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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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715 Stadium Dr.
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Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
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Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
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Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
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Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
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Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

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Ram Verma
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1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
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Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Xiao-Jun Yang
State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China
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Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang, Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Department of Mathematical Sciences
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An iterative algorithm of poles assignment for LDP systems *

Lingling Lv †, Zhe Zhang ‡, Lei Zhang §, Xianxing Liu ¶

Abstract

The problem of poles assignment and robust poles assignment in linear discrete-time periodic (LDP) system via periodic state feedback is discussed in this paper. Based on a numerical solution to the periodic Sylvester matrix equation, an iterative algorithm of computing the periodic feedback gain can be obtained. By optimizing the free parameter matrix in the proposed algorithm, according to robustness principle, an algorithm on the minimum norm and robust poles assignment for the LDP systems is presented. Two numerical examples are worked out to illustrate the effect of the proposed approaches.

Keywords: Linear discrete-time periodic (LDP) systems; poles assignment; robustness.

1 Introduction

Linear discrete-time periodic (LDP) systems are important bridges connecting time-varying systems and time-invariant systems. In fact, Many natural and engineering phenomena can be reduced to a composite of periodic systems thus applications of periodic systems would be found in different field, where periodic controllers could be used to dealing with the problem in which time-invariant controllers is helpless(for example, [1–3]). Moreover, another major role of the periodic controller is to improve the performance of the closed-loop system, which has also been extensively studied(one can see [4, 5] and references therein). Therefore, researches on LDP systems have attracted more and more attentions.

Since poles assignment techniques to modify the dynamic response of linear systems are the most studied problems among modern control theory, the above mentioned advantages of periodic systems and periodic controllers provide sufficient impetus for the researchers to carry out the study of poles assignment for periodic systems (see [6–9] and literatures therein). Due to the constraints of the constant controller in the periodic system, it is advocated in [6] that linear periodic output feedback is adequate to assign poles of a linear periodic discrete-time system. By utilizing a computational method on Sylvester equation, [7] proposes a complete parametric approach for pole assignment via periodic output feedback, in which parameter existed in the feedback gain could be used to accomplish some properties of plant system, robustness for instance. Using gradient search methods on the defined cost function, a computational approach is proposed in [8] to solve the minimum norm and robust pole assignment problem for linear periodic discrete-time system. Based on the proposed algorithm for parametric pole assignment problem, [9] considers the robust and minimum norm pole assignment problem and an explicit algorithm is proposed.

In this paper, the problem of poles assignment and robust poles assignment in LDP systems via state feedback is considered. Based on an iterative algorithm proposed in [13] for periodic Sylvester matrix equation, an algorithm on the problem of poles assignment in periodic linear discrete-time system with periodic state

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†1. College of Environment and Planning, Henan University, Kaifeng, 475004, P. R. China. 2. Institute of Electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: lingling_lv@163.com (Lingling Lv).

‡Institute of electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: zhe_Zhang5218@163.com (Zhe Zhang)

§Computer and Information Engineering College, Henan University, Kaifeng 475004, P. R. China. Email: zhanglei@henu.edu.cn (Lei Zhang).

¶Computer and Information Engineering College, Henan University, Kaifeng 475004, P. R. China. Email: liuxianxing@henu.edu.cn (Xianxing Liu). Corresponding author.

feedback is presented. The algorithm can realize accurate configuration of the closed-loop poles and obtain the numerical solution of the control gain. After solving the basic poles assignment problem, it is tempting to think: can we improve this algorithm to achieve the robustness of the system? The answer is positive. By optimizing the free parameter matrix in the proposed algorithm, this paper presents an algorithm on the minimum norm and robust poles assignment for the periodic linear discrete-time system. This algorithm can significantly improve the robust performance of closed-loop system. Two numerical examples are worked out to illustrate the effect of the proposed approaches.

Here, we give descriptions of some symbols which will be encountered in the rest of this paper. $\mathrm{tr}(A)$ means the trace of matrix A. Norm $\|A\|$ is a Frobenius norm of matrix A. $\Lambda(A)$ means the eigenvalue set of matrix A and Φ_{A_k} denotes the monodromy matrix $A_{K-1}A_{K-2}\cdots A_0$.

2 Main Discussions

2.1 Poles Assignment with Periodic State Feedback

Consider the completely reachable LDP systems as:

$$q_{k+1} = A_k q_k + B_k u_k, \tag{1}$$

where state matrix $A_k \in \mathbb{R}^{n\times n}$ and input matrix $B_k \in \mathbb{R}^{n\times r}$ are K-periodic. Based on the periodic feedback law in the form of

$$u_k = F_k q_k, \tag{2}$$

where F_k is the K-periodic control gain, the closed-loop system can be obtained as

$$q_{k+1} = A_{c,k} q_k, \tag{3}$$

where $A_{c,k}$ denotes $(A_k + B_k F_k)$. Then the problem of poles assignment for periodic discrete-time linear system by control law (2) can be represented as

Problem 1 *Consider the completely reachable periodic discrete-time linear system (1), seek the periodic state feedback gain $F_k \in \mathbb{R}^{m\times n}, k \in \overline{0, K-1}$, such that the poles of corresponding periodic closed-loop system (3) are set to the predetermined position $\Gamma = \{\lambda_1, \cdots, \lambda_n\}$, where Γ should be symmetrical about the real axis.*

In the following, we will first present a new poles assignment algorithm via periodic state feedback, then give strict mathematical argument to show the correctness of the proposed algorithm.

Algorithm 1 *(Poles assignment with periodic state feedback)*

1. *Choose the appropriate K-periodic matrices $\widetilde{A}_k \in \mathbb{R}^{n\times n}, k \in \overline{0, K-1}$, satisfying $\Lambda(\Phi_{\widetilde{A}_k}) = \Gamma$. Further, choose $G_k \in \mathbb{R}^{r\times n}, k \in \overline{0, K-1}$ such that periodic matrix pairs $(\widetilde{A}_k, G_k)$ are completely observable and $\Lambda(\Phi_{\widetilde{A}_k}) \cap \Lambda(\Phi_{A_k}) = 0$;*

2. *Set tolerance ε, for arbitrary initial matrix $X_k(0) \in \mathbb{R}^{n\times n}$, $k \in \overline{0, K-1}$, calculate*

$$\begin{aligned} Q_k(0) &= B_k G_k + A_k X_k(0) - X_{k+1}(0)\widetilde{A}_k; \\ R_k(0) &= -A_k^{\mathrm{T}} Q_k(0) + Q_{k-1}(0)\widetilde{A}_{k-1}^{\mathrm{T}}; \\ P_k(0) &= -R_k(0); \\ j &:= 0; \end{aligned}$$

3. *While* $\|R_k(j)\| \leq \varepsilon, k \in \overline{0, K-1}$, *calculate*

$$\begin{aligned}
\alpha(j) &= \frac{\sum_{k=0}^{K-1} \operatorname{tr}\left[P_k^{\mathrm{T}}(j) R_k(j)\right]}{\sum_{k=0}^{K-1} \left\| -A_k P_k(j) + P_{k+1}(j) \widetilde{A}_k \right\|^2}; \\
X_k(j+1) &= X_k(j) + \alpha(j) P_k(j) \in \mathbb{R}^{n \times n}; \\
Q_k(j+1) &= B_k G_k + A_k X_k(j+1) - X_{k+1}(j+1) \widetilde{A}_k \in \mathbb{R}^{n \times n}; \\
R_k(j+1) &= -A_k^{\mathrm{T}} Q_k(j+1) + Q_{k-1}(j+1) \widetilde{A}_{k-1}^{\mathrm{T}}; \\
P_k(j+1) &= -R_k(j+1) + \frac{\sum_{k=0}^{K-1} \|R_k(j+1)\|^2}{\sum_{k=0}^{K-1} \|R_k(j)\|^2} P_k(j) \in \mathbb{R}^{n \times n}; \\
j &= j+1;
\end{aligned}$$

4. *Let* $X_k^* = X_k(j)$, *calculate the periodic state feedback gain* F_k *by*

$$F_k = G_k (X_k^*)^{-1}, k \in \overline{0, K-1}.$$

To verify the validity of the above algorithm, we would provide several necessary lemmas for the problem under discussion, whose correctness can be easily checked by detail computation or derivation, and their proof is omitted due to space limitations.

Lemma 1 *For* $k \geq 0$, *the following equation holds:*

$$\sum_{k=0}^{T-1} \operatorname{tr}\left[R_k^{\mathrm{T}}(j+1) P_k(j)\right] = 0$$

for all $\{R_k(j)\}$ *and* $\{P_k(j)\}$ *derived from Algorithm 1.*

Lemma 2 *For* $k \geq 0$, *the following equation holds:*

$$\sum_{k=0}^{T-1} \operatorname{tr}\left[R_k^{\mathrm{T}}(j) P_k(j)\right] = -\sum_{j=0}^{T-1} \|R_k(j)\|^2$$

for all $\{R_k(j)\}$ *and* $\{P_k(j)\}$ *generated by Algorithm 1.*

Lemma 3 *For* $k \geq 0$, *the following relation holds:*

$$\sum_{j \geq 0} \frac{\left(\sum_{k=0}^{T-1} \|R_k(j)\|^2\right)^2}{\sum_{k=0}^{K-1} \|P_k(j)\|^2} < \infty.$$

for all $\{R_k(j)\}$ *and* $\{P_k(j)\}$ *generated by Algorithm 1.*

Based on these lemmas, we can further draw the following conclusion.

Theorem 1 *The matrices* $X_k^*, k \in \overline{0, T-1}$ *generated by Algorithm 1 satisfy periodic Sylvester matrix equation*

$$A_k X_k - X_{k+1} \widetilde{A}_k + B_k G_k = 0, k \in \overline{0, K-1}. \tag{4}$$

Proof. To explain matrices $X_k, k \in \overline{0, K-1}$ generated by Algorithm 1 are solutions to equation (10), we first illustrate that this problem is related to the convergence of matrix sequence $\{R_k(j)\}, k \in \overline{0, T-1}$ generated by Algorithm 1.

According to equation (10), construct the following index function:

$$J = \sum_{k=0}^{K-1} \frac{1}{2} \left\| B_k G_k + A_k X_k - X_{k+1} \widetilde{A}_k \right\|^2 . \tag{5}$$

It is easily obtained that for $k \in \overline{0, K-1}$,

$$\frac{\partial J}{\partial X_k} = -A_k^T \left(B_k G_k + A_k X_k - X_{k+1} \widetilde{A}_k \right) + \left(B_{k-1} G_{k-1} + A_{k-1} X_{k-1} - X_k \widetilde{A}_{k-1} \right) \widetilde{A}_{k-1}^T$$

So far, if we can find matrices $X_k^*, k \in \overline{0, K-1}$ such that

$$\left. \frac{\partial J}{\partial X_k} \right|_{X_k = X_k^*} = 0,$$

then matrices $X_k^*, k \in \overline{0, K-1}$ must be the solution to equation (10) in the meaning of least squares. From the formulation of sequence $\{R_k(j)\}, k \in \overline{0, T-1}$ in Algorithm 1, we can see

$$R_k(j) = \left. \frac{\partial J}{\partial X_k} \right|_{X_k = X_k(j)} .$$

That is to say, if matrix sequence $\{R_k(j)\}, k \in \overline{0, T-1}$ can converge to zero, matrices $X_k^*, k \in \overline{0, K-1}$ generated by Algorithm 1 must satisfy periodic matrix equation (10).

In the remaining, we only need proof that, for $k \in \overline{0, K-1}$

$$\lim_{j \to \infty} \|R_k(j)\| = 0.$$

By Lemma 1 and the expressions of $P_k(j+1)$ in Algorithm 1, we have

$$\sum_{k=0}^{K-1} \|P_k(j+1)\|^2 = \sum_{k=0}^{K-1} \left\| -R_k(j+1) + \frac{\sum_{k=0}^{K-1} \|R_k(j+1)\|^2}{\sum_{k=0}^{K-1} \|R_k(j)\|^2} P_k(j) \right\|^2$$
$$= \left(\frac{\sum_{k=0}^{K-1} \|R_k(j+1)\|^2}{\sum_{k=0}^{K-1} \|R_k(j)\|^2} \right)^2 \sum_{k=0}^{K-1} \|P_k(j)\|^2 + \sum_{k=0}^{K-1} \|R_k(j+1)\|^2 .$$

Let

$$t_j = \frac{\sum_{k=0}^{K-1} \|P_k(j)\|^2}{\left(\sum_{k=0}^{K-1} \|R_k(j)\|^2 \right)^2}.$$

Then the preceding relation can be written as

$$t_{j+1} = t_j + \frac{1}{\sum_{k=0}^{K-1} \|R_k(j+1)\|^2}. \tag{6}$$

equivalently.
We now proceed by contradiction and assume that

$$\lim_{j \to \infty} \sum_{k=0}^{K-1} \|R_k(j)\|^2 \neq 0. \tag{7}$$

This relation implies that there exists a constant $\delta > 0$ such that

$$\sum_{k=0}^{K-1} \|R_k(j)\|^2 \geq \delta$$

for all $j \geq 0$. It follows from (6) and (7) that

$$t_{j+1} \leq t_0 + \frac{j+1}{\delta}.$$

And it shows that
$$\frac{1}{t_{j+1}} \geq \frac{\delta}{\delta t_0 + j + 1}.$$
So we have
$$\sum_{j=1}^{\infty} \frac{1}{t_j} \geq \sum_{j=1}^{\infty} \frac{\delta}{\delta t_0 + j + 1} = \infty.$$
However, it follows from Lemma 3 that
$$\sum_{j=1}^{\infty} \frac{1}{t_j} < \infty.$$
This gives a contradiction.

Thus, the correctness of the theorem has been proved. ■

As for the effectiveness of Algorithm 1, we have the following conclusion:

Theorem 2 *Consider completely reachable periodic discrete-time linear system (1), the K-periodic matrix F_k generated from Algorithm 1 is a solution of the problem of poles assignment with periodic state feedback.*

Proof. Notice that the poles of LDP system (1) are the poles of the monodromy matrix Φ_{A_k}. According to algorithm 1, $\Phi_{\widetilde{A_k}}$ possesses the desired pole set Γ. To assign the poles of the closed-loop system (3) to set Γ, we just need find n-order invertible matrices $X_k, k \in \overline{0, K-1}$, such that
$$X_{k+1}^{-1} A_{ck} X_k = \widetilde{A}_k, \tag{8}$$
namely
$$X_{k+1}^{-1}(A_k + B_k F_k) X_k = \widetilde{A}_k, \tag{9}$$
Pre-multiplying the above equation by matrix X_{k+1} gives
$$A_k X_k - X_{k+1}\widetilde{A}_k + B_k F_k X_k = 0, k \in \overline{0, K-1},$$
Let
$$G_k = F_k X_k,$$
then Problem 1 is converted to the problem of solving the periodic Sylvester matrix equation in the form of
$$A_k X_k - X_{k+1}\widetilde{A}_k + B_k G_k = 0, k \in \overline{0, K-1}. \tag{10}$$
The step 2-3 in Algorithm 1 involve the solution of this matrix equation, and its correctness has been proved in [13]. By solving the solution matrix X_k, the periodic feedback gain can be obtained as
$$F_k = G_k X_k^{-1}, k \in \overline{0, K-1}. \tag{11}$$
That is, the periodic feedback gain F_k derived from (11) is a solution to Problem 1. ■

Remark 1 *For the periodic matrix $\widetilde{A}_k$, it should satisfy $\Lambda(\Phi_{\widetilde{A}}) = \Gamma$. This requirement can be achieved by letting F_0 be the real Jordan canonical form of the desired pole set and $F_k, k \in \overline{1, K-1}$ be unit matrices of corresponding dimension.*

Remark 2 *If system (1) is completely reachable and $\Lambda(\Phi_{\widetilde{A}}) \cap \Lambda(\Phi_A) = 0$, then X_k will be invertible naturally. That's why the algorithm requires condition $\Lambda(\Phi_{\widetilde{A}}) \cap \Lambda(\Phi_A) = 0$.*

2.2 Robust Consideration

In the previous subsection, the iterative algorithm can provide infinite numerical solutions for the pole assignment problem via periodic state feedback by choose different parameter matrix G_k. Therefore, by adding some additional conditions to the feedback gain matrix $F_k, k \in \overline{0, K-1}$ and transforming matrix $X_k, k \in \overline{0, K-1}$, the free parameter matrix G_k can be used to achieve the robustness of the system. In general, the small feedback gain is robust. Because small gain means small control signals, that is beneficial to reduce noise amplification. At the same time, in the sense of poles assignment, the closed-loop poles to be configured should be not as sensitive as possible to disturbances in the system matrix. Thus, the following robust and minimum norm pole assignment problem via periodic state feedback is proposed.

Problem 2 *Consider the completely reachable linear periodic discrete-time system (1), seek the K-periodic state feedback gain $F_k \in \mathbb{R}^{m\times n}$, such that*

1. *the poles of corresponding periodic closed-loop system are set to the predetermined position $\Gamma = \{\lambda_1, \cdots, \lambda_n\}$;*
2. *The periodic feedback gain is as small as possible and the closed-loop poles are not as sensitive as possible to disturbances in the system matrix.*

In order to solve Problem 2, the index function in [8] is introduced as follows:

$$J(G_k) = \gamma \frac{1}{2}\sum_{k=0}^{K-1} \kappa_{\mathrm{F}}^2(X_k) + (1-\gamma)\frac{1}{2}\sum_{k=0}^{K-1} \|F_k\|^2, \tag{12}$$

where $0 \le \gamma \le 1$ is a weighting factor. It is noted that when $\gamma = 0$, $J(G_k)$ degenerates into the index function of the minimum norm problem; when $\gamma = 1$, $J(G_k)$ becomes a purely objective function to solve the robust problem. Obviously, the weight γ leads to the combination of these two problems. [8] gives explicit analytical expressions for the index function J and its gradient. So it's easy to minimize $J(G_k)$ by using any gradient-based search method. Therefore, we can present an algorithm for the problem of periodic robust and minimum norm poles assignment.

Algorithm 2 *(Robust and minimum norm poles assignment)*

1. *Choose the appropriate K-periodic matrices $\widetilde{A}_k \in \mathbb{R}^{n\times n}$ satisfying $\Lambda(\Phi_{\widetilde{A}_k}) = \Gamma$, and initialize $G_k \in \mathbb{R}^{r\times n}$ such that periodic matrix pairs $(\widetilde{A}_k, G_k)$ are completely observable and $\Lambda(\Phi_{\widetilde{A}_k}) \cap \Lambda(\Phi_{A_k}) = 0$;*
2. *Set tolerance ε, for arbitrary initial matrix $X_k(0) \in \mathbb{R}^{n\times n}$, $k \in \overline{0, K-1}$, calculate*

$$\begin{aligned} Q_k(0) &= B_k G_k + A_k X_k(0) - X_{k+1}(0)\widetilde{A}_k; \\ R_k(0) &= -A_k^{\mathrm{T}} Q_k(0) + Q_{k-1}(0)\widetilde{A}_{k-1}^{\mathrm{T}}; \\ P_k(0) &= -R_k(0); \\ j &:= 0; \end{aligned}$$

3. *While $\|R_k(j)\| \le \varepsilon, k \in \overline{0, K-1}$, calculate*

$$\begin{aligned} \alpha(j) &= \frac{\sum_{k=0}^{K-1} \mathrm{tr}\left[P_k^{\mathrm{T}}(j) R_k(j)\right]}{\sum_{k=0}^{K-1}\left\|-A_k P_k(j) + P_{k+1}(j)\widetilde{A}_k\right\|^2}; \\ X_k(j+1) &= X_k(j) + \alpha(j) P_k(j) \in \mathbb{R}^{n\times n}; \\ Q_k(j+1) &= B_k G_k + A_k X_k(j+1) - X_{k+1}(j+1)\widetilde{A}_k \in \mathbb{R}^{n\times n}; \\ R_k(j+1) &= -A_k^{\mathrm{T}} Q_k(j+1) + Q_{k-1}(j+1)\widetilde{A}_{k-1}^{\mathrm{T}}; \\ P_k(j+1) &= -R_k(j+1) + \frac{\sum_{k=0}^{K-1}\|R_k(j+1)\|^2}{\sum_{k=0}^{K-1}\|R_k(j)\|^2} P_k(j) \in \mathbb{R}^{n\times n}; \\ j &= j+1; \end{aligned}$$

4. *Based on gradient-based search methods and the index (12), choosing the appropriate weighting factor γ, solve the optimization problem*
$$\text{Minimize } J(G_k).$$
Denote the optimal decision matrix by $G_{opt,k}$;
5. *Substituting $G_{opt,k}$ into step 2-3 gives optimization solution $X_{opt,k}(j)$;*
6. *Let $X_{opt,k} = X_{opt,k}(j)$, calculate the robust and minimum norm periodic state feedback gain $F_{opt,k}$ by*
$$F_{opt,k} = G_{opt,k}X_{opt,k}^{-1}, k \in \overline{0, K-1}.$$

3 Numerical examples

Example 1 *Consider the completely reachable system described by*

$$q(t+1) = A(t)q(t) + B(t)u(t)$$

with

$$A_0 = \begin{bmatrix} 0 & e & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & e^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & e & 0 & 0 \\ 0 & 1-e^{-1} & 0 & e^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e-1 & 0 \\ 0 & 1-e^{-1} \\ 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e-1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find 2-periodic control law $u(t) = F(t)q(t)$ such that the poles of the periodic close-loop system are assigned at $\Gamma = \{0.5 \pm 0.5i, 0.7 \pm 0.7i, -0.6\}$. Specially, let

$$G(t) = \begin{bmatrix} e & 0 & 2 & 0 & 1 \\ 0.5 & -e^{-1} & 0 & 1 & 2 \end{bmatrix}, t = 0, 1$$
$$\widetilde{A}(t) = \begin{cases} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0.7 & 0 \\ 0 & 0 & -0.7 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.6 \end{bmatrix}, t = 0 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, t = 1 \end{cases}$$

The proposed Algorithm 1 applied to the example gives the following 2-periodic feedback gain:

$$F(0) = \begin{bmatrix} 2.8249 & -0.4278 & -2.6334 & 2.3210 & 0.4035 \\ 1.1033 & 0.2796 & -0.8349 & 1.4695 & 0.2045 \end{bmatrix},$$
$$F(1) = \begin{bmatrix} -0.2648 & -1.0196 & -0.7015 & -0.2593 & -0.0573 \\ 1.0698 & -1.7859 & 1.4382 & -0.7656 & -0.2827 \end{bmatrix}.$$

What can be verified is that the poles assignment is valid.

Example 2 *This example is borrowed from [12]. The desired close-loop eigenvalues set is* $\Gamma = \{0.5, 0.6, 0.7, -0.6, -0.7\}$. *Arbitrarily assigning the parameter matrix* G_k *as*

$$G(t) = \begin{bmatrix} 0.3 & 0.5 & 2.1 & 0 & 1.1 \\ 0.6 & 1.1 & 0.7 & 1.2 & 0.2 \end{bmatrix}, t = 0, 1$$

gives a group of feedback gains as follows:

$$F_{\text{rand}}(0) = \begin{bmatrix} 1.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 36.9007 & -19.7886 & 93.1374 & 19.1142 & -9.4571 \end{bmatrix},$$

$$F_{\text{rand}}(1) = \begin{bmatrix} -0.0045 & 0.0419 & -1.3397 & -0.0351 & 0.0476 \\ -0.8356 & 0.1582 & 1.9971 & 0.4532 & -0.5408 \end{bmatrix}.$$

Applying Algorithm 2 with $\gamma = 0.5$ *gives the following robust feedback gains:*

$$F_{\text{robu}}(0) = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -0.0289 & -2.6601 & -0.0603 & 2.9199 & 0.0054 \end{bmatrix},$$

$$F_{\text{robu}}(1) = \begin{bmatrix} -0.0332 & 0.0005 & -1.2358 & -0.0004 & 0.0200 \\ 0.0042 & -0.8145 & -0.0068 & 1.0742 & 0.0029 \end{bmatrix}.$$

Let the close-loop system matrices be perturbed by $\Delta_k \in \mathbb{R}^{n\times n}, k = 0, 1$, *which are random perturbations satisfying* $\|\Delta_k\| = 1, k = 0, 1$. *Then the close-loop system with perturbations can be represented as:*

$$A_{ck} + \mu\Delta_k, k = 0, 1,$$

where $\mu > 0$ *is a factor representing the disturbance level. According to [14], the following index can be adopted to measure the robustness of the corresponding close-loop system:*

$$d_\mu(\Delta_k) = \max_{1\le i\le 5}\{|\lambda_i\{(A_{c1} + \mu\Delta_1)(A_{c0} + \mu\Delta_0)\}|\},$$

where $\lambda_i\{A\}$ *denotes the i-th eigenvalue of matrix A. 3,000 randomized trials were performed at* μ *equal to 0.002, 0.003 and 0.005, respectively. The worst and the average value of* $d_\mu(\Delta_k)$ *corresponding to* F_{robu} *and* F_{rand} *respectively are listed in Table 1. Polar plots of the trials are depicted in Fig.1, where the left hand side refers to* F_{robu} *and the right hand side refers to* F_{rand}. *As we can see, in the presence of disturbances, the robust periodic feedback gain* F_{robu} *always performs better than* F_{rand}.

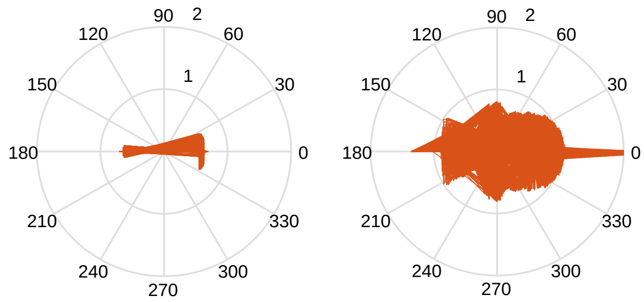
Table 1: Comparison between K_{robu} and K_{rand}

μ	μ=0.002		μ=0.003		μ=0.005	
d_μ	F_{robu}	F_{rand}	F_{robu}	F_{rand}	F_{robu}	F_{rand}
Worst	1.0237	3.3798	1.0197	4.7927	1.1561	10.9309
Mean	0.7262	1.3667	0.7244	1.5881	0.9022	2.5102

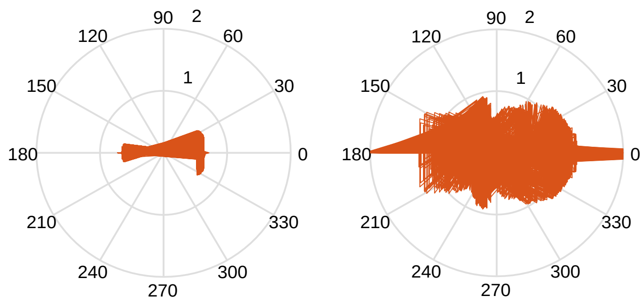
In terms of minimum norm, we compute the robust periodic feedback gains by minimize the index $J(G_k)$ *at* γ *equal to 0,0.5 and 1 respectively and the feedback norm* $\|F_0\|$, $\|F_1\|$ *together with* $\|F\| = \sqrt{\|F_0\|^2 + \|F_1\|^2}$. *The results can be see in Table 2.*

Table 2: Comparison between K_{robu} and K_{rand}

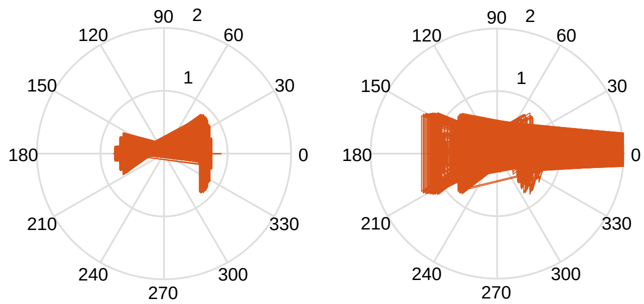
Factor	$\|F_0\|$	$\|F_1\|$	$\|F\|$
$\gamma = 0$	2.2230	2.2549	3.1665
$\gamma = 0.5$	4.0751	1.8292	4.4668
$\gamma = 1$	4.0727	1.8289	4.4645



(a) Perturbed eigenvalues of the close-loop system with $\mu = 0.002$



(b) Perturbed eigenvalues of the close-loop system with $\mu = 0.003$



(c) Perturbed eigenvalues of the close-loop system with $\mu = 0.005$

Figure 1: Perturbed eigenvalues of the close-loop system with different disturbance levels

4 Conclusions

Poles assignment with periodic state feedback and periodic robust and minimum norm poles assignment are discussed in this paper. Through mathematical derivation, the poles assignment problem is transformed into the solution to the periodic Sylvester matrix equation. Based on the recent method of solving the equation, an algorithm for solving the problem of poles assignment is presented. In this algorithm, the parameter matrix G_k can be used for further discussion on robustness. By analyzing the theory of robustness and the minimum norm, an index function of matrix G_k is adopted. Based on the gradient search algorithm, the optimization decision matrix is finally given, and the robust and minimum norm gain is thus obtained. Two examples demonstrate the effectiveness of the proposed approaches.

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C^*-ALGEBRA-VALUED MODULAR METRIC SPACES AND RELATED FIXED POINT RESULTS

BAHMAN MOEINI[1], ARSLAN HOJAT ANSARI[2], CHOONKIL PARK[3] AND DONG YUN SHIN[4]

ABSTRACT. In this paper, a concept of C^*-algebra-valued modular metric space is introduced which is a generalization of a modular metric space of Chistyakov (Folia Math. **14** (2008), 3-25). Next, some common fixed point theorems are proved for generalized contraction type mappings on such spaces. Also, to support of our results an application is provided for existence and uniqueness of solution for a system of integral equations.

1. INTRODUCTION

One of the main directions in obtaining possible generalizations of fixed point results is introduced in new types of spaces. The notion of modular spaces, as a generalization of metric spaces, was introduced by Nakano [18] and was intensively developed by Koshi and Shimogaki [12], Yamamuro [23] and others. Also, the theory of fixed points in the content of modular spaces was investigated by Khamsi *et al.* [11] and many authors generalized these results [1, 2, 9, 10, 15, 22].

In 2008, Chistyakov [3] introduced the notion of modular metric spaces generated by F-modular and developed the theory of this space. In 2010, Chistyakov [4] defined the notion of modular on an arbitrary set and developed the theory of metric spaces generated by modular which are t called the modular metric spaces. Recently, Mongkolkeha *et al.* [16, 17] have introduced some notions and established some fixed point results in modular metric spaces.

In [14], Ma *et al.* introduced the concept of C^*-algebra-valued metric spaces. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead of the set of real numbers. They showed that if $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space and $T : X \to X$ is a contractive mapping, i.e., there exists an $a \in \mathbb{A}$ with $\|a\| < 1$ such that

$$d(Tx, Ty) \preceq a^* d(x, y)a, \quad (\forall x, y \in X).$$

Then T has a unique fixed point in X. This line of research was continued in [7, 8, 13, 21, 24], where several other fixed point results were obtained in the framework of C^*-algebra valued metric, as well as (more general) C^*-algebra-valued b-metric spaces. Recently, Shateri [20] introduced the concept of C^*-algebra-valued modular space which is a generalization of a modular space and next proved some fixed point theorems for self-mappings with contractive or expansive conditions on such spaces.

In this paper, new type of modular metric space is introduced and by using some ideas of [19] some common fixed point results are proved for self-mappings with contractive

[0]Corresponding authors: baak@hanyang.ac.kr (Choonkil Park), dyshin@uos.ac.kr (Dong Yun Shin)

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conditions on such spaces. Also, some examples to elaborate and illustrate our results are constructed. Finally, as application, existence and uniqueness of solution for a type of system of nonlinear integral equations is established.

2. Basic notions

Let X be a nonempty set, $\lambda \in (0,\infty)$ and due to the disparity of the arguments, function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.1. [3] Let X be a nonempty set. A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a modular metric on X if it satisfies the following three axioms:

(i) given $x, y \in X$, $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda > 0$ and $x, y, z \in X$.

Then (X, ω) is called a modular metric space.

Recall that a Banach algebra $\mathbb{A}$ (over the field $\mathbb{C}$ of complex numbers) is said to be a C^*-algebra if there is an involution $*$ in $\mathbb{A}$ (i.e., a mapping $* : \mathbb{A} \to \mathbb{A}$ satisfying $a^{**} = a$ for each $a \in \mathbb{A}$) such that, for all $a, b \in \mathbb{A}$ and $\lambda, \mu \in \mathbb{C}$, the following holds:

(i) $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$;

(ii) $(ab)^* = b^* a^*$;

(iii) $\|a^* a\| = \|a\|^2$.

Note that, from (iii), it follows that $\|a\| = \|a^*\|$ for each $a \in \mathbb{A}$. Moreover, the pair $(\mathbb{A}, *)$ is called a unital $*$-algebra if $\mathbb{A}$ contains the unit element $1_{\mathbb{A}}$. A positive element of $\mathbb{A}$ is an element $a \in \mathbb{A}$ such that $a^* = a$ and its spectrum $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}} - a \text{ is noninvertible}\}$. The set of all positive elements will be denoted by $\mathbb{A}_+$. Such elements allow us to define a partial ordering '$\succeq$' on the elements of $\mathbb{A}$. That is,

$$b \succeq a \ \text{ if and only if } \ b - a \in \mathbb{A}_+.$$

If $a \in \mathbb{A}$ is positive, then we write $a \succeq \theta$, where θ is the zero element of $\mathbb{A}$. Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. From now on, by $\mathbb{A}$ we mean a unital C^*-algebra with unit element $1_{\mathbb{A}}$. Further, $\eth_+ = \{a \in \mathbb{A} : a \succeq \theta\}$ and $(a^*a)^{\frac{1}{2}} = |a|$.

Lemma 2.2. *[5] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit $1_{\mathbb{A}}$.*

(1) *For any $x \in \mathbb{A}_+$, we have $x \preceq 1_{\mathbb{A}} \Leftrightarrow \|x\| \le 1$.*

(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $1_{\mathbb{A}} - a$ is invertible and $\|a(1_{\mathbb{A}} - a)^{-1}\| < 1$.*

(3) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$. Then $ab \succeq \theta$.*

(4) *By $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$. If $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $1_{\mathbb{A}} - a \in \mathbb{A}'$ is an invertible operator, then*

$$(1_{\mathbb{A}} - a)^{-1} b \succeq (1_{\mathbb{A}} - a)^{-1} c.$$

Notice that in a C^*-algebra, if $\theta \preceq a, b$, one cannot conclude that $\theta \preceq ab$. For example, consider the C^*-algebra $\mathbb{M}_2(\mathbb{C})$ and set $a = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. Then $ab = \begin{pmatrix} -1 & 2 \\ -4 & 8 \end{pmatrix}$. Clearly $a, b \in \mathbb{M}_2(\mathbb{C})_+$, while ab is not.

In the following we begin to introduce and study a new type of modular metric space that is called a C^*-algebra-valued modular metric space.

Definition 2.3. Let X be a nonempty set. A function $\omega : (0,\infty) \times X \times X \to \mathbb{A}$ is said to be a C^*-algebra-valued modular metric (briefly, C^*.m.m) on X if it satisfies the following three axioms:

(i) given $x, y \in X$, $\omega_\lambda(x,y) = \theta$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \preceq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

The truple $(X, \mathbb{A}, \omega)$ is called a C^*.m.m space.

If instead of (i), we have the condition
(i') $\omega_\lambda(x,x) = \theta$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a C^*-algebra-valued pseudo modular metric (briefly, C^*.p.m.m) on X and if ω satisfies (i'), (iii) and
(i'') given $x, y \in X$, if there exists a number $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x,y) = \theta$, then $x = y$, then ω is called a C^*-algebra-valued strict modular metric (briefly, C^*.s.m.m) on X.

A C^*.m.m (or C^*.p.m.m, C^*.s.m.m) ω on X is said to be convex if, instead of (iii), we replace the following condition:

(iv) $\omega_{\lambda+\mu}(x,y) \preceq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Clearly, if ω is a C^*.s.m.m, then ω is a C^*.m.m, which in turn implies that ω is a C^*.p.m.m on X, and similar implications hold for convex ω. The essential property of a C^*.m.m ω on a set X is as follows: given $x, y \in X$, the function $0 < \lambda \to \omega_\lambda(x,y) \in \mathbb{A}$ is non increasing on $(0,\infty)$. In fact, if $0 < \mu < \lambda$, then we have

$$\omega_\lambda(x,y) \preceq \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y). \tag{2.1}$$

It follows that at each point $\lambda > 0$ the right limit $\omega_{\lambda+0}(x,y) := \lim_{\varepsilon\to+0}\omega_{\lambda+\varepsilon}(x,y)$ and the left limit $\omega_{\lambda-0}(x,y) := \lim_{\varepsilon\to+0}\omega_{\lambda-\varepsilon}(x,y)$ exist in $\mathbb{A}$ and the following two inequalities hold:

$$\omega_{\lambda+0}(x,y) \preceq \omega_\lambda(x,y) \preceq \omega_{\lambda-0}(x,y).$$

It can be checked that if $x_0 \in X$, then the set

$$X_\omega = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x,x_0) = \theta\}$$

is a C^*-algebra-valued metric space, called a C^*-algebra-valued modular space, where $d^0_\omega : X_\omega \times X_\omega \to \mathbb{A}$ is given by

$$d^0_\omega = \inf\{\lambda > 0 : \|\omega_\lambda(x,y)\| \le \lambda\} \text{ for all } x, y \in X_\omega.$$

Moreover, if ω is convex, then the set X_ω is equal to

$$X^*_\omega = \{x \in X : \exists\ \lambda = \lambda(x) > 0 \text{ such that } \|\omega_\lambda(x,x_0)\| < \infty\}$$

and $d^*_\omega : X^*_\omega \times X^*_\omega \to \mathbb{A}$ is given by

$$d^*_\omega = \inf\{\lambda > 0 : \|\omega_\lambda(x,y)\| \le 1\} \text{ for all } x, y \in X^*_\omega.$$

It is easy to see that if X is a real linear space, $\rho : X \to \mathbb{A}$ and

$$\omega_\lambda(x,y) = \rho\left(\frac{x-y}{\lambda}\right) \text{ for all } \lambda > 0 \text{ and } x, y \in X, \tag{2.2}$$

then ρ is a C^*-algebra valued modular (convex C^*-algebra-valued modular) on X if and only if ω is C^*.m.m (convex C^*.m.m, respectively) on X. On the other hand, assume that ω satisfies the following two conditions:

(i) $\omega_\lambda(\mu x, 0) = \omega_{\frac{\lambda}{\mu}}(x, 0)$ for all $\lambda, \mu > 0$ and $x \in X$;
(ii) $\omega_\lambda(x+z, y+z) = \omega_\lambda(x,y)$ for all $\lambda > 0$ and $x, y, z \in X$.

If we set $\rho(x) = \omega_1(x, 0)$ with (2.2), where $x \in X$, then $X_\rho = X_\omega$ is a linear subspace of X and the functional $\|x\|_\rho = d^0_\omega(x, 0),\ x \in X_\rho$ is an F-norm on X_ρ. If ω is convex, then $X^*_\rho \equiv X^*_\omega = X_\rho$ is a linear subspace of X and the functional $\|x\|_\rho = d^*_\omega(x, 0),\ x \in X^*_\rho$, is a norm on X^*_ρ.

Similar assertions hold if we replace C^*.m.m by C^*.p.m.m. If ω is C^*.m.m in X, then the set X_ω is a C^*.m.m space.

By the idea of property in C^*-algebra-valued metric spaces and C^*-algebra-valued modular spaces, we define the following:

Definition 2.4. Let X_ω be a C^*.m.m space.

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ with respect to $\mathbb{A}$ if
$\omega_\lambda(x_n, x) \to \theta$ as $n \to \infty$ for all $\lambda > 0$.
(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-Cauchy with respect to $\mathbb{A}$ if
$\omega_\lambda(x_m, x_n) \to \theta$ as $m, n \to \infty$ for all $\lambda > 0$.
(3) A subset C of X_ω is said to be ω-closed with respect to $\mathbb{A}$ if the limit of the ω-convergent sequence of C always belongs to C.
(4) X_ω is said to be ω-complete if any ω-Cauchy sequence with respect to $\mathbb{A}$ is ω-convergent.
(5) A subset C of X_ω is said to be ω-bounded with respect to $\mathbb{A}$ if for all $\lambda > 0$
$\delta_\omega(C) = \sup\{\|\omega_\lambda(x, y)\|;\ x, y \in C\} < \infty$.

Definition 2.5. Let X_ω be a C^*.m.m space. Let f, g be self-mappings of X_ω. A point x in X_ω is called a coincidence point of f and g if and only if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 2.6. Let X_ω be a C^*.m.m space. Two self-mappings f and g of X_ω are said to be weakly compatible if they commute at coincidence points.

Definition 2.7. Let X_ω be a C^*.m.m space. Two self-mappings f and g of X_ω are occasionally weakly compatible (owc) if and only if there is a point x in X_ω which is a coincidence point of f and g at which f and g commute.

Lemma 2.8. *[6] Let X_ω be a C^*.m.m space and f, g owc self-mappings of X_ω. If f and g have a unique point of coincidence, $w = fx = gx$, then w is a unique common fixed point of f and g.*

3. Main results

Theorem 3.1. *Let X_ω be a C^*.m.m space and $I, J, R, S, T, U : X_\omega \to X_\omega$ be self-mappings of X_ω such that the pairs (SR, I) and (TU, J) are occasionally weakly compatible. Suppose there exist $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$ such that the following assertion for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.1.1) $\omega_\lambda(SRx, TUy) \preceq a^*\omega_\lambda(Ix, Jy)a + b^*\omega_\lambda(SRx, Jy)b + c^*\omega_{2\lambda}(TUy, Ix)c;$
(3.1.2) $\|\omega_\lambda(SRx, TUy)\| < \infty.$

Then SR, TU, I and J have a common fixed point in X_ω. Furthermore, if the pairs $(S, R), (S, I), (R, I), (T, J), (T, U), (U, J)$ are commuting pairs of mappings, then I, J, R, S, T and U have a unique common fixed point in X_ω.

Proof. Since the pair (SR, I) and (TU, J) are occasionally weakly compatible, there exist $u, v \in X_\omega : SRu = Iu$ and $TUv = Jv$. Moreover, $SR(Iu) = I(SRu)$ and

$TU(Jv) = J(TUv)$. Now we can assert that $SRu = TUv$. If not then by (3.1.1)

$$\begin{aligned} \omega_\lambda(SRu, TUv) &\preceq a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(SRu, Jv)b + c^*\omega_{2\lambda}(TUv, Iu)c \\ &= a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*\omega_{2\lambda}(Jv, Iu)c \\ &= a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*\omega_{2\lambda}(Iu, Jv)c. \end{aligned} \tag{3.1}$$

By definition of C^*.m.m space and (2.1) and (3.1), we have

$$\begin{aligned} \omega_\lambda(SRu, TUv) &\preceq a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*(\omega_\lambda(Iu, Iu) + \omega_\lambda(Iu, Jv))c \\ &= a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*\omega_\lambda(Iu, Jv)c \\ &= a^*(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}a + b^*(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}b \\ &+c^*(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}c \\ &= (a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}})^*(a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}) \\ &+(b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}})^*(b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}) \\ &+(c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}})^*(c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}) \\ &= |a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}|^2 + |b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}|^2 + |c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}|^2 \\ &\preceq \|a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}\|^2 1_{\mathbb{A}} + \|b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}\|^2 1_{\mathbb{A}} + \|c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}\|^2 1_{\mathbb{A}}. \end{aligned}$$

Thus

$$\begin{aligned} \|\omega_\lambda(SRu, TUv)\| &\leq \|\omega_\lambda(Iu, Jv)\|(\|a\|^2 + \|b\|^2 + \|c\|^2) \\ &< \|\omega_\lambda(Iu, Jv)\|. \end{aligned}$$

So $\|\omega_\lambda(Iu, Jv)\| < \|\omega_\lambda(Iu, Jv)\|$, which is a contradiction. Hence $SRu = TUv$ and thus

$$SRu = Iu = TUv = Jv.$$

Moreover, assume that there is another point z such that $SRz = Iz$. Using (3.1.1),

$$\begin{aligned} \omega_\lambda(SRz, TUv) &\preceq a^*\omega_\lambda(Iz, Jv)a + b^*\omega_\lambda(SRz, Jv)b + c^*\omega_{2\lambda}(TUv, Iz)c \\ &= a^*\omega_\lambda(SRz, TUv)a + b^*\omega_\lambda(SRz, TUv)b + c^*\omega_{2\lambda}(SRz, TUv)c. \end{aligned} \tag{3.2}$$

By a similar way, $\|\omega_\lambda(SRz, TUv)\| < \|\omega_\lambda(SRz, TUv)\|(\|a\|^2 + \|b\|^2 + \|c\|^2)$, which is a contradiction. Hence we get

$$SRu = Iu = TUv = Jv. \tag{3.3}$$

Thus from (3.2) and (3.3), it follows that $SRu = SRz$. Hence $w = SRu = Iu$, for some $w \in X_\omega$, is the unique point of coincidence of SR and I. Then by Lemma 2.8, w is a unique common fixed point of SR and I. So $SRw = Iw = w$.

Similarly, there is another common fixed point $w' \in X_\omega : TUw' = Jw' = w'$.

For the uniqueness, suppose $w \neq w'$. Then by (3.1.1), we have

$$\begin{aligned} \omega_\lambda(SRw, TUw') &= \omega_\lambda(w, w') \\ &\preceq a^*\omega_\lambda(Iw, Jw'^*\omega_\lambda(SRw, Jw'^*\omega_{2\lambda}(TUw, Iw')c \\ &= a^*\omega_\lambda(w, w'^*\omega_\lambda(w, w'^*\omega_{2\lambda}(w, w')c. \end{aligned}$$

Thus $\|\omega_\lambda(w, w')\| < \|\omega_\lambda(w, w'^2 + \|b\|^2 + \|c\|^2)$, which is a contradiction. Hence $w = w'$. So w is a unique common fixed point of SR, TU, I and J.

Furthermore, if (S, R), (S, I), (R, I), (T, J), (T, U), (U, J) are commuting pairs, then

$$\begin{aligned} Sw &= S(SRw) = S(RS)w = SR(Sw) \\ Sw &= S(Iw) = S(RS)w = I(Sw) \\ Rw &= R(SRw) = RS(Rw) = SR(Rw) \\ Rw &= R(Iw) = (Rw), \end{aligned}$$

which show that Sw and Rw is a common fixed point of (SR, I), which gives $SRw = Sw = Rw = Iw = w$.

Similarly, we have $TUw = Tw = Uw = Jw = w$. Hence w is a unique common fixed point of $S,\ R,\ I,\ J,\ T,\ U$. □

Corollary 3.2. *Let X_ω be a C^*.m.m space and $I, J, S, T : X_\omega \to X_\omega$ be self-mappings of X_ω such that the pairs (S, I) and (T, J) are occasionally weakly compatible. Suppose there exist $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$ such that the following assertions for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.2.1) $\omega_\lambda(Sx, Ty) \preceq a^*\omega_\lambda(Ix, Jy)a + b^*\omega_\lambda(Sx, Jy)b + c^*\omega_{2\lambda}(Ty, Ix)c$;

(3.2.2) $\|\omega_\lambda(Sx, Ty)\| < \infty$.

Then S, T, I and J have a unique common fixed point in X_ω.

Proof. If we put $R = U := I_{X_\omega}$ where I_{X_ω} is an identity mapping on X_ω, then the result follows from Theorem 3.1. □

Corollary 3.3. *Let X_ω be a C^*.m.m space and $S, T : X_\omega \to X_\omega$ be self-mappings of X_ω such that S and T are occasionally weakly compatible. Suppose there exist $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$ such that the following assertions for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.3.1) $\omega_\lambda(Tx, Ty) \preceq a^*\omega_\lambda(Sx, Sy)a + b^*\omega_\lambda(Tx, Sy)b + c^*\omega_{2\lambda}(Ty, Sx)c$;

(3.3.2) $\|\omega_\lambda(Tx, Ty)\| < \infty$.

Then S and T have a unique common fixed point in X_ω.

Proof. If we put $I = J := S$ and $S := T$ in (3.2.1) and (3.2.2), then the result follows from Theorem 3.1. □

Corollary 3.4. *Let X_ω be a C^*.m.m space and $S, T : X_\omega \to X_\omega$ be self-mappings of X_ω such that S and T are occasionally weakly compatible. Suppose there exists $a \in \mathbb{A}$ with $0 < \|a\| < 1$ such that the following assertions for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.4.1) $\omega_\lambda(Tx, Ty) \preceq a^*\omega_\lambda(Sx, Sy)a$;

(3.4.2) $\|\omega_\lambda(Tx, Ty)\| < \infty$.

Then S and T have a unique common fixed point in X_ω.

Proof. If we put $b = c := 0_{\mathbb{A}}$ in (3.3.1), then the result follows from Corollary 3.3. □

4. Examples

In this section we provide some nontrivial examples in favour of our results.

Example 4.1. Let $X = \mathbb{R}$ and consider $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual operation of addition, scalar multiplication and matrix multiplication. Define a norm on $\mathbb{A}$ by $\|A\| = \left(\sum_{i,j=1}^{2} |a_{ij}|^2\right)^{\frac{1}{2}}$ and $* : \mathbb{A} \to \mathbb{A}$, given by $A^* = A$ for all $A \in \mathbb{A}$, defines an involution on $\mathbb{A}$. Thus $\mathbb{A}$ becomes a C^*-algebra. For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{A} = M_2(\mathbb{R}),$$

we denote $A \preceq B$ if and only if $(a_{ij} - b_{ij}) \leq 0$ for all $i, j = 1, 2$.
Define $\omega : (0, \infty) \times X \times X \to \mathbb{A}$ by

$$\omega_\lambda(x, y) = \begin{pmatrix} |\frac{x-y}{\lambda}| & 0 \\ 0 & |\frac{x-y}{\lambda}| \end{pmatrix}$$

for all $x, y \in X$ and $\lambda > 0$. It is easy to check that ω satisfies all the conditions of Definition 2.3. So $(X, \mathbb{A}, \omega)$ is a C^*.m.m space.

Example 4.2. Let $X = \{\frac{1}{c^n} : n = 1, 2, \cdots\}$ where $0 < c < 1$ and $\mathbb{A} = M_2(\mathbb{R})$. Define $\omega : (0, \infty) \times X \times X \to \mathbb{A}$ by

$$\omega_\lambda(x, y) = \begin{pmatrix} \|\frac{x-y}{\lambda}\| & 0 \\ 0 & \alpha\|\frac{x-y}{\lambda}\| \end{pmatrix}$$

for all $x, y \in X$, $\alpha \geq 0$ and $\lambda > 0$. Then it is easy to check that ω is a C^*.m.m. space.

Example 4.3. Let $X = L^\infty(E)$ and $H = L^2(E)$, where E is a Lebesgue measurable set. By $B(H)$ we denote the set of bounded linear operators on the Hilbert space H. Clearly, $B(H)$ is a C^*-algebra with the usual operator norm.
Define $\omega : (0, \infty) \times X \times X \to B(H)$ by

$$\omega_\lambda(f, g) = \pi_{|\frac{f-g}{\lambda}|}, \quad (\forall f, g \in X).$$

Here $\pi_h : H \to H$ is the multiplication operator defined by

$$\pi_h(\phi) = h \cdot \phi$$

for $\phi \in H$. Then ω is a C^*.m.m and $(X_\omega, B(H), \omega)$ is an ω-complete C^*.m.m space. It suffices to verify the completeness of X_ω. For this, let $\{f_n\}$ be an ω-Cauchy sequence with respect to $B(H)$, that is, for an arbitrary $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|\omega_\lambda(f_m, f_n)\| = \|\pi_{|\frac{f_m - f_n}{\lambda}|}\| = \|\frac{f_m - f_n}{\lambda}\|_\infty \leq \varepsilon.$$

So $\{f_n\}$ is a Cauchy sequence in Banach space X. Hence there are a function $f \in X$ and $N_1 \in \mathbb{N}$ such that

$$\|\frac{f_n - f}{\lambda}\|_\infty \leq \varepsilon \quad (n \geq N_1),$$

which implies that

$$\|\omega_\lambda(f_n, f)\| = \|\pi_{|\frac{f_n - f}{\lambda}|}\| = \|\frac{f_n - f}{\lambda}\|_\infty \leq \varepsilon, \quad (n \geq N_1).$$

Consequently, the sequence $\{f_n\}$ is an ω-convergent sequence in X_ω and so X_ω is an ω-complete C^*.m.m space.

Example 4.4. Let $(X, \mathbb{A}, \omega)$ be C^*.m.m space defined as in Example 4.1. Define $S, T, I, J : X_\omega \to X_\omega$ by

$$Sx = Tx = 1, \quad Jx = 2 - x, \quad Ix = \begin{cases} \frac{x}{2} & if\ x \in (-\infty, 1), \\ 1 & if\ x = 1, \\ 0 & if\ x \in (1, \infty) \end{cases}$$

for all $x, y \in X_\omega = \mathbb{R}$ and $\lambda > 0$. Then we have

$$0 = \left\| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|\omega_\lambda(Sx, Ty)\| < \infty.$$

For all $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$, we get

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \omega_\lambda(Sx, Ty) \preceq a^*\omega_\lambda(Ix, Jy)a + b^*\omega_\lambda(Sx, Jy)b + c^*\omega_{2\lambda}(Ty, Ix)c$ for all $x, y \in X_\omega$ and $\lambda > 0$. Also clearly, the pairs (S, I) and (T, J) are occasionally weakly compatible. So all the conditions of Corollary 3.2 are satisfied and $x = 1$ is a unique common fixed point of S, T, I and J.

5. Application

Remind that if for $\lambda > 0$ and $x, y \in L^\infty(E)$, we define $\omega : (0,\infty)\times L^\infty(E)\times L^\infty(E) \to B(H)$ by

$$\omega_\lambda(x,y) = \pi_{|\frac{x-y}{\lambda}|},$$

where $\pi_h : H \to H$ is defined as in Example 4.3, then $(L^\infty(E)_\omega, B(H), \omega)$ is an ω-complete C^*.m.m space.

