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Journal of Computational Analysis and Applications

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SHARP INEQUALITIES BETWEEN TOADER AND NEUMAN MEANS*

WEI-MAO QIAN[1], ZAI-YIN HE[2], AND YU-MING CHU[3,**]

Abstract. In the article, we prove that the double inequalities

$$\alpha_1 Q(a,b)+(1-\alpha_1)N_{GA}(a,b)<T(a,b)<\beta_1 Q(a,b)+(1-\beta_1)N_{GA}(a,b),$$
$$\alpha_2 Q(a,b)+(1-\alpha_2)N_{QA}(a,b)<T(a,b)<\beta_2 Q(a,b)+(1-\beta_2)N_{QA}(a,b),$$
$$\alpha_3 C(a,b)+(1-\alpha_3)N_{GA}(a,b)<T(a,b)<\beta_3 C(a,b)+(1-\beta_3)N_{GA}(a,b),$$
$$\alpha_4 C(a,b)+(1-\alpha_4)N_{QA}(a,b)<T(a,b)<\beta_4 C(a,b)+(1-\beta_4)N_{QA}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 5/8$, $\beta_1\geq(16-\pi^2)/[(4\sqrt{2}-\pi)\pi]=0,7758\cdots$, $\alpha_2\leq 1/4$, $\beta_2\geq 1-2(\sqrt{2}\pi-4)/[(\sqrt{2}-\log(1+\sqrt{2}))\pi]=0.4708\cdots$, $\alpha_3\leq 5/14=0.3571\cdots$, $\beta_3\geq(16-\pi^2)/[(8-\pi)\pi]=0.4016\cdots$, $\alpha_4\leq 1/10$ and $\beta_4\geq 1-4(\pi-2)/[(4-\sqrt{2}-\log(1+\sqrt{2}))\pi]=0.1472\cdots$, where $Q(a,b)$, $C(a,b)$ and $T(a,b)$ are respectively the quadratic, contra-harmonic and Toader means, and $N_{GA}(a,b)$ and $N_{QA}(a,b)$ are the Neuman means.

1. Introduction

Let $p\in\mathbb{R}$, $r\in(0,1)$ and $a,b>0$ with $a\neq b$. Then the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1-32] of the first and second kinds, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$, contra-harmonic mean $C(a,b)$, second contra-harmonic mean $\overline{C}(a,b)$, centroidal mean $\widetilde{C}(a,b)$, Toader mean $T(a,b)$ [33-36], pth power mean $M_p(a,b)$ [37-43], and Schwab-Borchardt mean $SB(a,b)$ [44-48] of a and b are given by

$$\mathcal{K}(r)=\int_0^{\pi/2}\left(1-r^2\sin^2 t\right)^{-1/2}dt,\quad \mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt,$$

$$G(a,b)=\sqrt{ab},\quad A(a,b)=\frac{a+b}{2},\quad Q(a,b)=\sqrt{\frac{a^2+b^2}{2}},$$

$$C(a,b)=\frac{a^2+b^2}{a+b},\quad \overline{C}(a,b)=\frac{a^3+b^3}{a^2+b^2},\quad \widetilde{C}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)},$$

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2(t)+b^2\sin^2(t)}dt,$$
$$=\begin{cases}2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a>b,\\ 2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a<b,\end{cases}\tag{1.1}$$

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Key words and phrases. Toader mean, Neuman mean, geometric mean, arithmetic mean, quadratic mean, contra-harmonic mean.

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$$M_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0\end{cases}$$

and

$$SB(a,b)=\begin{cases}\frac{\sqrt{b^2-a^2}}{\arccos{(a/b)}}, & a<b,\\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}{(a/b)}}, & a>b,\end{cases}$$

respectively, where $\cosh^{-1}(x)=\log(x+\sqrt{x^2-1})$ is the inverse hyperbolic cosine functions.

Recently, the bivariate means have attracted the attention of many researchers [49-82]. Neuman [83] introduced the Neuman mean

$$N(a,b)=\frac{1}{2}\left[a+\frac{b^2}{SB(a,b)}\right],$$

provided the explicit formulae for $N_{AG}(a,b)(a,b)$, $N_{GA}(a,b)$, $N_{AQ}(a,b)$ and $N_{QA}(a,b)$ as follows

$$N_{AG}(a,b)=:N[A(a,b),G(a,b)]=\frac{1}{2}A(a,b)\left[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right],$$

$$N_{GA}(a,b)=:N[G(a,b),A(a,b)]=\frac{1}{2}A(a,b)\left[\sqrt{1-v^2}+\frac{\arcsin(v)}{v}\right], \tag{1.2}$$

$$N_{AQ}(a,b)=:N[A(a,b),Q(a,b)]=\frac{1}{2}A(a,b)\left[1+(1+v^2)\frac{\arctan(v)}{v}\right],$$

$$N_{QA}(a,b)=:N[Q(a,b),A(a,b)]=\frac{1}{2}A(a,b)\left[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v}\right], \tag{1.3}$$

where $v=(a-b)/(a+b)$, $\tanh^{-1}(x)=\log[(1+x)/(1-x)]/2$ and $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ are the inverse hyperbolic tangent and sine functions, respectively.

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$ and the inequalities

$$\begin{aligned}G(a,b)=M_0(a,b)<A(a,b)=M_1(a,b)<\widetilde{C}(a,b)\\ <Q(a,b)=M_2(a,b)<C(a,b)<\overline{C}(a,b)\end{aligned} \tag{1.4}$$

hold for all $a,b>0$ with $a\neq b$.

Barnard, Pearce and Richards [84], and Alzer and Qiu [85] proved that the double inequality

$$M_{3/2}(a,b)<T(a,b)<M_{\log 2/\log(\pi/2)}(a,b)$$

holds all $a,b>0$ with $a\neq b$.

In [86], the authors stated that the double inequality

$$\alpha Q(a,b)+(1-\alpha)A(a,b)<T(a,b)<\beta Q(a,b)+(1-\beta)A(a,b) \tag{1.5}$$

is valid for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 1/2$ and $\beta\geq(4-\pi)/[(\sqrt{2}-1)\pi]=0.6596\cdots$.

Neuman [83] presented the inequalities

$$\begin{aligned}G(a,b)<N_{AG}(a,b)<N_{GA}(a,b)<A(a,b)\\ <N_{QA}(a,b)<N_{AQ}(a,b)<Q(a,b),\end{aligned} \tag{1.6}$$

$$\alpha_1A(a,b)+(1-\alpha_1)G(a,b)<N_{GA}(a,b)<\beta_1A(a,b)+(1-\beta_1)G(a,b),$$

$$\alpha_2Q(a,b)+(1-\alpha_2)A(a,b)<N_{AQ}(a,b)<\beta_2Q(a,b)+(1-\beta_2)A(a,b),$$

$$\alpha_3A(a,b)+(1-\alpha_3)G(a,b)<N_{AG}(a,b)<\beta_3A(a,b)+(1-\beta_3)G(a,b),$$

$$\alpha_4Q(a,b)+(1-\alpha_4)A(a,b)<N_{QA}(a,b)<\beta_4Q(a,b)+(1-\beta_4)A(a,b)$$

for all $a,b>0$ with $a\neq b$ if $\alpha_1\leq 2/3$, $\beta_1\geq\pi/4$, $\alpha_2\leq 2/3$, $\beta_2\geq(\pi-2)/[4(\sqrt{2}-1)]=0.6890\cdots$, $\alpha_3\leq 1/3$, $\beta_3\geq 1/2$, $\alpha_4\leq 1/3$, $\beta_4\geq(\log(1+\sqrt{2})+\sqrt{2}-2)/[2(\sqrt{2}-1)]=0.3568\cdots$.

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Li, Qian and Chu [87] proved that the double inequalities

$$\alpha N_{AQ}(a,b)+(1-\alpha)A(a,b)<T(a,b)<\beta N_{AQ}(a,b)+(1-\beta)A(a,b),$$

$$Q[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a]<T(a,b)<Q[\mu a+(1-\mu)b,\mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 3/4$ and $\beta\geq 4(4-\pi)/[\pi(\pi-2)]=0.9753\cdots$, $\lambda\leq 1/2+\sqrt{2}/4=0.8535\cdots$ and $\mu\geq 1/2+\sqrt{16/\pi^2-1}/2=0.8940\cdots$ if $\lambda,\mu\in(1/2,1)$.

Qian, Song, Zhang and Chu [88] proved that the two-sided inequalities

$$\lambda_1\overline{C}(a,b)+(1-\lambda_1)A(a,b)<T(a,b)<\mu_1\overline{C}(a,b)+(1-\mu_1)A(a,b)$$

$$\overline{C}[\lambda_2 a+(1-\lambda_2)b,\lambda_2 b+(1-\lambda_2)a]<T(a,b)<\overline{C}[\mu_2 a+(1-\mu_2)b,\mu_2 b+(1-\mu_2)a]$$

are valid for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1\leq 1/8$, $\mu_1\geq 4/\pi-1=0.2732\cdots$, $\lambda_2\leq 1/2+\sqrt{2}/8=0.6767\cdots$ and $\mu_2\geq 1/2+\sqrt{(4-\pi)/(3\pi-4)}/2=0.6988\cdots$ if $\lambda_2,\mu_2\in(1/2,1)$.

In [89], Song, Qian and Chu found that the inequalities

$$\alpha_1A(a,b)+(1-\alpha_1)\widetilde{C}(a,b)<N_{QA}(a,b)<\beta_1A(a,b)+(1-\beta_1)\widetilde{C}(a,b), \tag{1.7}$$

$$A^{\alpha_2}(a,b)\widetilde{C}^{1-\alpha_2}(a,b)<N_{QA}(a,b)<A^{\beta_2}(a,b)\widetilde{C}^{1-\beta_2}(a,b),$$

$$\widetilde{C}[\alpha_3a+(1-\alpha_3)b,\alpha_3b+(1-\alpha_3)a]<N_{QA}(a,b)<\widetilde{C}[\beta_3a+(1-\beta_3)b,\beta_3b+(1-\beta_3)a]$$

take place if and only if $\alpha_1\geq 4-3[\sqrt{2}+\log(1+\sqrt{2})]/2=0.5566\cdots$, $\beta_1\leq 1/2$, $\alpha_2\geq 1-[\log(\sqrt{2}+\log(1+\sqrt{2}))-\log 2]/(2\log 2-\log 3)=0.5208\cdots$, $\beta_2\leq 1/2$, $\beta_3\geq 1/2+\sqrt{2}/4=0.8535\cdots$ and $\alpha_3\leq 1/2+\sqrt{6[\sqrt{2}+\log(1+\sqrt{2})]-12}/4=0.8329\cdots$ if $\alpha_3,\beta_3\in(1/2,1)$.

From (1.4)-(1.7) we clearly see that the inequalities

$$N_{GA}(a,b)<N_{QA}(a,b)<\frac{1}{2}A(a,b)+\frac{1}{2}\widetilde{C}(a,b) \tag{1.8}$$

$$<\frac{1}{2}A(a,b)+\frac{1}{2}Q(a,b)<T(a,b)<Q(a,b)<C(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

Motivated by inequality (1.8), in the article we deal with the optimality of the parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 such that the double inequalities

$$\alpha_1Q(a,b)+(1-\alpha_1)N_{GA}(a,b)<T(a,b)<\beta_1Q(a,b)+(1-\beta_1)N_{GA}(a,b),$$

$$\alpha_2Q(a,b)+(1-\alpha_2)N_{QA}(a,b)<T(a,b)<\beta_2Q(a,b)+(1-\beta_2)N_{QA}(a,b),$$

$$\alpha_3C(a,b)+(1-\alpha_3)N_{GA}(a,b)<T(a,b)<\beta_3C(a,b)+(1-\beta_3)N_{GA}(a,b),$$

$$\alpha_4C(a,b)+(1-\alpha_4)N_{QA}(a,b)<T(a,b)<\beta_4C(a,b)+(1-\beta_4)N_{QA}(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main results, we need several formulas and lemmas which we present in this section.

The following formulas for $\mathcal{K}(r)$ and $\mathcal{E}(r)$ can be found in the literature [90]:

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)},\quad \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r},$$

$$\frac{d\left[\mathcal{K}(r)-\mathcal{E}(r)\right]}{dr}=\frac{r\mathcal{E}(r)}{1-r^2},\quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{1+r},$$

$$\mathcal{K}(0^+)=\mathcal{E}(0^+)=\frac{\pi}{2},\quad \mathcal{K}(1^-)=\infty,\quad \mathcal{E}(1^-)=1.$$

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Lemma 2.1. *(See [90, Theorem 1.25]) Let $-\infty < a < b < \infty$, $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *The following statements are true:*

(1) The function $r \mapsto \left[\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, 1)$;

(2) The function $r \mapsto \mathcal{K}(r)$ is strictly increasing from $(0, 1)$ onto $(\pi/2, \infty)$;

(3) The function $r \mapsto [\mathcal{K}(r) - \mathcal{E}(r)]/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, +\infty)$;

(4) The function $r \mapsto \phi(r) = \left[3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)\right]/\sqrt{1+r^2}$ is strictly increasing from $(0, 1)$ onto $(\pi/2, 3\sqrt{2}/2)$.

Proof. Parts (1)-(3) can be found in [8, Theorem 3.21(1), (2) and Exercise 3.43(11)]. For part (4), it is not difficult to verify that

$$\phi(0^+) = \frac{\pi}{2}, \qquad \phi(1^+) = \frac{3\sqrt{2}}{2}, \tag{2.1}$$

$$\begin{aligned}\phi'(r) &= \frac{\mathcal{E}(r) - 2r^2\mathcal{E}(r) - \mathcal{K}(r) + 3r^2\mathcal{K}(r)}{r(1+r^2)^{3/2}} \\ &= \frac{r}{(1+r^2)^{3/2}}\left[\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}\right] + \frac{2r^3}{(1+r^2)^{3/2}}\left[\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}\right].\end{aligned} \tag{2.2}$$

It follows from (2.2) together with Lemma 2.2(1) and (3) that

$$\phi'(r) > 0 \tag{2.3}$$

for $r \in (0, 1)$.

Therefore, part (4) follows from (2.1) and (2.3). □

Lemma 2.3. *The function*

$$\varphi(r) = \frac{2r^2 + 1 - \frac{2}{\pi}\sqrt{1+r^2}\left[3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)\right]}{r^2}$$

is strictly decreasing from $(0, 1)$ onto $(3 - 6\sqrt{2}/\pi, 3/4)$.

Proof. Let $\varphi_1(r) = 2r^2 + 1 - 2\sqrt{1+r^2}\left[3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)\right]/\pi$, $\varphi_2(r) = r^2$. Then simple computations lead to

$$\varphi_1(0^+) = \varphi_2(0^+) = 0, \quad \varphi(r) = \frac{\varphi_1(r)}{\varphi_2(r)}, \tag{2.4}$$

$$\varphi(1^-) = 3 - \frac{6\sqrt{2}}{\pi}, \tag{2.5}$$

$$\frac{\varphi_1'(r)}{\varphi_2'(r)} = 2 - \frac{1}{\pi}\left\{\frac{3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)}{\sqrt{1+r^2}} + \sqrt{1+r^2}\left[\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2} + \mathcal{K}(r)\right]\right\}. \tag{2.6}$$

It is not difficult to verify that the function $r \mapsto \sqrt{1+r^2}$ is strictly increasing on $(0, 1)$. Then it follows from Lemma 2.2(1), (2) and (4) together with (2.6) that $\varphi_1'(r)/\varphi_2'(r)$ is strictly decreasing on $(0, 1)$ and

$$\varphi(0^+) = \lim_{r\to 0^+}\frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{3}{4}. \tag{2.7}$$

Therefore, Lemma 2.3 follows from Lemma 2.1, (2.4), (2.5) and (2.7) together with the monotonicity of $\varphi_1'(r)/\varphi_2'(r)$. □

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Lemma 2.4. *The function*

$$\psi(r)=\frac{3r^2+1-\frac{2}{\pi}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]}{r^2}$$

is strictly decreasing from $(0,1)$ *onto* $(4-6/\pi, 9/4)$.

Proof. Let $\psi_1(r)=3r^2+1-2\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]/\pi$, $\psi_2(r)=r^2$. Then simple computations lead to

$$\psi_1(0^+)=\psi_2(0^+)=0,\quad \psi(r)=\frac{\psi_1(r)}{\psi_2(r)}, \tag{2.8}$$

$$\psi(1^-)=4-\frac{6}{\pi}, \tag{2.9}$$

$$\frac{\psi_1'(r)}{\psi_2'(r)}=3-\frac{1}{\pi}\left[\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}+\mathcal{K}(r)\right]. \tag{2.10}$$

From Lemma 2.2(1), (2) and (2.10) we know that $\psi_1'(r)/\psi_2'(r)$ is strictly decreasing on $(0,1)$ and

$$\psi(0^+)=\lim_{r\to 0^+}\frac{\psi_1'(r)}{\psi_2'(r)}=\frac{9}{4}. \tag{2.11}$$

Therefore, Lemma 2.4 follows from Lemma 2.1, (2.8), (2.9) and (2.11) together with the monotonicity of $\psi_1'(r)/\psi_2'(r)$. □

3. Main Results

Theorem 3.1. *The double inequality*

$$\alpha_1 Q(a,b)+(1-\alpha_1)N_{GA}(a,b)<T(a,b)<\beta_1 Q(a,b)+(1-\beta_1)N_{GA}(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1\le 5/8$ *and* $\beta_1\ge(16-\pi^2)/[\pi(4\sqrt{2}-\pi)]=0.7758\cdots$.

Proof. Since $Q(a,b)$, $N_{GA}(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $a>b$. Let $r=(a-b)/(a+b)\in(0,1)$. Then from (1.1) and (1.2) one has

$$T(a,b)=\frac{2}{\pi}A(a,b)\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right], \tag{3.1}$$

$$N_{GA}(a,b)=\frac{1}{2}A(a,b)\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]. \tag{3.2}$$

It follows from (3.1) and (3.2) together with $Q(a,b)=A(a,b)\sqrt{1+r^2}$ that

$$\frac{T(a,b)-N_{GA}(a,b)}{Q(a,b)-N_{GA}(a,b)}=\frac{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-\frac{1}{2}\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]}{\sqrt{1+r^2}-\frac{1}{2}\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]}$$

$$=1-\frac{2r\sqrt{1+r^2}-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r\sqrt{1+r^2}-r\sqrt{1-r^2}-\arcsin(r)}:=1-F(r). \tag{3.3}$$

Let $f_1(r)=2r\sqrt{1+r^2}-4r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_1(r)=2r\sqrt{1+r^2}-r\sqrt{1-r^2}-\arcsin(r)$. Then simple computations lead to

$$f_1(0^+)=g_1(0^+)=0,\quad F(r)=\frac{f_1(r)}{g_1(r)}, \tag{3.4}$$

$$\frac{f_1'(r)}{g_1'(r)}=\frac{2r^2+1-\frac{2}{\pi}\sqrt{1+r^2}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]}{2r^2-\sqrt{1-r^4}+1}$$

$$= \frac{\varphi(r)}{(2r^2 - \sqrt{1-r^4}+1)/r^2}, \tag{3.5}$$

where $\varphi(r)$ is defined as in Lemma 2.3.

It is easy to verify that the function $r \mapsto (2r^2 - \sqrt{1-r^4}+1)/r^2$ is positive and strictly increasing on (0, 1), then (3.5) and Lemma 2.3 lead to the conclusion that $f_1'(r)/g_1'(r)$ is strictly decreasing on (0,1). Hence from Lemma 2.1 and (3.4) we know that $F(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r\to 0^+} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r\sqrt{1+r^2} - r\sqrt{1-r^2} - \arcsin(r)} = \frac{3}{8}, \tag{3.6}$$

$$\lim_{r\to 1^-} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r\sqrt{1+r^2} - r\sqrt{1-r^2} - \arcsin(r)} = \frac{4(\sqrt{2}\pi - 4)}{\pi(4\sqrt{2}-\pi)}. \tag{3.7}$$

Therefore, Theorem 3.1 follows from (3.3), (3.6) and (3.7) together with the monotonicity of $F(r)$. □

Theorem 3.2. *The double inequality*

$$\alpha_2 Q(a,b) + (1-\alpha_2)N_{QA}(a,b) < T(a,b) < \beta_2 Q(a,b) + (1-\beta_2)N_{QA}(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_2 \leq 1/4$ *and* $\beta_2 \geq 1 - 2(\sqrt{2}\pi - 4)/\left[(\sqrt{2} - \log(1+\sqrt{2}))\pi\right] = 0.4708\cdots$.

Proof. Since $Q(a,b)$, $N_{QA}(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $a > b$. Let $r = (a-b)/(a+b) \in (0,1)$. Then from (1.4) we have

$$N_{QA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]. \tag{3.8}$$

It follows from (3.1) and (3.8) together with $Q(a,b) = A(a,b)\sqrt{1+r^2}$ that

$$\frac{T(a,b) - N_{QA}(a,b)}{Q(a,b) - N_{QA}(a,b)} = \frac{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - \frac{1}{2}\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]}{\sqrt{1+r^2} - \frac{1}{2}\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]}$$

$$= 1 - \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{r\sqrt{1+r^2} - \sinh^{-1}(r)} := 1 - G(r). \tag{3.9}$$

Let $f_1(r) = 2r\sqrt{1+r^2} - 4r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_2(r) = r\sqrt{1+r^2} - arcsinh(r)$. Then simple computations lead to

$$f_1(0^+) = g_2(0^+) = 0, \quad G(r) = \frac{f_1(r)}{g_2(r)}, \tag{3.10}$$

$$\frac{f_1'(r)}{g_2'(r)} = \frac{2r^2 + 1 - \frac{2}{\pi}\sqrt{1+r^2}\left[3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)\right]}{r^2} = \varphi(r), \tag{3.11}$$

where $\varphi(r)$ is defined as in Lemma 2.3.

It follows from Lemma 2.3 and (3.11) that $f_1'(r)/g_2'(r)$ is strictly decreasing on (0,1). Then Lemma 2.1 and (3.10) lead to the conclusion that $G(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r\to 0^+} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{r\sqrt{1+r^2} - \sinh^{-1}(r)} = \frac{3}{4}, \tag{3.12}$$

$$\lim_{r\to 1^-} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{r\sqrt{1+r^2} - \sinh^{-1}(r)} = \frac{2(\sqrt{2}\pi - 4)}{\left[\sqrt{2} - \log(1+\sqrt{2})\right]\pi}. \tag{3.13}$$

Therefore, Theorem 3.2 follows from (3.9), (3.12) and (3.13) together with the monotonicity of $G(r)$. □

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Theorem 3.3. *The double inequality*

$$\alpha_3 C(a,b) + (1-\alpha_3) N_{GA}(a,b) < T(a,b) < \beta_3 C(a,b) + (1-\beta_3) N_{GA}(a,b),$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_3 \leq 5/14$ *and* $\beta_3 \geq (16-\pi^2)/[\pi(8-\pi)] = 0.4016\cdots$.

Proof. Without loss of generality, we assume that $a > b$. Let $r = (a-b)/(a+b) \in (0,1)$. Then it follows from (3.1), (3.2) and $C(a,b) = A(a,b)(1+r^2)$ that

$$\frac{T(a,b) - N_{GA}(a,b)}{C(a,b) - N_{GA}(a,b)} = \frac{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - \frac{1}{2}\left[\sqrt{1-r^2} + \frac{\arcsin(r)}{r}\right]}{1 + r^2 - \frac{1}{2}\left[\sqrt{1-r^2} + \frac{\arcsin(r)}{r}\right]}$$

$$= 1 - \frac{2r(1+r^2) - \frac{4}{\pi} r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2) - r\sqrt{1-r^2} - \arcsin(r)} := 1 - H(r). \tag{3.14}$$

Let $f_2(r) = 2r(1+r^2) - 4r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_3(r) = 2r(1+r^2) - r\sqrt{1-r^2} - \arcsin(r)$. Then simple computations lead to

$$f_2(0^+) = g_3(0^+) = 0, \quad H(r) = \frac{f_2(r)}{g_3(r)}, \tag{3.15}$$

$$\frac{f_2'(r)}{g_3'(r)} = \frac{3r^2 + 1 - \frac{2}{\pi}\left[3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)\right]}{3r^2 - \sqrt{1-r^2} + 1}$$

$$= \frac{\psi(r)}{(3r^2 - \sqrt{1-r^2} + 1)/r^2}, \tag{3.16}$$

where $\psi(r)$ is defined as in Lemma 2.4.

It is easy to verify that the function $r \mapsto (3r^2 - \sqrt{1-r^2} + 1)/r^2$ is positive and strictly increasing on (0, 1). Then from Lemma 2.4 and (3.16) we know that $f_2'(r)/g_3'(r)$ is strictly decreasing on (0,1). Hence Lemma 2.1 and (3.15) lead to the conclusion that $H(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r \to 0^+} \frac{2r(1+r^2) - \frac{4}{\pi} r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2) - r\sqrt{1-r^2} - \arcsin(r)} = \frac{9}{14}, \tag{3.17}$$

$$\lim_{r \to 1^-} \frac{2r(1+r^2) - \frac{4}{\pi} r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2) - r\sqrt{1-r^2} - \arcsin(r)} = \frac{8(\pi-2)}{\pi(8-\pi)}. \tag{3.18}$$

Therefore, Theorem 3.3 follows from (3.14), (3.17) and (3.18) together with the monotonicity of $H(r)$. □

Theorem 3.4. *The double inequality*

$$\alpha_4 C(a,b) + (1-\alpha_4) N_{QA}(a,b) < T(a,b) < \beta_4 C(a,b) + (1-\beta_4) N_{QA}(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_4 \leq 1/10$ *and* $\beta_4 \geq 1 - 4(\pi-2)/\left[(4 - \sqrt{2} - \log(1+\sqrt{2}))\pi\right] = 0.1472$.

Proof. Without loss of generality, we assume that $a > b$. Let $r = (a-b)/(a+b) \in (0,1)$. Then it follows from (3.1), (3.8) and $C(a,b) = A(a,b)(1+r^2)$ that

$$\frac{T(a,b) - N_{QA}(a,b)}{C(a,b) - N_{QA}(a,b)} = \frac{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - \frac{1}{2}\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]}{1 + r^2 - \frac{1}{2}\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]}$$

$$= 1 - \frac{2r(1+r^2) - \frac{4}{\pi} r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2) - r\sqrt{1+r^2} - \sinh^{-1}(r)} := 1 - J(r). \tag{3.19}$$

Let $f_2(r)=2r(1+r^2)-4r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_4(r)=2r(1+r^2)-r\sqrt{1+r^2}-\sinh^{-1}(r)$. Then simple computations lead to

$$f_2(0^+)=g_4(0^+)=0,\quad J(r)=\frac{f_2(r)}{g_4(r)},\tag{3.20}$$

$$\begin{aligned}\frac{f_2'(r)}{g_4'(r)}&=\frac{3r^2+1-\frac{2}{\pi}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]}{3r^2-\sqrt{1+r^2}+1}\\&=\frac{\psi(r)}{(3r^2-\sqrt{1+r^2}+1)/r^2},\end{aligned}\tag{3.21}$$

where $\psi(r)$ is defined as in Lemma 2.4.

It is easy to verify that the function $r\mapsto(3r^2-\sqrt{1+r^2}+1)/r^2$ is positive and strictly increasing on (0,1). Then from Lemma 2.4 and (3.21) we know that $f_2'(r)/g_4'(r)$ is strictly decreasing on (0,1). Hence Lemma 2.1 and (3.20) lead to the conclusion that $J(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r\to0^+}\frac{2r(1+r^2)-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2)-r\sqrt{1+r^2}-\sinh^{-1}(r)}=\frac{9}{10},\tag{3.22}$$

$$\lim_{r\to1^-}\frac{2r(1+r^2)-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2)-r\sqrt{1+r^2}-\sinh^{-1}(r)}=\frac{4(\pi-2)}{\left[4-\sqrt{2}-\log(1+\sqrt{2})\right]\pi}.\tag{3.23}$$

Therefore, Theorem 3.4 follows from (3.19), (3.22) and (3.23) together with the monotonicity of $J(r)$. □

Let $r_0=\log(1+\sqrt{2})$, $r^*=r^2/(1+\sqrt{1-r^2})^2$. Then (1.1) and Theorems 3.1-3.4 lead to Corollary 3.5 immediately.

Corollary 3.5. *The double inequalities*

$$\frac{\pi}{64}\left[10\sqrt{2}\sqrt{2-r^2}+3(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}}+\frac{\arcsin(r^*)}{r^*}\right)\right]<\mathcal{E}(r)$$
$$<\frac{\sqrt{2}(16-\pi^2)}{4(4\sqrt{2}-\pi)}\sqrt{2-r^2}+\frac{\sqrt{2}\pi-4}{2(4\sqrt{2}-\pi)}(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}}+\frac{\arcsin(r^*)}{r^*}\right),$$
$$\frac{\pi}{32}\left[2\sqrt{2}\sqrt{2-r^2}+3(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}}+\frac{\sinh^{-1}(r^*)}{r^*}\right)\right]<\mathcal{E}(r)$$
$$<\frac{\sqrt{2}(8-\pi(\sqrt{2}+r_0))}{4(\sqrt{2}-r_0)}\sqrt{2-r^2}+\frac{\sqrt{2}\pi-4}{4(\sqrt{2}-r_0)}(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}}+\frac{\sinh^{-1}(r^*)}{r^*}\right),$$
$$\frac{\pi}{112}\left[\frac{20(2-r^2)}{1+\sqrt{1-r^2}}+9(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}}+\frac{\arcsin(r^*)}{r^*}\right)\right]<\mathcal{E}(r)$$
$$<\frac{16-\pi^2}{2(8-\pi)}\frac{2-r^2}{1+\sqrt{1-r^2}}+\frac{\pi-2}{8-\pi}(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}}+\frac{\arcsin(r^*)}{r^*}\right),$$
$$\frac{\pi}{80}\left[\frac{4(2-r^2)}{1+\sqrt{1-r^2}}+9(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}}+\frac{\sinh^{-1}(r^*)}{r^*}\right)\right]<\mathcal{E}(r)$$
$$<\frac{8-\pi(\sqrt{2}+r_0)}{2(4-\sqrt{2}-r_0)}\frac{2-r^2}{1+\sqrt{1-r^2}}+\frac{\pi-2}{2(4-\sqrt{2}-r_0)}(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}}+\frac{\sinh^{-1}(r^*)}{r^*}\right).$$

hold for all $r\in(0,1)$.

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4. Results and discussion

In the article, we present the best possible parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 such that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)N_{GA}(a,b) < T(a,b) < \beta_1 Q(a,b) + (1-\beta_1)N_{GA}(a,b),$$

$$\alpha_2 Q(a,b) + (1-\alpha_2)N_{QA}(a,b) < T(a,b) < \beta_2 Q(a,b) + (1-\beta_2)N_{QA}(a,b),$$

$$\alpha_3 C(a,b) + (1-\alpha_3)N_{GA}(a,b) < T(a,b) < \beta_3 C(a,b) + (1-\beta_3)N_{GA}(a,b),$$

$$\alpha_4 C(a,b) + (1-\alpha_4)N_{QA}(a,b) < T(a,b) < \beta_4 C(a,b) + (1-\beta_4)N_{QA}(a,b)$$

hold for all $a, b > 0$ with $a \neq b$. Our results are the improvements and refinements of the previously results.

5. Conclusion

We present several sharp bounds for the Toader mean in terms of the Neuman mean, quadratic mean and contraharmonic mean, and give new bounds for the complete elliptic integral of the second kind $\mathcal{E}(r)$. Our approach may have further applications in the theory of bivariate means and special functions.

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ON STRONGLY STARLIKENESS OF STRONGLY CONVEX FUNCTIONS

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Abstract. In this paper we introduce an argument property which gives an interesting relation between the classes of strongly convex and strongly starlike functions of order α and type β in the open unit disk. Also, the sufficient condition of starlikeness under certain restrictions is obtained.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The function $f(z)$ is called strongly starlike of order β and type α and strongly convex of order β and type α, respectively if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \tag{1.2}$$

and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta, \tag{1.3}$$

where $\alpha \in [0,1)$ and $\beta \in (0,1]$. We denote by $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$ the classes of functions satisfy the conditions (1.2) and (1.3) respectively. We note that both $S^*(\alpha,1) = S^*(\alpha)$ and $C(\alpha,1) = C(\alpha)$, are the well known classes of starlike functions of order α and convex functions of order α.

MacGregor [2] Wilken and Feng [5] obtained the following result:

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$$f(z) \in C(\alpha) \;\Rightarrow\; f(z) \in \; S^*(\beta) \qquad (0 \leq \alpha < 1),$$

where

$$\beta := \beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1}))}\ , & \alpha \neq \frac{1}{2} \\ \frac{1}{2\log 2}, & \alpha = \frac{1}{2}. \end{cases} \tag{1.4}$$

Also, Nunokawa *et al.*[4] investigated a certain relation between $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$. In the present paper, we obtain a relationship between strongly convex and strongly starlike functions by using the result given by Nunokawa [3].

In our investigation, we need the following lemma:

Lemma 1.1. [3] *Let $P(z)$ be analytic in $\mathbb{U}$, $P(0) = 1$, $P(z) \neq 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg(P(z_0))| = \frac{\pi}{2}\delta,$$

where $0 < \delta$. Then we have

$$\frac{z_0 P'(z_0)}{P(z_0)} = ik\delta,$$

where

$$k \geq \; \frac{1}{2}\left(a + \frac{1}{a}\right) \;\; when \;\; \arg(P(z_0)) = \frac{\pi}{2}\delta$$

and

$$k \leq -\; \frac{1}{2}\left(a + \frac{1}{a}\right) \;\; when \;\; \arg(P(z_0)) = -\frac{\pi}{2}\delta,$$

where $(P(z_0))^{1/\delta} = \pm ia$ and $a > 0$.

2. Main Result

Theorem 2.1. *Let $f(z)$ be analytic function defined by* (1.1) *and also, let*

$$f(z) \in C(\alpha, \gamma) \quad (z \in \mathbb{U}), \tag{2.5}$$

where $0 \leq \alpha < 1$ and $0 < \gamma < 1$.
Then

$$f(z) \in S(\beta, \delta) \quad (z \in \mathbb{U}), \tag{2.6}$$

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where

$$\gamma = \frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)a_0^{\delta-1}\left(a_0^2+1\right)}{2\left(\beta+(1-\beta)a_0^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a_0^{\delta}\right)}\right), \tag{2.7}$$

β is defined by (1.4), $0<\delta<1$ and a_0 is the positive root of the equation:

$$\begin{aligned}(\beta-\alpha)\beta\left((1+\delta)\,x^2-(1-\delta)\right)+x^{\delta}\,(1-\beta)\,(2\beta-\alpha)\left(x^2-1\right)\\ +\,x^{2\delta}\,(1-\beta)^2\left((1-\delta)\,x^2-(1+\delta)\right)=0,\end{aligned} \tag{2.8}$$

which satisfies

$$a_0^{\delta} \geq \left(\frac{\beta}{1-\beta}\left(\sqrt{\csc^2\left(\frac{\pi}{2}\delta\right)+\left(\frac{\beta-\alpha}{\beta}\right)}-\csc\left(\frac{\pi}{2}\delta\right)\right)\right)^{1/\delta} \tag{2.9}$$

Proof. Let

$$p(z)=\frac{z\ f'(z)}{f(z)},\ p(0)=1 \quad \text{and} \quad p(z)\neq\beta \quad (z\in\mathbb{U}).$$

Then we have

$$1+\frac{z\ f''(z)}{f'(z)}=p(z)+\frac{zp'(z)}{p(z)}.$$

If there exists $z_0\in\mathbb{U}$ such that

$$|\arg\left(P(z)\right)|=|\arg\left(p(z)-\beta\right)|<\frac{\pi}{2}\delta$$

for $|z|<|z_0|$ and

$$|\arg\left(P(z_0)\right)|=|\arg\left(p(z_0)-\beta\right)|=\frac{\pi}{2}\delta,$$

where

$$P(z)=\frac{p(z)-\beta}{1-\beta}\ .$$

Since $P(0)=1$ and by using Lemma 1.1, we have

$$\frac{z_0\ P'(z_0)}{P(z_0)}=\frac{z_0p'(z_0)}{p(z_0)-\beta}=i\delta k.$$

The first case, if

$$\arg\left(P(z_0)\right)=\arg\left(p(z_0)-\beta\right)=\frac{\pi}{2}\delta,$$

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then we have

$$\begin{aligned}
&\arg\left(1+\frac{z\ f''(z)}{f'(z)}-\alpha\right)\\
&=\arg\left((p(z_0)-\beta)\left(1+\frac{z_0p'(z_0)/p(z_0)}{p(z_0)-\beta}+\frac{\beta-\alpha}{p(z_0)-\beta}\right)\right)\\
&=\frac{\pi}{2}\delta+\arg\left(1+\frac{i\delta k}{\beta+(1-\beta)(ia)^\delta}+\frac{\beta-\alpha}{(1-\beta)(ia)^\delta}\right)\\
&=\frac{\pi}{2}\delta+\arg\left(1+\frac{i\delta k}{\beta+(1-\beta)a^\delta e^{i\frac{\pi}{2}\delta}}+\frac{\beta-\alpha}{(1-\beta)a^\delta e^{i\frac{\pi}{2}\delta}}\right)\\
&=\arg\left(e^{i\frac{\pi}{2}\delta}+\frac{i\delta k}{\beta e^{-i\frac{\pi}{2}\delta}+(1-\beta)a^\delta}+\frac{(\beta-\alpha)}{(1-\beta)a^\delta}\right)\\
&\geq\arctan\left(\frac{\frac{\delta k(1-\beta)a^\delta+\delta k\beta\cos\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}+\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}}\right).
\end{aligned}$$

Since the function $h(k)$ defined by

$$h(k)=\arctan\left(\frac{\frac{\delta k(1-\beta)a^\delta+\delta k\beta\ \ \cos\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}+\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}}\right)$$

is an increasing function of k $(k\geq 1)$, we have

$$\begin{aligned}
&\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)\\
&\geq\arctan\left(\frac{\frac{\left(\delta\ \ (1-\beta)a^\delta+\delta\beta\cos\left(\frac{\pi}{2}\delta\right)\right)(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}+\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta\sin\left(\frac{\pi}{2}\delta\right)(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}}\right).
\end{aligned}$$

Also, the function $f(\theta)$ defined by

$$f(\theta)=\arctan\left(\frac{\frac{\delta(1-\beta)a^\delta(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}+\frac{\delta\beta(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}\cos\theta+\sin\theta}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\theta-\frac{\beta\delta(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}\sin\theta}\right)$$

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is an increasing and continuous function of θ $(0 < \theta < \frac{\pi}{2})$ when a^δ satisfies (2.9). Therefore, we have

$$\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) \geq \arctan\left(\frac{\delta(1-\beta)\,(a+1/a)\,a^\delta}{2\,(\beta+(1-\beta)a^\delta)\,((\beta-\alpha)+(1-\beta)a^\delta)}\right). \tag{2.10}$$

On the other hand, since the function $g(x)$ defined by

$$g(x) = \frac{\delta(1-\beta)\left(x+\frac{1}{x}\right)x^\delta}{2\,(\beta+(1-\beta)x^\delta)\,((\beta-\alpha)+(1-\beta)x^\delta)} \qquad (x>0), \tag{2.11}$$

takes its minimum value when x is defined by (2.8), we see that this contradicts the hypothesis of Theorem 2.1.

The second case, if

$$\arg(P(z_0)) = \arg(p(z_0)-\beta) = -\frac{\pi}{2}\delta,$$

then we have

$$\begin{aligned}
&\arg\left(1 + \frac{z\,f''(z)}{f'(z)} - \alpha\right) \\
&= \arg\left((p(z_0)-\beta)\left(1 + \frac{z_0p'(z_0)/p(z_0)}{p(z_0)-\beta} + \frac{\beta-\alpha}{p(z_0)-\beta}\right)\right) \\
&= -\frac{\pi}{2}\delta + \arg\left(1 + \frac{i\delta k}{\beta+(1-\beta)a^\delta e^{-i\frac{\pi}{2}\delta}} + \frac{\beta-\alpha}{(1-\beta)a^\delta e^{-i\frac{\pi}{2}\delta}}\right) \\
&= \arg\left(e^{-i\frac{\pi}{2}\delta} + \frac{i\;\delta k}{\beta\;e^{i\frac{\pi}{2}\delta}+(1-\beta)a^\delta} + \frac{(\beta-\alpha)}{(1-\beta)a^\delta}\right) \\
&= \arctan\left(\frac{\frac{\delta k(1-\beta)a^\delta+\delta k\beta\cos\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2} - \sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta} + \cos\left(\frac{\pi}{2}\delta\right) + \frac{\beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}}\right).
\end{aligned}$$

Since the function $h(k)$ defined by

$$h(k) = \arctan\left(\frac{\delta k(1-\beta)a^\delta + \delta k\beta\cos\left(\frac{\pi}{2}\delta\right) - \sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta} + \cos\left(\frac{\pi}{2}\delta\right) + \beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}\right)$$

is a decreasing function of k $(k \leq -1)$, we have

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$$\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)$$
$$\leq \arctan\left(\frac{-\frac{\left(\delta(1-\beta)a^{\delta}+\delta\beta\cos\left(\frac{\pi}{2}\delta\right)\right)(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^2}-\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^{\delta}}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta\sin\left(\frac{\pi}{2}\delta\right)(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^2}}\right).$$

Also, the function $f(\theta)$ defined by

$$f(\theta)=-\arctan\left(\frac{\frac{\delta(1-\beta)a^{\delta}(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^2}+\frac{\delta\beta(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^2}\cos\theta+\sin\theta}{\frac{\beta-\alpha}{(1-\beta)a^{\delta}}+\cos\theta-\frac{\beta\delta(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^2}\sin\theta}\right)$$

is a decreasing and continuous function of θ $(0<\theta<\frac{\pi}{2})$, when a^{δ} satisfies (2.9). Therefore, we have

$$\arg\left(1+\frac{z\ f''(z)}{f'(z)}-\alpha\right)$$
$$\leq -\arctan\left(\frac{\delta(1-\beta)\left(a+\frac{1}{a}\right)a^{\delta}}{2\left(\beta+(1-\beta)a^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a^{\delta}\right)}\right).$$

Also, by using the function $g(x)$ defind by (2.11) which contradicts hypothesis of Theorem 2.1. Therefore, it completes the proof of the theorem. □

Putting $f(z)$ instead of $zf'(z)$ in Theorem 2.1, we have the following corollary

Corollary 2.1. *Let $f(z)$ be analytic function defined by* (1.1) *and also, let*

$$f(z)\in S^{*}(\alpha,\gamma)\quad(z\in\mathbb{U}), \tag{2.12}$$

where $0\leq\alpha<1$ and $0<\gamma<1$. Then

$$\left|\arg\left(\frac{f(z)}{A(z)}-\beta\right)\right|<\frac{\pi}{2}\delta\quad(z\in\mathbb{U}), \tag{2.13}$$

where $A(z)=\int_0^z(f(t)/t)dt$ is Alexander operator defined by Alexander [1],

$$\gamma=\frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)a_0^{\delta-1}\left(a_0^2+1\right)}{2\left(\beta+(1-\beta)a_0^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a_0^{\delta}\right)}\right), \tag{2.14}$$

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β is defined by (1.4), $0 < \delta < 1$ *and* a_0 *is the positive root of the equation:*

$$(\beta-\alpha)\beta\left((1+\delta)x^2-(1-\delta)\right)+x^{\delta}(1-\beta)(2\beta-\alpha)\left(x^2-1\right) + x^{2\delta}(1-\beta)^2\left((1-\delta)x^2-(1+\delta)\right)=0. \tag{2.15}$$

which satisfies

$$a_0^{\delta} \geq \left(\frac{\beta}{1-\beta}\left(\sqrt{\csc^2\left(\frac{\pi}{2}\delta\right)+\left(\frac{\beta-\alpha}{\beta}\right)}-\csc\left(\frac{\pi}{2}\delta\right)\right)\right)^{1/\delta}. \tag{2.16}$$

Corollary 2.2. *Let $f(z)$ be analytic function defined by* (1.1) *and also, let*

$$f(z) \in C(\alpha,\gamma) \quad (z \in \mathbb{U}), \tag{2.17}$$

where $0 \leq \alpha < 1$ and $0 < \gamma < 1$. Then

$$f(z) \in S(\beta,\delta) \quad (z \in \mathbb{U}), \tag{2.18}$$

where

$$\gamma = \frac{2}{\pi}\ \arctan\left(\frac{\delta\sqrt{\beta(\beta-\alpha)}}{\left(\beta+\sqrt{\beta(\beta-\alpha)}\right)\left((\beta-\alpha)+\sqrt{\beta(\beta-\alpha)}\right)}\right), \tag{2.19}$$

and β is defined by (1.4).

Proof. Let $f(z) \in C(\alpha,\gamma)$. Since the inequality (2.10) is satisfied when a^{δ} satisfies (2.9), we have

$$\begin{aligned}&\frac{\delta(1-\beta)(a+1/a)\,a^{\delta}}{2\left(\beta+(1-\beta)a^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a^{\delta}\right)}\\ &\geq \frac{\delta(1-\beta)a^{\delta}}{\left(\beta+(1-\beta)a^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a^{\delta}\right)}.\end{aligned}$$

Then the function $k(x)$ defined by

$$k(x) = \frac{\delta(1-\beta)x}{\left(\beta+(1-\beta)x\right)\left((\beta-\alpha)+(1-\beta)x\right)} \qquad (x>0)$$

takes its minimum value when $x = \frac{\sqrt{\beta(\beta-\alpha)}}{1-\beta}$.

On the other hand , we have

$$\frac{\sqrt{\beta(\beta-\alpha)}}{1-\beta} \geq \left(\frac{\beta}{1-\beta}\left(\sqrt{\csc^2\left(\frac{\pi}{2}\delta\right)+\left(\frac{\beta-\alpha}{\beta}\right)}-\csc\left(\frac{\pi}{2}\delta\right)\right)\right).$$

Hence we have $f(z) \in S(\beta,\delta)$. ∎

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Invariance analysis of a four-dimensional system of fourth-order difference equations with variable coefficients

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Abstract

A class of a four-dimensional system of difference equations is considered. A Lie symmetry analysis is performed and symmetries are derived. We use the differential invariant approach to obtain exact solutions. The link between the similarity variables and these symmetries is clearly given. Furthermore, we show the existence of periodic solutions for some specific coefficients. This work considerably extends some findings by El-Dessoky and Hobiny [M. M. El-Dessoky and A. Hobiny, J. Computational Analysis and Applications, **26:8** (2019), 1428–1439].

Keywords: System of difference equation; invariance analysis; group invariant solutions; periodicity
MSC: 39A11, 39A05

1 Introduction

The group theoretical approach for finding exact solutions to differential equations is now well reported [2, 14] and its application to difference equations has sparked interest recently [6–8, 10–13]. This approach, commonly known as Lie symmetry analysis, permits one to lower the order of the difference equations via a convenient choice of canonical coordinates obtained using a group of transformations admitted by the equation. Its application to higher dimensional system of difference equations is somewhat new and the calculation one deals with when finding symmetries in the latter can become cumbersome. Hydon in [10] extends the idea of Maeda [16] by developing a systematic algorithm permitting one to obtain the Lie algebra of a difference equation. Several authors have studied difference equations from different approaches and some interesting results can be found in [3–5, 17]

In this paper, inspired by the work in [1] where the authors study the behavior and existence of solutions of

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_n y_{n-3}z_{n-2}t_{n-1}} \\ z_{n+1} &= \frac{z_{n-3}}{\pm 1 \pm x_{n-1}y_n z_{n-3}t_{n-2}},\ t_{n+1} = \frac{t_{n-3}}{\pm 1 \pm x_{n-2}y_{n-1}z_n t_{n-3}}, \end{aligned} \tag{1}$$

we utilize Hydon's idea in a slightly modified manner to investigate the solutions to

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{a_n + b_n x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1} = \frac{y_{n-3}}{c_n + d_n x_n y_{n-3}z_{n-2}t_{n-1}} \\ z_{n+1} &= \frac{z_{n-3}}{e_n + f_n x_{n-1}y_n z_{n-3}t_{n-2}},\ t_{n+1} = \frac{t_{n-3}}{g_n + h_n x_{n-2}y_{n-1}z_n t_{n-3}}, \end{aligned} \tag{2}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$, $(f_n)_{n\in\mathbb{N}_0}$, $(g_n)_{n\in\mathbb{N}_0}$ and $(h_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers. The solutions of (2) are derived after a series of steps. Firstly, we obtain the Lie algebra of (2). We make use of point symmetries and additional assumptions on the characteristics to allow us derive analytic expressions for the symmetry generators. Secondly, we lower the order via the invariants and finally, find the solutions. We have showed that results in [1] are special cases of our findings.

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1.1 Preliminaries

In this section, we commence with some background necessary for understanding symmetry analysis. Note that throughout this paper, we utilize definitions and notation in [10, 14]. The notion of symmetry is strongly related to the notion of group transformations. Basically, it is a group of transformations that map a solution of a given equation onto another solution. Suppose G is a group of transformations acting on a manifold $\mathcal{M}$. Certain subsets $\mathcal{H}$ of this group, called $\mathcal{H}$-invariant, transform solutions onto themselves. Often times, for system of difference equations, the difference invariants of $\mathcal{H}$ are the new variables of the much simpler difference equations equivalent to the original system of equations.
Let S^i be the forward shift operator that maps n to $n+i$. We shall assume that a system of fourth order ordinary difference equations is of the form

$$S^p(u^k) = \Omega_k(n, [u^k]), \quad k = 1, 2, 3, 4, \tag{3}$$

where $[u^i]$ denotes the dependent variable u^i and its shifts. The invertible mapping $(n, u^k) \mapsto (n, \tilde{u}^k = u^k + \varepsilon Q_k(n, [u^k]) + O(\varepsilon^2)),\ k = 1, 2, 3, 4,$ is a symmetry group of transformations if and only if it satisfies the following linearized symmetry condition

$$S^p(Q_k) - \mathcal{X}(\Omega_k) = 0, \quad k = 1, 2, 3, 4, \tag{4}$$

where $\mathcal{X}$ is the $(p-1)$st prolongation of the symmetry generator

$$X = \sum_{k=1}^{4} Q_k \frac{\partial}{\partial u^k}, \tag{5}$$

i. e.,

$$\mathcal{X} = X^{[p-1]} = \sum_{j=0}^{p-1} \sum_{k=1}^{4} S^j(Q_k) \frac{\partial}{\partial S_j(u^k)}. \tag{6}$$

We shall refer to $Q_k = Q_k(n, u_n)$ as characteristics and for simplicity we shall consider point transformations only, that is, $Q_k = Q_k(n, u^k)$.

Definition 1.1 *[14] Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \textit{for all} \qquad x \in M,$$

Without any lucky guess, the reduction of order can readily be done via the canonical coordinates [9]

$$s^k = \int \frac{du^k}{Q_k(n, u^k)}, \quad k = 1, 2, 3, 4. \tag{7}$$

Eventually, the constraining restrictions on the constants in the characteristics, Q_k, $k = 1, 2, 3, 4$, hint on a perfect choice of invariants.

2 Main results

To start, we consider the corresponding forward system

$$\begin{aligned} x_{n+4} = \Omega_1 = \frac{x_n}{A_n + B_n x_n y_{n+1} z_{n+2} t_{n+3}},\ y_{n+4} = \Omega_2 = \frac{y_n}{C_n + D_n x_{n+3} y_n z_{n+1} t_{n+2}} \\ z_{n+4} = \Omega_3 = \frac{z_n}{E_n + F_n x_{n+2} y_{n+3} z_n t_{n+1}},\ t_{n+4} = \Omega_4 = \frac{t_n}{G_n + H_n x_{n+1} y_{n+2} z_{n+3} t_n}, \end{aligned} \tag{8}$$

where $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$, $(C_n)_{n\in\mathbb{N}_0}$, $(D_n)_{n\in\mathbb{N}_0}$, $(E_n)_{n\in\mathbb{N}_0}$, $(F_n)_{n\in\mathbb{N}_0}$, $(G_n)_{n\in\mathbb{N}_0}$ and $(H_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers, equivalent to (2).

