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HERMITE-HADAMARD TYPE INEQUALITIES INVOLVING CONFORMABLE FRACTIONAL INTEGRALS*

YOUSAF KHURSHID[1,2], MUHAMMAD ADIL KHAN[2], AND YU-MING CHU[3,**]

ABSTRACT. In the article, we establish an identity and several new Hermite-Hadamard type inequalities for conformable fractional integrals. As applications, we provide some inequalities for certain bivariate means and present the error estimations for the trapezoidal formula. The given results are the generalization of previously results.

1. INTRODUCTION

A real-valued function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If inequality (1.1) holds in the reverse direction, then we say that f is a concave function on the interval I.

The word "convexity" is one of the most important, natural and fundamental notations in mathematics. Convex functions were presented by Johan Jensen over 100 years ago. Over the past few years, many generalizations and extensions have been made for convexity. These extensions and generalizations in the theory of inequalities have made valuable contributions in many areas of mathematics. Some new generalized concepts in this point of view are quasi-convex [1], strongly convex [2], approximately convex [3], logarithmically convex [4], midconvex functions [5], pseudo-convex [6], φ-convex [7], λ-convex [8], h-convex [9], delta-convex [10], Schur convex [11-17] and and others [18-24].

Let $I \subseteq \mathbb{R}$ be an interval and $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then the well-known Hermite-Hadamard inequality [25-33] for convex functions states that the double inequality

$$h\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)dx \leq \frac{h(a_1)+h(a_2)}{2} \tag{1.2}$$

holds for all $a_1, a_2 \in I$ with $a_1 \neq a_2$. If the function h is concave on I, then both the inequalities in (1.2) hold in the reverse direction. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some

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of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions h. These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. In the last 60 years, many efforts have gone on generalizations extensions and variants of Hermite-Hadamard's inequality (see [34-36]).

Recently, the authors in [37] defined the conformable fractional derivative as follows: for a function $h:[0,\infty)\to\mathbb{R}$ the (conformable) fractional derivative of order $0<\alpha\le 1$ of h at $s>0$ was defined by

$$D_\alpha(h)(s)=\lim_{\epsilon\to 0}\frac{h(s+\epsilon s^{1-\alpha})-f(s)}{\epsilon},$$

if the conformable fractional derivative of h of order α exists, then we say that h is α-differentiable. The fractional derivative at 0 is defined as $h^\alpha(0)=\lim_{s\to 0^+}h^\alpha(s)$.

Now we recall some results for the conformable fractional derivative.

Theorem 1.1. *Let $\alpha\in(0,1]$ and h_1,h_2 be α-differentiable at a point $s>0$. Then*

$$\frac{d_\alpha}{d_\alpha s}(s^n)=ns^{n-\alpha}$$

for all $n\in\mathbb{R}$;

$$\frac{d_\alpha}{d_\alpha s}(c)=0$$

for all constant $c\in\mathbb{R}$;

$$\frac{d_\alpha}{d_\alpha s}(a_1h_1(s)+a_2h_2(s))=a_1\frac{d_\alpha}{d_\alpha s}(h_1(s))+a_2\frac{d_\alpha}{d_\alpha s}(h_2(s))$$

for all constants $a_1,a_2\in\mathbb{R}$;

$$\frac{d_\alpha}{d_\alpha s}(h_1(s)h_2(s))=h_1(s)\frac{d_\alpha}{d_\alpha s}(h_2(s))+h_2(s)\frac{d_\alpha}{d_\alpha s}(h_1(s));$$

$$\frac{d_\alpha}{d_\alpha s}\left(\frac{h_1(s)}{h_2(s)}\right)=\frac{h_2(s)\frac{d_\alpha}{d_\alpha s}(h_1(s))-h_1(s)\frac{d_\alpha}{d_\alpha s}(h_2(s))}{(h_2(s))^2};$$

$$\frac{d_\alpha}{d_\alpha s}(h_1(h_2)(s))=h_1'(h_2(s))\frac{d_\alpha}{d_\alpha s}(h_2(s))$$

if h_1 differentiable at $h_2(s)$.

If in addition h_1 is differentiable, then one has

$$\frac{d_\alpha}{d_\alpha s}(h_1(s))=s^{1-\alpha}\frac{d}{ds}(h_1(s)).$$

Definition 1.2. ***(Conformable fractional integral)*** *Let $\alpha\in(0,1]$ and $0\le a_1<a_2$. Then the function $h_1:[a_1,a_2]\to\mathbb{R}$ is said to be α-fractional integrable on $[a_1,a_2]$ if the integral*

$$\int_{a_1}^{a_2}h_1(x)d_\alpha x:=\int_{a_1}^{a_2}h_1(x)x^{\alpha-1}dx$$

exists and is finite. All α-fractional integrable functions on $[a_1,a_2]$ is indicated by $L^1_\alpha([a_1,a_2])$.

Remark 1. *Let $\alpha \in (0,1]$. Then*

$$I_{\alpha}^{a_1}(h_1)(s) = I_1^{a_1}(s^{\alpha-1}h_1) = \int_{a_1}^{s} \frac{h_1(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral.

Anderson [38] established the conformable integral version of Hermite-Hadamard inequality as follows:

Theorem 1.3. *If $\alpha \in (0,1]$ and $h : [a_1, a_2] \to \mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha h$ is increasing, then one has*

$$\frac{\alpha}{a_2^{\alpha} - a_1^{\alpha}} \int_{a_1}^{a_2} h(x) d_\alpha x \leq \frac{h(a_1) + h(a_2)}{2}.$$

Moreover, if the function h is decreasing on $[a_1, a_2]$, then we have

$$h\left(\frac{a_1 + a_2}{2}\right) \leq \frac{\alpha}{a_2^{\alpha} - a_1^{\alpha}} \int_{a_1}^{a_2} h(x) d_\alpha x.$$

In particular, if $\alpha = 1$, then this reduces to the classical Hermite-Hadamard inequality.

Due to the great importance of Hermite-Hadamard inequality, in recent years many mathematician have shown their interest for generalizations, extensions and variants for this inequality. In the article, we deal with the conformable integral version of Hermite-Hadamard inequality investigated by Anderson [38]. We shall establish an identity for the left side of the inequality and discuss their particular case. By applying Jensen's inequality, power mean inequality and the convexity of the functions $x^{\alpha-1}$ and $-x^{\alpha}$ $(x > 0, \alpha \in (0,1])$ in the identity, we obtain inequalities for conformable integral version of Hermite-Hadamard inequality. By using particular classes of convex functions we find new inequalities for special bivariate means. We also apply the results for error estimations of the mid point formula, for some related results (see [39-42]).

2. Main Results

We begin this section with the following Lemma 2.1, which is needed for the establishment of our main results.

Lemma 2.1. *Let $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-fractional differentiable function. Then the identity*

$$h\left(\frac{a_1 + a_2}{2}\right) - \frac{\alpha}{a_2^{\alpha} - a_1^{\alpha}} \int_{a_1}^{a_2} h(x) d_\alpha x \tag{2.1}$$

$$= \frac{(a_2 - a_1)}{2(a_2^{\alpha} - a_1^{\alpha})} \Bigg[\int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2} \right)^{2\alpha-1} - a_1^{\alpha} \left(\frac{2-s}{2}a_1 + \frac{sa_2}{2} \right)^{\alpha-1} \right)$$

$$\times D_\alpha(h) \left(\frac{2-s}{2}a_1 + \frac{sa_2}{2} \right) s^{1-\alpha} d_\alpha s + \int_0^1 \left(\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2 \right)^{2\alpha-1} \right.$$

$$\left. - a_2^{\alpha} \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2 \right)^{\alpha-1} \right) \times D_\alpha(h) \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2 \right) s^{1-\alpha} d_\alpha s \Bigg].$$

holds if $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$.

Proof. Integrating by parts, we have

$$\int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{2\alpha-1}-a_1^{\alpha}\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha-1}\right)\times D_\alpha(h)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds$$

$$+\int_0^1 \left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{2\alpha-1}-a_2^{\alpha}\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha-1}\right)$$

$$\times D_\alpha(h)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds$$

$$=\int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds$$

$$+\int_0^1 \left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}-a_2^{\alpha}\right)h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds$$

$$=\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\frac{h\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)}{\frac{a_2-a_1}{2}}\Bigg|_0^1$$

$$-\int_0^1 \alpha\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha-1}\left(\frac{a_2-a_1}{2}\right)\frac{h\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)}{\frac{a_2-a_1}{2}}ds$$

$$+\left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}-a_2^{\alpha}\right)\frac{h\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)}{\frac{a_2-a_1}{2}}\Bigg|_0^1$$

$$-\int_0^1 \alpha\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha-1}\left(\frac{a_2-a_1}{2}\right)\frac{h\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)}{\frac{a_2-a_1}{2}}ds$$

$$=\frac{2}{a_2-a_1}\left[\left(\left(\frac{a_1+a_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)h\left(\frac{a_1+a_2}{2}\right)-\alpha\int_a^{\frac{a_1+a_2}{2}}h(s)d_\alpha s\right]$$

$$+\frac{2}{a_2-a_1}\left[\left(a_2^{\alpha}-\left(\frac{a_1+a_2}{2}\right)^{\alpha}\right)h\left(\frac{a_1+a_2}{2}\right)-\alpha\int_{\frac{a_1+a_2}{2}}^{a_2}h(s)d_\alpha s\right]$$

$$=\frac{2(a_2^{\alpha}-a_1^{\alpha})}{a_2-a_1}h\left(\frac{a_1+a_2}{2}\right)-\frac{2\alpha}{a_2-a_1}\int_{a_1}^{a_2}h(x)d_\alpha x,$$

where we have used the change of variable $x=(1-s)a_1+sa_2$ and then multiplying both sides by $\frac{a_2-a_1}{2(a_2^{\alpha}-a_1^{\alpha})}$ to get the desired result in (2.1). □

Remark 2. *By putting $\alpha=1$ in (2.1), we get the identity*

$$h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(x)dx$$

$$=\frac{a_2-a_1}{4}\left[\int_0^1 sh'\left(\frac{sa_2}{2}+\frac{2-s}{2}a_1\right)ds-\int_0^1(1-s)h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds\right].$$

Theorem 2.2. *Let $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-differentiable function on (a_1, a_2). Then the inequality*

$$\left| h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(x) d_\alpha x \right| \tag{2.2}$$
$$\leq \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left[\frac{|h'(a_1)|}{96} [13a_2^\alpha - 19a_1^\alpha] + \frac{|h'(a_2)|}{96} [19a_2^\alpha - 21a_1^\alpha] \right.$$
$$\left. - a_1^\alpha a_2^{\alpha-1} \left[\frac{2|h'(a_1)| + |h'(a_2)|}{12} \right] + (a_1 a_2^{\alpha-1} + a_1^{\alpha-1} a_2) \left[\frac{11|h'(a_1)| + 5|h'(a_2)|}{96} \right] \right]$$

holds if $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$ and $|h'|$ is convex on $[a_1, a_2]$.

Proof. Let $\varphi_1 = x^{\alpha-1}$ and $\varphi_2 = -x^\alpha$ $(x > 0, \alpha \in (0,1])$. Then we clearly see that the functions φ_1 and φ_2 are convex. Now using Lemma 2.1 and the convexity of φ_1, φ_2 and $|h'|$, we have

$$\left| h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(x) d_\alpha x \right|$$
$$\leq \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left[\int_0^1 \left(\left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right)^\alpha - a_1^\alpha \right) \left| h'\left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right) \right| ds \right.$$
$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2} a_1 + \frac{1+s}{2} a_2 \right)^\alpha \right) \left| h'\left(\frac{1-s}{2} a_1 + \frac{1+s}{2} a_2 \right) \right| ds \right]$$
$$= \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left[\int_0^1 \left(\left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right)^{\alpha+1-1} - a_1^\alpha \right) \left| h'\left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right) \right| ds \right.$$
$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2} a_1 + \frac{1+s}{2} a_2 \right)^\alpha \right) \left| h'\left(\frac{1-s}{2} a_1 + \frac{1+s}{2} a_2 \right) \right| ds \right]$$
$$\leq \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left[\int_0^1 \left(\left(\frac{2-s}{2} a_1 + \frac{s}{2} a_2 \right)^{\alpha-1} \left(\frac{2-s}{2} a_1 + \frac{s}{2} a_2 \right) - a_1^\alpha \right) \left| h'\left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right) \right| ds \right.$$
$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2} a_1^\alpha + \frac{1+s}{2} a_2^\alpha \right) \right) \left| h'\left(\frac{1-s}{2} a_1 + \frac{1+s}{2} a_2 \right) \right| ds \right]$$
$$\leq \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left[\int_0^1 \left(\left(\frac{2-s}{2} a_1^{\alpha-1} + \frac{s}{2} a_2^{\alpha-1} \right) \left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right) - a_1^\alpha \right) \left| h'\left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right) \right| ds \right.$$
$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2} a_1^\alpha + \frac{1+s}{2} a_2^\alpha \right) \right) \left| h'\left(\frac{1-s}{2} a_1 + \frac{1+s}{2} a_2 \right) \right| ds \right]$$
$$\leq \frac{b-a}{2(a_2^\alpha - a_1^\alpha)} \left[\int_0^1 \left(\left(\frac{2-s}{2} a_1^{\alpha-1} + \frac{s}{2} a_2^{\alpha-1} \right) \left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right) - a_1^\alpha \right) \left[\frac{2-s}{2} |h'(a_1)| \right. \right.$$
$$\left. \left. + \frac{s}{2} |h'(a_2)| \right] ds + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2} a_1^\alpha + \frac{1+s}{2} a_2^\alpha \right) \right) \left[\frac{1-s}{2} |h'(a_1)| + \frac{1+s}{2} |h'(a_2)| \right] ds \right].$$

Evaluating all the above integrals, we have the following

$$\frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left[\int_0^1 \left(\left(\frac{2-s}{2} a_1^{\alpha-1} + \frac{s}{2} a_2^{\alpha-1} \right) \left(\frac{2-s}{2} a_1 + \frac{s a_2}{2} \right) - a_1^\alpha \right) \left[\frac{2-s}{2} |h'(a_1)| \right. \right.$$

$$+\frac{s}{2}|h'(a_2)|\Bigg]ds+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)\left[\frac{1-s}{2}|h'(a_1)|+\frac{1+s}{2}|h'(a_2))|\right]ds$$

$$=\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Bigg[\frac{15}{32}a_1^\alpha|h'(a_1)|+\frac{11}{96}a_1^{\alpha-1}a_2|h'(a_1)|-\frac{7}{12}a_1^\alpha|h'(a_1)|+\frac{11}{96}a_1a_2^{\alpha-1}|h'(a_1)|$$

$$+\frac{5}{96}a_2^\alpha|h'(a_1)|-\frac{1}{6}a_1^\alpha a_2^{\alpha-1}|h'(a_1)|+\frac{11}{96}a_1^\alpha|h'(a_2)|+\frac{5}{96}a_1^{\alpha-1}a_2|h'(a_2)|-\frac{1}{6}a_1^\alpha|h'(a_2)|$$

$$+\frac{5}{96}a_1a_2^{\alpha-1}|h'(a_2)|+\frac{1}{32}a_2^\alpha|h'(a_2)|-\frac{1}{12}a_1^\alpha a_2^{\alpha-1}|h'(a_2)|+\frac{1}{4}a_2^\alpha|h'(a_1)|-\frac{1}{12}a_1^\alpha|h'(a_1)|$$

$$-\frac{1}{6}a_2^\alpha|h'(a_1)|+\frac{3}{4}a_2^\alpha|h'(a_2)|-\frac{1}{6}a_1^\alpha|h'(a_2)|-\frac{7}{12}a_2^\alpha|h'(a_2)|\Bigg]$$

$$=\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Bigg[\frac{|h'(a_1)|}{96}\left[13a_2^\alpha-19a_1^\alpha\right]+\frac{|h'(a_2)|}{96}\left[19a_2^\alpha-21a_1^\alpha\right]$$

$$-a_1^\alpha a_2^{\alpha-1}\left[\frac{2|h'(a_1)|+|h'(a_2)|}{12}\right]+(a_1a_2^{\alpha-1}+a_1^{\alpha-1}a_2)\left[\frac{11|h'(a_1)|+5|h'(a_2)|}{96}\right]\Bigg].$$