Let E be a Lebesgue measurable set, $X = L^\infty(E)$ and $H = L^2(E)$ be the Hilbert space. Consider the following system of nonlinear integral equations:

$$x(t) = w(t) + k_i(t, x(t)) + \mu \int_E n(t,s)h_j(s, x(s))ds \tag{5.1}$$

for all $t \in E$, where $w \in L^\infty(E)_\omega$ is known, $k_i(t, x(t)),\ n(t,s),\ h_j(s, x(s)),\ i, j = 1, 2$ and $i \neq j$ are real or complex valued functions that are measurable both in t and s on E and μ is a real or complex number, and assume the following conditions:

(a) $sup_{s\in E} \int_E |n(t,s)|dt = M_1 < +\infty$,

(b) $k_i(s, x(s)) \in L^\infty(E)_\omega$ for all $x \in L^\infty(E)_\omega$, and there exists $L_1 > 1$ such that for all $s \in E$,

$$|k_1(s, x(s)) - k_2(s, y(s))| \geq L_1|x(s) - y(s)| \text{ for all } x, y \in L^\infty(E)_\omega,$$

(c) $h_i(s, x(s)) \in L^\infty(E)_\omega$ for all $x \in L^\infty(E)_\omega$, and there exists $L_2 > 0$ such that for all $s \in E$,

$$|h_1(s, x(s)) - h_2(s, y(s))| \leq L_2|x(s) - y(s)| \text{ for all } x, y \in L^\infty(E)_\omega,$$

(d) there exists $x(t) \in L^\infty(E)_\omega$ such that

$$x(t) - w(t) - \mu \int_E n(t,s)h_1(s, x(s))ds = k_1(t, x(t)),$$

which implies

$$\begin{aligned} &k_1(t, x(t)) - w(t) - \mu \textstyle\int_E n(t,s)h_1(s, k_1(s, x(s)))ds \\ &= k_1(t, x(t) - w(t) - \mu \textstyle\int_E n(t,s)h_1(s, x(s))ds), \end{aligned}$$

(e) there exists $y(t) \in L^\infty(E)_\omega$ such that

$$y(t) - w(t) - \mu \int_E n(t,s)h_2(s, y(s))ds = k_2(t, y(t)),$$

which implies

$$\begin{aligned} &k_2(t, y(t)) - w(t) - \mu \textstyle\int_E n(t,s)h_i(s, k_2(s, y(s)))ds \\ &= k_2(t, y(t) - w(t) - \mu \textstyle\int_E n(t,s)h_2(s, y(s))ds). \end{aligned}$$

Theorem 5.1. *With the assumptions (a)-(e), the system of nonlinear integral equations (5.1) has a unique solution x^* in $L^\infty(E)_\omega$ for each real or complex number μ with $\frac{1+|\mu|L_2M_1}{L_1} < 1$.*

Proof. Define

$$Sx(t) = x(t) - w(t) - \mu \int_E n(t,s)h_1(s, x(s))ds,$$

$$Tx(t) = x(t) - w(t) - \mu \int_E n(t,s)h_2(s, x(s))ds,$$

$$Ix(t) = k_1(t, x(t)),\ \ Jx(t) = k_2(t, x(t)).$$

Set $a = \sqrt{\frac{1+|\mu|M_1L_2}{L_1}} \cdot 1_{B(H)}$, $b = c = 0_{B(H)}$. Then $a \in B(H)_+$ and $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 = \frac{1+|\mu|M_1L_2}{L_1} < 1$.

For any $h \in H$, we have

$$\|\omega_\lambda(Sx, Ty)\| = \sup_{\|h\|=1}(\pi_{|\frac{Sx-Ty}{\lambda}|}h, h)$$

$$= \sup_{\|h\|=1} \int_E \left[\frac{1}{\lambda}\Big|(x-y) + \mu \int_E n(t,s)(h_2(s, y(s) - h_1(s, x(s))ds\Big|\right] h(t)\overline{h(t)}dt$$

$$\leq \sup_{\|h\|=1} \int_E \left[\frac{1}{\lambda}\Big|(x-y) + \mu \int_E n(t,s)(h_2(s, y(s) - h_1(s, x(s))ds\Big|\right] |h(t)|^2 dt$$

$$\leq \frac{1}{\lambda} \sup_{\|h\|=1} \int_E |h(t)|^2 dt \Big[\|x-y\|_\infty + |\mu| M_1 L_2 \|x-y\|_\infty\Big]$$

$$\leq (\frac{1+|\mu|M_1L_2}{\lambda})\|x-y\|_\infty$$

$$\leq (\frac{1+|\mu|M_1L_2}{L_1})\|\frac{k_1(t,x(t)) - k_2(t,y(t))}{\lambda}\|_\infty$$

$$= (\frac{1+|\mu|M_1L_2}{L_1})\|\omega_\lambda(Ix, Jy)\|$$

$$= \|a\|^2 \|\omega_\lambda(Ix, Jy)\|.$$

Then

$$\|\omega_\lambda(Sx, Ty)\| \leq \|a\|^2\|\omega_\lambda(Ix, Jy)\| + \|b\|^2\|\omega_\lambda(Sx, Jy)\| + \|c\|^2\|\omega_{2\lambda}(Ty, Ix)\|$$

for all $x, y \in L^\infty(E)_\omega$ and $\lambda > 0$. Also by conditions (d) and (e) the pairs (S, I) and (T, J) are occasionally weakly compatible. Therefore, by Corollary 3.2, there exists a unique common fixed point $x^* \in L^\infty(E)_\omega$ such that $x^* = Sx^* = Tx^* = Ix^* = Jx^*$, which proves the existence of unique solution of (5.1) in $L^\infty(E)_\omega$. This completes the proof. □

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[1]Department of Mathematics, Hidaj Branch, Islamic Azad University, Hidaj, Iran
E-mail address: moeini145523@gmail.com

[2]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
E-mail address: analsisamirmath2@gmail.com

[3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

[4]Department of Mathematics, University of Seoul, Seoul 02504, Korea
E-mail address: dyshin@uos.ac.kr

Strong Convergence Theorems and Applications of a New Viscosity Rule for Nonexpansive Mappings

Waqas Nazeer[1], Mobeen Munir[1], Sayed Fakhar Abbas Naqvi[2], Chahn Yong Jung[3,*] and Shin Min Kang[4,5,*]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: nazeer.waqas@ue.edu.pk (W.N); mmunir@ue.edu.pk (M.M)

[2]Department of Mathematics, Lahore Leads University, Lahore 54810, Pakistan
e-mail: fabbas27@gmail.com

[3]Department of Business Administration, Gyeongsang National University, Jinju 52828, Korea
e-mail: bb5734@gnu.ac.kr

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan

[5]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

Abstract

We introduced new viscosity rule for nonexpansive mappings in Hilbert Spaces. The strong convergence theorem of the new rule is proved under certain assumptions imposed on the sequence of parameters. Moreover, applications of proposed viscosity rule are also given.

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1 Introduction

In this paper, we shall take H as a real Hilbert space, $\langle \cdot, \cdot \rangle$ as inner product, $\| \cdot \|$ as the induced norm, and C as a nonempty closed subset of H.

Definition 1.1. Let $T : H \to H$ be a mapping. T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Definition 1.2. A mapping $f : H \to H$ is called a *contraction* if for all $x, y \in H$ and $\theta \in [0, 1)$

$$\|fx - fy\| \leq \theta \|x - y\|.$$

* Corresponding authors

Definition 1.3. $P_c : H \to C$ is called a *metric projection* if for every $x \in H$ there exists a unique nearest point in C, denoted by $P_c x$, such that

$$\|x - P_c x\| \leq \|x - y\|, \quad \forall y \in C.$$

The following theorem gives the condition for a projection mapping to be nonexpansive.

Theorem 1.4. *Let C be a nonempty closed convex subset of the real Hilbert space H and $P_c : H \to H$ a metric projection. Then*
(a) $\|P_c x - P_c y\|^2 \leq \langle x - y, P_c x - P_c y\rangle$ *for all* $x, y \in H$.
(b) P_c *is a nonexpansive mapping, that is,* $\|x - P_c x\| \leq \|x - y\|$ *for all* $y \in C$.
(c) $\langle x - P_c x, y - P_c x\rangle \leq 0$ *for all* $x \in H$ *and* $y \in C$.

In order to verify the weak convergence of an algorithm to a fixed point of a nonexpansive mapping we need the demiclosedness principle:

Theorem 1.5. (The demiclosedness principle) ([2]) *Let C be a nonempty closed convex subset of the real Hilbert space H and $T : C \to C$ such that $x_n \rightharpoonup x^* \in C$ and $(I - T)x_n \to 0$. Then $x^* = Tx^*$. (Here $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively).*

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

Theorem 1.6. ([9]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*
(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.
Then $a_n \to 0$ as $n \to \infty$.

The following strong convergence theorem, which is also called the *viscosity approximation method*, for nonexpansive mappings in real Hilbert spaces is given by Moudafi [8] in 2000.

Theorem 1.7. *Let C be a noneempty closed convex subset of the real Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $F(T) := \{x \in H : T(x) = x\}$ is nonempty. Let f be a contraction of C into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1 + \epsilon_n} f(x_n) + \frac{1}{1 + \epsilon_n} T(x_n), \quad n \geq 0,$$

where the sequence $\{\epsilon_n\} \in (0, 1)$ satisfies
(1) $\lim_{n\to\infty} \epsilon_n = 0$,
(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$,
(3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.
Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall \in F(T).$$

In 2015, Xu et al. [9] applied the viscosity method on the midpoint rule for nonexpansive mappings and give the generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

This, using contraction, regularizes the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of T. Ke and Ma [6], motivated and inspired by the idea of Xu et al. [9], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(s_n x_n + (1-s_n)x_{n+1}\right)$$

and

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}).$$

In [3], Jung et al. presented the following viscosity rule

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n(w_n) + \beta_n f(w_n) + \gamma_n T(w_n), \\ w_n = \frac{x_n + x_{n+1}}{2}. \end{cases}$$

In [7], Kwun et al. proved the strong convergence of the following viscosity rule

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n(x_n) + \beta_n f(x_n) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right). \end{cases}$$

Our contribution in this direction is the following new viscosity rule

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right). \tag{1.1}$$

2 New viscosity rule

Theorem 2.1. *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by the new viscosity rule (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ *and* $\lim_{n\to\infty} \gamma_n \to 1$,

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iv) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality $\langle (I - f)x, y - x\rangle \geq 0$, $\forall y \in F(T)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.

Proof. This proof is divided into five steps.

STEP 1. ($\{x_n\}$ is bounded)

Taking an arbitrary point p of $F(T)$, we have

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&= \left\|\alpha_n\left(\frac{x_n+x_{n+1}}{2}\right)+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)+\gamma_n T\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\| \\
&= \left\|\alpha_n\left(\frac{x_n+x_{n+1}}{2}\right)-\alpha_n p+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)-\beta_n p\right. \\
&\quad \left.+\gamma_n T\left(\frac{x_n+x_{n+1}}{2}\right)+(\alpha_n+\beta_n-1)p\right\| \\
&\le \alpha_n\left\|\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\|+\beta_n\left\|f\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\|+\gamma_n\left\|T\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\| \\
&\le \frac{\alpha_n}{2}\|x_n-p\|+\frac{\alpha_n}{2}\|x_{n+1}-p\|+\beta_n\left\|f\left(\frac{x_n+x_{n+1}}{2}\right)-f(p)\right\| \\
&\quad +\beta_n\|f(p)-p\|+\gamma_n\left\|\frac{x_n+x_{n+1}}{2}-p\right\| \\
&\le \frac{\alpha_n}{2}\|x_n-p\|+\frac{\alpha_n}{2}\|x_{n+1}-p\|+\theta\beta\left\|\frac{x_n+x_{n+1}}{2}-p\right\|+\beta\|f(p)-p\| \\
&\quad +\gamma_n\left[\frac{1}{2}\|x_n-p\|+\frac{1}{2}\|x_{n+1}-p\|\right] \\
&= \left(\frac{\alpha_n+\gamma_n+\theta\beta_n}{2}\right)\|x_n-p\|+\left(\frac{\alpha_n+\gamma_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \\
&\quad +\frac{\gamma_n}{2}\|x_{n+1}-p\|+\beta_n\|f(p)-p\| \\
&= \left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_n-p\|+\left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \\
&\quad +\frac{\gamma_n}{2}\|x_{n+1}-p\|+\beta_n\|f(p)-p\|.
\end{aligned}$$

It follows that

$$\left(1-\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \le \left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_n-p\|+\beta_n\|f(p)-p\|$$

implies

$$(1+\beta_n(1-\theta))\|x_{n+1}-p\| \le (1-\beta_n(1-\theta))\|x_n-p\|+2\beta_n\|f(p)-p\|. \tag{2.1}$$

Since $\beta_n,\theta\in(0,1)$, $1-\beta_n(1-\theta)\ge 0$. Moreover, by (2.1) and $\alpha_n+\beta_n+\gamma_n=1$ we get

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\le \frac{1-\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\|x_n-p\|+\frac{2\beta_n}{1+\beta_n(1-\theta)}\|f(p)-p\| \\
&\le \left[1-\frac{2\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\right]\|x_n-p\|+\frac{2\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\left(\frac{1}{1-\theta}\|f(p)-p\|\right).
\end{aligned}$$

Thus, we have

$$\|x_{n+1} - p\| \leq \max\left\{\|x_n - p\|, \frac{1}{1-\theta}\|f(p) - p\|\right\}.$$

By induction we obtain

$$\|x_{n+1} - p\| \leq \max\left\{\|x_0 - p\|, \frac{1}{1-\theta}\|f(p) - p\|\right\}.$$

Hence, we concluded that $\{x_n\}$ is bounded. Consequently, $\{f(\frac{x_n+x_{n+1}}{2})\}$ and $\{T(\frac{x_n+x_{n+1}}{2})\}$ are bounded.

STEP 2. ($\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$)

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \Bigg\|\alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right) \\
&\quad - \left[\alpha_{n-1}\left(\frac{x_n + x_{n-1}}{2}\right) + \beta_{n-1} f\left(\frac{x_n + x_{n-1}}{2}\right) + \gamma_{n-1} T\left(\frac{x_{n-1} + x_n}{2}\right)\right]\Bigg\| \\
&= \Bigg\|\frac{\alpha_n}{2}(x_{n+1} - x_n) + \frac{\alpha_n}{2}(x_n - x_{n-1}) + \frac{1}{2}(\alpha_n - \alpha_{n-1})x_n + \frac{1}{2}(\alpha_n - \alpha_{n-1})x_{n-1} \\
&\quad + \beta_n\left(f\left(\frac{x_n + x_{n+1}}{2}\right) - f\left(\frac{x_n + x_{n-1}}{2}\right)\right) + (\beta_n - \beta_{n-1}) f\left(\frac{x_n + x_{n-1}}{2}\right) \\
&\quad + \gamma_n\left[T\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right)\right] + (\gamma_n - \gamma_{n-1}) T\left(\frac{x_{n-1} + x_n}{2}\right)\Bigg\| \\
&= \Bigg\|\frac{\alpha_n}{2}(x_{n+1} - x_n) + \frac{\alpha_n}{2}(x_n - x_{n-1}) + \frac{1}{2}(\alpha_n - \alpha_{n-1})(x_n + x_{n-1}) \\
&\quad + \beta_n\left(f\left(\frac{x_n + x_{n+1}}{2}\right) - f\left(\frac{x_n + x_{n-1}}{2}\right)\right) + (\beta_n - \beta_{n-1}) f\left(\frac{x_n + x_{n-1}}{2}\right) \\
&\quad + \gamma_n\left[T\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right)\right] \\
&\quad - \left[(\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})\right] T\left(\frac{x_{n-1} + x_n}{2}\right)\Bigg\| \\
&\leq \frac{\alpha_n}{2}\|x_{n+1} - x_n\| + \frac{\alpha_n}{2}\|x_n - x_{n-1}\| \\
&\quad + \frac{1}{2}|\alpha_n - \alpha_{n-1}|\left\|x_{n-1} + x_n - 2T\left(\frac{x_{n-1} + x_n}{2}\right)\right\| \\
&\quad + \beta_n\left\|f\left(\frac{x_n + x_{n+1}}{2}\right) - f\left(\frac{x_n + x_{n-1}}{2}\right)\right\| \\
&\quad + |\beta_n - \beta_{n-1}|\left\|f\left(\frac{x_n + x_{n-1}}{2}\right) - T\left(\frac{x_n + x_{n-1}}{2}\right)\right\| \\
&\quad + \gamma_n\left\|T\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right)\right\| \\
&\leq \frac{\alpha_n}{2}\|x_{n+1} - x_n\| + \frac{\alpha_n}{2}\|x_n - x_{n-1}\| + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right)M \\
&\quad + \theta\beta_n\left\|\frac{x_{n+1} + x_n}{2} - \frac{x_n - x_{n-1}}{2}\right\| + \gamma_n\left\|\frac{x_{n+1} + x_n}{2} - \frac{x_n - x_{n-1}}{2}\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_n}{2}\|x_{n+1} - x_n\| + \frac{\alpha_n}{2}\|x_n - x_{n-1}\| + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right)M \\
&\quad + \frac{\theta\beta_n}{2}\|x_{n+1} - x_n\| + \frac{\theta\beta_n}{2}\|x_n - x_{n-1}\| + \frac{\gamma_n}{2}\|x_{n+1} - x_n\| + \frac{\gamma_n}{2}\|x_n - x_{n-}\| \\
&= \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\|x_{n+1} - x_n\| + \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\|x_n - x_{n-1}\| \\
&\quad + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right)M,
\end{aligned}
$$

where $M > 0$ is a constant such that

$$
\begin{aligned}
M \geq \max\Bigg\{ &\sup_{n\geq 0}\left\|x_n + x_{n-1} - 2T\left(\frac{x_{n-1} + x_n}{2}\right)\right\|, \\
&\sup_{n\geq 0}\left\|f\left(\frac{x_n + x_{n-1}}{2}\right) - T\left(\frac{x_n + x_{n-1}}{2}\right)\right\|\Bigg\}.
\end{aligned}
$$

It gives

$$
\begin{aligned}
&\left(1 - \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\right)\|x_{n+1} - x_n\| \\
&\leq \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\|x_n - x_{n-1}\| + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right)M
\end{aligned}
$$

implies

$$
\begin{aligned}
&\left(1 - \frac{1 - \beta_n + \theta\beta_n}{2}\right)\|x_{n+1} - x_n\| \\
&\leq \frac{1 - \beta_n + \theta\beta_n}{2}\|x_n - x_{n-1}\| + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right)M
\end{aligned}
$$

implies

$$
\begin{aligned}
(1 + \beta_n(1-\theta))\|x_{n+1} - x_n\| \leq{}& (1 - \beta_n(1-\theta))\|x_n - x_{n-1}\| \\
&+ (|\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|)M.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq{}& \left(\frac{1 - \beta_n(1-\theta)}{1 + \beta_n(1-\theta)}\right)\|x_n - x_{n-1}\| \\
&+ \frac{M}{1 + \beta_n(1-\theta)}(|\alpha_n - \alpha_{n-1}| - 2|\beta_n - \beta_{n-1}|).
\end{aligned}
$$

Since $\theta, \beta_n \in (0,1)$, $1 + \beta_n(1-\theta) \geq 1$ and hence

$$
\frac{1 - \beta_n(1-\theta)}{1 + \beta_n(1-\theta)} \leq 1 - \beta_n(1-\theta).
$$

Thus

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq{}& \Big[1 - \beta_n(1-\theta)\Big]\|x_n - x_{n-1}\| \\
&+ \frac{M}{1 + \beta_n(1-\theta)}(|\alpha_n - \alpha_{n-1}| - 2|\beta_n - \beta_{n-1}|).
\end{aligned}
$$

Since
$$\sum_{n=0}^{\infty}\beta_n=\infty,\quad \sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,\quad \text{and}\quad \sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty,$$
by Theorem 1.6, we have $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

Step 3. ($\|x_n-Tx_n\|\to 0$ as $\to\infty$)

Consider

$$\begin{aligned}
&\|x_n-Tx_n\|\\
&=\left\|x_n-x_{n+1}+x_{n+1}-T\left(\frac{x_n+x_{n+1}}{2}\right)+T\left(\frac{x_n+x_{n+1}}{2}\right)-Tx_n\right\|\\
&\le \|x_n-x_{n+1}\|+\left\|x_{n+1}-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\left\|T\left(\frac{x_n+x_{n+1}}{2}\right)-Tx_n\right\|\\
&\le \|x_n-x_{n+1}\|+\left\|\alpha_n\left(\frac{x_n+x_{n+1}}{2}\right)+\beta_nf\left(\frac{x_n+x_{n+1}}{2}\right)\right.\\
&\quad\left.+\gamma_nT\left(\frac{x_n+x_{n+1}}{2}\right)-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\left\|\frac{x_n+x_{n+1}}{2}-x_n\right\|\\
&=\|x_n-x_{n+1}\|+\left\|\frac{\alpha_n}{2}(x_n+x_{n+1})+\beta_nf\left(\frac{x_n+x_{n+1}}{2}\right)\right.\\
&\quad\left.-(1-\gamma_n)T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\frac{1}{2}\|x_{n+1}-x_n\|\\
&\le \frac{3}{2}\|x_n-x_{n+1}\|+\left\|\frac{\alpha_n}{2}(x_n+x_{n+1})+\beta_nf\left(\frac{x_n+x_{n+1}}{2}\right)\right.\\
&\quad\left.-(\alpha_n+\beta_n)T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\
&\le \frac{3}{2}\|x_n-x_{n+1}\|+\frac{\alpha_n}{2}\left\|x_n+x_{n+1}-2T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\
&\quad+\beta_n\left\|f\left(\frac{x_n+x_{n+1}}{2}\right)-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\
&\le \frac{3}{2}\|x_{n+1}-x_n\|+\left(\frac{\alpha_n}{2}+\beta_n\right)M.
\end{aligned}$$

Then by $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\gamma_n=1$, we get
$$\|x_n-Tx_n\|\to 0.$$

Step 4. ($\lim_{n\to\infty}\sup\langle x^*-f(x^*),x^*-x_n\rangle\le 0$, where $x^*=P_{F(T)}f(x^*)$)

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\}\rightharpoonup p$. From $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ and Theorem 1.5 we have $p=Tp$. This, together with the property of the metric projection, implies that

$$\begin{aligned}
\lim_{n\to\infty}\sup\langle x^*-f(x^*),x^*-x_n\rangle &= \lim_{n\to\infty}\sup\langle x^*-f(x^*),x^*-x_{n_i}\rangle\\
&=\langle x^*-f(x^*),x^*-p\rangle\\
&\le 0.
\end{aligned}$$

STEP 5. ($x_n \to x^*$ as $n \to \infty$)
Now we again take $x^* \in F(T)$ as the unique fixed point of the contraction $P_{F(T)}f$. Consider

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \left\|\alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 \\
&= \left\|\alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) - \alpha_n x^* + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) - \beta_n x^*\right. \\
&\quad \left. + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right) + (\alpha_n + \beta_n - 1)x^*\right\|^2 \\
&= \alpha_n^2\left\|\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 + \beta_n^2\left\|f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 \\
&\quad + \gamma_n^2\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 \\
&\quad + 2\alpha_n\beta_n\left\langle \frac{x_n + x_{n+1}}{2} - x^*, f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\alpha_n\gamma_n\left\langle \frac{x_n + x_{n+1}}{2} - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\beta_n\gamma_n\left\langle f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\le \alpha_n^2\left\|\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 + \beta_n^2\left\|f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 + \gamma_n^2\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 \\
&\quad + 2\alpha_n\beta_n\left\langle \left(\frac{x_n + x_{n+1}}{2}\right) - x^*, f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\alpha_n\gamma_n\|x_n - x^*\|\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\| \\
&\quad + 2\beta_n\gamma_n\left\langle f\left(\frac{x_n + x_{n+1}}{2}\right) - f(x^*), T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\beta_n\gamma_n\left\langle f(x^*) - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\le (\alpha_n^2 + \gamma_n^2)\left\|\frac{x_n + x_{n+1}}{2} - x^*\right\|^2 + 2\alpha_n\gamma_n\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 \\
&\quad + 2\beta_n\gamma_n\left\|f\left(\frac{x_n + x_{n+1}}{2}\right) - f(x^*)\right\|\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\| + K_n \\
&\le (\alpha_n + \gamma_n)^2\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + 2\theta\beta_n\gamma_n\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + K_n \\
&\le \left((\alpha_n + \gamma_n)^2 + 2\theta\beta_n\gamma_n\right)\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + K_n \\
&\le \left((1 - \beta_n)^2 + 2\theta\beta_n\gamma_n\right)\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + K_n,
\end{aligned}$$

where

$$\begin{aligned} K_n = \beta_n^2 & \left\| f\left(\frac{x_{n+1}+x_n}{2}\right) - x^* \right\|^2 \\ & + 2\alpha_n\beta_n \left\langle \left(\frac{x_{n+1}+x_n}{2}\right) - x^*, f\left(\frac{x_{n+1}+x_n}{2}\right) - x^* \right\rangle \\ & + 2\beta_n\gamma_n \left\langle f(x^*) - x^*, T\left(\frac{x_{n+1}+x_n}{2}\right) - x^* \right\rangle. \end{aligned}$$

It follows that

$$\left[(1-\beta_n)^2 + 2\theta\beta_n\gamma_n\right] \left\| \frac{x_{n+1}+x_n}{2} - x^* \right\|^2 \geq \|x_{n+1} - x_n\|^2 - K_n$$

implies

$$\sqrt{(1-\beta_n)^2 + 2\theta\beta_n\gamma_n} \left\| \frac{x_{n+1}+x_n}{2} - x^* \right\| \geq \sqrt{\|x_{n+1} - x_n\|^2 - K_n}$$

implies

$$\frac{1}{2}\sqrt{(1-\beta_n)^2 + 2\theta\beta_n\gamma_n}(\|x_{n+1} - x^*\| + \|x_n - x^*\|) \geq \sqrt{\|x_{n+1} - x_n\|^2 - K_n}$$

implies

$$\begin{aligned} & \frac{1}{4}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2 \\ & \quad + 2\|x_{n+1} - x^*\|\|x_n - x^*\|) \\ & \geq \|x_{n+1} - x_n\|^2 - K_n \end{aligned}$$

implies

$$\begin{aligned} & \frac{1}{4}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2 \\ & \quad + (\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2)) \\ & \geq \|x_{n+1} - x_n\|^2 - K_n \end{aligned}$$

implies

$$\begin{aligned} & \left[1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)\right] \|x_{n+1} - x^*\|^2 \\ & \leq \left[\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)\right] \|x_n - x^*\|^2 + K_n. \end{aligned}$$

Thus we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ & \leq \frac{\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\ & = \frac{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n) - 1 + ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \|x_n - x^*\|^2 \\ & \quad + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\ & = \left[1 - \frac{1 - ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}\right] \|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}. \end{aligned}$$

Note that

$$0 < 1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n) < 1$$

implies

$$\frac{1 - ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \geq 1 - ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n).$$

Thus we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq [1 - (1 - ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n))]\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&= [(1-\beta_n)^2 - 2\theta\beta_n\gamma_n]\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&\leq (1-\beta_n)^2\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}.
\end{aligned}$$

Since $0 < 1 - \beta_n < 1$, this give $(1-\beta_n)^2 < (1-\beta_n)$ and

$$\|x_{n+1} - x^*\|^2 \leq (1-\beta_n)\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}. \tag{2.2}$$

By $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \gamma_n = 1$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&= \lim_{n\to\infty} \Bigg(\frac{\beta_n^2 \|f\big(\frac{x_{n+1}+x_n}{2}\big) - x^*\|^2 + 2\alpha_n\beta_n\big\langle\big(\frac{x_{n+1}+x_n}{2}\big) - x^*, f\big(\frac{x_{n+1}+x_n}{2}\big) - x^*\big\rangle}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&\qquad + \frac{2\beta_n\gamma_n\langle f(x^*) - x^*, T\big(\frac{x_{n+1}+x_n}{2}\big) - x^*\rangle}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \Bigg) \\
&\leq 0.
\end{aligned} \tag{2.3}$$

From (2.2), (2.3), and Theorem 1.6 we have $\lim_{n\to\infty} \|x_{n+1} - x^*\|^2 = 0$, which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

3 Applications

The scheme can be used to solve problems of system of variational inequalities and constrained convex minimization. Moreover, it can be applied to find a fixed point in K-mappings.

3.1 A more general system of variational inequalities

Let C be a nonempty closed convex subset of the real Hilbert Space H and $\{A_i\}_{i=1}^N : C \to H$ be a family of mappings. In [1] Cai and Bu considered the problem of finding

$(x_1^*, x_2^*, \ldots, x_N^*) \in C \times C \times \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, \\ \langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{3.1}$$

The equation (3.1) can be written as

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N) x_N^*, x - x_1^* \rangle \geq 0, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1}) x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad \vdots \\ \langle x_3^* - (I - \lambda_2 A_2) x_2^*, x - x_3^* \rangle \geq 0, \\ \langle x_2^* - (I - \lambda_1 A_1) x_1^*, x - x_2^* \rangle \geq 0, \end{cases}$$

which is a more general system of variational inequalities in Hilbert spaces with $\lambda_i > 0$ for all $i \in \{1, 2, 3, \ldots, N\}$. Moreover, we have some useful results:

Lemma 3.1. ([1]) *Let C be a nonempty closed convex subset of the real Hilbert spaces H. For $i \in \{1, 2, 3, \cdots, N\}$, let $A_i : C \to H$ be δ_i-inverse strongly monotone for some positive real number δ_i, namely,*

$$\langle A_i x - A_i y, x - y \rangle \geq \delta_i \|A_i x - A_i y\|^2, \forall x, y \in C$$

Let $G : C \to C$ be a mapping defined by

$$\begin{aligned} G(x) = P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x, \quad \forall x \in C. \end{aligned} \tag{3.2}$$

If $0 < \lambda_i \leq 2\delta_i$ for all $i \in \{1, 2, 3, \cdots, N\}$, then G is nonexpansive.

Lemma 3.2. ([5]) *Let C be a nonempty closed convex subject of the real Hilbert Spaces H. Let $A_i : C \to H$ be a nonlinear mapping,where $i \in \{1, 2, 3, ..., N\}$. For given $x_i^* \in C$, $i \in \{1, 2, 3, ..., N\}$, $(x_1^*, x_2^*, x_3^*, ..., x_N^*)$ is a solution of the problem (3.1) if and only if*

$$\begin{aligned} x_1^* &= P_C(I - \lambda_N A_N) x_N^*, x_i^* \\ &= P_C(I - \lambda_{i-1} A_{i-1}) x_{i-1}^*, \quad i = 2, 3, 4, ..., N, \end{aligned}$$

that is,

$$\begin{aligned} x_1^* = P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x_1^*, \quad \forall x \in C. \end{aligned}$$

From Lemma 3.2, we know that $x_1^* = G(x_1^*)$, that is, x_1^* is a fixed point of the mapping G, where G is defined by (3.2). Moreover, if we find the fixed point x_1^*, it is easy to get the other points by (3.3). Applying Theorem 2.1 we get the result.

Theorem 3.3. *Let C be a nonempty closed convex subject of the real Hilbert spaces H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n G\left(\frac{x_n + x_{n+1}}{2}\right),$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the conditions (i)-(iv).

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping G, which is also the unique solution of the variational inequality $\langle (I - f)x, y - x\rangle \geq 0$, $\forall y \in F(T)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(G)}f$, that is, $P_{F(G)}f(x^*) = x^*$.

3.2 The constrained convex minimization problem

Now, we consider the following constrained convex minimization problem;

$$\min_{x \in C} \phi(x), \tag{3.4}$$

where $\phi : C \to R$ is a real-valued convex function and assumes that the problem (3.4)is consistent. Let Ω denote its solution set. For the minimization problem (3.4), if ϕ is (Fréchet)differentiable, then we have the following lemma.

Lemma 3.4. (Optimality Condition) ([5]) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem* (3.4) *is that x^* solves the variational inequality*

$$\langle \nabla\phi(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{3.5}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C\Big(x^* - \lambda\nabla\phi(x^*)\Big)$$

for every constant $\lambda > 0$. If, in a addition ϕ is convex, then the optimality condition (3.5) *is also sufficient.*

It is well known that the mapping $P_C(I - \lambda A)$ is nonexpansive when the mapping A is δ-inverse-strongly monotone and $0 < \lambda < 2\delta$. We therefore have the following result.

Theorem 3.5. *Let C be a nonempty closed convex subset of the real Hilbert Space H. For the minimization problem* (3.4), *assume that ϕ is* (*Fréchet*) *differentiable and the gradient $\nabla\phi$ is a δ-inverse-strongly monotone mapping for some positive real number δ. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n P_C(I - \lambda\nabla\phi)\left(\frac{x_n + x_{n+1}}{2}\right),$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *satisfying the conditions* (i)-(iv).

Then $\{x_n\}$ *converges strongly to a solution* x^* *of the minimization problem* (3.4), *which is also the unique solution of the variational inequality* $\langle (I-f)x, y-x\rangle \geq 0$, $\forall y \in \Omega$.

In other words, x^* *is the unique fixed point of the contraction* $P_\Omega f$, *that is,* $P_\Omega f(x^*) = x^*$.

3.3 K-mapping

Kangtunyakarn and Suantai [4] in 2009 gave K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ as follows.

Definition 3.6. ([4]) Let C be a nonempty convex subset of real Banach Space. Let $\{T_i\}_{i=1}^N$ be a family of mappings of C into itself and let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, 3, ..., N$. We define a mapping $K : C \to C$ as follows;

$$\begin{cases} U_1 = \lambda_1 T_1 + (1-\lambda_1)I, \\ U_2 = \lambda_2 T_2 U_1 + (1-\lambda_2)U_1, \\ \quad\vdots \\ U_{N-1} = \lambda_{N-1}T_{N-1}U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\ U_N = \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}. \end{cases}$$

Such a mapping is called a K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

In 2014, Kangtunyakarn and Suwannaut [10] established the following result for K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

Lemma 3.7. ([10]) *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For* $i = 1, 2, 3, ..., N$, *let* $\{T_i\}_{i=1}^N$ *be a finite family of* K_i*-strictly pseudo-contractive mapping of* C *into itself with* $K_i \leq \omega_i$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, *namely, there exist constants* $K_i \in [0,1)$ *such that*

$$\|T_i x - T_i y\|^2 \leq \|x-y\|^2 + K_i\|(I-T_i)x - (I-T_i)y\|^2, \quad \forall x, y \in C.$$

Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the K-mapping generated by* $T_1, T_2, T_3, ..., T_N$ *and* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. *Then the following properties hold:*

(*a*) $F(K) = \bigcap_{i=1}^N F(T_i)$.

(*b*) K *is a nonexpansive mapping.*

On the bases of above lemma, we have the following results.

Theorem 3.8. *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For* $i = 1, 2, 3, ..., N$, *let* $\{T_i\}_{i=1}^N$ *be a finite family of* K_i*-strictly pseudo-contractive mapping of* C *into itself with* $K_i \leq \omega_i$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, T_3, ..., T_N$ *and* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. *Let* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0,1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be sequence generated by*

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n K\left(\frac{x_n + x_{n+1}}{2}\right),$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *satisfying the conditions* (i)-(iv).

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the mappings* $\{T_i\}_{i=1}^N$, *which is also the unique solution of the variational inequality* $\langle (I-f)x, y-x\rangle$, $\forall y \in F(K) = \bigcap_{i=1}^N F(T_i)$.

In other words, x^* *is the unique fixed point of the contraction* $P_{\bigcap_{i=1}^N F(T_i)} f$, *that is,* $P_{\bigcap_{i=1}^N F(T_i)} f(x^*) = x^*$.

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GENERALIZED STABILITY OF CUBIC FUNCTIONAL EQUATIONS WITH AN AUTOMORPHISM ON A QUASI-β NORMED SPACE

DONGSEUNG KANG[1] AND HOEWOON B. KIM[2]

[1]Mathematics Education, Dankook University, 152, Jukjeon, Suji, Yongin, Gyeonggi, 16890, Korea

E-mail address: dskang@dankook.ac.kr

[2]Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

E-mail address: kimho@math.oregonstate.edu

Abstract. We introduce a generalized cubic functional equation with an automorphism and investigate the generalized stability of the cubic functions as solutions to the generalized cubic functional equation on a quasi-β Banach space by the fixed point of the alternative method.

Keywords: Hyers-Ulam Stability, Cubic functional equations, Quasi-β normed space, Fixed Point, Functional equations

1. Introduction

In a talk before the Mathematics Club of the University of Wisconsin in 1940, a Polish-American mathematician, S. M. Ulam [25] proposed the stability problem of the linear functional equation $f(x+y) = f(x) + f(y)$ where any solution $f(x)$ of this equation is called a linear function.

To make the statement of the problem precise, *let G_1 be a group and G_2 a metric group with the metric $d(\cdot,\cdot)$. Then given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $F : G_1 \longrightarrow G_2$ with $d(f(x), F(x)) < \epsilon$ for all $x \in G_1$?*. In other words, the question would be generalized as "Under what conditions a mathematical object satisfying a certain property approximately must

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Correspondence: Hoewoon B. Kim, kimho@math.oregonstate.edu.

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be close to an object satisfying the property exactly?".
In 1941, the first, affirmative, and partial solution to Ulam's question was provided by D. H. Hyers [10]. In his celebrated theorem Hyers explicitly constructed the linear function (or additive function) in Banach spaces directly from a given approximate function satisfying the well-known weak Hyers inequality with a positive constant. The Hyers stability result was first generalized in the stability of additive mappings by Aoki [1] allowing the Cauchy difference to become unbounded. In 1978 Th. M. Rassias [16] also provided a generalization of Hyers' theorem with the possibly unbounded Cauchy difference for linear mappings. For the last decades, stability problems of various functional equations, not only linear case, have been extensively investigated and generalized by many mathematicians (see [4, 7, 9, 11, 17, 20, 21]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a quadratic functional equation and every solution of this functional equation is said to be a quadratic function or mapping (e.g. $f(x)=cx^2$). The Hyers-Ulam stability problem for the quadratic functional equation was first studied by Skof [23] in a normed space as the domain of a quadracitc mapping of the equation. Cholewa [6] noticed that the results of Skof still hold in abelian groups. In [7] Czerwik obtained the Hyers-Ulam-Rassias stability (or generalized Hyers-Ulam stability) of the quadratic functional equation. See [2, 15, 27] for more results on the equation (1.1). Also the quadratic equation (1.1) was generalized by Stetkær in [24] introducing an involution σ of an abelian group G, i.e., an automorphism $\sigma : G \to G$ with $\sigma^2 = I$ (I denotes the identity) and considering the following functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(y) \tag{1.2}$$

for all $x, y \in G$. As we already notice the equation (1.1) corresponds to the equation (1.2) with $\sigma = -I$.

Jun and Kim [11] considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

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since it should be easy to see that a function $f(x) = cx^3$ is a solution of the equation (1.3) as the quadratic equation case. In a year they [12] proved the generalized Hyers-Ulam stability of a different version of a cubic functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = f(x + y) + 4f(x - y). \tag{1.4}$$

Since then the stability of cubic functional equations has been investigated by a number of authors (see [5, 14] for details). In particular, Najati [14] investigated the following generalized cubic functional equation

$$f(sx + y) + f(sx - y) = sf(x + y) + sf(x - y) + 2(s^3 - s)f(x) \tag{1.5}$$

for a positive integer $s \geq 2$.

As we might notice there are various definitions for the stability of the cubic functional equations and here we consider the following definition of a generalized cubic functional equation

$$\begin{aligned} &f(ax + y) - f(x + ay) + a(a - 1)f(x - y) \\ &= (a - 1)(a + 1)^2 f(x) - (a - 1)(a + 1)^2 f(y) \end{aligned} \tag{1.6}$$

for all $a \in \mathbb{Z}$ $(a \neq 0, \pm 1)$ and generalized the equation (1.6) with the involution σ of a linear space X when $a = 2$;

$$f(2x + y) - f(x + 2y) + 2f(x + \sigma(y)) - 9f(x) + 9f(y) = 0. \tag{1.7}$$

In this paper we will study the generalized Hyers-Ulam stability problem of the equation (1.7).

In order to give our results in Section 3 it is convenient to state the definition of a generalized metric on a set X and a result on a fixed point theorem of the alternative by Diaz and Margolis [8].

Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1. *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for*

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each element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

In 2009, Rassias and Kim [18] investigated the Hyers-Ulam stability of Cauchy and Jensen type additive mappings in quasi-β-normed spaces with the following definition of a quasi-β-norm:

Definition 1.2. Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over a field $\mathbb{K}$. A *quasi-β-norm* $||\cdot||$ is a real-valued function on X satisfying the following properties:

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$
(2) $||\lambda x|| = |\lambda|^\beta ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$
(3) There is a constant $K \geq 1$ such that $||x + y|| \leq K(||x|| + ||y||)$ for all $x, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi-β-normed space* if $||\cdot||$ is a quasi-β-norm on X. A smallest possible constant K is called the modulus of concavity of $||\cdot||$. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $||\cdot||$ is called a (β, p)-norm $(0 < p \leq 1)$ if the property (3) of the Definition 1.2 takes the form $||x+y||^p \leq ||x||^p + ||y||^p$ for all $x, y \in X$. In this case, a quasi-β-Banach space is referred to as a (β, p)-Banach space; see [3, 18, 19] for datails.

In this paper, using the Fixed Point method we prove the generalized Hyers-Ulam stability of the generalized cubic functional equation (1.7) in a quasi-β-normed linear space we just defined above (Definition 1.2). In Section 2 we establish the general solution of the cubic functional equation (1.7) applying the symmetric n-additive mappings for the cubic functional equation (1.7) that will be explained in detail in the Section. Finally, we obtain, in Section 3, the generalized Hyers-Ulam stability of the generalized cubic functional equation (1.7) with the Fixed Point theorem of the Alternative.

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2. The General Solution with $\sigma = -I$

In this section we study the general solution of the cubic functional equation (1.7) with $\sigma = -I$ by introducing and applying n-additive symmetric mappings and their properties that are found in [22, 26]. Before we proceed, let us give some basic backgrounds of n-additive symmetric mappings. Let X and Y be real vector spaces and n a positive integer. A function $A_n : X^n \longrightarrow Y$ is called *n-additive* if it is additive in each of its variables. A function $A_n : X^n \longrightarrow Y$ is said to be *symmetric* if $A_n(x_1, x_2, \cdots, x_n) = A_n(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)})$ for every permutation $\{\sigma(1), \sigma(2), \cdots, \sigma(n)\}$ of $\{1, 2, \cdots, n\}$. If $A_n(x_1, x_2, \cdots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \cdots, x)$ and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$. Such a function $A^n(x)$ will be called a *monomial function of degree n* assuming $A^n(x) \not\equiv 0$. Moreover, the resulting function after substituting $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1}, x_{s+2}, \cdots = x_n = y$ in $A_n(x_1, x_2, \cdots, x_n)$ will be denoted by $A^{s,n-s}(x, y)$.

Theorem 2.1. *A function $f : X \longrightarrow Y$ is a solution of the functional equation (1.7) with $\sigma = -I$ if and only if f is of the form $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of the 3-additive symmetric mapping $A_3 : X^3 \longrightarrow Y$.*

Proof. Assume that f satisfies the functional equation (1.7). Taking $x = y = 0$ in the equation (1.7) it's not hard to have $f(0) = 0$ since $\sigma(0) = 0$. Substituting $y = 0$ in (1.7) also gives

$$f(2x) - f(x) + 2f(x) - 9f(x) = 0,$$

that is,

$$f(2x) = 2^3 f(x) \tag{2.1}$$

for all $x \in X$. Similarly, when $x = 0$ in the equation (1.7) we have

$$2f(y) + 2f(\sigma(y)) = 0,$$

i.e.,

$$f(y) + f(-y)) = 0 \tag{2.2}$$

for all $y \in X$ since $\sigma(y) = -y$. This observation leads us to $f(-y) = -f(y)$ for all $y \in X$ and hence f is an odd function. Rewriting the equation (1.7) as

$$f(x) - \frac{1}{9}f(2x+y) + \frac{1}{9}f(x+2y) - \frac{2}{9}f(x-y) - f(y) = 0 \tag{2.3}$$

and applying Theorems 3.5 and 3.6 in [26] we express f as

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0 \tag{2.4}$$

where A^0 is an arbitrary element in Y and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i \longrightarrow Y$ for $i = 1, 2, 3$. Since f is odd and $f(0) = 0$ it follows that

$$f(x) = A^3(x) + A^1(x)$$

for all $x \in X$. By the property (2.1) of f and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$ we should obtain $A^1(x) = 0$ for all $x \in X$. Therefore we conclude that $f(x) = A^3(x)$ for all $x \in X$.

Conversely, let us assume that $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of a 3-additive symmetric mapping $A_3 : X^3 \longrightarrow Y$. Noting that

$$A^3(qx+ry) = q^3A^3(x) + 3q^2rA^{2,1}(x,y) + 3qr^2A^{1,2}(x,y) + r^3A^3(y)$$

and calculating simple computation for the equation (1.7) with $\sigma = -I$ in term of $A^3(x)$, we show that the function f satisfies the cubic equation (1.7) with $\sigma = -I$, which completes the proof. □

3. General Hyers-Ulam Stability in a Quasi-β Banach Space: A Fixed Point Theorem of the Alternative Approach

In this section we will investigate the generalized Hyers-Ulam stability of the cubic functional equation (1.7) which is introduced earlier in previous sections

$$f(2x+y) - f(x+2y) + 2f(x+\sigma(y)) - 9f(x) + 9f(y) = 0.$$

for all $x, y \in X$ by the approach of the fixed point of the alternative. As we used the notations in the previous sections we assume that X is a vector space and $(Y, ||\cdot||)$ is a quasi-β-Banach space in this section. A set $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.

Theorem 3.1. *Suppose that a function $\phi : X^2 \longrightarrow \mathbb{R}_+$ is given and there exists a constant L with $0 < L < 1$ such that*

$$\phi(2x, 2y) \leq 2L\phi(x, y) \quad and \quad \phi(x + \sigma(x), y + \sigma(y)) \leq 2L\phi(x, y) \tag{3.1}$$

for all $x, y \in X$. Furthermore, let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$||f(2x + y) - f(x + 2y) + 2f(x + \sigma(y)) - 9f(x) + 9f(y)|| \leq \phi(x, y) \tag{3.2}$$

for all $x, y \in X$ where σ is an automorphism on X with $\sigma^2 = I$ where I is the identity.
Then there exists the unique generalized cubic function $C : X \longrightarrow Y$ defined by $C(x) := lim_{n\to\infty} \left(\frac{1}{2^{3n}}\right)(f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x)))$ such that

$$||f(x) - C(x)|| \leq \left(\frac{1 + L}{2^3(1 - L)}\right)\Phi(x) \tag{3.3}$$

for all $x \in X$ where $\Phi(x) = \max\{\phi(x, 0), \phi(0, x)\}$ for all $x \in X$.

Proof. First, we put $y = 0$ in the inequality (3.2) to obtain

$$||f(2x) - 2^3 f(x)|| \leq \phi(x, 0) \tag{3.4}$$

for $x \in X$ since $\sigma(0) = 0$. Similarly we substitute $x = 0$ into the inequality (3.2) again to have

$$||10f(y) - f(2y) + 2f(\sigma(y))|| \leq \phi(0, y) \tag{3.5}$$

for all $y \in X$. Combining the two inequalities (3.4) and (3.5) we note that

$$\begin{aligned} ||2f(x) + 2f(\sigma(x))|| &= ||10f(x) - f(2x) + 2f(\sigma(x)) + f(2x) - 2^3 f(x)|| \\ &\leq \phi(x, 0) + \phi(0, x) \end{aligned}$$

and hence we conclude that

$$||f(x) + f(\sigma(x))|| \leq \frac{1}{2}(\phi(x, 0) + \phi(0, x)) \tag{3.6}$$

Then we let $x = x + \sigma(x)$ in the above inequality (3.6) and we are able to get

$$||f(x + \sigma(x))|| \leq \frac{1}{4}(\phi(x + \sigma(x), 0) + \phi(0, x + \sigma(x))) \leq \frac{L}{2}(\phi(x, 0) + \phi(0, x)) \tag{3.7}$$

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We also define a function $T(f)$ from X to Y by $T(f)(x) = \frac{1}{2^3}(f(2x)+f(x+\sigma(x)))$ and we then consider the following estimation

$$\begin{aligned} ||T(f)(x) - f(x)|| &= \left\| \frac{1}{2^3}(f(2x) + f(x+\sigma(x))) - f(x) \right\| \\ &= \left\| \frac{1}{2^3}(f(2x) - 2^3 f(x)) + \frac{1}{2^3} f(x+\sigma(x)) \right\| \\ &\le \frac{1}{2^3}\phi(x,0) + \frac{1}{2^3}\left(\frac{L}{2}\right)(\phi(x,0)+\phi(0,x)) \\ &\le \frac{1}{2^3}(1+L)\Phi(x) \end{aligned} \tag{3.8}$$

This idea enables us to define a sequence $\{T^n(f)\}$ in Y for each $x \in X$ by

$$T^n(f)(x) = \frac{1}{2^{3n}}(f(2^n x) + (2^n-1)f(2^{n-1}x + 2^{n-1}\sigma(x)))$$

for a nonnegative integer n with $T^0(f) = f$ and we claim that it should be a Cauchy sequence in Y. In order to show this we use the inequalities (3.4), (3.7), and (3.8) to compute the following estimations;

(3.9)

$$\begin{aligned} &||T^n(f)(x) - T^{n-1}(f)(x)|| = ||\frac{1}{2^{3n}}(f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))) \\ &- \frac{1}{2^{3(n-1)}}(f(2^{n-1}x) + (2^{n-1}-1)f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\ &= ||\frac{1}{2^{3n}}(f(2^n x) + f(2^{n-1}x + 2^{n-1}\sigma(x)) + (2^n-2)f(2^{n-1}x + 2^{n-1}\sigma(x))) \\ &- \frac{1}{2^{3(n-1)}}(f(2^{n-1}x) + (2^{n-1}-1)f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\ &= ||\frac{1}{2^{3n}}(f(2^n x) + f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^3 f(2^{n-1}x)) \\ &+ \frac{1}{2^{3n}}((2^n-2)f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^2(2^n-2)f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\ &= ||\frac{1}{2^{3n}}(f(2^n x) + f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^3 f(2^{n-1}x)) \\ &+ \frac{1}{2}\left(\frac{2^n-2}{2^{3n}}\right)(2f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^3 f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\ &\le \frac{1}{2^{3n}}(\phi(2^{n-1}x,0) + \frac{L}{2}(\phi(2^{n-1}x,0) + \phi(0,2^{n-1}x))) \\ &+ \left(\frac{1}{2}\left(\frac{2^n-2}{2^{3n}}\right)\right)\left(\phi(2^{n-2}x + 2^{n-2}\sigma(x),0) + \frac{L}{2}(\phi(2^{n-2}x + 2^{n-2}\sigma(x),0) + \phi(0,2^{n-2}x + 2^{n-2}\sigma(x)))\right) \\ &\le \frac{(2L)^{n-1}}{2^{3n}}(1+L)\Phi(x) + \frac{2^{n-1}-1}{2^{3n}}(2L)^{n-1}(1+L)\Phi(x) = \frac{1}{2^3}(1+L)\left(\frac{L}{2}\right)^{n-1}\Phi(x) \end{aligned}$$

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for all $x \in X$ and all nonnegative integer n. Hence we note that

$$\|T^n(f)(x) - T^m(f)(x)\| \leq \frac{1+L}{2^3} \sum_{j=m}^{n-1} \left(\frac{L}{2}\right)^j \Phi(x) \tag{3.10}$$

for all $x \in X$ and $n > m \in \mathbb{N}$.

With this result in mind we consider the set $\Omega = \{g | g : X \longrightarrow Y, g(0) = 0\}$ and then define a generalized metric d on Ω as follows:

$$d(g, h) = \inf \{\lambda \in [0, \infty] : \|g(x) - h(x)\| \leq \lambda\Phi(x) \, \text{for all} \, x \in X\}$$

with $\inf \emptyset = \infty$. Then (S, d) is a complete generalized metric space; see Lemma 2.1 in [13]. Now we define a mapping $T : \Omega \longrightarrow \Omega$ by

$$T(g)(x) = \frac{1}{2^3}(g(2x) + g(x + \sigma(x))) \tag{3.11}$$

for all $x \in X$. We, then, will show that T is strictly contractive on Ω.

Given $g, h \in \Omega$, let $\lambda \in [0, \infty]$ be a constant with $d(g, h) \leq \lambda$. Then we have $\|g(x) - h(x)\| \leq \lambda\Phi(x)$ for all $x \in X$.

By the equation (3.1) we have

$$\begin{aligned}
&\|T(g)(x) - T(h)(x)\| = \frac{1}{2^3}\|g(2x) - h(2x) + g(x + \sigma(x)) - h(x + \sigma(x))\| \\
&\leq \frac{1}{2^3}\|g(2x) - h(2x)\| + \frac{1}{2^3}\|g(x + \sigma(x)) - h(x + \sigma(x))\| \\
&\leq \frac{\lambda}{2^3}\Phi(2x) + \frac{\lambda}{2^3}\Phi(x + \sigma(x)) \leq \frac{1}{2}L\lambda \leq L\lambda
\end{aligned}$$

for all $x \in G$, which implies

$$d(T(g), T(h)) \leq L\lambda.$$

Therefore we may conclude that

$$d(T(g), T(h)) \leq Ld(g, h)$$

for any $g, h \in \Omega$. Since L is a constant with $0 < L < 1$, T is strictly contractive as claimed.

Also the inequality (3.8) implies that

$$d(T(f), f) \leq \frac{1}{2^3}(1 + L) < \infty. \tag{3.12}$$

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By the Alternative of Fixed Point as we introduced in the Introduction Section, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of T such that $d(T^n(f), C) \to 0$ as $n \to \infty$, that is,

$$C(x) = \lim_{n\to\infty} T^n(f)(x)$$

for all $x \in X$. Then we will show that C is cubic and it would not be hard if we recall the approximation inequality (3.2) for f where we let $x = 2^n x$, $y = 2^n y$ and $x = 2^{n-1}(x + \sigma(x))$, $y = 2^{n-1}(y + \sigma(y))$, respectably, as follows;

$$\begin{aligned}
&\|C(2x + y) - C(x + 2y) + 2C(x + \sigma(y)) - 9C(x) + 9C(y)\| \\
&\leq \lim_{n\to\infty} \frac{1}{2^{3n}}\phi(2^n x, 2^n y) + \lim_{n\to\infty} \frac{2^n - 1}{2^{3n}}\phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y))) \\
&\leq \lim_{n\to\infty} \frac{(2L)^n}{2^{3n}}\phi(x, y) + \lim_{n\to\infty} \frac{(2^n - 1)(2L)^n}{2^{3n}}\phi(x, y) \\
&= \lim_{n\to\infty} \left(\frac{L}{2}\right)^n \phi(x, y) = 0
\end{aligned}$$

for all $x, y \in X$, which implies that C is cubic.

By the Alternative of Fixed Point theorem and the inequality (3.12) we get

$$d(f, C) \leq \frac{1}{1 - L} d(f, T(f)) \leq \frac{1 + L}{2^3(1 - L)}.$$

Hence the inequality (3.3) is true for all $x \in X$.

By the uniqueness of the fixed point of T, the cubic function C should be unique, which completes the proof. □

Let us give the classical Cauchy difference type stability of the generalized cubic functional equation (1.7) when $\sigma = -I$ from Theorem 3.1 as we see the following Corollary.

Corollary 3.2. *Let $\epsilon \geq 0$, $0 < p < \frac{1}{\beta}$ be a real number. Suppose $f : X \longrightarrow Y$ is a function satisfying $f(0) = 0$ and*

$$||f(2x + y) - f(x + 2y) + 2f(x - y) - 9f(x) + 9f(y)|| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then there exists the unique cubic function $C : X \longrightarrow Y$ defined by

$$C(x) = lim_{n\to\infty} \left(\frac{1}{2^{3n}}\right) f(2^n x)$$

satisfying

$$||f(x) - C(x)|| \leq \left(\frac{\epsilon(1+L)}{2^3(1-L)}\right)||x||^p \tag{3.13}$$

for all $x \in X$.

