n} f(z)\right)' - \alpha\right| =$
$\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1}\right| - \beta \left|\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1} - \alpha\right| \leq$
$\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda\right| + \left|\sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1}\right| - \beta|\alpha| + \beta\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda\right| + \beta\left|\sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1}\right| <$
$\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)} - \beta|\alpha| + \sum_{j=t+1}^\infty \frac{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < 0.$
Hence, by using the maximum modulus Theorem and (1.1), $f \in \mathcal{D}(\mu, \lambda, \alpha, \beta)$. Conversely, assume that
$\left|\frac{\lambda(1-\mu)\frac{D_z^{-\lambda} SR^{m,n} f(z)}{z} + \mu\left(D_z^{-\lambda} SR^{m,n} f(z)\right)'}{\lambda(1-\mu)\frac{D_z^{-\lambda} SR^{m,n} f(z)}{z} + \mu\left(D_z^{-\lambda} SR^{m,n} f(z)\right)' - \alpha}\right| = \left|\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1} - \alpha}\right| < \beta$, $z \in U$.
Since $Re(z) \leq |z|$ for all $z \in U$, we have $Re\left\{\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)} z^\lambda + \sum_{j=t+1}^\infty \frac{(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda-1} - \alpha}\right\} < \beta$. By choosing choose values of z on the real axis so that $\lambda (1-\mu) \frac{D_z^{-\lambda} SR^{m,n} f(z)}{z} + \mu \left(D_z^{-\lambda} SR^{m,n} f(z)\right)'$ is real and letting $z \to 1$ through real values, we obtain the desired inequality (2.1). ■

Corollary 2.2 *If $f \in \mathcal{A}_t$ be in $\mathcal{D}(\mu, \lambda, \alpha, \beta)$, then*

$$a_j \leq \sqrt{\frac{\left(\beta|\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}}, \quad j \geq t+1, \tag{2.2}$$

with equality only for functions of the form $F(z)$.

Theorem 2.3 *Let $f_1(z) = z$ and $f_j(z) = z - \sqrt{\frac{\left(\beta|\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}} z^j$, $j \geq t+1$, for $\mu \geq 0$, $\lambda \in \mathbb{N}$, $\alpha \in \mathbb{C} - \{0\}$ and $0 < \beta \leq 1$. Then $f(z)$ is in the class $\mathcal{D}(\mu, \lambda, \alpha, \beta)$ if and only if it can be expressed in the form*

$$f(z) = \sum_{j=1}^{\infty} \omega_j f_j(z), \tag{2.3}$$

where $\omega_j \geq 0$ and $\sum_{j=1}^\infty \omega_j = 1$.

Proof. Suppose $f(z)$ can be written as in (2.3). Then $f(z) = z - \sum_{j=t+1}^\infty \omega_j \sqrt{\frac{\left(\beta|\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}} z^j$.
Now, $\sum_{j=t+1}^\infty \sqrt{\frac{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}} \omega_j \sqrt{\frac{\left(\beta|\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}} = \sum_{j=t+1}^\infty \omega_j = 1 - \omega_1 \leq 1$. Thus $f \in \mathcal{D}(\mu, \lambda, \alpha, \beta)$.

Conversely, let $f \in \mathcal{D}(\mu, \lambda, \alpha, \beta)$. Then by using (2.2), setting $\omega_j = \sqrt{\frac{\left(\beta|\alpha| - \frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j) j^{m+1}\Gamma(n+j)}} a_j$, $j \geq t+1$ and $\omega_1 = 1 - \sum_{j=2}^\infty \omega_j$, we have $f(z) = \sum_{j=1}^\infty \omega_j f_j(z)$. And this completes the proof of Theorem 2.3. ■

3 Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$.

Theorem 3.1 *If $f\in\mathcal{D}(\mu,\lambda,\alpha,\beta)$, then*

$$r-\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}\le|f(z)| \tag{3.1}$$

$$\le r+\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}$$

holds if the sequence $\{\sigma_j(\mu,\lambda,\alpha,\beta)\}_{j=t+1}^{\infty}$ is non-decreasing, and

$$1-(t+1)\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t}\le|f'(z)| \tag{3.2}$$

$$\le 1+(t+1)\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t}$$

holds if the sequence $\{\frac{\sigma_j(\mu,\lambda,\beta)}{j}\}_{j=t+1}^{\infty}$ is non- decreasing, where $\sigma_j(\mu,\lambda,\beta)=\sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}}$.

The bounds in (3.1) and (3.2) are sharp, for $f(z)$ given by $f(z)=z+\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(t+\lambda+2)}{(\beta+1)(\lambda+\mu t+\mu)(t+1)^{m+1}\Gamma(n+t+1)}}z^{t+1}$, $z=\pm r$.

Proof. In view of Theorem 2.1, we have $\sum_{j=t+1}^{\infty}a_j\le\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(t+\lambda+2)}{(\beta+1)(\lambda+\mu t+\mu)(t+1)^{m+1}\Gamma(n+t+1)}}$. We obtain $|z|-|z|^{t+1}\sum_{j=t+1}^{\infty}a_j\le|f(z)|\le|z|+|z|^{t+1}\sum_{j=t+1}^{\infty}a_j$. Thus

$$r-\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}\le|f(z)| \tag{3.3}$$

$$\le r+\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}.$$

Hence (3.1) follows from (3.3). Further, $\sum_{j=t+1}^{\infty}ja_j\le\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(t+\lambda+2)}{(\beta+1)(\lambda+\mu t+\mu)(t+1)^{m+1}\Gamma(n+t+1)}}$. Hence (3.2) follows from $1-r^t\sum_{j=t+1}^{\infty}ja_j\le|f'(z)|\le1+r^t\sum_{j=t+1}^{\infty}ja_j$. ■

4 Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ are given in this section.

Theorem 4.1 *Let the function $f\in\mathcal{A}_t$ belong to the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$, Then $f(z)$ is close -to-convex of order δ, $0\le\delta<1$ in the disc $|z|<r$, where $r:=\inf_{j\ge t+1}\left[\sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}\right]^{\frac{1}{t}}$. The result is sharp, with extremal function $f(z)$ given by (2.2).*

Proof. For given $f\in\mathcal{A}_t$ we must show that

$$|f'(z)-1|<1-\delta. \tag{4.1}$$

By a simple calculation we have $|f'(z)-1| \leq \sum_{j=t+1}^{\infty} ja_j |z|^t$. The last expression is less than $1-\delta$ if $\sum_{j=t+1}^{\infty} \frac{j}{1-\delta} a_j |z|^t < 1$. Using the fact that $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty} \frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < 1$, (4.1) holds true if $\frac{j}{1-\delta}|z|^t \leq \sum_{j=t+1}^{\infty} \sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$.

Or, equivalently, $|z|^t \leq \sum_{j=t+1}^{\infty} \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$, which completes the proof. ■

Theorem 4.2 *Let $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$. Then*

1. f is starlike of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_1$ where,

$$r_1 = \inf_{j \geq t+1} \left\{ \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j+\delta-2)^2\Gamma(j+\lambda+1)}} \right\}^{\frac{1}{t}}.$$

2. f is convex of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_2$ where,

$$r_2 = \inf_{j \geq t+1} \left\{ \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j-1)^2\Gamma(j+\lambda+1)}} \right\}^{\frac{1}{t}}.$$

Each of these results is sharp for the extremal function $f(z)$ given by (2.3).

Proof. 1. For $0 \leq \delta < 1$ we need to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1-\delta. \tag{4.2}$$

We have $\left|\frac{zf'(z)}{f(z)}-1\right| \leq \left|\frac{\sum_{j=t+1}^{\infty}(j-1)a_j|z|^t}{1+\sum_{j=t+1}^{\infty}a_j|z|^t}\right|$. The last expresion is less than $1-\delta$ if $\sum_{j=t+1}^{\infty} \frac{(j+\delta-2)}{1-\delta} a_j |z|^t < 1$. Using the fact that $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty} \frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < 1$, (4.2) holds true if $\frac{j+\delta-2}{1-\delta}|z|^t < \sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$.

Or, equivalently, $|z|^t < \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j+\delta-2)^2\Gamma(j+\lambda+1)}}$, which yields the starlikeness of the family.

2. Using the fact that f is convex if and only zf' is starlike, we can prove (2) with a similar way of the proof of (1). The function f is convex if and only if

$$|zf''(z)| < 1-\delta. \tag{4.3}$$

We have $|zf''(z)| \leq \left|\sum_{j=t+1}^{\infty} j(j-1)a_j |z|^{t-1}\right| < 1-\delta$, i.e. $\sum_{j=t+1}^{\infty} \frac{j(j-1)}{1-\delta} a_j |z|^{t-1} < 1$. Using the fact that $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty} \frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < 1$, (4.3) holds true if $\frac{j(j-1)}{1-\delta}|z|^{t-1} < \sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$, or, equivalently, $|z|^{t-1} < \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j-1)^2\Gamma(j+\lambda+1)}}$, which yields the convexity of the family. ■

References

[1] A. Alb Lupas, *Aspects of univalent holomorphic functions involving Sălăgean operator and Ruscheweyh derivative,* J. of Concrete Applicable Math., 13 (2015), No.'s 1-2, 51-59.

[2] N.E.Cho, A.M.K. Aouf, *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Tr. J. of Mathematics, Vol. 20, 1996, 553-562.

[3] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[4] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013 (1983), 362-372.

Properties on a subclass of analytic functions defined by a fractional integral operator

Alb Lupaş Alina
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro, alblupas@gmail.com

Abstract

In this paper we have introduced and studied the subclass $\mathcal{L}(\lambda, d, \alpha, \beta)$ using the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{L}(\lambda, d, \alpha, \beta)$.

Keywords: Analytic functions, univalent functions, radii of starlikeness and convexity, neighborhood property, Salagean operator, Ruscheweyh operator.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}(p,t) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+t}^{\infty} a_j z^j, \ z \in U\}$, with $\mathcal{A}(1,1) = \mathcal{A}$ and $\mathcal{H}[a,t] = \{f \in \mathcal{H}(U) : f(z) = a + a_t z^t + a_{t+1} z^{t+1} + \ldots, \ z \in U\}$, where $p, t \in \mathbb{N}$, $a \in \mathbb{C}$.

Definition 1.1 *(Sălăgean [4]) For $f \in \mathcal{A}$, and $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$S^0 f(z) = f(z), \ S^1 f(z) = z f'(z), \ \ldots, \ S^{n+1} f(z) = z \left(S^n f(z)\right)', \quad z \in U.$$

Remark 1.1 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.*

Definition 1.2 *(Ruscheweyh [3]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$R^0 f(z) = f(z), \ R^1 f(z) = z f'(z), \ \ldots, \ (n+1) R^{n+1} f(z) = z \left(R^n f(z)\right)' + n R^n f(z), \quad z \in U.$$

Remark 1.2 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} a_j z^j$ for $z \in U$.*

Definition 1.3 *Let $m, n \in \mathbb{N}$. Denote by $SR_{\lambda}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the Sălăgean operator S^m and the Ruscheweyh derivative R^n, $SR^{m,n} f(z) = (S^m * R^n) f(z)$,for any $z \in U$ and each nonnegative integers m, n.*

Remark 1.3 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $SR^{m,n} f(z) = z + \sum_{j=2}^{\infty} j^m \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} a_j^2 z^j$, $z \in U$.*

Definition 1.4 *([2]) The fractional integral of order λ ($\lambda > 0$) is defined for a function f by $D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt$, where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.*

From Definition 1.3 and Definition 1.4, we get the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative, which has the following form $D_z^{-\lambda} SR^{m,n} f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=t+1}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda}$, for a function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$.

We follow the works from [1].

Definition 1.5 *Let the function $f \in \mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$ if it satisfies the following criterion:*

$$\left|\frac{1}{d}\left(\frac{z(D_z^{-\lambda}SR^{m,n}f(z))' + \alpha z^2(D_z^{-\lambda}SR^{m,n}f(z))''}{(1-\alpha)D_z^{-\lambda}SR^{m,n}f(z) + \alpha z(D_z^{-\lambda}SR^{m,n}f(z))'} - 1\right)\right| < \beta, \tag{1.1}$$

where $\lambda > 0$, $\in \mathbb{C} - \{0\}$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $m, n \in \mathbb{N}$, $z \in U$.

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ, $0 \le \delta < 1$, for these functions.

2 Coefficient Inequality

Theorem 2.1 *Let the function $f \in \mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$ if and only if*

$$\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2 + [\alpha(2\lambda - 2 + \beta|d|) + 1]j + [\alpha(\lambda-1)+1](\lambda - 1 + \beta|d|)\right\}a_j^2 \le$$

$$(\alpha\lambda + 1)(\beta|d| - \lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}, \tag{2.1}$$

where $\lambda > 0$, $\in \mathbb{C} - \{0\}$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $m, n \in \mathbb{N}$, $z \in U$.

Proof. Let $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$. Assume that inequality (2.1) holds true. Then we find that
$\left|\frac{z(D_z^{-\lambda}SR^{m,n}f(z))'+\alpha z^2(D_z^{-\lambda}SR^{m,n}f(z))''}{(1-\alpha)D_z^{-\lambda}SR^{m,n}f(z)+\alpha z(D_z^{-\lambda}SR^{m,n}f(z))'} - 1\right| =$
$\left|\frac{\frac{\lambda(\alpha\lambda+1)}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}\left\{\alpha j^2+[2\alpha(\lambda-1)+1]j+(\lambda-1)[\alpha(\lambda-1)+1]\right\}a_j^2 z^{j+\lambda}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}[\alpha j+\alpha(\lambda-1)+1]a_j^2 z^{j+\lambda}}\right| \le$
$\frac{\frac{\lambda(\alpha\lambda+1)}{\Gamma(\lambda+2)}+\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}\left\{\alpha j^2+[2\alpha(\lambda-1)+1]j+(\lambda-1)[\alpha(\lambda-1)+1]\right\}a_j^2\left|z^{j-1}\right|}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)}-\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}[\alpha j+\alpha(\lambda-1)+1]a_j^2|z^{j-1}|} \le \beta|d|.$
Choosing values of z on real axis and letting $z \to 1^-$, we have
$\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2 + [\alpha(2\lambda-2+\beta|d|)+1]j + [\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}a_j^2 \le$
$(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}$.
Conversely, assume that $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$, then we get the following inequality
$Re\left\{\frac{z(D_z^{-\lambda}SR^{m,n}f(z))'+\alpha z^2(D_z^{-\lambda}SR^{m,n}f(z))''}{(1-\alpha)D_z^{-\lambda}SR^{m,n}f(z)+\alpha z(D_z^{-\lambda}SR^{m,n}f(z))'} - 1\right\} > -\beta|d|$
$Re\left\{\frac{\frac{(\lambda+1)(\alpha\lambda+1)}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}\left[\alpha j^2+(2a\lambda-\alpha+1)j+\lambda(\alpha\lambda-\alpha+1)\right]a_j^2 z^{j+\lambda}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}[\alpha j+\alpha(\lambda-1)+1]a_j^2 z^{j+\lambda}} - 1 + \beta|d|\right\} > 0$
$Re\frac{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}a_j^2 z^{j+\lambda}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}[\alpha j+\alpha(\lambda-1)+1]a_j^2 z^{j+\lambda}} > 0.$
Since $Re(-e^{i\theta}) \ge -|e^{i\theta}| = -1$, the above inequality reduces to
$\frac{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}r^{\lambda+1}-\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}a_j^2 r^{j+\lambda j}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)}r^{\lambda+1}-\sum_{j=2}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}[\alpha j+\alpha(\lambda-1)+1]a_j^2 r^{j+\lambda}} > 0.$

Letting $r \to 1^-$ and by the mean value theorem we have desired inequality (2.1). This completes the proof of Theorem 2.1 ■

Corollary 2.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$.*
Then $a_j \le \sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}$, $j \ge 2$.

3 Distortion Theorems

Theorem 3.1 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then for $|z| = r < 1$, we have*
$r - \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2 \le |f(z)| \le r + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2.$
The result is sharp for the function $f(z)$ given by $f(z) = z + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}z^2.$

Proof. Given that $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$, from the equation (2.1) and since $2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}$ is non decreasing and positive for $j \geq 2$, then we have
$\sqrt{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}} \sum_{j=2}^{\infty} a_j \leq$
$\sum_{j=2}^{\infty} \sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}a_j \leq$
$\sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$, which is equivalent to, $\sum_{j=2}^{\infty} a_j \leq \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$.
We obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$,
$|f(z)| \leq |z| + \sum_{j=2}^{\infty} a_j |z|^j \leq r + \sum_{j=2}^{\infty} a_j r^j \leq r + r^2 \sum_{j=2}^{\infty} a_j \leq r + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2$.
Similarly, $|f(z)| \geq r - \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2$. This completes the proof of Theorem 3.1. ■

Theorem 3.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then for $|z| = r < 1$, we have*
$-\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r \leq |f'(z)| \leq \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r$.
The result is sharp for the function $f(z)$ given by $f(z) = z + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}z^2$.

Proof. We have $f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$ and
$|f'(z)| \leq 1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1} \leq 1 + \sum_{j=2}^{\infty} j a_j r^{j-1} \leq 1 + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r$.
Similarly, $|f'(z)| \geq 1 - \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r$. This completes the proof of Theorem 3.2. ■

4 Closure Theorems

Theorem 4.1 *Let the functions f_k, $k = 1, 2, ..., l$, defined by $f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j$, $a_{j,k} \geq 0$, be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = \sum_{k=1}^{l} \mu_k f_k(z)$, $\mu_k \geq 0$, is also in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$, where $\sum_{k=1}^{l} \mu_k = 1$.*

Proof. We can write $h(z) = \sum_{k=1}^{l} \mu_k z + \sum_{k=1}^{l} \sum_{j=2}^{\infty} \mu_k a_{j,k} z^j = z + \sum_{j=2}^{\infty} \sum_{k=1}^{l} \mu_k a_{j,k} z^j$. Furthermore, since the functions $f_k(z)$, $k = 1, 2, ..., l$, are in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$, then from Corollary 2.2 we have
$\sum_{j=2}^{\infty} \sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}a_j \leq$
$\sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$.
Thus it is enough to prove that
$\sum_{j=2}^{\infty} \sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}\left(\sum_{k=1}^{m} \mu_k a_{j,k}\right) =$
$\sum_{k=1}^{m} \mu_k \sum_{j=2}^{\infty} \sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}a_{j,k} \leq$
$\sum_{k=1}^{m} \mu_k \sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}} = \sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$. Hence the proof is complete. ■

Corollary 4.2 *Let the functions f_k, $k = 1, 2$, defined by $f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j$, $a_{j,k} \geq 0$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = (1-\zeta)f_1(z) + \zeta f_2(z)$, $0 \leq \zeta \leq 1$, is also in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$.*

Theorem 4.3 *Let $f_1(z) = z$, and* $f_j(z) = z + \sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}z^j$, $j \geq 2$.
Then the function $f(z)$ is in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$ if and only if it can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z)$, where $\mu_1 \geq 0$, $\mu_j \geq 0$, $j \geq 2$ and $\mu_1 + \sum_{j=2}^{\infty} \mu_j = 1$.