□

Remark 3. *By putting $\alpha=1$ in (2.2), we obtain the inequality which is proved by Kirmaci in [43]*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(x)dx\right|\le\frac{(a_2-a_1)(|h'(a_1)|+|h'(a_2)|)}{8}.$$

Theorem 2.3. *Let $q>1$, $\alpha\in(0,1]$, $a_1,a_2\in\mathbb{R}^+$ with $a_1<a_2$ and $h:[a_1,a_2]\to\mathbb{R}$ be an α-differentiable function on (a_1,a_2). Then the inequality*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right| \tag{2.3}$$

$$\le\frac{(a_2-a_1)}{2(a_2^\alpha-a_1^\alpha)}\Bigg[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|h'(a_1)|^q+A_3(\alpha)|h'(a_2)|^q\right\}^{\frac{1}{q}}$$

$$+(B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|h'(a_1)|^q+B_3(\alpha)|h'(a_2)|^q\right\}^{\frac{1}{q}}\Bigg]$$

holds if $D_\alpha(h)\in L_\alpha^1([a_1,a_2])$ and $|h'|^q$ is convex on $[a_1,a_2]$, where

$$A_1(\alpha)=\left[\frac{(a_1+a_2)^{\alpha+1}-(2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right]-a_1^\alpha,$$

$$B_1(\alpha)=a_2^\alpha-\left[\frac{(2a_2)^{\alpha+1}-(a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right],$$

$$A_2(\alpha)=\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$

$$-\frac{2a_1^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+a_1}{(\alpha+2)(a_2-a_1)}\right]-\frac{3a_1^\alpha}{4},$$

$$B_2(\alpha)=\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$

$$-\frac{a_2^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)}+\frac{a_2^{\alpha}}{4},$$

$$A_3(\alpha)=\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)-(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_1^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)}-\frac{a_1^{\alpha}}{4},$$

$$B_3(\alpha)=\frac{3a_2^{\alpha}}{4}+\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_2^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(\alpha+2)(a_2-a_1)+a_2}{(\alpha+2)(a_2-a_1)}\right].$$

Proof. It follows from Lemma 2.1 that

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2}h(x)d_\alpha x\right|$$
$$=\left|\frac{(a_2-a_1)}{2(a_2^{\alpha}-a_1^{\alpha})}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{2\alpha-1}-a_1^{\alpha}\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha-1}\right)D_\alpha(h)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds\right.\right.$$
$$\left.\left.+\int_0^1\left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{2\alpha-1}-a_2^{\alpha}\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha-1}\right)\right]D_\alpha(h)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds\right|$$
$$\leq\frac{(a_2-a_1)}{2(a_2^{\alpha}-a_1^{\alpha})}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right.$$
$$\left.+\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds\right].$$

From the power-mean inequality and the convexity $|h'|^q$ we get

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds$$
$$\leq\left(\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)ds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(\left(\frac{2-s}{2}a+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|^q ds\right)^{\frac{1}{q}},$$

$$\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds$$
$$\leq\left(\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)ds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|^q ds\right)^{\frac{1}{q}},$$

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|^q ds$$

$$\leq \int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)\left[\frac{2-s}{2}|h'(a_1)|^q + \frac{s}{2}|h'(a_2)|^q\right]ds$$
$$= |h'(a_1)|^q \int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)\frac{2-s}{2}ds$$
$$+|h'(a_2)|^q \int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)\frac{s}{2}ds$$
$$= |h'(a_1)|^q\left(\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]\right.$$
$$\left.-\frac{2a_1^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+a_1}{(\alpha+2)(a_2-a_1)}\right] - \frac{3a_1^\alpha}{4}\right)$$
$$+|h'(a_2)|^q\left(\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)-(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]\right.$$
$$\left.-\frac{2a_1^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)} - \frac{a_1^\alpha}{4}\right),$$

$$\int\limits_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|^q ds$$
$$\leq \int\limits_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)\left[\frac{1-s}{2}|h'(a_1)|^q + \frac{1+s}{2}|h'(a_2)|^q\right]ds$$
$$= |h'(a_1)|^q \int\limits_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)\frac{1-s}{2}ds$$
$$+|h'(a_2)|^q \int\limits_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)\frac{1+s}{2}ds$$
$$= |h'(a_1)|^q\left(\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]\right.$$
$$\left.-\frac{a_2^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)} + \frac{a_2^\alpha}{4}\right)$$
$$+|h'(a_2)|^q\left(\frac{3a_2^\alpha}{4} + \frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]\right.$$
$$\left.-\frac{2a_2^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(\alpha+2)(a_2-a_1)+a_2}{(\alpha+2)(a_2-a_1)}\right]\right),$$

where we have used the facts that

$$\int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)ds = \left[\frac{(a_1+a_2)^{\alpha+1}-(2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right] - a_1^\alpha,$$

$$\int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)ds = a_2^\alpha - \left[\frac{(2a_2)^{\alpha+1} - (a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right].$$

Hence, we get the desired inequality (2.3). □

Remark 4. *Let $\alpha = 1$. Then inequality (2.3) leads to*

$$\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)dx\right|$$
$$\leq \frac{1}{2}\left(\frac{a_2-a_1}{4}\right)^{1-\frac{1}{q}}\left[\{A_2(1)|h'(a_1)|^q + A_3(1)|h'(a_2)|^q\}^{\frac{1}{q}}\right.$$
$$\left. + \{B_2(1)|h'(a_1)|^q + B_3(1)|h'(a_2)|^q\}^{\frac{1}{q}}\right],$$

where

$$A_2(1) = \frac{(a_1+a_2)^2(4a_2-2a_1) - 8a_1^2(3a_2-2a_1) - 18a_1(a_2-a_1)^2}{24(a_2-a_1)^2},$$
$$B_2(1) = \frac{(a_1+a_2)^2(4a_2-2a_1) - 4a_2^3 - 6a_2(a_2-a_1)^2}{24(a_2-a_1)^2},$$
$$A_3(1) = \frac{(a_1+a_2)^2(2a_2-4a_1) - 8a_1^3 - 6a_1(a_2-a_1)^2}{24(a_2-a_1)^2},$$
$$B_3(1) = \frac{(a_1+a_2)^2(4a_2-2a_1) - 8a_2^2(4a_2-3a_1) + 18a_2(a_2-a_1)^2}{24(a_2-a_1)^2}.$$

Theorem 2.4. *Let $q > 1$, $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1,a_2] \to \mathbb{R}$ be an α-differentiable function on (a_1,a_2). Then the inequality*

$$\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)d_\alpha x\right| \tag{2.4}$$
$$\leq \frac{(a_2-a_1)}{2(a_2^\alpha - a_1^\alpha)}\left[A_1(\alpha)h'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right) + B_1(\alpha)h'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right)\right]$$

holds if $D_\alpha(h) \in L^1_\alpha([a_1,a_2])$ and $|h'|^q$ is concave on $[a_1,a_2]$, where

$$A_1(\alpha) = \left[\frac{(a_1+a_2)^{\alpha+1} - (2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right] - a_1^\alpha,$$
$$B_1(\alpha) = a_2^\alpha - \left[\frac{(2a_2)^{\alpha+1} - (a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right],$$
$$C_1(\alpha) = (a_1+a_2)^{\alpha+2}\left[\frac{(\alpha+2)-1}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\right]$$
$$-\frac{2a_1^{\alpha+2}}{(\alpha+2)(a_2-a_1)^2}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+1)}\right] + \frac{a_1^\alpha}{4}(3a_1+a_2),$$
$$C_2(\alpha) = \frac{(a_1+a_2)^{\alpha+2}}{2^{\alpha+1}(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+2)(a_2-a_1)}\right]$$
$$-a_2^{\alpha+1}\left[\frac{(a_1+a_2)+2(\alpha+2)(a_2-a_1)}{(\alpha+2)(a_2-a_1)^2(\alpha+1)}\right] + \frac{a_2^\alpha}{4}(a_1+3a_2).$$

Proof. It follows from [44] and the concavity of $|h'|^q$ that $|h'|$ is also concave. Making use of Lemma 2.1 and Jensen's integral inequality we get

$$\begin{aligned}
&\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right| \\
&=\left|\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds\right.\right. \\
&\quad\left.\left.+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds\right|\right. \\
&\le\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right. \\
&\quad\left.+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds\right],
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds \\
&\le\left(\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\right) \\
&\quad h'\left(\frac{\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds}{\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)ds}\right) \\
&=A_1(\alpha)h'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds \\
&\le\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right) \\
&\quad h'\left(\frac{\int_0^1\left(\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\right)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds}{\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)ds}\right) \\
&=B_1(\alpha)h'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right),
\end{aligned}$$

where we used the facts that

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)ds=A_1(\alpha)=\left[\frac{(a_1+a_2)^{\alpha+1}-(2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right]-a_1^\alpha,$$

$$\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)ds=B_1(\alpha)=a_2^\alpha-\left[\frac{(2a_2)^{\alpha+1}-(a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right],$$

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds$$

$$= C_1(\alpha) = (a_1+a_2)^{\alpha+2}\left[\frac{(\alpha+2)-1}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\right]$$
$$-\frac{2a_1^{\alpha+2}}{(\alpha+2)(a_2-a_1)^2}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+1)}\right]+\frac{a_1^\alpha}{4}(3a_1+a_2)$$

and

$$\int_0^1\left(\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\right)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds$$
$$= C_2(\alpha) = \frac{(a_1+a_2)^{\alpha+2}}{2^{\alpha+1}(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+2)(a_2-a_1)}\right]$$
$$-a_2^{\alpha+1}\left[\frac{(a_1+a_2)+2(\alpha+2)(a_2-a_1)}{(\alpha+2)(a_2-a_1)^2(\alpha+1)}\right]+\frac{a_2^\alpha}{4}(a_1+3a_2).$$

□

Remark 5. *Let $\alpha = 1$. Then inequality (2.4) becomes*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(x)dx\right|$$
$$\leq \frac{(a_2-a_1)}{8}\Bigg[h'\left(\frac{(a_1+a_2)^3(a_2-a_1)-2a_1^3(4a_2-2a_1)+3a_1(3a_1+a_2)(a_2-a_1)^2}{3(a_2-a_1)}\right)$$
$$+h'\left(\frac{(a_1+a_2)^3(a_2-a_1)(2a_2-a_1)-2a_2^2(7a_2-5a_1)+3a_2(a_1+3a_2)(a_2-a_1)^2}{3(a_2-a_1)}\right)\Bigg].$$

Theorem 2.5. *Let $q > 1$, $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-differentiable function. Then the inequality*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right| \tag{2.5}$$
$$\leq \frac{(a_2-a_1)}{2(a_2^\alpha-a_1^\alpha)}\left[D_1(\alpha)h'\left(\frac{F_1(\alpha)}{D_1(\alpha)}\right)+E_1(\alpha)h'\left(\frac{F_2(\alpha)}{E_1(\alpha)}\right)\right]$$

holds if $D_\alpha(h) \in L^1_\alpha([a_1, a_2])$ and $|h'|^q$ is concave on $[a_1, a_2]$, where

$$D_1(\alpha) = \frac{-5a_1^\alpha+2a_1^{\alpha-1}a_2+2a_2^{\alpha-1}a_1+a_2^\alpha}{12},$$
$$E_1(\alpha) = \frac{a_2^\alpha-a_1^\alpha}{4},$$
$$F_1(\alpha) = \frac{-27a_1^{\alpha+1}-2a_1^\alpha a_2+11a_1^2a_2^{\alpha-1}+5a_1a_2^\alpha+5a_1^{\alpha-1}a_2^2+3a_2^{\alpha+1}}{96},$$
$$F_2(\alpha) = \frac{a_1a_2^\alpha-a_1^{\alpha+1}+2a_2^{\alpha+1}-2a_1^\alpha a_2}{12}.$$

Proof. It follows from [44] and the concavity of $|h'|^q$ that $|h'|$ is also concave. Making use of Lemma 2.1 and Jensen's integral inequality one has

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right|$$
$$\leq \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right.$$

$$+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds\Bigg]$$

$$\leq\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1^\alpha+\frac{sa_2^\alpha}{2}\right)-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right.$$

$$\left.+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds\right]$$

$$\leq\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1^{\alpha-1}+\frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right.$$

$$\left.+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds\right],$$

$$\int_0^1\left(\left(\frac{2-s}{2}a_1^{\alpha-1}+\frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds$$

$$\leq\left(\int_0^1\left(\left(\frac{2-s}{2}a_1^{\alpha-1}+\frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)-a_1^\alpha\right)\right)$$

$$h'\left(\frac{\int_0^1\left(\left(\frac{2-s}{2}a_1^{\alpha-1}+\frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)-a_1^\alpha\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds}{\int_0^1\left(\left(\frac{2-s}{2}a_1^{\alpha-1}+\frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)-a_1^\alpha\right)ds}\right)$$

$$=D_1(\alpha)h'\left(\frac{F_1(\alpha)}{D_1(\alpha)}\right),$$

$$\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds$$

$$\leq\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)$$

$$h'\left(\frac{\int_0^1\left(\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)\right)\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)ds}{\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)ds}\right)$$

$$=E_1(\alpha)f'\left(\frac{F_2(\alpha)}{E_1(\alpha)}\right),$$

where we have used the facts that

$$\int_0^1\left(\left(\frac{2-s}{2}a_1^{\alpha-1}+\frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)-a_1^\alpha\right)ds$$

$$=D_1(\alpha)=\frac{-5a_1^\alpha+2a_1^{\alpha-1}a_2+2a_2^{\alpha-1}a_1+a_2^\alpha}{12},$$

$$\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)ds=E_1(\alpha)=\frac{a_2^\alpha-a_1^\alpha}{4},$$

$$\int_0^1\left(\left(\frac{2-s}{2}a_1^{\alpha-1}+\frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)-a_1^\alpha\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds=F_1(\alpha)$$

$$= \frac{-27a_1^{\alpha+1} - 2a_1^{\alpha}a_2 + 11a_1^2a_2^{\alpha-1} + 5a_1a_2^{\alpha} + 5a_1^{\alpha-1}a_2^2 + 3a_2^{\alpha+1}}{96}$$

and

$$\int_0^1 \left(\left(a_2^{\alpha} - \left(\frac{1-s}{2}a_1^{\alpha} + \frac{1+s}{2}a_2^{\alpha}\right)\right)\right)\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right) ds$$

$$= F_2(\alpha) = \frac{a_1a_2^{\alpha} - a_1^{\alpha+1} + 2a_2^{\alpha+1} - 2a_1^{\alpha}a_2}{12}.$$

□

Remark 6. *Let $\alpha = 1$. Then inequality (2.5) leads to*

$$h\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)dx \tag{2.6}$$

$$\leq \frac{a_2-a_1}{8}\left[\left|h'\left(\frac{a_2+2a_1}{3}\right)\right| + \left|h'\left(\frac{a_1+2a_2}{3}\right)\right|\right].$$

Note that inequality (2.6) is an improvement of the inequality obtained by Pearce and Pečarić in [44] due to $|h'|$ is concave on $[a_1, a_2]$ and

$$\frac{a_2-a_1}{8}\left[\left|h'\left(\frac{a_2+2a_1}{3}\right)\right| + \left|h'\left(\frac{a_1+2a_2}{3}\right)\right|\right]$$

$$= \frac{a_2-a_1}{4}\left[\frac{1}{2}\left|h'\left(\frac{a_2+2a_1}{3}\right)\right| + \frac{1}{2}\left|h'\left(\frac{a_1+2a_2}{3}\right)\right|\right] \leq \frac{a_2-a_1}{4}\left|h'\left(\frac{a_1+a_2}{2}\right)\right|.$$

3. Applications to Special Bivariate Means

Let $a, b > 0$ with $a \neq b$. Then the arithmetic mean $A(a, b)$ [45-50], logarithmic mean $L(a, b)$ [51-55] and (α, r)-th generalized logarithmic mean $L_{(\alpha,r)}(a, b)$ [56-59] are defined by

$$A(a,b) = \frac{a+b}{2}, \quad L(a,b) = \frac{b-a}{\log b - \log a}, \quad L_{(\alpha,r)}(a,b) = \left[\frac{\alpha(b^{r+\alpha} - a^{r+\alpha})}{(r+\alpha)(b^{\alpha} - a^{\alpha})}\right]^{1/r},$$

respectively. Recently, the bivariate means have been the subject of intensive research [60-74] and many remarkable inequalities for the bivariate means and related special functions can be found in the literature [75-97].