Proof. This proof follows from Theorem 3.1 by taking $\phi(x, y) = \epsilon(||x||^p + ||y||^p)$ for all $x, y \in X$ with $L = |2|^{p\beta-1}$.

□

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[1]Mathematics Education, Dankook University, 152, Jukjeon, Suji, Yongin, Gyeonggi, 16890, Korea

E-mail address: dskang@dankook.ac.kr

[2]Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

E-mail address: kimho@math.oregonstate.edu

Two quotient BI-algebras induced by fuzzy normal subalgebras and fuzzy congruence relations

Yinhua Cui[1] **and Sun Shin Ahn**[2,*]

[1]*Department of Mathematics, Yanbian University, Yanji 133002, P. R. China*

[2]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. In this paper, we discuss two quotient BI-algebras induced by fuzzy normal subalgebras and induced by fuzzy congruence relations, which are useful in the study of the structural theory of fuzzy quotient BI-algebras.

1. Introduction

Zadeh [14] introduced the notion of a fuzzy subset A of a set X as a function from X into $[0,1]$. Rosenfeld [11] applied this concept to the theory of groupoids and groups. Liu [7] introduced and studied the notion of the fuzzy ideals of a ring. Mukherjee and Sen [9] defined and examined the fuzzy prime ideals of a ring. The concept of fuzzy ideals was applied to several algebras: BN-algebras [2], BL-algebras [8], semirings [5] and semigroups [3]. Recently, Song et al. [13] discussed positive implicative superior ideals induced by superior mappings in BCK-algebras.

Saeid et al. [12] introduced a new algebra, called a BI-algebra, which is a generalization of a (dual) implication algebra, and they discussed ideals and congruence relations. Ahn et al. [1] introduced the notion of normal subalgebras in BI-algebras, and studied its analytic construction.

In this paper, we discuss two quotient BI-algebras induced by fuzzy normal subalgebras and induced by fuzzy congruence relations, which are useful in the study of the structural theory of fuzzy quotient BI-algebras.

2. Preliminaries

We recall some definitions and results discussed in [12].

An algebra $(X,*,0)$ of type (2, 0) is called a *BI-algebra* [12] if

(B1) $x*x=0$ for all $x\in X$,

(B2) $x*(y*x)=x$ for all $x,y\in X$.

We introduced a relation "$\leq$" on a BI-algebra X by $x\leq y$ if and only if $x*y=0$. We note that the relation "$\leq$" is not a partial order, since it is only reflexive. A non-empty subset S of a BI-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Since $x*x=0$ for all $x\in X$, it is clear that $0\in S$.

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* Correspondence: Tel: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).

[0]**E-mail**: cuiyh@ybu.edu.cn (Y. Cui); sunshine@dongguk.edu (S. S. Ahn).

Definition 2.1. Let $(X, *, 0)$ be a BI-algebra and let I be a non-empty subset of X. Then I is called an *ideal* [12] of X if

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$

for any $x, y \in X$. Obviously, $\{0\}$ and X are ideals of X. We call $\{0\}$ and X a *zero ideal* and a *trivial ideal*, respectively. An ideal I is said to be *proper* if $I \neq X$.

Example 2.2. Let $X := \{0, a, b, c\}$ be a BI-algebra [12] with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then it is easy to check that $I_1 := \{0, a, c\}$ is an ideal of X, but $I_2 := \{0, a, b\}$ is not an ideal of X, since $c * a = b \in I_2$ and $a \in I_2$, but $c \notin I_2$.

Proposition 2.3. [12] *Let I be an ideal of a BI-algebra X. If $y \in I$ and $x \leq y$, then $x \in I$.*

Proposition 2.4. [12] *Let X be a BI-algebra. Then*

(i) $x * 0 = x$,

(ii) $0 * x = 0$,

(iii) $x * y = (x * y) * y$,

(iv) *if* $y * x = x$, *then* $X = \{0\}$,

(v) *if* $x * (y * z) = y * (x * z)$, *then* $X = \{0\}$,

(vi) *if* $x * y = z$, *then* $z * y = z$ *and* $y * z = y$,

(vii) *if* $(x * y) * (z * u) = (x * z) * (y * u)$, *then* $X = \{0\}$,

for all $x, y, z, u \in X$.

Definition 2.5. A non-empty subset N of a BI-algebra X is said to be *normal* (or a *normal subalgebra*) [1] if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$.

Definition 2.6. A BI-algebra X is called a BI_1-*algebra* [1] if $x * y = 0 = y * x$ implies $x = y$ for all $x, y \in X$.

3. Quotient BI-algebras induced by fuzzy normal subalgebras

Definition 3.1. A fuzzy set μ in a BI-algebra X is called a *fuzzy subalgebra* of X if for any $x, y \in X$,

(F0) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$.

Example 3.2. Let $X := \{0, a, b, c\}$ be a BI-algebra [12] with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	b
c	c	0	c	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) > \mu(a) = \mu(b) > \mu(c)$. Then μ is a fuzzy subalgebra of X.

Proposition 3.3. Let μ be a fuzzy subalgebra of a BI-algebra X. Then $\mu(0) \geq \mu(x)$ for all $x \in X$.

Proof. By (B1), we have $x * x = 0$ for all $x \in X$. Using (F0), $\mu(0) = \mu(x * x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$ for all $x \in X$. □

We denote a notation $\prod^n x * x$ by $\prod^n x * x := \underbrace{x * (x * (x * (\cdots * (x * x}_{\text{n}})) \cdots)$ for any natural number n.

Proposition 3.4. *Let μ be a fuzzy subalgebra of a BI-algebra X and let $n \in \mathbb{N}$. Then*

(i) $\mu(\prod^n x * x) \geq \mu(x)$ *whenever n is odd,*

(ii) $\mu(\prod^n x * x) = \mu(x)$ *whenever n is even.*

Proof. Let $x \in X$ and n be an odd natural number. Then $n = 2k - 1$ for some positive integer k. Then $\mu(\prod^{2(k+1)-1} x * x) = \mu(\prod^{2k+1} x * x) = \mu(\prod^{2k-1} x * (x * (x * x))) = \mu(\prod^{2k-1} x * x) \geq \mu(x)$ which proves (i). Similarly we can prove the second part, but we omit it. □

Definition 3.5. A fuzzy set μ in a BI-algebra X is said to be *fuzzy normal* if it satisfies the inequality

$$\mu((x * a) * (y * b)) \geq \min\{\mu(x * y), \mu(a * b)\}$$

for all $a, b, x, y \in$X.

Example 3.6. Let $X := \{0, 1, 2, 3\}$ be a BI-algebra [1] set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) > \mu(1) > \mu(2) = \mu(3)$. Then it easy to see that μ is fuzzy normal of X.

Theorem 3.7 *Every fuzzy normal set μ in a BI-algebra X is a fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Since μ is fuzzy normal, we have $\mu(x * y) = \mu((x * y) * (0 * 0)) \geq \min\{\mu(x * 0), \mu(y * 0)\} = \min\{\mu(x), \mu(y)\}$, which shows that μ is a fuzzy subalgebra of X. □

The converse of Theorem 3.7 may not be true in general.

Example 3.8. Consider a BI-algebra $X = \{0, a, b, c\}$ and a fuzzy set μ as in Example 3.2. Then μ is a fuzzy subalgebra of X, but not fuzzy normal, since $\mu((c * b) * (c * c)) = \mu(c) \ngeq \mu(b) = \min\{\mu(c * c), \mu(b * c)\}$.

Definition 3.9. A fuzzy set μ in a BI-algebra X is called a *fuzzy normal subalgebra* of X if it is both a fuzzy subalgebra and a fuzzy normal subset of X.

Example 3.10. Consider a BI-algebra $X = \{0, 1, 2, 3\}$ as in Example 3.6. Define a fuzzy set $\nu : X \to [0, 1]$ by

$$\nu(x) := \begin{cases} 0.7 & \text{if } x \in \{0, 1\}, \\ 0.3 & \text{if } x \in \{2, 3\}. \end{cases}$$

It is easy to show that ν is a fuzzy normal subalgebra of X.

Proposition 3.11. *If a fuzzy set μ in a BI-algebra X is fuzzy normal, then $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$.*

Proof. Let $x, y \in X$. Using Proposition 3.3, we have $\mu(x*y) = \mu((x*y)*(x*x)) \geq \min\{\mu(x*x), \mu(y*x)\} = \mu(y*x)$. Interchanging x with y, we obtain $\mu(y * x) \geq \mu(x * y)$, which proves the proposition. □

Theorem 3.12. *Let μ be a fuzzy normal BI-algebra X. Then the set*

$$X_\mu := \{x \in X | \mu(x) = \mu(0)\}$$

is a normal subalgebra of X.

Proof. Let $a, b, x, y \in X$ be such that $x * y \in X_\mu$ and $a * b \in X_\mu$. Then $\mu(x * y) = \mu(0) = \mu(a * b)$. Since μ is fuzzy normal, we have $\mu((x * a) * (y * b)) \geq \min\{\mu(x * y), \mu(a * b)\} = \mu(0)$. It follows from Proposition 3.3 that $\mu((x * a) * (y * b)) = \mu(0)$. Hence $(x * a) * (y * b) \in X_\mu$. This completes the proof. □

Theorem 3.13. *The intersection of a family of fuzzy normal subalgebras of a BI-algebra X is also a fuzzy normal subalgebra of X.*

Proof. Let $\{\mu_\alpha | \alpha \in \Lambda\}$ be a family of fuzzy normal subalgebras and let $a, b, x, y \in X$. Then

$$\begin{aligned} \cap_{\alpha\in\Lambda}\mu_\alpha((x * a) * (y * b)) &= \inf_{\alpha\in\Lambda} \mu_\alpha((x * a) * (y * b)) \\ &\geq \inf_{\alpha\in\Lambda} \{\min\{\mu_\alpha(x * y), \mu_\alpha(a * b)\}\} \\ &= \min\{\inf_{\alpha\in\Lambda} \mu_\alpha(x * y), \inf_{\alpha\in\Lambda} \mu_\alpha(a * b)\} \\ &= \min\{\cap_{\alpha\in\Lambda}\mu_\alpha(x * y), \cap_{\alpha\in\Lambda}\mu_\alpha(a * b)\} \end{aligned}$$

which shows that $\cap_{\alpha\in\Lambda}\mu_\alpha$ is fuzzy normal of X. By Proposition 3.7, we know that $\cap_{\alpha\in\Lambda}\mu_\alpha$ is a fuzzy normal subalgebra of X. □

Suppose that μ is a fuzzy normal subalgebra of a BI-algebra X. Define a binary relation " $\sim^\mu$ " on X by putting $x \sim^\mu y$ if and only if $\mu(x * y) = \mu(0)$ for any $x, y \in X$.

Lemma 3.14. *The relation $\sim^\mu$ is an equivalence relation on a BI-algebra X.*

Proof. Using (B1), $\mu(x * x) = \mu(0)$ and so $x \sim^\mu x$, which means $\sim^\mu$ is reflexive. Suppose that $x \sim^\mu y$ for any $x, y \in X$. Then $\mu(0) = \mu(x * y)$. By Proposition 3.11, $\mu(y * x) = \mu(0)$. So $y \sim^\mu x$, which means $\sim^\mu$ is symmetric.

Suppose that $x \sim^\mu y$ and $y \sim^\mu z$ for any $x, y, z \in X$. Then $\mu(x * y) = \mu(0)$, $\mu(y * z) = \mu(0) = \mu(z * y)$ and

$$\begin{aligned}\mu(x * z) =& \mu((x * z) * 0) = \mu((x * z) * (y * y)) \\ \geq& \min\{\mu(x * y), \mu(z * y)\} \\ =& \min\{\mu(0), \mu(0)\} = \mu(0).\end{aligned}$$

Also since $\mu(0) \geq \mu(x)$ for all $x \in X$, $\mu(0) \geq \mu(x * z)$ and so $\mu(x * z) = \mu(0)$. Hence $x \sim^\mu z$. Therefore $\sim^\mu$ is an equivalence relation on a BI-algebra X. □

Lemma 3.15. *For all x, y, z in a BI-algebra X, $x \sim^\mu y$ implies $x * z \sim^\mu y * z$ and $z * x \sim^\mu z * y$.*

Proof. Let $x \sim^\mu y$. Then $\mu(x * y) = \mu(0)$. Since $x * x = 0$ and $\mu(0) \geq \mu(x)$ for all $x \in X$, we have

$$\begin{aligned}\mu((x * z) * (y * z)) \geq& \min\{\mu(x * y), \mu(z * z)\} \\ =& \min\{\mu(0), \mu(0)\} = \mu(0).\end{aligned}$$

Since $\mu(0) \geq \mu(x)$ for all $x \in X$, $\mu(0) \geq \mu((x * z) * (y * z))$. Therefore $\mu(0) = \mu((x * z) * (y * z))$, so $x * z \sim^\mu y * z$. By a similar way, we can prove that $z * x \sim^\mu z * y$. The proof is complete. □

Lemma 3.16. *Let X be a BI-algebra. For any $x, y, z, w \in X$, $x \sim^\mu y$ and $z \sim^\mu w$ imply $x * z \sim^\mu y * w$.*

Proof. Let $x \sim^\mu y$ and $z \sim^\mu w$ for any $x, y, z, w \in X$. Then $\mu(x * y) = \mu(0)$ and $\mu(z * w) = \mu(0)$. Hence $\mu((x * z) * (y * w)) \geq \min\{\mu(x * y), \mu(z * w)\} = \min\{\mu(0), \mu(0)\} = \mu(0)$. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, $\mu(0) \geq \mu((x * z) * (y * w))$. Thus $\mu(0) = \mu((x * z) * (y * w))$, so $x * z \sim^\mu y * w$. The proof is complete. □

The above Lemmas 3.14, 3.15 and 3.16 give the following theorem.

Theorem 3.17. *The relation " $\sim^\mu$ " is a congruence relation on a BI-algebra X.*

Denote by μ_x the equivalence class containing x, and let X/μ be the set of all equivalence classes with respect to $\sim^\mu$, that is, $\mu_x = \{y \in X | y \sim^\mu x\}$ and $X/\mu = \{\mu_x | x \in X\}$. Now we define a binary operation " $*$ " in X/μ by putting $\mu_x * \mu_y := \mu_{x*y}$. Theorem 3.17 guarantees that this operation is well defined.

Theorem 3.18. *Let μ be a fuzzy normal subalgebra in a BI_1-algebra X. Then $(X/\mu, *, \mu_0)$ is a BI_1-algebra.*

Proof. Let $\mu_x, \mu_y, \mu_z \in X/\mu$. Then $\mu_x * \mu_x = \mu_{x*x} = \mu_0$ and $\mu_x = \mu_{x*(y*x)} = \mu_x * \mu_{y*x} = \mu_x * (\mu_y * \mu_x)$. If $\mu_x * \mu_y = \mu_0$ and $\mu_y * \mu_x = \mu_0$, then $\mu_{x*y} = \mu_0 = \mu_{y*x}$ and so $x * y = 0 = y * x$. Hence $x = y$ and therefore $\mu_x = \mu_y$. Thus $(X/\mu, *, \mu_0)$ is a BI_1-algebra. □

Corollary 3.19. *Let μ be a fuzzy normal subalgebra in a BI-algebra. Then $(X/\mu; *, \mu_0)$ is a BI-algebra.*

This algebra X/μ is called the *quotient BI-algebra* of a BI-algebra X induced by a fuzzy normal subalgebra μ.

If μ is a fuzzy normal subalgebra in a BI-algebra X, then the set $X_\mu := \{x \in X | \mu(x) = \mu(0)\}$ is a normal subalgebra of X.

Theorem 3.20. *Let μ be a fuzzy normal subalgebra of a BI-algebra X. The mapping $\gamma : X \to X/\mu$, given by $\gamma(x) = \mu_x$, is a surjective homomorphism, and $ker\gamma = \{x \in X | \gamma(x) = \mu_0\} = X_\mu$.*

Proof. Let $\mu_x \in X/\mu$. Then there exists an element $x_0 \in \mu_x$, so $x_0 \in X$ such that $\gamma(x_0) = \mu_x$. Hence γ is surjective. For any $x, y \in X$, $\gamma(x * y) = \mu_{x*y} = \mu_x * \mu_y = \gamma(x) * \gamma(y)$. Thus γ is a homomorphism. And $ker\gamma = \{x \in X | \gamma(x) = \mu_0\} = \{x \in X | x \sim^\mu 0\} = \{x \in X | \mu(x) = \mu(0)\} = X_\mu$. □

Let X, Y be BI-algebras. If we define $(x_1, y_1) * (x_2, y_2) := (x_1 * x_2, y_1 * y_2)$ in $X \times Y$, then $(X \times Y, *, (0, 0))$ becomes a BI-algebra, and we call it a *product BI-algebra*.

Theorem 3.21. *Let μ (resp., ν) be a fuzzy normal subalgebra in a BI-algebra X (resp., Y). If we define $(\mu \times \nu)(x, y) := \min\{\mu(x), \nu(x)\}$ in $X \times Y$ for $x \in X, y \in Y$, then $\mu \times \nu$ is also a fuzzy normal subalgebra in $X \times Y$.*

Proof. Let μ (resp., ν) be a fuzzy normal subalgebra in X (resp., Y). Then

$$\begin{aligned}(\mu \times \nu)((x_1, y_1)*(x_2, y_2)) &= (\mu \times \nu)(x_1 * x_2, y_1 * y_2)\\ &= \min\{\mu(x_1 * x_2), \nu(y_1 * y_2)\}\\ &\geq \min\{\min\{\mu(x_1), \mu(x_2)\}, \min\{\nu(y_1), \nu(y_2)\}\}\\ &= \min\{\min\{\mu(x_1), \nu(y_1)\}, \min\{\mu(x_2), \nu(y_2)\}\}\\ &= \min\{(\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2)\}\end{aligned}$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$. Hence $\mu \times \nu$ is a fuzzy subalgebra of $X \times Y$. And

$$\begin{aligned}&(\mu \times \nu)(((x_1, y_1) * (a_1, b_1)) * ((x_2, y_2) * (a_2, b_2)))\\ &=(\mu \times \nu)((x_1 * a_1, y_1 * b_1) * (x_2 * a_2, y_2 * b_2))\\ &=(\mu \times \nu)((x_1 * a_1) * (x_2 * a_2), (y_1 * b_1) * (y_2 * b_2))\\ &= \min\{\mu((x_1 * a_1) * (x_2 * a_2)), \nu((y_1 * b_1) * (y_2 * b_2))\}\\ &\geq \min\{\min\{\mu(x_1 * x_2), \mu(a_1 * a_2)\}, \min\{\nu(y_1 * y_2), \nu(b_1 * b_2)\}\}\\ &= \min\{\min\{\mu(x_1 * x_2), \nu(y_1 * y_2)\}, \min\{\mu(a_1 * a_2), \nu(b_1 * b_2)\}\}\\ &= \min\{(\mu \times \nu)((x_1 * x_2), (y_1 * y_2)), (\mu \times \nu)((a_1 * a_2), (b_1 * b_2))\}\\ &= \min\{(\mu \times \nu)((x_1, y_1) * (x_2, y_2)), (\mu \times \nu)((a_1, b_1) * (a_2, b_2))\}.\end{aligned}$$

So $\mu \times \nu$ is fuzzy normal. Therefore $\mu \times \nu$ is also a fuzzy normal subalgebra of $X \times Y$. □

Proposition 3.22. *Let μ be a fuzzy normal subalgebra of a BI-algebra X. If J is a normal subalgebra of X, then J/μ is a normal subalgebra of X/μ.*

Proof. Let μ be a fuzzy normal subalgebra of X and J be a normal subalgebra of X. Then for any $x, y \in J$, $x * y \in J$. Let $\mu_x, \mu_y \in J/\mu$. Then $\mu_x * \mu_y = \mu_{x*y} \in J/\mu$. So $J/\mu = \{\mu_x | x \in J\}$ is a subalgebra of X/μ. For any $x * y, a * b \in J$, $(x * a) * (y * b) \in J$. For any $\mu_x * \mu_y, \mu_a * \mu_b \in J/\mu$, we have

$$\begin{aligned}(\mu_x * \mu_a) * (\mu_y * \mu_b) &= \mu_{x*a} * \mu_{y*b}\\ &= \mu_{(x*a)*(y*b)} \in J/\mu.\end{aligned}$$

Hence J/μ is a normal subalgebra of X/μ. □

Theorem 3.23. *If J^* is a normal subalgebra of X/μ, then there exists a normal subalgebra $J = \cup\{x \in X | \mu_x \in J^*\}$ in X such that $J/\mu = J^*$.*

Proof. Since J^* is a normal subalgebra of X/μ, we have $\mu_x * \mu_y = \mu_{x*y} \in J^*$ for any $\mu_x, \mu_y \in J^*$. Hence $x * y \in J$ for any $x, y \in J$. And $\mu_{x*a} * \mu_{y*b} = \mu_{(x*a)*(y*b)} \in J^*$ for any $\mu_{x*y}, \mu_{a*b} \in J^*$. Therefore $(x * a) * (y * b) \in J$ for any $x * y, a * b \in J$. Thus J is a normal subalgebra of X. By Theorem 3.20,

$$\begin{aligned} J/\mu =& \{\mu_j | j \in J\} \\ =& \{\mu_j | \exists \mu_x \in J^* \text{ such that } j \sim^{\mu} x\} \\ =& \{\mu_j | \exists \mu_x \in J^* \text{ such that } \mu_x = \mu_j\} \\ =& \{\mu_j | \mu_j \in J^*\} = J^*. \end{aligned}$$

This completes the proof. □

4. Quotient BI-algebras induced by fuzzy congruence relations

Definition 4.1. [10] A binary operation θ from $X \times X \to [0, 1]$ is a *fuzzy equivalence relation* on X if for all $x, y, z, u \in X$

(FC1) $\theta(x, x) = \sup\{\theta(y, z) | y, z \in X\} = \theta(0, 0)$,
(FC2) $\theta(x, y) = \theta(y, x)$,
(FC3) $\theta(x, z) \geq \min\{\theta(x, y), \theta(y, z)\}$.

Moreover, if it satisfies

(FC4) $\theta(x * u, y * u) \geq \theta(x, y), \theta(u * x, u * y) \geq \theta(x, y)$

for all $x, y, u \in X$, we say that θ is a *fuzzy congruence relation* on $(X, *, 0)$.

Let $FCo(X)$ be the set of all fuzzy congruence relations on a BI-algebra X.

Lemma 4.2. *If θ satisfies the condition (FC2) $\sim$ (FC4) above, then (FC1) is equivalent to $\theta(0, 0) \geq \theta(x, y)$ for all $x, y \in X$.*

Proof. Suppose that $\theta(0, 0) = \theta(x, x)$. By (FC2) and (FC3), we have $\theta(0, 0) = \theta(x, x) \geq \min\{\theta(x, y), \theta(y, x)\} = \theta(x, y)$ for all $x, y \in X$.

Conversely, assume that $\theta(0, 0) \geq \theta(x, y)$ for all $x, y \in X$. It follows from (FC4) that $\theta(0, 0) \leq \theta(x * 0, x * 0) = \theta(x, x)$ By assumption, we have $\theta(0, 0) = \theta(x, x)$. Hence (FC1) holds. □

Proposition 4.3. *Let θ be a fuzzy congruence relation on a BI-algebra X. Then $\theta(x, y) = \theta(x * y, 0)$ for all $x, y \in X$.*

Proof. By (FC4) and Lemma 4.2, we have $\min\{\theta(x, y), \theta(y, y)\} = \min\{\theta(x, y), \theta(0, 0)\} = \theta(x, y) \leq \theta(x * y, y * y) = \theta(x * y, 0)$ for all $x, y \in X$. On the other hand, $\theta(x * y, 0) = \theta(x * y, x * x) \geq \theta(y, x)$. Hence $\theta(x, y) = \theta(x * y, 0)$ □

For every element $x \in X$, we define $\theta_x := \{y \in X | \theta(x, y) = \theta(0, 0)\}$ of X and $X/\theta := \{\theta_x | x \in X\}$. Define an operation "$\bullet$" on the set X/θ by

$$\theta_x \bullet \theta_y := \theta_{x*y}.$$

This operation is well defined. In fact, if $\theta_x = \theta_{x'}$ and $\theta_y = \theta_{y'}$, then we have $\theta(x, x') = \theta(y, y') = \theta(0, 0)$. Since $\theta(0, 0) = \min\{\theta(x, x'), \theta(y, y')\} \leq \theta(x * y, x' * y') \leq \theta(0, 0)$, we have $\theta(x * y, x' * y') = \theta(0, 0)$ and so $\theta_{x*y} = \theta_{x'*y'}$. Hence $\bullet$ is well defined.

Theorem 4.4. *If $\theta \in FCo(X)$, where X is a BI-algebra, then $(X/\theta, \bullet, \theta_0)$ is a BI-algebra.*

Proof. Straightforward. □

Proposition 4.5. *Let $f : X \to Y$ be a homomorphism of BI-algebras. If θ is a fuzzy congruence relation of Y, then $\bar{\theta}(x, y) := \theta(f(x), f(y))$ is a fuzzy congruence relation of X.*

Proof. It is obvious that $\bar{\theta}$ is well-defined. Let $x, y, z, u \in X$. Then
(i) $\bar{\theta}(x, x) = \theta(f(x), f(x)) = \theta(0, 0)$.
(ii) $\bar{\theta}(x, y) = \theta(f(x), f(y)) = \theta(f(y), f(x)) = \bar{\theta}(y, x)$.
(iii) It can be shown that $\bar{\theta}(x, z) = \theta(f(x), f(z)) \geq \min\{\theta(f(x), f(y)), \theta(f(y), f(z))\} = \min\{\bar{\theta}(x, y), \bar{\theta}(y, z)\}$.
(iv) It can be shown that $\bar{\theta}(x*u, y*u) = \theta(f(x*u), f(y*u)) = \theta(f(x)*f(u), f(y)*f(u)) \geq \theta(f(x), f(y)) = \bar{\theta}(x, y)$. By a similar way, we have $\bar{\theta}(u * x, u * y) \geq \bar{\theta}(x, y)$. Thus $\bar{\theta}$ is a fuzzy congruence relation. □

Proposition 4.6. *Let θ be a fuzzy congruence relation of a BI-algebra X. Then the mapping $\gamma : X \to X/\theta$, given by $\gamma(x) := \theta_x$, is a surjective homomorphism.*

Proof. Let $\theta_x \in X/\theta$. Then there exists an element $x_0 \in \theta_x$ such that $\gamma(x_0) = \theta_x$. Hence γ is surjective. For any $x, y \in X$, $\gamma(x * y) = \theta_{x*y} = \theta_x \bullet \theta_y = \gamma(x) \bullet \gamma(y)$. Thus γ is a homomorphism. □

Theorem 4.7. *Let $f : (X, *, 0_X) \to (Y, *, 0_Y)$ be an epimorphism of BI_1-algebras and let θ be a fuzzy congruence relation of Y. If $\bar{\theta} = \theta \circ f$, then the quotient algebra $X/\bar{\theta} := (X/(\theta \circ f), \bullet_X, \bar{\theta}_{0_X})$ is isomorphic to the quotient algebra $Y/\theta := (Y/\theta, \bullet_Y, \theta_{0_Y})$.*

Proof. By Theorem 4.4 and Proposition 4.5, $X/(\theta \circ f) := (X/(\theta \circ f), \bullet_X, \bar{\theta}_{0_X})$ is a BI-algebra and $Y/\theta := (Y/\theta, \bullet_Y, \theta_{0_Y})$ is a BI-algebra. Define a map

$$\eta : X/(\theta \circ f) \to Y/\theta,\ (\theta \circ f)_x \mapsto \theta_{f(x)}$$

for all $x \in X$. Then the function η is well-defined. In fact, assume that $(\theta \circ f)_x = (\theta \circ f)_y$ for all $x, y \in X$. Then we have $\theta(f(x) *_Y f(y)) = \theta(f(x *_X y)) = (\theta \circ f)(x *_X y) = (\theta \circ f)(0_X) = \theta(f(0_X)) = \theta(0_Y)$ and $\theta(f(y) *_Y f(x)) = \theta(f(y *_X x)) = (\theta \circ f)(y *_X x) = (\theta \circ f)(0_X) = \theta(f(0_X)) = \theta(0_Y)$. Hence $\theta_{f(x)} = \theta_{f(y)}$.

For any $(\theta \circ f)_x, (\theta \circ f)_y \in X/(\theta \circ f)$, we have $\eta((\theta \circ f)_x \bullet_X (\theta \circ f)_y) = \eta((\theta \circ f)_{x*y}) = \theta_{f(x*_X y)} = \theta_{f(x)*_Y f(y)} = \theta_{f(x)} \bullet \theta_{f(y)} = \eta((\theta \circ f_x)) \bullet_Y \eta((\theta \circ f)_y)$. Therefore η is a homomorphism.

Let $\theta_a \in Y/\theta$. Then there exists $x \in X$ such that $f(x) = a$, since f is surjective. Hence $\eta((\theta \circ f)_x) = \theta_{f(x)} = \theta_a$ and so η is surjective.

Let $x, y \in X$ be such that $\theta_{f(x)} = \theta_{f(y)}$. Then we have $(\theta \circ f)(x *_X y) = \theta(f(x *_X y)) = \theta(f(x) *_Y f(y)) = \theta(0_Y) = \theta(f(0_X)) = (\theta \circ f)(0_X)$ and $(\theta \circ f)(y *_X x) = \theta(f(y *_X x)) = \theta(f(y) *_Y f(x)) = \theta(0_Y) = \theta(f(0_X)) = (\theta \circ f)(0_X)$. It follows that $(\theta \circ f)_x = (\theta \circ f)_y$. Thus η is injective. This completes the proof. □

The homomorphism $\pi : X \to X/\theta$, $x \to \theta_x$, is called the *natural homomorphism* of X onto X/θ. In Theorem 4.7, if we define natural homomorphisms $\pi_X : X \to X/\theta \circ f$ and $\pi_Y : Y \to Y/\theta$, then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ f$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \pi_X}\big\downarrow & & {\scriptstyle \pi_Y}\big\downarrow \\ X/(\theta \circ f) & \xrightarrow{\eta} & Y/\theta. \end{array}$$

The fuzzy subset θ_x of a BI-algebra X, which is defined by $\theta_x(y) := \theta(x, y)$, is called the fuzzy congruence class containing x and X/θ is the set of all fuzzy congruences classes θ_x.

Proposition 4.8. *Let θ be a fuzzy congruence relation in a BI-algebra X. Then θ_0 is a fuzzy ideal of X.*

Proof. Let $x, y \in X$. Then $\theta_0(0) = \theta(0, 0) \geq \theta(x, y)$ by Lemma 4.2. Put $y := 0$ in above inequality. Then $\theta_0(0) \geq \theta(x, 0) = \theta_0(x)$. By (FC3), (FC2) and Proposition 5.3, we have $\theta_0(y) = \theta(0, y) \geq \min\{\theta(0, x), \theta(x, y)\} = \min\{\theta(x, 0), \theta(x * y, 0)\} = \min\{\theta_0(x), \theta_0(x * y)\}$. Thus θ_0 is a fuzzy ideal of X. □

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General quadratic functional equations in quasi-β-normed spaces: solution, superstability and stability

Shahrokh Farhadabadi[1*], Choonkil Park[2*] and Sungsik Yun[3*]

[1]Young Researchers and Elite Club, Parand Brunch, Islamic Azad University, Parand, Iran
[2]Research Institute for Natural Sciences Hanyang University, Seoul 04763, Korea
[3]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
e-mail: shahrokh_math@yahoo.com; baak@hanyang.ac.kr; ssyun@hs.ac.kr

Abstract. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping where $\mathcal{X}$ is a quasi-α-normed space and $\mathcal{Y}$ is a quasi-β-normed space. The following quadratic functional equation

$$\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} x_j + \frac{2-n}{2}x_i\Big) = \frac{n^2}{4}\sum_{i=1}^{n} f(x_i), \qquad (n \geq 3) \tag{0.1}$$

is introduced and solved by giving its general solution.

Moreover, we prove the Hyers-Ulam stability of the functional equation (0.1) by using a direct method.

1. Introduction and preliminaries

Studying functional equations by focusing on their approximate and exact solutions conduces to one of the most substantial significant study brunches in functional equations, what we call *"the theory of stability of functional equations"*. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable* if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable.*

In 1940, the most preliminary form of stability problems was proposed by Ulam [35]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [13] for the Cauchy's functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţa [12] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function for the existence of a unique linear mapping. For more epochal information and various aspects about the stability of functional equations theory, we refer the reader to the monographs [10, 11, 14, 15, 18, 20, 25, 27, 29, 30, 31, 32, 33], which also include many interesting results concerning the stability of different functional equations in many various spaces.

Now we present some brief explanations about the functional equation (0.1) and also generally about quadratic functional equations. Consider the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

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*Corresponding authors.

which is called *the classic quadratic functional equation.* Obviously, the function $f(x) = cx^2$ is its solution and so it is called quadratic. There are some other different types of quadratic functional equations. For examples, the following n-dimensional quadratic functional equations

$$\sum_{k=2}^{n}\Big[\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f\Big(\sum_{\substack{i=1\\ i\neq i_1,\cdots,i_{n-k+1}}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\Big)\Big] + f\Big(\sum_{i=1}^{n} x_i\Big) = 2^{n-1}\sum_{i=1}^{n} f(x_i),$$

$$f\Big(\sum_{i=1}^{n} x_i\Big) + \sum_{1\le i<j\le n} f(x_i - x_j) = n\sum_{i=1}^{n} f(x_i)$$

have been introduced in [9] and [3], respectively. These n-dimensional versions are generalized forms of (1.1), but each in a different way.

In this paper, we introduce another n-dimensional version as follows:

$$\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} x_j + \frac{2-n}{2}x_i\Big) = \frac{n^2}{4}\sum_{i=1}^{n} f(x_i) \qquad (n \ge 3), \tag{1.2}$$

in which the simplest case (for $n = 3$) is the functional equation

$$f\left(x + y - \frac{z}{2}\right) + f\left(x + z - \frac{y}{2}\right) + f\left(y + z - \frac{x}{2}\right) = \frac{9}{4}\Big[f(x) + f(y) + f(z)\Big]. \tag{1.3}$$

Note that (1.2), for each fixed integer $n \ge 3$, is symmetric with respect to any permutation of the variables.

In the next section, we will show that (1.2) is equivalent to (1.1). Nevertheless (1.2) is not a generalization of (1.1), rather in fact it is a generalized form of (1.3).

The stability problem for the classic quadratic functional equation was first proved by F. Skof [34] and then generalized by Cholewa [6], Czerwik [7, 8] and others [2, 4, 22, 23, 25, 26]. Many stability problems for some other versions can be found in [3, 5, 16, 17, 19, 21].

Now we give briefly some useful definitions, preliminary and fundamental results of quasi-β-normed spaces. Throughout this paper β will be a fixed real number with $0 < \beta \le 1$ and $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. ([28]) Let $\mathcal{X}$ be a linear space over $\mathbb{K}$. A *quasi-β-norm* $\|\cdot\|$ is a real-valued function on $\mathcal{X}$ satisfying the following conditions:

($\mathcal{C}_1$) $\|x\| \ge 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if only if $x = 0$;

($\mathcal{C}_2$) $\|\lambda \cdot x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in \mathcal{X}$;

($\mathcal{C}_3$) There is a constant $\mathcal{K} \ge 1$ such that $\|x + y\| \le \mathcal{K}(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-β-normed space* and the smallest possible $\mathcal{K}$ is called the modulus of concavity of $\|\cdot\|$.

A complete quasi-β-normed space is a *quasi-β-Banach space.*

Definition 1.2. ([28]) Let $0 < p \le 1$ be a real number. A quasi-β-normed space $(\mathcal{X}, \|\cdot\|)$ is called a *(β, p)-normed space* if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

2. The general solution

In this section, we give the general solution of the functional equation (0.1) by proving the fact that it is equivalent to the functional equation (1.1), which implies that it is quadratic too.

First, we prove a useful lemma.

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathcal{X}$, then

$$f(rx) = r^2 f(x)$$

for all $x \in \mathcal{X}$ and all $r \in \mathbb{Q}$.

Proof. Let $n \geq 2$ be a natural number. Replacing (x, y) by $(0, 0), (0, -x)$ and $(x, (n-1)x)$ respectively, we get $f(0) = 0, f(x) = f(-x)$ and

$$f(nx) = 2f(x) + 2f((n-1)x) - f((n-2)x)$$

for all $x \in \mathcal{X}$. This simply implies that

$$\begin{aligned} f(-2x) = f(2x) &= 2f(x) + 2f(x) - f(0) = 4f(x), \\ f(-3x) = f(3x) &= 2f(x) + 2f(2x) - f(x) = 9f(x), \\ f(-4x) = f(4x) &= 2f(x) + 2f(3x) - f(2x) = 16f(x), \\ &\vdots \\ f(kx) &= k^2 f(x) \end{aligned}$$

for all $x \in \mathcal{X}$ and all $k \in \mathbb{Z}$. Putting $\frac{x}{k}$ instead of x in the above line, we obtain

$$f\left(\frac{x}{k}\right) = \frac{1}{k^2} f(x)$$

for all $x \in \mathcal{X}$ and all $k \in \mathbb{Z}$. Therefore, we can conclude for any $r \in \mathbb{Q}$ that

$$f(rx) = f\left(\frac{m}{n}x\right) = m^2 f\left(\frac{x}{n}\right) = \frac{m^2}{n^2} f(x) = r^2 f(x)$$

for all $x \in \mathcal{X}$ and all $r \in \mathbb{Q}$, which ends the proof. □

Theorem 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $n \geq 3$ be a fixed positive integer. A mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation*

$$\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j + \frac{2-n}{2} x_i\Big) = \frac{n^2}{4} \sum_{i=1}^{n} f(x_i) \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ if and only if f satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.2}$$

for all $x, y \in \mathcal{X}$.

Proof. Sufficiency. For the *'only if'* part of the proof, suppose that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (2.1), we will show that f satisfies the classic quadratic functional equation (2.2).

First we except the cases $n = 3, 4$ and investigate them separately. In the case $n = 3$, (2.1) is in the form

$$f\left(x + y - \frac{z}{2}\right) + f\left(x + z - \frac{y}{2}\right) + f\left(y + z - \frac{x}{2}\right) = \frac{9}{4}\Big[f(x) + f(y) + f(z)\Big] \tag{2.3}$$

for all $x, y, z \in \mathcal{X}$. Replacing (x, y, z) in (2.3), by $(0, 0, 0)$, $(x, 0, 0)$, $(x, x, 0)$, $(\frac{2}{3}x, \frac{2}{3}x, \frac{2}{3}x)$ and $\left(\frac{2}{3}x, -\frac{2}{3}x, y\right)$, respectively, we obtain $f(0) = 0$ and

$$f\Big(\frac{-1}{2}x\Big) = \frac{1}{4} f(x), \tag{2.4}$$

$$f\Big(\frac{1}{2}x\Big) = \frac{9}{4} f(x) - \frac{1}{2} f(2x), \tag{2.5}$$

$$f\Big(\frac{2}{3}x\Big) = \frac{4}{9} f(x), \tag{2.6}$$

$$f\Big(\frac{-1}{2}y\Big) + f(x+y) + f(y-x) = \frac{9}{4}\Big[f\Big(\frac{2}{3}x\Big) + f\Big(\frac{-2}{3}x\Big) + f(y)\Big] \tag{2.7}$$

for all $x, y \in \mathcal{X}$, respectively. By using (2.4) and (2.6), we can rewrite (2.7) as

$$f(x+y)+f(y-x)=f(x)+f(-x)+2f(y) \tag{2.8}$$

for all $x, y \in \mathcal{X}$. Letting $x=y$ in (2.8), we see that

$$f(2x)=3f(x)+f(-x) \tag{2.9}$$

for all $x \in \mathcal{X}$. From (2.4), (2.5) and (2.9), it follows that

$$\frac{1}{4}f(-x)=f\Big(\frac{1}{2}x\Big)=\frac{9}{4}f(x)-\frac{1}{2}f(2x)=\frac{9}{4}f(x)-\frac{3}{2}f(x)-\frac{1}{2}f(-x)$$

for all $x \in \mathcal{X}$, which implies that $f(x)=f(-x)$ for all $x \in \mathcal{X}$. So (2.8) can be rewritten as

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in \mathcal{X}$, which is exactly (2.2).

Now the case $n=4$. In this case we have the functional equation

$$\begin{aligned} &f(x+y+z-w)+f(x+y+w-z)+f(x+z+w-y) \\ &\qquad +f(y+z+w-x)=4\big[f(x)+f(y)+f(z)+f(w)\big] \end{aligned} \tag{2.10}$$

for all $x, y, z, w \in \mathcal{X}$. Replacing (x, y, z, w) in (2.10), by $(0,0,0,0)$, $(x,0,0,0)$, $(\frac{1}{2}x, \frac{1}{2}x, 0, 0)$ and $\Big(\frac{1}{2}x, -\frac{1}{2}x, y, 0\Big)$, respectively, we get $f(0)=0$, $f(x)=f(-x)$, $f(\frac{1}{2}x)=\frac{1}{4}f(x)$ and

$$f(-y)+f(y)+f(x+y)+f(y-x)=4\Big[f\Big(\frac{1}{2}x\Big)+f\Big(\frac{-1}{2}x\Big)+f(y)\Big]$$

for all $x, y \in \mathcal{X}$, respectively, which can easily be simplified to (2.2).

Now we assume that $n \geq 5$.

Replacing the variables in (2.1), by $(0, \cdots, 0)$, $(x, 0, \cdots, 0)$, $(x, x, 0, \cdots, 0)$, $(-x, \cdots, -x, 0, 0)$ and $(\frac{2}{n}x, \cdots, \frac{2}{n}x)$, respectively, we have $f(0)=0$ and

$$\begin{aligned} f\Big(\frac{2-n}{2}x\Big) &= \frac{(n-2)^2}{4}f(x), &(2.11)\\ f\Big(\frac{4-n}{2}x\Big) &= \frac{n^2}{4}f(x)+\frac{2-n}{2}f(2x), &(2.12)\\ f\Big(\frac{4-n}{2}x\Big) &= \frac{n^2}{4}f(-x)+\frac{2}{2-n}f\Big((2-n)x\Big), &(2.13)\\ f\Big(\frac{2}{n}x\Big) &= \frac{4}{n^2}f(x) &(2.14) \end{aligned}$$

for all $x \in \mathcal{X}$, respectively. Note that $n \geq 5$ might be either an odd or an even number. In the cases of oddness and evenness, if we respectively put

$$\begin{aligned} \Big(x_1, \cdots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}, \cdots, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y\Big),\\ \Big(x_1, \cdots, x_{\frac{n-2}{2}}, x_{\frac{n}{2}}, \cdots, x_{n-2}, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y, 0\Big) \end{aligned}$$

in (2.1), then we get

$$\begin{aligned} f\Big(\frac{2-n}{2}y\Big)+\frac{n-1}{2}\Big[f(x+y)+f(y-x)\Big] &= \frac{n^2(n-1)}{8}\Big[f\Big(\frac{2}{n}x\Big)+f\Big(\frac{-2}{n}x\Big)\Big]+\frac{n^2}{4}f(y),\\ f\Big(\frac{2-n}{2}y\Big)+\frac{n-2}{2}\Big[f(x+y)+f(y-x)\Big] &= \frac{n^2(n-2)}{8}\Big[f\Big(\frac{2}{n}x\Big)+f\Big(\frac{-2}{n}x\Big)\Big]+\frac{n^2-4}{4}f(y) \end{aligned}$$

for all $x, y \in \mathcal{X}$, which both by using (2.11) and (2.14), are easily simplified to (2.8). From (2.8), we obtain (2.9) again. By using (2.9), we can rewrite (2.12) as

$$f\Big(\frac{4-n}{2}x\Big)=\frac{n^2-6n+12}{4}f(x)+\frac{2-n}{2}f(-x) \tag{2.15}$$

for all $x \in \mathcal{X}$. Subtracting (2.13) from (2.15), we get

$$f\Big((2-n)x\Big) = \frac{(2-n)(n^2-6n+12)}{8}f(x) + \frac{(n-2)(n^2+2n-4)}{8}f(-x)$$

for all $x \in \mathcal{X}$. Putting $\frac{x}{2}$ instead of x, and then applying (2.11), we obtain

$$\begin{aligned} \frac{(n-2)^2}{4}f(x) &= \frac{(2-n)(n^2-6n+12)}{8}f\Big(\frac{x}{2}\Big) + \frac{(n-2)(n^2+2n-4)}{8}f\Big(\frac{-x}{2}\Big), \\ f(x) &= \frac{-n^2+6n-12}{2(n-2)}f\Big(\frac{x}{2}\Big) + \frac{n^2+2n-4}{2(n-2)}f\Big(\frac{-x}{2}\Big) \qquad (2.16) \end{aligned}$$

for all $x \in \mathcal{X}$. On the other hand from (2.9), we have

$$f(x) = 3f\Big(\frac{x}{2}\Big) + f\Big(\frac{-x}{2}\Big) \qquad (2.17)$$

for all $x \in \mathcal{X}$. Comparing (2.16) and (2.17), we conclude that $f(x) = f(-x)$ for all $x \in \mathcal{X}$, which simply transforms the form of (2.8) to (2.2).

Necessity. For the *'if'* part of the proof, suppose that $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (2.2). We show that f satisfies (2.1) too.

First, we prove the following

$$\begin{aligned} f\Big(x_2+\cdots+x_n+\frac{2-n}{2}x_1\Big) &= \frac{n}{2}f(x_2+\cdots+x_n) + \Big(\frac{2-n}{2}\Big)f(x_1+\cdots+x_n) \\ &+ \frac{n}{2}\Big(\frac{n}{2}-1\Big)f(x_1) \qquad (2.18) \end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed integer $n \geq 3$.

Let $k \in \mathbb{N}$. Replacing (x, y) in (2.2) by (x, kx_1) respectively, we get

$$\begin{aligned} f(x-x_1) &= 2f(x)+2f(x_1)-f(x+x_1), \qquad (2.19) \\ f(x-2x_1) &= 2f(x)+2f(2x_1)-f(x+2x_1), \qquad (2.20) \\ f(x-3x_1) &= 2f(x)+2f(3x_1)-f(x+3x_1), \qquad (2.21) \\ f(x-4x_1) &= 2f(x)+2f(4x_1)-f(x+4x_1) \qquad (2.22) \\ &\vdots \end{aligned}$$

for all $x, x_1, \in \mathcal{X}$. Replacing (x, y) in (2.2) by (x_1+x, kx_1), respectively, we get

$$\begin{aligned} f(x+2x_1) = f(x_1+x+x_1) &= 2f(x_1+x)+2f(x_1)-f(x), \\ f(x+3x_1) = f(x_1+x+2x_1) &= 2f(x_1+x)+2f(2x_1)-f(x-x_1), \\ f(x+4x_1) = f(x_1+x+3x_1) &= 2f(x_1+x)+2f(3x_1)-f(x-2x_1), \\ &\vdots \end{aligned}$$

for all $x, x_1, \in \mathcal{X}$. Continuous process of the above equations (2.20), (2.21), $\cdots$ and Lemma 2.1 generally lead to

$$f(x-kx_1) = (k+1)f(x)+k(k+1)f(x_1)-kf(x+x_1) \qquad (2.23)$$

for all $x, x_1 \in \mathcal{X}$, and all $k \in \mathbb{N}$. Replacing (x, y) in (2.2), by $\big(x-\frac{k}{2}x_1, \frac{k}{2}x_1\big)$, using (2.23) and Lemma 2.1, we obtain

$$\begin{aligned} f\Big(x-\frac{k}{2}x_1\Big) &= \frac{1}{2}f(x)+\frac{1}{2}f(x-kx_1)-f\Big(\frac{k}{2}x_1\Big) \\ &= \frac{k+2}{2}f(x)+\frac{k^2+2k}{4}f(x_1)-\frac{k}{2}f(x+x_1) \qquad (2.24) \end{aligned}$$

for all $x, x_1 \in \mathcal{X}$, and all $k \in \mathbb{N}$.

Now (2.23) and (2.24) imply that (2.18) holds. The first one proves it for any fixed even integer $n_e \geq 4$, and the second one proves it for any fixed odd integer $n_o \geq 3$. It is done simply by putting $x = x_2 + \cdots + x_n$ in both (2.23) and (2.24) and by $k = \frac{n_e - 2}{2}$, $k = n_o - 2$ in (2.23) and (2.24), respectively.

It follows from (2.18) that

$$\begin{array}{c} f\Big(x_2 + \cdots + x_n + \frac{2-n}{2}x_1\Big) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-1} + \frac{2-n}{2}x_n\Big) \end{array} = \frac{n}{2}\begin{bmatrix} f(x_2 + \cdots + x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-1}\Big) \end{bmatrix} + \frac{n}{2}\Big(\frac{n}{2} - 1\Big)\begin{bmatrix} f(x_1) \\ + \\ \vdots \\ + \\ f(x_n) \end{bmatrix} + \frac{n}{2}(2-n)\Big[f(x_1 + \cdots + x_n)\Big]$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed integer $n \geq 3$. This signifies that in order to get (2.1), it is just necessary to show the following

$$\begin{bmatrix} f(x_2 + \cdots + x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-1}\Big) \end{bmatrix} + (2-n)\Big[f(x_1 + \cdots + x_n)\Big] = \begin{bmatrix} f(x_1) \\ + \\ \vdots \\ + \\ f(x_n) \end{bmatrix} \tag{2.25}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed integer $n \geq 3$.

As it is clear, the proof of (2.25) directly depends on the specific value of $n \geq 3$. Nevertheless, we try to provide a general idea to prove it.

First assume that $n \geq 3$ is an odd number. In this case, by frequently using (2.2), the left hand side of (2.25) will be in the form

$$\frac{1}{2}\begin{bmatrix} f(x_1 + x_2 + 2x_3 + \cdots + 2x_n) \\ + \\ \vdots \\ + \\ f\Big(2x_1 + \cdots + 2x_{n-3} + x_{n-2} + x_{n-1} + 2x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1 - x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2} - x_{n-1}) \end{bmatrix} + f\Big(x_1 + \cdots + x_{n-1}\Big) + (2-n)\Big[f(x_1 + \cdots + x_n)\Big] \tag{2.26}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed odd number $n \geq 3$. Since we have

$$\frac{1}{2}f(x_1 + x_2 + 2x_3 + \cdots + 2x_n) = f(x_1 + \cdots + x_n) + f(x_3 + \cdots + x_n) - \frac{1}{2}f(x_1 + x_2)$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ from (2.2), (2.26) is simplified to

$$\begin{bmatrix} f(x_3 + \cdots + x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-3} + x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1 - x_2) - f(x_1 + x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2} - x_{n-1}) - f(x_{n-2} + x_{n-1}) \end{bmatrix} + f(x_1 + \cdots + x_{n-1}) + \Big(\frac{3-n}{2}\Big)\Big[f(x_1 + \cdots + x_n)\Big] \tag{2.27}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any odd fixed integer $n \geq 3$. For the case $n = 3$, (2.27) leads to the right hand side of (2.25), which means that the proof is complete for the case $n = 3$. So we assume that $n \geq 5$ and continue the proof. We come across two cases:

a) $\frac{n-1}{2}$, which is the number of the terms in the first term of (2.27), is an even integer;

b) $\frac{n-1}{2}$ is an odd integer.

In the case *a*), similar to the process in which we obtain (2.27) from (2.25), we get (2.28) from (2.27), as follows:

$$\begin{bmatrix} f(x_5+\cdots+x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1+\cdots+x_{n-5}+x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2}-x_{n-1})-f(x_{n-2}+x_{n-1}) \end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix} f(x_1+x_2-x_3-x_4)-f(x_1+\cdots+x_4) \\ + \\ \vdots \\ + \\ f(x_{n-4}+x_{n-3}-x_{n-2}-x_{n-1})-f(x_{n-4}+\cdots+x_{n-1}) \end{bmatrix}$$

$$+f(x_1+\cdots+x_{n-1})+\Big(\frac{5-n}{4}\Big)\Big[f(x_1+\cdots+x_n)\Big] \tag{2.28}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed $n = 5, 9, 13, \cdots$. In the case $n = 5$, (2.28) is in the form

$$f(x_5)+\frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ f(x_3-x_4)-f(x_3+x_4) \end{bmatrix} + \frac{1}{2}\Big[f(x_1+x_2-x_3-x_4)+f(x_1+\cdots+x_4)\Big]$$

for all $x_1, \cdots, x_5 \in \mathcal{X}$, which simply by using (2.2) gives the right hand side of (2.25). By continuing the process we can obtain the result for $n = 9, 13, \cdots$.

Similarly in the case *b*), we get (2.29) from (2.27), as follows:

$$\begin{bmatrix} f(x_5+\cdots+x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1+\cdots+x_{n-7}+x_{n-2}+x_{n-1}+x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2}-x_{n-1})-f(x_{n-2}+x_{n-1}) \end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix} f(x_1+x_2-x_3-x_4)-f(x_1+\cdots+x_4) \\ + \\ \vdots \\ + \\ f(x_{n-6}+x_{n-5}-x_{n-4}-x_{n-3})-f(x_{n-6}+\cdots+x_{n-3}) \end{bmatrix}$$

$$+f(x_1+\cdots+x_{n-3}+x_n)+f(x_1+\cdots+x_{n-1})+\Big(\frac{3-n}{4}\Big)\Big[f(x_1+\cdots+x_n)\Big] \tag{2.29}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed $n = 7, 11, 15, \cdots$. If we put $n = 7$, in (2.29), then we have

$$f(x_5+x_6+x_7)+\frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ f(x_3-x_4)-f(x_3+x_4) \\ + \\ f(x_5-x_6)-f(x_5+x_6) \end{bmatrix} + \frac{1}{2}f(x_1+x_2-x_3-x_4)$$

$$-\frac{1}{2}f(x_1+\cdots+x_4)+f(x_1+\cdots+x_4+x_7)+f(x_1+\cdots+x_6)-f(x_1+\cdots+x_7)$$

for all $x_1, \cdots, x_7 \in \mathcal{X}$, which could be simplified to the right hand side of (2.25). For $n = 11, 15, \cdots$, we should continue the process.

Even cases of n are similar and also easier and so we omit them and the proof is complete. □

3. Superstability of the general quadratic functional equation (0.1)

In this section, we provide a superstability theorem for the functional equation (0.1). In fact, $f : \mathcal{X} \to \mathcal{Y}$ will be put in a normed functional inequality instead of an equality which is obviously considered a harder condition for f, in order to be quadratic. From this point of view, one can say that the obtained result in the previous section will be gotten stronger and improved in this section.

Theorem 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and $\frac{1}{6} < l < 1$ be a fixed real number. If $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional inequality*

$$\begin{aligned} \Big\| f\Big(y+z-\frac{1}{2}x\Big) + f\Big(x+z-\frac{1}{2}y\Big) + f\Big(x+y-\frac{1}{2}z\Big) \\ -\frac{9}{4}f(y) - \frac{9}{4}f(z) - l\frac{9}{4}f(x)\Big\| \quad \leq \quad \Big\|(1-l)\frac{9}{4}f(x)\Big\| \end{aligned} \tag{3.1}$$

for all $x, y, z \in \mathcal{X}$, then f is a quadratic mapping.