2.1 Symmetries

To construct the characteristics of the system of fourth order difference equations (8), we must impose linearized symmetry criterion (4). This amounts to

$$S^4Q_1 + \frac{B_n x_n^2(t_{n+3}z_{n+2}(S^1Q_2) + t_{n+3}y_{n+1}(S^2Q_3) + y_{n+1}z_{n+2}(S^3Q_4)) - A_nQ_1}{(A_n + B_nx_ny_{n+1}z_{n+2}t_{n+3})^2} = 0, \tag{9a}$$

$$S^4Q_2 + \frac{D_n y_n^2(t_{n+2}x_{n+3}(S^1Q_3) + x_{n+3}z_{n+1}(S^2Q_4) + t_{n+2}z_{n+1}(S^3Q_1)) - C_nQ_2}{(C_n + D_nx_{n+3}y_nz_{n+1}t_{n+2})^2} = 0, \tag{9b}$$

$$S^4Q_3 + \frac{F_n z_n^2(t_{n+1}x_{n+2}(S^3Q_2) + x_{n+2}y_{n+3}(S^1Q_4) + t_{n+1}y_{n+3}(S^2Q_1)) - E_nQ_3}{(E_n + F_nx_{n+2}y_{n+3}z_nt_{n+1})^2} = 0, \tag{9c}$$

$$S^4Q_4 + \frac{H_n t_n^2(x_{n+1}z_{n+3}(S^2Q_2) + x_{n+1}y_{n+2}(S^3Q_3) + y_{n+2}z_{n+3}(S^1Q_1)) - G_nQ_4}{(G_n + H_nx_{n+1}y_{n+2}z_{n+3}t_n)^2} = 0. \tag{9d}$$

We act the operators $\partial/\partial x_n-[(\partial\Omega_1/\partial x_n)/(\partial\Omega_1/\partial y_{n+1})]\partial/\partial y_{n+1}$, $\partial/\partial y_n-[(\partial\Omega_2/\partial y_n)(\partial\Omega_2/\partial z_{n+1})]\partial/\partial z_{n+1}$, $\partial/\partial z_n-[(\partial\Omega_3/\partial z_n)(\partial\Omega_3/\partial y_{n+3})]\partial/\partial y_{n+3}$ and $\partial/\partial t_n-[(\partial\Omega_4/\partial t_n)(\partial\Omega_4/\partial y_{n+2})]\partial/\partial y_{n+2}$ on equations in (9), respectively, to get

$$(S^1Q_2)' - {Q^1}' + (1/z_{n+2})(S^2Q_3) + (1/t_{n+3})(S^3Q^4) + (2/x_n)Q_1 = 0 \tag{10a}$$

$$-{Q^2}' + (S^1Q_3)' + (2/y_n)Q_2 + (1/t_{n+2})(S^2Q_4) + (1/x_{n+3})(S^3Q_1) = 0 \tag{10b}$$

$$(S^3Q^2)' - {Q^3}' + (2/z_n)Q^3 + (1/t_{n+1})(S^1Q_4) + (1/x_{n+2})(S^2Q^1) = 0 \tag{10c}$$

$$(S^2Q^2)' - {Q_4}' + (1/z_{n+3})(S^3Q_3) + (2/t_n)Q_4 + (1/x_{n+1})(S^1Q^1) = 0 \tag{10d}$$

after simplification. Note that $'$ denotes the derivative with respect to the continuous variable.
Next, we differentiate equations in (10) with respect to x_n, y_n, z_n and t_n, respectively. The latter leads to the differential equations

$$\begin{aligned} &-{Q^1}'' + (2/x_n){Q^1}' - (2/x_n^2)Q^1 = 0,\ -{Q^2}'' + (2/y_n){Q^2}' - (2/y_n^2)Q^2 = 0,\\ &-{Q^3}'' + (2/z_n){Q^3}' - (2/z_n^2)Q^3 = 0, -{Q^4}'' + (2/t_n){Q^4}' - (2/t_n^2)Q^4 = 0 \end{aligned} \tag{11}$$

whose solutions are given by

$$\begin{aligned} &Q_1(n,x_n) = \alpha_1(n){x_n}^2 + \beta_1(n)x_n, \quad Q_2(n,y_n) = \alpha_2(n){y_n}^2 + \beta_2(n)y_n,\\ &Q_3(n,z_n) = \alpha_3(n){z_n}^2 + \beta_3(n)z_n, \quad Q_4(n,t_n) = \alpha_4(n){t_n}^2 + \beta_4(n)t_n, \end{aligned} \tag{12}$$

for some functions α_i and β_i, respectively.
We replace (12) and their shits in (9). Due to the fact that the α_i's and β_i's depend on the independent variable only, we equate all products of shifts of dependent variables x_n, y_n, z_n and t_n in the resulting equations to zero; this yields the 'final constraints' below

$$\beta_1(n) + \beta_2(n+1) + \beta_3(n+2) + \beta_4(n+3) = 0, \alpha_1(n) = \alpha_2(n) = \alpha_3(n) = \alpha_4(n) = 0, \tag{13}$$

with $\beta_1(n) = \beta_1(n+4), \beta_2(n) = \beta_2(n+4), \beta_3(n) = \beta_3(n+4), \beta_4(n) = \beta_4(n+4)$. The reader can easily verify that the functions satisfying the above constraints are of the forms:

$$\begin{aligned} &\alpha_j(n) = 0,\ j = 1,2,3,4;\ \beta_1(n) = c_1 + c_2(-i)^n + c_3(i)^n + c_4(-1)^n;\ \beta_2(n) = c_5 + c_6(-i)^n + c_7(i)^n + c_8(-1)^n;\\ &\beta_3(n) = c_9 + c_{10}(-i)^n + c_{11}(i)^n + c_{12}(-1)^n;\ \beta_4(n) = (ic_2 + c_6 - ic_{10})(-i)^n + (c_7 - ic_3 + ic_{11})(i)^n + (c_4 - c_8\\ &+ c_{12})(-1)^n - c_1 - c_5 - c_9, \end{aligned} \tag{14}$$

where the c_i's, $i = 1, \dots, 12$, are arbitrary constants. Consequently, thanks to (5), (12) and (14), we obtain twelve symmetry generators:

$$\begin{aligned}
&X_1 = x_n\partial x_n - t_n\partial t_n, X_2 = (-i)^n(x_n\partial x_n + it_n\partial t_n), X_3 = i^n(x_n\partial x_n - it_n\partial t_n), X_4 = (-1)^n(x_n\partial x_n + t_n\partial t_n),\\
&X_5 = y_n\partial y_n - t_n\partial t_n, X_6 = (-i)^n(y_n\partial y_n + t_n\partial t_n), X_7 = i^n(y_n\partial y_n + t_n\partial t_n), X_8 = (-1)^n(y_n\partial y_n - t_n\partial t_n),\\
&X_9 = z_n\partial z_n - t_n\partial t_n, X_{10} = (-i)^n(z_n\partial z_n - it_n\partial t_n), X_{11} = i^n(z_n\partial z_n + it_n\partial t_n), X_{12} = (-1)^n(z_n\partial z_n + t_n\partial t_n).
\end{aligned} \tag{15}$$

Note that for simplicity, we adopt the notation $\partial x = \partial/\partial x$.

2.2 Reduction of order via symmetries and formulas for solutions

Using any linear combinations of the symmetries in (15) that involves all four independent variables x_n, y_n, z_n and t_n, say $X = X_1 + X_2 + X_3 = x_n\partial x_n + y_n\partial y_n + z_n\partial z_n - 3t_n\partial t_n$, we derive the corresponding canonical coordinates

$$s_1(n) = \int \frac{dx_n}{x_n},\ s_2(n) = \int \frac{dy_n}{y_n},\ s_3(n) = \int \frac{dz_n}{z_n}, s_4(n) = \int \frac{dt_n}{-3t_n}. \tag{16}$$

Inspired by the form of the equations in the final constraints (13), we construct the invariants:

$$\begin{aligned}
\tilde{X}_n =&\beta_1(n)s_1(n) + \beta_2(n+1)s_2(n+1) + \beta_3(n+2)s_3(n+2) + \beta_4(n+3)s_4(n+3) = \ln|x_n y_{n+1} z_{n+2} t_{n+3}|\\
\tilde{Y}_n =&\beta_1(n+3)s_1(n+3) + \beta_2(n)s_2(n) + \beta_3(n+1)s_3(n+1) + \beta_4(n+2)s_4(n+2) = \ln|x_{n+3} y_n z_{n+1} t_{n+2}|\\
\tilde{Z}_n =&\beta_1(n+2)s_1(n+2) + \beta_2(n+3)s_2(n+3) + \beta_3(n)s_3(n) + \beta_4(n+1)s_4(n+1) = \ln|x_{n+2} y_{n+3} z_n t_{n+1}|\\
\tilde{T}_n =&\beta_1(n+1)s_1(n+1) + \beta_2(n+2)s_2(n+2) + \beta_3(n+3)s_3(n+3) + \beta_4(n)s_4(n) = \ln|x_{n+1} y_{n+2} z_{n+3} t_n|,
\end{aligned}$$

obtained by replacing $\beta_i(n+j)$ by $s_i(n+j)\beta_i(n+j)$ in the left hand sides of equations in (13).
Using Definition 1.1, the reader can easily confirm that $\tilde{X}_n$, $\tilde{Y}_n$, $\tilde{Z}_n$ and $\tilde{T}_n$ are invariant functions. For simplicity, we introduce the variables

$$X_n = \exp(-\tilde{X}_n),\ Y_n = \exp(-\tilde{Y}_n),\ Z_n = \exp(-\tilde{Z}_n),\ T_n = \exp(-\tilde{T}_n). \tag{17}$$

Thus

$$X_{n+1} = H_n + G_n T_n,\ Y_{n+1} = B_n + A_n X_n,\ Z_{n+1} = D_n + C_n Y_n,\ T_{n+1} = F_n + E_n Z_n \tag{18a}$$

and so

$$x_{n+4} = \frac{X_n}{Y_{n+1}} x_n,\ y_{n+4} = \frac{Y_n}{Z_{n+1}} y_n,\ z_{n+4} = \frac{Z_n}{T_{n+1}} z_n,\ t_{n+4} = \frac{T_n}{X_{n+1}} t_n. \tag{18b}$$

Straightforward iterations (using equation (18a)) yield

$$X_{n+4} = \Lambda_n^x + (\Theta_n^x)X_n,\ Y_{n+4} = \Delta_n^y + (\Theta_n^y)Y_n,\ Z_{n+4} = \Delta_n^z + (\Theta_n^z)Z_n,\ T_{n+4} = \Delta_n^t + (\Theta_n^t)T_n$$

that is

$$U_{4n+j} = U_j\left(\prod_{k_1=0}^{n-1} \Theta_{4k_1+j}^u\right) + \sum_{l=0}^{n-1}\left(\Lambda_{4l+j}^u \prod_{k_2=l+1}^{n-1} \Theta_{4k_2+j}^u\right), \tag{19a}$$

for $j = 0, 1, 2, 3$ and $(U, u) \in \{(X, x), (Y, y), (Z, z), (T, t)\}$, where

$$\begin{aligned}
\Lambda_n^x &= H_{n+3} + G_{n+3}F_{n+2} + G_{n+3}E_{n+2}D_{n+1} + G_{n+3}E_{n+2}C_{n+1}B_n, \Theta_n^x = G_{n+3}E_{n+2}C_{n+1}A_n;\\
\Lambda_n^y &= B_{n+3} + A_{n+3}H_{n+2} + A_{n+3}G_{n+2}F_{n+1} + A_{n+3}G_{n+2}E_{n+1}D_n, \Theta_n^y = A_{n+3}G_{n+2}E_{n+1}C_n;\\
\Lambda_n^z &= D_{n+3} + C_{n+3}B_{n+2} + C_{n+3}A_{n+2}H_{n+1} + C_{n+3}A_{n+2}G_{n+1}F_n, \Theta_n^z = C_{n+3}A_{n+2}G_{n+1}E_n;\\
\Lambda_n^t &= F_{n+3} + E_{n+3}D_{n+2} + E_{n+3}C_{n+2}B_{n+1} + E_{n+3}C_{n+2}A_{n+1}H_n, \Theta_n^t = E_{n+3}C_{n+2}A_{n+1}G_n;
\end{aligned} \tag{19b}$$

Also, straightforward iterations (using equation (18b)) yield

$$x_{4n+j} = x_j \prod_{k=0}^{n-1} \frac{X_{4k+j}}{Y_{4k+1+j}}, y_{4n+j} = y_j \prod_{k=0}^{n-1} \frac{Y_{4k+j}}{Z_{4k+1+j}}, z_{4n+j} = z_j \prod_{k=0}^{n-1} \frac{Z_{4k+j}}{T_{4k+1+j}}, t_{4n+j} = t_j \prod_{k=0}^{n-1} \frac{T_{4k+j}}{X_{4k+1+j}}, \quad (19c)$$

$j = 0, 1, 2, 3$. Combining equations in (19), we obtain the following solutions $\{x_n\}$ of the system of equations (8):

$$\begin{aligned} x_{4n+j} =& x_j \prod_{s=0}^{n-1} \left[\frac{X_j \left(\prod_{k_1=0}^{s-1} \Theta^x_{4k_1+j} \right) + \sum_{l=0}^{s-1} \left(\Lambda^x_{4l+j} \prod_{k_2=l+1}^{s-1} \Theta^x_{4k_2+j} \right)}{Y_{j+1} \left(\prod_{k_1=0}^{s-1} \Theta^y_{4k_1+j+1} \right) + \sum_{l=0}^{s-1} \left(\Lambda^y_{4l+j+1} \prod_{k_2=l+1}^{s-1} \Theta^y_{4k_2+j+1} \right)} \right], \quad j = 0, 1, 2, \\ x_{4n+3} =& x_3 \prod_{s=0}^{n-1} \left[\frac{X_3 \left(\prod_{k_1=0}^{s-1} \Theta^x_{4k_1+3} \right) + \sum_{l=0}^{s-1} \left(\Lambda^x_{4l+3} \prod_{k_2=l+1}^{s-1} \Theta^x_{4k_2+3} \right)}{Y_0 \left(\prod_{k_1=0}^{s} \Theta^y_{4k_1} \right) + \sum_{l=0}^{s} \left(\Lambda^y_{4l} \prod_{k_2=l+1}^{s} \Theta^y_{4k_2} \right)} \right], \end{aligned} \quad (20)$$

where Θ^u_n and Λ^u_n, $u \in \{x, y, z, t\}$ are defined in (19b); and $X_0 = 1/(x_0y_1z_2t_3)$, $X_1 = H_0+G_0/(t_0x_1y_2z_3)$, $X_2 = F_0G_1+H_1+(E_0G_1)/(t_1x_2y_3z_0)$ $X_3 = D_0E_1G_2+F_1G_2+H_2+(C_0E_1G_2)/(t_2x_3y_0z_1)$, $Y_0 = 1/(t_2x_3y_0z_1)$, $Y_1 = B_0 + A_0/(t_3x_0y_1z_2)$, $Y_2 = A_1H_0 + B_1 + (A_1G_0)/(t_0x_1y_2z_3)$, $Y_3 = A_2F_0G_1 + B_1 + (A_2E_0G_1)/(t_1x_2y_3z_0)$.

Recall that we forward shifted equation (2) thrice to obtain (8) whose solutions x_n is giving in (20). Now, we go backward thrice and replace the capital letters in the right hand sides of equations in (19b) with lower cases letters to get the solutions x_n corresponding to (8). In other words, solutions $\{x_n\}$ of the system of equations (2) is giving by

$$\begin{aligned} x_{4n-3} =& x_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{i=0}^{s-1} \theta^x_{4i} \right) + x_{-3}y_{-2}z_{-1}t_0 \sum_{l=0}^{s-1} \left(\lambda^x_{4l} \prod_{i=l+1}^{s-1} \theta^x_{4i} \right)}{(a_0 + b_0x_{-3}y_{-2}z_{-1}t_0) \left(\prod_{i=0}^{s-1} \theta^y_{4i+1} \right) + x_{-3}y_{-2}z_{-1}t_0 \sum_{l=0}^{s-1} \left(\lambda^y_{4l+1} \prod_{i=l+1}^{s-1} \theta^y_{4i+1} \right)} \\ x_{4n-2} =& x_{-2} \prod_{s=0}^{n-1} \frac{(g_0 + h_0t_{-3}x_{-2}y_{-1}z_0) \left(\prod_{i=0}^{s-1} \theta^x_{4i+1} \right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda^x_{4l+1} \prod_{i=l+1}^{s-1} \theta^x_{4i+1} \right)}{((a_1h_0 + b_1)t_{-3}x_{-2}y_{-1}z_0 + a_1g_0) \left(\prod_{i=0}^{s-1} \theta^y_{4i+2} \right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda^y_{4l+2} \prod_{i=l+1}^{s-1} \theta^y_{4i+2} \right)} \\ x_{4n-1} =& x_{-1} \prod_{s=0}^{n-1} \frac{((f_0g_1 + h_1)t_{-2}x_{-1}y_0z_{-3} + e_0g_1) \left(\prod_{i=0}^{s-1} \theta^x_{4i+2} \right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s-1} \left(\lambda^x_{4l+2} \prod_{i=l+1}^{s-1} \theta^x_{4i+2} \right)}{((a_0f_0g_1 + a_2h_1 + b_2)t_{-2}x_{-1}y_0z_{-3} + a_2e_0g_1) \left(\prod_{i=0}^{s-1} \theta^y_{4i+3} \right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s-1} \left(\lambda^y_{4l+3} \prod_{i=l+1}^{s-1} \theta^y_{4i+3} \right)} \\ x_{4n} =& x_0 \prod_{s=0}^{n-1} \frac{((d_0e_1g_2 + f_1g_2 + h_2)t_{-1}x_0y_{-3}z_{-2} + c_0e_1g_2) \left(\prod_{i=0}^{s-1} \theta^x_{4i+3} \right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s-1} \left(\lambda^x_{4l+3} \prod_{i=l+1}^{s-1} \theta^x_{4i+3} \right)}{\left(\prod_{i=0}^{s} \theta^y_{4i} \right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s} \left(\lambda^y_{4l} \prod_{i=l+1}^{s} \theta^y_{4i} \right)}. \end{aligned}$$

Similar computations yield

$$y_{4n-3} = y_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{i=0}^{s-1} \theta_{4i}^{y}\right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s-1} \left(\lambda_{4l}^{y} \prod_{i=l+1}^{s-1} \theta_{4i}^{y}\right)}{(c_0 + d_0t_{-1}x_0y_{-3}z_{-2}) \left(\prod_{i=0}^{s-1} \theta_{4i+1}^{z}\right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s-1} \left(\lambda_{4l+1}^{z} \prod_{i=l+1}^{s-1} \theta_{4i+1}^{z}\right)}$$

$$y_{4n-2} = y_{-2} \prod_{s=0}^{n-1} \frac{(a_0 + b_0t_0x_{-3}y_{-2}z_{-1}) \left(\prod_{i=0}^{s-1} \theta_{4i+1}^{y}\right) + t_0x_{-3}y_{-2}z_{-1} \sum_{l=0}^{s-1} \left(\lambda_{4l+1}^{y} \prod_{i=l+1}^{s-1} \theta_{4i+1}^{y}\right)}{((b_0c_1 + d_1)t_0x_{-3}y_{-2}z_{-1} + a_0c_1) \left(\prod_{i=0}^{s-1} \theta_{4i+2}^{z}\right) + t_0x_{-3}y_{-2}z_{-1} \sum_{l=0}^{s-1} \left(\lambda_{4l+2}^{z} \prod_{i=l+1}^{s-1} \theta_{4i+2}^{z}\right)}$$

$$y_{4n-1} = y_{-1} \prod_{s=0}^{n-1} \frac{((a_1h_0 + b_1)t_{-3}x_{-2}y_{-1}z_0 + a_1g_0) \left(\prod_{i=0}^{s-1} \theta_{4i+2}^{y}\right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda_{4l+2}^{y} \prod_{i=l+1}^{s-1} \theta_{4i+2}^{y}\right)}{((a_1c_2h_0 + b_1c_2 + d_2)t_{-3}x_{-2}y_{-1}z_0 + a_1c_2g_0) \left(\prod_{i=0}^{s-1} \theta_{4i+3}^{z}\right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda_{4l+3}^{z} \prod_{i=l+1}^{s-1} \theta_{4i+3}^{z}\right)}$$

$$y_{4n} = y_0 \prod_{s=0}^{n-1} \frac{((a_2f_0g_1 + a_2h_1 + b_2)t_{-2}x_{-1}y_0z_{-3} + a_2e_0g_1) \left(\prod_{i=0}^{s-1} \theta_{4i+3}^{y}\right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s-1} \left(\lambda_{4l+3}^{y} \prod_{i=l+1}^{s-1} \theta_{4i+3}^{y}\right)}{\left(\prod_{i=0}^{s} \theta_{4i}^{z}\right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s} \left(\lambda_{4l}^{z} \prod_{i=l+1}^{s} \theta_{4i}^{z}\right)}$$

$$z_{4n-3} = z_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{i=0}^{s-1} \theta_{4i}^{z}\right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s-1} \left(\lambda_{4l}^{z} \prod_{i=l+1}^{s-1} \theta_{4i}^{z}\right)}{(e_0 + f_0t_{-2}x_{-1}y_0z_{-3}) \left(\prod_{i=0}^{s-1} \theta_{4i+1}^{t}\right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s-1} \left(\lambda_{4l+1}^{t} \prod_{i=l+1}^{s-1} \theta_{4i+1}^{t}\right)}$$

$$z_{4n-2} = z_{-2} \prod_{s=0}^{n-1} \frac{(c_0 + d_0t_{-1}x_0y_{-3}z_{-2}) \left(\prod_{i=0}^{s-1} \theta_{4i+1}^{z}\right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s-1} \left(\lambda_{4l+1}^{z} \prod_{i=l+1}^{s-1} \theta_{4i+1}^{z}\right)}{((d_0e_1 + f_1)t_{-1}x_0y_{-3}z_{-2} + c_0e_1) \left(\prod_{i=0}^{s-1} \theta_{4i+2}^{t}\right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s-1} \left(\lambda_{4l+2}^{t} \prod_{i=l+1}^{s-1} \theta_{4i+2}^{t}\right)}$$

$$z_{4n-1} = z_{-1} \prod_{s=0}^{n-1} \frac{((b_0c_1 + d_1)t_0x_{-3}y_{-2}z_{-1} + a_0c_1) \left(\prod_{i=0}^{s-1} \theta_{4i+2}^{z}\right) + t_0x_{-3}y_{-2}z_{-1} \sum_{l=0}^{s-1} \left(\lambda_{4l+2}^{z} \prod_{i=l+1}^{s-1} \theta_{4i+2}^{z}\right)}{((b_0c_1e_2 + d_1e_2 + f_2)t_0x_{-3}y_{-2}z_{-1} + a_0c_1e_2) \left(\prod_{i=0}^{s-1} \theta_{4i+3}^{t}\right) + t_0x_{-3}y_{-2}z_{-1} \sum_{l=0}^{s-1} \left(\lambda_{4l+3}^{t} \prod_{i=l+1}^{s-1} \theta_{4i+3}^{t}\right)}$$

$$z_{4n} = z_0 \prod_{s=0}^{n-1} \frac{((a_1c_2h_0 + b_1c_2 + d_2)t_{-3}x_{-2}y_{-1}z_0 + a_1c_2g_0) \left(\prod_{i=0}^{s-1} \theta_{4i+3}^{z}\right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda_{4l+3}^{z} \prod_{i=l+1}^{s-1} \theta_{4i+3}^{z}\right)}{\left(\prod_{i=0}^{s} \theta_{4i}^{t}\right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s} \left(\lambda_{4l}^{t} \prod_{i=l+1}^{s} \theta_{4i}^{t}\right)}$$

$$t_{4n-3} = t_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{i=0}^{s-1} \theta_{4i}^{t}\right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda_{4l}^{t} \prod_{i=l+1}^{s-1} \theta_{4i}^{t}\right)}{(g_0 + h_0t_{-3}x_{-2}y_{-1}z_0) \left(\prod_{i=0}^{s-1} \theta_{4i+1}^{x}\right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda_{4l+1}^{x} \prod_{i=l+1}^{s-1} \theta_{4i+1}^{x}\right)}$$

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$$t_{4n-2} = t_{-2} \prod_{s=0}^{n-1} \frac{(e_0 + f_0 t_{-2} x_{-1} y_0 z_{-3}) \left(\prod_{i=0}^{s-1} \theta^t_{4i+1} \right) + t_{-2} x_{-1} y_0 z_{-3} \sum_{l=0}^{s-1} \left(\lambda^t_{4l+1} \prod_{i=l+1}^{s-1} \theta^t_{4i+1} \right)}{((f_0 g_1 + h_1) t_{-2} x_{-1} y_0 z_{-3} + e_0 g_1) \left(\prod_{i=0}^{s-1} \theta^x_{4i+2} \right) + t_{-2} x_{-1} y_0 z_{-3} \sum_{l=0}^{s-1} \left(\lambda^x_{4l+2} \prod_{i=l+1}^{s-1} \theta^x_{4i+2} \right)}$$

$$t_{4n-1} = t_{-1} \prod_{s=0}^{n-1} \frac{((d_0 e_1 + f_1) t_{-1} x_0 y_{-3} z_{-2} + c_0 e_1) \left(\prod_{i=0}^{s-1} \theta^t_{4i+2} \right) + t_{-1} x_0 y_{-3} z_{-2} \sum_{l=0}^{s-1} \left(\lambda^t_{4l+2} \prod_{i=l+1}^{s-1} \theta^t_{4i+2} \right)}{((d_0 e_1 g_2 + f_1 g_2 + h_2) t_{-1} x_0 y_{-3} z_{-2} + c_0 e_1 g_2) \left(\prod_{i=0}^{s-1} \theta^x_{4i+3} \right) + t_{-1} x_0 y_{-3} z_{-2} \sum_{l=0}^{s-1} \left(\lambda^x_{4l+3} \prod_{i=l+1}^{s-1} \theta^x_{4i+3} \right)}$$

$$t_{4n} = t_0 \prod_{s=0}^{n-1} \frac{((b_0 c_1 e_2 + d_1 e_2 + f_2) t_0 x_{-3} y_{-2} z_{-1} + a_0 c_1 e_2) \left(\prod_{i=0}^{s-1} \theta^t_{4i+3} \right) + t_0 x_{-3} y_{-2} z_{-1} \sum_{l=0}^{s-1} \left(\lambda^t_{4l+3} \prod_{i=l+1}^{s-1} \theta^t_{4i+3} \right)}{\left(\prod_{i=0}^{s} \theta^x_{4i} \right) + t_0 x_{-3} y_{-2} z_{-1} \sum_{l=0}^{s} \left(\lambda^x_{4l} \prod_{i=l+1}^{s} \theta^x_{4i} \right)}. \tag{21a}$$

Note that

$$\begin{aligned}
\lambda^x_n &= h_{n+3} + g_{n+3} f_{n+2} + g_{n+3} e_{n+2} d_{n+1} + g_{n+3} e_{n+2} c_{n+1} b_n, \theta^x_n = g_{n+3} e_{n+2} c_{n+1} a_n; \\
\lambda^y_n &= b_{n+3} + a_{n+3} h_{n+2} + a_{n+3} g_{n+2} f_{n+1} + a_{n+3} g_{n+2} e_{n+1} d_n, \theta^y_n = a_{n+3} g_{n+2} e_{n+1} c_n; \\
\lambda^z_n &= d_{n+3} + c_{n+3} b_{n+2} + c_{n+3} a_{n+2} h_{n+1} + c_{n+3} a_{n+2} g_{n+1} f_n, \theta^z_n = c_{n+3} a_{n+2} g_{n+1} e_n; \\
\lambda^t_n &= f_{n+3} + e_{n+3} d_{n+2} + e_{n+3} c_{n+2} b_{n+1} + e_{n+3} c_{n+2} a_{n+1} h_n, \theta^t_n = e_{n+3} c_{n+2} a_{n+1} g_n.
\end{aligned} \tag{21b}$$

2.3 Case where a_n, b_n, c_n, d_n, e_n, f_n, g_n and h_n are periodic of period four

Suppose $\{a_n\} = \{a_0, a_1, a_2, a_3, a_0, \dots\}$, $\{b_n\} = \{b_0, b_1, b_2, b_3, b_0, \dots\}$, $\{c_n\} = \{c_0, c_1, c_2, c_3, c_0, \dots\}$, $\{d_n\} = \{d_0, d_1, d_2, d_3, d_0, \dots\}$, $\{e_n\} = \{e_0, e_1, e_2, e_3, e_0, \dots\}$, $\{f_n\} = \{f_0, f_1, f_2, f_3, f_0, \dots\}$ and $\{g_n\} = \{g_0, g_1, g_2, g_3, \ g_0, \dots\}$. Equations in (21) simplify to

$$x_{4n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{(\theta^x_0)^s + x_{-3} y_{-2} z_{-1} t_0 (\lambda^x_0) \sum_{l=0}^{s-1} (\theta^x_0)^l}{(a_0 + b_0 x_{-3} y_{-2} z_{-1} t_0) \, (\theta^y_1)^s + x_{-3} y_{-2} z_{-1} t_0 (\lambda^y_1) \sum_{l=0}^{s-1} (\theta^y_1)^l}$$

$$x_{4n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{(g_0 + h_0 t_{-3} x_{-2} y_{-1} z_0)(\theta^x_1)^s + t_{-3} x_{-2} y_{-1} z_0 (\lambda^x_1) \sum_{l=0}^{s-1} (\theta^x_1)^l}{((a_1 h_0 + b_1) t_{-3} x_{-2} y_{-1} z_0 + a_1 g_0) \, [\theta^y_2]^s + t_{-3} x_{-2} y_{-1} z_0 (\lambda^y_2) \sum_{l=0}^{s-1} (\theta^y_2)^l}$$

$$x_{4n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{((f_0 g_1 + h_1) t_{-2} x_{-1} y_0 z_{-3} + e_0 g_1) \, (\theta^x_2)^s + t_{-2} x_{-1} y_0 z_{-3} \lambda^x_2 \sum_{l=0}^{s-1} (\theta^x_2)^l}{((a_0 f_0 g_1 + a_2 h_1 + b_2) t_{-2} x_{-1} y_0 z_{-3} + a_2 e_0 g_1) \, (\theta^y_3)^s + t_{-2} x_{-1} y_0 z_{-3} (\lambda^y_3) \sum_{l=0}^{s-1} (\theta^y_3)^l}$$

$$x_{4n} = x_0 \prod_{s=0}^{n-1} \frac{((d_0 e_1 g_2 + f_1 g_2 + h_2) t_{-1} x_0 y_{-3} z_{-2} + c_0 e_1 g_2) \, (\theta^x_3)^s + t_{-1} x_0 y_{-3} z_{-2} (\lambda^x_3) \sum_{l=0}^{s-1} (\theta^x_3)^l}{(\theta^y_0)^{s+1} + t_{-1} x_0 y_{-3} z_{-2} (\lambda^y_0) \sum_{l=0}^{s} (\theta^y_0)^l}$$

$$y_{4n-3} = y_{-3} \prod_{s=0}^{n-1} \frac{(\theta_0^y)^s + t_{-1}x_0y_{-3}z_{-2}(\lambda_0^y) \sum_{l=0}^{s-1} (\theta_0^y)^l}{(c_0 + d_0t_{-1}x_0y_{-3}z_{-2})(\theta_1^z)^s + t_{-1}x_0y_{-3}z_{-2}(\lambda_1^z) \sum_{l=0}^{s-1} (\theta_1^z)^l}$$

$$y_{4n-2} = y_{-2} \prod_{s=0}^{n-1} \frac{(a_0 + b_0t_0x_{-3}y_{-2}z_{-1})(\theta_1^y)^s + t_0x_{-3}y_{-2}z_{-1}(\lambda_1^y) \sum_{l=0}^{s-1} (\theta_1^y)^l}{((b_0c_1 + d_1)t_0x_{-3}y_{-2}z_{-1} + a_0c_1)(\theta_2^z)^s + t_0x_{-3}y_{-2}z_{-1}(\lambda_2^z) \sum_{l=0}^{s-1} (\theta_2^z)^l}$$

$$y_{4n-1} = y_{-1} \prod_{s=0}^{n-1} \frac{((a_1h_0 + b_1)t_{-3}x_{-2}y_{-1}z_0 + a_1g_0)(\theta_2^y)^s + t_{-3}x_{-2}y_{-1}z_0(\lambda_2^y) \sum_{l=0}^{s-1} (\theta_2^y)^l}{((a_1c_2h_0 + b_1c_2 + d_2)t_{-3}x_{-2}y_{-1}z_0 + a_1c_2g_0)(\theta_3^z)^s + t_{-3}x_{-2}y_{-1}z_0(\lambda_3^z) \sum_{l=0}^{s-1} (\theta_3^z)^l}$$

$$y_{4n} = y_0 \prod_{s=0}^{n-1} \frac{((a_2f_0g_1 + a_2h_1 + b_2)t_{-2}x_{-1}y_0z_{-3} + a_2e_0g_1)(\theta_3^y)^s + t_{-2}x_{-1}y_0z_{-3}(\lambda_3^y) \sum_{l=0}^{s-1} (\theta_3^y)^l}{(\theta_0^z)^{s+1} + t_{-2}x_{-1}y_0z_{-3}(\lambda_0^z) \sum_{l=0}^{s} (\theta_0^z)^l}$$

$$z_{4n-3} = z_{-3} \prod_{s=0}^{n-1} \frac{(\theta_0^z)^s + t_{-2}x_{-1}y_0z_{-3}(\lambda_0^z) \sum_{l=0}^{s-1} (\theta_0^z)^l}{(e_0 + f_0t_{-2}x_{-1}y_0z_{-3})(\theta_1^t)^s + t_{-2}x_{-1}y_0z_{-3}(\lambda_1^t) \sum_{l=0}^{s-1} (\theta_1^t)^l}$$

$$z_{4n-2} = z_{-2} \prod_{s=0}^{n-1} \frac{(c_0 + d_0t_{-1}x_0y_{-3}z_{-2})(\theta_1^z)^s + t_{-1}x_0y_{-3}z_{-2}(\lambda_1^z) \sum_{l=0}^{s-1} (\theta_1^z)^l}{((d_0e_1 + f_1)t_{-1}x_0y_{-3}z_{-2} + c_0e_1)(\theta_2^t)^s + t_{-1}x_0y_{-3}z_{-2}(\lambda_2^t) \sum_{l=0}^{s-1} (\theta_2^t)^l}$$

$$z_{4n-1} = z_{-1} \prod_{s=0}^{n-1} \frac{((b_0c_1 + d_1)t_0x_{-3}y_{-2}z_{-1} + a_0c_1)(\theta_2^z)^s + t_0x_{-3}y_{-2}z_{-1}(\lambda_2^z) \sum_{l=0}^{s-1} (\theta_2^z)^l}{((b_0c_1e_2 + d_1e_2 + f_2)t_0x_{-3}y_{-2}z_{-1} + a_0c_1e_2)(\theta_3^t)^s + t_0x_{-3}y_{-2}z_{-1}(\lambda_3^t) \sum_{l=0}^{s-1} (\theta_3^t)^l}$$

$$z_{4n} = z_0 \prod_{s=0}^{n-1} \frac{((a_1c_2h_0 + b_1c_2 + d_2)t_{-3}x_{-2}y_{-1}z_0 + a_1c_2g_0)(\theta_3^z)^s + t_{-3}x_{-2}y_{-1}z_0(\lambda_3^z) \sum_{l=0}^{s-1} (\theta_3^z)^l}{(\theta_0^t)^{s+1} + t_{-3}x_{-2}y_{-1}z_0(\lambda_0^t) \sum_{l=0}^{s} (\theta_0^t)^l}$$

$$t_{4n-3} = t_{-3} \prod_{s=0}^{n-1} \frac{(\theta_0^t)^s + t_{-3}x_{-2}y_{-1}z_0(\lambda_0^t) \sum_{l=0}^{s-1} (\theta_0^t)^l}{(g_0 + h_0t_{-3}x_{-2}y_{-1}z_0)(\theta_1^x)^s + t_{-3}x_{-2}y_{-1}z_0(\lambda_1^x) \sum_{l=0}^{s-1} (\theta_1^x)^l}$$

$$t_{4n-2} = t_{-2} \prod_{s=0}^{n-1} \frac{(e_0 + f_0t_{-2}x_{-1}y_0z_{-3})(\theta_1^t)^s + t_{-2}x_{-1}y_0z_{-3}(\lambda_1^t) \sum_{l=0}^{s-1} (\theta_1^t)^l}{((f_0g_1 + h_1)t_{-2}x_{-1}y_0z_{-3} + e_0g_1)(\theta_2^x)^s + t_{-2}x_{-1}y_0z_{-3}(\lambda_2^x) \sum_{l=0}^{s-1} (\theta_2^x)^l}$$

$$t_{4n-1} = t_{-1} \prod_{s=0}^{n-1} \frac{((d_0e_1 + f_1)t_{-1}x_0y_{-3}z_{-2} + c_0e_1)(\theta_2^t)^s + t_{-1}x_0y_{-3}z_{-2}(\lambda_2^t) \sum_{l=0}^{s-1} (\theta_2^t)^l}{((d_0e_1g_2 + f_1g_2 + h_2)t_{-1}x_0y_{-3}z_{-2} + c_0e_1g_2)(\theta_3^x)^s + t_{-1}x_0y_{-3}z_{-2}(\lambda_3^x) \sum_{l=0}^{s-1} (\theta_3^x)^l}$$

$$t_{4n} = t_0 \prod_{s=0}^{n-1} \frac{((b_0c_1e_2 + d_1e_2 + f_2)t_0x_{-3}y_{-2}z_{-1} + a_0c_1e_2)\left(\theta_3^t\right)^s + t_0x_{-3}y_{-2}z_{-1}(\lambda_3^t)\sum\limits_{l=0}^{s-1}\left(\theta_3^t\right)^l}{\left(\theta_0^x\right)^{s+1} + t_0x_{-3}y_{-2}z_{-1}(\lambda_0^x)\sum\limits_{l=0}^{s}\left(\theta_0^x\right)^l}, \qquad (22)$$

where θ_0^u, λ_0^u, $u = x, y, z, t$ are defined in (21b).

2.4 Case where a_n, b_n, c_n, d_n, e_n, f_n, g_n and h_n are constant

Suppose that $a_n = a$, $b_n = b$, $c_n = c$, $d_n = d$, $e_n = e$, $f_n = f$ and $g_n = g$. Equations in (22) simplify to

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\left[\frac{(aceg)^s + x_{-3}y_{-2}z_{-1}t_0(h + gf + ged + gecb)\sum\limits_{l=0}^{s-1}(aceg)^l}{(a + bx_{-3}y_{-2}z_{-1}t_0)(aceg)^s + x_{-3}y_{-2}z_{-1}t_0(b + ah + agf + aged)\sum\limits_{l=0}^{s-1}(aceg)^l}\right]$$

$$x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{(g + htx_{-2}y_{-1}z_0)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(h + gf + ged + gecb)\sum\limits_{l=0}^{s-1}(aceg)^l}{((ah + b)t_{-3}x_{-2}y_{-1}z_0 + ag)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(b + ah + agf + aged)\sum\limits_{l=0}^{s-1}(aceg)^l}$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{((fg + h)t_{-2}x_{-1}y_0z_{-3} + eg)(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(h + gf + ged + gecb)\sum\limits_{l=0}^{s-1}(aceg)^l}{((afg + a_2h + b)t_{-2}x_{-1}y_0z_{-3} + aeg)(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(b + ah + agf + aged)\sum\limits_{l=0}^{s-1}(aceg)^l}$$

$$x_{4n} = x_0\prod_{s=0}^{n-1}\frac{((deg + fg + h)t_{-1}x_0y_{-3}z_{-2} + ceg)(aceg)^s + t_{-1}x_0y_{-3}z_{-2}(h + gf + ged + gecb)\sum\limits_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1} + t_{-1}x_0y_{-3}z_{-2}(b + ah + agf + aged)\sum\limits_{l=0}^{s}(aceg)^l}$$

$$y_{4n-3} = y_{-3}\prod_{s=0}^{n-1}\frac{(aceg)^s + t_{-1}x_0y_{-3}z_{-2}(b + ah + agf + aged)\sum\limits_{l=0}^{s-1}(aceg)^l}{(c + dt_{-1}x_0y_{-3}z_{-2})(aceg)^s + t_{-1}x_0y_{-3}z_{-2}(d + cb + cah + cagf)\sum\limits_{l=0}^{s-1}(aceg)^l}$$

$$y_{4n-2} = y_{-2}\prod_{s=0}^{n-1}\frac{(a + bt_0x_{-3}y_{-2}z_{-1})(aceg)^s + t_0x_{-3}y_{-2}z_{-1}(b + ah + agf + aged)\sum\limits_{l=0}^{s-1}(aceg)^l}{((bc + d)t_0x_{-3}y_{-2}z_{-1} + ac)(aceg)^s + t_0x_{-3}y_{-2}z_{-1}(d + cb + cah + cagf)\sum\limits_{l=0}^{s-1}(aceg)^l}$$

$$y_{4n-1} = y_{-1}\prod_{s=0}^{n-1}\frac{((ah + b)t_{-3}x_{-2}y_{-1}z_0 + ag)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(b + ah + agf + aged)\sum\limits_{l=0}^{s-1}(aceg)^l}{((ach + bc + d)t_{-3}x_{-2}y_{-1}z_0 + acg)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(d + cb + cah + cagf)\sum\limits_{l=0}^{s-1}(aceg)^l}$$

$$y_{4n} = y_0\prod_{s=0}^{n-1}\frac{((afg + ah + b)t_{-2}x_{-1}y_0z_{-3} + aeg)(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(b + ah + agf + aged)\sum\limits_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1} + t_{-2}x_{-1}y_0z_{-3}(d + cb + cah + cagf)\sum\limits_{l=0}^{s}(aceg)^l}$$

$$z_{4n-3} = z_{-3}\prod_{s=0}^{n-1}\frac{(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(d + cb + cah + cagf)\sum\limits_{l=0}^{s-1}(aceg)^l}{(e + ft_{-2}x_{-1}y_0z_{-3})(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(f + ed + ecb + ecah)\sum\limits_{l=0}^{s-1}(aceg)^l}$$

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$$
\begin{aligned}
z_{4n-2} =& z_{-2}\prod_{s=0}^{n-1}\frac{\left(c+dt_{-1}x_0y_{-3}z_{-2}\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(d+cb+cah+cagf)\sum_{l=0}^{s-1}(aceg)^l}{\left((de+f)t_{-1}x_0y_{-3}z_{-2}+ce\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(f+ed+ecb+ecah)\sum_{l=0}^{s-1}(aceg)^l}\\
z_{4n-1} =& z_{-1}\prod_{s=0}^{n-1}\frac{\left((bc+d)t_0x_{-3}y_{-2}z_{-1}+ac\right)(aceg)^s+t_0x_{-3}y_{-2}z_{-1}(d+cb+cah+cagf)\sum_{l=0}^{s-1}(aceg)^l}{\left((bce+de+f)t_0x_{-3}y_{-2}z_{-1}+ace\right)(aceg)^s+t_0x_{-3}y_{-2}z_{-1}(f+ed+ecb+ecah)\sum_{l=0}^{s-1}(aceg)^l}\\
z_{4n} =& z_0\prod_{s=0}^{n-1}\frac{\left((ach+bc+d)t_{-3}x_{-2}y_{-1}z_0+acg\right)(aceg)^s+t_{-3}x_{-2}y_{-1}z_0(d+cb+cah+cagf)\sum_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1}+t_{-3}x_{-2}y_{-1}z_0(f+ed+ecb+ecah)\sum_{l=0}^{s}(aceg)^l}\\
t_{4n-3} =& t_{-3}\prod_{s=0}^{n-1}\frac{(aceg)^s+t_{-3}x_{-2}y_{-1}z_0(f+ed+ecb+ecah)\sum_{l=0}^{s-1}(aceg)^l}{\left(g+ht_{-3}x_{-2}y_{-1}z_0\right)(aceg)^s+t_{-3}x_{-2}y_{-1}z_0(h+gf+ged+gecb)\sum_{l=0}^{s-1}(aceg)^l}\\
t_{4n-2} =& t_{-2}\prod_{s=0}^{n-1}\frac{\left(e+ft_{-2}x_{-1}y_0z_{-3}\right)(aceg)^s+t_{-2}x_{-1}y_0z_{-3}(f+ed+ecb+ecah)\sum_{l=0}^{s-1}(aceg)^l}{\left((fg+h)t_{-2}x_{-1}y_0z_{-3}+eg\right)(aceg)^s+t_{-2}x_{-1}y_0z_{-3}(h+gf+ged+gecb)\sum_{l=0}^{s-1}(aceg)^l}\\
t_{4n-1} =& t_{-1}\prod_{s=0}^{n-1}\frac{\left((de+f)t_{-1}x_0y_{-3}z_{-2}+ce\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(f+ed+ecb+ecah)\sum_{l=0}^{s-1}(aceg)^l}{\left((deg+fg+h)t_{-1}x_0y_{-3}z_{-2}+ceg\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(h+gf+ged+gecb)\sum_{l=0}^{s-1}(aceg)^l}\\
t_{4n} =& t_0\prod_{s=0}^{n-1}\frac{\left((bce+de+f)t_0x_{-3}y_{-2}z_{-1}+ace\right)(aceg)^s+t_0x_{-3}y_{-2}z_{-1}(f+ed+ecb+ecah)\sum_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1}+t_0x_{-3}y_{-2}z_{-1}(h+gf+ged+gecb)\sum_{l=0}^{s}(aceg)^l}.
\end{aligned}
\tag{23}
$$

2.4.1 Case where $a=1,\ b=1,\ c=1,\ d=1,\ e=1,\ f=1,\ g=1$ and $h=1$

Here, $\theta^x=\theta^y=\theta^z=\theta^t=1$ and $\lambda^x=\lambda^y=\lambda^z=\lambda^t=4$. Thus, equations in (23) simplify to

$$
\begin{aligned}
x_{4n-3} =& x_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4sx_{-3}y_{-2}z_{-1}t_0}{1+(4s+1)x_{-3}y_{-2}z_{-1}t_0}\right], x_{4n-2}=x_{-2}\prod_{s=0}^{n-1}\left[\frac{1+(4s+1)t_{-3}x_{-2}y_{-1}z_0}{1+(4s+2)t_{-3}x_{-2}y_{-1}z_0}\right],\\
x_{4n-1} =& x_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_{-2}x_{-1}y_0z_{-3}}{1+(4s+3)t_{-2}x_{-1}y_0z_{-3}}\right], x_{4n}=x_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_{-1}x_0y_{-3}z_{-2}}{1+(4s+4)t_{-1}x_0y_{-3}z_{-2}}\right],\\
y_{4n-3} =& y_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4st_{-1}x_0y_{-3}z_{-2}}{1+(4s+1)t_{-1}x_0y_{-3}z_{-2}}\right], y_{4n-2}=y_{-2}\prod_{s=0}^{n-1}\left[\frac{1++(4s+1)t_0x_{-3}y_{-2}z_{-1}}{1+(4s+2)t_0x_{-3}y_{-2}z_{-1}}\right],\\
y_{4n-1} =& y_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_{-3}x_{-2}y_{-1}z_0}{1+(4s+3)t_{-3}x_{-2}y_{-1}z_0}\right], y_{4n}=y_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_{-2}x_{-1}y_0z_{-3}}{1+(4s+4)t_{-2}x_{-1}y_0z_{-3}}\right],
\end{aligned}
$$

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$$\begin{aligned}
z_{4n-3} =& z_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4st_{-2}x_{-1}y_0z_{-3}}{1+(4s+1)t_{-2}x_{-1}y_0z_{-3}}\right], z_{4n-2} = z_{-2}\prod_{s=0}^{n-1}\left[\frac{1+(4s+1)t_{-1}x_0y_{-3}z_{-2}}{1+(4s+2)t_{-1}x_0y_{-3}z_{-2}}\right],\\
z_{4n-1} =& z_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_0x_{-3}y_{-2}z_{-1}}{1+(4s+3)t_0x_{-3}y_{-2}z_{-1}}\right], z_{4n} = z_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_{-3}x_{-2}y_{-1}z_0}{1+(4s+4)t_{-3}x_{-2}y_{-1}z_0}\right],\\
t_{4n-3} =& t_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4st_{-3}x_{-2}y_{-1}z_0}{1+(4s+1)t_{-3}x_{-2}y_{-1}z_0}\right], t_{4n-2} = t_{-2}\prod_{s=0}^{n-1}\left[\frac{1+(4s+1)t_{-2}x_{-1}y_0z_{-3}}{1+(4s+2)t_{-2}x_{-1}y_0z_{-3}},\right]\\
t_{4n-1} =& t_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_{-1}x_0y_{-3}z_{-2}}{1+(4s+3)t_{-1}x_0y_{-3}z_{-2}}\right], t_{4n} = t_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_0x_{-3}y_{-2}z_{-1}}{1+(4s+4)t_0x_{-3}y_{-2}z_{-1}}\right].
\end{aligned} \tag{24}$$

2.5 Case where $a = c = h = -1$ and $b = d = e = f = g = 1$

Here, $\theta^x = \theta^y = \theta^z = \theta^t = 1$ and $\lambda^x = \lambda^y = \lambda^z = \lambda^t = 0$. Thus, equations in (23) simplify to Theorem 2.2 in [1].

2.6 Case where $a = c = e = g = -1$ and $b = d = f = h = 1$

Here, $\theta^x = \theta^y = \theta^z = \theta^t = 1$ and $\lambda^x = \lambda^y = \lambda^z = \lambda^t = 0$. Thus, equations in (23) simplify to Theorem 2.3 in [1].

2.7 Case where $a = b = c = d = e = f = g = 1$ and $h = -1$

Here, $\theta^x = \theta^y = \theta^z = \theta^t = 1$ and $\lambda^x = \lambda^y = \lambda^z = \lambda^t = 0$. Thus, equations in (23) simplify to Theorem 3.1 in [1].

3 Existence of four periodic solutions

If

$$x_{-3}y_{-2}z_{-1}t_0 = x_{-2}y_{-1}z_0t_{-3} = x_{-1}y_0z_{-3}t_{-2} = x_0y_{-3}z_{-2}t_{-1} = \frac{1-a}{b} = \frac{1-c}{d} = \frac{1-e}{f} = \frac{1-g}{h},$$

then

$$\theta^x = \theta^y = \theta^z = \theta^t = geca$$

and

$$\lambda^x = \lambda^y = \lambda^z = \lambda^t = \frac{b}{1-a}(1-geca).$$

Thus, equations in (23) simplify to

$$\begin{aligned}
&x_{4n-3} = x_{-3}, x_{4n-2} = x_{-2}, x_{4n-1} = x_{-1}, x_{4n} = x_0,\\
&y_{4n-3} = y_{-3}, y_{4n-2} = y_{-2}, y_{4n-1} = y_{-1}, y_{4n} = y_0,\\
&z_{4n-3} = z_{-3}, z_{4n-2} = z_{-2}, z_{4n-1} = z_{-1}, z_{4n} = z_0,\\
&t_{4n-3} = t_{-3}, t_{4n-2} = t_{-2}, t_{4n-1} = t_{-1}, t_{4n} = t_0
\end{aligned}$$

and therefore all solutions of (8) are periodic with period four.
Below are the figures of some numerical examples that illustrate two cases of systems where solutions are periodic with period four.

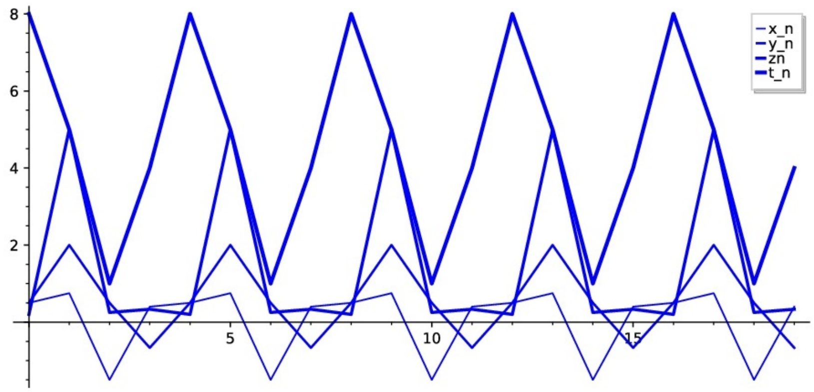

Figure 1: Periodic solutions of (8) when $a = 2$, $b = -1$, $c = 3$, $d = -2$, $e = 4$, $f = -3$, $g = 5$, $h = -4$ with initial conditions $x_0 = 0.5$, $x_1 = 0.75$, $x_2 = -3/2$, $x_3 = 0.4$, $y_0 = 0.5$, $y_1 = 2$, $y_2 = 0.5$, $y_3 = -2/3$, $z_0 = 1/5$, $z_1 = 5$, $z_2 = 0.25$, $z_3 = 1/3$, $t_0 = 8$, $t_1 = 5$, $t_2 = 1$, $t_3 = 4$.