Proof. Assume that $f(z)$ can be expressed in the form
$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) = z + \sum_{j=2}^{\infty} \sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}\mu_j z^j$. Thus
$\sum_{j=2}^{\infty} \sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}\cdot$
$\sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}\mu_j = \sum_{j=2}^{\infty} \mu_j = 1 - \mu_1 \leq 1$. Hence $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$.

Conversely, assume that $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$. Setting $\mu_j = \sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}a_j$, since $\mu_1 = 1 - \sum_{j=2}^{\infty}\mu_j$. Thus $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty}\mu_j f_j(z)$. Hence the proof is complete. ■

Corollary 4.4 *The extreme points of the class* $\mathcal{L}(d, \alpha, \beta)$ *are the functions* $f_1(z) = z$, *and* $f_j(z) = z + \sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}z^j$, $j \geq 2$.

5 Inclusion and Neighborhood Results

We define the δ- neighborhood of a function $f(z) \in \mathcal{A}$ by $N_\delta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ and $\sum_{j=2}^{\infty} j|a_j - b_j| \leq \delta\}$.

In particular, for $e(z) = z$, $N_\delta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ and $\sum_{j=2}^{\infty} j|b_j| \leq \delta\}$.

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{L}^\xi(\lambda, d, \alpha, \beta)$ if there exists a function $h(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$ such that $\left|\frac{f(z)}{h(z)} - 1\right| < 1 - \xi$, $z \in U$, $0 \leq \xi < 1$.

Theorem 5.1 *If* $\delta = \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$, *then* $\mathcal{L}(\lambda, d, \alpha, \beta) \subset N_\delta(e)$.

Proof. Let $f \in \mathcal{L}(\lambda, d, \alpha, \beta)$. Then in view of assertion of Corollary 2.2 and since $\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2 + [\alpha(2\lambda - 2 + \beta|d|) + 1]j + [\alpha(\lambda - 1) + 1](\lambda - 1 + \beta|d|)\right\} \geq \frac{2^{m-1}\Gamma(n+2)}{\Gamma(\lambda+3)}\{(\lambda + 1)[\alpha(\lambda + 1 + \beta|d|) + 1] + \beta|d|\}$ for $j \geq 2$, we get

$\sqrt{\frac{2^{m-1}\Gamma(n+2)}{\Gamma(\lambda+3)}\{(\lambda + 1)[\alpha(\lambda + 1 + \beta|d|) + 1] + \beta|d|\}}\sum_{j=2}^{\infty} a_j \leq$
$\sum_{j=2}^{\infty}\sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2 + [\alpha(2\lambda - 2 + \beta|d|) + 1]j + [\alpha(\lambda - 1) + 1](\lambda - 1 + \beta|d|)\}}a_j \leq$
$\sqrt{(\alpha\lambda + 1)(\beta|d| - \lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$, which implise $\sum_{j=2}^{\infty} a_j \leq \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$.

Applying assertion of Corollary 2.2, we obtain $\sum_{j=2}^{\infty} ja_j \leq \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}} = \delta$, so we have $f \in N_\delta(e)$. This completes the proof of the Theorem 5.1. ■

Theorem 5.2 *If* $h \in \mathcal{L}(\lambda, d, \alpha, \beta)$ *and*

$$\xi = 1 + \frac{\delta}{2}\sqrt{\frac{(\alpha\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)}{2^{m+1}(n + 1)\{(\lambda + 1)[\alpha(\lambda + 1 + \beta|d|) + 1] + \beta|d|\}}}, \tag{5.1}$$

then $N_\delta(h) \subset \mathcal{L}^\xi(d, \alpha, \beta)$.

Proof. Suppose that $f \in N_\delta(h)$, we then find that $\sum_{j=2}^{\infty} j|a_j - b_j| \leq \delta$, which readily implies the following coefficient inequality $\sum_{j=2}^{\infty}|a_j - b_j| \leq \frac{\delta}{2}$.

Next, since $h \in \mathcal{L}(d, \alpha, \beta)$, we have $\sum_{j=2}^{\infty} b_j \leq \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$ and we get $\left|\frac{f(z)}{h(z)} - 1\right| \leq \frac{\sum_{j=2}^{\infty}|a_j-b_j|}{1-\sum_{j=2}^{\infty}b_j} \leq \frac{\delta}{2\left(1-\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}\right)} = 1 - \xi$, provided that ξ is given by (5.1), thus $f \in \mathcal{L}^\xi(\lambda, d, \alpha, \beta)$, where ξ is given by (5.1). ■

6 Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 6.1 *Let the function* $f \in \mathcal{A}$ *be in the class* $\mathcal{L}(\lambda, d, \alpha, \beta)$. *Then* f *is univalent starlike of order* δ, $0 \leq \delta < 1$, *in* $|z| < r_1$, *where* $r_1 = \inf_j\left\{\frac{(1-\delta)^2\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$.

The result is sharp for the function $f(z)$ *given by*

$$f_j(z) = z + \sqrt{\frac{(\alpha\lambda + 1)(\beta|d| - \lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2 + [\alpha(2\lambda - 2 + \beta|d|) + 1]j + [\alpha(\lambda - 1) + 1](\lambda - 1 + \beta|d|)\}}}z^j, \quad j \geq 2. \tag{6.1}$$

Proof. It suffices to show that $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta$, $|z| < r_1$. Since $\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{\sum_{j=2}^{\infty}(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty}a_j z^{k-1}}|\right| \leq \frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}a_j|z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}a_j|z|^{j-1}} \leq 1-\delta$.

It is equivalent to $\sum_{j=2}^{\infty}(j-\delta)a_j|z|^{j-1} \leq 1-\delta$, using Theorem 2.1, we obtain

$|z| \leq \left\{\frac{(1-\delta)^2\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$. Hence the proof is complete. ■

Theorem 6.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda,d,\alpha,\beta)$. Then f is univalent convex of order δ, $0 \leq \delta \leq 1$, in $|z| < r_2$, where $r_2 = \inf_j \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$.*

The result is sharp for the function $f(z)$ given by (6.1).

Proof. It suffices to show that $\left|\frac{zf''(z)}{f'(z))}\right| \leq 1-\delta$, $|z| < r_2$. Since $\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{j=2}^{\infty}j(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty}ja_j z^{j-1}}\right| \leq \frac{\sum_{j=2}^{\infty}j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}ja_j|z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty}j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}ja_j|z|^{j-1}} \leq 1-\delta$, i.e. $\sum_{j=2}^{\infty}j(j-\delta)a_j|z|^{j-1} \leq 1-\delta$, using Theorem 2.1, we obtain $|z|^{j-1} \leq \frac{(1-\delta)}{j(j-\delta)}\sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}$, or

$|z| \leq \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$. Hence the proof is complete. ■

Theorem 6.3 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda,d,\alpha,\beta)$. Then f is univalent close-to-convex of order δ, $0 \leq \delta < 1$, in $|z| < r_3$, where $r_3 = \inf_j \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}\right\}^{\frac{1}{2(j-1)}}$.*

The result is sharp for the function $f(z)$ given by (6.1).

Proof. It suffices to show that $|f'(z)-1| \leq 1-\delta$, $|z| < r_3$. Then $|f'(z)-1| = \left|\sum_{j=2}^{\infty}ja_j z^{j-1}\right| \leq \sum_{j=2}^{\infty}ja_j|z|^{j-1}$. Thus $|f'(z)-1| \leq 1-\delta$ if $\sum_{j=2}^{\infty}\frac{ja_j}{1-\delta}|z|^{j-1} \leq 1$. Using Theorem 2.1, the above inequality holds true if $|z|^{j-1} \leq \frac{(1-\delta)}{j}\sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}$ or

$|z| \leq \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}\right\}^{\frac{1}{2(j-1)}}$. Hence the proof is complete. ■

References

[1] A. Alb Lupaş, *Properties on a subclass of univalent functions defined by using Sălăgean operator and Ruscheweyh derivative*, J. of Comput. Anal. Appl., 21 (2016), No.7, 1213-1217.

[2] N.E.Cho, A.M.K. Aouf, *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Tr. J. of Mathematics, Vol. 20, 1996, 553-562.

[3] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[4] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013 (1983), 362-372.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Normal criteria of meromorphic functions concerning holomorphic functions*

Da-Wei Meng[1], San-Yang Liu[1], and Hong-Yan Xu[1,2]

[1] School of Mathematics and statistics, Xidian University, Xi'an 710071, Shaanxi, P. R. China

[2] Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi, 333403, China

E-mail: Goths511@163.com(Da-Wei Meng); liusanyang@126.com(San-Yang Liu); xhyhhh@126.com

Abstract

In this paper, we mainly investigate the problem of normal families of meromorphic functions concerning shared functions, and obtain some normality criteria of meromorphic functions sharing a holomorphic function. Our results generalize or extend the previous theorems given by Ding J. J., Ding L. W. and Yuan W. J..

1 Introduction and main results

Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain D. In the sense of Montel, $\mathscr{F}$ is said to be normal in D, if for any sequence $\{f_n\} \subset \mathscr{F}$ there exists a

2010 Mathematics Subject Classification. Primary 30D35, 30D45.

*This work is supported by Natural Science Foundation of China (Grant No.11271227, No.11201360, No.61373174 and No.11561033), and the Fundamental Research Funds for the Central Universities of China (Grant No.800272125771).

Key words: meromorphic functions; normal family; sharing a holomorphic function.

subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞. For simplicity, we take $\to$ to stand for convergence, $\rightrightarrows$ for convergence spherically locally uniformly, and $\mathcal{M}(D)$ (resp. $\mathcal{A}(D)$) for the set of meromorphic (resp. holomorphic) functions on D. Let F and G two non-constant meromorphic functions defined in D. Then we say that f and g share a IM if $F-a$ and $G-a$ assume the same zeros ignoring multiplicity. The zeros of $F-a$ mean the poles of F when $a=\infty$.

In 1959, Hayman [9] proposed a conjecture: if $F\in\mathcal{M}(\mathbb{C})$ is transcendental, then F^nF' assumes every finite non-zero complex number infinitely often for any positive integer n. The conjecture is showed to be true by many authors, such as Hayman [10], Mues [17], Clunie [6], Bergweiler and Eremenko [2], Chen and Fang [4]. Accordingly, Hayman [10] conjectured that if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f\in\mathscr{F}$ satisfies $f^nf'\neq a$ for a positive integer n and a non-zero complex number a, then $\mathscr{F}$ is normal. This conjecture has been confirmed by some authors, such as Yang and Zhang [26] , Gu [8], Pang [20], Oshkin [18] and Pang [20]. In 2008, from the point of shared values, Zhang [29] concluded that if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each pair (f,g) of $\mathscr{F}$ satisfies that f^nf' and g^ng' share a finite non-zero complex number a IM for $n\geq 2$, then $\mathscr{F}$ is normal. Recently, Jiang and Gao [12] generalized Zhang's result based on the ideas of shared functions. For other generations, we can refer to [3, 15, 24].

For the case of $F^nF^{(k)}$, Zhang and Li [31] proved that if $F\in\mathcal{M}(\mathbb{C})$ is transcendental, then $F^nL[F]-a$ has infinitely many zeros for $n\geq 2$ and $a\neq 0,\infty$, where $L[F]=a_kF^{(k)}+a_{k-1}F^{(k-1)}+\cdots+a_0F$ in which a_i $(i=0,1,2,\cdots,k)$ are small functions of F. Pang and Zalcman [22] further obtained the corresponding normality criterion as follows: If $\mathscr{F}$ is the family of $\mathcal{A}(D)$ such that zeros of each $f\in\mathscr{F}$ have multiplicities at least k and such that each $f\in\mathscr{F}$ satisfies $f^nf^{(k)}\neq a$ for a non-zero complex number a, then $\mathscr{F}$ is normal. Recently, Meng and Hu [16] extended Pang's result, by replacing $f^nf^{(k)}\neq a$ into the condition that f^nf^k and g^ng^k share a IM. Similarly, we also have analogues related to some conditions of $f\left(f^{(k)}\right)^l$ for a positive integer l (refer to [1, 11, 13, 30]).

In 2013, considering the general case of $F^n(F^{(k)})^l$ from the view of shared values, Ding, Ding and Yuan [7] proved a normality criterion as follows: Let a be a non-zero value, if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each pair (f,g) of $\mathscr{F}$ satisfies that $f^n(f^{(k)})^l$ and $g^n(g^{(k)})^l$ share a non-zero value a, where each $f\in\mathcal{F}$ has only zeros of multiplicity at least $max(k,2)$, then $\mathscr{F}$ is normal. Naturally we ask: whether there exists normality theorem when a is a function?

Take four integers $k\geq 1$, $m\geq 0$, $n\geq 1$ and $l\geq 2$. Let a $(\not\equiv 0)$ be a holomorphic function in a domain D such that multiplicities of zeros of a are at most m and divisible by $n+l$. In this paper, we prove the following normality criterion:

2

Theorem 1.1. *Let $\mathscr{F}$ be the family of $\mathcal{M}(D)$ such that multiplicities of zeros of each $f \in \mathscr{F}$ are at least $k+m+1$ and such that multiplicities of poles of f are at least $m+1$ whenever f have zeros and poles. If each pair (f,g) of $\mathscr{F}$ satisfies that $f^n(f^{(k)})^l$ and $g^n(g^{(k)})^l$ share a IM, then $\mathscr{F}$ is normal in D.*

In special, when $k=1$, we may modify Theorem 1.1 as follows:

Theorem 1.2. *Suppose $a=a(z)$ as in Theorem 1.1, if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f \in \mathscr{F}$ satisfies that $f^n(f')^l \neq a$, then $\mathscr{F}$ is normal in D.*

Similar to the proof of Theorem 1.2, we conclude the following result:

Theorem 1.3. *Suppose $a=a(z)$ as in Theorem 1.1, if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f \in \mathscr{F}$ satisfies that $f^n(f'(z))^l = a$ implies $|f(z)| > A$ for a positive number A, then $\mathscr{F}$ is normal in D.*

As a matter of fact, Theorem 1.3 is inspired by the ideas of papers [11, 13] initially.

2 Preliminary lemmas

First of all, we introduce the following *Zalcman's lemma* [28]:

Lemma 2.1. *Take a positive integer k. Let $\mathscr{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that zeros of each $f \in \mathscr{F}$ are of multiplicity at least k. If $\mathscr{F}$ is not normal at a point $z_0 \in \Delta$, then for $0 \leq \alpha < k$, there exist a sequence $\{z_n\} \subset \Delta$ of complex numbers with $z_n \to z_0$; a sequence $\{f_n\}$ of $\mathscr{F}$; and a sequence $\{\rho_n\}$ of positive numbers with $\rho_n \to 0$ such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function $g(\xi)$ on $\mathbb{C}$. Moreover, the zeros of $g(\xi)$ are of multiplicity at least k, and the function $g(\xi)$ may be taken to satisfy the normalization $g^\sharp(\xi) \leq g^\sharp(0) = 1$ for any $\xi \in \mathbb{C}$. In particular, $g(\xi)$ has at most order 2.*

This Lemma is Pang's generalization [19, 21, 25] to the Main Lemma in [27] (where α is taken to be 0), with improvements due to Schwick [23], Chen and Gu [5].

Next, by applying the results from [1, 14, 31, 30] we can deduce the following lemma:

Lemma 2.2. *Let f be a transcendental meromorphic function in the complex plane. Let n, l, k be three positive integers and $a = a(z) \not\equiv 0$ be a polynomial. Then for $l \geq 2$, $f^n(f^{(k)})^l - a$ has infinitely many zeros.*

Finally, we investigate the zeros of $f^n(f^{(k)})^l - a$ if f is rational, and thus give Lemma 2.3 and 2.4:

Lemma 2.3. *Let $p \geq 0$, $n, k \geq 1$ and $l \geq 2$ be four integers, and let a be a non-zero polynomial of degree p. If f is a non-constant rational function which has only zeros of multiplicity at least $k+p+1$ and has only poles of multiplicity at least $p+1$, then $f^n(f^{(k)})^l - a$ has at least two distinct zeros.*

Proof. Firstly, we assume that f is a non-constant polynomial. It follows that $f^{(k)} \not\equiv 0$ from f has only zeros of multiplicity at least $k+p+1$. Hence we have

$$\deg\left(f^n(f^{(k)})^l\right) \geq n(k+p+1) + l(p+1) > p = \deg(a).$$

Therefore, it follows that $f^n(f^{(k)})^l - a$ is also a non-constant polynomial, and hence $f^n(f^{(k)})^l - a$ has at least one zero.

Further, we claim that $f^n(f^{(k)})^l - a$ has at least two distinct zeros if f is a non-constant polynomial. To the contrary, suppose that $f^n(f^{(k)})^l - a$ has only one zero z_0, which means

$$f^n(z)(f^{(k)})^l(z) - a(z) = A'(z - z_0)^d,$$

where A' is a non-zero constant and d is a positive integer. Since f is a non-constant polynomial which has only zeros of multiplicity at least $k+p+1$, we find $f^{(k)} \not\equiv 0$, and hence

$$d = \deg(f^n(f^{(k)})^l - a) > \deg(f^n) \geq n(k+p+1) \geq p+2.$$

By computing we find

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = A'd(d-1)...(d-p)(z-z_0)^{d-p-1},$$

hence $\left\{f^n(f^{(k))^l}\right\}^{(p+1)}$ has a unique zero z_0. Take a zero ξ_0 of f, then it is a zero of f^n with multiplicity at least $n(k+p+1)$. It follows that ξ_0 is a zero of $\left\{f^n(f^{(k)})^l\right\}^{(p)}$ and $\left\{f^n(f^{(k)})^l\right\}^{(p+1)}$, which further implies that $\xi_0 = z_0$. Therefore, we obtain $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = 0$. On the other hand, we get $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z) = b + A'd(d-1)...(d-p+1)(z-z_0)^{d-p}$, in which b is a non-zero constant such that $b = a^{(p)}(z)$. This yields that $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = b \neq 0$, which is contradictory to $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = 0$. The claim is proved.