Proof. Letting $(x, y, z) = (0, 0, 0)$ in (3.1), we get

$$\frac{18l-3}{4}\big\|f(0)\big\| \leq 0.$$

Since $l > \frac{1}{6}$, $\frac{18l-3}{4} > 0$ and so $f(0) = 0$. Letting $(x, y, z) = (0, x, y)$ in (3.1), we have

$$f\Big(x-\frac{1}{2}y\Big) + f\Big(y-\frac{1}{2}x\Big) + f(x+y) = \frac{9}{4}f(x) + \frac{9}{4}f(y) \tag{3.2}$$

for all $x, y \in \mathcal{X}$. Replacing (x, y) in (3.2) by $(x, 0)$, (x, x) and $(x, 2x)$, respectively, we obtain

$$f\Big(-\frac{1}{2}x\Big) = \frac{1}{4}f(x), \tag{3.3}$$

$$2f\Big(\frac{1}{2}x\Big) + f(2x) = \frac{9}{2}f(x), \tag{3.4}$$

$$f\Big(\frac{3}{2}x\Big) + f(3x) = \frac{9}{4}f(x) + \frac{9}{4}f(2x) \tag{3.5}$$

for all $x \in \mathcal{X}$, respectively. Using (3.3) and (3.4) we get $f(x) = f(-x)$ for all $x \in \mathcal{X}$. So (3.3) is rewritten as $f\big(\frac{1}{2}x\big) = \frac{1}{4}f(x)$ for all $x \in \mathcal{X}$. By using this, (3.5) could be simplified to $f(3x) = 9f(x)$, and so

$$f\Big(\frac{1}{3}x\Big) = \frac{1}{9}f(x) \tag{3.6}$$

for all $x \in \mathcal{X}$. Replacing (x, y) in (3.2) by $(x-y, -y)$ and $(y-x, -x)$, we have

$$\begin{aligned} f\Big(x-\frac{1}{2}y\Big) &= \frac{9}{4}f(x-y) + \frac{9}{4}f(y) - \frac{1}{4}f(x+y) - f(x-2y), \\ f\Big(y-\frac{1}{2}x\Big) &= \frac{9}{4}f(x-y) + \frac{9}{4}f(x) - \frac{1}{4}f(x+y) - f(y-2x) \end{aligned}$$

for all $x, y \in \mathcal{X}$. By putting these two equations in (3.2), we obtain

$$\frac{9}{2}f(x-y) + \frac{1}{2}f(x+y) = f(x-2y) + f(y-2x)$$

for all $x, y \in \mathcal{X}$. Now putting $(x, y) = \big(\frac{u+2v}{-3}, \frac{v+2u}{-3}\big)$ in the previous line, we get

$$\frac{9}{2}f\Big(\frac{u-v}{3}\Big) + \frac{1}{2}f(u+v) = f(u) + f(v)$$

for all $u, v \in \mathcal{X}$, which by (3.6) simply leads to (2.2) as desired. □

Theorem 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and n, k be fixed positive integers with $n \geq 4$ and $1 \leq k \leq n$. If $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional inequality*

$$\Big\|\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j + \frac{2-n}{2}x_i\Big) - \frac{n^2}{4}\sum_{\substack{i=1 \\ i\neq k}}^{n} f(x_i)\Big\| \leq \Big\|\frac{n^2}{4}f(x_k)\Big\| \tag{3.7}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, then f is a quadratic mapping.

Proof. Note that the functional inequality (3.7) is symmetric with respect to each variable. So we can take $k = 1$ and only prove this case and then conclude the statement for all cases with $1 \leq k \leq n$. From now on, assume that $k = 1$.

Letting $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ in (3.7), we obtain that

$$\frac{n^3 - (2n^2 + 4n)}{4}\big\|f(0)\big\| \leq 0.$$

Since $n^3 > (2n^2 + 4n)$ for all $n \geq 4$, $f(0) = 0$. Letting $x_1 = 0$ (or $x_k = 0$) in (3.7), we get

$$\sum_{i=2}^{n} f\Big(\sum_{\substack{j=2 \\ j\neq i}}^{n} x_j + \frac{2-n}{2}x_i\Big) + f\Big(\sum_{i=2}^{n} x_i\Big) = \frac{n^2}{4}\sum_{i=2}^{n} f(x_i) \tag{3.8}$$

for all $x_2, x_3, \cdots, x_n \in \mathcal{X}$.

First we investigate the case $n = 4$ separately. In this case, by putting $(x_2, x_3, x_4) = (x, y, z)$ in (3.8), we have

$$\begin{aligned} &f\big(y+z-x\big) + f\big(x+z-y\big) + f\big(x+y-z\big) \\ &\qquad + f\big(x+y+z\big) = 4[f(x) + f(y) + f(z)] \end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Replacing (x, y, z) in the above equation by $(x, 0, 0)$ and $(x, y, 0)$, we obtain $f(x) = f(-x)$ and

$$f(y-x) + f(x-y) + 2f(x+y) = 4f(x) + 4f(y)$$

for all $x, y \in \mathcal{X}$, which simply mean that (2.2) holds.

Now the case $n \geq 5$. Replacing $(x_2, x_3, \cdots, x_n)$ in (3.8) by $= (x, 0, \cdots, 0)$, $(x, x, 0, \cdots, 0)$ and $(x, \cdots, x, 0)$, respectively, we obtain

$$f\Big(\frac{2-n}{2}x\Big) = \frac{(n-2)^2}{4}f(x), \tag{3.9}$$

$$f\Big(\frac{4-n}{2}x\Big) = \frac{n^2}{4}f(x) + \frac{2-n}{2}f(2x), \tag{3.10}$$

$$f\Big(\frac{n-4}{2}x\Big) = \frac{n^2}{4}f(x) + \frac{2}{2-n}f\Big((n-2)x\Big) \tag{3.11}$$

for all $x \in \mathcal{X}$, respectively. In the case of evenness and oddness of $n \geq 5$, we respectively put

$$\begin{aligned} \Big(x_2, \cdots, x_{\frac{n}{2}}, x_{\frac{n+2}{2}}, \cdots, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y\Big), \\ \Big(x_2, \cdots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}, \cdots, x_{n-2}, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y, 0\Big) \end{aligned}$$

in (3.8), to get

$$\begin{aligned} f\Big(\frac{2-n}{2}y\Big) + \frac{n-2}{2}\Big[f(x+y) + f(y-x)\Big] &= \frac{n^2(n-2)}{8}\Big[f\Big(\frac{2}{n}x\Big) + f\Big(\frac{-2}{n}x\Big)\Big] + \frac{n^2-4}{4}f(y), \\ f\Big(\frac{2-n}{2}y\Big) + \frac{n-3}{2}\Big[f(x+y) + f(y-x)\Big] &= \frac{n^2(n-3)}{8}\Big[f\Big(\frac{2}{n}x\Big) + f\Big(\frac{-2}{n}x\Big)\Big] + \frac{n^2-8}{4}f(y) \end{aligned}$$

for all $x, y \in \mathcal{X}$, which both by (3.9) are simplified to

$$f(x+y) + f(y-x) = \frac{n^2}{4}\Big[f\Big(\frac{2}{n}x\Big) + f\Big(\frac{-2}{n}x\Big)\Big] + 2f(y) \tag{3.12}$$

for all $x, y \in \mathcal{X}$. Letting $y = 0$ in (3.12), we have

$$f(x) + f(-x) = \frac{n^2}{4}\Big[f\Big(\frac{2}{n}x\Big) + f\Big(\frac{-2}{n}x\Big)\Big]$$

for all $x \in \mathcal{X}$. By this, (3.12) is equivalent to

$$f(x+y) + f(y-x) = f(x) + f(-x) + 2f(y) \tag{3.13}$$

for all $x, y \in \mathcal{X}$. Now by (3.10), (3.11), (3.13) and a similar argument used in the last part of the proof of Theorem 3.2, we can obtain $f(x) = f(-x)$ for all $x, y \in \mathcal{X}$. This changes (3.13) to $f(x+y) + f(y-x) = 2f(x) + 2f(y)$ for all $x, y \in \mathcal{X}$, which finally ends the proof. □

4. Hyers-Ulam stability of the general quadratic functional equation (0.1)

In this section, we prove the Hyers-Ulam stability of the functional equation (0.1). Throughout this section $\mathcal{X}$ denotes a quasi-α-normed space and $\mathcal{Y}$ a quasi-β-Banach space. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$, we define the difference operator:

$$D_\lambda f(x_1, x_2, \cdots, x_n) := \sum_{i=1}^{n} f\Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda x_j + \frac{2-n}{2}\lambda x_i\Big) - \frac{\lambda^2 n^2}{4} \sum_{i=1}^{n} f(x_i)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$.

Theorem 4.1. *Let $\varphi : \mathcal{X}^n \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$, where $n \geq 3$ is a fixed integer. Denote by ϕ a function such that*

$$\phi(x_1, x_2, \cdots, x_n) := \sum_{m=0}^{\infty} \Big[\frac{4^{m\beta}}{n^{2m\beta}} \varphi\Big(\frac{n^m}{2^m} x_1, \frac{n^m}{2^m} x_2, \cdots, \frac{n^m}{2^m} x_n\Big)\Big]^p < \infty \tag{4.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying

$$\big\|D_1 f(x_1, x_2, \cdots, x_n)\big\|_{\mathcal{Y}} \leq \varphi(x_1, x_2, \cdots, x_n) \tag{4.2}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \frac{4^\beta}{n^{3\beta}} \sqrt[p]{\phi(x, x, \cdots, x)} \tag{4.3}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x_1 = x_2 = \cdots = x_n = 0$ in (4.2), we get $f(0) = 0$.

Letting $x_1 = x_2 = \cdots = x_n = x$ in (4.2), we have

$$\begin{aligned} \Big\|nf\Big(\frac{n}{2}x\Big) - \frac{n^3}{4} f(x)\Big\|_{\mathcal{Y}} &\leq \varphi(x, x, \cdots, x), \\ \Big\|\frac{4}{n^2} f\Big(\frac{n}{2}x\Big) - f(x)\Big\|_{\mathcal{Y}} &\leq \frac{4^\beta}{n^{3\beta}} \varphi(x, x, \cdots, x) \end{aligned}$$

for all $x \in \mathcal{X}$. Replacing x by $(\frac{n}{2})^i x$, we get

$$\Big\|\frac{4}{n^2} f\Big(\frac{n^{i+1}}{2^{i+1}} x\Big) - f\Big(\frac{n^i}{2^i} x\Big)\Big\|_{\mathcal{Y}} \leq \frac{4^\beta}{n^{3\beta}} \varphi\Big(\frac{n^i}{2^i} x, \cdots, \frac{n^i}{2^i} x\Big) \tag{4.4}$$

for all $x \in \mathcal{X}$ and all nonnegative integers i. Assume that m, l are positive integers with $m > l$. From the iterative method and (4.4), it follows that

$$\begin{aligned} \Big\|\frac{4^m}{n^{2m}} f\Big(\frac{n^m}{2^m} x\Big) - \frac{4^l}{n^{2l}} f\Big(\frac{n^l}{2^l} x\Big)\Big\|_{\mathcal{Y}}^p &\leq \sum_{i=l}^{m-1} \Big\|\frac{4^{i+1}}{n^{2i+2}} f\Big(\frac{n^{i+1}}{2^{i+1}} x\Big) - \frac{4^i}{n^{2i}} f\Big(\frac{n^i}{2^i} x\Big)\Big\|_{\mathcal{Y}}^p \\ &= \sum_{i=l}^{m-1} \frac{4^{i\beta p}}{n^{2i\beta p}} \Big\|\frac{4}{n^2} f\Big(\frac{n^{i+1}}{2^{i+1}} x\Big) - f\Big(\frac{n^i}{2^i} x\Big)\Big\|_{\mathcal{Y}}^p \\ &\leq \frac{4^{\beta p}}{n^{3\beta p}} \sum_{i=l}^{m-1} \Big[\frac{4^{i\beta}}{n^{2i\beta}} \varphi\Big(\frac{n^i}{2^i} x, \cdots, \frac{n^i}{2^i} x\Big)\Big]^p \end{aligned} \tag{4.5}$$

for all $x \in \mathcal{X}$, in which by (4.1) the right-hand side tends to zero as $m, l \to \infty$. This clarifies that the sequence $\Big\{\frac{4^m}{n^{2m}} f\left(\frac{n^m}{2^m} x\right)\Big\}$ is Cauchy in the complete space $\mathcal{Y}$ and therefore convergent. So we can define for all $x \in \mathcal{X}$, the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{Q}(x) := \lim_{m\to\infty} \frac{4^m}{n^{2m}} f\Big(\frac{n^m}{2^m} x\Big).$$

Now letting $l = 0$, passing the limit $m \to \infty$ in (4.5) and then using (4.1), we obtain (4.3), as desired.

Lastly, we prove that $\mathcal{Q}$ is unique. Let $\mathcal{Q}' : \mathcal{X} \to \mathcal{Y}$ be another quadratic mapping satisfying (4.3). Then we have

$$\begin{aligned}\big\|\mathcal{Q}(x) - \mathcal{Q}'(x)\big\|_{\mathcal{Y}}^p &\le \frac{4^{m\beta p}}{n^{2m\beta p}}\Big\|\mathcal{Q}\Big(\frac{n^m}{2^m}x\Big) - f\big(\frac{n^m}{2^m}x\big)\Big\|_{\mathcal{Y}}^p + \frac{4^{m\beta p}}{n^{2m\beta p}}\Big\|\mathcal{Q}'\Big(\frac{n^m}{2^m}x\Big) - f\Big(\frac{n^m}{2^m}x\Big)\Big\|_{\mathcal{Y}}^p \\ &\le 2\cdot\frac{4^{m\beta p}}{n^{2m\beta p}}\cdot\frac{4^{\beta p}}{n^{3\beta p}}\phi\Big(\frac{n^m}{2^m}x,\cdots,\frac{n^m}{2^m}x\Big) \\ &= 2\cdot\frac{4^{m\beta p}}{n^{2m\beta p}}\cdot\frac{4^{\beta p}}{n^{3\beta p}}\sum_{s=0}^{\infty}\Big[\frac{4^{s\beta}}{n^{2s\beta}}\varphi\Big(\frac{n^{m+s}}{2^{m+s}}x,\cdots,\frac{n^{m+s}}{2^{m+s}}x\Big)\Big]^p \\ &= 2\cdot\frac{4^{\beta p}}{n^{3\beta p}}\sum_{s=m}^{\infty}\Big[\frac{4^{s\beta}}{n^{2s\beta}}\varphi\Big(\frac{n^s}{2^s}x,\cdots,\frac{n^s}{2^s}x\Big)\Big]^p\end{aligned}$$

for all $x \in \mathcal{X}$. Now by (4.1), the right-hand side tends to zero as $m \to \infty$, and therefore $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ is unique and the proof is complete. □

Theorem 4.2. *Let $\varphi : \mathcal{X}^n \to [0,\infty)$ be a function with $\varphi(0,\cdots,0) = 0$, where $n \ge 3$ is a fixed integer. Denote by ϕ a function such that*

$$\phi(x_1,x_2,\cdots,x_n) := \sum_{m=0}^{\infty}\Big[\frac{n^{2m\beta}}{4^{m\beta}}\varphi\Big(\frac{2^{m+1}}{n^{m+1}}x_1,\cdots,\frac{2^{m+1}}{n^{m+1}}x_n\Big)\Big]^p < \infty \tag{4.6}$$

for all $x_1,x_2,\cdots,x_n \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying (4.2). Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \frac{1}{n^\beta}\sqrt[p]{\phi(x,x,\cdots,x)} \tag{4.7}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x_1 = x_2 = \cdots = x_n = 0$ in (4.2), we get $f(0) = 0$.

Letting $x_1 = x_2 = \cdots = x_n = \frac{2}{n}x$ in (4.2), we have

$$\begin{aligned}\Big\|nf(x) - \frac{n^3}{4}f\Big(\frac{2}{n}x\Big)\Big\|_{\mathcal{Y}} &\le \varphi\Big(\frac{2}{n}x,\frac{2}{n}x,\cdots,\frac{2}{n}x\Big), \\ \Big\|\frac{n^2}{4}f\Big(\frac{2}{n}x\Big) - f(x)\Big\|_{\mathcal{Y}} &\le \frac{1}{n^\beta}\varphi\Big(\frac{2}{n}x,\frac{2}{n}x,\cdots,\frac{2}{n}x\Big)\end{aligned}$$

for all $x \in \mathcal{X}$. By the same method used in the previous theorem, we can obtain

$$\Big\|\frac{n^{2m}}{4^m}f\Big(\frac{2^m}{n^m}x\Big) - \frac{n^{2l}}{4^l}f\Big(\frac{2^l}{n^l}x\Big)\Big\|_{\mathcal{Y}}^p \le \frac{1}{n^{\beta p}}\sum_{i=l}^{m-1}\Big[\frac{n^{2i\beta}}{4^{i\beta}}\varphi\Big(\frac{2^{i+1}}{n^{i+1}}x,\cdots,\frac{2^{i+1}}{n^{i+1}}x\Big)\Big]^p \tag{4.8}$$

for positive integers m, l with $m > l$ and all $x \in \mathcal{X}$, in which by (4.6) the right-hand side tends to zero as $m, l \to \infty$.

Now similar to the pervious theorem, the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ is definable as

$$\mathcal{Q}(x) := \lim_{m\to\infty}\frac{n^{2m}}{4^m}f\Big(\frac{2^m}{n^m}x\Big)$$

for all $x \in \mathcal{X}$, which by letting $l = 0$, passing the limit $m \to \infty$ in (4.8) and then using (4.6), satisfies (4.7).

The proof of the uniqueness of $\mathcal{Q}$ is similar to the previous theorem. □

Corollary 4.3. *Let ϑ be a nonnegative real number and q a positive real number with $q < 2\frac{\beta}{\alpha}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\big\|D_1 f(x_1,x_2,\cdots,x_n)\big\|_{\mathcal{Y}} \le \vartheta\big(\|x_1\|_{\mathcal{X}}^q + \cdots + \|x_n\|_{\mathcal{X}}^q\big) \tag{4.9}$$

for all $x_1,x_2,\cdots,x_n \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le 4^\beta\vartheta\frac{n^{1-\alpha q-\beta}}{\sqrt[p]{n^{p(2\beta-\alpha q)} - 2^{p(2\beta-\alpha q)}}}\|x\|_{\mathcal{X}}^q$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^q + \cdots + \|x_n\|_{\mathcal{X}}^q\big)$ and applying Theorem 4.1, we get the result. □

Corollary 4.4. *Let* ϑ *be a nonnegative real number and* q *a positive real number with* $q > 2\frac{\beta}{\alpha}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* (4.9). *Then there exists a unique quadratic mapping* $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le 2^{\alpha q}\vartheta \frac{n^{1-3\beta}}{\sqrt[p]{n^{p(\alpha q-2\beta)} - 2^{p(\alpha q-2\beta)}}}\|x\|_{\mathcal{X}}^q$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^q + \cdots + \|x_n\|_{\mathcal{X}}^q\big)$ and applying Theorem 4.2, we get the result. □

Corollary 4.5. *Let* ϑ *be a nonnegative real number and* $q_1, \ldots, q_n$ *positive real numbers with* $q_1 + \cdots + q_n < 2\frac{\beta}{\alpha}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying*

$$\big\|D_1 f(x_1, x_2, \cdots, x_n)\big\|_{\mathcal{Y}} \le \vartheta\big(\|x_1\|_{\mathcal{X}}^{q_1} \cdot \|x_2\|_{\mathcal{X}}^{q_2} \ldots \|x_n\|_{\mathcal{X}}^{q_n}\big) \tag{4.10}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. *Then there exists a unique quadratic mapping* $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \frac{4^{\beta}\vartheta n^{-\alpha(q_1+\cdots+q_n)-\beta}}{\sqrt[p]{n^{p(2\beta-\alpha(q_1+\cdots+q_n))} - 2^{p(2\beta-\alpha(q_1+\cdots+q_n))}}}\|x\|_{\mathcal{X}}^{(q_1+\cdots+q_n)}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^{q_1} \cdot \|x_2\|_{\mathcal{X}}^{q_2} \cdots \|x_n\|_{\mathcal{X}}^{q_n}\big)$ and applying Theorem 4.1, we get the result. □

Corollary 4.6. *Let* ϑ *be a nonnegative real number and* $q_1, \cdots, q_n$ *positive real numbers with* $q_1 + \cdots + q_n > 2\frac{\beta}{\alpha}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* (4.10). *Then there exists a unique quadratic mapping* $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \frac{2^{\alpha(q_1+\cdots+q_n)}\vartheta n^{-3\beta}}{\sqrt[p]{n^{p(\alpha(q_1+\cdots+q_n)-2\beta)} - 2^{p(\alpha(q_1+\cdots+q_n)-2\beta)}}}\|x\|_{\mathcal{X}}^{(q_1+\cdots+q_n)}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^{q_1} \cdot \|x_2\|_{\mathcal{X}}^{q_2} \cdots \|x_n\|_{\mathcal{X}}^{q_n}\big)$ and applying Theorem 4.2, we get the result. □

Note that in Corollary 4.5 and 4.6, we can put $q_1 = \cdots = q_n = q$ and make simpler results.

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On Impulsive Sequential Fractional Differential Equations

N. I. Mahmudov and B. Sami
Eastern Mediterranean University
Gazimagusa, TRNC, Mersin 10
Turkey

Abstract

This paper aims to study the existence of the solutions for an Impulsive Sequential Fractional Differential Equations of order $1 < q \leq 2$ involving separate boundary conditions. Our analysis relies on some fixed point theorems. In addition, an example is provided to illustrate the results of this study.

Keywords
Impulsive sequential fractional differential equations, Caputo fractional derivative, fixed point theorem.

1 Introduction

Fractional differential equations have recently proved to be strong tools in the modeling of many physical phenomena. It gives a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. Impulsive Fractional Differential Equations (IFDEs), and Sequential Fractional Differential Equations (SFDEs) have attracted the attention of many researchers, see [1]-[21]. To the best of our knowledge, the study of impulsive sequential fractional differential equations (ISFDE) supplemented with separated boundary conditions has yet to be initiated.

In [15] Tian and Bai studied the existence solutions for the following IFDEs with boundary conditions, by using Banach's fixed point theorem and Schauder's fixed point theorem:

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), & q \in (1, 2], t \in [0.1], t \neq t_k, \\ \Delta u|_{t=t_k} = \mathbf{I}_k(u(t_k)),\ \Delta u'|_{t=t_k} = \bar{\mathbf{I}}_k(u(t_k)), k = 1, 2, ..., p, & k = 1, ..., p, \\ u(0) + u^{'}(0) = 0, u(1) + u^{'}(\xi) = 0, \end{cases}$$

with the Caputo fractional derivative ${}^cD^q$, $f \in [0, 1] \times R \to R$ is a continuous function, $\mathbf{I}_k, \bar{\mathbf{I}}_k : R \to R, 0 = t_0 < t_1 < \cdots < t_k < ... < t_p < t_{p+1} = 1$.

In [16] Wang investigated the existence of the solutions of the problem which is given as follows :

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), & 1 < q \leq 2, t \in J', \\ \Delta u(t_k) = Q_k(u(t_k)),\ \Delta u'(t_k) = \mathbf{I}_k(u(t_k)), k = 1, ..., p, & k = 1, ..., p, \\ au(0) + bu^{'}(0) = x_0, cu(1) + du^{'}(1) = x_1. \end{cases}$$

Mahmudov and Unul, [17] provided existence of solutions for the following IFDEs of order q with mixed BVP :

$$\begin{cases} {}^cD_0^q u(t) = f(t, u(t)), & 1 < q \leq 2, t \in J', \\ \Delta u(t_k) = \mathbf{I}_k(u(t_k)) = u(t_k^+) - u(t_k^-), \Delta u'(t_k) = J_k(u(t_k)) = u'(t_k^+) - u'(t_k^-), & k = 1, ..., p, \\ u(0) + \mu_1 u^{'}(1) = \sigma_1, x(0) + \mu_2 x^{'}(1) = \sigma_2, \end{cases}$$

with ${}^cD^q$ is the Caputo derivative of order q, and $f \in (J \times R, R), \varphi_k, I_k \in C(R \times R), J = [0, 1], 0 = t_0 < t_1 < \cdots < t_k < ... < t_p < t_{p+1} = 1$. $\Delta u(t_k) = u\left(t_k^+\right) - u\left(t_k^-\right), \Delta u'(t_k) = u^{'}\left(t_k^+\right) - u'\left(t_k^-\right)$.

In [20], Ahmad and Ntouyas obtained new existence results by using standard fixed point theorems.

$$\begin{cases} {}^cD^{\xi}(D+\lambda)x(t) \in f_1(t,x(t)), & 0<t<1, n<\xi<n-1, \\ x(0)=0.x^{'}(0)=0, \\ x^{'}(0)=0,...,x^{n-1}(0)=0.x(1)=\alpha x(\sigma), \end{cases}$$

where $F:[0.1]\times R\to\mathcal{F}(R)$ is a multivalued map, $\mathcal{F}(R)$ is the family of all subsets of R.

Sequential fractional integral-differential were studied, in [20]:

$$\begin{cases} ({}^cD^q+\lambda\ {}^cD^{q-1})x(t)=f(t,x(t),{}^cD^{\beta}x(t),I^{\gamma}x(t)), & t\in[0,1], 2<q\le 3, 0<\beta,\gamma<1, k>0, \\ x(0)=0, x'(0)=0, \\ \sum_{i=1}^m a_i x(\zeta_i)=\lambda\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}x(s)ds, \delta\ge 1, 0<\eta<\zeta<...<\zeta<1, \end{cases}$$

Here $f:[0,1]\times R^3\to R$ is a given continuous function satisfying some natural conditions.

Alsaedi, et al, [21], used fixed point theorems to develop the existence theory for the following problem:

$$\begin{cases} ({}^cD^q+k\ {}^cD^{q-1})u(t)=f(t,u(t), & 1<q\le 2, t\in[0,T], \\ \alpha_1 u(0)+\sum_{i=1}^m a_i u(\eta_i)+\gamma_1 u(T)=\beta_1, \\ \alpha_2 u'(0)+\sum_{i=1}^m b_i u'(\eta_i)+\gamma_2 u'(T)=\beta_2, \alpha_3 u'(0)+\sum_{i=1}^m c_i u''(\eta_i)+\gamma_3 u^{''}(T)=\beta_3. \end{cases}$$

This paper is motivated from some recent papers treating the problem of the existence of solutions for ISFDEs with separated boundary conditions:

$$\begin{cases} ({}^cD^q+\lambda\ {}^cD^{q-1})x(t)=f(t,x(t)), 0<t<T & 1<p\le 2, \\ \alpha_1 x(0)+\beta_1 x^{'}(0)=\eta_1, \alpha_2 x(T)+\beta_2 x^{'}(T)=\eta_2, \\ \Delta x|_{t=t_k}=\varphi_k(x(t_k)),\ \Delta x'|_{t=t_k}=\varphi_k^*(x(t_k)), & k=1,...,p, \end{cases} \tag{1}$$

where ${}^cD^q$ is the Caputo derivative of order $q\in(1,2]$, and $f:[0T]\times R\to R$, $\beta,q,\eta_1,\alpha_1,\alpha_2,\beta_1,\beta_1,\eta_1,\eta_2\in R,\lambda\in R^+,\varphi_k,\varphi_k^*\in C(R,R)$, and $\Delta x|_{t=t_k}=x\left(t_k^+\right)-x\left(t_k^-\right)$, $\Delta x'|_{t=t_k}=x^{'}\left(t_k^+\right)-x'\left(t_k^-\right)$. $x\left(t_k^+\right)$ and $x\left(t_k^-\right)$ represent the right and the left hand limits of the function $x(t)$, at $t=t_k^+$,respectively.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and obtain the integral solution for the linear variants of the given problems. Section 3 contains the existence results for problem (1) obtained by applying Leray-Schauder's nonlinear alternative, Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 4, the main result is illustrated with the aid of an example.

2 Basic materials

The basic concepts of fractional calculus are presented in this section [13].

Denote that, $J=[0,T]$, $t_0=0, t_{p+1}=T$, $J_0=[0,t_1]$, $J_1=(t_1,t_2]$,..., $J_p=(t_p,T]$, $J'=J\backslash\{t_1,...,t_p\}$, $0=t_0<t_1<\cdots<t_p<t_{p+1}=T$, and insert the spaces:

$$PC(J)=\left\{x:J\to R\mid x\in C(J'),\text{and } x(t_k^+),x(t_k^-)\text{ exist, and } x(t_k^+)=x(t_k),1\le k\le p\right\},$$

with the norm $\|x\|_{PC}=\sup_{t\in J}|x(t)|$ and $PC(J)$ is a Banach space.

Definition 1 *The fractional integral of order $q>0$ of a function $\rho:[0,\infty)\to R$ is given by*

$$\mathbf{I}_{0^+}^q\rho(x)=\frac{1}{\Gamma(q)}\int_0^x\frac{\rho(r)}{(x-r)^{1-q}}dr,\ x>0\ ,q>0,$$

provided the right side is point-wise defined on $[0,\infty)$.

Definition 2 *The Caputo fractional derivative of order $q>0$, of a function $\rho:[0,\infty)\to R$ is defined by*

$$\mathbf{D}_{0^+}^{q}\rho(x)=\frac{1}{\Gamma(n-q)}\int_0^x(x-r)^{n-q-1}\rho^n dr,$$

whenever the right-hand side is defined on $[0,\infty)$.

Lemma 3 *Let $q>0$, then the differential equation*

$$^{c}D^{q}\rho(t)=0,$$

has solutions

$$\rho(x)=a_0+a_1x+a_2x^2+...+a_{n-1}x^{n-1},$$

where $a_i\in R,\ i=0,1,2,3,...,n-1;\ n=[\beta]+1.$

Lemma 4 *The set $F\subset PC\left([0,T],R^n\right)$ is relatively compact if and only if*

(i) F is bounded , that $\|x\|\le C$ for each $x\in F$ and some $C>0$;

(ii) F is quasi-equicontinuous in $[0,T]$. That is, to say that for any $\varepsilon>0$ there exist $\gamma>0$ such that if $x\in F;k\in N;s_1,s_2\in(t_{k-1},t_k]$, and $|s_1-s_2|<\gamma,\left|x(s_{1)}-x(s_2)\right|<\varepsilon$.

Lemma 5 *For $\rho\in PC(J,R)$, the solution of the following ISFDEs*

$$\begin{cases}({}^{c}D^{q}+\lambda\ {}^{c}D^{q-1})x(t)=\rho(t),\\ \Delta x|_{t=t_k}=\varphi_k(x(t_k)),\ \Delta x'|_{t=t_k}=\varphi_k^*(x(t_k)),\\ \alpha_1x(0)+\beta_1x^{'}(0)=\eta_1,\alpha_2x(T)+\beta_2x^{'}(T)=\eta_2,\quad k=1,...,p,\end{cases}\tag{2}$$

is given by

$$\begin{aligned}x\left(t\right)=&\int_0^t e^{-\lambda(t-s)}I^{q-1}\rho(s)ds+v_1(t)\int_0^T e^{-\lambda(T-s)}I^{q-1}\rho(s)ds\\&+v_2(t)I^{q-1}\rho(T)+v_3(t)\sum_{j=1}^{p}\varphi_j(x(t_j))+v_4(t)\sum_{k=1}^{p}\varphi_j^*(x(t_j)\\&+\sum_{j=1}^{p}z_{1j}\left(t\right)\varphi_j^*(x(t_j))+\sum_{j=k+1}^{p}z_{2j}\left(t\right)\varphi_j^*(x(t_j))-\sum_{j=k+1}^{p}\varphi_j(x(t_j))+z_3\left(t\right),\\&t\in\left[t_k,t_{k+1}\right),\ k=0,1,...,p,\end{aligned}\tag{3}$$

where

$$\begin{aligned}\Delta&=\left(\alpha_1-\lambda\beta_1\right)\alpha_2-\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)\alpha_1\neq0,\\ v_1(t)&=\frac{\left(\alpha_1e^{-\lambda t}-\alpha_1+\lambda\beta_1\right)\left(\alpha_2-\lambda\beta_2\right)}{\Delta},\\ v_2(t)&=\frac{\left(\alpha_1e^{-\lambda t}-\alpha_1+\lambda\beta_1\right)\beta_2}{\Delta},\\ v_3(t)&=\frac{\alpha_1\alpha_2}{\Delta}e^{-\lambda t}-\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta},\\ v_4(t)&=\frac{\alpha_1\alpha_2}{\lambda\Delta}e^{-\lambda t}-\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta},\\ z_{1,j}\left(t\right)&=-e^{\lambda t_j}e^{-\lambda t}\frac{\alpha_2\left(\alpha_1-\lambda\beta_1\right)}{\lambda\Delta}+e^{\lambda t_j}\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta},\\ z_{2,j}\left(t\right)&=e^{-\lambda t}\frac{1}{\lambda}e^{\lambda t_j}-\frac{1}{\lambda},\\ z_3\left(t\right)&=\left(e^{-\lambda t}\frac{\alpha_2}{\Delta}-\frac{\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\eta_1-\left(e^{-\lambda t}\frac{\alpha_1}{\Delta}-\frac{\left(\alpha_1-\lambda\beta_1\right)}{\Delta}\right)\eta_2.\end{aligned}$$

Proof. Assume that x is a solution of

$$({}^cD^q + \lambda\ {}^cD^{q-1})x(t) = \rho(t),$$

on $(t_k, t_{k+1}]$, $(k = 1, 2 \ldots, p)$. Applying the operator $\mathbf{I}^{q-1}$ operator to both sides of the above equation, we get

$$\begin{aligned}\mathbf{I}^{q-1}({}^cD^q + \lambda\ {}^cD^{q-1})x(t) &= \mathbf{I}^{q-1}\rho(t),\\ (D+\lambda)\,x(t) &= c_0 + \mathbf{I}^{q-1}\rho(t).\end{aligned}$$

This can be expressed as

$$e^{\lambda t}\left((D+\lambda)\,x\,(t)\right) = e^{\lambda t}\left(c_0 + \mathbf{I}^{q-1}\rho(t)\right),$$

Solving the above equation, we see that the general solution of (1) on each interval $(t_k, t_{k+1}]$, $(k = 1, 2 \ldots p)$, can be written as

$$x(t) = e^{-\lambda t}A_k + B_k + \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds, t \in J.$$

Next, solving the obtained linear equation on J_0, we get

$$x(t) = e^{-\lambda t}A_0 + B_0 + \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds,\ t \in J_0, \tag{4}$$

where A_0 and B_0 are arbitrary constants. Taking the derivative to (4), we get

$$x^{'}(t) = -\lambda e^{-\lambda t}A_0 - \lambda\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds + I^{q-1}\rho(t),\ t \in J_0. \tag{5}$$

Now, applying the boundary condition, we have

$$(\alpha_1 - \lambda\beta_1)\,A_0 + \alpha_1 B_0 = \eta_1. \tag{6}$$

In general, for $t \in [t_k, t_{k+1})$, we find

$$\begin{aligned}x(t) &= e^{-\lambda t}A_k + B_k + \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds, \qquad (7)\\ x^{'}(t) &= -\lambda e^{-\lambda t}A_k - \lambda\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds + \mathbf{I}^{q-1}\rho(t).\end{aligned}$$

Now, applying the boundary condition at $t_{k+1} = T$, we have

$$\left(\alpha_2 e^{-\lambda T} - \lambda\beta_2 e^{-\lambda T}\right)A_p + \alpha_2 B_p = \eta_2 - (\alpha_2 - \lambda\beta_2)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds - \beta_2\mathbf{I}^{q-1}\rho(T). \tag{8}$$

From $\Delta x'(t_k) = \varphi_k^*(x(t_k))$, we have

$$\begin{aligned}\varphi_k^*(x(t_k)) &= -\lambda e_k^{-\lambda t_k}A_k + \lambda e_{k-1}^{-\lambda t_k}A_{k-1},\\ A_k - A_{k-1} &= -\frac{1}{\lambda}e^{\lambda t_k}\varphi_k^*(x(t_k)),\ k = 1, ..., p. \qquad (9)\end{aligned}$$

Similarly, from $\Delta x(t_k) = \varphi_k(x(t_k))$, we get

$$\begin{aligned}\varphi_k(x(t_k)) &= e^{-\lambda t_k}A_k - e^{-\lambda t_k}A_{k-1} + B_k - B_{k-1},\\ B_k - B_{k-1} &= \varphi_k(x(t_k)) + \frac{1}{\lambda}\varphi_k^*(x(t_k)),\ k = 1, ..., p. \qquad (10)\end{aligned}$$

Next, it follows from (9) and (10) that

$$A_p - A_k = -\frac{1}{\lambda}\sum_{j=k+1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)), \tag{11}$$

$$B_p - B_k = \sum_{j=k+1}^{p}\varphi_j(x(t_j)) + \frac{1}{\lambda}\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)), \quad k = 0,1,...,p-1. \tag{12}$$

It follows that for $k=0$ from $(\alpha_1-\lambda\beta_1)A_0+\alpha_1 B_0=\eta_1$ that

$$(\alpha_1-\lambda\beta_1)A_p+\alpha_1 B_p=\eta_1-\frac{1}{\lambda}(\alpha_1-\lambda\beta_1)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j))+\alpha_1\sum_{j=1}^{p}\varphi_j(x(t_j))+\frac{1}{\lambda}\alpha_1\sum_{j=1}^{p}\varphi_j^*(x(t_j)).$$

Solving the last equation together(8), for A_p and B_p, we get

$$\begin{aligned}A_p &= \left(\frac{\alpha_1(\alpha_2-\lambda\beta_2)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds+\left(\frac{\alpha_1\beta_2}{\Delta}\right)I^{q-1}\rho(T)\\ &+\left(\frac{\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j))+\left(\frac{\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j))-\left(\frac{\alpha_2(\alpha_1-\lambda\beta_1)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j))\\ &+\frac{\alpha_2}{\Delta}\eta_1-\frac{\alpha_1}{\Delta}\eta_2,\end{aligned}$$

and

$$\begin{aligned}B_p &= -\left(\frac{(\alpha_1-\lambda\beta_1)(\alpha_2-\lambda\beta_2)}{\Delta}\right)\div\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds-\left(\frac{(\alpha_1-\lambda\beta_1)\beta_2}{\Delta}\right)I^{q-1}\rho(T)\\ &-\left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j))-\left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j))\\ &+\left(\frac{(\alpha_1-\lambda\beta_1)\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j))-\left(\frac{\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\eta_1+\left(\frac{(\alpha_1-\lambda\beta_1)}{\Delta}\right)\eta_2,\end{aligned}$$

where $\Delta=(\alpha_1-\lambda\beta_1)\alpha_2-\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)\alpha_1\neq 0$. Now, from the equations (11) and (12) it follows that

$$\begin{aligned}A_k &= A_p+\frac{1}{\lambda}\sum_{j=k+1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j)),\\ B_k &= B_p-\sum_{j=k+1}^{p}\varphi_j(x(t_j))-\frac{1}{\lambda}\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)),\ k=1,...,p-1.\end{aligned}$$

So

$$\begin{aligned}A_k &= \left(\frac{\alpha_1(\alpha_2-\lambda\beta_2)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds+\left(\frac{\alpha_1\beta_2}{\Delta}\right)I^{q-1}\rho(T)\\ &+\left(\frac{\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j))+\left(\frac{\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j))-\left(\frac{\alpha_2(\alpha_1-\lambda\beta_1)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j))\\ &+\left(\frac{\alpha_2}{\Delta}\right)\eta_1-\left(\frac{\alpha_1}{\Delta}\right)\eta_2+\left(\frac{1}{\lambda}\right)\sum_{j=k+1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j)).\end{aligned}$$

Multiplying the above equation by $e^{-\lambda t}$, we get

$$
\begin{aligned}
e^{-\lambda t}A_k &= \left(\frac{e^{-\lambda t}\alpha_1\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds+\left(\frac{e^{-\lambda t}\alpha_1\beta_2}{\Delta}\right)I^{q-1}\rho(T)\\
&+\left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\psi_j(x(t_j))+\left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\psi_j^*(x(t_j))-\left(\frac{e^{-\lambda t}\alpha_2\left(\alpha_1-\lambda\beta_1\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\psi_j^*\left(x(t_j)\right)\\
&+\left(\frac{e^{-\lambda t}\alpha_2}{\Delta}\right)\eta_1-\left(\frac{e^{-\lambda t}\alpha_1}{\Delta}\right)\eta_2+\left(\frac{e^{-\lambda t}}{\lambda}\right)\sum_{j=k+1}^{p}e^{\lambda t_j}\psi_j^*(x(t_j)).
\end{aligned}
$$

and

$$
\begin{aligned}
B_k &= -\left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds-\left(\frac{\left(\alpha_1-\lambda\beta_1\right)\beta_2}{\Delta}\right)\mathbf{I}^{q-1}\rho(T)\\
&-\left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j))-\left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j))\\
&+\left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j))-\left(\frac{\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\eta_1\\
&+\left(\frac{\left(\alpha_1-\lambda\beta_1\right)}{\Delta}\right)\eta_2-\sum_{j=k+1}^{p}\varphi_j(x(t_j))-\left(\frac{1}{\lambda}\right)\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)).
\end{aligned}
$$

Combining the last two equations, we get

$$
\begin{aligned}
e^{-\lambda t}A_k + B_k &= \left(\frac{e^{-\lambda t}\alpha_1\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds + \left(\frac{e^{-\lambda t}\alpha_1\beta_2}{\Delta}\right)I^{q-1}\rho(T) \\
&+\left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\psi_j(x(t_j)) + \left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\psi_j^*(x(t_j)) \\
&-\left(\frac{e^{-\lambda t}\alpha_2\left(\alpha_1-\lambda\beta_1\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\psi_j^*\left(x(t_j)\right) \\
&+\left(\frac{e^{-\lambda t}\alpha_2}{\Delta}\right)\eta_1 - \left(\frac{e^{-\lambda t}\alpha_1}{\Delta}\right)\eta_2 + \left(\frac{e^{-\lambda t}}{\lambda}\right)\sum_{j=k+1}^{p}e^{\lambda t_j}\psi_j^*(x(t_j)) \\
&-\left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds \\
&-\left(\frac{\left(\alpha_1-\lambda\beta_1\right)\beta_2}{\Delta}\right)\mathbf{I}^{q-1}\rho(T) - \left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j)) \\
&-\left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j)) \\
&+\left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j)) - \left(\frac{\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\eta_1 \\
&+\left(\frac{\left(\alpha_1-\lambda\beta_1\right)}{\Delta}\right)\eta_2 \\
&-\sum_{j=k+1}^{p}\varphi_j(x(t_j)) - \left(\frac{1}{\lambda}\right)\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)).
\end{aligned}
$$

$$
\begin{aligned}
e^{-\lambda t}A_k + B_k &= v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds + v_2(t)\mathbf{I}^{q-1}\rho(T) + v_3(t)\sum_{j=1}^{p}\varphi_j(x(t_j)) \\
&+v_4(t)\sum_{j=1}^{p}\varphi_j^*(x(t_j) + \sum_{j=1}^{p}z_{1,j}\left(t\right)\varphi_j^*(x(t_j)) + \sum_{j=k+1}^{p}z_{2,j}\left(t\right)\varphi_j^*(x(t_j)) \\
&-\sum_{j=k+1}^{p}\varphi_j(x(t_j)) + z_3(t).
\end{aligned} \tag{13}
$$

Inserting (13) into (7), thus we obtain the desired formula (3).

The converse of the lemma follows by direct computation. This completes the proof. ■

3 Main results

This section deals with the existence and uniqueness of solutions for the problem (1). Before stating and proving the main results, we introduce the following hypotheses.

(H$_1$) the function $f : J \times R \to R$ is jointly continuous .

(H$_2$) there exists a constant $L_f > 0$ such that

$$
\left|f\left(t,x\right)-f\left(t,y\right)\right| \le L_f\left|x-y\right|, \quad t\in J, \quad x,y\in R.
$$

(H$_3$) There exist a positive constants $L_{\varphi}, L_{\varphi^*}, M_{\varphi}, M_{\varphi^*}$ such that

$$|\varphi_k(x) - \varphi_k(y)| \leq L_{\varphi} |x-y|, |\varphi_k^*(x) - \varphi_k^*(y)| \leq L_{\varphi^*} |x-y|; |\varphi_k(x)| \leq M_{\varphi}, |\varphi_k^*(x)| \leq M_{\varphi^*}.$$

From (G$_1$)-(G$_3$) it follows that

$$\begin{aligned} |f(t,x)| &\leq L_f |x| + M_f, \quad t \in J, \quad x \in R, \quad M_f := \sup\{|f(t,0)| : 0 < t \leq T\}, \\ |\varphi_k(x)| &\leq L_{\varphi} |x| + M_{\varphi}, \quad |\varphi_k^*(x)| \leq L_{\varphi^*} |x| + M_{\varphi^*}. \end{aligned}$$

Theorem 6 *Suppose that (H$_1$), (H$_2$) and (H$_3$) hold. If*

$$\begin{aligned} L_{\mathfrak{T}} &:= \left(\frac{T^{q-1}}{\lambda \Gamma(q)} \left(1 - e^{-\lambda T}\right) (1 + \|\nu_1\|) + \frac{T^{q-1}}{\Gamma(q)} \|\nu_2\| \right) L_f \\ &+ (1 + \|\nu_3\|) p L_{\varphi} + (\|v_4\| + \|z_{1j}\| + \|z_{2j}\|) p L_{\varphi^*} < 1, \end{aligned} \tag{14}$$

then the equation (1) has a unique solution on J.

Proof. In view of Lemma 5, we can transform problem (1) into a fixed point problem. Consider the operator $\mathfrak{T} : PC(J,R) \to PC(J,R)$ defined by

$$\begin{aligned} (\mathfrak{T}x)(t) &:= \int_0^t e^{-\lambda(t-s)} \mathbf{I}^{q-1} f(s, x(s)) ds + v_1(t) \int_0^T e^{-\lambda(T-s)} \mathbf{I}^{q-1} f(s, x(s)) ds \\ &+ v_2(t) I^{q-1} f(T, x(T)) + v_3(t) \sum_{j=1}^p \varphi_j(x(t_j)) + v_4(t) \sum_{j=1}^p \varphi_j^*(x(t_j) \\ &+ \sum_{j=1}^p z_{1j}(t) \varphi_j^*(x(t_j)) + \sum_{j=k+1}^p z_{2j}(t) \varphi_j^*(x(t_j)) - \sum_{j=k+1}^p \varphi_j(x(t_j)) + z_3(t) \\ &, t \in J_k, \ k = 0, 1, ..., p. \end{aligned} \tag{15}$$

It is obvious that $\mathfrak{T}$ is well defined due to (H$_1$) and sends $PC(J,R)$ into itself.

Step 1. $\mathfrak{T}$ maps $B_r = \{x \in PC([0,T], R), \|x\| \leq r\}$ into itself for some $r > 0$.

Let

$$\begin{aligned} r &> (1 - L_{\mathfrak{T}})^{-1} \left(\frac{T^{q-1}}{\lambda \Gamma(q)} \left(1 - e^{-\lambda T}\right) (1 + \|\nu_1\|) + \frac{T^{q-1}}{\Gamma(q)} \|\nu_2\| \right) E_f \\ &+ (1 + \|\nu_3\|) p (L_{\varphi} r + M_{\varphi}) + (\|\nu_4\| + \|z_{1j}\| + \|z_{2j}\|) p (L_{\varphi^*} r + M_{\varphi^*}) + \|z_3\|. \end{aligned}$$

For $t \in J_k, k = 0, 1, ..., p$; $x \in B_r$, we have

$$\begin{aligned} |(\mathfrak{T}x)(t)| &\leq \frac{1}{\Gamma(q-1)} \int_0^t e^{-\lambda(t-s)} \left(\int_0^s (s-\tau)^{q-2} |f(\tau, x(\tau))| d\tau \right) ds \\ &+ \frac{|v_1(t)|}{\Gamma(q-1)} \int_0^T e^{-\lambda(T-s)} \left(\int_0^s (s-\tau)^{q-2} |f(\tau, x(\tau))| d\tau \right) ds \\ &+ \frac{|v_2(t)|}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} |f(s, x(s))| ds + |v_3(t)| \sum_{j=1}^p |\varphi_j(x(t_j))| \\ &+ |v_4(t)| \sum_{j=1}^p \left|\varphi_j^*(x(t_j)\right| + \sum_{j=1}^p |z_{1j}(t)| \left|\varphi_j^*(x(t_j))\right| \\ &+ \sum_{j=k+1}^p |z_{2j}(t)| \left|\varphi_j^*(x(t_j))\right| \sum_{j=k+1}^p |\varphi_j(x(t_j))| + |z_3(t)|, \end{aligned}$$

Thus

$$\begin{aligned}
|(\mathfrak{T}x)(t)| &\leq \frac{t^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda t}\right)(L_f r+M_f)+|v_1(t)|\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(L_f r+M_f)\\
&+|v_2(t)|\frac{T^{q-1}}{\Gamma(q)}(L_f r+M_f)+|v_3(t)|\,p\,(L_\varphi r+M_\varphi)+|v_4(t)|\,p\,(Lr+M_\varphi)\\
&+|z_{1j}(t)|\,p\,(L_\varphi r+M_\varphi)+|z_{2j}(t)|\,p\,(L_\varphi r+M_\varphi)+p\,(L_\varphi r+M_\varphi)+|z_3(t)|\\
&\leq \left(\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(1+\|\nu_1\|)+\frac{T^{q-1}}{\Gamma(q)}\|\nu_2\|\right)(L_f r+M_f)+(1+\|\nu_3\|)\,p\,(L_\varphi r+M_\varphi)\\
&+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,p\,(L_{\varphi^*}r+M_{\varphi^*})+\|z_3\|
\end{aligned}$$

We use the following estimation in what follows

$$\begin{aligned}
&\left|\frac{1}{\Gamma(q-1)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-\tau)^{q-2}\rho(\tau)\,d\tau\right)ds\right| \leq \frac{t^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda t}\right)\|\rho\|_{PC} \\
&= \frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)\|\rho\|_{PC}, \rho\in PC(J,R)
\end{aligned} \tag{16}$$

We obtain that

$$\begin{aligned}
|(\mathfrak{T}x)(t)| &\leq \left(\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(1+\|\nu_1\|)+\frac{T^{q-1}}{\Gamma(q)}\|\nu_2\|\right)(L_f r+M_f)+(1+\|\nu_3\|)\,p\,(L_\varphi r+M_\varphi)\\
&+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,p\,(L_{\varphi^*}r+M_{\varphi^*})+\|z_3\|<r.
\end{aligned}$$

This implies that $\mathfrak{T}x\in B_r$. Thus $\mathfrak{T}B_r\subset B_r$.

Step 2. $\mathfrak{T}$ is a contraction operator on $PC(J,R)$.

Let $x,y\in B_r$. Then For each $t\in J$, we have

$$\begin{aligned}
|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)| := &\left|\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}f(s,x(s))ds+v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}f(s,x(s))ds\right.\\
&+v_2(t)I^{q-1}f(T,x(T))+v_3(t)\sum_{j=1}^p\varphi_j(x(t_j))+v_4(t)\sum_{j=1}^p\varphi_j^*(x(t_j)\\
&\left.+\sum_{j=1}^p z_{1j}(t)\,\varphi_j^*(x(t_j))+\sum_{j=k+1}^p z_{2j}(t)\,\varphi_j^*(x(t_j))-\sum_{j=k+1}^p\varphi_j(x(t_j))+z_3(t)\right|\\
&-\left|\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}f(s,y(s))ds+v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}f(s,y(s))ds\right.\\
&+v_2(t)I^{q-1}f(T,y(T))+v_3(t)\sum_{j=1}^p\varphi_j(y(t_j))+v_4(t)\sum_{j=1}^p\varphi_j^*(y(t_j)\\
&\left.+\sum_{j=1}^p z_{1j}(t)\,\varphi_j^*(y(t_j))+\sum_{j=k+1}^p z_{2j}(t)\,\varphi_j^*(y(t_j))+\sum_{j=k+1}^p\varphi_j(y(t_j))+z_3(t)\right|,
\end{aligned}$$

$$
\begin{aligned}
|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)| &:= \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\left|f(s,x(s))-f(s,y(s))\right|ds \\
&+|v_1(t)|\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\left|f(s,x(s))-f(s,y(s))\right|ds \\
&+|v_2(t)|\,I^{q-1}\left|f(T,x(T))-f(T,y(T))\right|+|v_3(t)|\sum_{j=1}^{p}\left|\varphi_j(x(t_j))-\varphi_j(y(t_j))\right| \\
&+v_4(t)\sum_{j=1}^{p}\left|\varphi_j^*(x(t_j)-\varphi_j^*(y(t_j))\right|+\sum_{j=1}^{p}\left|z_{1j}\right|(t)\left|\varphi_j^*(x(t_j))-\varphi_j^*(y(t_j))\right| \\
&+\sum_{j=k+1}^{p}\left|z_{2j}\right|(t)\left|\varphi_j^*(x(t_j))-\varphi_j^*(y(t_j))\right|+\sum_{j=k+1}^{p}\left|\varphi_j(x(t_j))\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)| \leq &\left(\left(\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(1+\|\nu_1\|)+\frac{T^{q-1}}{\Gamma(q)}\|\nu_2\|\right)L_f\right. \\
&\left.+(1+\|\nu_3\|)\,pL_\varphi+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pL_{\varphi^*}\right)\|x-y\|_{PC} \\
&= L_{\mathfrak{T}}\|x-y\|_{PC}.
\end{aligned}
$$

Thus, $\mathfrak{T}$ is a contraction mapping on $PC(J,R)$ due to condition (14). By applying the well-known Banach's contraction mapping we see that the operator $\mathfrak{T}$ has a unique fixed point on $PC(J,R)$. Therefore, the problem (1) has a unique solution. This completes the proof. ■

The second result is based on a known result due to Krasnoselskii. We state the Krasnoselskii theorem which is needed to prove the existence of at least one solution of (1).

Theorem 7 . *Let M be a closed convex and nonempty subset of a Banach space X. Let $\mathfrak{T}_1$, $\mathfrak{T}_2$ be the operators such that:*

1. *$\mathfrak{T}_1x+\mathfrak{T}_2y\in M$ whenever $x,y\in M$;*
2. *$\mathfrak{T}_1$ is compact and continuous;*
3. *$\mathfrak{T}_2$ is a contraction mapping. Then there exists $z\in M$ such that $z=\mathfrak{T}_1z+\mathfrak{T}_2z$.*

Now, we replace (H_2) into the following condition:

$(\mathbf{H}_4)$ $|f(t,x)|\leq\mu(t)$ for $(t,x)\in J\times R$ where $\mu\in L^{\frac{1}{\sigma}}(J)$, $\sigma\in(0,q-1)$.

Theorem 8 *Suppose that (H_1),(H_3) and (H_4) hold. If*
$(1+\|\nu_3\|)\,pL_\varphi+(\|v_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pL_{\varphi^*}<1.$
Then (1) has at least one solution on J.