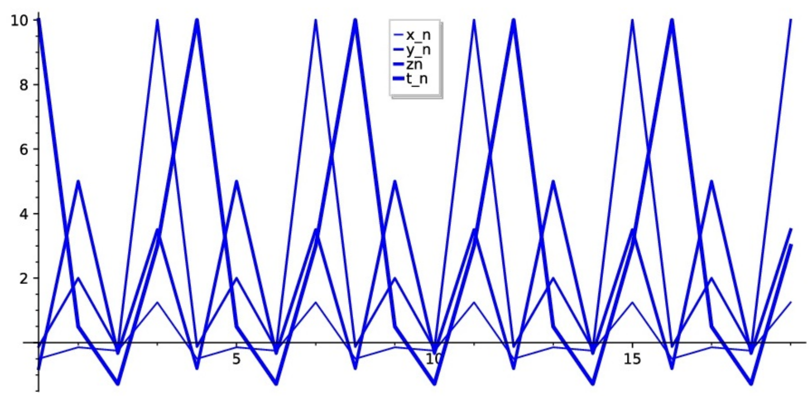

Figure 2: Periodic solutions of (8) when $a = 0.5$, $b = 0.5$, $c = 0.75$, $d = 0.25$, $e = 6$, $f = -5$, $g = -1$, $h = 2$ with initial conditions $x_0 = -0.5$, $x_1 = -1/7$, $x_2 = -1/4$, $x_3 = 1.25$, $y_0 = -0.125$, $y_1 = 2$, $y_2 = -1/5$, $y_3 = 10$, $z_0 = -0.8$, $z_1 = 5$, $z_2 = -1/3$, $z_3 = 3.5$, $t_0 = 10$, $t_1 = 0.5$, $t_2 = -1.28$, $t_3 = 3$.

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Dynamics of an anti-competitive system of difference equations

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Abstract

In this paper, we study the dynamical properties of an anti-competitive system of second-order rational difference equations. The proposed work is considerably extended and improve some exiting results in the literature.

Keywords and phrases: difference equations; boundedness and persistence; asymptotic behavior
2010 AMS: 39A10, 40A05

1 Introduction

In [1], Hamza *et al*. have investigated the global behavior of the difference equation: $x_{n+1} = \frac{Ax_{n-1}}{B+Cx_n^2},\ n = 0, 1, \cdots$, where $A,\ B,\ C$ and initial conditions $x_0,\ x_{-1}$ are positive real numbers. Motivated by the above studies, our aim in this paper is to investigate the dynamical properties of the following anti-competitive system of second-order rational difference equations:

$$x_{n+1} = \frac{\alpha + \beta y_{n-1}}{\gamma + \delta x_n^2},\ y_{n+1} = \frac{\alpha_1 + \beta_1 x_{n-1}}{\gamma_1 + \delta_1 y_n^2},\ n = 0, 1, \cdots, \tag{1}$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ \alpha_1,\ \beta_1,\ \gamma_1,\ \delta_1$ and the initial conditions $x_0,\ x_{-1},\ y_0,\ y_{-1}$ are positive real numbers. The rest of the paper is dedicated to investigate the boundedness and persistence, existence of unbounded solutions, existence and uniqueness of positive equilibrium point, local and global stability about the unique positive equilibrium point of the system (1).

2 Main results

2.1 Boundedness and persistence

Theorem 1. *If $\beta\beta_1 < \gamma\gamma_1$ then every solution $\{(x_n, y_n)/x_n, y_n > 0\}$ of the system (1) is bounded and persists.*

Proof. If $\{(x_n, y_n)/x_n, y_n > 0\}$ is a solution of the system (1) then

$$x_{n+1} \leq \frac{\alpha}{\gamma} + \frac{\beta}{\gamma} y_{n-1},\ y_{n+1} \leq \frac{\alpha_1}{\gamma_1} + \frac{\beta_1}{\gamma_1} x_{n-1},\ n = 0, 1, \cdots. \tag{2}$$

From (2), one get

$$x_{n+1} \leq \frac{\alpha}{\gamma} + \frac{\alpha_1\beta}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} x_{n-3},\ y_{n+1} \leq \frac{\alpha_1}{\gamma_1} + \frac{\alpha\beta_1}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} y_{n-3},\ n = 0, 1, \cdots. \tag{3}$$

Consider

$$\Phi_{n+1} = \frac{\alpha}{\gamma} + \frac{\alpha_1\beta}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} \Phi_{n-3},\ \xi_{n+1} = \frac{\alpha_1}{\gamma_1} + \frac{\alpha\beta_1}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} \xi_{n-3},\ n = 0, 1, \cdots. \tag{4}$$

The solution $\{(\Phi_n, \xi_n)\}$ of (4) is

$$\begin{aligned} \Phi_n &= r_1 \left(\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + r_2 \left(-\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + r_3 \left(\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + r_4 \left(-\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + \frac{\alpha\gamma_1 + \beta\alpha_1}{\gamma\gamma_1 - \beta\beta_1}, \\ \xi_n &= s_1 \left(\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + s_2 \left(-\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + s_3 \left(\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + s_4 \left(-\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + \frac{\alpha_1\gamma + \alpha\beta_1}{\gamma\gamma_1 - \beta\beta_1}, \end{aligned} \tag{5}$$

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where $r_1,\ r_2,\ r_3,\ r_4,\ s_1,\ s_2,\ s_3,\ s_4$ depend upon the initial values $\Phi_{-3},\ \Phi_{-2},\ \Phi_{-1},\ \Phi_0,\ \xi_{-3},\ \xi_{-2},\ \xi_{-1},\ \xi_0$. Assuming that $\beta\beta_1 < \gamma\gamma_1$ then (5) implies that Φ_n and ξ_n are bounded. Now consider the solution $\{(\Phi_n, \xi_n)\}$ of (5) such that

$$\begin{aligned} &\Phi_{-3} = x_{-3},\ \Phi_{-2} = x_{-2},\ \Phi_{-1} = x_{-1},\ \Phi_0 = x_0, \\ &\xi_{-3} = y_{-3},\ \xi_{-2} = y_{-2},\ \xi_{-1} = y_{-1},\ \xi_0 = y_0. \end{aligned} \tag{6}$$

From (3), (5) and (6) one get

$$x_n \le \frac{\alpha\gamma_1 + \beta\alpha_1}{\gamma\gamma_1 - \beta\beta_1} + \epsilon = U_1 + \epsilon,\ y_n \le \frac{\alpha_1\gamma + \alpha\beta_1}{\gamma\gamma_1 - \beta\beta_1} + \epsilon = U_2 + \epsilon, \tag{7}$$

where for large n, ϵ is a sufficiently small number. In addition from (1) and (7), we get

$$x_n \ge \frac{\alpha}{\gamma + \delta x_n^2} \ge \frac{\alpha(\gamma\gamma_1 - \beta\beta_1)^2}{\gamma(\gamma\gamma_1 - \beta\beta_1)^2 + \delta(\alpha\gamma_1 + \beta\alpha_1)^2} = L_1. \tag{8}$$

$$y_n \ge \frac{\alpha_1}{\gamma_1 + \delta_1 y_n^2} \ge \frac{\alpha_1(\gamma\gamma_1 - \beta\beta_1)^2}{\gamma_1(\gamma\gamma_1 - \beta\beta_1)^2 + \delta_1(\alpha_1\gamma + \beta_1\alpha)^2} = L_2. \tag{9}$$

Finally, from (7), (8) and (9) one get

$$L_1 \le x_n \le U_1,\ L_2 \le y_n \le U_2,\ n = 0, 1, \cdots. \tag{10}$$

□

2.2 Existence of unbounded solution

Theorem 2. *For solution $\{(x_n, y_n)/x_n, y_n > 0\}$ of the system (1), the following statements hold:*

(i) If $\beta\beta_1 > (\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)$ then $x_n \to \infty$ as $n \to \infty$.

(ii) If $\beta\beta_1 > (\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)$ then $y_n \to \infty$ as $n \to \infty$.

Proof. (i) If $\{(x_n, y_n)/x_n, y_n > 0\}$ is a solution of the system (1) then

$$x_{n+1} = \frac{\alpha + \beta y_{n-1}}{\gamma + \delta x_n^2} \ge \frac{\alpha + \beta y_{n-1}}{\gamma + \delta U_1^2} = \frac{\alpha}{\gamma + \delta U_1^2} + \frac{\beta}{\gamma + \delta U_1^2} y_{n-1}. \tag{11}$$

$$y_{n+1} = \frac{\alpha_1 + \beta_1 x_{n-1}}{\gamma_1 + \delta_1 y_n^2} \ge \frac{\alpha_1 + \beta_1 x_{n-1}}{\gamma_1 + \delta_1 U_2^2} = \frac{\alpha_1}{\gamma_1 + \delta_1 U_2^2} + \frac{\beta_1}{\gamma_1 + \delta_1 U_2^2} x_{n-1}. \tag{12}$$

From (12)

$$y_{n-1} \ge \frac{\alpha_1}{\gamma_1 + \delta_1 U_2^2} + \frac{\beta_1}{\gamma_1 + \delta_1 U_2^2} x_{n-3}. \tag{13}$$

Using (13) in (11), one get

$$x_{n+1} \ge \frac{\alpha}{\gamma + \delta U_1^2} + \frac{\beta\alpha_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} x_{n-3}. \tag{14}$$

Consider

$$\tau_{n+1} = \frac{\alpha}{\gamma + \delta U_1^2} + \frac{\beta\alpha_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} \tau_{n-3}. \tag{15}$$

The solution of (15) is

$$\begin{aligned} \tau_n \;=\; & c_1 \left(\sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + c_2 \left(-\sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + c_3 \left(\iota \sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + \\ & c_4 \left(-\iota \sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + \frac{\alpha(\gamma_1 + \delta_1 U_2^2) + \beta\alpha_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2) - \beta\beta_1}, \end{aligned}$$

where $c_1,\ c_2,\ c_3,\ c_4$ depends on $\tau_{-3},\ \tau_{-2},\ \tau_{-1},\ \tau_0$. Now if $\beta\beta_1 > (\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)$ then $\{\tau_n\}$ is divergent. Hence by comparison $x_n \to \infty$ as $n \to \infty$.

(ii) Similarly from (11), we have

$$x_{n-1} \geq \frac{\alpha}{\gamma+\delta U_1^2} + \frac{\beta}{\gamma+\delta U_1^2} y_{n-3}. \tag{16}$$

Using (16) in (12), we get

$$y_{n+1} \geq \frac{\alpha_1}{\gamma_1+\delta_1 U_2^2} + \frac{\beta_1\alpha}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} y_{n-3}. \tag{17}$$

Consider

$$\mu_{n+1} = \frac{\alpha_1}{\gamma_1+\delta_1 U_2^2} + \frac{\beta_1\alpha}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} \mu_{n-3}. \tag{18}$$

The solution of (18) is given by

$$\begin{aligned}\mu_n \;=\;& c_5\left(\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + c_6\left(-\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + c_7\left(\iota\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + \\ & c_8\left(-\iota\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + \frac{\alpha_1(\gamma+\delta U_1^2)+\beta_1\alpha}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)-\beta\beta_1},\end{aligned}$$

where $c_5,\ c_6,\ c_7,\ c_8$ depends on $\mu_{-3},\ \mu_{-2},\ \mu_{-1},\ \mu_0$. If $\beta\beta_1 > (\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)$ then $\{\mu_n\}$ is divergent. Hence by comparison $y_n \to \infty$ as $n \to \infty$. $\square$

2.3 Existence and uniqueness of positive equilibrium point

Theorem 3. *If*

$$\alpha_1+\beta_1 L_1 < \left(\gamma_1+\delta_1\left(\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}, \tag{19}$$

$$\alpha_1+\beta_1 U_1 > \left(\gamma_1+\delta_1\left(\frac{(\gamma+\delta U_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta U_1^2)U_1-\alpha}{\beta}, \tag{20}$$

and

$$\frac{\left(\gamma+3\delta {U_1}^2\right)\left(\gamma_1\beta^2+3\delta_1\left(\left(\gamma+\delta{U_1}^2\right)U_1-\alpha\right)^2\right)}{\beta^3\beta_1} < 1, \tag{21}$$

then the system (1) has a unique positive equilibrium point $\Omega = (\bar{x},\bar{y}) \in [L_1,U_1]\times[L_2,U_2]$.

Proof. Consider

$$x = \frac{\alpha+\beta y}{\gamma+\delta x^2}, \quad y = \frac{\alpha_1+\beta_1 x}{\gamma_1+\delta_1 y^2}. \tag{22}$$

From (22), we have

$$y = \frac{(\gamma+\delta x^2)x-\alpha}{\beta}, \quad x = \frac{(\gamma_1+\delta_1 y^2)y-\alpha_1}{\beta_1}.$$

Taking

$$F(x) = \frac{(\gamma_1+\delta_1(h(x))^2)h(x)-\alpha_1}{\beta_1} - x, \tag{23}$$

where

$$h(x) = \frac{(\gamma+\delta x^2)x-\alpha}{\beta}, \tag{24}$$

and $x \in [L_1,U_1]$. Now

$$F(L_1) \;=\; \frac{(\gamma_1+\delta_1(h(L_1))^2)h(L_1)-\alpha_1}{\beta_1} - L_1 = \frac{\left(\gamma_1+\delta_1\left(\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}-\alpha_1}{\beta_1} - L_1. \tag{25}$$

Assume that (19) hold then (25) implies that $F(L_1) > 0$. Also,

$$F(U_1) = \frac{(\gamma_1 + \delta_1 (h(U_1))^2) h(U_1) - \alpha_1}{\beta_1} - U_1 = \frac{\left(\gamma_1 + \delta_1 \left(\frac{(\gamma+\delta U_1^2)U_1 - \alpha}{\beta}\right)^2\right) \frac{(\gamma+\delta U_1^2)U_1-\alpha}{\beta} - \alpha_1}{\beta_1} - U_1. \quad (26)$$

Assuming (20) hold then from (26) one get $F(U_1) < 0$. Hence, $F(x)$ has at least one positive solution in $x \in [L_1, U_1]$. Furthermore,

$$F'(x) = h'(x)\frac{\gamma_1 + 3\delta_1 (h(x))^2}{\beta_1} - 1, \quad (27)$$

where

$$h'(x) = \frac{\gamma + 3\delta x^2}{\beta}. \quad (28)$$

Let $\bar{x}$ be a solution of equation $F(x) = 0$, then from (23), (24) and (28) one get

$$\bar{x} = \frac{(\gamma_1 + \delta_1 (h(\bar{x}))^2) h(\bar{x}) - \alpha_1}{\beta_1}, \quad (29)$$

$$h(\bar{x}) = \frac{(\gamma + \delta \bar{x}^2)\bar{x} - \alpha}{\beta}, \quad (30)$$

$$h'(\bar{x}) = \frac{\gamma + 3\delta \bar{x}^2}{\beta}. \quad (31)$$

In view of (30) and (31), equation (27) takes the following form

$$\begin{aligned} F'(\bar{x}) &= \frac{(\gamma + 3\delta\bar{x}^2)\left(\gamma_1\beta^2 + 3\delta_1\left((\gamma+\delta\bar{x}^2)\,\bar{x} - \alpha\right)^2\right)}{\beta^3\beta_1} - 1, \\ &\le \frac{(\gamma + 3\delta{U_1}^2)\left(\gamma_1\beta^2 + 3\delta_1\left((\gamma+\delta{U_1}^2)\,U_1 - \alpha\right)^2\right)}{\beta^3\beta_1} - 1. \end{aligned} \quad (32)$$

Assume that (21) hold then from (32) one get $F'(\bar{x}) < 0$. □

2.4 Local stability

Theorem 4. *For equilibrium Ω of the system (1), the following statements hold:*

(i) Ω of the system (1) is locally asymptotically stable if

$$\frac{2\delta U_1^2}{\gamma + \delta L_1^2}\left(1 + \frac{2\delta_1 U_2^2}{\gamma_1 + \delta_1 L_2^2}\right) + \frac{1}{\gamma_1 + \delta_1 L_2^2}\left(2\delta_1 U_2^2 + \frac{\beta\beta_1}{\gamma + \delta L_1^2}\right) < 1. \quad (33)$$

(ii) Ω of the system (1) is unstable if

$$\frac{2\delta L_1^2}{\gamma + \delta U_1^2}\left(1 + \frac{2\delta_1 L_2^2}{\gamma_1 + \delta_1 U_2^2}\right) + \frac{1}{\gamma_1 + \delta_1 U_2^2}\left(2\delta_1 L_2^2 + \frac{\beta\beta_1}{\gamma + \delta U_1^2}\right) > 1. \quad (34)$$

Proof. If $(\bar{x}, \bar{y})$ is an equilibrium point of the system (1) then

$$\bar{x} = \frac{\alpha + \beta\bar{y}}{\gamma + \delta\bar{x}^2}, \ \bar{y} = \frac{\alpha_1 + \beta_1\bar{x}}{\gamma_1 + \delta_1\bar{y}^2}. \quad (35)$$

Consider the following transformation in order to construct the corresponding linearized form of the system (1):

$$(x_{n+1}, x_n, y_{n+1}, y_n) \mapsto (f, f_1, g, g_1), \quad (36)$$

where

$$f=\frac{\alpha+\beta y_{n-1}}{\gamma+\delta x_n^2},\ f_1=x_n,\ g=\frac{\alpha_1+\beta_1 x_{n-1}}{\gamma_1+\delta_1 y_n^2},\ g_1=y_n. \tag{37}$$

The Jacobian matrix $J|_{(\bar{x},\bar{y})}$ about $(\bar{x},\bar{y})$ under the transformation (36) is given by

$$J|_{(\bar{x},\bar{y})}=\begin{pmatrix} a & 0 & 0 & b\\ 1 & 0 & 0 & 0\\ 0 & a_1 & b_1 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \tag{38}$$

where

$$a=-\frac{2\delta\bar{x}^2}{\gamma+\delta\bar{x}^2},\ b=\frac{\beta}{\gamma+\delta\bar{x}^2},\ a_1=\frac{\beta_1}{\gamma_1+\delta_1\bar{y}^2},\ b_1=-\frac{2\delta_1\bar{y}^2}{\gamma_1+\delta_1\bar{y}^2}. \tag{39}$$

The characteristic equation of $J|_{(\bar{x},\bar{y})}$ about $(\bar{x},\bar{y})$ is given by

$$\lambda^4-(a+b_1)\lambda^3+ab_1\lambda^2-a_1b=0. \tag{40}$$

Now,

$$\begin{aligned}|a|+|b_1|+|ab_1|+|a_1b| &= \frac{2\delta\bar{x}^2}{\gamma+\delta\bar{x}^2}+\frac{2\delta_1\bar{y}^2}{\gamma_1+\delta_1\bar{y}^2}+\frac{4\delta\delta_1\ \bar{x}^2\bar{y}^2}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)}+\frac{\beta\beta_1}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)},\\ &\leq \frac{2\delta U_1^2}{\gamma+\delta L_1^2}+\frac{2\delta_1 U_2^2}{\gamma_1+\delta_1 L_2^2}+\frac{4\delta\delta_1 U_1^2U_2^2}{(\gamma+\delta L_1^2)(\gamma_1+\delta_1 L_2^2)}+\frac{\beta\beta_1}{(\gamma+\delta L_1^2)(\gamma_1+\delta_1 L_2^2)},\\ &= \frac{2\delta U_1^2}{\gamma+\delta L_1^2}\left(1+\frac{2\delta_1 U_2^2}{\gamma_1+\delta_1 L_2^2}\right)+\frac{1}{\gamma_1+\delta_1 L_2^2}\left(2\delta_1 U_2^2+\frac{\beta\beta_1}{\gamma+\delta L_1^2}\right).\end{aligned} \tag{41}$$

Assuming that (33) hold then from (41) one gets $|a|+|b_1|+|ab_1|+|a_1b|<1$. Hence from Remark 1.3.1 of [2], Ω of (1) is locally asymptotically stable.

Proof (ii). Using same manipulations as for the proof of (i) and assume that (34) hold then

$$\begin{aligned}|a|+|b_1|+|ab_1|+|a_1b| &= \frac{2\delta\bar{x}^2}{\gamma+\delta\bar{x}^2}+\frac{2\delta_1\bar{y}^2}{\gamma_1+\delta_1\bar{y}^2}+\frac{4\delta\delta_1\ \bar{x}^2\bar{y}^2}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)}+\frac{\beta\beta_1}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)},\\ &\geq \frac{2\delta L_1^2}{\gamma+\delta U_1^2}+\frac{2\delta_1 L_2^2}{\gamma_1+\delta_1 U_2^2}+\frac{4\delta\delta_1 L_1^2L_2^2}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}+\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)},\\ &= \frac{2\delta L_1^2}{\gamma+\delta U_1^2}\left(1+\frac{2\delta_1 L_2^2}{\gamma_1+\delta_1 U_2^2}\right)+\frac{1}{\gamma_1+\delta_1 U_2^2}\left(2\delta_1 L_2^2+\frac{\beta\beta_1}{\gamma+\delta U_1^2}\right)>1.\end{aligned} \tag{42}$$

Hence Ω of system (1) is unstable. □

2.5 Global character

Now we will study the global dynamics of (1) about Ω by utilizing Theorem 1.16 of [3].

Theorem 5. *Ω of the system (1) is a global attractor.*

Proof. If $f(x,y)=\frac{\alpha+\beta y}{\gamma+\delta x^2}$ and $g(x,y)=\frac{\alpha_1+\beta_1 x}{\gamma_1+\delta_1 y^2}$ then it is easy to examine that $f(x,y)$ is non-increasing (resp. non-decreasing) in x (resp. y) $\forall\ (x,y)\in\left[\frac{\alpha(\gamma\gamma_1-\beta\beta_1)^2}{\gamma(\gamma\gamma_1-\beta\beta_1)^2+\delta(\alpha\gamma_1+\beta\alpha_1)^2},\frac{\alpha\gamma_1+\beta\alpha_1}{\gamma\gamma_1-\beta\beta_1}\right]\times\left[\frac{\alpha_1(\gamma\gamma_1-\beta\beta_1)^2}{\gamma_1(\gamma\gamma_1-\beta\beta_1)^2+\delta_1(\alpha_1\gamma+\beta_1\alpha)^2},\frac{\alpha_1\gamma+\alpha\beta_1}{\gamma\gamma_1-\beta\beta_1}\right]$. Also $g(x,y)$ is non-decreasing (resp. non-increasing) in x (resp. y) $\forall\ (x,y)\in\left[\frac{\alpha(\gamma\gamma_1-\beta\beta_1)^2}{\gamma(\gamma\gamma_1-\beta\beta_1)^2+\delta(\alpha\gamma_1+\beta\alpha_1)^2},\frac{\alpha\gamma_1+\beta\alpha_1}{\gamma\gamma_1-\beta\beta_1}\right]\times\left[\frac{\alpha_1(\gamma\gamma_1-\beta\beta_1)^2}{\gamma_1(\gamma\gamma_1-\beta\beta_1)^2+\delta_1(\alpha_1\gamma+\beta_1\alpha)^2},\frac{\alpha_1\gamma+\alpha\beta_1}{\gamma\gamma_1-\beta\beta_1}\right]$. Let (m_1,M_1,m_2,M_2) be a solution of the system

$$m_1=\frac{\alpha+\beta m_2}{\gamma+\delta M_1^2},\ M_1=\frac{\alpha+\beta M_2}{\gamma+\delta m_1^2}. \tag{43}$$

and

$$m_2=\frac{\alpha_1+\beta_1 m_1}{\gamma_1+\delta_1 M_2^2},\ M_2=\frac{\alpha_1+\beta_1 M_1}{\gamma_1+\delta_1 m_2^2}. \tag{44}$$

From (43) and (44), we get

$$\frac{m_1}{M_1} = \frac{(\alpha+\beta m_2)(\gamma+\delta m_1^2)}{(\gamma+\delta M_1^2)(\alpha+\beta M_2)}. \tag{45}$$

$$\frac{m_2}{M_2} = \frac{(\alpha_1+\beta_1 m_1)(\gamma_1+\delta_1 m_2^2)}{(\gamma_1+\delta_1 M_2^2)(\alpha_1+\beta_1 M_1)}. \tag{46}$$

Setting

$$\frac{m_1}{M_1} = a_1 \leq 1,\ \frac{m_2}{M_2} = a_2 \leq 1. \tag{47}$$

In view of (47), equations (45) and (46) then implies that

$$\begin{aligned} \beta\gamma(a_1-a_2)M_2 &= \alpha\delta(a_1-1)a_1M_1^2 + \beta\delta(a_1a_2-1)a_1M_1^2M_2 - \alpha\gamma(a_1-1), \\ \beta_1\gamma_1(a_2-a_1)M_1 &= \alpha_1\delta_1(a_2-1)a_2M_2^2 + \beta_1\delta_1(a_1a_2-1)a_2M_1M_2^2 - \alpha_1\gamma_1(a_2-1). \end{aligned} \tag{48}$$

So right-hand sides of (48) are less then or equal to zero, and thus

$$a_1 - a_2 \leq 0,\ a_2 - a_1 \leq 0.$$

This implies that

$$a_1 \leq a_2 \leq a_1,$$

which hold if and only if $a_1 = a_2$. In view of (48) it follows that $a_1 = a_2 = 1$ and thus $m_1 = M_1,\ m_2 = M_2$. Hence, by Theorem 1.16 of [3], Ω of the system (1) is a global attractor. $\square$

3 Conclusion

This work is related to the dynamical properties of an anti-competitive system of rational difference equations. We proved that if $\beta\beta_1 < \gamma\gamma_1$ then every solution $\{(x_n, y_n)/x_n, y_n > 0\}$ of the system (1) is bounded and persists. We proved that if $\alpha_1+\beta_1L_1 < \left(\gamma_1+\delta_1\left(\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}$, $\alpha_1+\beta_1U_1 > \left(\gamma_1+\delta_1\left(\frac{(\gamma+\delta U_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta U_1^2)U_1-\alpha}{\beta}$ and $\frac{(\gamma+3\delta {U_1}^2)\left(\gamma_1\beta^2+3\delta_1\left((\gamma+\delta {U_1}^2)U_1-\alpha\right)^2\right)}{\beta^3\beta_1} < 1$ then system (1) has a unique positive equilibrium point $\Omega = (\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$. Furthermore method of Linearization is used to study the local stability about the unique positive equilibrium point Ω. Linear stability analysis shows that Ω is locally asymptotically stable if $\frac{2\delta U_1^2}{\gamma+\delta L_1^2}\left(1+\frac{2\delta_1U_2^2}{\gamma_1+\delta_1L_2^2}\right) + \frac{1}{\gamma_1+\delta_1L_2^2}\left(2\delta_1U_2^2+\frac{\beta\beta_1}{\gamma+\delta L_1^2}\right) < 1$ and unstable if $\frac{2\delta L_1^2}{\gamma+\delta U_1^2}\left(1+\frac{2\delta_1L_2^2}{\gamma_1+\delta_1U_2^2}\right) + \frac{1}{\gamma_1+\delta_1U_2^2}\left(2\delta_1L_2^2+\frac{\beta\beta_1}{\gamma+\delta U_1^2}\right) > 1$. Finally global dynamics about Ω is also investigated.

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AN ITERATIVE SCHEME FOR SOLVING SPLIT SYSTEM OF MINIMIZATION PROBLEMS

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Abstract. In this paper, we propose iterative algorithm for solving split system of minimization problems. We prove strong convergence of the sequences generated by the proposed algorithms. The iterative schemes are proposed in such a way that the selection of the step-sizes does not need any prior information about the operator norm. We further give some example to numerically verify the efficiency and implementation of our method.

Keywords: Minimization problem, strong convergence, Moreau-Yosida approximate, Hilbert space.
AMS Subject Classification: 49J53, 49J52, 47J05, 90C25, 65K10.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given nonempty closed convex subsets C_i $(i = 1, \dots, N)$ and Q_i $(i = 1, \dots, M)$ of H_1 and H_2, respectively. The multiple-set split feasibility problem (MSSFP) which was introduced by Censor et al. [10] is formulated as finding a point

$$\bar{x} \in \bigcap_{i=1}^{N} C_i \text{ such that } A\bar{x} \in \bigcap_{j=1}^{M} Q_j. \tag{1.1}$$

In particular, if $N = M = 1$, then the MSSFP (1.1) is reduced to find a point

$$\bar{x} \in C \text{ such that } A\bar{x} \in Q. \tag{1.2}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively. The problem (1.2) is known as the split feasibility problem (SFP) which was first introduced by Censor and Elfving [9] for modeling inverse problems in finite-dimensional Hilbert spaces. Many authors studied the SFP, see for example in [5, 9, 13, 14, 17, 24], and MSSFP, see for example in [10, 15, 19, 34, 35], provided the solution exists. The SFP and MSSFP arises in many fields in the real world, such as image reconstruction, modeling inverse problems, radiation therapy treatment planning and signal processing, and medical care; for details see [6, 7, 8] and the references therein.

Throughout this paper, unless otherwise stated, we assume that H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is nonzero bounded linear operator, I denotes the identity operator on a Hilbert space and $\mathbb{R}$ denotes set of real numbers.

Let us consider the following problem: find $x \in H_1$ with the property that

$$\min_{x\in H_1} \{f(x) + g_\lambda(Ax)\}, \tag{1.3}$$

where $f : H_1 \to \mathbb{R}\cup\{+\infty\}$, $g : H_2 \to \mathbb{R}\cup\{+\infty\}$ are two proper, convex, lower-semicontinuous functions and g_λ is Moreau-Yosida approximate [26] of the function g of parameter λ given by $g_\lambda(y) = \min_{u\in H_2}\{g(u) + \frac{1}{2\lambda}\|y - u\|^2\}$. In [21], Moudafi and Thakur introduced a weakly convergent algorithm solving the (1.3) in case $\arg\min f \cap A^{-1}(\arg\min g) \neq \emptyset$. Note that if we take $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise], the indicator function of nonempty, closed and convex subset C of H_1 and $g = \delta_Q$, the indicator

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function of nonempty, closed and convex subset Q of H_2, then problem problem (1.3) is reduced to the following minimization problem:

$$\min_{x\in C}\left\{\frac{1}{2\lambda}\|(I-P_Q)(Ax)\|^2\right\} \tag{1.4}$$

which, when $C\cap A^{-1}(Q)\neq\emptyset$, is equivalent to the split feasibility problem (SEP). It should also be noticed that (1.3) is equivalent to the problem of finding a point $\bar{x}\in H_1$ with the property

$$\bar{x}\in\arg\min f \text{ such that } A\bar{x}\in\arg\min g. \tag{1.5}$$

Moudafi and Thakur [21] used the idea of Lopez et al. [17] to introduce a new way of selecting the step sizes given by

$$\theta_{\lambda\mu}(x)=\sqrt{\|A^*(I-\text{prox}_{\lambda g})Ax\|^2+\|(I-\text{prox}_{\lambda\mu f})x\|^2}$$

with $h_\lambda(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda g})Ax\|^2$ and $l_{\lambda\mu}(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda\mu f})x\|^2$ where $\text{prox}_{\lambda f}(x)=\arg\min\limits_{u\in H_1}\{f(u)+\frac{1}{2\lambda}\|u-y\|^2\}$ stands for the proximal mapping of f. They proposed the following split proximal algorithm, which generates, from an initial point $x_1\in H_1$ assume that x_n has been constructed and $\theta_\lambda(x_n)\neq 0$, then compute x_{n+1} via the rule

$$x_{n+1}=\text{prox}_{\lambda\mu_n f}\big(x_n-\mu_nA^*(I-\text{prox}_{\lambda g})Ax_n\big) \tag{1.6}$$

where stepsize $\mu_n=\rho_n\frac{h_\lambda(x_n)+l_{\lambda\mu_n}(x_n)}{\theta^2_{\lambda\mu_n}(x_n)}$ with $0<\rho_n<4$ and if $\theta_{\lambda\mu_n}(x_n)=0$, then $x_{n+1}=x_n$ is a solution of (1.5) and the iterative process stops; otherwise, we set $n:=n+1$ and go to (1.6). Based on Moudafi and Thakur [21] many iterative algorithms are proposed for solving split minimization problem (1.5), see eg, Shehu and Iyiola in [28, 29, 30, 31], Shehu and Ogbuisi in [27], Shehu et al. in [32], Abbas et al. in [1].

Very recently, Shehu and Iyiola [29] proposed algorithm for solving (1.5) as follows:

$$\begin{cases} u,x_1\in H_1,\\ z_n=(1-\alpha_n)x_n+\alpha_nu,\\ y_n=z_n-\rho_n\frac{h(z_n)+l(z_n)}{\theta^2(z_n)}\big((I-\text{prox}_{\lambda f})z_n+A^*(I-\text{prox}_{\lambda g})Az_n\big),\\ x_{n+1}=(1-\beta_n)z_n+\beta_ny_n,\end{cases} \tag{1.7}$$

where $l(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda f})x\|^2$, $h(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda g})Ax\|^2$ and $\theta(x)=\|(I-\text{prox}_{\lambda f})x+A^*(I-\text{prox}_{\lambda g})Ax\|$. It was shown that the sequence $\{x_n\}$ generated by iterative algorithm (1.7) converges strongly to the solution of problem (1.5) under the following conditions:

(a) : $0<\alpha_n<1$, $\lim\limits_{n\to\infty}\alpha_n=0$ and $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$.
(b) : $0<\beta\leq\beta_n\leq\delta<1$,
(c) : $0<\rho_n<4$, $\liminf\limits_{n\to\infty}\rho_n(4-\rho_n)>0$.

To prove the strong convergence of iterative algorithm (1.7) the authors used simpler alternative proof without recourse to 'two cases method' of proof studied by other authors [1, 27, 30, 31, 32] and is also different from the approaches used in the proofs of [21, 28].

Motivated and inspired by results in [10, 21, 29], in this paper, we introduce and study the following *split system of minimization problem* (SSMP): finding a point $\bar{x}\in H_1$ with the property

$$\bar{x}\in\bigcap_{i=1}^{N}(\arg\min f_i) \text{ such that } A\bar{x}\in\bigcap_{j=1}^{M}(\arg\min g_j) \tag{1.8}$$

where $f_i:H_1\to\mathbb{R}\cup\{+\infty\}$ and $g_j:H_2\to\mathbb{R}\cup\{+\infty\}$ are proper, lower semicontinuous convex functions, $\arg\min f_i=\{\bar{x}\in H_1: f_i(\bar{x})\leq f_i(x),\ \forall x\in H_1\}$, $\arg\min g_j=\{\bar{y}\in H_2: g_j(\bar{y})\leq g_j(y),\ \forall y\in H_2\}$ and $i\in\Phi=\{1,\dots,N\}$, $j\in\Psi=\{1,\dots,M\}$. The solution set Γ of problem (1.8) is denoted by

$$\Gamma=\Big\{\bar{x}\in H_1:\bar{x}\in\bigcap_{i=1}^{N}(\arg\min f_i) \text{ and } A\bar{x}\in\bigcap_{j=1}^{M}(\arg\min g_j)\Big\}.$$

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Minimizers of any proper, lower semicontinuous function are exactly fixed points of its proximal mappings and proximal mappings are nonexpansive mapping (whose set of fixed points is closed and convex), we have that the set of minimizers of any proper, lower semicontinuous function is closed and convex. Therefore, since A bounded linear operator the solution set Γ of problem (1.8) is closed convex set. We assume Γ is nonempty.

We propose an iterative scheme using extended form of selecting step sizes used to solve (1.5) to the context of solving split system of minimization problem (1.8). The iterative scheme is developed by computation of proximal of f_i at z_n and g_j at Az_n in a parallel setting under simple assumptions on step sizes. Moreover, the technique of the proof takes some steps of [29, 33] so that it takes few steps to complete the proof. Note that if $f_i = f$ for all $i \in \Phi$ and $g_j = g$ for all $j \in \Psi$, then problem (1.8) reduces to the problem of split minimization problem (1.5) considered in [1, 21, 27, 28, 29, 30, 31, 32].

This paper is organized in the following way. In Section 2, we collect some basic and useful lemmas for further study. In Section 3, we propose and analyze the convergence result of our algorithm. In Section 4, we give a numerical example to discuss performance of the proposed algorithm.

2. Preliminary

In order to prove our main results, we recall some basic definitions and lemmas, which will be needed in the sequel. The symbols " $\rightharpoonup$ " and " $\to$ " denote weak and strong convergence, respectively.

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The metric projection on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Lemma 2.1. *Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, then $z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in C.$$

Let $T : H \to H$. Then,

(I): T is L-Lipschitz if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in H.$$

If $L \in (0, 1)$, then we call T a contraction. If $L = 1$, then T is called a nonexpansive mapping.

(II): T is firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - ||(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H,$$

which is equivalent to

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in H.$$

If T is firmly nonexpansive, $I - T$ is also firmly nonexpansive.

(III): strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|x - y\|^2$$

for all $x, y \in H$.

(IV): inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|Tx - Ty\|^2$$

for all $x, y \in H$.

Note that the proximal mapping of f is nonexpansive and firmly nonexpansive mapping. The minimizers of any proper, lower semicontinuous function are exactly fixed points of its proximal mappings. Many properties of proximal operator can be found in [12] and the references therein.

Lemma 2.2. *Let H be a real Hilbert space. Then,*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \quad \forall x, y \in H$$

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The following facts will be used several times in the paper.

Lemma 2.3. [2] *Let H be a real Hilbert space. Then,*

$$\|(1-\alpha)x+\alpha y\|^2 = (1-\alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

$\forall \alpha \in \mathbb{R}$, $\forall x, y \in H$.

Let H be a real Hilbert space, $\{x_1, x_2, \ldots, x_d\} \subset H$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_d\} \subset [0,1]$ with $\sum_{i=1}^{d} \lambda_i = 1$. Then, from [2, 37] one can see that

$$\Big\|\sum_{i=1}^{d} \lambda_i x_i\Big\|^2 \leq \sum_{i=1}^{d} \lambda_i \|x_i\|^2,$$

i.e., convexity of $\|.\|^2$.

Lemma 2.4. [18] *Let $\{a_n\}$ be the sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \delta_n,$$

where $\{\delta_n\}$ is a sequence of real numbers bounded from above and $0 \leq \alpha_n \leq 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then it holds that

$$\limsup_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \delta_n.$$

3. Main result

First we introduce the following settings which is an extension of settings introduced by Moudafi and Thakur [21]. Let $\lambda > 0$. For $x \in H_1$,

(i): for each $i \in \Phi$, define

$$l_i(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f_i})x\|^2 \text{ and } \nabla l_i(x) = (I - \mathrm{prox}_{\lambda f_i})x,$$

(ii): $l(x)$ and $\nabla l(x)$ are defined as $l(x) = l_{i_x}(x)$ and so $\nabla l(x) = \nabla l_{i_x}(x)$ where i_x is in Φ such that

$$i_x \in \arg\max\{\|(I - \mathrm{prox}_{\lambda f_i})x\| : i \in \Phi\},$$

(iii): for each $j \in \Psi$, define

$$h_j(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g_j})Ax\|^2 \text{ and } \nabla h_j(x) = A^*(I - \mathrm{prox}_{\lambda g_j})Ax,$$

(iv): for each $j \in \Psi$, define

$$\theta_j(x) = \max\{\|\nabla h_j(x)\|, \|\nabla l(x)\|\}.$$

It is easy to see that, for $x \in H_1$

$$\|\nabla l_i(x)\| \leq \|\nabla l_{i_x}(x)\| = \|\nabla l(x)\|, \quad \forall i \in \Phi$$

and

$$l_i(x) = \frac{1}{2}\|\nabla l_i(x)\|^2, \quad \forall i \in \Phi.$$

In this section, we propose algorithm for solving SSMP (1.8) and we analyse the convergence of the iteration sequence generated by the algorithm by assuming that the solution set Γ is nonempty. In order to design the algorithm, we consider the parameter sequences satisfying the following conditions.

Condition 1

(C1) : $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(C2) : $0 < \beta \leq \beta_n \leq \delta < 1$,

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(C3) : $0 < \xi \leq \xi_n^j \leq 1$ such that $\sum_{j=1}^{M} \xi_n^j = 1$ for each $n \geq 1$.

(C4) : $0 < \delta \leq \delta_n^i \leq 1$ such that $\sum_{i=1}^{N} \delta_n^i = 1$ for each $n \geq 1$.

(C5) : $0 < \rho_n < 2\delta$, $\liminf_{n\to\infty} \rho_n(2\delta - \rho_n) > 0$.

Throughout this paper, unless otherwise stated, Condition 1 refers to conditions (C1)-(C5) above. Using the definitions of ∇l_i, l_i, l, ∇l, h_j, ∇h_j and θ_j given in **(i)-(iv)**, we are now in a position to introduce our algorithm.

Algorithm 1
Initialization: Choose u, $x_1 \in H_1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\rho_n\}$, $\{\delta_n^i\}$ and $\{\xi_n^j\}$ be real sequences satisfying Condition 1.

Step 1: Evaluate $z_n = (1-\alpha_n)x_n + \alpha_n u$.
Step 2: For each $j \in \Psi$ compute $\theta_j(z_n)$, $h_j(z_n)$ and $l(z_n)$.
Let $\Psi_n = \{j \in \Psi : \theta_j(z_n) \neq 0\}$.
If $\Psi_n = \emptyset$, then z_n is a solution of (1.8) and the iterative process stops, otherwise, go to Step 3.
Step 3: For each $j \in \Psi$ evaluate $\mu_n^j = \rho_n \eta_n^j$ where

$$\eta_n^j = \begin{cases} 0, & \text{if } j \notin \Psi_n \\ \frac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}, & \text{if } j \in \Psi_n. \end{cases}$$

Step 4: Evaluate

$$w_n = z_n - \Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n)$$

and

$$t_n = z_n - \sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n).$$

Step 5: Evaluate

$$y_n = \frac{w_n + t_n}{2}.$$

Step 6: Evaluate $x_{n+1} = (1-\beta_n)z_n + \beta_n y_n$.
Step 7: Set $n := n+1$ and go to Step 1.

Lemma 3.1. *If $\Psi_n = \emptyset$, then z_n is the solution of (1.8).*

Proof. Suppose $\Psi_n = \emptyset$ at some iteration n.
Then, from $\Psi_n = \{j \in \Psi : \theta_j(z_n) \neq 0\} = \emptyset$, we have

$$\begin{aligned} &\max\{\|\nabla h_j(z_n)\|, \|\nabla l(z_n)\|\} = 0, \forall j \in \Psi \\ &\Leftrightarrow \|\nabla h_j(z_n)\| = 0 = \|\nabla l(z_n)\|, \forall j \in \Psi, \\ &\Leftrightarrow \|\nabla h_j(z_n)\| = 0 = \|\nabla l_i(z_n)\|, \forall i \in \Phi, \forall j \in \Psi, \\ &\Leftrightarrow A^*(I - \text{prox}_{\lambda g_j})Az_n = 0 = (I - \text{prox}_{\lambda f_i})z_n, \forall i \in \Phi, \forall j \in \Psi, \end{aligned}$$

and this implies that $z_n \in \Gamma$. □

Remark 3.2. Note that we can also use $\theta_j(x) = \sqrt{\|\nabla h_j(x)\|^2 + \|\nabla l(x)\|^2}$ instead of $\theta_j(x) = \max\{\|\nabla h_j(x)\|, \|\nabla l(x)\|\}$ and the proof for convergence will be the same. It is clear to see that

$$\max\{\|\nabla h_j(x)\|, \|\nabla l(x)\|\} \leq \sqrt{\|\nabla h_j(x)\|^2 + \|\nabla l(x)\|^2}.$$

If Algorithm 1 does not stop, then we have the following strong convergence theorem for approximation of solution of problem (1.8).

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Theorem 3.3. *The sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $\bar{x} \in \Gamma$ where $\bar{x} = P_\Gamma u$.*

Proof. Let $\bar{x} = P_\Gamma u$. Since $\mathrm{prox}_{\lambda f_i}$ and $\mathrm{prox}_{\lambda g_j}$ are firmly nonexpansive, $I - \mathrm{prox}_{\lambda f_i}$ and $I - \mathrm{prox}_{\lambda g_j}$ are also firmly nonexpansive, and since $\bar{x}$ verifies (1.8) (since minimizers of any function are exactly fixed-points of its proximal mapping), we have for all $z \in H_1$

$$\begin{aligned} \langle \nabla l_i(z), z - \bar{x} \rangle &= \langle (I - \mathrm{prox}_{\lambda f_i})z, z - \bar{x} \rangle \\ &\geq \|(I - \mathrm{prox}_{\lambda f_i})z\|^2 = 2l_i(z) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \langle \nabla h_j(z), z - \bar{x} \rangle &= \langle A^*(I - \mathrm{prox}_{\lambda g_j})Az, z - \bar{x} \rangle \\ &= \langle (I - \mathrm{prox}_{\lambda g_j})Az, Az - A\bar{x} \rangle \\ &\geq \|(I - \mathrm{prox}_{\lambda g_j})Az\|^2 = 2h_j(z), \quad \forall j \in \Psi. \end{aligned} \tag{3.2}$$

Note that, for all $z \in H_1$, $\|\nabla l(z)\| \leq \theta_j(z)$, $\|\nabla h_j(z)\| \leq \theta_j(z)$, $\forall j \in \Psi$,

$$\sum_{i\in\Phi} \delta_n^i \|\nabla l_i(z)\|^2 \leq \|\nabla l(z)\|^2 \text{ and } \sum_{i\in\Phi} \delta_n^i l_i(z) \geq \zeta l(z).$$

Using convexity of $\|.\|^2$ together with (3.1), we have

$$\begin{aligned} \|w_n - \bar{x}\|^2 &= \|z_n - \Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n) - \bar{x}\|^2 \\ &= \|z_n - \bar{x}\|^2 + \Big\|\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n)\Big\|^2 \\ &\qquad - 2\Big\langle\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n), z_n - \bar{x}\Big\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big)^2 \sum_{i\in\Phi} \delta_n^i \big\|\nabla l_i(z_n)\big\|^2 \\ &\qquad - 2\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \langle \nabla l_i(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \Big(\sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2\Big) \sum_{i\in\Phi} \delta_n^i \big\|\nabla l_i(z_n)\big\|^2 \\ &\qquad - 2\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \langle \nabla l_i(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \Big(\sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2\Big) \sum_{i\in\Phi} \delta_n^i \big\|\nabla l_i(z_n)\big\|^2 \\ &\qquad - 4\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i l_i(z_n). \end{aligned} \tag{3.3}$$

Similarly, using convexity of $\|.\|^2$ together with (3.2), we have

$$\begin{aligned} &\|t_n - \bar{x}\|^2 = \|z_n - \sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n) - \bar{x}\|^2 \\ &= \|z_n - \bar{x}\|^2 + \|\sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n)\|^2 - 2\langle \sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2 \|\nabla h_j(z_n)\|^2 - 2\sum_{j\in\Psi} \xi_n^j \mu_n^j \langle \nabla h_j(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2 \|\nabla h_j(z_n)\|^2 - 4\sum_{j\in\Psi} \xi_n^j \mu_n^j h_j(z_n). \end{aligned} \tag{3.4}$$

Now,

$$\begin{aligned}
&\Big(\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\Big)\sum_{i\in\Phi}\delta_n^i\big\|\nabla l_i(z_n)\big\|^2-4\Big(\sum_{j\in\Psi}\xi_n^j\mu_n^j\Big)\sum_{i\in\Phi}\delta_n^i l_i(z_n)\\
&\le\Big(\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\Big)\|\nabla l(z_n)\|^2-4\Big(\sum_{j\in\Psi}\xi_n^j\mu_n^j\Big)\delta_n^{i_{z_n}}l(z_n)\\
&\le\Big(\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\Big)\|\nabla l(z_n)\|^2-4\zeta\Big(\sum_{j\in\Psi}\xi_n^j\mu_n^j\Big)l(z_n)\\
&=\Big(\sum_{j\in\Psi}\xi_n^j(\rho_n\eta_n^j)^2\Big)\|\nabla l(z_n)\|^2-4\zeta\Big(\sum_{j\in\Psi}\xi_n^j\rho_n\eta_n^j\Big)l(z_n)\\
&=\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}\Big)^2\|\nabla l(z_n)\|^2-4\zeta\sum_{j\in\Psi_n}\xi_n^j\rho_n\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}l(z_n)\\
&\le\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^4(z_n)}\theta_j^2(z_n)-4\zeta\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}l(z_n)\\
&=\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}-4\zeta\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\tfrac{l(z_n)}{h_j(z_n)+l(z_n)}\\
&=\rho_n\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta l(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)},
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
&\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\|\nabla h_j(z_n)\|^2-4\sum_{j\in\Psi}\xi_n^j\mu_n^j h_j(z_n)\\
&=\sum_{j\in\Psi}\xi_n^j(\rho_n\eta_n^j)^2\|\nabla h_j(z_n)\|^2-4\sum_{j\in\Psi}\xi_n^j\rho_n\eta_n^j h_j(z_n)\\
&=\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}\Big)^2\|\nabla h_j(z_n)\|^2-4\sum_{j\in\Psi_n}\xi_n^j\rho_n\tfrac{h_j(z_n)+(z_n)}{\theta_j^2(z_n)}h_j(z_n)\\
&\le\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^4(z_n)}\theta_j^2(z_n)-4\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}h_j(z_n)\\
&=\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(x_n)}-4\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\tfrac{h_j(z_n)}{h_j(z_n)+l(z_n)}\\
&\le\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}-4\zeta\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\tfrac{h_j(z_n)}{h_j(z_n)+l(z_n)}\\
&=\rho_n\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta h_j(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}.
\end{aligned}\tag{3.6}$$

From convexity of $\|.\|^2$ and (3.3)-(3.6), we have

$$\begin{aligned}
\|y_n-\bar{x}\|^2&=\|\tfrac12(w_n+t_n)-\bar{x}\|^2\le\tfrac12\|w_n-\bar{x}\|^2+\tfrac12\|t_n-\bar{x}\|^2\\
&\le\|z_n-\bar{x}\|^2+\tfrac{\rho_n}{2}\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta l(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\\
&\qquad+\tfrac{\rho_n}{2}\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta h_j(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\\
&=\|z_n-\bar{x}\|^2+\rho_n(\rho_n-2\zeta)\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}.
\end{aligned}\tag{3.7}$$

From (3.7) and (C5), we have

$$\|y_n-\bar{x}\|\le\|z_n-\bar{x}\|.\tag{3.8}$$

Using (3.8) and the definition of x_{n+1}, we get

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2&=\|(1-\beta_n)z_n+\beta_n y_n-\bar{x}\|^2\\
&=\|(1-\beta_n)(z_n-\bar{x})+\beta_n(y_n-\bar{x})\|^2\\
&=(1-\beta_n)\|z_n-\bar{x}\|^2+\beta_n\|y_n-\bar{x}\|^2-\beta_n(1-\beta_n)\|z_n-y_n\|^2\\
&\le\|z_n-\bar{x}\|^2-\beta_n(1-\beta_n)\|z_n-y_n\|^2.
\end{aligned}\tag{3.9}$$

From (3.9) and the definition of z_n, we get

$$\begin{aligned}\|x_{n+1}-\bar{x}\| \leq \|z_n-\bar{x}\| &= (1-\alpha_n)\|x_n-\bar{x}\|+\alpha_n\|u-\bar{x}\| \\ &\leq \max\{\|x_n-\bar{x}\|,\|u-\bar{x}\|\} \\ &\ \ \vdots \\ &\leq \max\{\|x_n-\bar{x}\|,\|u-\bar{x}\|\}\end{aligned} \tag{3.10}$$

which shows that $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{Ay_n\}$ and $\{z_n\}$ are all bounded. Now,

$$\tfrac{1}{\beta_n}(x_{n+1}-z_n) = \tfrac{1}{\beta_n}\big((1-\beta_n)z_n+\beta_n y_n-z_n\big) = y_n-z_n \tag{3.11}$$

and

$$\|y_n-z_n\|^2 = \tfrac{1}{\beta_n^2}\|x_{n+1}-z_n\|^2 = \tfrac{\alpha_n}{\beta_n}\Big(\tfrac{\|x_{n+1}-z_n\|^2}{\alpha_n\beta_n}\Big). \tag{3.12}$$

Using (3.9) and (3.11), we have

$$\|x_{n+1}-\bar{x}\|^2 \leq \|z_n-\bar{x}\|^2 - \tfrac{1-\beta_n}{\beta_n}\|x_{n+1}-z_n\|^2. \tag{3.13}$$

From the definition of z_n, we have

$$\begin{aligned}\|z_n-\bar{x}\|^2 &= \|(1-\alpha_n)x_n+\alpha_n u-\bar{x}\|^2 \\ &= (1-\alpha_n)^2\|x_n-\bar{x}\|^2+\alpha_n^2\|u-\bar{x}\|^2+2\alpha_n(1-\alpha_n)\langle x_n-\bar{x},u-\bar{x}\rangle \\ &= (1-\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n^2\|u-\bar{x}\|^2+2\alpha_n(1-\alpha_n)\langle x_n-\bar{x},u-\bar{x}\rangle\end{aligned} \tag{3.14}$$

Thus, (3.13) and (3.14) gives

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 \leq\ & (1-\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n^2\|u-\bar{x}\|^2 \\ & +2\alpha_n(1-\alpha_n)\langle x_n-\bar{x},u-\bar{x}\rangle - \tfrac{1-\beta_n}{\beta_n}\|x_{n+1}-z_n\|^2.\end{aligned} \tag{3.15}$$

That is,

$$\|x_{n+1}-\bar{x}\|^2 \leq (1-\alpha_n)\|x_n-\bar{x}\|^2 - \alpha_n\Gamma_n \tag{3.16}$$

where

$$\Gamma_n = -\alpha_n\|u-\bar{x}\|^2+2(1-\alpha_n)\langle \bar{x}-x_n,u-\bar{x}\rangle + \frac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1}-z_n\|^2.$$

We know that $\{x_n\}$ is bounded and so it is bounded below. Hence, Γ_n is bounded below. Furthermore, using Lemma 2.4 and (C1), we have

$$\limsup_{n\to\infty}\|x_n-\bar{x}\| \leq \limsup_{n\to\infty}(-\Gamma_n) = -\liminf_{n\to\infty}\Gamma_n. \tag{3.17}$$

Therefore, $\liminf\limits_{n\to\infty}\Gamma_n$ is a finite real number and by (C1), we have

$$\liminf_{n\to\infty}\Gamma_n = \liminf_{n\to\infty}\big(2\langle\bar{x}-x_n,u-\bar{x}\rangle+\tfrac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1}-z_n\|^2\big).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup p$ in H_1 and

$$\liminf_{n\to\infty}\Gamma_n = \liminf_{k\to\infty}\big(2\langle\bar{x}-x_{n_k},u-\bar{x}\rangle+\tfrac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1}-z_{n_k}\|^2\big). \tag{3.18}$$

Since $\{x_n\}$ is bounded and $\liminf\limits_{n\to\infty}\Gamma_n$ is finite, we have that $\frac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1}-z_{n_k}\|^2$ is bounded. Also, by (C2), we have $\frac{1-\beta_n}{\alpha_n\beta_n} \geq \frac{1-\delta}{\alpha_n\beta_n} > 0$ and so we have that $\frac{1}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1}-z_{n_k}\|^2$ is bounded. Observe from (C1) and (C2), we have

$$0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \leq \frac{\alpha_{n_k}}{\beta} \to 0,\quad k\to\infty.$$

Therefore, we obtain from (3.12) and $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0,\ \ k\to\infty$ that

$$\|y_{n_k}-z_{n_k}\| \to 0,\quad k\to\infty. \tag{3.19}$$

From the definition of x_{n+1}, we have

$$\|x_{n_k+1}-z_{n_k}\| = \beta_{n_k}\|y_{n_k}-z_{n_k}\| \to 0,\quad k\to\infty$$

and

$$\|z_{n_k} - x_{n_k}\| = \alpha_{n_k}\|u - x_{n_k}\| \to 0, \quad k \to \infty. \tag{3.20}$$

Hence,

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \to 0, \quad k \to \infty.$$

Now, using (3.7), we obtain

$$\begin{aligned} &\rho_{n_k}(2\zeta - \rho_{n_k}) \sum_{j\in\Psi_{n_k}} \xi^j_{n_k} \tfrac{(h_j(z_{n_k})+l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} \le \|z_{n_k} - \bar{x}\|^2 - \|y_{n_k} - \bar{x}\|^2 \\ &\le (\|z_{n_k} - \bar{x}\| - \|y_{n_k} - \bar{x}\|)(\|z_{n_k} - \bar{x}\| + \|y_{n_k} - \bar{x}\|) \\ &= \|z_{n_k} - y_{n_k}\|(\|z_{n_k} - \bar{x}\| + \|y_{n_k} - \bar{x}\|). \end{aligned} \tag{3.21}$$

Therefore, (3.19), (3.21) and (C5) gives

$$\rho_{n_k}(2\zeta - \rho_{n_k}) \sum_{j\in\Psi_{n_k}} \xi^j_n \frac{(h_j(z_{n_k}) + l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} \to 0, \quad k \to \infty. \tag{3.22}$$

Again using (C5) together with (3.22) yields

$$\sum_{j\in\Psi_{n_k}} \xi^j_n \frac{(h_j(z_{n_k}) + l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} \to 0, \quad k \to \infty. \tag{3.23}$$

Hence, in view of (3.23) and restriction condition imposed on ξ^j_n, we have

$$\frac{(h_j(z_{n_k}) + l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} \to 0, \quad k \to \infty \tag{3.24}$$

for all $j \in \Psi_{n_k}$.