Making use of Theorems 2.2 and 2.3 together with the convexity of the functions x^r and $1/x$ $(x > 0)$ we get some new inequalities for the arithmetic, logarithmic and generalized means immediately.

Theorem 3.1. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$. Then the inequality*

$$\left|A^r(a_1,a_2) - L^r_{(\alpha,r)}(a_1,a_2)\right| \tag{3.1}$$

$$\leq \frac{r(a_2-a_1)}{2(a_2^{\alpha} - a_1^{\alpha})}\left[\frac{|a_1|^{r-1}}{96}\left[13a_2^{\alpha} - 19a_1^{\alpha}\right] + \frac{|a_2|^{r-1}}{96}\left[19a_2^{\alpha} - 21a_1^{\alpha}\right]\right.$$

$$\left. -a_1^{\alpha}a_2^{\alpha-1}\left\{\frac{2|a_1|^{r-1} + |a_2|^{r-1}}{12}\right\}\right.$$

$$+(a_1 a_2^{\alpha-1} + a_1^{\alpha-1} a_2)\left\{\frac{11|a_1|^{r-1} + 5|a_2|^{r-1}}{192}\right\}\Bigg]$$

holds for all $r > 1$ and $\alpha \in (0,1]$.

Remark 7. *Let $\alpha = 1$. Then inequality (3.1) leads to*

$$|A^r(a_1, a_2) - L_r^r(a_1, a_2)| \leq \frac{r(a_2 - a_1)}{4} A(|a_1|^{r-1}, |a_2|^{r-1}),$$

which was proved by Kirmaci in [43].

Theorem 3.2. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $r > 1$. Then the inequality*

$$\left|A^r(a_1, a_2) - L^r_{(\alpha,r)}(a_1, a_2))\right|$$

$$\leq \frac{r(a_2 - a_1)}{2(a_2^\alpha - a_1^\alpha)}\Bigg[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|a_1|^{(r-1)q} + A_3(\alpha)|a_2|^{(r-1)q}\right\}^{\frac{1}{q}}$$

$$+ (B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|a_1|^{(r-1)q} + B_3(\alpha)|a_2|^{(r-1)q}\right\}^{\frac{1}{q}}\Bigg]$$

holds for all $q > 1$ and $\alpha \in (0,1]$.

Theorem 3.3. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$. The the inequality*

$$\left|A^r(a_1, a_2) - L^r_{(\alpha,r)}(a_1, a_2)\right| \tag{3.2}$$

$$\leq \frac{(a_2 - a_1)}{2(a_2^\alpha - a_1^\alpha)}\Bigg[\frac{|a_1|^{-2}}{96}\left[13a_2^\alpha - 19a_1^\alpha\right] + \frac{|a_2|^{-2}}{96}\left[19b^\alpha - 21a_1^\alpha\right]$$

$$-a_1^\alpha a_2^{\alpha-1}\left\{\frac{2|a_1|^{-2} + |a_2|^{-2}}{12}\right\} + (a_1 a_2^{\alpha-1} + a_1^{\alpha-1} a_2)\left\{\frac{11|a_1|^{-2} + 5|a_2|^{-2}}{192}\right\}\Bigg]$$

holds for all $\alpha \in (0,1]$.

Remark 8. *Let $\alpha = 1$. Then inequality (3.2) reduces to*

$$|A^r(a_1, a_2) - L_r^r(a_1, a_2)| \leq \frac{(a_2 - a_1)}{4} A(|a_1|^{-2}, |a_2|^{-2}),$$

which was proved by Kirmaci in [43].

Theorem 3.4. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$. Then the inequality*

$$\left|A^r(a_1, a_2) - L^r_{(\alpha,r)}(a_1, a_2))\right|$$

$$\leq \frac{(a_2 - a_1)}{2(b^\alpha - a_1^\alpha)}\Bigg[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|a_1|^{-2q} + A_3(\alpha)|a_2|^{-2q}\right\}^{\frac{1}{q}}$$

$$+ (B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|a_1|^{-2q} + B_3(\alpha)|a_2|^{-2q}\right\}^{\frac{1}{q}}\Bigg]$$

holds for all $q > 1$ and $\alpha \in (0,1]$.

4. Applications to Mid-Point Formula

Let P be the partition of the points $a_1 = y_0 < y_1 < ... < y_{n-1} < y_n = a_2$ of the interval $[a_1, a_2]$ and consider the quadrature formula

$$\int_a^b h(x) d_\alpha x = T_\alpha(h, P) + E_\alpha(h, P),$$

where

$$T_\alpha(h, P) = \sum_{i=0}^{n-1} h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha},$$

is the midpoint version and $E_\alpha(h, P)$ denotes the associated approximation error.

In this section, we shall present some new estimates for the midpoint formula.

Theorem 4.1. *Let $\alpha \in (0, 1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-differentiable function. Then the inequality*

$$|E_\alpha(h, P)| \leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)}{2\alpha} \left[\frac{|h'(y_i)|}{96} \left[13 y_{i+1}^\alpha - 19 y_i^\alpha\right] + \frac{|h'(y_{i+1})|}{96} \left[19 y_{i+1}^\alpha - 21 y_i^\alpha\right] \right.$$
$$\left. - y_i^\alpha y_{i+1}^{\alpha-1} \left[\frac{2|h'(y_i)| + |h'(y_{i+1})|}{12}\right] + (y_i y_{i+1}^{\alpha-1} + y_i^{\alpha-1} y_{i+1}) \left[\frac{11|h'(y_i)| + 5|h'(y_{i+1})|}{12}\right] \right]$$

holds if $D_\alpha(h) \in L^1_\alpha([a_1, a_2])$ and $|h'|$ is convex on $[a_1, a_2]$.

Proof. Applying Theorem 2.2 on the subinterval $[y_i, y_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition P, we have

$$\left| h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} - \int_{y_i}^{y_{i+1}} h(x) d_\alpha x \right|$$
$$\leq \frac{(y_{i+1} - y_i)}{2\alpha} \left[\frac{|h'(y_i)|}{96} \left[13 y_{i+1}^\alpha - 19 y_i^\alpha\right] + \frac{|h'(y_{i+1})|}{96} \left[19 y_{i+1}^\alpha - 21 y_i^\alpha\right] \right.$$
$$\left. - y_i^\alpha y_{i+1}^{\alpha-1} \left[\frac{2|h'(y_i)| + |h'(y_{i+1})|}{12}\right] + (y_i y_{i+1}^{\alpha-1} + y_i^{\alpha-1} y_{i+1}) \left[\frac{11|h'(y_i)| + 5|h'(y_{i+1})|}{12}\right] \right],$$

$$\left| \int_{a_1}^{a_2} h(x) d_\alpha x - T_\alpha(h, P) \right|$$
$$= \left| \sum_{i=0}^{n-1} \left\{ \int_{y_i}^{y_{i+1}} h(x) d_\alpha x - h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} \right\} \right|$$
$$\leq \sum_{i=0}^{n-1} \left| \left\{ \int_{y_i}^{y_{i+1}} h(x) d_\alpha x - h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} \right\} \right|$$
$$\leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)}{2\alpha} \left[\frac{|h'(y_i)|}{96} \left[13 y_{i+1}^\alpha - 19 y_i^\alpha\right] + \frac{|h'(y_{i+1})|}{96} \left[19 y_{i+1}^\alpha - 21 y_i^\alpha\right] \right.$$
$$\left. - y_i^\alpha y_{i+1}^{\alpha-1} \left[\frac{2|h'(y_i)| + |h'(y_{i+1})|}{12}\right] + (y_i y_{i+1}^{\alpha-1} + y_i^{\alpha-1} y_{i+1}) \left[\frac{11|h'(y_i)| + 5|h'(y_{i+1})|}{12}\right] \right].$$

□

Theorem 4.2. *Let $q > 1$, $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-differentiable function. Then the inequality*

$$|E_\alpha(h,P)| \leq \sum_{i=0}^{n-1} \frac{(y_{i+1}-y_i)}{2\alpha} \Bigg[(A_1(\alpha))^{1-\frac{1}{q}} \left\{ A_2(\alpha)|h'(y_i)|^q + A_3(\alpha)|h'(y_{i+1})|^q \right\}^{\frac{1}{q}} + (B_1(\alpha))^{1-\frac{1}{q}} \left\{ B_2(\alpha)|h'(y_i)|^q + B_3(\alpha)|h'(y_{i+1})|^q \right\}^{\frac{1}{q}} \Bigg]$$

holds if $D_\alpha(h) \in L^1_\alpha([a_1, a_2])$ and $|h'|^q$ is convex on $[a_1, a_2]$.

Proof. The proof is analogous to that of Theorem 4.1 only by using Theorem 2.3. □

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Neutrosophic BCC-ideals in BCC-algebras

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Abstract. The notions of a neutrosophic subalgebra and a neutrosohic ideal of a BCC-algebra are introduced and consider characterizations of a neutrosophic subalgebra and a neutrosophic ideal. We define the notion of a neutrosophic BCC-ideal of a BCC-algebra, and investigated some properties of it.

1. Introduction

Y. Kormori [8] introduced a notion of a BCC-algebras, and W. A. Dudek [4] redefined the notion of BCC-algebras by using a dual from of the ordinary definition of Y. Kormori. In [6], J. Hao introduced the notion of ideals in a BCC-algebra and studied some related properties. W. A. Dudek and X. Zhang [5] introdued a BCC-ideals in a BCC-algebra and described connections between such BCC-ideals and congruences. S. S. Ahn and S. H. Kwon [2] defined a topological BCC-algebra and investigated some properties of it.

Zadeh [10] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [3] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). Jun et. al [7] introduced the notions of a neutrosophic $\mathscr{N}$-subalgebras and a (closed) neutrosophic $\mathscr{N}$-ideal in a BCK/BCI-algebras and investigated some related properties. subalgebras

In this paper, we introduce the notions of a neutrosophic subalgebra and a neutrosohic ideal of a BCC-algebra and consider characterizations of a neutrosophic subalgebra and a neutrosophic ideal. We define the notion of a neutrosophic BCC-ideal of a BCC-algebra, and investigate some properties of it.

2. Preliminaries

By a BCC*-algebra* [4] we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: for all $x, y, z \in X$,

(a1) $((x * y) * (z * y)) * (x * z) = 0$,

(a2) $0 * x = 0$,

(a3) $x * 0 = x$,

(a4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call X a BCC*-algebra.* In X, we can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$. Then $\leq$ is a partial order on X.

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A BCC-algebra X has the following properties: for any $x, y \in X$,

(b1) $x * x = 0$,

(b2) $(x * y) * x = 0$,

(b3) $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebra [4]. Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(b4) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Let $(X, *, 0_X)$ and $(Y, *, 0_Y)$ be BCC-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for all $x, y \in X$. A non-empty subset S of a BCC-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of a BCI-algebra X is called an *ideal* [6] of X if it satisfies:

(c1) $0 \in I$,

(c2) $x * y, y \in I \Rightarrow x \in I$ for all $x, y \in X$.

I is called an *BCC-ideal* [5] of X if it satisfies (c1) and

(c3) $(x * y) * z, y \in I \Rightarrow x * z \in I$, for all $x, y, z \in X$.

Theorem 2.1. [6] *In a BCC-algebra, an ideal is a subalgebra.*

Theorem 2.2. [5] *In a BCC-algebra, a BCC-ideal is an ideal.*

Corollary 2.3. [5] *Any BCC-ideal of a BCC-algebra is a subalgebra.*

Definition 2.4. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosophic set A can be denoted by

$$A := \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.5. Let A be a neutrosophic set in a B-algebra X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$ and an (α, β, γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \leq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}.$$

For any family $\{a_i | i \in \Lambda\}$, we define

$$\bigvee \{a_i | i \in \Lambda\} := \begin{cases} \max\{a_i | i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i | i \in \Lambda\} & \text{otherwise} \end{cases}$$

and

$$\bigwedge\{a_i|i\in\Lambda\} := \begin{cases}\min\{a_i|i\in\Lambda\} & \text{if } \Lambda \text{ is finite,}\\ \inf\{a_i|i\in\Lambda\} & \text{otherwise.}\end{cases}$$

3. Neutrosophic BCC-ideals

In what follows, let X be a BCC-algebra unless otherwise specified.

Definition 3.1. A neutrosophic set A in a BCC-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $T_A(x*y)\le\max\{T_A(x),T_A(y)\}, I_A(x*y)\ge\min\{I_A(x),I_A(y)\},$ and $F_A(x*y)\le\max\{F_A(x),F_A(y)\}$, for any $x,y\in X$.

Proposition 3.2. *Every neutrosophic subalgebra of a BCC-algebra X satisfies the following conditions:*

(3.1) $T_A(0)\le T_A(x), I_A(0)\ge I_A(x)$, *and* $F_A(0)\le F_A(x)$ *for any* $x\in X$.