Proof. Let $B_r=\{x\in PC(J,R),\|x\|_{PC}\leq r\}$. We choose

$$
\begin{aligned}
r \geq &\frac{\|\mu\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q)}\left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}(1+\|v_1\|)+\frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\delta}\right)^{1-\sigma}}\|v_2\|\right) \\
&+(1+\|\nu_3\|)\,pL_\varphi+(\|v_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pL_{\varphi^*}.
\end{aligned}
$$

The operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ on B_r are defined as:

$$(\mathfrak{T}_1x)(t) = \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}f(s,x(s))ds + v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}f(s,x(s))ds \; + v_2(t)\mathbf{I}^{q-1}f(T,x(T)),$$

and

$$\begin{aligned}(\mathfrak{T}_2x)(t) &:= v_3(t)\sum_{j=1}^{p}\varphi_j(x(t_j)) + v_4(t)\sum_{j=1}^{p}\varphi_j^*(x(t_j) + \sum_{j=1}^{p} z_{1j}(t)\,\varphi_j^*(x(t_j)) \\ &+ \sum_{j=k+1}^{p} z_{2j}(t)\,\varphi_j^*(x(t_j)) - \sum_{j=k+1}^{p}\varphi_j(x(t_j)), t\in J_k, k=0,1,...,p.\end{aligned}$$

Step 1. $\mathfrak{T}_1x+\mathfrak{T}_2y\in B_r$ whenever $x,y\in B_r$.

For any $x,y\in B_r$ and $t\in J_k$, using the assumption (H$_4$) with the Holder inequality we get

$$\begin{aligned}&\left|\frac{1}{\Gamma(q-1)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-\tau)^{q-2}\left|f(\tau,x(\tau))\right|d\tau\right)ds\right| \\ &\leq \left|\frac{1}{\Gamma(q-1)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-\tau)^{\frac{q-2}{1-\sigma}}d\tau\right)^{1-\sigma}\left(\int_0^\tau \left|f(\tau,x(\tau))\right|^{\frac{1}{\sigma}}d\tau\right)^{\sigma}ds\right| \\ &\leq \frac{1}{\Gamma(q)}\frac{t^{q-\sigma-1}\left(1-e^{-\lambda t}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\left\|\mu\right\|_{L^{\frac{1}{\sigma}}},\end{aligned}$$

$$\left|\int_0^T e^{-\lambda(T-s)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,x(\tau))d\tau\right)ds\right| \leq \frac{1}{\Gamma(q)}\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\left\|\mu\right\|_{L^{\frac{1}{\sigma}}}.$$

and

$$\left|\frac{v_2(t)}{\Gamma(q-1)}\int_0^T (T-s)^{q-2}f(s,x(s))ds\right| \leq \frac{1}{\Gamma(q)}\frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\left\|\mu\right\|_{L^{\frac{1}{\sigma}}}.$$

Therefore,

$$\begin{aligned}\left\|\mathfrak{T}_1x+\mathfrak{T}_2y\right\|_{PC} &\leq \left\|\mu\right\|_{L^{\frac{1}{\sigma}}}\frac{1}{\Gamma(q)}\left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}(1+\|\nu_1\|) + \frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\|\nu_2\|\right) \\ &+ \left((1+\|\nu_3\|)\,pM_\varphi + (\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pM_{\varphi^*}\right) \leq r.\end{aligned}$$

Thus, $\|\mathfrak{T}_1x+\mathfrak{T}_2y\|\leq r$, so $\mathfrak{T}_1x+\mathfrak{T}_2y\in B_r$.

Step 2. $\mathfrak{T}_1$ is compact and continuous.

The continuity of f implies $\mathfrak{T}_1$ is continuous, also $\mathfrak{T}_1$ is uniformly bounded on B_r as

$$\left\|\mathfrak{T}_1x\right\|_{PC} \leq \left\|\mu\right\|_{L^{\frac{1}{\sigma}}}\frac{1}{\Gamma(q)}\left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}(1+\|\nu_1\|) + \frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\|\nu_2\|\right) \leq r.$$

For equicontinuity on $[0,t_1]$, let $x\in B_r$ and for any $s_1,s_2\in[0,t_1]$, $s_1<s_2$, we have

$$\begin{aligned}|(\mathfrak{T}_1x)(s_2)-(\mathfrak{T}_1x)(s_1)| &= \Bigg|\int_0^{s_2} e^{-\lambda(s_2-s)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,x(\tau))d\tau\right)ds \\ &+ v_1(s_2)\int_0^T e^{-\lambda(T-s)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,x(\tau))d\tau\right)ds \\ &+\frac{v_2(s_2)}{\Gamma(q-1)}\int_0^T (T-s)^{q-2}f(s,x(s))ds\Bigg| \\ &-\Bigg|\int_0^{s1} e^{-\lambda(s_1-s)}\left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)}f(\tau,x(\tau))d\tau\right)ds \\ & v_1(s_1)\int_0^T e^{-\lambda(T-s)}\left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)}f(\tau,x(\tau))d\tau\right)ds \\ &+\frac{v_2(s_1)}{\Gamma(q-1)}\int_0^T (T-s)^{q-2}f(s,x(s))ds\Bigg|,\end{aligned}$$

$$\begin{aligned}|(\mathfrak{T}_1x)(s_2)-(\mathfrak{T}_1x)(s_1)| &\le \left(e^{-\lambda(s_2)}-e^{-\lambda(s_1)}\right)\int_0^{s_1} e^{\lambda s}\left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)}d\tau\right)ds \\ &+\int_{s_1}^{s_2} e^{-\lambda(s_2-s)}\left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)}d\tau\right)ds \\ &+|v_1(s_2)-v_1(s_1)|\int_0^T e^{-\lambda(T-s)}\left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)}d\tau\right)ds \\ &+|v_2(s_2)-v_2(s_1)|\frac{v_2(s_1)}{\Gamma(q-1)}\int_0^T (T-s)^{q-2}ds.\end{aligned}$$

It tends to zero as $s_1\to s_2$. This implies that $\mathfrak{T}_1$ is equicontinuous on the interval $[0,t_1]$. In general, for the time interval $(t_k,t_{k+1}]$, we similarly obtain the same inequality, which yields that $\mathfrak{T}_1$ is equicontinuous on interval $(t_k,t_{k+1}]$. Together with the PC-type Arzela-Ascoli (Lemma 4) theorem, we can conclude that $\mathfrak{T}_1:B_r\to B_r$ is continuous and compact.

Step 3. It is clearly that $\mathfrak{T}_2$ is contraction mapping.

Thus all the assumptions of the Krasnoselskii theorem are satisfied. In consequence, the the Krasnoselskii theorem is applied and hence the problem (1) has at least one solution on J. ■

Our second existence result is based on the nonlinear alternative of Leray-Schauder type. Assume that

($\mathbf{H}_5$) There exist $\vartheta_f\in PC(J,R)$ and $\Psi:R^+\to R^+$ continuous and nondecreasing such that

$$|f(t,x)|\le\vartheta_f(t)\Psi(\|x\|),\text{for all }(t,x)\in J\times R,$$

($\mathbf{H}_6$) There exist an number $N>0$ such that

$$\frac{N}{L_{\mathfrak{T}}\|\vartheta\|\Psi(N)}>1.$$

Theorem 9 *Suppose that (H_1), (H_2), (H_5),(H_6) are hold. Then our BVP in (1) has at least one solution on J.*

Proof. Consider the operator $\mathfrak{T}:PC(J,R)\to PC(J,R)$ defined by (15). It can be easily shown that $\mathfrak{T}$ is continuous and compact. maps bounded sets into bounded sets in $PC(J,R)$. Repeating the same process in Step 2 of Theorem 8, we get

$$
\begin{aligned}
|(\mathfrak{T}x)(t)| &\leq \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\left|f(s,x(s))\right|ds+|v_1(t)|\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\left|f(s,x(s))\right|ds \\
&+|v_2(t)|\,\mathbf{I}^{q-1}\left|\rho(T)\right|+|v_3(t)|\sum_{j=1}^{p}|\varphi_j(x(t_j))|+|v_4(t)|\sum_{j=1}^{p}\left|\varphi_j^*(x(t_j)\right| \\
&+\sum_{j=1}^{p}|z_{1j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|z_{2j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|\varphi_j(x(t_j))|+|z_3(t)|,
\end{aligned}
$$

■

Theorem 10 ***Proof.***

$$
\begin{aligned}
&\leq \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\vartheta_f(s)\,\Psi(\|x\|)\,ds+|v_1(t)|\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\vartheta_f(s)\,\Psi(\|x\|)\,ds \\
&+|v_2(t)|\,\mathbf{I}^{q-1}\left|\rho(T)\right|\vartheta_f(s)\,\Psi(\|x\|)+|v_3(t)|\sum_{j=1}^{p}|\varphi_j(x(t_j))|+|v_4(t)|\sum_{j=1}^{p}\left|\varphi_j^*(x(t_j)\right| \\
&+\sum_{j=1}^{p}|z_{1j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|z_{2j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|\varphi_j(x(t_j))|+|z_3(t)|,
\end{aligned}
$$

$$
\begin{aligned}
|x(t)| \leq |(\mathfrak{T}x)(t)| &\leq \frac{1}{\Gamma(q)}\left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}(1+\|\nu_1\|)+\frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\|\nu_2\|\right)\|\vartheta\|\,\Psi(\|x\|) \\
&+(1+\|\nu_3\|)\,pM_{\varphi}+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pM_{\varphi^*}+\|z_3\|\,.
\end{aligned}
$$

Now, construct the set $\Lambda=\{x\in PC(J,R):\|x\|<N\}$. The operator $\mathfrak{T}:\overline{\Lambda}\to PC(J,R)$ is continuous and completely continuous. From the choice of Λ, there is no $x\in\partial\Lambda$ such that $x=\lambda\mathfrak{T}x$, $0\leq\lambda\leq1$. As a consequence of the nonlinear alternative of Leray–Schauder type, we deduce that $\mathfrak{T}$ has a fixed point $x\in\partial\Lambda$, which implies that the problem (1) has at least one solution. This completes the proof. ■

4 Example

In this section we give some examples to illustrate the usefulness of our main results.

Example 1. Consider the following ISFDE:

$$
\begin{aligned}
({}^cD^{\frac{3}{2}}+2\,{}^cD^{\frac{1}{2}})x(t) &= 0.01\left(t^2+\sin t+1+\tan^{-1}x(t)\right),\ t\in[0,1], \\
x(0)+x'(0) &= \eta_2, x(1)+x'(1)=\eta_2, \\
\Delta x(\frac{1}{4}) &= \Delta x(t_k)=0.01\frac{\|x\|}{1+\|x\|},\Delta x'(\frac{1}{4})=0.01\frac{\|x\|}{1+\|x\|},k=1,2,...,p.
\end{aligned}
\tag{17}
$$

Here $t\in[0,1]$, let $\alpha_1=1,\alpha_2=1,\beta_1=1,\beta_2=1$, $\beta=\frac{3}{2},\lambda=2,T=1,\eta_1,\eta_2=0$, $L_{\varphi},L_{\varphi^*},=0.01$, $f(t,x))=L\left(t^2+\sin t+1+\tan^{-1}x\right)$.

A simple calculations show that

$$
L_{\mathfrak{T}}:=\left(\frac{T^{\frac{3}{2}-1}}{2\Gamma(\frac{3}{2})}(1-e^{-2})\,(1+2.312)+\frac{1^{\frac{3}{2}-1}}{\Gamma(\frac{3}{2})}2.312\right)0.01+(1+1.312)\,0.01+(0.656+1.152+0.002)\,0.01<1,
$$

where we used the inequality $0.88 < \Gamma(\frac{3}{2}) < 0.89$.

To apply Theorem 6 we need to show conditions $(\mathrm{H}_1)-(\mathrm{H}_3)$ are satisfied. Indeed, f is jointly continuous and

(H_1) $|f(t,x)-f(t,y)| = 0.01\left|tan^{-1}x - tan^{-1}y\right| \leq 0.01|x-y|$.

(H_2) $L_{\mathfrak{T}} = 0.042 + 0.248 < 1$.

Therefore, by (6), ISFDE (17) has a unique solution on $[0,1]$.

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The Differentiability and Gradient for Fuzzy Mappings Based on The Generalized Difference of Fuzzy Numbers *

Shexiang Hai†, Fangdi Kong

[a] School of Science, Lanzhou University of Technology, Lanzhou, 730050, P.R. China

Abstract In this paper, the concepts of differentiability and gradient for fuzzy mappings are presented and discussed using the characteristic theorem for generalized difference of n dimensional fuzzy numbers. The relationships of gradient, *support-function-wise* gradient and *level-wise* gradient are characterized.
Keywords: Fuzzy numbers, Fuzzy mappings, Differentiability, Gradient.

1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh [1], many studies have focused on the theoretical aspects and applications of fuzzy sets. Soon after, Zadeh proposed the notion of fuzzy numbers in [2, 3, 4]. Since then, fuzzy numbers have been extensively investigated by many authors. Since then, fuzzy numbers have been extensively investigated by many authors. Fuzzy numbers are a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models.

As part of the development of theories about fuzzy numbers and its applications, researchers began to study the differentiability and integrability of fuzzy mappings. Initially, the derivative for fuzzy mappings from an open subset of a normed space into the n dimension fuzzy number space E^n was developed by Puri and Ralescu [5], which generalized and extended the concept of Hukuhara differentiability for set-valued mappings. In 1987, Kaleva [6] discussed the G-derivative, and obtained a sufficient condition for the H-differentiability of the fuzzy mappings from $[a, b]$ into E^n as well as a necessary condition for the H-differentiability of fuzzy mapping from $[a, b]$ into E^1. In 2003, Wang and Wu [7] put forward the concepts of directional derivative, differential and sub-differential of fuzzy mappings from R^n into E^1 by using Hukuhara difference. However, the Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [6] and the H-difference of two fuzzy numbers does not always exist [8]. The g-difference proposed in [8, 9] overcomes these shortcomings of the above discussed concepts and the g-difference of two fuzzy numbers always exists. Based on the novel generalizations of the Hukuhara difference for fuzzy sets, Bede [10] introduced and studied new generalized differentiability concepts for fuzzy valued functions in 2013.

The purpose of the present paper is to use the fuzzy g-difference introduced in [10] to define and study differentiability and gradient for fuzzy mappings. First of all, we give the preliminary terminology used in the present paper. And then, in Section 3, the differentiability and gradient were presented and the relations among gradient, *support-function-wise* gradient and *level-wise* gradient for fuzzy mappings are examined.

2. Preliminaries

In this section, basic definitions and operations for fuzzy numbers are presented [11, 12, 13, 14].

Throughout this paper, $F(R^n)$ denote the set of all fuzzy subsets on n dimensional Euclidean space R^n. A fuzzy subset $\widetilde{u}$ (in short, a fuzzy set) on R^n is a function $\widetilde{u} : R^n \to [0, 1]$. For each fuzzy sets $\widetilde{u}$, we

*This work is supported by National Natural Science Fund of China (11761047).
†Corresponding author. Tel.: +86 931 2973590. E-mail address: haishexiang@lut.cn.

denote its r-level set as $[\widetilde{u}]^r = \{x \in R^n : \widetilde{u}(x) \geq r\}$ for any $r \in (0,1]$. The support of $\widetilde{u}$ is denoted by supp$\widetilde{u} = \{x \in R^n : \widetilde{u}(x) > 0\}$. The closure of supp$\widetilde{u}$ defines the 0-level of $\widetilde{u}$, i.e. $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u})$. Here $cl(M)$ denotes the closure of set M. Fuzzy set $\widetilde{u} \in F(R^n)$ is called a fuzzy number if
(1) $\widetilde{u}$ is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$ such that $\widetilde{u}(x_0) = 1$,
(2) $\widetilde{u}$ is a convex fuzzy set, i.e., $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$,
(3) $\widetilde{u}$ is upper semicontinuous ,
(4) $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u}) = cl(\bigcup_{r\in(0,1]}[\widetilde{u}]^r)$ is compact.

We will denote E^n the set of fuzzy numbers [11, 12, 13].

It is clear that any $u \in R^n$ can be regarded as a fuzzy number $\widetilde{u}$ defined by

$$\widetilde{u}(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the fuzzy number $\widetilde{0}$ is defined as $\widetilde{0}(x) = 1$ if $x = 0$, and $\widetilde{0}(x) = 0$ otherwise.

Theorem 2.1.[6, 13] If $\widetilde{u} \in E^n$, then
(1) $[\widetilde{u}]^r$ is a nonempty compact convex subset of R^n for any $r \in (0,1]$,
(2) $[\widetilde{u}]^{r_1} \subseteq [\widetilde{u}]^{r_2}$, whenever $0 \leq r_2 \leq r_1 \leq 1$,
(3) if $r_k > 0$ and r_k is a nondecreasing sequence converging to $r \in (0,1]$, then $\bigcap_{k=1}^{\infty}[\widetilde{u}]^{r_n} = [\widetilde{u}]^r$.

Conversely, if $\{[A]^r \subseteq R^n : r \in [0,1]\}$ satisfies the conditions (1)-(3), then there exists a unique $\widetilde{u} \in E^n$ such that $[\widetilde{u}]^r = [A]^r$ for each $r \in (0,1]$ and $[\widetilde{u}]^0 = cl(\bigcup_{r\in(0,1]}[\widetilde{u}]^r) \subseteq A^0$.

Let $\widetilde{u}$, $\widetilde{v} \in E^n$ and $k \in R$. For any $x \in R^n$, the addition $\widetilde{u}+\widetilde{v}$ and scalar multiplication $k\widetilde{u}$ can be defined, respectively, as:

$$(\widetilde{u}+\widetilde{v})(x) = \sup_{s+t=x} \min\{\widetilde{u}(s), \widetilde{v}(t)\},$$

$$(k\widetilde{u})(x) = \widetilde{u}(\frac{x}{k}), k \neq 0,$$

$$(0\widetilde{u})(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

It is well known that for any $\widetilde{u}, \widetilde{v} \in E^n$ and $k \in R$, the addition $\widetilde{u}+\widetilde{v}$ and the scalar multiplication $k\widetilde{u}$ have the level sets

$$[\widetilde{u}+\widetilde{v}]^r = [\widetilde{u}]^r + [\widetilde{v}]^r = \{x+y : x \in [\widetilde{u}]^r, y \in [\widetilde{v}]^r\},$$

$$[k\widetilde{u}]^r = k[\widetilde{u}]^r = \{kx : x \in [\widetilde{u}]^r\},$$

for any $r \in [0,1]$.

The Hausdorff distance $D : E^n \times E^n \to [0, +\infty)$ on E^n is defined by

$$D(\widetilde{u}, \widetilde{v}) = \sup_{r\in[0,1]} d([\widetilde{u}]^r, [\widetilde{v}]^r),$$

where d is the Hausdorff metric given by

$$\begin{aligned} d([\widetilde{u}]^r, [\widetilde{v}]^r) &= \inf\{\varepsilon : [\widetilde{u}]^r \subset N([\widetilde{v}]^r, \varepsilon), [\widetilde{v}]^r \subset N([\widetilde{u}]^r, \varepsilon)\} \\ &= \max\{\sup_{a\in[\widetilde{u}]^r} \inf_{b\in[\widetilde{v}]^r} \|a-b\|, \sup_{b\in[\widetilde{v}]^r} \inf_{a\in[\widetilde{u}]^r} \|a-b\|\}. \end{aligned}$$

$N([\widetilde{u}]^r, \varepsilon) = \{x \in R^n : d(x, [\widetilde{u}]^r) = \inf_{y\in[\widetilde{u}]^r} d(x,y) \leq \varepsilon\}$ is the ε-neighborhood of $[\widetilde{u}]^r$. Then (E^n, D) is a complete metric space, and satisfies $D(\widetilde{u}+\widetilde{w}, \widetilde{v}+\widetilde{w}) = D(\widetilde{u}, \widetilde{v})$, $D(k\widetilde{u}, k\widetilde{v}) = |k|D(\widetilde{u}, \widetilde{v})$ for any $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E^n$ and $k \in R$.

Let $S^{n-1} = \{x \in R^n : \|x\| = 1\}$ be the unit sphere of R^n and $\langle \cdot, \cdot \rangle$ be the inner product in R^n, i.e. $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, where $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$. Suppose $\widetilde{u} \in E^n$, $r \in [0,1]$ and $x \in S^{n-1}$, the support function of $\widetilde{u}$ is defined by

$$\widetilde{u}^*(r, x) = \sup_{a \in [\widetilde{u}]^r} \langle a, x\rangle.$$

Theorem 2.2.[14] Suppose $\widetilde{u} \in E^n$, $r \in [0,1]$, then

$$[\widetilde{u}]^r = \{y \in R^n : \langle y, x\rangle \leq \widetilde{u}^*(r, x), x \in S^{n-1}\}.$$

The theorem below will give some basic properties of the support function.

Theorem 2.3.[14, 15] Suppose $\widetilde{u} \in E^n$, then
(1) $\widetilde{u}^*(r, x + y) \leq \widetilde{u}^*(r, x) + \widetilde{u}^*(r, y)$,
(2) $\widetilde{u}^*(r, x) \leq \sup_{a \in [\widetilde{u}]^r} \| a \|$, i.e. $\widetilde{u}^*(r, x)$ is bounded on S^{n-1} for each fixed $r \in [0,1]$,
(3) $\widetilde{u}^*(r, x)$ is nonincreasing and left continuous in $r \in [0,1]$, right continuous at $r = 0$, for each fixed $x \in S^{n-1}$,
(4) $\widetilde{u}^*(r, x)$ is Lipschitz continuous in x, i.e.

$$|\widetilde{u}^*(r, x) - \widetilde{u}^*(r, y)| \leq (\sup_{a \in [\widetilde{u}]^r} \|a\|)\|x - y\|,$$

(5) if $\widetilde{u}, \widetilde{v} \in E^n$, $r \in [0,1]$, then

$$d([\widetilde{u}]^r, [\widetilde{v}]^r) = \sup_{x \in S^{n-1}} |\widetilde{u}^*(r, x) - \widetilde{v}^*(r, x)|,$$

(6) $(\widetilde{u} + \widetilde{v})^*(r, x) = \widetilde{u}^*(r, x) + \widetilde{v}^*(r, x)$,
(7) $(k\widetilde{u})^*(r, x) = k\widetilde{u}^*(r, x)$, for any $k \geq 0$,
(8) $-\widetilde{u}^*(r, -x) \leq \widetilde{u}^*(r, x)$,
(9) $(-\widetilde{u})^*(r, x) = \widetilde{u}^*(r, -x)$.

Definition 2.1. [10] The generalized difference (g-difference for short) of two fuzzy numbers $\widetilde{u}, \widetilde{v} \in E^n$ is given by its level sets as

$$[\widetilde{u} \ominus_g \widetilde{v}]^r = cl(\bigcup_{\beta \geq r} ([\widetilde{u}]^\beta \ominus_{gH} [\widetilde{v}]^\beta)), \quad r \in [0,1],$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[\widetilde{u}]^\beta$ and $[\widetilde{v}]^\beta$.

Remark 2.1. A necessary condition for $\widetilde{u} \ominus_g \widetilde{v}$ to exist is that either $[\widetilde{u}]^r$ contains a translate of $[\widetilde{v}]^r$ or $[\widetilde{v}]^r$ contains a translate of $[\widetilde{u}]^r$ for any $r \in [0,1]$.

Theorem 2.4. [15] Let $\widetilde{u}, \widetilde{v} \in E^n$. If the g-difference $\widetilde{u} \ominus_g \widetilde{v}$ of $\widetilde{u}$ and $\widetilde{v}$ exists, then for any $r \in [0,1]$ and $x \in S^{n-1}$, we have

$$\begin{aligned}(\widetilde{u} \ominus_g \widetilde{v})^*(r, x) &= \begin{cases} \quad (1) & \sup_{\beta \geq r}(\widetilde{u}^*(\beta, x) - \widetilde{v}^*(\beta, x)), \\ \text{or } (2) & \sup_{\beta \geq r}((-\widetilde{v})^*(\beta, x) - (-\widetilde{u})^*(\beta, x)), \end{cases} \\ &= \begin{cases} \quad (1) & \sup_{\beta \geq r}(\widetilde{u}^*(\beta, x) - \widetilde{v}^*(\beta, x)), \\ \text{or } (2) & \sup_{\beta \geq r}(\widetilde{v}^*(\beta, -x) - \widetilde{u}^*(\beta, -x)). \end{cases}\end{aligned}$$

Theorem 2.5.[15] Let $\widetilde{u}, \widetilde{v} \in E^n$. Then
(1) if the g-difference exists, it is unique,
(2) $\widetilde{u} \ominus_g \widetilde{u} = 0$,
(3) $(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{v} = \widetilde{u}$, $(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{u} = \widetilde{v}$,
(4) $\widetilde{u} \ominus_g \widetilde{v} = -(\widetilde{v} \ominus_g \widetilde{u})$.

3. The differentiability and gradient for fuzzy mappings

In [5], Puri and Ralescu defined the g-derivative of fuzzy mappings from an open subset of a normed space into n-dimension fuzzy number space E^n by using Hukuhara difference. In [7], Wang and Wu defined the directional g-derivative of fuzzy mappings from R^n into E^1. Based on the generalizations of the Hukuhara difference for fuzzy sets, Bede [10] introduced and studied new generalized differentiability concepts for fuzzy valued functions from R into E^1. The new generalized differentiability concept is a useful and applicable tool dealing with fuzzy differential equations and fuzzy optimization problems. In the following, using the characteristic theorem for generalized difference of n dimensional fuzzy numbers introduced in [15], we define and study differentiability and gradient for fuzzy mappings.

Definition 3.1. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If g-difference $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there exist $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{D(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0), \sum_{j=1}^m \widetilde{u}_j (t_j - t_j^0))}{d(t, t_0)} = 0,$$

then we say that $\widetilde{F}$ is differentiable at t_0 and the fuzzy vector $(\widetilde{u}_1,\ \widetilde{u}_2, \cdots, \widetilde{u}_m)$ is the gradient of $\widetilde{F}$ at t_0, denoted by $\nabla\widetilde{F}(t_0)$, i.e., $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$.

Remark 3.1. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. Then the gradient $\nabla\widetilde{F}(t_0)$ exists at t_0 if and only if $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there are $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\widetilde{u}_j = \lim_{h \to 0} \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}.$$

Here the limit is taken in the metric space (E^n, D).

Theorem 3.1. The gradient $\nabla\widetilde{F}(t)$ of fuzzy mapping $\widetilde{F}: M \to E^n$ is unique if it exists.

Proof. Suppose we have two gradients $(\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ and $(\widetilde{v}_1,\ \widetilde{v}_2,\ \cdots,\ \widetilde{v}_m)$ for fuzzy mapping $\widetilde{F}$ at t_0. For any $\varepsilon > 0$, according to Remark 3.1, there exist two positive real numbers δ_1 and δ_2, when $|h| < \delta_1$, we have

$$D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{u}_j) < \frac{|h|}{2}\varepsilon \quad (j = 1, 2, \cdots, m),$$

when $|h| < \delta_2$, we have

$$D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{v}_j) < \frac{|h|}{2}\varepsilon \quad (j = 1, 2, \cdots, m).$$

Setting $|h| < \min(\delta_1, \delta_2)$, we obtain,

$$\begin{aligned} & D(\widetilde{u}_j, \widetilde{v}_j) \\ = & \tfrac{1}{|h|} D(h\widetilde{u}_j, h\widetilde{v}_j) \\ \leq & \tfrac{1}{|h|} D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{u}_j) \\ + & \tfrac{1}{|h|} D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{v}_j) \\ < & \varepsilon. \end{aligned}$$

Then $\widetilde{u}_j = \widetilde{v}_j$ $(j = 1, 2, \cdots, m)$, which implies that the gradient $\nabla\widetilde{F}(t)$ of fuzzy mapping $\widetilde{F}$ at t_0 is unique.

Definition 3.2. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If there exist $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{|\widetilde{F}(t)^*(r, x) - \widetilde{F}(t_0)^*(r, x) - \sum_{j=1}^m \widetilde{u}_j^*(r, x)(t_j - t_j^0)|}{d(t, t_0)} = 0,$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, then we say that $\widetilde{F}$ is *support-function-wise* differentiable (*s*-differentiable for short) at t_0 and the fuzzy vector $(\widetilde{u}_1,\ \widetilde{u}_2, \cdots, \widetilde{u}_m)$ is the *support-function-wise* gradient of $\widetilde{F}$ at t_0, denoted by $\nabla_s \widetilde{F}(t_0)$, i.e., $\nabla_s \widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$.

Remark 3.2. Let $\widetilde{F} : M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \mathrm{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \mathrm{int}M$. Then the *support-function-wise*-gradient $\nabla_s \widetilde{F}(t_0)$ exists at t_0 if and only if there are $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\widetilde{u}_j^*(r,x) = \lim_{h\to 0} \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0)^*(r,x) - \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)^*(r,x)}{h},$$

uniformly for any $r \in [0,1]$ any $x \in S^{n-1}$.

Theorem 3.2. The *support-function-wise* gradient $\nabla_s \widetilde{F}(t)$ of fuzzy mapping $\widetilde{F}$ is unique if it exists.

Theorem 3.3. If fuzzy mapping $\widetilde{F} : M \to E^n$ is *s*-differentiable at $t_0 \in intM$, then $-\widetilde{F}$ is *s*-differentiable at t_0 and

$$\nabla_s(-\widetilde{F}(t_0)) = -\nabla_s \widetilde{F}(t_0).$$

Proof. If $\widetilde{F} : M \to E^n$ is *s*-differentiable at t_0, then there exist $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t\to t_0} \frac{|\widetilde{F}(t)^*(r,x) - \widetilde{F}(t_0)^*(r,x) - \sum_{j=1}^m \widetilde{u}_j^*(r,x)(t_j - t_j^0)|}{d(t,t_0)} = 0,$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, where $t = (t_1,\ t_2,\ \cdots,\ t_m) \in \mathrm{int}M$, then

$$\widetilde{F}(t)^*(r,x) - \widetilde{F}(t_0)^*(r,x) = \sum_{j=1}^m \widetilde{u}_j^*(r,x)(t_j - t_j^0) + \circ(d(t,t_0)),$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$. It follows from Theorem 2.3 that

$$\begin{aligned} & (-\widetilde{F}(t))^*(r,x) - (-\widetilde{F}(t_0))^*(r,x) \\ = \ & \widetilde{F}(t)^*(r,-x) - \widetilde{F}(t_0)^*(r,-x) \\ = \ & \textstyle\sum_{j=1}^m \widetilde{u}_j^*(r,-x)(t_j - t_j^0) + \circ(d(t,t_0)) \\ = \ & \textstyle\sum_{j=1}^m (-\widetilde{u}_j)^*(r,x)(t_j - t_j^0) + \circ(d(t,t_0)), \end{aligned}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$. Thus

$$\lim_{t\to t_0} \frac{|(-\widetilde{F}(t))^*(r,x) - (-\widetilde{F}(t_0))^*(r,x) - \sum_{j=1}^m (-\widetilde{u}_j)^*(r,x)(t_j - t_j^0)|}{d(t,t_0)} = 0,$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, which implies that $-\widetilde{F}$ is *s*-differentiable at t_0 and $\nabla_s(-\widetilde{F}(t_0)) = -\nabla_s \widetilde{F}(t_0)$.

Theorem 3.4. Let $\widetilde{F} : M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \mathrm{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \mathrm{int}M$. If the *support-function-wise* gradient $\nabla_s \widetilde{F}(t)$ exists at $t_0 \in \mathrm{int}M$ and *g*-difference $\widetilde{F}(t_0 + h) \ominus_g \widetilde{F}(t_0)$ exists, then the gradient $\nabla \widetilde{F}(t)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j = \widetilde{v}_j \ \ (j = 1, 2, \cdots, m),$$

where $\nabla \widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$, $\nabla_s \widetilde{F}(t_0) = (\widetilde{v}_1,\ \widetilde{v}_2,\ \cdots,\ \widetilde{v}_m)$.

Proof. Let $\frac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h} = \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0)\ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)}{h} = (\widetilde{u}_j)_h \in E^n$. We can show that the class of sets

$$A_r = \{y \in R^n : \langle y, x\rangle \le \lim_{h\to 0}(\frac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x), x \in S^{n-1}\}$$

satisfies the conditions of Theorem 2.1.

(1) It follows from Theorem 2.1 that

$$[(\widetilde{u}_j)_h]^r = \{y \in R^n : \langle y, x\rangle \leq (\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x), x \in S^{n-1}\}$$

is a nonempty compact convex subset of R^n for any $r \in (0, 1]$, then

$$A_r = \{y \in R^n : \langle y, x\rangle \leq \lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x), x \in S^{n-1}\}$$

is also a nonempty compact convex subset of R^n for any $r \in (0, 1]$.

(2) When $0 \leq r_2 \leq r_1 \leq 1$, $[(\widetilde{u}_j)_h]^{r_1} \subseteq [(\widetilde{u}_j)_h]^{r_2}$, then

$$(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_1, x) \leq (\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_2, x).$$

for any $x \in S^{n-1}$. Thus,

$$\lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_1, x) \leq \lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_2, x),$$

which implies that

$$\begin{aligned} A_{r_1} &= \{y \in R^n : \langle y, x\rangle \leq \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h})^*(r_1, x), x \in S^{n-1}\} \\ &\subseteq \{y \in R^n : \langle y, x\rangle \leq \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h})^*(r_2, x), x \in S^{n-1}\} \\ &= A_{r_2}. \end{aligned}$$

(3) For any r_k increasing to $r \in (0, 1]$, since $\bigcap_{k=1}^{\infty}[(\widetilde{u}_j)_h]^{r_k} = [(\widetilde{u}_j)_h]^r$, that

$$\lim_{k\to\infty}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_k, x) = (\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x),$$

for any $x \in S^{n-1}$. Thus

$$\lim_{k\to\infty}\lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_k, x) = \lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x),$$

which implies that

$$\bigcap_{k=1}^{\infty} A_{r_n} = A_r.$$

Then, there are $\widetilde{u}_j \in E^n$, such that $[\widetilde{u}_j]^r = A_r$ and $[\widetilde{u}_j]^0 = \overline{\bigcup_{r\in(0,1]}[\widetilde{u}]^r} \subseteq A_0$ $(j = 1, 2, \cdots, m)$ for any $r \in (0, 1]$.

When $h > 0$, it follows from Theorem 2.3 that,

$$(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x) = \frac{1}{h}(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r, x),$$

for any $r \in [0, 1]$ and $x \in S^{n-1}$. For any $r \in [0, 1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r, x) = \sup_{\beta\geq r}(\widetilde{F}(t)^*(\beta, x) - \widetilde{F}(t_0)^*(\beta, x)),$$

then

$$\begin{aligned} \widetilde{u}_j^*(r, x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h})^*(r, x) \\ &= \lim_{h\to 0}\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\widetilde{v}_j^*(\beta, x). \end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $h < \delta$, we have

$$\begin{aligned} & D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\ = \; & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\ = \; & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |\sup_{\beta\geq r} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x)| \\ < \; & \varepsilon. \end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. On the other hand, for any $r \in [0,1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) = \sup_{\beta\geq r}(\widetilde{F}(t_0)^*(\beta,-x) - \widetilde{F}(t)^*(\beta,-x)),$$

we have from Theorem 2.3 and Theorem 2.5 that

$$\begin{aligned} \widetilde{u}_j^*(r,x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) \\ &= \lim_{h\to 0} \tfrac{1}{h}(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) \\ &= \lim_{h\to 0} \tfrac{1}{h}[-(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))]^*(r,x) \\ &= \lim_{h\to 0} \tfrac{1}{h}(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))^*(r,-x) \\ &= \lim_{h\to 0} \sup_{\beta\geq r} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r} \lim_{h\to 0} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x). \end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $h < \delta$, we have

$$\begin{aligned} & D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\ = \; & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\ = \; & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |\sup_{\beta\geq r} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x)| \\ < \; & \varepsilon. \end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. When $h < 0$, it follows from Theorem 2.3 and Theorem 2.5 that,

$$\begin{aligned} (\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) &= -\tfrac{1}{h}(-(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)))^*(r,x) \\ &= -\tfrac{1}{h}(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))^*(r,x), \end{aligned}$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. For any $r \in [0,1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) = \sup_{\beta\geq r}(\widetilde{F}(t)^*(\beta,x) - \widetilde{F}(t_0)^*(\beta,x)),$$

i.e.

$$(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))^*(r,x) = \sup_{\beta \geq r}(\widetilde{F}(t_0)^*(\beta,x) - \widetilde{F}(t)^*(\beta,x)),$$

then

$$\begin{aligned} \widetilde{u}_j^*(r,x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) \\ &= \lim_{h\to 0}\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x). \end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $-h < \delta$, we have

$$\begin{aligned} & D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\ = \;& \sup_{r\in[0,1]}\sup_{x\in S^{n-1}} |(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\ = \;& \sup_{r\in[0,1]}\sup_{x\in S^{n-1}} |\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x)| \\ < \;& \varepsilon. \end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. On the other hand, for any $r \in [0,1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) = \sup_{\beta \geq r}(\widetilde{F}(t_0)^*(\beta,-x) - \widetilde{F}(t)^*(\beta,-x)),$$

we have from Theorem 2.3 that

$$\begin{aligned} \widetilde{u}_j^*(r,x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) \\ &= \lim_{h\to 0}[-\tfrac{1}{h}(\widetilde{F}(t)\ominus_g \widetilde{F}(t_0))^*(r,x)] \\ &= \lim_{h\to 0}[-\tfrac{1}{h}\sup_{\beta\geq r}(\widetilde{F}(t_0)^*(\beta,-x) - \widetilde{F}(t)^*(\beta,-x))] \\ &= \lim_{h\to 0}\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\lim_{h\to 0}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x). \end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $-h < \delta$, we have

$$\begin{aligned} & D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\ = \;& \sup_{r\in[0,1]}\sup_{x\in S^{n-1}} |(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\ = \;& \sup_{r\in[0,1]}\sup_{x\in S^{n-1}} |\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x)| \\ < \;& \varepsilon. \end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$.

The converse result of Theorem 3.4 is not necessarily true, and hence the g-differentiability and the s-differentiability are not equivalent concepts.

Definition 3.3. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If for any $r \in [0,1]$, $\widetilde{F}_r(t) \ominus_{gH} \widetilde{F}_r(t_0)$ ($\widetilde{F}_r(t) = [\widetilde{F}(t)]^r$) exist and there exist $\widetilde{u}_j \in E^n$ ($j = 1, 2, \cdots, m$), such that

$$\lim_{t \to t_0} \frac{d(\widetilde{F}_r(t) \ominus_{gH} \widetilde{F}_r(t_0), \sum_{j=1}^m [\widetilde{u}_j]^r (t_j - t_j^0))}{d(t, t_0)} = 0,$$

uniformly for any $r \in [0,1]$, then we say that $\widetilde{F}$ is *level-wise* differentiable at t_0 and the fuzzy vector $(\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ is the *level-wise* gradient of $\widetilde{F}$ at t_0, denoted by $\nabla_l \widetilde{F}(t_0)$, i.e., $\nabla_l \widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$.

Remark 3.3. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. Then the *level-wise* gradient $\nabla_l \widetilde{F}(t)$ exists at t_0 if and only if for any $r \in [0,1]$, $\widetilde{F}_r(t) \ominus_{gH} \widetilde{F}_r(t_0)$ exist and there are $\widetilde{u}_j \in E^n$ ($j = 1, 2, \cdots, m$), such that

$$[\widetilde{u}_j]^r = \lim_{h \to 0} \frac{\widetilde{F}_r(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_{gH} \widetilde{F}_r(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h},$$

uniformly for any $r \in [0,1]$.

Here the limit is taken in the metric space $(\mathcal{K}_c^n, d)$.

Theorem 3.5. The *level-wise* gradient $\nabla_s \widetilde{F}(t)$ of fuzzy mapping $\widetilde{F}: M \to E^n$ is unique if it exists.

Theorem 3.6. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. If the *level-wise* gradient $\nabla_l \widetilde{F}(t_0)$ exists at $t_0 \in \text{int}M$ and g-difference $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists, then the gradient $\nabla \widetilde{F}(t)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j = \widetilde{v}_j \quad (j = 1, 2, \cdots, m),$$

where $\nabla \widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$, $\nabla_l \widetilde{F}(t_0) = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_m)$.

Proof. According to Definition 2.1, for any $\varepsilon > 0$, there is $\delta > 0$, when $|h| < \delta$, we have

$$\begin{aligned} & D\left(\tfrac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}, \widetilde{v}_j\right) \\ = \; & \sup_{r \in [0,1]} d\left(cl\left(\bigcup_{\beta \geq r} \tfrac{\widetilde{F}_\beta(t) \ominus_{gH} \widetilde{F}_\beta(t_0)}{h}\right), [\widetilde{v}_j]^r\right) \\ \leq \; & \sup_{r \in [0,1]} \sup_{\beta \geq r} d\left(\tfrac{\widetilde{F}_\beta(t) \ominus_{gH} \widetilde{F}_\beta(t_0)}{h}, [\widetilde{v}_j]^\beta\right) \\ < \; & \varepsilon. \end{aligned}$$

Then, the gradient $\nabla \widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and $\widetilde{u}_j = \widetilde{v}_j$ ($j = 1, 2, \cdots, m$).

4. Conclusion

This article is to use the generalized difference of n dimensional fuzzy numbers introduced in Bede and Stefanini [10] to define the differentiability and gradient for fuzzy mappings. Additionally, we have examined the relationships between the concepts of gradient, *support-function-wise* gradient and *level-wise* gradient for fuzzy mappings. The results from our study can be applied directly to fuzzy differential equations. The next step for the continuation of the research direction proposed here is to investigate the sub-differential of n dimensional fuzzy mappings and applications in the convex fuzzy programming.

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Global Attractivity and Periodic Nature of a Higher order Difference Equation

M. M. El-Dessoky[1,2], Abdul Khaliq[3], Asim Asiri[1] and Ansar Abbas[3]
[1]Mathematics Department, Faculty of Science, King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,Mansoura University, Mansoura 35516, Egypt.
[3]Department of Mathematics, Riphah International University, Lahore,
Pakistan.
E-mails:dessokym@mans.edu.eg; khaliqsyed@gmail.com

ABSTRACT

Our aim in this paper is to study the global stability character and the periodic nature of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{b + cx_{n-k}}{dx_{n-s} + ex_{n-t}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-r},\ x_{-r+1},\ x_{-r+2},\ ...,\ x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s, t\}$ is nonnegative integer and $a,\ b,\ c,\ d,\ e$ are positive constants.Finally, some numerical examples are presented and graphed by Matlab.

Keywords: stability, periodic solutions, global attractor, difference equations.

Mathematics Subject Classification: 39A10; 40A05.

1. INTRODUCTION

Difference equations or discrete dynamical systems are diversed field which impact almost every branch of pure and applied mathematics. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vice versa. Recently many researchers have studied the global attractivity, boundedness character and the periodic nature of nonlinear difference equations see for example [1-42]. One of the reasons for this is a prerequisite for some approaches, which can be used in inspecting equations arising in real life situations that can be model mathematically. The theory of difference equations and dynamical systems is developed during the last thirty years and there is no doubt that it will continue to play an important role in mathematical models describing real life situations and in many applied sciences, such as biology, physiology, ecology, engineering, economics, physics, probability theory, genetics, computers and resource allocation.

It is very interesting and attractive for the researcher to study the behavior and solution of nonlinear rational difference equations .Most of the real life phenomana has been solved by using these equations, examples include in [3,7,11,12] . Recently, many researchers have investigated the asymptotic behavior and periodic nature of rational difference equations for example in [36] R. Khallaf Allah investigated the asymptotic behavior and periodic nature of the following difference equation

$$x_{n+1} = \frac{x_{n-2}}{1 \pm x_n x_{n-1} x_{n-2}}.$$

G. Ladas et. al [8], investigated the asymptotic behavior and boundedness of the solution of the difference equation

$$x_n = \frac{(\alpha + \beta x_n + \gamma x_{n-1)}}{(A + Bx_n + Cx_{n-1})}.$$

E. M. E. Zayed [33] studied thequalitative properties of the nonlinear difference equation

$$x_{n+1}=\frac{\alpha x_{n-\delta}}{\beta+\gamma x_{n-\tau}}.$$

Yalçınkaya [32] has studied the following difference equation

$$x_{n+1}=\alpha+\frac{x_{n-m}}{x_n^k}.$$

Taixiang Sun et al [39] considered the class of nonlinear delay difference equation

$$x_{n+1}=\frac{Af_1(x_n,...,x_{n-k})+Bf_2(x_n,...,x_{n-k})f_3(x_n,...,x_{n-k})+C}{\alpha f_1(x_n,...,x_{n-k})f_2(x_n,...,x_{n-k})+\beta f_3(x_n,...,x_{n-k})+\gamma}.$$

The goal of this paper is to determine the global stability character and the periodicity of the solutions of the difference equation

$$x_{n+1}=ax_{n-l}+\frac{b+cx_{n-k}}{dx_{n-s}+ex_{n-t}},\quad n=0,\ 1,\ ..., \tag{1}$$

where the initial conditions $x_{-r},\ x_{-r+1},\ x_{-r+2},...,\ x_0$ are arbitrary positive real numbers, $r=\max\{l,\ k,\ s,\ t\}$ is nonnegative integer and $a,\ b,\ c,\ d,\ e$ are positive constants.

" Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F:I^{r+1}\to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-r},\ x_{-r+1},\ ...,\ x_0\in I$, the difference equation

$$x_{n+1}=F(x_n,x_{n-1},...,x_{n-r}),\quad n=0,1,..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-r}^{\infty}$.

A point $\overline{x}\in I$ is called an equilibrium point of Eq. (2) if

$$\overline{x}=f(\overline{x},\ \overline{x},\ ...,\ \overline{x}).$$

That is, $x_n=\overline{x}$ for $n\geq 0$, is a solution of Eq. (2), or equivalently $\overline{x}$ is a fixed point of f.

Definition 1.1. *(Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.*

Definition 1.2. *(Stability) (i) The equilibrium point $\overline{x}$ of Eq. (2) is locally stable if for every $\epsilon>0$, there exists $\delta>0$ such that for all $x_{-r},\ x_{-r+1},\ ...,x_{-1},\ x_0\in I$ with*

$$|x_{-r}-\overline{x}|+|x_{-r+1}-\overline{x}|+...+|x_0-\overline{x}|<\delta,$$

we have

$$|x_n-\overline{x}|<\epsilon \quad \text{for all}\quad n\geq -r.$$

(ii) The equilibrium point $\overline{x}$ of Eq. (2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma>0$, such that for all $x_{-r},\ x_{-r+1},\ ...,x_{-1},\ x_0\in I$ with

$$|x_{-r}-\overline{x}|+|x_{-r+1}-\overline{x}|+...+|x_0-\overline{x}|<\gamma,$$

we have

$$\lim_{n\to\infty}x_n=\overline{x}.$$

(iii) The equilibrium point $\overline{x}$ *of Eq. (2) is global attractor if for all* $x_{-r},\ x_{-r+1},\ ...,x_{-1},\ x_0 \in I$*, we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ *of Eq. (2) is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq. (2).*
(v) The equilibrium point $\overline{x}$ *of Eq.(2) is unstable if* $\overline{x}$ *is not locally stable.*

The linearized equation of Eq. (2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{r} \frac{\partial F(\overline{x},\overline{x},...,\overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [26]: Assume that $p,\ q \in\ R$ and $r \in \{0,\ 1,\ 2,\ ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-r} = 0, \quad n = 0,\ 1,\ ....$$

Remark 1. *Theorem A can be easily extended to a general linear equations of the form*

$$x_{n+r} + p_1x_{n+r-1} + ... + p_rx_n = 0, \ \ n = 0, 1, ..., \tag{4}$$

where $p_1, p_2, ..., p_r \in R$ *and* $r \in \{1, 2, ...\}$*. Then Eq.(4) is asymptotically stable provided that*

$$\sum_{i=1}^{r} |p_i| < 1.$$

Consider the following equation

$$x_{n+1} = g(x_n,\ x_{n-1},\ ...,\ x_{n-K}), \qquad n = 0,\ 1,\ 2,\ .... \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.
Theorem B [27]: Let $[\alpha,\ \beta]$ be an interval of real numbers and assume that

$$g : [\alpha,\ \beta]^{k+1} \to [\alpha,\ \beta]$$

is a continuous function satisfying the following properties :

(a) $g(x_1,\ x_2,\ ...,\ x_{k+1})$ is non-increasing in one component (for example x_σ) for each x_r $(r \neq \sigma)$ in $[\alpha,\ \beta]$, and is non-increasing in the remaining components for each $x_\sigma \in [\alpha, \beta]$

(b) If $(m,\ M) \in [\alpha,\ \beta] \times [\alpha,\ \beta]$ is a solution of the system

$$M = g(m,\ m,\ ...,\ m,\ M,\ m,\ ...,\ m,\ m) \ \ and \ \ m = g(M,\ M,\ ...,\ M,\ m,\ M,\ ...,\ M,\ M),$$

then

$$m = M.$$

Then Eq. (5) has a unique equilibrium $\overline{x} \in [\alpha,\ \beta]$ and every solution of Eq. (5) converges to $\overline{x}$."

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ. (1)

In this section we study the local stability character of the solutions of Eq. (1). The equilibrium points of Eq. (1) are given by the relation

$$\overline{x} = a\overline{x} + \frac{b + c\overline{x}}{d\overline{x} + e\overline{x}}.$$

If $a \neq 1,\ d + e - ae - ad \neq 0$, then the positive equilibrium point of Eq. (1) is given by

$$\overline{x} = \frac{-c + \sqrt{4be + 4bd + c^2 - 4abe - 4abd}}{2(a-1)(d+e)}.$$

Let $f : (0,\ \infty)^4 \longrightarrow (0,\ \infty)$ be a function defined by

$$f(u_0,\ u_1,\ u_2,\ u_3) = au_0 + \frac{b + cu_1}{du_2 + eu_3}.$$

Then we see that at $\overline{x} = \frac{-c+\sqrt{4be+4bd+c^2-4abe-4abd}}{2(a-1)(d+e)}$

$$\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_0} = a = -c_0, \quad \frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_1} = \frac{2c(a(d+e)-1)}{\left(-c + \sqrt{(4be + 4bd + c^2 - 4abe - 4abd)}\right)(d+e)} = -c_1$$

$$\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_2} = -\frac{(2(a(d+e)-1))d\left(c\sqrt{(4be + 4bd + c^2 - 4abe - 4abd)} + 2ba(d+e) - c^2 - 2b\right)}{(d+e)^2\left(-c + \sqrt{4be + 4bd + c^2 - 4abe - 4abd}\right)^2} = -c_2,$$

$$\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_3} = -\frac{(2(a(d+e)-1))e(c\sqrt{(4be + 4bd + c^2 - 4abe - 4abd)} + 2ba(d+e) - c^2 - 2b)}{(d+e)^2\left(-c + \sqrt{4be + 4bd + c^2 - 4abe - 4abd}\right)^2} = -c_3$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + c_0 y_{n-l} + c_1 y_{n-k} + c_2 y_{n-s} + c_3 y_{n-t} = 0.$$

3. EXISTENCE OF PERIODIC SOLUTIONS

In this section we study the existence of periodic solutions of Eq. (1).

THEOREM 3.1. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(a+1) - 4a^2 b(d+e) > 0, \qquad k,\ l,\ s,\ t - \text{even}. \tag{7}$$

Proof: First suppose that there exists a prime period two solution

$$...,\ p,\ q,\ p,\ q,\ ...,$$

of Eq. (1). We will prove that Condition (7) holds.

We see from Eq. (1) (when $k,\ l,\ s,\ t-$even) that

$$p = aq + \frac{b + cq}{dq + eq}, \quad q = ap + \frac{b + cp}{dp + ep}$$

Then

$$p = aq + \frac{b + cq}{(d+e)q}, \quad q = ap + \frac{b + cp}{(d+e)p}$$

$$(d+e)pq = a(d+e)q^2 + b + cq, \tag{8}$$

and

$$(d+e)pq = a(d+e)p^2 + b + cp. \tag{9}$$

Subtracting (9) from (8) gives

$$0 = a(d+e)(p^2 - q^2) + c(p-q).$$

Since $p \neq q$, it follows that

$$p + q = -\frac{c}{a(d+e)}. \tag{10}$$

Again, adding (8) and (9) yields

$$2(d+e)pq = a(d+e)(p+q)^2 - 2a(d+e)pq + 2b + c(p+q) \tag{11}$$

It follows by (10), (11) and the relation $p^2 + q^2 = (p+q)^2 - 2pq$ for all $p,\ q \in R$ that

$$pq = \frac{b}{(a+1)(d+e)}. \tag{12}$$

Now it is clear from Eq. (10) and Eq. (12) that p and q are the two positive distinct roots of the quadratic equation

$$t^2 + \left(\tfrac{c}{a(d+e)}\right) t + \left(\tfrac{b}{(a+1)(d+e)}\right) = 0, \tag{13}$$

$$a(d+e)(a+1)t^2 + c(a+1)t + ab = 0,$$

and so

$$(c(a+1))^2 - 4a^2 b(d+e)(a+1) > 0$$

thus

$$c^2(a+1) - 4a^2 b(d+e) > 0$$

Therefore Inequality (7) holds.

Second suppose that Inequality (7) is true. We will show that Eq. (1) has a prime period two solution. Assume that

$$p = \frac{-c(a+1) + \sqrt{\beta}}{2a(a+1)(d+e)} = \frac{-cA + \sqrt{\beta}}{2aAB},$$

and

$$q = \frac{-cA - \sqrt{\beta}}{2aAB}, \quad \text{where} \quad A = (a+1), \quad B = (d+e)$$

where $\beta = c^2(a+1)^2 - 4a^2 b(a+1)(d+e)$.