For each $i \in \Phi$ and for each $j \in \Psi$, $\nabla h_j(.)$ and $\nabla l_i(.)$ are Lipschitz continuous with constant $\|A\|^2$ and 1, respectively. Since the sequence $\{z_n\}$ is bounded and

$$\|\nabla h_j(z_n)\| = \|\nabla h_j(z_n)\| = \|\nabla h_j(z_n) - \nabla h_j(\bar{x})\| \le \|A\|^2\|z_n - \bar{x}\|, \forall j \in \Psi,$$
$$\|\nabla l_i(z_n)\| = \|\nabla l_i(z_n)\| = \|\nabla l_i(z_n) - \nabla l_i(\bar{x})\| \le \|z_n - \bar{x}\|, \forall i \in \Phi,$$

we have the sequences $\{\|\nabla l_i(z_n)\|\}_{n=1}^{+\infty}$ and $\{\|\nabla h_j(z_n)\|\}_{n=1}^{+\infty}$ are bounded. Hence, the boundedness of $\{\|\nabla l_i(z_n)\|\}_{n=1}^{+\infty}$ for all $i \in \Phi$ gives $\{\|\nabla l(z_n)\|\}_{n=1}^{+\infty}$ is bounded. Thus, we have $\{\theta_j^2(z_n)\}_{n=1}^{+\infty}$ is bounded and hence $\{\theta_j^2(z_{n_k})\}_{k=1}^{+\infty}$ is bounded. Consequently, using (3.24), we have for each $j \in \Psi_{n_k}$

$$\lim_{k\to+\infty}(h_j(z_{n_k}) + l(z_{n_k})) = 0 \Leftrightarrow \lim_{k\to+\infty} h_j(z_{n_k}) = \lim_{k\to+\infty} l(z_{n_k}) = 0.$$

Since $\theta_j(z_{n_k}) = 0$ for each $j \notin \Psi_{n_k}$ and this results $h_j(z_{n_k}) = 0 = l(z_{n_k})$ for each $j \notin \Psi_{n_k}$. Hence, using $\lim_{n\to+\infty} h_j(z_{n_k}) = \lim_{k\to+\infty} l(z_n) = 0$ for each $j \in \Psi_{n_k}$ and $h_j(z_{n_k}) = 0 = l(z_{n_k})$ for each $j \notin \Psi_{n_k}$, we have

$$\lim_{k\to+\infty} h_j(z_{n_k}) = \lim_{k\to+\infty} l(z_{n_k}) = 0, \ \ \forall j \in \Psi.$$

From the definition of $l(z_{n_k})$, we can have $l_i(z_{n_k}) \le l(z_{n_k}), \ \ \forall i \in \Phi$. Therefore,

$$\lim_{k\to+\infty} h_j(z_{n_k}) = \lim_{k\to+\infty} l_i(z_{n_k}) = 0, \ \ \forall i \in \Phi, \forall j \in \Psi.$$

Since $x_{n_k} \to p$ and using (3.20), we have $z_{n_k} \to p$.

The lower-semicontinuity of $h_j(.)$ implies that

$$0 \le h_j(p) \le \liminf_{k\to\infty} h_j(z_{n_k}) = \lim_{k\to\infty} h_j(z_{n_k}) = 0, \ \ \forall j \in \Psi.$$

That is, $h_j(p) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g_j})Ap\|^2 = 0$ for all $j \in \Psi$, i.e., Ap is a fixed point of the proximal mapping of each g_j or equivalently, $0 \in \partial g_j(Ap)$ for all $j \in \Psi$. In other words, Ap is a minimizer of each g_j for all $j \in \Psi$. Likewise, the lower-semicontinuity of $l_i(.)$ implies that

$$0 \le l_i(p) \le \liminf_{k\to\infty} l_i(z_{n_k}) = \lim_{k\to\infty} l_i(z_{n_k}) = 0, \ \ \forall i \in \Phi.$$

That is, $l_i(p) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f_i})p\|^2 = 0$ for all $i \in \Phi$, i.e., p is a fixed point of the proximal mapping of each f_i or equivalently, $0 \in \partial f_i(p)$ for all $i \in \Phi$. In other words, p is a minimizer of each f_i for all $i \in \Phi$. Thus, $p \in \Gamma$.
Now, we obtain from (3.18), Lemma 2.1 and $\bar{x} = P_\Gamma u$ that

$$\begin{aligned}\liminf_{n\to\infty} \Gamma_n &= \liminf_{k\to\infty} \big(2\langle \bar{x} - x_{n_k}, u - \bar{x}\rangle + \tfrac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1} - z_{n_k}\|^2\big)\\ &\geq 2\liminf_{k\to\infty}\langle \bar{x} - x_{n_k}, u - \bar{x}\rangle\\ &\geq 2\langle \bar{x} - p, u - \bar{x}\rangle \geq 0.\end{aligned}$$

Then we have from (3.17) that

$$\limsup_{n\to\infty} \|x_n - \bar{x}\|^2 \leq \limsup_{n\to\infty}(-\Gamma_n) = -\liminf_{n\to\infty} \Gamma_n \leq 0.$$

Therefore, $\|x_n - \bar{x}\| \to 0$ and this implies that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. □

It is worth mentioning that our approach also works for approximation of solution of split minimization problem (1.5). Let Ω_1 denote the solution set of (1.5), i.e.,

$$\Omega_1 = \{\bar{x} \in H_1 : \bar{x} \in \arg\min f \text{ and } A\bar{x} \in \arg\min g\}.$$

For $x \in H_1$, set $l(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f})x\|^2$, $\nabla l(x) = (I - \mathrm{prox}_{\lambda f})x$, $h(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g})Ax\|^2$, $\nabla h(x) = A^*(I - \mathrm{prox}_{\lambda g})Ax$ and $\theta(x) = \max\{\|\nabla h(x)\|, \|\nabla l(x)\|\}$. Thus, the following Corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\rho_n\}$ are real sequences satisfying the following conditions:*

(a) : $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
(b) : $0 < \beta \leq \beta_n \leq \delta < 1$,
(c) : $0 < \rho_n < 2\delta$, $\liminf_{n\to\infty} \rho_n(2\delta - \rho_n) > 0$.

then the sequence $\{x_n\}$ generated by iterative algorithm

$$\begin{cases} u, x_1 \in H_1,\\ z_n = (1-\alpha_n)x_n + \alpha_n u,\\ \mu_n = \begin{cases} \rho_n 0, & \text{if } \theta(z_n) = 0\\ \rho_n \frac{h(z_n)+l(z_n)}{\theta^2(z_n)}, & \text{if } \theta(z_n) \neq 0. \end{cases}\\ y_n = z_n - \frac{1}{2}\mu_n\big(\nabla l(z_n) + \nabla h(z_n)\big),\\ x_{n+1} = (1-\beta_n)z_n + \beta_n y_n, \end{cases} \tag{3.25}$$

converges strongly to $\bar{x} \in \Omega_1$ where $\bar{x} = P_{\Omega_1}u$.

Proof. Setting $f_i = f$ for all $i \in \Phi$ and $g_j = g$ for all $j \in \Psi$ in Theorem 3.3, we obtain the desired result. □

Remark 3.5. Iterative algorithm (3.25) seems to share a similar structure with the proposed algorithm in [29]. However, the selection of the step-sizes and their restriction slightly different.

The feasibility problem (convex feasibility problem), equilibrium problem and inclusion problem can be converted to the fixed point problem of firmly nonexpansive mapping. We can apply our algorithm to solve split system of feasibility problems (MSSFPs), split system of equilibrium problems and split system of inclusion problems.

1. *Multiple-set split feasibility problem* (1.1) by replacing $\mathrm{prox}_{\lambda f_i}$ by projection mapping P_{C_i} and $\mathrm{prox}_{\lambda g_j}$ by projection mapping P_{Q_j} in the Algorithm 1, for all $i' \in \Phi'$, $i \in \Phi = \{1, 2, \dots, N\}$ and $j \in \Psi = \{1, 2, \dots, M\}$.

2. *Split system of equilibrium problem:* Let $f_i : H_1 \times H_1 \to \mathbb{R}$ and $g_j : H_2 \times H_2 \to \mathbb{R}$ be bifunctions where $i \in \Phi = \{1, \dots, N\}$, $j \in \Psi = \{1, \dots, M\}$. Split system of equilibrium problem of a problem of find $\bar{x} \in H_1$ such that
$$\begin{cases} f_i(\bar{x}, x) \geq 0, \quad \forall x \in H_1, \forall i \in \Phi, \\ g_j(A\bar{x}, u) \geq 0, \quad \forall u \in H_2, , \forall j \in \Psi. \end{cases} \tag{3.26}$$
Our iterative algorithm solves (3.26) by replacing proximal mappings by the resolvent operators associated to monotone equilibrium bifunctions, see [11, 3, 22].
3. *Split null point problem*: Let $T_i : H_1 \to 2^{H_1}$, $U_j : H_2 \to 2^{H_2}$ be maximal monotone mappings for all $i \in \Phi = \{1, \dots, N\}$ and $j \in \Psi = \{1, \dots, M\}$. The split system of inclusion problem is to find $\bar{x} \in H_1$ such that
$$\begin{cases} 0 \in T_i(\bar{x}), \quad \forall i \in \Phi, \\ 0 \in U_j(A\bar{x}), \quad \forall j \in \Psi. \end{cases} \tag{3.27}$$
Our iterative algorithm solves (3.27) by replacing proximal mappings by the resolvent operators associated to the maximal monotone operators, see, [4, 25, 16, 20, 23, 36].

Our algorithm works for several split type problems and avoids the computational cost of finding operator norm.

4. Numerical results

Now in this section we will consider SSMP (1.8) involving quadratic optimization problems. The algorithm has been coded in Matlab R2017a running on MacBook 1.1 GHz Intel Core m3 8 GB 1867 MHz LPDDR3. Let $H_1 = \mathbb{R}^p$ and $H_2 = \mathbb{R}^q$. Consider
$$f_i(x) = \frac{1}{2}x^T B_i x + x^T D_i, \quad i \in \Phi = \{1, \dots, N\},$$
$$g_1(u) = \|u\|_q \text{ and } g_2(u) = \sum_{k=1}^{q} h(u_k)$$
where for each $i \in \Phi$, B_i is invertible symmetric positive semidefinite $p \times p$ matrix and each D_i are vectors in $\mathbb{R}^p$, $u = (u_1, u_2, \dots, u_q) \in \mathbb{R}^q$, $\|.\|_q$ is the Euclidean norm in $\mathbb{R}^q$ and
$$h(u_k) = \max\{|u_k| - 1, 0\}$$
for $k = 1, 2, \dots, q$.

Now for $\lambda = 1$, the proximal operators are given by
$$\text{prox}_{\lambda f_i}(x) = (I + B_i)^{-1}(x - D_i), \quad i \in \Phi,$$
$$\text{prox}_{\lambda g_1}(u) = \begin{cases} \left(1 - \frac{1}{\|u\|_q}\right)u, & \|u\|_q \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$
and
$$\text{prox}_{\lambda g_2}(u) = (\text{prox}_{\lambda h}(u_1), \text{prox}_{\lambda h}(u_2), \dots, \text{prox}_{\lambda h}(u_q))$$
where
$$\text{prox}_{\lambda h}(u_k) = \begin{cases} u_k, & \text{if } |u_k| < 1 \\ \text{sign}(u_k), & \text{if } 1 \leq |u_k| \leq 2 \\ \text{sign}(u_k - 1), & \text{if } |u_k| > 2. \end{cases}$$
The proximal operator (4.1) is called the block soft thresholding obtained in de-noising model.

We set $D_i = 0$ (zero vector in $\mathbb{R}^p$) for all $i \in \Phi$. Let $N = 3$, $p = q$, A is identity $p \times p$ matrix and B_1, B_2 and B_3 are randomly generated invertible symmetric positive semidefinite $p \times p$ matrices. Hence, with this setting, it is clear to see that $\Gamma = \{0\}$. In all the experiments we took $\delta_n^i = \frac{i}{6}$ and $\xi_n^j = \frac{j}{3}$ for $i \in \Phi = \{1, 2, 3\}$, $j \in \Psi = \{1, 2\}$, $\rho_n = \frac{1}{10}$ as $0 < \rho_n < 2\zeta$ for $\zeta = \frac{1}{6}$. Table 1, 2 and 3 describe the average execution time in second (CPU-t(s)) and the number of iterations (Iter(n)) of our algorithm for this example. The stopping criteria in the tables 1, 2 and 3 is defined as $\frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} \leq$ TOL.

TABLE 1. For $p = q = 4$, $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = 0.9$, $u = (1, 1, 1, 1)$, $x_1 = 10u$.

Iter(n)	TOL	CPU-t(s)	x_n				$\|x_{n+1} - x_n\|$
			x_n^1	x_n^2	x_n^3	x_n^4	
1			10	10	10	10	9.0304
2			5.4841	5.4854	5.4842	5.4852	3.0046
3			3.9817	3.9834	3.9820	3.9826	1.5000
4			3.2317	3.2341	3.2314	3.2326	0.8991
5			2.7825	2.7839	2.7820	2.7831	0.5988
6			2.4828	2.4844	2.4827	2.4838	0.4277
7			2.2691	2.2702	2.2690	2.2699	0.3204
⋮			⋮	⋮	⋮	⋮	⋮
23			1.3796	1.3800	1.3796	1.3799	0.0323
24	10^{-3}	0.0352	1.3634	1.3638	1.3634	1.3637	0.0297

TABLE 2. For $p = q = 100$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.5$ and randomly generated starting points u and x_1 in $\mathbb{R}^{100}$.

Iter(n)	TOL	CPU-t(s)	$\|x_{n+1} - x_n\|$
1			6383.1845
2			1519.9088
3			554.2387
4			247.0358
5			124.2771
⋮			⋮
15			1.5845
16	10^{-4}	0.2923	0.5736

TABLE 3. For $p = q = 200$, $\alpha_n = \frac{1}{10(n+1)})$, $\beta_n = 0.1$ and randomly generated starting points u and x_1 in $\mathbb{R}^{200}$.

Iter(n)	TOL	CPU-t(s)	$\|x_{n+1} - x_n\|$
1			14554.8769
2			3475.8500
3			1270.6095
4			567.6027
5			286.1360
6	10^{-2}	0.0093	156.6170

From the tables 1-3 we can see that our proposed algorithm is efficient and easy to implement.

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Complex Korovkin Theory

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Abstract

Let K be a compact convex subspace of $\mathbb{C}$ and $C(K, \mathbb{C})$ the space of continuous functions from K into $\mathbb{C}$. We consider bounded linear functionals from $C(K, \mathbb{C})$ into $\mathbb{C}$ and bounded linear operators from $C(K, \mathbb{C})$ into itself. We assume that these are bounded by companion real positive linear entities, respectively. We study quantitatively the rate of convergence of the approximation of these linearities to the corresponding unit elements. Our results are inequalities of Korovkin type involving the complex modulus of continuity and basic test functions.

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1 Introduction

The study of the convergence of positive linear operators became more intensive and attractive when P. Korovkin (1953) proved his famous theorem (see [7], p. 14).

Korovkin's First Theorem. Let $[a, b]$ be a compact interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators L_n mapping $C([a, b])$ into itself. Assume that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$. Then $(L_n f)$ converges uniformly to f on $[a, b]$ for all functions of $f \in C([a, b])$.

So a lot of authors since then have worked on the theoretical aspects of the above convergence. But R. A. Mamedov (1959) (see [8]) was the first to put Korovkin's theorem in a quantitative scheme.

Mamedov's Theorem. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators in the space $C([a, b])$, for which $L_n 1 = 1$, $L_n(t, x) = x + \alpha_n(x)$, $L_n(t^2, x) = x^2 + \beta_n(x)$. Then it holds

$$\|L_n(f, x) - f(x)\|_\infty \leq 3\omega_1\left(f, \sqrt{d_n}\right),$$

where ω_1 is the first modulus of continuity and $d_n = \|\beta_n(x) - 2x\alpha_n(x)\|_\infty$.

An improvement of the last result was the following.

Shisha and Mond's Theorem. (1968, see [10]). Let $[a,b] \subset \mathbb{R}$ be a compact interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$. For $n = 1, 2, ...,$ suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, ...,$ it holds

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \cdot \|L_n 1 - 1\|_\infty + \|L_n(1) + 1\|_\infty \cdot \omega_1(f, \mu_n),$$

where

$$\mu_n := \left\| \left(L_n\left((t-x)^2 \right) \right)(x) \right\|_\infty^{\frac{1}{2}}.$$

Shisha-Mond inequality generated and inspired a lot of research done by many authors worldwide on the rate of convergence of a sequence of positive linear operators to the unit operator, always producing similar inequalities however in many different directions, e.g., see the important work of H. Censka of 1983 in [6], etc.

The author (see [1]) in his 1993 research monograph, produces in many directions best upper bounds for $|(L_n f)(x_0) - f(x_0)|$, $x_0 \in Q \subseteq \mathbb{R}^n$, $n \geq 1$, compact and convex, which lead for the first time to sharp/attained inequalities of Shisha-Mond type. The method of proving is probabilistic from the theory of moments. His pointwise approach is closely related to the study of the weak convergence with rates of a sequence of finite positive measures to the unit measure at a specific point.

The author in [3], pp. 383-412 continued this work in an abstract setting: Let X be a normed vector space, Y be a Banach lattice; $M \subset X$ is a compact and convex subset. Consider the space of continuous functions from M into Y, denoted by $C(M,Y)$; also consider the space of bounded functions $B(M,Y)$. He studied the rate of the uniform convergence of lattice homomorphisms $T : C(M,Y) \to C(M,Y)$ or $T : C(M,Y) \to B(M,Y)$ to the unit operator I. See also [2].

Also the author in [4], pp. 175-188 continued the last abstract work for bounded linear operators that are bounded by companion real positive linear operators. Here the invoved functions are from $[a,b] \subset \mathbb{R}$ into $(X, \|\cdot\|)$ a Banach space.

All the above have inspired and motivated the work of this article. Our results are of Shisha-Mond type, i.e., of Korovkin type.

Namely here let K be a convex and compact subset of $\mathbb{C}$ and l be a linear functional from $C(K,\mathbb{C})$ into $\mathbb{C}$, and let $\widetilde{l}$ be a positive linear functional from $C(K,\mathbb{R})$ into $\mathbb{R}$, such that $|l(f)| \leq \widetilde{l}(|f|)$, $\forall\, f \in C(K,\mathbb{C})$.

Clearly then l is a bounded linear functional. Initially we create a quantitative Korovkin type theory over the last described setting, then we transfer these results to related bounded linear operators with similar properties.

2 Background

We need

Theorem 1 *Let* $K \subseteq (\mathbb{C}, |\cdot|)$ *and* f *a function from* K *into* $\mathbb{C}$. *Consider the first complex modulus of continuity*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in K \\ |x-y|<\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{1}$$

We have:

(1)' If K *is open convex or compact convex, then* $\omega_1(f,\delta) < \infty$, $\forall\ \delta > 0$, *where* $f \in UC(K,\mathbb{C})$ *(uniformly continuous functions).*

(2)' If K *is open convex or compact convex, then* $\omega_1(f,\delta)$ *is continuous on* $\mathbb{R}_+$ *in* δ, *for* $f \in UC(K,\mathbb{C})$.

(3)' If K *is convex, then*

$$\omega_1(f, t_1 + t_2) \leq \omega_1(f, t_1) + \omega_1(f, t_2), \quad t_1, t_2 > 0, \tag{2}$$

that is the subadditivity property is true. Also it holds

$$\omega_1(f, n\delta) \leq n\omega_1(f,\delta) \tag{3}$$

and

$$\omega_1(f, \lambda\delta) \leq \lceil\lambda\rceil\, \omega_1(f,\delta) \leq (\lambda+1)\, \omega_1(f,\delta), \tag{4}$$

where $n \in \mathbb{N}$, $\lambda > 0$, $\delta > 0$, $\lceil\cdot\rceil$ *is the ceiling of the number.*

(4)' Clearly in general $\omega_1(f,\delta) \geq 0$ *and is increasing in* $\delta > 0$ *and* $\omega_1(f,0) = 0$.

(5)' If K *is open or compact, then* $\omega_1(f,\delta) \to 0$ *as* $\delta \downarrow 0$, *iff* $f \in UC(K,\mathbb{C})$.

(6)' It holds

$$\omega_1(f+g,\delta) \leq \omega_1(f,\delta) + \omega_1(g,\delta), \tag{5}$$

for $\delta > 0$, *any* $f, g : K \to \mathbb{C}$, $K \subset \mathbb{C}$ *is arbitrary.*

Proof. (1)' Here K is open convex. Let here $f \in UC(K,\mathbb{C})$, iff $\forall\ \varepsilon > 0$, $\exists\ \delta > 0 : |x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $\varepsilon_0 > 0$ then $\exists\ \delta_0 > 0 :$ $|x-y| \leq \delta_0$ with $|f(x) - f(y)| < \varepsilon_0$, hence $\omega_1(f,\delta_0) \leq \varepsilon_0 < \infty$.

Let $\delta > 0$ arbitrary and $x, y \in K : |x-y| \leq \delta$. Choose $n \in \mathbb{N} : n\delta_0 > \delta$, and set $x_i = x + \frac{i}{n}(y-x)$, $0 \leq i \leq n$. Notice that all $x_i \in K$. Then

$$|f(x) - f(y)| = \left|\sum_{i=0}^{n-1} (f(x_i) - f(x_{i+1}))\right| \leq$$

$$|f(x) - f(x_1)| + |f(x_1) - f(x_2)| + |f(x_2) - f(x_3)| + ... + |f(x_{n-1}) - f(y)| \leq$$

$$n\omega_1(f,\delta_0) \leq n\varepsilon_0 < \infty,$$

since $|x_i - x_{i+1}| = \frac{1}{n}|x-y| \leq \frac{1}{n}\delta < \delta_0$.

Thus $\omega_1(f,\delta) \leq n\varepsilon_0 < \infty$, proving the claim. If K is compact convex, then claim is obvious.

(2)' Let $x, y \in K$ and let $|x-y| \leq t_1 + t_2$, then there exists a point $z \in \overline{xy}$, $z \in K : |x-z| \leq t_1$ and $|y-z| \leq t_2$, where $t_1, t_2 > 0$.

Notice that

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq \omega_1(f, t_1) + \omega_1(f, t_2).$$

Hence

$$\omega_1(f, t_1 + t_2) \leq \omega_1(f, t_1) + \omega_1(f, t_2),$$

proving (3)'. Then by the obvious property (4)' we get

$$0 \leq \omega_1(f, t_1 + t_2) - \omega_1(f, t_1) \leq \omega_1(f, t_2),$$

and

$$|\omega_1(f, t_1 + t_2) - \omega_1(f, t_1)| \leq \omega_1(f, t_2).$$

Let $f \in UC(K, \mathbb{C})$, then $\lim_{t_2 \downarrow 0} \omega_1(f, t_2) = 0$, by property (5)'. Hence $\omega_1(f, \cdot)$ is continuous on $\mathbb{R}_+$.

(5)' ($\Rightarrow$) Let $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ with $\omega_1(f,\delta) \leq \varepsilon$. I.e. $\forall\, x, y \in K : |x-y| \leq \delta$ we get $|f(x) - f(y)| \leq \varepsilon$. That is $f \in UC(K, \mathbb{C})$.

($\Leftarrow$) Let $f \in UC(K, \mathbb{C})$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$: whenever $|x-y| \leq \delta$, $x, y \in K$, it implies $|f(x) - f(y)| \leq \varepsilon$. I.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0 : \omega_1(f,\delta) \leq \varepsilon$. That is $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$.

(6)' Notice that

$$|(f(x) + g(x)) - (f(y) + g(y))| \leq |f(x) - f(y)| + |g(x) - g(y)|.$$

That is property (6)' now is clear. ■

We need

Theorem 2 *([1], p. 208) Let $(V_1, \|\cdot\|)$, $(V_2, \|\cdot\|)$ be real normed vector spaces and $Q \subseteq V_1$ which is star- shaped relative to the fixed point x_0. Consider $f : Q \to V_2$ with the properties:*

$$f(x_0) = 0, \text{ and } \|s-t\| \leq h \text{ implies } \|f(s) - f(t)\| \leq w; \quad w, h > 0. \tag{6}$$

Then, there exists a maximal such function Φ, namely

$$\Phi(t) := \left\lceil \frac{\|t - x_0\|}{h} \right\rceil \cdot w \cdot \overrightarrow{i}, \tag{7}$$

where $\overrightarrow{i}$ is any unit vector in V_2.

That is

$$\|f(t)\| \leq \|\Phi(t)\|, \text{ all } t \in Q. \tag{8}$$

4

Corollary 3 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ be a compact convex subset, and $f \in C(K, \mathbb{C})$. Then*

$$|f(x) - f(x_0)| \leq \omega_1(f, \delta) \left\lceil \frac{|x - x_0|}{\delta} \right\rceil, \quad \delta > 0, \tag{9}$$

$\forall\ x, x_0 \in K$.

We make

Remark 4 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ be a compact subset and $g \in C(K, \mathbb{R})$.*

A linear functional I from $C(K, \mathbb{R})$ into $\mathbb{R}$ is positive, iff $I(g_1) \geq I(g_2)$, whenever $g_1 \geq g_2$, where $g_1, g_2 \in C(K, \mathbb{R})$.

Let us assume that I is a positive linear functional. Then by Riesz representation theorem, [9], p. 304, there exists a unique Borel measure μ on K such that

$$I(g) = \int_K g(t)\, d\mu(t), \tag{10}$$

$\forall\ g \in C(K, \mathbb{R})$.

We make

Remark 5 *Here initially we follow [5].*

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$.

We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt. \tag{11}$$

By triangle inequality we have

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(z(t))\, z'(t)\, dt \right| \leq \int_a^b |f(z(t))|\, |z'(t)|\, dt := \int_\gamma |f(z)|\, |dz|. \tag{12}$$

Inequalities (12) provide a typical example on linear functionals: clearly $\int_\gamma f(z)\, dz$ induces a linear functional from $C(\gamma, \mathbb{C})$ into $\mathbb{C}$, and $\int_\gamma |f(z)|\, |dz|$ involves a positive linear functional from $C(\gamma, \mathbb{R})$ into $\mathbb{R}$.

Thus, be given K a convex and compact subset of $\mathbb{C}$ and l be a linear functional from $C(K, \mathbb{C})$ into $\mathbb{C}$, it is not strange to assume that there exists a positive linear functional $\widetilde{l}$ from $C(K, \mathbb{R})$ into $\mathbb{R}$, such that

$$|l(f)| \leq \widetilde{l}(|f|), \quad \forall\ f \in C(K, \mathbb{C}). \tag{13}$$

Furthermore, we may assume that $\widetilde{l}(1(\cdot)) = 1$, where $1(t) = 1$, $\forall\ t \in K$, $l(c(\cdot)) = c, \forall c \in \mathbb{C}$ where $c(t) = c$, $\forall\ t \in K\cdot$

We call $\widetilde{l}$ the companion functional to l.

Here $\mathbb{C}$ is a vector space over the field of reals. The functional l is linear over $\mathbb{R}$ and the functional $\widetilde{l}$ is linear over $\mathbb{R}$.

Next we study approximation properties of $\left(l_n, \widetilde{l}_n\right)$ pairs, $n \in \mathbb{N}$.

3 Main Results - I

First about linear functionals:

We present the following quantitative approximation result of Korovkin type.

Theorem 6 *Here K is a convex and compact subset of $\mathbb{C}$ and l_n is a sequence of linear functionals from $C(K,\mathbb{C})$ into $\mathbb{C}$, $n \in \mathbb{N}$. There is a sequence of companion positive linear functionals $\widetilde{l}_n$ from $C(K,\mathbb{R})$ into $\mathbb{R}$, such that*

$$|l_n(f)| \leq \widetilde{l}_n(|f|), \ \ \forall \ f \in C(K,\mathbb{C}), \ \forall \ n \in \mathbb{N}. \tag{14}$$

Additionally, we assume that $\widetilde{l}_n(1(\cdot)) = 1$ and $l_n(c(\cdot)) = c, \forall c \in \mathbb{C} \ \forall \ n \in \mathbb{N}$.

Then

$$|l_n(f) - f(x_0)| \leq 2\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \ \ \forall \ n \in \mathbb{N}, \ \forall \ x_0 \in K, \tag{15}$$

$\forall \ f \in C(K,\mathbb{C})$.

Proof. We notice that

$$|l_n(f) - f(x_0)| = |l_n(f) - l_n(f(x_0)(\cdot))| =$$

$$|l_n(f(\cdot) - f(x_0)(\cdot))| \overset{(14)}{\leq} \widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) \overset{(\text{by } \delta>0, \ (9))}{\leq}$$

$$\widetilde{l}_n\left(\omega_1(f,\delta)\left\lceil\frac{|\cdot - x_0|}{\delta}\right\rceil\right) \leq \omega_1(f,\delta)\,\widetilde{l}_n\left(1(\cdot) + \frac{|\cdot - x_0|}{\delta}\right) =$$

$$\omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot)) + \frac{1}{\delta}\widetilde{l}_n(|\cdot - x_0|)\right] =$$

$$\omega_1(f,\delta)\left[1 + \frac{1}{\delta}\widetilde{l}_n(|\cdot - x_0|)\right] = 2\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \tag{16}$$

by choosing

$$\delta := \widetilde{l}_n(|\cdot - x_0|),$$

if $\widetilde{l}_n(|\cdot - x_0|) > 0$, that is proving (15).

Next, we consider the case of $\widetilde{l}_n(|\cdot - x_0|) = 0$. By Riesz representation theorem, see (10) there exists a probability measure μ such that

$$\widetilde{l}_n(g) = \int_K g(t)\,d\mu(t), \ \ \forall \ g \in C(K,\mathbb{R}). \tag{17}$$

That is, here it holds

$$\int_K |t - x_0|\,d\mu(t) = 0,$$

which implies $|t - x_0| = 0$, a.e, hence $t - x_0 = 0$, a.e, and $t = x_0$, a.e. Consequently $\mu(\{t \in K : t \neq x_0\}) = 0$. Hence $\mu = \delta_{x_0}$, the Dirac measure with support only $\{x_0\}$.

Therefore in that case $\widetilde{l}_n(g) = g(x_0)$, $\forall \ g \in C(K,\mathbb{R})$. Thus, it holds $\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right) = \omega_1(f,0) = 0$, and $\widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) = |f(x_0) - f(x_0)| = 0$, giving $|l_n(f) - f(x_0)| = 0$. That is (15) is again true. ■

Remark 7 *We have that*

$$\widetilde{l}_n\left(\left|\cdot - x_0\right|\right) = \int_K \left|t - x_0\right| d\mu\left(t\right)$$

(by Schwarz's inequality)

$$\leq \left(\int_K 1 d\mu\left(t\right)\right)^{\frac{1}{2}} \left(\int_K \left|t - x_0\right|^2 d\mu\left(t\right)\right)^{\frac{1}{2}} =$$

$$\left(\widetilde{l}_n\left(1\right)\right)^{\frac{1}{2}} \left(\int_K \left|t - x_0\right|^2 d\mu\left(t\right)\right)^{\frac{1}{2}} = \left(\widetilde{l}_n\left(\left|\cdot - x_0\right|^2\right)\right)^{\frac{1}{2}}. \tag{18}$$

We give

Corollary 8 *All as in Theorem 6. Then*

$$\left|l_n\left(f\right) - f\left(x_0\right)\right| \leq 2\omega_1\left(f, \left(\widetilde{l}_n\left(\left|\cdot - x_0\right|^2\right)\right)^{\frac{1}{2}}\right), \quad \forall\ n \in \mathbb{N},\ \forall\ x_0 \in K. \tag{19}$$

Conclusion 9 *All as in Theorem 6. By (15) and/or (19), as* $\widetilde{l}_n\left(\left|\cdot - x_0\right|\right) \to 0$, *or* $\widetilde{l}_n\left(\left|\cdot - x_0\right|^2\right) \to 0$, *as* $n \to +\infty$, *we obtain that* $l_n\left(f\right) \to f\left(x_0\right)$ *with rates,* $\forall$ $x_0 \in K$.

Next comes a more general quantitative approximation result of Korovkin type.

Theorem 10 *Here* K *is a convex and compact subset of* $\mathbb{C}$ *and* l_n *is a sequence of linear functionals from* $C\left(K, \mathbb{C}\right)$ *into* $\mathbb{C}$, $n \in \mathbb{N}$. *There is a sequence of companion positive linear functionals* $\widetilde{l}_n$ *from* $C\left(K, \mathbb{R}\right)$ *into* $\mathbb{R}$, *such that*

$$\left|l_n\left(f\right)\right| \leq \widetilde{l}_n\left(\left|f\right|\right), \quad \forall\ f \in C\left(K, \mathbb{C}\right), \ \forall\ n \in \mathbb{N}. \tag{20}$$

Additionally, we assume that

$$l_n\left(cg\right) = c\widetilde{l}_n\left(g\right), \quad \forall\ g \in C\left(K, \mathbb{R}\right), \ \forall\ c \in \mathbb{C}. \tag{21}$$

Then, for any $f \in C\left(K, \mathbb{C}\right)$, *we have*

$$\left|l_n\left(f\right) - f\left(x_0\right)\right| \leq \left|f\left(x_0\right)\right| \left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| + \left(\widetilde{l}_n\left(1\left(\cdot\right)\right) + 1\right) \omega_1\left(f, \widetilde{l}_n\left(\left|\cdot - x_0\right|\right)\right), \tag{22}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

(Notice if $\widetilde{l}_n\left(1\left(\cdot\right)\right) = 1$, *then (22) collapses to (15). So Theorem 10 generalizes Theorem 6).*

By (22), as $\widetilde{l}_n\left(1\left(\cdot\right)\right) \to 1$ *and* $\widetilde{l}_n\left(\left|\cdot - x_0\right|\right) \to 0$, *then* $l_n\left(f\right) \to f\left(x_0\right)$, *as* $n \to +\infty$, *with rates, and as here* $\widetilde{l}_n\left(1\left(\cdot\right)\right)$ *is bounded.*

Proof. We observe that

$$\begin{aligned}
&\left|l_n(f) - f(x_0)\right| = \left|l_n(f) - l_n(f(x_0)(\cdot)) + l_n(f(x_0)(\cdot)) - f(x_0)\right| \leq \\
&\left|l_n(f) - l_n(f(x_0)(\cdot))\right| + \left|f(x_0)\widetilde{l}_n(1(\cdot)) - f(x_0)\right| = \\
&\left|l_n(f(\cdot) - f(x_0)(\cdot))\right| + \left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot)) - 1\right| \leq \\
&\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot)) - 1\right| + \widetilde{l}_n(\left|f(\cdot) - f(x_0)(\cdot)\right|) \leq \\
&\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot)) - 1\right| + \widetilde{l}_n\left(\omega_1(f,\delta)\left\lceil\frac{|\cdot - x_0|}{\delta}\right\rceil\right) \leq \\
&\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot)) - 1\right| + \widetilde{l}_n(\omega_1(f,\delta))\left(1(\cdot) + \frac{|\cdot - x_0|}{\delta}\right) = \\
&\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot)) - 1\right| + \omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot)) + \frac{1}{\delta}\widetilde{l}_n(|\cdot - x_0|)\right] = \\
&\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot)) - 1\right| + \left(\widetilde{l}_n(1(\cdot)) + 1\right)\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right),
\end{aligned} \tag{23}$$

by choosing

$$\delta := \widetilde{l}_n(|\cdot - x_0|), \tag{24}$$

if $\widetilde{l}_n(|\cdot - x_0|) > 0$.

Next we consider the case of

$$\widetilde{l}_n(|\cdot - x_0|) = 0. \tag{25}$$

By Riesz representation theorem there exists a positive finite measure μ such that

$$\widetilde{l}_n(g) = \int_K g(t)\,d\mu(t), \ \ \forall\, g \in C(K, \mathbb{R}). \tag{26}$$

That is

$$\int_K |t - x_0|\,d\mu(t) = 0, \tag{27}$$

which implies $|t - x_0| = 0$, a.e., hence $t - x_0 = 0$, a.e, and $t = x_0$, a.e. on K. Consequently $\mu(\{t \in K : t \neq x_0\}) = 0$. That is $\mu = \delta_{x_0} M$ (where $0 < M := \mu(K) = \widetilde{l}_n(1(\cdot))$). Hence, in that case $\widetilde{l}_n(g) = g(x_0)M$. Consequently it holds $\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right) = 0$, and the right hand side of (22) equals $|f(x_0)||M - 1|$. Also, it is $\widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) = |f(x_0) - f(x_0)|M = 0$. Hence from the first part of this proof we get $|l_n(f) - l_n(f(x_0)(\cdot))| = 0$, and $l_n(f) = l_n(f(x_0)(\cdot)) = f(x_0)\widetilde{l}_n(1(\cdot)) = Mf(x_0)$.

Consequently the left hand side of (22) becomes

$$|l_n(f) - f(x_0)| = |Mf(x_0) - f(x_0)| = |f(x_0)||M - 1|.$$

So that (22) becomes an equality, and both sides equal $|f(x_0)||M - 1|$ in the extreme case of $\widetilde{l}_n(|\cdot - x_0|) = 0$. Thus inequality (22) is proved completely in all cases. ■

We make

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Remark 11 *By Schwartz's inequality we get*

$$\widetilde{l}_n\left(\left|\cdot - x_0\right|\right) \leq \left(\widetilde{l}_n\left(\left|\cdot - x_0\right|^2\right)\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(1\left(\cdot\right)\right)\right)^{\frac{1}{2}}. \tag{28}$$

We give

Corollary 12 *All as in Theorem 10. Then*

$$\left|l_n\left(f\right) - f\left(x_0\right)\right| \leq \left|f\left(x_0\right)\right|\left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| +$$

$$\left(\widetilde{l}_n\left(1\left(\cdot\right)\right) + 1\right)\omega_1\left(f, \left(\widetilde{l}_n\left(1\left(\cdot\right)\right)\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(\left|\cdot - x_0\right|^2\right)\right)^{\frac{1}{2}}\right), \tag{29}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

Next we give another version of our Korovkin type result.

Theorem 13 *Here all are as in Theorem 10. Then, for any $f \in C\left(K, \mathbb{C}\right)$, we have*

$$\left|l_n\left(f\right) - f\left(x_0\right)\right| \leq \left|f\left(x_0\right)\right|\left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| + \left(\widetilde{l}_n\left(1\left(\cdot\right)\right) + 1\right)\omega_1\left(f, \left(\widetilde{l}_n\left(\left|\cdot - x_0\right|^2\right)\right)^{\frac{1}{2}}\right), \tag{30}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

By (30), as $\widetilde{l}_n\left(1\left(\cdot\right)\right) \to 1$ and $\widetilde{l}_n\left(\left|\cdot - x_0\right|^2\right) \to 0$, then $l_n\left(f\right) \to f\left(x_0\right)$, as $n \to +\infty$, with rates, and as here $\widetilde{l}_n\left(1\left(\cdot\right)\right)$ is bounded.

Proof. Let $t, x_0 \in K$ and $\delta > 0$. If $\left|t - x_0\right| > \delta$, then

$$\left|f\left(t\right) - f\left(x_0\right)\right| \leq \omega_1\left(f, \left|t - x_0\right|\right) = \omega_1\left(f, \left|t - x_0\right|\delta^{-1}\delta\right) \leq \tag{31}$$

$$\left(1 + \frac{\left|t - x_0\right|}{\delta}\right)\omega_1\left(f, \delta\right) \leq \left(1 + \frac{\left|t - x_0\right|^2}{\delta^2}\right)\omega_1\left(f, \delta\right).$$

The estimate

$$\left|f\left(t\right) - f\left(x_0\right)\right| \leq \left(1 + \frac{\left|t - x_0\right|^2}{\delta^2}\right)\omega_1\left(f, \delta\right) \tag{32}$$

also holds trivially when $\left|t - x_0\right| \leq \delta$.

So (32) is true always, $\forall\ t \in K$, for any $x_0 \in K$.

We can rewrite

$$\left|f\left(\cdot\right) - f\left(x_0\right)\right| \leq \left(1 + \frac{\left|\cdot - x_0\right|^2}{\delta^2}\right)\omega_1\left(f, \delta\right). \tag{33}$$

As in the proof of Theorem 10 we have

$$\left|l_n\left(f\right) - f\left(x_0\right)\right| \leq ... \leq \left|f\left(x_0\right)\right|\left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| +$$

$$\widetilde{l}_n\left(\omega_1(f,\delta)\left(1(\cdot)+\frac{|\cdot-x_0|^2}{\delta^2}\right)\right)=$$
$$|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot))+\frac{1}{\delta^2}\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right]= \tag{34}$$
$$|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)\left(\widetilde{l}_n(1(\cdot))+1\right),$$

by choosing
$$\delta:=\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}, \tag{35}$$
if $\widetilde{l}_n\left(|\cdot-x_0|^2\right)>0$.

Next we consider the case of
$$\widetilde{l}_n\left(|\cdot-x_0|^2\right)=0. \tag{36}$$

By Riesz representation theorem there exists a positive finite measure μ such that
$$\widetilde{l}_n(g)=\int_K g(t)\,d\mu(t),\ \ \forall\ g\in C(K,\mathbb{R}). \tag{37}$$

That is
$$\int_K |t-x_0|^2\,d\mu(t)=0,$$
which implies $|t-x_0|^2=0$, a.e., hence $t-x_0=0$, a.e, and $t=x_0$, a.e. on K. Consequently $\mu(\{t\in K: t\neq x_0\})=0$. That is $\mu=\delta_{x_0}M$ (where $0<M:=\mu(K)=\widetilde{l}_n(1(\cdot))$). Hence, in that case $\widetilde{l}_n(g)=g(x_0)M$. Consequently it holds $\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)=0$, and the right hand side of (30) equals $|f(x_0)||M-1|$.

Also, it is $\widetilde{l}_n(|f(\cdot)-f(x_0)(\cdot)|)=|f(x_0)-f(x_0)|M=0$. Hence from the first part of this proof we get: $|l_n(f)-l_n(f(x_0)(\cdot))|=0$, and $l_n(f)=l_n(f(x_0)(\cdot))=f(x_0)\widetilde{l}_n(1(\cdot))=Mf(x_0)$.

Consequently the left hand side of (30) becomes
$$|l_n(f)-f(x_0)|=|f(x_0)||M-1|.$$

So that (30) is true again. The proof of the theorem is now complete. ■

Corollary 14 *Here all are as in Theorem 10. Then*
$$|l_n(f)-f(x_0)|\leq|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\left(\widetilde{l}_n(1(\cdot))+1\right)\cdot$$
$$\min\left\{\omega_1\left(f,\left(\widetilde{l}_n(1(\cdot))\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right),\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)\right\}, \tag{38}$$
$\forall\ x_0\in K$, $\forall\ n\in\mathbb{N}$.

Proof. By (29) and (30). ■

So (29) is better that (30) only if $\widetilde{l}_n(1(\cdot)) < 1$.

We need

Theorem 15 *Let $K \subseteq \mathbb{C}$ convex, $x_0 \in K^0$ (interior of K) and $f : K \to \mathbb{R}$ such that $|f(t) - f(x_0)|$ is convex in $t \in K$. Furthermore let $\delta > 0$ so that the closed disk $D(x_0, \delta) \subset K$. Then*

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K. \tag{39}$$

Proof. Let $g(t) := |f(t) - f(x_0)|$, $t \in K$, which is convex in $t \in K$ and $g(x_0) = 0$.

Then by Lemma 8.1.1, p. 243 of [1], we obtain

$$g(t) \leq \frac{\omega_1(g, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K. \tag{40}$$

We notice the following

$$|f(t_1) - f(x_0)| = |f(t_1) - f(t_2) + f(t_2) - f(x_0)| \leq$$

$$|f(t_1) - f(t_2)| + |f(t_2) - f(x_0)|,$$

hence

$$|f(t_1) - f(x_0)| - |f(t_2) - f(x_0)| \leq |f(t_1) - f(t_2)|. \tag{41}$$

Similarly, it holds

$$|f(t_2) - f(x_0)| - |f(t_1) - f(x_0)| \leq |f(t_1) - f(t_2)|. \tag{42}$$

Therefore for any $t_1, t_2 \in K : |t_1 - t_2| \leq \delta$ we get

$$\left| |f(t_1) - f(x_0)| - |f(t_2) - f(x_0)| \right| \leq |f(t_1) - f(t_2)| \leq \omega_1(f, \delta). \tag{43}$$

That is

$$\omega_1(g, \delta) \leq \omega_1(f, \delta). \tag{44}$$

The last and (40) imply

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K, \tag{45}$$

proving (39). ■

We continue with a convex Korovkin type result:

Theorem 16 *All as in Theorem 10. Let $x_0 \in K^0$ and assume that $|f(t) - f(x_0)|$ is convex in $t \in K$. Let $\delta > 0$, such that the closed disk $D(x_0, \delta) \subset K$. Then*

$$|l_n(f) - f(x_0)| \leq |f(x_0)| \left| \widetilde{l}_n(1(\cdot)) - 1 \right| + \omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \quad \forall\, n \in \mathbb{N}. \tag{46}$$

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Proof. As in the proof Theorem 10 we have

$$\left|l_n(f)-f(x_0)\right| \leq ... \leq \left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\widetilde{l}_n(\left|f(\cdot)-f(x_0)(\cdot)\right|) \overset{(39)}{\leq} \tag{47}$$

$$\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\frac{\omega_1(f,\delta)}{\delta}\widetilde{l}_n(\left|\cdot-x_0\right|)=$$

$$\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1\left(f,\widetilde{l}_n(\left|\cdot-x_0\right|)\right),$$

by choosing

$$\delta:=\widetilde{l}_n(\left|\cdot-x_0\right|)>0,$$

if the last is positive. The case of $\widetilde{l}_n(\left|\cdot-x_0\right|)=0$ is treated similarly as in the proof of Theorem 10. The theorem is proved. ■

Theorem 17 *All as in Theorem 16. Inequality (46) is sharp, in fact it is attained by* $f^*(t)=\overrightarrow{j}\left|t-x_0\right|$, *where* $\overrightarrow{j}$ *is a unit vector of* $(\mathbb{C},|\cdot|)$; $t,x_0\in K$.

Proof. Indeed, f^* here fulfills the assumptions of the theorem. We further notice that $f^*(x_0)=0$, and $\left|f^*(t)-f^*(x_0)\right|=\left|t-x_0\right|$ is convex in $t\in K$. The left hand side of (46) is

$$\left|l_n(f^*)-f^*(x_0)\right|=\left|l_n(f^*)\right|=\left|l_n\left(\overrightarrow{j}\left|\cdot-x_0\right|\right)\right| \overset{(21)}{=}$$

$$\left|\overrightarrow{j}\widetilde{l}_n(\left|\cdot-x_0\right|)\right|=\left|\widetilde{l}_n(\left|\cdot-x_0\right|)\right|. \tag{48}$$

The right hand side of (46) is

$$\omega_1\left(f^*,\widetilde{l}_n(\left|\cdot-x_0\right|)\right)=\omega_1\left(\overrightarrow{j}\left|\cdot-x_0\right|,\widetilde{l}_n(\left|\cdot-x_0\right|)\right)=$$

$$\sup_{\substack{t_1,t_2\in K\\ |t_1-t_2|\leq\widetilde{l}_n(|\cdot-x_0|)}}\left|\overrightarrow{j}\left|t_1-x_0\right|-\overrightarrow{j}\left|t_2-x_0\right|\right|=$$

$$\sup_{\substack{t_1,t_2\in K\\ |t_1-t_2|\leq\widetilde{l}_n(|\cdot-x_0|)}}\left|\left|t_1-x_0\right|-\left|t_2-x_0\right|\right|\leq \tag{49}$$

$$\sup_{\substack{t_1,t_2\in K\\ |t_1-t_2|\leq\widetilde{l}_n(|\cdot-x_0|)}}\left|t_1-t_2\right|=\widetilde{l}_n(\left|\cdot-x_0\right|).$$

Hence we have found that

$$\omega_1\left(f^*,\widetilde{l}_n(\left|\cdot-x_0\right|)\right)\leq\widetilde{l}_n(\left|\cdot-x_0\right|). \tag{50}$$

Clearly (46) is attained.

The theorem is proved. ■

4 Main Results - II

Next we give results on linear operators:

Let K be a compact convex subset of $\mathbb{C}$. Consider $L : C(K, \mathbb{C}) \to C(K, \mathbb{C})$ a linear operator and $\widetilde{L} : C(K, \mathbb{R}) \to C(K, \mathbb{R})$ a positve linear operator (i.e. for $f_1.f_2 \in C(K, \mathbb{R})$ with $f_1 \geq f_2$ we get $\widetilde{L}(f_1) \geq \widetilde{L}(f_2)$) both over the field of $\mathbb{R}$.

We assume that

$$|L(f)| \leq \widetilde{L}(|f|), \ \forall\ f \in C(K, \mathbb{C}),$$

(i.e. $|L(f)(z)| \leq \widetilde{L}(|f|)(z)$, $\forall\ z \in K$).