Proof. Straightforward. □

Example 3.3. Let $X:=\{0,1,2,3\}$ be a BCC-algebra [6] with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

Define a neutrosophic set A in X as follows:

$$T_A: X\to[0,1],\ x\mapsto\begin{cases}0.12 & \text{if } x\in\{0,1,2\}\\ 0.83 & \text{if } x=3,\end{cases}$$

$$I_A: X\to[0,1],\ x\mapsto\begin{cases}0.81 & \text{if } x\in\{0,1,2\}\\ 0.14 & \text{if } x=3,\end{cases}$$

and

$$F_A: X\to[0,1],\ x\mapsto\begin{cases}0.12 & \text{if } x\in\{0,1,2\}\\ 0.83 & \text{if } x=3.\end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Theorem 3.4. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha,\beta,\gamma\in[0,1]$ with $0\le\alpha+\beta+\gamma\le3$. Then A is a neutrosophic subalgebra of X if and only if all of (α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)}\ne\emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha,\beta,\gamma\in[0,1]$ be such that $0\le\alpha+\beta+\gamma\le3$ and $A^{(\alpha,\beta,\gamma)}\ne\emptyset$. Let $x,y\in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x)\le\alpha, T_A(y)\le\alpha, I_A(x)\ge\beta, I_A(y)\ge\beta$ and $F_A(x)\le\gamma, F_A(y)\le\gamma$. Using (NSS), we have $T_A(x*y)\le\max\{T_A(x),T_A(y)\}\le\alpha$, $I_A(x*y)\ge\min\{I_A(x),I_A(y)\}\ge\beta$, and $F_A(x*y)\le\max\{F_A(x),F_A(y)\}\le\gamma$. Hence $x*y\in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)}\ne\emptyset$. Assume that there exist $a_t,b_t,a_i,b_i\in X$ and $a_f,b_f\in X$ such that $T_A(a_t*b_t)>\max\{T_A(a_t),T_A(b_t)\}, I_A(a_i*b_i)<\min\{I_A(a_i),I_A(b_i)\}$

and $F_A(a_f * b_f) > \max\{F_A(a_f), F_A(b_f)\}$. Then $T_A(a_t * b_t) > \alpha_1 \geq \max\{T_A(a_t), T_A(b_t)\}, I_A(a_i * b_i) < \beta_1 \leq \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f * b_f) > \gamma_1 \geq \max\{F_A(a_f), F_A(b_f)\}$ for some $\alpha_1, \gamma_1 \in [0, 1)$ and $\beta_1 \in (0, 1]$. Hence $a_t, b_t, a_i, b_i \in A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f, b_f \in A^{(\alpha_1,\beta_1,\gamma_1)}$. But $a_t * b_t, a_i * b_i \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f * b_f \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, which is a contradiction. Hence $T_A(x * y) \leq \max\{T_A(x), T_A(y)\}, I_A(x * y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x * y) \leq \max\{T_A(x), T_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.5. *If $\{A_i|i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a BCC-algebra X, then $(\{A_i|i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 3.6. *Let A be a neutrosophic subalgebra of a BCC-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 0, \lim_{n\to\infty} I_A(a_n) = 1$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$.*

Proof. By Proposition 3.2, we have $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$, and $F_A(0) \leq F_A(x)$ for all $x \in X$. Hence we have $T_A(0) \leq T_A(a_n), I_A(0) \geq I_A(a_n)$, and $F_A(0) \leq F_A(a_n)$ for every positive integer n. Therefore $0 \leq T_A(0) \leq \lim_{n\to\infty} T_A(a_n) = 0, 1 = \lim_{n\to\infty} I_A(a_n) \leq I_A(0) \leq 1$, and $0 \leq F_A(0) \leq \lim_{n\to\infty} F_A(a_n) = 0$. Thus we have $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$. □

Proposition 3.7. *If every neutrosophic subalgebra A of a BCC-algebra X satisfies the condition*

(3.2) $T_A(x * y) \leq T_A(y), I_A(x * y) \geq I_A(y), F_A(x * y) \leq F_A(y)$, *for any* $x, y \in X$,

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.2) that $T_A(x) = T_A(x * 0) \leq T_A(0), I_A(x) = I_A(x * 0) \geq I_A(0)$, and $F_A(x) = F_A(x * 0) \leq F_A(0)$ for any $x \in X$. By Proposition 3.2, we have $T_A(x) = T_A(0), I_A(x) = I_A(0)$, and $F_A(x) = F_A(0)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Theorem 3.8. *Every subalgebra of a BCC-algebra X can be represented as an (α, β, γ)-level set of a neutrosophic subalgebra A of X.*

Proof. Let S be a subalgebra of a BCC-algebra X and let A be a neutrosophic subalgebra of X. Define a neutrosophic set A in X as follows:

$$T_A : X \to [0, 1],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in S \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$I_A : X \to [0, 1],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in S \\ \beta_2 & \text{otherwise,} \end{cases}$$

$$F_A : X \to [0, 1],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in S \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [0, 1)$ and $\beta_1, \beta_2 \in (0, 1]$ with $\alpha_1 < \alpha_2, \beta_1 > \beta_2, \gamma_1 < \gamma_2$, and $0 \leq \alpha_1 + \beta_1 + \gamma_1 \leq 3, 0 \leq \alpha_2 + \beta_2 + \gamma_2 \leq 3$. Obviously, $S = A^{(\alpha_1,\beta_1,\gamma_1)}$. We now prove that A is a neutrosophic subalgebra of X. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ because S is a subalgebra of X. Hence $T_A(x) = T_A(y) = T_A(x * y) = \alpha_1$,

$I_A(x) = I_A(y) = I_A(x*y) = \beta_1$, $F_A(x) = F_A(y) = F_A(x*y) = \gamma_1$ and so $T_A(x*y) \le \max\{T_A(x), T_A(y)\}$, $I_A(x*y) \ge \min\{I_A(x), I_A(y)\}$, $F_A(x*y) \le \max\{F_A(x), F_A(y)\}$. If $x \in S$ and $y \notin S$, then $T_A(x) = \alpha_1, T_A(y) = \alpha_2$, $I_A(x) = \beta_1, I_A(y) = \beta_2$, $F_A(x) = \gamma_1, F_A(y) = \gamma_2$ and so $T_A(x*y) \le \max\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x*y) \ge \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x*y) \le \max\{F_A(x), F_A(y)\} = \gamma_2$. Obviously, if $x \notin A$ and $y \notin A$, then $T_A(x*y) \le \max\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x*y) \ge \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x*y) \le \max\{F_A(x), F_A(y)\} = \gamma_2$. Therefore A is a neutrosophic subalgebra of X. □

Definition 3.9. A neutrosophic set A in a BCC-algebra X is said to be *neutrosophic ideal* of X if it satisfies:

(NSI1) $T_A(0) \le T_A(x), I_A(0) \ge I_A(x)$, and $F_A(0) \le F_A(x)$ for any $x \in X$;

(NSI2) $T_A(x) \le \max\{T_A(x*y), T_A(y)\}, I_A(x) \ge \min\{I_A(x*y), I_A(y)\}$, and $F_A(x) \le \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$.

Proposition 3.10. *Every neutrosophic ideal of a BCC-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be a neutrosophic ideal of X. Put $x := x*y$ and $y := x$ in (NSI2). Then we have $T_A(x*y) \le \max\{T_A((x*y)*x), T_A(x)\}, I_A(x*y) \ge \min\{I_A((x*y)*x), I_A(x)\}$, and $F_A(x*y) \le \max\{F_A((x*y)*x), F_A(x)\}$. It follows from (b2) and (NSI1) that $T_A(x*y) \le \max\{T_A((x*y)*x), T_A(x)\} = \max\{T_A(0), T_A(x)\} \le \max\{T_A(x), T_A(y)\}, I_A(x*y) \ge \min\{I_A((x*y)*x), I_A(x)\} = \max\{I_A(0), I_A(x)\} \ge \max\{I_A(x), I_A(y)\}$, and $F_A(x*y) \le \max\{F_A((x*y)*x), F_A(x)\} = \max\{F_A(0), F_A(x)\} \le \max\{F_A(x), F_A(y)\}$. Thus A is a neutrosophic subalgebra of X. □

Theorem 3.11. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha, \beta, \gamma \in [0,1]$ with $0 \le \alpha+\beta+\gamma \le 3$. Then A is a neutrosophic ideal of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are ideals of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic ideal of X. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $0 \le \alpha+\beta+\gamma \le 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in X$ be such that $x*y, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x*y) \le \alpha, T_A(y) \le \alpha, I_A(x*y) \ge \beta, I_A(y) \ge \beta$, and $F_A(x*y) \le \gamma, F_A(y) \le \gamma$. By Definition 3.9, we have $T_A(0) \le T_A(x) \le \max\{T_A(x*y), T_A(y)\} \le \alpha, I_A(0) \ge I_A(x) \ge \min\{I_A(x*y)), I_A(y)\} \ge \beta$, and $F_A(0) \le F_A(x) \le \max\{F_A(x*y), T_A(y)\} \le \gamma$. Hence $0, x \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is an ideal of X.

Conversely, suppose that there exist $a, b, c \in X$ such that $T_A(0) > T_A(a), I_A(0) < I_A(b)$, and $F_A(0) > F_A(c)$. Then there exist $a_t, c_t \in [0,1)$ and $b_t \in (0,1]$ such that $T_A(0) > a_t \ge T_A(a), I_A(0) < b_t \le I_A(b)$ and $F_A(0) > c_t \ge F_A(c)$. Hence $0 \notin A^{(a_t,b_t,c_t)}$, which is a contradiction. Therefore $T_A(0) \le T_A(x), I_A(0) \ge I_A(x)$ and $F_A(0) \le F_A(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_A(a_t) > \max\{T_A(a_t*b_t), T_A(b_t)\}, I_A(a_i) < \min\{I_A(a_i*b_i), I_A(b_i)\}$, and $F_A(a_f) > \max\{T_A(a_f*b_f), T_A(b_f)\}$. Then there exist $s_t, s_f \in [0,1)$ and $s_i \in (0,1]$ such that $T_A(a_t) > s_t \ge \max\{T_A(a_t*b_t), T_A(b_t)\}, I_A(a_i) < s_i \le \min\{I_A(a_i*b_i), I_A(b_i)\}$, and $F_A(a_f) > s_f \ge \max\{T_A(a_f*b_f), T_A(b_f)\}$. Hence $a_t*b_t, b_t, a_i*b_i, a_f*b_f \in A^{(s_t,s_i,s_f)}$, and $b_t, b_i, b_f \in A^{(s_t,s_i,s_f)}$. But $a_t, a_i \notin A^{(s_t,s_i,s_f)}$ and $a_f \notin A^{(s_t,s_i,s_f)}$. This is a contradiction. Therefore $T_A(x) \le \max\{T_A(x*y), T_A(y)\}, I_A(x) \ge \min\{I_A(x*y)), I_A(y)\}$ and $F_A(x) \le \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic ideal of X □

Proposition 3.12. *Every neutrosophic ideal A of a BCC-algebra X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow T_A(x) \leq T_A(y), I_A(x) \geq I_A(y), F_A(x) \leq F_A(y))$,

(ii) $(\forall x, y, z \in X)(x*y \leq z \Rightarrow T_A(x) \leq \max\{T_A(y), T_A(z)\}, I_A(x) \geq \min\{I_A(y), I_A(z)\}, F_A(x) \leq \max\{F_A(y), F_A(z)\})$.

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. Using (NSI2) and (NSI1), we have $T_A(x) \leq \max\{T_A(x*y), T_A(y)\} = \max\{T_A(0), T_A(y)\} = T_A(y), I_A(y) \geq \min\{I_A(x*y), I_A(y)\} = \min\{I_A(0), I_A(y)\} = I_A(y)$, and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\} = \max\{F_A(0), F_A(y)\} = F_A(y)$.
(ii) Let $x, y, z \in X$ be such that $x * y \leq z$. By (NSI2) and (NSI1). we get $T_A(x*y) \leq \max\{T_A((x*y)*z), T_A(z)\} = \max\{T_A(0), T_A(z)\} = T_A(z), I_A(x*y) \geq \min\{I_A((x*y)*z), I_A(z)\} = \min\{I_A(0), I_A(z)\} = I_A(z)$, and $F_A(x*y) \leq \max\{F_A((x*y)*z), F_A(z)\} = \max\{F_A(0), F_A(z)\} = F_A(z)$. Hence $T_A(x) \leq \max\{T_A(x*y), T_A(y)\} \leq \max\{T_A(y), T_A(z)\}, I_A(x) \geq \min\{I_A(x*y), I_A(y)\} \geq \min\{I_A(y), I_A(z)\}$, and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\} \leq \max\{F_A(y), F_A(z)\}$. □

The following corollary is easily proved by induction.

Corollary 3.13. *Every neutrosophic ideal A of a BCC-algebra X satisfies the following property:*

(3.3) $(\cdots(x*a_1)*\cdots)*a_n = 0 \Rightarrow T_A(x) \leq \bigvee_{k=1}^n T_A(a_k), I_A(x) \geq \bigwedge_{k=1}^n I_A(a_k), F_A(x) \leq \bigvee_{k=1}^n F_A(a_k)$, *for all* $x, a_1, \cdots, a_n \in X$.

Definition 3.14. Let A and B be neutrosophic sets of a set X. The *union* of A and B is defined to be a neutrosophic set

$$A\tilde{\cup}B := \{\langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle | x \in X\},$$

where $T_{A\cup B}(x) = \min\{T_A(x), T_B(x)\}, I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}, F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$, for all $x \in X$. The *intersection* of A and B is defined to be a neutrosophic set

$$A\tilde{\cap}B := \{\langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle | x \in X\},$$

where $T_{A\cap B}(x) = \max\{T_A(x), T_B(x)\}, I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}, F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$, for all $x \in X$.

Theorem 3.15. *The intersection of two neutrosophic ideals of a BCC-algebra X is a also a neutrosophic ideal of X.*

Proof. Let A and B be neutrosophic ideals of X. For any $x \in X$, we have $T_{A\cap B}(0) = \max\{T_A(0), T_B(0)\} \leq \max\{T_A(x), T_B(x)\} = T_{A\cap B}(x), I_{A\cap B}(0) = \min\{T_A(0), T_B(0)\} \geq \min\{I_A(x), I_B(x)\} = I_{A\cap B}(x)$, and $F_{A\cap B}(0) = \max\{F_A(0), F_B(0)\} \leq \max\{F_A(x), F_B(x)\} = F_{A\cap B}(x)$. Let $x, y \in X$. Then we have

$$\begin{aligned} T_{A\cap B}(x) &= \max\{T_A(x), T_B(x)\} \\ &\leq \max\{\max\{T_A(x*y), T_A(y)\}, \max\{T_B(x*y), T_B(y)\}\} \\ &= \max\{\max\{T_A(x*y), T_B(x*y)\}, \max\{T_A(y), T_B(y)\}\} \\ &= \max\{T_{A\cap B}(x*y), T_{A\cap B}(y)\}, \end{aligned}$$

$$\begin{aligned} I_{A\cap B}(x) &= \min\{I_A(x), I_B(x)\} \\ &\geq \min\{\min\{I_A(x*y), I_A(y)\}, \min\{I_B(x*y), I_B(y)\}\} \\ &= \min\{\min\{I_A(x*y), I_B(x*y)\}, \min\{I_A(y), I_B(y)\}\} \\ &= \min\{I_{A\cap B}(x*y), I_{A\cap B}(y)\}, \end{aligned}$$

and

$$\begin{aligned} F_{A\cap B}(x) &= \max\{F_A(x), F_B(x)\} \\ &\leq \max\{\max\{F_A(x*y), F_A(y)\}, \max\{F_B(x*y), F_B(y)\}\} \\ &= \max\{\max\{F_A(x*y), F_B(x*y)\}, \max\{F_A(y), F_B(y)\}\} \\ &= \max\{F_{A\cap B}(x*y), F_{A\cap B}(y)\}. \end{aligned}$$

Hence $A\tilde{\cap}B$ is a neutrosophic ideal of X. □

Corollary 3.16. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosophic ideals of a BCC-algebra X, then so is $\tilde{\cap}_{i\in\mathbb{N}}A_i$.*

The union of any set of neutrosophic ideals of a BCC-algebra X need not be a neutrosophic ideal of X.

Example 3.17. Let $X = \{0,1,2,3,4\}$ be a BCC-algebra [5] with the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Define neutrosophic sets A and B of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.12, & \text{if } x \in \{0,1\} \\ 0.74 & \text{otherwise,} \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.63, & \text{if } x \in \{0,1\} \\ 0.11 & \text{otherwise,} \end{cases}$$

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.12, & \text{if } x \in \{0,1\} \\ 0.74 & \text{otherwise,} \end{cases}$$

$$T_B : X \to [0,1],\ x \mapsto \begin{cases} 0.13, & \text{if } x \in \{0,2\} \\ 0.63 & \text{otherwise,} \end{cases}$$

$$I_B : X \to [0,1],\ x \mapsto \begin{cases} 0.75, & \text{if } x \in \{0,2\} \\ 0.14 & \text{otherwise,} \end{cases}$$

and

$$F_B : X \to [0,1],\ x \mapsto \begin{cases} 0.13, & \text{if } x \in \{0,2\} \\ 0.63 & \text{otherwise.} \end{cases}$$

It is easy to check that A and B are neutrosophic ideals of X. But $A\tilde{\cup}B$ is not a neutrosophic ideal of X, since $T_{A\cup B}(3) = \min\{T_A(3), T_B(3)\} = 0.63 \not\leq \max\{T_{A\cup B}(3*2), T_{A\cup B}(2)\} = \max\{T_{A\cup B}(1), T_{A\cup B}(2)\} = \max\{\min\{T_A(1), T_B(1)\}, \min\{T_A(2), T_B(2)\}\} = \max\{0.12, 0.13\} = 0.13$.

Definition 3.18. A neutrosophic set A in a BCC-algebra X is said to be a *neutrosophic BCC-ideal* of X if it satisfies (NSI1) and

(NSI3) $T_A(x*z) \leq \max\{T_A((x*y)*z), T_A(y)\}, I_A(x*z) \geq \min\{I_A((x*y)*z), I_A(y)\}$, and $F_A(x*z) \leq \max\{F_A((x*y)*z), F_A(y)\}$, for any $x, y, z \in X$.