We see from Inequality (7) that

$$(c(a+1))^2 - 4a^2 b(d+e)(a+1) > 0$$

then after dividing by $(a+1)$we see that

$$\Rightarrow \qquad c^2 > 4a^2 b(d+e)$$

Therefore p and q are distinct real numbers.
Set

$$\begin{aligned} x_{-l} &= p,\ x_{-l+1} = q, \quad ,x_{-k} = p,\ x_{-k+1} = q, \\ x_{-s} &= p,\ x_{-s+1} = q, \quad x_{-t} = p,\ x_{-t+1} = q \text{ and } \quad x_0 = p. \end{aligned}$$

We wish to show that

$$x_1 = x_{-1} = q \quad \text{and} \quad x_2 = x_0 = p.$$

It follows from Eq. (1) that

$$x_1 = ax_{-l} + \frac{b + cx_{-k}}{dx_{-s} + ex_{-t}} = ap + \frac{b + cp}{dp + ep} = ap + \frac{b + cp}{(d + e)p}$$

$$= ap + \frac{b + c(\frac{-cA+\sqrt{\beta}}{2aAB})}{(d + e)\left(\frac{-cA+\sqrt{\beta}}{2aAB}\right)}.$$

Multiplying the denominator and numerator of the right side by $2aAB$ gives

$$x_1 = ap + \frac{2abAB+c(-cA+\sqrt{\beta})}{(d+e)(-cA+\sqrt{\beta})},$$

Multiplying the denominator and numerator of the right side by $(-cA - \sqrt{\beta})$

and by Replacing $A = (a + 1)$, $B = (d + e)$ and $\beta = c^2(a + 1)^2 - 4a^2b(a + 1)(d + e)$in denominator and numerator of above equation gives

$$\begin{aligned} x_1 &= ap + \frac{2abAB(-cA-\sqrt{\beta})+c(c^2A^2-\beta)}{(d+e)(c^2A^2-\beta)}, \\ &= ap + \frac{2ab(a+1)(d+e)(-cA-\sqrt{\beta})+c(c^2(a+1)^2-c^2(a+1)^2+4a^2b(a+1)(d+e))}{(d+e)(c^2(a+1)^2-c^2(a+1)^2+4a^2b(a+1)(d+e))}, \end{aligned}$$

$$= ap + \frac{2ab(a+1)(d+e)(-cA-\sqrt{\beta})+4a^2bc(a+1)(d+e)}{4a^2b(a+1)(d+e)^2},$$

Dividing numerator and denominator by $(2ab(a + 1)(d + e))$ we get

$$= ap + \frac{-cA-\sqrt{\beta}+2ac}{2a(d+e)}$$

$$= \frac{2a^2(d+e)p-cA-\sqrt{\beta}+2ac}{2a(d+e)}$$

Now inserting the value of p we get

$$\begin{aligned} x_1 &= \frac{1}{2a(d + e)} \left(\frac{-ca(a+1)+a\sqrt{\beta}-c(a+1)^2-(a+1)\sqrt{\beta}+2ac(a+1)}{(a+1)}\right) \\ &= \frac{1}{2a(a + 1)(d + e)} \left(-\sqrt{\beta} - c(a + 1)^2 + ca(a + 1)\right) \\ &= \frac{-\sqrt{\beta} - c(a + 1)}{2a(a + 1)(d + e)} \end{aligned}$$

But $(a + 1) = A$ and $(d + e) = B$ we get

$$x_1 = \frac{-cA-\sqrt{\beta}}{2aAB} = q$$

Similarly as before one can easily show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -1.$$

Thus Eq. (1) has the positive prime period two solution

$$..., \ p, \ q, \ p, \ q, \ ...,$$

where p and q are the distinct roots of the quadratic equation (13) and the proof is completed.

The following Theorems can be proved similarly.

Theorem 3.2. *Eq. (1) has a prime period two solutions if and only if*

$$c^2 + 4b(d+e)(1-a) > 0 \qquad (l, \ k, \ s, \ t - odd).$$

Theorem 3.3. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(d-e)(1+a) - 4(b(ad+e)^2 - ec^2) > 0 \qquad (l, \ k, \ s - even \text{ and } t - odd).$$

Theorem 3.4. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(e-d)(1+a) - 4(b(ae+d)^2 - c^2 d) > 0 \qquad (l, \ k, \ t - even \text{ and } s - odd).$$

Theorem 3.5. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a) - 4a(ab(d+e) + c^2) > 0 \qquad (l, \ s, \ t - even \text{ and } k - odd).$$

Theorem 3.6. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(e-d) - 4bd^2(1-a) > 0 \qquad (l, \ k, \ s - odd \text{ and } t - even).$$

Theorem 3.7. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(d-e) - 4be^2(1-a) > 0 \qquad (l, \ k, \ t - odd \text{ and } s - even).$$

Theorem 3.8. *Eq. (1) has a prime period two solutions if and only if*

$$c^2 - 4(b(d+e)(a-1) + c^2) > 0 \qquad (l, \ s, \ t - odd \text{ and } k - even).$$

Theorem 3.9. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a) + 4b(d+e) > 0 \qquad (k, \ s, \ t - odd \text{ and } l - even).$$

Theorem 3.10. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a) - 4(c^2 - b(d+e)) > 0 \qquad (l, \ k - even \text{ and } s, \ t - odd).$$

Theorem 3.11. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a)(d-e) - 4(b(ad+e)^2 + ac^2 d) > 0 \qquad (l, \ s - even \text{ and } k, \ t - odd).$$

Theorem 3.12. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(e-d)-4e(be(a-1)+c^2)>0 \qquad (s,\ k-\text{even and } \ l,\ t-\text{odd}).$$

Theorem 3.13. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(d-e)-4d(bd(a-1)+c^2)>0 \qquad (l,\ s-\text{odd and } \ k,\ t-\text{even}).$$

Theorem 3.14. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(a+1)(e-d)-4(b(ae+d)^2+ac^2e)>0 \qquad (s,\ k-\text{odd and } \ l,\ t-\text{even}).$$

Theorem 3.15. *Eq. (1) has no prime period two solutions if one of the following statements holds*

$$(i)\ c\neq 0 \qquad (k,\ s,\ t-\text{even and } \ l-\text{odd}),$$

$$(ii)\ c\neq 0 \qquad (s,\ t-\text{even and } \ l,\ k-\text{odd}).$$

4. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ. (1)

In this section we investigate the global attractivity character of solutions of Eq. (1).

Theorem 4.1. *The equilibrium point $\overline{x}$ of Eq. (1) is global attractor.*

Proof: Let $p,\ q$ are a real numbers and assume that $f:[p,\ q]^4\longrightarrow[p,\ q]$ be a function defined by

$$f(u_0,\ u_1,\ u_2,\ u_3)=au_0+\frac{b+cu_1}{du_2+eu_3}.$$

We can easily see that the function $f(u_0,\ u_1,\ u_2,\ u_3)$ increasing in $u_0,\ u_1$ and decreasing in $u_2,\ u_3$.

Suppose that $(m,\ M)$ is a solution of the system

$$m=f(m,\ m,\ M,\ M) \quad \text{and} \quad M=f(M,\ M,\ m,\ m).$$

Then from Eq. (1), we see that

$$m=am+\frac{b+cm}{(d+e)M},\quad M=aM+\frac{b+cM}{(d+e)m},$$

That is

$$(1-a)m=\frac{b+cm}{(d+e)M},\quad (1-a)M=\frac{b+cM}{(d+e)m},$$

or,

$$b+cm=b+cM$$

Thus $m=M$. It follows by the Theorem B that $\overline{x}$ is a global attractor of Eq. (1) and then the proof is complete.

5. NUMERICAL EXAMPLES

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume $l = 5,\ k = 4,\ s = 3,\ t = 5,\ \ x_{-5} = 6,\ x_{-4} = 9,\ x_{-3} = 8,\ x_{-2} = 9,\ x_{-1} = 12,\ x_{-1} = 4,\ a = 0.1,\ b = 0.2,\ c = 0.9,\ d = 0.7\ \ e = 0.8$**.** [See Fig. 1]

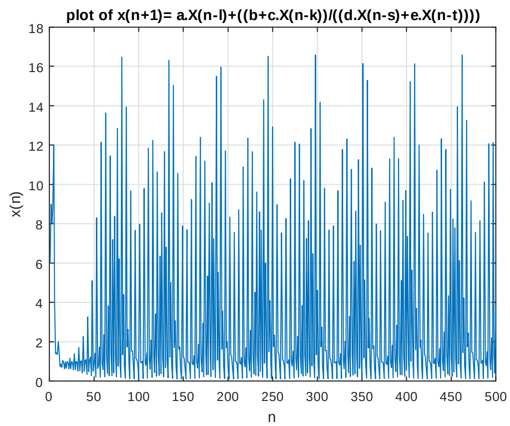

Figure 1.

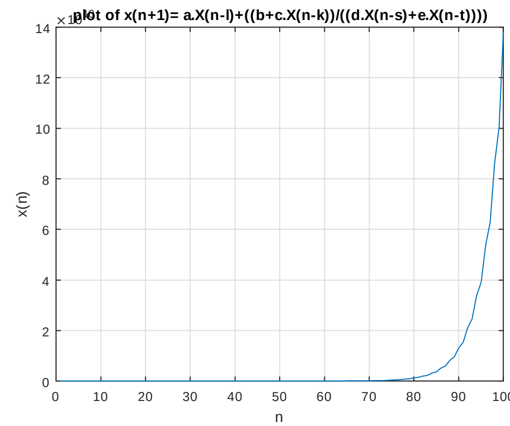

Figure 2.

Example 2. See Fig. 2, since $l = 1,\ k = 2,\ s = 1,\ t = 3,\ \ x_{-3} = 1.2,\ x_{-2} = 0.7,\ x_{-1} = 8.5,\ x_0 = 5,\ a = 1.6,\ b = 0.2,\ c = 0.9,\ d = 0.09,\ e = 0.01.$

Example 3. See Fig. 3, since $l = 1,\ k = 2, s = 1,\ t = 1\ x_{-3} = 12,\ x_{-2} = 7,\ x_{-1} = 8,\ x_0 = 3,\ a = 0.1,\ b = 0.2,\ c = 0.5,\ d = 0.6,\ e = 0.2.$

Example 4. Fig. 4. shows the solutions when $a = 0.1,\ b = 0.2,\ c = 0.5,\ d = 0.6, e = 0.9,\ l = 4,\ k = 2,\ s = 4,\ t = 2,\ \ x_{-4} = p,\ x_{-3} = q,\ x_{-2} = p,\ x_{-1} = q,\ x_0 = p.$

Since $\left(p, q = \frac{-c(a+1)\pm\sqrt{c^2(a+1)^2-4a^2b(a+1)(d+e)}}{2a(a+1)(d+e)}\right)$

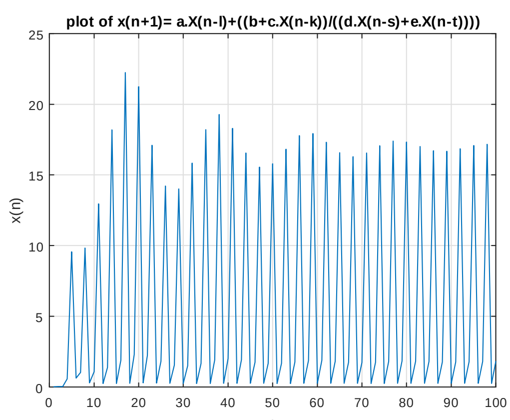

Figure 3.

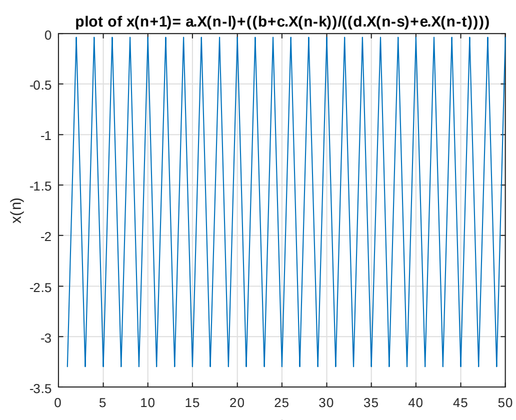

Figure 4.

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Asymptotic Representations for Fourier Approximation of Functions on the Unit Square *

Zhihua Zhang
College of Global Change and Earth System Science, Beijing Normal University, Beijing, China, 100875
E-mail: zhangzh@bnu.edu.cn

Abstract. In this paper, for any smooth function on $[0,1]^2$, we give an asymptotic representation of hyperbolic cross approximations of its Fourier series whose principal part is determined by the values of the function at vertexes of $[0,1]^2$ and present a novel approach to estimates of the upper bounds of approximation errors. At the same time, we also give an asymptotic formula of partial sum approximations whose principal part is determined by not only partial derivatives at vertexes of $[0,1]^2$, but also mean values on each side. Comparing asymptotic representations of these two kinds of approximation, we find that although in general the hyperbolic cross approximation is better than the partial sum approximation, the partial sum approximation possibly work better under some cases, and we also give the corresponding necessary and sufficient condition to characterize these cases.

1. Introduction

For a function f on $[0,1]^2$, regardless of how smooth it is, by the Riemann-Lebesgue lemma, we only know that its Fourier coefficients $c_{mn}(f) = o(1)$. In this paper, we first obtain a precise asymptotic formula of the Fourier coefficients (see Theorem 2.2) by using our novel decomposition formula of f:

$$f(x,y) = \begin{cases} q(x,y) + \tau(x,y) & (x,y) \in [0,1]^2, \\ q(x,y) & (x,y) \in \partial([0,1]^2), \end{cases}$$

where $q(x,y)$ is a combination of the boundary function and four simple polynomial factors $x, 1-x, y$, and $1-y$. After that, we will discuss further two kinds of Fourier approximations of functions on the unit square.

The sparse approximation has received much attention in recent years [1,6,7,8]. As an approximation tool, hyperbolic cross truncations of Fourier series has obvious advantages over partial sums of Fourier

*Zhihua Zhang is a full professor at Beijing Normal University, China. He has published more than 50 first-authored papers in applied mathematics, signal processing and climate change. His research is supported by National Key Science Program No.2013CB956604; Fundamental Research Funds for the Central Universities (Key Program) No.105565GK; Beijing Young Talent fund and Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry. Zhihua Zhang is an associate editor of "EURASIP Journal on Advances in Signal Processing" (Springer, SCI-indexed), an editorial board member in applied mathematics of "SpringerPlus" (Springer, SCI-indexed) and an editorial board member of "Journal of Applied Mathematics" (Hindawi).

series since the hyperbolic cross truncations [8]:

$$s_N^{(h)}(f;x,y) = \sum_{|m|=0}^{N} c_{m0}(f)\, e^{2\pi imx} + \sum_{|n|=1}^{N} c_{0n}(f)\, e^{2\pi iny} + \sum_{1\le|mn|\le N} c_{mn}(f)\, e^{2\pi i(mx+ny)} \tag{1.1}$$

can make full use of the decay of Fourier coefficients to reconstruct the target function f.

Throughout this paper, we always assume that $f \in C^{(3,3)}([0,1]^2)$ which means that $\frac{\partial f^{i+j}}{\partial x^i \partial y^j}(0 \le i,j \le 3)$ are continuous on $[0,1]^2$. We will show that, for the hyperbolic cross truncations of its Fourier series, the following asymptotic representation holds (see Theorem 3.1):

$$\| f - s_N^{(h)}(f) \|_2^2 = \frac{1}{4\pi^4}(f(0,0)+f(1,1)-f(0,1)-f(1,0))^2 \frac{\log^2 N_d}{N_d} + O\left(\frac{\log N_d}{N_d}\right), \tag{1.2}$$

where N_d is the number of Fourier coefficients in $s_N^{(h)}(f)$ and $\| F \|_2^2 = \int_0^1 \int_0^1 |F(x,y)|^2 \mathrm{d}x\mathrm{d}y$.

For the partial sum approximation of the Fourier series of f on $[0,1]^2$, we will give another asymptotic representation. The corresponding principal part will become more complicated. It depends on not only values of function f and its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at vertexes of $[0,1]^2$, but also the mean values of f on each side of the boundary $\partial([0,1]^2)$ (in detail, see Theorem 4.1).

Comparing asymptotic representations of two kinds of Fourier approximations, we find that for hyperbolic cross approximation, the approximation order is $\frac{\log^2 N_d}{N_d}$, while for the partial sum approximation, in general the approximation order is $\frac{1}{\sqrt{N_d}}$, and under some cases the approximation order is $\frac{1}{N_d}$. Moreover, we further give a corresponding necessary and sufficient condition for these cases (see Corollary 4.2).

2. Asymptotic representation of Fourier coefficients

Let $f \in C^{(3,3)}([0,1]^2)$. Expand f into Fourier series: $f(x,y) = \sum_{m,n} c_{mn}\, e^{2\pi i(mx+ny)}$, where

$$c_{mn}(f) = \int_0^1 \int_0^1 f(x,y)\, e^{-2\pi i(mx+ny)} \mathrm{d}x\mathrm{d}y$$

and $\sum_{m,n}$ means $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}$. We extend f from $[0,1]^2$ to $\mathbb{R}^2$. Then f is a function on the whole plane $\mathbb{R}^2$ with period 1 and f is discontinuous at the integral points $\{m,n\}_{m,n\in\mathbb{Z}}$. By the Riemann-Lebesgue lemma, we only know that $c_{mn}(f) = o(1)$ as $m \to 0$ or $n \to \infty$, where "o" means high-order infinitesimal. To obtain the precise asymptotic formula of Fourier coefficients, we construct a combination $q(x,y)$ of the boundary functions $f(x,0), f(x,1), f(0,y), f(1,y)$ and factors $x, (1-x), y, (1-y)$ such that the difference $f(x,y) - q(x,y)$ vanishes on the boundary $\partial([0,1]^2)$.

Now we define three functions as follows.

$$\begin{aligned} q_1(x,y) &= (f(x,0) - f(0,0)(1-x) - f(1,0)x)(1-y) + (f(x,1) - f(0,1)(1-x) - f(1,1)x)y, \\ q_2(x,y) &= (f(0,y) - f(0,0)(1-y) - f(0,1)y)(1-x) + (f(1,y) - f(1,0)(1-y) - f(1,1)y)x, \\ q_3(x,y) &= f(0,0)(1-x)(1-y) + f(0,1)(1-x)y + f(1,0)x(1-y) + f(1,1)xy. \end{aligned} \tag{2.1}$$

Then $q(x,y) = q_1(x,y) + q_2(x,y) + q_3(x,y)$ is the desired function, i.e., we have the following theorem.

Theorem 2.1. Let f be defined on $[0,1]^2$ and $q(x,y)$ be stated as above. Then $\tau(x,y)=f(x,y)-q(x,y)$ vanished on the boundary $\partial([0,1]^2)$.

From this, we deduce that if $f\in C^{(3,3)}([0,1]^2)$, then $\tau(x,y)\in C^{(3,3)}([0,1]^2)$ and satisfies that for $i=1,2,3$,

$$\begin{aligned}&\frac{\partial^i\tau}{\partial x^i}(x,0)=\frac{\partial^i\tau}{\partial x^i}(x,1)=0 \qquad (0\le x\le 1),\\&\frac{\partial^i\tau}{\partial y^i}(0,y)=\frac{\partial^i\tau}{\partial y^i}(1,y)=0 \qquad (0\le y\le 1).\end{aligned}\tag{2.2}$$

Now we further explain the relationship between $q(x,y)$ and $f(x,y)$. By (2.1), it follows that

$$\begin{aligned}\frac{\partial q}{\partial x}(x,y)&=\frac{\partial f}{\partial x}(x,0)(1-y)+\frac{\partial f}{\partial x}(x,1)y-f(0,y)+f(1,y)\\&\quad+(f(0,0)-f(1,0))(1-y)+(f(0,1)-f(1,1))y,\\\frac{\partial q}{\partial y}(x,y)&=\frac{\partial f}{\partial y}(0,y)(1-x)+\frac{\partial f}{\partial y}(1,y)x-f(x,0)+f(x,1)\\&\quad+(f(0,0)-f(0,1))(1-x)+(f(1,0)-f(1,1))x,\\\frac{\partial^2 q}{\partial x\partial y}(x,y)&=-\frac{\partial f}{\partial x}(x,0)+\frac{\partial f}{\partial x}(x,1)-\frac{\partial f}{\partial y}(0,y)+\frac{\partial f}{\partial y}(1,y)\\&\quad-f(0,0)+f(1,0)+f(0,1)-f(1,1),\\\frac{\partial^3 q}{\partial x^2\partial y}(x,y)&=-\frac{\partial^2 f}{\partial x^2}(x,0)+\frac{\partial^2 f}{\partial x^2}(x,1),\\\frac{\partial^3 q}{\partial x\partial y^2}(x,y)&=-\frac{\partial^2 f}{\partial y^2}(0,y)+\frac{\partial^2 f}{\partial y^2}(1,y),\\\frac{\partial^4 q}{\partial x^2\partial y^2}(x,y)&=0.\end{aligned}$$

From this, we get

$$\begin{aligned}&\frac{\partial^2 q}{\partial x\partial y}(1,1)-\frac{\partial^2 q}{\partial x\partial y}(1,0)-\frac{\partial^2 q}{\partial x\partial y}(0,1)+\frac{\partial^2 q}{\partial x\partial y}(0,0)=0,\\&\frac{\partial q}{\partial x}(1,y)-\frac{\partial q}{\partial x}(0,y)=\left(\frac{\partial f}{\partial x}(1,0)-\frac{\partial f}{\partial x}(0,0)\right)(1-y)+\left(\frac{\partial f}{\partial x}(1,1)-\frac{\partial f}{\partial x}(0,1)\right)y,\\&\frac{\partial q}{\partial y}(x,1)-\frac{\partial q}{\partial y}(x,0)=\left(\frac{\partial f}{\partial y}(0,1)-\frac{\partial f}{\partial y}(0,0)\right)(1-x)+\left(\frac{\partial f}{\partial y}(1,1)-\frac{\partial f}{\partial y}(1,0)\right)x.\end{aligned}\tag{2.3}$$

Since $c_{mn}(f)=c_{mn}(q)+c_{mn}(\tau)$ and $c_{mn}(q)=c_{mn}(q_1)+c_{mn}(q_2)+c_{mn}(q_3)$, by (2.1),

$$c_{mn}(q_1)=c_m(R(x,0))c_n(1-y)+c_m(R(x,1))c_n(y),\tag{2.4}$$

where

$$R(x,\nu)=f(x,\nu)-f(0,\nu)(1-x)-f(1,\nu)x \qquad (\nu=0,1).\tag{2.5}$$

Since $R(0,\nu)=R(1,\nu)$,

$$\begin{aligned}c_m(R(x,\nu))&=\int_0^1 R(x,\nu)\,e^{-2\pi imx}\mathrm{d}x=\frac{1}{2\pi im}\int_0^1\frac{\partial R}{\partial x}(x,\nu)\,e^{-2\pi imx}\mathrm{d}x\\&=\frac{1}{4\pi^2m^2}\left(\frac{\partial R}{\partial x}(1,\nu)-\frac{\partial R}{\partial x}(0,\nu)-\int_0^1\frac{\partial^2 R}{\partial x^2}(x,\nu)\,e^{-2\pi imx}\mathrm{d}x\right) \qquad (m\ne 0),\\c_0(R(x,\nu))&=\int_0^1 f(x,\nu)\mathrm{d}x-\frac{1}{2}(f(0,\nu)+f(1,\nu)).\end{aligned}$$

Noticing that $\frac{\partial R}{\partial x}(x,\nu)=\frac{\partial f}{\partial x}(x,\nu)+f(0,0)-f(1,\nu)$, we get

$$\frac{\partial R}{\partial x}(1,\nu)-\frac{\partial R}{\partial x}(0,\nu)=\frac{\partial f}{\partial x}(1,\nu)-\frac{\partial f}{\partial x}(0,\nu),$$

$$\frac{\partial^2 R}{\partial x^2}(x,\nu)=\frac{\partial^2 f}{\partial x^2}(x,\nu)\qquad(\nu=0,1).$$

Since mth Fourier coefficients of $(1-x)$ and x are $\frac{1}{2\pi im}$ and $-\frac{1}{2\pi im}$ $(m\neq 0)$, respectively, we get by (2.5)

$$c_m(R(x,\nu))=\frac{1}{4\pi^2m^2}\left(\frac{\partial f}{\partial x}(1,\nu)-\frac{\partial f}{\partial x}(0,\nu)-\int_0^1\frac{\partial^2 f}{\partial x^2}(x,\nu)\,e^{-2\pi imx}\mathrm{d}x\right)$$

while

$$\int_0^1\frac{\partial^2 f}{\partial x^2}(x,\nu)\,e^{-2\pi imx}\mathrm{d}x=-\frac{1}{2\pi im}\left(\frac{\partial^2 f}{\partial x^2}(1,\nu)-\frac{\partial^2 f}{\partial x^2}(0,\nu)-\int_0^1\frac{\partial^3 f}{\partial x^3}(x,\nu)\,e^{2\pi imx}\mathrm{d}x\right).$$

So

$$c_m(R(x,\nu))=\frac{1}{4\pi^2m^2}\left(\frac{\partial f}{\partial x}(1,\nu)-\frac{\partial f}{\partial x}(0,\nu)\right)+O\left(\frac{1}{m^3}\right)\qquad(m\neq 0),$$

$$c_0(R(x,\nu))=\int_0^1 f(x,\nu)\mathrm{d}x-\frac{1}{2}(f(0,0)+f(1,\nu)).$$

From this and (2.4), it follows that

$$c_{mn}(q_1)=-\frac{i}{8\pi^3m^2n}\left(\frac{\partial f}{\partial x}(1,0)+\frac{\partial f}{\partial x}(0,1)-\frac{\partial f}{\partial x}(0,0)-\frac{\partial f}{\partial x}(1,1)\right)+O\left(\frac{1}{m^3n}\right)\qquad(m\neq 0,n\neq 0),$$

$$c_{m0}(q_1)=\frac{1}{8\pi^2m^2}\left(\frac{\partial f}{\partial x}(1,0)-\frac{\partial f}{\partial x}(0,1)-\frac{\partial f}{\partial x}(0,0)+\frac{\partial f}{\partial x}(1,1)\right)+O\left(\frac{1}{m^3}\right)\qquad(m\neq 0),$$

$$c_{0n}(q_1)=-\frac{i}{2\pi n}\left(\int_0^1(f(x,0)-f(x,1))\mathrm{d}x-\frac{1}{2}(f(0,1)+f(1,0)-f(0,1)-f(1,1))\right)\qquad(n\neq 0).$$

Similarly, we have

$$c_{mn}(q_2)=-\frac{i}{8\pi^3mn^2}\left(\frac{\partial f}{\partial y}(1,0)+\frac{\partial f}{\partial y}(0,1)-\frac{\partial f}{\partial y}(0,0)-\frac{\partial f}{\partial y}(1,1)\right)+O\left(\frac{1}{mn^3}\right)\qquad(m\neq 0,n\neq 0),$$

$$c_{0n}(q_2)=\frac{1}{8\pi^2n^2}\left(\frac{\partial f}{\partial y}(0,1)-\frac{\partial f}{\partial y}(1,0)-\frac{\partial f}{\partial y}(0,0)+\frac{\partial f}{\partial y}(1,1)\right)+O\left(\frac{1}{n^3}\right)\qquad(n\neq 0),$$

$$c_{m0}(q_2)=-\frac{i}{2\pi m}\left(\int_0^1(f(0,y)-f(1,y))\mathrm{d}y-\frac{1}{2}(f(0,0)+f(0,1)-f(1,0)-f(1,1))\right)\qquad(m\neq 0).$$

and

$$c_{mn}(q_3)=\frac{1}{4\pi^2mn}(f(1,0)+f(0,1)-f(0,0)-f(1,1))\qquad(m\neq 0,\ n\neq 0),$$

$$c_{m0}(q_3)=-\frac{i}{4\pi m}(f(0,0)-f(0,1)-f(1,0)+f(1,1))\qquad(m\neq 0),$$

$$c_{0n}(q_3)=-\frac{i}{4\pi n}(f(0,0)-f(0,1)+f(1,0)-f(1,1))\qquad(n\neq 0).$$

From this, we get an asymptotic representation of $c_{mn}(q)$ by $q(x,y)=q_1(x,y)+q_2(x,y)+q_3(x,y)$. Finally, we write out the asymptotic representation of $c_{mn}(\tau)$.

Using the integration by parts, it follows by Theorem 2.1, (2,2) and (2.4) that

(i) For $m\neq 0,n\neq 0$,

$$c_{mn}(\tau)=\frac{1}{16\pi^4m^2n^2}\left(\frac{\partial^2 f}{\partial x\partial y}(1,1)-\frac{\partial^2 f}{\partial x\partial y}(1,0)-\frac{\partial^2 f}{\partial x\partial y}(0,1)+\frac{\partial^2 f}{\partial x\partial y}(0,0)\right)+O\left(\frac{1}{m^2n^2}\right)\left(\frac{1}{m}+\frac{1}{n}\right);$$

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(ii) For $m \neq 0$,

$$c_{m0}(\tau)=\frac{1}{4\pi^2m^2}\left(\int_0^1\left(\frac{\partial f}{\partial x}(1,y)-\frac{\partial f}{\partial x}(0,y)\right)\mathrm{d}y+\frac{1}{2}\left(\frac{\partial f}{\partial x}(0,0)-\frac{\partial f}{\partial x}(1,0)+\frac{\partial f}{\partial x}(0,1)-\frac{\partial f}{\partial x}(1,1)\right)\right)+O\left(\frac{1}{m^3}\right)$$

(iii) For $n \neq 0$,

$$c_{0n}(\tau)=\frac{1}{4\pi^2n^2}\left(\int_0^1\left(\frac{\partial f}{\partial y}(x,1)-\frac{\partial f}{\partial y}(x,0)\right)\mathrm{d}x+\frac{1}{2}\left(\frac{\partial f}{\partial y}(0,0)-\frac{\partial f}{\partial y}(0,1)+\frac{\partial f}{\partial y}(1,0)-\frac{\partial f}{\partial y}(1,1)\right)\right)+O\left(\frac{1}{n^3}\right).$$

From this and $c_{mn}(f) = c_{mn}(q) + c_{mn}(\tau)$, we get the following asymptotic representation of Fourier coefficients of $f(x, y)$.

Theorem 2.2. Let $f \in C^{(3,3)}([0, 1]^2)$. Then Fourier coefficients of $f(x, y)$ satisfy

(i) for $m \neq 0$, $n \neq 0$,

$$c_{mn}(f) = \frac{1}{4\pi^2mn}(-\alpha + i\frac{\beta}{2\pi m} + i\frac{\gamma}{2\pi n} + \frac{\delta}{4\pi^2mn}) + O\left(\frac{1}{m^2n}\right)\left(\frac{1}{m} + \frac{1}{n}\right),$$

where

$$\alpha = f(0,0) - f(0,1) - f(1,0) + f(1,1),$$

$$\beta = \tfrac{\partial f}{\partial x}(0,0) - \tfrac{\partial f}{\partial x}(0,1) - \tfrac{\partial f}{\partial x}(1,0) + \tfrac{\partial f}{\partial x}(1,1),$$

$$\gamma = \tfrac{\partial f}{\partial y}(0,0) - \tfrac{\partial f}{\partial y}(0,1) - \tfrac{\partial f}{\partial y}(1,0) + \tfrac{\partial f}{\partial y}(1,1),$$

$$\delta = \tfrac{\partial^2 f}{\partial x\partial y}(0,0) - \tfrac{\partial^2 f}{\partial x\partial y}(0,1) - \tfrac{\partial^2 f}{\partial x\partial y}(1,0) + \tfrac{\partial^2 f}{\partial x\partial y}(1,1);$$

(ii) for $m \neq 0$,

$$c_{m0}(f) = i\frac{a}{2\pi m} + \frac{b}{4\pi^2m^2} + O\left(\frac{1}{m^3}\right),$$

where

$$a = f(0,1) - f(1,0) - \textstyle\int_0^1(f(0,y) - f(1,y))\mathrm{d}y,$$

$$b = \textstyle\int_0^1\left(\tfrac{\partial f}{\partial x}(1,y) - \tfrac{\partial f}{\partial x}(0,y)\right)\mathrm{d}y;$$

(iii) for $n \neq 0$,

$$c_{0n}(f) = i\frac{c}{2\pi n} + \frac{d}{4\pi^2n^2} + O\left(\frac{1}{n^3}\right),$$

where

$$c = f(1,0) - f(0,1) - \textstyle\int_0^1(f(x,0) - f(x,1))\mathrm{d}x,$$

$$d = \textstyle\int_0^1\left(\tfrac{\partial f}{\partial y}(x,1) - \tfrac{\partial f}{\partial y}(x,0)\right)\mathrm{d}x + O\left(\tfrac{1}{n^3}\right).$$

Now we compute $|c_{mn}(f)|^2$. Since f is a real-valued function, it is clear that $\alpha, \beta, \gamma, \delta$ and a, b, c, d in Theorem 2.2 are all real numbers. So we get the following corollary.

Corollary 2.3. Let $f \in C^{(3,3)}([0, 1]^2)$. Then

$$|c_{mn}(f)|^2 = \tfrac{1}{16\pi^4m^2n^2}\left(\alpha^2 + \tfrac{\beta\gamma-\alpha\delta}{2\pi^2mn} + \tfrac{\beta^2}{4\pi^2m^2} + \tfrac{\gamma^2}{4\pi^2n^2}\right) + O\left(\tfrac{1}{m^3n^3}\right)\left(\tfrac{1}{m} + \tfrac{1}{n}\right),$$

$$|c_{m0}(f)|^2 = \tfrac{a^2}{4\pi^2m^2} + O\left(\tfrac{1}{m^4}\right),$$

$$|c_{0n}(f)|^2 = \tfrac{c^2}{4\pi^2n^2} + O\left(\tfrac{1}{n^4}\right),$$

where $\alpha, \beta, \gamma, \delta$ and a, b, c, d are stated as above.

3. Asymptotic representation of hyperbolic cross approximation

Let $f \in C^{(3,3)}([0,1]^2)$. We expand it into a Fourier series. Consider the hyperbolic cross truncations of its Fourier series:

$$s_N^{(h)}(f;x,y) = \sum_{|m|=0}^{N} c_{m0}(f)\, e^{2\pi imx} + \sum_{|n|=1}^{N} c_{0n}(f)\, e^{2\pi iny}$$

$$+ \sum_{|n|=1}^{N} \sum_{|m|\le \frac{N}{|n|}} c_{mn}(f)\, e^{2\pi i(mx+ny)},$$

where $c_{mn}(f) = \int_0^1 \int_0^1 f(x,y)\, e^{-2\pi i(mx+ny)} \mathrm{d}x\mathrm{d}y$. So

$$f(x,y) - s_N^{(h)}(f;x,y) = \sum_{|m|\ge N+1} c_{m0}(f)\, e^{2\pi imx} + \sum_{|n|=N+1}^{\infty} c_{0n}(f)\, e^{2\pi iny}$$

$$+ \sum_{|n|\ge N+1} \sum_{|m|=1}^{\infty} c_{mn}(f)\, e^{2\pi i(mx+ny)} + \sum_{|n|=1}^{N} \sum_{|m|> \frac{N}{|n|}} c_{mn}(f)\, e^{2\pi i(mx+ny)}.$$

Using the Parseval identity [4,5,9] of bivariate Fourier series,

$$\begin{aligned} \| f - s_N^{(h)} \|_2^2 &= \sum_{|n|\ge N+1} (|c_{0n}(f)|^2 + |c_{n0}(f)|^2) \\ &+ \sum_{|n|\ge N+1} \sum_{|m|=1}^{\infty} |c_{mn}(f)|^2 + \sum_{|n|=1}^{N} \sum_{|m|>\frac{N}{|n|}} |c_{mn}(f)|^2 \\ &=: P_N + Q_N + R_N. \end{aligned} \tag{3.1}$$

By Corollary 2.3,

$$|c_{mn}(f)|^2 = \frac{\alpha^2}{16\pi^4 m^2 n^2} + O\left(\frac{1}{m^3}\right)\frac{1}{n^2} + O\left(\frac{1}{n^3}\right)\frac{1}{m^2}.$$

We first compute R_N:

$$\begin{aligned} R_N &= \sum_{|n|=1}^{N} \sum_{|m|>\frac{N}{|n|}} |c_{mn}(f)|^2 \\ &= \tfrac{\alpha^2}{16\pi^4} \sum_{|n|=1}^{N} \tfrac{1}{n^2} \sum_{|m|>\frac{N}{|n|}} \tfrac{1}{m^2} + O(1) \sum_{|n|=1}^{N} \tfrac{1}{n^4} \sum_{|m|>\frac{N}{|n|}} \tfrac{1}{m^3} + O(1) \sum_{|n|=1}^{N} \tfrac{1}{n^3} \sum_{|m|>\frac{N}{|n|}} \tfrac{1}{m^4} \\ &=: R_N^{(1)} + R_N^{(2)} + R_N^{(3)}. \end{aligned} \tag{3.2}$$

Note that

$$R_N^{(1)} = \tfrac{\alpha^2}{16\pi^4} \sum_{|n|=1}^{N} \tfrac{1}{n^2} \sum_{|m|\ge \frac{N}{|n|}} \tfrac{1}{m^2},$$

$$\sum_{|m|\ge \frac{N}{|n|}} \tfrac{1}{m^2} = 2\int_{\frac{N}{|n|}}^{\infty} \tfrac{1}{t^2}\mathrm{d}t + O\left(\tfrac{n^2}{N^2}\right) = \tfrac{2|n|}{N} + O\left(\tfrac{n^2}{N^2}\right).$$

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This implies that

$$R_N^{(1)} = \frac{\alpha^2}{4\pi^4}\sum_{n=1}^{N}\frac{1}{nN} + O\left(\frac{1}{N}\right) = \frac{\alpha^2\log N}{4\pi^4 N} + O\left(\frac{1}{N}\right).$$

Similarly, $R_N^{(2)} = O\left(\frac{1}{N}\right)$ and $R_N^{(3)} = O\left(\frac{1}{N}\right)$. So

$$R_N = \frac{\alpha^2\log N}{4\pi^4 N} + O\left(\frac{1}{N}\right).$$

By $|c_{mn}|^2 = O\left(\frac{1}{m^2n^2}\right)$, it follows that

$$Q_N = O(1)\left(\sum_{|n|\geq N+1}\frac{1}{n^2}\right)\left(\sum_{|m|=1}\frac{1}{m^2}\right) + O\left(\frac{1}{N^2}\right) = O\left(\frac{1}{N}\right).$$

From $|c_{0n}(f)|^2 = O\left(\frac{1}{n^2}\right)$ and $|c_{m0}(f)|^2 = O\left(\frac{1}{m^2}\right)$, it is easy to deduce that

$$P_N = \sum_{|n|\geq N+1}|c_{0n}(f)|^2 + \sum_{|m|\geq N+1}|c_{m0}(f)| = O\left(\frac{1}{N}\right).$$

Therefore, by (3,1),

$$\| f - s_N^{(h)}(f) \|_2^2 = \frac{\alpha^2\log N}{4\pi^4 N} + O\left(\frac{1}{N}\right).$$

The number N_d of Fourier coefficients in the hyperbolic cross truncation $s_N^{(h)}(f)$ is equal to

$$N_d = 2N + 1 + \sum_{n_1=1}^{N}\left[\frac{N}{|n_1|}\right] = 2N\log N + O(N).$$

Theorem 3.1. Let $f \in C^{(3,3)}([0,1]^2)$. Then the asymptotic representation of the hyperbolic cross approximation of Fourier series of f is

$$\| f - s_N^{(h)}(f) \|_2^2 = \frac{\alpha^2\log^2 N_d}{4\pi^4 N_d}\left(1 + O\left(\frac{1}{\log N_d}\right)\right), \tag{3.3}$$

where N_d is the number of Fourier coefficients in hyperbolic cross truncation $s_N^{(h)}(f)$ and $\alpha = f(0,0) - f(0,1) - f(1,0) + f(1,1)$.

Corollary 3.2. Let $f \in C^{(2,2)}([0,1]^2)$. Then

(i) $\| f - s_N^{(h)}(f) \|_2^2 = O\left(\frac{\log N_d}{N_d}\right)$ if and only if $f(0,0) + f(1,1) = f(0,1) + f(1,0)$.

(ii) when $F(x,y) = f(x,y) + (f(0,1) + f(1,0) - f(0,0) - f(1,1))xy$,

$$\| F - s_N^{(h)}(F) \|_2^2 = O\left(\frac{\log N_d}{N_d}\right).$$

Now we show an approach to estimates of the bound of the term " O" in Theorem 3.1 using the Sobolev norm. For $\frac{\partial^6 f}{\partial x^3\partial y^3} \in C([0,1]^2)$, its Sobolev norm is defined as

$$M(f) = \max_{\substack{x,y\in\partial([0,1]^2)\\ i,j=0,1,2,3}}\left|\frac{\partial^{i+j} f}{\partial x^i\partial y^j}\right|.$$

By Theorem 2.1 and (2.2), and (2.4), we get

$$c_{mn}(\tau) = \int_0^1 \int_0^1 \tau(x,y)\, e^{-2\pi i(mx+ny)} \mathrm{d}x\mathrm{d}y = \frac{\delta}{16\pi^4 m^2 n^2} + J_{mn},$$

where

$$\delta = \frac{\partial^2 f}{\partial x \partial y}(0,0) - \frac{\partial^2 f}{\partial x \partial y}(0,1) - \frac{\partial^2 f}{\partial x \partial y}(1,0) + \frac{\partial^2 f}{\partial x \partial y}(1,1)$$

and

$$J_{mn} = \tfrac{1}{32\pi^5 m^2 n^3}\left(\tfrac{\partial^3 f}{\partial x \partial y^2}(1,1) - \tfrac{\partial^3 f}{\partial x \partial y^2}(0,1) - \tfrac{\partial^3 f}{\partial x \partial y^2}(1,0) + \tfrac{\partial^3 f}{\partial x \partial y^2}(0,0)\right)$$

$$-\tfrac{1}{32\pi^5 m^2 n^3}\int_0^1 \left(\tfrac{\partial^4 f}{\partial x \partial y^3}(1,y) - \tfrac{\partial^4 f}{\partial x \partial y^2}(0,y)\right) e^{-2\pi i n y}\mathrm{d}y$$

$$+\tfrac{1}{32 i\pi^5 m^3 n^2}\left(\tfrac{\partial^3 f}{\partial x^2 \partial y}(1,0) - \tfrac{\partial^3 f}{\partial x^2 \partial y}(0,0)\right)$$

$$-\tfrac{1}{32 i\pi^5 m^3 n^2}\int_0^1 \left(\tfrac{\partial^4 f}{\partial x^2 \partial y^2}(1,y) - \tfrac{\partial^4 f}{\partial x^2 \partial y^2}(0,y)\right) e^{-2\pi i n y}\mathrm{d}y$$

$$+\tfrac{1}{32 i\pi^5 m^3 n^2}\int_0^1 \left(\tfrac{\partial^4 f}{\partial x^3 \partial y}(x,1) - \tfrac{\partial^4 f}{\partial x^3 \partial y}(x,0)\right) e^{-2\pi i m x}\mathrm{d}x$$

$$-\tfrac{1}{32 i\pi^5 m^3 n^2}\int_0^1 \int_0^1 \tfrac{\partial^5 f}{\partial x^3 \partial y^2}(x,y)\, e^{-2\pi i(mx+ny)}\mathrm{d}x\mathrm{d}y$$

So

$$|J_{mn}| \le \frac{6M(f)}{32\pi^5 m^2 n^3} + \frac{7M(f)}{32\pi^5 m^3 n^2} \le \frac{13M(f)}{32\pi^5 m^2 n^2}\left(\frac{1}{m} + \frac{1}{n}\right).$$

For c_{m0} and c_{0n}, we have

$$c_{m0}(\tau) = \tfrac{1}{(2\pi m)^2}\left(\int_0^1 \left(\tfrac{\partial f}{\partial x}(1,y) - \tfrac{\partial f}{\partial x}(0,y)\right)\mathrm{d}y + \tfrac{1}{2}\beta\right) + T_m^{(1)},$$

$$c_{0n}(\tau) = \tfrac{1}{(2\pi n)^2}\left(\int_0^1 \left(\tfrac{\partial f}{\partial y}(x,1) - \tfrac{\partial f}{\partial y}(x,0)\right)\mathrm{d}x + \tfrac{1}{2}\gamma\right) + T_n^{(2)},$$

where

$$T_m^{(1)} = \tfrac{1}{(2\pi m)^3 i}\int_0^1 \left(\tfrac{\partial^2 f}{\partial x^2}(1,y) - \tfrac{\partial^2 f}{\partial x^2}(0,y)\right)\mathrm{d}y$$

$$-\tfrac{1}{2(2\pi m)^3 i}\left(\tfrac{\partial^2 f}{\partial x^2}(1,0) - \tfrac{\partial^2 f}{\partial x^2}(0,0) - \tfrac{\partial^2 f}{\partial x^2}(1,1) + \tfrac{\partial^2 f}{\partial x^2}(0,1)\right)$$

$$-\tfrac{1}{(2\pi m)^3 i}\int_0^1 \int_0^1 \left(\tfrac{\partial^3 f}{\partial x^3}(x,y) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial x^3}(x,0) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial x^3}(x,1)\right)\mathrm{d}x\mathrm{d}y.$$

$$T_n^{(2)} = \tfrac{1}{(2\pi n)^3 i}\int_0^1 \left(\tfrac{\partial^2 f}{\partial y^2}(x,1) - \tfrac{\partial^2 f}{\partial y^2}(x,0)\right)\mathrm{d}x$$

$$-\tfrac{1}{2(2\pi n)^3 i}\left(\tfrac{\partial^2 f}{\partial y^2}(0,1) - \tfrac{\partial^2 f}{\partial y^2}(0,0) - \tfrac{\partial^2 f}{\partial y^2}(1,1) + \tfrac{\partial^2 f}{\partial y^2}(1,0)\right)$$

$$-\tfrac{1}{(2\pi n)^3 i}\int_0^1 \int_0^1 \left(\tfrac{\partial^3 f}{\partial y^3}(x,y) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial y^3}(0,y) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial y^3}(1,y)\right)\mathrm{d}x\mathrm{d}y.$$

So

$$|T_m^{(1)}| \le \tfrac{6M(f)}{(2\pi m)^3},$$

$$|T_n^{(2)}| \le \tfrac{6M(f)}{(2\pi n)^3}.$$

Now we estimate $c_{mn}(q)$. Note that

$$c_m(R(x,\nu)) = \frac{1}{4\pi^2 m^2}\left(\frac{\partial f}{\partial x}(1,\nu) - \frac{\partial f}{\partial x}(0,\nu)\right) + L_m^{(\nu)} \qquad (\nu = 0,1),$$

where

$$L_m^{(\nu)} = \frac{1}{8\pi^3 m^3 i}\left(\frac{\partial^2 f}{\partial x^2}(1,\nu) - \frac{\partial^2 f}{\partial x^2}(0,\nu) - \int_0^1 \frac{\partial^3 f}{\partial x^3}(x,\nu)\, e^{-2\pi i m x} \mathrm{d}x\right) \qquad (\nu = 0,1).$$

Then $|L_m^{(\nu)}| \le \frac{5M(f)}{8\pi^3 m^3}$. This implies that

$$c_{mn}(q_1) = -\tfrac{\beta}{8\pi^3 m^2 n i} + H_{mn}^{(1)},$$

$$c_{mn}(q_2) = -\tfrac{\gamma}{8\pi^3 m n^2 i} + H_m^{(2)},$$

where

$$|H_{,mn}^{(1)}| \le \tfrac{5M(f)}{8\pi^4 m^3 n},$$

$$|H_{mn}^{(2)}| \le \tfrac{5M(f)}{8\pi^4 m n^3}.$$

From this and $c_{mn}(q_3) = \frac{\alpha}{4\pi^2 mn}$, we get

$$c_{mn}(q) = \frac{1}{4\pi^2 mn}\left(\alpha - \frac{\beta}{2\pi m i} - \frac{\gamma}{2\pi n i}\right) + H_{mn},$$

where $|H_{mn}| \le \frac{5M(f)}{8\pi^4 mn}\left(\frac{1}{m^2} + \frac{1}{n^2}\right)$.

Similarly, we may estimate $c_{m0}(q)$ and $c_{0n}(q)$. Using $c_{mn}(f) = c_{mn}(q) + c_{mn}(\tau)$ and the above estimates, we easily obtain the estimates of upper bounds of $|c_{mn}(f)|^2$. Again, using the method of argument in Theorem 3.1, we finally can give the bound of the term " O " in (3.3).

4. Asymptotic representation of square errors of partial sums

Let $f \in C^{(3,3)}([0,1]^2)$. Consider the partial sums of its Fourier series:

$$s_N(f;x,y) = \sum_{|m|\le N}\sum_{|n|\le N} c_{mn}(f)\, e^{2\pi i(mx+ny)}.$$

Then the square errors are equal to

$$\begin{aligned}\| f - s_N(f) \|_2^2 &= \sum_{|n|\ge N+1} |c_{0n}(f)|^2 + \sum_{|m|\ge N+1} |c_{m0}(f)|^2 \\ &\quad + \sum_{|n|\ge N+1}\sum_{|m|=1}^{\infty} |c_{mn}(f)|^2 + \sum_{|n|=1}^{N}\sum_{|m|\ge N+1} |c_{mn}(f)|^2 \\ &=: K_N + L_N + I_N + J_N.\end{aligned} \tag{4.1}$$

By Corollary 2,3 (ii) and (iii),

$$K_N = \tfrac{c^2}{2\pi^2 N} + O\left(\tfrac{1}{N^3}\right),$$

$$L_N = \tfrac{a^2}{2\pi^2 N} + O\left(\tfrac{1}{N^3}\right).$$

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By Corollary 2.3 (i),

$$|c_{mn}(f)|^2 = \frac{1}{16\pi^4 m^2 n^2}\left(\alpha^2 + \frac{\beta^2}{4\pi^2 m^2} + \frac{\gamma^2}{4\pi^2 n^2}\right) + O\left(\frac{1}{m^3 n^3}\right),$$

and so

$$I_N = \tfrac{\alpha^2}{48\pi^2}\left(\sum_{|n|>N} \tfrac{1}{n^2}\right) + \tfrac{\beta^2}{64\pi^6}\left(\sum_{|n|>N} \tfrac{1}{n^2}\right)\zeta(4) + \tfrac{\gamma^2}{192\pi^4}\left(\sum_{|n|>N} \tfrac{1}{n^4}\right) + O\left(\tfrac{1}{N^2}\right)$$

$$= \tfrac{1}{8\pi^2}\left(\tfrac{\alpha^2}{3} + \tfrac{\beta^2}{2\pi^4}\zeta(4)\right)\tfrac{1}{N} + O\left(\tfrac{1}{N^2}\right),$$

$$J_N = \tfrac{\alpha^2}{16\pi^4}\left(\sum_{|n|=1}^{N} \tfrac{1}{n^2}\right)\left(\sum_{|m|>N} \tfrac{1}{m^2}\right) + \tfrac{\beta^2}{64}\left(\sum_{|n|=1}^{N} \tfrac{1}{n^2}\right)\left(\sum_{|m|>N} \tfrac{1}{m^4}\right)$$

$$+\tfrac{\gamma^2}{64\pi^6}\left(\sum_{|n|=1}^{N} \tfrac{1}{n^4}\right)\left(\sum_{|m|>N} \tfrac{1}{m^2}\right) + O\left(\tfrac{1}{N^2}\right),$$

where $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann-Zeta function. Note that

$$\sum_{|n|=1}^{N} \tfrac{1}{n^2} = \sum_{|n|=1}^{\infty} \tfrac{1}{n^2} - \sum_{|n|>N} \tfrac{1}{n^2} = \tfrac{\pi^2}{3} + O\left(\tfrac{1}{N}\right),$$

$$\sum_{|n|=1}^{N} \tfrac{1}{n^4} = \sum_{|n|=1}^{\infty} \tfrac{1}{n^4} - \sum_{|n|>N} \tfrac{1}{n^4} = \zeta(4) + O\left(\tfrac{1}{N^3}\right).$$

Then

$$J_N = \frac{1}{8\pi^2}\left(\frac{\alpha^2}{3} + \frac{\gamma^2}{\pi^4}\zeta(4)\right)\frac{1}{N} + O\left(\frac{1}{N^2}\right).$$

Finally, by (4.1), we get the following theorem.

Theorem 4.1. Let $f \in C^{(3,3)}([0,1]^2)$. Then the partial sums $s_N(f)$ of its Fourier series satisfy

$$\| f - s_N(f) \|_2^2 = \left(\frac{a^2 + c^2}{2\pi^2} + \frac{\alpha^2}{24\pi^2} + \frac{\beta^2 + \gamma^2}{8\pi^6}\zeta(4)\right)\frac{1}{N} + O\left(\frac{1}{N^2}\right),$$

where $a, c, \alpha, \beta, \gamma$ are stated in Theorem 2.2 and $\zeta(4)$ is the Riemann-Zeta function.

Note that the number N_d of Fourier coefficients in the sum $s_N(f)$ is $(2N+1)^2$. From Theorem 4.1, it follows that

$$\| f - s_N(f) \|_2^2 \sim \frac{1}{\sqrt{N_d}}.$$

Again, by Theorem 4.1, we get the following corollary.

Corollary 4.2. Let $f \in C^{(3,3)}([0,1]^2)$. Then the partial sums $s_N(f)$ of its Fourier series satisfy

$$\| f - s_N(f) \|_2^2 = O\left(\frac{1}{N^2}\right)$$

if and only if $a = c = \alpha = \beta = \gamma = 0$, i.e.,

$$\begin{aligned}
&f(1,0) - f(0,1) = \textstyle\int_0^1 (f(x,0) - f(x,1))\mathrm{d}x, \\
&f(0,1) - f(1,0) = \textstyle\int_0^1 (f(0,y) - f(1,y))\mathrm{d}y, \\
&f(0,1) + f(1,0) = f(0,0) + f(1,1), \\
&\tfrac{\partial f}{\partial x}(0,1) + \tfrac{\partial f}{\partial x}(1,0) = \tfrac{\partial f}{\partial x}(0,0) + \tfrac{\partial f}{\partial x}(1,1), \\
&\tfrac{\partial f}{\partial y}(0,1) + \tfrac{\partial f}{\partial y}(1,0) = \tfrac{\partial f}{\partial y}(0,0) + \tfrac{\partial f}{\partial y}(1,1).
\end{aligned} \tag{4.2}$$

Since the number of Fourier coefficients is $2N + 1$ in $s_N(f)$, it is clear that when (4.2) holds,

$$\| f - s_N(f) \|_2^2 = O\left(\frac{1}{N_d}\right).$$

Comparing it with Theorem 3.1, we see that in this case the partial sum approximation is better than the hyperbolic cross approximation.

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Khatri-Rao Products and Selection Operators

Arnon Ploymukda and Pattrawut Chansangiam*
Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.

Abstract

We develop further theory for Khatri-Rao products of Hilbert space operators in connections with selection operators. We provide two constructions related to selection operators. Then we establish certain identities and inequalities involving Khatri-Rao and Tracy-Singh products. As consequences, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators.

Keywords: tensor product, Khatri-Rao product, Tracy-Singh product, operator matrix
Mathematics Subject Classifications 2010: 47A80, 15A69, 47A05.

1 Introduction

This paper concerns operator extensions of certain matrix products, namely, the Kronecker (tensor) product, the Tracy-Singh product, and the Khatri-Rao product. Fundamental theory for these matrix products are collected, for instance, in [1, 2, 4, 5, 10, 11, 12] and references therein. Denote by $M_{m,n}(\mathbb{C})$ the algebra of m-by-n complex matrices. Recall that the Kronecker product of $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is given by

$$A \hat{\otimes} B \;=\; [a_{ij}B]_{ij}\,.$$

Consider partitioned matrices A and B such that the (i,j)th block of A is A_{ij} and the (k,l)th block of B is B_{kl}. The Tracy-Singh product [9] of A and B is defined by

$$A \hat{\boxtimes} B \;=\; \left[\left[A_{ij} \hat{\otimes} B_{kl}\right]_{kl}\right]_{ij}\,. \qquad (1)$$

The Khatri-Rao product [3] is defined for two partitioned matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ as follows

$$A \hat{\boxdot} B \;=\; \left[A_{ij} \hat{\otimes} B_{ij}\right]_{ij}\,. \qquad (2)$$

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

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The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Recall that the tensor product of $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into $\mathcal{H}' \otimes \mathcal{K}'$ such that $(A \otimes B)(x \otimes y) = Ax \otimes By$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. Recently, the Tracy-Singh product and the Khatri-Rao product for matrices were generalized to those for operators acting on the direct sum of Hilbert spaces, see [6, 7, 8]. Fundamental algebraic and order properties of operator Khatri-Rao products are investigated in [8]. That paper also provides a construction of a unital positive linear map taking the Tracy-Singh product of two operators to their Khatri-Rao product. Such a linear map appears in the form $X \mapsto Z^*AZ$ where Z is an isometry, called a selection operator. See details in Section 2.

