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On common fixed point theorems of weakly compatible mappings in fuzzy metric spaces

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Abstract: The purpose of this paper is to obtain common fixed point theorem involving two pair of weakly compatible mappings in complete fuzzy metric spaces. Some related results and illustrative examples are also discussed.

Keywords: common fixed point; weakly compatible mapping; complete fuzzy metric space; coincidence point; point of coincidence

1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *contraction* if there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{1}$$

If the metric space (X, d) is complete, then the mapping satisfying (1) has a unique fixed point.

Rhoades [11] assumed a weakly contractive mapping $f : X \to X$ which satisfies the condition

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \tag{2}$$

where $x, y \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. Rhoades [11] obtained the following extension.

Theorem 1.1. ([11]) *Let* $T : X \to X$ *be a weakly contractive mapping, where* (X, d) *is a complete metric space. Then* T *has a unique fixed point.*

Dutta and Choudhury [7] introduced a new generalization of contraction principle in the following theorem.

Theorem 1.2. ([7]) *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a self-mapping satisfying the inequality*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{3}$$

for all $x, y \in X$, *where* $\phi, \varphi : [0, \infty) \to [0, \infty)$ *are both continuous and monotone nondecreasing functions with* $\psi(t) = \varphi(t) = 0$ *if and only if* $t = 0$. *Then* T *has a unique fixed point.*

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Common fixed point theorems of weakly compatible mappings

Several researchers have studied the existence of fixed points and common fixed points of mappings (see [1, 2, 3, 4, 5, 6, 8, 9, 10, 12]).

In this article, we give a fixed point theorem for contraction maps in complete fuzzy metric space, which improves and generalizes the above-mentioned result of Dutta and Choudhury.

We recall some definitions before giving the main result of this article.

Definition 1.3. *A binary operation* $* : [0,1]^2 \to [0,1]$ *is called a continuous t-norm if* $([0,1], *)$ *is an Abelian topological monoid, i.e.,*
   (1) $*$ *is associative and commutative;*
   (2) $*$ *is continuous;*
   (3) $a * 1 = a$ *for all* $a \in [0,1]$;
   (4) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 1.4. *A 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions:*
   (1) $M(x, y, t) > 0$,
   (2) $M(x, y, t) = 1$ *if and only if* $x = y$,
   (3) $M(x, y, t) = M(y, x, t)$,
   (4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
   (5) $M(x, y, .) : (0, \infty) \to [0,1]$ *is continuous,*
   for all $x, y, z \in X$ *and* $t, s > 0$.

Definition 1.5. *Let* f *and* g *be self-maps on a set* X. *If* $w = fx = gx$ *for some* $x \in X$, *then* x *is called coincidence point of* f *and* g, *and* w *is called a point of coincidence of* f *and* g.

Definition 1.6. *Let* f *and* g *be two self-maps on a set* X. *Then* f *and* g *are said to be weakly compatible if they commute at every coincidence point.*

2. MAIN RESULTS

Theorem 2.1. *Let* (X, M, t) *be a complete fuzzy metric space, and let* E *be a nonempty closed subset of* X. *Let* $S, T : E \to E$ *and* $I, J : E \to X$ *be mappings satisfying* $T(E) \subset I(E)$ *and* $S(E) \subset J(E)$ *and for every* $x, y \in X$,

$$\psi(M(Sx, Ty, t)) \leq \psi(M_{I,J}(x, y)) - \varphi(M_{I,J}(x, y)), \qquad (4)$$

where $\psi : [0, \infty) \to [0, \infty)$ *is a continuous and nondecreasing function such that* $\psi(t) = 0$ *if and only if* $t = 0$. $\varphi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$, *and*

$$M_{I,J}(x, y) = \max \Big\{ M(Ix, Jy, t), M(Ix, Sx, t), M(Jy, Ty, t), \qquad (5)$$
$$\frac{1}{2} \Big(M(Ix, Ty, t) + M(Jy, Sx, t) \Big) \Big\}.$$

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If one of $S(E)$, $T(E)$, $I(E)$, JE is a closed subset of X, then $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X. Moreover, if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $T(E) \subset I(E)$ and $S(E) \subset J(E)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n-1} = Sx_{2n-2} = Jx_{2n-1}, \ \ y_{2n} = Tx_{2n-1} = Ix_{2n}, \qquad n = 1, 2, \cdots.$$

Suppose that $y_{n_0} = y_{n_0+1}$ for some n_0. Then the sequence $\{y_n\}$ is constant for $n \geq n_0$. Indeed, let $n_0 = 2k$. Then $y_{2k} = y_{2k+1}$ and it follows from (4) that

$$
\begin{aligned}
\psi(M(y_{2k+1}, y_{2k+2}, t)) &= \psi(M(Sx_{2k}, Tx_{2k+1}, t)) \\
&\leq \psi(M_{I,J}(x_{2k}, x_{2k+1})) - \varphi(M_{I,J}(x_{2k}, x_{2k+1})),
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
M_{I,J}&(x_{2k}, x_{2k+1}) \\
&= \max \Big\{ M(y_{2k}, y_{2k+1}, t), M(y_{2k}, Sx_{2k}, t), M(y_{2k+1}, Tx_{2k+1}, t), \\
&\qquad \frac{1}{2}\Big(M(y_{2k}, Tx_{2k+1}, t) + M(y_{2k+1}, Sx_{2k}, t) \Big) \Big\} \\
&= \max \Big\{ 0, 0, M(y_{2k+1}, y_{2k+2}, t), \frac{1}{2}\Big(M(y_{2k}, y_{2k+2}, t) + 0 \Big) \Big\} \\
&= \max \Big\{ M(y_{2k+1}, y_{2k+2}, t), \frac{1}{2} M(y_{2k}, y_{2k+2}, t) \Big\} \\
&= M(y_{2k+1}, y_{2k+2}, t).
\end{aligned}
$$

By (6), we get

$$\psi(M(y_{2k+1}, y_{2k+2}, t)) \leq \psi(M(y_{2k+1}, y_{2k+2}, t)) - \varphi(M(y_{2k+1}, y_{2k+2}, t)),$$

and so $\varphi(M(y_{2k+1}, y_{2k+2}, t)) \leq 0$ and $y_{2k+1} = y_{2k+2}$.

Similarly, if $n_0 = 2k + 1$, then one easily obtains that $y_{2k+2} = y_{2k+3}$ and the sequence $\{y_n\}$ is constant (starting from some n_0). Therefore, $\{S, I\}$ and $\{T, J\}$ have a point of coincidence in X.

Now, suppose that $M(y_n, y_{n+1}, t) > 0$ for each n. We shall show that for each $n = 0, 1, \cdots$,

$$M(y_{n+1}, y_{n+2}, t) \leq M_{I,J}(x_n, x_{n+1}) = M(y_n, y_{n+1}, t). \tag{7}$$

Using (4), we obtain that

$$
\begin{aligned}
\psi(M(y_{2n+1}, y_{2n+2}, t)) &= \psi(M(Sx_{2n}, Tx_{2n+1}, t)) \\
&\leq \psi(M_{I,J}(x_{2n}, x_{2n+1})) - \varphi(M_{I,J}(x_{2n}, x_{2n+1})) \\
&< \psi(M_{I,J}(x_{2n}, x_{2n+1})).
\end{aligned}
\tag{8}
$$

On the other hand, the control function ψ is nondecreasing. Then

$$M(y_{2n+1}, y_{2n+2}, t) \leq M_{I,J}(x_{2n}, x_{2n+1}). \tag{9}$$

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Moreover, we have

$$
M_{I,J}(x_{2n}, x_{2n+1})
$$

$$
= \max \Big\{ M(y_{2n}, y_{2n+1}, t), M(y_{2n}, Sx_{2n}, t), M(y_{2n+1}, Tx_{2n+1}, t),
$$
$$
\frac{1}{2} \Big(M(y_{2n}, Tx_{2n+1}, t) + M(y_{2n+1}, Sx_{2n}, t) \Big) \Big\}
$$
$$
= \max \Big\{ M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t),
$$
$$
M(y_{2n+1}, y_{2n+2}, t), \frac{1}{2} M(y_{2n}, y_{2n+2}, t) \Big\}
$$
$$
\leq \max \Big\{ M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t),
$$
$$
\frac{1}{2} \Big(M(y_{2n}, y_{2n+1}, t) + M(y_{2n+1}, y_{2n+2}) \Big) \Big\}
$$
$$
\leq \max \Big\{ M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t) \Big\}.
$$

If $M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n}, y_{2n+1}, t)$, then by using the last inequality and (9), we have $M_{I,J}(x_{2n}, x_{2n+1}) = M(y_{2n+1}, y_{2n+2}, t)$ and (8) implies that

$$
\psi(M(y_{2n+1}, y_{2n+2}, t)) = \psi(M(Sx_{2n}, Tx_{2n+1}, t))
$$
$$
\leq \psi(M(y_{2n+1}, y_{2n+2}, t)) - \varphi(M(y_{2n+1}, y_{2n+2}, t)),
$$

which is only possible when $M(y_{2n+1}, y_{2n+2}, t) = 0$. It is a contradiction. Hence $M(y_{2n+1}, y_{2n+2}, t) \leq M(y_{2n}, y_{2n+1}, t)$ and $M_{I,J}(x_{2n}, x_{2n+1}) \leq M(y_{2n}, y_{2n+1}, t)$. By definition, $M_{I,J}(x_{2n}, x_{2n+1}) \geq M(y_{2n}, y_{2n+1}, t)$, and so (7) is proved for $\{M(y_{2n+1}, y_{2n+2}, t)\}$.

In a similar way, one can obtain that

$$
M(y_{2n+3}, y_{2n+2}, t) \leq M_{I,J}(x_{2n+2}, x_{2n+1}) = M(y_{2n+2}, y_{2n+1}, t).
$$

So (7) holds for each $n \in \mathbb{N}$.

It follows that the sequence $\{M(y_n, y_{n+1}, t)\}$ is nondecreasing and the limit

$$
\lim_{n \to \infty} M(y_n, y_{n+1}, t) = \lim_{n \to \infty} M_{I,J}(x_n, x_{n+1})
$$

exists. We denote this limit by d^*. We have $d^* \geq 0$.

Suppose that $d^* > 0$. Then

$$
\psi(M(y_{n+1}, y_{n+2}, t)) \leq \psi(M_{I,J}(x_n, x_{n+1})) - \varphi(M_{I,J}(x_n, x_{n+1})).
$$

Passing to the (upper) limit when $n \to \infty$, we get

$$
\psi(d^*) \leq \psi(d^*) - \liminf_{n \to \infty} \varphi(M_{I,J}(x_n, x_{n+1})) \leq \psi(d^*) - \varphi(d^*),
$$

i.e., $\varphi(d^*) \leq 0$. Using the properties of control functions, we get that $d^* = 0$, which is a contradiction. Hence we have $\lim_{n \to \infty} M(y_n, y_{n+1}, t) = 0$.

Now we show that $\{y_n\}$ is a Cauchy sequence in X.

It is enough to prove that $\{y_{2n}\}$ is a Cauchy sequence. Suppose the contrary. Then,

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for some $\epsilon > 0$, there exist subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the smallest index satisfying

$$n(k) > m(k) \text{ and } M(y_{n(k)}, y_{m(k)}, t) \geq \epsilon.$$

In particular, $M(y_{n(k)-2}, y_{m(k)}, t) < \epsilon$. Using the triangle inequality and the known relation $|d(x,z) - d(x,y)| \leq d(x,z)$, we obtain that

$$\lim_{k \to \infty} M(y_{2n(k)}, y_{2m(k)}, t) = \lim_{k \to \infty} M(y_{2n(k)}, y_{2m(k)-1}, t) = \lim_{k \to \infty} M(y_{2n(k)+1}, y_{2m(k)}, t)$$
$$= \lim_{k \to \infty} M(y_{2n(k)+1}, y_{2m(k)-1}, t) = \epsilon.$$

By the definition of $M(x, y, t)$ and by using the previous limits, we get that

$$\lim_{k \to \infty} M_{I,J}(x_{2n(k)}, x_{2m(k)-1}) = \epsilon.$$

Indeed,

$$M_{I,J}(x_{2n(k)}, x_{2m(k)-1})$$
$$= \max \left\{ M(y_{2n(k)}, y_{2m(k)-1}, t), M(y_{2n(k)}, y_{2n(k)+1}, t), M(y_{2m(k)-1}, y_{2m(k)}, t), \right.$$
$$\left. \frac{1}{2} \Big(M(y_{2n(k)}, y_{2m(k)}, t) + M(y_{2n(k)+1}, y_{2m(k)-1}, t) \Big) \right\}$$
$$\to \max \left\{ \epsilon, 0, 0, \frac{1}{2}(\epsilon + \epsilon) \right\} = \epsilon.$$

Applying (4), we obtain

$$\psi(M(y_{2n(k)+1}, y_{2m(k)}, t)) = \psi(M(Sx_{2n(k)}, Tx_{2m(k)-1}, t))$$
$$\leq \psi(M_{I,J}(x_{2n(k)}, x_{2m(k)-1})) - \varphi(M_{I,J}(x_{2n(k)}, x_{2m(k)-1})).$$

Passing to the limit $k \to \infty$, we obtain that $\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon)$, which is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in the complete metric (X, d). So there exists $u \in X$ such that $\lim_{n \to \infty} y_n = u$.

On the other hand, E is closed and $\{y_n\} \subset E$. Then $u \in E$. Suppose that $I(E)$ is closed. Then there exists $v \in E$ such that

$$u = Iv. \tag{10}$$

We claim that $Sv = u$. Using (4) and (10), we have

$$\psi(M(Sv, y_{2n}, t)) = \psi(M(Sv, Tx_{2n-1}, t)) \leq \psi(M_{I,J}(v, x_{2n-1})) - \varphi(M_{I,J}(v, x_{2n-1})), \tag{11}$$

where

$$M_{I,J}(v, x_{2n-1}) = \max \left\{ M(y_{2n-1}, u, t), M(u, Sv, t), M(y_{2n-1}, Tx_{2n-1}, t), \right.$$
$$\left. \frac{1}{2} \Big(M(y_{2n-1}, Sv, t) + M(u, Tx_{2n-1}, t) \Big) \right\}$$
$$\to \max \left\{ 0, M(u, Sv, t), 0, \frac{1}{2} M(u, Sv, t) \right\} = M(u, Sv, t).$$

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Passing to the limit when $n \to \infty$ in (11), we get

$$\psi(M(u, Sv, t)) \leq \psi(M(u, Sv, t)) - \varphi(M(u, Sv, t)).$$

It follows that

$$u = Sv. \tag{12}$$

Since $u = Sv \in SE \subset JE$, there exists $w \in E$ such that

$$u = Jw. \tag{13}$$

We claim that $Tw = u$. By (4), we get

$$\psi(M(u, Tw, t)) = \psi(M(Sv, Tw, t)) \leq \psi(M_{I,J}(v, w)) - \varphi(M_{I,J}(v, w)),$$

where

$$M_{I,J}(v, w) = \max \Big\{ M(u, u, t), M(Iv, Sv, t), M(Jw, Tw, t),$$

$$\frac{1}{2} \Big(M(Jw, Sv, t) + M(Iv, Tw, t) \Big) \Big\}$$

$$= \max \Big\{ 0, 0, M(u, Tw, t), \frac{1}{2} M(u, Tw, t) \Big\} = M(u, Tw, t).$$

Hence (2) implies that

$$\psi(M(u, Tw, t)) \leq \psi(M(u, Tw, t)) - \varphi(M(u, Tw, t)).$$

It follows that

$$u = Tw. \tag{14}$$

Combining (10) and (12) yields

$$u = Iv = Sv, \tag{15}$$

that is, u is a point of coincidence of I and S. Combining (13) and (14) yields

$$u = Jw = Tw, \tag{16}$$

that is, u is a point of coincidence of J and T.

To prove the uniqueness property of u, suppose that u' is another point of coincidence of I and S, that is,

$$u' = Iv' = Sv'$$

for some $v' \in E$. By (4), we have

$$\psi(M(u', u, t)) = \psi(M(Sv', Tw, t)) \leq \psi(M_{I,J}(v', w)) - \varphi(M_{I,J}(v', w)),$$

where

$$M_{I,J}(v', w) = \max \Big\{ M(u', u, t), 0, 0, \frac{1}{2} \Big(M(u', u, t) + M(u', u, t) \Big) \Big\}$$

$$= M(u', u, t).$$

It follows from (2) that $u' = u$.

Now, suppose that $\overline{u}$ is another point of coincidence of J and T, that is,

$$\overline{u} = jw' = Tw'$$

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for some $w' \in E$. Using (4), we obtain

$$\psi(M(\overline{u}, u, t)) = \psi(M(Sv, Tw, t')) \le \psi(M_{I,J}(v, w')) - \varphi(M_{I,J}(v, w')),$$

where

$$M_{I,J}(v, w') = \max\left\{ M(\overline{u}, u, t), 0, 0, \frac{1}{2}\Big(M(\overline{u}, u, t) + M(\overline{u}, u, t)\Big)\right\}$$
$$= M(\overline{u}, u, t).$$

It follows from (2) that $\overline{u} = u$.

Therefore, u is the unique point of coincidence of $\{S, I\}$ and $\{T, J\}$.

Now, if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then by (15) and (16), we have $Su = S(Iv) = I(Sv) = Iu = w_1$ and $Tu = T(Jw) = J(Tw) = Ju = w_2$. By (4), we have

$$\psi(M(w_1, w_2, t)) = \psi(M(Su, Tu, t)) \le \psi(M_{I,J}(u, u)) - \varphi(M_{I,J}(u, u)),$$

where

$$M_{I,J}(u, u) = \max\left\{ M(w_1, w_2, t), 0, 0, \frac{1}{2}\Big(M(w_1, w_2, t) + M(w_1, w_2, t)\Big)\right\}$$
$$= M(w_1, w_2, t).$$

It follows that $w_1 = w_2$, that is,

$$Su = Iu = Tu = Ju. \tag{17}$$

By (4) and (17), we have

$$\psi(M(Sv, Tu, t)) \le \psi(M_{I,J}(v, u) - \varphi(M_{I,J}(v, u)),$$

where

$$\begin{aligned}
M_{I,J}(v, u) &= \max\left\{ M(Iv, Ju, t), M(Iv, Sv, t), M(Ju, Tu, t),\right. \\
&\qquad \left. \frac{1}{2}\Big(M(Iv, Tu, t) + M(Sv, Tu, t)\Big)\right\} \\
&= \max\left\{ M(Sv, Tu, t), 0, 0, \frac{1}{2}\Big(M(Sv, Tu, t) + M(Sv, Tu, t)\Big)\right\} \\
&= M(Sv, Tu, t).
\end{aligned}$$

Therefore, we deduce that $Sv = Tu$, that is, $u = Tu$. It follows from (17) that

$$u = Su = Iu = Tu = Ju.$$

Then u is the unique common fixed point of S, I, J and T.

The rest of the proof is similar to the above case and so the rest will be omitted. $\square$

Example 2.2. *Let $X = [0, 1]$ be equipped with the natural metric $d(x, y) = |x - y|$. Now for $t \in [0, \infty)$ define*

$$M(x, y, t) = \begin{cases} 0 \text{ if } t = 0 \text{ and } x, y \in X \\ \frac{t}{t + |x - y|} \text{ if } t \ne 0 \text{ and } x, y \in X. \end{cases}$$

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Clearly, $(X, M, *)$ is a fuzzy metric on X, where $*$ is defined as $a * b = ab$. This fuzzy metric space is complete.

Let $E = \{0, \frac{1}{2}, 1\}$ and we define $T, S : E \to E$ as

$T0 = T1 = 0$ and $T\frac{1}{2} = 1$, $Sx = 0$.

We also define $I, J : E \to X$ as

$I0 = I1 = 0$ and $I\frac{1}{2} = 1$, $J0 = J1 = 0$ and $J\frac{1}{2} = 1$.

The functions $\psi : \varphi : [0, \infty) \to [0, \infty)$ are defined as

$\psi(t) = t$ and $\varphi(t) = \frac{t}{4}$.

Then

$$\psi(M(Sx, Ty, t)) \leq \psi(M_{I,J}(x, y)) - \varphi(M_{I,J}(x, y)).$$

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Analysis of latent CHIKV dynamics model with time delays

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Abstract

This paper proposes a latent Chikungunya viral infection model with saturated incidence rate. To take into account the time lag between the initial viral contacts uninfected monocytes and the production of new active CHIKV particles the model is incorporated by intracellular discrete or distributed time delays. We study the qualitative behavior of the model. Using the method of Lyapunov function, we established the global stability of the steady states of the model. The effect of the time delay on the stability of the steady states has also been shown by numerical simulations.

Keywords: Chikungunya virus infection; Latency; Time delay; Global stability; Lyapunov function.

1   Introduction

Mathematical analysis of viral infection models plays a substantial role in understanding the dynamics of human viruses (such as HIV, HCV, HBV, HTLV and Chikungunya virus). The models have been developed to mainly describe the relation among virus particles, uninfected target cells and infected cells [1]-[15]. The effect of Cytotoxic T Lymphocytes (CTL) immune response or humoral immune response has also been modeled (see e.g. [10]-[15]. Two main classes of mathematical models of viral infection have been proposed in the literature. The first class of models are given by ordinary differential equations. The second class of models is given by delay differential equations which incorporate the time lag between the initial viral contacts a target cell and the production of new active viruses. Modeling the virus dynamics with two types of infected cells, latently infected cells and actively infected cells has been studied by several researchers (see e.g. [2] and [14]). The latent viral infection model has been formulated as [2]:

$$\dot{S}(t) = \mu - aS(t) - bV(t)S(t), \tag{1}$$

$$\dot{L}(t) = (1-\rho)bV(t)S(t) - (\theta+\lambda)L(t), \tag{2}$$

$$\dot{I}(t) = \rho bV(t)S(t) + \lambda L(t) - \epsilon I(t), \tag{3}$$

$$\dot{V}(t) = mI(t) - rV(t), \tag{4}$$

where, S, L, I and V are the concentrations of uninfected cells, latently infected cells, actively infected cells and free virus particles. Parameters a and μ represent the death rate and birth rate constants of the uninfected cells, respectively. The uninfected cells become infected at rate bSV, where b is a constant. The parameters θ, ϵ and r denote the death rate constants of the latently infected cells, actively infected cells and free virus particles, respectively. An actively infected cell produces an average number m of virus particles. The parameter λ is the latent to active transmission rate constant. A fraction $(1-\rho)$ of infected cells is assumed to be latently infected cells and the remaining ρ becomes actively infected cells, where $0 < \rho < 1$.

Chikungunya virus (CHIKV) is an alphavirus and is transmitted to humans by Aedes aegypti and Aedes albopictus mosquitos. In the CHIKV literature, most of the mathematical models have been presented to describe the disease transmission in mosquito and human populations (see e.g. [17]-[22]). However, only few works have devoted for mathematical modeling of the dynamics of the CHIKV within host. In 2017, Wang and Liu [16] have presented a mathematical model for in host CHIKV infection model without considering the latent infection.

The objective of this paper is to propose a CHIKV infection model which improves the model presented in [16] by taking into account (i) two types of infected monocytes, latently infected monocytes and actively infected monocytes, (ii) two types of discrete or distributed time delays (iii) saturated incidence rate which is suitable to model the nonlinear dynamics of the CHIKV especially when its concentration is high. We investigate the nonnegativity and boundedness of the solutions of the CHIKV dynamics model. We show that the CHIKV dynamics is governed by one bifurcation parameter (the basic reproduction numbers $\mathcal{R}_0$). We use Lyapunov direct method to establish the global stability of the model's equilibria.

2 CHIKV model with discrete time delays

We consider a within-host CHIKV dynamics model with latently infected monocytes taking into account two discrete time delays.

$$\dot{S}(t) = \mu - aS(t) - \frac{bV(t)S(t)}{1 + \pi V(t)}, \tag{5}$$

$$\dot{L}(t) = \frac{(1-\rho)e^{-\delta_1 \tau_1} bV(t-\tau_1)S(t-\tau_1)}{1 + \pi V(t-\tau_1)} - (\theta + \lambda)L(t), \tag{6}$$

$$\dot{I}(t) = \frac{\rho e^{-\delta_2 \tau_2} bV(t-\tau_2)S(t-\tau_2)}{1 + \pi V(t-\tau_2)} + \lambda L(t) - \epsilon I(t), \tag{7}$$

$$\dot{V}(t) = mI(t) - rV(t) - qB(t)V(t), \tag{8}$$

$$\dot{B}(t) = \eta + cB(t)V(t) - \delta B(t), \tag{9}$$

where, S, L, I, V, and B are the concentrations of uninfected monocytes, latently infected monocytes, actively infected monocytes CHIKV particles and B cells, respectively. The CHIKV particles are attacked by the B cells at rate qVB. The B cells are produced at constant rate η, proliferated at rate cBV and die at rate δB. τ_1 denotes the time between the CHIKV contacts the uninfected monocytes and latent infection, while τ_2 denotes the time between monocytes infection and the production of active CHIKV particles. The probability of latently and actively infected monocytes surviving to the age of τ_1 and τ_2 are represented by $e^{-\delta_1 \tau_1}$ and $e^{-\delta_2 \tau_2}$, respectively, where δ_1 and δ_2 are. We consider the following initial conditions:

$$S(\vartheta) = \varphi_1(\vartheta), \ L(\vartheta) = \varphi_2(\vartheta), \ I(\vartheta) = \varphi_3(\vartheta), \ V(\vartheta) = \varphi_4(\vartheta), \ B(\vartheta) = \varphi_5(\vartheta),$$

$$\varphi_i(\vartheta) \geq 0, \vartheta \in [-\tau, 0] \ \text{and} \ \varphi_i \in C\left([-\tau, 0], \mathbb{R}_{\geq 0}\right), \quad i = 1, 2, ..., 5, \tag{10}$$

where $\tau = \max\{\tau_1, \tau_2\}$ and C is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_{\geq 0}$ with norm $\|\varphi_j\| = \sup_{-\tau \leq \vartheta \leq 0} |\varphi_j(\vartheta)|$. Then the uniqueness of the solution for $t > 0$ is guaranteed [23].

2.1 Preliminaries

In this subsection we show the nonnegativity and boundedness of solutions as well as the existence of the steady states of system (5)-(9).

Lemma 1. The solutions of system (5)-(9) with the initial states (10) are nonnegative and ultimately bounded.

Proof. From Eqs. (5) and (9) we have $\dot{S}\big|_{S=0} = \mu > 0$ and $\dot{B}\big|_{B=0} = \eta > 0$. Thus, $S(t) > 0$ and $B(t) > 0$ for all $t \geq 0$. Moreover, for $t \in [0, \tau]$ we have

$$L(t) = \varphi_2(0)e^{-(\theta+\lambda)t} + \int_0^t \left(\frac{(1-\rho)e^{-\delta_1\tau_1}bS(\omega-\tau_1)V(\omega-\tau_1)}{1+\pi V(\omega-\tau_1)} \right) e^{-(\theta+\lambda)(t-\omega)}d\omega \geq 0,$$

$$I(t) = \varphi_3(0)e^{-\epsilon t} + \int_0^t \left(\frac{\rho e^{-\delta_2\tau_2}bS(\omega-\tau_2)V(\omega-\tau_2)}{1+\pi V(\omega-\tau_2)} + \lambda L(\omega) \right) e^{-\epsilon(t-\omega)}d\omega \geq 0,$$

$$V(t) = \varphi_4(0)e^{-\int_0^t(c+qB(u))u} + \int_0^t mI(\omega)e^{-\int_\omega^t(c+qB(u))du}d\omega \geq 0.$$

By recursive argument, we get $L(t) \geq 0$, $I(t) \geq 0$ and $V(t) \geq 0$ for all $t \geq 0$.

Next, we establish the boundedness of the model's solutions. The nonnegativity of the model's solution implies that $\frac{dS(t)}{dt} \leq \mu - aS(t)$, which yields $\limsup_{t\to\infty} S(t) \leq \frac{\mu}{a}$. Let us define

$$X_1(t) = (1-\rho)e^{-\delta_1\tau_1}S(t-\tau_1) + L(t),$$

then

$$\dot{X}_1(t) = (1-\rho)e^{-\delta_1\tau_1}\left(\mu - aS(t-\tau_1) - \frac{bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)} \right) + (1-\rho)e^{-\delta_1\tau_1}\frac{bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)} - (\theta+\lambda)L(t)$$

$$\leq \mu(1-\rho)e^{-\delta_1\tau_1} - \sigma_1\left((1-\rho)e^{-\delta_1\tau_1}S(t-\tau_1) + L(t)\right) \leq \mu(1-\rho) - \sigma_1 X_1(t),$$

where $\sigma_1 = \min\{a, \theta+\lambda\}$. Then, $\limsup_{t\to\infty} X_1(t) \leq M_1$, and $\limsup_{t\to\infty} L(t) \leq M_1$, where $M_1 = \frac{\mu(1-\rho)}{\sigma_1}$. Let

$$X_2(t) = \rho e^{-\delta_2\tau_2}S(t-\tau_2) + I(t) + \frac{\epsilon}{2m}V(t) + \frac{\epsilon q}{2mc}B(t),$$

then we get

$$\dot{X}_2(t) = \rho e^{-\delta_2\tau_2}\left(\mu - aS(t-\tau_2) - \frac{bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)} \right) + \rho e^{-\delta_2\tau_2}\frac{bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)} + \lambda L(t) - \epsilon I(t)$$

$$+ \frac{\epsilon}{2m}(mI(t) - rV(t) - qV(t)B(t)) + \frac{\epsilon q}{2mc}(\eta + cB(t)V(t) - \delta B(t))$$

$$= \rho\mu e^{-\delta_2\tau_2} - \rho e^{-\delta_2\tau_2}aS(t-\tau_2) + \lambda L(t) - \frac{\epsilon}{2}I(t) + \frac{\epsilon q\eta}{2mc} - \frac{\epsilon r}{2m}V(t) - \frac{\epsilon q\delta}{2mc}B(t)$$

$$\leq \rho\mu + \lambda M_1 + \frac{\epsilon q\eta}{2mc} - \sigma_2\left(\rho e^{-\delta_2\tau_2}S(t-\tau_2) + I(t) + \frac{\epsilon}{2m}V(t) + \frac{\epsilon q}{2mc}B(t) \right)$$

$$= \rho\mu + \lambda M_1 + \frac{\epsilon q\eta}{2mc} - \sigma_2 X_2(t),$$

where $\sigma_2 = \min\{a, \frac{\epsilon}{2}, r, \delta\}$. It follows that $\limsup_{t\to\infty} I(t) \leq M_2$, $\limsup_{t\to\infty} V(t) \leq M_3$ and $\limsup_{t\to\infty} B(t) \leq M_4$, where $M_2 = \frac{\rho\mu+\lambda M_1}{\sigma_2} + \frac{\epsilon q\eta}{2mc\sigma_2}$, $M_3 = \frac{2m}{\epsilon}$ and $M_4 = \frac{2mc}{\epsilon q}$. This shows the ultimate boundedness of $S(t), L(t), I(t), V(t)$ and $B(t)$. $\square$

Lemma 2. For system (5)-(9) there exists a threshold parameter $\mathcal{R}_0 > 0$, such that

(i) if $\mathcal{R}_0 \leq 1$, then there exists only one positive steady state, virus-free steady state Q_0.

(i) if $\mathcal{R}_0 > 1$, then in addition to Q_0, there exists an endemic steady state Q_1

Proof.

To calculate the steady states we let the R.H.S of system (5)-(9) be equal zero

$$0 = \mu - aS - \frac{bVS}{1 + \pi V}, \tag{11}$$

$$0 = (1 - \rho)\frac{e^{-\delta_1 \tau_1} bVS}{1 + \pi V} - (\theta + \lambda)L, \tag{12}$$

$$0 = \frac{\rho e^{-\delta_2 \tau_2} bVS}{1 + \pi V} + \lambda L - \epsilon I, \tag{13}$$

$$0 = mI - rV - qVB, \tag{14}$$

$$0 = \eta + cBV - \delta B. \tag{15}$$

From Eqs. (11)-(15) we obtain

$$S = \frac{\mu(1 + \pi V)}{bV + a(1 + \pi V)}, \quad L = \frac{(1 - \rho)e^{-\delta_1 \tau_1} bVS}{(1 + \pi V)(\theta + \lambda)}, \quad I = \frac{b\beta VS}{\epsilon(1 + \pi V)(\theta + \lambda)}, \quad B = \frac{\eta}{\delta - cV}. \tag{16}$$

where $\beta = \lambda(1 - \rho)e^{-\delta_1 \tau_1} + \rho e^{-\delta_2 \tau_2}(\theta + \lambda)$. Substituting Eq. (16) into Eq. (14) we have

$$\left[\frac{m\mu b\beta}{\epsilon(\theta + \lambda)(bV + a(1 + \pi V))} - r - \frac{q\eta}{\delta - cV} \right] V = 0. \tag{17}$$

Equation (17) has two possibilities:

(i) $V = 0$ which gives the virus-free steady state $Q_0 = (S_0, L_0, I_0, V_0, B_0) = (\frac{\mu}{a}, 0, 0, 0, \frac{\eta}{\delta})$,

(ii) $V \neq 0$ which gives

$$\frac{m\mu b\beta}{\epsilon(\theta + \lambda)(bV + a(1 + \pi V))} - r - \frac{q\eta}{\delta - cV} = 0. \tag{18}$$

Equation (18) takes the form $P_1 V^2 - P_2 V + P_3 = 0$, where

$$P_1 = r\epsilon c(\theta + \lambda)(b + \pi a),$$
$$P_2 = -r\epsilon ca(\theta + \lambda) + m\mu bc(\beta) + \epsilon(r\delta + q\eta)(\theta + \lambda)(b + \pi a),$$
$$P_3 = m\rho b\mu\delta(\theta + \lambda)e^{-\delta_2 \tau_2} + m\mu b\lambda\delta(1 - \rho)e^{-\delta_1 \tau_1} - \epsilon a(r\delta + q\eta)(\theta + \lambda).$$

The constants P_1, P_2 and P_3 can be rewritten as

$$P_1 = r\epsilon c(\theta + \lambda)(b + \pi a),$$
$$P_2 = \frac{\epsilon ca(r\delta + q\eta)(\theta + \lambda)}{\delta}(\mathcal{R}_0 - 1) + \epsilon(r\delta + q\eta)(\theta + \lambda)(b + \pi a) + \frac{caq\epsilon\eta(\theta + \lambda)}{\delta},$$
$$P_3 = \epsilon a(r\delta + q\eta)(\theta + \lambda)(\mathcal{R}_0 - 1),$$

where

$$\mathcal{R}_0 = \frac{m\mu b\delta\beta}{\epsilon a(r\delta + q\eta)(\theta + \lambda)}. $$

Let

$$\Theta_1(V) = P_1 V^2 - P_2 V + P_3 = 0. \tag{19}$$

If $\mathcal{R}_0 > 1$, then $P_2 > 0$ and $P_3 > 0$. We have $\Theta_1(0) = P_3 > 0$, $\Theta_1\left(\frac{\delta}{c}\right) = -\frac{q\epsilon\eta(\theta + \lambda)(ca + \delta(b + \pi a))}{c} < 0$, and $\Theta_1'(0) = -P_2 < 0$. Then, Eq. (19) has two positive roots

$$V_1 = \frac{P_2 - \sqrt{P_2^2 - 4P_1 P_3}}{2P_1} < \frac{\delta}{c} \quad \text{and} \quad V_2 = \frac{P_2 + \sqrt{P_2^2 - 4P_1 P_3}}{2P_1} > \frac{\delta}{c}.$$

If $V = V_2$, then from Eq. (16) we get $B_2 = \frac{\eta}{\delta - cV_2} < 0$. Thus, when $\mathcal{R}_0 > 1$, a positive endemic steady state $Q_1 = (S_1, L_1, I_1, V_1, B_1)$ will appear, where

$$S_1 = \frac{\mu(1 + \pi V_1)}{bV_1 + a(1 + \pi V_1)}, \ L_1 = \frac{b\mu V_1(1 - \rho)e^{-\delta_1 \tau_1}}{(\theta + \lambda)(bV_1 + a(1 + \pi V_1))}, \ I_1 = \frac{b\mu\beta V_1}{\epsilon(\theta + \lambda)(bV_1 + a(1 + \pi V_1))},$$

$$V_1 = \frac{\rho_2 - \sqrt{\rho_2^2 - 4\rho_1\rho_3}}{2\rho_1}, \ B_1 = \frac{\eta}{\delta - cV_1}.$$

The parameter $\mathcal{R}_0$ represents the basic reproduction number. $\square$

2.2 Global stability

We define $H(x) = x - \ln x - 1$. Clearly, $H(1) = 0$ and $H(u) \geq 0$ for $u > 0$. Denote $(S, L, I, V, B) = (S(t), L(t), I(t), V(t), B(t))$.

Theorem 1. Suppose that $\mathcal{R}_0 \leq 1$, then Q_0 is globally asymptotically stable (GAS).

Proof. We define a Lyapunov functional Y_0 as:

$$Y_0(S, L, I, V, B) = \left(\frac{\beta}{\theta + \lambda}\right) S_0 H\left(\frac{S}{S_0}\right) + \frac{\lambda}{\theta + \lambda} L + I + \frac{\epsilon}{m} V + \frac{\epsilon q}{mc} B_0 H\left(\frac{B}{B_0}\right)$$

$$+ \frac{\lambda(1 - \rho)e^{-\delta_1 \tau_1}}{\theta + \lambda} \int_0^{\tau_1} \frac{bV(t - \vartheta)S(t - \vartheta)}{1 + \pi V(t - \vartheta)} d\vartheta + \rho e^{-\delta_2 \tau_2} \int_0^{\tau_2} \frac{bV(t - \vartheta)S(t - \vartheta)}{1 + \pi V(t - \vartheta)} d\vartheta. \quad (20)$$

Note that, $Y_0(S, L, I, V, B) > 0$ for all $S, L, I, V, B > 0$ and $Y_0(S_0, 0, 0, 0, B_0) = 0$. Calculating $\frac{dY_0}{dt}$ along the trajectories of (5)-(9) we get

$$\frac{dY_0}{dt} = \frac{\beta}{\theta + \lambda}\left(1 - \frac{S_0}{S}\right)\left(\mu - aS - \frac{bVS}{1 + \pi V}\right)$$

$$+ \frac{\lambda}{\theta + \lambda}\left(\frac{(1 - \rho)e^{-\delta_1 \tau_1}bV(t - \tau_1)S(t - \tau_1)}{1 + \pi V(t - \tau_1)} - (\theta + \lambda)L\right) + \frac{\rho e^{-\delta_2 \tau_2}bV(t - \tau_2)S(t - \tau_2)}{1 + \pi V(t - \tau_2)}$$

$$+ \lambda L - \epsilon I + \frac{\epsilon}{m}(mI - rV - qVB) + \frac{\epsilon q}{mc}\left(1 - \frac{B_0}{B}\right)(\eta + cBV - \delta B)$$

$$+ \frac{\lambda(1 - \rho)e^{-\delta_1 \tau_1}}{\theta + \lambda}\left(\frac{bVS}{1 + \pi V} - \frac{bV(t - \tau_1)S(t - \tau_1)}{1 + \pi V(t - \tau_1)}\right) + \rho e^{-\delta_2 \tau_2}\left(\frac{bVS}{1 + \pi V} - \frac{bV(t - \tau_2)S(t - \tau_2)}{1 + \pi V(t - \tau_2)}\right)$$

$$= -\frac{a\beta}{\theta + \lambda}\frac{(S - S_0)^2}{S} + \frac{\beta}{\theta + \lambda}\frac{bS_0 V}{1 + \pi V} - \frac{\epsilon r V}{m} - \frac{\epsilon q B_0 V}{m} + \frac{\epsilon q}{mc}\left(1 - \frac{B_0}{B}\right)(\delta B_0 - \delta B)$$

$$= -\frac{a\beta}{\theta + \lambda}\frac{(S - S_0)^2}{S} - \frac{\epsilon q \delta}{mc}\frac{(B - B_0)^2}{B} + \frac{\epsilon(r\delta + q\eta)}{m\delta}\left(\frac{m\mu b\delta\beta}{\epsilon a(r\delta + q\eta)(\theta + \lambda)(1 + \pi V)} - 1\right)V$$

$$= -\frac{a\beta}{\theta + \lambda}\frac{(S - S_0)^2}{S} - \frac{\epsilon q \delta}{mc}\frac{(B - B_0)^2}{B} + \frac{\epsilon(r\delta + q\eta)}{m\delta}(\mathcal{R}_0 - 1)V - \frac{\epsilon(r\delta + q\eta)\mathcal{R}_0\pi V^2}{m\delta(1 + \pi V)}. \quad (21)$$

Therefore, $\frac{dY_0}{dt} \leq 0$ holds if $\mathcal{R}_0 \leq 1$. Further, $\frac{dY_0}{dt} = 0$ if and only if $S = S_0$, $B = B_0$ and $V = 0$. By LaSalle's invariance principle, Q_0 is GAS. $\square$

In the next theorem we show the global stability of Q_1.

Theorem 2. Suppose that $\mathcal{R}_0 > 1$, then Q_1 is GAS.

Proof. Consider

$$Y_1(S, L, I, V, B) = \frac{\beta}{\theta + \lambda} S_1 H\left(\frac{S}{S_1}\right) + \frac{\lambda}{\theta + \lambda} L_1 H\left(\frac{L}{L_1}\right) + I_1 H\left(\frac{I}{I_1}\right) + \frac{\epsilon}{m} V_1 H\left(\frac{V}{V_1}\right) + \frac{\epsilon q}{mc} B_1 H\left(\frac{B}{B_1}\right)$$

$$+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta + \lambda} \frac{bS_1V_1}{1 + \pi V_1} \int_0^{\tau_1} H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1 V_1(1+\pi V(t-\vartheta))}\right) d\vartheta$$

$$+ \rho e^{-\delta_2\tau_2} \frac{bS_1V_1}{1 + \pi V_1} \int_0^{\tau_2} H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1 V_1(1+\pi V(t-\vartheta))}\right) d\vartheta.$$

We have $Y_1(S, L, I, V, B) > 0$ for all $S, L, I, V, B > 0$ and $Y_1(S_1, L_1, I_1, V_1, B_1) = 0$. Calculating $\frac{dY_1}{dt}$ along the trajectories of (5)-(9) we get

$$\frac{dY_1}{dt} = \frac{\beta}{\theta + \lambda}\left(1 - \frac{S_1}{S}\right)\left(\mu - aS - \frac{bVS}{1 + \pi V}\right)$$

$$+ \frac{\lambda}{\theta + \lambda}\left(1 - \frac{L_1}{L}\right)\left(\frac{(1-\rho)e^{-\delta_1\tau_1}bV(t-\tau_1)S(t-\tau_1)}{1 + \pi V(t-\tau_1)} - (\theta + \lambda)L\right)$$

$$+ \left(1 - \frac{I_1}{I}\right)\left(\frac{\rho e^{-\delta_2\tau_2}bV(t-\tau_2)S(t-\tau_2)}{1 + \pi V(t-\tau_2)} + \lambda L - \epsilon I\right) + \frac{\epsilon}{m}\left(1 - \frac{V_1}{V}\right)(mI - rV - qVB)$$

$$+ \frac{\epsilon q}{mc}\left(1 - \frac{B_1}{B}\right)(\eta + cBV - \delta B) + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta + \lambda}\left(\frac{bVS}{1 + \pi V} - \frac{bV(t-\tau_1)S(t-\tau_1)}{1 + \pi V(t-\tau_1)}\right)$$

$$+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta + \lambda}\frac{bS_1V_1}{1 + \pi V_1}\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS(1+\pi V(t-\tau_1))}\right) + \rho e^{-\delta_2\tau_2}\left(\frac{bVS}{1 + \pi V} - \frac{bV(t-\tau_2)S(t-\tau_2)}{1 + \pi V(t-\tau_2)}\right)$$

$$+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1 + \pi V_1}\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS(1+\pi V(t-\tau_2))}\right). \tag{22}$$

Applying

$$\mu = aS_1 + \frac{bS_1V_1}{1 + \pi V_1}, \quad \eta = \delta B_1 - cB_1V_1,$$

we obtain

$$\frac{dY_1}{dt} = \frac{\beta}{\theta + \lambda}\left(1 - \frac{S_1}{S}\right)(aS_1 - aS) + \frac{\beta}{\theta + \lambda}\frac{bS_1V_1}{1 + \pi V_1}\left(1 - \frac{S_1}{S}\right) + \frac{\beta}{\theta + \lambda}\frac{bS_1V}{1 + \pi V}$$

$$- \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta + \lambda}\frac{bV(t-\tau_1)S(t-\tau_1)L_1}{1 + \pi V(t-\tau_1)L} + \lambda L_1 - \rho e^{-\delta_2\tau_2}\frac{bV(t-\tau_2)S(t-\tau_2)I_1}{1 + \pi V(t-\tau_2)I} - \frac{\lambda LI_1}{I} + \epsilon I_1 - \frac{\epsilon IV_1}{V}$$

$$- \frac{r\epsilon V}{m} + \frac{r\epsilon V_1}{m} + \frac{\epsilon qBV_1}{m} + \frac{\epsilon q}{mc}\left(1 - \frac{B_1}{B}\right)(\delta B_1 - \delta B) - \frac{\epsilon qB_1V}{m} - \frac{\epsilon qB_1V_1}{m} + \frac{\epsilon qB_1V_1}{m}\left(\frac{B_1}{B}\right)$$

$$+ \frac{bS_1V_1}{1 + \pi V_1}\left[\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta + \lambda}\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS(1+\pi V(t-\tau_1))}\right) + \rho e^{-\delta_2\tau_2}\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS(1+\pi V(t-\tau_2))}\right)\right].$$

Using the equilibrium conditions for Q_1:

$$(1-\rho)e^{-\delta_1\tau_1}\frac{bS_1V_1}{1 + \pi V_1} = (\theta + \lambda)L_1, \quad \rho e^{-\delta_1\tau_1}\frac{bS_1V_1}{1 + \pi V_1} + \lambda L_1 = \epsilon I_1, \quad mI_1 = rV_1 + qB_1V_1,$$

we get

$$\epsilon I_1 = \frac{\beta}{\theta + \lambda}\frac{bS_1V_1}{(1 + \pi V_1)}, \quad \frac{r\epsilon V_1}{m} = \frac{\beta}{\theta + \lambda}\frac{bS_1V_1}{(1 + \pi V_1)} - \frac{\epsilon qB_1V_1}{m},$$

and

$$
\frac{dY_1}{dt} = -\frac{a\beta}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right)
$$

$$
+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right) + \left(\frac{\beta}{\theta+\lambda}\right)\frac{bS_1V_1}{1+\pi V_1}\left(\frac{(1+\pi V_1)V}{(1+\pi V)V_1}-\frac{V}{V_1}\right)
$$

$$
- \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L} + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}
$$

$$
- \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I} - \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{I_1L}{L_1I}
$$

$$
+ \frac{\lambda(1-\rho)}{\theta+\lambda}e^{-\delta_1\tau_1}\frac{bS_1V_1}{(1+\pi V_1)} + \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)} - \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V}
$$

$$
- \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} + \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)}
$$

$$
- \frac{2\epsilon q B_1V_1}{m} + \frac{\epsilon q BV_1}{m} + \frac{\epsilon q B_1V_1}{m}\left(\frac{B_1}{B}\right) - \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B}
$$

$$
+ \frac{bS_1V_1}{1+\pi V_1}\left[\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS(1+\pi V(t-\tau_1))}\right) + \rho e^{-\delta_2\tau_2}\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS(1+\pi V(t-\tau_2))}\right)\right].
$$

Using the following equalities:

$$
\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS(1+\pi V(t-\tau_1))}\right) = \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{LI_1}{L_1I}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right)
$$

$$
+ \ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}\right),
$$

$$
\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS(1+\pi V(t-\tau_2))}\right) = \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right)
$$

$$
+ \ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}\right),
$$

we get

$$
\frac{dY_1}{dt} = -\frac{a\beta}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{\beta}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left(-1+\frac{(1+\pi V_1)V}{(1+\pi V)V_1}-\frac{V}{V_1}+\frac{1+\pi V}{1+\pi V_1}\right)
$$

$$
+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right]
$$

$$
+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}+\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}\right)\right]
$$

$$
+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right]
$$

$$
+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{LI_1}{L_1I}+\ln\left(\frac{LI_1}{L_1I}\right)\right]
$$

$$
+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right]
$$

$$
+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}+\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}\right)\right]
$$

$$
+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right] - \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B} - \frac{\epsilon q B_1V_1}{m}\left[2-\frac{B}{B_1}-\frac{B_1}{B}\right]
$$

$$
\begin{aligned}
= & -\frac{a\beta(S-S_1)^2}{(\theta+\lambda)S} - \frac{\epsilon q\eta}{mcB_1}\frac{(B-B_1)^2}{B} - \frac{\beta}{\theta+\lambda}\frac{\pi b S_1(V-V_1)^2}{(1+\pi V)(1+\pi V_1)^2} - \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[H\left(\frac{S_1}{S}\right)\right. \\
& \left. + H\left(\frac{IV_1}{I_1V}\right) + H\left(\frac{1+\pi V}{1+\pi V_1}\right) + H\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}\right) + H\left(\frac{LI_1}{L_1I}\right)\right] \\
& - \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)}\left[H\left(\frac{S_1}{S}\right) + H\left(\frac{IV_1}{I_1V}\right) + H\left(\frac{1+\pi V}{1+\pi V_1}\right) + H\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}\right)\right]. \quad (23)
\end{aligned}
$$

It can be seen that if $\mathcal{R}_0 > 1$, then $\dfrac{dY_1}{dt} \leq 0$ for all $S, L, I, V, B > 0$ and $\dfrac{dY_1}{dt} = 0$ if and only if $S = S_1$, $L = L_1$, $I = I_1$, $V = V_1$, and $B = B_1$. It follows from LaSalle's invariance principle that, Q_1 is GAS. □

3 CHIKV model with delay-distributed

We suggest a dynamical model for within-host CHIKV infection with latently infected monocytes taking into account the distributed delays.

$$
\dot{S}(t) = \mu - aS(t) - \frac{bV(t)S(t)}{1+\pi V(t)}, \tag{24}
$$

$$
\dot{L}(t) = (1-\rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau - (\theta+\lambda)L(t), \tag{25}
$$

$$
\dot{I}(t) = \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau + \lambda L(t) - \epsilon I(t), \tag{26}
$$

$$
\dot{V}(t) = mI(t) - rV(t) - qV(t)B(t), \tag{27}
$$

$$
\dot{B}(t) = \eta + cB(t)V(t) - \delta B(t). \tag{28}
$$

where, $\xi_1(\tau)$ and $\xi_2(\tau)$ are probability distribution functions which satisfy $\xi_1(\tau) > 0$ and $\xi_2(\tau) > 0$, and

$$
\int_0^{\kappa_1}\xi_1(\tau)d\tau = \int_0^{\kappa_2}\xi_2(\tau)d\tau = 1, \quad \int_0^{\kappa_1}\xi_1(u)e^{nu}du < \infty, \quad \int_0^{\kappa_2}\xi_2(u)e^{nu}du < \infty, \tag{29}
$$

where n is a positive number. Let

$$
E = \int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}d\tau \quad \text{and} \quad K = \int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}d\tau
$$

Then $0 < E \leq 1$, $0 < K \leq 1$. The initial conditions for model (24)-(28) take the form

$$
\begin{aligned}
& S(\varphi) = \psi_1(\varphi), \ L(\varphi) = \psi_2(\varphi), \ I(\varphi) = \psi_3(\varphi), \\
& V(\varphi) = \psi_4(\varphi), \ B(\varphi) = \psi_5(\varphi), \\
& \psi_j(\varphi) \geq 0, \ \varphi \in [-\ell, 0], \ j = 1, ..., 5,
\end{aligned} \tag{30}
$$

where $\ell = \max\{\kappa_1, \kappa_2\}$, $\psi_j \in C([-\ell, 0], \mathbb{R}_{\geq 0})$. This guarantees the uniqueness of solution of the system [23].

3.1 Preliminaries

Lemma 3. The solutions of system (24)-(28) with the initial states (30) are nonnegative and ultimately bounded.

Proof. From Lemma 1 we have $S(t) > 0$ and $B(t) > 0$ for all $t \geq 0$. Moreover, one can show that for $t \geq 0$

$$L(t) = e^{-(\theta+\lambda)t}\psi_2(0) + (1-\rho)b \int_0^t e^{-(\theta+\lambda)(t-u)} \int_0^{\kappa_1} e^{-\delta_1\tau}\xi_1(\tau)\frac{S(u-\tau)V(u-\tau)}{1+\pi V(u-\tau)}d\tau du \geq 0,$$

$$I(t) = e^{-\epsilon t}\psi_3(0) + \rho b \int_0^t e^{-\epsilon(t-u)} \left(\int_0^{\kappa_2} e^{-\delta_2\tau}\xi_2(\tau)\frac{S(u-\tau)V(u-\tau)}{1+\pi V(u-\tau)}d\tau + \lambda L(u) \right) du \geq 0,$$

$$V(t) = e^{-\int_0^t (c+qB(u))u}\psi_4(0) + \int_0^t mI(\omega)e^{-\int_\omega^t (c+qB(u))du}d\omega \geq 0.$$

From (24), we have $\limsup_{t\to\infty} S(t) \leq \frac{\mu}{a}$. Let $T_1(t) = (1-\rho)\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}S(t-\tau)d\tau + L(t)$, then

$$\dot{T}_1(t) = (1-\rho)\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left(\mu - aS(t-\tau) - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau$$

$$+ (1-\rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau - (\theta+\lambda)L(t)$$

$$\leq \mu(1-\rho)E - \sigma_1\left((1-\rho)\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}S(t-\tau)d\tau + L(t)\right)$$

$$\leq \mu(1-\rho) - \sigma_1 T_1(t).$$

It follows that, $\limsup_{t\to\infty} T_1(t) \leq M_1$. Since $\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}S(t-\tau)d\tau > 0$, then $\limsup_{t\to\infty} L(t) \leq M_1$. Let

$$T_2(t) = \rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}S(t-\tau)d\tau + I(t) + \frac{\epsilon}{2m}V(t) + \frac{\epsilon q}{2mc}B(t),$$

then we have

$$\dot{T}_2(t) = \rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left(\mu - aS(t-\tau) - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau$$

$$+ \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau + \lambda L(t) - \epsilon I(t)$$

$$+ \frac{\epsilon}{2m}(mI(t) - rV(t) - qV(t)B(t)) + \frac{\epsilon q}{2mc}(\eta + cB(t)V(t) - \delta B(t))$$

$$\leq \mu\rho K + \lambda L_1 - \sigma_2\left(\rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}S(t-\tau)d\tau + I(t) + \frac{\epsilon}{2m}V(t) + \frac{\epsilon q}{2mc}B(t)\right)$$

$$\leq \mu\rho + \lambda L_1 - \sigma_2 T_2(t).$$

Then $\limsup_{t\to\infty} T_2(t) \leq M_2$. It follows that $\limsup_{t\to\infty} I(t) \leq M_2$, $\limsup_{t\to\infty} V(t) \leq M_3$ and $\limsup_{t\to\infty} B(t) \leq M_4$. Therefore $S(t), L(t), I(t), V(t),$ and $B(t)$ are ultimately bounded. $\square$

Lemma 4. For system (24)-(28) there exists a threshold parameter $\mathcal{R}_0^D > 0$, such that

(i) if $\mathcal{R}_0^D \leq 1$, then there exists only one positive steady state, virus-free steady state Q_0.

(i) if $\mathcal{R}_0^D > 1$, then in addition to Q_0, there exists an endemic steady state Q_1

Proof. Similar to the proof of Lemma 2 we can show that if $\mathcal{R}_0^D \leq 1$ then there exists $Q_0 = (S_0, 0, 0, 0, B_0)$, where $S_0 = \frac{\mu}{a}$ and $B_0 = \frac{\eta}{\delta}$, and if $\mathcal{R}_0^D > 1$ then there exists $Q_1 = (S_1, L_1, I_1, V_1, B_1)$, with

$$S_1 = \frac{\mu(1+\pi V_1)}{bV_1 + a(1+\pi V_1)}, \quad L_1 = \frac{E(1-\rho)b\mu V_1}{(\theta+\lambda)(bV_1 + a(1+\pi V_1))}, \quad I_1 = \frac{b\mu V_1\gamma}{\epsilon(\theta+\lambda)(bV_1 + a(1+\pi V_1))},$$

$$V_1 = \frac{P_2^D - \sqrt{(P_2^D)^2 - 4P_1^D P_3^D}}{2P_1^D} < \frac{\delta}{c}, \quad B_1 = \frac{\eta}{\delta - cV_1},$$

where

$$\gamma = E\lambda(1-\rho) + K\rho(\theta+\lambda), \quad P_1^D = rec(\theta+\lambda)(b+\pi a),$$

$$P_2^D = \frac{\epsilon ca\,(r\delta+q\eta)\,(\theta+\lambda)}{\delta}(\mathcal{R}_0^D - 1) + \epsilon\,(r\delta+q\eta)\,(\theta+\lambda)(b+\pi a) + \frac{caq\epsilon\eta(\theta+\lambda)}{\delta},$$

$$P_3^D = \epsilon a\,(r\delta+q\eta)\,(\theta+\lambda)(\mathcal{R}_0^D - 1).$$

The basic reproduction number for system (24)-(28) is defined as

$$\mathcal{R}_0^D = \frac{m\mu b\delta\gamma}{\epsilon a(r\delta+q\eta)(\theta+\lambda)}. \quad \square$$

3.2 Global stability

In this section we construct suitable Lyapunov functions to prove that the steady states Q_0 and Q_1 of system (24)-(28) are GAS.

Theorem 3. Suppose that $\mathcal{R}_0^D \leq 1$, then Q_0 is GAS.

Proof. Let us define $Y_0^D(S, L, I, V, B)$ as:

$$Y_0^D = \frac{\gamma}{\theta+\lambda}S_0 H\left(\frac{S}{S_0}\right) + \frac{\lambda}{\theta+\lambda}L + I + \frac{\epsilon}{m}V + \frac{\epsilon q}{mc}B_0 H\left(\frac{B}{B_0}\right)$$

$$+\frac{\lambda(1-\rho)b}{\theta+\lambda}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\int_0^\tau \frac{V(t-\vartheta)S(t-\vartheta)}{1+\pi V(t-\vartheta)}d\vartheta d\tau + \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\int_0^\tau \frac{V(t-\vartheta)S(t-\vartheta)}{1+\pi V(t-\vartheta)}d\vartheta d\tau. \quad (31)$$

Note that, $Y_0^D(S, L, I, V, B) > 0$ for all $S, L, I, V, B > 0$ and $Y_0^D(S_0, 0, 0, 0, B_0) = 0$. Calculating $\dfrac{dY_0^D}{dt}$ along the trajectories of (24)-(28) we get

$$\frac{dY_0^D}{dt} = \frac{\gamma}{\theta+\lambda}\left(1-\frac{S_0}{S}\right)\left(\mu - aS - \frac{bVS}{1+\pi V}\right)$$

$$+\frac{\lambda}{\theta+\lambda}\left((1-\rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau - (\theta+\lambda)L\right)$$

$$+\rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau + \lambda L - \epsilon I + \frac{\epsilon}{m}\left(mI - rV - qVB\right)$$

$$+\frac{\epsilon q}{mc}\left(1-\frac{B_0}{B}\right)(\eta + cBV - \delta B) + \frac{\lambda(1-\rho)}{\theta+\lambda}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau$$

$$+\rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau$$

$$= -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_0)^2}{S} + \left(\frac{\gamma}{(\theta+\lambda)}\right)\frac{bS_0 V}{1+\pi V} - \frac{\epsilon rV}{m} - \frac{\epsilon qB_0 V}{m} + \frac{\epsilon q}{mc}\left(1-\frac{B_0}{B}\right)(\delta B_0 - \delta B)$$

$$= -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_0)^2}{S} - \frac{\epsilon q\delta}{mc}\frac{(B-B_0)^2}{B} + \frac{\epsilon(r\delta+q\eta)}{m\delta}(\mathcal{R}_0^D - 1)V - \frac{\epsilon(r\delta+q\eta)\mathcal{R}_0^D\pi V^2}{m\delta(1+\pi V)}. \quad (32)$$

Therefore, $\dfrac{dY_0^D}{dt} \leq 0$ holds if $\mathcal{R}_0^D \leq 1$. Further, $\dfrac{dY_0^D}{dt} = 0$ if and only if $S = S_0$, $B = B_0$, $V = 0$. Applying LaSalle's invariance principle, we get that Q_0 is GAS . $\square$

Theorem 4. Suppose that $\mathcal{R}_0^D > 1$, then Q_1 is GAS.