We call $\widetilde{L}$ the companion operator of L.

Let $x_0 \in K$. Clearly, then $L(\cdot)(x_0)$ is a linear functional from $C(K, \mathbb{C})$ into $\mathbb{C}$, and $\widetilde{L}(\cdot)(x_0)$ is a positive linear functional from $C(K, \mathbb{R})$ into $\mathbb{R}$. Notice $L(f)(z) \in \mathbb{C}$ and $\widetilde{L}(|f|)(z) \in \mathbb{R}$, $\forall\ f \in C(K, \mathbb{C})$ (thus $|f| \in C(K, \mathbb{R})$). Here $L(f) \in C(K, \mathbb{C})$, and $\widetilde{L}(|f|) \in C(K, \mathbb{R})$, $\forall\ f \in C(K, \mathbb{C})$.

Notice that $C(K, \mathbb{C}) = UC(K, \mathbb{C})$, also $C(K, \mathbb{R}) = UC(K, \mathbb{R})$ (uniformly continuous functions).

By [3], p. 388, we have that $\widetilde{L}(|\cdot - x_0|^r)(x_0)$, $r > 0$, is a continuous function in $x_0 \in K$.

After this preparation we transfer the main results from section 3 to linear operators.

We have the following approximation results with rates of Korovkin type.

Theorem 18 *Here K is a convex and compact subset of $\mathbb{C}$ and L_n is a sequence of linear operators from $C(K, \mathbb{C})$ into itself, $n \in \mathbb{N}$. There is a sequence of companion positive linear operators $\widetilde{L}_n$ from $C(K, \mathbb{R})$ into itself, such that*

$$|L_n(f)| \leq \widetilde{L}_n(|f|), \ \forall\ f \in C(K, \mathbb{C}), \ \forall\ n \in \mathbb{N} \tag{51}$$

(i.e. $|L_n(f)(x_0)| \leq \left(\widetilde{L}_n(|f|)\right)(x_0)$, $\forall\ x_0 \in K$).

Additionally, we assume that

$$L_n(cg) = c\widetilde{L}_n(g), \ \ \forall\ g \in C(K, \mathbb{R}), \ \forall\ c \in \mathbb{C} \tag{52}$$

(i.e. $(L_n(cg))(x_0) = c\left(\widetilde{L}_n(g)\right)(x_0)$, $\forall\ x_0 \in K$).

Then, for any $f \in C(K, \mathbb{C})$, we have

$$|(L_n(f))(x_0) - f(x_0)| \leq |f(x_0)| \left|\widetilde{L}_n(1(\cdot))(x_0) - 1\right| +$$

$$\left(\widetilde{L}_n(1(\cdot))(x_0) + 1\right) \omega_1\left(f, \widetilde{L}_n(|\cdot - x_0|)(x_0)\right), \tag{53}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

Proof. By Theorem 10. ■

Corollary 19 *All as in Theorem 18. Then*

$$\|L_n(f)-f\|_{\infty,K}\leq\|f\|_{\infty,K}\left\|\widetilde{L}_n(1(\cdot))-1\right\|_{\infty,K}+$$

$$\left\|\widetilde{L}_n(1(\cdot))+1\right\|_{\infty,K}\omega_1\left(f,\left\|\widetilde{L}_n(|\cdot-x_0|)(x_0)\right\|_{\infty,K}\right), \tag{54}$$

$\forall\ n\in\mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))=1$, $\forall\ n\in\mathbb{N}$, *then*

$$\|L_n(f)-f\|_{\infty,K}\leq 2\omega_1\left(f,\left\|\widetilde{L}_n(|\cdot-x_0|)(x_0)\right\|_{\infty,K}\right), \tag{55}$$

$\forall\ n\in\mathbb{N}$.

As $\widetilde{L}_n(1(\cdot))\xrightarrow{u}1$, $\left\|\widetilde{L}_n(|\cdot-x_0|)(x_0)\right\|_{\infty,K}\xrightarrow{u}0$, *then (by (54))* $L_n(f)\xrightarrow{u}f$, *as* $n\to+\infty$, *where* u *means uniformly. Notice* $\widetilde{L}_n(1(\cdot))$ *is bounded, and all the suprema in (54) are finite.*

We continue with

Theorem 20 *Here all as in Theorem 18. Then, for any* $f\in C(K,\mathbb{C})$, *we have*

$$|(L_n(f))(x_0)-f(x_0)|\leq|f(x_0)|\left|\widetilde{L}_n(1(\cdot))(x_0)-1\right|+$$

$$\left(\widetilde{L}_n(1(\cdot))(x_0)+1\right)\omega_1\left(f,\left(\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right)^{\frac{1}{2}}\right), \tag{56}$$

$\forall\ x_0\in K$, $\forall\ n\in\mathbb{N}$.

Proof. By Theorem 13. ■

Corollary 21 *All as in Theorem 18. Then, for any* $f\in C(K,\mathbb{C})$, *we have*

$$\|L_n(f)-f\|_{\infty,K}\leq\|f\|_{\infty,K}\left\|\widetilde{L}_n(1(\cdot))-1\right\|_{\infty,K}+$$

$$\left\|\widetilde{L}_n(1(\cdot))+1\right\|_{\infty,K}\omega_1\left(f,\left\|\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right\|_{\infty,K}^{\frac{1}{2}}\right), \tag{57}$$

$\forall\ n\in\mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))=1$, *then*

$$\|L_n(f)-f\|_{\infty,K}\leq 2\omega_1\left(f,\left\|\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right\|_{\infty,K}^{\frac{1}{2}}\right), \tag{58}$$

$\forall\ n\in\mathbb{N}$.

As $\widetilde{L}_n(1(\cdot))\xrightarrow{u}1$, $\left\|\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right\|_{\infty,K}\xrightarrow{u}0$, *then (by (57))* $L_n(f)\xrightarrow{u}f$, *as* $n\to+\infty$.

We continue with a convex Korovkin type result:

Theorem 22 *All as in Theorem 18. Let a fixed* $x_0^* \in K^0$ *and assume that* $|f(t) - f(x_0^*)|$ *is convex in* $t \in K$*. Let* $\delta > 0$*,such that the closed disk* $D(x_0^*, \delta) \subset K$*. Then*

$$|(L_n(f))(x_0^*) - f(x_0^*)| \leq |f(x_0^*)| \left|\widetilde{L}_n(1(\cdot))(x_0^*) - 1\right|$$

$$+\omega_1\left(f, \widetilde{L}_n(|\cdot - x_0^*|)(x_0^*)\right), \quad \forall\, n \in \mathbb{N}. \tag{59}$$

As $\widetilde{L}_n(1(\cdot))(x_0^*) \to 1$*, and* $\widetilde{L}_n(|\cdot - x_0^*|)(x_0^*) \to 0$*, we get that* $(L_n(f))(x_0^*) \to f(x_0^*)$*, as* $n \to +\infty$*, a pointwise convergence.*

Proof. By Theorem 16. ■

Note: Theorem 22 goes throw if (51), (52) are valid only for the particular x_0^*.

We finish with

Proposition 23 *All as in Theorem 22. Inequality (59) is sharp, in fact it is attained by* $\overline{f}(t) = \overrightarrow{j}\,|t - x_0^*|$*, where* $\overrightarrow{j}$ *is a unit vector of* $\mathbb{C}$*;* $x_0^*, t \in K$*.*

Proof. By Theorem 17. ■

Note: Let K be a convex compact subset of a real normed vector space $(V, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ is a Banach space. We can consider bounded linear functionals and bounded operators on $C(K, X)$. This paper's methodology can be applied to this more general setting and produce a similar Korovkin theory in full strength.

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ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

INHO HWANG

Abstract. In this paper, we solve the additive ρ-functional inequalities

$$\|f(x+y)+f(x-y)-2f(x)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right)\right\|, \tag{0.1}$$

where ρ is a fixed non-Archimedean number with $|\rho|<1$, and

$$\left\|2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right\| \le \|\rho(f(x+y)+f(x-y)-2f(x))\|, \tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho|<|2|$.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([6]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\|=|r|\|x\| \qquad (r\in K, x\in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|,\|y\|\}, \qquad \forall x,y\in X$$

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holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

The stability problem of functional equations originated from a question of Ulam [13] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation.* In particular, every solution of the Cauchy equation is said to be an *additive mapping.* Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f(x+y) + f(x-y) = 2f(x)$ is called the *Jensen type additive functional equation.*

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* The stability of quadratic functional equation was proved by Skof [12] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 7, 8, **?**, 11]).

In this paper, we solve the additive ρ-functional inequalities (0.1) and (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \ne 1$.

2. Additive ρ-functional inequality (0.1) in non-Archimedean normed spaces

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the additive ρ-functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|f(x+y) + f(x-y) - 2f(x)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \\ &= |\rho|\|f(x+y) + f(x-y) - 2f(x)\| \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \\ &+ \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{|2|^r}\|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \le 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So $\|f(x) - 2f\left(\frac{x}{2}\right)\| \le \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\ &\le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &= \max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\le \max\left\{\frac{|2|^l}{|2|^{rl+r}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)+r}}\right\}2\theta\|x\|^r = \frac{2\theta}{|2|^{(r-1)l+r}}\|x\|^r \end{aligned} \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\|A(x+y)+A(x-y)-2A(x)\| = \lim_{n\to\infty}|2|^n\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)\right\|$$
$$\leq \lim_{n\to\infty}|2|^n|\rho|\left\|2f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)\right\|+\lim_{n\to\infty}\frac{|2|^n\theta}{|2|^{nr}}(\|x\|^r+\|y\|^r)$$
$$=|\rho|\left\|2A\left(\frac{x+y}{2}\right)+A(x-y)-2A(x)\right\|$$

for all $x, y \in X$. So

$$\|A(x+y)+A(x-y)-2A(x)\| \leq \left\|\rho\left(2A\left(\frac{x+y}{2}\right)+A(x-y)-2A(x)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive .

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$\|A(x)-T(x)\| = \left\|2^qA\left(\frac{x}{2^q}\right)-2^qT\left(\frac{x}{2^q}\right)\right\|$$
$$\leq \max\left\{\left\|2^qA\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|,\left\|2^qT\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|\right\}\leq\frac{2\theta}{|2|^{(r-1)q+r}}\|x\|^r,$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.4). □

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \leq \frac{2\theta}{|2|}\|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \leq \frac{2}{|2|}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|$$
$$\leq \max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|\right\}$$
$$=\max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right)-\frac{1}{2}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{2}f\left(2^mx\right)\right\|\right\}$$
$$\leq \max\left\{\frac{|2|^{lr}}{|2|^{l+1}},\cdots,\frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}2\theta\|x\|^r=\frac{2\theta}{|2|^{(1-r)l+1}}\|x\|^r$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Additive ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the additive ρ-functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies*

$$\left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right\| \le \|\rho(f(x+y) + f(x-y) - 2f(x))\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \le 0$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\|2f\left(\frac{x}{2}\right) - f(x)\| \le 0$ and so

$$2f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y) + f(x-y) - 2f(x)\| &= \left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right\| \\ &\le |\rho|\|f(x+y) + f(x-y) - 2f(x)\|\end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}\left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right\| &\le \|\rho(f(x+y) + f(x-y) - 2f(x))\| \\ &+ \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \theta\|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.3), we get $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.6)\\
&\quad \le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad = \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|2|^l}{|2|^{rl}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-1)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.3). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{|2|^r\theta}{|2|}\|x\|^r \qquad (3.7)$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{|2|^r\theta}{|2|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\right\| \qquad (3.8)\\
&\quad \le \max\left\{\left\|\frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^{l+1}} f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}} f\left(2^{m-1}x\right) - \frac{1}{2^m} f\left(2^m x\right)\right\|\right\}\\
&\quad = \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right) - \frac{1}{2} f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^m x\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|2|^{rl}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Competing interests

The author declares that he has no competing interests.

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Square root and 3rd root functional equations in C^*-algebras

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Abstract. In this paper, we introduce a square root functional equation and a 3rd root functional equation. We prove the Hyers-Ulam stability of the square root functional equation and of the 3rd root functional equation in C^*-algebras.

1. Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [7] concerning the stability of group homomorphisms. Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [6] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach.

Definition 1.1. [2] Let A be a C^*-algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.

The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [2]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [4]).

In this paper, we introduce a *square root functional equation*

$$S\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) = S(x) + S(y) \tag{1.1}$$

and a *3rd root functional equation*

$$T\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) = T(x) + T(y) \tag{1.2}$$

for all $x, y \in A^+$. Each solution of the square root functional equation is called a *square root mapping* and each solution of the 3rd root functional equation is called a *3rd root mapping.*

Note that the functions $S(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $T(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ in the set of non-negative real numbers are solutions of the functional equations (1.1) and (1.2), respectively.

In this paper, we prove the Hyers-Ulam stability of the functional equations (1.1) and (1.2) in C^*-algebras.

Throughout this paper, let A^+ and B^+ be the sets of positive elements in C^*-algebras A and B, respectively.

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2. Stability of the square root functional equation

In this section, we investigate the square root functional equation in C^*-algebras.

Lemma 2.1. *Let $S : A^+ \to B^+$ be a square root mapping satisfying* (1.1)*. Then S satisfies*

$$S(4^n x) = 2^n S(x) \tag{2.1}$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

Proof. Putting $x = y = 0$ in (1.1), we obtain $S(0) = 0$. Letting $y = 0$ in (1.1), we obtain

$$S(4^0 x) = S(x) = 2^0 S(x)$$

for all $x \in A^+$.

First of all, we use the induction on n to prove the equality (2.1) for all positive integers n.

Replacing y by x in (1.1), we get

$$S(4x) = 2S(x) \tag{2.2}$$

for all $x \in A^+$. So the equality (2.1) holds for $n = 1$.

Assume that

$$S(4^k x) = 2^k S(x) \tag{2.3}$$

holds for a positive integer k. Replacing x by $4x$ in (2.3) and using (2.2), we obtain

$$S(4^{k+1}x) = S(4^k \cdot 4x) = 2^k S(4x) = 2^{k+1} S(x)$$

for all $x \in A^+$. So the equality (2.1) holds for $n = k + 1$. Thus

$$S(4^n x) = 2^n S(x) \tag{2.4}$$

for all $x \in A^+$ and all positive integers n.

Next, replacing x by $4^{-n}x$ in (2.4), we obtain

$$S(x) = S(4^n \cdot 4^{-n}x) = 2^n S(4^{-n}x)$$

for all $x \in A^+$ and all positive integers n. So

$$S(4^n x) = 2^n S(x)$$

for all $x \in A^+$ and all negative integers n.

Therefore,

$$S(4^n x) = 2^n S(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$. □

We prove the Hyers-Ulam stability of the square root functional equation in C^*-algebras.

Theorem 2.2. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{4^j}, \frac{y}{4^j}\right) \quad < \quad \infty, \tag{2.5}$$

$$\left\| f\left(x + y + x^{\frac{1}{4}} y^{\frac{1}{2}} x^{\frac{1}{4}} + y^{\frac{1}{4}} x^{\frac{1}{2}} y^{\frac{1}{4}}\right) - f(x) - f(y) \right\| \quad \leq \quad \varphi(x, y) \tag{2.6}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to A^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, y) \tag{2.7}$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (2.6), we get

$$\|f(4x) - 2f(x)\| \le \varphi(x, x) \tag{2.8}$$

for all $x \in A^+$. It follows from (2.8) that

$$\left\|f(x) - 2f\left(\frac{x}{4}\right)\right\| \le \varphi\left(\frac{x}{4}, \frac{x}{4}\right)$$

for all $x \in A^+$. Hence

$$\left\|2^l f\left(\frac{x}{4^l}\right) - 2^m f\left(\frac{x}{4^m}\right)\right\| \le \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{4^j}, \frac{x}{4^j}\right) \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A^+$. It follows from (2.5) and (2.9) that the sequence $\{2^k f(\frac{x}{4^k})\}$ is Cauchy for all $x \in A^+$. Since B^+ is complete, the sequence $\{2^k f(\frac{x}{4^k})\}$ converges. So one can define the mapping $S : A^+ \to B^+$ by

$$S(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{4^k}\right)$$

for all $x \in A^+$.

By (2.8) and (2.9),

$$\begin{aligned}
&\left\|S\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - S(x) - S(y)\right\| \\
&= \lim_{k\to\infty} 2^k \left\|f\left(\frac{x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}}{4^k}\right) - f\left(\frac{x}{4^k}\right) - f\left(\frac{y}{4^k}\right)\right\| \\
&\le \lim_{k\to\infty} 2^k \varphi\left(\frac{x}{4^k}, \frac{y}{4^k}\right) = 0
\end{aligned}$$

for all $x, y \in A^+$. So

$$S\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - S(x) - S(y) = 0.$$

Hence the mapping $S : A^+ \to B^+$ is a square root mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.7). So there exists a square root mapping $S : A^+ \to B^+$ satisfying (1.1) and (2.7).

Now, let $S' : A^+ \to B^+$ be another square root mapping satisfying (1.1) and (2.7). Then we have

$$\begin{aligned}
\|S(x) - S'(x)\| &= 2^q \left\|S\left(\frac{x}{4^q}\right) - S'\left(\frac{x}{4^q}\right)\right\| \\
&\le 2^q \left\|S\left(\frac{x}{4^q}\right) - f\left(\frac{x}{4^q}\right)\right\| + 2^q \left\|S'\left(\frac{x}{4^q}\right) - f\left(\frac{x}{4^q}\right)\right\| \\
&\le \frac{2 \cdot 2^q}{2}\widetilde{\varphi}\left(\frac{x}{4^q}, \frac{x}{4^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in A^+$. So we can conclude that $S(x) = S'(x)$ for all $x \in A^+$. This proves the uniqueness of S. □

Corollary 2.3. *Let $p > \frac{1}{2}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\left\|f\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - f(x) - f(y)\right\| \le \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \tag{2.10}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{2\theta_1 + \theta_2}{4^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 2.2. Then we get the desired result. $\square$

Theorem 2.4. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0,\infty)$ satisfying* (2.6) *such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(4^j x, 4^j y) < \infty$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in A^+$.

Proof. It follows from (2.8) that

$$\left\| f(x) - \frac{1}{2}f(4x) \right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in A^+$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $0 < p < \frac{1}{2}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (2.10). *Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying* (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{2\theta_1 + \theta_2}{2 - 4^p}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 2.4. Then we get the desired result. $\square$

3. Stability of the 3rd root functional equation

In this section, we investigate the 3rd root functional equation in C^*-algebras.

Lemma 3.1. *Let $T : A^+ \to B^+$ be a 3rd root mapping satisfying* (1.2). *Then T satisfies*

$$T(8^n x) = 2^n T(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

Proof. The proof is similar to the proof of Lemma 2.1. $\square$

We prove the Hyers-Ulam stability of the 3rd root functional equation in C^*-algebras.

Theorem 3.2. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{8^j}, \frac{y}{8^j}\right) \quad < \quad \infty,$$

$$\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y) \right\| \quad \le \quad \varphi(x,y) \tag{3.1}$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to A^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, y)$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (3.1), we get

$$\|f(8x) - 2f(x)\| \leq \varphi(x, x) \tag{3.2}$$

for all $x \in A^+$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $p > \frac{1}{3}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y) \right\| \leq \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \tag{3.3}$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{8^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.2. Then we get the desired result. □

Theorem 3.4. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying* (3.1) *such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(8^j x, 8^j y) < \infty$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in A^+$.

Proof. It follows from (3.2) that

$$\left\| f(x) - \frac{1}{2}f(8x) \right\| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in A^+$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let $0 < p < \frac{1}{3}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (3.3). *Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying* (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{2 - 8^p}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.4. Then we get the desired result. □

4. Square root and 3rd root functional equations in C^*-algebras

We have defined a *square root functional equation*

$$S\left(x+y+x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}}+y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right)=S(x)+S(y)$$

and a *3rd root functional equation*

$$T\left(x+y+3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}}+3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right)=T(x)+T(y)$$

for all $x, y \in A^+$. Each solution of the square root functional equation is called a *square root mapping* and each solution of the 3rd root functional equation is called a *3rd root mapping*.

It was shown that each square root mapping $S : A^+ \to B^+$ satisfies $S(4^n x) = 2^n S(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$ and that each 3rd root mapping $T : A^+ \to B^+$ satisfies $T(8^n x) = 2^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$. Moreover, we prove that there exists a square root mapping near a given approximate square root mapping and that there exists a 3rd root mapping near a given approximate 3rd root mapping by using the Hyer-Ulam-Rassias approach.

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Competing interests

The authors declare that they have no competing interests.

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Approximation by Multivariate Sublinear and Max-product Operators, Revisited

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Abstract

Here we study quantitatively the approximation of multivariate function by general multivariate positive sublinear operators with applications to multivariate Max-product operators. These are of Bernstein type, of Favard-Szász-Mirakjan type, of Baskakov type, of sampling type, of Lagrange interpolation type and of Hermite-Fejér interpolation type. Our results are both: under the presence of smoothness and without any smoothness assumption on the function to be approximated.

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1 Background

Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x_0 := (x_{01}, ..., x_{0k}) \in Q$ be fixed. Let $f \in C^n(Q)$ and suppose that each nth partial derivative $f_\alpha = \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., k$, and $|\alpha| := \sum_{i=1}^k \alpha_i = n$, has relative to Q and the l_1-norm $\|\cdot\|$, a modulus of continuity $\omega_1(f_\alpha, h) \leq w$, where h and w are fixed positive numbers. Here

$$\omega_1(f_\alpha, h) := \sup_{\substack{x,y \in Q \\ \|x-y\|_{l_1} \leq h}} |f_\alpha(x) - f_\alpha(y)|. \tag{1}$$

The jth derivative of $g_z(t) = f(x_0 + t(z - x_0))$, $(z = (z_1, ..., z_k) \in Q)$ is given by

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})). \tag{2}$$

Consequently it holds

$$f(z_1, ..., z_k) = g_z(1) = \sum_{j=0}^{n} \frac{g_z^{(j)}(0)}{j!} + R_n(z, 0), \tag{3}$$

where

$$R_n(z,0) := \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{n-1}} \left(g_z^{(n)}(t_n) - g_z^{(n)}(0) \right) dt_n \right) ... \right) dt_1. \tag{4}$$

We apply Lemma 7.1.1, [1], pp. 208-209, to $(f_\alpha(x_0 + t(z - x_0)) - f_\alpha(x_0))$ as a function of z, when $\omega_1(f_\alpha, h) \leq w$.

$$|f_\alpha(x_0 + t(z - x_0)) - f_\alpha(x_0)| \leq w \left\lceil \frac{t \|z - x_0\|}{h} \right\rceil, \tag{5}$$

all $t \geq 0$, where $\lceil \cdot \rceil$ is the ceiling function.

For $\|z - x_0\| \neq 0$, it follows from (2)

$$|R_n(z,0)| \leq$$

$$\int_0^1 \int_0^{t_1} ... \int_0^{t_{n-1}} \left(\sum_{|\alpha|=n} \frac{n!}{\alpha_1!...\alpha_k!} |z_1 - x_{01}|^{\alpha_1} ... |z_k - x_{0k}|^{\alpha_k} w \left\lceil \frac{t_n \|z - x_0\|}{h} \right\rceil \right) dt_n...dt_1 \tag{6}$$

$$= \sum_{|\alpha|=n} \frac{n!}{\alpha_1!...\alpha_k!} \frac{\prod_{i=1}^{k} |z_i - x_{0i}|^{\alpha_i}}{\|z - x_0\|^n} w\Phi_n(\|z - x_0\|) = w\Phi(\|z - x_0\|),$$

since $\|z - x_0\| = \sum_{i=1}^{k} |z_i - x_{0i}|$. Above we denote (for $h > 0$ fixed):

$$\Phi_n(x) := \int_0^{|x|} \left\lceil \frac{t}{h} \right\rceil \frac{(|x| - t)^{n-1}}{(n-1)!} dt, \quad (x \in \mathbb{R}), \tag{7}$$

equivalently

$$\Phi_n(x) = \int_0^{|x|} \int_0^{x_1} ... \left(\int_0^{x_{n-1}} \left\lceil \frac{x_n}{h} \right\rceil dx_n \right) ...dx_1, \tag{8}$$

see [1], p. 210-211.

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Therefore we have

$$|R_n(z,0)| \leq w\Phi_n(\|z-x_0\|), \quad \text{for all } z \in Q. \tag{9}$$

Also we have $g_z(0) = f(x_0)$.

One obtains ([1], p. 210)

$$\Phi_n(x) = \frac{1}{n!}\left(\sum_{j=0}^{\infty}(|x| - jh)_+^n\right), \tag{10}$$

which is a polynomial spline function.

Furthermore we get ([1], pp. 210-211)

$$\Phi_n(x) \leq \Phi_{*n}(x) := \left(\frac{|x|^{n+1}}{(n+1)!h} + \frac{|x|^n}{2n!} + \frac{h|x|^{n-1}}{8(n-1)!}\right), \tag{11}$$

with equality only at $x = 0$.

Moreover, Φ_n is convex on $\mathbb{R}$ and strictly increasing on $\mathbb{R}_+$, $n \geq 1$.

In case of $Q := \{x \in \mathbb{R}^* : \|x\| \leq 1\}$, where $\|\cdot\|$ is the l_1-norm in $\mathbb{R}^k$ we have

$$0 \leq \|z - x_0\| \leq \|z\| + \|x_0\| \leq 1 + \|x_0\|, \ \forall\ z \in Q,$$

hence $\Phi_n(\|z-x_0\|) \leq \Phi_n(1+\|x_0\|)$, and by convexity of Φ_n we get

$$\frac{\Phi_n(\|z-x_0\|)}{\|z-x_0\|} \leq \frac{\Phi_n(1+\|x_0\|)}{(1+\|x_0\|)}, \tag{12}$$

$\forall\ z \in Q : \|z - x_0\| \neq 0$,

and hence

$$\Phi_n(\|z-x_0\|) \leq \|z-x_0\| \frac{\Phi_n(1+\|x_0\|)}{(1+\|x_0\|)}, \quad \forall\ z \in Q. \tag{13}$$

Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, $x_0 \in Q$ fixed, $f \in C^n(Q)$. Then for $j = 1, ..., n$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_k),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,k,\ |\alpha|:=\sum_{i=1}^k \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^k \alpha_i!}\right)\left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i}\right) f_\alpha(x_0). \tag{14}$$

If $f_\alpha(x_0) = 0$, for all $\alpha : |\alpha| = 1, ..., n$, then $g_z^{(j)}(0) = 0$, $j = 1, ..., n$, and by (3):

$$f(z) - f(x_0) = R_n(z,0), \tag{15}$$

that is

$$|f(z) - f(x_0)| \leq w\Phi_n(\|z-x_0\|), \ \forall\ z \in Q, \tag{16}$$

where $x_0 \in Q$ is fixed.

Using (11) we derive

$$\|f(z) - f(x_0)\| \leq w \left(\frac{\|z - x_0\|^{n+1}}{(n+1)!h} + \frac{\|z - x_0\|^{n}}{2n!} + h \frac{\|z - x_0\|^{n-1}}{8(n-1)!} \right), \quad \forall\, z \in Q. \tag{17}$$

We have proved the following fundamental result:

Theorem 1 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x_0 \in Q$ be fixed. Let $f \in C^n(Q)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x_0) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\|f(z) - f(x_0)\| \leq \left(\max_{\alpha:|\alpha=n|} \omega_1(f_\alpha, h) \right) \cdot$$

$$\left(\frac{\|z - x_0\|^{n+1}}{(n+1)!h} + \frac{\|z - x_0\|^{n}}{2n!} + h \frac{\|z - x_0\|^{n-1}}{8(n-1)!} \right), \quad \forall\, z \in Q. \tag{18}$$

In conclusion we have

Theorem 2 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x \in Q$ ($x = (x_1, ..., x_k)$) be fixed. Let $f \in C^n(Q)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\|f(t) - f(x)\| \leq \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h) \right) \cdot \tag{19}$$

$$\left(\frac{\|t - x\|^{n+1}}{(n+1)!h} + \frac{\|t - x\|^{n}}{2n!} + h \frac{\|t - x\|^{n-1}}{8(n-1)!} \right) \leq$$

$$\left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h) \right) \left(\frac{k^n \left(\sum_{i=1}^k |t_i - x_i|^{n+1} \right)}{(n+1)!h} + \frac{k^{n-1} \left(\sum_{i=1}^k |t_i - x_i|^{n} \right)}{2n!} \right.$$

$$\left. + \frac{hk^{n-2}}{8(n-1)!} \left(\sum_{i=1}^k |t_i - x_i|^{n-1} \right) \right), \ \forall\, t \in Q, \tag{20}$$

where $t = (t_1, ..., t_k)$.

Proof. By Theorem 1 and a convexity argument. ■

We need

Definition 3 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$. Here we denote*

$$C_+(Q) = \{f : Q \to \mathbb{R}_+ \text{ and continuous}\}.$$

Let $L_N : C_+(Q) \to C_+(Q)$, $N \in \mathbb{N}$, be a sequence of operators satisfying the following properties:

(i) (positive homogeneous)

$$L_N(\alpha f) = \alpha L_N(f), \ \ \forall\ \alpha \geq 0,\ f \in C_+(Q); \tag{21}$$

(ii) (monotonicity)
if $f, g \in C_+(Q)$ satisfy $f \leq g$, then

$$L_N(f) \leq L_N(g), \ \ \forall\ N \in \mathbb{N}, \tag{22}$$

and
(iii) (subadditivity)

$$L_N(f+g) \leq L_N(f) + L_N(g), \ \ \forall\ f, g \in C_+(Q). \tag{23}$$

We call L_N positive sublinear operators.

Remark 4 *(to Definition 3) Let $f, g \in C_+(Q)$. We see that $f = f - g + g \leq |f - g| + g$. Then $L_N(f) \leq L_N(|f-g|) + L_N(g)$, and $L_N(f) - L_N(g) \leq L_N(|f-g|)$.*

Similarly $g = g - f + f \leq |g - f| + f$, hence $L_N(g) \leq L_N(|f-g|) + L_N(f)$, and $L_N(g) - L_N(f) \leq L_N(|f-g|)$.

Consequently it holds

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f-g|)(x), \ \ \forall\ x \in Q. \tag{24}$$

In this article we treat $L_N : L_N(1) = 1$.

We observe that

$$|L_N(f)(x) - f(x)| = |L_N(f)(x) - L_N(f(x))(x)| \overset{(24)}{\leq}$$
$$L_N(|f(\cdot) - f(x)|)(x), \ \ \forall\ x \in Q. \tag{25}$$

We give

Theorem 5 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x \in Q$ be fixed. Let $f \in C^n(Q, \mathbb{R}_+)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Let $\{L_N\}_{N \in \mathbb{N}}$ positive sublinear operators mapping $C_+(Q)$ into itself, such that $L_N(1) = 1$. Then*

$$|L_N(f)(x) - f(x)| \leq \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right) \cdot$$
$$\left(\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^k L_N\left(|t_i - x_i|^{n+1}\right)(x)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^k L_N\left(|t_i - x_i|^n\right)(x)\right)\right.$$
$$\left. + \frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^k L_N\left(|t_i - x_i|^{n-1}\right)(x)\right)\right), \ \ \forall\ N \in \mathbb{N}. \tag{26}$$

Proof. By Theorem 2, see Definition 3, and by (25). ∎

We need

The Maximum Multiplicative Principle 6 *Here $\vee$ stands for maximum. Let $\alpha_i > 0$, $i = 1, ..., n$; $\beta_j > 0$, $j = 1, ..., m$. Then*

$$\vee_{i=1}^{n} \vee_{j=1}^{m} \alpha_i \beta_j = \left(\vee_{i=1}^{n} \alpha_i\right) \left(\vee_{j=1}^{m} \beta_j\right). \tag{27}$$

Proof. Obvious. ∎

We make

Remark 7 *In [4], p. 10, the authors introduced the basic Max-product Bernstein operators*

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad N \in \mathbb{N}, \tag{28}$$

where $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, $x \in [0,1]$, and $f : [0,1] \to \mathbb{R}_+$ is continuous.

In [4], p. 31, they proved that

$$B_N^{(M)}(|\cdot - x|)(x) \le \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1],\ \forall\ N \in \mathbb{N}. \tag{29}$$

And in [2] was proved that

$$B_N^{(M)}(|\cdot - x|^m)(x) \le \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1],\ \forall\ m, N \in \mathbb{N}. \tag{30}$$

We will also use

Corollary 8 *(to Theorem 5, case of $n = 1$) Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x \in Q$. Let $f \in C^1(Q, \mathbb{R}_+)$, $h > 0$. We assume that $\frac{\partial f(x)}{\partial x_i} = 0$, for $i = 1, ..., k$. Let $\{L_N\}_{N \in \mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into $C_+(Q) : L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \le \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, h\right)\right) \cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k} L_N\left((t_i - x_i)^2\right)(x)\right) + \frac{1}{2}\left(\sum_{i=1}^{k} L_N(|t_i - x_i|)(x)\right) + \frac{h}{8}\right], \tag{31}$$

$\forall\ N \in \mathbb{N}$.

In this article we study quantitatively the approximation properties of multivariate Max-product operators to the unit. These are special cases of positive sublinear operators. We give also general results regarding the convergence to the unit of positive sublinear operators. Special emphasis is given in our study about approximation under differentiability. Our work is motivated by [4].

2 Main Results

From now on $Q = [0,1]^k$, $k \in \mathbb{N} - \{1\}$, except otherwise specified.

We mention

Definition 9 *Let $f \in C_+\left([0,1]^k\right)$, and $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. We define the multivariate Max-product Bernstein operators as follows:*

$$B_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} p_{N_1,i_1}(x_1) p_{N_2,i_2}(x_2) ... p_{N_k,i_k}(x_k) f\left(\frac{i_1}{N_1}, ..., \frac{i_k}{N_k}\right)}{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} p_{N_1,i_1}(x_1) p_{N_2,i_2}(x_2) ... p_{N_k,i_k}(x_k)}, \quad (32)$$

$\forall\ x = (x_1, ..., x_k) \in [0,1]^k$. Call $N_{\min} := \min\{N_1, ..., N_k\}$.

The operators $B_{\overrightarrow{N}}^{(M)}(f)(x)$ are positive sublinear and they map $C_+\left([0,1]^k\right)$ into itself, and $B_{\overrightarrow{N}}^{(M)}(1) = 1$.

See also [4], p. 123 the bivariate case. We also have

$$B_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} p_{N_1,i_1}(x_1) p_{N_2,i_2}(x_2) ... p_{N_k,i_k}(x_k) f\left(\frac{i_1}{N_1}, ..., \frac{i_k}{N_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda} p_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \quad (33)$$

$\forall\ x \in [0,1]^k$, by the maximum multiplicative principle, see (27).

We make

Remark 10 *The coordinate Max-product Bernstein operators are defined as follows ($\lambda = 1, ..., k$):*

$$B_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda} p_{N_\lambda,i_\lambda}(x_\lambda) g\left(\frac{i_\lambda}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda} p_{N_\lambda,i_\lambda}(x_\lambda)}, \quad (34)$$

$\forall\ N_\lambda \in \mathbb{N}$, and $\forall\ x_\lambda \in [0,1]$, $\forall\ g \in C_+([0,1]) := \{g : [0,1] \to \mathbb{R}_+$ continuous$\}$.

Here we have

$$p_{N_\lambda,i_\lambda}(x_\lambda) = \binom{N_\lambda}{i_\lambda} x_\lambda^{i_\lambda}(1 - x_\lambda)^{N_\lambda - i_\lambda}, \quad \text{for all } \lambda = 1, ..., k;\ x_\lambda \in [0,1]. \quad (35)$$

In case of $f \in C_+\left([0,1]^k\right)$ is such that $f(x) := g(x_\lambda)$, $\forall\ x \in [0,1]^k$, where $x = (x_1, ..., x_\lambda, ..., x_k)$ and $g \in C_+([0,1])$, we get that

$$B_{\overrightarrow{N}}^{(M)}(f)(x) = B_{N_\lambda}^{(M)}(g)(x_\lambda), \quad (36)$$

by the maximum multiplicative principle (27) and simplification of (33).

Clearly it holds that

$$B_{\overrightarrow{N}}^{(M)}(f)(x) = f(x), \ \ \forall \ x = (x_1, ..., x_k) \in [0,1]^k : x_\lambda \in \{0,1\}, \ \lambda = 1, ..., k. \tag{37}$$

We present

Theorem 11 *Let $x \in [0,1]^k$, $k \in \mathbb{N} - \{1\}$, be fixed, and let $f \in C^n\left([0,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N} - \{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq 6\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right) \cdot \tag{38}$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}} + \frac{k^n}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right) + \frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right],$$

$\forall \ \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N} \to (\infty, ..., \infty)} B_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(36)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right) \cdot$$

$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^{k} B_{N_i}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^{k} B_{N_i}^{(M)}\left(|t_i - x_i|^n\right)(x_i)\right)\right. \tag{39}$$

$$\left. + \frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^{k} B_{N_i}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(30)}{\leq}$$

$$\left(\frac{6}{\sqrt{N_{\min}+1}}\right)\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\left[\frac{k^{n+1}}{(n+1)!h} + \frac{k^n}{2n!} + \frac{hk^{n-1}}{8(n-1)!}\right] =: (\xi).$$

Above notice $\sum_{i=1}^{k} B_{N_i}^{(M)}\left(|t_i - x_i|^n\right)(x_i) \overset{(30)}{\leq} \sum_{i=1}^{k} \frac{6}{\sqrt{N_i+1}} \leq \frac{6k}{\sqrt{N_{\min}+1}}$, etc.

Next we choose $h := \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{\sqrt{N_{\min}+1}}$.

We have

$$(\xi) = 6\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right) \cdot \tag{40}$$

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$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}+\frac{k^{n}}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)+\frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right],$$

proving the claim. ■

We also give

Proposition 12 *Let $x\in[0,1]^k$, $k\in\mathbb{N}-\{1\}$, be fixed and let $f\in C^1\left([0,1]^k,\mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i}=0$, for $i=1,...,k$. Then*

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right|\leq\left(\max_{i=1,...,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt[4]{N_{\min}+1}}\right)\right)\cdot \tag{41}$$

$$\left[\frac{3k^2}{\sqrt[4]{N_{\min}+1}}+\frac{3k}{\sqrt{N_{\min}+1}}+\frac{1}{8\left(\sqrt[4]{N_{\min}+1}\right)}\right],$$

$\forall\ \overrightarrow{N}\in\mathbb{N}^k$, where $N_{\min}:=\min\{N_1,...,N_k\}$.

Also it holds $\lim_{\overrightarrow{N}\to(\infty,...,\infty)}B_{\overrightarrow{N}}^{(M)}(f)(x)=f(x)$.

Proof. By (31) we get:

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right|\overset{(36)}{\leq}\left(\max_{i=1,...,k}\omega_1\left(\frac{\partial f}{\partial x_i},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k}B_{N_i}^{(M)}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k}B_{N_i}^{(M)}(|t_i-x_i|)(x_i)\right)+\frac{h}{8}\right] \tag{42}$$

(next we choose $h:=\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}$, then $h^2=\frac{1}{\sqrt{N_{\min}+1}}$)

$$\overset{(30)}{\leq}\left(\max_{i=1,...,k}\omega_1\left(\frac{\partial f}{\partial x_i},\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}\right)\right)\cdot \tag{43}$$

$$\left[3k^2\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}+3k\left(\frac{1}{\sqrt{N_{\min}+1}}\right)+\frac{1}{8}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}\right],$$

proving the claim. ■

We need

Theorem 13 *Let Q with $\|\cdot\|$ the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k\in\mathbb{N}-\{1\}$, and $f\in C_+(Q)$; $h>0$. We denote $\omega_1(f,h):=\sup_{\substack{x,y\in Q:\\ \|x-y\|\leq h}}|f(x)-f(y)|$, the modulus of continuity of f. Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into itself such that $L_N(1)=1$, $\forall\ N\in\mathbb{N}$. Then*

$$|L_N(f)(x)-f(x)|\leq\omega_1(f,h)\left(1+\frac{1}{h}L_N(\|t-x\|)(x)\right)\leq$$

$$\omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k} L_N(|t_i - x_i|)(x)\right)\right), \tag{44}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x \in Q$, *where* $x := (x_1, ..., x_k)$; $t = (t_1, ..., t_k) \in Q$.

Proof. We have that ([1], pp. 208-209)

$$|f(t) - f(x)| \leq \omega_1(f,h)\left\lceil \frac{\|t-x\|}{h} \right\rceil \leq \omega_1(f,h)\left(1+\frac{\|t-x\|}{h}\right), \tag{45}$$

$\forall\ t, x \in Q$.

By (25) we get:

$$|L_N(f)(x) - f(x)| \leq L_N(|f(t) - f(x)|)(x) \leq \tag{46}$$

$$\omega_1(f,h)\left(1+\frac{1}{h}L_N(\|t-x\|)(x)\right), \ \ \forall\ N \in \mathbb{N},$$

proving the claim. ■

We give

Theorem 14 *Let* $f \in C_+\left([0,1]^k\right)$, $k \in \mathbb{N} - \{1\}$. *Then*

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq (6k+1)\,\omega_1\left(f, \frac{1}{\sqrt{N_{\min}+1}}\right), \tag{47}$$

$\forall\ x \in [0,1]^k$, $\forall\ \overrightarrow{N} \in \mathbb{N}^k$, *where* $N_{\min} := \min\{N_1, ..., N_k\}$.

That is

$$\left\|B_{\overrightarrow{N}}^{(M)}(f) - f\right\|_\infty \leq (6k+1)\,\omega_1\left(f, \frac{1}{\sqrt{N_{\min}+1}}\right). \tag{48}$$

It holds that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} B_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$, *uniformly.*

Proof. We get that (use of (44))

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(36)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k} B_{N_i}^{(M)}(|t_i - x_i|)(x_i)\right)\right)$$

$$\overset{(29)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\frac{6k}{\sqrt{N_{\min}+1}}\right)\right) \tag{49}$$

(setting $h := \frac{1}{\sqrt{N_{\min}+1}}$)

$$= \omega_1\left(f, \frac{1}{\sqrt{N_{\min}+1}}\right)(6k+1), \ \ \forall\ x \in [0,1]^k, \ \forall\ \overrightarrow{N} \in \mathbb{N}^k,$$

proving the claim. ■

We continue with

Definition 15 *([4], p. 123) We define the bivariate Max-product Bernstein type operators:*

$$A_N^{(M)}(f)(x,y) := \frac{\bigvee_{i=0}^{N}\bigvee_{j=0}^{N-i}\binom{N}{i}\binom{N-i}{j}x^i y^j(1-x-y)^{N-i-j} f\left(\frac{i}{N},\frac{j}{N}\right)}{\bigvee_{i=0}^{N}\bigvee_{j=0}^{N-i}\binom{N}{i}\binom{N-i}{j}x^i y^j(1-x-y)^{N-i-j}}, \tag{50}$$

$\forall\,(x,y)\in\Delta := \{(x,y) : x\geq 0,\ y\geq 0,\ x+y\leq 1\}$, $\forall\, N\in\mathbb{N}$, *and* $\forall\, f\in C_+(\Delta)$.

Remark 16 *By [4], p. 137, Theorem 2.7.5 there,* $A_N^{(M)}$ *is a positive sublinear operator mapping* $C_+(\Delta)$ *into itself and* $A_N^{(M)}(1)=1$*, furthermore it holds*

$$\left|A_N^{(M)}(f)-A_N^{(M)}(g)\right| \leq A_N^{(M)}(|f-g|),\ \ \forall\, f,g\in C_+(\Delta),\ \forall\, N\in\mathbb{N}. \tag{51}$$

By [4], p. 125 we get that $A_N^{(M)}(f)(1,0)=f(1,0)$, $A_N^{(M)}(f)(0,1)=f(0,1)$, *and* $A_N^{(M)}(f)(0,0)=f(0,0)$.

By [4], p. 139, we have that ($(x,y)\in\Delta$)*:*

$$A_N^{(M)}(|\cdot-x|)(x,y) = B_N^{(M)}(|\cdot-x|)(x), \tag{52}$$

and

$$A_N^{(M)}(|\cdot-y|)(x,y) = B_N^{(M)}(|\cdot-y|)(y). \tag{53}$$

Working exactly the same way as (52), (53) are proved we also derive ($m\in\mathbb{N}$, $(x,y)\in\Delta$)*:*

$$A_N^{(M)}(|\cdot-x|^m)(x,y) = B_N^{(M)}(|\cdot-x|^m)(x), \tag{54}$$

and

$$A_N^{(M)}(|\cdot-y|^m)(x,y) = B_N^{(M)}(|\cdot-y|^m)(y). \tag{55}$$

We present

Theorem 17 *Let* $x:=(x_1,x_2)\in\Delta$ *be fixed, and* $f\in C^n(\Delta,\mathbb{R}_+)$, $n\in\mathbb{N}-\{1\}$. *We assume that* $f_\alpha(x)=0$*, for all* $\alpha : |\alpha|=1,...,n$. *Then*

$$\left|A_N^{(M)}(f)(x_1,x_2)-f(x_1,x_2)\right| \leq 6\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha,\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}\right)\right)\cdot \tag{56}$$

$$\left[\frac{2^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{n}{n+1}}+\frac{2^{n-1}}{n!}\left(\frac{1}{\sqrt{N+1}}\right)+\frac{2^{n-4}}{(n-1)!}\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{n+2}{n+1}}\right],$$

$\forall\, N\in\mathbb{N}$.

It holds $\lim_{N\to\infty} A_N^{(M)}(f)(x_1,x_2)=f(x_1,x_2)$.

Proof. By (26) we get (here $x := (x_1, x_2) \in \Delta$):

$$\left|A_N^{(M)}(f)(x_1,x_2) - f(x_1,x_2)\right| \leq \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right) \cdot$$

$$\left[\frac{2^n}{(n+1)!h}\left(\sum_{i=1}^{2} A_N^{(M)}\left(|t_i - x_i|^{n+1}\right)(x)\right) + \frac{2^{n-2}}{n!}\left(\sum_{i=1}^{2} A_N^{(M)}\left(|t_i - x_i|^{n}\right)(x)\right)\right.$$

$$\left. + \frac{h2^{n-5}}{(n-1)!}\left(\sum_{i=1}^{2} A_N^{(M)}\left(|t_i - x_i|^{n-1}\right)(x)\right)\right] \overset{\text{(by (54), (55))}}{=} \tag{57}$$

$$\left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right)\left[\frac{2^n}{(n+1)!h}\left(\sum_{i=1}^{2} B_N^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \right.$$

$$\left.\frac{2^{n-2}}{n!}\left(\sum_{i=1}^{2} B_N^{(M)}\left(|t_i - x_i|^{n}\right)(x_i)\right) + \frac{h2^{n-5}}{(n-1)!}\left(\sum_{i=1}^{2} B_N^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \tag{58}$$

$$\overset{(30)}{\leq} \frac{6\left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right)}{\sqrt{N+1}}\left[\frac{2^{n+1}}{(n+1)!h} + \frac{2^{n-1}}{n!} + \frac{h2^{n-4}}{(n-1)!}\right] =: (\xi).$$

Next we choose $h := \left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{\sqrt{N+1}}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{\sqrt{N+1}}$.

We have

$$(\xi) = 6\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}\right)\right) \cdot \tag{59}$$

$$\left[\frac{2^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{n}{n+1}} + \frac{2^{n-1}}{n!}\left(\frac{1}{\sqrt{N+1}}\right) + \frac{2^{n-4}}{(n-1)!}\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{n+2}{n+1}}\right],$$

proving the claim. ■

We also give

Theorem 18 *Let $x := (x_1, x_2) \in \Delta$ be fixed, and $f \in C^1(\Delta, \mathbb{R}_+)$. We assume that $\frac{\partial f}{\partial x_i}(x) = 0$, for $i = 1, 2$. Then*

$$\left|A_N^{(M)}(f)(x_1,x_2) - f(x_1,x_2)\right| \leq \left(\max_{i=1,2} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt[4]{N+1}}\right)\right) \cdot \tag{60}$$

$$\left[\frac{12}{\sqrt[4]{N+1}} + \frac{6}{\sqrt{N+1}} + \frac{1}{8}\left(\frac{1}{\sqrt[4]{N+1}}\right)\right],$$

$\forall\ N \in \mathbb{N}$.

It holds $\lim_{N\to\infty} A_N^{(M)}(f)(x_1,x_2) = f(x_1,x_2)$.

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Proof. By (31) we get (here $x := (x_1, x_2) \in \Delta$):

$$\left|A_N^{(M)}(f)(x_1, x_2) - f(x_1, x_2)\right| \leq \left(\max_{i=1,2} \omega_1\left(\frac{\partial f}{\partial x_i}, h\right)\right) \cdot$$

$$\left[\frac{1}{h}\left(\sum_{i=1}^{2} A_N^{(M)}\left((t_i - x_i)^2\right)(x)\right) + \frac{1}{2}\left(\sum_{i=1}^{2} A_N^{(M)}(|t_i - x_i|)(x)\right) + \frac{h}{8}\right] \tag{61}$$

$$\overset{(\text{by } (54), (55))}{=} \left(\max_{i=1,2} \omega_1\left(\frac{\partial f}{\partial x_i}, h\right)\right)\left[\frac{1}{h}\left(\sum_{i=1}^{2} B_N^{(M)}\left((t_i - x_i)^2\right)(x_i)\right) + \right.$$

$$\left.\frac{1}{2}\left(\sum_{i=1}^{2} B_N^{(M)}(|t_i - x_i|)(x_i)\right) + \frac{h}{8}\right]$$

(next we choose $h := \left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{2}}$, then $h^2 = \frac{1}{\sqrt{N+1}}$)

$$\overset{(30)}{\leq} \left(\max_{i=1,2} \omega_1\left(\frac{\partial f}{\partial x_i}, \left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{2}}\right)\right) \cdot \tag{62}$$

$$\left[12\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{2}} + \left(\frac{6}{\sqrt{N+1}}\right) + \frac{1}{8}\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{2}}\right],$$

proving the claim. ■

We further obtain

Theorem 19 *Let $f \in C_+(\Delta)$. Then*

$$\left|A_N^{(M)}(f)(x_1, x_2) - f(x_1, x_2)\right| \leq 13\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \tag{63}$$

$\forall\ (x_1, x_2) \in \Delta$, $\forall\ N \in \mathbb{N}$.

That is

$$\left\|A_N^{(M)}(f) - f\right\|_{\infty,\Delta} \leq 13\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \tag{64}$$

$\forall\ N \in \mathbb{N}$.

It holds that $\lim_{N\to\infty} A_N^{(M)}(f) = f$, uniformly, $\forall\ f \in C_+(\Delta)$.

Proof. Using (44) ($x := (x_1, x_2) \in \Delta$) we get:

$$\left|A_N^{(M)}(f)(x_1, x_2) - f(x_1, x_2)\right| \leq$$

$$\omega_1(f, h)\left(1 + \frac{1}{h}\left(\sum_{i=1}^{2} A_N^{(M)}(|t_i - x_i|)(x)\right)\right) \overset{(\text{by } (52), (53))}{=}$$

$$\omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{2} B_N^{(M)}(|t_i-x_i|)(x_i)\right)\right) \overset{(29)}{\leq}$$

$$\omega_1(f,h)\left(1+\frac{2}{h}\cdot\frac{6}{\sqrt{N+1}}\right) \tag{65}$$

(setting $h := \frac{1}{\sqrt{N+1}}$)

$$= 13\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \ \ \forall \ (x_1, x_2) \in \Delta, \ \forall \ N \in \mathbb{N},$$

proving the claim. ■

We make

Remark 20 *The Max-product truncated Favard-Szász-Mirakjan operators*

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x)\, f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{66}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, *see also [4], p. 11.*

By [4], p. 178-179, we get that

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ \forall \ N \in \mathbb{N}. \tag{67}$$

And from [2] we have

$$T_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ \forall \ N, m \in \mathbb{N}. \tag{68}$$

We make

Definition 21 *Let $f \in C_+\left([0,1]^k\right)$, $k \in \mathbb{N} - \{1\}$, and $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. We define the multivariate Max-product truncated Favard-Szász-Mirakjan operators as follows:*

$$T_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k} s_{N_1,i_1}(x_1)\, s_{N_2,i_2}(x_2)\, ...s_{N_k,i_k}(x_k)\, f\left(\frac{i_1}{N_1}, ..., \frac{i_k}{N_k}\right)}{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k} s_{N_1,i_1}(x_1)\, s_{N_2,i_2}(x_2)\, ...s_{N_k,i_k}(x_k)}, \tag{69}$$

$\forall \ x = (x_1, ..., x_k) \in [0,1]^k$. *Call $N_{\min} := \min\{N_1, ..., N_k\}$.*

The operators $T_{\overrightarrow{N}}^{(M)}(f)(x)$ are positive sublinear mapping $C_+\left([0,1]^k\right)$ into itself, and $T_{\overrightarrow{N}}^{(M)}(1) = 1$.