Secondly, we assume that f has poles, and then set

$$f(z) = \frac{A(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.1}$$

where A is a non-zero constant, α_i distinct zeroes of f with $s \geq 0$, and β_j distinct poles of f with $t \geq 1$. For simplicity, we put

$$m_1 + m_2 + \cdots + m_s = M \geq (k+p+1)s, \tag{2.2}$$

$$n_1 + n_2 + \cdots + n_t = N \geq (p+1)t. \tag{2.3}$$

From (2.1), we obtain

$$f^{(k)}(z) = \frac{(z-\alpha_1)^{m_1-k}(z-\alpha_2)^{m_2-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}(z-\beta_2)^{n_2+k}\cdots(z-\beta_t)^{n_t+k}}, \tag{2.4}$$

where g is a polynomial of degree $\leq kl(s+t-1)$. From (2.1) and (2.4), we get

$$f^n(z)(f^{(k)})^l(z) = \frac{A^n(z-\alpha_1)^{M_1}(z-\alpha_2)^{M_2}\cdots(z-\alpha_s)^{M_s}g^l(z)}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.5}$$

in which

$$M_i = (n+l)m_i - kl,\ i = 1, 2, \cdots, s,$$

$$N_j = (n+l)n_j + kl,\ j = 1, 2, \cdots, t.$$

Differentiating (2.5) yields

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = \frac{(z-\alpha_1)^{M_1-p-1}(z-\alpha_2)^{M_2-p-1}\cdots(z-\alpha_s)^{M_s-p-1}g_0(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.6}$$

where $g_0(z)$ is a polynomial of degree $\leq (p+kl+1)(s+t-1)$.

We claim that $f^n(f^{(k)})^l - a$ has at least one zero if f is a non-polynomial rational function. In order to prove this claim, suppose the contrary holds, thus we set

$$f^n(z)(f^{(k)})^l(z) = a(z) + \frac{C}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.7}$$

where C is a non-zero constant. Subsequently, (2.7) yields

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = \frac{g_1(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.8}$$

where $g_1(z)$ is a polynomial of degree $\leq (p+1)(t-1)$. Comparing (2.6) with (2.8), we get

$$(p+1)(t-1) \geq \deg(g_1) \geq (n+l)M - kls - (p+1)s,$$

and hence

$$M < \frac{p+kl+1}{n+l}s + \frac{p+1}{n+l}t. \tag{2.9}$$

On the other hand, from (2.5) and (2.7) we have

$$(n+l)N + klt + p = (n+l)M - kls + \deg(g^l).$$

Since $\deg(g^l) \leq kl(s+t-1)$, we find

$$(n+l)N \leq (n+l)M - kls + kl(s+t-1) - klt - p,$$

implying that $(n+l)N < (n+l)M$, and thus we have

$$N < M. \tag{2.10}$$

By (2.9), (2.10) and noting that $M \geq (k+p+1)s$, $N \geq (p+1)t$, we deduce that

$$M < \frac{p+kl+1}{n+l}s + \frac{p+1}{n+l}t \leq \left\{ \frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} \right\} M. \tag{2.11}$$

Noting that $l \geq 2$ we immediately obtain

$$\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} = \frac{(1+l)k+2p+2}{(n+l)(k+p+1)} \leq 1.$$

Hence it follows from (2.11) that $M < M$, which is a contradiction. The claim is proved.

Now we suppose that $f^n(f^{(k)})^l - a$ has only one zero z_0, where f is a non-polynomial rational function, then we find

$$f^n(z)(f^{(k)})^l(z) = a(z) + \frac{C'(z-z_0)^d}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.12}$$

where C' is a non-zero constant and d is a positive integer. We distinguish two cases to deduce contradictions.

Case 1. $p \geq d$. Since $p \geq d$, the expression (2.5) together with (2.12) imply that

$$(n+l)N + klt + p = (n+l)M - kls + \deg(g^l).$$

Therefore, we can also conclude (2.10), that is, $N < M$. Differentiating (2.12), we obtain

$$\left\{ f^n(f^{(k)})^l \right\}^{(p+1)}(z) = \frac{g_2(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}},$$

where $g_2(z)$ is a polynomial of degree at most $(p+1)t-(p+1)+d$, and hence

$$(p+1)t-(p+1)+d \geq \deg(g_2) \geq (n+l)M - kls - (p+1)s.$$

Then we have

$$\frac{p+1-d}{n+l} \leq \frac{p+kl+1}{n+l}s + \frac{p+1}{n+1}t - M \leq \left\{ \frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} - 1 \right\} M \tag{2.13}$$

since $M \geq (k+p+1)s, N \geq (p+1)t, M > N$. It follows that

$$\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} < 1$$

since $l \geq 2$. Therefore, from (2.13) we conclude that $\frac{p+1-d}{n+l} < 0$, a contradiction with the assumption $p \geq d$.

Case 2. $d > p$. The expression (2.12) yields

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = \frac{(z-z_0)^{d-p-1}g_3(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.14}$$

where $g_3(z)$ is a polynomial with $\deg(g_3) \le (p+1)t$. We claim that $z_0 \ne \alpha_i$ for each i. Otherwise, if $z_0 = \alpha_i$ for some i, then (2.12) yields

$$a^{(p)}(z_0) = \left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = \left\{f^n(f^{(k)})^l\right\}^{(p)}(\alpha_i) = 0$$

because each α_i is a zero of $f^n(f^{(k)})^l$ of multiplicity $> n(k+p+1) \ge p+2$. This is impossible since $\deg(a) = p$. Hence $(z-z_0)^{d-p-1}$ is a factor of the polynomial g_0 in (2.6). By (2.6) and (2.14), we conclude that

$$(p+1)t \ge \deg(g_3) \ge (n+l)M - kls - (p+1)s,$$

which is equivalent to

$$M \le \frac{p+kl+1}{n+l}s + \frac{p+1}{n+l}t. \tag{2.15}$$

If $d \ne (n+l)N + klt + p$, then (2.5) together with (2.12) implies

$$(n+l)N + klt + p \le (n+l)M - kls + \deg(g^l),$$

so we get $N < M$ from $\deg(g^l) \le kl(s+t-1)$. Therefore, by using the facts $M \ge (k+p+1)s, N \ge (p+1)t$, (2.15) implies a contradiction

$$M < \left\{\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l}\right\} M \le M.$$

Hence $d = (n+l)N + klt + p$.

Now we must have $N \ge M$, otherwise, when $N < M$, we can deduce the contradiction $M < M$ from (2.15). Comparing (2.6) with (2.14), we find

$$(p+kl+1)(s+t-1) \ge \deg(g_0) \ge d-p-1$$

since $(z-z_0)^{l-p-1}|g_0$, and hence

$$(n+l)N + klt + p = d \le (p+kl+1)s + (p+kl+1)t - kl,$$

which further yields

$$N < \frac{p+k+1}{n+1}s + \frac{p+1}{n+1}t.$$

Since $M \ge (k+p+1)s$ and $N \ge (p+1)t$, it follows from that

$$N < \frac{p+kl+1}{(n+l)(k+p+1)}M + \frac{1}{n+l}N.$$

Hence $N \geq M$ yields

$$N < \left\{ \frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} \right\} N. \tag{2.16}$$

Since $l \geq 2$, we obtain consequently

$$\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} \leq 1.$$

Hence (2.16) yields $N < N$. This is a contradiction. Proof of Lemma 2.3 is completed. □

Lemma 2.4. *Let $p \geq 0$,$n \geq 1$ and $l \geq 2$ be three integers such that p is divisible by $n+l$, and let a be a non-zero polynomial of degree p. If f is a non-constant rational function, then $f^n(f')^l - a$ has at least one zero.*

Proof. If f is a non-constant polynomial, then $f' \not\equiv 0$. We consequently conclude that

$$\deg\left(f^n(f')^l\right) = (n+l)\deg(f) - l \neq p$$

since p is divisible by $n+l$. It follows that $f^n(f')^l - a$ is also a non-constant polynomial, so that $f^n(f')^l - a$ has at least one zero.

If f has poles, we can express f by (2.1) again, and then, by differentiating (2.1), we deduce that

$$f'(z) = \frac{(z-\alpha_1)^{m_1-1}(z-\alpha_2)^{m_2-1}\cdots(z-\alpha_s)^{m_s-1}h(z)}{(z-\beta_1)^{n_1+1}(z-\beta_2)^{n_2+1}\cdots(z-\beta_t)^{n_t+1}}, \tag{2.17}$$

where $h(z)$ is a polynomial of form

$$h(z) = (M-N)z^{s+t-1} + \cdots.$$

From (2.1) and (2.17), we obtain

$$f^n(f')^l = \frac{P}{Q},$$

in which

$$P(z) = A^n(z-\alpha_1)^{(n+l)m_1-l}(z-\alpha_2)^{(n+l)m_2-l}\cdots(z-\alpha_s)^{(n+l)m_s-l}h^l(z),$$

$$Q(z) = (z-\beta_1)^{(n+l)n_1+l}(z-\beta_2)^{(n+l)n_2+l}\cdots(z-\beta_t)^{(n+l)n_t+l}.$$

We suppose, to the contrary, that $f^nf' - a$ has no zero. When $M \neq N$, we have

$$f^nf' = a + \frac{B}{Q} = \frac{P}{Q},$$

where B is a non-zero constant. Therefore, we obtain

$$\deg(P) = \deg(Qa+B) = \deg(Q) + p.$$

Note $deg(h^l) \geq l(s+t-1)$ implies that

$$(n+l)M - ls + l(s+t-1) \geq (n+l)N + lt + p,$$

or equivalently

$$(n+l)M - N \geq (p+l),$$

which further yields $M > N$ and $deg(h) = s+t-1$. It follows that

$$(n+l)M - ls + l(s+t-1) = (n+l)N + lt + p.$$

Thus we immediately obtain

$$M - N = \frac{p+l}{n+l},$$

which is impossible since $M-N$ is an integer. Therefore, $f^n f' - a$ has at least one zero. □

3 Proof of Theorem 1.1

Without loss of generality, we may assume that $D = \{z \in \mathbb{C} \mid |z| < 1\}$. For any point z_0 in D, either $a(z_0) = 0$ or $a(z_0) \neq 0$ holds. For simplicity, we assume $z_0 = 0$ and distinguish two cases.

Case 1. $a(0) \neq 0$. To the contrary, we suppose that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence $\{f_j\}$ of $\mathscr{F}$; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ $(j \to \infty)$ such that

$$g_j(\xi) = \rho_j^{-\frac{lk}{n+l}} f_j(z_j + \rho_j \xi)$$

converges uniformly to a non-constant meromorphic function $g(\xi)$ in $\mathbb{C}$ with respect to the spherical metric. Moreover, $g(\xi)$ is of order at most 2. By Hurwitz's theorem, the zeros of $g(\xi)$ have at least multiplicity $k+m+1$.

On every compact subset of $\mathbb{C}$ which contains no poles of g, we have uniformly

$$\begin{aligned} & f_j^n(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^l - a(z_j+\rho_j\xi) \\ = \ & g_j^n(\xi)(g_j^{(k)}(\xi))^l - a(z_j+\rho_j\xi) \rightrightarrows g^n(\xi)(g^{(k)}(\xi))^l - a(0). \end{aligned} \tag{3.1}$$

If $g^n(g^{(k)})^l \equiv a(0)$, then g has no zeros and poles. Then there exist constants c_i such that $(c_1, c_2) \neq (0,0)$, and

$$g(\xi) = e^{c_0 + c_1\xi + c_2\xi^2}$$

since g is a non-constant meromorphic function of order at most 2. Obviously, this is contrary to the case $g^n(g^{(k)})^l \equiv a(0)$. Hence we have $g^n(g^{(k)})^l \not\equiv a(0)$.

By Lemma 2.2 and Lemma 2.3, the function $g^n(g^{(k)})^l - a(0)$ has two distinct zeros ξ_0 and ξ_0^*. We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and such that $g^n(g^{(k)})^l - a(0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_0 and ξ_0^*, where

$$D_1 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0| < \delta\},\ D_2 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0^*| < \delta\}.$$

By (3.1) and Hurwitz's theorem, there exist points $\xi_j \in D_1$, $\xi_j^* \in D_2$ such that

$$f_j^n(z_j + \rho_j\xi_j)(f_j^{(k)}(z_j + \rho_j\xi_j))^l - a(z_j + \rho_j\xi_j) = 0,$$

and

$$f_j^n(z_j + \rho_j\xi_j^*)(f_j^{(k)}(z_j + \rho_j\xi_j^*))^l - a(z_j + \rho_j\xi_j^*) = 0$$

for sufficiently large j.

By the assumption in Theorem 1.1, $f_1^n(f_1^{(k)})^l$ and $f_j^n(f_j^{(k)})^l$ share a IM for each j. It follows

$$f_1^n(z_j + \rho_j\xi_j)(f_1^{(k)}(z_j + \rho_j\xi_j))^l - a(z_j + \rho_j\xi_j) = 0,$$

and

$$f_1^n(z_j + \rho_j\xi_j^*)(f_1^{(k)}(z_j + \rho_j\xi_j^*))^l - a(z_j + \rho_j\xi_j^*) = 0.$$

By letting $j \to \infty$, and noting $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, we obtain

$$f_1^n(0)(f_1^{(k)}(0))^l - a(0) = 0.$$

Since the zeros of $f_1^n(\xi)(f_1^{(k)}(\xi))^l - a(\xi)$ has no accumulation points, in fact we have

$$z_j + \rho_j\xi_j = 0,\ z_j + \rho_j\xi_j^* = 0,$$

or equivalently

$$\xi_j = -\frac{z_j}{\rho_j}, \qquad \xi_j^* = -\frac{z_j}{\rho_j}.$$

This contradicts with the facts that $\xi_j \in D_1$, $\xi_j^* \in D_2$, $D_1 \cap D_2 = \emptyset$. Thus $\mathscr{F}$ is normal at $z_0 = 0$.

Case 2. $a(0) = 0$. We assume that $z_0 = 0$ is a zero of a of multiplicity p. Then we have $p \le m$ by the assumption. Write $a(z) = z^p b(z)$, in which $b(0) = b_p \neq 0$. Since multiplicities of all zeros of a are divisible by $n + l$, then $d = \frac{p}{n+l}$ is just a positive integer. Thus we obtain a new family of $\mathcal{M}(D)$ as follows

$$\mathscr{H} = \left\{ \frac{f(z)}{z^d} \;\middle|\; f \in \mathscr{F} \right\}.$$

We claim that $\mathscr{H}$ is normal at 0.

Otherwise, if $\mathscr{H}$ is not normal at 0, then by lemma 2.1 there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence $\{h_j\}$ of $\mathscr{H}$; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ $(j \to \infty)$ such that

$$g_j(\xi) = \rho_j^{-\frac{lk}{n+l}} h_j(z_j + \rho_j \xi) \tag{3.2}$$

converges uniformly to a non-constant meromorphic function $g(\xi)$ in $\mathbb{C}$ with respect to the spherical metric, where $g^\sharp(\xi) \leq 1$, $\mathrm{ord}(g) \leq 2$, and h_j has the following form

$$h_j(z) = \frac{f_j(z)}{z^d}.$$

We will deduce contradiction by distinguishing two cases.

Subcase 2.1. There exists a subsequence of $\frac{z_j}{\rho_j}$, for simplicity we still denote it as $\frac{z_j}{\rho_j}$, such that $\frac{z_j}{\rho_j} \to c$ as $j \to \infty$, where c is a finite number. Thus we have

$$F_j(\xi) = \frac{f_j(\rho_j \xi)}{\rho_j^{\frac{lk}{n+l}+d}} = \frac{(\rho_j \xi)^d h_j(z_j + \rho_j(\xi - \frac{z_j}{\rho_j}))}{(\rho_j)^d (\rho_j)^{\frac{lk}{n+l}}} \rightrightarrows \xi^d g(\xi - c) = h(\xi),$$

and

$$F_j^n(\xi)(F_j^{(k)}(\xi))^l - \frac{a(\rho_j \xi)}{\rho_j^p} = \frac{f_j^n(\rho_j \xi)(f_j^{(k)}(\rho_j \xi))^l - a(\rho_j \xi)}{\rho_j^p} \rightrightarrows h^n(\xi)(h^{(k)}(\xi))^l - b_p \xi^p. \tag{3.3}$$

Noting $p \leq m$, it follows from Lemma 2.2 and Lemma 2.3 that $h^n(\xi)(h^{(k)}(\xi))^l - b_p \xi^p$ has at least two distinct zeros. Similar to the proof of Case1, we can obtain a contradiction.

Subcase 2.2. There exists a subsequence of $\frac{z_j}{\rho_j}$, for simplicity we still denote it as $\frac{z_j}{\rho_j}$, such that $\frac{z_j}{\rho_j} \to \infty$ as $j \to \infty$. Then we deduce that

$$\begin{aligned} f_j^{(k)}(z_j + \rho_j \xi) &= \Big\{(z_j + \rho_j \xi)^d h_j(z_j + \rho_j \xi)\Big\}^{(k)} \\ &= (z_j + \rho_j \xi)^d h_j^{(k)}(z_j + \rho_j \xi) + \sum_{i=1}^{k} a_i (z_j + \rho_j \xi)^{d-i} h_j^{(k-i)}(z_j + \rho_j \xi) \\ &= (z_j + \rho_j \xi)^d \rho_j^{-\frac{nk}{n+l}} g_j^{(k)}(\xi) + \sum_{i=1}^{k} a_i (z_j + \rho_j \xi)^{d-i} \rho_j^{-\frac{nk}{n+l}+i} g_j^{(k-i)}(\xi), \end{aligned}$$

in which $a_i (i = 1, 2, ..., k)$ are all constants. Thus the expansion of $(f_j^{(k)}(z_j + \rho_j \xi))^l$ can be stated as

$$(g_j^{(k)}(\xi))^l (z_j + \rho_j \xi)^{ld} (\rho_j \xi)^{-\frac{nlk}{n+l}} + \sum_{l_0 < l} \prod_{i=0}^{k} (a_i g_j^{(k-i)}(\xi))^{l_i} (z_j + \rho_j \xi)^{ld} (\rho_j \xi)^{-\frac{nlk}{n+l}} \left(\frac{\rho_j}{z_j + \rho_j \xi} \right)^{\sum_{i=1}^{k} i l_i},$$

where $l_i (i = 0, 1, ..., k)$ are arbitrary non-negative integers satisfying $\sum_{i=0}^{k} l_i = l$.

Since $\frac{z_j}{\rho_j} \to \infty$, $b(z_j+\rho_j\xi) \to b_p$ as $j \to \infty$, it follows that

$$\begin{aligned}
& b_p \frac{f_j^n(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^l}{a(z_j+\rho_j\xi)} - b_p \\
= \ & \frac{b_p(z_j+\rho_j\xi)^{(n+l)d} g_j^n(\xi)(g_j^{(k)}(\xi))^l}{b(z_j+\rho_j\xi)(z_j+\rho_j\xi)^p} \\
& + \sum_{l_0<l} \frac{b_p(z_j+\rho_j\xi)^{(n+l)d} g_j^n(\xi)\prod_{i=0}^k (a_i g_j^{(k-i)}(\xi))^{l_i}}{b(z_j+\rho_j\xi)(z_j+\rho_j\xi)^p} \left(\frac{\rho_j}{z_j+\rho_j\xi}\right)^{\sum_{i=1}^k il_i} - b_p \\
\rightrightarrows \ & g^n(\xi)(g^{(k)}(\xi))^l - b_p \qquad (3.4)
\end{aligned}$$

on every compact subset of $\mathbb{C}$ which contains no poles of g.