Proof. Consider

$$Y_1^D(S, L, I, V, B) = \frac{\gamma}{\theta+\lambda}S_1 H\left(\frac{S}{S_1}\right) + \frac{\lambda}{\theta+\lambda}L_1 H\left(\frac{L}{L_1}\right) + I_1 H\left(\frac{I}{I_1}\right) + \frac{\epsilon}{m}V_1 H\left(\frac{V}{V_1}\right)$$

$$+\frac{\epsilon q}{mc}B_1 H\left(\frac{B}{B_1}\right) + \frac{\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1 V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\int_0^\tau H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1 V_1\,(1+\pi V(t-\vartheta))}\right)d\vartheta d\tau$$

$$+\frac{\rho bS_1 V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\int_0^\tau H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1 V_1\,(1+\pi V(t-\vartheta))}\right)d\vartheta d\tau.$$

We have $Y_1^D(S, L, I, V, B) > 0$ for all $S, L, I, V, B > 0$ and $Y_1^D(S_1, L_1, I_1, V_1, B_1) = 0$. Calculating $\dfrac{dY_1^D}{dt}$ along the trajectories of (24)-(28) we get

$$
\begin{aligned}
\frac{dY_1^D}{dt} &= \frac{\gamma}{\theta + \lambda}\left(1 - \frac{S_1}{S}\right)\left(\mu - aS - \frac{bVS}{1 + \pi V}\right) \\
&+ \frac{\lambda}{\theta + \lambda}\left(1 - \frac{L_1}{L}\right)\left((1 - \rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t - \tau)S(t - \tau)}{1 + \pi V(t - \tau)}d\tau - (\theta + \lambda)L\right) \\
&+ \left(1 - \frac{I_1}{I}\right)\left(\rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t - \tau)S(t - \tau)}{1 + \pi V(t - \tau)}d\tau + \lambda L - \epsilon I\right) + \frac{\epsilon}{m}\left(1 - \frac{V_1}{V}\right)(mI - rV - qVB) \\
&+ \frac{\epsilon q}{mc}\left(1 - \frac{B_1}{B}\right)(\eta + cBV - \delta B) + \frac{\lambda(1 - \rho)}{(\theta + \lambda)}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left(\frac{bVS}{1 + \pi V} - \frac{bV(t - \tau)S(t - \tau)}{1 + \pi V(t - \tau)}\right)d\tau \\
&+ \frac{\lambda(1 - \rho)}{(\theta + \lambda)}\frac{bS_1V_1}{1 + \pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\ln\left(\frac{V(t - \tau)S(t - \tau)(1 + \pi V)}{VS(1 + \pi V(t - \tau))}\right)d\tau \\
&+ \rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left(\frac{bVS}{1 + \pi V} - \frac{bV(t - \tau)S(t - \tau)}{1 + \pi V(t - \tau)}\right)d\tau \\
&+ \frac{\rho bS_1V_1}{1 + \pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\ln\left(\frac{V(t - \tau)S(t - \tau)(1 + \pi V)}{VS(1 + \pi V(t - \tau))}\right)d\tau.
\end{aligned}
\tag{33}
$$

Applying

$$
\mu = aS_1 + \frac{bS_1V_1}{1 + \pi V_1}, \ \ \eta = \delta B_1 - cB_1V_1,
$$

we obtain

$$
\begin{aligned}
\frac{dY_1^D}{dt} &= \frac{\gamma}{\theta + \lambda}\left(1 - \frac{S_1}{S}\right)(aS_1 - aS) + \left(\frac{\gamma}{\theta + \lambda}\right)\frac{bS_1V_1}{1 + \pi V_1}\left(1 - \frac{S_1}{S}\right) \\
&+ \frac{\gamma}{\theta + \lambda}\frac{bS_1V}{1 + \pi V} - \frac{\lambda(1 - \rho)b}{\theta + \lambda}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t - \tau)S(t - \tau)L_1}{(1 + \pi V(t - \tau))L}d\tau + \lambda L_1 \\
&- \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t - \tau)S(t - \tau)I_1}{(1 + \pi V(t - \tau))I}d\tau - \frac{\lambda LI_1}{I} + \epsilon I_1 - \frac{\epsilon IV_1}{V} - \frac{r\epsilon V}{m} + \frac{r\epsilon V_1}{m} + \frac{\epsilon qBV_1}{m} \\
&+ \frac{\epsilon q}{mc}\left(1 - \frac{B_1}{B}\right)(\delta B_1 - \delta B) - \frac{\epsilon qB_1V}{m} - \frac{\epsilon qB_1V_1}{m} + \frac{\epsilon qB_1V_1}{m}\left(\frac{B_1}{B}\right) \\
&+ \frac{\lambda(1 - \rho)}{\theta + \lambda}\frac{bS_1V_1}{1 + \pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\ln\left(\frac{V(t - \tau)S(t - \tau)(1 + \pi V)}{VS(1 + \pi V(t - \tau))}\right)d\tau \\
&+ \frac{\rho bS_1V_1}{1 + \pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\ln\left(\frac{V(t - \tau)S(t - \tau)(1 + \pi V)}{VS(1 + \pi V(t - \tau))}\right)d\tau.
\end{aligned}
$$

The components of the steady state Q_1 satisfy

$$
E(1 - \rho)\frac{bS_1V_1}{1 + \pi V_1} = (\theta + \lambda)L_1, \ \ K\rho\frac{bS_1V_1}{1 + \pi V_1} + \lambda L_1 = \epsilon I_1, \ \ mI_1 = rV_1 + qB_1V_1,
$$

then

$$
\epsilon I_1 = \frac{\gamma}{\theta + \lambda}\frac{bS_1V_1}{(1 + \pi V_1)}, \qquad \frac{r\epsilon V_1}{m} = \frac{\gamma}{\theta + \lambda}\frac{bS_1V_1}{(1 + \pi V_1)} - \frac{\epsilon qB_1V_1}{m},
$$

and

$$
\begin{aligned}
\frac{dY_1^D}{dt} ={}& -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right) \\
&+ K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right) + \left(\frac{\gamma}{\theta+\lambda}\right)\frac{bS_1V_1}{1+\pi V_1}\left(\frac{(1+\pi V_1)V}{(1+\pi V)V_1}-\frac{V}{V_1}\right) \\
&- \frac{\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}d\tau + \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} \\
&- \frac{\rho bS_1V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}d\tau - \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{I_1L}{L_1I} \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} + K\rho\frac{bS_1V_1}{(1+\pi V_1)} - \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} - K\rho\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} + K\rho\frac{bS_1V_1}{(1+\pi V_1)} - \frac{2\epsilon q B_1V_1}{m} + \frac{\epsilon q B V_1}{m} + \frac{\epsilon q B_1V_1}{m}\left(\frac{B_1}{B}\right) \\
&- \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B} + \frac{\lambda(1-\rho)}{(\theta+\lambda)}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS(1+\pi V(t-\tau))}\right)d\tau \\
&+ \frac{\rho bS_1V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS(1+\pi V(t-\tau))}\right)d\tau.
\end{aligned}
$$

Utilizing the following equalities

$$
\begin{aligned}
\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS(1+\pi V(t-\tau))}\right) ={}& \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right) + \ln\left(\frac{LI_1}{L_1I}\right) \\
&+ \ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right), \\
\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS(1+\pi V(t-\tau))}\right) ={}& \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right) \\
&+ \ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{dY_1^D}{dt} ={}& -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{\gamma}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left(-1+\frac{(1+\pi V_1)V}{(1+\pi V)V_1}-\frac{V}{V_1}+\frac{1+\pi V}{1+\pi V_1}\right) \\
&+ \frac{E\lambda(1-\rho)}{(\theta+\lambda)}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + \frac{E\lambda(1-\rho)}{(\theta+\lambda)}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right] \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{1}{E}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left[1-\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right. \\
&\left.+ \ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right)\right]d\tau \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right] + \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{LI_1}{L_1I}+\ln\left(\frac{LI_1}{L_1I}\right)\right] \\
&+ K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + K\rho\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right] \\
&+ K\rho\frac{bS_1V_1}{1+\pi V_1}\frac{1}{K}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left[1-\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right. \\
&\left.+ \ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right)\right]d\tau \\
&+ K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right] - \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B} - \frac{\epsilon q B_1V_1}{m}\left[2-\frac{B}{B_1}-\frac{B_1}{B}\right]
\end{aligned}
$$

$$= -a\left(\frac{\gamma}{\theta+\lambda}\right)\frac{(S-S_1)^2}{S} - \left(\frac{\gamma}{\theta+\lambda}\right)\frac{\pi b S_1(V-V_1)^2}{(1+\pi V)(1+\pi V_1)^2}$$

$$-\frac{\epsilon q\eta}{mcB_1}\frac{(B-B_1)^2}{B} - \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left[H\left(\frac{S_1}{S}\right) + H\left(\frac{IV_1}{I_1V}\right) + H\left(\frac{1+\pi V}{1+\pi V_1}\right) + H\left(\frac{LI_1}{L_1I}\right)\right.$$

$$\left. +\frac{1}{E}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}H\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right)d\tau\right]$$

$$-K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left[H\left(\frac{S_1}{S}\right) + H\left(\frac{IV_1}{I_1V}\right) + H\left(\frac{1+\pi V}{1+\pi V_1}\right)\right.$$

$$\left. +\frac{1}{K}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}H\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right)d\tau\right].$$

It can be seen that if $\mathcal{R}_0^D > 1$, then $S_1, L_1, I_1, V_1, B_1 > 0$ and $\frac{dY_1^D}{dt} \leq 0$ for all $S, L, I, V, B > 0$. We have $\frac{dY_1^D}{dt} = 0$ if and only if $S = S_1$, $L = L_1$, $I = I_1$, $V = V_1, B = B_1$ and $H = 0$. Then using from LaSalle's invariance principle, we show that Q_1 is GAS. $\square$

4   Numerical simulations

Next we conduct numerical simulations for system (5)-(9). The values of the parameters are listed in Table 1. We let $\tau_i = \tau_1 = \tau_2$. The following initial conditions are used:

$$\varphi_1(\vartheta) = 1.7, \varphi_2(\vartheta) = 0.4, \varphi_3(\vartheta) = 0.6, \varphi_4(\vartheta) = 0.6, \varphi_5(\vartheta) = 1.6, \quad \vartheta \in [-\tau_i, 0]$$

In Figures 1-5, we show the evolution of the five states of the system S, L, I, V and B with respect to the time. The effect of τ_i on the stability of Q_0 and Q_1 is also shown. We can see that, for smaller values of τ_i e.g. $\tau_i = 0.0, 0.5, 1.0$ and 2.0, the corresponding values of $\mathcal{R}_0$ satisfy $\mathcal{R}_0 > 1$, and the trajectory of the system converges to the steady states Q_1. This confirm the results of Theorem 2 that Q_1 is GAS. On the the other hand, when τ_i become larger e.g. $\tau_i = 3.0$ and 5.0, then $\mathcal{R}_0 < 1$, and the system has one steady state Q_0 and according to Theorem 1 it is GAS. For this case, the concentrations of the uninfected monocytes and B cells return to their values $S_0 = \frac{\mu}{a} = 2.2885$ and $B_0 = \frac{\eta}{\delta} = 1.1207$, respectively, while the CHIKV particles are cleared from the body.

Let τ^{cr} be the critical value of the parameter τ_i, such that

$$\mathcal{R}_0 = \frac{bm\delta\mu(\lambda(1-\rho)e^{-\delta_1\tau^{cr}} + \rho(\theta+\lambda)e^{-\delta_1\tau^{cr}})}{\epsilon a(r\delta + q\eta)(\theta+\lambda)} = 1.$$

Using the data given in Table 1 we obtain $\tau^{cr} = 2.01206$. The value of $\mathcal{R}_0$ for different values of τ_i are listed in Table 2. We can observed that as τ_i is increased then $\mathcal{R}_0$ is decreased. Moreover, we have the following cases:

(i) if $0 \leq \tau_i < 2.01206$, then Q_1 exists and it is GAS,

(ii) if $\tau_i \geq 2.01206$, then Q_0 is GAS. It is clearly seen that, an increasing in time delay will stabilize the system around Q_0. Biologicaly, the time delay has a similar effect as the antiviral treatment which can be used to eliminate the CHIKV. We observe that, when the delay period is sufficiently long the CHIKV replication will be cleared.

Table 1: The values of the parameters of model (5)-(9).

Parameter	Value	Parameter	Value
μ	1.826	m	2.02
π	0.1	q	0.5964
c	1.2129	r	0.4418
a	0.7979	η	1.402
θ	0.5	δ_1	0.5
λ	0.1	τ_1	$varied$
ϵ	0.4441	τ_2	$varied$
δ	1.251	b	0.5
ρ	0.5		

Table 2: The values of steady states, $\mathcal{R}_0$ for model (5)-(9) with different values of τ_i.

τ_i	Steady states	R_0	
0.0	$Q_1 = (1.6788, 0.4054, 0.6390, 0.6152, 2.7772)$	2.7347	
0.5	$Q_1 = (1.7636, 0.2718, 0.4284, 0.4986, 2.1694)$	2.1298	
1.0	$Q_1 = (1.8827, 0.1637, 0.2580, 0.3562, 1.7120)$	1.6587	
1.5	$Q_1 = (2.0497, 0.0750, 0.1182, 0.1895, 1.3729)$	1.2918	
2.0	$Q_1 = (2.2819, 0.0016, 0.0025, 0.0046, 1.1257)$	1.0060	
2.01206	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	1.0000	
2.5	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.7835	
3.0	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.6102	
3.5	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.4752	
4.0	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.3701	
4.5	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.2882	
5.0	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.2245	
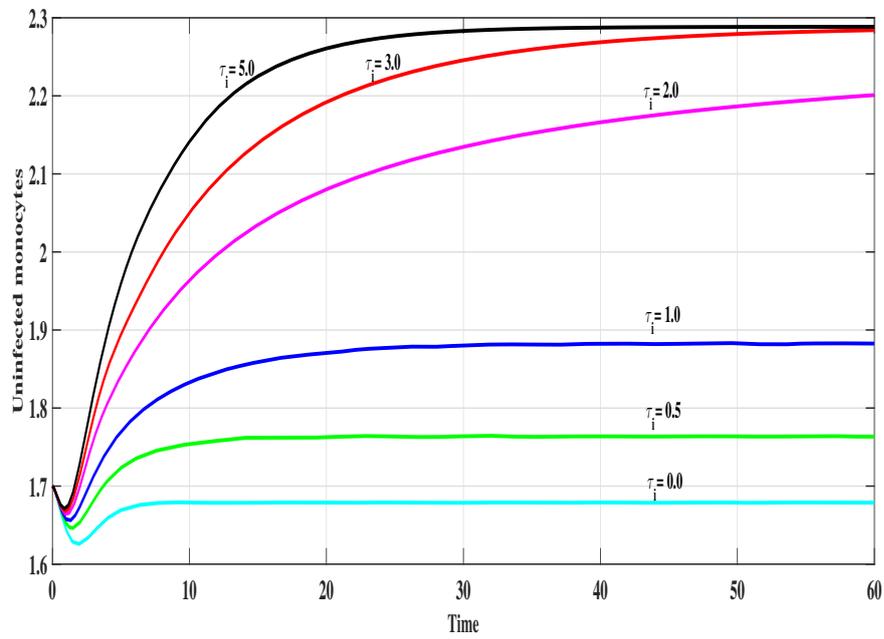
Figure 1: The evolution of uninfected monocytes.
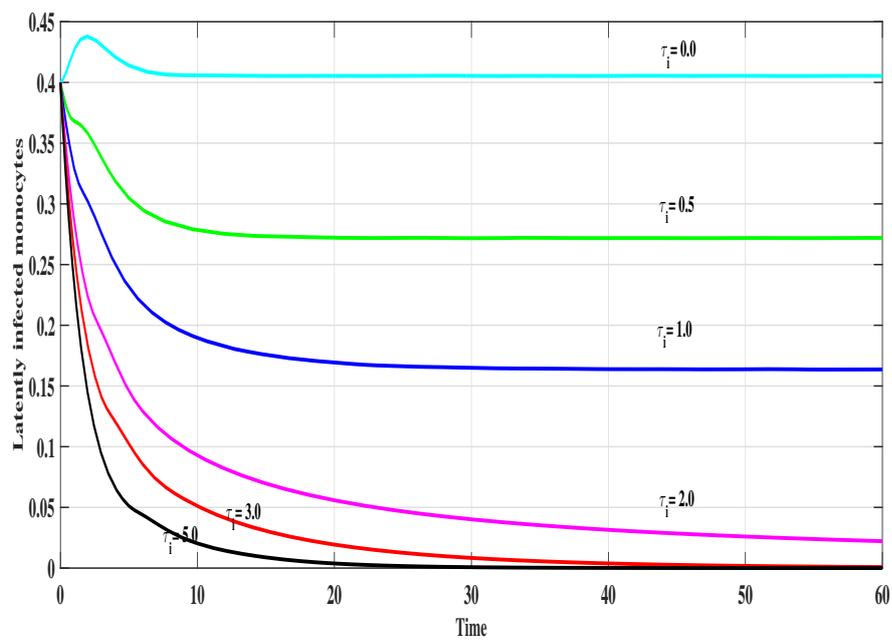
Figure 2: The evolution of latently infected monocytes.
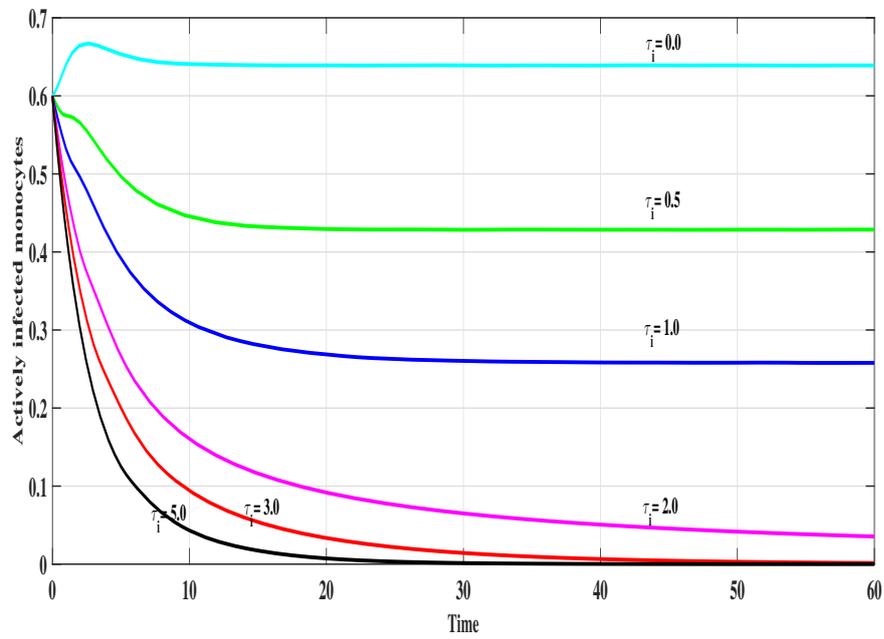
Figure 3: The evolution of actively infected monocytes.
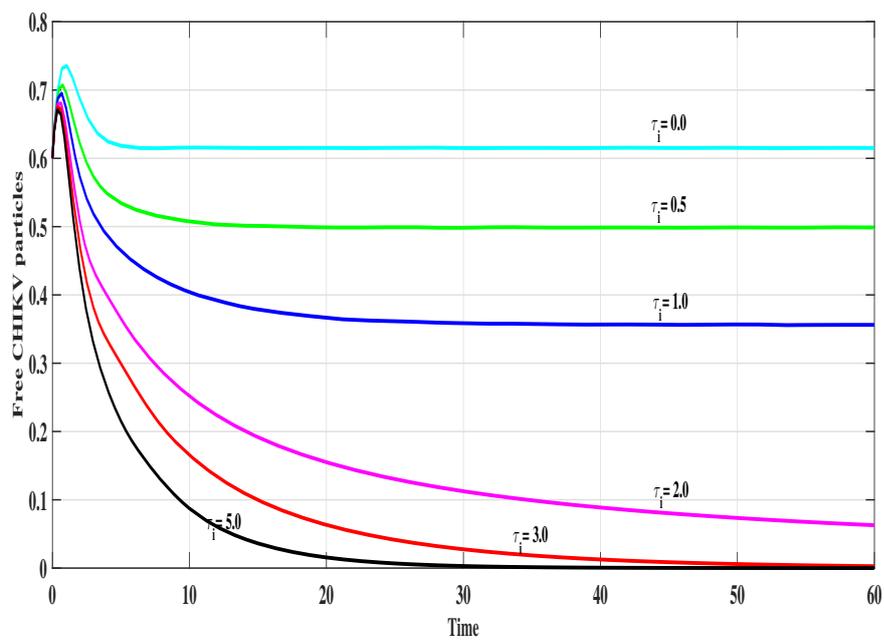
Figure 4: The evolution of free CHIKV particles.
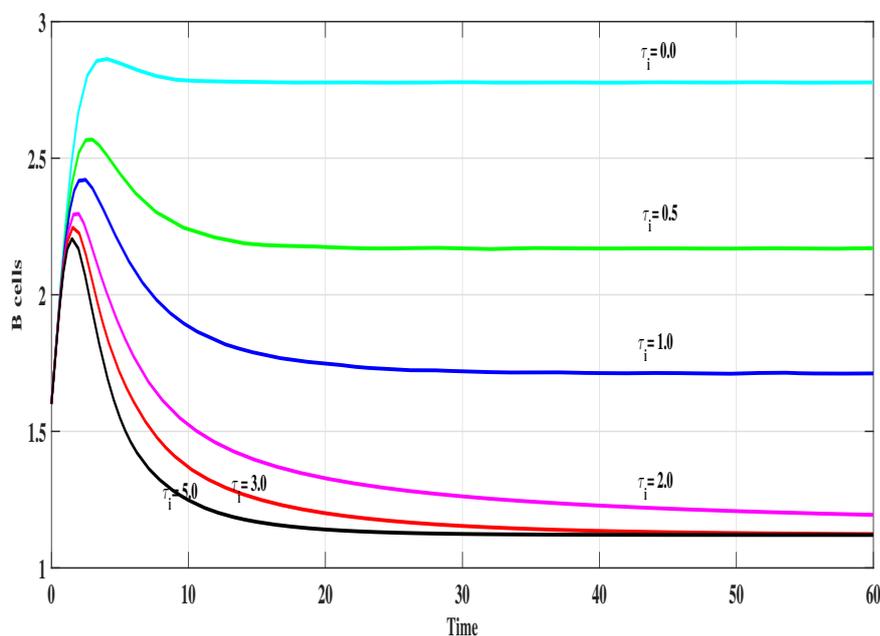
Figure 5: The evolution of B cells.

5   Acknowledgment

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Dynamical behavior of MERS-CoV model with discrete delays

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Abstract

A nonlinear mathematical model for Middle East Respiratory Syndrome Corona Virus (MERS-CoV) with two discrete time delays is proposed and analyzed. We show that the solutions of the model are nonnegative and bounded. We derive the basic reproduction number for the MERS-CoV model, R_0. we prove that if $R_0 \leq 1$ then there exists a disease-free equilibrium P_0 and $R_0 > 1$ then in addition to P_0 the model has an endemic equilibrium P^*. Utilizing Lyapunov method, the global asymptotic stability of disease-free equilibrium of the proposed model is obtained. The dynamical behaviour of the model is also shown by numerical simulations.

Keywords: Infectious diseases; global stability; Lyapunov functional.

1 Introduction

Mathematical of infectious diseases have received the attention of several researchers during the past decides. Some of the models are given by a set of ODEs (see e.g. [1]-[12]). For some disease such as influenza, on adequate contact with an infective, a susceptible individual becomes exposed, that is, infected but not infective. This individual stays in exposed class for a certain latent period before becoming infective. This period can been described as delays on the spread of infectious diseases, and thus, delays should be incorporated into infection term in the system. As a result, the models are given by DDEs (see e.g. [13]-[19]). There are two types of time delays: (i) discrete delay, where the time delay is assumed to be constant (see e.g. [13]-[15]), (ii) distributed delays, where the time delay is assumed to be random parameter taken from probability distributed function (see e.g. [16]-[19]). Recently, Chowell et al. [20] have studied the spread of a Middle East Respiratory Syndrome

Corona virus (MERS-CoV) by using a SEIR-type compartmental transmission model as:

$$\frac{dS}{dt} = -\frac{\beta\,S\,(I_i + I_s + \ell\,H)}{N} - \alpha, \tag{1}$$

$$\frac{dE_i}{dt} = \alpha - k\,E_i, \tag{2}$$

$$\frac{dE_s}{dt} = \frac{\beta\,S\,(I_i + I_s + \ell\,H)}{N} - k\,E_s, \tag{3}$$

$$\frac{dI_i}{dt} = k\,\rho_{c,i}\,E_i - \gamma_a\,I_i - \gamma_{I,i}\,I_i \tag{4}$$

$$\frac{dA_i}{dt} = k\,(1 - \rho_{c,i})\,E_i, \tag{5}$$

$$\frac{dI_s}{dt} = k\,\rho_{c,s}\,E_s - \gamma_a\,I_s - \gamma_{I,s}\,I_s, \tag{6}$$

$$\frac{dA_s}{dt} = k\,(1 - \rho_{c,s})\,E_s, \tag{7}$$

$$\frac{dH}{dt} = \gamma_a\,(I_i + I_s) - \gamma_r\,H, \tag{8}$$

$$\frac{dR}{dt} = \gamma_r H + \gamma_{I,i}\,I_i + \gamma_{I,s}\,I_s. \tag{9}$$

In model (1)-(9), the populations divided into 9 compartment: susceptible individuals S, individuals exposed to the zoonotic reservoir E_i or to infectious humans E_s, infectious and symptomatic individuals arising from reservoir I_i, or from human-to-human transmission I_s, asymptomatic and non-infectious individuals arising from environmental/animal exposure A_i or arising from human-to-human transmission A_s, hospitalized individuals H, and removed individuals after recovery or disease-induced death R [20]. Susceptible individuals are infected uniformly at random from the zoonotic reservoir at rate α. The parameter β is the mean human-to-human transmission rate per day, ℓ is relative transmissibility of hospitalized cases, $\frac{1}{k}$ mean latent period (days), $\rho_{c,i}$ is proportion of symptomatic and infectious cases among index cases, $\rho_{s,i}$ denote to proportion of symptomatic and infectious cases among secondary cases, $\rho_{h,i}$ proportion of hospitalized individuals among symptomatic and infectious index cases, $\rho_{h,s}$ is proportion of hospitalized individuals among symptomatic and infectious secondary cases, $\frac{1}{\gamma_{I,i}}$ represent the mean infectious period among primary cases (days), $\frac{1}{\gamma_{I,s}}$ is the mean infectious period among secondary cases (days), $\frac{1}{\gamma_a}$ is the mean time from symptom onset to hospital admission (days) and $\frac{1}{\gamma_r}$ denote to mean length of hospital stay (days). Chowell et al., assume that the asymptomatic individuals do not contribute to the transmission process. Moreover, the basic properties of model (1)-(9) are not well studied. Therefore, the aim of this paper is to study the effect of asymptomatic individuals on the transmission of MERS-CoV. Our proposed model is a modification of model (1)-(9) by incorporate the asymptomatic individuals as a carrier individuals. We assume that the first scenario describes only the carrier cases and the second one describes the infected cases which demonstrate symptoms. We introduce two types of discrete time delays into the MERS-CoV model. We study the basic properties of the model such as nonnegativity and boundedness of the solutions, stability analysis of the equilibria. At the end we perform some numerical simulations.

2   The MERS-CoV model

In this section, we propose a MERS-CoV model with two discrete delays . Let us define

$$\Upsilon(t) = S(t)(\beta I_c(t) + \gamma I_m(t) + \ell H(t)).$$

Then we propose the following model:

$$\dot{S}(t) = b - \Upsilon(t) - d_1 S(t), \tag{10}$$

$$\dot{E}_c(t) = p\, e^{-\mu_1 \tau_1} \Upsilon(t - \tau_1) - (k\rho_1 + d_2) E_c(t), \tag{11}$$

$$\dot{E}_m(t) = (1 - p) e^{-\mu_2 \tau_2} \Upsilon(t - \tau_2) - (k\rho_2 + d_3) E_m(t), \tag{12}$$

$$\dot{I}_c(t) = k\, \rho_1 E_c(t) - \gamma_a I_c(t) - q I_c(t) - \gamma_1 I_c(t) - d_4 I_c(t), \tag{13}$$

$$\dot{I}_m(t) = k\, \rho_2 E_m(t) - \gamma_a I_m(t) - \gamma_2 I_m(t) + q I_c(t) - d_5 I_m(t), \tag{14}$$

$$\dot{H}(t) = \gamma_a (I_c(t) + I_m(t)) - \gamma_r H(t) - d_6 H(t), \tag{15}$$

$$\dot{R}(t) = \gamma_1 I_c(t) + \gamma_2 I_m(t) + \gamma_r H(t) - d_7 R(t), \tag{16}$$

where S is susceptible individuals, E_c exposed individuals to carrier, E_m exposed individuals to infected, I_c carrier individuals, I_m infected individuals, H hospitalized infected and R recovered individuals. The parameters $\tau_1 \geq 0$ and $\tau_2 \geq 0$ represents for the time between contact the susceptible individuals with exposed to carrier E_c and exposed to infected E_m, respectively. The factors $e^{-\mu_1 \tau_1}$ and $e^{-\mu_2 \tau_2}$ are the probability that an individuals survives during the delay periods $[0, \tau_1]$ and $[0, \tau_2]$, respectively. The other parameters are defined in section 6. The initial conditions of system (10)-(16) are given by

$$S(\theta) = \varphi_1(\theta), \ E_c(\theta) = \varphi_2(\theta), \ E_m(\theta) = \varphi_3(\theta),$$
$$I_c(\theta) = \varphi_4(\theta), \ I_m(\theta) = \varphi_5(\theta), \ H(\theta) = \varphi_6(\theta), \ R(\theta) = \varphi_7(\theta), \tag{17}$$
$$\varphi_i(\theta) \geq 0, \ \theta \in [-\varrho, 0], \ i = 1, ..., 7,$$

where, $\varrho = \max\{\tau_1, \tau_2]$ and $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_7(\theta)) \in C([-\varrho, 0], \mathbf{R}^7_{\geq 0})$ where C is the Banach space of continuous functions mapping the interval $[-\varrho, 0]$ into $\mathbf{R}^7_{\geq 0}$. By the fundamental theory of functional differential equations [21], system (10)-(16) has a unique solution satisfying the initial conditions.

3 Nonnegativity and boundedness

In this section, we will study the nonnegativity and boundedness of the model's solutions.

Theorem 1. The solutions of system (10)-(16) are nonnegative and there exist a positive number Q such that the compact set:
$\Gamma = \{(S, E_c, E_m, I_c, I_m, H, R) \in \mathbf{R}^7_{\geq 0} : 0 \leq S, E_c, E_m, I_c, I_m, H, R \leq Q\}$
is positively invariant.

Proof First, we show the nonnegativity solutions, we will write the system in the matrix form as $\dot{Y} = \phi(Y)$, where $Y = (S, E_c, E_m, I_c, I_m, H, R)^T$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7)^T$. Then,

$$
\begin{pmatrix}
\phi_1(Y) \\
\phi_2(Y) \\
\phi_3(Y) \\
\phi_4(Y) \\
\phi_5(Y) \\
\phi_6(Y) \\
\phi_7(Y)
\end{pmatrix}
=
\begin{pmatrix}
b - \Upsilon(t) - d_1 S(t) \\
p\, e^{-\mu_1 \tau_1} \Upsilon(t - \tau_1) - k\, \rho_1 E_c(t) - d_2 E_c(t) \\
(1 - p)\, e^{-\mu_2 \tau_2} \Upsilon(t - \tau_2) - k\, \rho_2 E_m(t) - d_3 E_m(t) \\
k\, \rho_1 E_c(t) - \gamma_a I_c(t) - q I_c(t) - \gamma_1 I_c(t) - d_4 I_c(t) \\
k\, \rho_2 E_m(t) - \gamma_a I_m(t) - \gamma_2 I_m(t) + q I_c(t) - d_5 I_m(t) \\
\gamma_a (I_c(t) + I_m(t)) - \gamma_r H(t) - d_6 H(t) \\
\gamma_1 I_c(t) + \gamma_2 I_m(t) + \gamma_r H(t) - d_7 R(t)
\end{pmatrix}.
$$

It is easy to see that functions ϕ_i satisfies the following condition

$$\phi_i(Y(t))|_{Y_i(t) = 0, Y(t) \in \mathbf{R}^7_{>0}} \geq 0.$$

Due to lemma 2 in [22], any solution of (10)-(16) with initial (17) is such that $Y(t) \in \mathbf{R}_{\geq 0}^7$ for all $t \geq 0$. Next, we prove the ultimate bound of the solutions of system (10)-(16). Let us define
$L(t) = p\,e^{-\mu_1 \tau_1} S(t - \tau_1) + (1 - p)\,e^{-\mu_2 \tau_2} S(t - \tau_2) + E_c(t) + E_m(t) + I_c(t) + I_m(t) + H(t) + R(t).$
Then,

$$
\begin{aligned}
\dot{L}(t) =\ & p\,e^{-\mu_1 \tau_1}\,(b - \Upsilon(t - \tau_1) - d_1 S(t - \tau_1)) \\
& + (1 - p)\,e^{-\mu_2 \tau_2}\,(b - \Upsilon(t - \tau_2) - d_1\,S(t - \tau_2)) \\
& + p\,e^{-\mu_1 \tau_1}\,\Upsilon(t - \tau_1) - k\,\rho_1\,E_c(t) - d_2\,E_c(t) \\
& + (1 - p)e^{-\mu_2 \tau_2}\,\Upsilon(t - \tau_2) - k\rho_2 E_m(t) - d_3 E_m(t) \\
& + k\,\rho_1\,E_c(t) - \gamma_a\,I_c(t) - q\,I_c(t) - \gamma_1\,I_c(t) - d_4\,I_c(t) \\
& + k\,\rho_2\,E_m(t) - \gamma_a\,I_m(t) - \gamma_2\,I_m(t) + q\,I_c(t) - d_5\,I_m(t) + \gamma_a\,(I_c(t) + I_m(t)) \\
& - \gamma_r\,H(t) - d_6\,H(t) + \gamma_1\,I_c(t) + \gamma_2\,I_m(t) + \gamma_r\,H(t) - d_7\,R(t), \\
=\ & (p\,e^{-\mu_1 \tau_1} + (1 - p)\,e^{-\mu_2 \tau_2})\,b - p\,e^{-\mu_1 \tau_1}\,d_1\,S(t - \tau_1) - (1 - p)\,e^{-\mu_2 \tau_2} d_1\,S(t - \tau_2) \\
& - d_2 E_c(t) - d_3\,E_m(t) - d_4 I_c(t) - d_5 I_m(t) - d_6 H(t) - d_7 R(t) \\
\leq\ & b - \bar{d}\,L(t),
\end{aligned}
$$

where $\bar{d} = min\{d_i\}, \ i = 1, .., 7$. It follows that, $\limsup_{t \to \infty} L(t) \leq Q$, where $Q = \frac{b}{\bar{d}}$. Then,
$\limsup_{t \to \infty} S(t) \leq Q, \limsup_{t \to \infty} E_c(t) \leq Q, \limsup_{t \to \infty} E_m(t) \leq Q, \limsup_{t \to \infty} I_c(t) \leq Q, \limsup_{t \to \infty} I_m(t) \leq Q, \limsup_{t \to \infty} H(t) \leq Q,$ and $\limsup_{t \to \infty} R(t) \leq Q.$ $\square$

4 Equilibria and biological thresholds

To calculated the equilibria of model (10)-(16), we put the R.H.S of Eqs. (10)-(16) equals zero, we get

$$b - S\,(d_1 + \beta\,I_c + \gamma\,I_m + \ell\,H) = 0, \tag{18}$$

$$p\,e^{-\mu_1 \tau_1} S\,(\beta\,I_c + \gamma\,I_m + \ell\,H) - a_1\,E_c = 0, \tag{19}$$

$$(1 - p)\,e^{-\mu_2 \tau_2} S(\,(\beta\,I_c + \gamma\,I_m + \ell\,H) - a_2\,E_m, = 0, \tag{20}$$

$$\lambda_1\,E_c - a_3\,I_c = 0, \tag{21}$$

$$\lambda_2\,E_m - a_4\,I_m + q\,I_c = 0, \tag{22}$$

$$\gamma_a\,(I_c + I_m) - a_5\,H = 0, \tag{23}$$

$$\gamma_1\,I_c + \gamma_2\,I_m + \gamma_r\,H - d_7\,R = 0, \tag{24}$$

where

$$
\begin{aligned}
a_1 &= k\,\rho_1 + d_2, & a_2 &= k\,\rho_2 + d_3 \\
a_3 &= \gamma_a + \gamma_1 + q + d_4, & a_4 &= \gamma_a + \gamma_2 + d_5, \\
a_5 &= \gamma_r + d_6, & \lambda_1 &= k\rho_1, \ \lambda_2 = k\rho_2.
\end{aligned}
$$

Solving system (18)-(24), we find that the system has two equilibria

- The disease-free equilibrium

$$P_0 = (S_0,\,0,\,0,\,0,\,0,\,0,\,0) = \left(\frac{b}{d_1},\,0,\,0,\,0,\,0,\,0,\,0\right). \tag{25}$$

- The endemic equilibrium

$$P^* = (S^*, E_c^*, E_m^*, I_c^*, I_m^*, H^*, R^*), \tag{26}$$

where

$$S^* = \frac{a_0}{A_1}, \quad E_c^* = \frac{p(A_2 - a_0\, d_1\, e^{\mu_2\, \tau_2})}{a_1\, A_3}, \quad E_m^* = \frac{(1-p)\,(A_4 - a_0\, d_1\, e^{\mu_1\, \tau_1})}{a_2\, A_3},$$

$$I_c^* = \frac{\lambda_1\, p\,(A_2 - a_0\, d_1\, e^{\mu_2\tau_2})}{a_1\, a_3\, A_3}, \quad I_m^* = \frac{A_5 + a_2\,(A_6 - 2\, a_1\, a_3\, A_7)}{a_1\, a_2\, a_3\, a_4\, A_3},$$

$$H^* = \frac{\gamma_a\,(2\, a_1\, a_3\,(a_2\, A_{10} + A_9) - A_8)}{a_0\, A_3}, \quad R^* = \frac{A_{11} - a_1\, a_3\,(A_{12} + a_2\, A_{13})}{d_1\, a_0\, A_3},$$

and

$$a_0 = a_1\, a_2\, a_3\, a_4\, a_5,$$

$$A_1 = ((a_4\beta + \gamma\, q)a_5 + \gamma_a\ell(q + a_4))a_2\lambda_1\, p\, e^{(-\mu_1\tau_1)} + (1-p)\lambda_2\, a_3\, a_1\,(a_5\gamma + \ell\gamma_a)\, e^{(-\mu_2\tau_2)},$$

$$A_2 = ((a_4\beta + \gamma\, q)a_5 + \gamma_a\ell(q + a_4))a_2\lambda_1\, p\, b\, e^{(-\mu_1\tau_1 + \mu_2\tau_2)} + (1-p)\lambda_2\, b\, a_3\, a_1\,(a_5\gamma + \ell\gamma_a),$$

$$A_3 = ((a_4\beta + \gamma\, q)a_5 + \gamma_a\ell(q + a_4))a_2\lambda_1\, p\, e^{(\mu_2\tau_2)} + (1-p)\lambda_2\, a_3\, a_1\,(a_5\gamma + \ell\gamma_a)\, e^{(\mu_1\tau_1)},$$

$$A_4 = ((a_4\beta + \gamma\, q)a_5 + \gamma_a\ell(q + a_4))a_2\lambda_1\, p\, b + (1-p)\lambda_2\, a_3\, a_1\, b\,(a_5\gamma + \ell\gamma_a)\, e^{(\mu_1\tau_1 - \mu_2\tau_2)},$$

$$A_5 = b\,\lambda_2^2\, a_1^2\, a_3^2\,(p-1)^2\,(a_5\gamma + \ell\gamma_a)\, e^{\mu_1\tau_1 - \mu_2\tau_2},$$

$$A_6 = q\, b\, \lambda_1^2\,((a_4\beta + \gamma\, q)\, a_5 + \ell\,\gamma_a\,(q + a_4))\, p^2 a_2\, e^{-\mu_1\tau_1 + \mu_2\tau_2},$$

$$A_7 = -\frac{1}{2}\, d_1\lambda_2\, a_1\, a_3\, a_4\, a_5\,(p-1)\, e^{\mu_1\tau_1} + \lambda_1\left(\frac{1}{2}\, d_1\, e^{\mu_2\tau_2}\, q\, a_2\, a_4\, a_5\right.$$

$$\left. + b\left(\left(\frac{1}{2}\, a_4\beta + \gamma\, q\right)\, a_5 + \ell\gamma_a\left(q + \frac{1}{2}a_4\right)\right)(p-1)\lambda_2\right)\, p,$$

$$A_8 = b\, \lambda_1^2\,((a_4\beta + \gamma\, q)\, a_5 + \ell\,\gamma_a\,(q + a_4))\,(q + a_4)\, p^2 a_2^2\, e^{-\mu_1\tau_1 + \mu_2\tau_2},$$

$$A_9 = -\frac{1}{2}(b\lambda_2^2\, a_1\, a_3\,(p-1)^2\,(a_5\gamma + \ell\gamma_a)\, e^{\mu_1\tau_1 - \mu_2\tau_2}),$$

$$A_{10} = -\frac{1}{2}\, d_1\lambda_2\, a_1\, a_3\, a_4 a_5\,(p-1)\, e^{\mu_1\tau_1} + \lambda_1\frac{1}{2}\, d_1\, a_2\, a_4\, a_5\,(q + a_4)\, e^{\mu_2\tau_2}$$

$$+ b\,(p-1)\left(\left(\left(\frac{1}{2}\beta + \frac{1}{2}\gamma\right)\, a_4 + \gamma\right)\, a_5 + \ell\gamma_a\,(q + a_4)\right)\lambda_2\, p,$$

$$A_{11} = b\, \lambda_1^2\,((a_4\gamma_1 + \gamma_2\, q)\, a_5 + \gamma_r\gamma_a\,(q + a_4))\, p^2\, a_2^2\,((a_4\beta + \gamma\, q)\, a_5 + \ell\gamma_a\,(q + a_4))\, e^{-\mu_1\tau_1 + \mu_2\tau_2},$$

$$A_{12} = -b\lambda_2^2\, a_1\, a_3\,(p-1)^2\,(a_5\gamma_2 + \gamma_a\gamma_r)\,(a_5\gamma + \ell\,\gamma_a)\, e^{\mu_1\tau_1 - \mu_2\tau_2},$$

$$A_{13} = -d_1\lambda_2\, a_1\, a_3\, a_4\, a_5\,(p-1)\,(a_5\,\gamma_2 + \gamma_a\gamma_r)\, e^{\mu_1\tau_1} + \lambda_1\,(((a_4\gamma_1 + \gamma_2\, q)\, a_5$$

$$+ \gamma_r\gamma_a\,(q + a_4))\, a_4\, a_2\, a_5\, d_1\, e^{\mu_2\tau_2} + \lambda_2\, b\,(((\beta\gamma_2 + \gamma\,\gamma_1)\, a_4 + 2\gamma\,\gamma_2\, q)\, a_5^2$$

$$+ \gamma_a\,(((\beta + \gamma)\gamma_r + \ell(\gamma_1 + \gamma_2))\, a_4 + 2\, q\,(\gamma\,\gamma_r + \gamma_2\ell))\, a_5 + 2\ell\gamma_a^2\gamma_r\,(q + a_4))\,(p-1))\, p.$$

4.1 Calculating the basic reproduction number

We will apply the next generation method [23] to determine the basic reproduction number R_0 for system (10)-(16). We follow the following steps

(i) We evaluated the matrix F at P_0 as:

$$F = \begin{pmatrix} 0 & 0 & p\, e^{-\mu_1\tau_1}\, \beta\, \frac{b}{d_1} & p\, e^{-\mu_1\tau_1}\, \gamma\, \frac{b}{d_1} & p\, e^{-\mu_1\tau_1}\, \ell\, \frac{b}{d_1} \\ 0 & 0 & (1-p)\, e^{-\mu_2\tau_2}\, \beta\, \frac{b}{d_1} & (1-p)\, e^{-\mu_2\tau_2}\, \gamma\, \frac{b}{d_1} & (1-p)\, e^{-\mu_2\tau_2}\, \ell\, \frac{b}{d_1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) We get the matrix V at P_0 as:

$$V = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ -\lambda_1 & 0 & a_3 & 0 & 0 \\ 0 & -\lambda_2 & -q & a_4 & 0 \\ 0 & 0 & -\gamma_a & -\gamma_a & a_5 \end{pmatrix}.$$

(iii) Finally, the basic reproduction number is given by

$$R_0 = \rho(FV^{-1}) = \frac{A_1 \, S_0}{a_0}.$$

4.2   Existence of equilibria

Theorem 2. For system (10)-(16), we have

(i) If $R_0 \le 1$, then there exists only one positive equilibria P_0.

(ii) If $R_0 > 1$, then there exist two positive equilibria P_0 and P^*.

Proof We have

$$S^* = \frac{a_0}{A_1} = \frac{S_0}{R_0},$$

$$E_c^* = \frac{p\,(A_2 - a_0\,d_1\,e^{\mu_2\,\tau_2})}{a_1\,A_3} = \frac{p}{a_1 A_3}\left(\frac{A_2}{e^{\mu_2 \tau_2}} - a_0\,d_1\right) = \frac{p}{a_1 A_3}(b\,A_1 - a_0\,d_1) = \frac{p}{a_1 A_3}(R_0 - 1),$$

$$E_m^* = \frac{(1-p)(A_4 - a_0\,d_1\,e^{\mu_1\tau_1})}{a_2\,A_3} = \frac{(1-p)}{a_2\,A_3}\left(\frac{A_4}{e^{\mu_1\tau_1}} - a_0\,d_1\right),$$

$$= \frac{(1-p)}{a_2\,A_3}(b\,A_1 - a_0\,d_1) = \frac{(1-p)}{a_2\,A_3}(R_0 - 1).$$

From Eq. (13)-(16), we have

$$I_c^* = \frac{\lambda_1}{a_3} E_c^* = \frac{\lambda_1}{a_3}\frac{p}{a_1 A_3}(R_0 - 1) = \frac{\lambda_1\,p}{a_1\,a_3\,A_3}(R_0 - 1), \tag{27}$$

$$I_m^* = \frac{1}{a_4}(\lambda_2\,E_m^* + q\,I_c^*) = C_1(R_0\,-1), \tag{28}$$

$$H^* = \frac{1}{a_5}(\gamma_a(I_c^* + I_m^*)) = C_2\,(R_0\,-1)), \tag{29}$$

$$R^* = \frac{1}{d_7}(\gamma_1 I_c^* + \gamma_2 I_m^* + \gamma_r H^*) = C_3(R_0 - 1), \tag{30}$$

where,

$$\begin{aligned} C_1 &= \frac{1}{a_4}(\lambda_2\,\frac{(1-p)}{a_2\,A_3} + q\,\frac{\lambda_1\,p}{a_1\,a_3\,A_3}), \\ C_2 &= \frac{\gamma_a}{a_5}(\frac{\lambda_1\,p}{a_1\,a_3\,A_3} + \frac{1}{a_4}(\lambda_2\,\frac{(1-p)}{a_2\,A_3} + q\,\frac{\lambda_1\,p}{a_1\,a_3\,A_3})), \\ C_3 &= \frac{1}{d_7}(\gamma_1\,\frac{\lambda_1\,p}{a_1\,a_3\,A_3} + \gamma_2\,C_1\,+\gamma_r\,C_2\,). \end{aligned} \tag{31}$$

$\square$

5 Global stability analysis of P_0

In this section, we use Lyapunov function and LaSalle's invariance principle to establish the global stability of P_0.

Theorem 3. For system (10)-(16), if $R_0 \leq 1$, then P_0 is GAS.

Proof We define the following Lyapunov functional

$$W_0 = S_0 \left(\frac{S}{S_0} - 1 - \ln \frac{S}{S_0} \right) + \varepsilon_1 E_c + \varepsilon_2 E_m + \varepsilon_3 I_c + \varepsilon_4 I_m + \varepsilon_5 H$$

$$+ \varepsilon_6 \int_0^{\tau_1} S(t-s)(\beta I_c(t-s) + \gamma I_m(t-s) + \ell H(t-s)) \, ds$$

$$+ \varepsilon_7 \int_0^{\tau_2} S(t-s)(\beta I_c(t-s) + \gamma I_m(t-s) + \ell H(t-s)) \, ds.$$

The time derivative of W_0 along the trajectory of system (10)-(16) is given by

$$\begin{aligned}
\frac{dW_0}{dt} &= \left(1 - \frac{S_0}{S(t)} \right) (b - S(t)(\beta I_c(t) + \gamma I_m(t) + \ell H(t)) - d_1 S(t)) \\
&\quad + \varepsilon_1 (p\, e^{-\mu_1 \tau_1}\, S(t-\tau_1)(\beta I_c(t-\tau_1) + \gamma I_m(t-\tau_1) + \ell H(t-\tau_1)) - a_1 E_c(t)) \\
&\quad + \varepsilon_2 ((1-p)\, e^{-\mu_2 \tau_2}\, S(t-\tau_2)(\beta I_c(t-\tau_2) + \gamma I_m(t-\tau_2) + \ell H(t-\tau_2)) - a_2 E_m(t)) \\
&\quad + \varepsilon_3 (\lambda_1 E_c(t) - a_3 I_c(t)) + \varepsilon_4 (\lambda_2 E_m(t) - a_4 I_m(t) + q I_c(t)) + \varepsilon_5 (\gamma_a(I_c(t) + I_m) - a_5 H(t)) \\
&\quad + \varepsilon_6 \{ (S(t)(\beta I_c(t) + \gamma I_m(t) + \ell H(t))) - (S(t-\tau_1)(\beta I_c(t-\tau_1) + \gamma I_m(t-\tau_1) + \ell H(t-\tau_1))) \} \\
&\quad + \varepsilon_7 \{ (S(t)(\beta I_c(t) + \gamma I_m(t) + \ell H(t))) - (S(t-\tau_2)(\beta I_c(t-\tau_2) + \gamma I_m(t-\tau_2) + \ell H(t-\tau_2))) \},
\end{aligned} \tag{32}$$

The parameters $\varepsilon_i, \ i = 1, ..., 7$ are chosen such that

$$\varepsilon_6 + \varepsilon_7 = 1, \tag{33}$$

$$p\, \varepsilon_1\, e^{-\mu_1 \tau_1} - \varepsilon_6 = 0, \tag{34}$$

$$(1-p)\, \varepsilon_2\, e^{-\mu_2 \tau_2} - \varepsilon_7 = 0, \tag{35}$$

$$-\varepsilon_1 a_1 + \lambda_1 \varepsilon_3 = 0, \tag{36}$$

$$-\varepsilon_2 a_2 + \lambda_2 \varepsilon_4 = 0, \tag{37}$$

$$-a_3\, \varepsilon_3 + q\, \varepsilon_4 + \gamma_a\, \varepsilon_5 + \beta S_0 = 0, \tag{38}$$

$$-a_4\, \varepsilon_4 + \gamma_a\, \varepsilon_5 + \gamma S_0 = 0. \tag{39}$$

Solving Eqs. (33)-(39), we get

$$\varepsilon_5 = \frac{G(1 - R_0) + \ell S_0}{a_5},$$

where

$$G = \frac{a_1 a_2 a_3 a_4 a_5}{\gamma_a (\lambda_1 p a_2 (a_4 + q) e^{-\mu_1 \tau_1} + \lambda_2 e^{-\mu_2 \tau_2} a_1 a_3 (1-p))}.$$

We can see that $\varepsilon_5 > 0$ if $R_0 \leq 1$.
From Eqs. (34)-(38) we get

$$\varepsilon_4 = \frac{1}{a_4} \left(\gamma_a \, \varepsilon_5 + \gamma S_0 \right) > 0.$$

$$\varepsilon_3 = \frac{1}{a_3} \left(q\varepsilon_4 + \gamma_a \varepsilon_5 + \beta S_0 \right) > 0.$$

$$\varepsilon_2 = \frac{\lambda_2 \varepsilon_4}{a_2} > 0.$$

$$\varepsilon_1 = \frac{\lambda_1 \, \varepsilon_3}{a_1} > 0.$$

$$\varepsilon_7 = (1 - p) \, \varepsilon_2 \, e^{-\mu_2 \tau_2} > 0.$$

$$\varepsilon_6 = p \, \varepsilon_1 \, e^{-\mu_1 \tau_1} > 0,$$

Thus, Eq. (32) becomes

$$\frac{dW_0}{dt} = -b\frac{(S - S_0)^2}{S} + (\ell S_0 - a_5 \varepsilon_5) \, H, \tag{40}$$

we have

$$\ell S_0 - a_5 \varepsilon_5 = G(R_0 - 1).$$

. Then

$$\frac{dW_0}{dt} = -b\frac{(S - S_0)^2}{S} + \frac{G}{a_5} \, (R_0 - 1) \, H, \tag{41}$$

From Eq (41), $\frac{dW_0}{dt} \leq 0$ if $R_0 \leq 1$. Then, $\frac{dW_0}{dt}$ equal to zero if $S = S_0$ and $H = 0$. Let $\Omega = \{(S, E_c, E_m, I_c, I_m, H, R) : S = S_0, H = 0\}$. From system (10)-(16), if $H = 0$, then $\dot{H} = 0$ and $0 = \gamma_a(I_c + I_m)$. Since, $I_c \geq 0$, $I_m \geq 0$ then $I_c = 0, I_m = 0, \Rightarrow \dot{I}_c = \dot{I}_m = 0$. From system (10)-(16), we have $0 = \dot{I}_c = \lambda_1 E_c \Rightarrow E_c = 0$. Similarly, we have $0 = \dot{I}_m = \lambda_2 E_m \Rightarrow E_m = 0$. Finally, $\dot{R}(t) = -d_7 R$ it follows that $R \to 0$ as $t \to \infty$. From LaSalle's invariance principle, P_0 is GAS in Γ. $\square$

6 Numerical simulations and discussions

In this section, we introduce the numerical results of system (10)-(16). We consider the following initial conditions

IC $: S(\theta) = 600, \ E_c(\theta) = 30, \ E_m(\theta) = 80, \ I_c(\theta) = 3, \ I_m(\theta) = 12, \ H(\theta) = 8, \ R(\theta) = 40,$

$\theta \in [-\max\{\tau_1, \tau_2\}, 0]$. we use the values of the parameters in Table 1. In addition we choose $\mu_1 = \mu_2 = 1$.

We study the following cases:

6.1 Effect of parameters β, γ and ℓ on the stability of equilibria:

In this case, we fix the values $\tau_1 = \tau_2 = 0.01$. Figure 1 shows the evaluation of system states for two scenarios:

i) $R_0 \leq 1$. We choose $\beta = 0.002$, $\gamma = 0.0001$, and $\ell = 0.0001$ then we compute $R_0 = 0.23$. We can see from the figure that the states of the system approach $P_0 = (1000, 0, 0, 0, 0, 0, 0)$. This means that according to Theorem 3 P_0 is GAS.

ii) $R_0 > 1$. We choose $\beta = 0.02$, $\gamma = 0.001$, and $\ell = 0.001$ then we compute $R_0 = 2.37$ and $P^* = (421.6, 55.4, 129.2, 5.08, 20.9, 14.5, 72.4)$. Then P^* exists and this confirm the results of Theorem 2. Figure 1 shows that the states of the system converge to P^*.

Table 1: The parameters values of MERS-CoV model

Symbol	Parameter	Value
b	Rate of generation of new susceptible individuals	100
β	Rate constant of transmission for carriers	Varied
γ	Rate constant of symptomatically infected individuals	Varied
ℓ	Relative transmissibility of hospitalized cases	Varied
γ_a	Mean time from carrier and infected to hospital admission (days)	0.3
d_1	Death rate of susceptible individuals	0.1
d_2	Death rate of exposed to carrier	0.2
d_3	Death rate of exposed to infected individuals	0.2
d_4	Death rate of carrier individuals	0.2
d_5	Death rate of infected individuals	0.3
d_6	Death rate of hospitalized individuals	0.4
d_7	Death rate of recovered individuals	0.1
k	Mean latent period	0.19
$\rho_1 = \rho_2$	Proportion of carrier and infected cases	0.58
$\gamma_1 = \gamma_2$	Mean infectious period	0.2
γ_r	Mean length of hospital stay	0.14
p	Rate of infected individual who becomes carrier	0.3
q	Rate of carrier individual who becomes infected	0.5

6.2 Effect of the time delays on the asymptotic behaviour of the equilibria:

In this case, we take the values $\beta = 0.02$, $\gamma = 0.001$, and $\ell = 0.001$. Let us consider the case $\tau_1 = \tau_2 = \tau$. In Table 2, we present the values of R_0 and the equilibria of system (10)-(16) with different values of τ. From

Table 2: Values of R_0 and steady states of system (10)-(16) with different values of τ

τ	R_0	Steady states
0.067	2.24	$P^* = (446.37, 50.07, 116.83, 4.6, 18.97, 13.09, 65.46)$
0.082	2.21	$P^* = (453.12, 48.73, 113.69, 4.47, 18.97, 12.74, 63.70)$
0.67	1.23	$P^* = (815.79, 9.12, 21.27, 0.84, 3.45, 2.38, 11.92)$
0.8735928143	1.00	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
1.2	0.72	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
1.5	0.5	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
2.5	0.19	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
3.1	0.11	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
3.5	0.07	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$

Table 2, we can observe that the value of R_0 is decreased as τ is increased. Moreover, for small values of τ, P^* exists and for large values of τ the system moved from P^* to P_0 with is GAS. Figures 2 shows the effect of the parameter τ on the evaluation of the states of the system. We can see that as the time delay parameter is increased, the number of susceptible individuals are increased and tend to its normal number, while the number

of individuals in other groups is reduced and tends to zero. It means that, the time delay play the role of controlling the disease transmission.
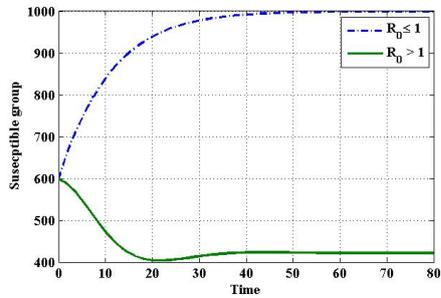
(a) Evaluation of $S(t)$.
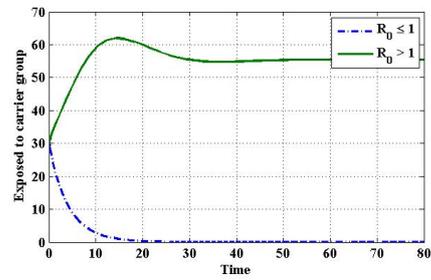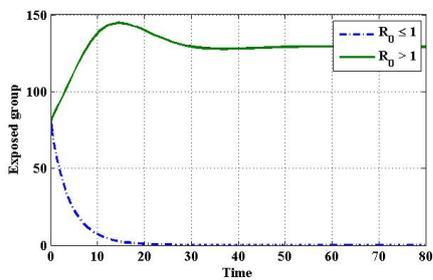
(b) Evaluation of $E_c(t)$.
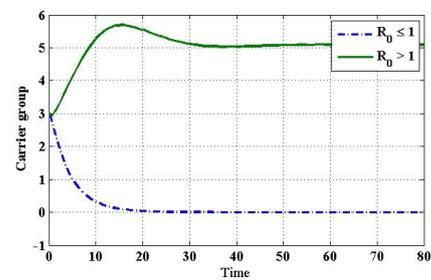
(c) Evaluation of $E_m(t)$.
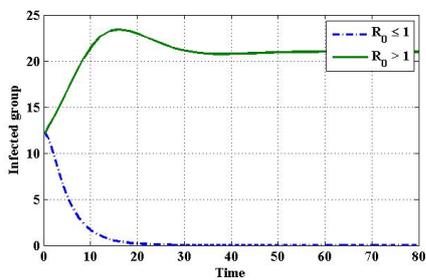
(d) Evaluation of $I_c(t)$.

(e) Evaluation of carrier $I_m(t)$.
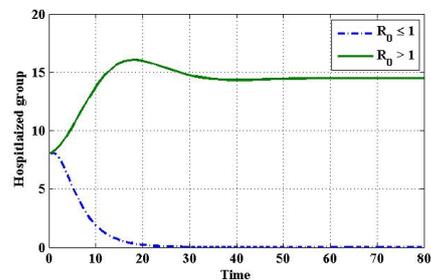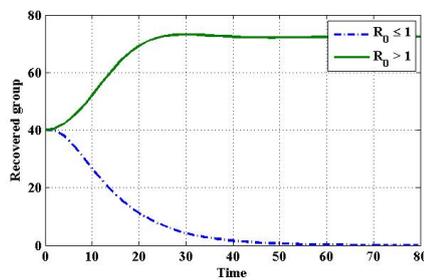
(f) Evaluation of $H(t)$.

(g) Evaluation of $R(t)$.

Figure 1: The evaluations of the system states (10)-(16) with two delays $\tau_1 = \tau_2 = 0.01$.
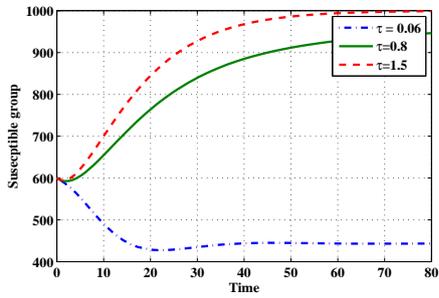
.

(a) Evaluation of $S(t)$.
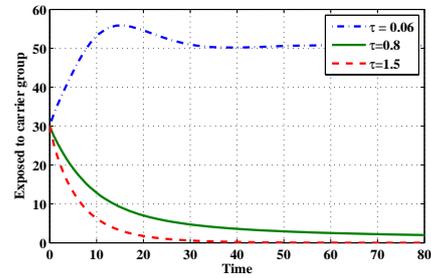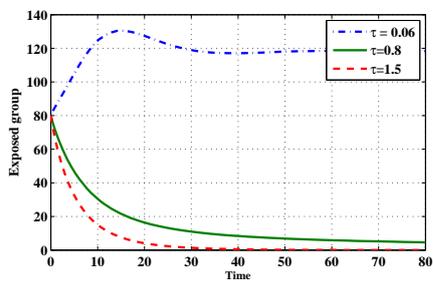
(b) Evaluation of $E_c(t)$.
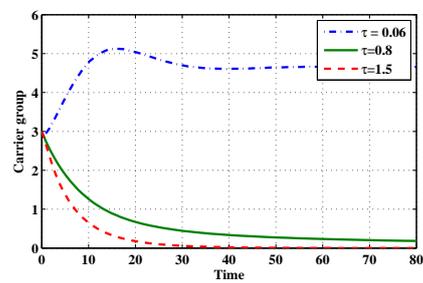
(c) Evaluation of $E_m(t)$.
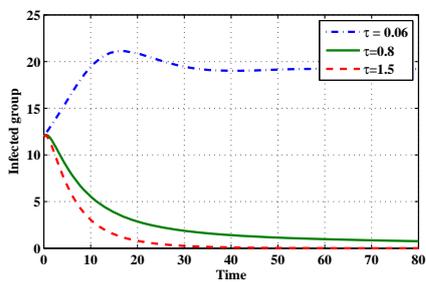
(d) Evaluation of $I_c(t)$.

(e) Evaluation of carrier $I_m(t)$.
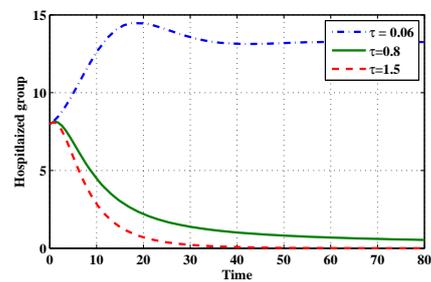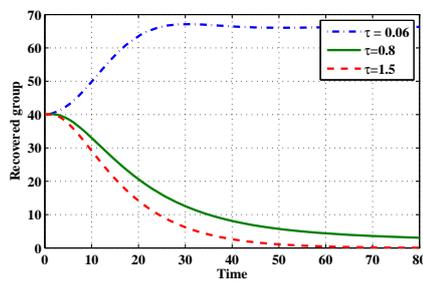
(f) Evaluation of $H(t)$.

(g) Evaluation of $R(t)$.

Figure 2: The evaluations of system (10) -(16) with different values of τ.

7 Conclusion

We have proposed a MERS-CoV model with two times delay. We have obtained the biological threshold, the basic reproduction number R_0. The existence of the model's equilibria has been proven. The global asymptotic stability of the disease free equilibria P_0 has been investigated by constructing Lyapunov functional and using LaSalle's invariance principle. To support our theoretical results, we have presented the numerical simulations.

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CONVEXITY AND HYPERCONVEXITY IN FUZZY METRIC SPACE

EBRU YİĞİT AND HAKAN EFE

ABSTRACT. In this paper, firstly we give the definion of fuzzy convex metric, in a different way. Then we introduce the concept of hyperconvexity in fuzzy metric space and prove that every fuzzy hyperconvex metric space is complete. Also it is proved that for $m-$seperable fuzzy metric spaces, fuzzy $m-$hyperconvexity is equivalent to fuzzy hyperconvexity.