The present paper contains further development on operator Khatri-Rao products in relations with Tracy-Singh products and selection operators. First, we provide two constructions related to selection operators (see Section 3). Consequently, we establish some operator identities and inequalities involving Khatri-Rao and Tracy-Singh products (see Section 4). Finally, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators (see Section 5).

2 Tracy-Singh products and Khatri-Rao products for operators

Throughout this paper, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex separable Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, let us denote by $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ the space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ and abbreviate $\mathbb{B}(\mathcal{X}, \mathcal{X})$ to $\mathbb{B}(\mathcal{X})$. Capital letters always denote a Hilbert space operator. In particular, I and O stand for the identity and the zero operator, respectively.

In order to define Tracy-Singh products of operators, we fix the following decompositions

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{j=1}^{q} \mathcal{K}_j, \quad \mathcal{K}' = \bigoplus_{i=1}^{p} \mathcal{K}'_i. \tag{3}$$

where all of $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. For each j and l, let $M_j : \mathcal{H}_j \to \mathcal{H}$ and $N_l : \mathcal{K}_l \to \mathcal{K}$ be the canonical injections. For each i and k, let $P_i : \mathcal{H}' \to \mathcal{H}'_i$ and $Q_k : \mathcal{K}' \to \mathcal{K}'_k$ be the canonical projections. Given $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, put $A_{ij} = P_i A M_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}'_i)$ for each i, j. Thus we can write A in the operator-matrix form $A = [A_{ij}]_{i,j=1}^{m,n}$. Similarly, given $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, let $B_{kl} = Q_k B N_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}'_k)$ for each $k = 1, \dots, p$ and $l = 1, \dots, q$. We can identify B with the operator matrix $B = [B_{kl}]_{k,l=1}^{p,q}$.

Definition 1. *The Tracy-Singh product of A and B is defined to be the bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$ represented by*

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}. \tag{4}$$

If both factor A and B consist of only one block, then $A \boxtimes B = A \otimes B$.

Lemma 2 ([6])**.** *The following properties of the Tracy-Singh product for operators hold (provided that each term is well-defined):*

1. *Compatibility with adjoints:* $(A \boxtimes B)^* = A^* \boxtimes B^*$.
2. *Mixed-product property:* $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.
3. *Monotonicity: if* $A \geqslant B \geqslant 0$ *and* $C \geqslant D \geqslant 0$*, then* $A \boxtimes B \geqslant C \boxtimes D \geqslant 0$.

From now on, we fix the decomposition (3), and assume $n = q$ and $m = p$.

Definition 3. *The Khatri-Rao product of* $A = [A_{ij}]_{i,j=1}^{m,n}$ *and* $B = [B_{ij}]_{i,j=1}^{m,n}$ *is defined to be a bounded linear operator from* $\bigoplus_{j=1}^{n} \mathcal{H}_j \otimes \mathcal{K}_j$ *to* $\bigoplus_{i=1}^{m} \mathcal{H}'_i \otimes \mathcal{K}'_i$ *represented by the operator matrix*

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n}. \tag{5}$$

Lemma 4 ([8])**.** *For* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$*, we have* $(A \boxdot B)^* = A^* \boxdot B^*$.

Fix an ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ of Hilbert spaces. Define the ordered pair (Z_1, Z_2) of selection operators associated with $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ by [8]:

$$Z_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}. \tag{6}$$

Here, for each $r = 1, ..., m$

$$E_r = \left[E_{gh}^{(r)}\right]_{g,h=1}^{m,m} : \bigoplus_{k=1}^{m} \mathcal{H}'_k \otimes \mathcal{K}'_k \to \bigoplus_{l=1}^{m} \mathcal{H}'_r \otimes \mathcal{K}'_l$$

with $E_{gh}^{(r)}$ is an identity operator if $g = h = r$ and the others are zero operators. For each $s = 1, ..., n$, the operator F_s is defined by

$$F_s = \left[F_{gh}^{(s)}\right]_{g,h=1}^{n,n} : \bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i \to \bigoplus_{j=1}^{n} \mathcal{H}_s \otimes \mathcal{K}_j$$

with $F_{gh}^{(s)}$ is an identity operator if $g = h = s$ and the others are zero operators. From the construction, the operator Z_i is an isometry and $Z_i Z_i^* \leqslant I$ for $i = 1, 2$. When $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $Z_1 = Z_2$.

Lemma 5 ([8])**.** *Let* (Z_1, Z_2) *be the ordered pair of selection operators associated with the ordered tuple* $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$*. For any operator matrices* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$*, we have*

$$A \boxdot B = Z_1^*(A \boxtimes B)Z_2. \tag{7}$$

For the case $\mathcal{H} = \mathcal{H}'$ *and* $\mathcal{K} = \mathcal{K}'$*, we have* $Z_1 = Z_2 := Z$ *and hence for any* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$*,*

$$A \boxdot B = Z^*(A \boxtimes B)Z. \tag{8}$$

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3 Two constructions related to selection operators

In this section, we construct certain operators related to selection operators.

Theorem 6. *Let (Z_1, Z_2) be the ordered pair of selection operators associated with an ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$. Then there exist operators*

$$V : \bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m}\mathcal{H}_i'\otimes\mathcal{K}_j' \to \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{m}\mathcal{H}_i'\otimes\mathcal{K}_j',$$
$$W : \bigoplus_{i=1}^{n-1}\bigoplus_{j=1}^{n}\mathcal{H}_i\otimes\mathcal{K}_j \to \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n}\mathcal{H}_i\otimes\mathcal{K}_j$$

such that $Z_1^*V = 0$, $Z_2^*W = 0$, $Z_1Z_1^* + VV^* = I$ and $Z_2Z_2^* + WW^* = I$. If, in addition, $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $V = W$.

Proof. Let

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} \tag{9}$$

where

$$V^{(r)} = \left[V_{kl}^{(r)}\right]_{k,l=1}^{m,m^2-1} : \bigoplus_{i=1}^{m}\bigoplus_{\substack{j=1\\ i+j<m^2}}^{m} \mathcal{H}_i'\otimes\mathcal{K}_i' \to \bigoplus_{i=1}^{m}\mathcal{H}_r'\otimes\mathcal{K}_i'$$

for $r = 1, ..., m$, with $V_{kl}^{(r)}$ is an identity operator if $k \neq r$ and $l = m(r-1)+k$ and the others are zero operators. Note that

$$\begin{aligned} &E_1^*V_1 \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

For each r, we have

$$V_rV_r^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}.$$

Then we obtain

$$Z_1^*V = \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{bmatrix} = E_1^*V_1 + E_2^*V_2 + \ldots + E_m^*V_m = 0,$$

$$\begin{aligned} & Z_1Z_1^* + VV^* \\ &= \begin{bmatrix} E_1E_1^* & E_1E_2^* & \cdots & E_1E_m^* \\ E_2E_1^* & E_2E_2^* & \cdots & E_2E_m^* \\ \vdots & \vdots & \ddots & \vdots \\ E_mE_1^* & E_mE_2^* & \cdots & E_mE_m^* \end{bmatrix} + \begin{bmatrix} V_1V_1^* & V_1V_2^* & \cdots & V_1V_m^* \\ V_2V_1^* & V_2V_2^* & \cdots & V_2V_m^* \\ \vdots & \vdots & \ddots & \vdots \\ V_mEV_1^* & V_mV_2^* & \cdots & V_mV_m^* \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}. \end{aligned}$$

Now, let

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_m \end{bmatrix} \tag{10}$$

where

$$W^{(s)} = \left[W_{kl}^{(s)} \right]_{k,l=1}^{n,n^2-1} : \bigoplus_{i=1}^{n} \bigoplus_{\substack{j=1 \\ i+j<n^2}}^{n} \mathcal{H}_i \otimes \mathcal{K}_i \to \bigoplus_{i=1}^{n} \mathcal{H}_s \otimes \mathcal{K}_i$$

for $s = 1, ..., n$, with $W_{kl}^{(s)}$ is an identity operator if $k \neq s$ and $l = n(s-1)+k$ and others are zero operators. A direct computation shows that $Z_2^*W = 0$ and $Z_2Z_2^* + WW^* = I$. When $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $V_i = W_i$ for all $i = 1, \ldots, m$, i.e. $V = W$. □

Theorem 7. *Fix the decomposition* (3) *with* $n = q$ *and* $m = p$. *Suppose further that* $\mathcal{H}_i = \mathcal{X}$, $\mathcal{K}_i = \mathcal{Y}$, $\mathcal{H}'_j = \mathcal{X}'$ *and* $\mathcal{K}'_j = \mathcal{Y}'$ *for all* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$. *Let* (Z_1, Z_2) *be the ordered pair of associated selection operators. Then there exist operators*

$$\begin{aligned} Q_1 &: \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^{m} \mathcal{X}' \otimes \mathcal{Y}' \to \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} \mathcal{X}' \otimes \mathcal{Y}', \\ Q_2 &: \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n} \mathcal{X} \otimes \mathcal{Y} \to \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \mathcal{X} \otimes \mathcal{Y} \end{aligned}$$

such that $Z_i^* Q_i = 0, Q_i^* Q_i = I$ *and* $Z_i Z_i^* + Q_i Q_i^* = I$ *for* $i = 1, 2$. *If, in addition,* $\mathcal{H} = \mathcal{H}'$ *and* $\mathcal{K} = \mathcal{K}'$, *we have* $Q_1 = Q_2$.

Proof. Consider

$$Q_1 = \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} F_2 & F_3 & \cdots & F_n \\ F_3 & F_4 & \cdots & F_1 \\ \vdots & \vdots & \ddots & \vdots \\ F_1 & F_2 & \cdots & F_{n-1} \end{bmatrix}. \tag{11}$$

Then calculations reveal that

$$Z_1^* Q_1 = \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\begin{aligned} Q_1^* Q_1 &= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} \\ &= \begin{bmatrix} \sum E_i^* E_i & 0 & \cdots & 0 \\ 0 & \sum E_i^* E_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i^* E_i \end{bmatrix} = I, \end{aligned}$$

$$\begin{aligned} & Q_1^* Q_1 + Z_1 Z_1^* \\ &= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i \neq 1} E_i E_j^* & 0 & \cdots & 0 \\ 0 & \sum_{i \neq 2} E_i E_j^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i \neq m} E_i E_j^* \end{bmatrix} + \begin{bmatrix} E_1 E_1^* & 0 & \cdots & 0 \\ 0 & E_2 E_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_m E_m^* \end{bmatrix} \\ &= \begin{bmatrix} \sum E_i E_i^* & 0 & \cdots & 0 \\ 0 & \sum E_i E_i^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i E_i^* \end{bmatrix} = I. \end{aligned}$$

Similarly, we have $Z_2^* Q_2 = 0$, $Q_2^* Q_2 = I$ and $Z_2 Z_2^* + Q_2 Q_2^* = I$. When $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $E_i = F_i$ for all $i = 1, \ldots, m$, i.e. $Q_1 = Q_2$. □

4 Operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators

In this section, we apply the construction in Section 3 to establish certain operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators.

Theorem 8. *Let* (Z_1, Z_2) *be the ordered pair of selection operators associated with an ordered tuple* $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$. *Let* V *and* W *be operator matrices defined by* (9) *and* (10). *For any operator matrices* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, *we have*

$$AA^* \boxdot BB^* = (A \boxdot B)(A^* \boxdot B^*) + Z_1^*(A \boxtimes B)WW^*(A^* \boxtimes B^*)Z_1, \quad (12)$$
$$A^*A \boxdot B^*B = (A^* \boxdot B^*)(A \boxdot B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. \quad (13)$$

Proof. Since $AA^* \in \mathbb{B}(\mathcal{H}')$ and $BB^* \in \mathbb{B}(\mathcal{K}')$, the ordered pair of selection operators associated with $(\mathcal{H}', \mathcal{H}', \mathcal{K}', \mathcal{K}')$ is given by (Z_1, Z_1). By using Lemmas 2 and 5, and Theorem 6, we get

$$\begin{aligned} AA^* \boxdot BB^* &= Z_1^* (AA^* \boxtimes BB^*) Z_1 \\ &= Z_1^*(A \boxtimes B)(A \boxtimes B)^* Z_1 \\ &= Z_1^*(A \boxtimes B)(Z_2 Z_2^* + WW^*)(A \boxtimes B)^* Z_1 \\ &= Z_1^*(A \boxtimes B)Z_2 Z_2^*(A \boxtimes B)^* Z_1 + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^* Z_1 \\ &= (A \boxdot B)(A \boxdot B)^* + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^* Z_1. \end{aligned}$$

Now, for inequality (13), note that $A^*A \in \mathbb{B}(\mathcal{H})$ and $B^*B \in \mathbb{B}(\mathcal{K})$. In this case, the pair of associated selection operators is (Z_2, Z_2). It follows that

$$\begin{aligned} A^*A \boxdot B^*B &= Z_2^* (A^*A \boxtimes B^*B) Z_2 \\ &= Z_2^*(A \boxtimes B)^*(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*(Z_1 Z_1^* + VV^*)(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^* Z_1 Z_1^*(A \boxtimes B)Z_2 + Z_2^*(A \boxtimes B)^* VV^*(A \boxtimes B)Z_2 \\ &= (A^* \boxdot B^*)(A \boxdot B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. \end{aligned}$$ □

Corollary 9. *Let* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ *be operator matrices. Then*

$$AA^* \boxdot BB^* \geqslant (A \boxdot B)(A^* \boxdot B^*). \quad (14)$$

Proof. It follows immediately from Theorem 8. □

Theorem 10. *Assume the hypothesis of Theorem 7. For any* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, *we have*

$$AA^* \boxdot BB^* = (A \boxdot B)(A^* \boxdot B^*) + Z_1^*(A \boxtimes B)Q_2 Q_2^*(A^* \boxtimes B^*)Z_1, \quad (15)$$
$$A^*A \boxdot B^*B = (A^* \boxdot B^*)(A \boxdot B) + Z_2^*(A^* \boxtimes B^*)Q_1 Q_1^*(A \boxtimes B)Z_2, \quad (16)$$

where Q_1 *and* Q_2 *are operator matrices in* (11).

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Proof. The proof is similar to that of Theorem 8. Instead of Theorem 6, we apply Theorem 7. □

5 Characterizations of the mixed product property for Khatri-Rao products

In general, the mixed product property

$$(A \boxdot B)(C \boxdot D) \; = \; AC \boxdot BD$$

does not hold for compatible operator matrices A, B, C, D. It is interesting to find necessary and sufficient conditions for which this property holds. Indeed, we have the following assertions.

Theorem 11. *Assume the notations in Theorem 8. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following statements are equivalent:*

(i) $AC \boxdot BD \; = \; (A \boxdot B)(C \boxdot D)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$,

(ii) $AA^* \boxdot BB^* \; = \; (A \boxdot B)(A^* \boxdot B^*)$,

(iii) $Z_1^*(A \boxtimes B)W \; = \; 0$.

Proof. It is clear that (i)⇒(ii). To prove (ii)⇒(iii), suppose (ii). By Theorem 8, (ii) holds only if

$$[Z_1^*(A \boxtimes B)W]\,[W^*(A^* \boxtimes B^*)Z_1] = 0,$$

i.e., $Z_1^*(A \boxtimes B)W = 0$.
(iii)⇒(i): Assume the condition (iii) holds. Note that by Theorem 6 we have

$$Z_1^*(A \boxtimes B)(I - Z_2 Z_2^*) \; = \; Z_1^*(A \boxtimes B)WW^* \; = \; 0,$$

and hence $Z_1^*(A \boxtimes B) = Z_1^*(A \boxtimes B)Z_2 Z_2^*$. For any $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ and $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$, we have by Lemmas 2 and 5 that

$$\begin{aligned} AC \boxdot BD \; &= \; Z_1^*(AC \boxtimes BD)Z_1 \\ &= \; Z_1^*(A \boxtimes B)(C \boxtimes D)Z_1 \\ &= \; Z_1^*(A \boxtimes B)Z_2 Z_2^*(C \boxtimes D)Z_1 \\ &= \; (A \boxdot B(C \boxdot D). \end{aligned}$$

Thus we arrive at (i). □

Theorem 12. *Assume the notations in Theorem 8. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following statements are equivalent:*

(i) $CA \boxdot DB \; = \; (C \boxdot D)(A \boxdot B)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$,

(ii) $A^*A \boxdot B^*B \; = \; (A^* \boxdot B^*)(A \boxdot B)$,

(iii) $V^*(A \boxtimes B)Z_2 = 0.$

Proof. Clearly,(i)⇒(ii). The assertion (ii)⇒(iii) follows from Theorem 8. Now, suppose that (iii) holds. Then $VV^*(A \boxtimes B)Z_2 = 0$. Using Theorem 6, we get

$$(I - Z_1Z_1^*)(A \boxtimes B)Z_1 = VV^*(A \boxtimes B)Z_1 = 0$$

which implies $(A \boxtimes B)Z_1 = Z_1Z_1^*(A \boxtimes B)Z_1$. For any $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ and $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$, we have by Lemmas 2 and 5 that

$$\begin{aligned} CA \boxdot DB &= Z_2^*(CA \boxtimes DB)Z_2 \\ &= Z_2^*(C \boxtimes D)(A \boxtimes B)Z_2 \\ &= Z_2^*(C \boxtimes D)Z_1Z_1^*(A \boxtimes B)Z_2 \\ &= (C \boxdot D)(A \boxdot B). \end{aligned}$$
□

Theorem 13. *Assume the hypothesis of Theorem 7. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following conditions are equivalent:*

(i) $AC \boxdot BD = (A \boxdot B)(C \boxdot D)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$,

(ii) $AA^* \boxdot BB^* = (A \boxdot B)(A^* \boxdot B^*)$,

(iii) $Z_1^*(A \boxtimes B)Q_2 = 0.$

Proof. The proof is similar to that of Theorem 11. □

Theorem 14. *Assume the hypothesis of Theorem 7. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following statements are equivalent:*

(i) $CA \boxdot DB = (C \boxdot D)(A \boxdot B)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$,

(ii) $A^*A \boxdot B^*B = (A^* \boxdot B^*)(A \boxdot B)$,

(iii) $Q_1^*(A \boxtimes B)Z_2 = 0.$

Proof. The proof is similar to that of Theorem 12. □

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Some new coupled fixed point theorems in partially ordered complete Menger probabilistic G-metric spaces

Gang Wang, Chuanxi Zhu*, Zhaoqi Wu

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China

gangwang0904@126.com (G. Wang), chuanxizhu@126.com (C. X. Zhu)

Abstract. In this paper, we study the mapping satisfying mixed g-monotone property in partially ordered complete Menger probabilistic G-metric spaces. By weakening the notion of Ψ, we prove some new coupled coincidence point theorems and coupled common fixed point theorems. Finally, we provide an example to illustrate our results.

Keywords: partially ordered; coupled fixed point; mixed g-monotone mapping; Menger PGM-space

1 Introduction

The notions of mixed monotone mappings and coupled fixed point were first introduced by Bhaskar and Lakshmikantham [1], which was extended to the partially ordered metric spaces. Since then, some results have been presented about the existence and uniqueness of coupled fixed points (see [2]-[8]). In 2009, Lakshmikantham and *Ćirić* [7] introduced the concept of a mixed g-monotone mapping, which generalized and extended the notion of mixed monotone mappings and the coupled fixed point in [1]. In 2010, Jachymski [9] established a fixed point theorem for φ-contractions and gave a characterization of a function φ, satisfying probabilistic φ-contraction. On the other hand, Choudhury and Das [2] gave a fixed point theorem by using an altering distance function. In addition, by taking advantage of the notion of the notion of φ-contractive mapping in Menger PM-spaces, some fixed point theorems were brought by Ktbi and Gopal [6]. And Jin [10] put forward a new fixed point theorems for φ-contraction in KM fuzzy metric spaces. For other results in the direction, we refer to [11]-[14].

†Corresponding author: Chuanxi Zhu. Email: chuanxizhu@126.com.

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In this paper, we generalize the results of other scholars ([8],[14]) by weakening the notion of Ψ in [4]. We study compatibility of the mappings g and T, where T is a mixed g-monotone mapping. We also establish some new coupled coincidences point theorems and coupled common fixed point theorems in partially ordered Menger probabilistic G-metric spaces. Finally, an example is given to illustrate our main results.

2 preliminaries

At this stage, we recall some well-known definitions and results in the theory of partially ordered set and PGM-space.

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+$ be the set of all nonnegative real numbers, $\mathbb{Z}^+$ be the set of all positive integers.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing and left continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 1$. We will denote $\mathcal{D}$ by the set of all distributions function.

Let H denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 1, \text{ if } t > 0. \end{cases}$$

Definition 2.1 ([9]). A function $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, $t-norm$) if the following conditions are satisfied for any $a, b, c, d \in [0,1]$:

$(\triangle - 1)$ $\triangle(a,1) = a$;

$(\triangle - 2)$ $\triangle(a,b) = \triangle(b,a)$;

$(\triangle - 3)$ $\triangle(a,b) \geq \triangle(c,d)$, for $a \geq c, b \geq d$;

$(\triangle - 4)$ $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

Definition 2.2 ([2]). Let Φ denote the class of all functions $\phi : R^+ \to R^+$ satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$;

(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$;

(iii) ϕ is left continuous in $(0, +\infty)$;

(iv) ϕ is continuous at 0.

Definition 2.3 ([8]). Let Ψ denote the class of all functions $\psi : R^+ \to R^+$ satisfies the following conditions:

(1)ψ is non-decreasing;
(2) $\psi(t+s) \leq \psi(t)+\psi(s)$ for all $t, s \in [0,1)$.

Remark 2.1 ([8]). Ψ also satisfies that Ψ is continuous and $\Psi(t) = 0$ if and only if $t = 0$. It is easy to see that the notion of Ψ is stronger than Definition 2.3 in [8]. And it is obvious that the following condition holds:
(3) $\psi(p+q+t+s) \leq \psi(p)+\psi(q)+\psi(t)+\psi(s)$ for all $p, q, t, s \in [0,1)$.

Definition 2.4 ([18]). A Menger probabilistic G-metric space (briefly, a PGM-space) is a triple $(X, G^*, \triangle)$, where X is a nonempty set, $\triangle$ is a continuous t-norm, and G^* is a mapping from $X \times X \times X$ into $\mathscr{D}^+$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x,y,z)) satisfying the following conditions:
(PGM-1) $G^*_{x,y,z}(t) = 1$ for $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;
(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for $x, y, z \in X$ with $z \neq y$ and $t > 0$;
(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \cdots$ (symmetry in all three variables);
(PGM-4) $G^*_{x,y,z}(t+s) \geq \triangle(G^*_{x,a,a}(t), G^*_{a,y,z}(s))$ for $x, y, z, a \in X$ and $s, t > 0$.

Definition 2.5 ([1]). Let $(X, G^*, \triangle)$ be a PGM-space, and $\{x_n\}$ is a sequence in X. (1) $\{x_n\}$ is said to be convergent to $x \in X$ (write $x_n \to x$), if for any $\varepsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\varepsilon,\lambda}$ such that $x_n \in N_{x_0}(\varepsilon, \lambda)$ whenever $n > M_{\varepsilon,\lambda}$;
(2) $\{x_n\}$ is said to be *Cauchy* sequence, if for any $\varepsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\varepsilon,\lambda}$ such that $G^*_{x_n,x_m,x_l} > 1-\delta$ whenever $n, m, l > M_{\varepsilon,\lambda}$;
(3) $(X, G^*, \triangle)$ is said to be complete, if every *Cauchy* sequence in X converges to a point in X.

Definition 2.6 ([7]). Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if

$$g(F(x,y)) = F(g(x), g(y)) \quad \text{for all } x, y \in X.$$

Definition 2.7 ([7]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ is said to possess the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1)$$

Definition 2.8 ([11]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ is said to have the mixed $g-$monotone property if F is monotone $g-$non-decreasing in its first argument and is monotone $g-$non-decreasing in its second argument, that is, for any $x, y \in X$.

$$x_1, x_2 \in X,\ g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X,\ g(y_1) \leq g(y_2) \Rightarrow F(x, y_2) \leq F(x, y_1).$$

3 Coupled coincidence point results in partially ordered complete Menger probabilistic G-metric spaces

In this section, We begin with the following definition which is useful to prove some new coupled coincidence point theorems and coupled fixed point theorems in partially ordered complete Menger probabilistic G-metric spaces.

Definition 3.1 Let $(X, G^*, \triangle)$ be a Menger PGM-space with $\triangle$ (a continuous $t-norm$), $T : X^4 \to X$ and $g : X \to X$ be two mappings satisfying the following condition:

$$\begin{aligned}\psi(\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\phi(\lambda t))} - 1) \leq \frac{1}{4}\psi(&\frac{1}{G^*_{g(x),g(u),g(a)}(\phi(t))} - 1 + \frac{1}{G^*_{g(y),g(v),g(b)}(\phi(t))} - 1 \\ &+ \frac{1}{G^*_{g(z),g(p),g(c)}(\phi(t))} - 1 + \frac{1}{G^*_{g(w),g(q),g(d)}(\phi(t))} - 1).\end{aligned} \tag{3.1}$$

for all $t > 0$, and $x, y, z, w, u, v, p, q, a, b, c, d \in X, g(x) \leq g(u) \leq g(a), g(y) \geq g(v) \geq g(b), g(z) \leq g(p) \leq g(c)$ and $g(w) \geq g(q) \geq g(d)$, where $\lambda \in (0, 1), \psi \in \Psi$ and $\phi \in \Phi$. Then mappings T and g are said to satisfy ψ-contractive condition.

Theorem 3.1 Let$(X, \leq)$ be a partially ordered set and $(X, G^*, \triangle)$ be a complete PGM-space with a continuous $t - norm$. suppose that $T : X^4 \to X$ and $g : X \to X$ are the mappings with mixed $g-monotone$ property and satisfy ψ-contractive condition, such that $G^*_{g(x),g(u),g(a)} > 0, G^*_{g(y),g(v),g(b)} > 0, G^*_{g(z),g(p),g(c)} > 0, G^*_{g(w),g(q),g(d)} > 0$. Suppose $T(X^4) \subseteq g(X)$, g is continuous and commutes with T. Assuming that either

(a) T is continuous, or

(b) X has the following properties:

(I) If a non-decreasing sequence $x_n \to x, z_n \to z$, then $x_n \leq x, z_n \leq z$ for all n;

(II) If a non-increasing sequence $y_n \to y, w_n \to w$, then $y_n \leq y, w_n \leq w$ for all n.

If there exist $x_0, y_0, z_0, w_0 \in X$, such that $g(x_0) \leq T(x_0, y_0, z_0, w_0), g(z_0) \leq T(z_0, w_0, x_0, y_0)$, $g(y_0) \geq T(y_0, z_0, w_0, x_0)$ and $g(w_0) \geq T(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$, such that

$$g(x) = T(x, y, z, w), g(y) = T(y, z, w, x), g(z) = T(z, w, x, y), g(w) = T(w, x, y, z),$$

that is, T and g have a coupled coincidence point.

Proof Let $x_0, y_0, z_0, w_0 \in X$, such that $g(x_0) \leq T(x_0, y_0, z_0, w_0), g(z_0) \leq T(z_0, w_0, x_0, y_0)$ and $g(y_0) \geq T(y_0, z_0, w_0, x_0), g(w_0) \geq T(w_0, x_0, y_0, z_0)$, since $T(X^4) \subseteq g(X)$, we can choose $x_1, y_1, z_1, w_1 \in X$, such that

$$g(x_1) = T(x_0, y_0, z_0, w_0), g(y_1) = T(y_0, z_0, w_0, x_0), \tag{3.2}$$

$$g(z_1) = T(z_0, w_0, x_0, y_0), g(w_1) = T(w_0, x_0, y_0, z_0). \tag{3.3}$$

Continuing this process we can construct sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X, such that

$$g(x_{n+1}) = T(x_n, y_n, z_n, w_n), g(y_{n+1}) = T(y_n, z_n, w_n, x_n) \quad \text{for all } n \geq 0,$$
$$g(z_{n+1}) = T(z_n, w_n, x_n, y_n), g(w_{n+1}) = T(w_n, x_n, y_n, z_n) \quad \text{for all } n \geq 0,$$

we shall show that

$$g(x_n) \leq g(x_{n+1}),\ g(y_n) \geq g(y_{n+1}),\ g(z_n) \leq g(z_{n+1}),\ g(w_n) \geq g(w_{n+1}). \tag{3.4}$$

We shall use the mathematical induction to show that (3.4) holds.

Let $n = 0$, since

$$g(x_0) \leq T(x_0, y_0, z_0, w_0), g(y_0) \geq T(y_0, z_0, w_0, x_0),$$
$$g(z_0) \leq T(z_0, w_0, x_0, y_0), g(w_0) \geq T(w_0, x_0, y_0, z_0),$$

by (3.2) and (3.3), we have

$$g(x_0) \leq g(x_1), \quad g(y_0) \geq g(y_1), \quad g(z_0) \leq g(z_1), \quad g(w_0) \geq g(w_1).$$

Thus (3.4) holds for $n = 0$.

Now we suppose that (3.4) holds for some $n = i, i \in \mathbb{Z}^+$, we get

$$g(x_i) \leq g(x_{i+1}),\ g(y_i) \geq g(y_{i+1}), g(z_i) \leq g(z_{i+1}),\ g(w_i) \geq g(w_{i+1}).$$

Let $n = i+1$, owing to the property of mixed g-monotone, we have

$$g(x_{i+2}) = T(x_{i+1}, y_{i+1}, z_{i+1}, w_{i+1}) \geq T(x_i, y_{i+1}, z_i, w_{i+1}) \geq T(x_i, y_i, z_i, w_i) = g(x_{i+1}),$$
$$g(y_{i+2}) = T(y_{i+1}, z_{i+1}, w_{i+1}, x_{i+1}) \leq T(y_i, z_{i+1}, w_i, x_{i+1}) \leq T(y_i, z_n, w_i, x_i) = g(y_{i+1}).$$

Similarly, we obtain

$$g(z_{i+2}) \geq g(z_{i+1}), \quad g(w_{i+2}) \leq g(w_{i+1}).$$

By the mathematical induction, we conclude that (3.4) holds for all $n > 0$. Therefore

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq ... \leq g(x_n) \leq g(x_{n+1}) \leq \cdots;$$
$$g(y_0) \geq g(y_1) \geq g(y_2) \geq ... \geq g(y_n) \geq g(y_{n+1}) \leq \cdots;$$
$$g(z_0) \leq g(z_1) \leq g(z_2) \leq ... \leq g(z_n) \leq g(z_{n+1}) \leq \cdots;$$
$$g(w_0) \geq g(w_1) \geq g(w_2) \geq ... \geq g(w_n) \geq g(w_{n+1}) \leq \cdots.$$

In view of the fact, we have

$$\sup_{t\in\mathbb{R}} G^*_{g(x_2),g(x_1),g(x_0)}(t) = 1, \ \sup_{t\in\mathbb{R}} G^*_{g(y_2),g(y_1),g(y_0)}(t) = 1,$$
$$\sup_{t\in\mathbb{R}} G^*_{g(z_2),g(z_1),g(z_0)}(t) = 1, \ \sup_{t\in\mathbb{R}} G^*_{g(w_2),g(w_1),g(w_0)}(t) = 1,$$

and by (ii) of Definition 2.2, we can find some $t > 0$, such that

$$G^*_{g(x_2),g(x_1),g(x_0)}(\phi(t)) > 0, \ G^*_{g(y_2),g(y_1),g(y_0)}(\phi(t)) > 0,$$
$$G^*_{g(z_2),g(z_1),g(z_0)}(\phi(t)) > 0, \ G^*_{g(w_2),g(w_1),g(w_0)}(\phi(t)) > 0,$$

for

$$g(x_0) \leq g(x_1) \leq g(x_2), g(y_0) \geq g(y_1) \geq g(y_2),$$
$$g(z_0) \leq g(z_1) \leq g(z_2, g(w_0) \geq g(w_1) \geq g(w_2),$$

which implies that

$$G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda})) > 0, G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda})) > 0,$$
$$G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda})) > 0, G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda})) > 0.$$

Then by (3.1), we get

$$\begin{aligned}
\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))} - 1) &= \psi(\frac{1}{G^*_{T(x_2,y_2,z_2,w_2),T(x_1,y_1,z_1,w_1),T(x_0,y_0,z_0,w_0)}(\phi(t))} - 1) \\
&\leq \frac{1}{4}\psi(G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda})) - 1 + G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda})) - 1 \\
&\quad + G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda})) - 1 + G^*_{(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda})) - 1).
\end{aligned} \tag{3.5}$$

Similarly,

$$\psi(\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1) \le \frac{1}{4}\psi(G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))-1 + G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))-1), \tag{3.6}$$

$$\psi(\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1) \le \frac{1}{4}\psi(G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))-1 + G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))-1), \tag{3.7}$$

$$\psi(\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1) \le \frac{1}{4}\psi(G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))-1 + G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))-1). \tag{3.8}$$

From (3.5)-(3.8), we have

$$\begin{aligned}&\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1)\\&\quad+\psi(\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1)\\&\le\psi(\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))}-1).\end{aligned}$$

By (3) of Remark 2.1, we have

$$\begin{aligned}&\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1)\\&\le\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1)\\&\quad+\psi(\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1),\end{aligned}$$

which implies that

$$\begin{aligned}&\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi((t)))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1)\\&\le\psi(\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))}-1).\end{aligned}$$

Using the fact that ψ is non-decreasing, we get

$$\begin{aligned}&\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1\\&\le\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))}-1.\end{aligned}$$

From the above inequalities we deduce that

$$G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))>0,G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))>0,$$
$$G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))>0,G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))>0,$$

and

$$G^*_{g(x_3),g(x_2),g(x_1)}(\phi(\frac{t}{\lambda}))>0,G^*_{g(y_3),g(y_2),g(y_1)}(\phi(\frac{t}{\lambda}))>0,$$
$$G^*_{g(z_3),g(z_2),g(z_1)}(\phi(\frac{t}{\lambda}))>0,G^*_{g(w_3),g(w_2),g(w_1)}(\phi(\frac{t}{\lambda}))>0.$$

Again, by using (3.1), we have

$$\begin{aligned}&\frac{1}{G^*_{g(x_4),g(x_3),g(x_2)}(\phi(t))}-1+\frac{1}{G^*_{g(y_4),g(y_3),g(y_2)}(\phi(t))}-1+\frac{1}{G^*_{g(z_4),g(z_3),g(z_2)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_4),g(w_3),g(w_2)}(\phi(t))}-1\\&\le\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(\frac{t}{\lambda}))}-1\\&\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(\frac{t}{\lambda}))}-1\\&\le\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda^2}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda^2}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda^2}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda^2}))}-1.\end{aligned}$$

Repeating the above procedure successively, we obtain

$$\begin{aligned}&\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(t))}-1+\frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(t))}-1+\frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(t))}-1\\&\le\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda^n}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda^n}))}-1.\end{aligned}$$

If we replace x_0 with x_k in the previous inequalities, then for all $n > k$, we get

$$\begin{aligned}
&\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1 \\
&\quad + \frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1 \\
&\leq \frac{1}{G^*_{g(x_{k+2}),g(x_{k+1}),g(x_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1 + \frac{1}{G^*_{g(y_{k+2}),g(y_{k+1}),g(y_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1 \\
&\quad + \frac{1}{G^*_{g(z_{k+2}),g(z_{k+1}),g(z_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1 + \frac{1}{G^*_{g(w_{k+2}),g(w_{k+1}),g(w_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1.
\end{aligned}$$

Since $\phi(\frac{\lambda^k t}{\lambda^{n-k}}) \to \infty$ as $n \to \infty$ for all $0 < k < n$, we have

$$\begin{aligned}
&\lim_{n\to\infty} G^*_{g(x_{k+2}),g(x_{k+1}),g(x_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1, \quad \lim_{n\to\infty} G^*_{g(y_{k+2}),g(y_{k+1}),g(y_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1, \\
&\lim_{n\to\infty} G^*_{g(z_{k+2}),g(z_{k+1}),g(z_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1, \quad \lim_{n\to\infty} G^*_{g(w_{k+2}),g(w_{k+1}),g(w_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty}(\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1) \\
&\leq \lim_{n\to\infty}(\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1 \\
&\quad + \frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1) \leq 0, \\
&\lim_{n\to\infty}(\frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1) \\
&\leq \lim_{n\to\infty}(\frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1 \\
&\quad + \frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1) \leq 0,
\end{aligned}$$

similarly

$$\lim_{n\to\infty}(\frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1) \leq 0, \quad \lim_{n\to\infty}(\frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1) \leq 0,$$

which implies that

$$\lim_{n\to\infty}(G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t)) = 1, \lim_{n\to\infty}(G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t)) = 1, \tag{3.9}$$

$$\lim_{n\to\infty}(G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t)) = 1, \lim_{n\to\infty}(G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t)) = 1. \tag{3.10}$$

Now, let $\epsilon > 0$ be given, by (i) and (iv) of Definition 2.2, we can find $k \in \mathbb{Z}^+$ such that $\phi(\lambda^k t) < \epsilon$, it follows from (3.9) and (3.10) that

$$\begin{aligned}
&\lim_{n\to\infty}(G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\epsilon)) \geq \lim_{n\to\infty}(G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t)) = 1, \\
&\lim_{n\to\infty}(G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\epsilon)) \geq \lim_{n\to\infty}(G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t)) = 1,
\end{aligned}$$

similarly,

$$\lim_{n\to\infty}(G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\epsilon)) \geq 1, \quad \lim_{n\to\infty}(G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\epsilon)) \geq 1.$$

By using Menger triangle inequality, we obtain

$$G^*_{g(x_{n+p}),g(x_{n+1}),g(x_n)}(\epsilon) \geq \triangle(G^*_{g(x_{n+p}),g(x_{n+p-1}),g(x_{n+p-1})}(\frac{\epsilon}{p}), \triangle(G^*_{g(x_{n+p-1}),g(x_{n+p-2}),g(x_{n+p-2})}(\frac{\epsilon}{p}) \cdots, G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\frac{\epsilon}{p})).$$

Thus, letting $n \to \infty$ and making use of (3.9) and (3.10), for any integer, we get

$$\lim_{n\to\infty} G^*_{g(x_{n+p}),g(x_{n+1}),g(x_n)}(\epsilon) = 1 \quad \text{for every } \epsilon > 0.$$

Hence $g(x_n)$ is a Cauchy sequence. Similarly, we can prove that $g(y_n), g(z_n), g(w_n)$ are also Cauchy sequences. Since $(X, G^*, \triangle)$ is complete, there exist $x, y, z, w \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x, \quad \lim_{n\to\infty} g(y_n) = y, \quad \lim_{n\to\infty} g(z_n) = z, \quad \lim_{n\to\infty} g(w_n) = w. \tag{3.11}$$

From (3.11) and the continuity of g, we have

$$\lim_{n\to\infty} g(g(x_n)) = g(x), \quad \lim_{n\to\infty} g(g(y_n)) = g(y), \quad \lim_{n\to\infty} g(g(z_n)) = g(z), \quad \lim_{n\to\infty} g(g(w_n)) = g(w).$$

From (3.2), (3.3) and the commutativity of T and g, we have

$$g(g(x_{n+1})) = g(T(x_n, y_n, z_n, w_n)) = T(g(x_n), g(y_n), g(z_n), g(w_n)), \tag{3.12}$$

$$g(g(y_{n+1})) = g(T(y_n, z_n, w_n, x_n)) = T(g(y_n), g(z_n), g(w_n), g(x_n)), \tag{3.13}$$

$$g(g(z_{n+1})) = g(T(z_n, w_n, x_n, y_n)) = T(g(z_n), g(w_n), g(x_n), g(y_n)), \tag{3.14}$$

$$g(g(w_{n+1})) = g(T(w_n, x_n, y_n, z_n)) = T(g(w_n), g(x_n), g(y_n), g(z_n)). \tag{3.15}$$

Now,we show that

$$g(x) = T(x, y, z, w), \; g(y) = T(y, z, w, x), \; g(z) = T(z, w, x, y), \; g(w) = T(w, x, y, z).$$

Suppose that the assumption (a) holds. Taking the limit of (3.11) as $n \to \infty$, by (3.12) $\sim$ (3.15) and the continuity of T, we get

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} T(g(x_n, y_n, z_n, w_n)) = T(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(z_n), \lim_{n\to\infty} g(w_n)) \\ &= T(x, y, z, w), \end{aligned}$$

$$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} T(g(y_n, z_n, w_n, x_n)) = T(\lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(z_n), \lim_{n\to\infty} g(w_n), \lim_{n\to\infty} g(x_n))$$
$$= T(y, z, w, x).$$

Similarly,

$$g(z) = T(z, w, x, y),\ g(w) = T(w, x, y, z).$$

Thus we prove that

$$g(x) = T(x, y, z, w),\ g(y) = T(y, z, w, x),\ g(z) = T(z, w, x, y),\ g(w) = T(w, x, y, z).$$

Suppose now that (b) holds, since

$$G^*_{g(x),T(x,y,z,w),T(x,y,z,w)}(\epsilon) \geq \triangle(G^*_{g(x),g(g(x_{n+1})),g(g(x_{n+1}))}(\frac{\epsilon}{2}), G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\frac{\epsilon}{2})). \quad (3.16)$$

and using (i) of Definition 2.2, we find some $s > 0$ such that $\phi(s) < \frac{\epsilon}{2}$, since

$$\lim_{n\to\infty} g(g(x_n)) = g(x), \lim_{n\to\infty} g(g(y_n)) = g(y), \lim_{n\to\infty} g(g(z_n)) = g(z), \lim_{n\to\infty} g(g(w_n)) = g(w).$$

then there exists $n_0 \in \mathbb{Z}^+$, such that

$$G^*_{g(g(x_n)),g(x),g(x)}(\phi(s)) > 0, \quad G^*_{g(g(y_n)),g(y),g(y)}(\phi(s)) > 0,$$
$$G^*_{g(g(z_n)),g(z),g(z)}(\phi(s)) > 0, \quad G^*_{g(g(w_n)),g(w),g(w)}(\phi(s)) > 0.$$

for all $n > n_0$. Since $\{g(x_n)\}, \{g(z_n)\}$ is non-decreasing and as $\{g(y_n)\}, \{g(w_n)\}$ is non-increasing and

$$g(x_n) \to x,\ g(y_n) \to y,\ g(z_n) \to z,\ g(w_n) \to w.$$

By (3.1) and (3.12)-(3,15), we get

$$\psi(\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(s))} - 1) = \psi(\frac{1}{G^*_{T(g(x_n),g(y_n),g(z_n),g(w_n)),T(x,y,z,w),T(x,y,z,w)}(\phi(s))} - 1)$$
$$\leq \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}(\phi(\frac{s}{\lambda}))} - 1 + \frac{1}{G^*_{g(g(y_n)),g(y),g(y)}(\phi(\frac{s}{\lambda}))} - 1 + \frac{1}{G^*_{g(g(z_n)),g(z),g(z)}(\phi(\frac{s}{\lambda}))} - 1$$
$$+ \frac{1}{G^*_{g(g(w_n)),g(w),g(w)}(\phi(\frac{s}{\lambda}))} - 1).$$

By the same way, we obtain

$$\psi(\frac{1}{G^*_{g(g(y_{n+1})),T(y,z,w,x),T(y,z,w,x)}(\phi(s))} - 1)$$
$$\leq \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}\phi(\frac{s}{\lambda})} - 1 + \frac{1}{G^*_{g(g(y_n)),g(y),g(y)}\phi(\frac{s}{\lambda})} - 1 + \frac{1}{G^*_{g(g(z_n)),g(z),g(z)}\phi(\frac{s}{\lambda})} - 1$$
$$+ \frac{1}{G^*_{g(g(w_n)),g(w),g(w)}\phi(\frac{s}{\lambda})} - 1),$$

$$\begin{aligned}
&\psi(\frac{1}{G^*_{g(g(z_{n+1})),T(z,w,x,y),T(z,w,x,y)}(\phi(s))}-1)\\
&\leq \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(y_n)),g(y),g(y)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(z_n)),g(z),g(z)}\phi(\frac{s}{\lambda})}-1\\
&\quad+\frac{1}{G^*_{g(g(w_n)),g(w),g(w)}\phi(\frac{s}{\lambda})}-1),\\
&\psi(\frac{1}{G^*_{g(g(w_{n+1})),T(w,x,y,z),T(w,x,y,z)}(\phi(s))}-1)\\
&\leq \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(y_n)),g(y),g(y)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(z_n)),g(z),g(z)}\phi(\frac{s}{\lambda})}-1\\
&\quad+\frac{1}{G^*_{g(g(w_n)),g(w),g(w)}\phi(\frac{s}{\lambda})}-1).
\end{aligned}$$

By the above inequalities and (3) of Remark 2.1, we have

$$\begin{aligned}
&\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(\frac{\epsilon}{2}))}-1\leq\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(s))}-1\\
&\leq\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(s))}-1+\frac{1}{G^*_{g(g(y_{n+1})),T(y,z,w,x),T(y,z,w,x)}(\phi(s))}-1\\
&\quad+\frac{1}{G^*_{g(g(z_{n+1})),T(z,w,x,y),T(z,w,x,y)}(\phi(s))}-1+\frac{1}{G^*_{g(g(w_{n+1})),T(w,x,y,z),T(w,x,y,z)}(\phi(s))}-1 \qquad (3.17)\\
&\leq\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}(\phi(\frac{s}{\lambda}))}-1+\frac{1}{G^*_{g(g(y_n)),g(y),g(y)}(\phi(\frac{s}{\lambda}))}-1+\frac{1}{G^*_{g(g(z_n)),g(z),g(z)}(\phi(\frac{s}{\lambda}))}-1\\
&\quad+\frac{1}{G^*_{g(g(w_n)),g(w),g(w)}(\phi(\frac{s}{\lambda}))}-1.
\end{aligned}$$

Letting $n\to\infty$ in above inequalities (3.17), we obtain

$$\lim_{n\to\infty}G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\frac{\epsilon}{2})=1. \qquad (3.18)$$

From (3.16) and (3.18), we get $G^*_{g(x),T(x,y,z,w),T(x,y,z,w)}(\epsilon)=1$ for every $\epsilon>0$, which implies that $g(x)=T(x,y,z,w)$. Similarly, we show that $g(y)=T(y,z,w,x), g(z)=T(z,w,x,y), g(w)=T(w,x,y,z)$. Thus we prove that g and T have a coupled coincidence point.

Corollary 3.1 Let $(X,\leq)$ be a partially ordered set and $(X,G^*,\triangle)$ be a complete PGM-space with a continuous $t-norm$. Assume that $T:X^4\to X$ has the mixed monotone property, and satisfying the following:

$$\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\frac{t}{2})}-1\leq\frac{1}{4}(\frac{1}{G^*_{x,u,a}(t)}-1+\frac{1}{G^*_{y,v,b}(t)}-1+\frac{1}{G^*_{z,p,c}(t)}-1+\frac{1}{G^*_{w,q,d}(t)}-1)$$

for $t>0$, $G^*_{x,u,a}(t)>0, G^*_{y,v,b}(t)>0, G^*_{z,p,c}(t)>0, G^*_{w,q,d}(t)>0$ and $x,y,z,w,u,v,p,q,a,b,c,d\in X$ satisfying $x\leq u\leq a, z\leq p\leq c$, $y\geq v\geq b$ and $w\geq q\geq d$. Suppose that either

(a) T is continuous, or

(b) X has the following properties:

(I) if non-decreasing sequences $\{x_n\} \to x, \{z_n\} \to z$, then $x_n \leq x, z_n \leq z$ for all n,

(II) if non-increasing sequences $\{y_n\} \to y, \{w_n\} \to w$, then $y_n \leq y, w_n \leq w$ for all n.

If there exist $x_0, y_0, z_0, w_0 \in X$, such that $x_0 \leq T(x_0, y_0, z_0, w_0), z_0 \leq T(z_0, w_0, x_0, y_0)$ and $y_0 \geq T(y_0, z_0, w_0, x_0), w_0 \geq T(w_0, x_0, y_0, z_0)$, then their exist $x, y, z, w \in X$, such that

$$x = T(x, y, z, w), y = T(y, z, w, x), z = T(z, w, x, y), w = T(w, x, y, z),$$

that is, T has a coupled coincidence point.

Proof Taking $g = I_X$(the identity mapping on X), $\lambda = \frac{1}{2}$ and $\phi(t) = \varphi(t) = t$ for all $t \geq 0$ in Theorem 3.1, we can easily obtain the above corollary.

4 Coupled common fixed point results in partially ordered complete Menger probabilistic G-metric spaces

In the section, we prove the existence and uniqueness theorem of a coupled fixed point in partially ordered complete Menger probabilistic G-metric spaces.