Since all zeros of $f_j \in \mathscr{F}$ have at least multiplicity $k+m+1$, then multiplicities of zeros of g are at least $k+1$. Then from Lemma 2.2 and Lemma 2.3, the function $g^n(\xi)(g^{(k)}(\xi))^l - b_p$ has at least two distinct zeros. With similar discussion to the proof of Case1, we can get a contradiction.

Hence the claim is proved, that is, $\mathscr{H}$ is normal at $z_0 = 0$. Therefore, for any sequence $\{f_t\} \subset \mathscr{F}$ there exist $\Delta_r = \{z : |z| < r\}$ and a subsequence $\{h_{t_k}\}$ of $\{h_t(z) = f_t(z)/z^d\} \subset \mathscr{H}$ such that $h_{t_k} \rightrightarrows I$ or ∞ in Δ_r, where I is a meromorphic function. Next we distinguish two cases.

Case A. Assume $f_{t_k}(0) \neq 0$ when k is sufficiently large. Then $I(0) = \infty$, and hence for arbitrary $R > 0$, there exists a positive number δ with $0 < \delta < r$ such that $|I(z)| > R$ when $z \in \Delta_\delta$. Hence when k is sufficiently large, we have $|h_{t_k}(z)| > \frac{R}{2}$, which means that $\frac{1}{f_{t_k}}$ is holomorphic in Δ_δ. In fact, when $|z| = \frac{\delta}{2}$,

$$\left|\frac{1}{f_{t_k}(z)}\right| = \left|\frac{1}{h_{t_k}(z)z^d}\right| \leq M = \frac{2^{d+1}}{R\delta^d}.$$

By applying maximum principle, we have

$$\left|\frac{1}{f_{t_k}(z)}\right| \leq M$$

for $z \in \Delta_{\delta/2}$. It follows from Motel's normal criterion that there exists a convergent subsequence of $\{f_{t_k}\}$, that is, $\mathscr{F}$ is normal at 0.

Case B. There exists a subsequence of f_{t_k}, for simplicity we still denote it as f_{t_k}, such that $f_{t_k}(0) = 0$. Then we get $I(0) = 0$ since $h_{t_k}(z) = \frac{f_{t_k}(z)}{z^d} \rightrightarrows I(z)$, and hence there exists a positive number ρ with $0 < \rho < r$ such that $I(z)$ is holomorphic in Δ_ρ and has a unique zero $z = 0$ in Δ_ρ. Therefore, we have $f_{t_k}(z) \rightrightarrows z^d I(z)$ in Δ_ρ since h_{t_k} converges spherically locally uniformly to a holomorphic function I in Δ_ρ. Thus $\mathscr{F}$ is normal at 0.

Similarly, we can prove that $\mathscr{F}$ is normal at arbitrary $z_0 \in D$, hence $\mathscr{F}$ is normal in D.

4 Proof of Theorem 1.2

Similar to the proof of Theorem 1.1, we assume that $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $z_0 = 0$, and then distinguish two cases by either $a(0) = 0$ or $a(0) \neq 0$.

Case 1. $a(0) \neq 0$. To the contrary, we suppose that $\mathscr{F}$ is not normal at 0. By using the notations in the proof of Theorem 1.1, we also obtain

$$\begin{aligned} & f_j^n(z_j + \rho_j\xi)(f_j'(z_j + \rho_j\xi))^l - a(z_j + \rho_j\xi) \\ = \; & g_j^n(\xi)(g_j'(\xi))^l - a(z_j + \rho_j\xi) \rightrightarrows g^n(\xi)(g'(\xi))^l - a(0), \end{aligned} \tag{4.1}$$

where $g^n(g^{(k)})^l \not\equiv a(0)$.

By Lemma 2.2 and Lemma 2.4, the function $g^n(g')^l - a(0)$ has a zero η_0. By (4.1) and Hurwitz's theorem, there exist points $\eta_j \to \eta_0$ $(j \to \infty)$ such that for sufficiently large j, $z_j + \rho_j\eta_j \in D$ and

$$f_j^n(z_j + \rho_j\eta_j)(f_j'(z_j + \rho_j\eta_j))^l - a(z_j + \rho_j\eta_j) = 0,$$

which contradicts the assumption that $f^n(f')^l \neq a$.

Case 2. $a(0) = 0$. By using the notations in the proof of Theorem 1.1, we also get the formulas (3.1)–(3.4). Therefore, with the similar method in Case 1, we can prove that $\mathscr{F}$ is normal at $z_0 = 0$, and hence $\mathscr{F}$ is normal in D.

5 Proof of Theorem 1.3

We also take the assumptions in the proof of Theorem 1.1, distinguishing two cases as follows:

Case 1. $a(0) \neq 0$. Similar to the proof of proof of Theorem 1.2, we get that $g^n(g')^l - a(0)$ has a zero η_0. By Hurwitz's theorem, there exist points $\eta_j \to \eta_0$ $(j \to \infty)$ such that for sufficiently large j, $z_j + \rho_j\eta_j \in D$ and $f_j^n(z_j + \rho_j\eta_j)(f_j'(z_j + \rho_j\eta_j))^l = a(z_j + \rho_j\eta_j)$. Consequently, we have $|g_j(\eta_j)| = |\rho_j^{-\frac{lk}{n+l}} f_j(z_j + \rho_j\eta_j)| \geq |\rho_j^{-\frac{lk}{n+l}}|A$, which implies that $g(\eta_0) = \infty$. This contradicts the assumption that $g^n(g'(\eta_0))^l = a(0)$. Hence $\mathscr{F}$ is normal at $z_0 = 0$.

Case 2. $a(0) = 0$. We also obtain the formulas (3.1)–(3.4), thus we can prove that $\mathscr{F}$ is normal by using the similar method of Case 1.

6 Acknowledgements

We would like to repress their hearty thanks to the referee for his/her valuable comments and suggestions made to this paper. This research is supported by Natural Science Foundation of China (Grant No.11271227, No.11201360) and the Fundamental Research Funds for the Central Universities of China (Grant No.800272125771).

References

[1] Alotaibi, A., On the zeros of $af\left(f^{(k)}\right)^n - 1$ for $n \geq 2$, Comput. Methods Funct. Theory 4(1) (2004), 227-235.

[2] Bergweiler, W. and Eremenko, A., On the singularities of the inverse to a meromorphic function of finite order, Rev, Mat. Iberoamericana 11 (1995), 355-373.

[3] Chang, J. M. and Fang, M. L., Normality and shared functions of holomorphic functions and their derivatives, Michigan Math. J. 53(2005), 625-645.

[4] Chen, H. H. and Fang, M. L., On the value distribution of $f^n f'$, Sci. China Ser. A 38 (1995), 789-798.

[5] Chen, H. H. and Gu, Y. X., An improvement of Marty's criterion and its applications, Sci. China Ser. A 36 (1993), 674-681.

[6] Clunie, J., On a result of Hayman, J. London Math. Soc. 42 (1967), 389-392.

[7] Ding, J. J., Ding, L. W. and Yuan, W. J., Normal families of meromorphic functions concerning shared values, Complex Var. Elliptic Equ. 58(1) (2013), 113-121.

[8] Gu, Y. X., Sur les familles normales de fonctions méromorphes, Sci. Sinica 21 (1978), 431-445.

[9] Hayman, W., Picard value of meromorphic functions and their derivatives, Annals of Mathematics 70 (1959), 9-42.

[10] Hayman, W., Research problems in function theory, Athlone Press (University of London), London, 1967.

[11] Hu, P. C. and Meng, D. W., Normal criteria of meromorphic fucntions with multiple zeros, J. Math. Anal. Appl. 357 (2009), 323-329.

[12] Jiang, Y. B. and Gao, Z. S., Normal families of meromorphic functions sharing values and functions, J. Inequal. Appl. 72 (2011).

[13] Jiang, Y. B. and Gao, Z. S., Normal families of meromorphic functions sharing a holomorphic function and the converse of the Bloch principle, Acta Math. Sci. 4 (2012), 1503-1512.

[14] Li, G. W. and Gao, L. Y., On the distribution of $f^m(f^{(k)})^n - \varphi$, J. Jinan Univ. 29 (2008), 424-426.

[15] Li, Y. T. and Gu, Y. X., On normal families of meromorphic functions, J. Math. Anal. Appl. 354 (2009), 421-425.

[16] Meng, D. W. and Hu, P. C., Normality criteria of meromorphic functions sharing a holomorphic function, Bull. Malays. Math. Sci. Soc. 38 (2015), 1331-1347.

[17] Mues, E., Über ein problem von Hayman, Math. Z. 164 (1979), 239-259.

[18] Oshkin, I. B., On a test for the normality of families of holomorphic functions, Uspehi Mat. Nauk 37(2) (1982), 221-222; Russian Math. Surveys 37(2) (1982), 237-238.

[19] Pang, X. C., Normality conditions for differential polynomials (in Chinese), Kexue Tongbao 33 (1988), 1690-1693.

[20] Pang, X. C., Bloch principle and normality criterion, Sci. Sinica Ser. A 11 (1988), 1153-1159; Sci. China Ser. A 32 (1989), 782-791.

[21] Pang, X. C., On normal criterion of meromorphic functions, Sci. China Ser. A 33 (1990), 521-527.

[22] Pang, X. C. and L. Zalcmam, On theorems of Hayman and clunie, NewZealand J. Math. 28 (1999), 71-75.

[23] Schwick, W., Normal criteria for families of meromorphic function, J. Anal. Math. 52 (1989), 241-289.

[24] Xu, Y. and Chang, J. M., Normality criteria and multiple values II, Annales Polonici Mathematici 102.1(2011), 91-99.

[25] Xue, G. F. and Pang, X. C., A criterion for normality of a family of meromorphic functions (Chinese), J. East China Norm. Univ. Natur. Sci. Ed. 2 (1988), 15-22.

[26] Yang, L. and Zhang, G. H., Recherches sur la normalité des familes de fonctions analytiques à des valeurs multiples, I. Un nouveau critère et quelques applications, Scientia Sinica, Series A 14 (1965), 1258-1271; II. Géneralisations, ibid., 15 (1966), 433-453.

[27] Zalcman, L., A heuristic principle in complex function theory, Amer. Math. Monthly 82 (1975), 813-817.

[28] Zalcman, L., Normal families: New perspectives, Bull. Amer. Math. Soc. 35 (1998) 215-230.

[29] Zhang, Q. C., Some normality criteria of meromorphic functions, Complex Var. Elliptic Equ. 53(8) (2008), 791-795.

[30] Zhang, Z. F. and Song, G. D., On the zeros of $f\left(f^{(k)}\right)^n - a(z)$, Chinese Ann. Math. Ser. A 19(2) (1998), 275-282.

[31] Zhang, Z. L. and Li, W., Picard exceptional values for two class differential polynomials, Acta Math. Sinica 34 (1994), 828-835.

Mixed Weakly Monotone Mappings and its Application to System of Integral Equations via Fixed Point Theorems

Deepak Singh

Department of Applied Sciences

NITTTR, (Under Ministry of HRD, Govt. of India) Bhopal,462002, India.

E-mail: dk.singh1002@gmail.com

Om Prakash Chauhan

Department of Applied Mathematics, Jabalpur Engineering College, Jabalpur, (M.P.), India.

E-mail: chauhaan.op@gmail.com

Afrah A N Abdou[1]

Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

E-mail: aabdou@kau.edu.sa

Garima Singh

University Institute of Technology, Barkatullah University, Bhopal, (M.P.), India.

E-mail: drgarimasingh@ymail.com

Abstract: In this note, the notion of a mixed weakly monotone pair of mappings is invoked to prove some coupled common fixed point theorems in partially ordered b-metric spaces. Results proved in this note are authenticated by some innovative examples. Demonstrative surfaces leading to better understanding of calculation done. Moreover as an application our results are utilized to establish existence of solutions of system of integral equations which is also substantiated by an example.

Key Words. Cauchy sequence, Partially ordered b-metric Space, mixed weakly monotone mappings, coupled common fixed point, integral equation.

2010 AMS Subject classification. 47H10, 54H25.

1 Introduction and Preliminaries

In 1989, Bakhtin [2] introduced the notion of b-metric spaces and studied the concept of b-metric spaces as a generalization of metric spaces. Also he proved the Banach contraction principle in b-metric spaces. After that the study of fixed point theorems in b-metric spaces is followed by some other mathematicians (see [1], [5], [13]).

In 2011, Ran and Rarings [12] introduced the existence of fixed point in partially ordered metric spaces and studied

[1]Corresponding Author

some applications to matrix equations.

Guo and Lakshmikantham [8] introduced the concept of coupled fixed point. later on Bhaskar and Lakshmikantham [3] introduced the notions of a mixed monotone mappings and then established some coupled fixed point theorems for mixed monotone mappings. They also discussed the existence and uniqueness of the solution for periodic boundary value problems.

In 2009, Lakshmikantham and Ciric [9] defined g- monotone property and proved coupled coincidence and coupled common fixed points theorems for nonlinear mappings satisfying certain contractive conditions in partially ordered metric spaces. Some remarkable contributions on this line can be seen in [4], [10], [11].

In 2012, Gordji et al. [7], proved some coupled fixed point theorems for a contractive-type mappings with the mixed weakly monotone property in partially ordered metric spaces. In this article, utilizing the notion of a mixed weakly monotone pair of mappings we prove coupled common fixed points theorems for mappings on partially ordered b-metric spaces.

First we recall some basic definitions, notions, lemmas, and examples which will be needed in the sequel.

Definition 1.1. *[5] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$*;*

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space. If $(X, \precsim)$ is still a partially ordered set, then $(X, \precsim, d)$ is called a partially ordered b-metric space.

Definition 1.2. *[3] An element $(x, y) \in X \times X$ is called coupled fixed point of a mapping $F : X \times X \to X$ if* $x = F(x, y)$ *and* $y = F(y, x)$.

Definition 1.3. *[3] Let $(X, d, \precsim)$ be a partially ordered set and $f : X \times X \to X$ be mapping. We say that f has the mixed monotone property on X if, for all $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \precsim x_2 \Rightarrow f(x_1, y) \precsim f(x_2, y)$$

$$\text{and} \quad y_1, y_2 \in X, y_1 \precsim y_2 \Rightarrow f(x, y_1) \succsim f(x, y_2).$$

Definition 1.4. *[7] let $(X, \precsim)$ be a partially ordered set and $f, g : X \times X \to X$ be mappings. We say that a pair (f, g) has the mixed weakly monotone property on X, if for all $x, y \in X$,we have*

$$x \precsim kf(x, y), f(y, x) \precsim y \ \text{ implies } \ f(x, y) \precsim g(f(x, y), f(y, x)), g(f(y, x), f(x, y)) \precsim f(y, x)$$

$$\text{and} \quad x \precsim g(x, y), g(y, x) \precsim y \ \text{ implies } \ g(x, y) \precsim f(g(x, y), g(y, x)), f(g(y, x), g(x, y)) \precsim g(y, x).$$

For mixed weakly monotone property related examples one is suggested to refer [7]

Remark 1.1. *[7] Let $(X, \precsim)$ be a partially ordered set, $f : X \times X \to X$ be a map with the mixed monotone property on X. Then for all $n \in \mathbb{N}$, the pair (f^n, f^n) has the mixed weakly monotone property on X.*

Lemma 1.1. *[7] Let (X, d) be a metric space. Then $X \times X$ is a metric space with the metric D_d given by $D_d((x, y), (u, v)) = d(x, u) + d(y, v)$, for all $x, y, u, v \in X$.*

2 Main result

In this section, some fixed point theorems for contraction conditions described by rational expressions are proved.

Lemma 2.1. *Let (X, d) be a b-metric space. Then $X \times X$ is a b-metric space with the b-metric D given by $D((x, y), (u, v)) = d(x, u) + d(y, v)$, for all $x, y, u, v, w, t \in X$.*

Proof. For all $x, y, u, v, w, t \in X$, we have $D((x, y), (u, v)) \in [0, \infty)$ and
$D((x, y), (u, v)) = 0$ if and only if $d(x, u) + d(y, v) = 0$
if and only if $x = u, y = v$, that is $(x, y) = (u, v)$ and

$$\begin{aligned} D((x, y), (u, v)) &= d(x, u) + d(y, v) \\ &= d(u, x) + d(v, y) = D((u, v), (x, y)). \\ Also, \quad D((x, y), (u, v)) &= d(x, u) + d(y, v) \\ &\leq s[d(x, w) + d(w, u)] + s[d(y, t) + d(t, v)] \\ &\leq s[d(x, w) + d(y, t)] + s[d(w, u) + d(t, v)] \\ &\leq s[D((x, y), (w, t)) + D((w, t), (u, v))]. \end{aligned}$$

Hence, D is a b-metric on $X \times X$. □

Let $(X, d, \precsim)$ be a partially ordered complete metric space. We consider the product space $X \times X$ with the following partial order, for all $(x, y), (u, v) \in X \times X$

$$(x, y) \precsim (u, v) \Leftrightarrow x \precsim u, y \succsim v.$$

Also let $(X \times X, D)$ be a b-metric space with the following metric

$$D((x, y), (u, v)) = d(x, u) + d(y, v), \quad for\ all\ (x, y), (u, v) \in X \times X.$$

Theorem 2.1. *Let $(X, d, \precsim)$ be a partially ordered complete b-metric space. Let $f, g : X \times X \to X$ be the mappings such that the pair (f, g) has the mixed weakly monotone property on X. Suppose there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + 2s(\gamma + \delta) < \frac{1}{2}$ such that*

$$\begin{aligned} d(f(x,y), g(u,v)) \precsim \alpha \frac{[1 + D((x,y),(f(x,y),f(y,x)))].D((u,v),(g(u,v),g(v,u)))}{[1 + D((x,y),(u,v))]} + \beta D((x,y),(u,v)) \\ + \gamma[D((x,y),(f(x,y),f(y,x))) + D((u,v),(g(u,v),g(v,u)))] \\ + \delta[D((u,v),(f(x,y),f(y,x))) + D((x,y),(g(u,v),g(v,u)))] \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \precsim u$ and $y \succsim v$ and D is defined as in Lemma 2.1. Let $x_0, y_0 \in X$ be such that $x_0 \precsim f(x_0, y_0), y_0 \succsim f(y_0, x_0)$ or $x_0 \precsim g(x_0, y_0), y_0 \succsim g(y_0, x_0)$. If f or g is continuous, then f and g have a coupled common fixed point in X.