1. INTRODUCTION

The concept of convex metric space has been studied by many authors, in some different ways [7, 9, 11, 14, 15]. After that, some authors examined this concept for fuzzy metric space by using the definition of fuzzy metric which is introduced by George and Veeremani [1], for example; Thanithamil [4] introduced the convex structure in fuzzy metric spaces and Vosoughi and Hosseni [8] gave the definion of metrically convex fuzzy metric space $(X, M, *)$. The other common concept for metric space is hyperconvexity which was introduced by Aronszajn and Panitch-pakdi [10] in 1956. Since then many interesting works have been appeared for hyperconvex spaces [5, 11, 13].

In this paper, we give the notion of fuzzy convex metric space by using the closed balls, in a different way. Also, we introduce a new notion for fuzzy metric space which is called fuzzy hyperconvex metric space. One of the main result of this paper is that every fuzzy hyperconvex metric space is complete. Also, the fuzzy $m-$hyperconvexity is introduced for any cardinal $m \geq 3$, which is a weaker property than fuzzy hyperconvexity. The definition $m-$seperability for fuzzy metric space is used, so the other result for this paper is that for any $m-$seperable fuzzy metric spaces, fuzzy $m-$hyperconvexity is equivalent to fuzzy hyperconvexity.

2. PRELIMINARIES

Definition 1. [6] *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is continuous t-norm if* $*$ *satisfies the following conditions:*

(i) $*$ *is commutative and associative;*

(ii) $*$ *is continuous;*

(iii) $a * 1 = a$ *for all* $a \in [0,1]$;

(iv) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for* $a, b, c, d \in [0,1]$.

Remark 1. [1] *(i) For any* $r_1 \in (0,1)$ *with* $r_1 > r_2$, *there exist* $r_3 \in (0,1)$ *such that* $r_1 * r_3 \geq r_2$.

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(ii) For any $r_4 \in (0,1)$, there exist $r_5 \in (0,1)$ such that $r_5 * r_5 \geq r_4$.

Definition 2. [1] *The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:*

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
(FM-5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 1. [1] *(Induced fuzzy metric). Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ for all $\forall a, b \in [0, 1]$ and let M be fuzzy set on $X \times X \times (0, \infty)$ as follows:*

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}, \ k, m, n \in \mathbb{R}^+.$$

Then $(X, M, *)$ is a fuzzy metric space. In this example by taking $k = m = n = 1$, we get

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

We call this fuzzy metric induced by a metric d the standard fuzzy metric.

Definition 3. [1] *Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The open ball and the closed ball with center x and radius r with respect to t are defined as follows, respectively*

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

$$\bar{B}_M(x, r, t) = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

Remark 2. [1] *Every open ball is an open set and every closed ball is a closed set in a fuzzy metric space $(X, M, *)$.*

Theorem 1. [1] *Let $(X, M, *)$ be a fuzzy metric space. Define*

$$\tau_M = \{A \subset X : \forall \ x \in A, \ \exists r \in (0, 1) \ and \ t > 0 \ \ni \ B_M(x, r, t) \subset A \}.$$

Then τ_M is a topology on X.

Definition 4. [1] *Let $(X, M, *)$ be a fuzzy metric space. Then*
(a) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m \geq n_0$.
(b) $(X, M, *)$ is called complete if every Cauchy sequence is convergent with respect to τ_M.

Definition 5. [3] *Let $(X, M, *)$ be a fuzzy metric space. A collection of sets $\{F_n\}_{n \in \mathbb{N}}$ is said to have fuzzy diameter zero if and only if for each pair $r, t > 0$, ($r \in (0, 1)$ and $t > 0$), there exists $n \in \mathbb{N}$ such that $M(x, y, t) > 1 - r$ for all $x, y \in F_n$.*

Remark 3. [3] *A non-empty subset F of a fuzzy metric space X has fuzzy diameter zero if and only if F is a singleton set.*

Theorem 2. [3] *A necessary and sufficient condition that a fuzzy metric space* $(X, M, *)$ *be complete is that every nested sequence of non-empty closed sets* $\{F_n\}_{n=1}^{\infty}$ *with fuzzy diameter zero have non-empty intersection. And the element* $x \in \bigcap\limits_{n \in \mathbb{N}} F_n$ *is unique.*

Definition 6. [12] *Let* $(X, M, *)$ *be a fuzzy metric space. Let the mappings* $\delta_A(t) : (0, \infty) \longrightarrow [0, 1]$ *be defined as*

$$\delta_A(t) = \inf_{x,y \in A} \sup_{\varepsilon < t} M(x, y, \varepsilon).$$

The constant $\delta_A = \sup\limits_{t > 0} \delta_A(t)$ *will be called fuzzy diameter of set* A. *If* $\delta_A = 1$ *the set* A *will be called F-strongly bounded.*

Definition 7. [11] *Let* (X, d) *be a metric space. We say that* X *is metrically convex if for any points* $x_1, x_2 \in X$ *and positive numbers* α *and* β *such that* $d(x_1, x_2) \leq \alpha + \beta$, *there exists* $z \in X$ *such that* $d(x_1, z) \leq \alpha$ *and* $d(x_2, z) \leq \beta$, *or equivalently* $z \in \bar{B}(x_1, \alpha) \cap \bar{B}(x_2, \beta)$.

Definition 8. [11] *Let* (X, d) *be a metric space and* Γ *be an index set. The metric space* X *is said to has the ball intersection property (BIP in short) if* $\bigcap\limits_{\alpha \in \Gamma} \bar{B}_\alpha \neq \varnothing$ *for any collection of closed balls* $(\bar{B}_\alpha)_{\alpha \in \Gamma}$ *such that* $\bigcap\limits_{\alpha \in \Gamma_f} \bar{B}_\alpha \neq \varnothing$, *for any finite subset* $\Gamma_f \subset \Gamma$.

Definition 9. [11] *Let* (X, d) *be a metric space and* Γ *be an index set. The metric space* X *is said to be hyperconvex if* $\bigcap\limits_{\alpha \in \Gamma} \bar{B}(x_\alpha, r_\alpha) \neq \varnothing$ *for any collection of points* $\{x_\alpha\}_{\alpha \in \Gamma}$ *in* X *and positive numbers* $\{r_\alpha\}_{\alpha \in \Gamma}$ *such that* $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ *for any* α *and* β *in* Γ.

Example 2. [11] *The real line* $\mathbb{R}$ *is hyperconvex with the usual metric* d.

Example 3. [11] *The infinite dimensional Banach space* l_∞ *is hyperconvex.*

Definition 10. [10] *A metric space* (X, d) *is called* $m-$*seperable if it contains a dense subset of cardinal* $< m$.

3. MAIN RESULTS

Before we give the definition of fuzzy metrically convexity, we give the following Lemma for the definition to be clear.

Lemma 1. *Let* $(X, M, *)$ *be a fuzzy metric space,* $x_1, x_2 \in X$, $r_1, r_2 \in (0, 1)$ *and* $t_1, t_2 \in (0, \infty)$. *If* $\bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \neq \varnothing$ *then* $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$ *for any* $x_1, x_2 \in X$ *and each pair of* $r_1, t_1 > 0$ *and* $r_2, t_2 > 0$.

Proof. Let $\bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \neq \varnothing$. Then there exists $z \in X$ such that

$$
\begin{aligned}
z \quad &\in \quad \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \\
&\implies \quad z \in \bar{B}_M(x_1, r_1, t_1) \text{ and } z \in \bar{B}_M(x_2, r_2, t_2) \\
&\implies \quad M(x_1, z, t_1) \geq (1 - r_1) \text{ and } M(x_2, z, t_2) \geq (1 - r_2).
\end{aligned}
$$

By the Definition 1-(vi) we have $M(x_1, z, t_1) * M(x_2, z, t_2) \geq (1 - r_1) * (1 - r_2)$ and by the condition (FM-4) of fuzzy metric we get $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$. $\square$

The converse of Lemma 1 may not be true. Example 4 explain this situation.

Example 4. *Let $X = \mathbb{N}$. Define $a * b = a.b$ for all $\forall a, b \in [0, 1]$ and let M be fuzzy set on $\mathbb{N} \times \mathbb{N} \times (0, \infty)$ as follows:*

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}.$$

In this case we know that M is a fuzzy metric on $\mathbb{N}$. If we choose $t_1 = 1$, $t_2 = 1$, $r_1 = 0.3$, $r_2 = 0.5$, $x_1 = 3$ and $x_2 = 10$ then the inequality $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$ is satisfied but $\bar{B}_M(3, 0.3, 1) \cap \bar{B}_M(10, 0.5, 1) = \varnothing$ and so we can not find any point $z \in X$ such that $M(x_1, z, t_1) \geq (1 - r_1)$ and $M(x_2, z, t_2) \geq (1 - r_2)$.

Consequently, when the converse of Lemma 1 also be true, we give Definition 11.

Definition 11. *Let $(X, M, *)$ be a fuzzy metric space. We say that X is fuzzy metrically convex if for any points $x_1, x_2 \in X$ and for each pair $r_1, t_1 > 0$ and $r_2, t_2 > 0$ ($r_1, r_2 \in (0, 1)$ and $t_1, t_2 \in (0, \infty)$) such that $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$, there exists $z \in X$ such that $M(x_1, z, t_1) \geq (1 - r_1)$ and $M(x_2, z, t_2) \geq (1 - r_2)$ or equivalently $z \in \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2)$.*

Example 5. *Let the metric space (X, d) be metrically convex. Define continuous t-norm as $a * b = a.b$ for all $\forall a, b \in [0, 1]$ and let M be fuzzy set on $X \times X \times (0, \infty)$ as follows:*

$$M(x, y, t) = e^{\frac{-d(x,y)}{t}}.$$

Then the 3-tuple $(X, M, *)$ is a fuzzy metric space and under these conditions $(X, M, *)$ is fuzzy metrically convex. Indeed, let (X, d) be metrically convex then for any points $x_1, x_2 \in X$ and positive numbers α and β such that $d(x_1, x_2) \leq \alpha + \beta$, there exists $z \in X$ such that $d(x_1, z) \leq \alpha$ and $d(x_2, z) \leq \beta$, or equivalently $z \in B(x_1, \alpha) \cap B(x_2, \beta)$. Take $\alpha = -t_1 \ln(1 - r_1)$ and $\beta = -t_2 \ln(1 - r_2)$. By the choices of α, β, the inequality $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$ is satisfied and also r_1, $r_2 \in (0, 1)$. By using the metrically convexity of (X, d);

$$
\begin{aligned}
d(x_1, z) \quad &\leq \quad -t_1 \ln(1 - r_1) \text{ and } d(x_2, z) \leq -t_2 \ln(1 - r_2) \\
&\implies \quad -d(x_1, z) \geq t_1 \ln(1 - r_1) \text{ and } -d(x_2, z) \geq t_2 \ln(1 - r_2) \\
&\implies \quad e^{-d(x_1, z)} \geq e^{t_1 \ln(1 - r_1)} \text{ and } e^{-d(x_2, z)} \geq e^{t_2 \ln(1 - r_2)} \\
&\implies \quad e^{\frac{-d(x_1, z)}{t_1}} \geq (1 - r_1) \text{ and } e^{\frac{-d(x_2, z)}{t_2}} \geq (1 - r_2) \\
&\implies \quad M(x_1, z, t_1) \geq (1 - r_1) \text{ and } M(x_2, z, t_2) \geq (1 - r_2).
\end{aligned}
$$

This implies that $z \in \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2)$, then the fuzzy metric space $(X, M, *)$ is fuzzy metrically convex.

Definition 12. *Let $(X, M, *)$ be a metric space, Γ be an index set, $r_i \in (0, 1)$ and $t_i \in (0, \infty)$ for all $i \in \Gamma$. The fuzzy metric space X is said to has the ball intersection property (BIP in short) if $\bigcap\limits_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for any collection of closed balls $(\bar{B}_M(x_i, r_i, t_i))_{i \in \Gamma}$ such that $\bigcap\limits_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for any finite subset $\Gamma_f \subset \Gamma$.*

Definition 13. *Let $(X, M, *)$ be a metric space, Γ be an index set, $r_i \in (0, 1)$ and $t_i \in (0, \infty)$ for all $i \in \Gamma$. The fuzzy metric space X is said to be fuzzy hyperconvex*

if for any indexed class of closed balls $\bar{B}_M(x_i, r_i, t_i)$ in X, satifying the condition that

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$$

for all $i, j \in \Gamma$, the intersection $\bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$.

Theorem 3. *Let $(X = \mathbb{R}, d)$ be the usual metric space. Consider the standard fuzzy metric M where $M(x, y, t) = \frac{t}{t + d(x,y)}$ with $a * b = \min\{a, b\}$ for all $a, b \in (0, 1)$. Then $(X, M, *)$ is fuzzy metrically hyperconvex (or fuzzy hyperconvex).*

Proof. Since $(\mathbb{R}, d)$ is hyperconvex, for any collection of closed balls $\bar{B}(x_\alpha, r_\alpha)$ satisfying the condition that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any α and β in Γ, the intersection $\bigcap_{\alpha \in \Gamma} \bar{B}(x_\alpha, r_\alpha) \neq \varnothing$. Now choose $R_\alpha = \frac{r_\alpha}{t_\alpha + r_\alpha}$ ve $R_\beta = \frac{r_\beta}{t_\beta + r_\beta}$. It is clear that R_α, $R_\beta \in (0, 1)$ and by these choices and the minimum t-norm, $M(x_\alpha, x_\beta, t_\alpha + t_\beta) \geq (1 - R_\alpha) * (1 - R_\beta) = \min\{(1 - R_\alpha), (1 - R_\beta)\} = (1 - R_\alpha)$ (without lost generality we can take $R_\alpha \geq R_\beta$) is satisfied. By the hyperconvexity of $(\mathbb{R}, d)$;

$$\bigcap_{\alpha \in \Gamma} \bar{B}(x_\alpha, r_\alpha) \quad \neq \quad \varnothing, \text{ for all } \alpha \in \Gamma \text{ , then there exsists } z \in X \text{ such that}$$

$$\begin{aligned}
z \quad &\in \quad \bigcap_{\alpha \in \Gamma} \bar{B}(x_\alpha, r_\alpha) \\
\implies \quad &d(x_\alpha, z) \leq r_\alpha \\
\implies \quad &t_\alpha + d(x_\alpha, z) \leq t_\alpha + r_\alpha \\
\implies \quad &\frac{t_\alpha}{t_\alpha + d(x_\alpha, z)} \geq \frac{t_\alpha}{t_\alpha + r_\alpha} \\
\implies \quad &M(x_\alpha, z, t_\alpha) \geq 1 - R_\alpha \\
\implies \quad &z \in \bar{B}_M(x_\alpha, r_\alpha, t_\alpha), \text{ for all } \alpha \in \Gamma \\
\implies \quad &z \in \bigcap_{i \in \Gamma} \bar{B}_M(x_\alpha, r_\alpha, t_\alpha).
\end{aligned}$$

So $(\mathbb{R}, M, *)$ is fuzzy hyperconvex. $\qquad\square$

Example 6. *In particular if we take $t = 1$ in the Theorem 3 M becomes $M(x, y) = \frac{1}{1 + d(x,y)}$. M is stationary fuzzy metric on $\mathbb{R}$ with the continuous minimum t-norm and $(\mathbb{R}, M, *)$ is fuzzy hyperconvex.*

Proposition 1. *If the space $(X, M, *)$ is fuzzy hyperconvex then it has the ball intersection property.*

Proof. Let $(X, M, *)$ be fuzzy hyperconvex and be $\bigcap_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for any finite subset $\Gamma_f \subset \Gamma$. Then it follows that

$$\bigcap_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) \quad \neq \quad \varnothing, \text{ then there exists } z \in X \text{ such that}$$

$$\begin{aligned}
z \quad &\in \quad \bigcap_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) \text{ for } i = \{1, 2, ..., n\} \\
\implies \quad &z \in \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \cap ... \cap \bar{B}_M(x_n, r_n, t_n) \\
\implies \quad &M(x_1, z, t_1) \geq (1 - r_1), M(x_2, z, t_2) \geq (1 - r_2), ..., M(x_n, z, t_n) \geq (1 - r_n)
\end{aligned}$$

so $M(x_i, z, t_i) \geq (1 - r_i)$ and $M(x_j, z, t_j) \geq (1 - r_j)$ for arbitrary $i, j \in \Gamma_f$. By the condition (FM-4)

$$(3.1) \qquad M(x_i, x_j, t_i + t_j) \geq M(x_i, z, t_i) * M(x_j, z, t_j) \geq (1 - r_i) * (1 - r_j).$$

Since $(X, M, *)$ is fuzzy hyperconvex and the inequality (3.1) is satisfied for all $i, j \in \Gamma$, then $\bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for all $i \in \Gamma$ and so $(X, M, *)$ has the ball intersection property. $\square$

Theorem 4. *Any fuzzy metric space $(X, M, *)$ which has the ball intersection property is complete. In particular any fuzzy hyperconvex metric space is complete.*

Proof. Let $(X, M, *)$ be a fuzzy metric space which has ball intersection property and let $\{x_n\}$ be a Cauchy sequence in X. For any $n \geq 1$, take the set

$$r_n = \sup_{t_n > 0} \left\{ \inf_{m \geq n} \left\{ \sup_{s < t_n} \{M(x_n, x_m, s)\} \right\} \right\}.$$

Consider the collection of closed balls $\left(\bar{B}_M(x_n, r_n, t_n) \right)_{n \geq 1}$. Since $\{x_n\}$ is Cauchy and by the choice of r_n, for $m \geq n$ we have $M(x_n, x_m, t_n) \geq 1 - r_n$ i.e $\{r_n\}$ has fuzzy diameter zero. Now we examine this situation for any finite index $n_1 < n_2 < ... < n_k$. For $n_1 < n_2 < ... < n_k$, we have

$$M(x_{n_1}, x_{n_k}, t_{n_1}) \geq 1 - r_{n_1}, \ M(x_{n_2}, x_{n_k}, t_{n_2}) \geq 1 - r_{n_2}, \ ..., \ M(x_{n_k}, x_{n_k}, t_{n_k}) \geq 1 - r_{n_k}$$

which means that

$$
\begin{aligned}
x_{n_1}, x_{n_2}, ..., x_{n_k} &\in \bar{B}_M(x_{n_1}, r_{n_1}, t_{n_1}) \\
x_{n_2}, ..., x_{n_k} &\in \bar{B}_M(x_{n_2}, r_{n_2}, t_{n_2}) \\
&... \\
x_{n_k} &\in \bar{B}_M(x_{n_k}, r_{n_k}, t_{n_k})
\end{aligned}
$$

therefore

$$x_{n_k} \in \bar{B}_M(x_{n_1}, r_{n_1}, t_{n_1}) \cap \bar{B}_M(x_{n_2}, r_{n_2}, t_{n_2}) \cap ... \cap \bar{B}_M(x_{n_k}, r_{n_k}, t_{n_k}).$$

Since X has the ball intersection property, then we may conclude that $\bigcap_{n \geq 1} \bar{B}_M(x_n, r_n, t_n) \neq \varnothing$ for any $n \in \mathbb{N}$. Since $\{x_n\}$ is a Cauchy sequence and $\{r_n\}$ has fuzzy diameter zero, the intersection $\bigcap_{n \geq 1} \bar{B}_M(x_n, r_n, t_n)$ is reduced to one point z which is the limit of the sequence $\{x_n\}$. So indeed, the point $z \in \bigcap_{n \geq 1} \bar{B}_M(x_n, r_n, t_n)$ then for each pair of $r_n, t_n > 0$ there exists $n_1 \in \mathbb{N}$ such that $M(x_n, z, t_n) > 1 - r_n$ for all $n \geq n_1$. Therefore, $M(x_n, z, t_n)$ converges to 1 when $n \longrightarrow \infty$, for each $t_n > 0$ and $(X, M, *)$ is complete. $\square$

Proposition 2. *Fuzzy hyperconvexity is equivalent to the ball intersection property and fuzzy metrically convexity.*

Proof. If $(X, M, *)$ is fuzzy hyperconvex, by Proposition 1 X satisfies the ball intersection property and it is easy to see that X is fuzzy convex metric space. Conversely, if two closed balls $\bar{B}_M(x_i, r_i, t_i)$ and $\bar{B}_M(x_j, r_j, t_j)$ satisfy the relation $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$, they must intersect since X has ball intersection property. $\square$

Now we give the definition of fuzzy $m-$hyperconvexity. Note that fuzzy m-hyperconvexity is a weaker property than fuzzy hyperconvexity. The definitions of fuzzy hyperconvexity and fuzzy $m-$hyperconvexity can be considered structurally similar.

Definition 14. *Let $(X, M, *)$ be a metric space, Γ be an index set such that $card(\Gamma) < m$, $r_i \in (0, 1)$ and $t_i \in (0, \infty)$ for all $i \in \Gamma$. The fuzzy metric space X is said to be fuzzy $m-$hyperconvex if for any indexed class of closed balls $\bar{B}_M(x_i, r_i, t_i)$ in X, satifying the condition that*

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$$

for all $i, j \in \Gamma$, the intersection $\bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$.

Proposition 3. *(i) It is clear that fuzzy hyperconvexity is stronger than fuzzy $m-$hyperconvexity, which is stronger than fuzzy $n-$hyperconvexity if $n < m$.*
(ii) It is easy to see that every fuzzy metric space $(X, M, *)$ is fuzzy $1-$hyperconvex.

Theorem 5. *For $m = 3$, fuzzy $3-$hyperconvexity is equivalent to fuzzy metrically convexity.*

Proof. Let $(X, M, *)$ be fuzzy $3-$hyperconvex. Since $card(\Gamma) < m = 3$, the index set Γ is $\Gamma = \{1, 2\}$. It follows that for any points $x_1, x_2 \in X$ and for each pair $r_1, t_1 > 0$ and $r_2, t_2 > 0$ ($r_1, r_2 \in (0, 1)$ and $t_1, t_2 \in (0, \infty)$) such that $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$, there exists $z \in X$ such that $M(x_1, z, t_1) \geq (1 - r_1)$ and $M(x_2, z, t_2) \geq (1 - r_2)$. This means that $(X, M, *)$ is fuzzy metrically convex.

Conversely, Let $(X, M, *)$ be fuzzy metrically convex. Then for $\Gamma = \{1, 2\}$, we have $\bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \neq \varnothing$. So for any indexed class of closed balls $\bar{B}_M(x_i, r_i, t_i)$ in X, satifying the condition that

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$$

for all $i, j \in \Gamma = \{1, 2\}$, the intersection $\bigcap_{i=1}^{2} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$. So $(X, M, *)$ is fuzzy $3-$hyperconvex. $\square$

Definition 15. *A fuzzy metric space $(X, M, *)$ is called $m-$seperable if it contains a dense subset of cardinal $(K) < m$ where $K \subset \Gamma$, Γ is index set. (This definition is the same with Definition 10 except for the spaces.)*

Note that when $n < m$, $m-$seperability is weaker than $n-$seperability for any fuzzy metric space $(X, M, *)$. $m-$seperability for a finite cardinal m means that the fuzzy metric space $(X, M, *)$ is a finite set, and at the same time it contains at most $m - 1$ points.

Theorem 6. *If the fuzzy metric space $(X, M, *)$ is fuzzy $m-$hyperconvex and at the same time $m-$seperable, then it is fuzzy hyperconvex.*

Proof. Consider an arbitrary indexed family of closed balls $\bar{B}_M(x_i, r_i, t_i)$ satisfying the condition that $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$, for all $i, j \in \Gamma$. Let X be fuzzy m-hyperconvex and let $\{p_k\}$, $k \in K$ with $card(K) < m$, be an indexed set of points, which is dense in X. Take the pair of $r_k^{'}$, $t_k^{'} > 0$ as follows, respectively

(3.2) $r_k^{'}, t_k^{'} = $ $''$the infimum of all $r \in (0, 1)$ and the infimum of all $t > 0$

    such that $\exists\, i \in \Gamma$ with $\bar{B}_M(x_i, r_i, t_i) \subset \bar{B}_M(p_k, r, t)''$.

Now we claim that the class of closed balls $\bar{B}_M(p_k, r_k', t_k')$, $k \in K$, satisfies the requirement of fuzzy $m-$hyperconvexity. So indeed, take any indices $k, l \in K$ and arbitrary $\varepsilon \in (0, 1)$ and arbitrary $\varepsilon' > 0$. By (3.2) there exist $i, j \in \Gamma$ such that

$$(3.3) \qquad \bar{B}_M(x_i, r_i, t_i) \subset \bar{B}_M(p_k, r_k' + \varepsilon, t_k' + \varepsilon')$$

and

$$(3.4) \qquad \bar{B}_M(x_j, r_j, t_j) \subset \bar{B}_M(p_l, r_l' + \varepsilon, t_l' + \varepsilon').$$

Since X is fuzzy $m-$hyperconvex, there exist a point q in $\bar{B}_M(x_i, r_i, t_i) \cap \bar{B}_M(x_j, r_j, t_j)$, at the same time by (3.3), (3.4) q is in $\bar{B}_M(p_k, r_k' + \varepsilon, t_k' + \varepsilon') \cap \bar{B}_M(p_l, r_l' + \varepsilon, t_l' + \varepsilon')$. Then,

$$
\begin{aligned}
q \quad &\in \quad \bar{B}_M(p_k, r_k' + \varepsilon, t_k' + \varepsilon') \cap \bar{B}_M(p_l, r_l' + \varepsilon, t_l' + \varepsilon') \\
&\implies \quad M(p_k, q, t_k' + \varepsilon') \geq 1 - (r_k' + \varepsilon) \text{ and } M(p_l, q, t_l' + \varepsilon') \geq 1 - (r_l' + \varepsilon) \\
(3.5) \quad &\implies \quad M(p_k, p_l, t_k' + t_l' + 2\varepsilon') \geq \left[1 - (r_k' + \varepsilon)\right] * \left[1 - (r_l' + \varepsilon)\right].
\end{aligned}
$$

Since $\varepsilon \in (0, 1)$ and $\varepsilon' > 0$ are arbitrary, by (3.5) we find the requirement for $m-$hyperconvexity for the collection of closed balls $\bar{B}_M(p_k, r_k', t_k')$, $k \in K$. So, there is a point x in $\bigcap_{k \in K} \bar{B}_M(p_k, r_k', t_k')$.

Now we need to show that $x \in \bar{B}_M(x_i, r_i, t_i)$, for all $i \in \Gamma$, i.e. $M(x, x_i, t_i) \geq 1 - r_i$ to see the fuzzy hyperconvexity of X. For this, take an arbitrary $\varepsilon \in (0, 1)$ and arbitrary $\varepsilon' > 0$. Since the set $\{p_k\}$, $k \in K$ is dense in X, there exists a point p_k for each $x_i \in X$ such that

$$(3.6) \qquad M(x_i, p_k, \varepsilon') > 1 - \varepsilon.$$

Therefore

$$\bar{B}_M(x_i, r_i, t_i) \subset \bar{B}_M(p_k, r_i + \varepsilon, t_i + \varepsilon').$$

Due to the choices of r_k' and t_k', it follows that $\bar{B}_M(p_k, r_k', t_k') \subset \bar{B}_M(p_k, r_i + \varepsilon, t_i + \varepsilon')$ and so, we get that $r_k' \leq r_i + \varepsilon$ and $t_k' \leq t_i + \varepsilon'$. Therefore, by the triangle inequality for fuzzy metric (i.e. the condition (FM-4)),

$$
\begin{aligned}
M(x, x_i, t_i + 2\varepsilon') \quad &\geq \quad M(x, p_k, t_i + \varepsilon') * M(p_k, x_i, \varepsilon') \\
&\geq \quad M(x, p_k, t_k') * M(p_k, x_i, \varepsilon') \\
&> \quad (1 - r_k') * (1 - \varepsilon) \\
&\geq \quad [1 - (r_i + \varepsilon)] * (1 - \varepsilon).
\end{aligned}
$$

Since ε and ε' is arbitrary, $M(x, x_i, t_i) \geq 1 - r_i$. This means that $x \in \bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i)$ and so X is fuzzy hyperconvex. $\qquad \square$

Remark 4. *It is clear that if the fuzzy metric space $(X, M, *)$ is $m-$seperable and the space X has finite number of points, then we can not mention fuzzy $m-$hyperconvexity. So indeed, since fuzzy $m-$hyperconvexity ($m \geq 3$) implies the fuzzy metrically convexity (Proposition 2), X can not be a finite set except when the set is reduced to a single point.*

Consequently, Teorem 6 indicate this situation i.e. if the fuzzy metric space $(X, M, *)$ is fuzzy $m-$ hyperconvex and $m-$seperable then $(X, M, *)$ is fuzzy hyperconvex.

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ON GENERALIZATIONS OF A REVERSE HARDY-HILBERT'S TYPE INEQUALITY

ZHENGPING ZHANG, GAOWEN XI

ABSTRACT. By introducing a parameter α and using the expression of the β function establishing the inequality of the weight coefficient, a generalizations of the reverse Hardy-Hilbert's type inequality is proved. As applications, some equivalent form and a number of particular cases are obtained.

1. INTRODUCTION

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, and $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=0}^{\infty} a_n^p \right)^{1/p} \left(\sum_{n=0}^{\infty} b_n^q \right)^{1/q}, \qquad (1.1)$$

and

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < pq \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}}, \qquad (1.2)$$

where the constant $\frac{\pi}{\sin \frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

For (1.1), Yang et al. [7], [8], [9], [10] and [11] gave some strengthened versions and extensions as follows:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n}+3)} \right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n}+3)} \right] b_n^2 \right\}^{1/2}; \quad (1.3)$$

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} \quad &< \quad \left\{ \sum_{n=0}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{ln2 - C}{(2n+1)^{1+1/p}} \right] a_n^p \right\}^{1/p} \\ &\times \left\{ \sum_{n=0}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{\ln 2 - C}{(2n+1)^{1+1/q}} \right] b_n^q \right\}^{1/q}, \qquad (1.4)\end{aligned}$$

1

where $\ln 2 - C = 0.1159315^+$ (C is the Euler constant).

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^p\right\}^{1/p}$$

$$\times \left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}b_n^q\right\}^{1/q}, \tag{1.5}$$

where the constant $B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible ($2 - \min\{p, \ q\} < \lambda \le 2$).

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{\lambda}{p}, \ \frac{\lambda}{q}\right)\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{p-1-\lambda}a_n^p\right\}^{1/p}$$

$$\times \left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{q-1-\lambda}b_n^q\right\}^{1/q}. \tag{1.6}$$

where the constant $B(\frac{\lambda}{p}, \ \frac{\lambda}{q})$ is the best possible ($0 < \lambda \le \min\{p, \ q\}$).

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}\right)\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p\right\}^{1/p}$$

$$\times \left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q\right\}^{1/q}, \tag{1.7}$$

where the constant $B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$ is the best possible ($r > 1, \frac{1}{r}+\frac{1}{s} = 1, \ t \in [0, \ 1]$, $2 - \min\{r, \ s\}t < \lambda \le 2 - \min\{r, \ s\}t + \min\{r, \ s\}$).

In [5] and [6], Xi gave a generalizations and reinforcements of inequalities (1.2:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max(m^\lambda, n^\lambda)} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}$$

$$\times \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.8}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda+A, n^\lambda+B\}} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}$$

$$\times \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.9}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0, 2-\min\{p,q\} < \lambda \le 2, 0 \le A \le B \le \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$.

For the reverse Hardy-Hilbert's inequality, recently, Yang [12] gave a reverse form of inequalities (1.5), (1.6) and (1.7) for $\lambda = 2$. In [4], Xi gave an extension of the above Yang's work for $1.5 \le \lambda < 3$:

ON GENERALIZATIONS OF A REVERSE HARDY-HILBERT'S TYPE INEQUALITY 3

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^\lambda} > \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p\right\}^{1/p}$$

$$\times\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}b_n^q\right\}^{1/q}, \tag{1.10}$$

where $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $1.5 \le \lambda < 3$ and $a_n \ge 0, b_n > 0$, such that $0 < \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}a_n^p}{2n+3-\lambda} < \infty$, $0 < \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda} < \infty$.

In this paper, by introducing a parameter α and using the expression of the β function establishing the inequality of the weight coefficient. The purpose of this paper is to give a generalization of inequality (1.10).

For this, we need the following expression of the β function $B(p, q)$ (see [3])

$$B(p, q) = B(q, p) = \int_0^{\infty}\frac{1}{(1+u)^{p+q}}u^{p-1}du \quad (p, q > 0), \tag{1.11}$$

and the following inequality [8]:

$$\int_0^{\infty} f(x)dx + \frac{1}{2}f(0) < \sum_{m=0}^{\infty} f(m) < \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0) \tag{1.12}$$

where $f(x) \in C^3[0, \infty)$, and $\int_0^{\infty} f(x)dx < \infty$, $(-1)^n f^{(n)}(x) > 0, f^{(n)}(\infty) = 0(n = 0, 1, 2, 3)$.

2. MAIN RESULTS

Lemma 2.1. *Let N_0 be the set of non-negative integers, N be the set of positive integers and R be the set of real numbers. The weight coefficient $\omega_\lambda(n,\alpha)$ is defined by*

$$\omega_\lambda(n,\alpha) = \sum_{m=0}^{\infty}\frac{1}{(m+n+\alpha)^\lambda}, \qquad n \in N_0, \quad 1.5 \le \lambda < 3, \alpha \ge 1.$$

Then we have

$$\frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right] < \omega_\lambda(n,\alpha)$$

$$< \frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}. \tag{2.1}$$

Proof If $n \in N_0$, let $f(x) = \frac{1}{(x+n+\alpha)^\lambda}$, $x \in [0, \infty)$. By (1.12), we obtain

$$\omega_\lambda(n,\alpha) > \int_0^{\infty}\frac{dx}{(x+n+\alpha)^\lambda} + \frac{1}{2(n+\alpha)^\lambda} = \frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}} + \frac{1}{2(n+\alpha)^\lambda}.$$

$$\omega_\lambda(n,\alpha) < \int_0^{\infty}\frac{1}{(x+n+\alpha)^\lambda}dx + \frac{1}{2(n+\alpha)^\lambda} + \frac{\lambda}{12(n+\alpha)^{\lambda+1}}$$

$$= \frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}} + \frac{1}{2(n+\alpha)^\lambda} + \frac{\lambda}{12(n+\alpha)^{\lambda+1}}.$$

Since we find

$$\left[\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}} + \frac{1}{2(n+\alpha)^{\lambda}}\right]\left[2(n+\alpha)^{\lambda-1} - (\lambda-1)(n+\alpha)^{\lambda-2}\right] = \frac{2}{\lambda-1} - \frac{\lambda-1}{2(n+\alpha)^2},$$

$$\left[\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}} + \frac{1}{2(n+\alpha)^{\lambda}} + \frac{1}{12(n+\alpha)^{\lambda+1}}\right]\left[2(n+\alpha)^{\lambda-1} - (\lambda-1)(n+\alpha)^{\lambda-2}\right]$$
$$= \frac{2}{\lambda-1} - \frac{2\lambda-3}{6(n+\alpha)^2} - \frac{\lambda(\lambda-1)}{12(n+\alpha)^3}.$$

Then we obtain

$$\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}} + \frac{1}{2(n+\alpha)^{\lambda}} = \frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}\left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2}\right],$$
$$\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}} + \frac{1}{2(n+\alpha)^{\lambda}} + \frac{1}{12(n+\alpha)^{\lambda+1}} = \frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}$$
$$\times\left[1 - \frac{(2\lambda-3)(\lambda-1)}{12(n+\alpha)^2} - \frac{\lambda(\lambda-1)^2}{24(n+\alpha)^3}\right].$$

Sine $1.5 \leq \lambda < 3$, $\alpha \geq 1$, so $\frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)} > 0, \frac{(2\lambda-3)(\lambda-1)}{12(n+\alpha)^2} \geq 0, \frac{\lambda(\lambda-1)^2}{24(n+\alpha)^3} > 0$. Then we have (2.1). The lemma is proved.

Theorem 2.2. *Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1.5 \leq \lambda < 3$, $\alpha \geq 1$, and $a_n \geq 0, b_n > 0$, such that $0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2}\right]a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}b_n^q < \infty$. Then we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+\alpha)^{\lambda}} > \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2}\right]a_n^p\right\}^{1/p}$$
$$\times \left\{\sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}b_n^q\right\}^{1/q}. \qquad (2.2)$$

Proof By the reverse Hölder 's inequality [2], we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+\alpha)^{\lambda}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\frac{\lambda}{p}}} \cdot \frac{b_n}{(m+n+\alpha)^{\frac{\lambda}{q}}}$$
$$\geq \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m^p}{(m+n+\alpha)^{\lambda}}\right\}^{\frac{1}{p}} \cdot \left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{b_n^q}{(m+n+\alpha)^{\lambda}}\right\}^{\frac{1}{q}}$$
$$= \left\{\sum_{m=0}^{\infty} \omega_{\lambda}(m,\alpha)a_m^p\right\}^{\frac{1}{p}} \cdot \left\{\sum_{n=0}^{\infty} \omega_{\lambda}(n,\alpha)b_n^q\right\}^{\frac{1}{q}}.$$

Since $0 < p < 1$ and $q < 0$, then by (2.1), we obtain (2.2). The theorem is proved.

In Theorem 2.2, for $\alpha = 1$ we have

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Corollary 2.3. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1.5 \le \lambda < 3$ *and* $a_n \ge 0, b_n > 0$, *such that*
$0 < \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} a_n^p}{2n+3-\lambda} < \infty$, $0 < \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+3-\lambda} < \infty$. *Then we have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} > \frac{2}{\lambda-1} \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p \right\}^{1/p}$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} b_n^q \right\}^{1/q}. \tag{2.3}$$

Remark. Inequality (2.3) is inequality (1.10). Hence, inequality (2.2) is an extension inequality (1.10)

Theorem 2.4. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1.5 \le \lambda < 3$, $\alpha \ge 1$, *and* $a_n \ge 0$, *such that*
$0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1} < \infty$. *Then we have*

$$\sum_{n=0}^{\infty} \left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^{p}$$

$$> \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p. \tag{2.4}$$

Inequalities (2.4) and (2.2) are equivalent.

Proof Let

$$b_n = \left[\frac{(n+1)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^{p-1}, \qquad n \in N_0.$$

By (2.2), we have

$$\left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} \right\}^p = \left\{ \sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^{p} \right\}^p$$

$$= \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+\alpha)^{\lambda}} \right\}^p$$

$$\ge \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} \right\}^{p-1}.$$

Then we obtain

$$\sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} = \sum_{n=0}^{\infty} \left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^{p}$$

$$\ge \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p. \tag{2.5}$$

If $\sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} = \infty$, then in view of

$$0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] \le \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1} < \infty$$

and (2.5), we have

$$\sum_{n=0}^{\infty} \left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^\lambda} \right]^p > \left(\frac{2}{\lambda-1} \right)^p$$

$$\times \sum_{n=0}^{\infty} \frac{(n+\alpha)^{\lambda-2}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p;$$

if $0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{\lambda-2} b_n^q}{2n+2\alpha-\lambda+1} < \infty$, then by (2.2), we find

$$\sum_{n=0}^{\infty} \left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^\lambda} \right]^p > \left(\frac{2}{\lambda-1} \right)^p$$

$$\times \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p.$$

Hence we obtain (2.4).

On the other-hand, by the reverse Hölder 's inequality [2], we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+\alpha)^\lambda} = \left[\sum_{n=0}^{\infty} (n+\alpha)^{\frac{\lambda-2}{q}} (2n+2\alpha-\lambda+1)^{\frac{1}{q}} \sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^\lambda} \right]$$

$$\times \left[\frac{b_n}{(n+\alpha)^{\frac{\lambda-2}{q}} (2n+2\alpha-\lambda+1)^{\frac{1}{q}}} \right]$$

$$\ge \left\{ \sum_{n=0}^{\infty} \left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^\lambda} \right]^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} \right\}^{\frac{1}{q}}.$$

Hence by (2.4), it follows

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+\alpha)^\lambda} > \frac{2}{\lambda-1} \left\{ \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} \right\}^{\frac{1}{q}}.$$

Then, (2.4) and (2.2) are equivalent. The theorem is proved.

In (2.4), for $\alpha = 1$, we have

Corollary 2.5. *Let* $1.5 \leq \lambda < 3$, $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $0 < \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} a_n^p}{2n-\lambda+3} < \infty$, *Then we have*

$$\sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n-\lambda+3} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^{\lambda}} \right]^p$$

$$> \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n-\lambda+3} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p. \tag{2.6}$$

Inequalities (2.6) and (2.3) are equivalent.

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Dunkl generalization of q-Szász-Mirakjan-Kantrovich type operators and approximation

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Abstract

The purpose of this paper is to introduce a modification of q-Dunkl generalization of exponential functions. These types of operators modification enables better error estimation on the interval $[\frac{1}{2}, \infty)$ than the classical ones. We obtain some approximation results via well known Korovkin's type theorem. Convergence properties by using the modulus of continuity and the rate of convergence of the operators for functions belonging to the Lipschitz class are also presented.

Keywords and phrases: q-integers; Dunkl analogue; Szász operator; q- Szász-Mirakjan-Kantrovich; modulus of continuity; Peetre's K-functional.

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1. Introduction and preliminaries

In 1912, S.N Bernstein [1] introduced the following sequence of operators $B_n : C[0,1] \to C[0,1]$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \ x \in [0,1] \qquad (1.1)$$

for $n \in \mathbb{N}$ and $f \in C[0,1]$.

In 1950, for $x \geq 0$, Szász [17] introduced the operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \ f \in C[0,\infty). \qquad (1.2)$$

In the field of approximation theory, the application of q-calculus emerged as a new area in the field of approximation theory. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupaş by applying the idea of q-integers [5]. In 1997 Phillips [14] considered another q-analogue of the classical Bernstein polynomials. Later on, many authors introduced q-generalizations of various operators and investigated several approximation properties [6, 7, 8, 9, 10, 11, 12, 13].

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We now present some basic definitions and concept details of the q-calculus which are used in this paper.

Let $k \in \mathbb{N}_0$ and $q \in (0,1)$ then q-integer $[k]_q$ is defined as

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q} & \text{if } q \neq 1, \\ \\ k & \text{if } q = 1. \end{cases}$$

The q-factorial $[k]_q!$ is defined as

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q & \text{if } k \in \mathbb{N}, \\ \\ 1 & \text{if } k = 0, \end{cases}$$

and for $k \in \mathbb{N}$, q-binomial coefficient $\begin{bmatrix} k \\ r \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} k \\ r \end{bmatrix}_q = \frac{[k]_q!}{[r]_q![k-r]_q!}, \ 1 \leq r \leq k,$$

with $\begin{bmatrix} k \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} k \\ r \end{bmatrix}_q = 0$ for $r > k$.

There are two q-analogue of the exponential function e^x

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \ |x| < \frac{1}{1-q}, \ |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]_q!} = (1+(1-q)x)_q^{\infty}, \ |q| < 1,$$

where

$$(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^j x).$$

Our investigation is to construct a linear positive operators generated by generalization of exponential function for defined by [15]

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)}.$$

Here

$$\gamma_\mu(2k) = \frac{2^{2k}k!\Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)},$$

and

$$\gamma_\mu(2k+1) = \frac{2^{2k+1}k!\Gamma\left(k+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)}.$$

The recursion formula for γ_μ is given by

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k), \ k = 0,1,2,\cdots,$$

where $\mu > -\frac{1}{2}$ and

$$\theta_k = \begin{cases} 0 & \text{if } k \in 2\mathbb{N} \\ 1 & \text{if } k \in 2\mathbb{N} + 1. \end{cases}$$

Sucu [16] defined a Dunkl analogue of Szász operators via a generalization of the exponential function [15] as follows:

$$S_n^*(f; x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k + 2\mu\theta_k}{n}\right), \tag{1.3}$$

where $x \geq 0, \ f \in C[0, \infty), \mu \geq 0, \ n \in \mathbb{N}$.

Cheikh et al., [2] stated the q-Dunkl classical q-Hermite type polynomials and gave definitions of q-Dunkl analogues of exponential functions and recursion relations for $\mu > -\frac{1}{2}$ and $0 < q < 1$.

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)}, \ x \in [0, \infty) \tag{1.4}$$

$$E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)}, \ x \in [0, \infty) \tag{1.5}$$

$$\gamma_{\mu,q}(n + 1) = \left(\frac{1 - q^{2\mu\theta_{n+1} + n + 1}}{1 - q}\right) \gamma_{\mu,q}(n), \ n \in \mathbb{N}, \tag{1.6}$$

$$\theta_n = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}, \\ 1 & \text{if } n \in 2\mathbb{N} + 1. \end{cases}$$

An explicit formula for $\gamma_{\mu,q}(n)$ is

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1}, q^2)_{[\frac{n+1}{2}]}(q^2, q^2)_{[\frac{n}{2}]}}{(1 - q)^n} \gamma_{\mu,q}(n), \ n \in \mathbb{N}.$$

And some of the special cases of $\gamma_{\mu,q}(n)$ are defined as:

$$\gamma_{\mu,q}(0) = 1, \ \gamma_{\mu,q}(1) = \frac{1 - q^{2\mu+1}}{1 - q}, \ \gamma_{\mu,q}(2) = \left(\frac{1 - q^{2\mu+1}}{1 - q}\right)\left(\frac{1 - q^2}{1 - q}\right),$$

$$\gamma_{\mu,q}(3) = \left(\frac{1 - q^{2\mu+1}}{1 - q}\right)\left(\frac{1 - q^2}{1 - q}\right)\left(\frac{1 - q^{2\mu+3}}{1 - q}\right),$$

$$\gamma_{\mu,q}(4) = \left(\frac{1 - q^{2\mu+1}}{1 - q}\right)\left(\frac{1 - q^2}{1 - q}\right)\left(\frac{1 - q^{2\mu+3}}{1 - q}\right)\left(\frac{1 - q^4}{1 - q}\right).$$

In [4], Gürhan Içöz gave the Dunkl generalization of Szász operators via q-calculus as:

$$D_{n,q}(f; x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1 - q^{2\mu\theta_k + k}}{1 - q^n}\right), \tag{1.7}$$

for $\mu > \frac{1}{2}, \ x \geq 0, \ 0 < q < 1$ and $f \in C[0, \infty)$.

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Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated.

Motivated essentially by by Içöz [4] the recent investigation of Dunkl generalization of Szász-Mirakjan operators via q-calculus we have showed that our modified operators have better error estimation than [4]. We have proved several approximation results. We have successfully extended these results and modifying the results of papers [4].

2. CONSTRUCTION OF OPERATORS AND MOMENTS ESTIMATION

We modify the q Dunkl analogue of Szász-operators [4].

Let $\{r_{[n]_q}(x)\}$ be a sequence of real-valued continuous functions defined on $[0,\infty)$ with $0 \le r_{[n]_q} < \infty$. Then we define

$$D^{\star}_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right). \qquad (2.1)$$

Now, if we replace $r_{[n]_q}(x)$ as

$$r_{[n]_q}(x) = x - \frac{1}{2[n]_q}, \text{ where } \frac{1}{2n} \le x < \frac{1}{1-q^n} \text{ and } n \in \mathbb{N}. \qquad (2.2)$$

Then for any $\frac{1}{2n} \le x < \frac{1}{1-q^n}$, $0 < q < 1$, $\mu > \frac{1}{2n}$ and $n \in \mathbb{N}$ we have

$$D^{*}_{n,q}(f;x) = \frac{1}{e_{\mu,q}\left(\frac{2[n]_q x-1}{2}\right)} \sum_{k=0}^{\infty} \frac{(2[n]_q x-1)^k}{2^k \gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right). \qquad (2.3)$$

where $e_{\mu,q}(x)$, $\gamma_{\mu,q}$ are defined in (1.4),(1.6) by [16] and $f \in C_\zeta[0,\infty)$ with $\zeta \ge 0$ and

$$C_\zeta[0,\infty) = \{f \in C[0,\infty) :\mid f(t)\mid \le M(1+t)^\zeta, \text{ for some } M > 0, \ \zeta > 0\}. \qquad (2.4)$$

Lemma 2.1. *Let $D^{*}_{n,q}(.\ ;\ .)$ be the operators given by (2.3). Then for each $\frac{1}{2n} \le x < \frac{1}{1-q^n}$, $n \in \mathbb{N}$, we have we have the following identities:*

(1) $D^{*}_{n,q}(1;x) = 1$,

(2) $D^{*}_{n,q}(t;x) = r_{[n]_q}(x) = x - \frac{1}{2[n]_q}$,

(3) $x^2 + \left(q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q[n]_q r_{[n]_q}(x))}{e_{\mu,q}([n]_q r_{[n]_q}(x))} - 1\right) \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q[n]_q r_{[n]_q}(x))}{e_{\mu,q}([n]_q r_{[n]_q}(x))} - 1\right) \le D^{*}_{n,q}(t^2;x) \le x^2 + ([1+2\mu]_q - 1)\frac{x}{[n]_q} - \frac{1}{4[n]_q^2}(2[1+2\mu]_q - 1)$.

Proof. (1) $D^{*}_{n,q}(1;x) = \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k)} = 1$.

(2)

$$
\begin{aligned}
D_{n,q}^*(t;x) &= \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k)} \left(\frac{1 - q^{2\mu\theta_k + k}}{1 - q^n} \right) \\
&= \frac{1}{[n]_q e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=1}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k-1)} \\
&= x - \frac{1}{2[n]_q}
\end{aligned}
$$

(3)

$$
\begin{aligned}
D_{n,q}^*(t^2;x) &= \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k)} \left(\frac{1 - q^{2\mu\theta_k + k}}{1 - q^n} \right)^2 \\
&= \frac{1}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k-1)} \left(\frac{1 - q^{2\mu\theta_k + k}}{1 - q} \right) \\
&= \frac{1}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{k+1}}{\gamma_\mu(k)} \left(\frac{1 - q^{2\mu\theta_{k+1} + k+1}}{1 - q} \right).
\end{aligned}
$$

From [4] we know that

$$
[2\mu\theta_{k+1} + k + 1]_q = [2\mu\theta_k + k]_q + q^{2\mu\theta_k + k}[2\mu(-1)^k + 1]_q, \tag{2.5}
$$

Now by separating to the even and odd terms and using (2.5), we get

$$
\begin{aligned}
D_{n,q}^*(t^2;x) &= \frac{1}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{k+1}}{\gamma_\mu(k)} \left(\frac{1 - q^{2\mu\theta_{k+1} + k+1}}{1 - q} \right) \\
&+ \frac{[1+2\mu]_q}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{2k+1}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k}+2k} \\
&+ \frac{[1-2\mu]_q}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{2k+2}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k+1}+2k+1}.
\end{aligned}
$$

We know the inequality

$$
[1 - 2\mu]_q \le [1 + 2\mu]_q. \tag{2.6}
$$

Therefore by using (2.6) we have

$$
\begin{aligned}
D_{n,q}^*(t^2;x) &\ge (r_{[n]_q}(x))^2 + \frac{r_{[n]_q}(x)[1-2\mu]_q}{[n]_q e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{(q[n]_q r_{[n]_q}(x))^{2k}}{\gamma_\mu(2k)} \\
&+ \frac{q^{2\mu} r_{[n]_q}(x)[1-2\mu]_q}{[n]_q e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{(q[n]_q r_{[n]_q}(x))^{2k+1}}{\gamma_\mu(2k+1)} \\
&\ge (r_{[n]_q}(x))^2 + q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q[n]_q r_{[n]_q}(x))}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \frac{r_{[n]_q}(x)}{[n]_q}.
\end{aligned}
$$

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Similarly on the other hand we have

$$D_{n,q}^*(t^2; x) \leq (r_{[n]_q}(x))^2 + [1 + 2\mu]_q \frac{r_{[n]_q}(x)}{[n]_q}.$$

Which completes the proof.

$\square$

Lemma 2.2. *Let the operators* $D_{n,q}^*(.\ ;\ .)$ *be given by* (2.3). *Then for each* $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$, *we have*

(1) $D_{n,q}^*(t - x; x) = -\frac{1}{2[n]_q}$,

(2) $D_{n,q}^*((t - x)^2; x) \leq [1 + 2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2} (2[1 + 2\mu]_q - 1)$.

3. MAIN RESULTS

We obtain the Korovkin's type approximation properties for our operators defined by (2.3).

Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+ = [0, \infty)$, which is linear normed space with

$$\| f \|_{C_B} = \sup_{x \geq 0} | f(x) | .$$

Let

$$H := \{f : x \in [0, \infty), \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty\}.$$

Remark 3.1. By lemma 2.1, it is clear that the positive liner operators $D_{n,q}^*$ given by (2.3) preserve a linear functions, that is for $\phi(y) = cy + d$, $c, d \in \mathbb{R}$(*Real numbers*), $D_{n,q}^*(\phi; x) = \phi(x)$ for all $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$.

Now, fix $b > \frac{1}{2}$ and consider the lattice homomorphism $H_b : C[0, \infty] \to C[0, b]$ defined by $H_b(f) = f\big|_{[0,b]}$ for every $f \in C[0, \infty]$, where $f\big|_{[0,b]}$ denotes the restriction of the domain of f to the interval $[0, b]$. In this case for each $j = 0, 1, 2$, we have

$$\lim H_b\left(D_{n,q}^*(e_j)\right) = H_b(e_j) \qquad uniformly \quad on \quad \left[\frac{1}{2}, b\right]. \qquad (3.1)$$

Thus, by using (3.1) and with the universal Korovkin-type property with respect to the monotone operators. And hence we have the following Korovkin-type approximation result.

Theorem 3.2. *Let* $D_{n,q}^*(.\ ;\ .)$ *be the operators defined by* (2.3). *Then for any function* $f \in C_\zeta[0, \infty) \cap H$, $\zeta \geq 2$,

$$\lim_{n \to \infty} D_{n,q}^*(f; x) = f(x)$$

is uniformly on each compact subset of $[0, \infty)$, *where* $x \in \left[\frac{1}{2}, b\right]$, $\frac{1}{2} < b < \infty$.

Proof. The proof is based on Lemma 2.1 and well known Korovkin's theorem regarding the convergence of a sequence of linear and positive operators, so it is enough to prove the conditions

$$\lim_{n\to\infty} D^*_{n,q}((t^j; x) = x^j, \ j = 0, 1, 2, \ \{\text{as } n \to \infty\}$$

uniformly on $[0, 1]$.
Clearly $\frac{1}{[n]_q} \to 0 \ (n \to \infty)$ we have

$$\lim_{n\to\infty} D^*_{n,q}(t; x) = x, \ \lim_{n\to\infty} D^*_{n,q}(t^2; x) = x^2.$$

Which complete the proof. □

We recall the weighted spaces of the functions on $\mathbb{R}^+$, which are defined as follows:

$$P_\rho(\mathbb{R}^+) \ = \ \{f :| f(x) |\leq M_f\rho(x)\},$$
$$Q_\rho(\mathbb{R}^+) \ = \ \{f : f \in P_\rho(\mathbb{R}^+) \cap C[0, \infty)\},$$
$$Q_\rho^k(\mathbb{R}^+) \ = \ \left\{f : f \in Q_\rho(\mathbb{R}^+) \text{ and } \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k(k \text{ is a constant})\right\},$$

where $\rho(x) = 1 + x^2$ is a weight function and M_f is a constant depending only on f. Note that $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm $\| f \|_\rho = \sup_{x\geq 0} \frac{|f(x)|}{\rho(x)}$.

Lemma 3.3. *([3]) The linear positive operators L_n, $n \geq 1$ act from $Q_\rho(\mathbb{R}^+) \to P_\rho(\mathbb{R}^+)$ if and only if*

$$\| L_n(\varphi; x) \|\leq K\varphi(x),$$

where $\varphi(x) = 1 + x^2$, $x \in \mathbb{R}^+$ and K is a positive constant.

Theorem 3.4. *([3]) Let $\{L_n\}_{n\geq 1}$ be a sequence of positive linear operators acting from $Q_\rho(\mathbb{R}^+) \to P_\rho(\mathbb{R}^+)$ and satisfying the condition*

$$\lim_{n\to\infty} \| L_n(\rho^\tau) - \rho^\tau \|_\varphi = 0, \ \tau = 0, 1, 2.$$

Then for any function $f \in Q_\rho^k(\mathbb{R}^+)$, we have

$$\lim_{n\to\infty} \| L_n(f; x) - f \|_\varphi = 0.$$

Theorem 3.5. *Let $D^*_{n,q}(. \ ; \ .)$ be the operators defined by (2.3). Then for each function $f \in Q_\rho^k(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} \| D^*_{n,q}(f; x) - f \|_\rho = 0.$$

Proof. From Lemma 2.1 and Theorem 3.4 for $\tau = 0$, the first condition is fulfilled. Therefore

$$\lim_{n\to\infty} \| D^*_{n,q}(1; x) - 1 \|_\rho = 0.$$

Similarly From Lemma 2.1 and Theorem 3.4 for $\tau = 1, 2$ we have that

$$\sup_{x\in[0,\infty)} \frac{| D^*_{n,q}(t; x) - x |}{1 + x^2} \ \leq \ \frac{1}{2[n]_q} \sup_{x\in[0,\infty)} \frac{1}{1 + x^2}$$
$$= \ \frac{1}{2[n]_q},$$

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which imply that

$$\lim_{n \to \infty} \parallel D_{n,q}^*(t; x) - x \parallel_\rho = 0.$$

$$\sup_{x \in [0,\infty)} \frac{| D_{n,q}^*(t^2; x) - x^2 |}{1 + x^2} \leq \frac{| [1 + 2\mu]_q - 1 |}{[n]_q} \sup_{x \in [0,\infty)} \frac{x}{1 + x^2}$$

$$+ \frac{1}{4[n]_q^2} | [1 + 2\mu]_q - 1 | \sup_{x \in [0,\infty)} \frac{1}{1 + x^2}$$

which imply that

$$\lim_{n \to \infty} \parallel D_{n,q}^*(t^2; x) - x^2 \parallel_\rho = 0.$$

This complete the proof. □

4. Rate of Convergence

Here we calculate the rate of convergence of operators (2.3) by means of modulus of continuity and Lipschitz type maximal functions.

Let $f \in C_B[0, \infty]$, the space of all bounded and continuous functions on $[0, \infty)$ and $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f, \delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

$$\omega(f, \delta) = \sup_{|t - x| \leq \delta} | f(t) - f(x) |, \ t \in [0, \infty). \tag{4.1}$$

It is known that $\lim_{\delta \to 0+} \omega(f, \delta) = 0$ for $f \in C_B[0, \infty)$ and for any $\delta > 0$ one has

$$| f(t) - f(x) | \leq \left(\frac{| t - x |}{\delta} + 1 \right) \omega(f, \delta). \tag{4.2}$$

Theorem 4.1. *Let $D_{n,q}^*(. ; .)$ be the operators defined by (2.3). Then for $f \in C_B[0, \infty)$, $x \geq \frac{1}{2n}$ and $n \in \mathbb{N}$ we have*

$$| D_{n,q}^*(f; x) - f(x) | \leq 2\omega \left(f; \delta_{n,x} \right),$$

where $C_B[0, \infty)$ is the space of uniformly continuous bounded functions on $\mathbb{R}^+$, $\omega(f, \delta)$ is the modulus of continuity of the function $f \in C_B[0, \infty)$ defined in (4.1) and

$$\delta_{n,x} = \sqrt{[1 + 2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2} (2[1 + 2\mu]_q - 1)}. \tag{4.3}$$

Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality we can easily get

$$| D_{n,q}^*(f; x) - f(x) | \leq \left\{ 1 + \frac{1}{\delta} \left(D_{n,q}^*(t - x)^2; x \right)^{\frac{1}{2}} \right\} \omega(f; \delta)$$

if we choose $\delta = \delta_{n,x}$ and by applying the result (2) of Lemma 2.2 complete the proof. □

Remark 4.2. For the operators $D_{n,q}(.\;;\;.)$ defined by (1.7) we may write that, for every $f \in C_B[0,\infty)$, $x \geq 0$ and $n \in \mathbb{N}$

$$\mid D_{n,q}(f;x) - f(x) \mid \leq 2\omega\left(f;\lambda_{n,x}\right), \tag{4.4}$$

where by [4] we have

$$\lambda_{n,x} = \sqrt{D_{n,q}((t-x)^2;x)} \leq \sqrt{[1+2\mu]_q \frac{x}{[n]_q}}. \tag{4.5}$$

Now we claim that the error estimation in Theorem 4.1 is better than that of (4.4) provided $f \in C_B[0,\infty)$ and $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$. Indeed, for $x \geq \frac{1}{2n}$, $\mu \geq \frac{1}{2n}$ and $n \in \mathbb{N}$, it is guarantees that

$$D^*_{n,q}((t-x)^2;x) \leq D_{n,q}((t-x)^2;x), \tag{4.6}$$

$$[1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right) \leq [1+2\mu]_q \frac{x}{[n]_q}. \tag{4.7}$$

Which imply that

$$\sqrt{[1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)} \leq \sqrt{[1+2\mu]_q \frac{x}{[n]_q}}. \tag{4.8}$$

Now we give the rate of convergence of the operators $D^*_{n,q}(f;x)$ defined in (2.3) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$.

Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \{f : \mid f(\zeta_1) - f(\zeta_2) \mid \leq M \mid \zeta_1 - \zeta_2 \mid^\nu \;\; (\zeta_1, \zeta_2 \in [0,\infty))\} \tag{4.9}$$

Theorem 4.3. *Let $D^*_{n,q}(.\;;\;.)$ be the operator defined in (2.3). Then for each $f \in Lip_M(\nu)$, $(M > 0,\; 0 < \nu \leq 1)$ satisfying (4.9) we have*

$$\mid D^*_{n,q}(f;x) - f(x) \mid \leq M\left(\delta_{n,x}\right)^{\frac{\nu}{2}}$$

where $\delta_{n,x}$ is given in Theorem 4.1.

Proof. We prove it by using (4.9) and Hölder inequality.

$$\begin{aligned}
\mid D^*_{n,q}(f;x) - f(x) \mid &\leq \mid D^*_{n,q}(f(t) - f(x);x) \mid \\
&\leq D^*_{n,q}(\mid f(t) - f(x) \mid;x) \\
&\leq \mid MD^*_{n,q}(\mid t - x \mid^\nu;x).
\end{aligned}$$

Therefore

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$$| D_{n,q}^*(f;x) - f(x) |$$

$$\leq M \frac{[n]_q}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)} \left| \frac{1 - q^{2\mu\theta_k + k}}{1 - q^n} - x \right|^\nu dt$$

$$\leq M \frac{[n]_q}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \left(\frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)} \right)^{\frac{2-\nu}{2}}$$

$$\times \left(\frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)} \right)^{\frac{\nu}{2}} \left| \frac{1 - q^{2\mu\theta_k + k}}{1 - q^n} - x \right|^\nu dt$$

$$\leq M \left(\frac{n}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)} dt \right)^{\frac{2-\nu}{2}}$$

$$\times \left(\frac{[n]_q}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)} \left| \frac{1 - q^{2\mu\theta_k + k}}{1 - q^n} - x \right|^2 dt \right)^{\frac{\nu}{2}}$$

$$= M \left(D_{n,q}^*(t-x)^2; x \right)^{\frac{\nu}{2}}.$$

Which complete the proof. □

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $\mathbb{R}^+ = [0,\infty)$ and

$$C_B^2(\mathbb{R}^+) = \{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\}, \qquad (4.10)$$

with the norm

$$\| g \|_{C_B^2(\mathbb{R}^+)} = \| g \|_{C_B(\mathbb{R}^+)} + \| g' \|_{C_B(\mathbb{R}^+)} + \| g'' \|_{C_B(\mathbb{R}^+)}, \qquad (4.11)$$

also

$$\| g \|_{C_B(\mathbb{R}^+)} = \sup_{x \in \mathbb{R}^+} | g(x) | . \qquad (4.12)$$

Theorem 4.4. *Let $D_{n,q}^*(. ; .)$ be the operator defined in (2.3). Then for any $g \in C_B^2(\mathbb{R}^+)$ we have*

$$| D_{n,q}^*(f;x) - f(x) | \leq \left\{ \left(-\frac{1}{2[n]_q} \right) + \frac{\delta_{n,x}}{2} \right\} \| g \|_{C_B^2(\mathbb{R}^+)},$$

where $\delta_{n,x}$ is given in Theorem 4.1.