We also have

$$T_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k} s_{N_1,i_1}(x_1)\, s_{N_2,i_2}(x_2)\,...s_{N_k,i_k}(x_k)\, f\left(\frac{i_1}{N_1},...,\frac{i_k}{N_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda} s_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \quad (70)$$

$\forall\ x \in [0,1]^k$, by the maximum multiplicative principle, see (27).

We make

Remark 22 *The coordinate Max-product truncated Favard-Szász-Mirakjan operators are defined as follows ($\lambda = 1,...,k$):*

$$T_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda} s_{N_\lambda,i_\lambda}(x_\lambda)\, g\left(\frac{i_\lambda}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda} s_{N_\lambda,i_\lambda}(x_\lambda)}, \quad (71)$$

$\forall\ N_\lambda \in \mathbb{N}$, and $\forall\ x_\lambda \in [0,1]$, $\forall\ g \in C_+([0,1])$.

Here we have

$$s_{N_\lambda,i_\lambda}(x_\lambda) = \frac{(N_\lambda x_\lambda)^{i_\lambda}}{i_\lambda!},\ \ \lambda = 1,...,k;\ \ x_\lambda \in [0,1]. \quad (72)$$

In case of $f \in C_+\left([0,1]^k\right)$ such that $f(x) := g(x_\lambda)$, $\forall\ x \in [0,1]^k$, where $x = (x_1,...,x_\lambda,...,x_k)$ and $g \in C_+([0,1])$, we get that

$$T_{\overrightarrow{N}}^{(M)}(f)(x) = T_{N_\lambda}^{(M)}(g)(x_\lambda), \quad (73)$$

by the maximum multiplicative principle (27) and simplification of (70).

We present

Theorem 23 *Let $x \in [0,1]^k$, $k \in \mathbb{N} - \{1\}$, be fixed, and let $f \in C^n\left([0,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N} - \{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1,...,n$. Then*

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq 3\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{n+1}}\right)\right)\right)\cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n}{n+1}} + \frac{k^n}{2n!}\left(\frac{1}{\sqrt{N_{\min}}}\right) + \frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n+2}{n+1}}\right], \quad (74)$$

$\forall\ \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min} := \min\{N_1,...,N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} T_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(73)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\cdot$$

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$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^{k} T_{N_i}^{(M)}\left(|t_i-x_i|^{n+1}\right)(x_i)\right)+\frac{k^{n-1}}{2n!}\left(\sum_{i=1}^{k} T_{N_i}^{(M)}\left(|t_i-x_i|^{n}\right)(x_i)\right)\right. \tag{75}$$

$$\left.+\frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^{k} T_{N_i}^{(M)}\left(|t_i-x_i|^{n-1}\right)(x_i)\right)\right] \overset{(68)}{\leq}$$

$$\frac{3}{\sqrt{N_{\min}}}\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha,h)\right)\left[\frac{k^{n+1}}{(n+1)!h}+\frac{k^n}{2n!}+\frac{hk^{n-1}}{8(n-1)!}\right]=:(\xi).$$

Above notice that $\sum_{i=1}^{k} T_{N_i}^{(M)}\left(|t_i-x_i|^{n}\right)(x_i) \overset{(68)}{\leq} \sum_{i=1}^{k}\frac{3}{\sqrt{N_i}} \leq \frac{3k}{\sqrt{N_{\min}}}$, etc.

Next we choose $h:=\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{n+1}}$, then $h^n=\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n}{n+1}}$ and $h^{n+1}=\frac{1}{\sqrt{N_{\min}}}$.

We have

$$(\xi)=3\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha,\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{n+1}}\right)\right)\right)\cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n}{n+1}}+\frac{k^n}{2n!}\left(\frac{1}{\sqrt{N_{\min}}}\right)+\frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n+2}{n+1}}\right], \tag{76}$$

proving the claim. ■

We also give

Proposition 24 *Let $x\in[0,1]^k$, $k\in\mathbb{N}-\{1\}$, be fixed and let $f\in C^1\left([0,1]^k,\mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i}=0$, for $i=1,...,k$. Then*

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right|\leq\left(\max_{i=1,\dots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt[4]{N_{\min}}}\right)\right)\cdot$$

$$\left[\frac{3k^2}{2}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right)+\frac{3k}{2}\left(\frac{1}{\sqrt{N_{\min}}}\right)+\frac{1}{8}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right)\right], \tag{77}$$

$\forall\ \overrightarrow{N}\in\mathbb{N}^k$, where $N_{\min}:=\min\{N_1,...,N_k\}$.

Also it holds $\lim_{\overrightarrow{N}\to(\infty,\dots,\infty)} T_{\overrightarrow{N}}^{(M)}(f)(x)=f(x)$.

Proof. By (31) we get:

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right|\overset{(73)}{\leq}\left(\max_{i=1,\dots,k}\omega_1\left(\frac{\partial f}{\partial x_i},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k} T_{N_i}^{(M)}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k} T_{N_i}^{(M)}\left(|t_i-x_i|\right)(x_i)\right)+\frac{h}{8}\right] \tag{78}$$

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(next we choose $h := \left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{2}}$, then $h^2 = \frac{1}{\sqrt{N_{\min}}}$)

$$\overset{(68)}{\leq} \left(\max_{i=1,...,k} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt[4]{N_{\min}}}\right)\right) \cdot$$

$$\left[\frac{3k^2}{2}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right) + \frac{3k}{2}\left(\frac{1}{\sqrt{N_{\min}}}\right) + \frac{1}{8}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right)\right], \tag{79}$$

proving the claim. ∎

It follows

Theorem 25 *Let $f \in C_+\left([0,1]^k\right)$, $k \in \mathbb{N} - \{1\}$. Then*

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq (3k+1)\,\omega_1\left(f, \frac{1}{\sqrt{N_{\min}}}\right), \tag{80}$$

$\forall\ x \in [0,1]^k$, $\forall\ \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min} := \min\{N_1, ..., N_k\}$.

That is

$$\left\|T_{\overrightarrow{N}}^{(M)}(f) - f\right\|_{\infty} \leq (3k+1)\,\omega_1\left(f, \frac{1}{\sqrt{N_{\min}}}\right). \tag{81}$$

It holds that $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} T_{\overrightarrow{N}}^{(M)}(f) = f$, uniformly.

Proof. We get that (use of (44))

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(73)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\sum_{i=1}^{k} T_{N_i}^{(M)}(|t_i - x_i|)(x)\right)\right)$$

$$\overset{(67)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\frac{3k}{\sqrt{N_{\min}}}\right)\right) \tag{82}$$

(setting $h := \frac{1}{\sqrt{N_{\min}}}$)

$$= \omega_1\left(f, \frac{1}{\sqrt{N_{\min}}}\right)(3k+1), \quad \forall\ x \in [0,1]^k,\ \forall\ \overrightarrow{N} \in \mathbb{N}^k,$$

proving the claim. ∎

We make

Remark 26 *We mention the truncated Max-product Baskakov operator (see [4], p. 11)*

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x)\, f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1],\ f \in C_+([0,1]),\ \forall\ N \in \mathbb{N}, \tag{83}$$

where

$$b_{N,k}(x)=\binom{N+k-1}{k}\frac{x^k}{(1+x)^{N+k}}. \tag{84}$$

From [4], pp. 217-218, we get ($x\in[0,1]$)

$$\left(U_N^{(M)}(|\cdot-x|)\right)(x)\leq\frac{12}{\sqrt{N+1}},\ \ N\geq 2,\ N\in\mathbb{N}. \tag{85}$$

And as in [2], we obtain ($m\in\mathbb{N}$)

$$\left(U_N^{(M)}(|\cdot-x|^m)\right)(x)\leq\frac{12}{\sqrt{N+1}},\ \ N\geq 2,\ N\in\mathbb{N},\ \forall\ x\in[0,1]. \tag{86}$$

Definition 27 *Let $f\in C_+\left([0,1]^k\right)$, $k\in\mathbb{N}-\{1\}$, and $\overrightarrow{N}=(N_1,...,N_k)\in\mathbb{N}^k$. We define the multivariate Max-product truncated Baskakov operators as follows:*

$$U_{\overrightarrow{N}}^{(M)}(f)(x):=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}b_{N_1,i_1}(x_1)\,b_{N_2,i_2}(x_2)...b_{N_k,i_k}(x_k)\,f\left(\frac{i_1}{N_1},...,\frac{i_k}{N_k}\right)}{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}b_{N_1,i_1}(x_1)\,b_{N_2,i_2}(x_2)...b_{N_k,i_k}(x_k)}, \tag{87}$$

$\forall\ x=(x_1,...,x_k)\in[0,1]^k$. Call $N_{\min}:=\min\{N_1,...,N_k\}$.

The operators $U_{\overrightarrow{N}}^{(M)}(f)(x)$ are positive sublinear mapping $C_+\left([0,1]^k\right)$ into itself, and $U_{\overrightarrow{N}}^{(M)}(1)=1$.

We also have

$$U_{\overrightarrow{N}}^{(M)}(f)(x):=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}b_{N_1,i_1}(x_1)\,b_{N_2,i_2}(x_2)...b_{N_k,i_k}(x_k)\,f\left(\frac{i_1}{N_1},...,\frac{i_k}{N_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda}b_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{88}$$

$\forall\ x\in[0,1]^k$, by the maximum multiplicative principle, see (27).

We make

Remark 28 *The coordinate Max-product truncated Baskakov operators are defined as follows ($\lambda=1,...,k$):*

$$U_{N_\lambda}^{(M)}(g)(x_\lambda):=\frac{\bigvee_{i_\lambda=0}^{N_\lambda}b_{N_\lambda,i_\lambda}(x_\lambda)\,g\left(\frac{i_\lambda}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda}b_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{89}$$

$\forall\ N_\lambda\in\mathbb{N}$, and $\forall\ x_\lambda\in[0,1]$, $\forall\ g\in C_+([0,1])$.

Here we have

$$b_{N_\lambda,i_\lambda}(x_\lambda)=\binom{N_\lambda+i_\lambda-1}{i_\lambda}\frac{x_\lambda^{i_\lambda}}{(1+x_\lambda)^{N+i_\lambda}},\ \lambda=1,...,k;\ x_\lambda\in[0,1].$$

In case of $f \in C_{+}\left([0,1]^{k}\right)$ *such that* $f(x) := g(x_{\lambda})$, $\forall\ x \in [0,1]^{k}$, *where* $x = (x_1, ..., x_{\lambda}, ..., x_k)$ *and* $g \in C_{+}([0,1])$, *we get that*

$$U_{\overrightarrow{N}}^{(M)}(f)(x) = U_{N_{\lambda}}^{(M)}(g)(x_{\lambda}), \tag{90}$$

by the maximum multiplicative principle (27) and simplification of (89).

We present

Theorem 29 *Let* $x \in [0,1]^{k}$, $k \in \mathbb{N} - \{1\}$, *be fixed, and let* $f \in C^{n}\left([0,1]^{k}, \mathbb{R}_{+}\right)$, $n \in \mathbb{N} - \{1\}$. *We assume that* $f_{\alpha}(x) = 0$, *for all* $\alpha : |\alpha| = 1, ..., n$. *Then*

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq 12\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_{\alpha}, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right) \cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}} + \frac{k^{n}}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right) + \frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right], \tag{91}$$

$\forall\ \overrightarrow{N} \in (\mathbb{N} - \{1\})^{k}$, *where* $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N} \to (\infty, ..., \infty)} U_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(90)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_{\alpha}, h)\right) \cdot$$

$$\left[\frac{k^{n}}{(n+1)!h}\left(\sum_{i=1}^{k}U_{N_i}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^{k}U_{N_i}^{(M)}\left(|t_i - x_i|^{n}\right)(x_i)\right)\right. \tag{92}$$

$$\left. + \frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^{k}U_{N_i}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(86)}{\leq}$$

$$\frac{12}{\sqrt{N_{\min}+1}}\left(\max_{\alpha:|\alpha|=n}\omega_1(f_{\alpha}, h)\right)\left[\frac{k^{n+1}}{(n+1)!h} + \frac{k^{n}}{2n!} + \frac{hk^{n-1}}{8(n-1)!}\right] =: (\xi).$$

Above notice that $\sum_{i=1}^{k}U_{N_i}^{(M)}\left(|t_i - x_i|^{n}\right)(x_i) \overset{(86)}{\leq} \sum_{i=1}^{k}\frac{12}{\sqrt{N_i+1}} \leq \frac{12k}{\sqrt{N_{\min}+1}}$, etc.

Next we choose $h := \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}$, then $h^{n} = \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{\sqrt{N_{\min}+1}}$.

We have

$$(\xi) = 12\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_{\alpha}, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right) \cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}+\frac{k^{n}}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)+\frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right], \tag{93}$$

proving the claim. ■

We also give

Proposition 30 *Let $x \in [0,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed and let $f \in C^1\left([0,1]^k, \mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i}=0$, for $i=1,...,k$. Then*

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \leq \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt[4]{N_{\min}+1}}\right)\right)\cdot \tag{94}$$

$$\left[\frac{6k^2}{\sqrt[4]{N_{\min}+1}}+\frac{6k}{\sqrt{N_{\min}+1}}+\frac{1}{8\left(\sqrt[4]{N_{\min}+1}\right)}\right],$$

$\forall\ \overrightarrow{N} \in (\mathbb{N}-\{1\})^k$, where $N_{\min}:=\min\{N_1,...,N_k\}$.

Also it holds $\lim_{\overrightarrow{N}\to(\infty,\ldots,\infty)} U_{\overrightarrow{N}}^{(M)}(f)(x)=f(x)$.

Proof. By (31) we get:

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(90)}{\leq} \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k}U_{N_i}^{(M)}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k}U_{N_i}^{(M)}\left(|t_i-x_i|\right)(x_i)\right)+\frac{h}{8}\right] \tag{95}$$

(next we choose $h:=\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}$, then $h^2=\frac{1}{\sqrt{N_{\min}+1}}$)

$$\overset{(85)}{\leq}\left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt[4]{N_{\min}+1}}\right)\right)\cdot$$

$$\left[\frac{6k^2}{\sqrt[4]{N_{\min}+1}}+\frac{6k}{\sqrt{N_{\min}+1}}+\frac{1}{8\left(\sqrt[4]{N_{\min}+1}\right)}\right], \tag{96}$$

proving the claim. ■

It follows

Theorem 31 *Let $f \in C_+\left([0,1]^k\right)$, $k \in \mathbb{N}-\{1\}$. Then*

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \leq (12k+1)\,\omega_1\left(f,\frac{1}{\sqrt{N_{\min}+1}}\right), \tag{97}$$

$\forall\ x \in [0,1]^k$, $\forall\ \overrightarrow{N} \in (\mathbb{N}-\{1\})^k$, where $N_{\min}:=\min\{N_1,...,N_k\}$.

That is

$$\left\| U_{\overrightarrow{N}}^{(M)}(f) - f \right\|_{\infty} \leq (12k+1)\,\omega_1\left(f, \frac{1}{\sqrt{N_{\min}+1}}\right). \tag{98}$$

It holds that $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} U_{\overrightarrow{N}}^{(M)}(f) = f$, *uniformly.*

Proof. We get that (use of (44))

$$\left| U_{\overrightarrow{N}}^{(M)}(f)(x) - f(x) \right| \overset{(90)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\sum_{i=1}^{k} U_{N_i}^{(M)}(|t_i - x_i|)(x_i)\right)\right)$$

$$\overset{(85)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\frac{12k}{\sqrt{N_{\min}+1}}\right)\right) \tag{99}$$

(setting $h := \frac{1}{\sqrt{N_{\min}+1}}$)

$$= \omega_1\left(f, \frac{1}{\sqrt{N_{\min}+1}}\right)(12k+1), \quad \forall\, x \in [0,1]^k, \ \forall\, \overrightarrow{N} \in (\mathbb{N} - \{1\})^k,$$

proving the claim. ■

We make

Remark 32 *Here we mention the Max-product truncated sampling operators (see [4], p. 13) defined by*

$$W_N^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \quad x \in [0,\pi], \tag{100}$$

$f : [0,\pi] \to \mathbb{R}_+$, *continuous,*

and

$$K_N^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \quad x \in [0,\pi], \tag{101}$$

$f : [0,\pi] \to \mathbb{R}_+$, *continuous.*

By convention we talk $\frac{\sin(0)}{0} = 1$, *which implies for every* $x = \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$ *that we have* $\frac{\sin(Nx-k\pi)}{Nx-k\pi} = 1$.

We define the Max-product truncated combined sampling operators

$$M_N^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \rho_{N,k}(x)\, f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \rho_{N,k}(x)}, \quad x \in [0,\pi], \tag{102}$$

$f \in C_+([0,\pi])$, *where*

$$M_N^{(M)}(f)(x) := \begin{cases} W_N^{(M)}(f)(x), \text{ if } \rho_{N,k}(x) := \frac{\sin(Nx-k\pi)}{Nx-k\pi}, \\ K_N^{(M)}(f)(x), \text{ if } \rho_{N,k}(x) := \left(\frac{\sin(Nx-k\pi)}{Nx-k\pi}\right)^2. \end{cases} \tag{103}$$

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By [4], p. 346 and p. 352 we get

$$\left(M_N^{(M)}\left(|\cdot - x|\right)\right)(x) \leq \frac{\pi}{2N}, \tag{104}$$

and by [3] ($m \in \mathbb{N}$) we have

$$\left(M_N^{(M)}\left(|\cdot - x|^m\right)\right)(x) \leq \frac{\pi^m}{2N}, \ \ \forall\ x \in [0,\pi]\,,\ \forall\ N \in \mathbb{N}. \tag{105}$$

We give

Definition 33 *Let $f \in C_+\left([0,\pi]^k\right)$, $k \in \mathbb{N} - \{1\}$, and $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. We define the multivariate Max-product truncated combined sampling operators as follows:*

$$M_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}\rho_{N_1,i_1}(x_1)\,\rho_{N_2,i_2}(x_2)...\rho_{N_k,i_k}(x_k)\,f\left(\frac{i_1\pi}{N_1},\frac{i_2\pi}{N_2},...,\frac{i_k\pi}{N_k}\right)}{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}\rho_{N_1,i_1}(x_1)\,\rho_{N_2,i_2}(x_2)...\rho_{N_k,i_k}(x_k)}, \tag{106}$$

$\forall\ x = (x_1, ..., x_k) \in [0,\pi]^k$. *Call $N_{\min} := \min\{N_1, ..., N_k\}$.*

The operators $M_{\overrightarrow{N}}^{(M)}(f)(x)$ are positive sublinear mapping $C_+\left([0,\pi]^k\right)$ into itself, and $M_{\overrightarrow{N}}^{(M)}(1) = 1$.

We also have

$$M_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}\rho_{N_1,i_1}(x_1)\,\rho_{N_2,i_2}(x_2)...\rho_{N_k,i_k}(x_k)\,f\left(\frac{i_1\pi}{N_1},\frac{i_2\pi}{N_2},...,\frac{i_k\pi}{N_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda}\rho_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{107}$$

$\forall\ x \in [0,\pi]^k$, *by the maximum multiplicative principle, see (27).*

We make

Remark 34 *The coordinate Max-product truncated combined sampling operators are defined as follows ($\lambda = 1, ..., k$):*

$$M_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda}\rho_{N_\lambda,i_\lambda}(x_\lambda)\,g\left(\frac{i_\lambda\pi}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda}\rho_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{108}$$

$\forall\ N_\lambda \in \mathbb{N}$, *and* $\forall\ x_\lambda \in [0,\pi]$, $\forall\ g \in C_+([0,\pi])$.

Here we have ($\lambda = 1, ..., k$; $x_\lambda \in [0,\pi]$)

$$\rho_{N_\lambda,i_\lambda}(x_\lambda) = \left\{\begin{array}{l} \frac{\sin(N_\lambda x_\lambda - i_\lambda\pi)}{N_\lambda x_\lambda - i_\lambda\pi}, \quad \text{if } M_{N_\lambda}^{(M)} = W_{N_\lambda}^{(M)}, \\ \left(\frac{\sin(N_\lambda x_\lambda - i_\lambda\pi)}{N_\lambda x_\lambda - i_\lambda\pi}\right)^2, \quad \text{if } M_{N_\lambda}^{(M)} = K_{N_\lambda}^{(M)}. \end{array}\right\}. \tag{109}$$

In case of $f \in C_+\left([0,\pi]^k\right)$ *such that* $f(x) := g(x_\lambda)$, $\forall\ x \in [0,\pi]^k$, *where* $x = (x_1, ..., x_\lambda, ..., x_k)$ *and* $g \in C_+([0,\pi])$, *we get that*

$$M_{\overrightarrow{N}}^{(M)}(f)(x) = M_{N_\lambda}^{(M)}(g)(x_\lambda), \tag{110}$$

by the maximum multiplicative principle (27) and simplification of (107).

We present

Theorem 35 *Let* $x \in [0,\pi]^k$, $k \in \mathbb{N}-\{1\}$, *be fixed, and let* $f \in C^n\left([0,\pi]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N}-\{1\}$. *We assume that* $f_\alpha(x) = 0$, *for all* $\alpha : |\alpha| = 1, ..., n$. *Then*

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq \frac{(k\pi)^{n-1}}{2}\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{(N_{\min})^{\frac{1}{n+1}}}\right)\right)\cdot \tag{111}$$

$$\left[\frac{(k\pi)^2}{(n+1)!}\frac{1}{(N_{\min})^{\frac{n}{n+1}}} + \frac{k\pi}{2n!N_{\min}} + \frac{1}{8(n-1)!(N_{\min})^{\frac{n+2}{n+1}}}\right],$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$, *where* $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} M_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(110)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\cdot$$

$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^k M_{N_i}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^k M_{N_i}^{(M)}\left(|t_i - x_i|^n\right)(x_i)\right)\right. \tag{112}$$

$$\left.+\frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^k M_{N_i}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(105)}{\leq}$$

$$\frac{1}{2N_{\min}}\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\left[\frac{k^{n+1}\pi^{n+1}}{(n+1)!h} + \frac{k^n\pi^n}{2n!} + \frac{hk^{n-1}\pi^{n-1}}{8(n-1)!}\right] =: (\xi).$$

Above notice that $\sum_{i=1}^k M_{N_i}^{(M)}(|t_i - x_i|^n)(x_i) \overset{(105)}{\leq} \sum_{i=1}^k \frac{\pi^n}{2N_i} \leq \frac{k\pi^n}{2N_{\min}}$, etc.

Next we choose $h := \left(\frac{1}{N_{\min}}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{N_{\min}}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{N_{\min}}$.

We have

$$(\xi) = \frac{(k\pi)^{n-1}}{2}\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{(N_{\min})^{\frac{1}{n+1}}}\right)\right)\cdot \tag{113}$$

$$\left[\frac{(k\pi)^2}{(n+1)!}\frac{1}{(N_{\min})^{\frac{n}{n+1}}} + \frac{k\pi}{2n!N_{\min}} + \frac{1}{8(n-1)!(N_{\min})^{\frac{n+2}{n+1}}}\right],$$

proving the claim. ■

We also give

Proposition 36 *Let $x \in [0,\pi]^k$, $k \in \mathbb{N}-\{1\}$, be fixed and let $f \in C^1([0,\pi],\mathbb{R}_+)$. We assume that $\frac{\partial f(x)}{\partial x_i}=0$, for $i=1,...,k$. Then*

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \leq \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt{N_{\min}}}\right)\right)\cdot$$

$$\left[\frac{(k\pi)^2}{4\sqrt{N_{\min}}}+\frac{k\pi}{4N_{\min}}+\frac{1}{8\left(\sqrt{N_{\min}}\right)}\right], \tag{114}$$

$\forall\ \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min}:=\min\{N_1,...,N_k\}$.

Also it holds $\lim_{\overrightarrow{N}\to(\infty,\ldots,\infty)} M_{\overrightarrow{N}}^{(M)}(f)(x)=f(x)$.

Proof. By (31) we get:

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(110)}{\leq} \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k}M_{N_i}^{(M)}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k}M_{N_i}^{(M)}(|t_i-x_i|)(x_i)\right)+\frac{h}{8}\right] \tag{115}$$

(next we choose $h:=\left(\frac{1}{N_{\min}}\right)^{\frac{1}{2}}$, then $h^2=\frac{1}{N_{\min}}$)

$$\overset{(105)}{\leq} \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt{N_{\min}}}\right)\right)\cdot$$

$$\left[\frac{(k\pi)^2}{4\sqrt{N_{\min}}}+\frac{k\pi}{4N_{\min}}+\frac{1}{8\left(\sqrt{N_{\min}}\right)}\right], \tag{116}$$

proving the claim. ■

It follows

Theorem 37 *Let $f \in C_+\left([0,\pi]^k\right)$, $k \in \mathbb{N}-\{1\}$. Then*

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \leq \left(\frac{k\pi}{2}+1\right)\omega_1\left(f,\frac{1}{N_{\min}}\right), \tag{117}$$

$\forall\ x \in [0,\pi]^k$, $\forall\ \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min}:=\min\{N_1,...,N_k\}$.

That is

$$\left\|M_{\overrightarrow{N}}^{(M)}(f)-f\right\|_\infty \leq \left(\frac{k\pi}{2}+1\right)\omega_1\left(f,\frac{1}{N_{\min}}\right). \tag{118}$$

It holds $\lim_{\overrightarrow{N}\to(\infty,\ldots,\infty)} M_{\overrightarrow{N}}^{(M)}(f)=f$, uniformly.

Proof. We get that (use of (44))

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(110)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k} M_{N_i}^{(M)}(|t_i-x_i|)(x_i)\right)\right)$$

$$\overset{(104)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\frac{k\pi}{2N_{\min}}\right)\right) \tag{119}$$

(setting $h:=\frac{1}{N_{\min}}$)

$$=\omega_1\left(f,\frac{1}{N_{\min}}\right)\left(\frac{k\pi}{2}+1\right), \quad \forall\, x\in[0,\pi]^k, \ \forall\, \overrightarrow{N}\in\mathbb{N}^k,$$

proving the claim. ■

We make

Remark 38 *Let $f\in C_+([-1,1])$. Let the Chebyshev knots of second kind $x_{N,k}=\cos\left(\left(\frac{N-k}{N-1}\right)\pi\right)\in[-1,1]$, $k=1,...,N$, $N\in\mathbb{N}-\{1\}$, which are the roots of $\omega_N(x)=\sin(N-1)t\sin t$, $x=\cos t\in[-1,1]$. Notice that $x_{N,1}=-1$ and $x_{N,N}=1$.*

Define

$$l_{N,k}(x):=\frac{(-1)^{k-1}\,\omega_N(x)}{(1+\delta_{k,1}+\delta_{k,N})(N-1)(x-x_{N,k})}, \tag{120}$$

$N\geq 2$, $k=1,...,N$, and $\omega_N(x)=\prod_{k=1}^{N}(x-x_{N,k})$ and $\delta_{i,j}$ denotes the Kronecher's symbol, that is $\delta_{i,j}=1$, if $i=j$, and $\delta_{i,j}=0$, if $i\neq j$.

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ±1, are defined by ([4], p. 12)

$$L_N^{(M)}(f)(x)=\frac{\bigvee_{k=1}^{N} l_{N,k}(x)\,f(x_{N,k})}{\bigvee_{k=1}^{N} l_{N,k}(x)}, \quad x\in[-1,1]. \tag{121}$$

By [4], pp. 297-298 and [3], we get that

$$L_N^{(M)}(|\cdot-x|^m)(x)\leq\frac{2^{m+1}\pi^2}{3(N-1)}, \tag{122}$$

$\forall\, x\in(-1,1)$ and $\forall\, m\in\mathbb{N}$; $\forall\, N\in\mathbb{N}$, $N\geq 4$.

We see that $L_N^{(M)}(f)(x)\geq 0$ is well defined and continuous for any $x\in[-1,1]$. Following [4], p. 289, because $\sum_{k=1}^{N} l_{N,k}(x)=1$, $\forall\, x\in[-1,1]$, for any x there exists $k\in\{1,...,N\}: l_{N,k}(x)>0$, hence $\bigvee_{k=1}^{N} l_{N,k}(x)>0$. We have that $l_{N,k}(x_{N,k})=1$, and $l_{N,k}(x_{N,j})=0$, if $k\neq j$. Furthermore it holds $L_N^{(M)}(f)(x_{N,j})=f(x_{N,j})$, all $j\in\{1,...,N\}$, and $L_N^{(M)}(1)=1$.

By [4], pp. 289-290, $L_N^{(M)}$ are positive sublinear operators.

We give

Definition 39 *Let* $f \in C_+\left(\left[-1,1\right]^k\right)$, $k \in \mathbb{N}-\{1\}$, *and* $\overrightarrow{N}=(N_1,...,N_k) \in (\mathbb{N}-\{1\})^k$. *We define the multivariate Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints* ± 1, *as follows:*

$$L_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=1}^{N_1}\bigvee_{i_2=1}^{N_2}...\bigvee_{i_k=1}^{N_k} l_{N_1,i_1}(x_1)\, l_{N_2,i_2}(x_2)...l_{N_k,i_k}(x_k)\, f(x_{N_1,i_1},x_{N_2,i_2},...,x_{N_k,i_k})}{\bigvee_{i_1=1}^{N_1}\bigvee_{i_2=1}^{N_2}...\bigvee_{i_k=1}^{N_k} l_{N_1,i_1}(x_1)\, l_{N_2,i_2}(x_2)...l_{N_k,i_k}(x_k)}, \tag{123}$$

$\forall\ x=(x_1,...,x_k) \in [-1,1]^k$. *Call* $N_{\min} := \min\{N_1,...,N_k\}$.

The operators $L_{\overrightarrow{N}}^{(M)}(f)(x)$ *are positive sublinear mapping* $C_+\left(\left[-1,1\right]^k\right)$ *into itself, and* $L_{\overrightarrow{N}}^{(M)}(1)=1$.

We also have

$$L_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=1}^{N_1}\bigvee_{i_2=1}^{N_2}...\bigvee_{i_k=1}^{N_k} l_{N_1,i_1}(x_1)\, l_{N_2,i_2}(x_2)...l_{N_k,i_k}(x_k)\, f(x_{N_1,i_1},x_{N_2,i_2},...,x_{N_k,i_k})}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=1}^{N_\lambda} l_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{124}$$

$\forall\ x=(x_1,...,x_\lambda,...,x_k) \in [-1,1]^k$, *by the maximum multiplicative principle, see (27). Notice that* $L_{\overrightarrow{N}}^{(M)}(f)(x_{N_1,i_1},...,x_{N_k,i_k}) = f(x_{N_1,i_1},...,x_{N_k,i_k})$. *The last is also true if* $x_{N_1,i_1},...,x_{N_k,i_k} \in \{-1,1\}$.

We make

Remark 40 *The coordinate Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints* ± 1, *are defined as follows* ($\lambda=1,...,k$)*:*

$$L_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=1}^{N_\lambda} l_{N_\lambda,i_\lambda}(x_\lambda)\, g(x_{N_\lambda,i_\lambda})}{\bigvee_{i_\lambda=1}^{N_\lambda} l_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{125}$$

$\forall\ N_\lambda \in \mathbb{N}$, $N_\lambda \geq 2$, *and* $\forall\ x_\lambda \in [-1,1]$, $\forall\ g \in C_+([-1,1])$.

Here we have ($\lambda=1,...,k$; $x_\lambda \in [-1,1]$)

$$l_{N_\lambda,i_\lambda}(x_\lambda) = \frac{(-1)^{i_\lambda-1}\,\omega_{N_\lambda}(x_\lambda)}{(1+\delta_{i_\lambda,1}+\delta_{i_\lambda,N_\lambda})(N_\lambda-1)(x_\lambda-x_{N_\lambda,i_\lambda})}, \tag{126}$$

$N_\lambda \geq 2$, $i_\lambda=1,...,N_\lambda$ *and* $\omega_{N_\lambda}(x_\lambda) = \prod_{i_\lambda=1}^{N_\lambda}(x_\lambda - x_{N_\lambda,i_\lambda})$; *where* $x_{N_\lambda,i_\lambda} = \cos\left(\left(\frac{N_\lambda-i_\lambda}{N_\lambda-1}\right)\pi\right) \in [-1,1]$, $i_\lambda=1,...,N_\lambda$ ($N_\lambda \geq 2$) *are roots of* $\omega_{N_\lambda}(x_\lambda) = \sin(N_\lambda-1)t_\lambda \sin t_\lambda$, $x_\lambda=\cos t_\lambda$. *Notice that* $x_{N_\lambda,1}=-1$, $x_{N_\lambda,N_\lambda}=1$.

In case of $f \in C_+\left(\left[-1,1\right]^k\right)$ *such that* $f(x) := g(x_\lambda)$, $\forall\ x \in [-1,1]^k$, *where* $x=(x_1,...,x_\lambda,...,x_k)$ *and* $g \in C_+([-1,1])$, *we get that*

$$L_{\overrightarrow{N}}^{(M)}(f)(x) = L_{N_\lambda}^{(M)}(g)(x_\lambda), \tag{127}$$

by the maximum multiplicative principle (27) and simplification of (124).

We present

Theorem 41 *Let $x \in (-1,1)^k$, $k \in \mathbb{N} - \{1\}$, be fixed, and let $f \in C^n\left([-1,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N} - \{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left| L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x) \right| \leq \frac{(2k)^{n-1}\pi^2}{3} \left(\max_{\alpha:|\alpha|=n} \omega_1 \left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min} - 1}} \right) \right) \cdot \tag{128}$$

$$\left[\frac{8k^2}{(n+1)!\,(N_{\min}-1)^{\frac{n}{n+1}}} + \frac{2k}{n!\,(N_{\min}-1)} + \frac{1}{4\,(n-1)!\,(N_{\min}-1)^{\frac{n+2}{n+1}}} \right],$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$; $N_i \geq 4$, $i = 1, ..., k$, and $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} L_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left| L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x) \right| \overset{(127)}{\leq} \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h) \right) \cdot$$

$$\left[\frac{k^n}{(n+1)!h} \left(\sum_{i=1}^k L_{N_i}^{(M)} \left(|t_i - x_i|^{n+1} \right)(x_i) \right) + \frac{k^{n-1}}{2n!} \left(\sum_{i=1}^k L_{N_i}^{(M)} \left(|t_i - x_i|^n \right)(x_i) \right) \right. \tag{129}$$

$$\left. + \frac{hk^{n-2}}{8\,(n-1)!} \left(\sum_{i=1}^k L_{N_i}^{(M)} \left(|t_i - x_i|^{n-1} \right)(x_i) \right) \right] \overset{(122)}{\leq}$$

$$\frac{\pi^2}{3\,(N_{\min}-1)} \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h) \right) \left[\frac{k^{n+1}2^{n+2}}{(n+1)!h} + \frac{k^n 2^{n+1}}{2n!} + \frac{hk^{n-1}2^n}{8\,(n-1)!} \right] =: (\xi).$$

Above we notice that $\sum_{i=1}^k L_{N_i}^{(M)}(|t_i - x_i|^n)(x_i) \overset{(122)}{\leq} \sum_{i=1}^k \frac{2^{n+1}\pi^2}{3(N_i-1)} \leq \frac{2^{n+1}\pi^2 k}{3(N_{\min}-1)}$, etc.

Next we choose $h := \left(\frac{1}{N_{\min}-1}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{N_{\min}-1}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{N_{\min}-1}$.

We have

$$(\xi) = \frac{\pi^2}{3} \left(\max_{\alpha:|\alpha|=n} \omega_1 \left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min}-1}} \right) \right) \cdot \tag{130}$$

$$\left[\frac{k^{n+1}2^{n+2}}{(n+1)!} \frac{1}{(N_{\min}-1)^{\frac{n}{n+1}}} + \frac{k^n 2^n}{n!\,(N_{\min}-1)} + \frac{k^{n-1}2^{n-1}}{4\,(n-1)!} \frac{1}{(N_{\min}-1)^{\frac{n+2}{n+1}}} \right],$$

proving the claim. ■

We also give

Proposition 42 *Let $x \in (-1,1)^k$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^1\left([-1,1]^k, \mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i} = 0$, for $i = 1,...,k$. Then*

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}-1}}\right)\right) \cdot \tag{131}$$

$$\left[\frac{(4/3)(k\pi)^2}{\sqrt{N_{\min}-1}} + \frac{(2/3)k\pi^2}{(N_{\min}-1)} + \frac{1}{8\left(\sqrt{N_{\min}-1}\right)}\right],$$

$\forall\ \overrightarrow{N} = (N_1,...,N_k) \in \mathbb{N}^k$; $N_i \geq 4$, $i = 1,...,k$, and $N_{\min} := \min\{N_1,...,N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} L_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (31) we get:

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(127)}{\leq} \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, h\right)\right) \cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k} L_{N_i}^{(M)}\left((t_i - x_i)^2\right)(x_i)\right) + \frac{1}{2}\left(\sum_{i=1}^{k} L_{N_i}^{(M)}(|t_i - x_i|)(x_i)\right) + \frac{h}{8}\right] \tag{132}$$

(next we choose $h := \left(\frac{1}{N_{\min}-1}\right)^{\frac{1}{2}}$, then $h^2 = \frac{1}{N_{\min}-1}$)

$$\overset{(122)}{\leq} \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}-1}}\right)\right) \cdot$$

$$\left[\frac{(4/3)(k\pi)^2}{\sqrt{N_{\min}-1}} + \frac{(2/3)k\pi^2}{(N_{\min}-1)} + \frac{1}{8\left(\sqrt{N_{\min}-1}\right)}\right], \tag{133}$$

proving the claim. ■

It follows

Theorem 43 *Let any $x \in [-1,1]^k$, $k \in \mathbb{N} - \{1\}$, and let $f \in C_+\left([-1,1]^k\right)$. Then*

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq \left(1 + \frac{4\pi^2 k}{3}\right)\omega_1\left(f, \frac{1}{(N_{\min}-1)}\right), \tag{134}$$

and

$$\left\|L_{\overrightarrow{N}}^{(M)}(f) - f\right\|_\infty \leq \left(1 + \frac{4\pi^2 k}{3}\right)\omega_1\left(f, \frac{1}{(N_{\min}-1)}\right), \tag{135}$$

$\forall\ \overrightarrow{N} = (N_1,...,N_k) \in \mathbb{N}^k$; $N_i \geq 4$, $i = 1,...,k$, and $N_{\min} := \min\{N_1,...,N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} L_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$, $\forall\ x := (x_1,...,x_k) \in [-1,1]^k$, uniformly.

Proof. We get that (use of (44))

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(127)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k} L_{N_i}^{(M)}(|t_i-x_i|)(x)\right)\right)$$

$$\overset{(122)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k}\frac{2^2\pi^2}{3(N_i-1)}\right)\right) \leq \omega_1(f,h)\left(1+\frac{1}{h}\left(\frac{4\pi^2 k}{3(N_{\min}-1)}\right)\right) \tag{137}$$

(setting $h := \frac{1}{N_{\min}-1}$)

$$= \omega_1\left(f, \frac{1}{(N_{\min}-1)}\right)\left(1+\frac{4\pi^2 k}{3}\right), \quad \forall\, x \in (-1,1)^k,$$

proving the claim. ■

We make

Remark 44 *The Chebyshev knots of first kind* $x_{N,k} := \cos\left(\frac{(2(N-k)+1)}{2(N+1)}\pi\right) \in (-1,1)$, $k \in \{0,1,...,N\}$, $-1 < x_{N,0} < x_{N,1} < ... < x_{N,N} < 1$, *are the roots of the first kind Chebyshev polynomial* $T_{N+1}(x) := \cos((N+1)\arccos x)$, $x \in [-1,1]$.

Define ($x \in [-1,1]$)

$$h_{N,k}(x) := (1-x\cdot x_{N,k})\left(\frac{T_{N+1}(x)}{(N+1)(x-x_{N,k})}\right)^2, \tag{138}$$

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (seep. 12 of [4]) are defined by

$$H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall\, N \in \mathbb{N}, \tag{139}$$

for $f \in C_+([-1,1])$, $\forall\, x \in [-1,1]$.

By [4], p. 287, we have

$$H_{2N+1}^{(M)}(|\cdot - x|)(x) \leq \frac{2\pi}{N+1}, \quad \forall\, x \in [-1,1], \forall\, N \in \mathbb{N}. \tag{140}$$

And by [3], we get that

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^m\pi}{N+1}, \quad \forall\, x \in [-1,1], \forall\, m, N \in \mathbb{N}. \tag{141}$$

Notice $H_{2N+1}^{(M)}(1) = 1$, *and* $H_{2N+1}^{(M)}$ *maps* $C_+([-1,1])$ *into itself, and it is a positive sublinear operator. Furthermore it holds* $\bigvee_{k=0}^{N} h_{N,k}(x) > 0$, $\forall\, x \in [-1,1]$. *We also have* $h_{N,k}(x_{N,k}) = 1$, *and* $h_{N,k}(x_{N,j}) = 0$, *if* $k \neq j$, *and* $H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, *for all* $j \in \{0,1,...,N\}$, *see [4], p. 282.*

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We need

Definition 45 *Let $f \in C_+\left([-1,1]^k\right)$, $k \in \mathbb{N}-\{1\}$, and $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. We define the multivariate Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind, as follows:*

$$H_{2\overrightarrow{N}+1}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} h_{N_1,i_1}(x_1) h_{N_2,i_2}(x_2) ... h_{N_k,i_k}(x_k) f(x_{N_1,i_1}, x_{N_2,i_2}, ..., x_{N_k,i_k})}{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} h_{N_1,i_1}(x_1) h_{N_2,i_2}(x_2) ... h_{N_k,i_k}(x_k)}, \tag{142}$$

$\forall\ x = (x_1, ..., x_k) \in [-1,1]^k$. Call $N_{\min} := \min\{N_1, ..., N_k\}$.

The operators $H_{2\overrightarrow{N}+1}^{(M)}(f)(x)$ are positive sublinear mapping $C_+\left([-1,1]^k\right)$ into itself, and $H_{2\overrightarrow{N}+1}^{(M)}(1) = 1$.

We also have

$$H_{2\overrightarrow{N}+1}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} h_{N_1,i_1}(x_1) h_{N_2,i_2}(x_2) ... h_{N_k,i_k}(x_k) f(x_{N_1,i_1}, x_{N_2,i_2}, ..., x_{N_k,i_k})}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda} h_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{143}$$

$\forall\ x = (x_1, ..., x_\lambda, ..., x_k) \in [-1,1]^k$, by the maximum multiplicative principle, see (27). Notice that $H_{2\overrightarrow{N}+1}^{(M)}(f)(x_{N_1,i_1}, ..., x_{N_k,i_k}) = f(x_{N_1,i_1}, ..., x_{N_k,i_k})$.

We make

Remark 46 *The coordinate Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind, are defined as follows ($\lambda = 1, ..., k$):*

$$H_{2N_\lambda+1}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda} h_{N_\lambda,i_\lambda}(x_\lambda) g(x_{N_\lambda,i_\lambda})}{\bigvee_{i_\lambda=0}^{N_\lambda} h_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{144}$$

$\forall\ N_\lambda \in \mathbb{N}$, and $\forall\ x_\lambda \in [-1,1]$, $\forall\ g \in C_+([-1,1])$.

Here we have ($\lambda = 1, ..., k$; $x_\lambda \in [-1,1]$)

$$h_{N_\lambda,i_\lambda}(x_\lambda) = (1 - x_\lambda \cdot x_{N_\lambda,i_\lambda})\left(\frac{T_{N_\lambda+1}(x_\lambda)}{(N_\lambda+1)(x_\lambda - x_{N_\lambda,i_\lambda})}\right)^2, \tag{145}$$

where the Chebyshev knots $x_{N_\lambda,i_\lambda} = \cos\left(\frac{(2(N_\lambda - i_\lambda)+1)}{2(N_\lambda+1)}\pi\right) \in (-1,1)$, $i_\lambda \in \{0, 1, ..., N_\lambda\}$, $-1 < x_{N_\lambda,0} < x_{N_\lambda,1} < ... < x_{N_\lambda,N_\lambda} < 1$ are the roots of the first kind Chebyshev polynomial $T_{N_\lambda+1}(x_\lambda) = \cos((N_\lambda+1)\arccos x_\lambda)$, $x_\lambda \in [-1,1]$.

In case of $f \in C_+\left([-1,1]^k\right)$ such that $f(x) := g(x_\lambda)$, $\forall\ x \in [-1,1]^k$ and $g \in C_+([-1,1])$, we get that

$$H_{2\overrightarrow{N}+1}^{(M)}(f)(x) = H_{2N_\lambda+1}^{(M)}(g)(x_\lambda), \tag{146}$$

by the maximum multiplicative principle (27) and simplification of (143).

We present

Theorem 47 *Let $x \in [-1,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^n\left([-1,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N}-\{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left|H_{2\overrightarrow{N}+1}^{(M)}(f)(x) - f(x)\right| \leq 2^{n-2}k^{n-1}\pi\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min}+1}}\right)\right) \cdot \tag{147}$$
$$\left[\frac{8k^2}{(n+1)!\,(N_{\min}+1)^{\frac{n}{n+1}}} + \frac{2k}{n!\,(N_{\min}+1)} + \frac{1}{4\,(n-1)!\,(N_{\min}+1)^{\frac{n+2}{n+1}}}\right],$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$, and $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} H_{2\overrightarrow{N}+1}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|H_{2\overrightarrow{N}+1}^{(M)}(f)(x) - f(x)\right| \overset{(146)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right) \cdot$$
$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^{k} H_{2N_i+1}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(H_{2N_i+1}^{(M)}\left(|t_i - x_i|^n\right)(x_i)\right)\right. \tag{148}$$
$$\left.+\frac{hk^{n-2}}{8\,(n-1)!}\left(\sum_{i=1}^{k} H_{2N_i+1}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(141)}{\leq}$$
$$\left(\frac{\pi}{N_{\min}+1}\right)\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\left[\frac{k^{n+1}2^{n+1}}{(n+1)!h} + \frac{k^n 2^n}{2n!} + \frac{hk^{n-1}2^{n-1}}{8\,(n-1)!}\right] =: (\xi).$$

Next we choose $h := \left(\frac{1}{N_{\min}+1}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{N_{\min}+1}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{N_{\min}+1}$.

We have

$$(\xi) = \pi\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min}+1}}\right)\right) \cdot \tag{149}$$
$$\left[\frac{(2k)^{n+1}}{(n+1)!\,(N_{\min}+1)^{\frac{n}{n+1}}} + \frac{2^{n-1}k^n}{n!\,(N_{\min}+1)} + \frac{2^{n-2}k^{n-1}}{4\,(n-1)!\,(N_{\min}+1)^{\frac{n+2}{n+1}}}\right],$$

proving the claim. ■

We also give

Proposition 48 *Let $x \in [-1,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^1\left([-1,1]^k, \mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i} = 0$, for $i = 1, ..., k$. Then*

$$\left|H_{2\overrightarrow{N}+1}^{(M)}(f)(x) - f(x)\right| \leq \left(\max_{i=1,...,k}\omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}+1}}\right)\right) \cdot \tag{150}$$

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$$\left[\frac{2k^2\pi}{\sqrt{N_{\min}+1}}+\frac{k\pi}{(N_{\min}+1)}+\frac{1}{8\left(\sqrt{N_{\min}+1}\right)}\right],$$

$\forall\ \overrightarrow{N}=(N_1,...,N_k)\in\mathbb{N}^k,\ N_{\min}:=\min\{N_1,...,N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,\dots,\infty)} H^{(M)}_{2\overrightarrow{N}+1}(f)(x)=f(x)$.

Proof. By (31) we get

$$\left|H^{(M)}_{2\overrightarrow{N}+1}(f)(x)-f(x)\right|\overset{(146)}{\leq}\left(\max_{i=1,\dots,k}\omega_1\left(\frac{\partial f}{\partial x_i},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k}H^{(M)}_{2N_i+1}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k}H^{(M)}_{2N_i+1}\left(|t_i-x_i|\right)(x_i)\right)+\frac{h}{8}\right] \tag{151}$$

(next we choose $h:=\frac{1}{\sqrt{N_{\min}+1}}$, then $h^2=\frac{1}{N_{\min}+1}$)

$$\overset{(141)}{\leq}\left(\max_{i=1,\dots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt{N_{\min}+1}}\right)\right)\cdot$$

$$\left[\frac{2k^2\pi}{\sqrt{N_{\min}+1}}+\frac{k\pi}{(N_{\min}+1)}+\frac{1}{8\left(\sqrt{N_{\min}+1}\right)}\right], \tag{152}$$

proving the claim. ■

It follows

Theorem 49 *Let* $f\in C_+\left([-1,1]^k\right)$, $k\in\mathbb{N}-\{1\}$. *Then*

$$\left|H^{(M)}_{2\overrightarrow{N}+1}(f)(x)-f(x)\right|\leq(2k\pi+1)\,\omega_1\left(f,\frac{1}{N_{\min}+1}\right), \tag{153}$$

$\forall\ x\in[-1,1]^k$, *and* $\forall\ \overrightarrow{N}=(N_1,...,N_k)\in\mathbb{N}^k$, *where* $N_{\min}:=\min\{N_1,...,N_k\}$.

That is

$$\left\|H^{(M)}_{2\overrightarrow{N}+1}(f)-f\right\|_{\infty}\leq(2k\pi+1)\,\omega_1\left(f,\frac{1}{N_{\min}+1}\right), \tag{154}$$

We get that

$$\lim_{\overrightarrow{N}\to(\infty,\dots,\infty)}H^{(M)}_{2\overrightarrow{N}+1}(f)=f, \tag{155}$$

uniformly.

Proof. We get that (use of (44))

$$\left| H_{2\overrightarrow{N}+1}^{(M)}(f)(x) - f(x) \right| \overset{(146)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\sum_{i=1}^{k} H_{2N_i+1}^{(M)}(|t_i - x_i|)(x_i)\right)\right)$$

$$\overset{(140)}{\leq} \omega_1(f,h)\left(1 + \frac{k}{h}\left(\frac{2\pi}{(N_{\min}+1)}\right)\right) \tag{156}$$

(setting $h := \frac{1}{N_{\min}+1}$)

$$= \omega_1\left(f, \frac{1}{N_{\min}+1}\right)(1 + 2k\pi), \quad \forall\, x \in [-1,1]^k,$$

proving the claim. ■

We make

Remark 50 *Let $\theta_{\overrightarrow{N}}^{(M)}$ denote any of the Max-product multivariate operators studied in this article: $B_{\overrightarrow{N}}^{(M)}$, $T_{N}^{(M)}$, $U_{\overrightarrow{N}}^{(M)}$, $T_{\overrightarrow{N}}^{(M)}$, $M_{\overrightarrow{N}}^{(M)}$, $L_{\overrightarrow{N}}^{(M)}$ and $H_{2\overrightarrow{N}+1}^{(M)}$. We observe that an important contraction property holds:*

$$\left\| \theta_{\overrightarrow{N}}^{(M)}(f) \right\|_{\infty} \leq \|f\|_{\infty}, \tag{157}$$

and

$$\left\| \theta_{\overrightarrow{N}}^{(M)}\left(\theta_{\overrightarrow{N}}^{(M)}(f)\right) \right\|_{\infty} \leq \left\| \theta_{\overrightarrow{N}}^{(M)}(f) \right\|_{\infty} \leq \|f\|_{\infty}, \tag{158}$$

i.e.

$$\left\| \left(\theta_{\overrightarrow{N}}^{(M)}\right)^2(f) \right\|_{\infty} \leq \|f\|_{\infty}, \tag{159}$$

and in general holds

$$\left\| \left(\theta_{\overrightarrow{N}}^{(M)}\right)^n(f) \right\|_{\infty} \leq \left\| \left(\theta_{\overrightarrow{N}}^{(M)}\right)^{n-1}(f) \right\|_{\infty} \leq \ldots \leq \|f\|_{\infty}, \quad \forall\, n \in \mathbb{N}. \tag{160}$$

We need the following Holder's type inequality:

Theorem 51 *Let Q, with the l_1-norm $\|\cdot\|$, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and $L : C_+(Q) \to C_+(Q)$, be a positive sublinear operator and $f, g \in C_+(Q)$, furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L((f(\cdot))^p)(s_*)$, $L((g(\cdot))^q)(s_*) > 0$ for some $s_* \in Q$. Then*

$$L(f(\cdot)\,g(\cdot))(s_*) \leq (L((f(\cdot))^p)(s_*))^{\frac{1}{p}}\,(L((g(\cdot))^q)(s_*))^{\frac{1}{q}}. \tag{161}$$

Proof. Let $a, b \geq 0$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. The Young's inequality says

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{162}$$

Then

$$\frac{f(s)}{(L((f(\cdot))^p)(s_*))^{\frac{1}{p}}} \cdot \frac{g(s)}{(L((g(\cdot))^q)(s_*))^{\frac{1}{q}}} \leq$$

$$\frac{(f(s))^p}{p(L((f(\cdot))^p)(s_*))} + \frac{(g(s))^q}{q(L((g(\cdot))^q)(s_*))}, \quad \forall\, s \in Q. \tag{163}$$

Hence it holds

$$\frac{L(f(\cdot)\,g(\cdot))(s_*)}{(L((f(\cdot))^p)(s_*))^{\frac{1}{p}}\,(L((g(\cdot))^q)(s_*))^{\frac{1}{q}}} \leq \tag{164}$$

$$\frac{(L((f(\cdot))^p))(s_*)}{p(L((f(\cdot))^p)(s_*))} + \frac{(L((g(\cdot))^q))(s_*)}{q(L((g(\cdot))^q)(s_*))} = \frac{1}{p} + \frac{1}{q} = 1, \quad \text{for } s_* \in Q,$$

proving the claim. ■

By (161), under the assumption $L_N\left(\|\cdot - x\|^{n+1}\right)(x) > 0$, and $L_N(1) = 1$, we obtain

$$L_N(\|\cdot - x\|^n)(x) \leq \left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}, \tag{165}$$

in case of $n = 1$ we derive

$$L_N(\|\cdot - x\|)(x) \leq \sqrt{\left(L_N\left(\|\cdot - x\|^2\right)(x)\right)}. \tag{166}$$

We give

Theorem 52 *Let Q with $\|\cdot\|$ the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, and $f \in C_+(Q)$. Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into itself, such that $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$. We assume further that $L_N(\|t - x\|)(x) > 0$, $\forall\, N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq 2\omega_1(f, L_N(\|t - x\|)(x)), \tag{167}$$

$\forall\, N \in \mathbb{N}$, $x = (x_1, ..., x_k) \in Q$; $t = (t_1, ..., t_k) \in Q$, where

$$\omega_1(f, h) := \sup_{\substack{x,y\in Q: \\ \|x-y\|\leq h}} |f(x) - f(y)|. \tag{168}$$

If $L_N(\|t - x\|)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. By Theorem 13. ■

We need

Theorem 53 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, and let $x \in Q$ ($x = (x_1, ..., x_k)$) be fixed. Let $f \in C^n(Q)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$.*

Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into $C_+(Q)$, such that $L_N(1)=1$, $\forall\ N\in\mathbb{N}$. Then

$$|L_N(f)(x)-f(x)|\leq\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha,h)\right)\cdot$$

$$\left[\frac{L_N\left(\|\cdot-x\|^{n+1}\right)(x)}{(n+1)!h}+\frac{L_N\left(\|\cdot-x\|^{n}\right)(x)}{2n!}+\frac{h}{8(n-1)!}L_N\left(\|\cdot-x\|^{n-1}\right)(x)\right], \tag{169}$$

$\forall\ N\in\mathbb{N}$.