Theorem 3.2 In addition to the hypotheses of Theorem 3.1, suppose that for every $(x, y, z, w), (x^*, y^*, z^*, w^*) \in X^4$ there exists a $(u, v, p, q) \in X^4$, such that $(T(u, v, p, q), T(v, p, q, u), T(p, q, u, v), T(q, u, v, p))$ are comparable to $(T(x, y, z, w), T(y, z, w, x), T(z, w, x, y), T(w, x, y, z))$ and $(T(x^*, y^*, z^*, w^*), T(y^*, z^*, w^*, x^*), T(z^*, w^*, x^*, y^*), T(w^*, x^*, y^*, z^*))$. Then T and g have a unique coupled common fixed point, that is, there exists a unique $(x, y, z, w) \in X^4$, such that

$$x = g(x) = T(x, y, z, w), y = g(y) = T(y, z, w, x), z = g(z) = T(z, w, x, y), w = g(w) = T(w, x, y, z).$$

Proof From Theorem 3.1, the set of coupled coincidences is non-empty, we shall first show that if (x, y, z, w) and (x^*, y^*, z^*, w^*) are coupled coincidence points, that is, if

$$g(x) = T(x, y, z, w),\ g(y) = T(y, z, w, x),\ g(z) = T(z, w, x, y),\ g(w) = T(w, x, y, z)$$

and

$$g(x^*) = T(x^*, y^*, z^*, w^*), g(y^*) = T(y^*, z^*, w^*, x^*),$$
$$g(z^*) = T(z^*, w^*, x^*, y^*), g(w^*) = T(w^*, x^*, y^*, z^*),$$

then

$$g(x) = g(x^*),\quad g(y) = g(y^*),\quad g(z) = g(z^*),\quad g(w) = g(w^*). \tag{4.1}$$

By assumption, there exists a $(u, v, p, q) \in X^4$, such that $(T(u, v, p, q), T(v, p, q, u), T(p, q, u, v), T(q, u, v, p))$ is comparable to $(T(x, y, z, w), T(y, z, w, x), T(z, w, x, y), T(w, x, y, z))$ and $(T(x^*, y^*, z^*, w^*), T(y^*, z^*, w^*, x^*), T(z^*, w^*, x^*, y^*), T(w^*, x^*, y^*, z^*))$. Putting $u_0 = u, v_0 = v, p_0 = p, q_0 = q$ and $u_1, v_1, p_1, q_1 \in X$, such that $g(u_1) = T(u_0, v_0, p_0, q_0), g(v_1) = T(v_0, p_0, q_0, u_0), g(p_1) = T(p_0, q_0, u_0, v_0)$, $g(q_1) = T(q_0, u_0, v_0, p_0)$. The proof of Theorems is similar to Theorem 3.1. We inductively define sequences $\{g(u_n)\}, \{g(v_n)\}, \{g(p_n)\}, \{g(q_n)\}$, such that

$$g(u_{n+1}) = T(u_n, v_n, p_n, q_n),\ g(v_{n+1}) = T(v_n, p_n, q_n, u_n),$$
$$g(p_{n+1}) = T(p_n, q_n, u_n, v_n),\ g(q_{n+1}) = T(q_n, u_n, v_n, p_n).$$

Similarly, setting $x_0 = x, y_0 = y, z_0 = z, w_0 = w$, and $x_0^* = x^*, y_0^* = y^*, z_0^* = z^*, w_0^* = w^*$. We also define sequences $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}, \{g(w_n)\}$ and $\{g(x_n^*)\}, \{g(y_n^*)\}, \{g(z_n^*)\}, \{g(w_n^*)\}$, then it is easy to show that

$$g(x_n) = T(x, y, z, w),\quad g(y_n) = T(y, z, w, x),\quad g(z_n) = T(z, w, x, y),\quad g(w_n) = T(w, x, y, z)$$

and

$$g(x_n^*) = T(x^*, y^*, z^*, w^*),\ g(y_n^*) = T(y^*, z^*, w^*, x^*),$$
$$g(z_n^*) = T(z^*, w^*, x^*, y^*),\ g(w_n^*) = T(w^*, x^*, y^*, z^*).$$

Since $(T(x, y, z, w), T(y, z, w, x), T(z, w, x, y), T(w, x, y, z)) = (g(x_1), g(y_1), g(z_1), g(w_1)) = (g(x), g(y), g(z), g(w))$ and $(T(u, v, p, q), T(v, p, q, u), T(p, q, u, v), T(q, u, v, p)) = (g(u_1), g(v_1), g(p_1), g(q_1))$ are comparable, then we have $g(x) \leq g(u_1), g(z) \leq g(p_1)$, $g(y) \geq g(v_1)$ and $g(w) \geq g(q_1)$. It is easy to show that $(g(x), g(y), g(w), g(z))$ and $(g(u_n), g(v_n), g(p_n), g(q_n))$ are comparable, that is, $g(x) \leq g(x_n), g(z) \leq g(z_n)$, $g(y) \geq g(y_n)$ and $g(w) \geq g(w_n)$, for all $n \geq 1$. Following the proof of Theorem 3.1, we can find some $t > 0$ such that

$$G^*_{g(x),g(u_n,g(u_n)}(\phi(\frac{t}{\lambda})) > 0,\ G^*_{g(y),g(v_n,g(v_n)}(\phi(\frac{t}{\lambda})) > 0 \ \text{ for all } n \geq 0,$$
$$G^*_{g(z),g(p_n,g(p_n)}(\phi(\frac{t}{\lambda})) > 0,\ G^*_{g(z),g(q_n,g(q_n)}(\phi(\frac{t}{\lambda})) > 0 \ \text{ for all } n \geq 0.$$

Thus from (3.1)

$$\psi(\frac{1}{G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\phi(t))} - 1) = \psi(\frac{1}{G^*_{T(x,y,z,w),T(u_n,v_n,p_n,q_n),T(u_n,v_n,p_n,q_n)}(\phi(t))} - 1)$$
$$\leq \frac{1}{4}\psi(\frac{1}{G^*_{x,u_n,u_n}(\phi(\frac{t}{\lambda}))} - 1 + \frac{1}{G^*_{y,v_n,v_n}(\phi(\frac{t}{\lambda}))} - 1 + \frac{1}{G^*_{z,p_n,p_n}(\phi(\frac{t}{\lambda}))} - 1 + \frac{1}{G^*_{w,q_n,q_n}(\phi(\frac{t}{\lambda}))} - 1).$$

By Remark 2.4, we get

$$\begin{aligned}
&\frac{1}{G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\phi(t))}-1+\frac{1}{G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\phi(t))}-1+\frac{1}{G^*_{g(z),g(p_{n+1}),g(p_{n+1})}(\phi(t))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_{n+1}),g(q_{n+1})}(\phi(t))}-1\\
&\leq\frac{1}{G^*_{g(x),g(u_n),g(u_n)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y),g(v_n),g(v_n)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z),g(p_n),g(p_n)}(\phi(\frac{t}{\lambda}))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_n),g(q_n)}(\phi(\frac{t}{\lambda}))}-1 \qquad (4.2)\\
&\vdots\\
&\leq\frac{1}{G^*_{g(x),g(u_0),g(u_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(y),g(v_0),g(v_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(z),g(p_0),g(p_0)}(\phi(\frac{t}{\lambda^n}))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_0),g(q_0)}(\phi(\frac{t}{\lambda^n}))}-1.
\end{aligned}$$

We replace u_k with u_0 in (4.2), we get

$$\begin{aligned}
&\frac{1}{G^*_{g(x),g(u_{n+1}),g(u_{n+1})}\phi(\lambda^k t)}-1+\frac{1}{G^*_{g(y),g(v_{n+1}),g(v_{n+1})}\phi(\lambda^k t)}-1+\frac{1}{G^*_{g(z),g(p_{n+1}),g(p_{n+1})}\phi(\lambda^k t)}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_{n+1}),g(q_{n+1})}\phi(\lambda^k t)}-1\\
&\leq\frac{1}{G^*_{g(x),g(u_k),g(u_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1+\frac{1}{G^*_{g(y),g(v_k),g(v_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1+\frac{1}{G^*_{g(z),g(p_k),g(p_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_k),g(q_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1,
\end{aligned}$$

for all $n>k$. Letting $n\to\infty$, we obtain

$$\lim_{n\to\infty}G^*_{g(x),g(u_{n+1},g(u_{n+1})(\phi(\lambda^k t))}=1,\ \lim_{n\to\infty}G^*_{g(y),g(v_{n+1},g(v_{n+1})(\phi(\lambda^k t))}=1.$$
$$\lim_{n\to\infty}G^*_{g(z),g(p_{n+1},g(p_{n+1})(\phi(\lambda^k t))}=1,\ \lim_{n\to\infty}G^*_{g(w),g(q_{n+1},g(q_{n+1})(\phi(\lambda^k t))}=1.$$

Let $\epsilon>0$ be given. By (i) and (iv) of Definition 2.2, there exists $k\in\mathbb{Z}^+$, such that $\phi(\lambda^k t)<\frac{\epsilon}{2}$. Then we have

$$\lim_{n\to\infty}G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\frac{\epsilon}{2})\geq\lim_{n\to\infty}G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\phi(\lambda^k t))=1, \qquad (4.3)$$
$$\lim_{n\to\infty}G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\frac{\epsilon}{2})\geq\lim_{n\to\infty}G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\phi(\lambda^k t))=1. \qquad (4.4)$$

Similarly, we prove that

$$\lim_{n\to\infty}G^*_{g(x^*),g(u_{n+1}),,g(u_{n+1})}(\frac{\epsilon}{2})=1,\ \lim_{n\to\infty}G^*_{g(y^*),g(v_{n+1}),,g(v_{n+1})}(\frac{\epsilon}{2})=1. \qquad (4.5)$$
$$\lim_{n\to\infty}G^*_{g(z^*),g(p_{n+1}),,g(p_{n+1})}(\frac{\epsilon}{2})=1,\ \lim_{n\to\infty}G^*_{g(w^*),g(q_{n+1}),,g(q_{n+1})}(\frac{\epsilon}{2})=1. \qquad (4.6)$$

By using Menger triangle inequality, and (4.3)-(4.6), we get

$$G^*_{g(x),g(u_{n+1}),g(x^*)}(\epsilon) \geq \triangle(G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\frac{\epsilon}{2}), G^*_{g(u_{n+1}),g(u_{n+1}),g(x^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty,$$
$$G^*_{g(y),g(v_{n+1}),g(y^*)}(\epsilon) \geq \triangle(G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\frac{\epsilon}{2}), G^*_{g(v_{n+1}),g(v_{n+1}),g(y^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty,$$
$$G^*_{g(z),g(p_{n+1}),g(z^*)}(\epsilon) \geq \triangle(G^*_{g(z),g(p_{n+1}),g(p_{n+1})}(\frac{\epsilon}{2}), G^*_{g(p_{n+1}),g(p_{n+1}),g(z^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty,$$
$$G^*_{g(w),g(q_{n+1}),g(w^*)}(\epsilon) \geq \triangle(G^*_{g(w),g(q_{n+1}),g(q_{n+1})}(\frac{\epsilon}{2}), G^*_{g(q_{n+1}),g(q_{n+1}),g(w^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty.$$

Hence $g(x) = g(x^*), g(y) = g(y^*), g(z) = g(z^*), g(w) = g(w^*)$, thus (4.1) holds. Since $g(x) = T(x,y,z,w), g(y) = T(y,z,w,x), g(z) = T(z,w,x,y), g(w) = T(w,x,y,z)$, by commutativity of T and g, we have

$$g(g(x)) = g(T(x,y,z,w)) = T(g(x),g(y),g(z),g(w)), \tag{4.7}$$
$$g(g(y)) = g(T(y,z,w,x)) = T(g(y),g(z),g(w),g(x)), \tag{4.8}$$
$$g(g(z)) = g(T(z,w,x,y)) = T(g(z),g(w),g(x),g(y)), \tag{4.9}$$
$$g(g(w)) = g(T(w,x,y,z)) = T(g(w),g(x),g(y),g(z)). \tag{4.10}$$

Denote $g(x) = \alpha, g(y) = \beta, g(z) = \gamma, g(w) = \sigma$. From (4.7)-(4.10), we obtain

$$g(\alpha) = T(\alpha,\beta,\gamma,\sigma), g(\beta) = T(\beta,\gamma,\sigma,\alpha), g(\gamma) = T(\gamma,\sigma,\alpha,\beta), g(\sigma) = T(\sigma,\alpha,\beta,\gamma), \tag{4.11}$$

thus $(\alpha,\beta,\gamma,\sigma)$ is a coupled coincidence point. Owing to (4.1) with $x^* = \alpha, y^* = \beta, z^* = \gamma$, and $w^* = \sigma$, it follows

$$g(\alpha) = g(x),\ g(\beta) = g(y),\ g(\gamma) = g(z),\ g(\sigma) = g(w),$$

that is

$$g(\alpha) = \alpha, g(\beta) = \beta, g(\gamma) = \gamma, g(\sigma) = \sigma. \tag{4.12}$$

From (4.11) and (4.12), we have

$$\alpha = g(\alpha) = T(\alpha,\beta,\gamma,\sigma), \beta = g(\beta) = T(\beta,\gamma,\sigma,\alpha), \gamma = g(\gamma) = T(\gamma,\sigma,\alpha,\beta), \sigma = g(\sigma) = T(\sigma,\alpha,\beta,\gamma).$$

Therefore, $(\alpha,\beta,\gamma,\sigma)$ is a coupled common fixed point of T and g. Suppose that $(\alpha^*,\beta^*,\gamma^*,\sigma^*)$ is another coupled common fixed point. By (4.1), we have

$$\alpha^* = g(\alpha^*) = g(x) = x, \beta^* = g(\beta^*) = g(y) = y, \gamma^* = g(\gamma^*) = g(z) = z, \sigma^* = g(\sigma^*) = g(w) = w,$$

which implies that T and g has a unique coupled common fixed point.

This completes the proof.

5 An example

In this section, an example are presented to verify the effectiveness and applicability of Theorem 3.1.

Example 5.1 Let $X = [0,1]$ be given. Define $G(x,y,z) = |x-y| + |y-z| + |z-x|$. A mapping $T : X^4 \to X$ define by $T(x_1,x_2,x_3,x_4) = \frac{x_1+x_2+x_3+x_4}{16}$. And $g : X \to X$ define by $g(x) = \frac{x}{2}$. Define

$$G^*_{x,y,z}(t) = \begin{cases} \frac{t}{t+G(x,y,z)}, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

for $x_1,x_2,x_3,x_4,x,y,z \in X$, where $T(X^4) \subset g(X)$. Then $(X, G^*, \triangle_m)$ is a complete Menger PGM-space with a continuous t-norm $\triangle_m$. Let $\lambda = \frac{1}{2}$, $\varphi(t) = \frac{9t}{10}$ and $\phi(t) = \frac{t}{2}$ be given for all $t > 0$. Then we have

$$\begin{aligned} &\psi(\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\phi(\lambda t))} - 1) = \psi(\frac{1}{\phi(\lambda t)}(G(T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)))) \\ &= \frac{9}{40t}(|x+y+z+w-u-v-p-q| + |u+v+p+q-a-b-c-d| \\ &\quad + |a+b+c+d-x-y-z-w|), \end{aligned} \tag{5.1}$$

$$\begin{aligned} &\frac{1}{4}\psi(\frac{1}{G^*_{g(x),g(u),g(a)}\phi(t)} - 1 + \frac{1}{G^*_{g(y),g(v),g(b)}\phi(t)} - 1 + \frac{1}{G^*_{g(z),g(p),g(c)}\phi(t)} - 1 + \frac{1}{G^*_{g(w),g(q),g(d)}\phi(t)} - 1) \\ &= \frac{9}{40t}(|x-u| + |u-a| + |a-x| + |y-v| + |v-b| + |b-y| + |z-p| + |p-c| + |c-z| \\ &\quad + |w-q| + |q-d| + |d-w|). \end{aligned} \tag{5.2}$$

By (5.1) and (5.2), we obtain

$$\begin{aligned} \psi(\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\phi(\lambda t))} - 1) \leq &\frac{1}{4}\psi(\frac{1}{G^*_{g(x),g(u),g(a)}\phi(t)} - 1 + \frac{1}{G^*_{g(y),g(v),g(b)}\phi(t)} - 1 \\ &+ \frac{1}{G^*_{g(z),g(p),g(c)}\phi(t)} - 1 + \frac{1}{G^*_{g(w),g(q),g(d)}\phi(t)} - 1), \end{aligned}$$

which implies that T and g satisfy ψ-contractive condition. Thus, all the conditions of Theorem 3.1 are satisfied. And (0,0,0,0) is the coupled coincidence point of T and g.

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FOURIER SERIES OF SUMS OF PRODUCTS OF HIGHER-ORDER EULER FUNCTIONS

TAEKYUN KIM[1], DAE SAN KIM[2], GWAN-WOO JANG[3], AND JONGKYUM KWON[4,*]

Abstract. In this paper, we consider three types of functions given by sums of products of higher-order Euler functions and derive their Fourier series expansions. Moreover, we express each of them in terms of Bernoulli functions.

1. Introduction

Let r be a nonnegative integer. The Euler polynomials $E_m^{(r)}(x)$ of order r are defined by the generating function (see [2, 9–12, 17, 19])

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{m=0}^{\infty} E_m^{(r)}(x)\frac{t^m}{m!}. \tag{1.1}$$

When $x = 0$, $E_m^{(r)} = E_m^{(r)}(0)$ are called the Euler numbers of order r. For $r = 1$, $E_m(x) = E_m^{(1)}(x)$, and $E_m = E_m^{(1)}$ are called Euler polynomials and numbers, respectively.

From (1.1), it is immediate to see that

$$\frac{d}{dx}E_m^{(r)}(x) = mE_{m-1}^{(r)}(x),\ m \geq 1, E_m^{(r)}(x+1)+E_m^{(r)}(x) = 2E_m^{(r-1)}(x),\ m \geq 0. \tag{1.2}$$

These in turn imply that

$$E_m^{(r)}(1) = 2E_m^{(r-1)} - E_m^{(r)},\ (m \geq 0), \tag{1.3}$$

and

$$\int_0^1 E_m^{(r)}(x)dx = \frac{2}{m+1}\left(E_{m+1}^{(r-1)} - E_{m+1}^{(r)}\right),\ (m \geq 0). \tag{1.4}$$

For any real number x, the fractional part of x is denoted by

$$< x >= x - [x] \in [0, 1). \tag{1.5}$$

We will need the following facts about the Fourier series expansion of the Bernoulli function $B_m(< x >)$:

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* corresponding author.

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}, \tag{1.6}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.7}$$

In the present paper, we will study the following three types of sums of products of higher-order Euler functions and find Fourier series expansions for them. Furthermore, we will express them in terms of Bernoulli functions. In the following, we let r, s be positive integers.

(1) $\alpha_m(< x >) = \sum_{k=0}^{m} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >),\ (m \geq 1)$;
(2) $\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >),\ (m \geq 1)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >),\ (m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1, 20]).

As to $\gamma_m(< x >)$, we note that the polynomial identity (1.8) follows immediately from the Fourier series expansion of $\gamma_m(< x >)$ in Theorems 4.1 and 4.2:

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(x)E_{m-k}^{(s)}(x) \\ &= \frac{1}{m} \sum_{k=0}^{m} \binom{m}{k} \Big\{ \Lambda_{m-k+1} + \frac{2(H_{m-1} - H_{m-k})}{m-k+1} \\ &\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)}) \Big\} B_k(x), \end{aligned} \tag{1.8}$$

where, for each integer $l \geq 2$,

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{2}{k(l-k)} \left(2E_k^{(r-1)} E_{l-k}^{(s-1)} - E_k^{(r)} E_{l-k}^{(s-1)} - E_k^{(r-1)} E_{l-k}^{(s)}\right), \tag{1.9}$$

and $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers.

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is noteworthy that from the Fourier series expansion of the function

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle) \tag{1.10}$$

we can derive the famous Faber-Pandharipande-Zagier identity (see [4, 7, 8]) and the Miki's identity (see [3, 5, 7, 8, 18]). Hence our problem here is a natural extension of the previous works which lead to a simple proof for the important Faber-Pandharipande-Zagier and Miki's identities (see [15]). Some related recent works can be found in [6, 13–16].

2. The function $\alpha_m(<x>)$

Let $\alpha_m(x) = \sum_{k=0}^{m} E_k^{(r)}(x)E_{m-k}^{(s)}(x)$, $(m \geq 1)$. Then we will consider the function

$$\alpha_m(<x>) = \sum_{k=0}^{m} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),\ (m \geq 1), \tag{2.1}$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(<x>)e^{-2\pi i n x}dx = \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx. \tag{2.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned}
\alpha_m'(x) &= \sum_{k=0}^{m} \left(kE_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x) + (m-k)E_k^{(r)}(x)E_{m-k-1}^{(s)}(x) \right) \\
&= \sum_{k=1}^{m} kE_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x) + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)E_{m-k-1}^{(s)}(x) \\
&= \sum_{k=0}^{m-1}(k+1)E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) \\
&= (m+1)\sum_{k=0}^{m-1} E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.4}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+2} \right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.6}$$

For $m \geq 1$, we put

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m} \left(E_k^{(r)}(1)E_{m-k}^{(s)}(1) - E_k^{(r)}E_{m-k}^{(s)} \right) \\
&= \sum_{k=0}^{m} \left((2E_k^{(r-1)} - E_k^{(r)})(2E_{m-k}^{(s-1)} - E_{m-k}^{(s)}) - E_k^{(r)}E_{m-k}^{(s)} \right) \\
&= 2\sum_{k=0}^{m} \left(2E_k^{(r-1)}E_{m-k}^{(s-1)} - E_k^{(r)}E_{m-k}^{(s-1)} - E_k^{(r-1)}E_{m-k}^{(s)} \right).
\end{aligned} \tag{2.7}$$

We now see that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0, \tag{2.8}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

Next, we want to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}[\alpha_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m, \end{aligned} \tag{2.10}$$

from which by induction on m, we can easily derive that

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}. \tag{2.11}$$

Case 2 : $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.12}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise C^∞. In addition, $\alpha_m(< x >)$ is continuous for those positive integers m with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for a positive integer m. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned} \alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >) \\ &+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

We are now going to state our first result.

Theorem 2.1. *For each positive integer l, we let*

$$\Delta_l = 2\sum_{k=0}^{l}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)} - E_k^{(r)}E_{l-k}^{(s-1)} - E_k^{(r-1)}E_{l-k}^{(s)}\right).$$

Assume that $\Delta_m = 0$, for a positive integer m. Then we have the following.
(a) $\sum_{k=0}^{m} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >)$ has the Fourier series expansion

$$\begin{aligned}&\sum_{k=0}^{m} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >)\\ =&\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned} \tag{2.14}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$(b)\sum_{k=0}^{m} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >) = \frac{1}{m+2}\sum_{j=0, j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >), \tag{2.15}$$

for all x in $\mathbb{R}$.

Assume next that $\Delta_m \neq 0$, for a positive integer m. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^{∞}, and discontinuous with jump discontinuities at integers.
Then the Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \notin \mathbb{Z}$,
and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.16}$$

for $x \in \mathbb{Z}$.

Now, we are going to state our second result.

Theorem 2.2. *For each positive integer l, we let*

$$\Delta_l = 2\sum_{k=0}^{l}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)} - E_k^{(r)}E_{l-k}^{(s-1)} - E_k^{(r-1)}E_{l-k}^{(s)}\right).$$

Assume that $\Delta_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}&(a)\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\ &= \begin{cases}\sum_{k=0}^{m} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >), & \text{for } x \notin \mathbb{Z},\\ \sum_{k=0}^{m} E_k^{(r)}E_{m-k}^{(s)} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{2.17}$$

$$(b)\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=\sum_{k=0}^{m}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),\ for\ x\notin\mathbb{Z}; \tag{2.18}$$

$$\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=\sum_{k=0}^{m}E_k^{(r)}E_{m-k}^{(s)}+\frac{1}{2}\Delta_m,\ for\ x\in\mathbb{Z}. \tag{2.19}$$

3. The function $\beta_m(<x>)$

Let $\beta_m(x)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(x)E_{m-k}^{(s)}(x),\ (m\geq 1)$. Then we will consider the function

$$\beta_m(<x>)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),\ (m\geq 1),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx}, \tag{3.1}$$

where

$$B_n^{(m)}=\int_0^1\beta_m(<x>)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx. \tag{3.2}$$

Before continuing further, we need to note the following.

$$\begin{aligned}\beta_m'(x)&=\sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}E_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x)+\frac{(m-k)}{k!(m-k)!}E_k^{(r)}(x)E_{m-k-1}^{(s)}(x)\right\}\\&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}E_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x)\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}E_k^{(r)}(x)E_{m-k-1}^{(s)}(x)\\&=\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k^{(r)}(x)E_{m-1-k}^{(s)}(x)\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k^{(r)}(x)E_{m-1-k}^{(s)}(x)\\&=2\beta_{m-1}(x).\end{aligned} \tag{3.3}$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x), \tag{3.4}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Big(\beta_{m+1}(1) - \beta_{m+1}(0)\Big). \tag{3.5}$$

For $m \geq 1$, we set

$$\begin{aligned}\Omega_m &= \beta_m(1) - \beta_m(0)\\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\big(E_k^{(r)}(1)E_{m-k}^{(s)}(1) - E_k^{(r)}E_{m-k}^{(s)}\big)\\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\Big((2E_k^{(r-1)} - E_k^{(r)})(2E_{m-k}^{(s-1)} - E_{m-k}^{(s)}) - E_k^{(r)}E_{m-k}^{(s)}\Big)\\ &= \sum_{k=0}^{m} \frac{2}{k!(m-k)!}\big(2E_k^{(r-1)}E_{m-k}^{(s-1)} - E_k^{(r)}E_{m-k}^{(s-1)} - E_k^{(r-1)}E_{m-k}^{(s)}\big).\end{aligned} \tag{3.6}$$

Now, it is immediate to see that

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.7}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.8}$$

We now would like to determine the Fourier coefficients $B_n^{(m)}$.

Case 1:$n \neq 0$

$$\begin{aligned}B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\Big[\beta_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{2}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m,\end{aligned} \tag{3.9}$$

from which by induction on m gives

$$B_n^{(m)} = -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}. \tag{3.10}$$

Case 2: $n = 0$

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.11}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. Further, $\beta_m(< x >)$ is continuous for those positive integers m with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for a positive integer m. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x}\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \Big(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi n x}}{(2\pi i n)^j} \Big)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >)\\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.12}$$

Now, we are going to state our first result.

Theorem 3.1. *For each positive integer l, we let*

$$\Omega_l = \sum_{k=0}^{l} \frac{2}{k!(l-k)!} \big(2E_k^{(r-1)} E_{l-k}^{(s-1)} - E_k^{(r)} E_{l-k}^{(s-1)} - E_k^{(r-1)} E_{l-k}^{(s)} \big). \tag{3.13}$$

Assume that $\Omega_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >)$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x},
\end{aligned} \tag{3.14}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}
(b) \ &\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >)\\
&= \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >),
\end{aligned} \tag{3.15}$$

for all $x \in \mathbb{R}$.

Assume next that $\Omega_m \neq 0$, for a positive integer m. Then, $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.16}$$

for $x \in \mathbb{Z}$.

Next, we are going to state our second result.

Theorem 3.2. *For each positive integer l, we let*

$$\Omega_l=\sum_{k=0}^{l}\frac{2}{k!(l-k)!}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)}-E_k^{(r)}E_{l-k}^{(s-1)}-E_k^{(r-1)}E_{l-k}^{(s)}\right). \tag{3.17}$$

Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}(a)\ &\frac{1}{2}\Omega_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}\\ &=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}E_{m-k}^{(s)}+\frac{1}{2}\Omega_m, & \textit{for } x\in\mathbb{Z}.\end{cases}\end{aligned} \tag{3.18}$$

$$\begin{aligned}(b)\ &\sum_{j=0}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)\\ &=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>), \quad \textit{for } x\notin\mathbb{Z};\\ &\sum_{j=0,j\neq 1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)\\ &=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}E_{m-k}^{(s)}+\frac{1}{2}\Omega_m, \quad \textit{for } x\in\mathbb{Z}.\end{aligned} \tag{3.19}$$

4. The function $\gamma_m(<x>)$

Let $\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(x)E_{m-k}^{(s)}(x)$, $(m\geq 2)$. Then we will consider the function

$$\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx}, \tag{4.1}$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx. \tag{4.2}$$

To proceed further, we need to observe the following.

$$\begin{aligned}\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{m-k} E_{k-1}^{(r)}(x) E_{m-k}^{(s)}(x) + \sum_{k=1}^{m-1} \frac{1}{k} E_k^{(r)}(x) E_{m-k-1}^{(s)}(x) \\ &= \sum_{k=0}^{m-2} \frac{1}{m-1-k} E_k^{(r)}(x) E_{m-1-k}^{(s)}(x) + \sum_{k=1}^{m-1} \frac{1}{k} E_k^{(r)}(x) E_{m-1-k}^{(s)}(x) \\ &= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} E_k^{(r)}(x) E_{m-1-k}^{(s)}(x) + \frac{1}{m-1} E_{m-1}^{(s)}(x) + \frac{1}{m-1} E_{m-1}^{(r)}(x) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} E_{m-1}^{(s)}(x) + \frac{1}{m-1} E_{m-1}^{(r)}(x).\end{aligned} \tag{4.3}$$

From this, we easily see that

$$\left(\frac{1}{m}\big(\gamma_{m+1}(x) - \frac{1}{m(m+1)} E_{m+1}^{(r)}(x) - \frac{1}{m(m+1)} E_{m+1}^{(s)}(x)\big)\right)' = \gamma_m(x), \tag{4.4}$$

and

$$\begin{aligned}&\int_0^1 \gamma_m(x)dx \\ &= \frac{1}{m}\Big[\gamma_{m+1}(x) - \frac{1}{m(m+1)} E_{m+1}^{(r)}(x) - \frac{1}{m(m+1)} E_{m+1}^{(s)}(x)\Big]_0^1 \\ &= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(E_{m+1}^{(r)}(1) - E_{m+1}^{(r)}(0)) \\ &- \frac{1}{m(m+1)}(E_{m+1}^{(s)}(1) - E_{m+1}^{(s)}(0))\Big) \\ &= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)}) \\ &- \frac{2}{m(m+1)}(E_{m+1}^{(s-1)} - E_{m+1}^{(s)})\Big).\end{aligned} \tag{4.5}$$

Let $\Lambda_1 = 0$, and for $m \geq 2$, we let

$$\begin{aligned}\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big(E_k^{(r)}(1) E_{m-k}^{(s)}(1) - E_k^{(r)} E_{m-k}^{(s)}\Big) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big((2E_k^{(r-1)} - E_k^{(r)})(2E_{m-k}^{(s-1)} - E_{m-k}^{(s)}) - E_k^{(r)} E_{m-k}^{(s)}\Big) \\ &= \sum_{k=1}^{m-1} \frac{2}{k(m-k)} \big(2E_k^{(r-1)} E_{m-k}^{(s-1)} - E_k^{(r)} E_{m-k}^{(s-1)} - E_k^{(r-1)} E_{m-k}^{(s)}\big).\end{aligned} \tag{4.6}$$

Then we have

$$\gamma_m(0) = \gamma_m(1) \iff \Lambda_m = 0, \ (m \geq 2), \tag{4.7}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}\Big(E_{m+1}^{(r-1)} - E_{m+1}^{(r)}\Big) - \frac{2}{m(m+1)}\Big(E_{m+1}^{(s-1)} - E_{m+1}^{(s)}\Big)\right). \tag{4.8}$$

We now want to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\
&+ \frac{1}{2\pi in}\int_0^1 \{(m-1)\gamma_{m-1}(x) + \frac{1}{m-1}E_{m-1}^{(r)}(x) + \frac{1}{m-1}E_{m-1}^{(s)}(x)\}e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{1}{2\pi in(m-1)}\int_0^1 E_{m-1}^{(r)}(x)e^{-2\pi inx}dx \\
&+ \frac{1}{2\pi in(m-1)}\int_0^1 E_{m-1}^{(s)}(x)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Big(\Phi_m^{(r)} + \Phi_m^{(s)}\Big),
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\Phi_m^{(r)} &= \sum_{k=1}^{m-1}\frac{2(m-1)_{k-1}}{(2\pi in)^k}\Big(E_{m-k}^{(r-1)} - E_{m-k}^{(r)}\Big), \\
&\int_0^1 E_l^{(r)}(x)e^{-2\pi inx}dx \\
&= \begin{cases} -\sum_{k=1}^{l}\frac{2(l)_{k-1}}{(2\pi in)^k}\Big(E_{l-k+1}^{(r-1)} - E_{l-k+1}^{(r)}\Big), & \text{for } n \neq 0, \\ \frac{2}{l+1}\Big(E_{l+1}^{(r-1)} - E_{l+1}^{(r)}\Big), & \text{for } n = 0. \end{cases}
\end{aligned} \tag{4.10}$$

Thus we have shown that

$$C_n^{(m)} = \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Big(\Phi_m^{(r)} + \Phi_m^{(s)}\Big). \tag{4.11}$$

An easy induction on m now gives

$$C_n^{(m)} = -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}(\Phi_{m-j+1}^{(r)} + \Phi_{m-j+1}^{(s)}). \tag{4.12}$$

To find a more explicit expression for $C_n^{(m)}$, we need to observe the following.

$$
\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Phi_{m-j+1}^{(r)} \\
=&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \sum_{k=1}^{m-j} \frac{2(m-j)_{k-1}}{(2\pi in)^k} (E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
=&2\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}} (E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
=&2\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=j+1}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^k} (E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}) \\
=&2\sum_{k=2}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^k} (E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}) \sum_{j=1}^{k-1} \frac{1}{m-j} \\
=&2\sum_{k=1}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^k} (E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}) (H_{m-1} - H_{m-k}) \\
=&\frac{2}{m} \sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} \frac{E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}}{m-k+1} (H_{m-1} - H_{m-k}).
\end{aligned}
\tag{4.13}
$$

Recalling that $\Lambda_1 = 0$, we get the following expression of $C_n^{(m)}$: for $n \neq 0$,

$$
\begin{aligned}
C_n^{(m)} =& -\frac{1}{m} \sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} \Big(\Lambda_{m-k+1} + \frac{2(H_{m-1} - H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)}) \Big).
\end{aligned}
\tag{4.14}
$$

Case 2: $n = 0$

$$
C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m} \left(\Lambda_{m+1} - \frac{2}{m(m+1)} (E_{m+1}^{(r-1)} + E_{m+1}^{(s-1)} - E_{m+1}^{(r)} - E_{m+1}^{(s)}) \right).
\tag{4.15}
$$

$\gamma_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Furthermore, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integer $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$, for an integer $m \geq 2$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$

converges uniformly to $\gamma_m(<x>)$, and

$$\begin{aligned}
&\gamma_m(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}(E_{m+1}^{(r-1)}+E_{m+1}^{(s-1)}-E_{m+1}^{(r)}-E_{m+1}^{(s)})\right)\\
&-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}\Big(\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big)\Big\}e^{2\pi inx}\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}(E_{m+1}^{(r-1)}+E_{m+1}^{(s-1)}-E_{m+1}^{(r)}-E_{m+1}^{(s)})\right)\\
&+\frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}\Big(-k!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}\Big)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}(E_{m+1}^{(r-1)}+E_{m+1}^{(s-1)}-E_{m+1}^{(r)}-E_{m+1}^{(s)})\right)\\
&+\frac{1}{m}\sum_{k=2}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases}\\
&=\frac{1}{m}\sum_{k=0,k\neq 1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}\tag{4.16}$$

Now, we can state our first result.

Theorem 4.1. *For each integer $l\geq 2$, we let*

$$\Lambda_l=\sum_{k=1}^{l-1}\frac{2}{k(l-k)}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)}-E_k^{(r)}E_{l-k}^{(s-1)}-E_k^{(r-1)}E_{l-k}^{(s)}\right),\tag{4.17}$$

with $\Lambda_1=0$. Assume that $\Lambda_m=0$, for an integer $m\geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>)$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} + E_{m+1}^{(s-1)} - E_{m+1}^{(r)} - E_{m+1}^{(s)})\right) \\
&- \frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\big(\Lambda_{m-k+1} + \frac{2(H_{m-1}-H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)})\big)\Big\} e^{2\pi i n x},
\end{aligned} \tag{4.18}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}
&(b) \ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >) \\
&= \frac{1}{m}\sum_{k=0, k\neq 1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1} + \frac{2(H_{m-1}-H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)})\Big\} B_k(< x >)
\end{aligned} \tag{4.19}$$

for all $x \in \mathbb{R}$.

Assume next that $\Lambda_m \neq 0$, for an integers $m \geq 2$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.20}$$

for $x \in \mathbb{Z}$.

We can now state our second result.

Theorem 4.2. *For each integer $l \geq 2$, let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{2}{k(l-k)}(2E_k^{(r-1)}E_{l-k}^{(s-1)} - E_k^{(r)}E_{l-k}^{(s-1)} - E_k^{(r-1)}E_{l-k}^{(s)}), \tag{4.21}$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$. Then we have the following.

$$\begin{aligned}
&(a) \ \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} + E_{m+1}^{(s-1)} - E_{m+1}^{(r)} - E_{m+1}^{(s)})\right) \\
&- \frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\big(\Lambda_{m-k+1} + \frac{2(H_{m-1}-H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)})\big)\Big\} e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}E_{m-k}^{(s)} + \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.22}$$

$$\begin{aligned}(b)\ & \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1} \\ & \times (E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>) \\ & =\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),\ for\ x\notin\mathbb{Z}; \\ & \frac{1}{m}\sum_{k=0,k\neq 1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1} \\ & \times (E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>) \\ & =\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}E_{m-k}^{(s)}+\frac{1}{2}\Lambda_m,\ for\ x\in\mathbb{Z}.\end{aligned} \tag{4.23}$$

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[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

[2] Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

[3] Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
E-mail address: jgw5687@naver.com

[4,*] Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
E-mail address: mathkjk26@gnu.ac.kr

Some symmetric identities for (p, q)-Euler zeta function

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper we obtain several symmetric identities of the (p, q)-Euler zeta function. We also give some new interesting properties, explicit formulas, a connection with (p, q)-Euler numbers and polynomials.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, (p, q)-Euler numbers and polynomials, (p, q)-analogue of Euler zeta function.

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1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-10]). The Euler numbers and the Euler polynomials have been extensively worked in many different contexts in such branches of mathematics as, for instance, complex analytic number theory, elementary number theory, differential topology, q-adic analytic number theory and quantum physics. In this paper, we obtain symmetric properties of the (p, q)-Euler zeta function. As applications of these properties, we study some interesting identities for the (p, q)-Euler polynomials and numbers.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -2, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We remember that the classical Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by the following generating functions

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

and

$$\left(\frac{2}{e^t + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.2}$$

respectively.

Some interesting properties of the (p, q)-Euler numbers and polynomials were first investigated by Ryoo[6]. The (p, q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It is clear that (p, q)-number contains symmetric property, and this number is q-number when $p = 1$. In particular, we can see $\lim_{q \to 1}[n]_{p,q} = n$ with $p = 1$.

By using (p, q)-number, we introduced the (p, q)-Euler polynomials and numbers, which generalized the previously known numbers and polynomials, including the Carlitz's type q-Euler numbers and polynomials. We begin by recalling here the Carlitz's type (p, q)-Euler numbers and polynomials(see [2]).

Definition 1. For $0 < q < p \leq 1$, the Carlitz's type (p, q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t) = \sum_{n=0}^{\infty} E_{n,p,q} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m]_{p,q} t}. \tag{1.1}$$

and

$$F_{p,q}(t, x) = \sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m+x]_{p,q} t}, \tag{1.2}$$

respectively.

The following elementary properties of Carlitz's type (p, q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are readily derived from (1.1) and (1.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see reference [6].

Theorem 2. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q}^{(h)}(x) = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \frac{1}{1 + q^{l+1} p^{n-l+h}}.$$

Theorem 3 (Distribution relation). For any positive integer m(=odd), we have

$$E_{n,p,q}(x) = \frac{[2]_q}{[2]_{q^m}} [m]_{p,q}^n \sum_{a=0}^{m-1} (-1)^a q^a E_{n,p^m,q^m} \left(\frac{a+x}{m}\right), \quad n \in \mathbb{N}_0.$$

Next, we introduce Carlitz's type (h, p, q)-Euler polynomials $E_{n,p,q}^{(h)}(x)$. The Carlitz's type (h, p, q)-Euler polynomials $E_{n,p,q}^{(h)}(x)$ are defined by

$$E_{n,p,q}^{(h)}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m p^{hm} [m + x]_{p,q}^n.$$

By (p, q)-number, we have the following theorem.

Theorem 4. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{p,q}^{n-l} q^{xl} E_{l,p,q}^{(n-l)}.$$

By using Carlitz's type (p, q)-Euler numbers and polynomials, (p, q)-Euler zeta function and Hurwitz (p, q)-Euler zeta functions are defined. These functions interpolate the Carlitz's type (p, q)-Euler numbers $E_{n,p,q}$, and polynomials $E_{n,p,q}(x)$, respectively. From (1.1), we note that

$$\begin{aligned} \left.\frac{d^k}{dt^k} F_{p,q}(t)\right|_{t=0} &= [2]_q \sum_{m=0}^{\infty} (-1)^n q^m [m]_{p,q}^k \\ &= E_{k,p,q}, (k \in \mathbb{N}). \end{aligned}$$

By using the above equation, we are now ready to define (p, q)-Euler zeta function.

Definition 5. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\zeta_{p,q}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_{p,q}^s}. \tag{1.3}$$

Note that $\zeta_{p,q}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $p = 1, q \to 1$, then $\zeta_{p,q}(s) = \zeta_E(s)$ which is the Euler zeta function(see [3, 4]). Relation between $\zeta_{p,q}(s)$ and $E_{k,p,q}$ is given by the following theorem.

Theorem 6. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q}(-k) = E_{k,p,q}.$$

Observe that $\zeta_{p,q}(s)$ function interpolates $E_{k,p,q}$ numbers at non-negative integers. By using (1.2), we note that

$$\frac{d^k}{dt^k} F_{p,q}(t,x)\bigg|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [m+x]_{p,q}^k \tag{1.4}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!}\right)\bigg|_{t=0} = E_{k,p,q}(x), \text{ for } k \in \mathbb{N}. \tag{1.5}$$

By (1.4) and (1.5), we are now ready to define the Hurwitz (p,q)-Euler zeta function.

Definition 7. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$\zeta_{p,q}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[n+x]_{p,q}^s}. \tag{1.6}$$

Note that $\zeta_{p,q}(s,x)$ is a meromorphic function on $\mathbb{C}$.

Obverse that, if $p = 1$ and $q \to 1$, then $\zeta_{p,q}(s,x) = \zeta_E(s,x)$ which is the Hurwitz Euler zeta function(see [3, 4]). Relation between $\zeta_{p,q}(s,x)$ and $E_{k,p,q}(x)$ is given by the following theorem.

Theorem 8. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q}(-k,x) = E_{k,p,q}(x).$$

Observe that $\zeta_{p,q}(-k,x)$ function interpolates $E_{k,p,q}(x)$ numbers at non-negative integers.

2. Symmetric properties about (p,q)-analogue of Euler zeta functions

In this section, we are going to obtain the main results of (p,q)-Euler zeta function. We also establish some interesting symmetric identities for (p,q)-Euler polynomials by using (p,q)-Euler zeta function.

Observe that $[xy]_{p,q} = [x]_{p^y,q^y}[y]_{p,q}$ for any $x, y \in \mathbb{C}$.
By substitute $w_1 x + \frac{w_1 i}{w_2}$ for x in Definition 7, replace p by p^{w_2} and replace q by q^{w_2}, respectively, we derive

$$\begin{aligned}
&\zeta_{p^{w_2},q^{w_2}}\left(s, w_1 x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_2}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_2 n}}{[w_1 x + \frac{w_1 i}{w_2} + n]_{p^{w_2},q^{w_2}}^s} \\
&= [2]_{q^{w_2}} [w_2]_{p,q}^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_2 n}}{[w_1 w_2 x + w_1 i + w_2 n]_{p,q}^s}.
\end{aligned}$$

Since for any non-negative integer m and odd positive integer w_1, there exist unique non-negative integer r such that $m = w_1 r + j$ with $0 \leq j \leq w_1 - 1$. Hence, this can be written as

$$\begin{aligned}
&\zeta_{p^{w_2},q^{w_2}}\left(s, w_1 x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_2}} [w_2]_{p,q}^s \sum_{\substack{w_1 r + j = 0 \\ 0 \leq j \leq w_1 - 1}}^{\infty} \frac{(-1)^{w_1 r + j} q^{w_2(w_1 r + j)}}{[w_2(w_1 r + j) + w_1 w_2 x + w_1 i]_{p,q}^s} \\
&= [2]_{q^{w_2}} [w_2]_{p,q}^s \sum_{j=0}^{w_1 - 1} \sum_{r=0}^{\infty} \frac{(-1)^{w_1 r + j} q^{w_2(w_1 r + j)}}{[w_1 w_2 (r + x) + w_1 i + w_2 j]_{p,q}^s}.
\end{aligned}$$

It follows from the above equation that

$$\begin{aligned}
&[2]_{q^{w_1}} [w_1]_{p,q}^s \sum_{i=0}^{w_2 - 1} (-1)^i q^{w_1 i} \zeta_{p^{w_2},q^{w_2}}\left(s, w_1 x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_1}} [2]_{q^{w_2}} [w_1]_{p,q}^s [w_2]_{p,q}^s \sum_{i=0}^{w_2 - 1} \sum_{j=0}^{w_1 - 1} \sum_{r=0}^{\infty} \frac{(-1)^{r+i+j} q^{(w_1 w_2 r + w_1 i + w_2 j)}}{[w_1 w_2 (r + x) + w_1 i + w_2 j]_q^s}.
\end{aligned} \tag{2.1}$$

From the similar method, we can have that

$$\begin{aligned}
\zeta_{p^{w_1},q^{w_1}}\left(s, w_2 x + \frac{w_2 j}{w_1}\right) &= [2]_{q^{w_1}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_1 n}}{[w_2 x + \frac{w_2 j}{w_1} + n]_{p^{w_1},q^{w_1}}^s} \\
&= [2]_{q^{w_1}} [w_1]_{p,q}^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_1 n}}{[w_1 w_2 x + w_2 j + w_1 n]_{p,q}^s}.
\end{aligned}$$

After some calculations in the above, we have

$$\begin{aligned}
&[2]_{q^{w_2}} [w_2]_{p,q}^s \sum_{j=0}^{w_1 - 1} (-1)^j q^{w_2 j} \zeta_{p^{w_1},q^{w_1}}^{(h)}\left(s, w_2 x + \frac{w_2 j}{w_1}\right) \\
&= [2]_{q^{w_1}} [2]_{q^{w_2}} [w_1]_{p,q}^s [w_2]_{p,q}^s \sum_{i=0}^{w_2 - 1} \sum_{j=0}^{w_1 - 1} \sum_{r=0}^{\infty} \frac{(-1)^{r+i+j} q^{(w_1 w_2 r + w_1 i + w_2 j)}}{[w_1 w_2 (r + x) + w_1 i + w_2 j]_{p,q}^s}.
\end{aligned} \tag{2.2}$$

Thus, we have the following theorem from (2.1) and (2.2).

Theorem 9. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and w_1, w_2 : odd positive integers. Then one has

$$\begin{aligned}
&[2]_{q^{w_1}} [w_1]_{p,q}^s \sum_{i=0}^{w_2 - 1} (-1)^i q^{w_1 i} \zeta_{p^{w_2},q^{w_2}}\left(s, w_1 x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_2}} [w_2]_{p,q}^s \sum_{j=0}^{w_1 - 1} (-1)^j q^{w_2 j} \zeta_{p^{w_1},q^{w_1}}\left(s, w_2 x + \frac{w_2 j}{w_1}\right).
\end{aligned}$$

In Theorem 9, we get the following formulas for the (p, q)-tangent zeta function.

Corollary 10. Let $w_2 = 1$ in Theorem 9. Then we get

$$\zeta_{p,q}(s, x) = [w_1]_{p,q}^{-s} \sum_{j=0}^{w_1 - 1} (-1)^j q^j \zeta_{p^{w_1},q^{w_1}}\left(s, \frac{x + j}{w_1}\right).$$

Corollary 11. Let $w_1 = 2, w_2 = 1$ in Theorem 9. Then we have

$$\zeta_{p^2,q^2}\left(s, \frac{x}{2}\right) - q \zeta_{p^2,q^2}\left(s, \frac{x + 1}{2}\right) = [2]_{q^2} [2]_q^{-1} [2]_{p,q}^s \zeta_{p,q}(s, x).$$

For $n \in \mathbb{N}$, we have

$$\zeta_{p,q}(-n,x) = E_{n,p,q}(x), (\text{see Theorem 8}).$$

By substituting $E_{n,p,q}(x)$ for $\zeta_{p,q}(s,x)$ in Theorem 9, we can derive that

$$\begin{aligned}
&[2]_{q^{w_1}}[w_1]_{p,q}^{-n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\zeta_{p^{w_2},q^{w_2}}\left(-n, w_1x+\frac{w_1 i}{w_2}\right)\\
&=[2]_{q^{w_1}}[w_1]_{p,q}^{-n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&[2]_{q^{w_2}}[w_2]_{p,q}^{-n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\zeta_{p^{w_1},q^{w_1}}\left(-n, w_2x+\frac{w_2 j}{w_1}\right)\\
&=[2]_{q^{w_2}}[w_2]_{p,q}^{-n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right).
\end{aligned}$$

Thus, we obtain the following theorem from Theorem 9.

Theorem 12. Let w_1, w_2 be any odd positive integer. Then for non-negative integers n, one has

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\
&=[2]_{q^{w_2}}[w_1]_{p,q}^{n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right).
\end{aligned}$$

Considering $w_1 = 1$ in the Theorem 12, we obtain as below equation(see Theorem 3).

$$E_{n,p,q}(x) = \frac{[2]_q}{[2]_{q^{w_2}}}[w_2]_{p,q}^n\sum_{j=1}^{w_2-1}(-1)^j q^j E_{n,p^{w_2},q^{w_2}}\left(\frac{x+j}{w_2}\right).$$

We obtain another result by applying the addition theorem for the Carlitz's type (h,p,q)-tangent polynomials $E^{(h)}_{n,p,q}(x)$. From the Theorem 12, we have

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\
&=[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{l=0}^{n}\binom{n}{l}q^{w_1(n-l)i}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l[i]^l_{p^{w_1},q^{w_1}}\\
&=[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}q^{(n-l)w_1 i}[i]^l_{p^{w_1},q^{w_1}}.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\
&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]^l_{p,q}[w_2]^{n-l}_{p,q}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\mathcal{E}_{n,l,p^{w_1},q^{w_1}}(w_2),
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&[2]_{q^{w_2}}[w_1]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right)\\&=[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]_{p,q}^l[w_1]_{p,q}^{n-l}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_1},q^{w_1}}(w_2x)\mathcal{E}_{n,l,p^{w_2},q^{w_2}}(w_1).\end{aligned}\tag{2.4}$$

where $\mathcal{E}_{n,l,p,q}(k)=\sum_{i=0}^{k-1}(-1)^i q^{(1+n-l)i}[i]_{p,q}^l$ is called as the sums of powers.

Hence, from (2.3) and (2.4), we have the following theorem.

Theorem 13. Let w_1, w_2 be any odd positive integer. Then we have

$$\begin{aligned}&[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]_{p,q}^l[w_1]_{p,q}^{n-l}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_1},q^{w_1}}(w_2x)\mathcal{E}_{n,l,p^{w_2},q^{w_2}}(w_1)\\&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]_{p,q}^l[w_2]_{p,q}^{n-l}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\mathcal{E}_{n,l,p^{w_1},q^{w_1}}(w_2).\end{aligned}$$

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ADDITIVE (ρ_1, ρ_2)-FUNCTIONAL INEQUALITIES IN COMPLEX BANACH SPACES

CHOONKIL PARK, DONG YUN SHIN*, AND GEORGE A. ANASTASSIOU

Abstract. In this paper, we introduce and solve the following additive (ρ_1, ρ_2)-functional inequalities

$$\begin{aligned}\|f(x+y+z)-f(x)-f(y)-f(z)\| &\geq \|\rho_1(f(x+y-z)-f(x)-f(y)+f(z))\| \\ &+ \|\rho_2(f(x-y+z)-f(x)+f(y)-f(z))\|, \qquad (0.1)\end{aligned}$$

where ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1| \cdot |\rho_2| > 1$, and

$$\begin{aligned}\|f(x+y-z)-f(x)-f(y)+f(z)\| &\geq \|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\ &+ \|\rho_2(f(x-y+z)-f(x)+f(y)-f(z))\| \qquad (0.2)\end{aligned}$$

where ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1| > 1$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [28] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

Park [18, 19] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 7, 10, 11, 15, 17, 20, 21, 24, 25, 26, 27, 30, 31, 32]).

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 9] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

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*Corresponding author (Dong Yun Shin).

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 22]).

In Section 2, we solve the additive (ρ_1, ρ_2)-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the direct method.

In Section 4, we solve the additive (ρ_1, ρ_2)-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 5, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the direct method.

Throughout this paper, let X be a real or complex normed space with norm $\|\cdot\|$ and Y a complex Banach space with norm $\|\cdot\|$. Assume that ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1| \cdot |\rho_2| > 1$.

2. Additive (ρ_1, ρ_2)-functional inequality (0.1): a fixed point method

In this section, we solve and investigate the additive (ρ_1, ρ_2)-functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} \|f(x+y+z) - f(x) - f(y) - f(z)\| &\geq \|\rho_1(f(x+y-z) - f(x) - f(y) + f(z))\| \\ &\quad + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Since $|\rho_1| \cdot |\rho_2| > 1$, $|\rho_1| > 1$ or $|\rho_2| > 1$.

(i) Assume that $|\rho_1| > 1$. Letting $z = 0$ in (4.1), we get

$$(1 - |\rho_1|)\|f(x+y) - f(x) - f(y)\| \geq |\rho_2|\|f(x-y) - f(x) + f(y)\|$$

for all $x, y \in X$. So $f(x+y) = f(x) + f(y)$ for all $x, y \in X$, since $|\rho_1| > 1$. So f is additive.

(ii) Assume that $|\rho_2| > 1$. Letting $y = 0$ in (4.1), we get

$$(1 - |\rho_2|)\|f(x+z) - f(x) - f(z)\| \geq |\rho_1|\|f(x-z) - f(x) + f(z)\|$$

for all $x, z \in X$. So $f(x+z) = f(x) + f(z)$ for all $x, z \in X$, since $|\rho_2| > 1$. So f is additive. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (2.1) in complex Banach spaces.

Since $|\rho_1| \cdot |\rho_2| > 1$, $|\rho_1| > 1$ or $|\rho_2| > 1$. One can exchange y and z and from now on, one can assume that $|\rho_1| > 1$.

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2}\varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|\rho_1(f(x+y-z) - f(x) - f(y) + f(z))\| + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \\ &\leq \|f(x+y+z) - f(x) - f(y) - f(z)\| + \varphi(x, y, z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)(|\rho_1|-1)}\varphi(x, x, 0)$$

for all $x \in X$.

Proof. Letting $z = 0$ and $y = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \le \frac{1}{|\rho_1|-1}\varphi(x, x, 0) \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x, x, 0),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x, x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \\ &\le 2\varepsilon\frac{L}{2}\varphi(x, x, 0) = L\varepsilon\varphi(x, x, 0)\end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.4) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{1}{|\rho_1|-1}\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \le \frac{L}{2(|\rho_1|-1)}\varphi(x, x, 0)$$

for all $x \in X$ So $d(f, Jf) \le \frac{L}{2(|\rho_1|-1)}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu\varphi(x, x, 0)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|A(x+y+z) - A(x) - A(y) - A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\ge \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\| f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y+z) - A(x) - A(y) - A(z)\| &\ge \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
&+ \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|\rho_1(f(x+y-z) - f(x) - f(y) + f(z))\| + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \\
&\le \|f(x+y+z) - f(x) - f(y) - f(z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \qquad (2.6)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{(2^r - 2)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 2.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \qquad (2.7)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)(|\rho_1| - 1)} \varphi(x, x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{1}{2(|\rho_1| - 1)} \varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{(2 - 2^r)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

Remark 2.6. If ρ_1 and ρ_2 are real numbers such that $|\rho_1| \cdot |\rho_2| > 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive (ρ_1, ρ_2)-functional inequality (0.1): a direct method

In this section, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (2.1) in complex Banach spaces by using the direct method.

Theorem 3.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{3.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0) \tag{3.2}$$

for all $x \in X$.

Proof. Letting $z = y$ and $x = 0$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \le \frac{1}{|\rho_1| - 1} \varphi(x, x, 0) \tag{3.3}$$

and so

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \frac{1}{|\rho_1| - 1} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$. Thus

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\le \sum_{j=l}^{m-1} \frac{2^j}{|\rho_1| - 1} \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.2).

It follows from (2.3) and (3.1) that

$$\begin{aligned}
&\|A(x+y+z) - A(x) - A(y) - A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\geq \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\| f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y+z) - A(x) - A(y) - A(z)\| &\geq \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\leq \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\leq \frac{2^q}{|\rho_1| - 1} \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, 0\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2^r - 2)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2.3) and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{3.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (3.3) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2(|\rho_1| - 1)} \varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}(|\rho_1| - 1)} \varphi(2^j x, 2^j x, 0) \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.6).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2 - 2^r)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

4. Additive (ρ_1, ρ_2)-functional inequality (0.2): a fixed point method

In this section, we solve and investigate the additive (ρ_1, ρ_2)-functional inequality (0.2) in complex Banach spaces.

From now on, assume that $\rho_1| > 1$.

Lemma 4.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} \|f(x + y - z) - f(x) - f(y) + f(z)\| &\geq \|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\| \\ &+ \|\rho_2 (f(x - y + z) - f(x) + f(y) - f(z))\| \end{aligned} \tag{4.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (4.1).

Letting $z = 0$ in (4.1), we get

$$(1 - |\rho_1|)\|f(x + y) - f(x) - f(y)\| \geq |\rho_2|\|f(x - y) - f(x) + f(y)\|$$

for all $x, y \in X$. So $f(x + y) = f(x) + f(y)$ for all $x, y \in X$, since $|\rho_1| > 1$. So f is additive. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (4.1) in complex Banach spaces.

Theorem 4.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2} \varphi(x, y, z) \tag{4.2}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\| + \|\rho_2 (f(x - y + z) - f(x) + f(y) - f(z))\| \\ &\leq \|f(x + y - z) - f(x) - f(y) + f(z)\| + \varphi(x, y, z) \end{aligned} \tag{4.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)(|\rho_1|-1)}\varphi(x, x, 0)$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (4.3), we get

$$\|f(2x) - 2f(x)\| \le \frac{1}{|\rho_1|-1}\varphi(x, x, 0) \tag{4.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x, x, 0), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x, x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \\ &\le 2\varepsilon\frac{L}{2}\varphi(x, x, 0) = L\varepsilon\varphi(x, x, 0) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (4.4) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{1}{|\rho_1|-1}\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \le \frac{L}{2(|\rho_1|-1)}\varphi(x, x, 0)$$

for all $x \in X$ So $d(f, Jf) \le \frac{L}{2(|\rho_1|-1)}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{4.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (4.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu\varphi(x, x, 0)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

It follows from (4.2) and (4.3) that

$$\begin{aligned}
&\|A(x+y-z) - A(x) - A(y) + A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\ge \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\| f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y-z) - A(x) - A(y) + A(z)\| &\ge \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&+ \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 4.1, the mapping $A : X \to Y$ is additive. □

Corollary 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|\rho_1(f(x+y+z) - f(x) - f(y) - f(z))\| + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \\
&\le \|f(x+y-z) - f(x) - f(y) + f(z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \qquad (4.6)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{(2^r-2)(|\rho_1|-1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 4.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \qquad (4.7)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 4.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (4.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2(|\rho_1| - 1)} \varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 4.2. □

Corollary 4.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2 - 2^r)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

Remark 4.6. If ρ_1 and ρ_2 are real numbers such that $|\rho_1| > 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

5. Additive (ρ_1, ρ_2)-functional inequality (0.2): a direct method

In this section, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (4.1) in complex Banach spaces by using the direct method.

Theorem 5.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{5.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0) \tag{5.2}$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (4.3), we get

$$\|f(2x) - 2f(x)\| \leq \frac{1}{|\rho_1| - 1} \varphi(x, x, 0) \tag{5.3}$$

and so

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \frac{1}{|\rho_1| - 1} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$. Thus

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{2^j}{|\rho_1| - 1} \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \end{aligned} \tag{5.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.4), we get (5.2).

It follows from (5.4) and (5.1) that

$$\begin{aligned}
&\|A(x+y-z) - A(x) - A(y) + A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\geq \lim_{n\to\infty} 2^n |\rho_1| \left\|f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\|f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y-z) - A(x) - A(y) + A(z)\| &\geq \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&+ \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 4.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (5.2). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\leq \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\leq \frac{2^q}{|\rho_1| - 1} \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, 0\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. □

Corollary 5.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2^r - 2)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Theorem 5.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (4.3) and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{5.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2(|\rho_1|-1)} \varphi(x,x,0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}(|\rho_1|-1)} \varphi(2^j x, 2^j x, 0) \end{aligned} \tag{5.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.6) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.6), we get (5.6).

The rest of the proof is similar to the proof of Theorem 5.1. □

Corollary 5.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2-2^r)(|\rho_1|-1)} \|x\|^r$$

for all $x \in X$.

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Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea
E-mail address: dyshin@uos.ac.kr

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
E-mail address: ganastss@memphis.edu

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