Proof. Let $g \in C_B^2(\mathbb{R}^+)$, then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi) \frac{(t-x)^2}{2}, \ \psi \in (x,t).$$

By applying linearity property on $D_{n,q}^*$, we have

$$D_{n,q}^*(g,x) - g(x) = g'(x) D_{n,q}^* ((t-x); x) + \frac{g''(\psi)}{2} D_{n,q}^* ((t-x)^2; x),$$

which imply that

$$\mid D^*_{n,q}(g;x) - g(x) \mid \le \left(-\frac{1}{2[n]_q}\right) \parallel g' \parallel_{C_B(\mathbb{R}^+)} + \left([1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)\right) \frac{\parallel g'' \parallel_{C_B(\mathbb{R}^+)}}{2}.$$

From (4.11) we have $\parallel g' \parallel_{C_B[0,\infty)} \le \parallel g \parallel_{C_B^2[0,\infty)}$.

$$\mid D^*_{n,q}(g;x) - g(x) \mid \le \left(-\frac{1}{2[n]_q}\right) \parallel g \parallel_{C_B^2(\mathbb{R}^+)} + \left([1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)\right) \frac{\parallel g \parallel_{C_B^2(\mathbb{R}^+)}}{2}.$$

This completes the proof from 2 of Lemma 2.2. $\qquad\square$

Acknowledgement

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Pointwise error estimates for spherical hybrid interpolation *

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Abstract

This paper studies pointwise error estimates for spherical hybrid interpolation, which combines spherical polynomials together with spherical radial basis functions constructed by a strictly positive definite zonal kernel. The study is first carried out in the native space associated with the kernel, and then refined error estimates for a target function in a still smaller space are established.

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1 Introduction

In recent years, fitting a surface to scattered data arising from sampling an unknown function defined on an underlying manifold comes up frequently in applied problems. If the underlying manifold is $\mathbb{S}^2$, the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, then there are applications to astrophysics, meteorology, geodesy, geophysics and other areas (see [5, 6, 27]). Amongst approaches for scattered data interpolation and approximation on $\mathbb{S}^2$, many authors have used spherical polynomials or spherical radial basis functions (see [5, 6, 9, 12, 18, 20, 25, 26, 27, 13, 2]). Motivated by the fact that the spherical radial basis functions are helpful to handle scattered data and rapid changes, at the same time, the spherical polynomials contribute to handle the slowly varying large-scale features, a hybrid interpolation scheme was given in [23].

The hybrid interpolation scheme combines spherical radial basis functions together with spherical polynomials, that is a little different from interpolation by radial basis functions constructed from conditionally positive definite kernels (in which case a polynomial part is needed to make the theory work, see [8]). Sloan and Sommariva [23] restricted their attention to the case of strictly positive definite kernels, so that the polynomial component is voluntary rather than forced.

This paper studies the hybrid interpolation problem in an appropriate native space $\mathcal{N}_\phi$ of continuous functions on $\mathbb{S}^2$, which is defined by an underlying strictly positive definite kernel ϕ. We use the method in [23] to get the pointwise error estimate for the hybrid interpolation.

It is known that if ϕ is smooth, the native space $\mathcal{N}_\phi$ is small in the sense that it is composed of very smooth functions. That is so called "native space barrier" problem and there are several literatures focus on it. We refer the readers to [10, 11, 15, 16, 17] for more details. In this paper, we combine the approach which was used by Levesley and Sun in [10] with the techniques in [24], and embed the smooth radial basis functions in a larger native space generated by a less smooth kernel ψ and still use the hybrid interpolation associated with the smooth kernel ϕ to interpolate the target function from the larger native space. In the process of obtaining the corresponding error estimates, we will use the "norming set" method developed by Jetter in [9] and a special case of the general Bernstein-type inequality established by Ditzian [4].

This paper is organized as follows. In Section 2, we give some notations and preliminary results. The hybrid interpolation is introduced and the crucial condition for the scheme to be well defined

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and discussed in Subsection 2.2, and native space and Sobolev space are introduced in Subsection 2.3. Finally, the pointwise errors are estimated in Section 3.

In the following, we adopt the following convention regarding symbols. Let C be a positive constant, whose value will be different at different occurrence even within the same formula. The symbol $A \sim B$ means that there exist positive constant C_1 and C_2 such that $C_1 B \leq A \leq C_2 B$.

2　Preliminaries

Let $\mathbb{S}^2$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, i.e.,

$$\mathbb{S}^2 := \left\{ x := (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

For integer $l \geq 0$, the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial with degree l is called a spherical harmonic of degree l. The class of all spherical harmonics with degree l is denoted by $\mathcal{H}_l$, and it is well know that spherical harmonics of different degrees are orthogonal with respect to the $L_2(\mathbb{S}^2)$ inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^2} f(x)g(x)d\omega(x),$$

where $d\omega$ denotes surface measure on $\mathbb{S}^2$. Hence, if we choose an orthogonal basis $\{Y_{l,k} : k = 1, \ldots, 2l+1\}$ for each $\mathcal{H}_l$, then the set $\{Y_{l,k} : l = 0, 1, \ldots, \ k = 1, \ldots, 2l+1\}$ is an orthogonal basis for $L_2(\mathbb{S}^2)$. The class of all spherical harmonics with total degree $l \leq L$ is denoted by $\mathcal{P}_L$. Of course, $\mathcal{P}_L = \bigoplus_{l=0}^{L} \mathcal{H}_l$, and the dimension of $\mathcal{H}_l$ is $2l+1$ and that of $\mathcal{P}_L$ is $(L+1)^2$.

We denote by $L_p(\mathbb{S}^2)$ the space of p-integrable functions on $\mathbb{S}^2$ endowed with the norms

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{S}^2)} := \operatorname*{ess\,sup}_{x \in \mathbb{S}^2} |f(x)|, \quad p = \infty,$$

and

$$\|f\|_p := \|f\|_{L_p(\mathbb{S}^2)} := \left\{ \int_{\mathbb{S}^2} |f(x)|^p d\omega(x) \right\}^{1/p} < \infty, \quad 1 \leq p < \infty.$$

The well known addition formula is given by (see [14])

$$\sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y),$$

where P_l is the Legendre polynomial with degree l and dimension three, which is normalized such that $P_l(1) = 1$, and satisfies the orthogonality relation (see [14])

$$\int_{-1}^{1} P_k(t)P_j(t)dt = \frac{2}{2l+1} \delta_{k,j},$$

where the symbol $\delta_{k,j}$ denotes the usual Kronecker symbol.

The addition formula also yields the following useful relation

$$\sum_{k=1}^{2l+1} |Y_{l,k}(x)Y_{l,k}(y)| \leq \sum_{k=1}^{2l+1} Y_{l,k}^2(x) = \frac{2l+1}{4\pi}, \quad x, y \in \mathbb{S}^2. \tag{2.1}$$

2.1　Strictly positive definite kernel

Definition 2.1 (see [27]). *A continuous and symmetric function $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$ is called positive definite kernel, if, for any $N \in \mathbb{N}_+$, $\alpha = (\alpha_i)_{i=1,\ldots,N} \in \mathbb{R}^N$ and $\{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, we have*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i, x_j) \geq 0.$$

When for any N distinct points $\{x_1, \ldots, x_N\}$, the above quadratic form is positive for all $\alpha = (\alpha_i)_{i=1,\ldots,N} \in \mathbb{R}^N/\{0\}$, then ϕ is called strictly positive definite kernel.

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A kernel ϕ is called rotational invariant, if $\phi(\rho x, \rho y) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$ and for all rotations ρ. It can be shown that a continuous rotational invariant kernel depends only on the distance between x and y, that is, there is a function $\varphi : [-1, 1] \to \mathbb{R}$, such that $\varphi(xy) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$ (see [22]). Therefore, a rotational invariant kernel is also called a zonal kernel in the literature.

Schoenberg characterized the positive definite zonal kernels in [21] and the notation of strictly positive definiteness on spheres was first introduced by Xu and Cheney [28]. It is important to characterize all the strictly positive definite functions on spheres and such an endeavor has been taken by Ron and Sun in [19]. In [3], Chen et al. established a necessary and sufficient condition for strictly positive definite zonal kernels: the kernel ϕ is strictly positive definite and zonal if and only if

$$\phi(x, y) = \sum_{l=0}^{\infty} a_l \sum_{k=1}^{2l+1} Y_{l,k}(x) Y_{l,k}(y) = \sum_{l=0}^{\infty} \frac{(2l+1)a_l}{4\pi} P_l(x \cdot y),$$

with $a_l \geq 0$ for all l, $\sum_{l=0}^{\infty} l a_l < \infty$ and $a_l > 0$ for infinitely many even values of l and infinitely many odd values of l.

2.2 The hybrid interpolation

Assume that we are given a strictly positive definite kernel $\phi(\cdot, \cdot)$ and a set of distinct points $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$. Then for a target function $f \in C(\mathbb{S}^2)$ we can take the hybrid interpolation for f in the form

$$I_{X,L}f = \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k},$$

where we fix $L \geq 0$ as the desired degree of the polynomial component of the hybrid interpolation and the coefficients $\{\alpha_j\}_{j=1}^{N}$, $\{\beta_{l,k}\}_{k=1,\ldots,2l+1,\, l=0,\ldots,L}$ are determined by the interpolation conditions

$$I_{X,L}f(x_i) = f(x_i), \ \ i = 1, \ldots, N, \tag{2.2}$$

and also (in order to give a square linear system) the side conditions

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0, \ \ \forall p \in \mathcal{P}_L.$$

In order to give the conditions which will make sure that the interpolation is exist and unique, we shall impose a condition on the point set X.

Definition 2.2 (see [23, Definition 3.1]). The set $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is said to be $\mathcal{P}_L$-unisolvent if

$$p \in \mathcal{P}_L, \ p(x_j) = 0 \text{ for } j = 1, \ldots, N \Rightarrow p = 0.$$

For the analysis of the interpolation error in the later sections it is convenient to define a finite-dimensional space $V_{X,L}$ within the interpolation $I_{X,L}f$ lies.

$$V_{X,L} := \left\{ \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + q : q \in \mathcal{P}_L, \ \alpha_j \in \mathbb{R} \text{ for } j = 1, \ldots, N, \text{ and } \sum_{j=1}^{N} \alpha_j p(x_j) = 0, \ \forall p \in \mathcal{P}_L \right\}.$$

The following Theorem 2.1 gives a crucial condition for the interpolation to be well defined, whose proof can be find in [23].

Theorem 2.1 Let $\phi(\cdot, \cdot)$ be a strictly positive definite kernel, let $L \geq 0$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ be a set of distinct points which is $\mathcal{P}_L$-unisolvent. Then for each $f \in C(\mathbb{S}^2)$ there exists a unique $I_{X,L}f \in V_{X,L}$ that satisfies the interpolation conditions in (2.2).

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2.3 Native space and Sobolev space

Here and in the other sections we assume that the strictly positive definite kernel ϕ is zonal and has the expansion

$$\phi(x,y) = \sum_{l=0}^{\infty} a_l \sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) \tag{2.3}$$

with $a_l > 0$ for all l, $\sum_{l=0}^{\infty} la_l < \infty$, in which case the series of the right side in (2.3) converges uniformly for $x,y \in \mathbb{S}^2$.

For $f,g \in L_2(\mathbb{S}^2)$, they can be represented by their Fourier series

$$f = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \hat{f}_{l,k} Y_{l,k}, \qquad g = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \hat{g}_{l,k} Y_{l,k},$$

respectively. With respect to the inner product expressed as (see [27])

$$(f,g)_{\mathcal{N}_\phi} = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{\hat{f}_{l,k}\hat{g}_{l,k}}{a_l},$$

the native space $\mathcal{N}_\phi$, which is the subspace of $L_2(\mathbb{S}^2)$, can be defined by

$$\mathcal{N}_\phi := \left\{ f \in L_2(\mathbb{S}^2) : \|f\|_{\mathcal{N}_\phi}^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{|\hat{f}_{l,k}|^2}{a_l} < \infty \right\}.$$

It is easy to verify that the native space $\mathcal{N}_\phi$ is a reproducing kernel Hilbert space with reproducing kernel $\phi(\cdot,\cdot)$, that is,

$$(f,\phi(\cdot,x))_{\mathcal{N}_\phi} = f(x), \ x \in \mathbb{S}^2, \ f \in \mathcal{N}_\phi.$$

When $a_l \sim (l+1)^{-2s}$ for $l = 0,1,\ldots$, the native space $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s:

$$H_s := \left\{ f \in L_2(\mathbb{S}^2) : \|f\|_{H_s}^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} (l+1)^{2s} |\hat{f}_{l,k}|^2 < \infty \right\},$$

and the Sobolev embedding theorem in [7] implies that if $s > 1$, then the space H_s is continuously embedded in $C(\mathbb{S}^2)$, so that H_s is a reproducing kernel Hilbert space.

3 Pointwise error estimates

As we can see that the uniqueness result in Theorem 2.1 ensures the existence and uniqueness of the lagrangians $l_j := l_{j,X,L} : \mathbb{S}^2 \to \mathbb{R}$, which is defined by

$$l_j \in V_{X,L}, \ l_j(x_i) = \delta_{i,j}, \ i,j = 1,\ldots,N.$$

The following Theorem 3.1 is a little different from the obtained result in [23] and it is the difference that helps us to extend the error estimates for hybrid interpolation to L_p norm in the next section.

Theorem 3.1 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel defined in (2.3), and let $X = \{x_1,\ldots,x_N\} \subset \mathbb{S}^2$ be a $\mathcal{P}_L$-unisolvent set of distinct points on $\mathbb{S}^2$. For $f \in \mathcal{N}_\phi$, let $I_{X,L}f \in V_{X,L}$ be the hybrid interpolation defined in Section 2.2. Then for a fixed $x \in \mathbb{S}^2$, we have*

$$|f(x) - I_{X,L}f(x)| \le \|f - I_{X,L}f\|_{\mathcal{N}_\phi} P_{\phi,X,L}(x),$$

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where $P_{\phi,X,L}$ is the power function defined by

$$P_{\phi,X,L}(x) = \left(\phi(x,x) - 2 \sum_{j=1}^{N} l_j(x)\phi(x,x_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} l_i(x)l_j(x)\phi(x_i,x_j) \right)^{1/2}.$$

Proof. With the help of the reproducing property of ϕ, we can rewrite the form of $I_{X,L}f$ as

$$I_{X,L}f(x) = \sum_{j=1}^{N} f(x_j)l_j(x) = \sum_{j=1}^{N} (f,\phi(\cdot,x_j))_{\mathcal{N}_\phi} l_j(x) = \left(f, \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x) \right)_{\mathcal{N}_\phi}, \quad x \in \mathbb{S}^2.$$

Since we also have $f(x) = (f,\phi(\cdot,x))_{\mathcal{N}_\phi}, \quad x \in \mathbb{S}^2$, from the reproducing property of ϕ, for $I_{X,L}f \in V_{X,L}$, we have, since $V_{X,L} \subset \mathcal{N}_\phi$,

$$\left(I_{X,L}f, \phi(\cdot,x) - \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x) \right)_{\mathcal{N}_\phi} = I_{X,L}f(x) - \sum_{j=1}^{N} l_j(x)I_{X,L}f(x_j) = 0,$$

here the Lagrange representation of $I_{X,L}f \in V_{X,L}$ ensures that

$$\sum_{j=1}^{N} l_j(x)I_{X,L}f(x_j) = I_{X,L}f(x), \quad \forall x \in \mathbb{S}^2.$$

So the pointwise error turns into

$$f(x) - I_{X,L}f(x) = \left(f - I_{X,L}f, \phi(\cdot,x) - \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x) \right)_{\mathcal{N}_\phi}, \tag{3.4}$$

and by the Cauchy-Schwarz inequality, we have

$$|f(x) - I_{X,L}f(x)| \le \|f - I_{X,L}f\|_{\mathcal{N}_\phi} P_{\phi,X,L}(x), \quad x \in \mathbb{S}^2,$$

where $P_{\phi,X,L}$ is the power function defined by

$$P_{\phi,X,L}(x) = \left\| \phi(\cdot,x) - \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x) \right\|_{\mathcal{N}_\phi}, \quad x \in \mathbb{S}^2.$$

On using the definition $\| \cdot \| = (\cdot,\cdot)_{\mathcal{N}_\phi}^{1/2}$ and the reproducing property of ϕ, the power function turns into

$$P_{\phi,X,L}(x) = \left(\phi(x,x) - 2 \sum_{j=1}^{N} l_j(x)\phi(x,x_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} l_i(x)l_j(x)\phi(x_i,x_j) \right)^{1/2},$$

completing the proof of Theorem 3.1.

The following Lemma 3.1 is taken from [27] and it is also established by Sloan and Sommariva in [23].

Lemma 3.1 (see [23, Lemma 5.3]). Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, and assume that $X = \{x_1,\ldots,x_N\} \subset \mathbb{S}^2$ is a $\mathcal{P}_L$- unisolvent set of distinct points on $\mathbb{S}^2$. For a fixed $x \in \mathbb{S}^2$, we define the quadratic functional $\mathcal{L}_x : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{L}_x(\alpha) := \phi(x,x) - 2 \sum_{j=1}^{N} \alpha_j\phi(x,x_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j\phi(x_i,x_j), \quad \alpha = (\alpha_1,\ldots,\alpha_N).$$

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Then the minimum of $\mathcal{L}_x(\alpha)$ on the set

$$
M_{x,L} := \left\{ \alpha \in \mathbb{R}^N : \sum_{j=1}^{N} \alpha_j p(x_j) = p(x), \ \forall p \in \mathcal{P}_L \right\},
$$

is achieved by the vector $(l_1(x), \ldots, l_N(x))$, that is, $\mathcal{L}_x(l_1(x), \ldots, l_N(x)) \leq \mathcal{L}_x(\alpha)$, for all $\alpha \in M_{x,L}$.

Follows from Theorem 3.1 and 3.1, we can easily obtain the next Theorem 3.2.

Theorem 3.2 *Under the conditions of Theorem 3.1, for a fixed $x \in \mathbb{S}^2$, we have*

$$
|f(x) - I_{X,L}f(x)| \leq \|f - I_{X,L}f\|_{\mathcal{N}_\phi} \left(\mathcal{L}_x(\alpha) \right)^{1/2},
$$

for any real number $\alpha_j := \alpha_j(x)$, $j = 1, \ldots, N$, such that $\sum_{j=1}^{N} \alpha_j p(x_j) = p(x)$, for all $p \in \mathcal{P}_L$, and

$$
\mathcal{L}_x(\alpha) := \phi(x,x) - 2 \sum_{j=1}^{N} \alpha_j \phi(x,x_j) + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i, x_j).
$$

The error estimates are general expressed in terms of the mesh norm of $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, which is defined by

$$
h_X := \sup_{x \in \mathbb{S}^2} \inf_{x_j \in X} d(x, x_j),
$$

where $d(x, x_j) = \arccos(x \cdot x_j)$ is the geodesic distance between x_j and x.

Next, we state the following Lemma 3.2, whose proof can be found in [27, Corollary 17.12].

Lemma 3.2 *Suppose that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ has mesh norm $h_X \leq \frac{1}{2L}$ for some integer $L \geq 1$. Then there exist functions $\alpha_j : \mathbb{S}^2 \to \mathbb{R}$ for $j = 1, \ldots, N$ such that*
(i) $\sum_{j=1}^{N} \alpha_j(x) p(x_j) = p(x)$, $\quad \forall p \in \mathcal{P}_L$, $\forall x \in \mathbb{S}^2$,
(ii) $\sum_{j=1}^{N} |\alpha_j(x)| \leq 2$, $\quad \forall x \in \mathbb{S}^2$.

With the above obtained results we can provide the following crucial result about the pointwise error estimate for the hybrid interpolation.

Theorem 3.3 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel defined by (2.3) and $a_l \sim (l+1)^{-2s}$, $s > 1$. Assume that integer $L \geq 1$ and that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $1/(2L+2) < h_X \leq 1/(2L)$. For $f \in \mathcal{N}_\phi$, let $I_{X,L}f \in V_{X,L}$ be the hybrid interpolation defined in Section 2.2. Then for a fixed $x \in \mathbb{S}^2$, we have*

$$
|f(x) - I_{X,L}f(x)| \leq C h_X^{s-1} \|f - I_{X,L}f\|_{\mathcal{N}_\phi}.
$$

Proof. Because $h_X \leq \frac{1}{2L}$, it follows that for each $x \in \mathbb{S}^2$ there exists $\alpha = \alpha(x) \in \mathbb{R}^N$ satisfying (i) and (ii) in Lemma 3.2. For (i), it means that a polynomial $p \in \mathcal{P}_L$ that vanishes at $x_1, \ldots, x_N$ must vanish identically, which verify that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a $\mathcal{P}_L$-*unisolvent* set of distinct points on $\mathbb{S}^2$. By using Theorem 3.2, we only have to give the estimate of the factor $\left(\mathcal{L}_x(\alpha) \right)^{1/2}$,

$$
\mathcal{L}_x(\alpha) := \phi(x,x) - 2 \sum_{j=1}^{N} \alpha_j \phi(x,x_j) + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i, x_j)
$$

$$
= \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) a_l \left[P_l(x \cdot x) - \sum_{j=1}^{N} \alpha_j P_l(x \cdot x_j) - \sum_{j=1}^{N} \alpha_j \left(P_l(x \cdot x_j) - \sum_{i=1}^{N} \alpha_i P_l(x_i \cdot x_j) \right) \right],
$$

in which the terms with $l \leq L$ vanish by property (i) of Lemma 3.2. Hence

$$
\mathcal{L}_x(\alpha) := \frac{1}{4\pi} \sum_{l=L+1}^{\infty} (2l+1) a_l \left(P_l(x \cdot x) - 2 \sum_{j=1}^{N} \alpha_j P_l(x \cdot x_j) + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j P_l(x_i \cdot x_j) \right),
$$

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and since $|P_l(z)| \leq 1$, $\sum_{j=1}^{N} |\alpha_j| \leq 2$ and $a_l \sim (l+1)^{-2s}$, we have

$$
\begin{aligned}
|\mathcal{L}_x(\alpha)| &\leq \frac{1}{4\pi} \sum_{l=L+1}^{\infty} (2l+1)a_l \left(1 + 2\sum_{j=1}^{N} |\alpha_j| + \sum_{i=1}^{N}\sum_{j=1}^{N} |\alpha_i||\alpha_j| \right) \\
&\leq C \sum_{l=L+1}^{\infty} (2l+1)a_l \leq C \sum_{l=L+1}^{\infty} (l+1)^{-2s+1} \\
&\leq C \int_{L}^{\infty} (x+1)^{-2s+1} dx = C(L+1)^{-2s+2} \leq Ch_X^{2s-2}.
\end{aligned}
$$

With the help of Theorem 3.2, we see that

$$
|f(x) - I_{X,L}f(x)| \leq Ch_X^{s-1}\|f - I_{X,L}f\|_{\mathcal{N}_\phi}.
$$

This completes the proof of Theorem 3.3.

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INVESTIGATING DYNAMICS OF THE RATIONAL DIFFERENCE EQUATION
$$x_{n+1} = \frac{x_{n-1}}{A + Bx_n x_{n-1}}$$

MALEK GHAZEL, TAHER S. HASSAN, AND AHMED M. MOSALLEM

ABSTRACT. This paper is devoted to investigate the dynamics of the rational difference equation
$$x_{n+1} = \frac{x_{n-1}}{A + Bx_n x_{n-1}}$$
with arbitrary initial conditions A and B as nonzero real numbers. The solution is obtained and analytical study and asymptotic behavior are investigated. The forbidden set is determined. The existence of periodic and oscillatory solutions are discussed. Our results are illustrated with numerical simulations.

1. INTRODUCTION

The study of difference equation has been of great interest and many spectacular developments have been witnessed in the last decade. They are also used to present many numerical schemes in an easiest manner [1–16]. This is largely due to the fact that it appears as direct mathematical models describing real life situations in physics and engineering [5, 6], biology [8], game theory [7, 9, 10, 12, 13, 19] and economy [14, 15]. Therefore, the study of behavior and global stability of nonlinear difference equations is of paramount importance and rational difference equations are one of the most practical classes of equations. Immense literature is available on the second order difference equations of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}},$$

where α, β, γ, A, B and C and the initial conditions x_{-1}, x_0 are real numbers. In a particular case when $\gamma = C = 0$, this equation is known as the first order Riccati difference equation which can also be written in the form $x_{n+1} = a + \dfrac{b}{x_n}$. The results such as Agarwal *et al* [17], investigated the global stability, periodic nature and solved some particular cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Elsayed [18] studied the dynamical behavior and gave the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-2}x_{n-5}}.$$

Aloqeili [11] found the solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [20] determined the global stability and obtained the positive solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

1

Elabbasy *et al* [21, 22] obtained the solution in some particular cases and studied the global stability, periodicity of the following difference equations

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}} \text{ and } x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \Pi_{p=0}^{k} x_{n-p}}.$$

The problem of existence of solutions for a given difference equation is of great importance. The primary aim is to find the set $\mathcal{F}$ of all initial values at which the solution of the given equation is not defined for all natural number n. The set of this nature is called the forbidden set of the equation. In order to avoid the appearance of the forbidden set, the common assumption used by some authors, while studying rational difference equations, is to choose positive initial values and coefficients. The interest of this problem has increased in the literature recently [23–25]. Azizi [26] found the forbidden set of the second order rational Riccati difference equation. Also, Balibrea *et al* [27] gave sufficient conditions for a rational difference equation of order two to be not uniformly eventually positive outside a bounded set. Camouzis *et al* [28] described the forbidden set of the difference equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}.$$

In [29] Sedaghat studied the existence of solutions of certain singular difference equations. Stevic̀ [30] studied the domains for which the solutions of some equations and systems of difference equations are not well-defined.

The study of existence of oscillatory solutions (periodic or aperiodic) of difference equations is in a great concern and it is extremely useful in the behavior of mathematical models describing real live situations, for some results in this area. Ladas [31] studied the oscillation of positive solutions about the positive steady state N in the delay logistic difference equation

$$N_{n+1} = N_n \exp\left(r - r \sum_{j=0}^{m} p_j N_{n-j} \right),$$

which describes that the population growth is not continuous but seasonal.

Matti [32] studied the oscillations in some nonlinear economic relationships modeled by a difference equations. Sedaghat [33] studied the oscillations and chaos in a discrete model of combat. See also related results [34–37].

Motivated by above, in this paper, we will present complete analytical study and asymptotic behavior of the solutions of the more general second order difference equation

(1.1) $$x_{n+1} = \frac{x_{n-1}}{A + Bx_n x_{n-1}}, \ x_0 = c \text{ and } x_{-1} = d,$$

with arbitrary parameters A and B. To the best of our knowledge, the analysis for convergence, oscillation and periodicity of equation (1.1) have not been considered till now and other results extend and improve existing results in the literature, especially those established in [11, 20].

Throughout the paper we use the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\prod_{p=n}^{m} a_p = 1$ and $\sum_{p=n}^{m} a_p = 0$, where $(a_p)_p$ is a sequence of real numbers and $m < n$ for $m, n \in \mathbb{Z}$ and the cases when $AB = 0$ and $A + B \neq 0$ are trivial, therefore we will assume that $A \neq 0$ and $B \neq 0$.

2. Stability analysis of the equilibrium points

Before stating stability analysis of the equilibrium points, we begin with the following theorem which will given equilibrium points of Eq. (1.1).

Theorem 1. *Let* $(x_n)_{n \geq -1}$ *be a solution of Eq.* (1.1).

(1) *If $B(1 - A) \le 0$, then the Eq. (1.1) has a unique equilibrium point $\bar{x}_1 = 0$.*
(2) *If $B(1 - A) > 0$, then the Eq. (1.1) has exactly three equilibrium points*

$$\bar{x}_1 = 0 \quad and \quad \bar{x}_{2,3} = \pm\sqrt{\frac{1 - A}{B}}.$$

Proof. Let $\bar{x}$ be a equilibrium point of Eq. (1.1). It is easy to see that

$$\bar{x} = 0 \quad \text{or} \quad \bar{x}^2 = \frac{1 - A}{B}.$$

This completes the proof. ∎

Now, we will prove the following stability analysis of the equilibrium points for equation (1.1).

Theorem 2. *Let $(x_n)_{n \ge -1}$ be a solution of Eq. (1.1). Then:*
(1) *For $A < 0$, the characteristic equation about the equilibrium point $\bar{x}_1$ has no real roots.*
(2) *For $0 < A < 1$, the equilibrium point $\bar{x}_1$ is a repeller.*
(3) *For $A = 1$, the equilibrium point $\bar{x}_1$ is nonhyperbolic.*
(4) *For $A > 1$, the equilibrium point $\bar{x}_1$ is locally asymptotically stable.*

Moreover, for $B(1 - A) > 0$,
(i) *The equilibrium points $\bar{x}_{2,3}$ are nonhyperbolic.*
(ii) *If $0 < |A| < 1$, then the equilibrium points $\bar{x}_{2,3}$ are unstable.*

Proof. Denote by $U := (u_0, u_1)$ an arbitrary point in the good set of Eq. (1.1) and $\bar{x}$ be an equilibrium point of Eq. (1.1), recall that the characteristic equation about the equilibrium point $\bar{x}$ is defined as

$$(2.2) \qquad \lambda^2 - q_0\lambda - q_1 = 0,$$

where $q_k = \dfrac{\partial F}{\partial u_k}(\bar{x}, \bar{x})$, $k = 0, 1$ with $F(u_0, u_1) = \dfrac{u_1}{A + Bu_0u_1}$. Since the partial derivative of the function F are

$$(2.3) \qquad \frac{\partial F}{\partial u_0} = \frac{-Bu_1}{(A + Bu_0u_1)^2} \quad \text{and} \quad \frac{\partial F}{\partial u_1} = \frac{A}{(A + Bu_0u_1)^2},$$

so, for the equilibrium point $\bar{x}_1 = 0$, the coefficients of the characteristic equation are $q_0 = \dfrac{\partial F}{\partial u_0}(0, 0) = 0$ and $q_1 = \dfrac{\partial F}{\partial u_1}(0, 0) = \dfrac{1}{A}$. Hence the characteristic equation about the equilibrium point $\bar{x}_1$ is

$$(2.4) \qquad \lambda^2 - \frac{1}{A} = 0.$$

Thus, we have the following cases:

(1) If $A < 0$, then the Eq. (2.4) has no real roots.

(2) If $0 < A < 1$, then the real roots of Eq. (2.4) are $\pm\sqrt{\dfrac{1}{A}}$, their absolute values are greater than one which implies that the equilibrium point $\bar{x}_1$ is a repeller.

(3) If $A = 1$, then the real roots of Eq. (2.4) are ± 1, so $\bar{x}_1$ is nonhyperbolic.

(4) If $A > 1$, then all real roots of Eq. (2.4) have absolute value less than one, so $\bar{x}_1$ is locally asymptotically stable.

In the case when $B(1 - A) > 0$, two new equilibrium points appear $\bar{x}_2$ and $\bar{x}_3$. According to the Eq. (2.3), the coefficients q_0 and q_1 of their characteristic equations are the same and they are given as $q_0 = A - 1$ and $q_1 = A$, so the characteristic equation about $\bar{x}_k$, $k = 2, 3$ is

$$\lambda^2 - (A - 1)\lambda - A = 0,$$

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which has -1 and A as real roots, then $\bar{x}_2$ and $\bar{x}_3$ are nonhyperbolic. Furthermore, if $|A| < 1$, then $\bar{x}_2$ and $\bar{x}_3$ are unstable. This completes the proof. ∎

3. ANALYTICAL EXPRESSIONS OF $(x_n)_{n \geq -1}$

In this section, we give some analytical expressions of the sequence $(x_n)_{n \geq -1}$, where $(x_n)_{n \geq -1}$ is a solution of Eq. (1.1).

Theorem 3. *Let $(x_n)_{n \geq -1}$ be a solution of Eq. (1.1). Then for all integer $n \in \mathbb{N}$,*

$$(3.5) \qquad x_{2n-1} = d \prod_{p=0}^{n-1} \left(\frac{A^{2p} + Bcd \sum_{k=0}^{2p-1} A^k}{A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k} \right),$$

and

$$(3.6) \qquad x_{2n} = c \prod_{p=0}^{n-1} \left(\frac{A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k}{A^{2p+2} + Bcd \sum_{k=0}^{2p+1} A^k} \right).$$

Proof. We show it by induction. First we have

$$x_{-1} = d \prod_{p=0}^{-1} \left(\frac{A^{2p} + Bcd \sum_{k=0}^{2p-1} A^k}{A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k} \right) = d$$

and

$$x_0 = c \prod_{p=0}^{-1} \left(\frac{A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k}{A^{2p+2} + Bcd \sum_{k=0}^{2p+1} A^k} \right) = c$$

This shows that (3.5) and (3.6) hold for $n = 0$. Assume (3.5) and (3.6) hold with n replaced by some $k \in \mathbb{N}$. From Eq. (1.1) we get

$$
\begin{aligned}
x_{2(k+1)-1} &= x_{2k+1} = \frac{x_{2k-1}}{A + Bx_{2k}x_{2k-1}} \\
&= \left(d \prod_{p=0}^{n-1} \left(A^{2p} + Bcd \sum_{k=0}^{2p-1} A^k \right) \prod_{p=0}^{n-1} \left(A^{2p+2} + Bcd \sum_{k=0}^{2p+1} A^k \right) \right) \Big/ \\
&\quad \left(\prod_{p=0}^{n-1} \left(A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k \right) \right. \\
&\quad \left. \left[A \prod_{p=0}^{n-1} \left(A^{2p+2} + Bcd \sum_{k=0}^{2p+1} A^k \right) + Bcd \prod_{p=0}^{n-1} \left(A^{2p} + Bcd \sum_{k=0}^{2p-1} A^k \right) \right] \right) \\
&= d \prod_{p=0}^{k} \left(\frac{A^{2p} + Bcd \sum_{k=0}^{2p-1} A^k}{A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k} \right).
\end{aligned}
$$

and

$$
\begin{aligned}
x_{2(k+1)} &= x_{2k+1+1} = \frac{x_{2k}}{A + Bx_{2k+1}x_{2k}} \\
&= \left(c \prod_{p=0}^{k-1} \left(A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k \right) \prod_{p=0}^{k} \left(A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k \right) \right) \Big/ \\
&\quad \left(\prod_{p=1}^{k} \left(A^{2p} + Bcd \sum_{k=0}^{2p-1} A^k \right) \right. \\
&\quad \left. \left[A \prod_{p=0}^{k} \left(A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k \right) + Bcd \prod_{p=0}^{k-1} \left(A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k \right) \right] \right) \\
&= c \prod_{p=0}^{k} \left(\frac{A^{2p+1} + Bcd \sum_{k=0}^{2p} A^k}{A^{2p+2} + Bcd \sum_{k=0}^{2p+1} A^k} \right).
\end{aligned}
$$

This shows that (3.5) and (3.6) hold for $k + 1$. Therefore, (3.5) and (3.6) hold for $n \in \mathbb{N}$. This completes the proof. ∎

Corollary 4. *Let $(x_n)_{n \geq -1}$ be a solution of Eq. (1.1). Then:*
    (1) *for $A \neq 1$,*

$$
(3.7) \qquad x_{2n-1} = d \prod_{p=0}^{n-1} \frac{(A - 1 + Bcd)A^{2p} - Bcd}{(A - 1 + Bcd)A^{2p+1} - Bcd},
$$

    and

$$
x_{2n} = c \prod_{p=0}^{n-1} \frac{(A - 1 + Bcd)A^{2p+1} - Bcd}{(A - 1 + Bcd)A^{2p+2} - Bcd}.
$$

(2) *for $A = 1$,*

(3.8)
$$x_{2n-1} = d \prod_{p=0}^{n-1} \Big(\frac{1 + 2pBcd}{1 + (2p+1)Bcd} \Big),$$

and

$$x_{2n} = c \prod_{p=0}^{n-1} \Big(\frac{1 + (2p+1)Bcd}{1 + (2p+2)Bcd} \Big).$$

Proof. It is sufficient to use in the (3.5) and (3.6), the identity

$$\sum_{k=0}^{p} x^k = \frac{1 - x^{p+1}}{1 - x},$$

where p is a nonnegative integer and x is a real numbers different of one, and the proof is directly obtained. ∎

4. Main Results

4.1. The forbidden set. The determination of the set of all initial conditions through which the solution of a given difference equation is defined for all $n \in \mathbb{N}$ is in general a problem of great difficulty. This problem leads to introduce the notion of forbidden set.

Definition 1. Consider a difference equation of order k in $\mathbb{N}$

(4.9) $x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-(k-1)})$ for $n \in \mathbb{N}$,

where $F = F(u_0, u_1, ..., u_{k-1})$ is a function that maps on some subset Ω in $\mathbb{R}^k$, and let $(x_0, x_{-1}, \dots, x_{-k+1}) \in \Omega$ be the vector of initial conditions of the Eq. (4.9). The forbidden set of Eq. (4.9) is the set denoted $\mathcal{F}$ defined as the set of all vectors of initial conditions $(x_0, x_{-1}, ..., x_{-k+1})$ through which the solution of Eq. (4.9) is not defined for all positive integer n. The good set $\mathcal{G}$ is the complementary in Ω of the forbidden set, consequently, the solution $(x_n)_n$ of Eq. (4.9) is well defined for all $n \in \mathbb{N}$ if and only if $(x_0, x_{-1}, ..., x_{-k+1}) \in \mathcal{G}$.

When we obtain the analytic expression of the solution for a given difference equation, the determination of the forbidden set becomes more easy to obtain. However it can be gotten in some particular cases by the mean of substitution, in the following Theorem, we give the forbidden set in the case when $A = 1$.

Theorem 5. *Let $(x_n)_{n \geq -1}$ be a solution of the Eq. (1.1) and $\mathcal{F}$ be the forbidden set of the sequence $(x_n)_{n \geq -1}$. If $A = 1$, then*

$$\mathcal{F} = \Big\{ (c, d) \in \mathbb{R}^2 \ such \ that \ cd \in \Big\{ \frac{-1}{nB}, \ n \in \mathbb{N} \Big\} \Big\}.$$

Proof. The sequence $(x_n)_{n \geq -1}$ satisfies the equation

$$Bx_{n+1}x_n = \frac{Bx_nx_{n-1}}{A + Bx_nx_{n-1}}, \quad x_{-1} = d, \ x_0 = c,$$

Hence,

(4.10) $A + Bx_{n+1}x_n = A + 1 - \dfrac{A}{A + Bx_nx_{n-1}}.$

Let $(y_n)_{n \geq 0}$ be the sequence defined as

(4.11) $y_n := A + Bx_nx_{n-1},$

So the Eq. (4.10) can be written as

$$y_{n+1} = A + 1 - \frac{A}{y_n},$$

which is a first order Ricatti difference equation. If $A = 1$, then

$$y_{n+1} = 2 - \frac{1}{y_n} \quad \text{for } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$, for y_n to exist a necessary and sufficient condition are that for all integer $0 \leq k \leq n - 1$, $y_k \neq 0$,

$$y_0 \neq 0 \text{ is equivalent to } cd \neq \frac{-1}{B},$$

$$y_1 \neq 0 \text{ is equivalent to } y_0 \neq 0 \text{ and } y_0 \neq \frac{1}{2},$$

and

$$y_2 \neq 0 \text{ iff } y_0 \notin \left\{ 0, \frac{1}{2}, \frac{2}{3} \right\}.$$

By induction, we can easily prove that for all $n \in \mathbb{N}$,

$$y_n \neq 0 \text{ iff for all } k \leq n + 1, \quad y_0 \neq \frac{k-1}{k}.$$

So the forbidden set of the sequence $Y = (y_n)_{n \geq 0}$ is $\mathcal{F}_Y = \{ \frac{n-1}{n}, \quad n \in \mathbb{N} \}$. Now, let $n \in \mathbb{N}$, $y_0 = \frac{n-1}{n}$ is equivalent to $1 + Bcd = \frac{n-1}{n}$ which is equivalent to $cd = \frac{-1}{nB}$. Thus, the forbidden set of the sequence $(x_n)_{n \geq -1}$ is given by

$$\mathcal{F} = \left\{ (c, d) \in \mathbb{R}^2 \text{ such that } cd \in \left\{ \frac{-1}{nB}, n \in \mathbb{N} \right\} \right\}.$$

The proof is complete. ∎

This results can be immediately found by using Corollary 4. Also, in the case when $A \neq 1$, the forbidden set $\mathcal{F}$ of Eq. (1.1) can be easily obtained by using Corollary 4 as in the following theorem.

Theorem 6. *Let $(x_n)_{n \geq -1}$ be a solution of the Eq. (1.1). Suppose that $A \neq 1$, then the forbidden set of the sequence $(x_n)_{n \geq -1}$ is*

$$\mathcal{F} = \left\{ (c, d) \in \mathbb{R}^2 \text{ such that } \begin{cases} A = -1 \quad and \quad cd = \dfrac{1}{B} \\ or \\ A \neq -1 \quad and \quad cd \in \left\{ \dfrac{(1-A)A^n}{B(A^n - 1)}, n \in \mathbb{N} \right\}, \end{cases} \right\}.$$

4.2. Convergence. In this section, we study the asymptotic behavior of a solution of the difference Eq. (1.1).

4.2.1. *The case when $0 < |A| < 1$.*

Theorem 7. *Let $(x_n)_{n \geq -1}$ be a solution of the Eq. (1.1). Assume that $|A| < 1$, then the subsequences (x_{2n-1}) and (x_{2n}) converge.*

Proof. Using Corollary 4, we obtain

$$
\begin{aligned}
x_{2n-1} &= d \prod_{p=0}^{n-1} \frac{(A-1+Bcd)A^{2p} - Bcd}{(A-1+Bcd)A^{2p+1} - Bcd} \\
&= d \prod_{p=0}^{n-1} \frac{1 - \frac{A-1+Bcd}{Bcd} A^{2p}}{1 - \frac{A-1+Bcd}{Bcd} A^{2p+1}} \\
&= d \prod_{p=0}^{n-1} U_p,
\end{aligned}
$$

where

$$
U_p := \frac{1 - \alpha A^{2p}}{1 - \alpha A^{2p+1}} \text{ with } \alpha := \frac{A-1+Bcd}{Bcd}.
$$

One of the following cases holds: For p big enough, U_p is always is in $(0,1)$ or lies greater than one, this allows us to apply of the Taylor expansion to the sequence $(U_p)_{p\geq 0}$ which gives that

$$
U_p \text{ is asymptotically equivalent to } 1 - \alpha(A-1)A^{2p},
$$

which is the general term of convergent infinite product, thus (x_{2n-1}) converges.
Again by using Corollary 4, we get

$$
\begin{aligned}
x_{2n} &= c \prod_{p=0}^{n-1} \frac{(A-1+Bcd)A^{2p+1} - Bcd}{(A-1+Bcd)A^{2p+2} - Bcd} \\
&= c \prod_{p=0}^{n-1} \frac{1 - \frac{A-1+Bcd}{Bcd} A^{2p+1}}{1 - \frac{A-1+Bcd}{Bcd} A^{2p+2}} \\
&= c \prod_{p=0}^{n-1} T_p,
\end{aligned}
$$

where

$$
T_p := \frac{1 - \alpha A^{2p+1}}{1 - \alpha A^{2p+2}} \text{ with } \alpha = \frac{A-1+Bcd}{Bcd}.
$$

Hence,

$$
T_p \text{ is asymptotically equivalent to } 1 - \alpha(1-A)A^{2p+1},
$$

the last term is the general term of convergent infinite product, then $(x_{2n})_n$ converges. This completed the proof. ∎

4.2.2. *The case when $A = -1$.*

Lemma 8. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1). Assume that $A = -1$, then*
  (1) *The subsequence $(x_{2n-1})_n$ converges iff $Bcd \in (-\infty, 0) \cup [2, \infty)$.*
  (2) *The subsequence $(x_{2n})_n$ converges iff $Bcd \in (0, 2]$.*

Proof.   1. Replacing A by -1 in Corollary (4), for the subsequence $(x_{2n-1})_n$, we obtain

$$
\begin{aligned}
x_{2n-1} &= d\left(\frac{-2}{2-2Bcd}\right)^n \\
&= \frac{d}{(Bcd-1)^n},
\end{aligned}
$$

then

$$(x_{2n-1})_n \text{ converges iff } \begin{cases} |Bcd - 1| > 1, \\ or \\ Bcd - 1 = 1, \end{cases}$$

the last system is equivalent to $Bcd \in (-\infty, 0) \cup [2, \infty)$.

2. To prove the second part of the Theorem, we replace A by (-1) in Corollary (4) for the subsequence $(x_{2n})_n$, we get

$$\begin{aligned} x_{2n} &= c\Big(\frac{2Bcd - 2}{-2}\Big)^n \\ &= c(1 - Bcd)^n, \end{aligned}$$

then

$$(x_{2n})_n \text{ converges iff } \begin{cases} |Bcd - 1| < 1, \\ or \\ Bcd - 1 = 1, \end{cases}$$

the last system holds iff $Bcd \in (0, 2]$. As a result, the proof is completed.

∎

Remark 1. Using the computation in the proof of Lemma (8), we can easily deduce that when $A = -1$, we have

(1) If $Bcd \in (-\infty, 0) \cup (2, \infty)$, then (x_{2n-1}) converges to zero and $(|x_{2n}|)$ goes to infinity.
(2) If $Bcd \in (0, 2)$, then $(|x_{2n-1}|)$ goes to infinity and (x_{2n}) converges to zero.
(3) If $Bcd = 2$, then the subsequences (x_{2n-1}) and (x_{2n}) are constant, $x_{2n-1} = d$ and $x_{2n} = c$.

The following theorem is now proved.

Theorem 9. *Let $(x_n)_{n \geq -1}$ be a solution of the Eq. (1.1). Assume that $A = -1$, then*

The whole sequence $(x_n)_{n \geq -1}$ converges iff $B > 0$ and $c = d = \pm\sqrt{\dfrac{2}{B}}$.

In this case, $(x_n)_{n \geq -1}$ is constant and equal $\pm\sqrt{\dfrac{2}{B}}$.

4.2.3. *The case when $A = 1$.*

Theorem 10. *Let $(x_n)_{n \geq -1}$ be a solution of the Eq. (1.1). Assume that $A = 1$, then $(x_n)_{n \geq -1}$ converges to zero.*

Proof. Replacing A by 1, then by Eq. (3.8),

$$\begin{aligned} x_{2n-1} &= d\prod_{p=0}^{n-1}\Big(\frac{1 + 2pBcd}{1 + (2p+1)Bcd}\Big) \\ &= d\prod_{p=0}^{n-1} V_p, \end{aligned}$$

where $(V_p)_{p \geq 1}$ is the sequence defined as

(4.12) $$V_p = 1 - \frac{Bcd}{1 + (2p+1)Bcd}.$$

It can be easily verified that there exists a positive integer r_0 such that for all $p \geq r_0$, we have $V_p \in (0, 1)$. Therefore, if p is big enough, the x_{2n-1} is then written in infinite series form as

$$(4.13) \qquad x_{2n-1} = d \Big(\prod_{p=0}^{r_0-1} V_p \Big) \exp \Big(\sum_{p=r_0}^{n-1} \ln V_p \Big).$$

We have $\ln V_p$ is equivalent to $\dfrac{-1}{2p}$ which is a general term divergence infinite series, since for all $p \geq r_0$ $V_p \in (0,1)$, then the infinite series $\displaystyle\sum_{p \geq r_0} \ln V_p$ goes to $-\infty$, consequently $(x_{2n-1})_n$ converges to zero.

Although the proof of the convergence of the subsequence $(x_{2n})_n$ to zero can be done similarly, we describe in order to use its notations in the sequel, from Eq. (3.7), we can see that

$$
\begin{aligned}
x_{2n} &= c \prod_{\substack{p=0}}^{n-1} \Big(\frac{1 + (2p+1)Bcd}{1 + (2p+2)Bcd} \Big) \\
&= c \prod_{p=0}^{n-1} W_p,
\end{aligned}
$$

where $(W_p)_{p \geq 0}$ is the sequence defined as

$$(4.14) \qquad W_p = 1 - \frac{Bcd}{1 + (2p+2)Bcd}.$$

Similarly, it can be easily checked that there exist a positive integer s_0 such that for all $p \geq s_0$, we have $W_p \in (0,1)$. Hence if p is big enough, the subsequence x_{2n} is then written as

$$(4.15) \qquad x_{2n} = c \Big(\prod_{p=0}^{s_0-1} W_p \Big) \exp \Big(\sum_{p=s_0}^{n-1} \ln W_p \Big).$$

We have $\ln W_p$ is equivalent to $\dfrac{-1}{2p}$ which is a general term divergence infinite series, since for all $p \geq s_0$ $W_p \in (0,1)$, then the infinite series $\displaystyle\sum_{p=s_0}^{n-1} \ln W_p$ goes to $-\infty$, consequently $(x_{2n})_n$ converges to zero. This complete the proof of Theorem.   ∎

4.2.4. *The case when $|A| > 1$.*

Theorem 11. *Let $(x_n)_{n \geq -1}$ be a solution of the Eq. (1.1). Assume that $|A| > 1$, then*

(1) *The subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converges.*
(2) *The whole sequence $(x_n)_{n \geq -1}$ converge iff*

$$
\begin{cases}
A - 1 + Bcd \neq 0 \\
or \\
(1-A)B > 0 \ and \ c = d = \pm \sqrt{\dfrac{1-A}{B}}.
\end{cases}
$$

Proof. We distinguish two cases:

(1) (I) If $A - 1 + Bcd \neq 0$, then using Corollary (4),

$$\begin{aligned}
x_{2n-1} &= d \prod_{p=0}^{n-1} \left(\frac{(A - 1 + Bcd)A^{2p} - Bcd}{(A - 1 + Bcd)A^{2p+1} - Bcd} \right) \\
&= d \prod_{p=0}^{n-1} \left(\frac{1 - \dfrac{Bcd}{(A - 1 + Bcd)A^{2p}}}{A(1 - \dfrac{Bcd}{(A - 1 + Bcd)A^{2p+1}})} \right) \\
&= \frac{d}{A^n} \prod_{p=0}^{n-1} Y_p,
\end{aligned}$$

where $(Y_p)_{p \geq 0}$ is the sequence defined as

$$Y_p = \frac{1 - \dfrac{\beta}{A^{2p}}}{1 - \dfrac{\beta}{A^{2p+1}}} \text{ and } \beta = \frac{Bcd}{A - 1 + Bcd}.$$

It can be easily verified that for p big enough, always Y_p is in the interval $(0, 1)$ or lies in the interval $(1, \infty)$. The Taylor expansion applied to the sequence $(Y_p)_{p \geq 0}$ gives

$$(Y_p)_{p \geq 0} \text{ is equivalent to } 1 + \beta(\frac{1}{A} - 1)\frac{1}{A^{2p}},$$

the last term is a general term of convergent infinite product so $(x_{2n-1})_n$ converges to zero. An easy calculus gives that

$$x_{2n} = \frac{c}{A^n} \prod_{p=0}^{n-1} Z_p,$$

where $(Z_p)_{p \geq 0}$ is the sequence defined as

$$Z_p = \frac{1 - \dfrac{\beta}{A^{2p+1}}}{1 - \dfrac{\beta}{A^{2p+2}}},$$

we have

$$(Z_p)_{p \geq 0} \text{ is asymptotically equivalent to } 1 + \beta(\frac{1}{A} - 1)\frac{1}{A^{2p+1}},$$

the last term is a general term of convergent infinite product, so $(x_{2n})_n$ converges to zero.

(II) If $A - 1 + Bcd = 0$, then the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1} = d$ and $x_{2n} = c$, so they converge. By the calculus in the preview part of the proof, if $A - 1 + Bcd \neq 0$, then the whole sequence $(x_n)_{n \geq -1}$ converges to zero. When $A - 1 + Bcd = 0$ that is

(4.16)
$$cd = \frac{1 - A}{B},$$

the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant equal d and c respectively, then the whole sequence $(x_n)_{n \geq -1}$ converges if and only if $c = d$, using Eq. (4.16) the last proposition is equivalent to $c = d = \pm\sqrt{\dfrac{1 - A}{B}}$, for this to can hold it is necessary and sufficient that $(1 - A)B > 0$. Hence, the proof is achieved.

∎

4.3. Oscillation about the equilibrium point $\overline{x}_1 = 0$. In this section, we study the oscillation the solution of difference Eq. (1.1) about the equilibrium point $\overline{x}_1 = 0$.

Theorem 12. *Let $(x_n)_{n \geq -1}$ be a solution of the Eq. (1.1). Assume that $|A| < 1$, then the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converge, then*

(1) *For $|A| < 1$*

$$(x_n)_{n \geq -1} \text{ is oscillatory about zero iff } cd\Big(\prod_{p=0}^{n_0-1} U_p \Big)\Big(\prod_{p=0}^{m_0-1} T_p \Big) < 0,$$

where $(U_p)_p$, $(T_p)_p$ are the sequences defined in the proof of Theorem (7) and n_0, m_0, are integers such that, for all $p \geq n_0$, U_p is positive and for all $p \geq m_0$, T_p is positive.

(2) *For $A = -1$, $(x_n)_{n \geq -1}$ is oscillatory about zero.*

(3) *For $A = 1$,*

$$(x_n)_{n \geq -1} \text{ is oscillatory about zero iff } cd \prod_{p=0}^{r_0-1} V_p \prod_{p=0}^{s_0-1} W_p < 0,$$

where $(V_p)_{p \geq 1}$, $(W_p)_{p \geq 0}$, r_0 and s_0 are defined in the proof of Theorem (10).

(4) *For $|A| > 1$,*

$$(x_n)_{n \geq -1} \text{ is oscillatory about zero iff}$$

$$\begin{cases} A - 1 + Bcd = 0 \text{ and } cd < 0, \\ or \\ A - 1 + Bcd \neq 0 \text{ and } \begin{cases} A < -1, \\ or \\ A > 1 \text{ and } cd \prod_{p=0}^{p_0-1} Y_p \prod_{p=0}^{q_0-1} Z_p < 0, \end{cases} \end{cases}$$

where $(Y_p)_{p \geq 1}$, $(Z_p)_{p \geq 0}$, p_0 and q_0 are defined in the proof of Theorem (11).

Proof.   1. For $|A| < 1$ The sequences $(x_{2n-1})_n$ and $(x_{2n})_n$ have a constant signs which are these of

$$d \prod_{p=0}^{n_0-1} U_p \text{ and } c \prod_{p=0}^{m_0-1} T_p,$$

respectively, so we can immediately obtain the aimed result.

2. For $A = -1$, in this case $x_{2n-1} = \dfrac{d}{(Bcd-1)^n}$ and $x_{2n-1} = c(1 - Bcd)^n$. Hence, if $Bcd - 1 < 0$, then $(x_{2n-1})_n$ and therefore $(x_n)_n$ is oscillatory about zero. If $Bcd - 1 > 0$, then $(x_{2n})_n$ and therefore $(x_n)_n$ is oscillatory about zero.

3. For $A = 1$, the Eq. (4.13) and (4.15) give

$$x_{2n-1} = d \prod_{p=0}^{r_0-1} V_p \exp\Big(\sum_{p=r_0}^{n-1} \ln V_p \Big),$$

and

$$x_{2n} = c \prod_{p=0}^{q_0-1} W_p \exp\Big(\sum_{p=q_0}^{n-1} \ln W_p \Big),$$

in this case the sequences $(x_{2n-1})_n$ and $(x_{2n})_n$ have a constant signs which are these of

$$d \prod_{p=0}^{r_0-1} V_p \text{ and } c \prod_{p=0}^{q_0-1} W_p,$$

respectively, we find that $(x_n)_n$ is oscillatory about zero iff

$$cd \prod_{p=0}^{r_0-1} V_p \prod_{p=0}^{q_0-1} W_p < 0.$$

4. For $|A| > 1$, if $A - 1 + Bcd = 0$, then the subsequences are constant $(x_{2n-1})_n$ and $(x_{2n})_n$ equal d and c respectively, so $(x_n)_n$ is oscillatory about zero iff $cd < 0$. If $A - 1 + Bcd \neq 0$, the the sequence $(x_n)_{n\geq -1}$ converges to zero and we have

$$x_{2n-1} = \frac{d}{A^n} \prod_{p=0}^{n-1} Y_p \text{ and } x_{2n} = \frac{c}{A^n} \prod_{p=0}^{n-1} Z_p,$$

where $(Y_p)_p$ and $(Z_p)_p$ are the sequences defined in the proof of Theorem (11). It has been seen that there exists integers p_0 and q_0 such that for all $p \geq p_0$, Y_p is positive and for all $p \geq q_0$, Z_p is positive, then for n big enough, the sign of $d \prod_{p=0}^{n-1} Y_p$ and $c \prod_{p=0}^{n-1} Z_p$ are constant. Then, we have the following cases:

(a) When $A < -1$, the sequence $(x_{2n-1})_n$ and consequently $(x_n)_{n\geq -1}$ are oscillatory about zero.

(b) When $A > 1$, the sign of x_{2n-1} is that of $d \prod_{p=0}^{p_0-1} Y_p$ and the sign of x_{2n} is that of $c \prod_{p=0}^{q_0-1} Z_p$.

Thus, we can immediately have the target result and the proof is complete.
∎

4.4. Periodicity. Firstly, we recall the following Lemma, which describes sufficient conditions for Eq. (1.1) to have a periodic solution.

Lemma 13. *Let $(x_n)_{n\geq -k+1}$ be a solution of Eq. (1.1). Suppose that there are real numbers l_r, $r = 0, 1, ..., p - 1$ such that*

$$\lim_{n\to\infty} x_{pn+r} = l_r \text{ for all } r = 0, 1, ..., p - 1.$$

Finally, let $(y_n)_{n\geq -k+1}$ be the periodic-p sequence such that

$$y_r = l_r \text{ for all } r = 0, 1, ..., p - 1.$$

Then $(y_n)_{n\geq -k+1}$ is a periodic-p solution of Eq. (1.1).

Note that the zero sequence is a solution of Eq. (1.1) corresponding to the initial conditions $x_{-1} = 0$ and $x_0 = 0$, this solution is called trivial solution of of Eq. (1.1). The periodicity results are given by the following Theorem

Theorem 14. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1).*

(1) *For $|A| < 1$, Eq. (1.1) has a nontrivial periodic-2 solution.*

(2) *For $A = -1$, Eq. (1.1) has a nontrivial periodic-2 solution if and only if $cd = \dfrac{2}{B}$.*

(3) *For $A = 1$, Eq. (1.1) has no nontrivial periodic-2 solution.*

(4) *For $|A| > 1$, Eq. (1.1) has a nontrivial periodic-2 solution if and only if $cd = \dfrac{1-A}{B}$.*

Proof.   1. If $|A| < 1$, then by Theorem (7), the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converge, let l_1 and l_0 be their limits respectively. Applying Lemma (13), it follow that the sequence

$$l_1, l_0, l_1, l_0, ...$$

is a periodic-2 solution of Eq. (1.1).

2. Suppose that $A = -1$, we distinguish two cases:

    (a) If $Bcd \neq 2$, then using Lemma (8), every solution of Eq. (1.1) is unbounded, so Eq. (1.1) has no periodic solutions.

    (b) If $Bcd = 2$, then using Lemma (8), the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1} = d$ and $x_{2n} = c$, therefore $(x_n)_{n \geq -1}$ is the periodic-2 solution

$$d, c, d, c, \dots .$$

3. If $A = 1$, then by using the proof of Theorem (10), every solution of of Eq. (1.1) converges to zero, so Eq. (1.1) has no nontrivial solution.

4. If $|A| > 1$, we distinguish two cases:

    (a) If $A - 1 + Bcd \neq 0$, then by using the proof of Theorem (11), every solution of Eq. (1.1) converges to zero, so Eq. (1.1) has no nontrivial solution.

    (b) If $A - 1 + Bcd = 0$, then by Theorem (11), the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1} = d$ and $x_{2n} = c$, consequently $(x_n)_{n \geq -1}$ is the periodic-2 solution $d, c, d, c, \dots$

This achieves the proof. ∎

5. Numerical simulation

(1) The case $|A| < 1$ is illustrated in Fig. (1), in which we set $A = \dfrac{1}{2}$, $B = 4$, $c = 3$ and $d = 2$. The subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converge. This is coherent with Theorem (7).

(2) In Fig. (2) (case $A = -1$ and $Bcd \in (-\infty, 0) \cup (2, \infty)$, we choose $A = -1$, $B = \dfrac{1}{2}$, $c = 1$ and $d = -2$. The subsequence $(x_{2n-1})_n$ converges to zero and the subsequence $(|x_{2n}|)_n$ goes to infinity and oscillates about zero which matches Lemma (8), Remark (1) and Theorem (12).

(3) The case $A = -1$ and $Bcd \in (0, 2)$ is studied using the parameters values $A = -1$, $B = \dfrac{1}{2}$, $c = 3$ and $d = 1$. The subsequence $(|x_{2n-1}|)_n$ goes to infinity and the subsequence $(x_{2n})_n$ converges to zero as depicted in Fig. (3) which is coherent to Lemma (8), Remark (1) and Theorem (12).

(4) In order to illustrate the case $A = -1$ and $Bcd = 2$, we choose $A = -1$, $B = \dfrac{1}{2}$, $c = 1$ and $d = 4$. In Fig. (4), it is shown that the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1} = d$ and $x_{2n} = c$ which agrees Lemma (8) and Remark (1), consequently $(x_n)_{n \geq -1}$ is the periodic-2 solution

$$d, c, d, c, \dots .$$

This is in harmony with Theorem (14).

(5) The case $A = 1$ is investigated using the parameters values $A = 1$, $B = 3$, $c = 0.5$ and $d = 3$. In Fig. (5), the simulation results show that the whole sequence $(x_n)_{n \geq -1}$ converges to zero which matches Theorem (10).

(6) The case $|A| > 1$ and $A - 1 + Bcd \neq 0$ can be taken by choosing $A = 5$, $B = 1$, $c = 3$ and $d = 0.5$. The whole sequence $(x_n)_{n \geq -1}$ converges to zero as depicted in Fig. (6) which is coherent to Theorem (11).