Proof. By (19) and (25). ■

It follows

Theorem 54 *All as in Theorem 53. Additionally assume that $L_N\left(\|\cdot-x\|^{n+1}\right)(x)>0$, $\forall\ N\in\mathbb{N}$. Then*

$$|L_N(f)(x)-f(x)|\leq\frac{1}{2n!}\left(3+\frac{n}{4(n+1)}\right)\cdot$$

$$\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha,\frac{1}{(n+1)}\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right)\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}, \tag{170}$$

$\forall\ N\in\mathbb{N}$, $x=(x_1,...,x_k)\in Q$, ω_1 as in (168) for f_α.

If $L_N\left(\|\cdot-x\|^{n+1}\right)(x)\to 0$, then $L_N(f)(x)\to f(x)$, as $N\to+\infty$.

Proof. By Theorem 51 notice also that

$$L_N\left(\|\cdot-x\|^{n-1}\right)(x)\leq\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{n-1}{n+1}}. \tag{171}$$

We choose

$$h:=\frac{1}{(n+1)}\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}>0. \tag{172}$$

That is

$$(h(n+1))^{n+1}=L_N\left(\|\cdot-x\|^{n+1}\right)(x). \tag{173}$$

We apply (169) to have (see also (165) and (171)).

$$|L_N(f)(x)-f(x)|\leq\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha,h)\right)\cdot$$

$$\left[\frac{L_N\left(\|\cdot-x\|^{n+1}\right)(x)}{(n+1)!h}+\frac{\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{2n!}+\right. \tag{174}$$

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$$\frac{h}{8(n-1)!} L_N\left(\left(\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n-1}{n+1}}\right] =$$

$$\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \frac{1}{(n+1)}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right) \cdot$$

$$\left[\frac{h^n (n+1)^{n+1}}{(n+1)!} + \frac{h^n (n+1)^n}{2n!} + \frac{h^n (n+1)^{n-1}}{8(n-1)!}\right] =$$

$$\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \frac{1}{(n+1)}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right) \cdot$$

$$\left[\frac{(n+1)^{n+1}}{(n+1)!} + \frac{(n+1)^n}{2n!} + \frac{(n+1)^{n-1}}{8(n-1)!}\right] \frac{1}{(n+1)^n}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}} =$$

$$\left[\frac{3}{2n!} + \frac{n}{8(n+1)!}\right]\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \frac{1}{(n+1)}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right) \cdot$$

$$\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}, \tag{175}$$

proving the claim. ■

Final application for $n = 1$ follows:

Corollary 55 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, and let $x \in Q$ ($x = (x_1, ..., x_k)$) be fixed. Let $f \in C^1(Q)$. We assume that $\frac{\partial f}{\partial x_i}(x) = 0$, $i = 1, ..., k$. Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into $C_+(Q)$, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Assume that $L_N\left(\|\cdot - x\|^2\right)(x) > 0$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{25}{16}\left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{2}\left(L_N\left(\|\cdot - x\|^2\right)(x)\right)^{\frac{1}{2}}\right)\right) \cdot$$

$$\left(L_N\left(\|\cdot - x\|^2\right)(x)\right)^{\frac{1}{2}}, \tag{176}$$

$\forall\ N \in \mathbb{N}$.

If $L_N\left(\|\cdot - x\|^2\right)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

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NEW DYNAMIC INEQUALITIES ON TIME SCALES BY USING THE SNEAK-OUT PRINCIPLE

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ABSTRACT. In this paper, we extend and improve some dynamic inequalities by using the sneak-out principle with different exponents on time scales. The main results can be used to formulate the corresponding discrete inequalities of Bennett and G-Erdmann type.

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1. INTRODUCTION

In 1967 Littlewood [9] formulated some problems concerning elementary inequalities for infinite series in connection with some work on general theory of orthogonal series. One of the simplest (non-trivial) examples is the following inequality

$$\sum_{n=1}^{\infty} a_n^3 \left(\sum_{k=1}^{n} a_k^2 A_k \right) \leq K \sum_{n=1}^{\infty} a_n^4 A_n^2, \tag{1.1}$$

where a_n is a non-negative sequence and $A_n = \sum_{k=1}^{n} a_k$. One of such problems that has been proposed by Littlewood is to seek to know whether a constant K exists such that the inequality (1.1) holds. In other words, is it possible to get the term A_k out from the inner sum in (1.1) and if this happened what is the smallest value of K which preserves on the direction of the inequality? Bennett [4] proved this for the special case when the sequence a_n is decreasing, and he showed that the inequality (1.1) holds with $K = 2$. His proof based on the fact that $a_n \leq nA_n$ (noting that a_n is decreasing) and the application of Cauchy's inequality and the classical discrete Hardy's inequality. The generalization of the Littlewood inequality (1.1) which has not been considered before is given by

$$\sum_{n=1}^{\infty} a_n^{p(p-1)+1} A_k^{p-2} \left(\sum_{k=1}^{n} a_k^p A_k \right) \leq K \sum_{n=1}^{\infty} \left[a_n^p A_n \right]^p, \quad p > 1, \tag{1.2}$$

where K is a positive constant. Motivated by the work of Littlewood [9] Bennett and G-Erdmann [5] considered the inequality

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \leq K(\alpha, p) \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p, \tag{1.3}$$

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and determined the value of K for different values of p and α. In particular, Bennett and G-Erdmann [5, Theorem 8] proved that if $\alpha \geq 1$ and $p \geq 1$, then

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \leq (1+\alpha p)^p \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p, \tag{1.4}$$

where g_n is a non-negative sequence and $A_n = \sum_{k=1}^{n} a_k$, for any $n \in \mathbb{N}$. In [5, Theorem 9] the authors proved that if $p \geq 1$ and $0 \leq \alpha \leq 1$, then

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \leq (1+p)^p \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p. \tag{1.5}$$

Also in [5, Theorem 10] they proved that if $p \geq 1$ and $-1/p < \alpha \leq 0$, then

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \geq \left(\frac{1+\alpha p}{1+p+\alpha p} \right)^p \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p. \tag{1.6}$$

Motivated by the above work, we believe that the study of dynamic inequalities will help in proving several results for classical integral inequalities and inequalities involving discrete sequences. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. We assume that the reader has a good background in time scale calculus. For dynamic inequalities on time scales, we refer the reader to the books [2, 3] and the papers [1, 7, 10, 11, 12, 13]. For instance, we recall some related results.

Saker, O'Regan and Agarwal [13] proved a new inequality of Hardy type of the form

$$\int_a^{\infty} \frac{(A^{\sigma}(t))^p}{(\sigma(t)-a)^{\gamma}} \Delta t \leq \left(\frac{p}{\gamma-1} \right)^p \int_a^{\infty} \frac{(\sigma(t)-a)^{\gamma(p-1)}}{(t-a)^{(\gamma-1)p}} g^p(t) \Delta t, \quad p,\ \gamma > 1, \tag{1.7}$$

where $A(t) := \int_a^t g(s)\Delta s$, for $t \in [a, \infty)_{\mathbb{T}}$ and employed it in the proof of the extension of (1.2) on time scales. In particular they proved that if p, $\gamma > 1$ and g is a nonnegative rd-continuous and decreasing function, then

$$\int_a^{\infty} \frac{(a(t))^{p(p-1)+1}}{(A^{\sigma}(t))^{2-p}} \left(\int_a^{\sigma(t)} a^p(s) A^{\sigma}(s) \Delta s \right) \Delta t \leq \frac{p\gamma^p}{(p-1)} \int_a^{\infty} \left[a^p(t) A^{\sigma}(t) \right]^p \Delta t, \tag{1.8}$$

where $A(t) = \int_a^t a(s)\Delta s$, for $t \in [a, \infty)_{\mathbb{T}}$. Bohner and Saker in [7] employed the Minkowski inequality [6, Theorem 6.16] on time scales

$$\left(\int_a^b |h(t)| \, |u(t)+\upsilon(t)|^p \, \Delta t \right)^{1/p} \leq \left[\int_a^b |h(t)| \, |u(t)|^p \, \Delta t \right]^{\frac{1}{p}} + \left[\int_a^b |h(t)| \, |\upsilon(t)|^p \, \Delta t \right]^{\frac{1}{p}}, \tag{1.9}$$

where a, $b \in \mathbb{T}$, u, $\upsilon \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, $p > 1$ and established the time scale versions of the inequalities (1.4), (1.5) and (1.6). In more precisely, they proved

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that if $a(t)$, $g(t)$ are nonnegative rd-continuous functions on $[t_0,\infty)_{\mathbb{T}}$, then for $\alpha \geq 1$ and $p \geq 1$

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leq (1+\alpha p)^p \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^\infty g(s)\Delta s\right)^p \Delta t, \tag{1.10}$$

where

$$\Psi(t) = \int_t^\infty \left(A^\sigma(s)\right)^\alpha g(s)\Delta s \quad \text{and} \quad A(t) = \int_{t_0}^t a(s)\Delta s,$$

and if $0 \leq \alpha \leq 1$, $p \geq 1$, then

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leq (1+p)^p \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^\infty g(s)\Delta s\right)^p \Delta t. \tag{1.11}$$

Also in [7] they proved that if $-1/p < \alpha \leq 0$ and $p \geq 1$, then

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \geq \left(\frac{1+\alpha p}{1+p+\alpha p}\right)^p \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^\infty g(s)\Delta s\right)^p \Delta t. \tag{1.12}$$

Our aim in this paper is to apply the sneak-out principle which is given in the inequalities (1.10) and (1.11) to prove some new inequalities with different exponents for the given values of α. Also we prove a new dynamic inequality which as special case improves the inequality (1.12).

2. Main Results

Before we prove our main results, we briefly introduce some basic definitions and results concerning the delta calculus on time scales that will be used in the sequel; for more details we refer the reader to the book [6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. Recall the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable function f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.1}$$

The chain rule formula on time scales [6] is given by (here $x : \mathbb{T} \to (0, \infty)$ is assumed to be differentiable)

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_{0}^{1} [hx^{\sigma} + (1-h)x]^{\gamma-1} \, dh x^{\Delta}(t), \quad \gamma \in \mathbb{R}. \tag{2.2}$$

In this paper we will use the (delta) integral which we can define as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. The integration by parts formula on time scales reads

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.3}$$

Hölder's inequality [6, Theorem 6.13] states that any two rd-continuous functions $u, v : \mathbb{T} \to \mathbb{R}$ satisfy

$$\int_a^b |u(t)v(t)| \, \Delta t \le \left[\int_a^b |u(t)|^q \, \Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p \, \Delta t\right]^{\frac{1}{p}}, \tag{2.4}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \in \mathbb{T}$. Throughout this paper, we will assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions and the integrals considered are assumed to exist.

The following dynamic inequality of Copson's type on time scales [3], will be used later to prove the main results.

Theorem 2.1. *Assume that $a : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and define $A(t) = \int_{t_0}^t a(s)\Delta s$, $t \in \mathbb{T}$. Let $\varphi : \mathbb{T} \to \mathbb{R}^+$ and define*

$$\bar{\Phi}(t) := \int_t^{\infty} a(s)\varphi(s)\Delta s, \quad t \in \mathbb{T}. \tag{2.5}$$

If $k > 1$ and $0 \le c < 1$, then

$$\int_{t_0}^{\infty} \frac{a(t)}{(A^{\sigma}(t))^c} \left(\bar{\Phi}(t)\right)^k \Delta t \le \left(\frac{k}{1-c}\right)^k \int_{t_0}^{\infty} a(t) \left(A^{\sigma}(t)\right)^{k-c} \varphi^k(t) \, \Delta t. \tag{2.6}$$

Our main results are given in the following. For simplicity, we define

$$\Omega(t) := \int_t^{\infty} g(s)\Delta s, \quad \text{and} \quad \Psi(t) := \int_t^{\infty} (A^{\sigma}(s))^{\alpha} g(s)\Delta s, \ t \in \mathbb{T}. \tag{2.7}$$

Theorem 2.2. *Let $t_0 \in \mathbb{T}$, $\alpha \ge 1$, $p \ge 1$ and q, $r > 1$ such that $r > q$ and $(r-q)/(p-q) > 1$. Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \le K_1(\alpha, p, q, r) \left(\int_{t_0}^{\infty} \left((A^{\sigma}(t))^{\alpha} \Omega(t)\right)^{2r-q} \Delta t\right)^{\frac{p}{2r-q}}, \tag{2.8}$$

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where

$$\begin{aligned} K_1(\alpha,p,q,r) \ :&= \left[\frac{(1+\alpha r)^{r(p-q)}}{(1+\alpha q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \\ &\times \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{r-q}}(t)\,\Delta t\right)^{\frac{p-q}{2r-q}} \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{2(r-q)}}(t)\,\Delta t\right)^{\frac{2(r-p)}{2r-q}}. \end{aligned}$$

Proof. We first observe that

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t = \int_{t_0}^{\infty}\left(a^{\frac{p-q}{r-q}}(t)\Psi^{\frac{r(p-q)}{r-q}}(t)\right)\left(a^{\frac{r-p}{r-q}}(t)\Psi^{\frac{q(r-p)}{r-q}}(t)\right)\Delta t.$$

Applying Hölder's inequality (2.4) with indices $(r-q)/(p-q)$ and $(r-q)/(r-p)$, we obtain

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \le \left(\int_{t_0}^{\infty} a(t)\Psi^r(t)\Delta t\right)^{\frac{p-q}{r-q}} \left(\int_{t_0}^{\infty} a(t)\Psi^q(t)\Delta t\right)^{\frac{r-p}{r-q}}.$$

By using (1.10) to the two integrals on the right-hand side with $p=r$ and also with $p=q$, we get that

$$\begin{aligned} \int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \ \le& \ (1+\alpha r)^{\frac{r(p-q)}{r-q}} \left(\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha r}\left(\int_t^{\infty} g(s)\Delta s\right)^r \Delta t\right)^{\frac{p-q}{r-q}} \\ &\times (1+\alpha q)^{\frac{q(r-p)}{r-q}} \left(\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha q}\left(\int_t^{\infty} g(s)\Delta s\right)^q \Delta t\right)^{\frac{r-p}{r-q}}. \end{aligned}$$

Applying Hölder's inequality (2.4) with indices $(2r-q)/r$ and $(2r-q)/(r-q)$ to the integral

$$\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha r}(\Omega(t))^r\,\Delta t,$$

also applying it again on the integral

$$\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha q}(\Omega(t))^q\,\Delta t,$$

with indices $(2r-q)/q$ and $(2r-q)/2(r-q)$ and combining the result, we get that

$$\int_{t_0}^{\infty} a(t)\left(\Psi(t)\right)^p\Delta t \le K_1(\alpha,p,q,r)\left(\int_{t_0}^{\infty}\left(\left(A^\sigma(t)\right)^{\alpha}\Omega(t)\right)^{2r-q}\Delta t\right)^{\frac{p}{2r-q}},$$

which is the desired inequality (2.8). The proof is complete. □

Proceeding as in the proof of Theorem 2.2 and using inequality (1.11) instead of (1.10), we can obtain the following result.

Theorem 2.3. *Let $t_0 \in \mathbb{T}$, $0 \le \alpha \le 1$, $p \ge 1$ and q, $r > 1$ such that $r > q$ and $(r-q)/(p-q) > 1$. Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \le K_2(p,q,r)\left(\int_{t_0}^{\infty}\left(\left(A^\sigma(t)\right)^{\alpha}\Omega(t)\right)^{2r-q}\Delta t\right)^{\frac{p}{2r-q}}, \tag{2.9}$$

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where

$$\begin{aligned} K_2(p,q,r) \ :&= \left[\frac{(1+r)^{r(p-q)}}{(1+q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \\ &\times \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{r-q}}(t)\,\Delta t\right)^{\frac{p-q}{2r-q}} \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{2(r-q)}}(t)\,\Delta t\right)^{\frac{2(r-p)}{2r-q}}. \end{aligned}$$

The next result follows from Theorem 2.2 by choosing $r = p$ and $q = p - 1$.

Corollary 2.1. *Let $p \geq 1$ and $\alpha \geq 1$. Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leq K_1(\alpha,p)\left(\int_{t_0}^{\infty} \left((A^{\sigma}(t))^{\alpha}\,\Omega(t)\right)^{p+1}\Delta t\right)^{\frac{p}{p+1}}, \tag{2.10}$$

where

$$K_1(\alpha,p) = (1+\alpha p)^p\left(\int_{t_0}^{\infty} a^{p+1}(t)\,\Delta t\right)^{\frac{1}{p+1}}.$$

Remark 2.1. *In Theorem 2.2 when $\mathbb{T} = \mathbb{R}$, we have that*

$$\Psi(t) = \int_t^{\infty} A^{\alpha}(s)g(s)ds, \quad A(t) = \int_{t_0}^{t} a(s)ds \quad \text{and} \quad \Omega(t) = \int_t^{\infty} g(s)ds, \quad t \in \mathbb{R},$$

and then from (2.8) we obtain the following new integral inequality

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)dt \leq K_1(\alpha,p,q,r)\left(\int_{t_0}^{\infty} A^{\alpha(2r-q)}(t)\,(\Omega(t))^{2r-q}\,dt\right)^{\frac{p}{2r-q}}, \tag{2.11}$$

where

$$\begin{aligned} K_1(\alpha,p,q,r) \ :&= \left[\frac{(1+\alpha r)^{r(p-q)}}{(1+\alpha q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \\ &\times \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{r-q}}(t)\,dt\right)^{\frac{p-q}{2r-q}} \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{2(r-q)}}(t)\,dt\right)^{\frac{2(r-p)}{2r-q}}. \end{aligned}$$

Remark 2.2. *In Theorem 2.2 when $\mathbb{T} = \mathbb{N}$ and $n_0 = 1$, we have that*

$$\Psi(n) = \sum_{k=n}^{\infty} A^{\alpha}(k)g(k), \ A(n) = \sum_{k=1}^{n} a(k), \quad n \in \mathbb{N},$$

and then from (2.8), we get the following discrete inequality of Bennett and G-Erdmann [5] *type*

$$\sum_{n=1}^{\infty} a(n)\Psi^p(n) \leq K_1(\alpha,p,q,r)\left(\sum_{n=1}^{\infty} A^{\alpha(2r-q)}(n)\left(\sum_{k=n}^{\infty} g(k)\right)^{2r-q}\right)^{\frac{p}{2r-q}}, \tag{2.12}$$

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where

$$K_1(\alpha,p,q,r) \;:=\; \left[\frac{(1+\alpha r)^{r(p-q)}}{(1+\alpha q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \times\left(\sum_{n=1}^{\infty} a^{\frac{2r-q}{r-q}}(n)\right)^{\frac{p-q}{2r-q}}\left(\sum_{n=1}^{\infty} a^{\frac{2r-q}{2(r-q)}}(n)\right)^{\frac{2(r-p)}{2r-q}}.$$

Remark 2.3. *Setting $r=p$ and $q=p-1$ in (2.12) yields the following inequality*

$$\sum_{n=1}^{\infty} a(n)\Psi^p(n) \le K_1(\alpha,p)\left(\sum_{n=1}^{\infty} A^{\alpha(p+1)}(n)\left(\sum_{k=n}^{\infty} g(k)\right)^{p+1}\right)^{\frac{p}{p+1}}, \tag{2.13}$$

where

$$K_1(\alpha,p)=(1+\alpha p)^p\left(\sum_{n=1}^{\infty} a^{p+1}(n)\right)^{\frac{1}{p+1}}.$$

An improvement of the dynamic inequality (1.12) is obtained in the following Theorem.

Theorem 2.4. *Let $t_0\in\mathbb{T}$, $-1/p<\alpha\le 0$, $p\ge 1$ and $q,\ r>1$ such that $r>q$ and $(r-q)/(p-q)>1$. Then*

$$\begin{aligned}(2.14)\qquad &\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha p}(\Omega(t))^p\,\Delta t\\ \le\;& K_3(\alpha,p,q,r)\left[\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-r)}(\Psi(t))^r\,\Delta t\right]^{\frac{p-q}{r-q}}\\ &\times\left[\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-q)}(\Psi(t))^q\,\Delta t\right]^{\frac{r-p}{r-q}},\end{aligned}$$

where

$$K_3(\alpha,p,q,r):=\left(\frac{1+r+\alpha p}{1+\alpha p}\right)^{\frac{r(p-q)}{r-q}}\left(\frac{1+q+\alpha p}{1+\alpha p}\right)^{\frac{q(r-p)}{r-q}}.$$

Proof. In this proof for brivity, we set

$$b(t):=\left(A^{\sigma}(t)\right)^{\alpha} g(t).$$

Then the left hand side of (2.14) can be written in the form

$$\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha p}\Omega^p(t)\Delta t=\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha p}\left(\int_t^{\infty}\frac{b(s)}{\left(A^{\sigma}(s)\right)^{\alpha}}\Delta s\right)^p\Delta t. \tag{2.15}$$

Integrating the term $\int_t^{\infty}\left(A^{\sigma}(s)\right)^{-\alpha} b(s)\Delta s$ by parts, with $u^{\Delta}(s)=b(s)$ and $\upsilon^{\sigma}(s)=\left(A^{\sigma}(s)\right)^{-\alpha}$, we have

$$\int_t^{\infty}\left(A^{\sigma}(s)\right)^{-\alpha} b(s)\Delta s=u(s)\left(A(s)\right)^{-\alpha}\Big|_t^{\infty}-\int_t^{\infty} u(s)\left(\left(A(s)\right)^{-\alpha}\right)^{\Delta}\Delta s,$$

where $u(t)=-\int_t^\infty b(s)\Delta s=-\Psi(t)$, and so (note that $A(t)\le A^\sigma(t)$ and $-\alpha>0$)

$$\begin{aligned}\int_t^\infty \left(A^\sigma(s)\right)^{-\alpha} b(s)\Delta s &= \Psi(t)\left(A(t)\right)^{-\alpha}+\int_t^\infty \Psi(s)\left(\left(A(s)\right)^{-\alpha}\right)^\Delta \Delta s\\ &\le \Psi(t)\left(A^\sigma(t)\right)^{-\alpha}+\int_t^\infty \Psi(s)\left(\left(A(s)\right)^{-\alpha}\right)^\Delta \Delta s.\end{aligned}$$

Using the following inequality (see [7, Lemma 2.2])

$$(f^\gamma(t))^\Delta \le f^\Delta(t)\left(f^\sigma(t)\right)^{\gamma-1}, \quad if\ \ 0\le\gamma\le 1,\ f^\Delta>0, \tag{2.16}$$

with $f=A$ and $\gamma=-\alpha$, we observe that

$$\left(\left(A(s)\right)^{-\alpha}\right)^\Delta \le \frac{a(s)}{\left(A^\sigma(s)\right)^{\alpha+1}}, \quad \text{(note that } 0\le-\alpha\le 1).$$

This gives us

$$\int_t^\infty \left(A^\sigma(s)\right)^{-\alpha} b(s)\Delta s \le \Psi(t)\left(A^\sigma(t)\right)^{-\alpha}+\int_t^\infty \frac{a(s)\Psi(s)}{\left(A^\sigma(s)\right)^{\alpha+1}}\Delta s. \tag{2.17}$$

Substitute (2.17) into (2.15) and using the Minkowski inequality [8, Theorem 2.1]

$$\begin{aligned}&\int_a^b |h(t)|\left|u(t)+v(t)\right|^p \Delta t \\ \le\ & \left[\left(\int_a^b |h(t)|\left|u(t)\right|^r \Delta t\right)^{\frac{1}{r}}+\left(\int_a^b |h(t)|\left|v(t)\right|^r \Delta t\right)^{\frac{1}{r}}\right]^{\frac{r(p-q)}{r-q}}\\ &\times\left[\left(\int_a^b |h(t)|\left|u(t)\right|^q \Delta t\right)^{\frac{1}{q}}+\left(\int_a^b |h(t)|\left|v(t)\right|^q \Delta t\right)^{\frac{1}{q}}\right]^{\frac{q(r-p)}{r-q}}.\end{aligned} \tag{2.18}$$

for $r>q$ such that $r,\ q>1$ and $(r-q)/(p-q)>1$, we obtain

$$\begin{aligned}&\int_{t_0}^\infty a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^\infty \frac{b(s)}{\left(A^\sigma(s)\right)^\alpha}\Delta s\right)^p \Delta t\\ \le\ & \int_{t_0}^\infty a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\Psi(t)\left(A^\sigma(t)\right)^{-\alpha}+\int_t^\infty \frac{a(s)\Psi(s)}{\left(A^\sigma(s)\right)^{\alpha+1}}\Delta s\right)^p \Delta t\\ \le\ & \left[\left(\int_{t_0}^\infty a(t)\left(A^\sigma(t)\right)^{\alpha(p-r)}\left(\Psi(t)\right)^r \Delta t\right)^{\frac{1}{r}}+\left(\int_{t_0}^\infty a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\check{\Phi}(t)\right)^r \Delta t\right)^{\frac{1}{r}}\right]^{\frac{r(p-q)}{r-q}}\\ &\times\left[\left(\int_{t_0}^\infty a(t)\left(A^\sigma(t)\right)^{\alpha(p-q)}\left(\Psi(t)\right)^q \Delta t\right)^{\frac{1}{q}}+\left(\int_{t_0}^\infty a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\check{\Phi}(t)\right)^q \Delta t\right)^{\frac{1}{q}}\right]^{\frac{q(r-p)}{r-q}},\end{aligned}$$

where

$$\check{\Phi}(t):=\int_t^\infty \frac{a(s)\Psi(s)}{\left(A^\sigma(s)\right)^{\alpha+1}}\Delta s.$$

Applying Theorem 2.1 with $0 < c = -\alpha p < 1$, and $\varphi(t) = \Psi(t)/\left(A^\sigma(t)\right)^{\alpha+1}$, we have

$$\begin{aligned} &\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\check{\Phi}(t)\right)^r \Delta t \\ \leq\ & \left(\frac{r}{1+\alpha p}\right)^r \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{r+\alpha p}\left(\frac{\Psi(t)}{\left(A^\sigma(t)\right)^{\alpha+1}}\right)^r \Delta t \\ =\ & \left(\frac{r}{1+\alpha p}\right)^r \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-r)}\left(\Psi(t)\right)^r \Delta t, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\check{\Phi}(t)\right)^q \Delta t \\ \leq\ & \left(\frac{q}{1+\alpha p}\right)^q \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-q)}\left(\Psi(t)\right)^q \Delta t. \end{aligned} \tag{2.20}$$

From (2.19) and (2.20), we get that

$$\begin{aligned} &\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^\infty \frac{b(s)}{\left(A^\sigma(s)\right)^\alpha}\Delta s\right)^p \Delta t \\ \leq\ & \left[\left(\frac{1+r+\alpha p}{1+\alpha p}\right)\left(\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-r)}\left(\Psi(t)\right)^r \Delta t\right)^{\frac{1}{r}}\right]^{\frac{r(p-q)}{r-q}} \\ &\times\left[\left(\frac{1+q+\alpha p}{1+\alpha p}\right)\left(\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-q)}\left(\Psi(t)\right)^r \Delta t\right)^{\frac{1}{q}}\right]^{\frac{q(r-p)}{r-q}} \\ =\ & \left(\frac{1+r+\alpha p}{1+\alpha p}\right)^{\frac{r(p-q)}{r-q}}\left[\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-r)}\left(\Psi(t)\right)^r \Delta t\right]^{\frac{p-q}{r-q}} \\ &\times\left(\frac{1+q+\alpha p}{1+\alpha p}\right)^{\frac{q(r-p)}{r-q}}\left[\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-q)}\left(\Psi(t)\right)^q \Delta t\right]^{\frac{r-p}{r-q}}, \end{aligned}$$

which is the desired inequality (2.14). The proof is complete. □

Remark 2.4. *As a special case of (2.14) when $r = p$, we get the inequality (1.12) which has been proved by Bohner and Saker.*

Remark 2.5. *In Theorem 2.4 if $\mathbb{T} = \mathbb{N}$ and $r = p$, then inequality (2.14) reduces to the discrete inequality (1.6) due to Bennett and G-Erdmann.*

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ADDITIVE-QUADRATIC FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES AND ITS STABILITY

CHANG IL KIM AND GILJUN HAN*

Abstract. In this paper, we investigate the functional inequality

$$\begin{aligned}&N(f(2x+y)+f(2x-y)-6f(x)-2f(-x)-f(y)-f(-y),t)\\&\geq N(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y),kt)\end{aligned}$$

for some fixed real number k and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction

In 1940, Ulam proposed the following stability problem (cf. [28]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [13] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [22] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians (see [3], [4], [5], [10], and [18]).

In 2008, for the first time, Mirmostafaee and Moslehian [15], [16] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of the stability for the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

In [11], Glányi showed that if a mapping $f : X \longrightarrow Y$ satisfies the following functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.3}$$

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

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Glányi [12] and Fechner [9] proved the Hyers-Ulam stability of (1.3). Park, Cho, and Han [21] proved the Hyers-Ulam stability of the following functional inequality:

$$\|f(x)+f(y)+f(z)\| \leq \|f(x+y+z)\|. \tag{1.4}$$

Further, Park [20] proved the generalized Hyers-Ulam stability of the Cauchy additive functional inequality (1.4) in fuzzy Banach spaces using the fixed point method if f is an odd mapping.

In this paper, we investigate the following functional inequality

$$\begin{aligned} &N(f(2x+y)+f(2x-y)-6f(x)-2f(-x)-f(y)-f(-y),t) \\ &\geq N(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y),kt) \end{aligned} \tag{1.5}$$

for some fixed nonzero real number k and prove the generalized Hyers-Ulam stability for (1.5) in fuzzy Banach spaces by fixed point methods.

2. PRELIMINARIES

In this paper, we use the definition of fuzzy normed spaces given in [2], [16], and [17].

Definition 2.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space.*

Let (X, N) be a fuzzy normed space and $\{x_n\}$ a sequence in X. Then (i) $\{x_n\}$ is said to be *Cauchy in* (X, N) if for any $\varepsilon > 0$, there exists an $m \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \varepsilon$ for all $n \geq m$, all positive integer p, and all $t > 0$ and (ii) $\{x_n\}$ is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$.

Sequences of fuzzy numbers using the fuzzy metric or the fuzzy norm was studied by Das [6], [7], Tripathy et al. [23], Tripathy and Borgohain [24], [25], Tripathy and Dutta [26], Tripathy and Debnath [27] and others.

Example 2.2. For example, it is well known that for any normed space $(X, ||\cdot||)$ and any nonnegative real number ε, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0,1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+\varepsilon||x||}, & \text{if } t > 0 \end{cases},$$

is a fuzzy norm on X([16], [17], and [18]).

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It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

In 1996, Isac and Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 2.3. [8] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

3. Solutions of (1.5)

In this section, we investigate the solution of (1.5) in fuzzy spaces. For any mapping $f : X \longrightarrow Y$, let

$$A_f(x, y) = f(2x + y) + f(2x - y) - 6f(x) - 2f(-x) - f(y) - f(-y),$$
$$B_f(x, y) = f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y),$$
$$C_f(x, y) = f(x + y) - f(x) - f(y),\ D_f(x, y) = f(x - y) - f(x) + f(y),$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2},\ f_e(x) = \frac{f(x) + f(-x)}{2}.$$

Then f_o is an odd mapping and f_e is an even mapping. By (N5), we can easily prove the following lemma.

Lemma 3.1. *Let $\alpha_i : [0, \infty) \longrightarrow [0, \infty)(i = 1, 2, \cdots, n)$ be mappings and r a real number with $r > 1$ and $y, z, z_1, z_2, \cdot, \cdot, \cdot, z_n \in Y$. Then we have the following :*
(1) *If $N(y, t) \geq \min\{N(z, r^k t), N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$ for all $t > 0$ and all $k \in \mathbb{N}$, then*

$$N(y, t) \geq \min\{N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$.
(2) *If $N(y, t) \geq \min\{N(y, rt), N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$ for all $t > 0$ and $\alpha_i(i = 1, 2, \cdots, n)$ is non-decreasing, then*

$$N(y, t) \geq \min\{N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$.
(3) *If $N(y, t) \geq N(y, rt)$ for all $t > 0$, then $y = 0$.*

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We establish the following theorem using Lemma 3.1 :

Theorem 3.2. *Let $f : X \longrightarrow Y$ be an odd mapping. Suppose that a and b are real numbers with $a > 4$ and $b > 2$. Then f is an additive mapping if and only if f satisfies the following inequality*

$$N(A_f(x,y),t) \geq \min\{N(B_f(x,y),at), N(B_f(y,2x),bt)\} \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$.

Proof. Suppose that f is a solution of (3.1). Letting $x = 0$ and $y = 0$ in (3.1), we get $f(0) = 0$. Letting $y = 0$ in (3.1), by (N2), we get

$$f(2x) = 2f(x) \tag{3.2}$$

for all $x \in X$. Letting $y = 2y$ in (3.1), by (3.2), we have

$$N(B_f(x,y),t) \geq \min\{N(B_f(x,2y),2at), N(B_f(y,x),bt)\} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Putting $x = 2x + y$ and $y = x$ in (3.3), we get

$$\begin{aligned} &N(f(3x+y)+f(x+y)-2f(2x+y),t) \\ &\geq \min\{N(f(4x+y)+f(y)-2f(2x+y),2at), N(f(3x+y)-f(x+y)-2f(x),bt)\} \\ &\geq \min\Big\{N(f(4x+y)+f(y)-2f(2x+y),2at), N\Big(f(2x+y)-f(x+y)-f(x),\frac{b}{4}t\Big), \\ &N\Big(f(3x+y)+f(x+y)-2f(2x+y),\frac{b}{2}t\Big)\Big\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$ and all $t > 0$. Since $b > 2$, by (3.4) and Lemma 3.1, we have

$$\begin{aligned} &N(f(3x+y)+f(x+y)-2f(2x+y),t) \\ &\geq \min\Big\{N(f(4x+y)+f(y)-2f(2x+y),2at), N\Big(f(2x+y)-f(x+y)-f(x),\frac{b}{4}t\Big)\Big\} \end{aligned} \tag{3.5}$$

for all $x, y \in X$ and all $t > 0$. Letting $x = x + y$ and $y = x$ in (3.3), by (3.5), we get

$$\begin{aligned} &N(f(2x+y)+f(y)-2f(x+y),t) \\ &\geq \min\{N(f(3x+y)-f(x-y)-2f(x+y),2at), N(f(2x+y)-f(y)-2f(x),bt)\} \\ &\geq \min\Big\{N\Big(f(3x+y)+f(x+y)-2f(2x+y),\frac{a}{2}t\Big), N\Big(f(2x+y)+f(y)-2f(x+y),\frac{a}{2}t\Big), \\ &N\Big(f(x+y)-f(x-y)-2f(y),\frac{a}{2}t\Big), N(f(2x+y)-f(y)-2f(x),bt)\Big\} \\ &\geq \min\Big\{N\Big(f(4x+y)+f(y)-2f(2x+y),a^2t\Big), N\Big(f(2x+y)-f(x+y)-f(x),\frac{ab}{8}t\Big), \\ &N\Big(f(2x+y)+f(y)-2f(x+y),\frac{a}{2}t\Big), N\Big(f(x+y)-f(x-y)-2f(y),\frac{a}{2}t\Big), \\ &N(f(2x+y)-f(y)-2f(x),bt)\Big\} \end{aligned} \tag{3.6}$$

for all $x, y \in X$ and all $t > 0$. Since $a > 4$, by (3.6) and Lemma 3.1, we have

$$\begin{aligned} &N(f(2x+y)+f(y)-2f(x+y),t) \geq \min\Big\{N\Big(f(4x+y)+f(y)-2f(2x+y),a^2t\Big), \\ &N\Big(f(2x+y)-f(x+y)-f(x),\frac{ab}{8}t\Big), N\Big(f(x+y)-f(x-y)-2f(y),\frac{a}{2}t\Big), \\ &N(f(2x+y)-f(y)-2f(x),bt)\Big\} \end{aligned} \tag{3.7}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = 2y$ in (3.7), by (3.2), we have

$$\begin{aligned}
&N(f(x+y)+f(y)-f(x+2y),t)\\
&\geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t), N\Big(2f(x+y)-f(x+2y)-f(x),\frac{ab}{4}t\Big),\\
&N(f(x+2y)-f(x-2y)-4f(y),at), N(C_f(x,y),bt)\Big\}\\
&\geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t), N\Big(f(y)+f(x+y)-f(x+2y),\frac{ab}{8}t\Big),\\
&N(f(x+2y)-f(x-2y)-4f(y),at), N\Big(C_f(x,y),\min\Big\{\frac{ab}{8},b\Big\}t\Big)\Big\}\\
&\geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t), N\Big(f(y)+f(x+y)-f(x+2y),\frac{a}{4}t\Big),\\
&N\Big(f(x-2y)-f(x-y)+f(y),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.8}$$

for all $x, y \in X$ and all $t > 0$, because $b > 2$. Since $a > 4$, by (3.8), we have

$$\begin{aligned}
&N(f(x+y)+f(y)-f(x+2y),t) \geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t),\\
&N\Big(f(x-2y)-f(x-y)+f(y),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.9}$$

for all $x, y \in X$ and all $t > 0$. Interchanging x and y in (3.9), we have

$$\begin{aligned}
&N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N(f(x+2y)+f(x)-2f(x+y),a^2t),\\
&N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\\
&\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^2}{2}t\Big), N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big),\\
&N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\\
&\geq \min\Big\{N\Big(f(2x+y)-f(x+y)-f(x),\frac{a^4}{2}t\Big), N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{8}t\Big),\\
&N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.10}$$

for all $x, y \in X$ and all $t > 0$. Hence by Lemma 3.1 and (3.10), we have

$$\begin{aligned}
&N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big),\\
&N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{8}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.11}$$

for all $x, y \in X$ and all $t > 0$, because $a > 4$. By (3.11), we have

$$\begin{aligned}
&N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N\Big(f(2x+y)-f(x+y)-f(x),\frac{a^2}{2^4}t\Big),\\
&N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^4}{2^5}t\Big), N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{8}t\Big),\\
&N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.12}$$

for all $x, y \in X$ and all $t > 0$. Thus by Lemma 3.1 and (3.12), we have

$$\begin{aligned}&N(f(2x+y)-f(x+y)-f(x),t)\\ &\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^4}{2^5}t\Big),N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{2^3}t\Big),\\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\end{aligned} \tag{3.13}$$

for all $x, y \in X$ and all $t > 0$. Interchanging x and y in (3.13), we have

$$\begin{aligned}&N(f(x+2y)-f(x+y)-f(y),t)\\ &\geq \min\Big\{N\Big(f(2x+y)-f(x+y)-f(x),\frac{a^4}{2^5}t\Big),N\Big(f(2x-y)-f(x-y)-f(x),\frac{a^3}{2^3}t\Big),\\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\\ &\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^8}{2^{10}}t\Big),N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{2^3}t\Big),\\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\end{aligned} \tag{3.14}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1 and (3.14), we get

$$\begin{aligned}&N(f(x+2y)-f(x+y)-f(y),t) \geq \min\Big\{N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{2^3}t\Big),\\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\\ &\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^6}{2^6}t\Big),N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),\\ &N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\end{aligned} \tag{3.15}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1 and (3.15), we get

$$\begin{aligned}&N(f(x+2y)-f(x+y)-f(y),t) \geq \min\Big\{N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),\\ &N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\end{aligned} \tag{3.16}$$

for all $x, y \in X$ and all $t > 0$. Interchanging x and y in (3.16), we have

$$\begin{aligned}&N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),\\ &N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\end{aligned} \tag{3.17}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = y - x$ in (3.17), we get

$$\begin{aligned}&N(C_f(x,y),t) \geq \min\Big\{N\Big(f(2x-y)-f(x)-f(x-y),\min\Big\{\frac{a}{4},b\Big\}t\Big),\\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\\ &\geq \min\Big\{N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\end{aligned} \tag{3.18}$$

for all $x, y \in X$ and all $t > 0$. Since $\min\{\frac{a}{4}, b\} > 1$, by Lemma 3.1 and (3.18), we have

$$N(C_f(x,y),t) \geq N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big) \geq N\Big(C_f(x,y),\Big[\min\Big\{\frac{a}{4},b\Big\}\Big]^2t\Big)$$

for all $x, y \in X$ and all $t > 0$ and hence by Lemma 3.1, f is an additive mapping.

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The converse is trivial. □

Theorem 3.3. *Let $f : X \longrightarrow Y$ be an even mapping. Suppose that k is a real number with $k > 1$. Then f is a solution of the following functional equation*

$$N(A_f(x,y),t) \geq N(B_f(x,y),kt) \tag{3.19}$$

for all $x, y \in X$ if and only if f is a quadratic mapping.

Proof. Suppose that f is a solution of (3.19). Letting $x = 0$ and $y = 0$ in (1.5), we have

$$N(f(0),t) \geq N\Big(f(0),4kt\Big)$$

for all $t > 0$ and sicne $4k > 1$, by Lemma 3.1, we get $f(0) = 0$. Letting $y = 0$ in (3.19), by (N2), we get

$$f(2x) = 4f(x) \tag{3.20}$$

for all $x \in X$. Now, letting $x = 2x$ in (3.19), by (3.20), we have

$$\begin{aligned} N(f(4x+y)+f(4x-y)-32f(x)-2f(y),t) &\geq N(A_f(x,y),kt) \\ &\geq N(B_f(x,y),k^2t) \end{aligned} \tag{3.21}$$

for all $x, y \in X$. Letting $y = 2y$ in (3.21), by (3.19), we have

$$N(A_f(x,y),t) \geq N(B_f(2y,x),4k^2t) = N(A_f(y,x),4k^2t) \geq N(B_f(x,y),4k^3t) \tag{3.22}$$

for all $x, y \in X$. Letting $x = 2x$ in (3.22), by (3.19), we have

$$N(f(4x+y)+f(4x-y)-32f(x)-2f(y),t) \geq N(B_f(x,y),4k^4t)$$

for all $x, y \in X$. Hence by induction, we get

$$N(f(4x+y)+f(4x-y)-32f(x)-2f(y),t) \geq N(B_f(x,y),4^nk^{n+3}t)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $k > 1$, by Lemma 3.1 and (N5), we have

$$f(4x+y)+f(4x-y)-32f(x)-2f(y) = 0$$

for all $x, y \in X$. Hence f is a quadratic mapping. □

4. The generalized Hyers-Ulam stability for (1.5)

Now, we will prove the generalized Hyers-Ulam stability for (1.5) in fuzzy normed spaces.

Theorem 4.1. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'(\phi(2x,2y),t) \geq N'(4L\phi(x,y),t) \tag{4.1}$$

for all $x, y \in X$, $t > 0$ and some real number L with $0 < L < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(A_f(x,y),t) \geq \min\{N(B_f(x,y),kt), N'(\phi(x,y),t)\} \tag{4.2}$$

for all $x, y \in X$, $t > 0$ and some real number k with $k > 32$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{1}{2(1-2L)}t\Big) \geq \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\} \tag{4.3}$$

for all $x \in X$ and all $t > 0$.

Proof. By (4.2), we get

$$\begin{aligned} N(A_{f_o}(x,y),t) \geq \min\Big\{&N\Big(B_{f_o}(x,y), \frac{k}{2}t\Big), N\Big(B_{f_e}(x,y), \frac{k}{2}t\Big), \\ &N'(\phi(x,y),t), N'(\phi(-x,-y),t)\Big\} \end{aligned} \tag{4.4}$$

for all $x, y \in X$, $t > 0$ and

$$\begin{aligned} N(A_{f_e}(x,y),t) \geq \min\Big\{&N\Big(B_{f_o}(x,y), \frac{k}{2}t\Big), N\Big(B_{f_e}(x,y), \frac{k}{2}t\Big), \\ &N'(\phi(x,y),t), N'(\phi(-x,-y),t)\Big\} \end{aligned} \tag{4.5}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = 0$ in (4.4) and (4.5), by (N2), we have

$$N(2f_o(2x) - 4f_o(x), t) \geq \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\} \tag{4.6}$$

and

$$N(2f_e(2x) - 8f_e(x), t) \geq \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\} \tag{4.7}$$

for all $y \in X$ and all $t > 0$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \geq \phi_o(x,t), \forall x \in X, \forall t > 0\},$$

where $\phi_o(x,t) = \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\}$. Then (S,d) is a complete metric space([19]). Define a mapping $J_o : S \longrightarrow S$ by $J_o g(x) = \frac{1}{2}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4.1), we have

$$N(J_o g(x) - J_o h(x), 2cLt) = N\Big(g(2x) - h(2x), 4cLt\Big) \geq \phi_o(2x, 4Lt) \geq \phi_o(x,t)$$

for all $x \in X$ and all $t > 0$. Hence $d(J_o g, J_o h) \leq 2Ld(g,h)$ for any $g, h \in S$ and by (4.6), we have $d(J_o f_o, f_o) \leq \frac{1}{4} < \infty$. By Theorem 2.3, there exists a mapping $P : X \longrightarrow Y$ which is a fixed point of J_o such that

$$N\Big(f_o(x) - P(x), \frac{1}{4(1-2L)}t\Big) \geq \phi_o(x,t) \tag{4.8}$$

for all $x \in X$ and all $t > 0$. Moreover, $d(J_o^n f_o, A) \to 0$ as $n \to \infty$. That is,

$$P(x) = N - \lim_{n\to\infty} \frac{f_o(2^n x)}{2^n}$$

for all $x \in X$. Now, define a mapping $J_e : S \longrightarrow S$ by $J_e g(x) = \frac{1}{4}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4.1), we have

$$N(J_e g(x) - J_e h(x), cLt) = N\Big(g(2x) - h(2x), 4cLt\Big) \geq \phi_o(2x, 4Lt) \geq \phi_o(x,t)$$

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for all $x \in X$ and $t > 0$. Hence $d(J_e g, J_e h) \leq Ld(g, h)$ for any $g, h \in S$ and by (4.7), we have $d(J_e f_e, f_e) \leq \frac{1}{8} < \infty$. By Theorem 2.3, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of J_e such that

$$N\Big(f_e(x) - Q(x), \frac{1}{8(1-L)}t\Big) \geq \phi_o(x, t) \tag{4.9}$$

for all $x \in X$ and all $t > 0$. Moreover, $d(J_e^n f_e, A) \to 0$ as $n \to \infty$. That is,

$$Q(x) = N - \lim_{n\to\infty} \frac{f_e(2^n x)}{2^{2n}} \tag{4.10}$$

for all $x \in X$. Replacing x, and y by $2^n x$ and $2^n y$ in (4.5), respectively, by (4.1), we have

$$\begin{aligned} N\Big(\frac{1}{2^{2n}}A_{f_e}(2^n x, 2^n y), t\Big) &\geq \min\Big\{N\Big(\frac{1}{2^n}B_{f_o}(2^n x, 2^n y), 2^{n-1}kt\Big), \\ N\Big(\frac{1}{2^{2n}}B_{f_e}(2^n x, 2^n y), \frac{k}{2}t\Big)&, N'\Big(\phi(x, y), \frac{1}{L^n}t\Big), N'\Big(\phi(-x, -y), \frac{1}{L^n}t\Big)\Big\} \end{aligned} \tag{4.11}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. By (N4) and (4.11), we have

$$\begin{aligned} &N(A_Q(x, y), t) \\ &\geq \min\Big\{N\Big(A_Q(x, y) - \frac{1}{2^{2n}}A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big), N\Big(\frac{1}{2^{2n}}A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big)\Big\} \\ &\geq \min\Big\{N\Big(A_Q(x, y) - \frac{1}{2^{2n}}A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big), N\Big(\frac{1}{2^n}B_{f_o}(2^n x, 2^n y), 2^{n-2}kt\Big), \\ &N\Big(\frac{1}{2^{2n}}B_{f_e}(2^n x, 2^n y), \frac{k}{4}t\Big), N'\Big(\phi(x, y), \frac{1}{2L^n}t\Big), N'\Big(\phi(-x, -y), \frac{1}{2L^n}t\Big)\Big\} \\ &\geq \min\Big\{N\Big(A_Q(x, y) - \frac{1}{2^{2n}}A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big), N\Big(\frac{1}{2^n}B_{f_o}(2^n x, 2^n y), 2^{n-2}kt\Big), \\ &N\Big(\frac{1}{2^{2n}}B_{f_e}(2^n x, 2^n y) - B_Q(x, y), \frac{k}{8}t\Big), N\Big(B_Q(x, y), \frac{k}{8}t\Big), \\ &N'\Big(\phi(x, y), \frac{1}{2L^n}t\Big), N'\Big(\phi(-x, -y), \frac{1}{2L^n}t\Big)\Big\} \end{aligned} \tag{4.12}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. By (N4), we have

$$\begin{aligned} &N\Big(\frac{1}{2^n}B_{f_o}(2^n x, 2^n y), 2^n t\Big) \\ &\geq \min\Big\{N\Big(\frac{1}{2^n}B_{f_o}(2^n x, 2^n y) - B_P(x, y), 2^{n-1}t\Big), N\Big(B_P(x, y), 2^{n-1}t\Big)\Big\} \end{aligned} \tag{4.13}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4.13), by (N5), we have

$$\lim_{n\to\infty} N\Big(\frac{1}{2^{2n}}B_{f_o}(2^n x, 2^n y), t\Big) = 1 \tag{4.14}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4.12), by (4.10) and (4.14), we have

$$N(A_Q(x, y), t) \geq N\Big(B_Q(x, y), \frac{k}{8}t\Big) \tag{4.15}$$

for all $x, y \in X$ and all $t > 0$. Since f_e is even, by (4.10), Q is even and hence by (4.15) and Theorem 3.3, Q is a quadratic mapping. By (4.5) and (4.7), we have

$$\begin{aligned}
&N(B_{f_e}(x,2y),t) \geq \min\Big\{N\Big(A_{f_e}(y,x),\frac{t}{2}\Big), N\Big(8f_e(y)-2f_e(2y),\frac{t}{2}\Big)\Big\}\\
&\geq \min\Big\{N\Big(B_{f_o}(y,x),\frac{k}{4}t\Big), N\Big(B_{f_e}(y,x),\frac{k}{4}t\Big), N'\Big(\phi(y,x),\frac{t}{2}\Big),\\
&N'\Big(\phi(-y,-x),\frac{t}{2}\Big), N'\Big(\phi(y,0),\frac{t}{2}\Big), N'\Big(\phi(-y,0),\frac{t}{2}\Big)\Big\}
\end{aligned} \tag{4.16}$$

for all $x, y \in X$ and $t > 0$. By (4.7) and (4.16), we have

$$\begin{aligned}
&N(B_{f_e}(x,y),t) = N(4B_{f_e}(x,y),4t)\\
&\geq \min\{N(B_{f_e}(2x,2y),2t), N(4B_{f_e}(x,y)-B_{f_e}(2x,2y),2t)\}\\
&\geq \min\Big\{N\Big(B_{f_o}(y,2x),\frac{k}{2}t\Big), N\Big(B_{f_e}(y,2x),\frac{k}{2}t\Big), \Phi_1(x,y,t)\Big\}\\
&\geq \min\Big\{N\Big(B_{f_o}(y,2x),\frac{k}{2}t\Big), N\Big(B_{f_o}(x,y),\frac{k^2}{8}t\Big), N\Big(B_{f_e}(x,y),\frac{k^2}{8}t\Big), \Phi_2(x,y,t)\Big\}
\end{aligned} \tag{4.17}$$

for all $x, y \in X$ and all $t > 0$, where

$$\begin{aligned}
\Phi_1(x,y,t) = \min\Big\{&N'(\phi(y,2x),t), N'(\phi(-y,-2x),t), N'(\phi(x+y,0),t),\\
&N'(\phi(-x-y,0),t), N'(\phi(x-y,0),t), N'(\phi(-x+y,0),t),\\
&N'\Big(\phi(x,0),\frac{t}{2}\Big), N'\Big(\phi(-x,0),\frac{t}{2}\Big), N'\Big(\phi(y,0),\frac{t}{2}\Big), N'\Big(\phi(-y,0),\frac{t}{2}\Big)\Big\}
\end{aligned}$$

and

$$\Phi_2(x,y,t) = \min\Big\{\Phi_1(x,y,t), N'\Big(\phi(x,y),\frac{k}{4}t\Big), N'\Big(\phi(-x,-y),\frac{k}{4}t\Big)\Big\},$$

because $k > 32$. By Lemma 3.1 and (4.17), we have

$$N(B_{f_e}(x,y),t) \geq \min\Big\{N\Big(B_{f_o}(y,2x),\frac{k}{2}t\Big), N\Big(B_{f_o}(x,y),\frac{k^2}{8}t\Big), \Phi_2(x,y,t)\Big\} \tag{4.18}$$

for all $x, y \in X$ and all $t > 0$ and hence by (4.4) and (4.18), we have

$$\begin{aligned}
N(A_{f_o}(x,y),t) \geq \min\Big\{&N\Big(B_{f_o}(x,y),\frac{k}{2}t\Big), N\Big(B_{f_o}(y,2x),\frac{k^2}{4}t\Big),\\
&\Phi_1\Big(x,y,\frac{k}{2}t\Big), N'(\phi(x,y),t), N'(\phi(-x,-y),t)\Big\}
\end{aligned} \tag{4.19}$$

for all $x, y \in X$, $t > 0$ and replacing x and y by $2^n x$ and $2^n y$ in (4.19), respectively, by (4.1), we have

$$\begin{aligned}
&N\Big(A_{f_o}(2^n x, 2^n y), 2^n t\Big)\\
&\geq \min\Big\{N(B_{f_o}(2^n x, 2^n y), 2^{n-1}kt), N(B_{f_o}(2^n y, 2^{n+1}x), 2^{n-2}k^2 t),\\
&\Phi_1\Big(x,y,\frac{k}{2(2L)^n}t\Big), N'\Big(\phi(x,y),\frac{1}{(2L)^n}t\Big), N'\Big(\phi(-x,-y),\frac{1}{(2L)^n}t\Big)\Big\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Similar to Q, we have

$$N(A_P(x,y),t) \geq \min\Big\{N\Big(B_P(x,y),\frac{k}{8}t\Big), N\Big(B_P(y,2x),\frac{k^2}{16}t\Big)\Big\} \tag{4.20}$$

for all $x, y \in X$ and all $t > 0$. Cleraly, P is an odd mapping and since $k > 32$, by Theorem 3.2, P is an additive mapping. Let $F = P + Q$. Then $F : X \longrightarrow Y$ is an additive-quadratic mapping. By (4.8) and (4.9), we have (4.3).