Lemma 3.19. *Every neutrosophic BCC-ideal of a BCC-algebra X is a neutrosophic ideal of X.*

Proof. Let A be a neutrosophic BCC-ideal of a BCC-algebra X. Put $z := 0$ in (NSI3). By (a3), we have $T_A(x*0) = T_A(x) \leq \max\{T_A((x*y)*0), T_A(y)\} = \max\{T_A(x*y), T_A(y)\}, I_A(x*0) = I_A(x) \geq \min\{I_A((x*y)*0), I_A(y)\} = \min\{I_A(x*y), I_A(y)\}$, and $F_A(x*0) = F_A(x) \leq \max\{F_A((x*y)*0), F_A(y)\} = \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$. Hence A is a neutrosophic ideal of X. □

Corollary 3.20. *Every neutrosophic BCC-ideal of a BCC-algebra X is a neutrosophic subalgebra of X.*

The converse of Proposition 3.10 and Lemma 3.19 need not be true in general (see Example 3.21).

Example 3.21. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra as in Example 3.17. Define a neutrosophic set A of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.13 & \text{if } x \in \{0,1,2,3\} \\ 0.83 & \text{if } x = 4, \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.82 & \text{if } x \in \{0,1,2,3\} \\ 0.11 & \text{if } x = 4, \end{cases}$$

and

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.13 & \text{if } x \in \{0,1,2,3\} \\ 0.83 & \text{if } x = 4, \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X, but not a neutrosophic ideal of X, since $T_A(4) = 0.83 \not\leq \max\{T_A(4*3), T_A(3)\} = \max\{T_A(3), T_A(3)\} = 0.13$. Consider a neutrosophic set B of X which is given by

$$T_B : X \to [0,1],\ x \mapsto \begin{cases} 0.14 & \text{if } x \in \{0,1\}, \\ 0.84 & \text{if } x \in \{2,3,4\} \end{cases}$$

$$I_B : X \to [0,1],\ x \mapsto \begin{cases} 0.85 & \text{if } x \in \{0,1\} \\ 0.12 & \text{if } x \in \{2,3,4\}, \end{cases}$$

and

$$F_B : X \to [0,1],\ x \mapsto \begin{cases} 0.14 & \text{if } x \in \{0,1\} \\ 0.84 & \text{if } x \in \{2,3,4\}. \end{cases}$$

It is easy to show that B is a neutrosophic ideal of X, but not a neutrosophic BCC-ideal of X, since $T_B(4*3) = T_B(3) = 0.84 \not\leq \max\{T_B((4*1)*3), T_B(1)\} = \max\{T_B(0), T_B(1)\} = 0.14$.

Example 3.22. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra [5] with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Define a neutrosophic set A of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{0,1,2,3,4\} \\ 0.55 & \text{if } x = 5, \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.54 & \text{if } x \in \{0,1,2,3\,4\} \\ 0.42 & \text{if } x = 5, \end{cases}$$

and

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{0,1,2,3,4\} \\ 0.55 & \text{if } x = 5. \end{cases}$$

It is easy to check that A is a neutrosophic BCC-ideal of X.

Theorem 3.23. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Then A is a neutrosophic BCC-ideal of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are BCC-ideals of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Similar to Theorem 3.11. □

Proposition 3.24. *Let A be a neutrosophic BCC-ideal of a BCC-algebra X. Then $X_T := \{x \in X | T_A(x) = T_A(0)\}, X_T := \{x \in X | I_A(x) = I_A(0)\}$, and $X_F := \{x \in X | F_A(x) = F_A(0)\}$ are BCC-ideals of X.*

Proof. Clearly, $0 \in X_T$. Let $(x * y) * z, y \in X_T$. Then $T_A((x * y) * z) = T_A(0)$ and $T_A(y) = T_A(0)$. It follows from (NSI3) that $T_A(x * z) \leq \max\{T_A((x * y) * z), T_A(y)\} = T_A(0)$. By (NSI1), we get $T_A(x * z) = T_A(0)$. Hence $x * z \in X_T$. Therefore X_T is a BCC-ideal of X. By a similar way, X_I and X_F are BCC-ideals of X. □

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Global Dynamics and Bifurcations of Two Second Order Difference Equations in Mathematical Biology

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Abstract. We investigate the global behavior of two difference equations with exponential nonlinearities

$$x_{n+1} = be^{-cx_n} + px_{n-1}, \quad n = 0, 1, \ldots$$

where the parameters b, c are positive real numbers and $p \in (0, 1)$ and

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots$$

where the parameters a, b are positive numbers. The the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. The two equations are well known mathematical models in biology which behavior was studied by other authors and resulted in partial global dynamics behavior. In this paper, we complete the results of other authors and give the global dynamics of both equations. In order to obtain our results we will prove several results on global attractivity and boundedness and unboundedness for general second order difference equations

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots$$

which are of interest on their own.

Keywords. attractivity, difference equation, invariant sets, period doubling, periodic solutions, stable set .

AMS 2010 Mathematics Subject Classification: 39A20, 39A28, 39A30, 92D25

1 Introduction and Preliminaries

We investigate the global behavior of the system of difference equations

$$x_{n+1} = be^{-cx_n} + py_n, \quad y_{n+1} = x_n, \quad n = 0, 1, \ldots \tag{1}$$

where the parameters b and c are positive real numbers, $p \in (0, 1)$, and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. This system can be rewritten in the form of the second order difference equation

$$x_{n+1} = be^{-cx_n} + px_{n-1}, \quad n = 0, 1, \ldots \tag{2}$$

In [5], the authors originally studied this model to describe the synchrony of ovulation cycles of the Glaucous-winged Gulls. The model assumed that there is an infinite breeding season as well as the number of gulls available to breed is infinite. The value of c is a positive number representing the colony density. The parameter b is the number of birds per day ready to begin ovulating. The parameter p is the probability that a bird will begin to ovulate and $1 - e^{-cx_n}$ is the probability of delaying ovulation. In making the model, the authors assumed that the delay only occurs for birds entering the system, not birds switching between different segments of the cycle. Note the authors state that the bifurcation of two-cycle solutions is the same as ovulation synchrony with the value of c increasing. In [5], they used the local bifurcation theory to come to the conclusion that there exists a unique equilibrium such that for sufficiently small values of c, the equilibrium branch is locally asymptotically stable. Additionally, for large enough values of c, there exists a two-cycle branch that will be locally asymptotically stable. In this paper we will improve these results by making them global. Using the results of Camouzis and Ladas, see [2] and [6], we are able to find the global dynamics of (1), which was not completed in [5]. We will show that Equation (1) exhibits global period doubling bifurcation described by Theorem 5.1 in [11], which shows that global dynamics of Equation (1) changes from global asymptotic stability of the unique equilibrium solution to the global asymptotic stability of the minimal period-two solution within its basin of attraction, as the parameter passes through the critical value.

By using a similar method, we investigate the dynamics of

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots \tag{3}$$

where the parameters a, b are positive real numbers and the the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. As it was mentioned in [8], Equation (3) could be considered as a mathematical model in biology where a represent the

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constant immigration and b represent the population growth rate. In this paper we find a simpler equivalent condition to $\frac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}}e^{\frac{a+\sqrt{a^2+4a}}{2}} < b$ in [8] for the existence of a minimal period-two solution. We split the results into the two cases of $b \geq e^a$ and $b < e^a$. While using a similar method as in [9] to establish the existence of a period-two solution when $b < e^a$, we are able to find the global dynamics of Equation (3). By using new results for general second order difference equation we will prove the existence of unbounded solutions for the case when $b \geq e^a$. Similar as for Equation (1) we will show that Equation (3) exhibits global period doubling bifurcation described by Theorem 5.1 in [11]. In addition, we give the precise description of the basins of attractions of all attractors of both Equations (1) and (3).
The rest of the paper is organized as follows. In the rest of this section we introduce some known results about monotone systems in the plane needed for the proofs of the main results as well as some new results about the existence of unbounded solutions. Section 2 gives the global dynamics of Equation (1) and Section 3 gives the global dynamics of Equation (3).

The next result, which is combination of two theorems from [2] and [6], is important for the global dynamics of general second order difference equation.

Theorem 1 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is either non-increasing in the first variable and non-decreasing in the second variable or non-decreasing in both variables. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, \ n = 0, 1, \ldots \tag{4}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

We now give some basic notions about monotone maps in the plane.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$ where $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

If T is differentiable map on a nonempty set $\mathcal{R}$, a sufficient condition for T to be strongly monotone with respect to the SE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \tag{5}$$

provided that $\mathcal{R}$ is open and convex.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction. *Basin of attraction* of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The next five results, from [12, 11], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [14, 13].

Theorem 2 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 4 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 3 *For the curve $\mathcal{C}$ of Theorem 2 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period-two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 2 reduces just to $|\lambda| < 1$. This follows from a change of variables [14] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 4 *(A) Assume the hypotheses of Theorem 2, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{6}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \mathrm{int}\, Q_2(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \mathrm{int}\, Q_4(\overline{\mathrm{x}})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 5 *In addition to the hypotheses of part (B) of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 2 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Remark 1 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (4) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (4) is a strictly competitive map on $I \times I$, see [11].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (4) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots.$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [12].

Remark 2 The characteristic equation of Equation (4) at an equilibrium point $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{7}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decraesing in first and increasing in second variable. Thus the applicability of Theorems 2-5 depends on the existence or nonexistence of minimal period-two solutions.

We now present theorems relating to the existence of unbounded solutions of Equation (4). The original result was obtained in [4]. Here we give an improved version of Theorem 2.1 in [4] taking out the extraneous conditions of requiring a continuity of f and the existence of an equilibrium solution. Additionally, we have extended the results in [4] to obtain a theorem in which the function f is nondecreasing in both arguments.

Theorem 6 *Assume that the function $f : I \times I \to I$ is nonincreasing in the the first variable and nondecreasing in the second variable, where $I \subset R$ is an interval. Assume there exists numbers $L, U \in I$ such that $L < U$ which satisfy*

$$f(U, L) \leq L \tag{8}$$

and

$$f(L, U) \geq U, \tag{9}$$

where at least one inequality is strict. If $x_{-1} \leq L$ and $x_0 \geq U$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_{2n-1} \leq L \quad \text{and} \quad x_{2n} \geq U, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) has no minimal period-two solution then,

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

Similarily, if $x_{-1} \geq U$ and $x_0 \leq L$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_{2n-1} \geq U \quad \text{and} \quad x_{2n} \leq L, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) has no minimal period-two solution then,

$$\lim_{n\to\infty} x_{2n-1} = \infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n} = -\infty.$$

Proof. Assume that $x_{-1} \leq L$ and $x_0 \geq U$. Then by using the monotonicity of f (nonincreasing in the first variable and nondecreasing in the second variable) and conditions (8) and (9) we obtain

$$x_1 = f(x_0, x_{-1}) \leq f(U, L) \leq L$$

and

$$x_2 = f(x_1, x_0) \geq f(L, U) \geq U.$$

By using induction it follows that $x_{2n-1} \leq L$ and $x_{2n} \geq U$ for all $n = 0, 1, \ldots$ where at least one inequality is strict. In view of Theorem 1 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. Assume that f is a continuous function and there is no minimal period-two solution. We will consider a few cases based on the properties of the interval I. First suppose there exist $a \in \mathbb{R}$ such that $I = [a, \infty)$ and $a < L$. Then $\{x_{2n-1}\}_{n=0}^{\infty}$ will be convergent as the subsequence is bounded in $[a, L]$. If $\{x_{2n}\}_{n=0}^{\infty}$ converges, this would create a contradiction as there would exist a minimal period-two solution. Therefore,

$$\lim_{n\to\infty} x_{2n} = \infty.$$

Next suppose that for some $b \in \mathbb{R}$, both $I = (-\infty, b]$ and $U < b$. Here $\{x_{2n}\}_{n=0}^{\infty}$ will be convergent as the subsequence is bounded in the interval of $[U, b]$. So $\{x_{2n-1}\}_{n=0}^{\infty}$ cannot converge as there is no minimal period-two solution resulting in

$$\lim_{n\to\infty} x_{2n-1} = -\infty.$$

If $I = (-\infty, \infty)$, then similar to the two cases above, at most one subsequence can converge as there is no minimal period-two solution. So either

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

with the option of both occurring. Finally, we will prove that I cannot be $I = [a, b]$ where $a, b \in \mathbb{R}$. Suppose that $I = [a, b]$ such that $a < L < U < b$ and $a, b \in \mathbb{R}$. Since $x_n \in [a, b]$ for all n, both subsequences would be convergent. As $\lim_{n\to\infty} x_{2n-1} = p < \lim_{n\to\infty} x_{2n} = q$ for some $p, q \in \mathbb{R}$, there exists a period-two solution, which is a contradiction. The case when $x_{-1} \geq U$ and $x_0 \leq L$ will follow similarly to the proof used here. $\square$

Many examples of the use of Theorem 6 are provided in [4].

Theorem 7 *Assume that $f : I \times I \to I$ is a function which is nondecreasing in both variables, where $I \subset R$ is an interval. Assume there exists numbers $L, U \in I$ such that $L < U$ where*

$$f(L, L) \leq L \tag{10}$$

and

$$f(U,U) \geq U \tag{11}$$

are satisfied, where at least one inequality is strict. If $x_{-1}, x_0 \leq L$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (4) *satisfies*

$$x_n \leq L, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) *has no minimal period-two solution, then either x_n converges to an equilibrium point or*

$$\lim_{n\to\infty} x_{2n-1} = -\infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n} = -\infty.$$

If $x_{-1}, x_0 \geq U$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_n \geq U, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) *has no period-two solution, then either x_n converges to an equilibrium point or*

$$\lim_{n\to\infty} x_{2n-1} = \infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n} = \infty.$$

Proof. Assume that $x_{-1}, x_0 \leq L$. Then by using the monotonicity of f (both variables are nondecreasing) and conditions (10) and (11) we obtain

$$x_1 = f(x_0, x_{-1}) \leq f(L, L) \leq L \quad \text{and} \quad x_2 = f(x_1, x_0) \leq f(L, L) \leq L.$$

By using induction it follows that $x_{2n-1}, x_{2n} \leq L$ for all $n = 0, 1, \ldots$ with at least one inequality being strict. In view of Theorem 1 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. We can assume that f is continuous and that there is no minimal period-two solution. We can choose the value of L such that at most one equilibrium is included in the region. Note the subsequences may converge to the equilibrium point if present. We will break this proof into cases for different intervals I assuming that the subsequences do not converge to an equilibrium point. First suppose that either $I = [a, \infty)$ or $I = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < L < U < b$. As both subsequences are less than L, then $x_n \in [a, L]$ for every n. As a consequence, both subsequences will be convergent. Thus, $\lim_{n\to\infty} x_{2n-1} = p$ and $\lim_{n\to\infty} x_{2n} = q$. If $p = q$, we get a contradiction as the subsequences do not converge to an equilibrium point. Otherwise, $p \neq q$, so (p, q) is a period-two solution, which is a contradiction as well. Thus, for $I = [a, \infty)$ or $I = [a, b]$, there must be an equilibrium point present. Next suppose that either $I = (-\infty, a]$ or $I = (-\infty, \infty)$. Now $x_n \in (-\infty, L]$ for all n. At least one subsequence must be decreasing as the subsequences do not converge to an equilibrium point. Furthermore since there is no period-two solution, the subsequences cannot be bounded below resulting in either

$$\lim_{n\to\infty} x_{2n} = -\infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

with the possibility of both options occurring.
Now assume that $x_{-1}, x_0 \geq U$. Then by using the monotonicity of f and conditions (10) and (11) we obtain

$$x_1 = f(x_0, x_{-1}) \geq f(U, U) \geq U$$

and

$$x_2 = f(x_1, x_0) \geq f(U, U) \geq U.$$

By using induction it follows that $x_{2n-1}, x_{2n} \geq U$ for all $n = 0, 1, \ldots$ with at least one inequality being strict. In view of Theorem 1 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. Assume that f is continuous and that there is no minimal period-two solution. We can choose the value of U such that at most one equilibrium is included in the region. Note the subsequences may converge to the equilibrium point if present. We will break this proof into cases for different intervals I assuming that the subsequences do not converge to an equilibrium point. First suppose that either $I = (-\infty, b]$ or $I = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < L < U < b$. As both subsequences are greater than U, then $x_n \in [U, b]$ for every n. As a consequence, both subsequences will be convergent. Thus, $\lim_{n\to\infty} x_{2n-1} = p$ and $\lim_{n\to\infty} x_{2n} = q$. If $p = q$, we get a contradiction as the subsequences do not converge to an equilibrium point. Otherwise, $p \neq q$, so (p, q) is a period-two solution, which is a contradiction as well. Thus, for $I = (-\infty, b]$ or $I = [a, b]$, there must be an equilibrium point present. Next suppose that either $I = [a, \infty]$ or $I = (-\infty, \infty)$. Now $x_n \in [U, \infty)$ for all n. At least one subsequence must be increasing as the subsequences do not converge to an equilibrium point. Furthermore since there is no period-two solution, the subsequences cannot be bounded above resulting in either

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = \infty.$$

with the option of both occurring. □

Now we give few examples which illustrate all possible scenarios of Theorem 7.