(7) Fig. (7) illustrates the case $|A| > 1$ and $A - 1 + Bcd = 0$, we choose $A = 5$, $B = 1$, $c = 2$ and $d = -2$, the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant: $x_{2n-1} = d$ and $x_{2n} = c$, we obtain a periodic-2 solution. This case is justified analytically in the proofs of Theorems (11), (12) and (14).
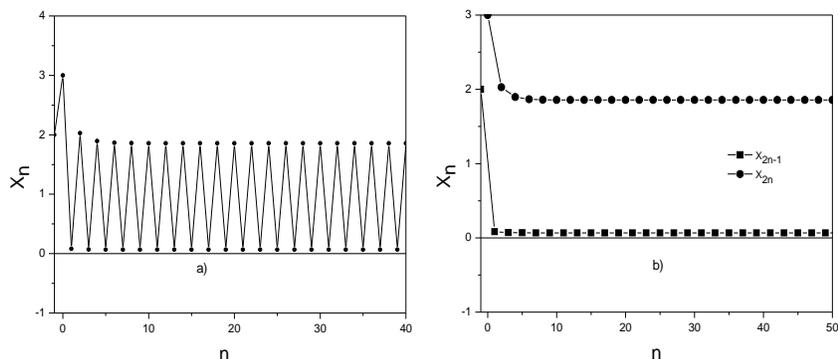
FIGURE 1. $|A| < 1$, $A - 1 + Bcd \neq 0$: $(x_{2n-1})_n$ and $(x_{2n})_n$ converge.
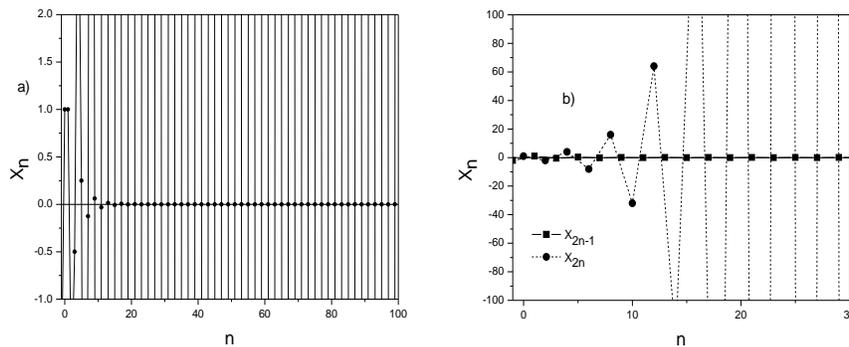
FIGURE 2. $A = -1$ and $Bcd \in (-\infty, 0) \cup (2, \infty)$: $(x_{2n-1})_n$ converges to zero and $(|x_{2n}|)_n$ goes to infinity, the solution is unbounded.
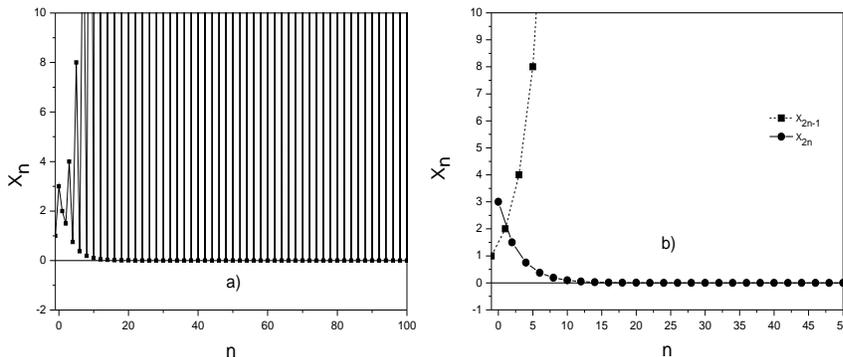
FIGURE 3. $A = -1$ and $Bcd \in (0, 2)$: $(|x_{2n-1}|)_n$ goes to infinity and $(x_{2n})_n$ converges to zero, the solution is unbounded.
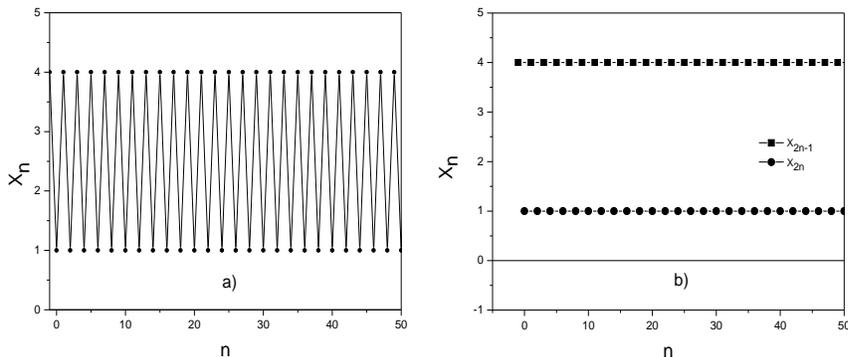
FIGURE 4. $A = -1$ and $Bcd = 2$: $(x_{2n-1})_n$ and $(x_{2n})_n$ are constants, $(x_n)_n$ is periodic-2 solution.
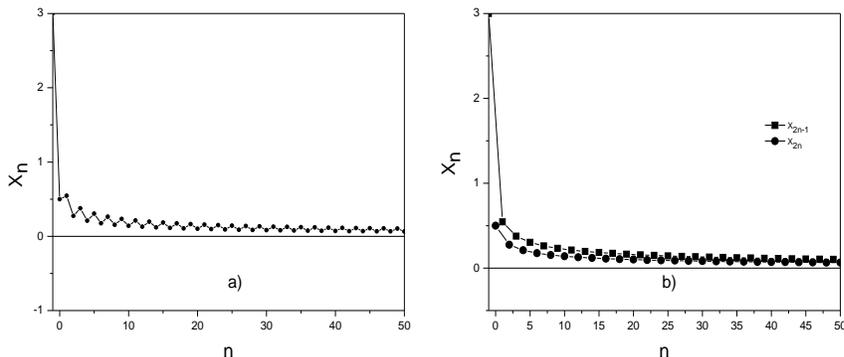
FIGURE 5. $A = 1$: the solution converges to zero.
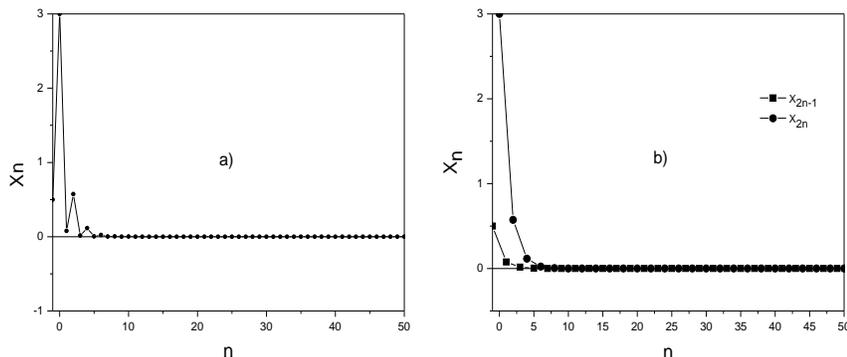
FIGURE 6. $|A| > 1$ and $A - 1 + Bcd \neq 0$: the solution converges to zero.
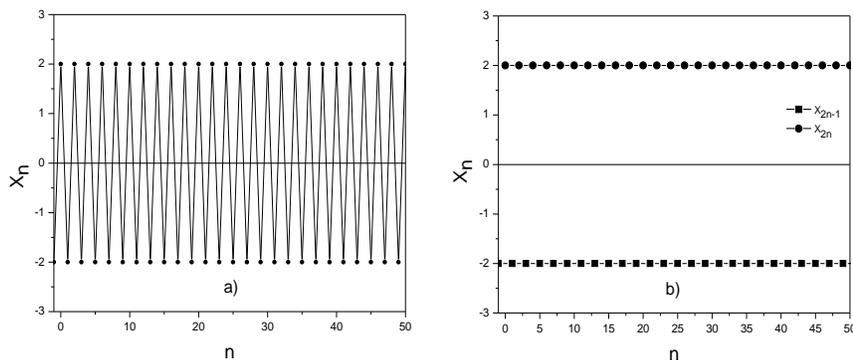
FIGURE 7. $|A| > 1$ and $A - 1 + Bcd = 0$: $(x_{2n-1})_n$ and $(x_{2n})_n$ are constants, $(x_n)_n$ is periodic-2 solution.

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L_p approximation errors for hybrid interpolation on the unit sphere *

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Abstract

This paper discusses L_p approximation error estimates for hybrid interpolation on the unit sphere. This interpolation scheme is integrated by spherical polynomials and radial basis functions. The smooth radial basis functions generated by a strictly positive definite zonal kernel are embedded in a larger native space generated by a less smooth kernel, and the error estimates for hybrid interpolation to a target function from the larger native space are given. In a sense, the results of this paper show that the hybrid interpolation associated with the smooth kernel enjoys the same order of error estimate as hybrid interpolation associated with the less smooth kernel for a target function from the rough native space.

MSC(2000): 41A17, 41A30

Keywords: Sphere; Interpolation; Approximation; Error

1 Introduction

Recently, fitting spherical scattered data comes up in many application areas, such as astrophysics, meteorology, geodesy, geophysics, and so on [5, 6, 29]. As interpolation or approximation tools, spherical polynomials or spherical radial basis functions were used to handle spherical scattered data in more studies [5, 6, 11, 14, 20, 22, 27, 28, 29, 15, 2]. Since spherical polynomials can handle the slowly varying large-scale features, and spherical radial basis functions are helpful to handle scattered and rapidly changed data, Sloan and Sommariv [25] introduced a hybrid interpolation scheme, which combines spherical radial basis functions together with spherical polynomials, and restricts the radial basis functions to the case of strictly positive definite kernels, so that the polynomial component is voluntary rather than forced.

This paper studies the hybrid interpolation in an appropriate native space $\mathcal{N}_\phi$ of continuous functions on the unit sphere, which is defined by a underlying strictly positive definite kernel ϕ. We apply the approach used by Hubbert and Morton [9, 10] to obtain error estimates in L_p norm. However, if the target function is from a subspace of the native space $\mathcal{N}_\phi$, we then adopt the inf-sup condition [26] and the method of constructing a convolution kernel to improve the error estimates.

So called "native space barrier" problem means that if ϕ is smooth, then the native space $\mathcal{N}_\phi$ is small. There have been much literature to focus on it, for example, [12, 13, 17, 18, 19]. In this paper, we employ the approach in [12] and the techniques in [26], and embed the smooth radial basis functions in a larger native space generated by a less smooth kernel ψ. At same time, we utilize the hybrid interpolation associated with the smooth kernel ϕ to interpolate the target function from the larger native space. In the process of error estimates, the "norming set" method developed by Jetter [11] and a special case of the general Bernstein-type inequality in [4] are used.

This paper is organized as follows. Section 2 is preliminary, which is related to introducing notations, hybrid interpolation and its crucial condition, native space, and Sobolev space. The L_p approximation error estimates are established in Section 3. In Section 4, for a target function f

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in a subspace of the original native space, we improve the global L_p-error estimates. In Section 5, we still use the hybrid interpolation defined in Section 2 to interpolate and approximate a target function f from a larger native space generated by a less smooth kernel.

2   Preliminaries

This paper uses C to denote a positive constant, whose value may be different at different occurrence even within the same formula. The symbol $A \sim B$ means that there exist positive constant C_1 and C_2 such that $C_1 B \leq A \leq C_2 B$.

We use $\mathbb{S}^2 := \left\{ x := (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \right\}$ to denote the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, and denote by $L_p(\mathbb{S}^2)$ the space of p-integrable functions on $\mathbb{S}^2$ endowed with the norms $\|f\|_\infty := \|f\|_{L_\infty(\mathbb{S}^2)} := \text{esssup}_{x \in \mathbb{S}^2} |f(x)|(p = \infty)$, and $\|f\|_p := \|f\|_{L_p(\mathbb{S}^2)} := \left\{ \int_{\mathbb{S}^2} |f(x)|^p d\omega(x) \right\}^{1/p} < \infty (1 \leq p < \infty)$. The so called spherical harmonic with degree l is the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial with degree $l \geq 0$. The class of all spherical harmonics with degree l is denoted by $\mathcal{H}_l$, and the class of all spherical harmonics with total degree $l \leq L$ is denoted by $\mathcal{P}_L$. Clearly, spherical harmonics with different degrees are orthogonal with respect to the $L_2(\mathbb{S}^2)$ inner product: $\langle f, g \rangle := \int_{\mathbb{S}^2} f(x)g(x)d\omega(x)$, where $d\omega$ is surface measure on $\mathbb{S}^2$.

The famous addition formula $\sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y)$ yields the following useful relation [16]:

$$\sum_{k=1}^{2l+1} |Y_{l,k}(x)Y_{l,k}(y)| \leq \sum_{k=1}^{2l+1} Y_{l,k}^2(x) = \frac{2l+1}{4\pi}, \quad x, y \in \mathbb{S}^2. \tag{2.1}$$

Here P_l is the Legendre polynomial with degree l and dimension three, which is normalized such that $P_l(1) = 1$, and satisfies the orthogonality relation: $\int_{-1}^{1} P_k(t)P_j(t)dt = \frac{2}{2l+1}\delta_{k,j}$, where the symbol $\delta_{k,j}$ denotes the usual Kronecker symbol.

The definition of strictly positive definite kernel is given by

Definition 2.1 *(see [29]). A continuous and symmetric function $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$ is called positive definite kernel, if, for any $N \in \mathbb{N}_+$, $\alpha = (\alpha_i)_{i=1,\ldots,N} \in \mathbb{R}^N$ and $\{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, we have*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i, x_j) \geq 0.$$

When for any N distinct points $\{x_1, \ldots, x_N\}$, the above quadratic form is positive for all $\alpha = (\alpha_i)_{i=1,\ldots,N} \in \mathbb{R}^N/\{0\}$, then ϕ is called strictly positive definite kernel.

We say that a kernel ϕ is called rotational invariant if $\phi(\rho x, \rho y) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$ and for all rotations ρ. So a continuous rotational invariant kernel depends only on the distance between x and y [24], that is, there is a function $\varphi : [-1, 1] \to \mathbb{R}$, such that $\varphi(xy) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$. Therefore, a rotational invariant kernel is also called a zonal kernel. In [23], Schoenberg characterized the positive definite zonal kernels. In [30], Xu and Cheney introduced the notation of strictly positive definiteness on the sphere. Clearly, it is important to characterize all the strictly positive definite functions on the sphere, and such an endeavor has been taken by Ron and Sun in [21]. In [3], Chen et al. established a necessary and sufficient condition for strictly positive definite zonal kernels: the kernel ϕ is strictly positive definite and zonal if and only if

$$\phi(x, y) = \sum_{l=0}^{\infty} a_l \sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \sum_{l=0}^{\infty} \frac{(2l+1)a_l}{4\pi} P_l(x \cdot y),$$

with $a_l \geq 0$ for all l, $\sum_{l=0}^{\infty} la_l < \infty$ and $a_l > 0$ for infinitely many even values of l and infinitely many odd values of l.

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For given strictly positive definite kernel $\phi(\cdot, \cdot)$, a set of distinct points $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, and target function $f \in C(\mathbb{S}^2)$, we take the hybrid interpolation for f in the form

$$I_{X,L}f = \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k},$$

where we fix $L \geq 0$ as the desired degree of the polynomial component of the hybrid interpolation and the coefficients $\{\alpha_j\}_{j=1}^{N}$, $\{\beta_{l,k}\}_{k=1,\ldots,2l+1,\, l=0,\ldots,L}$ are determined by the interpolation conditions

$$I_{X,L}f(x_i) = f(x_i), \ \ i = 1, \ldots, N, \tag{2.2}$$

and also (in order to give a square linear system) the side conditions $\sum_{j=1}^{N} \alpha_j p(x_j) = 0, \ \forall p \in \mathcal{P}_L$.

Now we give a condition on the point set X, which makes sure that the interpolation is exist and unique.

Definition 2.2 (see [25, Definition 3.1]). The set $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is said to be $\mathcal{P}_L$ -unisolvent if

$$p \in \mathcal{P}_L, \ p(x_j) = 0 \ for \ j = 1, \ldots, N \Rightarrow p = 0.$$

In order to analyze the interpolation error in the later sections it is convenient to define a finite-dimensional space $V_{X,L}$ within the interpolation $I_{X,L}f$ lies.

$$V_{X,L} := \left\{ \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + q : q \in \mathcal{P}_L, \ \alpha_j \in \mathbb{R} \text{ for } j = 1, \ldots, N, \text{ and} \sum_{j=1}^{N} \alpha_j p(x_j) = 0, \ \forall p \in \mathcal{P}_L \right\}.$$

The following Theorem 2.1 gives a crucial condition for the interpolation to be well defined, whose proof can be find in [25].

Theorem 2.1 Let $\phi(\cdot, \cdot)$ be a strictly positive definite kernel, and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ be a set of distinct points which is $\mathcal{P}_L$ -unisolvent for $L \geq 0$. Then for each $f \in C(\mathbb{S}^2)$ there exists a unique $I_{X,L}f \in V_{X,L}$ that satisfies the interpolation conditions in (2.2).

In this paper, we assume that the strictly positive definite kernel ϕ is zonal and has the expansion

$$\phi(x, y) = \sum_{l=0}^{\infty} a_l \sum_{k=1}^{2l+1} Y_{l,k}(x) Y_{l,k}(y) \tag{2.3}$$

with $a_l > 0$ for all l, $\sum_{l=0}^{\infty} l a_l < \infty$, in which case the series of the right side in (2.3) converges uniformly for $x, y \in \mathbb{S}^2$.

For $f, g \in L_2(\mathbb{S}^2)$, they can be represented by their Fourier series $f = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \hat{f}_{l,k} Y_{l,k}$ and $g = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \hat{g}_{l,k} Y_{l,k}$, respectively. With respect to the inner product expressed as (see [29]) $(f, g)_{\mathcal{N}_\phi} = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{\hat{f}_{l,k} \hat{g}_{l,k}}{a_l}$, the native space $\mathcal{N}_\phi$, which is the subspace of $L_2(\mathbb{S}^2)$, can be defined by

$$\mathcal{N}_\phi := \left\{ f \in L_2(\mathbb{S}^2) : \|f\|_{\mathcal{N}_\phi}^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{|\hat{f}_{l,k}|^2}{a_l} < \infty \right\}.$$

It is easy to verify that the native space $\mathcal{N}_\phi$ is a reproducing kernel Hilbert space with reproducing kernel $\phi(\cdot, \cdot)$, that is, $(f, \phi(\cdot, x))_{\mathcal{N}_\phi} = f(x)$, $x \in \mathbb{S}^2$, $f \in \mathcal{N}_\phi$.

When $a_l \sim (l+1)^{-2s}$ for $l = 0, 1, \ldots$, the native space $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s:

$$H_s := \left\{ f \in L_2(\mathbb{S}^2) : \|f\|_{H_s}^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} (l+1)^{2s} |\hat{f}_{l,k}|^2 < \infty \right\},$$

and the Sobolev embedding theorem in [7] implies that if $s > 1$, then the space H_s is continuously embedded in $C(\mathbb{S}^2)$, so that H_s is a reproducing kernel Hilbert space.

The error estimates are general expressed in terms of the mesh norm of $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, which is defined by $h_X := \sup_{x \in \mathbb{S}^2} \inf_{x_j \in X} d(x, x_j)$, where $d(x, x_j) = \arccos(x \cdot x_j)$ is the geodesic distance between x_j and x.

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3 Global error estimates for L_p norm

We first give the following three lemmas, which can be found in [9] and [10].

Lemma 3.1 *Let $d \geq 1$ be an integer and set $M := 2\sqrt{d}$ and $\delta_d := \frac{1}{4(d+1)^{3/2}}$. Let M_1 be an arbitrary positive number, $\theta \in (0, \frac{\pi}{3})$ and set $h_0 := \frac{\theta}{M+M_1+\delta_d}$. Then for any $h \in (0, h_0)$, there exists a set of points $Z_h \subset \mathbb{S}^d$ such that $\mathbb{S}^d = \bigcup_{z \in Z_h} D(z, Mh)$. Here we denote by $D(x_0, \gamma)$ the spherical cap with center x_0 and angle γ, i.e., $D(x_0, \gamma) := \{x \in \mathbb{S}^d : x \cdot x_0 > \cos \gamma\}$, and then denote by $A(x_0, \gamma)$ the volume of $D(x_0, \gamma)$, i.e., $A(x_0, \gamma) := \Omega_d \int_0^\gamma \sin^{d-1} \theta d\theta$, where Ω_d denotes the volume of $\mathbb{S}^d$. Let F_A denote the characteristic function of a set $A \subset \mathbb{S}^d$. There exists a positive integer Q independent of h such that*

$$\sum_{z \in Z_h} F_{D(z, M'h)} \leq Q, \quad where \ M' = M + M_1.$$

Further, the cardinality of Z_h is bounded above by $C_Q h^{-d}$, where C_Q is independent of h.

Lemma 3.2 *Let $z \in \mathbb{S}^d$ and $X = \{x_i\}_{i=1}^N$ denote a set of distinct points on $\mathbb{S}^d$. Let $s \in [k, k+1]$, where $k > \frac{d}{2}$ is a positive integer. There exist positive numbers C_1 and C_2 such that if we let $M_1 > \max\{C_1 - 2d^{1/2}, 0\}$ be a fixed positive number and let*

$$h_0 = \frac{C_2}{3M_2}, \quad where \ M_2 = 2d^{1/2} + M_1,$$

then, assuming that X has mesh norm $h := h_X \in (0, h_0)$, there exists an extension operator $E_{D(z, M_2h)} : H_s(D(z, M_2h)) \longrightarrow H_s(\mathbb{S}^d)$ satisfying
(1) $(E_{D(z, M_2h)} f)|_{D(z, M_2h)} = f$, for all $f \in H_s(D(z, M_2h))$,
(2) there exists a positive constant C, independent of h and z such that

$$\|E_{D(z, M_2h)} f\|_{H_s(\mathbb{S}^d)} \leq C \|f\|_{H_s(D(z, M_2h))},$$

for all $f \in H_s(D(z, M_2h))$ such that $f(\xi) = 0$ for $\xi \in X \bigcap D(z, M_2h)$.

Lemma 3.3 *Let $s > 0$ and let M_1 be any positive number. Let $h \in (0, h_0)$ and let Z_h denote the corresponding quasi-uniform mesh for $\mathbb{S}^d$ from Lemma 3.1. Then, for any $f \in H_s(\mathbb{S}^d)$, we have*

$$\sum_{z \in Z_h} \|f\|_{H_s(D(z, M_2h))}^2 \leq Q \|f\|_{H_s(\mathbb{S}^d)}^2,$$

where Q is the constant (independent of h) from Lemma 3.1.

We are now ready to state the main results for the error estimates of the hybrid interpolation in L_p norm.

Theorem 3.1 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in (2.3) and $a_l \sim (l+1)^{-2s}$. Assume that integer $L \geq 1$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $1/(2L+2) < h_X \leq 1/(2L)$. For $f \in \mathcal{N}_\phi$, let $I_{X,L} f \in V_{X,L}$ be the hybrid interpolation defined in Section 2. Then we have*

$$\|f - I_{X,L} f\|_{L_p(\mathbb{S}^2)} \leq C h_X^{\frac{2}{p}+s-1} \|f - I_{X,L} f\|_{\mathcal{N}_\phi}, \quad p \in [2, +\infty),$$

and

$$\|f - I_{X,L} f\|_{L_p(\mathbb{S}^2)} \leq C h_X^s \|f - I_{X,L} f\|_{\mathcal{N}_\phi}, \quad p \in [1, 2),$$

where the constant C is independent of f and h_X.

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Proof. For the case $\mathbb{S}^2$, we can take $d = 2$ in Lemma 3.1, Lemma 3.2 and Lemma 3.3. By using Lemma 3.1, for arbitrary $1 \le p < \infty$, we have

$$\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} = \int_{\mathbb{S}^2} |(f - I_{X,L}f)(\xi)|^p d\omega(\xi) \le \sum_{z \in Z_h} \int_{D(z,Mh)} |(f - I_{X,L}f)(\xi)|^p d\omega(\xi), \quad (3.4)$$

where $M = 2^{3/2}$. This step motivates us to consider the error estimates locally. In particular, $f - I_{X,L}f$ is continuous on $\overline{D(z,Mh)}$, which is a compact subset of $\mathbb{S}^2$, so there exists a point $\xi_z \in \overline{D(z,Mh)}$ where $f - I_{X,L}f$ attains its *maximum*. Now we can write

$$\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} \le \sum_{z \in Z_h} |(f - I_{X,L}f)(\xi)|^p \int_{D(z,Mh)} d\omega(\xi) \le Ch^2_X \sum_{z \in Z_h} |(f - I_{X,L}f)(\xi)|^p, \quad (3.5)$$

where the constant C satisfies $A(z, Mh) \le Ch^2_X$. We know that $f - I_{X,L}f \in \mathcal{N}_\phi$ and $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s. Now, rather than consider $f - I_{X,L}f$, we choose instead to consider the restriction $f - I_{X,L}f\big|_{D(z,M_2h)}$, where $M_2 = 2^{3/2} + M_1$.

We should choose a suitable M_1 to fit the conditions of Lemma 3.2, because we can find constant C_1, C_2 such that

$$\frac{2C_2L}{3} > C_1, \qquad \frac{2C_2L}{3} > 2^{3/2}.$$

So set $h_0 = \frac{1}{2L}$, $M_1 = \frac{2C_2L}{3} - 2^{3/2}$, and $M_2 = \frac{2C_2L}{3}$, then it is easy to prove that Lemma 3.2 holds. If we let $v_z := f - I_{X,L}f\big|_{D(z,M_2h)}$ and use Lemma 3.2, we have

(E1) $E_{D(z,M_2h)}v_z \in H_s(\mathbb{S}^2)$,

(E2) $E_{D(z,M_2h)}v_z(\xi) = 0$ for all $\xi \in X \bigcap D(z, M_2h)$,

(E3) there exists a positive constant C, independent of h_X and z such that

$$\left\| E_{D(z,M_2h)}v_z \right\|_{H_s(\mathbb{S}^2)} \le C\|v_z\|_{H_s(D(z,M_2h))}.$$

Hence, with the help of Theorem and (E3) we can obtain

$$|(f - I_{X,L}f)(\xi_z)| = \left| E_{D(z,M_2h)}v_z(\xi_z) \right| \le Ch^{s-1}_X \left\| E_{D(z,M_2h)}v_z \right\|_{\mathcal{N}_\phi}$$
$$\le Ch^{s-1}_X \left\| E_{D(z,M_2h)}v_z \right\|_{H_s} \le Ch^{s-1}_X \|v_z\|_{H_s(D(z,M_2h))}.$$

Substituting this into (3.5) gives

$$\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} \le Ch^{2+p(s-1)}_X \sum_{z \in Z_h} \|v_z\|^p_{H_s(D(z,M_2h))}. \quad (3.6)$$

For $p \in [2, \infty)$ we use Jensen's inequality [1] $\sum_{i=1}^N a^p_i \le \left(\sum_{i=1}^N a^2_i \right)^{\frac{p}{2}}$ followed by Lemma 3.3 to give

$$\begin{aligned}
\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} &\le Ch^{2+p(s-1)}_X \left(\sum_{z \in Z_h} \left\| f - I_{X,L}f\big|_{D(z,M_2h)} \right\|^2_{H_s(D(z,M_2h))} \right)^{p/2} \\
&\le Ch^{2+p(s-1)}_X \left(\|f - I_{X,L}f\|^2_{H_s(\mathbb{S}^2)} \right)^{p/2} \le Ch^{2+p(s-1)}_X \|f - I_{X,L}f\|^p_{\mathcal{N}_\phi}.
\end{aligned}$$

Finally, taking the p-th root gives

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \le Ch^{\frac{2}{p}+s-1}_X \|f - I_{X,L}f\|_{\mathcal{N}_\phi}, \quad p \in [2, +\infty), \quad (3.7)$$

where the constant C is independent of f and h_X.

For $p \in [1, 2)$ we conduct the same steps as above, however we replace Jensen's inequality with

$$\sum_{i=1}^N a^p_i \le N^{1-\frac{p}{2}} \left(\sum_{i=1}^N a^2_i \right)^{\frac{p}{2}}.$$

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Moreover, we use the fact that the cardinality of Z_h is bounded by $C_Q h^{-2}$ (see Lemma 3.1), and we obtain

$$
\begin{aligned}
\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} &\leq Ch_X^{ps} \left(\sum_{z \in Z_h} \left\| f - I_{X,L}f \right|_{D(z,M_2h)} \right\|^2_{H_s(D(z,M_2h))} \right)^{p/2} \\
&\leq Ch_X^{ps} \left(\|f - I_{X,L}f\|^2_{H_s(\mathbb{S}^2)} \right)^{p/2} \leq Ch_X^{ps} \|f - I_{X,L}f\|^p_{\mathcal{N}_\phi}.
\end{aligned}
$$

Finally, taking the p-th root gives

$$
\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^s \|f - I_{X,L}f\|_{\mathcal{N}_\phi}, \quad p \in [1,2), \tag{3.8}
$$

where the constant C is independent of f and h_X.

Combining (3.7) and (3.8) yields Theorem 3.1.

4   Inf-sup condition and improved global error estimates

As we can see that the factor $\|f - I_{X,L}f\|_{\mathcal{N}_\phi}$ in Theorem 3.1 may be harder to estimate than factor $\|f\|_{\mathcal{N}_\phi}$. Considering the fact that the hybrid interpolation defined in Section 2 is different from the interpolation scheme only by radial basis functions constructed from strictly positive definite kernels or conditionally positive definite kernels (see [10,]), we should find the other method to characterize the relationship between $\|f - I_{X,L}f\|_{\mathcal{N}_\phi}$ and $\|f\|_{\mathcal{N}_\phi}$. The following Inf-sup condition is quoted from [26], whose method is helpful to "tidy up" the error results in Theorem 3.1.

Theorem 4.1 (see [26, Theorem 6.1]). Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in (2.3) and $a_l \sim (l+1)^{-2s}$. Then there exist constants $\gamma > 0$ and $\tau > 0$ depending only on s such that for all $L \geq 1$ and all $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ satisfying $h_X \leq \tau/L$, the following inequality holds:

$$
\sup_{v \in R_X \setminus \{0\}} \frac{(p,v)_{\mathcal{N}_\phi}}{\|v\|_{\mathcal{N}_\phi}} \geq \gamma \|p\|_{\mathcal{N}_\phi}, \quad p \in \mathcal{P}_L, \tag{4.9}
$$

where $R_X = span\{\phi(\cdot, x_1), \ldots, \phi(\cdot, x_N)\}$.

In order to use the same method in [26], we simply denote that $I_{X,L}f = u_{X,L} + p_{X,L}$, where $p_{X,L} = \sum_{l=0}^L \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k}$, and $u_{X,L} = \sum_{j=1}^N \phi(\cdot, x_j)$.

For a given $f \in \mathcal{N}_\phi$, the interpolation conditions $I_{X,L}f(x_i) = f(x_i)$, $i = 1, \ldots, N$, and the side conditions $\sum_{j=1}^N \alpha_j q(x_j) = 0$, $\forall q \in \mathcal{P}_L$, are equivalent to

$$
(u_{X,L}, v_X)_{\mathcal{N}_\phi} + (p_{X,L}, v_X)_{\mathcal{N}_\phi} = (f, v_X)_{\mathcal{N}_\phi}, \quad v_X \in R_X, \tag{4.10}
$$

and

$$
(q, u_{X,L})_{\mathcal{N}_\phi} = 0, \quad \forall q \in \mathcal{P}_L. \tag{4.11}
$$

Now we can write the target function $f \in \mathcal{N}_\phi$ in an analogous way to $I_{X,L}f$ as $f := u + p$, where $p \in \mathcal{P}_L$ and $u \in \mathcal{N}_\phi$ are defined by $(p,q)_{\mathcal{N}_\phi} = (f,q)_{\mathcal{N}_\phi}, q \in \mathcal{P}_L$, which means that p is the $\mathcal{N}_\phi$-orthogonal project of f onto $\mathcal{P}_L$.

Similar to (4.10) and (4.11), we have

$$
(u, v)_{\mathcal{N}_\phi} + (p, v)_{\mathcal{N}_\phi} = (f, v)_{\mathcal{N}_\phi}, \quad v_X \in R_X, \tag{4.12}
$$

and

$$
(q, u)_{\mathcal{N}_\phi} = 0, \quad q \in \mathcal{P}_L. \tag{4.13}
$$

By subtracting (4.10) from (4.12) (with v replaced by v_X) and (4.11) from (4.13), we can obtain

$$
(u - u_{X,L}, v_X)_{\mathcal{N}_\phi} + (p - p_{X,L}, v_X)_{\mathcal{N}_\phi} = 0, \quad v_X \in R_X, \tag{4.14}
$$

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and

$$(q, u - u_{X,L})_{\mathcal{N}_\phi} = 0, \quad q \in \mathcal{P}_L. \tag{4.15}$$

Now we define $\tilde{u}_X \in R_X$ to be the $\mathcal{N}_\phi$-orthogonal project of u onto R_X, that is,

$$(\tilde{u}_X, v_X)_{\mathcal{N}_\phi} = (u, v_X)_{\mathcal{N}_\phi}, \quad v_X \in R_X. \tag{4.16}$$

From (4.14), (4.15) and (4.16), we clearly have

$$(\tilde{u}_X - u_{X,L}, v_X)_{\mathcal{N}_\phi} + (p - p_{X,L}, v_X)_{\mathcal{N}_\phi} = 0, \quad v_X \in R_X, \tag{4.17}$$

and

$$(q, \tilde{u}_X - u_{X,L})_{\mathcal{N}_\phi} = (q, \tilde{u}_X - u)_{\mathcal{N}_\phi}, \quad q \in \mathcal{P}_L. \tag{4.18}$$

With the help of Theorem 4.1, we have

$$
\begin{aligned}
\|p - p_{X,L}\|_{\mathcal{N}_\phi} &\leq \frac{1}{\gamma} \sup_{v_X \in R_X \setminus \{0\}} \frac{(p - p_{X,L}, v_X)_{\mathcal{N}_\phi}}{\|v_X\|_{\mathcal{N}_\phi}} \\
&= \frac{1}{\gamma} \sup_{v_X \in R_X \setminus \{0\}} \frac{(u_{X,L} - \tilde{u}_X, v_X)_{\mathcal{N}_\phi}}{\|v_X\|_{\mathcal{N}_\phi}} \leq \frac{1}{\gamma} \|u_{X,L} - \tilde{u}_X\|_{\mathcal{N}_\phi}.
\end{aligned}
$$

By using (4.17) with $v_X = \tilde{u}_X - u_{X,L}$ and (4.18), we also have

$$
\begin{aligned}
\|\tilde{u}_X - u_{X,L}\|_{\mathcal{N}_\phi}^2 &= -(p - p_{X,L}, \tilde{u}_X - u_{X,L})_{\mathcal{N}_\phi} = -(p - p_{X,L}, \tilde{u}_X - u)_{\mathcal{N}_\phi} \\
&\leq \|p - p_{X,L}\|_{\mathcal{N}_\phi} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \leq \frac{1}{\gamma} \|u_{X,L} - \tilde{u}_X\|_{\mathcal{N}_\phi} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}.
\end{aligned}
$$

So we obtain that

$$\|\tilde{u}_X - u_{X,L}\|_{\mathcal{N}_\phi} \leq \frac{1}{\gamma} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \leq C \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}, \tag{4.19}$$

and

$$\|p - p_{X,L}\|_{\mathcal{N}_\phi} \leq \frac{1}{\gamma^2} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \leq C \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}. \tag{4.20}$$

With the above inequalities (4.19) and (4.20), we can establish the following Theorem 4.2, which indicates the relationship between $\|f - I_{X,L}f\|_{\mathcal{N}_\phi}$ and $\|f\|_{\mathcal{N}_\phi}$.

Theorem 4.2 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in (2.3) and $a_l \sim (l+1)^{-2s}$, $s > 1$. Assume that integer $L \geq 1$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $h_X \leq \tau/L$, where τ is as in Theorem 4.1. For $f \in \mathcal{N}_\phi$, let $I_{X,L}f \in V_{X,L}$ be the hybrid interpolation defined in Section 2. Then we have*

$$\|f - I_{X,L}f\|_{\mathcal{N}_\phi} \leq C \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_\phi} \leq C \|f\|_{\mathcal{N}_\phi}.$$

Proof. Using the representation $I_{X,L}f = u_{X,L} + p_{X,L}$, $f = u + p$ and (4.19), (4.20) we have

$$
\begin{aligned}
\|f - I_{X,L}f\|_{\mathcal{N}_\phi} &\leq \|u - u_{X,L}\|_{\mathcal{N}_\phi} + \|p - p_{X,L}\|_{\mathcal{N}_\phi} \\
&\leq \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} + \|\tilde{u}_X - u_{X,L}\|_{\mathcal{N}_\phi} + \|p - p_{X,L}\|_{\mathcal{N}_\phi} \leq C \|\tilde{u}_X - u\|_{\mathcal{N}_\phi},
\end{aligned}
$$

and also we have $\|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \leq \|u\|_{\mathcal{N}_\phi} = \|f - p\|_{\mathcal{N}_\phi} = \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_\phi}$, which yields $\|f - I_{X,L}f\|_{\mathcal{N}_\phi} \leq C \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_\phi} \leq C \|f\|_{\mathcal{N}_\phi}$, and the proof of Theorem 4.2 is completed.

Combining Theorem 4.2 with Theorem 3.1, we can easily verify the following Corollary 4.1.

Corollary 4.1 *Under the conditions of Theorem 3.1 apart from the mesh norm $1/(2L+2) < h_X \leq \min\{1/(2L), \tau/L\}$, where τ is as in Theorem 4.1. For $f \in \mathcal{N}_\phi$, let $I_{X,L}f \in V_{X,L}$ be the hybrid interpolation defined in Section 2. Then we have*

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq C h_X^{\frac{2}{p}+s-1} \|f\|_{\mathcal{N}_\phi}, \quad p \in [2, +\infty),$$

and

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq C h_X^s \|f\|_{\mathcal{N}_\phi}, \quad p \in [1, 2),$$

where the constant C is independent of f and h_X.

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In the rest part of this section, unlike the above arguments we used to perform the "cleaner" error estimates in Corollary 4.1, we will show that improved global error estimates are available, provided that the target function f belongs to a certain subspace of $\mathcal{N}_\phi$, which defined by $\mathcal{N}_{\phi*\phi}$. This procedure is the same as in [10] and the following Definition 4.1 is about the convolution kernel of ϕ, which generates the corresponding native space $\mathcal{N}_{\phi*\phi}$.

Definition 4.1 *Let ϕ be a strictly positive definite zonal kernel that defined in (2.3) We define the convolution kernel of ϕ by*

$$(\phi * \phi)(x, y) := \int_{\mathbb{S}^2} \phi(x, z)\phi(y, z)d\omega(z), \ \ x, y \in \mathbb{S}^2.$$

Working in terms of Fourier expansions, we have $(\phi * \phi)(x, y) := \sum_{l=0}^{\infty} a_l^2 \sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y)$. Executing the same arguments as in Section 2, we know that the native space $\mathcal{N}_{\phi*\phi}$ associated with kernel $(\phi * \phi)(\cdot, \cdot)$ can be defined by

$$\mathcal{N}_{\phi*\phi} := \left\{ f \in L_2(S^2) : \|f\|_{\mathcal{N}_{\phi*\phi}}^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{|\hat{f}_{l,k}|^2}{a_l^2} < \infty \right\},$$

and it is a reproducing kernel Hilbert space with the reproducing kernel $(\phi * \phi)(\cdot, \cdot)$.

When $a_l \sim (l+1)^{-2s}$ for $l = 0, 1, \ldots$ and $s > 1$, we know that the native space $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s. So $\mathcal{N}_{\phi*\phi} \cong H_{2s} \subset H_s \cong \mathcal{N}_\phi$, where $\cong$ denotes norm equivalence. Obviously, we see $\mathcal{N}_{\phi*\phi} \subset \mathcal{N}_\phi$.

The following Lemma 4.1 gives a crucial inequality, which is helpful to improve the global error estimates of the hybrid interpolation for a target function $f \in \mathcal{N}_{\phi*\phi}$.

Lemma 4.1 *Let $u \in \mathcal{N}_{\phi*\phi}$ and $\tilde{u}_X \in R_X$ be the $\mathcal{N}_\phi$-orthogonal project of u onto R_X, which has the property as in (4.16), then we have*

$$\|\tilde{u}_X - u\|_{\mathcal{N}_\phi}^2 \le \|u\|_{\mathcal{N}_{\phi*\phi}} \cdot \|\tilde{u}_X - u\|_{L_2(\mathbb{S}^2)}, \tag{4.21}$$

where R_X is the same as in Theorem 4.1.

Proof. By using (4.16), the definition of $(\cdot, \cdot)_{\mathcal{N}_{\phi*\phi}}$, and Cauchy-Schwarz inequality respectively, we have

$$\|\tilde{u}_X - u\|_{\mathcal{N}_\phi}^2 = (u, \tilde{u}_X - u)_{\mathcal{N}_\phi} = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{\hat{u}_{l,k} \cdot \left(\hat{u}_{l,k} - \widehat{(\tilde{u}_X)}_{l,k}\right)}{a_l}$$

$$\le \left(\sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{|\hat{u}_{l,k}|^2}{a_l^2} \right)^{1/2} \left(\sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \left(\hat{u}_{l,k} - \widehat{(\tilde{u}_X)}_{l,k} \right)^2 \right)^{1/2} \le \|u\|_{\mathcal{N}_{\phi*\phi}} \cdot \|\tilde{u}_X - u\|_{L_2(\mathbb{S}^2)}$$

With this in place we can provide the following improved global error estimates.

Theorem 4.3 *Under the conditions of Corollary 4.1 and assume further that $f \in \mathcal{N}_{\phi*\phi}$, we have*

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \le Ch_X^{\frac{2}{p}+2s-1}\|f\|_{\mathcal{N}_{\phi*\phi}}, \ \ p \in [2, +\infty), \tag{4.22}$$

and

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \le Ch_X^{2s}\|f\|_{\mathcal{N}_{\phi*\phi}}, \ \ p \in [1, 2), \tag{4.23}$$

where the constant C is independent of f and h_X.

Proof. First we have, from Theorem 3.1, with $p = 2$, that

$$\|\tilde{u}_X - u\|_{L_2(\mathbb{S}^2)} \le Ch_X^s\|\tilde{u}_X - u\|_{\mathcal{N}_\phi}.$$

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Substituting this into (4.21) gives

$$\|\tilde{u}_X - u\|^2_{\mathcal{N}_\phi} \le C h_X^s \|u\|_{\mathcal{N}_{\phi * \phi}} \cdot \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}. \tag{4.24}$$

So,

$$\|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \le C h_X^s \|u\|_{\mathcal{N}_{\phi * \phi}}. \tag{4.25}$$

Using the same procedure as in the proof of Theorem 4.2, we see that

$$\|f - I_{X,L}f\|_{\mathcal{N}_\phi} \le \|u - u_{X,L}\|_{\mathcal{N}_\phi} + \|p - p_{X,L}\|_{\mathcal{N}_\phi} \le C\|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \le C h_X^s \|u\|_{\mathcal{N}_{\phi * \phi}}.$$

Clearly,

$$\|u\|_{\mathcal{N}_{\phi * \phi}} = \|f - p\|_{\mathcal{N}_{\phi * \phi}} = \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_{\phi * \phi}} \le \|f\|_{\mathcal{N}_{\phi * \phi}}, \tag{4.26}$$

which implies

$$\|f - I_{X,L}f\|_{\mathcal{N}_\phi} \le C h_X^s \|f\|_{\mathcal{N}_{\phi * \phi}}. \tag{4.27}$$

With the help of Theorem 3.1 we see

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \le C h_X^{\frac{2}{p} + 2s - 1} \|f\|_{\mathcal{N}_{\phi * \phi}}, \quad p \in [2, +\infty),$$

and

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \le C h_X^{2s} \|f\|_{\mathcal{N}_{\phi * \phi}}, \quad p \in [1, 2),$$

where the constant C is independent of f and h_X.

5   Hybrid interpolation for rough native space

In order to generate a larger native space than $\mathcal{N}_\phi$, we should give a new kernel defined in the form

$$\psi(x, y) = \sum_{l=0}^{\infty} b_l \sum_{k=1}^{2l+1} Y_{l,k}(x) Y_{l,k}(y), \tag{5.28}$$

with $b_l > 0$ for all l, and $\sum_{l=0}^{\infty} l b_l < \infty$.

With respect to the inner product expressed as $(f, g)_{\mathcal{N}_\psi} = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{\hat{f}_{l,k} \hat{g}_{l,k}}{b_l}$, the native space $\mathcal{N}_\psi$ may alternatively be characterized as the following set

$$\mathcal{N}_\psi := \left\{ f \in L_2(\mathbb{S}^2) : \|f\|^2_{\mathcal{N}_\psi} = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{|\hat{f}_{l,k}|^2}{b_l} < \infty \right\}.$$

Assuming that $b_l > a_l$, for all $l = 0, 1, \ldots$, we then see that $\mathcal{N}_\phi \subset \mathcal{N}_\psi$.

Next, we will consider the error estimates for the hybrid interpolation of a target function $f \in \mathcal{N}_\psi \supset \mathcal{N}_\phi$. Obviously, if we take the hybrid interpolation associated with the less smooth kernel ψ in the form $I_{X,L,\psi}f = \sum_{j=1}^{N} \alpha_j \psi(\cdot, x_j) + \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k}$, then Theorem 3.1 above still holds for $f \in \mathcal{N}_\psi$. However, motivated by the idea in [12], we still take the initial hybrid interpolation $I_{X,L,\phi}f$ in the form

$$I_{X,L,\phi}f = \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k}, \tag{5.29}$$

and consider the error estimate $\|f - I_{X,L,\phi}f\|_{L_p(\mathbb{S}^2)}$.

Lemma 5.1 *Let α be a nonnegative real number, and let M be the multiplier operator defined on $\mathcal{P}_L$ (embedded in $C(\mathbb{S}^2)$) by $M(p) = \sum_{l=0}^{L} (\lambda_l)^\alpha \sum_{k=1}^{2l+1} c_{l,k} Y_{l,k}$, where $p = \sum_{l=0}^{L} \sum_{k=1}^{2l+1} c_{l,k} Y_{l,k}$. Then we have $\|M(p)\| \le C(\lambda_L)^\alpha \|p\|$, where C is a constant independent of p and L.*

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Lemma 5.1 is a special case of Theorem 3.2 in [4].

Lemma 5.2 *Let $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ satisfying $h_X \leq 1/(2L)$, then for any linear functional σ on $\mathcal{P}_L$ (embedded in $C(\mathbb{S}^2)$), such that $\|\sigma\|_* = 1$, there exist N real numbers $\alpha_j := \alpha_j(x)$ (x is fixed) with $\sum_{j=1}^N |\alpha_j| \leq 2$, so that $\sigma(f) = \sum_{j=1}^N \alpha_j \delta_j(f)$ for all $f \in \mathcal{P}_L$, where δ_j denotes the point evaluation functional at the point x_j in X.*

The proof of the following Lemma 5.3 can be found in [29, Corollary 17.12].

Lemma 5.3 *Suppose that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ has mesh norm $h_X \leq \frac{1}{2L}$ for some integer $L \geq 1$. Then there exist functions $\alpha_j : \mathbb{S}^2 \to \mathbb{R}$ for $j = 1, \ldots, N$ such that*
(i) $\sum_{j=1}^N \alpha_j(x)p(x_j) = p(x)$, $\forall p \in \mathcal{P}_L$, $\forall x \in \mathbb{S}^2$,
(ii) $\sum_{j=1}^N |\alpha_j(x)| \leq 2$, $\forall x \in \mathbb{S}^2$.

The following Theorem 5.1 is about the pointwise error estimate $|f(x) - I_{X,L,\phi}f(x)|$, by which we can obtain the global error estimate $\|f - I_{X,L,\phi}f\|_{L_p(\mathbb{S}^2)}$.

Theorem 5.1 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel defined by (2.3), let $\psi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in (5.28), $b_l/a_l = \lambda_l$ for $l \geq 1$ and $b_l \sim (l+1)^{-2s}, s > 1, l \geq 0$. Assume that integer $L \geq 1$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $1/(2L+2) < h_X \leq 1/(2L)$. For $f \in \mathcal{N}_\psi$, let $I_{X,L,\phi}f \in V_{X,L}$ be the hybrid interpolation defined in (5.29). Then for a fixed $x \in \mathbb{S}^2$, we have*

$$|f(x) - I_{X,L,\phi}f(x)| \leq Ch_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}.$$

Proof. For $\forall f \in \mathcal{N}_\psi$, we simply take the hybrid interpolation associated with the smooth kernel ϕ by $I_{X,L,\phi}f(x) = u_{X,L,\phi} + p_{X,L}$, where $u_{X,L,\phi} = \sum_{j=1}^N \alpha_j \phi(\cdot, x_j)$, $x_j \in X = \{x_1, x_2, \ldots, x_N\}$, and $p_{X,L} = \sum_{l=0}^L \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k}$, such that $I_{X,L,\phi}f(x_j) = f(x_j)(j = 1, 2, \ldots, N)$.

However, if we just use the hybrid interpolation associated with the less smooth kernel ψ, we have $I_{X,L,\psi}f(x) = u_{X,L,\psi} + p'_{X,L}$, where $u_{X,L,\psi} = \sum_{j=1}^N \gamma_j \psi(\cdot, x_j)$, $x_j \in X = \{x_1, x_2, \ldots, x_N\}$, and $p'_{X,L} = \sum_{l=0}^L \sum_{k=1}^{2l+1} \beta'_{l,k} Y_{l,k}$.

First, we consider the estimate of $\|\psi(\cdot, x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi}$. Using the same method as that in [12], we have

$$\|\psi(\cdot, x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi} = \sup_{\substack{v \in \mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi} = 1}} (\psi(\cdot, x) - u_{X,L,\psi}, v)_{\mathcal{N}_\psi}$$

$$= \sup_{\substack{v \in \mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi} = 1}} \sum_{l=0}^\infty b_l^{-1} \sum_{k=1}^{2l+1} \hat{v}_{l,k} \left(b_l \sum_{j=1}^N \gamma_j Y_{l,k}(x_j) - b_l Y_{l,k}(x) \right)$$

$$= \sup_{\substack{v \in \mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi} = 1}} \sum_{l=0}^\infty \sum_{k=1}^{2l+1} \hat{v}_{l,k} \left(\sum_{j=1}^N \gamma_j Y_{l,k}(x_j) - Y_{l,k}(x) \right).$$

By using Lemma 5.3, for a fixed x, we see that there exist N real numbers γ_j with $\sum_{j=1}^N |\gamma_j| \leq 2$ such that

$$\sum_{j=1}^N \gamma_j Y_{l,k}(x_j) = Y_{l,k}(x), \quad l = 0, 1, \ldots, L, \tag{5.30}$$

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which yields

$$\|\psi(\cdot,x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi} = \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \sum_{l=L+1}^\infty \sum_{k=1}^{2l+1} \hat{v}_{l,k}\left(\sum_{j=1}^N \gamma_j Y_{l,k}(x_j) - Y_{l,k}(x)\right)$$

$$= \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^N \gamma_j \sum_{l=L+1}^\infty \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x_j) - \sum_{l=L+1}^\infty \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x)\right]$$

$$\le \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^N |\gamma_j|\left|\sum_{l=L+1}^\infty \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x_j)\right|\right] + \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left|\sum_{l=L+1}^\infty \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x)\right|.$$

By using the Cauchy-Schwarz inequality, (5.30) and the relation in (2.1), we see that

$$\|\psi(\cdot,x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi} \le \sum_{j=1}^N |\gamma_j| \max_{x_j\in X}\left(\sum_{l=L+1}^\infty b_l \sum_{k=1}^{2l+1} Y_{l,k}^2(x_j)\right)^{\frac{1}{2}} \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}}\left(\sum_{l=L+1}^\infty \sum_{k=1}^{2l+1} \frac{\hat{v}_{l,k}^2}{b_l}\right)^{\frac{1}{2}}$$

$$+\left(\sum_{l=L+1}^\infty b_l \sum_{k=1}^{2l+1} Y_{l,k}^2(x)\right)^{\frac{1}{2}} t \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}}\left(\sum_{l=L+1}^\infty \sum_{k=1}^{2l+1} \frac{\hat{v}_{l,k}^2}{b_l}\right)^{\frac{1}{2}}$$

$$\le 2\left(\sum_{l=L+1}^\infty b_l \frac{2l+1}{4\pi}\right)^{\frac{1}{2}} + \left(\sum_{l=L+1}^\infty b_l \frac{2l+1}{4\pi}\right)^{\frac{1}{2}} \le C_1\left(\sum_{l=L+1}^\infty (2l+1)b_l\right)^{\frac{1}{2}}$$

$$\le C_1\left(\sum_{l=L+1}^\infty (l+1)^{-2s+1}\right)^{\frac{1}{2}} \le C_1(L+1)^{-s+1} \le C_1 h_X^{s-1}. \tag{5.31}$$

Next we consider the estimate of $\|\psi(\cdot,x) - u_{X,L,\phi}\|_{\mathcal{N}_\psi}$, in which we will use Lemma 5.1 and Lemma 5.2.

$$\|\psi(\cdot,x) - u_{X,L,\phi}\|_{\mathcal{N}_\psi} = \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} (\psi(\cdot,x) - u_{X,L,\phi}, v)_{\mathcal{N}_\psi}$$

$$= \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \sum_{l=0}^\infty b_l^{-1} \sum_{k=1}^{2l+1} \hat{v}_{l,k}\left(a_l \sum_{j=1}^N \alpha_j Y_{l,k}(x_j) - b_l Y_{l,k}(x)\right)$$

$$= \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \sum_{l=0}^\infty b_l^{-1} a_l \sum_{k=1}^{2l+1} \hat{v}_{l,k}\left(\sum_{j=1}^N \alpha_j Y_{l,k}(x_j) - \frac{b_l}{a_l} Y_{l,k}(x)\right).$$

Let T_L be the multiplier operator defined on $\mathcal{P}_L$ (embedded in $C(\mathbb{S}^2)$) by $T_L(Y_{l,k}) = \frac{b_l}{a_l} Y_{l,k}$, for each $l = 0,1,\ldots,L$ and all $k = 1,2,\ldots,2l+1$, and extended linearly throughout $\mathcal{P}_L$. Let σ be the linear functional on $\mathcal{P}_L$ defined by $\sigma = \delta_x \circ T_L$. That is $\sigma(p) = (T_L(p))(x)$ for each $p \in \mathcal{P}_L$. By Lemma 5.1 with $\alpha = 1$ and the assumption that $b_l/a_l = \lambda_l$, $l \ge 1$, we have

$$|\sigma(p)| = |(T_L(p))(x)| \le \|T_L(p)\| \le C\lambda_L\|p\| = C\frac{b_L}{a_L}\|p\|,$$

in which C is a constant independent of p and L. Then by Lemma 5.2, there exist N real numbers α_j with $\sum_{j=1}^N |\alpha_j| \le 2C\frac{b_L}{a_L}$ such that

$$\sum_{j=1}^N \alpha_j Y_{l,k}(x_j) = \frac{b_l}{a_l} Y_{l,k}(x), \quad l = 0,1,\ldots,L. \tag{5.32}$$

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With the help of (5.32), we see that

$$
\|\psi(\cdot,x) - u_{X,L,\phi}\|_{\mathcal{N}_\psi} = \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \sum_{l=L+1}^{\infty} b_l^{-1} a_l \sum_{k=1}^{2l+1} \hat{v}_{l,k} \left(\sum_{j=1}^{N} \alpha_j Y_{l,k}(x_j) - \frac{b_l}{a_l} Y_{l,k}(x) \right)
$$

$$
= \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^{N} \alpha_j \sum_{l=L+1}^{\infty} b_l^{-1} a_l \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x_j) - \sum_{l=L+1}^{\infty} \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x) \right]
$$

$$
\leq \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^{N} |\alpha_j| \left| \sum_{l=L+1}^{\infty} \frac{a_l}{b_l} \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x_j) \right| \right] + \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left| \sum_{l=L+1}^{\infty} \sum_{k=1}^{2l+1} \hat{v}_{l,k} Y_{l,k}(x) \right|.
$$

Using the Cauchy-Schwarz inequality, we see that

$$
\|\psi(\cdot,x) - u_{X,L,\phi}\|_{\mathcal{N}_\psi} \leq \sum_{j=1}^{N} |\alpha_j| \max_{x_j\in X} \left(\sum_{l=L+1}^{\infty} \frac{a_l^2}{b_l} \sum_{k=1}^{2l+1} Y_{l,k}^2(x_j) \right)^{\frac{1}{2}} \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left(\sum_{l=L+1}^{\infty} \sum_{k=1}^{2l+1} \frac{\hat{v}_{l,k}^2}{b_l} \right)^{\frac{1}{2}}
$$

$$
+ \left(\sum_{l=L+1}^{\infty} b_l \sum_{k=1}^{2l+1} Y_{l,k}^2(x) \right)^{\frac{1}{2}} \sup_{\substack{v\in\mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi}=1}} \left(\sum_{l=L+1}^{\infty} \sum_{k=1}^{2l+1} \frac{\hat{v}_{l,k}^2}{b_l} \right)^{\frac{1}{2}}.
$$

With the help of $\sum_{j=1}^{N} |\alpha_j| \leq 2C \frac{b_L}{a_L}$, $b_l > a_l$ and the relation in (2.1), we have

$$
\begin{aligned}
\|\psi(\cdot,x) - u_{X,L,\phi}\|_{\mathcal{N}_\psi} &\leq 2C \frac{b_L}{a_L} \left(\sum_{l=L+1}^{\infty} \frac{a_l^2}{b_l} \frac{2l+1}{4\pi} \right)^{\frac{1}{2}} + \left(\sum_{l=L+1}^{\infty} b_l \frac{2l+1}{4\pi} \right)^{\frac{1}{2}} \\
&\leq 2C \frac{b_L}{a_L} \left(\sum_{l=L+1}^{\infty} b_l \frac{2l+1}{4\pi} \right)^{\frac{1}{2}} + \left(\sum_{l=L+1}^{\infty} b_l \frac{2l+1}{4\pi} \right)^{\frac{1}{2}} \\
&\leq C_2 \left(\sum_{l=L+1}^{\infty} (2l+1) b_l \right)^{\frac{1}{2}} \leq C_2 \left(\sum_{l=L+1}^{\infty} (l+1)^{-2s+1} \right)^{\frac{1}{2}} \\
&\leq C_2 (L+1)^{-s+1} \leq C_2 h_X^{s-1}. \quad\quad (5.33)
\end{aligned}
$$

With the above obtained results, we can provide the following pointwise error estimate:

$$
\begin{aligned}
|f(x) - I_{X,L,\phi} f(x)| &= \left| (f - I_{X,L,\phi} f, \psi(\cdot,x))_{\mathcal{N}_\psi} \right| \\
&= \left| (f - I_{X,L,\phi} f, \psi(\cdot,x) - u_{X,L,\phi})_{\mathcal{N}_\psi} + (f - I_{X,L,\phi} f, u_{X,L,\psi})_{\mathcal{N}_\psi} \right. \\
&\quad \left. + (f - I_{X,L,\phi} f, u_{X,L,\phi} - u_{X,L,\psi})_{\mathcal{N}_\psi} \right| := |I_1 + I_2 + I_3|.
\end{aligned}
$$

It is easy to verify that

$$
\begin{aligned}
I_2 &:= (f - I_{X,L,\phi} f, u_{X,L,\psi})_{\mathcal{N}_\psi} = \left(f - I_{X,L,\phi} f, \sum_{j=1}^{N} \gamma_j \psi(\cdot,x_j) \right)_{\mathcal{N}_\psi} \\
&= \sum_{j=1}^{N} \gamma_j \left(f(x_j) - I_{X,L,\phi} f(x_j) \right) = 0.
\end{aligned}
$$

With the help of (5.33), we have

$$
\begin{aligned}
|I_1| &:= \left| (f - I_{X,L,\phi} f, \psi(\cdot,x) - u_{X,L,\phi})_{\mathcal{N}_\psi} \right| \leq \|f - I_{X,L,\phi} f\|_{\mathcal{N}_\psi} \|\psi(\cdot,x) - u_{X,L,\phi}\|_{\mathcal{N}_\psi} \\
&\leq C_2 h_X^{s-1} \|f - I_{X,L,\phi} f\|_{\mathcal{N}_\psi}.
\end{aligned}
$$

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We denote $I_4 := (f - I_{X,L,\phi}f, \psi(\cdot, x) - u_{X,L,\psi})_{\mathcal{N}_\psi}$ so that we have $I_3 = I_4 - I_1$.

With the help of (5.31) we can see that

$$|I_4| = \left| (f - I_{X,L,\phi}f, \psi(\cdot, x) - u_{X,L,\psi})_{\mathcal{N}_\psi} \right| \leq \|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi} \|\psi(\cdot, x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi}$$
$$\leq C_1 h_X^{s-1} \|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi},$$

which yields $|I_3| \leq (C_1 + C_2)h_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}$. Then

$$|f(x) - I_{X,L,\phi}f(x)| \leq |I_1| + |I_2| + |I_3| \leq (C_1 + 2C_2)h_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}$$
$$\leq Ch_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}.$$

This completes the proof of Theorem 5.1.

Having the pointwise error estimate in Theorem 5.1, we can perform the same steps in Theorem 3.1, where the local-global strategy is the key to establish the error estimates. So we are now ready to state the error estimates of the hybrid interpolation for a target function $f \in \mathcal{N}_\psi$ for L_p norm.

Theorem 5.2 *Under the conditions of Theorem 5.1, we have*

$$\|f - I_{X,L,\phi}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^{\frac{2}{p}+s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}, \quad p \in [2, +\infty), \tag{5.34}$$

and

$$\| f - I_{X,L,\phi}f \|_{L_p(\mathbb{S}^2)} \leq Ch_X^{s}\| f - I_{X,L,\phi}f \|_{\mathcal{N}_\psi}, \quad p \in [1, 2), \tag{5.35}$$

where the constant C is independent of f and h_X.

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Some best approximation formulas and inequalities for the Bateman's $G-$function

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Abstract

In the paper, the authors established two best approximation formulas for the Bateman's $G-$function. Also, they studied the completely monotonicity of some functions involving $G(x)$. Some new inequalities are deduced for the function and its derivative such as

$$\frac{1}{2} \ln \left[1 + \frac{2x + a}{x^2 + 2x + \frac{4}{3}} \right] < G(x + 2) < \frac{1}{2} \ln \left[1 + \frac{2x + b}{x^2 + 2x + \frac{4}{3}} \right], \qquad x > 0$$

where $a = 3$ and $b = \frac{e^4 - 16}{12}$ are the best possible constants. Our results improve some recent inequalities about the function $G(x)$.