Proof. We construct two Cauchy sequence in X. Let $x_0, y_0 \in X$, be such that $x_0 \precsim f(x_0, y_0), y_0 \succsim f(y_0, x_0)$.
Put $x_1 = f(x_0, y_0), y_1 = f(y_0, x_0), x_2 = g(x_1, y_1), y_2 = g(y_1, x_1)$
Continuing this, way $x_{2n+1} = f(x_{2n}, y_{2n}), y_{2n+1} = f(y_{2n}, x_{2n}),$

$$x_{2n+2} = g(x_{2n+1}, y_{2n+1}), y_{2n+2} = g(y_{2n+1}, x_{2n+1}) \ \ for \ \ all \ n \in N.$$

From the choice of x_0, y_0 and the (f, g) has mixed weakly monotone property, we have

$$\begin{aligned} x_1 =& f(x_0, y_0) \precsim g(f(x_0, y_0), f(y_0, x_0) = g(x_1, y_1) = x_2 \Rightarrow x_1 \precsim x_2, \\ x_2 =& g(x_1, y_1) \precsim f(g(x_1, y_1), g(y_1, x_1) = f(x_2, y_2) = x_3 \Rightarrow x_2 \precsim x_3. \\ Similarly, \quad y_1 =& f(y_0, x_0) \succsim g(f(y_0, x_0), f(x_0, y_0) = g(y_1, x_1) = y_2 \Rightarrow y_1 \succsim y_2, \\ y_2 =& g(y_1, x_1) \succsim f(g(y_1, x_1), g(x_1, y_1) = f(y_2, x_2) = y_3 \Rightarrow y_2 \succsim y_3. \end{aligned}$$

Therefore, we acquire

$$x_0 \precsim x_1 \precsim x_2 \precsim \dots x_n \precsim x_{n+1} \precsim \dots$$

$$y_0 \succsim y_1 \succsim y_2 \succsim \dots y_n \succsim y_{n+1} \succsim \dots$$

the sequences $\{x_n\}$ and $\{y_n\}$ are monotone increasing and decreasing respectively. Applying (2.1), we obtain

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) = d(f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1})) \\ & \precsim \alpha \tfrac{[1+D((x_{2n},y_{2n}),(f(x_{2n},y_{2n}),f(y_{2n},x_{2n})))]D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))}{[1+D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))]} \\ & \quad + \beta D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\ & \quad + \gamma[D((x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) + D((x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1})))] \\ & \quad + \delta[D((x_{2n+1}, y_{2n+1}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) + D((x_{2n}, y_{2n}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1})))] \end{aligned}$$

$$
\begin{aligned}
d(x_{2n+1}, x_{2n+2}) \precsim \alpha & \frac{[1 + D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))]D((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))}{[1 + D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))]} \\
& + \beta D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) + \\
& \gamma[D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) + D((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))] + \\
& \delta[D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1})) + D((x_{2n}, y_{2n}), (x_{2n+2}, y_{2n+2}))]
\end{aligned}
$$

$$
\begin{aligned}
d(x_{2n+1}, x_{2n+2}) \precsim & (\alpha + \gamma)D((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})) + \\
& (\beta + \gamma)D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) + \delta D((x_{2n}, y_{2n}), (x_{2n+2}, y_{2n+2})) \\
\precsim & (\alpha + \gamma)D((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})) + \\
& (\beta + \gamma)D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) + \\
& s\delta[D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) + D((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))] \\
\precsim & (\alpha + \gamma)[d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2})] + \\
& (\beta + \gamma)[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})] + \\
& s\delta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]
\end{aligned}
$$

$$
\begin{aligned}
d(x_{2n+1}, x_{2n+2}) \precsim & (\alpha + \gamma + s\delta)[d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2})] + \\
& (\beta + \gamma + s\delta)[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})] \quad for \;\; all \;\; n \in N.
\end{aligned} \tag{2.2}
$$

Similarly, we have

$$
\begin{aligned}
d(y_{2n+1}, y_{2n+2}) \precsim & (\alpha + \gamma + s\delta)[d(y_{2n+1}, y_{2n+2}) + d(x_{2n+1}, x_{2n+2})] + \\
& (\beta + \gamma + s\delta)[d(y_{2n}, y_{2n+1}) + d(x_{2n}, x_{2n+1})] \quad for \;\; all \;\; n \in N.
\end{aligned} \tag{2.3}
$$

Thus it follows from (2.2) and (2.3) that

$$
\begin{aligned}
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \precsim & 2(\alpha + \gamma + s\delta)[d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2})] + \\
& 2(\beta + \gamma + s\delta)[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})]
\end{aligned}
$$

or

$$
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \precsim \frac{2(\beta + \gamma + s\delta)}{1 - 2(\alpha + \gamma + s\delta)}[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})] \quad for \;\; all \;\; n \in N. \tag{2.4}
$$

Moreover, if we apply (2.1), then we have

$$
\begin{aligned}
& d(x_{2n+2}, x_{2n+3}) = d(g(x_{2n+1}, y_{2n+1}), f(x_{2n+2}, y_{2n+2})) \\
& \precsim \alpha \tfrac{[1+D((x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1})))]D((x_{2n+2}, y_{2n+2}), (f(x_{2n+2}, y_{2n+2}), f(y_{2n+2}, x_{2n+2})))}{[1+D((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))]} \\
& + \beta D((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})) + \gamma[D((x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))) \\
& + D((x_{2n+2}, y_{2n+2}), (f(x_{2n+2}, y_{2n+2}), f(y_{2n+2}, x_{2n+2})))] + \delta[D((x_{2n+2}, y_{2n+2}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1})))
\end{aligned}
$$

$$+D((x_{2n+1},y_{2n+1}),(f(x_{2n+2},y_{2n+2}),f(y_{2n+2},x_{2n+2})))]$$

$$d(x_{2n+2},x_{2n+3}) \precsim \alpha\frac{[1+D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))}{[1+D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]}$$
$$+\beta D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+$$
$$\gamma[D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))]+$$
$$\delta[D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}))+D((x_{2n+1},y_{2n+1}),(x_{2n+3},y_{2n+3}))]$$

$$\begin{aligned} d(x_{2n+2},x_{2n+3}) \precsim & (\alpha+\gamma)D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))+ \\ & (\beta+\gamma)D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+\delta D((x_{2n+1},y_{2n+1}),(x_{2n+3},y_{2n+3})) \\ \precsim & (\alpha+\gamma)D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))+ \\ & (\beta+\gamma)D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+ \\ & s\delta[D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))] \\ & (\alpha+\gamma+s\delta)[D(x_{2n+2},y_{2n+2}),d(x_{2n+3},y_{2n+3})]+ \\ & (\beta+\gamma+s\delta)[D(x_{2n+1},y_{2n+1}),d(x_{2n+2},y_{2n+2})] \end{aligned}$$

$$\begin{aligned} d(x_{2n+2},x_{2n+3}) \precsim & (\alpha+\gamma+s\delta)[d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3})]+ \\ & (\beta+\gamma+s\delta)[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})], \quad for \ \ all \ \ n\in N. \end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned} d(y_{2n+2},y_{2n+3}) \precsim & (\alpha+\gamma+s\delta)[d(y_{2n+2},y_{2n+3})+d(x_{2n+2},x_{2n+3})]+ \\ & (\beta+\gamma+s\delta)[d(y_{2n+1},y_{2n+2})+d(x_{2n+1},x_{2n+2})], \quad for \ \ all \ \ n\in N. \end{aligned} \tag{2.6}$$

Thus it follows from (2.5) and (2.6) that

$$\begin{aligned} d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3}) \precsim & 2(\alpha+\gamma+s\delta)[d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3})]+ \\ & 2(\beta+\gamma+s\delta)[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})] \end{aligned}$$

or

$$d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3}) \precsim \frac{2(\beta+\gamma+s\delta)}{1-2(\alpha+\gamma+s\delta)}[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})], \quad for \ \ all \ \ n\in N. \tag{2.7}$$

Moreover, it follows from (2.4) and (2.7) that

$$d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3}) \precsim \Big[\frac{2(\beta+\gamma+s\delta)}{1-2(\alpha+\gamma+s\delta)}\Big]^2[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})], \quad for \ \ all \ \ n\in N. \tag{2.8}$$

Let $\lambda = \frac{2(\beta+\gamma+s\delta)}{1-2(\alpha+\gamma+s\delta)}$. Then $0 \leq \lambda < 1$ and

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) &\precsim \lambda[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})] + \\
&\precsim \lambda^3[d(x_{2n-2}, x_{2n-1}) + d(y_{2n-2}, y_{2n-1})] \\
&\precsim \lambda^5[d(x_{2n-4}, x_{2n-3}) + d(y_{2n-4}, y_{2n-3})] \\
&\ \vdots \\
&\precsim \lambda^{2n+1}[d(x_0, x_1) + d(y_0, y_1)] \\
and\ \ d(x_{2n+2}, x_{2n+3}) + d(y_{2n+2}, y_{2n+3}) &\precsim \lambda[d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2})] + \\
&\precsim \lambda.\lambda^{2n+1}[d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \lambda^{2n+2}[d(x_0, x_1) + d(y_0, y_1)] \quad for \ \ all \ \ n \in N.
\end{aligned}$$

Now, for all $m, n \geq 1$ with $n \leq m$, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2m+1}) + d(y_{2n+1}, y_{2m+1}) &\precsim s[d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2m+1})] \\
&\quad + s[d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, x_{2m+1})] \\
&\precsim s[(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}] + s[x_{2n+2}, x_{2m+1}) + d(y_{2n+2}, y_{2m+1}] \\
&\precsim s[(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}] + s^2[x_{2n+2}, x_{2n+3}) + d(y_{2n+2}, y_{2n+3}] \\
&\quad + s^2[d(x_{2n+3}, x_{2m+1}) + d(y_{2n+3}, y_{2m+1})] \\
&\precsim s[(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}] + s^2[x_{2n+2}, x_{2n+3}) + d(y_{2n+2}, y_{2n+3}] \\
&\quad + \ldots + s^{2(m-n)}[d(x_{2m}, x_{2m+1}) + d(y_{2m}, y_{2m+1})] \\
&\precsim s\lambda^{2n+1}[d(x_0, x_1) + d(y_0, y_1)] + s^2\lambda^{2n+2}[d(x_0, x_1) + d(y_0, y_1)] \\
&\quad + \ldots + s^{2(m-n)}\lambda^{2m}[d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim s\lambda^{2n+1}[1 + s\lambda + (s\lambda)^2 + \ldots + (s\lambda)^{2(m-n)-1}][d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \frac{s\lambda^{2n+1}}{1 - s\lambda}[d(x_0, x_1) + d(y_0, y_1)].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
d(x_{2n}, x_{2m+1}) + d(y_{2n}, y_{2m+1}) &\precsim s\lambda^{2n}[1 + s\lambda + (s\lambda)^2 + \ldots + (s\lambda)^{2(m-n)}][d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \frac{s\lambda^{2n}}{1 - s\lambda}[d(x_0, x_1) + d(y_0, y_1)].
\end{aligned}$$

$$\begin{aligned}
d(x_{2n}, x_{2m}) + d(y_{2n}, y_{2m}) &\precsim s\lambda^{2n}[1 + s\lambda + (s\lambda)^2 + \ldots + (s\lambda)^{2(m-n)-1}][d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \frac{s\lambda^{2n}}{1 - s\lambda}[d(x_0, x_1) + d(y_0, y_1)]
\end{aligned}$$

and

$$\begin{aligned} d(x_{2n+1}, x_{2m}) + d(y_{2n+1}, y_{2m}) &\precsim s\lambda^{2n+1}[1 + s\lambda + (s\lambda)^2 + \ldots + (s\lambda)^{2(m-n-1)}][d(x_0, x_1) + d(y_0, y_1)] \\ &\precsim \frac{s\lambda^{2n+1}}{1 - s\lambda}[d(x_0, x_1) + d(y_0, y_1)]. \end{aligned}$$

Hence for all $m, n \geq 1$ with $n \leq m$, it follows that

$$d(x_n, x_m) + d(y_n, y_m) \precsim \frac{s\lambda^n}{1 - s\lambda}[d(x_0, x_1) + d(y_0, y_1)].$$

Since $0 \leq \lambda < 1$, we can conclude that

$d(x_n, x_m) + d(y_n, y_m) \to 0$ as $n \to \infty$, which implies that $d(x_n, x_m) \to 0$ and $d(y_n, y_m) \to 0$ as $m, n \to \infty$. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X.

Since (X, d) be a partially ordered complete b-metric space, then there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Suppose that f is a continuous then we have

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}, y_{2n}) = f(x, y)$$

$$and\ \ y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(y_{2n}, x_{2n}) = f(y, x).$$

this implies (x, y) is coupled fixed point of f.

Taking $u = x$ and $v = y$ in (2.1), we have

$$\begin{aligned} &d(f(x, y), g(x, y)) + d(f(y, x), g(y, x)) \\ &\quad \precsim \alpha\frac{[1 + D((x, y), (f(x, y), f(y, x)))]D((x, y), (g(x, y), g(y, x)))}{[1 + D((x, y), (x, y))]} + \beta D((x, y), (x, y)) \\ &\quad + \gamma[D((x, y), (f(x, y), f(y, x))) + D((x, y), (g(x, y), g(y, x)))] \\ &\quad k + \delta[D((x, y), (f(x, y), f(y, x))) + D((x, y), (g(x, y), g(y, x)))] \\ &\quad + \alpha\frac{[1 + D((y, x), (f(y, x), f(x, y)))]D((y, x), (g(y, x), g(x, y)))}{[1 + D((y, x), (y, x))]} + \beta D((y, x), (y, x)) \\ &\quad + \gamma[D((y, x), (f(y, x), f(x, y))) + D((y, x), (g(y, x), g(x, y)))] \\ &\quad + \delta[D((y, x), (f(y, x), f(x, y))) + D((y, x), (g(y, x), g(x, y)))] \\ &\quad \precsim \alpha[1 + D((x, y), (x, y))]D((x, y), (g(x, y), g(y, x))) \\ &\quad + \gamma[D((x, y), (x, y))) + D((x, y), (g(x, y), g(y, x)))] \\ &\quad + \delta[D((x, y), (x, y))) + D((x, y), (g(x, y), g(y, x)))] \\ &\quad + \alpha[1 + D((y, x), (y, x))]D((y, x), (g(y, x), g(x, y))) \\ &\quad + \gamma[D((y, x), (y, x)) + D((y, x), (g(y, x), g(x, y)))] \\ &\quad + \delta[D((y, x), (y, x)) + D((y, x), (g(y, x), g(x, y)))]. \end{aligned}$$

Hence, we have

$$\begin{aligned} d(x,g(x,y))+d(y,,g(y,x)) &\precsim (\alpha+\gamma+\delta)[D((x,y),(g(x,y),g(y,x)))+\\ &D((y,x),(g(y,x),g(x,y)))]\\ &\precsim 2(\alpha+\gamma+\delta)[d(x,g(x,y))+d(y,g(y,x))]\end{aligned}$$

Since $2(\alpha+\gamma+\delta)<1$, we get $d(x,g(x,y))=0,\;\; d(y,g(y,x))=0 \;\;\Rightarrow x=g(x,y),\;\; y=g(y,x))$. This implies (x,y) is a coupled fixed point of g. Hence (x,y) is a coupled common fixed point of f and g when f is continuous.

Similarly, we can prove that (x,y) is a coupled common fixed point of f and g when g is continuous. □

Next result is proved, relaxing continuity.

Theorem 2.2. *Let $(X,d,\precsim)$ be a partially ordered complete b-metric space. Assume that X has the following property:*

1 if $\{x_n\}$ is a increasing sequence with $x_n\to x$, then $x_n\precsim x$ for all $n\in N$;

2 if $\{y_n\}$ is a decreasing sequence with $y_n\to y$, then $y_n\succsim y$ for all $n\in N$.

Let $f,g:X\times X\to X$ be the mappings such that the pair (f,g) has the mixed weakly monotone property on X. Also, Suppose there exists $\alpha,\beta,\gamma,\delta\in[0,1)$ with $\alpha+\beta+2s(\gamma+\delta)<\frac{1}{2}$ such that

$$\begin{aligned} d(f(x,y),g(u,v)) &\precsim \alpha\frac{[1+D((x,y),(f(x,y),f(y,x)))].D((u,v),(g(u,v),g(v,u)))}{[1+D((x,y),(u,v))]}+\beta D((x,y),(u,v))\\ &+\gamma[D((x,y),(f(x,y),f(y,x)))+D((u,v),(g(u,v),g(v,u)))]\\ &+\delta[D((u,v),(f(x,y),f(y,x)))+D((x,y),(g(u,v),g(v,u)))]\end{aligned}$$

for all $x,y,u,v\in X$ with $x\precsim u$ and $y\succsim v$ and D is defined as in Lemma 2.1. Let $x_0,y_0\in X$ be such that $x_0\precsim f(x_0,y_0), y_0\succsim f(y_0,x_0)$ or $x_0\precsim g(x_0,y_0), y_0\succsim g(y_0,x_0)$, then f and g have a coupled common fixed point in X.

Proof. Following the proof of Theorem 2.1, we only have to show that

$$f(x,y)=g(x,y)=x, f(y,x)=g(y,x)=y.$$

It is clear that

$$\begin{aligned}
&D((x,y),(f(x,y),f(y,x))) \precsim s[D((x,y),(x_{2n+2},y_{2n+2}))+D((x_{2n+2},y_{2n+2}),(f(x,y),f(y,x)))]\\
&\quad = sD((x,y),(x_{2n+2},y_{2n+2}))+sD((g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(f(x,y),f(y,x)))\\
&\quad = sD((x,y),(x_{2n+2},y_{2n+2}))+sd(f(x,y),g(x_{2n+1},y_{2n+1}))+sd(f(y,x),g(y_{2n+1},x_{2n+1}))\\
&\quad \precsim sD((x,y),(x_{2n+2},y_{2n+2}))+\\
&\quad s\alpha\Big\{\frac{[1+D((x,y),(f(x,y),f(y,x)))]D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))}{1+D((x,y),(x_{2n+1},y_{2n+1}))}+\\
&\quad \beta D((x,y),(x_{2n+1},y_{2n+1}))+\\
&\quad \gamma[D((x,y),(f(x,y),f(y,x)))+D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))]+\\
&\quad \delta[D((x_{2n+1},y_{2n+1}),(f(x,y),f(y,x)))+D((x,y),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))]\Big\}\\
&\quad +s\alpha\Big\{\frac{[1+D((x,y),(f(x,y),f(y,x)))]D((y_{2n+1},x_{2n+1}),(g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1})))}{1+D((y,x),(y_{2n+1},x_{2n+1}))}+\\
&\quad \beta D((y,x),(y_{2n+1},x_{2n+1}))+\\
&\quad \gamma[D((y,x),(f(y,x),f(x,y)))+D((y_{2n+1},x_{2n+1}),(g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1})))]+\\
&\quad \delta[D((y_{2n+1},x_{2n+1}),(f(y,x),f(x,y)))+D((y,x),(g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1})))]\Big\}.
\end{aligned} \tag{2.9}$$

Letting $n \to \infty$ in (2.9), we obtain

$$\begin{aligned}
d(x,f(x,y)+d(y,f(y,x))) &\precsim s\gamma D((x,y),(f(x,y),f(y,x)))+\\
&\quad s\delta D((y_{2n+1},x_{2n+1}),(f(x,y),f(y,x)))+\\
&\quad s\gamma D((y,x),(f(y,x),f(x,y)))+\\
&\quad s\delta D((y_{2n+1},x_{2n+1}),(f(y,x),f(x,y)))\\
&\precsim 2s(\gamma+\delta)[d(x,f(x,y)+d(y,f(y,x)))]
\end{aligned}$$

and, since $2s(\gamma+\delta) < \frac{1}{2}$, we have $d(x,f(x,y))+d(y,f(y,x))=0$ and so $f(x,y)=x, \quad f(y,x)=y$. Similarly we can show that $g(x,y)=x \ and \ g(y,x)=y$. Therefore (x,y) is a coupled common fixed point of f and g. □

Following example establishes validity of Theorem 2.1.