Now, we show the uniqueness of F. Let H be another additive-quadratic mapping with (4.3). Since F and H are addiitve-quadratic mappings, we have

$$F(x) = \frac{1+2^n}{2^{2n+1}}F(2^n x) + \frac{1-2^n}{2^{2n+1}}F(-2^n x),\ H(x) = \frac{1+2^n}{2^{2n+1}}H(2^n x) + \frac{1-2^n}{2^{2n+1}}H(-2^n x),$$

for all $x \in X$ and all positive integer n. Hence by (4.3), (N3) and (N4), we have

$$\begin{aligned}
&N(F(x) - H(x), t)\\
&\geq \min\Big\{N\Big(F(2^n x) - H(2^n x), \frac{2^{2n}}{1+2^n}t\Big), N\Big(F(-2^n x) - H(-2^n x), \frac{2^{2n}}{2^n - 1}t\Big)\Big\}\\
&\geq \min\Big\{N\Big(F(2^n x) - f(2^n x), \frac{2^{2n-1}}{1+2^n}t\Big), N\Big(f(2^n x) - H(2^n x), \frac{2^{2n-1}}{1+2^n}t\Big),\\
&N\Big(F(-2^n x) - f(-2^n x), \frac{2^{2n-1}}{2^n - 1}t\Big), N\Big(f(-2^n x) - H(-2^n x), \frac{2^{2n-1}}{2^n - 1}t\Big)\Big\}\\
&\geq \min\Big\{\phi_o\Big(2^n x, \frac{2^{2n}(1-2L)}{1+2^n}t\Big), \phi_o\Big(2^n x, \frac{2^{2n}(1-2L)}{2^n - 1}t\Big)\Big\}\\
&\geq \min\Big\{\phi_o\Big(x, \frac{1-2L}{(L)^n + (2L)^n}t\Big), \phi_o\Big(x, \frac{1-2L}{(2L)^n\Big(1 - \frac{1}{2^n}\Big)}t\Big)\Big\}
\end{aligned}$$

for all $x \in X$, $t > 0$, and all $n \in \mathbb{N}$. Since $0 < L < \frac{1}{2}$, letting $n \to \infty$ in the above inequality, we have $F(x) = H(x)$ for all $x \in X$. □

By Theorem 4.1, we can show that the following corollaries:

Corollary 4.2. *Let ε and p be real numbers with $\varepsilon \geq 0$ and $0 < p < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(A_f(x,y), t) \geq \min\Big\{N(B_f(x,y), kt),\ \frac{t}{t + \varepsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}\Big\} \tag{4.21}$$

for all $x, y \in X$, all $t > 0$ and some real number k with $k > 32$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \frac{(2 - 2^{2p})t}{(2 - 2^{2p})t + \varepsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.3. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function with (4.1). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(rA_f(x,y) + B_f(x,y), t) \geq \min\{N(B_f(x,y), t), N'(\phi(x,y), t)\} \tag{4.22}$$

for all $x, y \in X$, all $t > 0$ and some real numbers r with $|r| > 64$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{1}{2(1-2L)}t\Big) \geq \min\{N'(\phi(x,0), t), N'(\phi(-x,0), t)\}$$

for all $x \in X$ and all $t > 0$.

Proof. By (N5) and (4.22), we have

$$\begin{aligned} N(A_f(x,y),t) &\geq \min\Big\{N\Big(rA_f(x,y)+B_f(x,y),\frac{|r|}{2}t\Big), N\Big(B_f(x,y),\frac{|r|}{2}t\Big)\Big\} \\ &\geq \min\Big\{N\Big(B_f(x,y),\frac{|r|}{2}t\Big), N'\Big(\phi(x,y),\frac{|r|}{2}t\Big)\Big\} \\ &\geq \min\Big\{N\Big(B_f(x,y),\frac{|r|}{2}t\Big), N'\Big(\phi(x,y),t\Big)\Big\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Hence we have the results. □

Corollary 4.4. *Let ε and p be real numbers with $\varepsilon \geq 0$ and $0 < p < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(rA_f(x,y)+B_f(x,y),t) \geq \min\Big\{N(B_f(x,y),t),\ \ \frac{t}{t+\varepsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}\Big\} \tag{4.23}$$

for all $x, y \in X$, all $t > 0$ and some real number r with $|r| > 64$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x)-F(x),t) \geq \frac{(2-2^{2p})t}{(2-2^{2p})t+\varepsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Related with Theorem 4.1, we can also have the following theorem. The proof is similar to that of Theorem 4.1.

Theorem 4.5. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2},\frac{y}{2}\Big),t\Big) \geq N'\Big(\frac{L}{2}\phi(x,y),t\Big) \tag{4.24}$$

for all $x, y \in X$, $t > 0$ and some real number L with $0 < L < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and (4.2). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x)-F(x),\frac{L}{2(1-L)}t\Big) \geq \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\phi_o(x,t) = \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\}$. Letting $x = \frac{x}{2}$ in (4.6) and (4.7), by (4.24), we have

$$N\Big(2f_o(x)-4f_o\Big(\frac{x}{2}\Big),\frac{L}{2}t\Big) \geq \phi_o(x,t) \tag{4.25}$$

and

$$N\Big(2f_e(x)-8f_e\Big(\frac{x}{2}\Big),\frac{L}{2}t\Big) \geq \phi_o(x,t) \tag{4.26}$$

for all $y \in X$ and $t > 0$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x)-h(x),ct) \geq \phi_o(x,t), \forall x \in X, \forall t > 0\}.$$

Then (S,d) is a complete metric space([19]). Define a mapping $J_o : S \longrightarrow S$ by $J_o g(x) = 2g\Big(\frac{x}{2}\Big)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (4.1), we have

$$N\Big(J_o g(x) - J_o h(x), cLt\Big) = N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), c\frac{L}{2}t\Big) \ge \phi_o\Big(\frac{x}{2}, \frac{L}{2}t\Big) \ge \phi_o(x,t)$$

for all $x \in X$ and $t > 0$. Hence $d(J_o g, J_o h) \le Ld(g,h)$ for any $g, h \in S$. By (4.25), we have $d(J_o f_o, f_o) \le \frac{L}{4} < \infty$ and by Theorem 2.3, there exists a mapping $P : X \longrightarrow Y$ which is a fixed point of J_o such that

$$N\Big(f_o(x) - P(x), \frac{L}{4(1-L)}t\Big) \ge \phi_o(x,t)$$

for all $x \in X$, all $t > 0$ and $d(J_o^n f_o, A) \to 0$ as $n \to \infty$.

Now, define a mapping $J_e : S \longrightarrow S$ by $J_e g(x) = 4g\Big(\frac{x}{2}\Big)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (4.1), we have

$$N(J_e g(x) - J_e h(x), 2cLt) = N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), c\frac{L}{2}t\Big) \ge \phi_o\Big(\frac{x}{2}, \frac{L}{2}t\Big) \ge \phi_o(x,t)$$

for all $x \in X$ and $t > 0$. Hence $d(J_e g, J_e h) \le 2Ld(g,h)$ and by (4.26), we have $d(J_e f_e, f_e) \le \frac{L}{4} < \infty$. By Theorem 2.3, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of J_e such that

$$N\Big(f_e(x) - Q(x), \frac{L}{4(1-2L)}t\Big) \ge \phi_o(x,t)$$

for all $x \in X$, all $t > 0$ and $d(J_e^n f_e, A) \to 0$ as $n \to \infty$.

The rest of the proof is similar to Theorem 4.1. □

By Theorem 4.5, we can show that the following corollaries:

Corollary 4.6. *Let ε and p be real numbers with $\varepsilon \ge 0$ and $p > 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.21). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

$$N(f(x) - F(x), t) \ge \frac{(2^{2p} - 2)t}{(2^{2p} - 2)t + \varepsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.7. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function with (4.24). Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.22). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

$$N(f(x) - F(x), \frac{L}{2(1-L)}t) \ge \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.8. *Let ε and p be real numbers with $\varepsilon \ge 0$ and $p > 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.23). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

$$N(f(x) - F(x), t) \ge \frac{(2^{2p} - 2)t}{(2^{2p} - 2)t + \varepsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

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NEW CHARACTERIZATIONS OF WEIGHTS IN HARDY'S TYPE INEQUALITIES VIA OPIAL'S DYNAMIC INEQUALITIES

S. H. SAKER, M. M. OSMAN AND I. ABOHELA

ABSTRACT. In this paper, we prove some new characterizations of weights in some Hardy-type inequalities on time scales. The results as special cases contain the results due to Beesack and Heinig, Leindler and Bloom and Kerman. Some new integral and discrete inequalities related to Copson's, Flett's, Bliss's and Bennett's will be formulated. The main results will be proved by using new generalizations of Opial's type inequalities, Hölder's inequality, Minkowski's inequality and the chain rule on time scales.

Keywords: Hardy's inequality, Opial's inequality, time scales.

AMS Classif: 26A15, 26D10, 26D15, 39A13, 34A40.

1. INTRODUCTION

During the last decades the inequality

$$\left(\int_a^b r(t)\left(\int_a^t f(\tau)d\tau\right)^q dt\right)^{1/q} \leq C\left(\int_a^b s(t)\, f^p(t)dt\right)^{1/p}, \; 1<p\leq q<\infty, \tag{1.1}$$

with two different positive weighted functions defined in $[a,b]\subset\mathbb{R}^+$ has been studied by several authors, we refer the reader to the papers [11, 23, 37, 38] and the books [20, 24]. The main idea is to give a relation between the functions r and s and to find the optimal value of the constant C such that the inequality (1.1) holds. A systematic investigation of this type of inequality of Hardy's type with two different weights started in the late fifties and early sixties by Beesack [7]. In particular Beesack proved that

$$\int_a^b r(t)\left(\int_0^t f(\tau)d\tau\right)^p dt \leq \int_a^b s(t)\, f^p(t)dt, \tag{1.2}$$

where r and s satisfy the Euler-Lagrange differential equation

$$\frac{d}{dt}\left(s(t)\left(y'(t)\right)^{p-1}\right)+r(t)\, y^{p-1}(t)=0.$$

Also Beesack and Heinig [8] proved that if $0<p<1$ and $\int_0^\infty r(t)\left(\int_0^t f(\tau)d\tau\right)^p dt<\infty$, then

$$\int_0^\infty r(t)\left(\int_0^t f(\tau)d\tau\right)^p dt \geq p^p\int_0^\infty r^{1-p}(t)\left(\int_t^\infty r(\tau)\,d\tau\right)^p f^p(t)dt, \tag{1.3}$$

and if $\int_0^\infty r(t)\left(\int_t^\infty f(\tau)d\tau\right)^p dt<\infty$, then

$$\int_0^\infty r(t)\left(\int_t^\infty f(\tau)d\tau\right)^p dt \geq p^p\int_a^\infty r^{1-p}(t)\left(\int_0^t r(\tau)\,d\tau\right)^p f^p(t)dt. \tag{1.4}$$

Bloom and Kerman [10] proved that if $1<p<\infty$, $f\geq 0$ and $\int_0^\infty (s(t)\,f(t))^p\,dt<\infty$, then

$$\int_0^\infty \left(r(t)\int_0^t f(\tau)d\tau\right)^p dt \leq C\int_0^\infty (s(t)\,f(t))^p\,dt, \tag{1.5}$$

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holds if and only if

$$\int_t^\infty \left(s^{-1}(\tau)\int_\tau^\infty r^p(x)\,dx\right)^{p'} d\tau \le C\int_t^\infty r^p(\tau)\,d\tau.$$

By using a new approach depends on the application of Opial's type inequalities Agarwal et al. [4] proved that if r, s are nonnegative measurable functions on (a,b) and $p>0$, $k>1$, then

$$\int_a^b r(t)\left(\int_a^t f(\tau)d\tau\right)^{p+1} dt \le (p+1)\,K_1(p,1,k)\left[\int_a^b s(t)\,f^k(t)dt\right]^{\frac{p+1}{k}}, \tag{1.6}$$

where

$$K_1(p,1,k)=\left(\frac{1}{p+1}\right)^{\frac{1}{k}}\left(\int_a^b (R(t,b))^{\frac{k}{k-1}}(s(t))^{\frac{-1}{k-1}}\left(\int_a^t s^{\frac{-1}{k-1}}(\tau)\,d\tau\right)^p dt\right)^{\frac{k-1}{k}},$$

and $R(t,b)=\int_t^b r(\tau)\,d\tau$.

In the last decades the study of discrete results on l^p analogues for L^p-bounds has been proved by some authors. One of the reasons for this upsurge of interest in discrete cases is due to the fact that the discrete operators may even behave differently from their continuous counterparts. So it was natural to look on the discrete results on l^p analogues for the above L^p-results. We mention here that in some special cases it is possible to translate or adapt almost straightforward the objects and results from the continuous setting to the discrete setting or vice versa, however, in some other cases that is far from be trivial. But l^p-bounds for discrete analogues of more complicated operators are not implied by results in the continuous setting, and moreover the discrete analogues are resistant to conventional methods. The main challenge here is that there are no general methods to study these questions and the methods should to be developed starting from the basic definitions in the discrete space. For example, Leindler [22] established the discrete versions of (1.3) and (1.4), and proved that if $0<p\le 1$, $a_n\ge 0$ and $\lambda_n>0$, then

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=1}^{n}a_k\right)^p \ge p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=n}^{\infty}\lambda_k\right)^p a_n^p, \tag{1.7}$$

and

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=n}^{\infty}a_k\right)^p \ge p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=1}^{n}\lambda_k\right)^p a_n^p. \tag{1.8}$$

In recent years the study of dynamic equations and inequalities on time scales has received a lot of attention in the literature and has become a major field in pure and applied mathematics. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$, to avoid proving results twice, once for differential inequality and once again for difference inequality. This idea goes back to its founder Stefan Hilger [19] who started the study of dynamic equations on time scales. Since the integral and discrete inequalities are important in the analysis of qualitative properties of solutions of differential and difference equations, we also believe that the dynamic Hardy type inequalities with weights on time scales will play the same effective role in the analysis of qualitative properties of dynamic equations with boundary conditions like oscillation, nonoscillation and distribution of zeros of solutions. For related dynamic inequalities on time scales, we refer the reader to the papers [26, 27, 32, 33] and the books [2,3]. Our technique in this paper will overcame the lack of calculus in the discrete space where there is no power rules and also there is no chain rule which are the main tools used in the proofs of the continuous case.

The aim of this paper is to prove some new dynamic inequalities by employing some Opial's type inequalities on an arbitrary time scale $\mathbb{T}$ which contain the integral and discrete inequalities (1.3)–(1.6) as special cases. For applications of the main results we get some well-known dynamic inequalities as special cases. The paper is divided into two sections. In Section 2, we introduce some preliminaries on time scales and establish some basic lemmas that will be needed in the proofs. In Section 3, we prove the main results and formulate some discrete results to show the application of the new results.

2. Preliminaries and Some Basic Lemmas

In this section, we present some basic definitions and results concerning the delta calculus on time scales; for more details we refer the reader to the book [14]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right-dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. Also, the set of functions that are differentiable and whose derivative is rd–continuous is denoted by $C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. Without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$. Recall of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable functions f and g

$$(2.1) \qquad (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$

The first chain rule that we will use in this paper is

$$(2.2) \qquad (f^\gamma(t))^\Delta = \gamma \int_0^1 [hf^\sigma + (1-h)f]^{\gamma-1} \, dh f^\Delta(t), \quad \gamma \in \mathbb{R},$$

which is a simple consequence of Keller's chain rule [14, Theorem 1.90]. The second chain rule that we will use in this paper is given in the following. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and

$$(2.3) \qquad f^\Delta(g(t)) = f'(g(d))\, g^\Delta(t), \quad \text{for} \quad d \in [t, \sigma(t)].$$

In this paper we will refer to the (delta) integral which we can define as follows. If $F^\Delta(t) = f(t)$, then the Cauchy (delta) integral of f is defined by $\int_{t_0}^t f(s)\Delta s := F(t) - F(t_0)$. It can be shown (see [14]) that if $f \in C_{rd}(\mathbb{T})$, then the Cauchy integral $F(t) := \int_{t_0}^t f(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $F^\Delta(t) = f(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$. Integration on discrete time scales is defined by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t).$$

The integration by parts formula on time scales reads

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.4}$$

Hölder's inequality [5, Theorem 6.2] states that for $f,\ g \in C_{rd}([a,\ b]_{\mathbb{T}},\ \mathbb{R})$, we have

$$\int_a^b |f(t)g(t)|\,\Delta t \le \left[\int_a^b |f(t)|^p\,\Delta t\right]^{1/p}\left[\int_a^b |g(t)|^q\,\Delta t\right]^{1/q}, \tag{2.5}$$

where $p > 1$, $1/p + 1/q = 1$ and $a,\ b \in \mathbb{T}$. This inequality is reversed if $0 < p < 1$ and $\int_a^b |g(t)|^q\,\Delta t > 0$, and it is also reversed if $p < 0$ and $\int_a^b |f(t)|^p\,\Delta t > 0$.

Throughout this paper, we will assume that $r(t)$, $s(t)$ and $f(t)$ are nonnegative rd-continuous functions and the integrals considered are assumed to exist. In order to prove our main results in Section 3, we need the following lemmas.

Lemma 2.1. *Assume $F : \mathbb{T} \to \mathbb{R}$ is differentiable and positive. If F^{Δ} is always positive, then*

$$\left(F^{\lambda}\right)^{\Delta} \ge F^{\Delta}\left(F^{\sigma}(t)\right)^{\lambda-1}, \quad \textit{if} \quad \lambda \ge 1, \tag{2.6}$$

and

$$\left(F^{\lambda}\right)^{\Delta} \le F^{\Delta}\left(F^{\sigma}(t)\right)^{\lambda-1}, \quad \textit{if} \quad 0 \le \lambda \le 1, \tag{2.7}$$

Proof. If F is increasing and $\lambda \ge 1$, then $F^{\lambda-1}$ is increasing and thus $\left(F^{\lambda-1}\right)^{\Delta} > 0$ so that

$$\left(F^{\lambda}\right)^{\Delta} = \left(FF^{\lambda-1}\right)^{\Delta} = F^{\Delta}\left(F^{\sigma}(t)\right)^{\lambda-1} + F\left(F^{\lambda-1}\right)^{\Delta} \ge 0.$$

This shows (2.6), and (2.7) follows similarly. The proof is complete. □

Lemma 2.2. *Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$. If $p > 0$, then*

$$\int_a^b r(t)\left(\int_a^{\sigma(t)} f(\tau)\,\Delta\tau\right)^{p+1}\Delta t \le (p+1)\int_a^b R(t,b)\left(F^{\sigma}(t)\right)^p F^{\Delta}(t)\,\Delta t, \tag{2.8}$$

where

$$R(t,b) = \int_t^b r(\tau)\,\Delta\tau, \quad \textit{and} \quad F(t) = \int_a^t f(\tau)\,\Delta\tau. \tag{2.9}$$

Proof. From (2.9) and applying integration by parts (2.4) with $u^{\Delta}(t) = R^{\Delta}(t,b)$ and $v^{\sigma}(t) = \left(F^{\sigma}(t)\right)^{p+1}$, we obtain

$$\begin{aligned}\int_a^b r(t)\left(\int_a^{\sigma(t)} f(\tau)\,\Delta\tau\right)^{p+1}\Delta t &= \int_a^b \left(-R^{\Delta}(t,b)\right)\left(F^{\sigma}(t)\right)^{p+1}\Delta t \\ &= -R(t,b)\,F^{p+1}(t)\Big|_a^b + \int_a^b R(t,b)\left(F^{p+1}(t)\right)^{\Delta}\Delta t.\end{aligned}$$

Using the fact that $R(b,b) = 0$ and $F(a) = 0$, we have

$$\int_a^b r(t)\left(\int_a^{\sigma(t)} f(\tau)\,\Delta\tau\right)^{p+1}\Delta t = \int_a^b R(t,b)\left(F^{p+1}(t)\right)^{\Delta}\Delta t. \tag{2.10}$$

By the chain rule (2.2) and the fact that $F^{\Delta}(t) = f(t) \ge 0$ yields

$$\begin{aligned}\left(F^{p+1}(t)\right)^{\Delta} &= (p+1)\int_0^1 \left[hF^{\sigma}(t) + (1-h)F(t)\right]^p F^{\Delta}(t) \\ &\le (p+1)\int_0^1 \left[hF^{\sigma}(t) + (1-h)F^{\sigma}(t)\right]^p F^{\Delta}(t) \\ &= (p+1)\left(F^{\sigma}(t)\right)^p F^{\Delta}(t).\end{aligned}$$

Substituting into (2.10), we get (2.8). The proof is complete. □

Lemma 2.3. *Let $\mathbb{T}$ be a time scale with $a,\, b \in \mathbb{T}$. If $p > 0$, then*

$$\int_a^b r(t)\left(\int_t^b f(\tau)\,\Delta\tau\right)^{p+1}\Delta t \leq (p+1)\int_a^b R(a,\sigma(t))\,\bar{F}^p(t)\,f(t)\,\Delta t, \tag{2.11}$$

where

$$R(a,t) = \int_a^t r(\tau)\,\Delta\tau, \quad \textit{and} \quad \bar{F}(t) = \int_t^b f(\tau)\,\Delta\tau. \tag{2.12}$$

Proof. From (2.12) and applying integration by parts (2.4) with $v^\Delta(t) = R^\Delta(a,t)$ and $u(t) = \bar{F}^{p+1}(t)$, we obtain

$$\begin{aligned}\int_a^b r(t)\left(\int_t^b f(\tau)\,\Delta\tau\right)^{p+1}\Delta t &= \int_a^b R^\Delta(a,t)\,\bar{F}^{p+1}(t)\,\Delta t \\ &= R(a,t)\;\bar{F}^{p+1}(t)\big|_a^b - \int_a^b R(a,\sigma(t))\left(\bar{F}^{p+1}(t)\right)^\Delta\Delta t.\end{aligned}$$

Using the fact that $R(a,a) = 0$ and $\bar{F}(b) = 0$, we have

$$\int_a^b r(t)\left(\int_t^b f(\tau)\,\Delta\tau\right)^{p+1}\Delta t = -\int_a^b R(a,\sigma(t))\left(\bar{F}^{p+1}(t)\right)^\Delta\Delta t. \tag{2.13}$$

By the chain rule (2.3) and the fact that $\bar{F}^\Delta(t) = -f(t) \leq 0$ and $t \leq d$, we see that

$$\left(\bar{F}^{p+1}(t)\right)^\Delta = (p+1)\,\bar{F}^p(d)\,F^\Delta(t) \geq (p+1)\,\bar{F}^p(t)\,\bar{F}^\Delta(t).$$

Substituting into (2.13), we get (2.11). The proof is complete. □

3. Main Results

In this section, we prove the main results.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_{\mathbb{T}}$, $0 < p < 1$. If*

$$\int_a^\infty r(t)\left(\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^p\Delta t < \infty,$$

then

$$\int_a^\infty r(t)\left(\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^p\Delta t \geq p^p\int_a^\infty r^{1-p}(t)\left(\int_t^\infty r(\tau)\,\Delta\tau\right)^p f^p(t)\Delta t. \tag{3.1}$$

Proof. Define $F(t) = \int_a^t f(\tau)\Delta\tau$. Integrating the left hand side of (3.1) by parts (2.4) with $u^\Delta(t) = r(t)$ and $v^\sigma(t) = (F^\sigma(t))^p$, we obtain

$$\begin{aligned}\int_a^\infty r(t)\,(F^\sigma(t))^p\,\Delta t &= u(t)\,F^p(t)|_a^\infty - \int_a^\infty u(t)\,(F^p(t))^\Delta\,\Delta t \\ &= \int_a^\infty(-u(t))\,(F^p(t))^\Delta\,\Delta t, \end{aligned} \tag{3.2}$$

where $u(t) = -\int_t^\infty r(\tau)\,\Delta\tau$. From (2.3), we have (note that $F^\Delta(t) = f(t) \geq 0$ and $d \leq \sigma(t)$)

$$(F^p(t))^\Delta = pF^{p-1}(d)F^\Delta(t) \geq p\,(F^\sigma(t))^{p-1}f(t). \tag{3.3}$$

Substitute (3.3) into (3.2) and applying Hölder's inequality (2.5) to get

$$\begin{aligned}\int_a^{\infty} r(t)(F^{\sigma}(t))^p\Delta t &\geq p\int_a^{\infty} f(t)(F^{\sigma}(t))^{p-1}\left(\int_t^{\infty} r(\tau)\Delta\tau\right)\Delta t\\ &= p\int_a^{\infty} f(t)r^{-1/p'}(t)\left(\int_t^{\infty} r(\tau)\Delta\tau\right)r^{1/p'}(t)(F^{\sigma}(t))^{p-1}\Delta t\\ &\geq p\left\{\int_a^{\infty} r^{1-p}(t)\left(\int_t^{\infty} r(\tau)\Delta\tau\right)^p f^p(t)\Delta t\right\}^{1/p}\\ &\quad\times\left\{\int_a^{\infty} r(t)(F^{\sigma}(t))^p\Delta t\right\}^{1/p'},\end{aligned}$$

and consequently, we obtain

$$\left\{\int_a^{\infty} r(t)(F^{\sigma}(t))^p\Delta t\right\}^{1/p}\geq p\left\{\int_a^{\infty} r^{1-p}(t)\left(\int_t^{\infty} r(\tau)\Delta\tau\right)^p f^p(t)\Delta t\right\}^{1/p},$$

which is (3.1). The proof is complete. □

Remark 3.1. *If $\mathbb{T}=\mathbb{R}$, then inequality (3.1) reduces to the Beesack and Heinig integral inequality (1.3).*

Remark 3.2. *If $\mathbb{T}=\mathbb{N}$, then inequality (3.1) reduces to the Leindler discrete inequality (1.7).*

Here, we state the Minkowski inequality [29, Lemma 2.6] on time scales which is needed in the proof of our next main result.

Lemma 3.1. *Let $\mathbb{T}$ be a time scale with $a, b\in\mathbb{T}$ and let f, g be nonnegative rd-continuous functions on $[a,b]_{\mathbb{T}}$. If $\gamma\geq 1$, then*

$$\left(\int_a^b f(x)\left(\int_a^{\sigma(x)} g(t)\Delta t\right)^{\gamma}\Delta x\right)^{1/\gamma}\leq\int_a^b g(t)\left(\int_t^b f(x)\Delta x\right)^{1/\gamma}\Delta t. \tag{3.4}$$

Theorem 3.2. *Let $\mathbb{T}$ be a time scale with $a\in[0,\infty)_{\mathbb{T}}$, $0<p<1$. If*

$$\int_a^{\infty} r(t)\left(\int_t^{\infty} f(\tau)\Delta\tau\right)^p\Delta t<\infty,$$

then

$$\int_a^{\infty} r(t)\left(\int_t^{\infty} f(\tau)\Delta\tau\right)^p\Delta t\geq p^p\int_a^{\infty} r^{1-p}(t)\left(\int_a^{\sigma(t)} r(\tau)\Delta\tau\right)^p f^p(t)\Delta t. \tag{3.5}$$

Proof. Define $\bar{F}(t):=\int_t^{\infty} f(\tau)\Delta\tau$. Since

$$\bar{F}^p(t)=-\int_t^{\infty}\left(\bar{F}^p(\tau)\right)^{\Delta}\Delta\tau, \tag{3.6}$$

so, from (2.3), we have (note that $\bar{F}^{\Delta}(\tau)=-f(\tau)\leq 0$, and $d\geq\tau$)

$$\left(\bar{F}^p(\tau)\right)^{\Delta}=p\bar{F}^{p-1}(d)\bar{F}^{\Delta}(\tau)\leq -p\bar{F}^{p-1}(\tau)f(\tau). \tag{3.7}$$

Substitute (3.7) into (3.6) gives

$$\bar{F}^p(t)\geq p\int_t^{\infty}\bar{F}^{p-1}(\tau)f(\tau)\Delta\tau.$$

Applying Minkowski's inequality and Hölder's inequality to get

$$\begin{aligned}
\int_a^\infty \bar{F}^p(t)\,r(t)\Delta t &\geq p\int_a^\infty r(t)\left(\int_t^\infty \bar{F}^{p-1}(\tau)f(\tau)\Delta\tau\right)\Delta t \\
&\geq p\int_a^\infty f(\tau)r^{-1/p'}(\tau)\left(\int_a^{\sigma(\tau)} r(t)\,\Delta t\right)\bar{F}^{p-1}(\tau)r^{1/p'}(\tau)\,\Delta\tau \\
&= p\left\{\int_a^\infty r^{1-p}(\tau)\left(\int_a^{\sigma(\tau)} r(t)\,\Delta t\right)^p f^p(\tau)\Delta\tau\right\}^{1/p} \\
&\quad\times\left\{\int_a^\infty \bar{F}^p(\tau)r(\tau)\,\Delta\tau\right\}^{1/p'},
\end{aligned}$$

and consequently, we obtain

$$\left(\int_a^\infty \bar{F}^p(t)\,r(t)\Delta t\right)^{1/p} \leq p\left(\int_a^\infty r^{1-p}(\tau)\left(\int_a^{\sigma(\tau)} r(t)\,\Delta t\right)^p f^p(\tau)\Delta\tau\right)^{1/p},$$

which is the desired inequality (3.5). The proof is complete. □

Remark 3.3. *If $\mathbb{T}=\mathbb{R}$, then inequality (3.5) reduces to the Beesack and Heinig integral inequality (1.4).*

Remark 3.4. *If $\mathbb{T}=\mathbb{N}$, then inequality (3.5) reduces to the Leindler discrete inequality (1.8). (See also [28, Remark 3.5])*

Theorem 3.3. *Let $\mathbb{T}$ be a time scale with $a\in[0,\infty)_{\mathbb{T}}$, $1<p<\infty$. If*

$$\int_a^\infty (s(t)\,f(t))^p\,\Delta t<\infty,$$

and

$$\int_t^\infty\left(s^{-1}(\tau)\int_\tau^\infty r^p(x)\,\Delta x\right)^{p'}\Delta\tau\leq C\int_t^\infty r^p(\tau)\,\Delta\tau<\infty, \tag{3.8}$$

then

$$\int_a^\infty\left(r(t)\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^p\Delta t\leq C\int_a^\infty (s(t)\,f(t))^p\,\Delta t, \tag{3.9}$$

Proof. Assume first that (3.8) holds and define $F(t)=\int_a^t f(\tau)\Delta\tau$. Integrating the left hand side of (3.9) by parts (2.4) with $u^\Delta(t)=r^p(t)$ and $v^\sigma(t)=(F^\sigma(t))^p$, we obtain

$$\begin{aligned}
\int_a^\infty r^p(t)\,(F^\sigma(t))^p\,\Delta t &= u(t)\,F^p(t)|_a^\infty-\int_a^\infty u(t)\,(F^p(t))^\Delta\,\Delta t \\
&= \int_a^\infty(-u(t))\,(F^p(t))^\Delta\,\Delta t,
\end{aligned}$$

where $u(t)=-\int_t^\infty r^p(\tau)\,\Delta\tau$. From (2.3), we have

$$(F^p(t))^\Delta\leq p\,(F^\sigma(t))^{p-1}f(t),$$

and so

$$\begin{aligned}
\int_a^\infty r^p(t)\,(F^\sigma(t))^p\,\Delta t &\leq p\int_a^\infty f(t)\,(F^\sigma(t))^{p-1}\left(\int_t^\infty r^p(\tau)\,\Delta\tau\right)\Delta t \\
&= p\int_a^\infty s(t)\,f(t)\,(F^\sigma(t))^{p-1}\left(s^{-1}(t)\int_t^\infty r^p(\tau)\,\Delta\tau\right)\Delta t.
\end{aligned}$$

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If we assume that $\int_a^\infty (s(t) f(t))^p \Delta t = 1$, then Hölder's inequality (2.5) gives

$$\begin{aligned}\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t &\leq p\left\{\int_a^\infty (s(t) f(t))^p \Delta t\right\}^{1/p} \\ &\quad \times \left\{\int_a^\infty (F^\sigma(t))^p \left(s^{-1}(t)\int_t^\infty r^p(\tau)\Delta\tau\right)^{p'} \Delta t\right\}^{1/p'} \\ &= p\left\{\int_a^\infty (F^\sigma(t))^p \left(s^{-1}(t)\int_t^\infty r^p(\tau)\Delta\tau\right)^{p'} \Delta t\right\}^{1/p'}.\end{aligned}$$

Using integration by parts with $u^\Delta(t) = \left(s^{-1}(t)\int_t^\infty r^p(\tau)\Delta\tau\right)^{p'}$ and $v^\sigma(t) = (F^\sigma(t))^p$ to get

$$\begin{aligned}&\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t \\ &\leq p\left\{\int_a^\infty \left[\int_t^\infty \left(s^{-1}(\tau)\int_\tau^\infty r^p(x)\Delta x\right)^{p'} \Delta\tau\right](F^p(t))^\Delta \Delta t\right\}^{1/p'}.\end{aligned}$$

Using (3.8) and integration by parts again with $u(t) = \int_t^\infty r^p(\tau)\Delta\tau$ and $v^\Delta(t) = (F^p(t))^\Delta$, we obtain

$$\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t \leq C\left\{\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t\right\}^{1/p'} < \infty,$$

and so $\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t \leq C$. The proof is complete. □

To prove the next results, need the following two theorems which are adapted from [35] and [1].

Theorem 3.4. *If $p(t), q(t) \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$ are positive functions such that $\int_a^t (p(\tau))^{-1/(k-1)}\Delta\tau < \infty$, and $y \in C_{rd}^1([a,b]_{\mathbb{T}},\mathbb{R})$ with $y(a) = 0$, then for $k > 1$, $\lambda > 0$ and $0 < \gamma < k$, we have*

$$\int_a^b q(t)\left|y(t)\right|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \leq K_1(\lambda,\gamma,k)\left[\int_a^b p(t)\left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k}, \tag{3.10}$$

where

$$K_1(\lambda,\gamma,k) := \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_a^b \left(\frac{q^k(t)}{p^\gamma(t)}\right)^{\frac{1}{k-\gamma}}\left(\int_a^t p^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^{\frac{\lambda(k-1)}{(k-\gamma)}} \Delta t\right]^{\frac{k-\gamma}{k}}.$$

If $y \in C_{rd}^1([a,b]_{\mathbb{T}},\mathbb{R})$ with $y(b) = 0$, then for $k > 1$, $\lambda > 0$ and $0 < \gamma < k$, we have that

$$\int_a^b q(t)\left|y(t)\right|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \leq K_2(\lambda,\gamma,k)\left[\int_a^b p(t)\left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k}, \tag{3.11}$$

where

$$K_2(\lambda,\gamma,k) := \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_a^b \left(\frac{q^k(t)}{p^\gamma(t)}\right)^{\frac{1}{k-\gamma}}\left(\int_t^b p^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^{\frac{\lambda(k-1)}{(k-\gamma)}} \Delta t\right]^{\frac{k-\gamma}{k}}.$$

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Theorem 3.5. *If* $p(t), q(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ *are positive functions and* $y \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ *such that* $y(a) = 0$, *then for* $\lambda \geq 1$, $\gamma \geq 0$ *and* $k > \gamma + 1$, *we have that*

$$\int_a^b q(t) \left|(y^\lambda)^\Delta(t)(y^\Delta(t))^\gamma\right| \Delta t \leq G_1(\lambda, \gamma, k) \left\{ \int_a^b p(t) \left|y^\Delta(t)\right|^k \Delta t \right\}^{\frac{\lambda+\gamma}{k}}, \tag{3.12}$$

where

$$G_1(\lambda, \gamma, k) := c \left\{ \int_a^b (q(t))^{\frac{k}{k-\gamma-1}} (p(t))^{\frac{-k\gamma}{(k-1)(k-\gamma-1)}} \left(R^{\frac{k\lambda-\lambda-\gamma}{k-\gamma-1}}\right)^\Delta (t) \Delta t \right\}^{\frac{k-\gamma-1}{k}},$$

with

$$c = \lambda \left(\frac{k-\gamma-1}{k\lambda-\lambda-\gamma}\right)^{\frac{k-\gamma-1}{k}} \left(\frac{\gamma+1}{\lambda+\gamma}\right)^{\frac{\gamma+1}{k}}, \quad \textit{and} \quad R(t) = \int_a^t \frac{\Delta\tau}{(p(\tau))^{\frac{1}{k-1}}}.$$

From (2.6), inequality (3.12) becomes as follow: If $p(t), q(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ are positive functions and $y \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $y^\Delta > 0$ satisfies $y(a) = 0$, then for $\lambda \geq 1$, $\gamma \geq 0$ and $k > \gamma + 1$

$$\int_a^b q(t) \left|y^\sigma(t)\right|^{\lambda-1} \left|y^\Delta(t)\right|^{\gamma+1} \Delta t \leq G_1(\lambda, \gamma, k) \left\{ \int_a^b p(t) \left|y^\Delta(t)\right|^k \Delta t \right\}^{\frac{\lambda+\gamma}{k}}, \tag{3.13}$$

where $G_1(\lambda, \gamma, k)$ is defined as in (3.12).

Theorem 3.6. *Let* $\mathbb{T}$ *be a time scale with* $a, b \in \mathbb{T}$. *If* $p > 0$ *and* $k > 1$, *then*

$$\int_a^b r(t) \left(\int_a^{\sigma(t)} f(\tau) \Delta\tau\right)^{p+1} \Delta t \leq (p+1) G_1(p+1, k) \left[\int_a^b s(t) (f(t))^k \Delta t\right]^{\frac{p+1}{k}}, \tag{3.14}$$

where

$$G_1(p+1, k) := \left[\int_a^b (R(t,b))^{\frac{k}{k-1}} \left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau) \Delta\tau\right)^{p+1}\right)^\Delta \Delta t\right]^{\frac{k-1}{k}},$$

and $R(t,b)$ *is defined as in (2.9).*

Proof. Applying Opial's inequality (3.13) with $y(t) = F(t)$, $q(t) = R(t,b)$, $p(t) = s(t)$, $\lambda = p+1$ and $\gamma = 0$, we obtain

$$\int_a^b R(t,b) (F^\sigma(t))^p F^\Delta(t) \Delta t \leq G_1(p+1, k) \left[\int_a^b s(t) (f(t))^k \Delta t\right]^{\frac{p+1}{k}}. \tag{3.15}$$

The result follows from (2.8) and (3.15). The proof is complete. $\square$

Theorem 3.7. *Let* $\mathbb{T}$ *be a time scale with* $a, b \in \mathbb{T}$. *If* $p > 0$ *and* $k > 1$, *then*

$$\int_a^b r(t) \left(\int_t^b f(\tau) \Delta\tau\right)^{p+1} \Delta t \leq (p+1) K_2(p, 1, k) \left[\int_a^b s(t) (f(t))^k \Delta t\right]^{\frac{p+1}{k}}, \tag{3.16}$$

where

$$K_2(p, 1, k) := \left(\frac{1}{p+1}\right)^{\frac{1}{k}} \left[\int_a^b (R(a, \sigma(t)))^{\frac{k}{k-1}} s^{\frac{-1}{k-1}}(t) \left(\int_t^b s^{\frac{-1}{k-1}}(\tau) \Delta\tau\right)^p \Delta t\right]^{\frac{k-1}{k}},$$

and $R(a,t)$ *is defined as in (2.12).*

Proof. Applying Opial's inequality (3.11) with $y(t) = F(t)$, $q(t) = R(a, \sigma(t))$, $p(t) = s(t)$, $\lambda = p$ and $\gamma = 1$, we obtain

$$\int_a^b R(a, \sigma(t)) \bar{F}^p(t) f(t) \Delta t \le K_2(p, 1, k) \left[\int_a^b s(t) (f(t))^k \Delta t \right]^{\frac{p+1}{k}}. \tag{3.17}$$

The result follows from (2.11) and (3.17). The proof is complete. □

The next result follows from Theorems 3.6 and 3.7 by choosing $k = p + 1$.

Corollary 3.1. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $k > 1$, then*

$$\int_a^b r(t) \left(\int_a^{\sigma(t)} f(\tau) \Delta\tau \right)^k \Delta t \le k G_1(k) \int_a^b s(t) (f(t))^k \Delta t, \tag{3.18}$$

where

$$G_1(k) := \left[\int_a^b (R(t, b))^{\frac{k}{k-1}} \left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau) \Delta\tau \right)^k \right)^{\Delta} \Delta t \right]^{\frac{k-1}{k}},$$

and

$$\int_a^b r(t) \left(\int_t^b f(\tau) \Delta\tau \right)^k \Delta t \le k K_2(k) \int_a^b s(t) (f(t))^k \Delta t, \tag{3.19}$$

where

$$K_2(k) := \left(\frac{1}{k} \right)^{\frac{1}{k}} \left[\int_a^b (R(a, \sigma(t)))^{\frac{k}{k-1}} s^{\frac{-1}{k-1}}(t) \left(\int_t^b s^{\frac{-1}{k-1}}(\tau) \Delta\tau \right)^{k-1} \Delta t \right]^{\frac{k-1}{k}}.$$

Remark 3.5. *Note that Theorems 3.6 and 3.7 are consequences of the weighted Hardy-type inequality due to Saker et al. [29, 36] with $p + 1 = q$ and $k = p$.*

As special cases of Theorems 3.6 and 3.7 when $\mathbb{T} = \mathbb{N}$, we have the following new discrete results

Corollary 3.2. *Let $\{x_n\}$, $\{\lambda_n\}$ and $\{w_n\}$ be nonnegative sequences. If $p > 0$ and $k > 1$, then*

$$\sum_{n=1}^N r_n \left(\sum_{i=1}^n x_i \right)^{p+1} \le (p+1) G_1(p+1, k) \left(\sum_{n=1}^N s_n x_n^k \right)^{\frac{p+1}{k}},$$

where

$$G_1(p+1, k) := \left[\sum_{n=1}^N (R(n, N))^{\frac{k}{k-1}} \Delta \left(\sum_{i=1}^n (s_i)^{\frac{-1}{k-1}} \right)^{p+1} \right]^{\frac{k-1}{k}},$$

with $R(n, N) = \sum_{i=n}^N r_i$.

Corollary 3.3. *Let $\{x_n\}$, $\{\lambda_n\}$ and $\{w_n\}$ be nonnegative sequences. If $p > 0$ and $k > 1$, then*

$$\sum_{n=1}^N r_n \left(\sum_{i=n}^N x_i \right)^{p+1} \le (p+1) K_2(p, 1, k) \left(\sum_{n=1}^N s_n x_n^k \right)^{\frac{p+1}{k}},$$

where

$$K_2(p, 1, k) := \left(\frac{1}{p+1} \right)^{\frac{1}{k}} \left[\sum_{n=1}^N (R(1, n+1))^{\frac{k}{k-1}} (s_n)^{\frac{-1}{k-1}} \left(\sum_{i=n}^N (s_i)^{\frac{-1}{k-1}} \right)^p \right]^{\frac{k-1}{k}},$$

with $R(1, n+1) = \sum_{i=1}^n r_i$.

By making suitable substitutions for the two weighted functions $r(t)$ and $s(t)$, we get some extensions related to the dynamic inequalities due to Řehák [25] and Saker et al. [30, 31] respectively. Also when $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=\mathbb{R}$, we get consequences due to Bennett [6], Bliss [9] and Flett [15]. For illustrations, we will present these special cases in the following examples.

Example 3.1. *If $r(t)=(\sigma(t)-a)^{-k}$ and $s(t)=1$, then inequality (3.18) reduces to the following extension of the Hardy-type inequality due to Řehák [25, Theorem 2.1]*

$$\int_a^\infty\left(\frac{1}{\sigma(t)-a}\int_a^{\sigma(t)}f(\tau)\Delta\tau\right)^k\Delta t\leq kR_1\int_a^\infty f^k(t)\Delta t,$$

where

$$R_1:=\left[\int_a^\infty(R(t,\infty))^{\frac{k}{k-1}}\left((t-a)^k\right)^\Delta\Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.2. *If we choose $r(t)=1/t^\gamma$ and $s(t)=1/t^{\gamma-k}$, $\gamma>1$ in Corollary 3.1, we get the inequality*

$$\int_a^\infty\frac{1}{t^\gamma}\left(\int_a^{\sigma(t)}f(\tau)\Delta\tau\right)^k\Delta t\leq kR_2\int_a^\infty\frac{1}{t^{\gamma-k}}f^k(t)\Delta t,$$

which is related to the inequality due to Saker and O'Regan [30, Theorem 2.2], where

$$R_2:=\left[\int_a^\infty(R(t,\infty))^{\frac{k}{k-1}}\left(\left(\int_a^t\left(\frac{1}{\tau^{\gamma-k}}\right)^{\frac{-1}{k-1}}\Delta\tau\right)^k\right)^\Delta\Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.3. *If we choose $r(t)=1/\sigma^\gamma(t)$ and $s(t)=1/\sigma^{\gamma-k}(t)$ in Corollary 3.1, we have the inequality*

$$\int_a^\infty\frac{1}{\sigma^\gamma(t)}\left(\int_t^\infty f(\tau)\Delta\tau\right)^k\Delta t\leq kR_3\int_a^\infty\frac{1}{\sigma^{\gamma-k}(t)}f^k(t)\Delta t,$$

which is related to the inequality due to Saker and O'Regan [30, Theorem 2.1], where

$$R_3:=\left(\frac{1}{k}\right)^{\frac{1}{k}}\left[\int_a^\infty(R(a,\sigma(t)))^{\frac{k}{k-1}}(\sigma(t))^{\frac{\gamma-k}{k-1}}\left(\int_t^\infty(\sigma(\tau))^{\frac{\gamma-k}{k-1}}\Delta\tau\right)^{k-1}\Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.4. *If we take $f(t)=\lambda(t)g(t)$,*

$$r(t)=\frac{\lambda(t)}{(\Lambda^\sigma(t))^\gamma},\ s(t)=\lambda^{1-k}(t)(\Lambda^\sigma(t))^{k-\gamma},\ k\geq\gamma>1,$$

in Corollary 3.1, we have the inequality

$$\int_a^b\frac{\lambda(t)}{(\Lambda^\sigma(t))^\gamma}\left(\int_a^{\sigma(t)}\lambda(\tau)g(\tau)\Delta\tau\right)^k\Delta t\leq kR_4\int_a^b\lambda(t)(\Lambda^\sigma(t))^{k-\gamma}g^k(t)\Delta t,$$

which is related to the inequality due to Saker et al. [31, Theorem 2.1], where $\Lambda(t)=\int_a^t\lambda(\tau)\Delta\tau$ and

$$R_4:=\left[\int_a^b(R(t,b))^{\frac{k}{k-1}}\left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^k\right)^\Delta\Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.5. *If we choose* $r(t) = t^{-1-(p+1)\lambda}/t^{p+1}$ *and* $s(t) = t^{-1-k\lambda}$, $\lambda > -1$ *in Theorem 3.6, we obtain the inequality*

$$\int_a^b t^{-1-(p+1)\lambda}\left(\frac{\int_a^{\sigma(t)} f(\tau)\Delta\tau}{t}\right)^{p+1}\Delta t \le (p+1)\,R_5\left[\int_a^b t^{-1-k\lambda} f^k(t)\Delta t\right]^{\frac{p+1}{k}},$$

which is related to the inequalities due to Flett [15] and Bliss [9], Hardy and Littlewood [18] (with $\lambda = -1/k$*), where*

$$R_5 := \left[\int_a^b (R(t,b))^{\frac{k}{k-1}}\left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^{p+1}\right)^{\Delta}\Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.6. *If we take*

$$r_n = \frac{\lambda_n}{\Lambda_n^{1-\frac{(p+1)}{k}(1-c)}},\ s_n = \lambda_n^{1-k}\Lambda_n^{k-c},\ c > 1 \text{ and } x_n = \lambda_n y_n,$$

in Corollary 3.2, we get the inequality

$$\sum_{n=1}^{N}\lambda_n\Lambda_n^{\frac{(p+1)(1-c)}{k}-1}\left(\sum_{i=1}^{n}\lambda_i y_i\right)^{p+1} \le (p+1)\,R_6\left(\sum_{n=1}^{N}\lambda_n\Lambda_n^{k-c}y_n^k\right)^{\frac{p+1}{k}},$$

which is related to Bennett's inequality [6, Corollary 7], where $\Lambda_n = \sum_{i=1}^{n}\lambda_i$ *and*

$$R_6 := \left[\sum_{n=1}^{N}(R(n,N))^{\frac{k}{k-1}}\Delta\left(\sum_{i=1}^{n}(s_i)^{\frac{-1}{k-1}}\right)^{p+1}\right]^{\frac{k-1}{k}},$$

with $R(n,N) = \sum_{i=n}^{N}\lambda_i\Lambda_i^{\frac{(p+1)(1-c)}{k}-1}$.

Remark 3.6. *As an application, we can apply Opial's inequalities together with a Hardy-type inequality (3.16) on time scales to establish some lower bounds of the distance between zeros of a solution and/or its derivatives for the fourth-order dynamic equation (see [13, Theorem 5.1])*

$$\left(r(t)y^{\Delta^3}(t)\right)^{\Delta} - \left(p(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y^{\sigma}(t) = 0, \quad t \in [a,b]_{\mathbb{T}}. \tag{3.20}$$

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