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1   Introduction.

In 2010, Mortici [21] presented the following Lemma which is considered as a powerful tool to constructing asymptotic expansions and to measure the rate of convergence:

Lemma 1.1. *If $\{\tau_s\}_{s \in N}$ is convergent to zero and there exist h in R and $k > 1$ such that*

$$\lim_{s \to \infty} s^k (\tau_s - \tau_{s+1}) = h, \tag{1}$$

then we get

$$\lim_{s \to \infty} s^{k-1} \tau_s = \frac{h}{k-1}.$$

It is clear from lemma (1.1) that, the sequence $\{\tau_s\}_{s \in N}$ will converge more quickly when the value of k is large in the relation (1). This Lemma has been applied successfully to produce several approximations and inequalities in several papers such as [6], [7], [11], [15], [16], [22], [24], [28]. In this paper, Lemma (1.1) will be an effectively tool in producing best approximations of the Bateman's $G-$function defined by [9]

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq 0, -1, -2, ... \tag{2}$$

where $\psi(x)$ is the digamma or Psi function which is defined by

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

and $\Gamma(x)$ is the classical Euler gamma given for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-w} w^{x-1} dw.$$

The hypergeometric representation of the function $G(x)$ is given by

$$G(x) = \frac{1}{x} \, _2F_1\left(1, 1; 1+x; 1/2\right) \tag{3}$$

and it satisfies the following relations [9]:

$$G(x+1) + G(x) = \frac{2}{x} \tag{4}$$

and

$$G(x) = 2 \int_0^\infty \frac{e^{-xw}}{1 + e^{-w}} dw, \qquad x > 0. \tag{5}$$

Qiu and Vuorinen [30] established the inequality

$$\frac{x + (6 - 4\ln 4)}{x^2} < G(x) < \frac{x + 1/2}{x^2}, \qquad x > 1/2 \tag{6}$$

and Mortici [23] presented the general inequality

$$0 < \psi(x+j) - \psi(x) \leq \psi(j) + \gamma - j + j^{-1}, \qquad x \geq 1; \ j \in (0, 1) \tag{7}$$

where γ is Euler$-$Mascheroni constant (also called Euler's constant) defined by

$$\gamma = \lim_{m \to \infty} \left(-\ln m + \sum_{w=1}^m \frac{1}{w}\right).$$

Mahmoud and Agarwal [17] deduced the following asymptotic formula for $x \to \infty$

$$G(x) - \frac{1}{x} \sim \sum_{w=1}^{\infty} \frac{(2^{2w} - 1) B_{2w}}{w} x^{-2w}, \tag{8}$$

where $B_w's$ are the Bernoulli numbers [1] defined by the generating series

$$\sum_{w=0}^{\infty} B_w \frac{v^w}{w!} = \frac{v}{e^v - 1}.$$

They also presented the following double inequality

$$\frac{1}{x} + \frac{1}{2x^2 + \frac{3}{2}} < G(x) < \frac{x + 1/2}{x^2}, \qquad x > 0 \tag{9}$$

which improves the lower bound of the inequality (6) for $x > \left(\frac{9 - 12 \ln 2}{16 \ln 2 - 11}\right)^{1/2}$. In [18] Mahmoud and Almuashi proved the following inequality

$$\sum_{w=1}^{2r} \frac{(2^{2w} - 1)}{w} \frac{B_{2w}}{x^{2w}} < G(x) - x^{-1} < \sum_{w=1}^{2r-1} \frac{(2^{2w} - 1)}{w} \frac{B_{2w}}{x^{2w}}, \qquad r \in \mathbb{N} \tag{10}$$

where $\frac{(2^{2w}-1)}{w} B_{2w}$ are the best possible constants. In [19], Mahmoud, Talat and Moustafa presented the following approximations of the Bateman's $G-$function

$$G(x) \approx \ln\left(1 + \frac{1}{x + c}\right) + \frac{2}{x(x + 1)}, \qquad c \in [1, 2], \qquad x > 0 \tag{11}$$

and they deduced the following double inequality

$$\ln\left(1 + \frac{1}{x + \alpha_2}\right) + \frac{2}{x(x + 1)} < G(x) < \ln\left(1 + \frac{1}{x + \alpha_1}\right) + \frac{2}{x(x + 1)}, \qquad x > 0 \tag{12}$$

where the constants $\alpha_1 = 1$ and $\alpha_2 = \frac{4}{e^2 - 4}$ are the best possible constants.

Recently, Mahmoud, Talat, Moustafa and Agarwal [20] improved the double inequality (9) by

$$\frac{1}{x} + \frac{1}{2x^2 + a} < G(x) < \frac{1}{x} + \frac{1}{2x^2 + b}, \qquad x > 0 \tag{13}$$

where $a = 1$ and $b = 0$ are the best possible constants.

A function T defined on an interval I is said to be completely monotonic if it possesses derivatives $T^{(s)}(x)$ for all $s = 0, 1, 2, ...$ such that

$$(-1)^s T^{(s)}(x) \geq 0 \qquad x \in I; \ s = 0, 1, 2, ... \tag{14}$$

Such functions occur in several areas such as numerical analysis, elasticity and probability theory, for more details see [2], [5], [12]-[14], [26], [27], [29]. According to Bernstein theorem [31],

the necessary and sufficient condition for the function $T(x)$ to be completely monotonic for $0 < x < \infty$ is that

$$T(x) = \int_0^\infty e^{-xt} d\lambda(t), \tag{15}$$

where $\lambda(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

In this paper, we presented two best approximation formulas of the Bateman's $G-$function and some completely monotonic functions involving it. Some new inequalities of $G(x)$ and its derivative will be deduced, which improve some pervious results.

2   Auxiliary Results

We can easily prove the following simple modification of Lemma (1.1):

Lemma 2.1. *If $\{\tau_s\}_{s \in N}$ is convergent to zero and there exist $h \in R$, $m \in N$ and $k > 1$ such that $\lim_{s \to \infty} s^k(\tau_s - \tau_{s+m}) = h$, then we get $\lim_{s \to \infty} s^{k-1}\tau_s = \frac{h/m}{k-1}$.*

Proof. Using the relation

$$
\begin{aligned}
\lim_{s \to \infty} s^k(\tau_s - \tau_{s+m}) &= \lim_{s \to \infty} s^k \sum_{i=0}^{m-1}(\tau_{s+i} - \tau_{s+i+1}) = \lim_{s \to \infty} \sum_{i=0}^{m-1}\left(\frac{s}{s+i}\right)^k (s+i)^k(\tau_{s+i} - \tau_{s+i+1}) \\
&= \sum_{i=0}^{m-1} \lim_{s \to \infty}\left(\frac{s}{s+i}\right)^k (s+i)^k(\tau_{s+i} - \tau_{s+i+1}) = \sum_{i=0}^{m-1} \lim_{s \to \infty}(s+i)^k(\tau_{s+i} - \tau_{s+i+1}) \\
&= \sum_{i=0}^{m-1} \lim_{v \to \infty} v^k(\tau_v - \tau_{v+1}) = m \lim_{v \to \infty} v^k(\tau_v - \tau_{v+1}),
\end{aligned}
$$

then $\lim_{v \to \infty} v^k(\tau_v - \tau_{v+1}) = \frac{h}{m}$. Applying Lemma (1.1) to get $\lim_{s \to \infty} s^{k-1}\tau_s = \frac{h/m}{k-1}$. □

Lemma 2.2.

1. *For $x > x_0 \approx 4.02361$, we have $N(x) = \ln\left(\frac{(x+1)(3-\sqrt{6}+3x)}{(x+2)(-\sqrt{6}+3x)}\right) - \frac{\left(1+\sqrt{\frac{2}{3}}\right)}{x(x+1)} < 0$.*

2. *For $x > x_\lambda \approx 2.02059$, we have $M(x) = \ln\left(\frac{(x+\frac{4}{e^2-4})(3+\sqrt{6}+3x)}{(x+1+\frac{4}{e^2-4})(\sqrt{6}+3x)}\right) - \frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)} > 0$.*

3. *For $x > 0$, we have $H(x) = \ln\left(\frac{(\sqrt{6}+3x)^2(13+12x+3x^2)}{(3+\sqrt{6}+3x)^2(4+6x+3x^2)}\right) + 2\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)} > 0$.*

Proof.

1. For $x > \sqrt{\frac{2}{3}}$, $N'(x) = \frac{9x^3-(9+14\sqrt{6})x^2-(6+11\sqrt{6})x-2\sqrt{6}}{x^2(x+1)^2(x+2)(3x-\sqrt{6})(3x+3-\sqrt{6})} \triangleq \frac{n_1(x)}{n_2(x)}$, where the polynomial $n_2(x)$ is positive for $x > \sqrt{\frac{2}{3}}$ and $n_1(x)$ is a polynomial of degree 3 has only one positive real root $x_1 \approx 5.49455$ and $n_1(x) > (<) 0$ for $x > (<) x_1$. Then $N'(x) > 0$ for $x > x_1$ with

$\lim_{x\to\infty} N(x) = 0$ and hence $N(x) < 0$ for $x > x_1$. Also, $N(x)$ is decreasing on $\left(\sqrt{\frac{2}{3}}, x_1\right)$ with $N(4.023) \approx 0.0000005 > 0$ and $N(4.024) \approx -0.0000003 < 0$. Then $N(x)$ has only one real root $x_0 \approx 4.02361 \in \left(\sqrt{\frac{2}{3}}, x_1\right)$ and $N(x) < 0$ for $x_0 < x < x_1$. Hence, $N(x) < 0$ for $x > x_0$.

2. For $x > 0$, $M'(x) = \frac{m(x)}{x^2(1+x)^2(\sqrt{6}+3x)(3+\sqrt{6}+3x)(4+(e^2-4)x)(e^2+(e^2-4)x)}$, where

$$
\begin{aligned}
m(x) &= 4\sqrt{6}e^2 + \left(-16\sqrt{6} + (-12+20\sqrt{6})e^2 + \sqrt{6}e^4\right)x + \left(576 - 216e^2 + 18e^4\right)x^5 \\
&+ \left(144 - 16\sqrt{6} - (72+20\sqrt{6})e^2 + (3+9\sqrt{6})e^4\right)x^2 \\
&+ \left(384 + 224\sqrt{6} - (144+160\sqrt{6})e^2 + (12+20\sqrt{6})e^4\right)x^3 \\
&+ \left(432 + 384\sqrt{6} - (252+144\sqrt{6})e^2 + (27+12\sqrt{6})e^4\right)x^4
\end{aligned}
$$

and

$$
\begin{aligned}
m'(x) &= 5\left(576 - 216e^2 + 18e^4\right)x^4 + \left(-16\sqrt{6} + (-12+20\sqrt{6})e^2 + \sqrt{6}e^4\right) \\
&+ 4\left(432 + 384\sqrt{6} + (-252-144\sqrt{6})e^2 + (27+12\sqrt{6})e^4\right)x^3 \\
&+ 3\left(384 + 224\sqrt{6} + (-144-160\sqrt{6})e^2 + (12+20\sqrt{6})e^4\right)x^2 \\
&+ 2\left(144 - 16\sqrt{6} + (-72-20\sqrt{6})e^2 + (3+9\sqrt{6})e^4\right)x.
\end{aligned}
$$

The polynomial $m'(x)$ of fourth degree has only one positive real root $x_\alpha \approx 2.57862$ also $m'(x) < 0$ for $x > x_\alpha$ and $m'(x) > 0$ for $0 < x < x_\alpha$. Hence $m(x)$ is increasing on $(0, x_\alpha)$ and is decreasing on (x_α, ∞) with $m(0) > 0$, $m(3.453) \approx 22.157 > 0$ and $m(3.455) \approx -6.01919 < 0$. Then $m(x)$ has only one positive real root $x_\beta \approx 3.45457$ with $m(x) < 0$ for $x > x_\beta$ and $m(x) > 0$ for $0 < x < x_\beta$. Now $M(x)$ is decreasing on (x_β, ∞) and $\lim_{x\to\infty} M(x) = 0$, then $M(x) > 0$ for $x > x_\beta$. Also, $M(x)$ is increasing on $(0, x_\beta)$ with $M(2.0205) \approx -0.0000006 < 0$ and $M(2.0206) \approx 0.0000001 > 0$, then $M(x)$ has only one positive real root $x_\lambda \approx 2.02059$. Hence, $M(x) > 0$ for $x > x_\lambda$.

3.
$$
H'(x) = \frac{-4h(x)}{x^2(1+x)^2(\sqrt{6}+3x)(3+\sqrt{6}+3x)(4+6x+3x^2)(13+12x+3x^2)},
$$
where

$$
h(x) = 26\sqrt{6} + (-78+193\sqrt{6})x + (-300+477\sqrt{6})x^2 + (-324+498\sqrt{6})x^3
$$

$$
+(-126+234\sqrt{6})x^4 + 36\sqrt{6}x^5 > 0, \qquad x > 0.
$$

Hence $H'(x) < 0$ for all $x > 0$ with $\lim_{x\to\infty} H(x) = 0$, then $H(x) > 0$ for $x > 0$.

$\square$

The following result is considered as a method presented by Elbert and Laforgia in [8] (see also, [4], [25] and [32]):

Corollary 2.3. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x \to \infty} K(x) = 0$. Then $K(x) > 0$, if $K(x) > K(x + 1)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x + 1)$ for all $x > a$.*

This result has the following simple modification [20]:

Corollary 2.4. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x \to \infty} K(x) = 0$. Then for $m \in \mathbb{N}$, $K(x) > 0$, if $K(x) > K(x+m)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+m)$ for all $x > a$.*

3 First formula of the best approximations and some its related inequalities

With the help of Mortici's technique in Lemma(1.1), we provide the first best approximation formula of the Bateman's $G-$function.

Lemma 3.1. *The best approximation*

$$G(n) \approx \ln(1 + \frac{1}{n+a}) + \frac{b}{n(n+c)} \tag{16}$$

holds for

$$a = \frac{\theta_1 + \theta_2 - 5}{9}, \quad b = a + 1 \quad and \quad c = \frac{\theta_1^2 + \theta_2^2 + 2\theta_1 + 2\theta_2 - 21}{54}, \tag{17}$$

where $\theta_1, \theta_2 = \sqrt[3]{91 \pm 63\sqrt{2}}$ and the sequence $G(n) - \ln(1 + \frac{1}{n+a}) - \frac{b}{n(n+c)}$ converges to zero with speed estimated by n^{-5}.

Proof. Define the error sequence by $v_n = G(n) - \ln\left(1 + \frac{1}{n+a}\right) - \frac{b}{n(n+c)}$. Using the functional equation (4), we get

$$
\begin{aligned}
v_n - v_{n+2} &= \sum_{r=3}^{\infty} \frac{(-1)^{r-1}}{n^r} \left\{ b[c^{r-1} + 2^{r-1} - (2 + c)^{r-1}]/c + [(a + 3)^r - (a + 2)^r - (a + 1)^r \right. \\
&+ \left. a^r]/r - 2 \right\} \\
&= \frac{4(a - b + 1)}{n^3} - \frac{2(7 + 3a(a + 3) - 3b(c + 2))}{n^4} \\
&+ \frac{4(a(16 + a(9 + 2a)) - 2(-5 + b(4 + c(3 + c))))}{n^5} \\
&- \frac{326/3 + 10a(3 + a)(7 + a(3 + a)) - 10b(2 + c)(4 + c(2 + c))}{n^6} + O(n^{-7}).
\end{aligned}
$$

According to Lemma (2.1), the three parameters a, b and c which produce the fastest convergence of the sequence v_n satisfy the system

$$a - b + 1 = 0$$
$$3a^2 + 9a - 3b(c + 2) + 7 = 0$$
$$a^3 + \frac{9}{2}a^2 + 8a + 5 - b(c^2 + 3c + 4) = 0.$$

Now, the values of a, b and c determined in (17) form solution of this system and the sequence v_n converges to zero with speed estimated by n^{-5}. $\qquad \square$

Now we will prove the complete monotonicity of some functions involving the function $G(x)$ depending on the approximation formula (16).

Lemma 3.2.

1. For the values of a and c in (17), the function $L_1(x) = \ln\left(1 + \frac{1}{x+a}\right) + \frac{a+1}{x(x+c)} - G(x)$ is completely monotonic on $(0, \infty)$.

2. The function $L_2(x) = \ln\left(1 + \frac{1}{x - \sqrt{\frac{2}{3}}}\right) + \frac{(1-\sqrt{\frac{2}{3}})}{x(x+1)} - G(x)$ is completely monotonic on $\left(\sqrt{\frac{2}{3}}, \infty\right)$.

3. The function $L_3(x) = G(x) - \ln\left(1 + \frac{1}{x + \sqrt{\frac{2}{3}}}\right) - \frac{(1+\sqrt{\frac{2}{3}})}{x(x+1)}$ is completely monotonic on $(0, \infty)$.

Proof. 1. Using the formula [1]

$$\frac{1}{x^k} = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-xt} dt, \qquad k \in \mathbb{N} \tag{18}$$

and the integral representation (5) of $G'(x)$, we get

$$L_1'(x) = \int_0^\infty \frac{e^{-(x+a+1)t}}{1+e^t} \nu_1(t) dt,$$

where

$$\nu_1(t) = \sum_{k=0}^\infty \frac{\left[(2 - \frac{(a+1)}{c})(a+2)^k + \frac{(a+1)}{c}[(a+1-c)^k + (a+2-c)^k - (a+1)^k] - \frac{2^{k+1}}{(k+1)}\right]}{k!} t^{k+1}$$

$$= -0.0316 t^5 - 0.0381 t^6 - 0.243 t^7 + \sum_{k=7}^\infty \frac{(a+2)^k [C_1(k)] - \frac{(a+1)}{c}(a+1)^k - \frac{2^{k+1}}{(k+1)}}{k!} t^{k+1}$$

with

$$C_1(k) = \left(2 - \frac{a+1}{c}\right) + \frac{a+1}{c}\left[\left(\frac{a+1-c}{a+2}\right)^k + \left(\frac{a+2-c}{a+2}\right)^k\right].$$

The sequences $(\frac{a+1-c}{a+2})^k$ and $(\frac{a+2-c}{a+2})^k$ are decreasing for $k \geq 7$, hence

$$C_1(k) < \left(2 - \frac{a+1}{c}\right) + \frac{a+1}{c}\left[\left(\frac{a+1-c}{a+2}\right)^7 + \left(\frac{a+2-c}{a+2}\right)^7\right] \approx -0.05248 < 0$$

and consequently $\nu_1(t) < 0$. Then $-L_1'(x)$ is completely monotonic. The function $L_1(x)$ is decreasing on $(0, \infty)$ and $\lim_{x \to \infty} L_1(x) = 0$, then $L_1(x) > 0$ and hence $L_1(x)$ is completely monotonic on $(0, \infty)$.

2.

$$L_2'(x) = \int_0^\infty \frac{e^{-(x+1)t}}{1+e^t} \nu_2(t)dt,$$

where

$$\nu_2(t) = \sum_{k=3}^\infty \frac{2^{k+1}\left[\left(\frac{1}{\sqrt{6}}\right)^{k+1} - \left(\frac{1}{\sqrt{6}}+1\right)^{k+1} + (k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right)\right]}{(k+1)!} t^{k+1}.$$

Now, consider the following sequence for $k = 3, 4, 5, ...$

$$\begin{aligned}
C_2(k) &= \left(\frac{1}{\sqrt{6}}\right)^{k+1} - \left(\frac{1}{\sqrt{6}}+1\right)^{k+1} + (k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right) \\
&= -\sum_{r=0}^k \binom{k+1}{r}\left(\frac{1}{\sqrt{6}}\right)^r + (k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right) \\
&< -\sum_{r=0}^2 \binom{k+1}{r}\left(\frac{1}{\sqrt{6}}\right)^r + (k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right) < -\frac{1}{12}(k-2)(k-3) < 0.
\end{aligned}$$

Hence $\nu_2(t) < 0$ and $-L_2'(x)$ is completely monotonic. The function $L_2(x)$ is decreasing on $\left(\sqrt{\frac{2}{3}}, \infty\right)$ with $\lim_{x\to\infty} L_2(x) = 0$ and then $L_2(x) > 0$. Hence $L_2(x)$ is completely monotonic on $\left(\sqrt{\frac{2}{3}}, \infty\right)$.

3.

$$L_3'(x) = \int_0^\infty \frac{e^{-\left(x+\sqrt{\frac{2}{3}}+1\right)t}}{1+e^t}\nu_3(t)dt,$$

where

$$\nu_3(t) = \sum_{k=3}^\infty \frac{2^{k+1}\left[\left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\left(1+\frac{1}{\sqrt{6}}\right)^k + \frac{1}{(k+1)} - \left(\frac{1}{2}+\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right)^k\right]}{k!} t^{k+1}.$$

The sequence

$$\begin{aligned}
C_3(k) &= \left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\left(1+\frac{1}{\sqrt{6}}\right)^k + \frac{1}{k+1} = \left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\sum_{r=0}^k \binom{k}{r}\left(\frac{1}{\sqrt{6}}\right)^r + \frac{1}{k+1} \\
&< \left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\sum_{r=0}^2 \binom{k}{r}\left(\frac{1}{\sqrt{6}}\right)^r + \frac{1}{k+1} \\
&< \frac{(k-3)((\sqrt{6}-3)k^2 + (3-3\sqrt{6})k - (12+4\sqrt{6}))}{72(k+1)} < 0, \qquad k = 3, 4, 5, ...
\end{aligned}$$

Then $\nu_3(t) < 0$ and hence $-L_3'(x)$ is completely monotonic. The function $L_3(x)$ is decreasing for all $x > 0$ with $\lim_{x\to\infty} L_3(x) = 0$. and hence $L_3(x) > 0$ for all $x > 0$. Then $L_3(x)$ is completely monotonic on $(0, \infty)$.

$\square$

From the complete monotonicity of the functions $L_1(x)$, $L_2(x)$ and $L_3(x)$, we deduce the following result:

Lemma 3.3. *1.*

$$\ln\left(1+\frac{1}{x+\sqrt{\frac{2}{3}}}\right)+\frac{\left(1+\sqrt{\frac{2}{3}}\right)}{x(x+1)} < G(x) < \ln\left(1-\frac{1}{x-\sqrt{\frac{2}{3}}}\right)+\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)}, \qquad (19)$$

where the upper bound holds for $x > \sqrt{\frac{2}{3}}$ and the lower bound holds for $x > 0$.

2.

$$G(x) < \ln\left(1+\frac{1}{x+a}\right)+\frac{1+a}{x(x+c)}, \qquad x > 0 \qquad (20)$$

where the values of a and c are in (17)

Remark 1. From Lemma (2.2), we can conclude that the inequality (19) improves the lower bound of the inequality (12) for $x > x_\lambda \simeq 2.02059$ and improves its upper bound for $x > x_0 \simeq 4.02361$.

Lemma 3.4. *The following inequality holds*

$$\ln\left(1+\frac{1}{x+\sqrt{\frac{2}{3}}}\right)+\frac{\left(1+\sqrt{\frac{2}{3}}\right)}{x(x+1)}+\frac{1}{6\sqrt{6}x^4} < G(x) < \ln\left(1+\frac{1}{x-\sqrt{\frac{2}{3}}}\right)+\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)}-\frac{1}{6\sqrt{6}x^4},$$

where the upper bound holds for $x > \sqrt{\frac{2}{3}}$ and the lower bound holds for $x \geq 2$.

Proof. Consider the function

$$T(x) = \ln\left(1+\frac{1}{x-\sqrt{\frac{2}{3}}}\right)+\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)}-\frac{1}{6\sqrt{6}x^4}-G(x), \qquad x > \sqrt{\frac{2}{3}}$$

and use the functional equation (4) to obtain

$$T'(x+2) - T'(x) = \frac{2l(x)}{81x^5(1+x)^2(2+x)^5(3+x)^2 \sum_{i=0}^{3}\left(x+i-\sqrt{\frac{2}{3}}\right)},$$

where

$$\begin{aligned}
l(x) &= \left(25920 - 10080\sqrt{6}\right) + \left(197856 - 75408\sqrt{6}\right)x \\
&\quad + \left(677952 - 257008\sqrt{6}\right)x^2 + \left(1367472 - 520800\sqrt{6}\right)x^3 \\
&\quad + \left(1777284 - 666478\sqrt{6}\right)x^4 + \left(1535268 - 502094\sqrt{6}\right)x^5 \\
&\quad + \left(879720 - 127639\sqrt{6}\right)x^6 + \left(321960 + 145938\sqrt{6}\right)x^7 \\
&\quad + \left(66960 + 180403\sqrt{6}\right)x^8 + \left(4596 + 95742\sqrt{6}\right)x^9 \\
&\quad + \left(-864 + 28431\sqrt{6}\right)x^{10} + \left(-144 + 4590\sqrt{6}\right)x^{11} + 315\sqrt{6}x^{12} > 0, \ x > 0.
\end{aligned}$$

Then $T'(x+2) - T'(x) > 0$ for $x > \sqrt{\frac{2}{3}}$ and also $\lim_{x \to \infty} T'(x) = 0$. Using Corollary (2.4), we get that $T'(x) < 0$ for all $x > \sqrt{\frac{2}{3}}$. Hence $T(x)$ is decreasing on $\left(\sqrt{\frac{2}{3}}, \infty\right)$ with $\lim_{x \to \infty} T(x) = 0$, thus $T(x) > 0$ for all $x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$. Now consider the function

$$Q(x) = G(x) - \ln\left(1 + \frac{1}{x + \sqrt{\frac{2}{3}}}\right) - \frac{\left(1 + \sqrt{\frac{2}{3}}\right)}{x(x+1)} - \frac{1}{6\sqrt{6}x^4}, \quad x > 0.$$

Then

$$Q'(x+2) - Q'(x) = \frac{2u(x-2)}{81x^5(1+x)^2(2+x)^5(3+x)^2 \sum_{i=0}^{3}\left(x + i + \sqrt{\frac{2}{3}}\right)},$$

where

$$
\begin{aligned}
u(x) &= \left(-207466560 + 113432160\sqrt{6}\right) + \left(-585268704 + 582357840\sqrt{6}\right)x \\
&+ \left(-729011328 + 1250421968\sqrt{6}\right)x^2 + \left(-523396080 + 1539421184\sqrt{6}\right)x^3 \\
&+ \left(-235893516 + 1231511026\sqrt{6}\right)x^4 + \left(-67175076 + 680979590\sqrt{6}\right)x^5 \\
&+ \left(-10943256 + 268473813\sqrt{6}\right)x^6 + \left(-465384 + 76331554\sqrt{6}\right)x^7 \\
&+ \left(195912 + 15574939\sqrt{6}\right)x^8 + \left(44364 + 2228562\sqrt{6}\right)x^9 \\
&+ \left(4032 + 212571\sqrt{6}\right)x^{10} + \left(144 + 12150\sqrt{6}\right)x^{11} + 315\sqrt{6}x^{12} > 0, \quad x \geq 0.
\end{aligned}
$$

Thus $Q'(x+2) - Q'(x) > 0$ for $x \geq 2$ and also $\lim_{x \to \infty} Q'(x) = 0$. Using Corollary (2.4), we obtain that $Q'(x) < 0$ for all $x \geq 2$. and then $Q(x)$ is decreasing on $[2, \infty)$ with $\lim_{x \to \infty} Q(x) = 0$. Then $Q(x) > 0$ for all $x \geq 2$. $\qquad \square$

4 Second formula of the best approximations and some of its related inequalities

In this section, we will present the best constants of the approximation of formula

$$G(n) \approx \frac{1}{2}\ln\left[1 + \frac{P_1(n)}{P_2(n)}\right] + \frac{2}{n(n+1)}, \quad n \in \mathbb{N}$$

where $P_1(n)$ and $P_2(n)$ are two polynomials of degrees one and two (resp.). Also, some inequalities of the function $G(x)$ will provided, which improve some results of the previous section.

Lemma 4.1. *The best approximation of the formula*

$$G(n) \approx \frac{1}{2}\ln\left[1 + \frac{\alpha n + \beta}{n^2 + \rho x + \sigma}\right] + \frac{2}{n(n+1)}, \quad n \in \mathbb{N} \tag{21}$$

holds for $\alpha = 2$, $\beta = 3$, $\rho = 2$ and $\sigma = 4/3$ and the sequence $G(n) - \frac{1}{2}\ln\left[1 + \frac{\alpha n + \beta}{n^2 + \rho x + \sigma}\right] - \frac{2}{n(n+1)}$ converges to zero with speed estimated by n^{-5}.

Proof. Consider the error sequence $\chi_n = G(n) - \frac{1}{2}\ln\left[1 + \frac{\alpha n + \beta}{n^2 + \rho x + \sigma}\right] - \frac{2}{n(n+1)}$, then we have

$$
\begin{aligned}
\chi_n - \chi_{n+2} &= \frac{1}{n^2}(2 - \alpha) + \frac{1}{n^3}(-2(5 + \beta) + \alpha(2 + \alpha + 2\rho)) \\
&+ \frac{1}{n^4}(38 - \alpha^3 - 3\alpha^2(1 + \rho) + 3\beta(2 + \rho) + \alpha(-4 + 3\beta - 3\rho(2 + \rho) + 3\sigma)) \\
&+ \frac{1}{n^5}(\alpha^4 + 4\alpha^3(1 + \rho) + 2(-65 + \beta^2 - 2\beta(4 + \rho(3 + \rho) - \sigma)) \\
&+ \alpha^2 (8 - 4\beta + 6\rho(2 + \rho) - 4\sigma) + 4\alpha(2 - \beta(3 + 2\rho) + \rho(4 + \rho(3 + \rho) - 2\sigma) - 3\sigma)) \\
&+ \frac{1}{n^6}(422 - \alpha^5 - 5\alpha^4(1 + \rho) + 5\beta(2 + \rho)(4 - \beta + \rho(2 + \rho) - 2\sigma) \\
&+ 5/3\alpha^3(-8 + 3\beta - 6\rho(2 + \rho) + 3\sigma) + 5\alpha^2(\beta(4 + 3\rho) - 2(1 + \rho)(2 + \rho(2 + \rho)) \\
&+ (4 + 3\rho)\sigma) - \alpha(16 + 5\beta^2 - 5\beta(8 + \rho(8 + 3\rho) - 2\sigma) - 40\sigma + 5(\rho(2 + \rho)(4 \\
&+ \rho(2 + \rho)) - \rho(8 + 3\rho)\sigma + \sigma^2))) + O(n^{-7}).
\end{aligned}
$$

According to Lemma (2.1), the fastest convergence of the sequence χ_n satisfies if $\alpha = 2$, $\beta = 3$, $\rho = 2$ and $\sigma = 4/3$ with speed estimated by n^{-5}. $\qquad\square$

Lemma 4.2. *For $x > -1$, the function*

$$R(x) = (e^{2G(x+2)} - 1)(x^2 + 2x + \frac{4}{3}) - 2x \tag{22}$$

is strictly decreasing and convex. As consequence, we have

$$\frac{1}{2}\ln\left[1 + \frac{2x + 3}{x^2 + 2x + \frac{4}{3}}\right] < G(x + 2) < \frac{1}{2}\ln\left[1 + \frac{2x + \frac{e^4 - 16}{12}}{x^2 + 2x + \frac{4}{3}}\right], \qquad x > 0 \tag{23}$$

where the constants 3 and $\frac{e^4 - 16}{12}$ are the best possible.

Proof.

$$\frac{1}{2}R'(x) = -x - 2 + [(x^2 + 2x + \frac{4}{3})G'(x + 2) + (x + 1)]e^{2G(x+2)},$$

$$\frac{1}{2}R''(x) = -1 + 2[(x^2 + 2x + \frac{4}{3})G'(x + 2) + (x + 1)]G'(x + 2)e^{2G(x+2)}$$

$$+[2(x + 1)G'(x + 2) + (x^2 + 2x + \frac{4}{3})G''(x + 2) + 1]e^{2G(x+2)}$$

and

$$
\begin{aligned}
\frac{1}{2e^{2G(x+2)}}R'''(x) &= 4(x^2 + 2x + \frac{4}{3})(G'(x + 2))^3 + 12(x + 1)(G'(x + 2))^2 \\
&+ 6(x^2 + 2x + \frac{4}{3})G'(x + 2)G''(x + 2) \\
&+ 6(x + 1)G''(x + 2) + 6G'(x + 2) + (x^2 + 2x + \frac{4}{3})G'''(x + 2) \triangleq U(x).
\end{aligned}
$$

Now,

$$
\begin{aligned}
U(x+2) - U(x) =\ & 16(x+2)(G'(x+2))^3 + 4(x+2)G'''(x+2) + 24(x+2)G'(x+2)G''(x+2) \\
& + \frac{8(4+x)(62+66x+24x^2+3x^3)}{(x+2)^2(x+3)^2}(G'(x+2))^2 \\
& + \frac{8(1448+2564x+2008x^2+915x^3+258x^4+42x^5+3x^6)}{(x+2)^4(x+3)^4}G'(x+2) \\
& + \frac{4(4+x)(62+66x+24x^2+3x^3)}{(x+2)^2(x+3)^2}G''(x+2) \\
& + \frac{1}{3(2+x)^6(3+x)^6}4(-33056-88128x-92112x^2-46976x^3-10548x^4 \\
& + 360x^5+705x^6+138x^7+9x^8).
\end{aligned}
$$

Also, let $V(x) = \frac{U(x+2)-U(x)}{4(x+2)}$, then

$$
\begin{aligned}
V(x+2) - V(x) =\ & \frac{-4}{(2+x)^3(3+x)^2(4+x)^3(5+x)^2}[34576+91136x+105392x^2+68520x^3 \\
& +27152x^4+6698x^5+1005x^6+84x^7+3x^8](G'(x+2))^2 \\
& - \frac{4}{(2+x)^5(3+x)^4(4+x)^5(5+x)^4}[376528768+1642942016x+3297590048x^2 \\
& + 4031614688x^3+3354474592x^4+2010658592x^5+896184192x^6 \\
& + 302070808x^7+77457190x^8+15051780x^9 \\
& + 2183975x^{10}+229624x^{11}+16548x^{12}+732x^{13}+15x^{14}]G'(x+2) \\
& - \frac{2}{(2+x)^3(3+x)^2(4+x)^3(5+x)^2}(34576+91136x+105392x^2 \\
& + 68520x^3+27152x^4+6698x^5+1005x^6+84x^7+3x^8)G''(x+2) \\
& + \frac{2}{3(2+x)^7(3+x)^6(4+x)^7(5+x)^6}[(331346962432+5157549202432x \\
& + 24078469545984x^2+60544326323200x^3+99175110059776x^4 \\
& + 116067402353280x^5+102360341211232x^6+70356164081536x^7 \\
& + 38541166023024x^8+17080773307136x^9+6184124004420x^{10} \\
& + 1839407553792x^{11}+450403876283x^{12}+90669453918x^{13}+14930598072x^{14} \\
& + 1992453932x^{15}+212255598x^{16}+17635104x^{17}+1101756x^{18}+48708x^{19} \\
& + 1359x^{20}+18x^{21})]
\end{aligned}
$$

Using the completely monotonicity of the functions $X_1(x) = \frac{1}{x} - G(x) + \sum_{k=1}^{2m-1} \frac{(2^{2k}-1)B_{2k}}{kx^{2k}}$ and $X_2(x) = G(x) - \frac{1}{x} - \frac{1}{2x^2+1}$ for $x > 0$ (see [20]), we get the following inequalities: $G'(x) > -\frac{x+1}{x^3}$, $(G'(x))^2 > \left[\frac{4x^4+4x^3+4x^2+1}{x^2(2x^2+1)^2}\right]^2$ and $G''(x) > \frac{2(8x^6+12x^5+12x^4-2x^3+6x^2+1)}{x^3(2x^2+1)^3}$ for $x > 0$. Hence,

$$
V(x+2) - V(x) < F(x), \qquad x > 0
$$

where

$$F(x) = \frac{-2A(x+1)}{3(2+x)^8(3+x)^6(4+x)^7(5+x)^6(9+8x+2x^2)^4}$$

with

$$
\begin{aligned}
A(x) \;=\;& 74586749184 + 1263034988928x + 9398610597600x^2 + 42943724513952x^3 \\
+\;& 138441396784472x^4 + 339577282357568x^5 + 663837528239296x^6 \\
+\;& 1065966556249164x^7 + 1434284269631783x^8 + 1638094930661455x^9 \\
+\;& 1600662870950288x^{10} + 1344078197917456x^{11} + 971670315225407x^{12} \\
+\;& 604754235543331x^{13} + 323563802759956x^{14} + 148404632743888x^{15} \\
+\;& 58109372496201x^{16} + 19315095938361x^{17} + 5408999070416x^{18} \\
+\;& 1263403407224x^{19} + 242838160053x^{20} + 37707313393x^{21} + 4608156812x^{22} \\
+\;& 426325500x^{23} + 28044100x^{24} + 1167972x^{25} + 23136x^{26}.
\end{aligned}
$$

Using $A(x) > 0$ for all $x > 0$, then we obtain $F(x) < 0$ for all $x > -1$ and hence $V(x+2) - V(x) < 0$ for all $x > -1$. Using the asymptotic expansion (8) and its derivatives, we have

$$G'(x) = \frac{-1}{x^2} - \frac{1}{x^3} + \frac{1}{x^5} - \frac{3}{x^7} + \frac{17}{x^9} + O(x^{-11}) \tag{24}$$

$$G''(x) = \frac{2}{x^3} + \frac{3}{x^4} - \frac{5}{x^6} + \frac{21}{x^8} - \frac{153}{x^{10}} + O(x^{-12}), \tag{25}$$

and

$$G'''(x) = \frac{-6}{x^4} - \frac{12}{x^5} + \frac{30}{x^7} - \frac{168}{x^9} + \frac{1530}{x^{11}} + O(x^{-13}). \tag{26}$$

Then

$$\lim_{x\to\infty} V(x) = \lim_{x\to\infty}\left(\frac{64x^{25}}{(2+x)^{27}(3+x)^6} + O(x^{-9})\right) = 0$$

and hence $V(x) > 0$ for all $x > -1$. Now, $U(x+2) - U(x) > 0$ with

$$\lim_{x\to\infty} U(x) = \lim_{x\to\infty}\left(\frac{-64x^{21}}{3(x+2)^{27}} + O(x^{-7})\right) = 0$$

and so $U(x) < 0$ for all $x > -1$. Thus, $R'''(x) < 0$ and

$$\lim_{x\to\infty} R''(x) = \lim_{x\to\infty}\left(\frac{128}{15x^5} - \frac{448}{9x^6} + \frac{2368}{21x^7} + O(x^{-8})\right) = 0.$$

Then $R''(x) > 0$ for all $x > -1$ and so the function $R(x)$ is convex for $x \in (-1, \infty)$. Also,

$$\lim_{x\to\infty} R'(x) = \lim_{x\to\infty}\left(\frac{-32}{15x^4} + \frac{448}{45x^5} - \frac{1184}{63x^6} - \frac{688}{63x^7} + \frac{11104}{81x^8} + O(x^{-9})\right) = 0$$

and thus $R'(x) < 0$ for all $x > -1$. Hence we conclude that $R(x)$ is decreasing on $(-1, \infty)$ with $R(0) = \frac{e^4 - 16}{12}$ and

$$\lim_{x\to\infty} R(x) = \lim_{x\to\infty}\left(3 + \frac{32}{45x^3} - \frac{112}{45x^4} + \frac{1184}{315x^5} + O(x^{-6})\right) = 3.$$

Then

$$3 < (e^{2G(x+2)} - 1)(x^2 + 2x + \frac{4}{3}) - 2x < \frac{e^4 - 16}{12},$$

where the constants 3 and $\frac{e^4-16}{12}$ are the best possible. $\qquad\square$

Lemma 4.3. *For every $x \geq 0$, we have*

$$\frac{1}{2} \ln\left[\frac{4x + a}{(x^2 + 6x + \frac{28}{3})e^{-\frac{4}{(x+2)(x+3)}} - (x^2 + 2x + \frac{4}{3})} \right] \leq G(x+2)$$

$$< \frac{1}{2} \ln\left[\frac{4x + b}{(x^2 + 6x + \frac{28}{3})e^{-\frac{4}{(x+2)(x+3)}} - (x^2 + 2x + \frac{4}{3})} \right] \tag{27}$$

where $a = \frac{7e^{\frac{10}{3}} - e^4}{12}$ and $b = 12$ are the best possible constants.

Proof. For $x \geq 0$, consider $f(x) = R(x+2) - R(x)$, where $R(x)$ defined in (22). Then $f'(x) = R'(x+2) - R'(x)$ and $R(x)$ is convex function for $x \in (-1, \infty)$. Hence $f(x)$ is increasing with $f(0) = \frac{7e^{\frac{10}{3}} - e^4}{12} - 12$ and $\lim_{x\to\infty} f(x) = 0$, where $\lim_{x\to\infty} R(x) = 3$. Then $\frac{7e^{\frac{10}{3}} - e^4}{12} - 12 \leq f(x) < 0$ or

$$\frac{7e^{\frac{10}{3}} - e^4}{12} \leq e^{2G(x+2)}[(x^2 + 6x + \frac{28}{3})e^{-\frac{4}{(x+2)(x+3)}} - (x^2 + 2x + \frac{4}{3})] - 4x < 12,$$

where $\frac{7e^{\frac{10}{3}} - e^4}{12}$ and 12 are the best possible constants. $\qquad\square$

Lemma 4.4. *For every $x > 0$, then we have*

$$\frac{(x + \alpha)e^{-2G(x+2)} - (x+1)}{(x^2 + 2x + \frac{4}{3})} < G'(x+2) < \frac{(x + \beta)e^{-2G(x+2)} - (x+1)}{(x^2 + 2x + \frac{4}{3})}, \tag{28}$$

where $\alpha = \frac{(2\pi^2 - 15)e^4}{144}$ and $\beta = 2$ are the best possible constants.

Proof. The function $R(x)$ defined in (22) is convex for $x \in (-1, \infty)$ and hence $R'(x)$ is increasing. Then $R'(0) < R'(x) < \lim_{x\to\infty} R'(x)$ with $R'(0) = \frac{(2\pi^2 - 15)e^4}{72} - 4$ and $\lim_{x\to\infty} R'(x) = 0$. Hence

$$\frac{(2\pi^2 - 15)e^4}{144} < -x + [(x^2 + 2x + \frac{4}{3})G'(x+2) + (x+1)]e^{2G(x+2)} < 2,$$

where $\frac{(2\pi^2 - 15)e^4}{144}$ and 2 are the best possible constants. $\qquad\square$

Lemma 4.5. *The following inequality holds*

$$\frac{1}{2} \ln\left[1 + \frac{2x + 3}{x^2 + 2x + \frac{4}{3}} \right] + \frac{2}{x(x+1)} < G(x) < \frac{1}{2} \ln\left[1 + \frac{2x + 3}{x^2 + 2x + \frac{48}{e^4 - 16}} \right] + \frac{2}{x(x+1)}, \tag{29}$$

where the upper bound holds for $x > x_\delta \approx 0.575833$ and the lower bound holds for $x > 0$.

Proof. Consider the function $F(x) = G(x+2) - \frac{1}{2}\ln\left[1 + \frac{2x+3}{x^2+2x+\frac{4}{3}}\right]$, then

$$F'(x+2) - F'(x) = G'(x+4) - G'(x+2)$$
$$- \frac{54(5+2x)(56+140x+103x^2+30x^3+3x^4)}{(4+6x+3x^2)(13+12x+3x^2)(28+18x+3x^2)(49+24x+3x^2)}.$$

Using the functional equation (4) and its derivative, we get

$$F'(x+2) - F'(x) =$$
$$\frac{32(5+2x)(1057+1680x+1011x^2+270x^3+27x^4)}{(2+x)^2(3+x)^2(4+6x+3x^2)(13+12x+3x^2)(28+18x+3x^2)(49+24x+3x^2)}.$$

Thus $F'(x+2) - F'(x) > 0$, for $x > 0$ and also $\lim_{x\to\infty} F'(x) = 0$. Using Corollary (2.4), we get that $F'(x) < 0$ for all $x > 0$. Then $F(x)$ is decreasing function on $(0, \infty)$ with $\lim_{x\to\infty} F(x) = 0$, thus $F(x) > 0$ for $x > 0$. Now, let

$$S(x) = G(x+2) - \frac{1}{2}\ln\left[1 + \frac{2x+3}{x^2+2x+\frac{48}{e^4-16}}\right]$$

and then

$$S'(x+2) - S'(x) = \frac{-8(5+2x)W(x)}{(e^4-16)^4(2+x)^2(3+x)^2(x^2+2x+\frac{48}{e^4-16})(x^2+4x+\frac{3e^4}{e^4-16})D(x)},$$

where

$$D(x) = \left((x+2)^2 + 2(x+2) + \frac{48}{e^4-16}\right)\left((x+2)^2 + 4(x+2) + \frac{3e^4}{e^4-16}\right) > 0, \qquad x > 0$$

and

$$\begin{aligned}
W(x) &= \left(21233664 - 6856704e^4 + 720576e^8 - 28944e^{12} + 324e^{16}\right) \\
&+ \left(100270080 - 26173440e^4 + 2361600e^8 - 84240e^{12} + 900e^{16}\right)x \\
&+ \left(152764416 - 35570688e^4 + 2920320e^8 - 96948e^{12} + 1005e^{16}\right)x^2 \\
&+ \left(106332160 - 23142400e^4 + 1795200e^8 - 57040e^{12} + 580e^{16}\right)x^3 \\
&+ \left(37257216 - 7818240e^4 + 587520e^8 - 18204e^{12} + 183e^{16}\right)x^4 \\
&+ \left(6389760 - 1320960e^4 + 97920e^8 - 3000e^{12} + 30e^{16}\right)x^5 \\
&+ \left(425984 - 88064e^4 + 6528e^8 - 200e^{12} + 2e^{16}\right)x^6.
\end{aligned}$$

Then

$$\begin{aligned}
W'(x) &= \left(100270080 - 26173440e^4 + 2361600e^8 - 84240e^{12} + 900e^{16}\right) \\
&+ 2\left(152764416 - 35570688e^4 + 2920320e^8 - 96948e^{12} + 1005e^{16}\right)x \\
&+ 3\left(106332160 - 23142400e^4 + 1795200e^8 - 57040e^{12} + 580e^{16}\right)x^2 \\
&+ 4\left(37257216 - 7818240e^4 + 587520e^8 - 18204e^{12} + 183e^{16}\right)x^3 \\
&+ 5\left(6389760 - 1320960e^4 + 97920e^8 - 3000e^{12} + 30e^{16}\right)x^4 \\
&+ 6\left(425984 - 88064e^4 + 6528e^8 - 200e^{12} + 2e^{16}\right)x^5
\end{aligned}$$

and

$$
\begin{aligned}
W''(x) \;=\; & 2\left(152764416 - 35570688e^4 + 2920320e^8 - 96948e^{12} + 1005e^{16}\right) \\
& +6\left(106332160 - 23142400e^4 + 1795200e^8 - 57040e^{12} + 580e^{16}\right)x \\
& +12\left(37257216 - 7818240e^4 + 587520e^8 - 18204e^{12} + 183e^{16}\right)x^2 \\
& +20\left(6389760 - 1320960e^4 + 97920e^8 - 3000e^{12} + 30e^{16}\right)x^3 \\
& +30\left(425984 - 88064e^4 + 6528e^8 - 200e^{12} + 2e^{16}\right)x^4 > 0, \quad x > 0.
\end{aligned}
$$

Thus $W'(x)$ is increasing on $(0,\infty)$ which implies that $W'(x) > W'(0.1) > 0$. Then $W(x)$ is increasing on $(0.1,\infty)$ with $W(0.57583) \approx -475.425 < 0$ and $W(0.57584) \approx 1147.33 > 0$. Hence $W(x)$ has only one positive root on $(0.57583,\infty)$ say $x_\delta \approx 0.575833$ and then $W(x) > 0$ for $x > 0.575833$. Now, $S'(x+2) - S'(x) < 0$ for $x > 0.575833$ and also $\lim_{x\to\infty} S'(x) = 0$. Using Corollary (2.4), then $S'(x) > 0$ for all $x > 0.575833$ or $S(x)$ is increasing on $(0.575833,\infty)$ with $\lim_{x\to\infty} S(x) = 0$. Thus $S(x) < 0$ for all $x > 0.575833$. $\qquad\square$

Remark 2. Using the inequalities

$$
1 + (2x + 3)/(x^2 + 2x + 48/(e^4 - 16)) < (1 + 1/(x + 1))^2, \qquad x > 0
$$

and

$$
1 + (2x + 3)/(x^2 + 2x + 4/3) > (1 + 1/(x + 4/(e^2 - 4)))^2, \qquad x > x_\mu
$$

where $x_\mu = \frac{-112 + 68e^2 - 7e^4 - \sqrt{52480 - 30208e^2 + 6672e^4 - 664e^6 + 25e^8}}{6(32 - 12e^2 + e^4)} \simeq 0.465586$, we can conclude that the inequality (29) improves the lower bound of the inequality (12) for $x > x_\mu$ and improves its upper bound for $x > 0$.

Remark 3. The inequality (29) improves the lower bound of the inequality (19) for $x > 0$.

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A new q-extension of Euler polynomial of the second kind and some related polynomials

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Abstract : We define q-Euler polynomials of the second kind using q-analogue within exponential function. We have some basic properties of this polynomials such as addition, alternative finite sum, and symmetry property. We also investigate some relations of q-Euler, q-Bernoulli, and q-tangent polynomials using q-Euler polynomials of the second kind including two parameters.

Key words : q-Euler polynomials of the second kind, q-Euler polynomials, q-Bernoulli polynomials, q-tangent polynomials

2000 Mathematics Subject Classification : 11B68, 11B75, 12D10

1. Introduction

The main aim of this paper is to extend Euler numbers and polynomials of the second kind, and study some of their properties. Our paper is organised as follows: in Section 2, we define q-Euler numbers and polynomials of the second kind. From this definition we investigate some interesting properties of these numbers and polynomials using q-analogue of exponential function. In Section 3, we consider q-Euler polynomials of the second kind in two parameters and make some relations between q-Euler polynomials of the second kind and q-Euler , q-Bernoulli, q-tangent polynomials. Furthermore, we derive a symmetric relation, multiple q-derivative, and multiple q-integral.

For any $n \in \mathbb{C}$, the q-number is defined by

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{0 \leq i \leq n} q^i = 1 + q + q^2 + \cdots + q^{n-1}.$$

An intensive and somewhat surprising interest in q-numbers appeared in many areas of mathematics and applications including q-difference equations, special functions, q-combinatorics, q-integrable systems, variational q-calculus, q-series, and so on. In this paper, we introduce some basic definitions and theorems(see [1-18]).

Definition 1.1.[1,3-5,10-13] The Gaussian binomial coefficients are defined by

$$\binom{m}{r}_q = \begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{cases} 0 & \text{if } r > m \\ \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & \text{if } r \leq m \end{cases},$$

where m and r are non-negative integers. For $r = 0$ the value is 1 since the numerator and the denominator are both empty products. Like the classical binomial coefficients, the Gaussian binomial coefficients are center-symmetric. There are analogues of the binomial formula, and this definition has a number of properties.

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Theorem 1.2.[5] Let n, k be non-negative integers. Then we get

$$(i) \qquad \prod_{k=0}^{n-1}(1 + q^k t) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k,$$

$$(ii) \qquad \prod_{k=0}^{n-1} \frac{1}{(1 - q^k t)} = \sum_{k=0}^{\infty} \begin{bmatrix} n + k - 1 \\ k \end{bmatrix}_q t^k.$$

Definition 1.3.[1,4,12-13] Let z be any complex numbers with $|z| < 1$. Two forms of q-exponential functions are defined by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad e_{q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

Definition 1.4.[4,10-11,13] The q-derivative operator of any function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \qquad x \neq 0,$$

and $D_q f(0) = f'(0)$. We can prove that f is differentiable at 0, and it is clear that $D_q x^n = [n]_q x^{n-1}$.

Definition 1.5.[4,10-11,13] We define the q-integral as

$$\int_0^b f(x) d_q x = (1 - q) b \sum_{j=0}^{\infty} q^j f(q^j b).$$

If this function, $f(x)$, is differentiable on the point x, the q-derivative in Definition 1.4 goes to the ordinary derivative in the classical analysis when $q \to 1$.

In 1961, L.Calitz introduced several properties of the Bernoulli and Euler polynomials of the second kind(see [6]). Euler numbers of the second kind was expanded, and C. S. Ryoo have studied these numbers and polynomials of the second kind in [17]. He also developed several properties of these numbers and polynomials.

Definition 1.6.[7-8, 6, 17] The classical Euler numbers, $\widetilde{E}_n$, and the classical Euler polynomials, $\widetilde{E}_n(x)$, of the second kind are defined by means of the generating functions

$$\sum_{n=0}^{\infty} \widetilde{E}_n \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}}, \quad \sum_{n=0}^{\infty} \widetilde{E}_n(x) \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}} e^{tx}.$$

Theorem 1.7.[17] For any positive integer n, we have

$$(i) \qquad \text{For any positive integer } m(=\text{odd}),$$

$$\widetilde{E}_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i \widetilde{E}_n \left(\frac{2i + x + 1 - m}{m} \right) \text{ for } n \geq 0,$$

$$(ii) \qquad \widetilde{E}_l(x + y) = \sum_{n=0}^{l} \binom{l}{n} \widetilde{E}_n(x) y^{l-n},$$

$$(iii) \qquad \widetilde{E}_n(x) = (-1)^n \widetilde{E}_n(-x).$$

2. Some basic properties of the q-Euler polynomials of the second kind

In this section, we define the q-Euler numbers and polynomials of the second kind, and investigate basic properties of these numbers and polynomials. Furthermore, we find the alternative finite sum which is related to the q-Euler numbers and polynomials of second kind.

Definition 2.1. Let n be any non-negative integer. For $|q| < 1, x \in \mathbb{C}$, we define q-Euler numbers and polynomials of the second kind as

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)},$$

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} e_q(tx).$$

Substituting $x = 0$ in the q-Euler polynomials of the second kind, they can be simplified as follows:

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(0) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} = \frac{1}{\cosh_q(t)},$$

where $\widetilde{\mathcal{E}}_{n,q}$ is q-Euler numbers of the second kind. If $q \to 1$, then we can find the classical Euler polynomials of the second kind in $\widetilde{\mathcal{E}}_{n,q}(x)$(see [6,17]).

Theorem 2.2. Let $|q| < 1, x$ be any complex numbers. Then, we have

$$\widetilde{\mathcal{E}}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q} x^{n-k}.$$

Proof. From the generating function of the q-Euler polynomials of second kind, $\widetilde{\mathcal{E}}_{n,q}(x)$, we can find

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} e_q(tx) = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}(x) x^{n-k} \right) \frac{t^n}{[n]_q!},$$

which gives the required result.

Theorem 2.3. For $|q| < 1$, the following holds:

$$D_q \widetilde{\mathcal{E}}_{n,q}(x) = [n]_q \widetilde{\mathcal{E}}_{n-1,q}(x).$$

Proof. Considering q-derivative of x^{n-k} in Theorem 2.2, we get

$$D_q \widetilde{\mathcal{E}}_{n,q}(x) = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q [n-k]_q \widetilde{\mathcal{E}}_{k,q} x^{n-k-1}.$$

Transforming a binomial operation of q and using Theorem 2.2 again, we obtain

$$D_q \widetilde{\mathcal{E}}_{n+1,q}(x) = [n+1]_q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q} x^{n-k} = [n+1]_q \widetilde{\mathcal{E}}_{n,q}(x).$$

The required relation now follows at once.

Theorem 2.4. Let n be any non-negative integer. Then, the following holds:

$$\int_0^x \widetilde{\mathcal{E}}_{n,q}(x)d_qx = \frac{\widetilde{\mathcal{E}}_{n+1,q}(x) - \widetilde{\mathcal{E}}_{n+1,q}}{[n+1]_q},$$

where $\widetilde{\mathcal{E}}_{n,q}(0) = \widetilde{\mathcal{E}}_{n,q}$ is q-Euler numbers of the second kind.

Proof. Using q-integral in Theorem 2.2, we have

$$\int_0^x \widetilde{\mathcal{E}}_{n,q}(x)d_qx = \int_0^x \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}x^{n-k}d_qx = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}\frac{1}{[n-k+1]_q}x^{n-k+1}\Big|_0^x$$

$$= \frac{1}{[n+1]_q}\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}x^{n-k+1}\Big|_0^x = \frac{1}{[n+1]_q}\left(\widetilde{\mathcal{E}}_{n+1,q}(x) - \widetilde{\mathcal{E}}_{n+1,q}\right),$$

and we obtain the required relation at once.

Corollary 2.5. In Theorem 2.4, we get

$$\int_a^b \widetilde{\mathcal{E}}_{n,q}(x)d_qx = \frac{\widetilde{\mathcal{E}}_{n+1,q}(b) - \widetilde{\mathcal{E}}_{n+1,q}(a)}{[n+1]_q}.$$

Now we find some properties of q-exponential function to obtain the next theorem. From Definition 1.3 and Theorem 1.2, we find that

(i) $\qquad [n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!,$

(ii) $\qquad e_q(t)e_{q^{-1}}(t) = \sum_{n=0}^\infty \frac{t^n}{[n]_q!} \sum_{n=0}^\infty \frac{t^n}{[n]_{q^{-1}}!} = \sum_{n=0}^\infty \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}\right)\frac{t^n}{[n]_q!}$

$$= \sum_{n=0}^\infty \left(\prod_{k=0}^{n-1}(1+q^k)\right)\frac{t^n}{[n]_q!},$$

(iii) $\qquad e_q(t)e_{q^{-1}}(-t) = \sum_{n=0}^\infty \frac{t^n}{[n]_q!} \sum_{n=0}^\infty \frac{(-t)^n}{[n]_{q^{-1}}!} = \sum_{n=0}^\infty \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}}\right)\frac{t^n}{[n]_q!}$

$$= \sum_{n=0}^\infty \left(\prod_{k=0}^{n-1}(1-q^k)\right)\frac{t^n}{[n]_q!},$$

(iv) $\qquad e_q(-t)e_{q^{-1}}(t) = \sum_{n=0}^\infty \frac{(-t)^n}{[n]_q!} \sum_{n=0}^\infty \frac{t^n}{[n]_{q^{-1}}!} = \sum_{n=0}^\infty \left((-1)^n\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}(-1)^k\right)\frac{t^n}{[n]_q!}$

$$= \sum_{n=0}^\infty \left((-1)^n\prod_{k=0}^{n-1}(1-q^k)\right)\frac{t^n}{[n]_q!},$$

(v) $\qquad e_q(-t)e_{q^{-1}}(-t) = \sum_{n=0}^\infty \frac{(-t)^n}{[n]_q!} \sum_{n=0}^\infty \frac{(-t)^n}{[n]_{q^{-1}}!} = \sum_{n=0}^\infty \left((-1)^n\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}\right)\frac{t^n}{[n]_q!}$

$$= \sum_{n=0}^\infty \left((-1)^n\prod_{k=0}^{n-1}(1+q^k)\right)\frac{t^n}{[n]_q!}.$$

Theorem 2.6. For $|q| < 1$, we find

(i) $\quad \displaystyle\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^k + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^{n-l} \right) \widetilde{\mathcal{E}}_{l,q} = 2(-1)^n,$

(ii) $\quad \displaystyle\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^k + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^{n-l} \right) \widetilde{\mathcal{E}}_{l,q}(x)$

$$= 2\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^{n-l} x^l.$$

Proof. (i) Loading $e_q(t)e_q(-t) + e_q(-t)e_q(-t) \neq 0$ for the generating function of q-Euler numbers of the second kind, one obtains

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} (e_q(t)e_q(-t) + e_q(-t)e_q(-t)) = 2e_q(-t),$$

and we can transform such as

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} (e_q(t)e_q(-t) + e_q(-t)e_q(-t))$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^n \right) \frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^k + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^{n-l} \right) \widetilde{\mathcal{E}}_{l,q} \right\} \frac{t^n}{[n]_q!}$$

$$= 2\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{[n]_q!}.$$

The required relation now follows at once.

(ii) We omit a proof of the q-Euler polynomials of the second kind due to its similarity to (i).

Corollary 2.7. For $q \to 1$, in Theorem 2.6, one holds

(i) $\quad \displaystyle\sum_{l=0}^{n} \binom{n}{l} \left(\sum_{k=0}^{n-l} \binom{n-l}{k} (-1)^k + \sum_{k=0}^{n-l} \binom{n-l}{k} (-1)^{n-l} \right) \widetilde{E}_l = 2(-1)^n,$

(ii) $\quad \displaystyle\sum_{l=0}^{n} \binom{n}{l} \left(\sum_{k=0}^{n-l} \binom{n-l}{k} (-1)^k + \sum_{k=0}^{n-l} \binom{n-l}{k} (-1)^{n-l} \right) \widetilde{E}_l(x) = 2\sum_{l=0}^{n} \binom{n}{k} (-1)^{n-l} x^l,$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind and $\widetilde{E}_n$ is the classical Euler numbers of the second kind(see [16]).

Theorem 2.8. Let $|q| < 1$. Then we have

(i) $\quad \displaystyle\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1} (1+q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1} (1-q^k) \right) \widetilde{\mathcal{E}}_{l,q} = 2q^{\binom{n}{2}},$

(ii) $\quad \displaystyle\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1} (1+q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1} (1-q^k) \right) \widetilde{\mathcal{E}}_{l,q}(x) = 2\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{n-l}{2}} x^l.$

Proof. (i) For $e_q(t)e_{q^{-1}}(t) + e_q(-t)e_{q^{-1}}(t) \neq 0$, we have

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(t) + e_q(-t)e_{q^{-1}}(t)) = 2e_{q^{-1}}(t).$$

To obtain the result, we can calculate the above equation as

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(t) + e_q(-t)e_{q^{-1}}(t))$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1}(1+q^k) + (-1)^n \prod_{k=0}^{n-1}(1-q^k) \right) \frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1+q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1}(1-q^k) \right) \widetilde{\mathcal{E}}_{l,q} \frac{t^n}{[n]_q!}$$

$$= 2\sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{[n]_q!}.$$

The required relation now follows on comparing the coefficients of t^n on both sides.

(ii) Using the same method as (i) we can find the required result, so we omit the proof.

Corollary 2.9. In Theorem 2.8, we can see

(i) $\qquad q^{\binom{n}{2}} = \frac{1}{2} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1+q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1}(1-q^k) \right) \widetilde{\mathcal{E}}_{l,q},$

(ii) $\qquad \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{n-l}{2}} x^l = \frac{1}{2} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1+q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1}(1-q^k) \right) \widetilde{\mathcal{E}}_{l,q}(x).$

Theorem 2.10. For $|q| < 1, k \in \mathbb{N}$, one holds

(i) $\qquad \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1-q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1}(1+q^k) \right) \widetilde{\mathcal{E}}_{l,q} = 2(-1)^n q^{\binom{n}{2}},$

(ii) $\qquad \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1-q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1}(1+q^k) \right) \widetilde{\mathcal{E}}_{l,q}(x) = 2 \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^{n-l} q^{\binom{n-l}{2}} x^l.$

Proof. (i) Let $e_q(t)e_{q^{-1}}(-t) + e_q(-t)e_{q^{-1}}(-t) \neq 0$. From the generating function of q-Euler numbers of the second kind, we can find

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(-t) + e_q(-t)e_{q^{-1}}(-t)) = 2e_{q^{-1}}(-t),$$

or, equivalently,

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(-t) + e_q(-t)e_{q^{-1}}(-t))$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1}(1-q^k) + (-1)^n \prod_{k=0}^{n-1}(1+q^k) \right) \frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1-q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1}(1+q^k) \right) \widetilde{\mathcal{E}}_{l,q} \right\} \frac{t^n}{[n]_q!} = 2\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, the proof is complete.