Examplex 2.1. *Consider $(\mathbb{R},d,\leq)$, where $\leq$ represents the b-metric with usual order relation and metric $d(x,y)=(|x-y|)^2=(x-y)^2$ on $\mathbb{R}$, where $s=2$.*

Let $f,g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be two functions defined by

$$f(x,y)=\frac{10x-5y+115}{120}, \quad g(x,y)=\frac{16x-8y+184}{192}.$$

Then the pair (f, g) has the mixed weakly monotone property.

In order to verify the condition (2.1)*, first we notice that*

$$0 \leq \alpha \frac{[1 + D((x,y),(f(x,y),f(y,x)))].D((u,v),(g(u,v),g(v,u)))}{[1 + D((x,y),(u,v))]},$$

$$0 \leq \gamma[D((x,y),(f(x,y),f(y,x))) + D((u,v),(g(u,v),g(v,u)))],$$

$$0 \leq \delta[D((u,v),(f(x,y),f(y,x))) + D((x,y),(g(u,v),g(v,u)))] \quad for \ \ all \ \ x, y \in R.$$

Thus it is sufficient to show that $d(f(x,y), g(u,v)) \leq \beta D((x,y),(y,v))$.

$$\begin{aligned} Now, \quad d(f(x,y), g(u,v)) =& (|f(x,y) - g(u,v)|)^2 \\ =& \left(\left|\frac{10x - 5y + 115}{120} - \frac{16u - 8v + 184}{192}\right|\right)^2 \\ \leq& \left(\frac{1}{12}|x-u| + \frac{1}{24}|y-v|\right)^2 \leq \left(\frac{1}{12}(|x-u| + |y-v|)\right)^2 \\ \leq& \left[\frac{1}{3}(|x-u|^2 + |y-v|^2)\right], \quad \forall x, y \in R \\ \leq& \beta D((x,y),(y,v)). \end{aligned}$$

For $\beta = \frac{1}{3}$ *and choosing* $\alpha, \gamma, \delta \geq 0$ *such that* $\alpha + \beta + 2s(\gamma + \delta) \leq \frac{1}{2}$*, Thus condition* (2.1) *is satisfied. Following Figure 1 and Figure 2 show that* $(1, 1)$ *is the coupled fixed point of mapping* f.

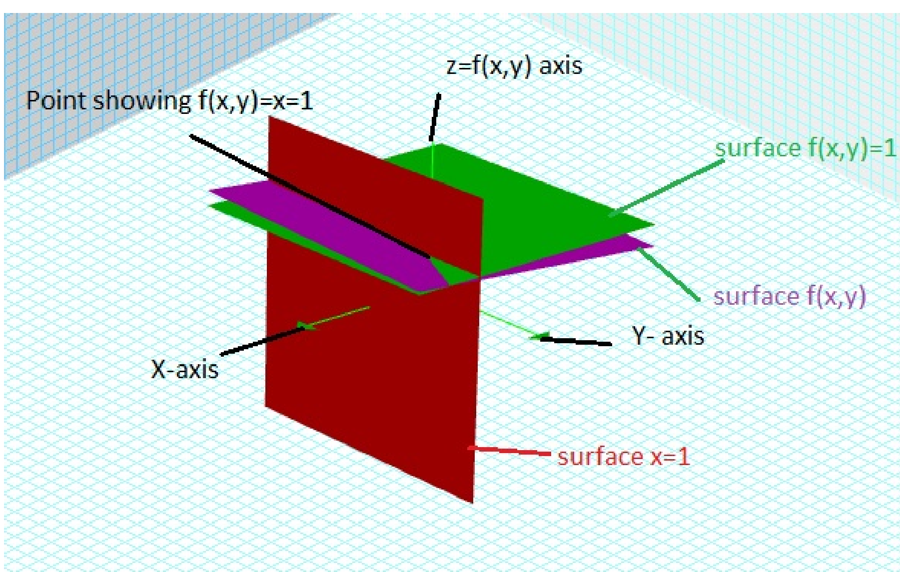


Figure 1: [Figure showing x=f(x,y)]

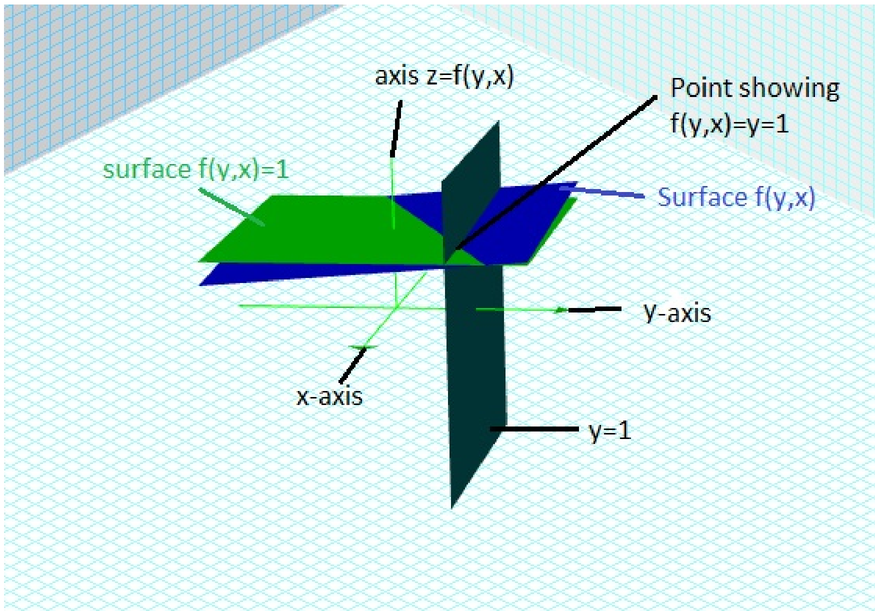


Figure 2: [Figure showing y=f(y,x)]

Next two Figure 3 and Figure 4 are demonstrating that $(1,1)$ *is coupled fixed point of mapping* g *also.*

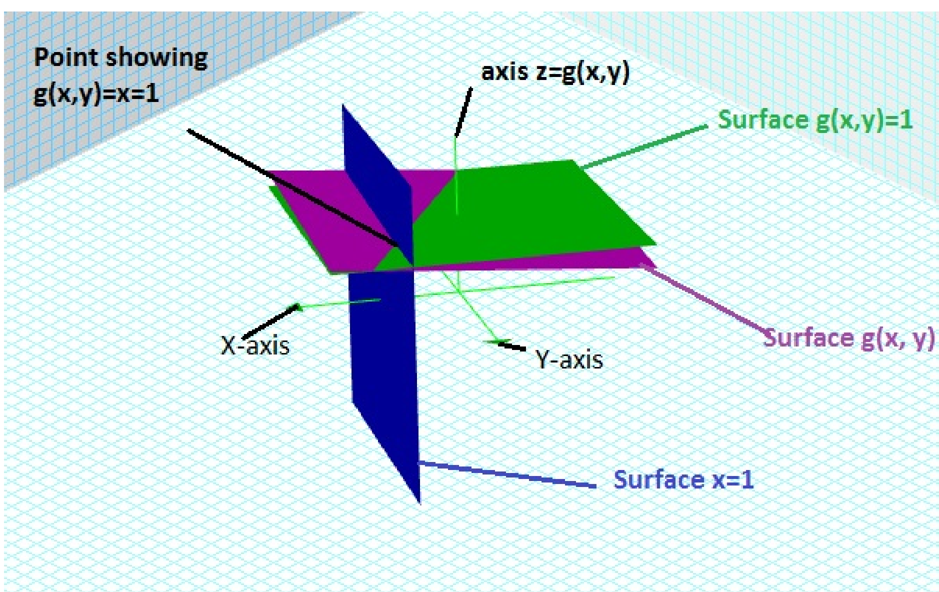


Figure 3: [Figure showing x=g(x,y)]

Thus we conclude that $(1,1)$ *is a coupled common fixed point of* f *and* g.

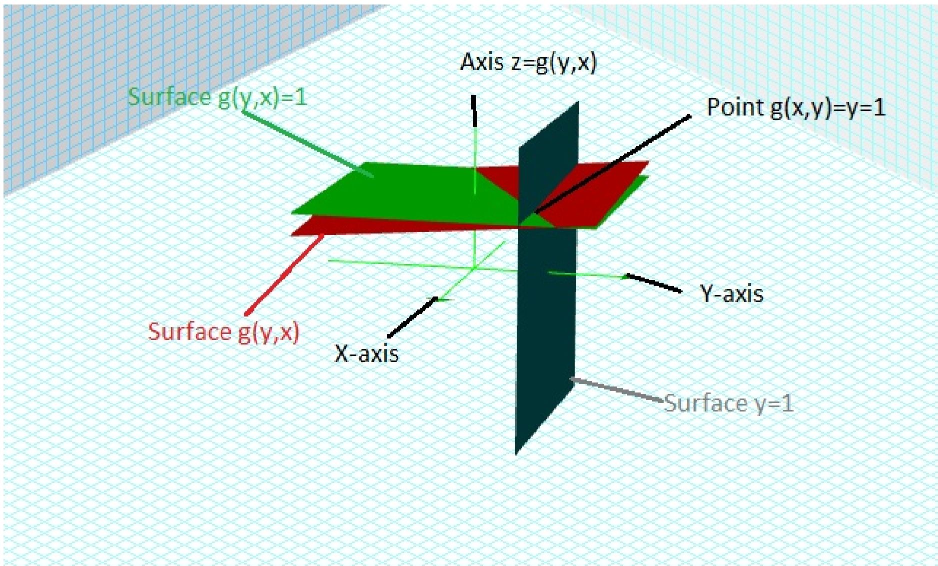


Figure 4: [Figure showing y=g(y,x)]

Theorem 2.3. *In Theorems 2.1 and 2.2, if X is a total ordered set with ordering $\precsim$, then a coupled common fixed point of f and g is unique and $x = y$.*

Proof. If $(x^*, y^*) \in X \times X$ is another coupled common fixed point of f and g, then, invoking (2.1), we have

$$\begin{aligned} d(x,x^*) + d(y,y^*) &= d(f(x,y), g(x^*,y^*)) + d(f(y,x), g(y^*,x^*)) \\ &\precsim \alpha\frac{[1 + D((x,y),(f(x,y),f(y,x)))].D((x^*,y^*),(g(x^*,y^*),g(y^*,x^*)))}{[1 + D((x,y),(x^*,y^*))]} + \\ &\beta D((x,y),(x^*,y^*)) + \\ &\gamma[D((x,y),(f(x,y),f(y,x))) + D((x^*,y^*),(g(x^*,y^*),g(y^*,x^*)))] + \\ &\delta[D((x^*,y^*),(f(x,y),f(y,x))) + D((x,y),(g(x^*,y^*),g(y^*,x^*)))] + \\ &\alpha\frac{[1 + D((y,x),(f(y,x),f(x,y)))].D((y^*,x^*),(g(y^*,x^*),g(x^*,y^*)))}{[1 + D((y,x),(y^*,x^*))]} + \\ &\beta D((y,x),(y^*,x^*)) + \\ &\gamma[D((y,x),(f(y,x),f(x,y))) + D((y^*,x^*),(g(y^*,x^*),g(x^*,y^*)))] + \\ &\delta[D((y^*,x^*),(f(y,x),f(x,y))) + D((y,x),(g(y^*,x^*),g(x^*,y^*)))] \\ &= 2\beta(d(x,x^*) + d(y,y^*)) + \\ &2\delta(d(x^*,f(x,y)) + d(y^*,f(y,x)) + d(x,g(x^*,y^*)) + d(y,g(y^*,x^*))) \\ &= (2\beta + 4\delta)(d(x,x^*) + d(y,y^*)) \end{aligned}$$

$$and \ \ hence \ \ d(x,x^*) + d(y,y^*) = (2\beta + 4\delta)(d(x,x^*) + d(y,y^*))$$

Since, $(2\beta + 4\delta) \leq \frac{1}{2}$, we have $d(x, x^*) + d(y, y^*) = 0$ which implies that $x = x^*$ and $y = y^*$. On the other hand, we have

$$\begin{aligned} d(x, y) &= d(f(x, y), g(y, x)) \\ &\preceq \alpha \frac{[1 + D((x, y), (f(x, y), f(y, x)))].D((y, x), (g(y, x), g(x, y)))}{[1 + D((x, y), (y, x))]} + \beta D((x, y), (y, x)) \\ &+ \gamma[D((x, y), (f(x, y), f(y, x))) + D((y, x), (g(y, x), g(x, y)))] \\ &+ \delta[D((y, x), (f(x, y), f(y, x))) + D((x, y), (g(y, x), g(x, y)))] \\ &\preceq (\beta + 2\delta)(d(x, y) + d(y, x)) \\ &\preceq (2\beta + 4\delta)d(x, y). \end{aligned}$$

Since $(2\beta + 4\delta) \leq \frac{1}{2}$, we have $d(x, y) = 0$ and $x = y$. This complete the proof. □

Let $f : X \times X \to X$ be a mapping. Now we denote $f^{n+1}(x, y) = f(f^n(x, y), f^n(y, x))$, for all $x, y \in X$ and $n \in N$ k8

3 Application to metric space

Taking $s = 1$, $f = g$ and $\alpha = \gamma = \delta = 0$ in Theorem 2.1, we get the following:

Corollary 3.1. *Let $(X, d, \precsim)$ be a partially ordered complete metric space. Let $f : X \times X \to X$ be the mapping such that f has the mixed monotone property on X. Suppose there exists $\beta \in [0, 1)$ with $\beta < \frac{1}{2}$ such that*

$$d(f(x, y), f(u, v)) \precsim \beta D((x, y), (u, v))$$

for all $x, y, u, v \in X$ with $x \precsim u$ and $y \succsim v$ and D is defined as in Lemma 2.1. Let $x_0, y_0 \in X$ be such that $x_0 \precsim f(x_0, y_0), y_0 \succsim f(y_0, x_0)$. If f is continuous, then f has a coupled fixed point in X.

3.1 Application to system of integral equations

Consider the following system of integral equations:

$$\begin{aligned} u(t) &= p(t) + \int_0^T \lambda(t, s)[f_1(s, u(s)) + f_2(s, v(s))]ds \\ v(t) &= p(t) + \int_0^T \lambda(t, s)[f_1(s, v(s)) + f_2(s, u(s))]ds. \end{aligned} \tag{3.1}$$

We consider the space $X = C([0, T], \mathbb{R})$ of continuous functions defined on $[0, T]$. Obviously, the space with the metric given by

$$d(u, v) = \max_{t \in [0, T]} |u(t) - v(t)|, \quad u, v \in C([0, T], \mathbb{R})$$

is a complete metric space. Consider on $X = C([0,T],\mathbb{R})$ the natural partial order relation, that is,

$$u, v \in C([0,T],\mathbb{R}), \quad u \leq v \iff u(t) \leq v(t), \ \ t \in [0,T].$$

Theorem 3.1. *Consider the problem (3.1) and assume that the following conditions are satisfied:*

(i) $f_1,\ f_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ *are continuous;*

(ii) $p : [0,T] \to \mathbb{R}$ *is continuous;*

(iii) $\lambda : [0,T] \times \mathbb{R} \to [0,\infty)$ *is continuous;*

(iv) there exists $c > 0$ *and* $\beta \in [0,1)$ *with* $\beta < \frac{1}{2}$ *such that for all* $u, v \in \mathbb{R},\ v \geq u$,

$$0 \leq f_1(s,v) - f_1(s,u) \leq c\ \beta(v-u)$$

$$0 \leq f_2(s,u) - f_2(s,v) \leq c\ \beta(v-u);$$

(v) assume that $c\ \max_{t\in[0,1]} \int_0^1 \lambda(t,s)ds \leq 1$;

(vi) there exists $x_0, y_0 \in X$ *such that*

$$x_0(t) \geq p(t) + \int_0^T \lambda(t,s)[f_1(s,x_0(s)) + f_2(s,y_0(s))]ds$$
$$y_0(t) \leq p(t) + \int_0^T \lambda(t,s)[f_1(s,y_0(s)) + f_2(s,x_0(s))]ds.$$

Then the system of integral equation (3.1) has a unique solution in X^2 *with* $(X = C([0,T],\mathbb{R}))$.

Proof. Consider the mapping $F : X \times X \to X$ defined by

$$F(u,v)(t) = p(t) + \int_0^T \lambda(t,s)[f_1(s,u(s)) + f_2(s,v(s))]ds, \tag{3.2}$$

for all $u, v \in X$ and $t \in [0,T]$. Now, we shall show that all the conditions of Corollary 3.1 are satisfied. From the condition (iv) of the Theorem 3.1, it is easy to prove that F has mixed monotone property.