Example 1 Consider the difference equation

$$x_{n+1} = x_n^2 + x_{n-1}^2, \quad n = 1, 2, \ldots$$

where $x_{-1}, x_0 \in \mathbb{R}^+$, and $x_n \geq 0$ for $n = 1, 2, \ldots$. Here $f(u, v) = u^2 + v^2$ is increasing in both variables. The equilibrium points are $\overline{x}_0 = 0$ and $\overline{x}_+ = 1/2$. The linearized difference equation is $z_{n+1} = 2\overline{x}z_n + 2\overline{x}z_{n-1}$ and the characteristic equation is $\lambda^2 = 2\overline{x}\lambda + 2\overline{x}$. The zero equilibrium $\overline{x}_0$ is locally asymptotically stable. For the equilibrium point $\overline{x}_+$, $\lambda^2 = \lambda + 1$, so that $\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. As $\frac{1+\sqrt{5}}{2} > 1$ and $\frac{1-\sqrt{5}}{2} \in (-1, 0)$, then $\overline{x}_+$ is a saddle point. There is no minimal period-two solution as

$$\phi = \psi^2 + \phi^2 \quad \text{and} \quad \psi = \phi^2 + \psi^2$$

implies $\phi = \psi$. Now we want to find a $L < U$ that satisfies the conditions (10) and (11). Condition (10) $f(L, L) \leq L$ implies $2L^2 \leq L$, which simplifies to $L \leq 1/2$. As well, $f(U, U) \geq U$ if $2U^2 \geq U$, which simplifies to $U \geq 1/2$. We can choose at least one of these inequalities to be strict. From Theorem 7, we can conclude that every solution with $x_1, x_0 \leq L$ converges to 0, while every solution with $x_{-1}, x_0 \geq U$ is eventually increasing and tends toward ∞. As $L < 1/2 < U$ are arbitrary this conclusion holds for every case where $x_{-1}, x_0 \leq L$ or $x_{-1}, x_0 \geq U$. These results do not give conclusions when $x_{-1} \leq L$ and $x_0 \geq U$ or $x_{-1} \geq U$ and $x_0 \leq L$. In this case one may use theory of monotone maps as in [3].

Example 2 Consider the difference equation

$$x_{n+1} = x_n^2 + x_{n-1}^2 + a, \quad n = 1, 2, \ldots$$

where $a > 1/8$, $x_n \geq 0$, and $x_{-1}, x_0 \in \mathbb{R}$. Here $f(u, v) = u^2 + v^2 + a$ is increasing in both variables. There is no equilibrium points as the discriminant of the equilibrium equation $1 - 8a < 0$ and no minimal period-two solution exists as

$$\phi = \psi^2 + \phi^2 + a \quad \text{and} \quad \psi = \phi^2 + \psi^2 + a$$

implies $\phi = \psi$. We can find U that satisfies the conditions (10) and (11) of Theorem 7. As $f(U, U) \geq U$ simplifies to $2U^2 + a \geq U$, which always holds, every solution will be eventually increasing and tends to ∞.

Example 3 Consider the difference equation

$$x_{n+1} = x_n^5 + x_{n-1}^5, \quad n = 1, 2, \ldots$$

where $x_{-1}, x_0 \in \mathbb{R}$. The function $f(u, v) = u^5 + v^5$ is increasing in both variables. The equilibrium points are $\overline{x}_0 = 0$ and $\overline{x}_\pm = \pm 1/\sqrt[4]{2}$. The characteristic equation at the equilibrium solution $\bar{x}$ is $\lambda^2 = 5\overline{x}^4\lambda + 5\overline{x}^4$. For the equilibrium point $\overline{x}_0$, $\lambda^2 = 0$ so that $\lambda_{1,2} = 0$ and $\overline{x}_0$ is locally asymptotically stable. For the equilibrium point $\overline{x}_\pm$, $\lambda^2 = 5/2\lambda + 5/2$, so that $\lambda_{1,2} = \frac{5 \pm \sqrt{65}}{4}$. As $\frac{5+\sqrt{65}}{4} > 1$ and $\frac{5-\sqrt{65}}{4} \in (-1, 0)$, then the equilibrium points $\overline{x}_\pm$ are saddle points. There is no minimal period-two solution as

$$\phi = \psi^5 + \phi^5 \quad \text{and} \quad \psi = \phi^5 + \psi^5$$

implies $\phi = \psi$.

Now we want to find $L < U$ that satisfies the conditions of Theorem 7. Clearly $f(L, L) \leq L$ if $2L^5 \leq L$, which simplifies to $L \leq 1/\sqrt[4]{2}$ if $L > 0$ and to $L \leq -1/\sqrt[4]{2}$ if $L < 0$. As well, $f(U, U) \geq U$ if $2U^5 \geq U$, which simplifies to $U \geq 1/\sqrt[4]{2}$. We can choose at least one of these inequalities to be strict. From Theorem 7, we can conclude that every solution with $x_1, x_0 \leq L, L > 0$ converges to 0, while every solution with $x_{-1}, x_0 \geq U$ is eventually increasing and tends toward ∞. As $L < 1/\sqrt[4]{2} < U$ are arbitrary we conclude that

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{when} \quad \bar{x}_- < x_{-1}, x_0 < \bar{x}_+, \\ \infty & \text{when} \quad x_{-1}, x_0 > \bar{x}_+, \\ -\infty & \text{when} \quad x_{-1}, x_0 < \bar{x}_-. \end{cases}$$

Theorem 7 does not apply when $x_{-1} \leq L$ and $x_0 \geq U$ or $x_{-1} \geq U$ and $x_0 \leq L$. In this cases one can use the results from [3].

Example 4 Consider the difference equation

$$x_{n+1} = \frac{ax_n^2}{1 + x_n^2} + \frac{bx_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 1, 2, \ldots$$

where $a, b > 0$ and $x_{-1}, x_0 \in \mathbb{R}$. The function $f(u, v) = \frac{au^2}{1+u^2} + \frac{bv^2}{1+v^2}$ is increasing in both variables. One equilibrium point is $\overline{x}_0 = 0$. The non-zero equilibrium point satisfies the quadratic equation $1 + \overline{x}^2 - (a + b)\overline{x} = 0$ which has real solutions if $(a + b)^2 - 4 \geq 0$. If $a + b < 2$, then there only exist $\overline{x}_0$, if $a + b = 2$, then there exists $\overline{x}_0$ and $\overline{x}$, and if $a + b > 2$, then there exist three equilibrium points $\overline{x}_0 < \overline{x}_- < \overline{x}_+$. The characteristic equation at the equilibrium solution $\bar{x}$ is $\lambda^2 = \frac{2a\overline{x}}{(1+\bar{x}^2)^2}\lambda + \frac{2b\overline{x}}{(1+\bar{x}^2)^2}$. For the equilibrium point $\overline{x}_0$, $\lambda^2 = 0$ so that $\lambda_{1,2} = 0$ and thus, $\overline{x}_0$ is locally asymptotically stable. The conditions for local stability of the equilibrium points $\bar{x}_\pm$ are quite involved and can be found in [1]. In particular $\overline{x}_-$ will either be a saddle point, repeller, or non-hyperbolic depending on whether $2a(a+b)+(a-b)\sqrt{(a+b)^2-4}$ is greater than, less than, or equal to 0, and the equilibrium point $\overline{x}_+$ is either locally asymptotically stable or non-hyperbolic when it exists.

Now we want to find a $L < U$ that satisfies the conditions (10) and (11) of Theorem 7. First $f(L,L) \le L$ if $\frac{(a+b)L^2}{1+L^2} \le L$, which simplifies to $0 \le 1 + L^2 - (a+b)L$. This will occur when $L < L_-$ or $L > L_+$ where we can set $L_- = \overline{x}_-$ and $L_+ = \overline{x}_+$. As well, $f(U,U) \ge U$ if $\frac{(a+b)U^2}{1+U^2} \ge U$, which simplifies to $0 \ge 1 + U^2 - (a+b)U$. This occurs when $U_- < U < U_+$ where we can set $U_- = \overline{x}_-$ and $U_+ = \overline{x}_+$. For both L and U to exist, we need $L < L_-$ to satisfy $L < U$. From Theorem 7, we can conclude that every solution with $x_1, x_0 \le L$ converges to 0, while every solution with $x_{-1}, x_0 \ge U$ converges to $\overline{x}_+$. Note that in the region where L and U exist, no minimal period-two solutions exists. All the period-two solutions are located in the region which is the union of the second and the fourth quadrant with respect to $\overline{x}_-$.

2 Global Dynamics of Equation (1)

In this section we present the global dynamics of Equation (1).

2.1 Local stability results

We begin by observing that the function $f(u,v) = be^{-cu} + pv$ is decreasing in the first variable and increasing in the second variable and so by Theorem 1, for every solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic.

Equation (1) has a unique positive equilibrium point $\overline{x}e^{c\overline{x}} = \frac{b}{1-p}$ where $0 < \overline{x} < \frac{b}{1-p}$. Note that $\frac{\partial f}{\partial u}(\overline{x},\overline{x}) = -cbe^{-c\overline{x}} = -c(1-p)\overline{x}$ and $\frac{\partial f}{\partial v}(\overline{x},\overline{x}) = p$. The characteristic equation of Equation (1) is

$$\lambda^2 + (1-p)c\overline{x}\lambda - p = 0.$$

Applying local stability test [10] we obtain

Lemma 1 *Equation* (1) *has a unique positive equilibrium solution* $\overline{x}e^{c\overline{x}} = \frac{b}{1-p}$.
i) If $\overline{x} < \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is locally asymptotically stable.*
ii) If $\overline{x} > \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is a saddle point.*
iii) If $\overline{x} = \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is non-hyperbolic of the stable type (with eigenvalues* $\lambda_1 = -1$ *and* $\lambda_2 = p$*).*

Proof.
i) Equilibrium point $\overline{x}$ is locally asymptotically stable if

$$|(1-p)c\overline{x}| < 1 - p < 2.$$

As $p \in (0,1)$ then $1 - p < 2$ holds. As $(1-p)c\overline{x} > 0$, then $\overline{x}$ is stable if

$$(1-p)c\overline{x} < 1 - p \Leftrightarrow c\overline{x} < 1 \Leftrightarrow \overline{x} < \frac{1}{c}.$$

Therefore, the equilibrium $\overline{x}$ is locally asymptotically stable if $\overline{x} < \frac{1}{c}$
ii) If $|(1-p)c\overline{x}| > |1-p|$, then the equilibrium point $\overline{x}$ is a saddle point. As $(1-p)c\overline{x}$ is positive, we obtain

$$(1-p)c\overline{x} > 1 - p \Leftrightarrow c\overline{x} > 1 \Leftrightarrow \overline{x} > \frac{1}{c}.$$

So the equilibrium point $\overline{x}$ is a saddle point if $\overline{x} > \frac{1}{c}$.
iii) The equilibrium point $\overline{x}$ is non-hyperbolic if

$$|(1-p)c\overline{x}| = |1-p|.$$

We see that $c\overline{x} = 1 \Leftrightarrow \overline{x} = \frac{1}{c}$. The characteristic equation at the equilibrium becomes

$$\lambda^2 + (1-p)\lambda - p = 0,$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = p$. □

2.2 Periodic solutions

In this section we present results about existence and uniqueness of the minimal period-two solution of Equation (1).

Theorem 8 *If $\overline{x} > \frac{1}{c}$, then Equation* (1) *has a unique minimal period-two solution:*

$$\phi, \psi, \phi, \psi, \ldots (\phi \neq \psi, \phi > 0 \text{ and } \psi > 0).$$

Proof. Let $\{\phi, \psi\}$ be a minimal period-two solution of Equation (1), where ϕ and ψ are distinct positive real numbers. Then we have

$$\phi = be^{-c\psi} + p\phi, \quad \psi = be^{-c\phi} + p\psi, \tag{12}$$

where $\phi \neq \psi$. This implies

$$\psi = \frac{be^{-c\phi}}{1-p}, \quad \phi = be^{\frac{-cbe^{-c\phi}}{1-p}} + p\phi.$$

Let $F(\phi) = be^{\frac{-cbe^{-c\phi}}{1-p}} + (p-1)\phi$. The equilibrium point $\overline{x} = \frac{b}{1-p}e^{-c\overline{x}}$ will be a zero of F as

$$F(\overline{x}) = be^{\frac{-cbe^{-c\overline{x}}}{1-p}} + (p-1)\overline{x} = be^{-c\overline{x}} + (p-1)\overline{x} = 0.$$

Note that $F(0) = be^{\frac{-cb}{1-p}} > 0$ since $b > 0$. Additionally, as ϕ approaches ∞, then $F(\phi)$ approaches $-\infty$. Notice graphically, the the function F begins above the x-axis and ends approaching $-\infty$. As the function F crosses the x-axis at least once at $\overline{x}$, then F must cross the x-axis at least three times when $F'(\overline{x}) > 0$. This will result in the existence of a minimal period-two solution. We want to prove that $F'(\overline{x}) > 0$ holds true for some values of parameters. Observe that the derivative of F is

$$F'(\phi) = \frac{b^2c^2}{1-p}e^{-c\phi}e^{\frac{-cbe^{-c\phi}}{1-p}} + (p-1)$$

so that when $\overline{x}$ is substituted $F'(\overline{x}) = \overline{x}bc^2e^{-c\overline{x}} + (p-1)$. Then $F'(\overline{x}) > 0$ when $\overline{x} > \frac{1}{c}$ as

$$F'(\overline{x}) = \overline{x}bc^2e^{-c\overline{x}} + (p-1) > 0 \Leftrightarrow c^2\overline{x} > \frac{1-p}{b}e^{c\overline{x}} \Leftrightarrow c^2\overline{x} > \frac{1}{\overline{x}} \Leftrightarrow \overline{x} > \frac{1}{c}.$$

Thus when $\overline{x} > \frac{1}{c}$, there will be a minimal period-two solution.