(ii) We omit the proof of the q-Euler polynomials because we can derive it in the same method as (i).

Corollary 2.11. In Theorem 2.10, we get

(i)
$$(-1)^n q^{\binom{n}{2}} = \frac{1}{2} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1} (1 - q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1} (1 + q^k) \right) \widetilde{\mathcal{E}}_{l,q},$$

(ii)
$$\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^{n-l} q^{\binom{n-l}{2}} x^l = \frac{1}{2} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1} (1 - q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1} (1 + q^k) \right) \widetilde{\mathcal{E}}_{l,q}(x).$$

Theorem 2.12. For $x \in \mathbb{C}$, we hold

(i)
$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1 + (-1)^k) \widetilde{\mathcal{E}}_{n-k,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases},$$

(ii)
$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1 + (-1)^k) \widetilde{\mathcal{E}}_{n-k,q}(x) = 2x^n.$$

Proof. (i) From Definition 2.1, we can represent q-Euler numbers, $\widetilde{\mathcal{E}}_{n,q}$, as

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} (1 + (-1)^n) \frac{t^n}{[n]_q!} = 2.$$

Now using the Cauchy's product, we find the relation,

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1 + (-1)^k) \widetilde{\mathcal{E}}_{n-k,q} \right) \frac{t^n}{[n]_q!} = 2,$$

and the proof is done.

(ii) We omit a proof of (ii) since we can obtain (ii) using Cauchy's product and the method of coefficient comparison for Definition 2.1 using the same method (i).

Theorem 2.13. Let $x \in \mathbb{C}$ and $|q| < 1$. Then, the following holds:

(i)
$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1 + (-1)^k) \widetilde{\mathcal{E}}_{n-k,q} = 2 \sum_{k=0}^{\left[\frac{n}{2} \right]} \begin{bmatrix} n \\ n - 2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q},$$

(ii)
$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1 + (-1)^k) \widetilde{\mathcal{E}}_{n-k,q}(x) = 2 \sum_{k=0}^{\left[\frac{n}{2} \right]} \begin{bmatrix} n \\ n - 2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q}(x),$$

where $[x]$ is the greatest integer not exceeding x.

Proof. (i) In Theorem 2.12. (i), the left-side is changed as:

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} (1 + (-1)^n) \frac{t^n}{[n]_q!} = 2 \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \frac{t^{2n}}{[2n]_q!}$$

$$= 2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} 2n - k \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q} \right) \frac{t^{2n-k}}{[2n - k]_q!} = 2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{n}{2} \right]} \begin{bmatrix} n \\ n - 2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q} \right) \frac{t^n}{[n]_q!}.$$

The required relation now follows on comparing the coefficients of t^n on both sides.

(ii) Now following the same procedure as (i), we find (ii).

Corollary 2.14. From the Theorem 2.12 and Theorem 2.13, the following relations hold:

$$
\text{(i)} \quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \begin{bmatrix} n \\ n-2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases},
$$

$$
\text{(ii)} \quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \begin{bmatrix} n \\ n-2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q}(x) = x^n,
$$

where $[x]$ is the greatest integer not exceeding x.

Theorem 2.15. For $x \in \mathbb{C}$, the following relation holds

$$
\widetilde{\mathcal{E}}_{n,q}(x) = (-1)^n \widetilde{\mathcal{E}}_{n,q}(-x).
$$

Proof. Replacing t, x with $-t, -x$, respectively, we get

$$
\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(-x) \frac{(-t)^n}{[n]_q!} = \frac{2}{e_q(-t) + e_q(t)} e_q(tx) = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!},
$$

which on comparing the coefficients immediately gives the required relation.

Corollary 2.16. Putting $x = 1$ in Theorem 2.15, we see

$$
\widetilde{\mathcal{E}}_{n,q}(1) = (-1)^n \widetilde{\mathcal{E}}_{n,q}(-1).
$$

3. Some special properties of the q-Euler polynomials of the second kind

In this section, we define the q-Euler polynomials of the second kind in two parameters. From these polynomials, we can find some relations between these polynomials and other polynomials. We can also observe a symmetric property of the q-Euler polynomials of the second kind.

Definition 3.1. Let $x, y \in \mathbb{C}$. We then define the q-Euler polynomials of the second kind in two parameters as:

$$
\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x, y) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} e_q(tx) e_q(ty).
$$

For $y = 0$, we can see that $\widetilde{\mathcal{E}}_{n,q}(x, 0) = \widetilde{\mathcal{E}}_{n,q}(x)$.

Theorem 3.2. Let x be any complex numbers. Then we hold

$$
\text{(i)} \quad \widetilde{\mathcal{E}}_{n,q}(x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}(x) y^{n-k},
$$

$$
\text{(ii)} \quad \widetilde{\mathcal{E}}_{n,q}(x, y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-l} \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q x^{l-k} y^k.
$$

Proof. From Definition 3.1, we find

$$
\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x, y) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} e_q(tx) e_q(ty)
$$

$$
= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} y^n \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}(x) y^{n-k} \right) \frac{t^n}{[n]_q!}.
$$

The required relation now follows immediately.

Theorem 3.3. For $x \in \mathbb{C}$, we hold

$$\widetilde{\mathcal{E}}_{n,q}(x,1) + \widetilde{\mathcal{E}}_{n,q}(x,-1) = 2x^n.$$

Proof. Setting $y = 1$ and -1, we can get

$$\sum_{n=0}^{\infty} \left(\widetilde{\mathcal{E}}_{n,q}(x,1) + \widetilde{\mathcal{E}}_{n,q}(x,-1) \right) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} e_q(tx)e_q(t) + \frac{2}{e_q(t) + e_q(-t)} e_q(tx)e_q(-t)$$

$$= \frac{2}{e_q(t) + e_q(-t)} e_q(tx) \left(e_q(t) + e_q(-t) \right) = 2e_q(tx) = 2 \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!},$$

and the proof is complete on comparing the coefficient of both sides.

Corollary 3.4. From Theorem 3.3, we see

$$x^n = \frac{1}{2} \left(\widetilde{\mathcal{E}}_{n,q}(x,1) + \widetilde{\mathcal{E}}_{n,q}(x,-1) \right).$$

To investigate some relations of other polynomials, we define q-Euler, q-Bernoulli, and q-tangent polynomials. These polynomials have a lot of properties, applications, and identities.

Definition 3.5. We define q-tangent polynomials, $\mathcal{T}(x)$; q-Euler polynomials, $\mathcal{E}(x)$; and q-Bernoulli polynomials, $\mathcal{B}(x)$ as

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(t) + 1} e_q(tx), \quad |t| < \pi,$$

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t) + 1} e_q(tx), \quad |t| < \frac{\pi}{2},$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{t}{e_q(t) - 1} e_q(tx), \quad |t| < 2\pi, \quad (\text{see } [7, 14\text{-}15]).$$

Theorem 3.6. For $x, y \in \mathbb{C}$, the following relation holds:

$$\widetilde{\mathcal{E}}_{n,q}(x,y) = \frac{1}{[2]_q} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\frac{\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} \right) \mathcal{E}_{l,q}(my),$$

where $\mathcal{E}_{n,q}(x)$ is q-Euler polynomials.

Proof. Transforming the q-Euler polynomials of the second kind containing two parameters, we have

$$\frac{2}{e_q(t) + e_q(-t)} e_q(tx)e_q(ty) = \left(\frac{[2]_q}{e_q(\frac{t}{m}) + 1} e_q(ty) \right) \left(\frac{e_q(\frac{t}{m}) + 1}{[2]_q} \right) \left(\frac{2}{e_q(t) + e_q(-t)} e_q(tx) \right).$$

For the relation between q-Euler polynomials of the second kind containing two parameters and q-Euler polynomials, we get

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{1}{[2]_q} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{E}_{l,q}(my) \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} \right) \frac{t^n}{[n]_q!}$$

$$+ \frac{1}{[2]_q} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{E}_{l,q}(my) \frac{\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} \right) \frac{t^n}{[n]_q!},$$

which, on comparing the coefficients, immediately gives the required relation.

Corollary 3.7. For $q \to 1$ in Theorem 3.6, we have

$$\widetilde{E}_n(x,y) = \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{\widetilde{E}_{n-l}(x)}{m^l} + \sum_{k=0}^{n-l} \binom{n-l}{k} \frac{\widetilde{E}_k(x)}{m^{n-k}} \right) E_l(my),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind, and $E_n(x)$ is the classical Euler polynomials(see [17]).

Theorem 3.8. Let $x,y \in \mathbb{C}$ and $|q| < 1$. Then we get

$$\widetilde{\mathcal{E}}_{n,q}(x,y) = \frac{1}{[2]_q} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\frac{2^l \widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{2^{n-k} \widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} \right) \mathcal{T}_{l,q}(\frac{m}{2}y),$$

where $\mathcal{T}_{n,q}(x)$ is q-tangent polynomials.

Proof. To obtain the relation between q-Euler polynomials of the second kind and q-tangent polynomials, we can make

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{[2]_q}{e_q(\frac{2t}{m})+1} e_q(ty) \right) \left(\frac{e_q(\frac{2t}{m})+1}{[2]_q} \right) \left(\frac{2}{e_q(t)+e_q(-t)} e_q(tx) \right).$$

From the above equation, we hold

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{1}{[2]_q} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{T}_{l,q}(\frac{m}{2}y) \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{2^{n-k} \widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} \right) \frac{t^n}{[n]_q!}$$

$$+ \frac{1}{[2]_q} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-l,q}(x) \frac{2^l \mathcal{T}_{l,q}(\frac{m}{2}y)}{m^l} \right) \frac{t^n}{[n]_q!},$$

which, on comparing the coefficients, the required relation at once.

Corollary 3.9. For $q \to 1$ in Theorem 3.8, we see

$$\widetilde{E}_n(x,y) = \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{2^l \widetilde{E}_{n-l}(x)}{m^l} + \sum_{k=0}^{n-l} \binom{n-l}{k} \frac{2^{n-k} \widetilde{E}_k(x)}{m^{n-k}} \right) T_l(\frac{m}{2}y),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind, and $T_l(x)$ is the classical tangent polynomials(see [6, 13-14, 17]).

Theorem 3.10. Let $|q| < 1$ and x,y be any complex numbers. Then we hold

$$[n]_q \widetilde{\mathcal{E}}_{n-1,q}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} - \frac{[n-l]_q \widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} \right) \mathcal{B}_{l,q}(my).$$

Proof. Multiplying the generating function of q-Euler polynomials of the second kind in two param-

eters by a suitable function, we have

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(\frac{t}{m})-1} e_q(ty) \right) \left(\frac{e_q(\frac{t}{m})-1}{t} \right) \left(\frac{2}{e_q(t)+e_q(-t)} e_q(tx) \right)$$

$$= \left(\sum_{n=0}^{\infty} B_{n,q}(my) \frac{t^n}{m^n [n]_q!} \right) \left(\sum_{n=0}^{\infty} \frac{t^{n-1}}{m^n [n]_q!} - \frac{1}{t} \right) \left(\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{[n]_q} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{B}_{l,q}(my) \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} \right) \frac{t^{n-1}}{[n-1]_q!}$$

$$- \sum_{n=0}^{\infty} \left(\frac{1}{[n]_q} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{[n-l]_q \widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} \mathcal{B}_{l,q}(my) \right) \frac{t^{n-1}}{[n-1]_q!}.$$

Comparing the coefficients of both sides leads to

$$\sum_{n=0}^{\infty} [n]_q \widetilde{\mathcal{E}}_{n-1,q}(x,y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} - \frac{[n-l]_q \widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} \right) \mathcal{B}_{l,q}(my) \right\} \frac{t^n}{[n]_q!},$$

which gives the required result.

Corollary 3.11. For $q \to 1$, in Theorem 3.10, we see

$$[n]_q \widetilde{E}_{n-1}(x,y) = \sum_{l=0}^{n} \binom{n}{l} \left(\sum_{k=0}^{n-l} \binom{n-l}{k} \frac{\widetilde{\mathcal{E}}_k(x)}{m^{n-k}} - \frac{(n-l)\widetilde{\mathcal{E}}_{n-l}(x)}{m^l} \right) B_l(my),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind, and $B_n(x)$ is the classical Bernoulli polynomials(see [6-9, 12, 15, 17]).

Theorem 3.12. For q, x and $y \in \mathbb{C}$, the q-Euler polynomials of the second kind have

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{b}{a} \right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,q}\left(\frac{a}{b}x \right) \widetilde{\mathcal{E}}_{k,q}\left(\frac{b}{a}y \right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{a}{b} \right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,q}\left(\frac{b}{a}x \right) \widetilde{\mathcal{E}}_{k,q}\left(\frac{a}{b}y \right).$$

Proof. Consider that

$$A = \frac{4e_q(tx)e_q(ty)}{\left(e_q(\frac{b}{a}t) + e_q(-\frac{b}{a}t) \right) \left(e_q(\frac{a}{b}t) + e_q(-\frac{a}{b}t) \right)}.$$

The form A can turn to

$$A = \frac{2e_q(tx)}{\left(e_q(\frac{b}{a}t) + e_q(-\frac{b}{a}t) \right)} \frac{2e_q(ty)}{\left(e_q(\frac{a}{b}t) + e_q(-\frac{a}{b}t) \right)}$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}\left(\frac{a}{b}x \right) \frac{\left(\frac{b}{a}t\right)^n}{[n]_q!} \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}\left(\frac{b}{a}y \right) \frac{\left(\frac{a}{b}t\right)^n}{[n]_q!} \qquad (3.1)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{b}{a} \right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,q}\left(\frac{a}{b}x \right) \widetilde{\mathcal{E}}_{k,q}\left(\frac{b}{a}y \right) \right) \frac{t^n}{[n]_q!},$$

or, equivalently,

$$
\begin{aligned}
A &= \frac{2e_q(tx)}{\left(e_q(\frac{a}{b}t) + e_q(-\frac{a}{b}t)\right)} \frac{2e_q(ty)}{\left(e_q(\frac{b}{a}t) + e_q(-\frac{b}{a}t)\right)} \\
&= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}\left(\frac{b}{a}x\right) \frac{\left(\frac{a}{b}t\right)^n}{[n]_q!} \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}\left(\frac{a}{b}y\right) \frac{\left(\frac{b}{a}t\right)^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{a}{b}\right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,q}\left(\frac{b}{a}x\right) \widetilde{\mathcal{E}}_{k,q}\left(\frac{a}{b}y\right) \right) \frac{t^n}{[n]_q!},
\end{aligned}
\tag{3.2}
$$

and the theorem is proved in (3.1) and (3.2).

Corollary 3.13. If $q \to 1$ in Theorem 3.12, then we see

$$
\sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{a}\right)^{n-2k} \widetilde{E}_{n-k}\left(\frac{a}{b}x\right) \widetilde{E}_k\left(\frac{b}{a}y\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{b}\right)^{n-2k} \widetilde{E}_{n-k}\left(\frac{b}{a}x\right) \widetilde{E}_k\left(\frac{a}{b}y\right),
$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind(see [6, 17]).

Theorem 3.14. For $k \in \mathbb{N}$, we get

$$
D_q^{(k)} \widetilde{\mathcal{E}}_{n,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q [k]_q! \widetilde{\mathcal{E}}_{n-k,q}(x).
$$

Proof. To make this result we have to remember Theorem 2.2,

$$
\widetilde{\mathcal{E}}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q} x^{n-k},
$$

and Theorem 2.3,

$$
D_q^{(1)} \widetilde{\mathcal{E}}_{n,q}(x) = [n]_q \widetilde{\mathcal{E}}_{n-1,q}(x).
$$

Repeating q-derivative for Theorem 2.3, we can see

$$
\begin{aligned}
D_q^{(2)} \widetilde{\mathcal{E}}_{n,q}(x) &= [n]_q[n-1]_q \widetilde{\mathcal{E}}_{n-2,q}(x) \\
D_q^{(3)} \widetilde{\mathcal{E}}_{n,q}(x) &= [n]_q[n-1]_q[n-2]_q \widetilde{\mathcal{E}}_{n-3,q}(x) \\
D_q^{(4)} \widetilde{\mathcal{E}}_{n,q}(x) &= [n]_q[n-1]_q[n-2]_q[n-3]_q \widetilde{\mathcal{E}}_{n-4,q}(x) \\
&\cdots .
\end{aligned}
$$

For $k \in \mathbb{N}$, we find q-derivative of k-times such as

$$
\begin{aligned}
D_q^{(k)} \widetilde{\mathcal{E}}_{n,q}(x) &= [n]_q[n-1]_q[n-2]_q[n-3]_q \cdots [n-k+1]_q \widetilde{\mathcal{E}}_{n-k,q}(x) \\
&= \begin{bmatrix} n \\ k \end{bmatrix}_q [k]_q! \widetilde{\mathcal{E}}_{n-k,q}(x),
\end{aligned}
$$

and it is proved.

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Regularized moving least squares approximation with Laplace-Beltrami operator on the sphere [*]

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Abstract

This paper proposes a pointwise approximation scheme on the unit sphere $\mathbb{S}^2$, which aims to handle spherical scattered data with high level noise by using a regularized moving least squares (RMLS) approximation with Laplace-Beltrami operator. The pointwise errors of approximation by the RMLS are estimated, and some numerical examples are designed to examine the obtained theoretical results. Also, the given numerical examples illustrate that the different choices for the order of Laplace-Beltrami operator and regularized parameter can provide different pointwise approximation results.

MSC(2000): 65D32, 65H10, 41A17

Keywords Sphere; Moving least squares; Pointwise approximation; Laplace-Beltrami operator

1 Introduction

In recent years, scattered data approximation on the unit sphere $\mathbb{S}^2$ has been widely applied to astrophysics, meteorology, geodesy, geophysics, and other areas (see [9, 11, 36]). In the approximation, spherical polynomials and spherical radial basis functions are usually taken as approximation tools. We refer the reader to [12, 13, 14, 16, 17, 25, 24, 26, 36, 30, 31, 32].

Moving least squares (MLS) scheme for scattered data approximation in Euclidean space $\mathbb{R}^n$ was proposed in [34] and [36] several years ago, which is actually a scheme to approximate target functions by using polynomials. Its simplest form coincides with Shepards interpolation method [29]. And, in terms of Backus-Gilbert method and constructive way of computing MLS approximation, it is also called as Backus-Gilbert optimal [4, 5, 6, 8]. Afterwards MLS was extended by McLain [21, 22], Franke [10], and Lancaster [15]. Meanwhile, MLS approximation has been frequently applied to potential energy surfaces [20], surface reconstruction [15], and partial differential equations [7]. In addition, several errors of approximation of MLS were estimated by different ways in [2, 3, 18, 34, 36].

In the MLS method, the highlight is process of MLS approximation involving the local polynomial reproduction property and the key ingredient in depriving error estimates. Based on the idea, Wendland [35] studied MLS approximation on the sphere to estimate the function value $f(x)$ by solving a local weighted least squares problem for every point x on the sphere which seems to be better than the method of spherical radial basis functions (SBFs) interpolation, because it does not require solving a large linear system. Li [19] developed a theoretical analysis of the generalization performances of regularized least square regression algorithm with spherical polynomial kernels. An et al. [1] considered regularized least squares scheme by using spherical designs, which can be used to handle the data set with high level noise when the regularized operator is chosen as spherical Laplace-Beltrami operator.

Based on the denoising effect of the Laplace-Beltrami operator and the superiority of MLS approximation, this paper proposes an approximation scheme of regularized moving least squares

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(RMLS) with the Laplace-Beltrami operator. A pointwise approximation issue on the unit sphere $\mathbb{S}^2$, which aims to handle data set with high level noise, is considered, and the errors of approximation by the proposed RMLS are estimated. The numerical experiments are designed to further show the effectiveness of the proposed new method.

This paper is organized as follows. In Section 2, the necessary backgrounds about spherical harmonics, sphere function spaces, and Laplace-Beltrami operator are reviewed. The MLS approximation on the sphere is stated in Section 3. In Section 4, we propose the RMLS approximation scheme. Section 5 devotes to give the error estimation for RMLS approximation. Numerical experiments are given in Section 6 to demonstrate the effectiveness of RMLS approximation scheme for data with high level noise.

2   Preliminaries

2.1   Spherical harmonics, sphere function spaces, and sphere point sets

In this section, we introduce some notations about spherical harmonics, sphere function spaces, and sphere point sets, which can be found in [23] and [33].

Let $\mathbb{S}^2$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, i.e.,

$$\mathbb{S}^2 := \left\{ x := (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\|_2^2 := x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

For integer $l \geq 0$, the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial with degree l is called a spherical harmonic with degree l. The class of all spherical harmonics with degree l is denoted by $\mathcal{H}_l$, and it is well-known that spherical harmonics of different degree are orthogonal with respect to the inner product $\langle f, g \rangle_{L_2} := \int_{\mathbb{S}^2} f(x)g(x)d\omega(x)$, where $d\omega$ denotes surface measure on $\mathbb{S}^2$. Hence, if we choose an orthogonal basis $\{Y_{l,k} : k = 1, \ldots, 2l+1\}$ for each $\mathcal{H}_l$, then the set $\{Y_{l,k} : l = 0, 1, \ldots, \ k = 1, \ldots, 2l+1\}$ is an orthogonal basis for $L_2(\mathbb{S}^2)$. The class of all spherical harmonics with total degree $l \leq L$ is denoted by $\mathcal{P}_L$. Of course, $\mathcal{P}_L = \bigoplus_{l=0}^{L} \mathcal{H}_l$, and the dimension of $\mathcal{H}_l$ is $2l+1$ and that of $\mathcal{P}_L$ is $(L+1)^2$. The well-known addition formula is given by (see [23])

$$\sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y), \tag{2.1}$$

where P_l is the Legendre polynomial with degree l and dimension 3, which is normalized such that $P_l(1) = 1$.

For every $p \in \mathcal{P}_L$, we have

$$p(x) = \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \alpha_{l,k} Y_{l,k}(x), \tag{2.2}$$

where $\alpha_{l,k} = \int_{\mathbb{S}^2} p(y) Y_{l,k}(y) d\omega(y)$.

We denote by $C(\mathbb{S}^2)$ the space of continuous functions on $\mathbb{S}^2$ endowed with the uniform (supremum) norm $\|f\|_{C(\mathbb{S}^2)} := \sup_{x \in \mathbb{S}^2} |f(x)|$. The geodesic distance between two points on the unit sphere $\mathbb{S}^2$ is defined by $d(x, y) := \arccos(\langle x, y \rangle) (x, y \in \mathbb{S}^2)$, where $\langle x, y \rangle$ denotes the Euclidean inner product.

Let $x_1, x_2, \ldots, x_N \in \mathbb{S}^2$, which are pairwise distinct. Then $X := \{x_1, x_2, \ldots, x_N\}$ is called as a centers set. The mesh norm of X is denoted by

$$h_{X,\mathbb{S}^2} := \sup_{x \in \mathbb{S}^2} \min_{x_i \in X} d(x, x_i),$$

and the separation radius is defined to be

$$q_{X,\mathbb{S}^2} := \frac{1}{2} \min_{i \neq j} d(x_i, x_j)$$

that is half of the smallest geodesic distance between any two distinct points in X. It is easy to see that $h_{X,\mathbb{S}^2} \geq q_{X,\mathbb{S}^2}$, where equality can hold only for a uniform distribution of points on the circle $\mathbb{S}^1$.

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For a given point set X, if there exists a constant $c_1 > 1$, such that

$$q_{X,\mathbb{S}^2} \leq h_{X,\mathbb{S}^2} \leq c_1 q_{X,\mathbb{S}^2}, \qquad (2.3)$$

then X is called quasi-uniform.

In addition, the set X is said to be $\mathcal{P}_L$-unisolvent (see [32]), if

$$p \in \mathcal{P}_L, \quad p(x_i) = 0 \ \text{ for } \ i = 1, 2, \ldots, N \Rightarrow p = 0.$$

2.2 Laplace-Beltrami operator

In this subsection, we introduce Laplace-Beltrami operator on $\mathbb{S}^2$ (see [23, 33]). The Laplace-Beltrami operator is defined by

$$\Delta f := \sum_{i=1}^{3} \frac{\partial^2 g(x)}{\partial x_i^2}\bigg|_{\|x\|_2=1}, \quad g(x) := f\left(\frac{x}{\|x\|_2}\right).$$

In fact, the Laplace-Beltrami operator is the angular part of the Laplace operator in three dimensions

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Giving point $x := (x_1, x_2, x_3)$ on $\mathbb{S}^2$, then the related spherical polar coordinate system is (θ, φ), $0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi$, in terms of polar coordinate transformation

$$x_1 = \sin\theta\cos\varphi, \quad x_2 = \sin\theta\sin\varphi, \quad x_3 = \cos\theta,$$

the Laplace-Beltrami operator acting as a differential operator can be written by

$$\Delta := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

The literature has pointed an intrinsic characterization of spherical harmonics, which is every element of $\mathcal{H}_l$ is an eigenfunction corresponding to the eigenvalue $-l(l+1)$ of the Laplace-Beltrami operator Δ, namely that

$$\Delta Y_{l,k}(x) = -l(l+1)Y_{l,k}(x).$$

In fact, Δ is a semi-positive operator, and for any $s > 0$ we can define $(-\Delta)^s$ as

$$(-\Delta)^s Y_{l,k} = (l(l+1))^s Y_{l,k}(x) = \beta_l Y_{l,k}(x).$$

So, for $p(x) \in \mathcal{P}_L$, $(-\Delta)^s p(x)$ can be represented by

$$(-\Delta)^s p(x) = \sum_{l=0}^{L}\beta_l\sum_{k=1}^{2l+1}Y_{l,k}(x)\langle Y_{l,k}, p\rangle = \sum_{l=0}^{L}\beta_l\int_{\mathbb{S}^2}\frac{2l+1}{4\pi}P_l(x,y)p(y)d\omega(y),$$

where $\beta_\mu = (\mu(\mu+1))^s$, $\mu = 0, 1, \ldots, L$, and P_l is the Legendre polynomial with degree l.

3 Moving least squares

Moving least squares (MLS) approximation has been frequently applied to potential energy surfaces [20], surface reconstruction [15], and partial differential equations [7]. In order to propose regularized moving least squares (RMLS) in the next section, we should first review some details about MLS approximation on the sphere [35].

The issue of MLS approximation on the sphere has been given some detailed discussions by Wendland in [34, 35, 36]. Suppose an unknown continuous function $f \in C(\mathbb{S}^2)$ and $x \in \mathbb{S}^2$, we can construct an approximation of $f(x)$ from values $\{f(x_i)\}_{i=1}^N$ of f on a given point set

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$X = \{x_1, \ldots, x_N\} \subseteq \mathbb{S}^2$. Then the approximate value $p^*(x)$ of $f(x)$ can be obtained by the solution of following minimization problem

$$\min\left\{\sum_{i=1}^{N}(f(x_i) - p(x_i))^2 w(x, x_i) : p \in \mathcal{P}\right\}, \tag{3.4}$$

where $\mathcal{P} \subseteq C(\mathbb{S}^2)$ is a finite dimensional subspace, usually spanned by spherical harmonics, and $w : \mathbb{S}^2 \times \mathbb{S}^2 \to [0, \infty)$ is a continuous function. Since we consider a local process, we choose $w(x, y)$ as a radial and compactly supported function, even if it is not really necessary. So Wendland [34, 35, 36] chose continuous function $\phi : [0, \infty) \to [0, \infty)$ with

- $\phi(r) > 0, 0 \le r < 1$,
- $\phi(r) = 0, r \ge 1$,

and define

$$\theta_\delta(x, y) := \frac{1}{\phi\left(\frac{d(x,y)}{\delta}\right)}, \quad x, y \in \mathbb{S}^2, \tag{3.5}$$

where $\delta > 0$ is a scale. Then above weight function $w(x, x_i)$ has the following form

$$w(x, x_i) = \frac{1}{\theta_\delta(x, x_i)} = \phi\left(\frac{d(x,y)}{\delta}\right).$$

For $X = \{x_1, x_2, \ldots, x_N\}$, we further define the index set $I(x)$ as

$$I(x) := I(x, \delta, X) = \{i \in \{1, 2, \ldots, N\} : d(x, x_i) < \delta\}, \tag{3.6}$$

which contains the subscripts of points within the spherical cap of radius δ centered at x. And we choose $\mathcal{P} = \mathcal{P}_L$. Then the MLS approximation (3.4) takes the form (see [18, 34, 35, 36])

$$s_{f,X}(x) = \sum_{i \in I(x)} a_i^*(x) f(x_i),$$

where the coefficients $a_i^*(x)$ are determined by minimizing

$$\frac{1}{2}\sum_{i \in I(x)} a_i^2(x)\theta_\delta(x, x_i) \tag{3.7}$$

under the constraints

$$\sum_{i \in I(x)} a_i(x)p(x_i) = p(x), \quad p \in \mathcal{P}_L. \tag{3.8}$$

If X satisfies certain conditions, then we have the following theorem [35].

Theorem 3.1 *Assume that $Z = \{x_i \in X : i \in I(x, \delta, X)\}$ is $\mathcal{P}_L$-unisolvent. Then the minimization problem (3.7) with constraint (3.8) has an unique solution $a_i^*(x)$:*

$$a_i^*(x) = \phi\left(\frac{d(x, x_i)}{\delta}\right)\sum_{\mu=0}^{L}\sum_{\nu=1}^{2\mu+1}\lambda_{\mu,\nu}Y_{\mu,\nu}(x_i),$$

where $i \in I(x), x_i \in Z$, and the Lagrange multipliers $\lambda_{l,k}$ have unique solution by solving the following system of equations:

$$\sum_{\mu=0}^{L}\sum_{\nu=1}^{2\mu+1}\lambda_{\mu,\nu}\sum_{i \in I(x)}\phi\left(\frac{d(x, x_i)}{\delta}\right)Y_{\mu,\nu}(x_i)Y_{l,k}(x_i) = Y_{l,k}(x)$$

with $0 \le l \le L, 1 \le k \le 2\mu + 1$.

Since $Z = \{x_{i_1}, x_{i_2}, \ldots, x_{i_M}\} = \{x_i, i \in I(x)\} \subseteq X$ involves the choice of scale δ, so it is also an interesting research direction. From [36] we know that if x lies in a region with a high data density, then the δ should be chosen small. However, we should choose a bigger δ, since our method is local. Therefore, we often choose

$$\delta = \delta_X = C_1 h_X, \tag{3.9}$$

where C_1 is a constant.

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4   Regularized moving least squares with Laplace-Beltrami operator

In this section, we propose a category of local polynomial approximation on the unit sphere $\mathbb{S}^2$ in terms of an improvement of MLS, and give the model of the RMLS.

For an unknown continuous function $f \in C(\mathbb{S}^2)$, $X = \{x_1, x_2, \cdots, x_N\} \subseteq \mathbb{S}^2$, and $x \in \mathbb{S}^2$, we can get an approximate value $p(x)$ of $f(x)$ from values $\{f(x_i)\}_{i=1}^N$ by the solution p of following minimization problem

$$\min \left\{ \sum_{i=1}^N \left((f(x_i) - p(x_i))^2 + \lambda \left((-\Delta)^s p(x_i) \right)^2 \right) \phi \left(\frac{d(x, x_i)}{\delta} \right) : p \in \mathcal{P}_L \right\}, \qquad (4.10)$$

where $(-\Delta)^s$ and ϕ is defined as above, $\lambda > 0$ is a regularization parameter.

Similar to [18, 34, 35, 36], we want to use polynomial local reconstruction to estimate approximation order. So, the new approximation form can be constructed and it is the same as the solution of (4.10). We construct the new approximation form:

$$s_{f,X}(x) = \sum_{i \in I(x)} a_i^*(x) f(x_i), \qquad (4.11)$$

where the coefficients are determined by minimizing

$$\frac{1}{2} \sum_{i \in I(x)} a_i^2(x) \theta_\delta(x, x_i) \qquad (4.12)$$

under the constraints

$$\sum_{i \in I(x)} a_i(x) p(x_i) = q(x), \ \ p \in \mathcal{P}_L, \qquad (4.13)$$

where

$$q(x) = \sum_{\mu=0}^L \sum_{\nu=1}^{2\mu+1} (1 + \lambda \beta_\mu^2)^{-1} \hat{p}_{\mu,\nu} Y_{\mu,\nu}(x),$$

$\hat{p}_{\mu,\nu}$ is the Fourier coefficient of p, and $\beta_\mu = (\mu(\mu+1))^s$. The following (2) of Theorem 4.1 shows that the constructed approximation form (4.11) and constrained optimization problems (4.12)-(4.13) are valid.

In the following, we focus on how to solve the new constrained optimization problem, where $Z = \{x_{i_1}, x_{i_2}, \ldots, x_{i_M}\} = \{x_i, i \in I(x)\} \subseteq X$. We need the following notations:

$$f = (f(x_1), f(x_2), \ldots, f(x_M))^T;$$

$$a = (a_1^*(x), a_2^*(x), \ldots, a_M^*(x))^T;$$

$$\alpha = (\alpha_{0,1}, \ldots, \alpha_{L,2L+1})^T;$$

$$\varphi = (Y_{0,1}(x), \ldots, Y_{L,2L+1}(x))^T;$$

$$W = \operatorname{diag} \left\{ \phi \left(\frac{d(x, x_1)}{\delta} \right), \phi \left(\frac{d(x, x_2)}{\delta} \right), \ldots, \phi \left(\frac{d(x, x_M)}{\delta} \right) \right\};$$

$$B = \operatorname{diag}\{\beta_0, \beta_1, \beta_1, \beta_1, \ldots, \underbrace{\beta_\mu, \ldots, \beta_\mu}_{2\mu+1}, \ldots, \beta_{2L+1}, \ldots, \beta_{2L+1}\};$$

$$Y = \begin{pmatrix} Y_{0,1}(x_1) & Y_{1,1}(x_1) & Y_{1,2}(x_1) & \cdots & Y_{L,2L+1}(x_1) \\ Y_{0,1}(x_2) & Y_{1,1}(x_2) & Y_{1,2}(x_2) & \cdots & Y_{L,2L+1}(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{0,1}(x_M) & Y_{1,1}(x_M) & Y_{1,2}(x_M) & \cdots & Y_{L,2L+1}(x_M) \end{pmatrix}.$$

The following Theorem 4.1 will give the concrete form of the solution of RMLS approximation, and proves that the solution of (4.10) is equivalent to the solution of the minimization problem (4.12) with constraint (4.13). Namely, the constructions (4.11)-(4.13) are valid.

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Theorem 4.1 *The following statements hold.*

(1). For a given point set $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, if $Z = \{x_i \in X : i \in I(x, \delta, X)\}$ is $\mathcal{P}_L$-unisolvent, then the minimization problem (4.12) with constraint (4.13) has unique solution $a_i^*(x)$:

$$a_i^*(x) = \phi\left(\frac{d(x, x_i)}{\delta}\right) \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} Y_{\mu,\nu}(x_i), \; i \in I(x), \tag{4.14}$$

where $z_{l,k}$ can be obtained by solving the following system of equations:

$$\sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) Y_{\mu,\nu}(x_i) Y_{l,k}(x_i) = (1 + \lambda\beta_l^2)^{-1} Y_{l,k}(x) \tag{4.15}$$

with $0 \leq l \leq L, 1 \leq k \leq 2l+1, \lambda > 0$;

(2). The solution of (4.10) is equivalent to the solution of the minimization problem (4.12) with constraint (4.13).

Proof. We first prove (1). Similar to [34], [35] and [36], we introduce Lagrange multiplies $z = (\hat{z}_{0,1}, \ldots, \hat{z}_{L,2L+1})$ to solve the optimal problem (4.12) with constraint (4.13). Let

$$J = \frac{1}{2} \sum_{i \in I(x)} a_i^2(x) \theta_\delta(x, x_i) - \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} \left(\sum_{i \in I(x)} a_i(x) Y_{\mu,\nu}(x_i) - (1 + \lambda\beta_\mu^2)^{-1} Y_{\mu,\nu}(x) \right),$$

where $z_{\mu,\nu} = \hat{z}_{\mu,\nu} \hat{p}_{\mu,\nu}$. We solve partial derivatives about $a_i(x)$ and $z_{l,k}$ for J, respectively,

$$\frac{\partial J}{\partial a_i(x)} = a_i(x) \theta_\delta(x, x_i) - \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} Y_{\mu,\nu}(x_i) = 0, \; i \in I(x),$$

$$\frac{\partial J}{\partial z_{l,k}} = - \sum_{i \in I(x)} a_i(x) Y_{l,k}(x_i) + (1 + \lambda\beta_l^2)^{-1} Y_{l,k}(x) = 0, \; 0 \leq l \leq L, 1 \leq k \leq 2l+1,$$

then, solving the above equations, we can get (4.14) and (4.15).

In order to prove equivalent conditions, the solution (4.14) of the optimal problem (4.12) under constraint (4.13) can be written as matrix form

$$a = WY(Y^T WY + \lambda B^T Y^T WYB)^{-1} \varphi.$$

Next, we prove the uniqueness of the solution. In fact, we only need to prove that $Y^T WY + \lambda B^T Y^T WYB$ is a positive definite matrix. For any vector

$$r = (r_{0,1}, \ldots, r_{L,2L+1})^T \in \mathbb{R}^{(L+1)^2},$$

and for $i \in I(x), w(x, x_i) > 0$,

$$r^T(Y^T WY + \lambda B^T Y^T WYB)r = (Yr)^T W(Yr) + r^T \lambda B^T Y^T WYBr$$

$$= \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} Y_{l,k}\right)^2 + \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} r_{l,k} \beta_l Y_{l,k}\right)^2$$

$$\geq 0.$$

Since the entries of diagonal line of matrix B are not all 0, and Z is $\mathcal{P}_L$-unisolvent, we see that

$$\sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} Y_{l,k}\right)^2 + \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} r_{l,k} \beta_l Y_{l,k}\right)^2 = 0,$$

implies $r = 0$. So $Y^T WY + \lambda B^T Y^T WYB$ is a positive definite matrix.

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We now prove property (2). Let $\{Y_{0,1}, \ldots, Y_{L,2L+1}\}$ be a set of the spherical harmonics in $\mathcal{P}_L$. Then the minimizer of (4.10) can be written as

$$p^*(x) = \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} \alpha_{\mu,\nu} Y_{\mu,\nu}(x).$$

Thus

$$\alpha^T = f^T W Y (Y^T W Y + \lambda B^T Y^T W Y B)^{-1},$$

hence

$$
\begin{aligned}
p^*(x) &= \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} \alpha_{\mu,\nu} Y_{\mu,\nu}(x) = \alpha^T \varphi \\
&= f^T W Y (Y^T W Y + \lambda B^T Y^T W Y B)^{-1} \varphi = f^T a \\
&= s_{f,X}(x).
\end{aligned}
$$

The proof of Theorem 4.1 is complete. $\square$

The following Theorem 4.2 devotes that $p(x)$ of the solution of the minimization problem (4.12) with constraint (4.13) uniformly converges to "s-smoothed" solution f_s:

$$f_s(x) := \sum_{l=0}^{\infty} \frac{1}{1 + \lambda(l(l+1))^{2s}} \sum_{k=1}^{2l+1} \hat{f}_{l,k} Y_{l,k}(x) = \sum_{l=0}^{\infty} \frac{1}{1 + \lambda(l(l+1))^{2s}} \frac{1}{4\pi} \int_{\mathbb{S}^2} P_l(x \cdot y) f(y) d\omega(y),$$

where the last equation uses the addition theorem (2.1).

Theorem 4.2 *Assume that the order of Laplace-Beltrami operator $s > 1/2$, $p(x)$ is the solution of the minimization problem (4.12) with constraints (4.13), and L is the order of $p(x)$, then we have $\lim_{L \to \infty} \|p - f_s\|_{C(\mathbb{S}^2)} = 0$.*

This theorem has been proved in [1].

5   Error estimates

In this section, we will give an error estimate for RMLS approximation, which ensures the fact that RMLS approximation scheme is reasonable (see Theorem 6 below). But before starting the error analysis, we need to collect a few auxiliary results. The following Lemma 4 indicates the local polynomial reproduction on the sphere, which is quoted from [35]. We also refer the reader to [18] and [34] for a general form of the local polynomial reproduction property, and it plays an important role in the error estimates for RMLS approximation.

Lemma 5.1 *There exist constants $h_0, C_2, C_3 > 0$ such that for every point set $X = \{x_1, x_2, \ldots, x_N\} \subseteq \mathbb{S}^2$ with $h_{X,\mathbb{S}^2} \leq h_0$ and every $x \in \mathbb{S}^2$, there exist $a_1^X(x), a_2^X(x), \ldots, a_N^X(x)$ satisfying that*
(1) $\sum_{i=1}^{N} a_i^X(x) p(x_i) = q(x), \text{for any } p \in \mathcal{P}_L$;
(2) $a_i^X(x) = 0, \text{if } d(x, x_i) > C_2 h_{X,\mathbb{S}^2}$;
(3) $\sum_{i=1}^{N} |a_i^X(x)| \leq C_3$,
where

$$q(x) = \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} (1 + \lambda \beta_\mu^2)^{-1} \hat{p}_{\mu,\nu} Y_{\mu,\nu}(x).$$

The following Lemma 5.2 is quoted from [35], which shows that $|I(x)|$ is uniformly bounded in terms of packing argument from [27], and it plays an important role in the error estimates for RMLS and MLS approximation.

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Lemma 5.2 *Assume that $X = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{S}^2$ is quasi-uniform with $h_{X,\mathbb{S}^2} \leq h_0$, $I(x) := \{i \in \{1, 2, \ldots, N\} : d(x, x_i) < \delta\}$, and $\delta = C_1 h_{X,\mathbb{S}^2}$. Then*

$$|I(x)| \leq \frac{q_X + \delta}{q_X} \leq (1 + c_1 C_1),$$

where the c_1 and C_1 are constants which associate with (2.3) and (3.9), respectively.

From Lemma 5.1 and Lemma 5.2, we use the similar techniques of [35], and obtain the following Theorem 5.1, which is an error estimate for RMLS approximation.

Theorem 5.1 *Let C_1, C_3, δ be given in (3.9), Lemma 5.1, and Lemma 5.2. Suppose that $X = \{x_1, x_2, \ldots, x_N\} \subseteq \mathbb{S}^2$ is quasi-uniform, and $s_{X,f}$ is the RMLS approximation of $f \in C(\mathbb{S}^2)$ by minimization (4.12) under the constraint (4.13). Then there exist constants h_0 and C which are independent of f and X, such that for every X with $h_{X,\mathbb{S}^2} \leq h_0$ and every $x \in \mathbb{S}^2$, the error between f and $s_{X,f}$ can be bounded by*

$$|f(x) - s_{X,f}(x)| \leq C c_f C_1^{l+1} h_{X,\mathbb{S}^2}^{l+1}.$$

Proof. Let $q \in \mathcal{P}_L$ and $B(x, \delta) = \{y \in \mathbb{S}^2; d(x, y) \leq \delta\}$. We adopt the standard arguments to estimate the error of RMLS approximation:

$$
\begin{aligned}
|f(x) - s_{X,f}(x)| &= |f(x) - q(x) + q(x) - s_{X,f}(x)| \leq |f(x) - q(x)| + |q(x) - s_{X,f}(x)| \\
&\leq \|f(x) - q(x)\|_{\infty, B(x,\delta)} + \sum_{i \in I(x)} |a_i^*(x)| \|f(x) - p(x)\|_{\infty, B(x,\delta)}
\end{aligned}
$$

where the relationship between $p(x)$ and $q(x)$ is $q(x) = \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} (1 + \lambda \beta_\mu^2)^{-1} \hat{p}_{\mu,\nu} Y_{\mu,\nu}(x)$, $\hat{p}_{\mu,\nu}$ is the Fourier coefficient of p, and $\beta_\mu = (\mu(\mu+1))^s$. So we can write

$$\max\left\{\|f(x) - q(x)\|_{\infty, B(x,\delta)}, \|f(x) - p(x)\|_{\infty, B(x,\delta)}\right\} := \|f(x) - G(x)\|_{\infty, B(x,\delta)},$$

then

$$|f(x) - s_{X,f}(x)| \leq (1 + \sum_{i \in I(x)} |a_i^*(x)|) \|f(x) - G(x)\|_{\infty, B(x,\delta)}.$$

For $\sum_{i \in I(x)} |a_i^*(x)|$, using Cauchy inequality we have

$$\sum_{i \in I(x)} |a_i^*(x)| \leq \left(\sum_{i \in I(x)} |a_i^*(x)|^2 \theta_\delta(x, x_i)\right)^{1/2} \left(\sum_{i \in I(x)} \phi(\frac{d(x, x_i)}{\delta})\right)^{1/2}. \tag{5.16}$$

Now we prove that the first term of the right of (5.16) is bounded. According to $h_{X,\mathbb{S}^2} \leq h_0$, we can get $a_i(x)$ that reproduces spherical harmonics and vanishes if $d(x, x_i) > \frac{\delta}{2}$. We set $\widetilde{I(x)} = \{i : d(x, x_i) \leq \frac{\delta}{2}\}$, then, by the minimization condition it is not difficult for us to obtain that

$$
\begin{aligned}
\sum_{i \in I(x)} |a_i^*(x)|^2 \theta_\delta(x, x_i) &\leq \sum_{i \in \widetilde{I(x)}} |a_i(x)|^2 \theta_\delta(x, x_i) \leq \frac{1}{\min_{i \in \widetilde{I(x)}} \phi(\frac{d(x, x_i)}{\delta})} \sum_{i \in \widetilde{I(x)}} |a_i(x)|^2 \\
&\leq \left(\sum_{i=1}^{N} |a_i(x)|\right)^2 \frac{1}{\min_{y \in Z} \phi(\frac{d(x, y)}{\delta})} \leq C_3^2 \frac{1}{\min_{y \in Z} \phi(\frac{d(x, y)}{\delta})},
\end{aligned}
$$

and

$$\sum_{i \in \widetilde{I(x)}} \phi(\frac{d(x, x_i)}{\delta}) \leq |I(x)| \|\phi\|_\infty.$$

From Lemma 5.2, we see that $|I(x)|$ is uniformly bounded, which implies that (5.16) is bounded. Therefore, there exists a constant C, such that $\left(1 + \sum_{i \in I(x)} |a_i^*(x)|\right) < C$.

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Next, we prove that $\|f(x) - G(x)\|_{\infty,Z}$ is bounded. According to [35], without loss of generality, we suppose that $x = (0,0,1)^T$. Then

$$B(x,\delta) = \{y \in \mathbb{S}^2 : d(x,y) < \delta\} = \{y \in \mathbb{S}^2 : y_3 > \cos\delta\}.$$

We define the bijective map $T : U \to B(x,\delta)$ by $\tilde{y} \to (\tilde{y}, \sqrt{1 - \|\tilde{y}\|_2^2})^T$, where $U = \{\tilde{y} \in R^2 : \|\tilde{y}\|_2^2 < 1 - \cos^2\delta\}$. Obviously, the inverse of T is $T^{-1}(y) = \tilde{y} = (y_1, y_2)^T$. Then, the Taylor expansion of g around $\tilde{x} = 0$ is

$$g(\tilde{y}) = \sum_{|\alpha| \le l} \frac{g^{(\alpha)}(0)}{\alpha!} \tilde{y}^\alpha + \sum_{|\alpha| = l+1} \frac{g^{(\alpha)}(\xi)}{\alpha!} \tilde{y}^\alpha.$$

So

$$f(y) = g \circ T^{-1}(y) = \sum_{|\alpha| \le l} c_\alpha y^\alpha + \sum_{|\alpha| = l+1} \frac{g^{(\alpha)}(\xi)}{\alpha!} \tilde{y}^\alpha,$$

and

$$G(y) = \sum_{|\alpha| \le l} c_\alpha y^\alpha.$$

Hence

$$\begin{aligned}
|f(y) - G(y)| &\le c_f \|\tilde{y}\|_2^{l+1} = c_f(1 - y_3^2)^{(l+1)/2} \le c_f(1 - \cos^2\delta)^{(l+1)/2} = c_f(\sin\delta)^{l+1} \\
&\le c_f \delta^{l+1} = c_f C_1 h_{X,\mathbb{S}^2}^{l+1}.
\end{aligned}$$

Therefore,

$$|f(x) - s_{X,f}(x)| \le C\|f - G\|_{\infty, B(x,\delta)} \le Cc_f C_1^{l+1} h_{X,\mathbb{S}^2}^{l+1}.$$

The proof of Theorem 5.1 is complete.   □

6   Numerical experiments

In order to further validate our theoretical results derived in the previous sections, this section presents some numerical experiments handling data set with high level noise. In our experiments, we choose two test functions, where the Franke function $f(x,y,z)$ is chosen as the first test function which has been frequently used in the other literature (for example, [28, 35]),

$$\begin{aligned}
f_1(x,y,z) \; = \; & \frac{3}{4} \exp\left(-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4} - \frac{(9z-2)^2}{4}\right) \\
& + \frac{3}{4} \exp\left(-\frac{(9x+1)^2}{49} - \frac{(9y+1)^2}{10} - \frac{(9z+1)^2}{10}\right) \\
& + \frac{1}{2} \exp\left(-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4} - \frac{(9z-5)^2}{4}\right) \\
& - \frac{1}{5} \exp\left(-(9x-4)^2 - (9y-7)^2 - (9z-5)^2\right), \quad (x,y,z) \in \mathbb{S}^2.
\end{aligned}$$

This function is shown in the Figure 2 (a), and it is $C^\infty(\mathbb{S}^2)$. The second test function is spherical cap function which is a sum of the Franke function f_1 and an other function f_{cap} (see [38]), which is defined by $f_2 := f_1 + f_{cap}$, where

$$f_{cap} := \begin{cases} \rho\cos\left(\dfrac{\pi\arccos(\langle x_c, x\rangle)}{2r}\right), & x \in C(x_c, r); \\ 0, & \text{otherwise,} \end{cases}$$

and ρ is a positive number. We set $x_c = (-\frac{1}{2}, -\frac{1}{2}, \sqrt{\frac{1}{2}})$, $\rho = 2$, and $r = \frac{1}{2}$ in the experiment. This function is shown in the Figure 3 (a).

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In the RMLS approximation, the weight function plays an important role. We choose a famous radial basis function $\phi(r)$ as weight function in our numerical experiments, that is

$$\phi(r) = (1 - r)_+^4 (4r + 1),$$

which is called Wendland function (see [37]). The uniform error of the approximation is estimated by

$$\|f - p\|_{C(\mathbb{S}^2)} \approx \max_{x_i \in X} |f(x_i) - p(x_i)|.$$

In our numerical experiments, we choose X to be a set of 1024 points generated from the equal area algorithm [30], which is shown in Figure 1.
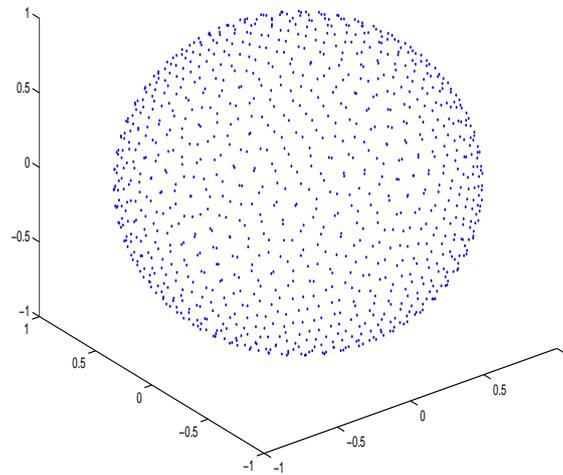
Figure 1: A set of 1024 points generated from the equal area algorithm

Next, we consider two groups of numerical experiments reconstructing the test function f_1 and f_2 in terms of RMLS and MLS, where the data set X has been contaminated by high levels of noise. In the experiment 1 and 2, $X = \{x_1, x_2, \ldots, x_N\}$, and $N = 1024$, meanwhile, 30% noise have been used in X, where the noise is a sample of a normal random variable with mean 0 and standard deviation $\sigma = 0.1$. In order to achieve uniform standard of comparison, we take polynomial degree $L = 2$ and scale $\delta = 0.25$.

Experiment 1. We want to reconstruct the Frank function f_1 from contaminated data and compare approximation results of RMLS ($\lambda = 0.2$) and MLS ($\lambda = 0$), meanwhile, s is set as 2.

Figure 2 illustrates that RMLS exists more obvious advantages than MLS when we reconstruct test function f_1 from data set with high level noise. The Figure 2 (a) shows original function f_1, the Figure 2 (b) reports f_1 with high level noise, and the Figure 2 (c) reveals approximation result of RMLS for reconstructing f_1, and the uniform error of RMLS approximation is 0.0868. At last, the Figure 2 (d) shows approximation result of MLS for reconstructing f_1, and the uniform error of MLS approximation is 0.1363.

As we known, the test function f_1 called Franke function is $C^\infty(\mathbb{S}^2)$, however, test function f_2 is continuous on the unit sphere $\mathbb{S}^2$ but not differentiable on the boundary of spherical cap $C(x_c, r)$. In order to show the effect of RMLS approximation for reconstructing function, we reconstruct f_2 from data set with high level noise in the following experiments.

Experiment 2. Test function f_2 is reconstructed from data set with high level noise, and its designing approach is similar with Experiment 1. First of all, we fix the order of Laplace-Beltrami
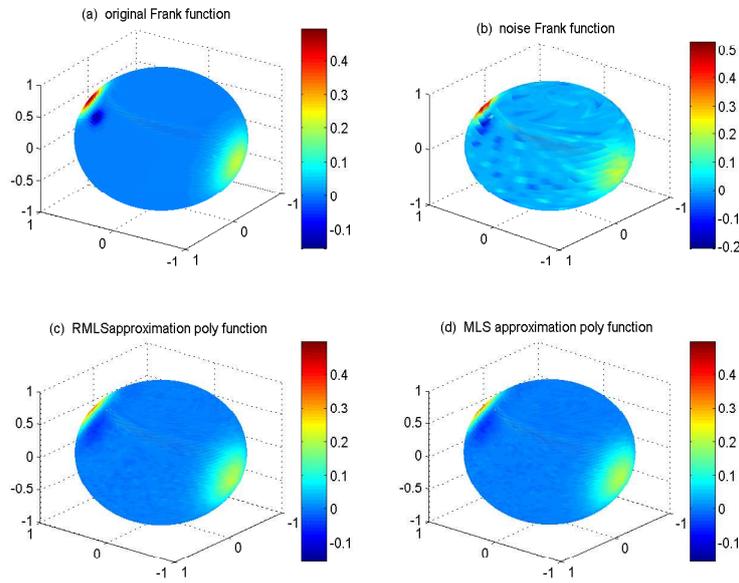
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Figure 2: A result of test function f_1 in experiment 1
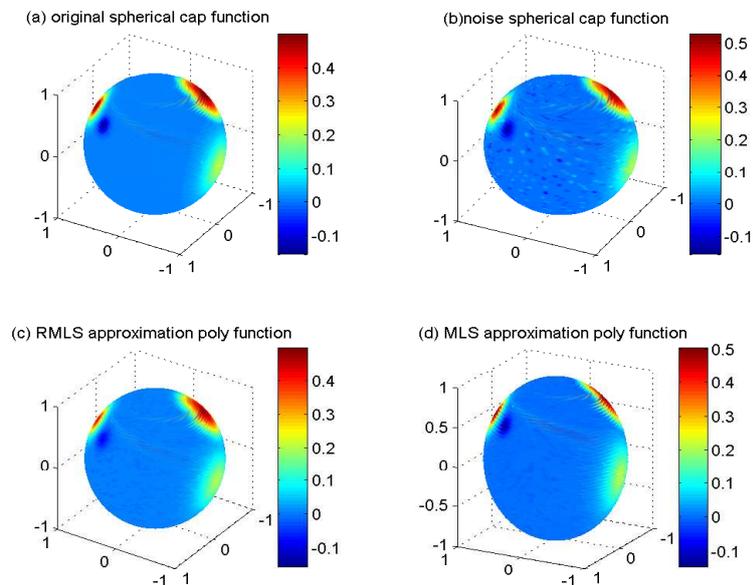
Figure 3: A result of test function f_2 in experiment 2

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Table 1: The uniform error of MLS and RMLS for f_1 and f_2 when λ and s were changed

λ	uniform error for f_1		uniform error for f_2	
	$s = 2$	$s = 4$	$s = 2$	$s = 4$
0	0.1363	0.1363	0.1303	0.1303
0.2	0.0868	0.0805	0.0991	0.0958
0.4	0.0909	0.0881	0.0990	0.0818
0.6	0.0936	0.0839	0.0851	0.0755
0.8	0.0936	0.0832	0.0836	0.0801
1.0	0.0900	0.0933	0.0905	0.0829
1.2	0.0983	0.0929	0.0932	0.0932
1.4	0.0847	0.0902	0.0934	0.0758
1.6	0.0834	0.0845	0.0899	0.0791
1.8	0.1033	0.0854	0.0982	0.0863
2.0	0.0900	0.0844	0.0833	0.0830

operator $s = 4$. Secondly, in order to compare RMLS with MLS, we let $\lambda = 0$ (MLS) and $\lambda = 1.4$ (RMLS). At last, we show the superiority in terms of uniform error.

Figure 3 illustrates that RMLS exists more obvious advantages than MLS when we reconstruct test function f_2 from data set with high level noise. The Figure 3 (a) shows original function f_2, the Figure 3 (b) reports f_2 with high level noise, and the Figure 3 (c) reveals approximation result of RMLS for reconstructing f_2, and the uniform error of RMLS approximation 0.0758. Finally, the Figure 3 (d) shows approximation result of MLS for reconstructing f_2, and the uniform error of MLS approximation 0.1330.

Table 1 gives the values of uniform error for Experiment 1 and 2, when we choose different regularized parameter λ and order of Lplace-Beltrami operator s for (4.10). The results indicate that the choosing method of λ and s are uncertain, and the optimal combination of λ and s is $\lambda = 0.2, s = 4$ for approximation f_1. However, the optimal combination of λ and s is $\lambda = 1.4, s = 4$ for approximation f_2 . The different choices for the order of Lplace-Beltrami operator and regularlized parameter can provide different pointwise approximation results.

From what has been discussed above, the RMLS is better than the MLS for recoving a function from data set with high level noise. However, the choice of λ and s is critical. How to automatically choose the proper λ and s is a challenging problem.

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Chaos Control and Function Projective Synchronization of Noval Chaotic Dynamical System

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ABSTRACT

In this paper, a Noval chaotic dynamical system is proposed and the basic properties of the system are investigated. Linear feedback control technique is used to suppress chaos. The controlled system is stable under some conditions on the parameters of the system determined by Lyapunov direct method. In addition, a function projective synchronization of two identical Noval system is presented. Lyapunov method of stability is used to prove the asymptotic stability of solutions for the error dynamical system. Numerical simulations results are included to show the effectiveness of the proposed schemes.

1. INTRODUCTION

Chaos has been developed and thoroughly studied over the past two decades. A chaotic system is a nonlinear deterministic system that displays complex and unpredictable behavior. The sensitive dependence on the initial conditions and on the system's parameter variation is a prominent characteristic of chaotic behavior. Research efforts have investigated chaos control and chaos synchronization problems in many physical chaotic systems.

Controlling chaos has become a challenging topic in nonlinear dynamics. Feedback control methods are used to control chaos by stabilizing a desired unstable periodic solution which is embedded in a chaotic attractor [1-12].

Generalized synchronization is another interesting chaos synchronization technique. Li considered a new type of projective synchronization method, called a modified projective synchronization (MPS). Chen et al. introduced another new projective synchronization which is called a function projective synchronization (FPS), where the response of the synchronized dynamical states synchronizes up to scaling function factor [11-29].

The object of this paper is to study the function project synchronization (FPS) of two identical Noval chaotic system with known parameters.

The paper is organized as follows. In Section 2, presented the model of Noval chaotic system. In Section 3, the dissipation, symmetry, equilibrium points and lyapunov exponents. In Section 4, the feedback control method is applied to Noval system and numerical simulations are presented to show the effectiveness of the proposed method. In Section 5, the proposed scheme is applied to function projective synchronize two identical Noval chaotic systems. Also numerical simulations are presented in order to validate the proposed synchronization approach. Finally, in Section 6 the conclusion of the paper is given.

2. THE MODEL OF NOVAL CHAOTIC SYSTEM

The Noval chaotic system [30] is described by the following system of differential equations:

$$
\begin{aligned}
\dot{x} &= (-a + \frac{1}{b})x + xy + z \\
\dot{y} &= -by - x^2 \\
\dot{z} &= -x - cz
\end{aligned}
\tag{1}
$$

Where the parameters a, b, c are positive real constants.
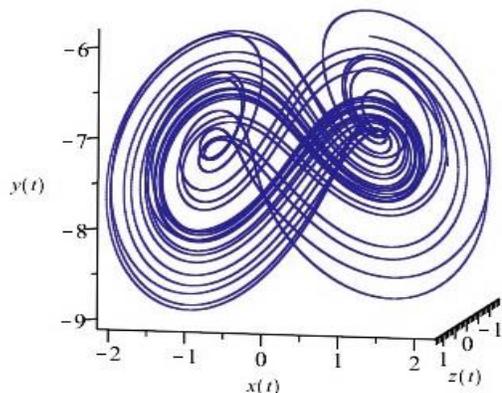A new chaotic attractor for the parameters $a = 2$, $b = 0.1$, $c = 1$ is shown in Fig. 1.

Figure 1: Noval Chaotic System at $a = 2, b = 0.1, c = 1$.

3. DYNAMICAL BEHAVIOR OF THE NOVAL CHAOTIC SYSTEM

3.1. The dissipation

The divergence of Noval system is given by;

$$
\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a + \frac{1}{b} + y - b - c.
$$

When $y < a + b + c - \frac{1}{b}$, then Noval system is dissipative.

3.2. Symmetry

The relation of $(x, y, z) \rightarrow (-x, y, -z)$ is transformed, the system remains unchanged. The system trajectory in the x,z plane symmetry of y axis.

3.3. Equilibrium points and stability

By putting the right side of equation of system (1) equal to zero, that is;

$$
\begin{aligned}
(-a + \frac{1}{b})x + xy + z &= 0 \\
-by - x^2 &= 0 \\
-x - cz &= 0
\end{aligned}
$$

This system has three equilibrium points:
$\mathbf{P}_1 = (0, \ 0, \ 0)$, $\mathbf{P}_{2,3} = (\pm \sqrt{1 - ab - b/c}, \ a - 1/b + 1/c, \ \mp \frac{1}{c}\sqrt{1 - ab - b/c})$

The eigenvalues at each equilibrium point can be obtained as shown in Table 1. And all the equilibrium points are unstable, since at least one eigenvalue has positive real part for each equilibrium point.