Now, for $x, y, u, v \in X$ with $x \geq u,\ y \leq v$ we have

$$\begin{aligned}
d(F(x,y),F(u,v)) &= \max_{t\in[0,T]} |F(x,y)(t) - F(u,v)(t)| \\
&= \max_{t\in[0,T]} \Big| \int_0^T \lambda(t,s)\Big[f_1(s,x(s)) + f_2(s,y(s))\Big]ds - \int_0^T \lambda(t,s)\Big[f_1(s,u(s)) + f_2(s,v(s))\Big]ds \Big| \\
&\leq \max_{t\in[0,T]} \Big[\int_0^T |f_1(s,x(s)) - f_1(s,u(s))|.|\lambda(t,s)|ds + \int_0^T |f_1(s,y(s)) - f_1(s,v(s))|.|\lambda(t,s)|ds \Big] \\
&= \max_{t\in[0,T]} c\ \beta \Big[\int_0^T |x(s) - u(s)|.|\lambda(t,s)|ds + \int_0^T |y(s) - v(s)|.|\lambda(t,s)|ds \Big] \\
&\leq \Big[\max_{t\in[0,T]} |x(t) - u(t)| + \max_{t\in[0,T]} |y(t) - v(t)| \Big] c\ \beta \int_0^T |\lambda(t,s)|ds \\
&\leq \beta(d(x,u) + d(y,v)) \\
&= \beta D((x,y),(u,v)).
\end{aligned}$$

Which implies $d(F(x,y),F(u,v)) \leq \beta D((x,y),(u,v))$.

Which is just the contractive condition given in Corollary 3.1. Therefore, from Corollary 3.1, we deduce that, F has a coupled fixed point (x,y) in X, that is the system of integral equations has a solution. □

The following example shows that the superiority of Theorem3.1

Examplex 3.1. *Consider the following integral equation in $X = C([0,1],R)$*

$$F(u,v)(t) = \frac{t^2+8}{5} + \int_0^1 \frac{s^2}{35(t+4)}\Big[u(s) + \frac{1}{v(s)+1}\Big]ds. \tag{3.3}$$

It is easy to verify that the aforesaid equation is the special case of equation 3.2, in which

$$p(t) = \frac{t^2+8}{5},\ \ \lambda(t,s) = \frac{s^2}{35(t+4)},\ \ f_1(s,t) = t,\ \ f_2(s,t) = \frac{1}{t+1}.$$

Indeed, the function p, λ, f_1 and f_2 are continuous. Hence the assumption (i)-(iii) are fulfilled. Further, for all $u, v \in R, v \geq u$ there exist $C = 16 > 0$ and $\beta = \frac{1}{4} \in [0,1)$ with $\beta < \frac{1}{2}$ such that

$$0 \leq f_1(s,v) - f_1(s,u) \leq c\beta(v-u),$$

$$0 \leq f_2(s,u) - f_2(s,v) \leq c\beta(v-u).$$

Thus the condition (iv) of Theorem 3.1 is satisfied. For condition (v), we have

$$c \max_{t\in[0,1]} \int_0^T \lambda(t,s)ds = 16 \max_{t\in[0,1]} \int_0^1 \frac{s^2}{35(t+4)}ds = \max_{t\in[0,1]} \frac{16}{105(t+4)} \leq 1$$

shows the validity of condition (v).

Consider $x_0(t) = 1$ and $y_0(t) = 1$, then we get

$$\begin{aligned} p(t) + \int_0^1 \lambda(t,s)[f_1(s,x_0(s)) + f_2(s,y_0(s))]ds &= \frac{t^2+8}{5} + \int_0^1 \frac{s^2}{35(t+4)}[f_1(s,1) + f_2(s,1)]ds \\ &= \frac{t^2+8}{5} + \int_0^1 \frac{s^2}{35(t+4)}\Big[1 + \frac{1}{2}\Big]ds \\ &= \frac{t^2+8}{5} + \frac{3}{70(t+4)}\Big(\frac{s^3}{3}\Big)_0^1 \\ &= \frac{t^2+8}{5} + \frac{3}{70(t+4)} \geq 1 \end{aligned}$$

that is $x_0 \geq f(x_0,y_0)$.

Similarly, one can show that $y_0 \leq f(y_0,x_0)$. It follows that all the conditions are satisfied. Thus the integral equation (3.3) *has a solution in X^2 with $X = C([0,1],R)$.*

References

[1] H. Aydi, Monica-F Bota, E. Karapinar, S. Mitrovic, A fixed point theorem for set-valued quasi contractions in b-metric spaces, *Fixed Point Theory and Applications*,2012:88(12012).

[2] I. A. Bakhtin, The contraction principle in quasi metric spaces, *Functional Analysis*,30,26-37(1989).

[3] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.*,65,1379-1393(2006).

[4] B. S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, *Nonlinear Anal.*,73,2524-2531(2010).

[5] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*,1,5-11(1993).

[6] N. V. Dung, On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces, *Fixed Point Theory and Applications*,2013:48(2013).

[7] M. E. Gordji, E. Akbartabar, Y. J. Cho, M. Ramezani, Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces, *Fixed Point Theory Appl.* ,2012:95(2012).

[8] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal.*,11,623-632(1987).

[9] V. Lakshmikantham, L. Ciric, Coupled fixed points theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.*,70,4341-4349(2009).

[10] N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Anal.*,74,983-992(2011).

[11] H. K. Nashine, B. Samet, C. Vetro,Coupled coincidence points for compatible mappings satisfying mixed monotone property, *J. Nonlinear Sci. Appl.*,5(2):104-114(2012).

[12] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Am. Math. Soc.*,132:1435-1443(2004).

[13] D. Singh, O.P. Chauhan, N. Singh, V. Joshi, Common fixed point theorems in complex valued b- metric spaces, *J. Math. Comput. Sci.*, 5,(3),412-429(2013).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES AND ITS STABILITY

GILJUN HAN AND CHANG IL KIM*

Abstract. In this paper, we investigate the functional inequality

$$N(f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y), t)$$
$$\geq N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), kt)$$

for some fixed nonzero real number k and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces by fixed point methods.

1. Introduction

In 1940, Ulam proposed the following stability problem (cf. [25]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [12] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [24] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians (see [4], [5], [6], [9], and [19]).

In 2001, Rassias [23] introduced the following cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) = 0 \tag{1.1}$$

and every solution of the cubic functional equation is called *a cubic mapping* and in ([14]), the following cubic functional equation was investigated

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.2}$$

Katsaras [15] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a vector space in different points of view. In particular, Bag and Samanta [2], following Cheng and Mordeson [3], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [16].

In [10], Glányi showed that if a mapping $f : X \longrightarrow Y$ satisfies the following functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 39B62, 39B72, 54A40, 47H10.

Key words and phrases. Hyers-Ulam stability, fuzzy normed space, fixed point theorem.

* Corresponding author.

The second author was supported by the research fund of Dankook University in 2018.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

Glányi [11] and Fechner [8] proved the Hyers-Ulam stability of (1.3). Park, Cho, and Han [22] proved the Hyers-Ulam stability of the following functional inequality:

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|. \tag{1.4}$$

Further, Park [21] proved the generalized Hyers-Ulam stability of the Cauchy additive functional inequality (1.4) in fuzzy Banach spaces using the fixed point method if f is an odd mapping.

In this paper, we investigate the following functional inequality related by (1.1) and (1.2)

$$\begin{aligned} &N(f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y), t) \\ &\geq N(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), kt) \end{aligned} \tag{1.5}$$

for some fixed nonzero real number k and prove the generalized Hyers-Ulam stability for (1.5) in fuzzy Banach spaces by fixed point methods.

2. PRELIMINARIES

In this paper, we use the definition of fuzzy normed spaces given in [2], [17], and [18].

Definition 2.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is called *a fuzzy norm on X* if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,

(N1) $N(x, t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for any $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space*.

Let (X, N) be a fuzzy normed space and $\{x_n\}$ a sequence in X. Then (i) $\{x_n\}$ is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $n \geq m$, all positive integer p, and any $t > 0$ and (ii) $\{x_n\}$ is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$.

Example 2.2. For example, it is well known that for any normed space $(X, ||\cdot||)$ and any nonnegative real number ϵ, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x, t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+\epsilon||x||}, & \text{if } t > 0 \ , \end{cases}$$

is a fuzzy norm on X([17], [18], and [19]).

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

In 1996, Isac and Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 2.3. [7] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

3. Solutions of (1.5)

In this section, we investigate the solution and prove the generalized Hyers-Ulam stability of the functional inequality (1.5) in fuzzy Banach spaces. For any mapping $f : X \longrightarrow Y$, let

$$A_f(x, y) = f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)$$

and

$$B_f(x, y) = f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y).$$

By (N5), we can easily shown the following lemma.

Lemma 3.1. *Let $\alpha_i : [0, \infty) \longrightarrow [0, \infty) (i = 1, 2, \cdots, n)$ be a mapping and r a positive real numbers with $r > 1$ and $y, z, z_1, z_2, \cdot, z_n \in Y$. Suppose that*

$$N(y, t) \geq \min\{N(z, r^n t), N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$ and all $n \in \mathbb{N}$. Then

$$N(y, t) \geq \min\{N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$.

By Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let r be a real number with $r > 1$ and $y \in Y$. Suppose that*

$$N(y, t) \geq N(y, rt)$$

for all $t > 0$. Then $y = 0$.

Using Lemma 3.1 and Corollary 3.2, we will prove the following theorem :

Theorem 3.3. *Let $f : X \longrightarrow Y$ be a mapping. Suppose that k is a real number with $k > 4$. Then f is cubic if and only if f is a solution of (1.5).*

Proof. Letting $x = 0$ and $y = 0$ in (1.5), we have

$$N(f(0), t) \geq N\Big(f(0), \frac{3}{7}kt\Big)$$

for all $t > 0$ and sicne $\frac{3}{7}k > 1$, by Corollary 3.2, we get $f(0) = 0$.

Suppose that f is a solution of (1.5). Letting $x=0$ in (1.5), we have

$$N(f(2y)-9f(y)-f(-y),t)\geq N(f(y)+f(-y),kt) \tag{3.1}$$

for all $y\in X$ and all $t>0$ and letting $y=-x$ in (1.5), we have

$$N(f(2x)-3f(x)+5f(-x),t)\geq N(f(3x)-2f(2x)-11f(x),kt) \tag{3.2}$$

for all $x\in X$ and all $t>0$. Letting $y=x$ in (1.5), we have

$$N(f(3x)-3f(2x)-3f(x),t)\geq N(f(3x)-2f(2x)-11f(x),kt) \tag{3.3}$$

for all $x\in X$ and all $t>0$. By (3.1) and (3.2), we get

$$\begin{aligned}&N(6f(x)+6f(-x),t)\\ &\geq \min\Big\{N\Big(f(2x)-9f(x)-f(-x),\frac{t}{2}\Big),N\Big(f(2x)-3f(x)+5f(-x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(f(x)+f(-x),\frac{kt}{2}\Big),N\Big(f(3x)-2f(2x)-11f(x),\frac{kt}{2}\Big)\Big\}\end{aligned}$$

for all $x\in X$ and all $t>0$ and so we obtain

$$\begin{aligned}&N(f(x)+f(-x),t)\\ &\geq \min\{N(f(x)+f(-x),3kt),N(f(3x)-2f(2x)-11f(x),3kt)\}\end{aligned} \tag{3.4}$$

for all $x\in X$ and all $t>0$. For any $x\in X$, let

$$G(x)=f(3x)-2f(2x)-11f(x),\;\; H(x)=f(x)+f(-x)$$

for all $x\in X$. By (3.1), (3.4), and (N5), we have

$$\begin{aligned}N(f(2x)-8f(x),t)&\geq \min\Big\{N\Big(f(2x)-9f(x)-f(-x),\frac{t}{2}\Big),N\Big(H(x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(H(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{3kt}{2}\Big),N\Big(G(x),\frac{3kt}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(H(x),\frac{kt}{2}\Big),N\Big(G(x),\frac{3kt}{2}\Big)\Big\}\end{aligned} \tag{3.5}$$

for all $x\in X$ and all $t>0$. Further, by (3.3), (3.5), and (N5), we have

$$\begin{aligned}N(G(x),t)&\geq \min\Big\{N\Big(f(3x)-3f(2x)-3f(x),\frac{t}{2}\Big),N\Big(f(2x)-8f(x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{kt}{4}\Big),N\Big(G(x),\frac{3kt}{4}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\end{aligned} \tag{3.6}$$

for all $x\in X$ and all $t>0$ and since $k>4$, by (3.6) and (N5), we have

$$\begin{aligned}N(G(x),t)&\geq \min\Big\{N\Big(G(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{k^2t}{2^2}\Big),N\Big(H(x),\frac{k^2t}{2^3}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{k^2t}{2^2}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\end{aligned}$$

for all $x\in X$ and all $t>0$. Hence by induction, we get

$$N(G(x),t)\geq \min\Big\{N\Big(G(x),\frac{k^nt}{2^n}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\} \tag{3.7}$$

for all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$. By Lemma 3.1 and (3.7), we obtain

$$N(G(x), t) \geq N\Big(H(x), \frac{kt}{4}\Big)$$

for all $x \in X$ and all $t > 0$. By (3.4) and (N5), we have

$$\begin{aligned} N(G(x), t) &\geq N\Big(H(x), \frac{kt}{4}\Big) \\ &\geq \min\Big\{N\Big(H(x), \frac{(3k)^2 t}{12}\Big), N\Big(G(x), \frac{(3k)^2 t}{12}\Big)\Big\} \\ &\geq \min\Big\{N\Big(H(x), \frac{(3k)^3 t}{12}\Big), N\Big(G(x), \frac{(3k)^3 t}{12}\Big), N\Big(G(x), \frac{(3k)^2 t}{12}\Big)\Big\} \\ &\geq \min\Big\{N\Big(H(x), \frac{(3k)^3 t}{12}\Big), N\Big(G(x), \frac{(3k)^2 t}{12}\Big)\Big\} \end{aligned} \tag{3.8}$$

for all $x \in X$ and all $t > 0$. By induction and (3.8), we get

$$N(G(x), t) \geq N\Big(G(x), \frac{(3k)^2 t}{12}\Big) \tag{3.9}$$

for all $x \in X$ and all $t > 0$. By (3.9) and Corollary 3.2, we get

$$G(x) = f(3x) - 2f(2x) - 11f(x) = 0 \tag{3.10}$$

for all $x \in X$. By (3.4) and (3.10), we get

$$N(H(x), t) \geq N(H(x), 3kt) \tag{3.11}$$

for all $x \in X$ and by Corollary 3.2, we have

$$H(x) = f(x) + f(-x) = 0 \tag{3.12}$$

for all $x \in X$. Hence f is an odd mapping. Further, by (3.5), (3.10), and (3.12), we get

$$f(2x) = 8f(x) \tag{3.13}$$

for all $x \in X$. Now, letting $x = 2y$ in (1.5), by (3.13), we have

$$\begin{aligned} &N(8f(x+y) - 3f(2x+y) + 24f(x) - f(2x-y) - 6f(y), t) \\ &\geq N(f(4x+y) + f(4x-y) - 2f(2x+y) - 2f(2x-y) - 96f(x), kt) \end{aligned} \tag{3.14}$$

for all $x, y \in X$ and all $t > 0$ and letting $y = -y$ in (3.14), by (3.12), we have

$$\begin{aligned} &N(8f(x-y) - 3f(2x-y) + 24f(x) - f(2x+y) + 6f(y), t) \\ &\geq N(f(4x+y) + f(4x-y) - 2f(2x+y) - 2f(2x-y) - 96f(x), kt) \end{aligned} \tag{3.15}$$

for all $x, y \in X$ and all $t > 0$. By (3.14) and (3.15), we have

$$N(4A_f(x, y), t) \geq N\Big(A_f(2x, y), \frac{kt}{2}\Big)$$

for all $x, y \in X$ and all $t > 0$ and so we have

$$N(A_f(x, y), t) \geq N(A_f(2x, y), 2kt) \tag{3.16}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = 2y$ in (1.5), we have

$$\begin{aligned} &N(f(x+4y) - 3f(x+2y) + 3f(x) - f(x-2y) - 48f(y), t) \\ &\geq N(8f(x+y) + 8f(x-y) - 2f(x+2y) - 2f(x-2y) - 12f(x), kt) \end{aligned} \tag{3.17}$$

for all $x, y \in X$ and all $t > 0$ and interchang x and y in (3.17), by (3.12) and (1.5), we get

$$\begin{aligned}
&N(f(4x+y) - 3f(2x+y) + 3f(y) + f(2x-y) - 48f(x), t)\\
&\geq N(8f(x+y) - 8f(x-y) - 2f(2x+y) + 2f(2x-y) - 12f(y), kt)\\
&= N(-2A_f(x,y) - 4B_f(y,-x), kt)\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(B_f(y,-x), \frac{kt}{6}\Big)\Big\} \qquad (3.18)\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(A_f(y,-x), \frac{k^2t}{6}\Big)\Big\}\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(A_f(y,x), \frac{k^2t}{6}\Big)\Big\}
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = -y$ in (3.18), we get

$$\begin{aligned}
&N(f(4x-y) - 3f(2x-y) - 3f(y) + f(2x+y) - 48f(x), t)\\
&\geq \min\Big\{N\Big(A_f(x,-y), \frac{kt}{6}\Big), N\Big(A_f(-y,-x), \frac{k^2t}{6}\Big)\Big\} \qquad (3.19)\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(A_f(y,x), \frac{k^2t}{6}\Big)\Big\}
\end{aligned}$$

for all $x, y \in X$. By (3.16), (3.18), (3.19), and (N5), we have

$$N(A_f(x,y),t) \geq N(A_f(2x,y), 2kt) \geq \min\Big\{N\Big(A_f(x,y), \frac{k^2t}{6}\Big), N\Big(A_f(y,x), \frac{k^3t}{6}\Big)\Big\}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1 and induction, we get

$$A_f(x,y) = f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) = 0$$

for all $x, y \in X$. Thus f is a cubic mapping. □

By Theorem 3.3, we have the following corollaries :

Corollary 3.4. *Let $f : X \longrightarrow Y$ be a mapping. Suppose that a, b are real numbers with $|a| > 4|b| > 0$. Then f is cubic if and only if f satisfies the following inequlaity*

$$N(aB_f(x,y), t) \geq N(bA_f(x,y), t) \qquad (3.20)$$

for all $x, y \in X$ and all $t > 0$.

Corollary 3.5. *Let $f : X \longrightarrow Y$ be a mapping. Suppose that a is a real number with $|a| > 8$. Then f is cubic if and only if f satisfies the following inequlaity*

$$N(aB_f(x,y) + A_f(x,y), t) \geq N(A_f(x,y), t) \qquad (3.21)$$

for all $x, y \in X$ and all $t > 0$.

Proof. By (3.21) and (N5), we have

$$\begin{aligned}
N(B_f(x,y), t) &= N(aB_f(x,y), |a|t)\\
&\geq \min\Big\{N\Big(aB_f(x,y) + A_f(x,y), \frac{|a|}{2}t\Big), N\Big(A_f(x,y), \frac{|a|}{2}t\Big)\Big\}\\
&= N\Big(A_f(x,y), \frac{|a|}{2}t\Big)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Hence by Theorem 3.3, we have the result. □

Using the fuzzy norm $N_X : X \times \mathbb{R} \longrightarrow [0,1]$ in Exmaple 2.2, we have the following corollary :

Corollary 3.6. *Let $(X, || \cdot ||)$ be a normed space and $f : X \longrightarrow Y$ a mapping. Suppose that a is a real number with $|a| > 8$. Then f is cubic if and only if f satisfies the following inequlaity*

$$\|aB_f(x,y) + A_f(x,y)\| \leq \|A_f(x,y)\| \tag{3.22}$$

for all $x, y \in X$.