Next we want to prove that the period-two solution is unique. Rewritting (12) we obtain

$$\phi e^{c\psi} = \frac{b}{1-p} = \psi e^{c\phi} \Leftrightarrow \phi e^{-c\phi} = \psi e^{-c\psi}.$$

Let $g(x) = xe^{-cx}$. As $g'(x) = e^{-cx}(1-cx)$, then the global maximum of g is attatined at $x = \frac{1}{c}$. For each y value there will be two corresponding x values when $g(x) < g(\frac{1}{c}) = \frac{1}{ce}$. This will happen when

$$xe^{-cx} < \frac{1}{ce} \Leftrightarrow e^{cx} - ecx > 0.$$

Let $G(x) = e^{cx} - ecx$ and notice that $G(0) = 1$. The derivative of G will be $G'(x) = c(e^{cx} - e)$. Notice $G'(x) \leq 0$ when $e^{cx} \leq e$ such that $x \leq \frac{1}{c}$, and $G'(x) > 0$ when $x > \frac{1}{c}$. Thus, $G(x) > 0$ on $[0, \frac{1}{c}) \cup (\frac{1}{c}, \infty)$ where $G(\frac{1}{c}) = 0$ is a global minimum. Thus when the period-two solution exists, it is unique. □

2.3 Global stability results

In view of Theorem 1 every bounded solution of Equation (1) converges to either an equilibrium solution or a minimal period-two solution.

Lemma 2 *The solutions of Equation* (1) *are bounded.*

Proof. Equation (1) implies

$$x_{n+1} = be^{-cx_n} + px_{n-1} \leq b + px_{n-1}, \quad n = 0, 1, \ldots.$$

Consider the difference equation of

$$u_{n+1} = b + pu_{n-1}, \quad n = 0, 1, \ldots. \tag{13}$$

The solution of Equation (13) is $u_n = \frac{b}{1-p} + C_1(\sqrt{p})^n + C_2(-\sqrt{p})^n$. As $n \to \infty$, then $u_n \to \frac{b}{1-p}$. In view of difference inequality result, see [7] $x_n \leq u_n \leq \frac{b}{1-p} + \epsilon = \mathcal{U}$ for $n = 0, 1, \ldots$ and some $+\epsilon > 0$ when $x_0 \leq u_0$. □

Theorem 9 *(i) If $\bar{x} > \frac{1}{c}$, then the equilibrium solution $\bar{x}$ is a saddle point and the minimal period-two solution $\{\phi, \psi\}, \phi < \psi$ is globally asymptotically stable within the basin of attraction $\mathcal{B}(\phi, \psi) = [0, \infty)^2 \setminus \mathcal{W}^s(\bar{x}, \bar{x})$, where $\mathcal{W}^s(\bar{x}, \bar{x})$ is the global stable manifold of $(\bar{x}, \bar{x})$.*
(ii) If $\bar{x} \le \frac{1}{c}$, then the equilibrium solution $\bar{x}$ is globally asymptotically stable.

Proof. Using Theorem 1 every bounded solution of Equation (1) converges to an equilibrium solution or period-two solution. By Lemma 2, every solution of Equation (1) is bounded so that all solutions converge to either an equilibrium solution or to the unique period-two solution $\{\phi, \psi\}, \phi < \psi$. When $\overline{x} > \frac{1}{c}$, then $\overline{x}$ is a saddle point, by Lemma 1 part (ii), and has the global stable $\mathcal{W}^s(\overline{x}, \overline{x})$ and global unstable manifolds $\mathcal{W}^u(\overline{x}, \overline{x})$, where $\mathcal{W}^s(\overline{x}, \overline{x})$ is the graph of a non-decreasing function and $\mathcal{W}^u(\overline{x}, \overline{x})$ is the graph of a non-increasing function, which has endpoints at (ϕ, ψ) and (ψ, ϕ). Every initial point (x_{-1}, x_0) which starts south east of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ψ, ϕ), while every initial point (x_{-1}, x_0) which starts north west of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ϕ, ψ), see Theorems 2, 4. In this case in view of Theorem 1 global attractivity of period-two solution implies its local stability since the convergence is monotonic.

When $\overline{x} \le \frac{1}{c}$, the equilibrium solution is locally and so globally asymptotically stable by Lemma 1 part (i) and part (iii) . □

Remark 3 For instance, case $i)$ of Theorem 9 holds when $b = 1, p = .5, c = 2$, case $ii)$ holds when $b = 1, p = .5, c = 1$ and when $b = 1, p = (e-1)/e, c = 1$.

3 Global Dynamics of Equation (3)

In this section we present global dynamics of Equation (3).

3.1 Local stability results

First, notice that the function $f(u, v) = a + bve^{-u}$ is decreasing in the first variable and increasing in the second variable. By Theorem 1, for all solutions $\{x_n\}_{n=-1}^{\infty}$ of Equation (3) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic.

Equation (3) has a unique positive equilibrium point $\overline{x} = \frac{a}{1-be^{-\overline{x}}}$ where $a < \overline{x}$. Note that $\frac{\partial f}{\partial u}(\overline{x}, \overline{x}) = -b\overline{x}e^{-\overline{x}}$ and $\frac{\partial f}{\partial v}(\overline{x}, \overline{x}) = be^{-\overline{x}}$. The characteristic equation of Equation (3) is

$$\lambda^2 + b\overline{x}e^{-\overline{x}}\lambda - be^{-\overline{x}} = 0.$$

Lemma 3 *Equation (3) has a unique positive equilibrium solution $\overline{x} = \frac{a}{1-be^{-\overline{x}}}$.*

i) If $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution $\overline{x}$ is locally asymptotically stable.

ii) If $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution $\overline{x}$ is a saddle point.

iii) If $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution $\overline{x}$ is non-hyperbolic of stable type (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = be^{-\overline{x}}$).

Proof.
i) The equilibrium point $\overline{x}$ is locally asymptotically stable if

$$\left|b\overline{x}e^{-\overline{x}}\right| < 1 - be^{-\overline{x}} < 2.$$

As $be^{-\overline{x}} > 0$, then $1 - be^{-\overline{x}} < 2$ holds true. So rearranging the other inequality we obtain

$$b\overline{x}e^{-\overline{x}} < 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) < 1 \Leftrightarrow \overline{x} + 1 < \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} < \frac{e^{\overline{x}}}{b} - 1.$$

Therefore, the equilibrium $\overline{x}$ is locally asymptotically stable if $\overline{x} < \frac{e^{\overline{x}}}{b} - 1$. As $\overline{x} = a + b\overline{x}e^{-\overline{x}}$ we have

$$e^{\overline{x}} = \frac{b\overline{x}}{\overline{x} - a}. \tag{14}$$

Then we can equivalently write the condition to be locally asymptotically stable as

$$\begin{aligned} \overline{x} < \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} < \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} < \frac{\overline{x}}{\overline{x}-a} - 1 \\ \Leftrightarrow \overline{x}^2 - \overline{x}a - a < 0 \Leftrightarrow \overline{x} < \frac{a+\sqrt{a^2+4a}}{2}. \end{aligned}$$

ii) If

$$\left|b\overline{x}e^{-\overline{x}}\right| > \left|1 - be^{-\overline{x}}\right|,$$

then the equilibrium solution $\overline{x}$ is a saddle point. Note that $be^{-\overline{x}} < 1$ since

$$be^{-\overline{x}} < 1 \Leftrightarrow \frac{\overline{x}-a}{\overline{x}} < 1 \Leftrightarrow \frac{-a}{\overline{x}} < 0$$

always holds as $a > 0$. The condition for $\overline{x}$ to be a saddle point yields

$$b\overline{x}e^{-\overline{x}} > 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) > 1 \Leftrightarrow \overline{x}+1 > \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} > \frac{e^{\overline{x}}}{b} - 1.$$

So the equilibrium point $\overline{x}$ is a saddle point if $\overline{x} > \frac{e^{\overline{x}}}{b} - 1$. By using (14), the inequality can then equivalently be written as

$$\overline{x} > \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} > \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} > \frac{\overline{x}}{\overline{x}-a} - 1 \Leftrightarrow \overline{x}^2 - \overline{x}a - a > 0 \Leftrightarrow \overline{x} > \frac{a+\sqrt{a^2+4a}}{2}.$$

iii) The equilibrium point $\overline{x}$ is non-hyperbolic point if

$$\left|b\overline{x}e^{-\overline{x}}\right| = \left|1 - be^{-\overline{x}}\right|.$$

We see that

$$b\overline{x}e^{-\overline{x}} = 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) = 1 \Leftrightarrow \overline{x}+1 = \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} = \frac{e^{\overline{x}}}{b} - 1.$$

In view of (14) this can be rewritten as

$$\overline{x} = \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} = \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} = \frac{\overline{x}}{\overline{x}-a} - 1 \Leftrightarrow \overline{x}^2 - \overline{x}a - a = 0 \Leftrightarrow \overline{x} = \frac{a+\sqrt{a^2+4a}}{2}.$$

The characteristic equation at the equilibrium point will become

$$\lambda^2 + (1 - be^{-\overline{x}})\lambda - be^{-\overline{x}} = 0,$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = be^{-\overline{x}} \in (0,1)$. $\square$

3.2 Periodic solutions

In this section we present results about existence and uniqueness of minimal period-two solutions of Equation (3).

Theorem 10 *Assume that $b < e^a$. If $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then Equation (3) has minimal period-two solution:*

$$\phi, \psi, \phi, \psi, \ldots \quad (\phi \neq \psi \text{ and } \phi > 0, \psi > 0).$$

Proof. We want to find for which values of $\overline{x}$ there exists a minimal period-two solution (ϕ, ψ) where ϕ and ψ are distinct positive real numbers. A period-two solution satisfies

$$\phi = a + b\phi e^{-\psi}, \quad \psi = a + b\psi e^{-\phi}, \tag{15}$$

where ϕ and ψ are distinct real numbers. Rewritting ψ and then substituting into ϕ we obtain

$$\psi = \frac{a}{1 - be^{-\phi}}, \quad \phi = a + b\phi e^{-\frac{a}{1-be^{-\phi}}}. \tag{16}$$

Let $F(\phi) = a + \phi(be^{-\frac{a}{1-be^{-\phi}}} - 1)$. The equilibrium point $\overline{x} = \frac{e^{\overline{x}}(\overline{x}-a)}{b}$ will be a zero of F as

$$F(\overline{x}) = a + \overline{x}(be^{-\frac{a}{1-be^{-\overline{x}}}} - 1) = a + \overline{x}(be^{-\overline{x}} - 1) = 0.$$

Now

$$F(a) = a + a(be^{-\frac{a}{1-be^{-a}}} - 1) = abe^{-\frac{a}{1-be^{-a}}}$$

is positive as a and b are positive constants. As ϕ approaches ∞, then F approaches $-\infty$ assuming that $b < e^a$. When $F'(\overline{x}) > 0$ then F will cross the $x-$axis at least three times resulting in a minimal period-two solution. Thus, we want to prove when $F'(\overline{x}) > 0$ holds. Taking the derivative of F we have

$$F'(\phi) = (be^{-\frac{a}{1-be^{-\phi}}} - 1) + \frac{\phi ab^2 e^{-\phi}}{(1-be^{-\phi})^2} e^{-\frac{a}{1-be^{-\phi}}}$$

so that $F'(\overline{x}) = \frac{-a}{\overline{x}} + \frac{\overline{x}^3 b^2 e^{-2\overline{x}}}{a}$. Then $F'(\overline{x}) > 0$ hold true when

$$\frac{-a}{\overline{x}} + \frac{\overline{x}^3 b^2 e^{-2\overline{x}}}{a} > 0 \Leftrightarrow \overline{x}^4 b^2 e^{-2\overline{x}} > a^2 \Leftrightarrow \overline{x}^2 b e^{-\overline{x}} > a \Leftrightarrow \overline{x}^2 b > \frac{a\overline{x}b}{\overline{x}-a} \Leftrightarrow \overline{x}(\overline{x}-a) > a\overline{x}^2 - \overline{x}a - a > 0.$$

Thus, when $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, there will be a minimal period-two solution.

When $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then

$$\frac{a}{1-be^{-\overline{x}}} > \frac{a+\sqrt{a^2+4a}}{2} \Leftrightarrow \frac{2a}{a+\sqrt{a^2+4a}} > 1 - be^{-\overline{x}}$$

$$\Leftrightarrow \frac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}} e^{\overline{x}} < b \Leftrightarrow \frac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}} e^{\frac{a+\sqrt{a^2+4a}}{2}} < b.$$

Next we want to prove that the minimal period-two solution is unique. By rewriting (15) we find that

$$\phi(1-be^{-\psi}) = a = \psi(1-be^{-\phi}) \Leftrightarrow \frac{\phi}{1-be^{-\phi}} = \frac{\psi}{1-be^{-\psi}}.$$

Let $g(x) = \frac{x}{1-be^{-x}}$. Using $g'(x) = \frac{1-be^{-x}(x+1)}{(1-be^{-x})^2}$ to find the critical points we get that $1 - be^{-x}(x+1) = 0 \Leftrightarrow e^x = b(x+1)$. There exists a unique value of m where $\frac{1}{m+1} = be^{-m}$ for which this holds. Using the first-derivative theorem we can check that m is a local minima. Note it suffices to check the numerator of $g'(m-1)$ as the denominator is always positive. Using the fact that $\frac{1}{m+1} = be^{-m}$

$$1 - be^{-(m-1)}m < 0 \Leftrightarrow \frac{1}{m} < be^{-(m-1)} \Leftrightarrow \frac{1}{m} < \frac{e}{m+1} \Leftrightarrow \frac{m+1}{m} < e.$$

This proves that $g'(m-1) < 0$. Next using the same method taking the numerator of $g'(m+1)$ we see that

$$1 - be^{-(m+1)}(m+2) > 0 \Leftrightarrow \frac{1}{m+2} > be^{-(m+1)} \Leftrightarrow \frac{1}{m+2} > \frac{e^{-1}}{m+1} \Leftrightarrow \frac{m+1}{m+2} > e^{-1}.$$

This proves that $g'(m+1) > 0$. As the derivative changes from negative to positive around the critical point, it will be a local minima. Note that $g(a) > 0$ and as x approaches ∞, $g(x)$ approaches ∞. Since m is the only critical point, each y value will have two x values with the exception at m. This results in the fact that there can only be one period-two solution. □

Proposition 1 *If $b \geq e^a$, there are no minimal period-two solutions.*

Proof. Assume that $\{\phi, \psi\}$ is a period-two solution. Then $\{\phi, \psi\}$ satisfies (15) and so it satisfies (16) as well.

Let $F(\phi) = a + \phi(be^{-\frac{a}{1-be^{-\phi}}} - 1)$. The equilibrium point $\overline{x} = \frac{e^{\overline{x}}(\overline{x}-a)}{b}$ will be a zero of F as

$$F(\overline{x}) = a + \overline{x}(be^{-\frac{a}{1-be^{-\overline{x}}}} - 1) = a + \overline{x}(be^{-\overline{x}} - 1) = 0.$$

We see that

$$F(a) = a + a(be^{-\frac{a}{1-be^{-a}}} - 1) = abe^{-\frac{a}{1-be^{-a}}}$$

which is a positive value as a and b are positive constants. As ϕ approaches ∞, then F approaches ∞ as $b \geq e^a$. As the function begins above the x-axis at a and approaches ∞, F will cross the x-axis an even number of times. Since $F(\overline{x}) = 0$ is one of the points that lie on the x-axis and the only equilibrium point, there cannot be a minimal period-two solution. □

3.3 Global stability results

By Theorem 1 every bounded solution of Equation (1) converges to either an equilibrium solution or a minimal period-two solution.

Lemma 4 *The solutions of Equation* (3) *are bounded if* $b < e^a$.

Proof. By Equation (3),

$$x_{n+1} = a + bx_{n-1}e^{-x_n} \leq a + bx_{n-1}, \quad n = 0, 1, ...$$

Consider the difference equation of

$$u_{n+1} = a + bu_{n-1}, \quad n = 0, 1, ... \tag{17}$$

Suppose that $b < e^a$. The solution of Equation (17) is $u_n = \frac{a}{1-b} + C_1(\sqrt{b})^n + C_2(-\sqrt{b})^n$. As $n \to \infty$, then $u_n \to \frac{a}{1-b}$. In view of difference inequality result, see [7] $x_n \leq u_n \leq \frac{a}{1-b} + \epsilon = \mathcal{U}$ for $n = 0, 1, ...$ when $x_0 \leq u_0$, where $\epsilon > 0$. □

Theorem 11 *Consider Equation* (3).