Table 1. Eigenvalues and stability of equilibrium points

Equilibrium points	Eigenvalues	Stable/Unstable
$\mathbf{P}_1$	$\lambda_1 = -0.1$, $\lambda_2 = -0.887$, $\lambda_3 = 7.887$	Unstable
$\mathbf{P}_2$	$\lambda_1 = -0.7446$, $\lambda_2 = 0.32 + 1.33i$, $\lambda_3 = 0.32 - 1.33i$	Unstable
$\mathbf{P}_3$	$\lambda_1 = -0.7446$, $\lambda_2 = 0.32 + 1.33i$, $\lambda_3 = 0.32 - 1.33i$	Unstable

3.4. Lyapunov exponents and its dimension

By using singular value decomposition method and we may get three Lyapunov exponents of system,$\lambda_1 = 0.13$, $\lambda_2 = 0$, $\lambda_3 = -0.52$. and the Lyapunov dimension of the new chaotic system is as follows:

$$D_L = j + \frac{1}{|\lambda_{j+1}|} \sum_{i=1}^{j} \lambda_i = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|} = 2 + \frac{0.13 + 0}{|-0.52|} = 2.254$$

Thus, the Lyapunov dimension is the fractal dimension, shows that the system is a chaotic system

4. CONTROLLING NOVAL SYSTEM

In order to control the Noval system to the unstable fixed point $(x_i,\ y_i,\ z_i)$, we introduce the feedback control to guide the chaotic trajectory $(x(t),\ y(t),\ z(t))$ to the unstable fixed point $(x_i,\ y_i,\ z_i)$.
let system (1) be controlled by the following form:

$$
\begin{aligned}
\dot{x} &= (-a + \frac{1}{b})x + xy + z - k_{i1}(x - x_i) \\
\dot{y} &= -by - x^2 - k_{i2}(y - y_i) \\
\dot{z} &= -x - cz - k_{i3}(z - z_i)
\end{aligned}
\tag{2}
$$

where $i = 1,\ 2,\ 3$.

4.1. *First*

For $i = 1$, the controlled system (2) has one equilibrium point $(x_1,\ y_1,\ z_1) = (0,\ 0,\ 0)$. Let system (2) be controlled by a linear feedback control of the form:

$$
\begin{aligned}
\dot{x} &= (-a + \frac{1}{b})x + xy + z - k_{11}(x - x_1) \\
\dot{y} &= -by - x^2 - k_{12}(y - y_1) \\
\dot{z} &= -x - cz - k_{13}(z - z_1)
\end{aligned}
\tag{3}
$$

The controlled system (3) has one equilibrium point $(x_1,\ y_1,\ z_1)$. We linearize (3) about this equilibrium point. Then the linearized system is given by:

$$
\begin{cases}
\dot{X} = (-a + \frac{1}{b} - k_{11} + y_1)X + x_1 Y + Z \\
\dot{Y} = -(b + k_{12})Y - 2x_1 X \\
\dot{Z} = -X - (c + k_{13})Z
\end{cases}
\tag{4}
$$

where $(x_1,\ y_1,\ z_1) = (0,\ 0,\ 0)$, that is;

$$
\begin{cases}
\dot{X} = (-a + \frac{1}{b} - k_{11})X + Z \\
\dot{Y} = -(b + k_{12})Y \\
\dot{Z} = -X - (c + k_{13})Z
\end{cases}
\tag{5}
$$

To prove the asymptotic stability we use the direct method of Lyapunov. Define the Lyapunov function for system (5) by:

$$V(X, \ Y, \ Z) = \frac{1}{2}(X^2 + Y^2 + Z^2) \tag{6}$$

The function V satisfied:

i $V(0, \ 0, \ 0) = 0$

ii $V(X, \ Y, \ Z) > 0$ for $X, \ Y$ and Z in the neighbourhood of the origin.

So, $V(X, \ Y, \ Z)$ is positive definite. Also, we have:

$$\frac{dV}{dt} = -\{(a - \frac{1}{b} + k_{11})X^2 + (b + k_{12})Y^2 + (c + k_{13})Z^2\} \tag{7}$$

therefore, the derivative $\frac{dV}{dt} \leq 0$ if,

$$k_{11} \geq \frac{1}{b} - a, \quad k_{12} \geq -b, \quad k_{13} \geq -c \tag{8}$$

i.e. dV/dt is negative definite under condition (8). We deduce the following lemma,

LEMMA 4.1. *The equilibrium solution* $(x_1, \ y_1, \ z_1)$ *of the controlled system (3) is asymptotically stable such that the feedback control gain K satisfy:* $k_{11} \geq \frac{1}{b} - a$ *and* $k_{12} = k_{13} = 0$.

4.2. Second

we introduce the conventional feedback control to guide the chaotic trajectory $(x(t), \ y(t), \ z(t))$ to the second unstable equilibrium point $(x_2, \ y_2, \ z_2) = (\sqrt{1 - ab - b/c}, \ a - \frac{1}{b} + \frac{1}{c}, \ -\frac{1}{c}\sqrt{1 - ab - b/c})$

$$\begin{cases} \dot{x} = (-a + \frac{1}{b})x + xy + z - k_{21}(x - x_2) \\ \dot{y} = -by - x^2 - k_{22}(y - y_2) \\ \dot{z} = -x - cz - k_{23}(z - z_2) \end{cases} \tag{9}$$

The controlled system (9) has one equilibrium point (x_2, y_2, z_2). We linearize (9) about this equilibrium point. Then the linearized system is given by:

$$\begin{cases} \dot{X} = (-a + \frac{1}{b} - k_{21} + y_2)X + x_2Y + Z \\ \dot{Y} = -(b + k_{22})Y - 2x_2X \\ \dot{Z} = -X - (c + k_{23})Z \end{cases} \tag{10}$$

where $(x_2, y_2, z_2) = (\sqrt{1 - ab - b/c}, a - \frac{1}{b} + \frac{1}{c}, -\frac{1}{c}\sqrt{1 - ab - b/c})$, that is;

$$\begin{cases} \dot{X} = (\frac{1}{c} - k_{21})X + (\sqrt{1 - ab - b/c})Y + Z \\ \dot{Y} = -(b + k_{22})Y - 2(\sqrt{1 - ab - b/c})X \\ \dot{Z} = -X - (c + k_{23})Z \end{cases} \tag{11}$$

To prove the asymptotic stability we use the direct method of Lyapunov. Define the Lyapunov function for system(10) by:

$$V(X, \ Y, \ Z) = \frac{1}{2}(X^2 + Y^2 + Z^2) \tag{12}$$

The function V satisfied:

i $V(0,0,0) = 0$

ii $V(X, Y, Z) > 0$ for X, Y and Z in the neighbourhood of the origin.

So, $V(X, Y, Z)$ is positive definite. Also, we have:

$$\frac{dV}{dt} = -\{2(k_{21} - \frac{1}{c})X^2 + (b + k_{22})Y^2 + 2(c + k_{23})Z^2\} \tag{13}$$

therefore, the derivative $\frac{dV}{dt} \leq 0$ if,

$$k_{21} \geq \frac{1}{c}, \quad k_{22} \geq -b, \quad k_{23} \geq -c \tag{14}$$

i.e. $\frac{dV}{dt}$ is negative definite under condition (14). We deduce the following lemma,

LEMMA 4.2. *The equilibrium solution* $(x_2, \ y_2, \ z_2)$ *of the controlled system (9) is asymptotically stable such that the feedback control gain K has the simple choice $k_{21} \geq \frac{1}{c}$ and $k_{22} = k_{23} = 0$.*

4.3. *Third*

we introduce the conventional feedback control to guide the chaotic trajectory $(x(t), y(t), z(t))$ to the third unstable equilibrium point $(x_3, \ y_3, \ z_3) = (-\sqrt{1 - ab - b/c}, \ a - \frac{1}{b} + \frac{1}{c}, \ \frac{1}{c}\sqrt{1 - ab - b/c})$

$$\begin{cases} \dot{x} = (-a + \frac{1}{b})x + xy + z - k_{31}(x - x_3) \\ \dot{y} = -by - x^2 - k_{32}(y - y_3) \\ \dot{z} = -x - cz - k_{33}(z - z_3) \end{cases} \tag{15}$$

The controlled system (14) has one equilibrium point (x_3, y_3, z_3). We linearize (14) about this equilibrium point. Then the linearized system is given by:

$$\begin{cases} \dot{X} = (-a + \frac{1}{b} - k_{31} + y_3)X + x_3Y + Z \\ \dot{Y} = -(b + k_{32})Y - 2x_3X \\ \dot{Z} = -X - (c + k_{33})Z \end{cases} \tag{16}$$

where $(x_3, y_3, z_3) = (-\sqrt{1 - ab - b/c}, a - \frac{1}{b} + \frac{1}{c}, \frac{1}{c}\sqrt{1 - ab - b/c})$, that is;

$$\begin{cases} \dot{X} = (\frac{1}{c} - k_{31})X - (\sqrt{1 - ab - b/c})Y + Z \\ \dot{Y} = -(b + k_{32})Y + 2(\sqrt{1 - ab - b/c})X \\ \dot{Z} = -X - (c + k_{33})Z \end{cases} \tag{17}$$

To prove the asymptotic stability we use the direct method of Lyapunov. Define the Lyapunov function for system(16) by:

$$V(X, \ Y, \ Z) = \frac{1}{2}(X^2 + Y^2 + Z^2) \tag{18}$$

The function V satisfied:

i $V(0,0,0) = 0$

ii $V(X, Y, Z) > 0$ for X, Y and Z in the neighbourhood of the origin.

So, $V(X, Y, Z)$ is positive definite. Also, we have:

$$\frac{dV}{dt} = -\{2(k_{31} - \frac{1}{c})X^2 + (b + k_{32})Y^2 + 2(c + k_{33})Z^2\} \tag{19}$$

therefore, the derivative $\frac{dV}{dt} \leq 0$ if,

$$k_{31} \geq \frac{1}{c}, \quad k_{32} \geq -b, \quad k_{33} \geq -c \tag{20}$$

i.e. $\frac{dV}{dt}$ is negative definite under condition (20). We deduce the following lemma,

LEMMA 4.3. *The equilibrium solution* $(x_3, \ y_3, \ z_3)$ *of the controlled system (15) is asymptotically stable such that the feedback control gain* K *has the simple choice* $k_{31} \geq \frac{1}{c}$ *and* $k_{32} = k_{33} = 0$.

5. THE SCHEME OF GENERALIZED FUNCTION PROJECTIVE SYNCHRONIZATION OF CHAOTIC SYSTEMS

The chaotic (master and slave) systems can be given in the following form:

$$\dot{X} = F(X) \tag{21}$$

$$\dot{Y} = G(Y) + U(X, Y, t) \tag{22}$$

Where $X = (x_1, x_2, \ldots, x_n)^T, Y = (y_1, y_2, \ldots, y_n)^T \in \mathbf{R}^n$ are state vectors of the system (20) and (21), respectively; $F, G : \mathbf{R}^n \to \mathbf{R}^n$ are two continuous vector functions and $U : (\mathbf{R}^n, \mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}^n$ is a controller which will be designed later.

DEFINITION 5.1. *For the master system (20) and the slave system (21), there is said to be generalized function projective synchronization (GFPS) if there exists a vector function* $U(X, Y, t)$ *such that;* $\lim_{t \to +\infty} \|Y - \Lambda(X)X\| = 0$ *where* $\Lambda(X) = diag\{h_1(X), h_2(X), \ldots, h_n(X)\}$ *where* $h_i(X)$ *are continuous functions,* $\|.\|$ *represents a vector norm induced by the matrix norm.*

REMARK 1. *We define* $e = Y - \Lambda(X)X$ *which is called the error vector between systems (20) and (21) for GFPS, where* $e = (e_1, e_2, \ldots, e_n)^T$, *and*
$e_i = Y_i - h_i(X)X_i, (i = 1, 2, \ldots, n)$

REMARK 2.

If $\Lambda = \sigma \mathbf{I}, \sigma \in \mathbf{R}$, *the GFPS problem will be reduced to projective synchronization, where* $\mathbf{I}$ *is an* $n \times n$ *identity matrix. In particular if* $\sigma = 1$ *and* -1 *the problem is further simplified to complete synchronization and antiphase synchronization, respectively. And if* $\Lambda = diag\{a_1, a_2, \ldots, a_n\}$, *the modified projective synchronization will appear.*

We will study the FPS of novel system with known parameters and determine controller function for the FPS of the derive and response systems. Our aim is to design a controller and make the response system trace the drive system and become ultimately. The Noval system as a drive system is given as below;

$$\begin{cases} \dot{x}_1 &= (-a + \frac{1}{b})x_1 + x_1 y_1 + z_1 \\ \dot{y}_1 &= -by_1 - x_1^2 \\ \dot{z}_1 &= -x_1 - cz_1 \end{cases} \tag{23}$$

the Noval system as the response system is also given by;

$$\begin{cases} \dot{x}_2 &= (-a + \frac{1}{b})x_2 + x_2 y_2 + z_2 + u_1 \\ \dot{y}_2 &= -by_2 - x_2^2 + u_2 \\ \dot{z}_2 &= -x_2 - cz_2 + u_3 \end{cases} \tag{24}$$

According to the FPS scheme presented in the previous section, without loss of generality, we choose the scaling function matrix
$\Lambda(X) = diag\{d_{11}x_1 + d_{12}, d_{21}y_1 + d_{22}, d_{31}z_1 + d_{32}\}$ where $d_{ij}(i = 1, 2, 3; j = 1, 2)$ are constant numbers. The error vector can be defined as

$$\begin{cases} e_x & = x_2 - (d_{11}x_1 + d_{12})x_1 \\ e_y & = y_2 - (d_{21}y_1 + d_{22})y_1 \\ e_z & = z_2 - (d_{31}z_1 + d_{32})z_1 \end{cases} \quad (25)$$

The error dynamical system between (23) and (24) is;

$$\begin{cases} \dot{e}_x & = (-a + \frac{1}{b})e_x + x_2y_2 + z_2 - d_{11}(-a + \frac{1}{b})x_1^2 - 2d_{11}x_1z_1 \\ & \quad -2d_{11}x_1^2y_1 - d_{12}x_1y_1 - d_{12}z_1 + u_1 \\ \dot{e}_y & = -be_y - x_2^2 + d_{21}by_1^2 + 2d_{21}y_1x_1^2 + d_{22}x_1^2 + u_2 \\ \dot{e}_z & = -ce_z - x_2 + d_{31}cz_1^2 + 2d_{31}z_1x_1 + d_{32}x_1 + u_3 \end{cases} \quad (26)$$

we can get the controller

$$\begin{cases} u_1 & = \frac{-2}{b}e_x - x_2y_2 - z_2 + d_{11}(-a + \frac{1}{b})x_1^2 + 2d_{11}x_1z_1 + 2d_{11}x_1^2y_1 + d_{12}x_1y_1 + d_{12}z_1 \\ u_2 & = x_2^2 - d_{21}by_1^2 - 2d_{21}y_1x_1^2 - d_{22}x_1^2 \\ u_3 & = x_2 - d_{31}cz_1^2 - 2d_{31}z_1x_1 - d_{32}x_1 \end{cases} \quad (27)$$

then the error dynamical system is described by

$$\begin{cases} \dot{e}_x & = -(a + \frac{1}{b})e_x \\ \dot{e}_y & = -be_y \\ \dot{e}_z & = -ce_z \end{cases} \quad (28)$$

for this choice, the closed loop system (28) has three negative eigenvalues $-(a + \frac{1}{b}), -b, -c$ which implies that the error state e_x, e_y and e_z converge to zero as time t tends to infinity.
Hence the FPS between the identical Noval chaotic system is achieved.

5.1. Numerical Results

In this section, some numerical simulation results are presented to verify the previous analytical results where $a = 2, b = 0.1, c = 1$. Figure 2: shows the convergence of the trajectory of the controlled system to the unstable equilibrium point $(x_1, y_1, z_1) = (0, 0, 0)$ of the uncontrolled system (1). Figure 3: shows the convergence of the trajectory of the controlled system to the unstable equilibrium point $(x_2, y_2, z_2) = (\sqrt{1 - ab - b/c}, a - \frac{1}{b} + \frac{1}{c}, -\frac{1}{c}\sqrt{1 - ab - b/c})$ of the uncontrolled system (1). Figure 4: shows the convergence of the trajectory of thec ontrolled system to the unstable equilibrium point $(x_3, y_3, z_3) = (-\sqrt{1 - ab - b/c}, a - \frac{1}{b} + \frac{1}{c}, \frac{1}{c}\sqrt{1 - ab - b/c})$ of the uncontrolled system (1).
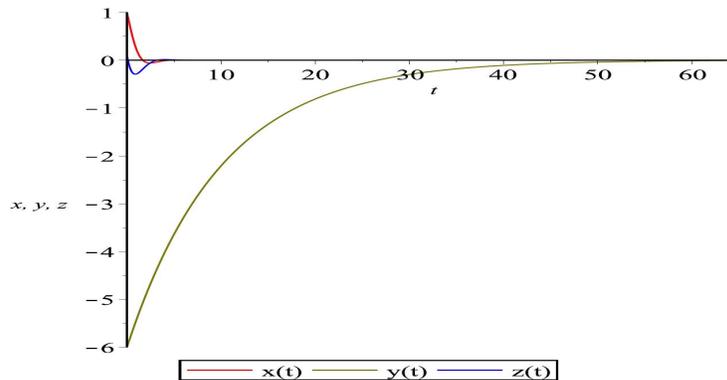
Figure 2: The time responses for the states of the controlled Noval system to a fixed point (x_1, y_1, z_1).
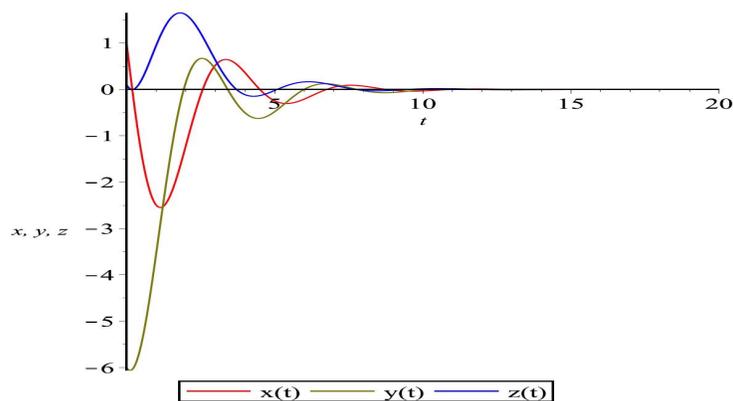
Figure 3: The time responses for the states of the controlled Noval system to a fixed point (x_2, y_2, z_2).
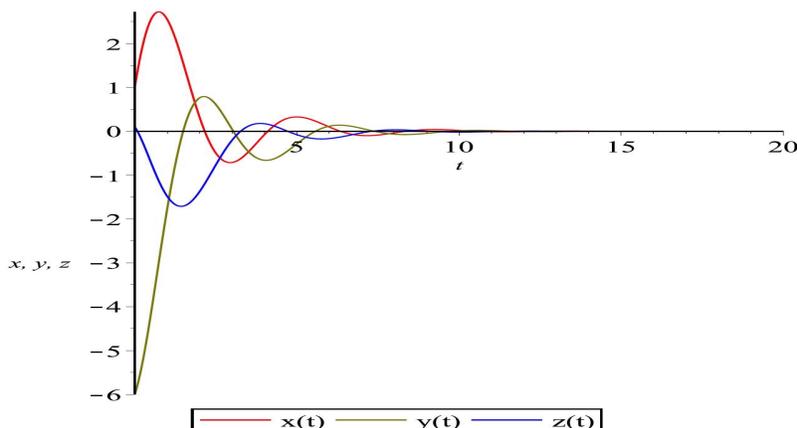
Figure 4: The time responses for the states of the controlled Noval system to a fixed point (x_3, y_3, z_3).

The initial values of the drive system and response system are taken as:

$$(x_1(0), y_1(0), z_1(0))^T = (1, -6, 0.1)^T, (x_2(0), y_2(0), z_2(0))^T = (10, 12, -3)^T.$$

We choose the scaling function factors as:

$$h_1 = x_1 + 2, h_2 = -2y_1 - 2 \ and \ h_3 = z_1 - 2.$$

Figure 5: show the FPS between two identical Noval systems. When the scaling factors are simplified as $h_i = 1 \ (i = 1, 2, 3)$, the complete synchronization between two identical Noval systems are shown in Figure 6. Furthermore, when the scaling factors are simplified as $h_i = -1 \ (i = 1, 2, 3)$, the anti synchronization between two identical Noval systems are shown in Figure 7. Finally, when the scaling factors are simplified as $h_1 = 1.5, h_2 = 2 \ and \ h_3 = 2.5$, the modified projective synchronization (MPS) between two identical Noval

systems are shown in Figure 8.
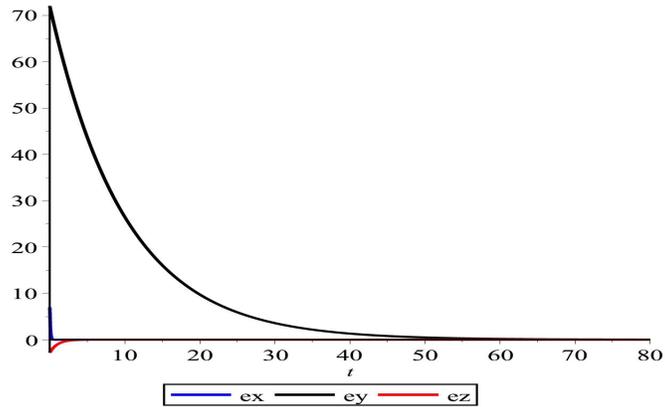
Figure 5: The behaviour of the trajectories e_x, e_y and e_z of the error system tends to zero for FPS.
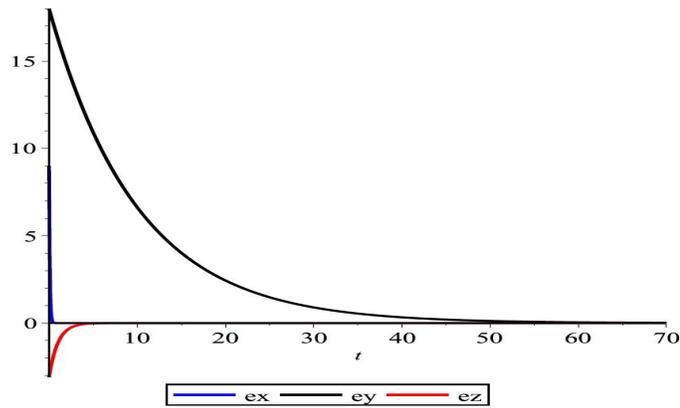
Figure 6: The behaviour of the trajectories e_x, e_y and e_z of the error system tends to zero for complete synchronization
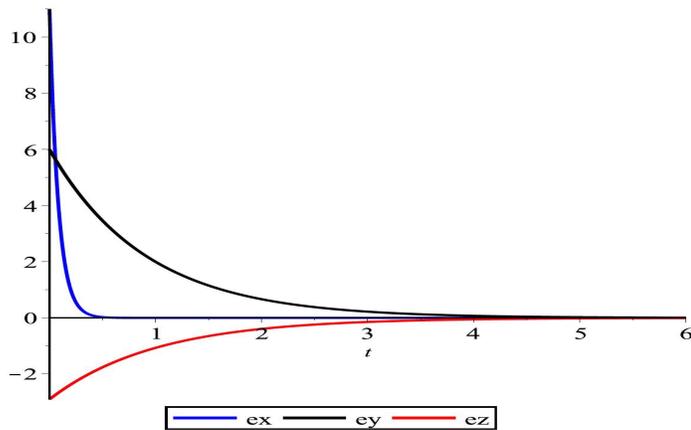
Figure 7: The behaviour of the trajectories e_x, e_y and e_z of the error system tends to zero for anti synchronization
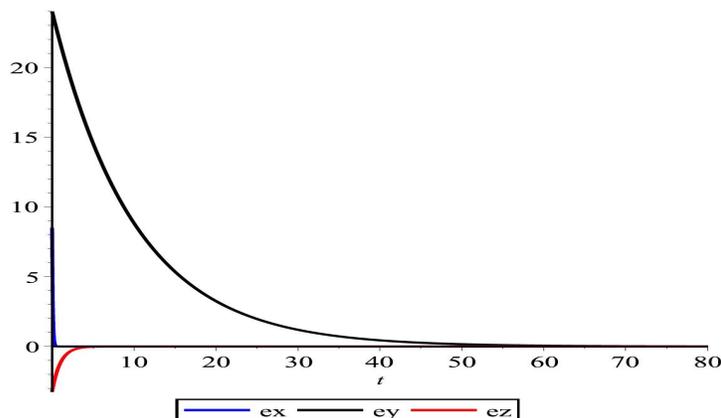
Figure 8: The behaviour of the trajectories e_x, e_y and e_z of the error system tends to zero for MPS.

6. CONCLUSION

The paper has studied the noval chaotic dynamical system, including some basic dynamical properties, Lyapunov exponents, Lyapunov dimension. A feedback control has been proposed to the noval chaotic dynamical system. The controlling conditions are derived from the Lyapunov direct method. The function projective synchronization has been used to synchronize two identical chaotic systems with known parameters. By the Lyapunov stability theory, the sufficient condition of the function projective synchronization has been obtained. Finally, numerical simulations are provided to verify the effectiveness of the results obtained.

Acknowledgments

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UMBRAL CALCULUS APPROACH TO r-STIRLING NUMBERS OF THE SECOND KIND AND r-BELL POLYNOMIALS

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ABSTRACT. In this paper, we would like to use umbral calculus in order to derive some properties, recurrence relations and identities related to r-Stirling numbers of second kind and r-Bell polynomials. In particular, we will express the r-Bell polynomials as linear combinations of many well-known families of special polynomials.

1. INTRODUCTION

The Stirling numbers $S_2(n, k)$ of the second kind counts the number of partitions of the set $[n] = \{1, 2, \cdots, n\}$ into k nonempty disjoint subsets.

The $S_2(n, k), (n, k \geq 0)$ are given by the recurrence relation

$$S_2(n, k) = kS_2(n-1, k) + S_2(n-1, k-1), \ (n, k \geq 1), \tag{1.1}$$

with the initial conditions

$$S_2(n, 0) = \delta_{0n}, S_2(0, k) = \delta_{0k}. \tag{1.2}$$

They are also given by

$$x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k, \tag{1.3}$$

with $(x)_0 = 1, (x)_k = x(x-1)\cdots(x-k+1)$, for $k \geq 1$, and by

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k)\frac{t^n}{n!}. \tag{1.4}$$

More explicitly, they are given by

$$S_2(n, k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j} j^n$$

$$= \frac{1}{k!} \triangle^k 0^n, \ (n \geq k), \tag{1.5}$$

where $\triangle^k 0^n = \triangle^k x^n|_{x=0}$, and $\triangle f(x) = f(x+1) - f(x)$ is the forward difference operator. For these well known facts, one may refer to [3,4].

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2   Umbral calculus approach to r-Stirling numbers of the second kind and r-Bell polynomials

Let r be any positive integer. These 'classical' Stirling numbers $S_2(n,k)$ of the second kind were generalized to the r-Stirling numbers $S_{2,r}(n,k)$ of the second kind (see, [1,2,7]). The $S_{2,r}(n,k)$ enumerates the number of partitions of the set $[n] = \{1, 2, \cdots, n\}$ into k nonempty disjoint subsets in such a way that $1, 2, \cdots, r$ are in distinct subsets.

They are given by the recurrence relation

$$S_{2,r}(n,k) = kS_{2,r}(n-1,k) + S_{2,r}(n-1,k-1), \ (n > r), \tag{1.6}$$

with the initial conditions

$$S_{2,r}(n,k) = 0, (n < r); S_{2,r}(n,k) = \delta_{kr}, (n = r). \tag{1.7}$$

The $S_{2,r}(n,k)$ are also given by

$$(x+r)^n = \sum_{k=0}^{n} S_{2,r}(n+r, k+r)(x)_k, \tag{1.8}$$

and by

$$\frac{1}{k!} e^{rt}(e^t - 1)^k = \sum_{n=k}^{\infty} S_{2,r}(n+r, k+r)\frac{t^n}{n!}. \tag{1.9}$$

Analogously to the classical case, they are explicitly given by

$$\begin{aligned} S_{2,r}(n+r, k+r) &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j}(r+j)^n \\ &= \frac{1}{k!} \triangle^k r^n, \ (n \geq k), \end{aligned} \tag{1.10}$$

where $\triangle^k r^n = \triangle^k x^n|_{x=r}$.

For more details about r-Stirling numbers of the second kind, one may refer to [1,2,7].

The Bell polynomials $Bel_n(x)$ (also called exponential or Touchard polynomials) are defined by

$$e^{x(e^t - 1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \ (see \ [3, 4, 8, 9]). \tag{1.11}$$

Then it is immediate to see that

$$Bel_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k. \tag{1.12}$$

For $x = 1$, $Bel_n = Bel_n(1) = \sum_{k=0}^{n} S_2(n,k)$ are called Bell numbers so that

$$e^{e^t - 1} = \sum_{n=0}^{\infty} Bel_n \frac{t^n}{n!}. \tag{1.13}$$

Further, the Bell polynomial is given by Dobinski's formula

$$Bel_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k. \tag{1.14}$$

On the other hand, the r-Bell polynomials $Bel_{n,r}(x)$ are defined by

$$e^{rt}e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_{n,r}(x)\frac{t^n}{n!}, \ (see \ [5]). \tag{1.15}$$

Then it is easy to see that

$$Bel_{n,r}(x) = \sum_{k=0}^{n} S_{2,r}(n+r,k+r)x^k. \tag{1.16}$$

Moreover, they satisfy the generalized Dobinski's formula

$$Bel_{n,r}(x) = e^{-x}\sum_{k=0}^{\infty} \frac{(k+r)^n}{k!}x^k. \tag{1.17}$$

When $x = 1$, $Bel_{n,r} = Bel_{n,r}(1) = \sum_{k=0}^{n} S_{2,r}(n+r,k+r)$ are called r-Bell numbers so that

$$e^{e^t-1+rt} = \sum_{n=0}^{\infty} Bel_{n,r}\frac{t^n}{n!}. \tag{1.18}$$

We note here, in passing, that r-Bell numbers were called in another name, namely extended Bell numbers,(see [6]).

In this paper, we would like to use umbral calculus in order to derive some properties, recurrence relations and identities related to r-Stirling numbers of the second kind and r-Bell polynomials. In particular, we will express the r-Bell polynomials as linear combinations of many well-known families of special polynomials.

2. Review on umbral calculus

Here we will go over some of the basic facts about umbral calculus. For a complete treatment, the reader may refer to [4].

Let $\mathcal{F}$ be the algebra of all formal power series in the single variable t with the coefficients in the field $\mathbb{C}$ of complex numbers:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k\frac{t^k}{k!} \ \bigg| \ a_k \in \mathbb{C} \right\}. \tag{2.1}$$

Let $\mathbb{P} = \mathbb{C}[x]$ denote the ring of polynomials in x with the coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. For $L \in \mathbb{P}^*$, $p(x) \in \mathbb{P}$, $< L\,|\,p(x) >$ denotes the action of the linear functional L on $p(x)$. For $f(t) = \sum_{k=0}^{\infty} a_k\frac{t^k}{k!} \in \mathcal{F}$, the linear functional $< f(t)\,|\,\cdot >$ on $\mathbb{P}$ is defined by

$$< f(t)\,|\,x^n >= a_n, \ (n \geq 0). \tag{2.2}$$

For $L \in \mathbb{P}^*$, let $f_L(t) = \sum_{k=0}^{\infty} \left\langle L|x^k \right\rangle \frac{t^k}{k!} \in \mathcal{F}$. Then we evidently have $\left\langle f_L(t)|x^n \right\rangle = \left\langle L|x^n \right\rangle$, and the map $L \to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathcal{F}$. Thus $\mathcal{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in t. So an element $f(t) \in \mathcal{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra, the study of which is the umbral calculus.

4   Umbral calculus approach to r-Stirling numbers of the second kind and r-Bell polynomials

The order $o(f(t))$ of $0 \neq f(t) \in \mathcal{F}$ is the smallest integer k such that the coefficients of t^k does not vanish. In particular, for $0 \neq f(t) \in \mathcal{F}$, it is called an invertible series if $o(f(t)) = 0$ and a delta series if $o(f(t)) = 1$.

Let $f(t), g(t) \in \mathcal{F}$, with $o(g(t)) = 0$, $o(f(t)) = 1$. Then there exists a unique sequence of polynomials $S_n(x)$ ($\deg S_n(x) = n$) such that $\left\langle g(t)f(t)^k | S_n(x) \right\rangle = n! \delta_{n,k}$, for $n, k \geq 0$. Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, which is concisely denoted by $S_n(x) \sim (g(t), f(t))$.

It is known that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}, \tag{2.3}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

Let $p_n(x) \sim (1, f(t)), q_n(x) \sim (1, l(t))$. Then the transfer formula says that

$$q_n(x) = x \left(\frac{f(t)}{l(t)} \right)^n x^{-1} p_n(x), \ (n \geq 1). \tag{2.4}$$

Let $S_n(x) \sim (g(t)), f(t))$. Then we have the Sheffer identity:

$$S_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} S_k(x) p_{n-k}(y), \tag{2.5}$$

where $p_n(x) = g(t)S_n(x) \sim (1, f(t))$.

The derivative of $S_n(x)$ is given by

$$\frac{d}{dx} S_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} \left\langle \bar{f}(t) | x^{n-k} \right\rangle S_k(x), \ (n \geq 1). \tag{2.6}$$

Also, we have the recurrence formula:

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} S_n(x). \tag{2.7}$$

Assume that $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$. Then

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \tag{2.8}$$

where

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k | x^n \right\rangle. \tag{2.9}$$

Finally, we also need the following: for any $h(t) \in \mathcal{F}$, $p(x) \in \mathbb{P}$,

$$\left\langle h(t) | xp(x) \right\rangle = \left\langle \partial_t h(t) | p(x) \right\rangle. \tag{2.10}$$

3. Main Results

As we can see from (1.15) and (2.3), we see that

$$Bel_{n,r}(x) \sim \left(\frac{1}{(1+t)^r}, \log((1+t)) \right) = (g(t), f(t)). \qquad (3.1)$$

Let $n \geq 1$. Then, using (2.10), we have

$$
\begin{aligned}
Bel_{n,r}(y) &= \left\langle \sum_{m=0}^{\infty} Bel_{m,r}(y) \frac{t^m}{m!} \Big| x^n \right\rangle \\
&= \left\langle e^{rt} e^{y(e^t-1)} \Big| x^n \right\rangle \\
&= \left\langle \partial_t (e^{rt} e^{y(e^t-1)}) \Big| x^{n-1} \right\rangle \\
&= \left\langle r e^{rt} e^{y(e^t-1)} + e^{rt} e^{y(e^t-1)} y e^t \Big| x^{n-1} \right\rangle \\
&= r \left\langle e^{rt} e^{y(e^t-1)} \Big| x^{n-1} \right\rangle + y \left\langle e^{(r+1)t} e^{y(e^t-1)} \Big| x^{n-1} \right\rangle \\
&= r Bel_{n-1,r}(y) + y Bel_{n-1,r+1}(y).
\end{aligned}
\qquad (3.2)
$$

Thus we obtain the following recurrence relation for r-Bell polynomials.

Theorem 3.1. *For all integers $n \geq 1$, we have the recurrence relation.*

$$Bel_{n,r}(x) = r Bel_{n-1,r}(x) + x Bel_{n-1,r+1}(x), \ (n \geq 1).$$

From (2.6), we have

$$
\begin{aligned}
\frac{d}{dx} Bel_{n,r}(x) &= \sum_{k=0}^{n-1} \binom{n}{k} \left\langle e^t - 1 \Big| x^{n-k} \right\rangle Bel_{k,r}(x) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (1 - \delta_{n,k}) Bel_{k,r}(x) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} Bel_{k,r}(x), \ (n \geq 1).
\end{aligned}
\qquad (3.3)
$$

Using (2.7), we obtain

$$
\begin{aligned}
Bel_{n+1,r}(x) &= (x + r\frac{1}{1+t})(1+t) Bel_{n,r}(x) \\
&= x(1+t) Bel_{n,r}(x) + r Bel_{n,r}(x) \\
&= x Bel_{n,r}(x) + x \frac{d}{dx} Bel_{n,r}(x) + r Bel_{n,r}(x),
\end{aligned}
\qquad (3.4)
$$

from which it follows that

$$
\begin{aligned}
&\frac{d}{dx} Bel_{n,r}(x) \\
&= \frac{Bel_{n+1,r}(x)}{x} - \frac{r Bel_{n,r}(x)}{x} - Bel_{n,r}(x).
\end{aligned}
\qquad (3.5)
$$

This agrees with the result in [2].

6   Umbral calculus approach to r-Stirling numbers of the second kind and r-Bell polynomials

Noting that $p_n(x) = g(t)Bel_{n,r}(x) \sim (1, \log(1+t))$, we have $p_n(x) = Bel_n(x)$. Hence from (2.5), we get the following Sheffer identity

$$Bel_{n,r}(x+y) = \sum_{k=0}^{n} \binom{n}{k} Bel_{k,r}(x) Bel_{n-k}(y). \tag{3.6}$$

$$
\begin{aligned}
Bel_{n,r}(y) &= \left\langle e^{rt} e^{y(e^t-1)} | x^n \right\rangle \\
&= \left\langle e^{rt} | e^{y(e^t-1)} x^n \right\rangle \\
&= \left\langle e^{rt} | \sum_{m=0}^{\infty} Bel_m(y) \frac{t^m}{m!} x^n \right\rangle \\
&= \sum_{m=0}^{n} \binom{n}{m} Bel_m(y) \left\langle e^{rt} | x^{n-m} \right\rangle \\
&= \sum_{m=0}^{n} \binom{n}{m} Bel_m(y) r^{n-m}.
\end{aligned}
\tag{3.7}
$$

Hence we get

$$Bel_{n,r}(x) = \sum_{m=0}^{n} \binom{n}{m} r^{n-m} Bel_m(x). \tag{3.8}$$

Here we apply the transfer formula in (2.4) to $x^n \sim (1,t)$, $\frac{1}{(1+t)^r} Bel_{n,r}(x) \sim (1, \log((1+t))$. For $n \geq 1$, we have

$$
\begin{aligned}
\frac{1}{(1+t)^r} Bel_{n,r}(x) &= x \left(\frac{t}{\log(1+t)} \right)^n x^{-1} x^n \\
&= x \sum_{k=0}^{\infty} b_k^{(n)} \frac{t^k}{k!} x^{n-1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} b_k^{(n)} x^{n-k}.
\end{aligned}
\tag{3.9}
$$

Here $b_k^{(n)}$ are the Bernoulli numbers of the second kind of order n defined by

$$\left(\frac{t}{\log(1+t)} \right)^n = \sum_{k=0}^{\infty} b_k^{(n)} \frac{t^k}{k!}. \tag{3.10}$$

Here, as is well known, $b_k^{(n)} = B_k^{(k-n+1)}(1)$, with $B_k^{(n)}(x)$ denoting the Bernoulli polynomials of order n. Thus we obtain

$$Bel_{n,r}(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} b_k^{(n)} (1+t)^r x^{n-k}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} b_k^{(n)} \sum_{l=0}^{r} \binom{r}{l} t^l x^{n-k} \tag{3.11}$$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{r} \binom{n-1}{k} \binom{r}{l} (n-k)_l b_k^{(n)} x^{n-k-l}.$$

As $\frac{1}{(1+t)^r} Bel_{n,r}(x) = Bel_n(x) = \sum_{j=0}^{n} S_2(n,j) x^j$, we can proceed as follows.

$$Bel_{n,r}(x) = (1+t)^r Bel_n(x)$$

$$= \sum_{k=0}^{\infty} \binom{r}{k} t^k Bel_n(x)$$

$$= \sum_{k=0}^{n} \binom{r}{k} t^k \sum_{j=0}^{n} S_2(n,j) x^j$$

$$= \sum_{k=0}^{n} \binom{r}{k} \sum_{j=k}^{n} S_2(n,j)(j)_k x^{j-k} \tag{3.12}$$

$$= \sum_{k=0}^{n} \binom{r}{k} \sum_{l=0}^{n-k} S_2(n,k+l)(k+l)_k x^l$$

$$= \sum_{l=0}^{n} \left(\sum_{k=0}^{n-l} \binom{r}{k} (k+l)_k S_2(n,k+l) \right) x^l.$$

Also, from $Bel_{n,r}(x) = (1+t)^r Bel_n(x)$,

$$(1+t)^s Bel_{n,r}(x) = Bel_{n,r+s}(x), \quad (s \geq 0). \tag{3.13}$$

In particular, for $s = 1$, we have

$$Bel_{n,r+1}(x) = Bel_{n,r}(x) + \frac{d}{dx} Bel_{n,r}(x). \tag{3.14}$$

Hence in addition to (3.3) and (3.4) we obtain another expression for the derivative of $Bel_{n,r}(x)$, namely

$$\frac{d}{dx} Bel_{n,r}(x) = Bel_{n,r+1}(x) - Bel_{n,r}(x). \tag{3.15}$$

Combining this with (3.3), we get

$$Bel_{n,r+1}(x) = \sum_{k=0}^{n} \binom{n}{k} Bel_{k,r}(x). \tag{3.16}$$

We are now going to summarize the results obtained so far as the following three theorems. Theorem 2 follows from (3.3), (3.5) and (3.15), Theorem 3 from (3.6), (3.8) and (3.16), and Theorem 4 from (3.11) and (3.12).

Theorem 3.2. *For all integers $n \geq 1$, the derivative of r-Bell polynomials can be given as follows:*

$$\frac{d}{dx} Bel_{n,r}(x) = \sum_{k=0}^{n-1} \binom{n}{k} Bel_{k,r}(x)$$

$$= \frac{Bel_{n+1,r}(x)}{x} - \frac{rBel_{n,r}(x)}{x} - Bel_{n,r}(x)$$

$$= Bel_{n,r+1}(x) - Bel_{n,r}(x).$$

Theorem 3.3. *For all integers $n \geq 0$, the following identities hold true.*

$$Bel_{n,r}(x+y) = \sum_{k=0}^{n} \binom{n}{k} Bel_{k,r}(x) Bel_{n-k}(y),$$

$$Bel_{n,r}(x) = \sum_{m=0}^{n} \binom{n}{m} r^{n-m} Bel_m(x),$$

$$Bel_{n,r+1}(x) = \sum_{k=0}^{n} \binom{n}{k} Bel_{k,r}(x).$$

Theorem 3.4. *For all integers $n \geq 0$, we have the following expressions of r-Bell polynomials.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n-1} \sum_{l=0}^{r} \binom{n-1}{k} \binom{r}{l} (n-k)_l b_k^{(n)} x^{n-k-l}$$

$$= \sum_{l=0}^{n} \left(\sum_{k=0}^{n-l} \binom{r}{k} (k+l)_k S_2(n, k+l) \right) x^l,$$

where $b_k^{(n)}$ are the Bernoulli numbers of the second kind of order n given by (3.10).

From now on, we will apply the formula (2.9) in order to express $Bel_{n,r}(x)$ as linear combinations of well-known families of special polynomials. For this, let us remind you of the fact in (3.1), namely

$$Bel_{n,r}(x) \sim \left(\frac{1}{(1+t)^r}, \log(1+t) \right). \tag{3.17}$$

Noting that the Bernoulli polynomial $B_n(x)$ is Sheffer for $\left(\frac{e^t-1}{t}, t \right)$, we write $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} B_k(x)$. Then

$$C_{n,k} = \left\langle \frac{e^{e^t-1}-1}{e^t-1} \Big| \frac{1}{k!} e^{rt} (e^t-1)^k x^n \right\rangle$$

$$= \left\langle \frac{e^{e^t-1}-1}{e^t-1} \Big| \sum_{l=k}^{\infty} S_{2,r}(l+r, k+r) \frac{t^l}{l!} x^n \right\rangle \tag{3.18}$$

$$= \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \left\langle \frac{e^{e^t-1}-1}{e^t-1} \Big| x^{n-l} \right\rangle.$$

Here we observe that

$$
\begin{aligned}
&\Big\langle \frac{e^{e^t-1}-1}{e^t-1} \Big| x^{n-l}\Big\rangle \\
&= \Big\langle \frac{e^{e^t-1}-1}{t} \Big| \frac{t}{e^t-1}x^{n-l}\Big\rangle \\
&= \Big\langle \frac{e^{e^t-1}-1}{t} \Big| \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}x^{n-l} \\
&= \sum_{m=0}^{n-l} \binom{n-l}{m} B_m \Big\langle \frac{e^{e^t-1}-1}{t} \Big| x^{n-l-m}\Big\rangle \\
&= \sum_{m=0}^{n-l} \frac{1}{n-l-m+1}\binom{n-l}{m} B_m \Big\langle e^{e^t-1}-1 \Big| x^{n-l-m+1}\Big\rangle \\
&= \sum_{m=0}^{n-l} \frac{1}{n-l-m+1}\binom{n-l}{m} B_m Bel_{n-l-m+1}.
\end{aligned}
\tag{3.19}
$$

Thus we see that

$$
\begin{aligned}
C_{n,k} =& \frac{1}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{l}\binom{n-l+1}{m}S_{2,r}(l+r,k+r) \\
& \times B_m Bel_{n-l-m+1}.
\end{aligned}
\tag{3.20}
$$

Finally, we obtain

$$
\begin{aligned}
Bel_{n,r}(x) =& \frac{1}{n+1}\sum_{k=0}^{n}\Big(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{l}\binom{n-l+1}{m}S_{2,r}(l+r,k+r) \\
& \times B_m Bel_{n-l-m+1}\Big)B_k(x).
\end{aligned}
\tag{3.21}
$$

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k}E_k(x)$. Here $E_n(x)$ are the Euler polynomials with $E_n(x) \sim (\frac{e^t+1}{2}, t)$. Then

$$
\begin{aligned}
C_{n,k} =& \frac{1}{2}\Big\langle e^{e^t-1}+1 \Big| \frac{1}{k!}e^{rt}(e^t-1)^k x^n\Big\rangle \\
=& \frac{1}{2}\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\Big\langle e^{e^t-1}+1 \Big| x^{n-l}\Big\rangle \\
=& \frac{1}{2}\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)(Bel_{n-l}+\delta_{n,l}).
\end{aligned}
\tag{3.22}
$$

Hence we get

$$
Bel_{n,r}(x) = \frac{1}{2}\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)(Bel_{n-l}+\delta n,l)\right)E_k(x).
\tag{3.23}
$$

We summarize the expressions of $Bel_{n,r}(x)$ in (3.21) and (3.23) as a theorem.

Theorem 3.5. *For all integers $n \geq 0$, we have the following expressions.*

$$Bel_{n,r}(x) = \frac{1}{n+1} \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n+1}{l} \binom{n-l+1}{m} S_{2,r}(l+r,k+r) \right.$$

$$\times B_m Bel_{n-l-m+1} \right) B_k(x)$$

$$= \frac{1}{2} \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r,k+r)(Bel_{n-l} + \delta n, l) \right) E_k(x).$$

Write $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k}(x)_k$, where $(x)_n$ are the falling factorials with $(x)_n \sim (1, e^t - 1)$. Then

$$C_{n,k} = \left\langle e^{rt} | \frac{1}{k!} (e^{e^t - 1} - 1)^k x^n \right\rangle$$

$$= \left\langle e^{rt} | \sum_{l=k}^{\infty} S_2(l,k) \frac{1}{l!} (e^t - 1)^l x^n \right\rangle$$

$$= \sum_{l=k}^{n} S_2(l,k) \left\langle e^{rt} | \sum_{m=l}^{\infty} S_2(m,l) \frac{t^m}{m!} x^n \right\rangle \qquad (3.24)$$

$$= \sum_{l=k}^{n} S_2(l,k) \sum_{m=l}^{n} \binom{n}{m} S_2(m,l) \left\langle e^{rt} | x^{n-m} \right\rangle$$

$$= \sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) r^{n-m}.$$

Thus we have

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) r^{n-m} \right) (x)_k. \qquad (3.25)$$

As in (3.24), we let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k}(x)_k$. But here we compute the coefficients $C_{n,k}$ in a way different from (3.24). Then

$$C_{n,k} = \frac{1}{k!} \left\langle e^{rt} | (e^{e^t - 1} - 1)^k x^n \right\rangle$$

$$= \frac{1}{k!} \left\langle e^{rt} | \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{l(e^t - 1)} x^n \right\rangle$$

$$= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\langle e^{rt} | \sum_{m=0}^{\infty} Bel_m(l) \frac{t^m}{m!} x^n \right\rangle \qquad (3.26)$$

$$= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sum_{m=0}^{n} \binom{n}{m} Bel_m(l) r^{n-m}$$

$$= \frac{1}{k!} \sum_{l=0}^{k} \sum_{m=0}^{n} (-1)^{k-l} \binom{k}{l} \binom{n}{m} r^{n-m} Bel_m(l).$$

Hence we obtain

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-l}}{k!} \binom{k}{l} \binom{n}{m} r^{n-m} Bel_m(l) \right) (x)_k. \qquad (3.27)$$

Combining (3.25) and (3.27), we get the following theorem.

Theorem 3.6. *For all integers $n \geq 0$, we have the following expressions.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) r^{n-m} \right) (x)_k$$

$$= \sum_{k=0}^{n} \left(\sum_{l=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-l}}{k!} \binom{k}{l} \binom{n}{m} r^{n-m} Bel_m(l) \right) (x)_k.$$

We recall here that the Abel polynomial $A_n(x;a)(a \neq 0)$ is the associated sequence for te^{at}, namely $A_n(x;a) \sim (1, te^{at})$. Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} A_k(x;a)$. Then

$$C_{n,k} = \left\langle e^{rt} e^{ak(e^t-1)} | \frac{1}{k!} (e^t-1)^k x^n \right\rangle$$

$$= \left\langle e^{rt} e^{ak(e^t-1)} | \sum_{l=k}^{\infty} S_2(l,k) \frac{t^l}{l!} x^n \right\rangle$$

$$= \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \left\langle e^{rt} | e^{ak(e^t-1)} x^{n-l} \right\rangle$$

$$= \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \left\langle e^{rt} | \sum_{m=0}^{\infty} Bel_m(ak) \frac{t^m}{m!} x^{n-l} \right\rangle \qquad (3.28)$$

$$= \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \sum_{m=0}^{n-l} \binom{n-l}{m} Bel_m(ak) r^{n-l-m}$$

$$= \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_2(l,k) r^{n-l-m} Bel_m(ak).$$

Thus we have the following result.

Theorem 3.7. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_2(l,k) r^{n-l-m} Bel_m(ak) \right) A_k(x;a),$$

where $A_n(x;a)$ are the Abel polynomials.

The ordered Bell polynomials $Ob_n(x)$ are the Appell polynomial with $Ob_n(x) \sim (2 - e^t, t)$. Write $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} Ob_k(x)$. Then

$$C_{n,k} = \left\langle 2 - e^{e^t-1} | \frac{1}{k!} e^{rt} (e^t-1)^k x^n \right\rangle$$

$$= \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \left\langle 2 - e^{e^t-1} | x^{n-l} \right\rangle \qquad (3.29)$$

$$= \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) (2\delta_{n,l} - Bel_{n-l}).$$

Hence we obtain the following theorem.

Theorem 3.8. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\big(2\delta_{n,l} - Bel_{n-l}\big)\right)Ob_k(x),$$

where $Ob_n(x)$ are the ordered Bell polynomials.

In (3.29), we saw that the ordered Bell polynomials $Ob_m(x)$ are given by generating function

$$\frac{1}{2-e^t}e^{xt} = \sum_{m=0}^{\infty}Ob_m(x)\frac{t^m}{m!}. \tag{3.30}$$

More generally, the ordered Bell polynomials $Ob_m^{(\alpha)}(x)$ of order α are defined by

$$\left(\frac{1}{2-e^t}\right)^{\alpha}e^{xt} = \sum_{m=0}^{\infty}Ob_m^{(\alpha)}(x)\frac{t^m}{m!}. \tag{3.31}$$

For $x = 0$, $Ob_m^{(\alpha)} = Ob_m^{(\alpha)}(0)$ are called the ordered Bell numbers of order α and given by

$$\left(\frac{1}{2-e^t}\right)^{\alpha} = \sum_{m=0}^{\infty}Ob_m^{(\alpha)}\frac{t^m}{m!}. \tag{3.32}$$

Let $Bel_{n,r}(x) = \sum_{k=0}^{n}C_{n,k}L_k^{(\alpha)}(x)$. Here $L_n^{(\alpha)}(x)$ are the Laguerre polynomials of order α with $L_n^{(\alpha)}(x) \sim \left((1-t)^{-\alpha-1}, \frac{t}{t-1}\right)$. Then

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle(2-e^t)^{-\alpha-1}e^{rt}\left(\frac{e^t-1}{e^t-2}\right)^k\bigg|x^n\right\rangle\\
&= (-1)^k\left\langle(2-e^t)^{-(k+\alpha+1)}\bigg|\frac{1}{k!}e^{rt}(e^t-1)^k x^n\right\rangle\\
&= (-1)^k\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\left\langle(2-e^t)^{-(k+\alpha+1)}\bigg|x^{n-l}\right\rangle\\
&= (-1)^k\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)Ob_{n-l}^{(k+\alpha+1)}.
\end{aligned} \tag{3.33}$$

Then we have the following theorem.

Theorem 3.9. *For all integers $n \geq 0$, we have the following expression.*

$$\begin{aligned}
Bel_{n,r}(x) = \sum_{k=0}^{n}\bigg(\sum_{l=k}^{n}(-1)^k\binom{n}{l}S_{2,r}(l+r,k+r)\\
\times Ob_{n-l}^{(k+\alpha+1)}\bigg)L_k^{(\alpha)}(x),
\end{aligned}$$

where $Ob_n^{(\alpha)}(x)$ and $L_n^{(\alpha)}(x)$ are the higher-order ordered Bell polynomials and the Laguerre polynomials of order α, respectively.

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} D_k(x)$, where $D_n(x)$ are the Daehee polynomials with $D_n(x) \sim \left(\frac{e^t-1}{t}, e^t - 1\right)$. Then

$$
\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle \frac{t}{e^t-1}e^{rt}\frac{e^{e^t-1}-1}{t}\left(e^{e^t-1}-1\right)^k \Big| x^n \right\rangle \\
&= \frac{1}{k!}\frac{1}{n+1}\left\langle \frac{t}{e^t-1}e^{rt}(e^{e^t-1}-1)\left(e^{e^t-1}-1\right)^k \Big| x^{n+1} \right\rangle \\
&= \frac{k+1}{n+1}\left\langle \frac{t}{e^t-1}e^{rt} \Big| \frac{1}{(k+1)!}\left(e^{e^t-1}-1\right)^{k+1} x^{n+1} \right\rangle \\
&= \frac{k+1}{n+1}\left\langle \frac{t}{e^t-1}e^{rt} \Big| \sum_{l=k+1}^{\infty} S_2(l, k+1)\frac{1}{l!}\left(e^t-1\right)^l x^{n+1} \right\rangle \\
&= \frac{k+1}{n+1}\sum_{l=k+1}^{n+1} S_2(l, k+1)\left\langle \frac{t}{e^t-1}e^{rt} \Big| \sum_{m=l}^{\infty} S_2(m, l)\frac{t^m}{m!} x^{n+1} \right\rangle \\
&= \frac{k+1}{n+1}\sum_{l=k+1}^{n+1} S_2(l, k+1)\sum_{m=l}^{n+1} \binom{n+1}{m} S_2(m, l)\left\langle \frac{t}{e^t-1}e^{rt} \Big| x^{n+1-m} \right\rangle \\
&= \sum_{l=k+1}^{n+1}\sum_{m=l}^{n+1} \frac{k+1}{n+1}\binom{n+1}{m} S_2(l, k+1)S_2(m, l)B_{n+1-m}(r).
\end{aligned}
$$

(3.34)

Thus we have the following theorem.

Theorem 3.10. *For all integers $n \geq 0$, we have the following expression.*

$$
Bel_{n,r}(x) = \sum_{k=0}^{n}\left(\sum_{l=k+1}^{n+1}\sum_{m=l}^{n+1}\frac{k+1}{n+1}\binom{n+1}{m} S_2(l, k+1)S_2(m, l)B_{n+1-m}(r)\right) D_k(x),
$$

where $D_n(x)$ are the Daehee polynomials.

Write $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} H_k^{(\nu)}(x)$. Here $H_n^{(\nu)}(x)$ are the Hermite polynomials with $H_n^{(\nu)}(x) \sim (e^{\frac{\nu t^2}{2}}, t)$. Then

$$
\begin{aligned}
C_{n,k} &= \left\langle e^{\frac{\nu(e^t-1)^2}{2}} \Big| \frac{1}{k!}e^{rt}\left(e^t-1\right)^k x^n \right\rangle \\
&= \sum_{l=k}^{n}\binom{n}{l} S_{2,r}(l+r, k+r)\left\langle e^{\frac{\nu(e^t-1)^2}{2}} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=k}^{n}\binom{n}{l} S_{2,r}(l+r, k+r)\left\langle \sum_{m=0}^{\infty}\frac{\nu^m}{m!}\frac{(e^t-1)^{2m}}{2^m} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=k}^{n}\binom{n}{l} S_{2,r}(l+r, k+r)\sum_{m=0}^{\left[\frac{n-l}{2}\right]}\frac{(2m)!\nu^m}{m!2^m}\left\langle \frac{1}{(2m)!}(e^t-1)^{2m} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=k}^{n}\binom{n}{l} S_{2,r}(l+r, k+r)\sum_{m=0}^{\left[\frac{n-l}{2}\right]}\frac{(2m)!}{m!}(\frac{\nu}{2})^m\left\langle \sum_{i=2m}^{\infty} S_2(i, 2m)\frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=k}^{n}\sum_{m=0}^{\left[\frac{n-l}{2}\right]}\binom{n}{l}\frac{(2m)!}{m!}(\frac{\nu}{2})^m S_{2,r}(l+r, k+r)S_2(n-l, 2m).
\end{aligned}
$$

(3.35)

Hence we obtain the following result.

Theorem 3.11. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=0}^{n} \sum_{m=0}^{[\frac{n-l}{2}]} \binom{n}{l} \frac{(2m)!}{m!} (\frac{\nu}{2})^m S_{2,r}(l+r, k+r) \right.$$

$$\left. \times S_2(n-l, 2m) \right) H_k^{(\nu)}(x),$$

where $H_n^{(\nu)}(x)$ are the Hermite polynomials.

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} p_k(x)$. Here $p_n(x) = x^n y_{n-1}(\frac{1}{x}) \sim (1, t - \frac{1}{2}t^2)$, where $y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k$ are called Bessel polynomials and satisfy the differential equation

$$x^2 y'' + (2x+2)y' + n(n+1)y = 0. \tag{3.36}$$

$$C_{n,k} = \left(-\frac{1}{2}\right)^k \left\langle (e^t - 3)^k \Big| \frac{1}{k!} e^{rt} (e^t - 1)^k x^n \right\rangle$$

$$= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \left\langle (e^t - 3)^k \Big| x^{n-l} \right\rangle$$

$$= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \left\langle \sum_{m=0}^{k} \binom{k}{m} (-3)^{k-m} e^{mt} \Big| x^{n-l} \right\rangle \tag{3.37}$$

$$= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \sum_{m=0}^{k} \binom{k}{m} (-3)^{k-m} m^{n-l}$$

$$= \left(\frac{3}{2}\right)^k \sum_{l=k}^{n} \sum_{m=0}^{k} \binom{n}{l} \binom{k}{m} (-\frac{1}{3})^m m^{n-l} S_{2,r}(l+r, k+r).$$

Hence we have the following result.

Theorem 3.12. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=0}^{k} \binom{n}{l} \binom{k}{m} \left(\frac{3}{2}\right)^k (-\frac{1}{3})^m m^{n-l} \right.$$

$$\left. \times S_{2,r}(l+r, k+r) \right) p_k(x),$$

where $p_n(x) = x^n y_{n-1}(\frac{1}{x})$, with $y_n(x)$ the Bessel polynomials.

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} b_k(x)$, where $b_n(x)$ are the Bernoulli polynomials of the second kind with $b_n(x) \sim \left(\frac{t}{e^t - 1}, e^t - 1 \right)$.

$$
\begin{aligned}
C_{n,k} &= \left\langle \frac{e^t - 1}{e^{e^t - 1} - 1} e^{rt} \Big| \frac{1}{k!} (e^{e^t - 1} - 1)^k x^n \right\rangle \\
&= \left\langle \frac{e^t - 1}{e^{e^t - 1} - 1} e^{rt} \Big| \sum_{l=k}^{\infty} S_2(l,k) \frac{1}{l!} (e^t - 1)^l x^n \right\rangle \\
&= \sum_{l=k}^{n} S_2(l,k) \left\langle \frac{e^t - 1}{e^{e^t - 1} - 1} e^{rt} \Big| \sum_{m=l}^{\infty} S_2(m,l) \frac{t^m}{m!} x^n \right\rangle \\
&= \sum_{l=k}^{n} S_2(l,k) \sum_{m=l}^{n} \binom{n}{m} S_2(m,l) \left\langle e^{rt} \Big| \frac{e^t - 1}{e^{e^t - 1} - 1} x^{n-m} \right\rangle \\
&= \sum_{l=k}^{n} S_2(l,k) \sum_{m=l}^{n} \binom{n}{m} S_2(m,l) \left\langle e^{rt} \Big| \sum_{i=0}^{\infty} B_i \frac{1}{i!} (e^t - 1)^i x^{n-m} \right\rangle \qquad (3.38) \\
&= \sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) \sum_{i=0}^{n-m} B_i \left\langle e^{rt} \Big| \sum_{j=i}^{\infty} S_2(j,i) \frac{t^j}{j!} x^{n-m} \right\rangle \\
&= \sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) \sum_{i=0}^{n-m} B_i \sum_{j=i}^{n-m} \binom{n-m}{j} S_2(j,i) r^{n-m-j} \\
&= \sum_{l=k}^{n} \sum_{m=l}^{n} \sum_{i=0}^{n-m} \sum_{j=i}^{n-m} \binom{n}{m} \binom{n-m}{j} S_2(l,k) S_2(m,l) S_2(j,i) r^{n-m-j} B_i.
\end{aligned}
$$

Thus we get the final result of this paper.

Theorem 3.13. *For all integers $n \geq 0$, we have the following expression.*

$$
\begin{aligned}
Bel_{n,r}(x) = \sum_{k=0}^{n} \Bigg(& \sum_{l=k}^{n} \sum_{m=l}^{n} \sum_{i=0}^{n-m} \sum_{j=i}^{n-m} \binom{n}{m} \binom{n-m}{j} \\
& \times S_2(l,k) S_2(m,l) S_2(j,i) r^{n-m-j} B_i \Bigg) b_k(x),
\end{aligned}
$$

where $b_n(x)$ are the Bernoulli polynomials of the second kind.

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