4. The generalized Hyers-Ulam stability for (1.5)

Now, we will prove the generalized Hyers-Ulam stability for (1.5) in fuzzy normed spaces.

Theorem 4.1. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'(\phi(2x,2y),t) \geq N'(8L\phi(x,y),t) \tag{4.1}$$

for all $x, y \in X$, $t > 0$ and some real number L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(B_f(x,y),t) \geq \min\{N(A_f(x,y),kt), N'(\phi(x,y),t)\} \tag{4.2}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), \frac{1}{8(1-L)}t) \geq \Psi(x,t) \tag{4.3}$$

for all $x \in X$, $t > 0$, and some real number k with $k > 4$, where $\Psi(x,t) = \min\Big\{N'\Big(\phi(x.x), \frac{3kt}{4}\Big), N'\Big(\phi(x.-x), \frac{3t}{2}\Big), N'\Big(\phi(0,x), \frac{t}{2}\Big)\Big\}$.

Proof. Letting $x = 0$ in (4.2), by (N2), we have

$$N(f(2y) - 9f(y) - f(-y),t) \geq \min\{N(H(y),kt), N'(\phi(0,y),t)\} \tag{4.4}$$

for all $y \in X$ and $t > 0$ and letting $y = -x$ in (4.2), by (N2), we have

$$N(f(2x) + 5f(-x) - 3f(x),t) \geq \min\{N(G(x),kt), N'(\phi(x.-x),t)\} \tag{4.5}$$

for all $x \in X$ and $t > 0$. Letting $y = x$ in (4.2), we have

$$N(f(3x) - 3f(2x) - 3f(x),t) \geq \min\{N(G(x),kt), N'(\phi(x.x),t)\} \tag{4.6}$$

for all $x \in X$ and all $t > 0$. By (4.4) and (4.5), we get

$$\begin{aligned} &N(H(x),t) \\ &\geq \min\{N(H(x),3kt), N(G(x),3kt), N'(\phi(0,x),3t), N'(\phi(x.-x),3t)\} \end{aligned} \tag{4.7}$$

for all $x \in X$ and all $t > 0$. Similar to the proof of Theorem 3.3, by (4.7), we have

$$N(H(x),t) \geq \min\{N(G(x),3kt), N'(\phi(0,x),3t), N'(\phi(x.-x),3t)\} \tag{4.8}$$

for all $x \in X$ and all $t > 0$. By (4.4), (4.8), and (N5), we get

(4.9)
$$\begin{aligned}&N(f(2x)-8f(x),t) \geq \min\Big\{N\Big(H(x),\frac{t}{2}\Big), N\Big(f(2x)-9f(x)-f(-x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(H(x),\frac{t}{2}\Big), N\Big(H(x),\frac{kt}{2}\Big), N'\Big(\phi(0,x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{3kt}{2}\Big), N'\Big(\phi(x.-x),\frac{3t}{2}\Big), N'\Big(\phi(0,x),\frac{t}{2}\Big)\Big\}\end{aligned}$$

for all $x \in X$ and all $t > 0$ and by (4.6), (4.9), and (N5), we get

(4.10)
$$\begin{aligned}&N(G(x),t) \geq \min\Big\{N\Big(f(3x)-3f(2x)+5f(-x),\frac{t}{2}\Big), N\Big(f(2x)-8f(x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{kt}{2}\Big), N'\Big(\phi(x.x),\frac{t}{2}\Big), N'\Big(\phi(x.-x),\frac{3t}{4}\Big), N'\Big(\phi(0,x),\frac{t}{4}\Big)\Big\}\end{aligned}$$

for all $x \in X$ and all $t > 0$. Since $k > 4$, by (4.10) and (N5), we obtain

$$N(G(x),t) \geq \min\Big\{N'\Big(\phi(x.x),\frac{t}{2}\Big), N'\Big(\phi(x.-x),\frac{3t}{4}\Big), N'\Big(\phi(0,x),\frac{t}{4}\Big)\Big\} \tag{4.11}$$

for all $x \in X$ and all $t > 0$. By (4.9), (4.11), and (N5), we get

$$\begin{aligned}&N(f(2x)-8f(x),t)\\ &\geq \min\Big\{N'\Big(\phi(x.x),\frac{3kt}{4}\Big), N'\Big(\phi(x.-x),\frac{9kt}{8}\Big), N'\Big(\phi(0,x),\frac{3kt}{8}\Big),\\ &N'\Big(\phi(x.-x),\frac{3t}{2}\Big), N'\Big(\phi(0,x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N'\Big(\phi(x.x),\frac{3kt}{4}\Big), N'\Big(\phi(x.-x),\frac{3t}{2}\Big), N'\Big(\phi(0,x),\frac{t}{2}\Big)\Big\}\end{aligned} \tag{4.12}$$

for all $x \in X$ and all $t > 0$.

Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x)-h(x),ct) \geq \Psi(x,t), \forall x \in X, \forall t > 0\}.$$

Then (S,d) is a complete metric space(See [20]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = \frac{1}{8}g(2x)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4.1), we have

$$N(Jg(x)-Jh(x),ct) = N(g(2x)-h(2x),8ct) \geq \Psi(2x,8t) \geq \Psi(x,\frac{t}{L})$$

for all $x \in X$ and $t > 0$. Hence $N(Jg(x)-Jh(x),cLt) \geq \Psi(x,t)$ for all $x \in X$ and $t > 0$ and thus $d(Jg,Jh) \leq Ld(g,h)$ for any $g, h \in S$. Moreover, by (4.12), we have $d(Jf,f) \leq \frac{1}{8} < \infty$. By Theorem 2.3, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^nf,A) \to 0$ as $n \to \infty$. That is,

$$C(x) = N - \lim_{n\to\infty} \frac{f(2^nx)}{2^{3n}} \tag{4.13}$$

for all $x \in X$. Replacing x, and y by 2^nx and 2^ny in (4.2), respectively, by (4.1), we have

$$\begin{aligned}&N(B_f(2^nx,2^n),2^{3n}t)\\ &\geq \min\Big\{N(A_f(2^nx,2^ny),2^{3n}t), N'\Big(\phi(x,y),\frac{1}{L^n}t\Big)\Big\}\end{aligned} \tag{4.14}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x, y \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4.14), C is a solution of (1.5) and so by Theorem 3.3, C is a cubic mapping. Since $d(f, Jf) \leq \frac{1}{8}$, by Theorem 2.3, we have (4.17).

Now, we show the uniqueness of C. Let C_0 be another cubic mapping with (4.17). Then for any positive integer n,

$$C(x) = \frac{C(2^n x)}{2^{3n}},\ C_0(x) = \frac{C_0(2^n x)}{2^{3n}}$$

for all $x \in X$. Hence by (4.17), (N3) and (N4), we have

$$N(C(x) - C_0(x), t) = N(C(2^n x) - C_0(2^n x), 2^{3n} t) \geq \Psi(2^n x, 2^{3n} 8(1 - L)t)$$
$$\geq \Psi\Big(x, \frac{8(1 - L)t}{L^n}\Big)$$

for all $x \in X$, $t > 0$, and all $n \in \mathbb{N}$. Hence, letting $n \to \infty$ in the above inequality, we have $C(x) = C_0(x)$ for all $x \in X$. □

By Corollary 3.5 and Theorem 4.1, we can show that the following corollaries:

Corollary 4.2. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $0 < p < \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(B_f(x, y), t) \geq \min\Big\{N(A_f(x, y), kt),\ \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p \|y\|^p)}\Big\} \tag{4.15}$$

for all $x, y \in X$, all $t > 0$ and some real number k with $k > 4$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{(8 - 2^{2p})t}{(8 - 2^{2p})t + 2\epsilon \|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.3. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function with (4.1) Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(aB_f(x, y) + A_f(x, y), t) \geq \min\{N(A_f(x, y), t), N'(\phi(x, y), t)\} \tag{4.16}$$

for all $x, y \in X$, all $t > 0$ and some real numbers a with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), \frac{1}{8(1 - L)} t) \geq \Psi(x, t) \tag{4.17}$$

*for all $x \in X$, $t > 0$, and some fixed real number k with $k > 4$,
where $\Psi(x, t) = \min\Big\{N'\Big(\phi(x.x), \frac{3|a|t}{8}\Big), N'\Big(\phi(x. - x), \frac{3t}{2}\Big), N'\Big(\phi(0, x), \frac{t}{2}\Big)\Big\}$.*

Proof. By (N5) and (4.16), we have

$$N(B_f(x, y), t) \geq \min\Big\{N\Big(aB_f(x, y) + A_f(x, y), \frac{|a|}{2} t\Big), N\Big(A_f(x, y), \frac{|a|}{2} t\Big)\Big\}$$
$$\geq \min\Big\{N\Big(A_f(x, y), \frac{|a|}{2} t\Big), N'\Big(\phi(x, y), \frac{|a|}{2} t\Big)\Big\}$$
$$\geq \min\Big\{N\Big(A_f(x, y), \frac{|a|}{2} t\Big), N'\Big(\phi(x, y), t\Big)\Big\}$$

for all $x, y \in X$, all $t > 0$. Hence we have the results. □

Corollary 4.4. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $0 < p < \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &N(aB_f(x,y) + A_f(x,y), t) \\ &\geq \min\Big\{N(A_f(x,y),t),\ \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}\Big\} \end{aligned} \tag{4.18}$$

for all $x, y \in X$, all $t > 0$ and some real number a with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{(8 - 2^{2p})t}{(8 - 2^{2p})t + 2\epsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Related with Theorem 4.1, we can also have the following theorem. And the proof is similar to that of Theorem 4.1.

Theorem 4.5. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2}, \frac{y}{2}\Big), t\Big) \geq N'\Big(\frac{L}{8}\phi(x,y), t\Big) \tag{4.19}$$

for all $x, y \in X$, $t > 0$ and some L with $0 \leq L < 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.2). Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N\Big(f(x) - C(x), \frac{L}{1-L}t\Big) \geq \Psi_0(x,t) \tag{4.20}$$

for all $x \in X$, $t > 0$, and some fixed real number k with $k > 4$,
where $\Psi_0(x,t) = \min\Big\{N'\Big(\phi(x.x), 6kt\Big), N'\Big(\phi(x.-x), 12t\Big), N'\Big(\phi(0,x), 4t\Big)\Big\}$.

Proof. By (4.12) in Theorem 4.1, we get

$$\begin{aligned} &N\Big(f(x) - 8f\Big(\frac{x}{2}\Big), t\Big) \\ &\geq \min\Big\{N'\Big(\phi(x.x), \frac{6kt}{L}\Big), N'\Big(\phi(x.-x), \frac{12t}{L}\Big), N'\Big(\phi(0,x), \frac{4t}{L}\Big)\Big\} \end{aligned} \tag{4.21}$$

for all $x \in X$ and all $t > 0$.

Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \geq \Psi_0(x,t), \forall x \in X, \forall t > 0\}.$$

Then (S, d) is a complete metric space(See [20]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 8g\Big(\frac{1}{2}x\Big)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4.19), we have

$$N(Jg(x) - Jh(x), ct) = N\Big(8g\Big(\frac{1}{2}x\Big) - 8h\Big(\frac{1}{2}x\Big), ct\Big) \geq \Psi_0\Big(\frac{1}{2}x, \frac{t}{8}\Big) \geq \Psi_0\Big(x, \frac{t}{L}\Big)$$

for all $x \in X$ and $t > 0$. Hence $N(Jg(x) - Jh(x), cLt) \geq \Psi_0(x,t)$ for all $x \in X$ and $t > 0$ and thus $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$. Moreover, by (4.21) we have $d(f, Jf) \leq L < \infty$. The rest of the proof is similar to Theorem 4.1. □

By Corollary 3.6 and Theorem 4.5, we can show that the following corollaries:

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Corollary 4.6. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $p > \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(B_f(x,y),t) \geq \min\Big\{N(A_f(x,y),kt),\ \frac{t}{t+\epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}\Big\} \tag{4.22}$$

for all $x, y \in X$, all $t > 0$ and some real number k with $k > 4$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x)-C(x),t) \geq \frac{(2^{2p}-8)t}{(2^{2p}-8)t+2\epsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.7. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function with (4.1) Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(aB_f(x,y)+A_f(x,y),t) \geq \min\{N(A_f(x,y),t), N'(\phi(x,y),t)\} \tag{4.23}$$

for all $x, y \in X$, all $t > 0$ and some real numbers a, b with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x)-C(x),\frac{L}{1-L}t) \geq \Psi_0(x,t) \tag{4.24}$$

for all $x \in X$, $t > 0$, and some fixed real number k with $k > 4$,
where $\Psi_0(x,t) = \min\Big\{N'\Big(\phi(x.x),3|a|t\Big), N'\Big(\phi(x.-x),12t\Big), N'\Big(\phi(0,x),4t\Big)\Big\}$.

Corollary 4.8. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $p > \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &N(aB_f(x,y)+A_f(x,y),t) \\ &\geq \min\Big\{N(A_f(x,y),t),\ \frac{t}{t+\epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}\Big\} \end{aligned} \tag{4.25}$$

for all $x, y \in X$, all $t > 0$ and some real number a with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x)-C(x),t) \geq \frac{(2^{2p}-8)t}{(2^{2p}-8)t+2\epsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2**(1950), 64-66.

[2] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11**(2003), 687-705.

[3] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces, Bull. Calcutta Math. Soc.* **86***(1994), 429-436.*

[4] P.W.Cholewa, Remarkes on the stability of functional equations*, Aequationes Math.* **27***(1984), 76-86.*

[5] K Cieplinski, Applications of fixed point theorems to the hyers-ulam stability of functional equation-A survey*, Ann. Funct. Anal.* **3***(2012), no. 1, 151-164.*

[6] S. Czerwik, On the stability of the quadratic mapping in normed spaces*, Bull. Abh. Math. Sem. Univ. Hamburg* **62***(1992), 59-64.*

[7] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc.* **74** *(1968), 305-309.*

[8] W. Fechner, Stability of a functional inequalty associated with the Jordan-Von Neumann functional equation*, Aequationes Math.* **71***(2006), 149-161.*

[9] P. Găvruta, A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184**(1994), 431-436. .

[10] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Mathematicae, **62**(2001), 303-309.

[11] A. Gilányi, *On a problem by K. Nikoden, Mathematical Inequalities and Applications*, **5**(2002), 701-710.

[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27**(1941), 222-224.

[13] G. Isac and Th. M. Rassias, *Stability of ψ-additive mappings, Appications to nonlinear analysis*, Internat. J. Math. and Math. Sci. **19**(1996), 219-228.

[14] K. W. Jun and H. M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274**(2002), 867-878.

[15] A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets Syst **12**(1984), 143-154.

[16] I. Kramosil and J. Michálek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11**(1975), 336-344.

[17] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets Syst. **159**(2008), 720-729.

[18] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets Syst. **159**(2008), 730-738.

[19] M. Mirzavaziri and M. S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bulletin of the Brazilian Mathematical Society **37**(2006), no. 3, 361-376

[20] M. S. Moslehian and T. H. Rassias, *Stability of functional equations in non-Archimedean spaces*, Applicable Anal. Discrete Math. **1**(2007), 325-334.

[21] C. Park, *Fuzzy Stability of Additive Functional Inequalities with the Fixed Point Alternative*, J. Inequal. Appl. **2009**(2009), 1-17.

[22] C. Park, Y. S. Cho, and M. H. Han, *Functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**(2007), 1-13.

[23] J. M. Rassias, *Solution of the Ulam stability problem for cubic mappings*, Glasnik Matematički, **36**(2001), 63-72.

[24] Th. M. Rassias, *On the stability of the linear mapping in Banach sapces*, Proc. Amer. Math. Sco. **72**(1978), 297-300.

[25] S. M. Ulam, Problems in modern mathematics, Science Editions John Wiley and Sons, Inc., New York, 1964.

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, Korea
E-mail address: gilhan@dankook.ac.kr

Department of Mathematics Education, Dankook University, 126, Jukjeon, Yongin, Gyeonggi, South Korea 448-701, KOREA
E-mail address: kci206@hanmail.net

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 3, 2019

Modified Halpern's iteration without assumptions on fixed point set in metric space, Kanyarat Cheawchan and Atid Kangtunyakarn,……………………………………………………………393

Convergence of double acting iterative scheme for a family of generalized φ-weak contraction mappings in CAT(0) spaces, Kyung Soo Kim,……………………………………………….404

On solution of a system of differential equations via fixed point theorem, Muhammad Nazam, Muhammad Arshad, Choonkil Park, Ozlem Acar, Sungsik Yun, and George A. Anastassiou,417

Some equalities and inequalities for K-g-frames, Zhong-Qi Xiang and Yin-Suo Jia,………427

AQ-functional equation in matrix non-Archimedean fuzzy normed spaces, Jung-Rye Lee, George A. Anastassiou, Choonkil Park, Murali Ramdoss, and Vithya Veeramani,…………438

Existence of continuous selection for some special kind of multivalued mappings, G. Poonguzali, Muthiah Marudai, George A. Anastassiou, and Choonkil Park,…………………………….447

Refined stability of set-valued functional equations, Hong-Mei Liang, Hark-Mahn Kim, and Hwan-Yong Shin,………………………………………………………………………………453

Approximate Cauchy-Jensen and bi-quadratic mappings in 2-Banach spaces, Won-Gil Park and Jae-Hyeong Bae,……………………………………………………………………………….463

Birkhoff Normal Forms, KAM theory and continua of periodic points for certain planar system, M. R. S. Kulenović, E. Pilav, and N. Mujić,…………………………………………………470

Durrmeyer type (p, q)-Baskakov operators for functions of one and two variables, Qing-Bo Cai and Guorong Zhou,……………………………………………………………………………..481

A subclass of analytic functions defined by a fractional integral operator, Alb Lupaș Alina,502

Properties on a subclass of analytic functions defined by a fractional integral operator, Alb Lupaș Alina,…………………………………………………………………………………………506

Normal criteria of meromorphic functions concerning holomorphic functions, Da-Wei Meng, San-Yang Liu, and Hong-Yan Xu,……………………………………………………………511

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 3, 2019

(continued)

Mixed Weakly Monotone Mappings and its Application to System of Integral Equations via Fixed Point Theorems, Deepak Singh, Om Prakash Chauhan, Afrah A N Abdou, and Garima Singh,……………………………………………………………………………………527

Functional inequalities in fuzzy normed spaces and its stability, Giljun Han, Chang Il Kim,544