(i) If $b < e^a$ and $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then there exists a period-two solution that is locally asymptotically stable and the equilibrium point, $\overline{x}$, that is is a saddle point. The unique period-two solution attracts all solutions which start off the global stable manifold of $\mathcal{W}^s(E(\overline{x}, \overline{x}))$.

(ii) If $b < e^a$ and $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution, $\overline{x}$, is globally asymptotically stable.

(iii) If $b < e^a$ and $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution, $\overline{x}$, is non-hyperbolic of the stable type and is global attractor.

Proof.

(i) Using Theorem 1 every bounded solution of Equation (3) converges to an equilibrium solution or period-two solution. By Lemma 4, when $b < e^a$ every solution of Equation (3) is bounded such that all solutions will converge to either an equilibrium solution or period-two solution. If $b < e^a$ and $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then $\overline{x}$ will be a saddle point by Lemma 3 part (ii), and there will be a minimal period-two solution by Theorem 10. In view of Theorems 2, 4 there exist the global stable manifold $\mathcal{W}^s(\overline{x}, \overline{x})$ and global unstable manifold $\mathcal{W}^u(\overline{x}, \overline{x})$, where $\mathcal{W}^s(\overline{x}, \overline{x})$ is the graph of a non-decreasing function and $\mathcal{W}^u(\overline{x}, \overline{x})$ is the graph of a non-increasing function, which has endpoints at (ϕ, ψ) and (ψ, ϕ). Every initial point (x_{-1}, x_0) which starts south east of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ψ, ϕ), while every initial point (x_{-1}, x_0) which starts north west of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ϕ, ψ).

(ii) When $b < e^a$ and $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, then $\overline{x}$ is locally asymptotically stable by Lemma 3 part (i). Since $[a, U]^2$ is invariant box and $(\overline{x}, \overline{x})$ is the only fixed point then, by Theorem 2.1 in [11] is global attractor and so globally asymptotically stable.

(iii) Moreover, when $b < e^a$ and $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, $\overline{x}$ will be non-hyperbolic of the stable type by Lemma 3 part (iii). Since $[a, U]^2$ is invariant box and $(\overline{x}, \overline{x})$ is the only fixed point then, by Theorem 2.1 in [11] is global attractor and so globally asymptotically stable.

□

Theorem 12 *If $b \geq e^a$, then Equation* (3) *has unbounded solutions.*

Proof. We will use Theorem 6 to prove this theorem. The conditions of (8) and (9) of Theorem 6 become

$$f(U, L) = a + bLe^{-U} \leq L \quad \text{and} \quad f(L, U) = a + bUe^{-L} \geq U.$$

These inequalities can be reduced to

$$a \leq L(1 - be^{-U}) \quad \text{and} \quad a \geq U(1 - be^{-L}).$$

Any value of L and U such that $\frac{U}{1-be^{-U}} \leq \frac{L}{1-be^{-L}}$ will satisfy the theorem. Let $G(x) = \frac{x}{1-be^{-x}}$. There is a vertical asymptote at $1 - be^{-x} = 0$ that is at $x = \ln(b)$. In interval $(\ln(b), \infty)$ we can find L and U that satisfies these inequalities. As $b \geq e^a$ then $\ln(b) \geq a$ so that $(\ln(b), \infty)$ is part of the domain of difference equation (3). An example of where this holds is when $L = a + \epsilon$. Using the fact that $b \geq e^a$ and ϵ is small, then $b \geq e^{a+\epsilon}$. By condition (9) the inequality holds true as

$$a + bUe^{-(a+\epsilon)} \geq U \Leftrightarrow e^{a+\epsilon} \leq \frac{bU}{U - a}.$$

We will use condition (8) and $b \geq e^a$ to find the criteria for U based on our L. Thus,

$$a + b(a+\epsilon)e^{-U} \leq (a+\epsilon) \Leftrightarrow e^U \geq \frac{b(a+\epsilon)}{\epsilon} \Leftrightarrow e^U \geq \frac{e^a(a+\epsilon)}{\epsilon} \Leftrightarrow U \geq a + \ln\left(\frac{a+\epsilon}{\epsilon}\right).$$

Let U be such that $U > a + \ln\left(\frac{a+\epsilon}{\epsilon}\right)$. It holds that $U \geq L$. Overall, as f is continuous and there is no minimal period-two solution by Proposition 1, using Theorem (6) some solutions will approach ∞. □

Remark 4 For instance, case i) of Theorem 11 holds when $a = 1, b = 2$, case ii) holds when $a = 4, b = 2$ and case iii) holds when $a = 2, b = \frac{\sqrt{3}-1}{\sqrt{3}+1}e^{1+\sqrt{3}}$, and the conditions of Theorem 12 holds when $a = .5, b = 2$.

In conclusion, Equations (1) and (3) exhibit the global period doubling bifurcation described by Theorem 5.1 in [11]. Checking the conditions of Theorem 5.1 in [11] is exactly the content of Lemmas 1-3 and Theorems 10-12.

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Bounds for the Real Parts and Arguments of Normalized Analytic Functions Defined by the Srivastava-Attiya Operator

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Abstract

In this paper, we derive some bounds for the real parts and arguments of the functionals given by

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)}, \quad \frac{J_{s,b}(f)(z)}{z} \quad \text{and} \quad \frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)} \quad (z \in \mathbb{D}),$$

where $J_{s,b}$ is the widely-investigated Srivastava-Attiya operator defined on the class of normalized analytic functions f in the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

with suitable real parameters s and b. These results reduce upon specialization to some well-known inclusion relationships for several classes of functions with given geometric properties. We also make a comparison between one of the results obtained here and an already known result for some specific cases.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

The general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ is defined by

$$\Phi(z, s, b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^s}$$

$$\big(b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re\{s\} > 1 \quad \text{when} \quad |z| = 1\big).$$

It is known that the function $\Phi(z, s, b)$ reduces to such more familiar functions of Analytic Number Theory as the Riemann and the Hurwitz Zeta functions, Lerch's Zeta function, the Polylogarithmic function and the Lipschitz-Lerch Zeta function (see, for details, [12]).

Srivastava and Attiya [11] introduced the linear operator $J_{s,b} : \mathcal{A} \to \mathcal{A}$ defined by

$$J_{s,b}(f)(z) = G_{s,b}(z) * f(z) \qquad (b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}),$$

where the symbol $*$ denotes the Hadamard product (or convolution) of analytic functions and the function $G_{s,b}$ is defined by

$$G_{s,b}(z) = (b+1)^s[\Phi(z, s, b) - b^{-s}].$$

For a $f \in \mathcal{A}$ of the form given by (1.1), we get

$$J_{s,b}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s a_n z^n \qquad (z \in \mathbb{D}). \tag{1.2}$$

Srivastava and Attiya [11] showed that (see also the recent work by Srivastava *et al.* [13])

$$J_{0,b}(f)(z) = f(z),$$

$$J_{1,0}(f)(z) = \int_0^z \frac{f(t)}{t} \mathrm{d}t =: A(f)(z),$$

$$J_{1,\gamma}(f)(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) \mathrm{d}t =: \mathcal{J}_\gamma(f)(z) \qquad (\gamma > -1)$$

and

$$J_{\sigma,1}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n =: I^\sigma(f)(z) \qquad (\sigma > 0),$$

where A, $\mathcal{J}_\gamma$ and I^σ are the familiar Alexander [1], Bernardi [2] and Jung-Kim-Srivastava [4] integral operators, respectively.

From the equation (1.2), we can obtain the following recurrence relation:

$$zJ'_{s+1,b}(f)(z) = (1+b)J_{s,b}(f)(z) - bJ_{s+1,b}(f)(z). \tag{1.3}$$

For $\alpha \in [0, 1)$ and $\beta \in (0, 1]$, let $\Omega_{\alpha,\beta}$ denote a subset of $\mathbb{C}$ defined by

$$\Omega_{\alpha,\beta} = \left\{w : w \in \mathbb{C} \quad \text{and} \quad |\arg(w - \alpha)| < \frac{\pi}{2}\,\beta\right\}.$$

We denote by $\mathcal{S}^*(\alpha, \beta)$ and $\mathcal{C}(\alpha, \beta)$ the classes of functions $f \in \mathcal{A}$ satisfying the following conditions:

$$\frac{zf'(z)}{f(z)} \in \Omega_{\alpha,\beta} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \in \Omega_{\alpha,\beta} \qquad (\forall\ z \in \mathbb{D}),$$

respectively. The function f in the classes $\mathcal{S}^*(\alpha,\beta)$ and $\mathcal{C}(\alpha,\beta)$ is called starlike of order β and type α in $\mathbb{D}$ and strongly convex of order β and type α in $\mathbb{D}$, respectively. We note that

$$\mathcal{S}^*(\alpha,1)\equiv\mathcal{S}^*(\alpha)\quad\text{and}\quad\mathcal{C}(\alpha,1)\equiv\mathcal{C}(\alpha),$$

which are the well-known classes of starlike functions of order α in $\mathbb{D}$ and convex functions of order α in $\mathbb{D}$.

Wilken and Feng [15] showed that $f\in\mathcal{C}(\alpha,1)$ implies that $f\in\mathcal{S}^*(\beta,1)$, where

$$\beta:=\beta(\alpha)=\begin{cases}\dfrac{1-2\alpha}{2^{2-2\alpha}[1-2^{2\alpha-1}]} & \left(\alpha\neq\dfrac{1}{2}\right)\\[2ex] \dfrac{1}{2\log 2} & \left(\alpha=\dfrac{1}{2}\right).\end{cases}\tag{1.4}$$

Nunokawa *et al.* [8] investigated relations between $\gamma\in(0,1)$ and $\delta\in(0,1)$ so that $\mathcal{S}^*(\alpha,\gamma)$ implies that $\mathcal{C}(\beta,\delta)$, where β is given by (1.4). We will discuss this relation in Section 4.

The relation given above can be represented by using the operator $J_{s,b}$ as follows:

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)}\in\Omega_{\alpha,\gamma}\Longrightarrow\frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)}\in\Omega_{\beta,\delta}\qquad(z\in\mathbb{D}),\tag{1.5}$$

for $s=-1$ and $b=0$.

In the present paper, we will consider the implication given in (1.5) for suitable values of s and b in $\mathbb{R}$. We also consider other similar problems associated with (1.5), which are related to the forms given by

$$\frac{J_{s,b}(f)(z)}{z}\quad\text{and}\quad\frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)}.$$

We say that f is subordinate to F in $\mathbb{D}$, written as $f\prec F$ or as $f(z)\prec F(z)$ in $\mathbb{D}$, if and only if $f(z)=F\big(\omega(z)\big)$ for some Schwarz function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in\mathbb{D}$. It is well known that, if F is univalent in $\mathbb{D}$, then $f\prec F$ is equivalent to $f(0)=F(0)$ and $f(\mathbb{D})\subset F(\mathbb{D})$ (see, for details, [10, p. 36]).

Let $\psi:\mathbb{C}^2\to\mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the following differential subordination:

$$\psi\big(p(z),zp'(z)\big)\prec h(z)\qquad(z\in\mathbb{D}),$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant of the solutions of the differential subordination (or, simply, a dominant) if $p\prec q$ in $\mathbb{D}$ for all solutions p. A function $\widetilde{q}$ is called best dominant if $\widetilde{q}\prec q$ in $\mathbb{D}$ for all dominants q.

We recall the following lemmas which are required in our present investigation.

Lemma 1. (see Hallenbeck and Ruscheweyh [3]; see also [6, p. 71]) *Let h be convex in $\mathbb{D}$ with $h(0)=a$, $\gamma\neq 0$ and $\Re\{\gamma\}\geqq 0$. If p is analytic in $\mathbb{D}$ with the form given by*

$$p(z)=a+c_nz^n+c_{n+1}z^{n+1}\cdots\qquad(n\in\mathbb{N}:=\{1,2,3,\cdots\})$$

and

$$p(z)+\frac{zp'(z)}{\gamma}\prec h(z)\quad(z\in\mathbb{D}),\tag{1.6}$$

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in \mathbb{D}),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1}dt.$$

The function q is convex and is the best dominant of (1.6).

Lemma 2. (see Miller and Mocanu [5]) *If $-1 \leqq B < A \leqq 1$, $\beta > 0$ and the complex number γ satisfies the inequality*:

$$\Re\{\gamma\} \geqq -\frac{(1-A)\beta}{1-B},$$

then the following differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{D})$$

has a univalent solution in $\mathbb{D}$ given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{(A-B)\beta/B}\mathrm{d}t} - \dfrac{\gamma}{\beta} & (B \neq 0) \\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)\mathrm{d}t} - \dfrac{\gamma}{\beta} & (B = 0). \end{cases}$$

If the function $p(z)$ given by

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic in $\mathbb{D}$ and satisfies the following subordination condition:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{D}), \tag{1.7}$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{D}),$$

and q is the best dominant of (1.7).

The generalized hypergeometric function ${}_{\mathfrak{q}}F_{\mathfrak{s}}$ is defined by

$${}_{\mathfrak{q}}F_{\mathfrak{s}}(z) = {}_{\mathfrak{q}}F_{\mathfrak{s}}(\alpha_1, \cdots, \alpha_{\mathfrak{q}}; \beta_1, \cdots, \beta_{\mathfrak{s}}; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_{\mathfrak{q}})_n}{(\beta_1)_n \cdots (\beta_{\mathfrak{s}})_n} \frac{z^n}{n!} \qquad (z \in \mathbb{D}), \tag{1.8}$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \cdots, \mathfrak{q})$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- := \{0, -1, -2, \cdots\}$ $(j = 1, \cdots, \mathfrak{s})$, $\mathfrak{q} \leqq \mathfrak{s} + 1$, $\mathfrak{q}, \mathfrak{s} \in \mathbb{N}_0$, and $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_0 = 1 \quad \text{and} \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1) \qquad (n \in \mathbb{N}),$$

$\Gamma(z)$ being the Gamma function of the argument z.

We recall following well-known identities for the Gaussian hypergeometric function ${}_2F_1$, that is, the special case of (1.8) when $\mathfrak{q} - 1 = \mathfrak{s} = 1$:

Lemma 3. (see [14, pp. 285 and 293]) *For real or complex numbers a, b and c ($c \notin \mathbb{Z}_0^-$), the following identities hold true*:

(i) $\displaystyle\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1\,(a,b;c;z)$
when $\Re\{c\} > \Re\{b\} > 0$;

(ii) ${}_2F_1\,(a,b;c;z) = {}_2F_1\,(b,a;c;z)$;

(iii) ${}_2F_1\,(a,b;c;z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$.

The following lemmas will also be required in our present investigation.

Lemma 4. (see Wilken and Feng [15]) *Let* ν *be a positive measure on* $[0,1]$ *and let* h *be a complex-valued function defined on* $\mathbb{D}\times[0,1]$ *such that* $h(\cdot,t)$ *is analytic in* $\mathbb{D}$ *for each* $t\in[0,1]$ *and that* $h(z,\cdot)$ *is* ν*-integrable on* $[0,1]$ *for all* $z\in\mathbb{D}$. *In addition, suppose that* $\Re\{h(z,t)\}>0$, $h(-r,t)$ *is real and*

$$\Re\left\{\frac{1}{h(z,t)}\right\} \geqq \frac{1}{h(-r,t)} \qquad (|z|\leqq r<1;\ t\in[0,1]).$$

If the function H *is defined by*

$$H(z) = \int_0^1 h(z,t)\mathrm{d}\nu(t),$$

then

$$\Re\left\{\frac{1}{H(z)}\right\} \geqq \frac{1}{H(-r)} \qquad (|z|\leqq r<1).$$

Lemma 5. (see Nunokawa [7]) *Let the function* P *be analytic in* $\mathbb{D}$, $P(0)=1$, $P(z)\neq 0$ *in* $\mathbb{D}$ *and suppose that there exists a point* $z_0\in\mathbb{D}$ *such that*

$$\left|\arg\left(P(z)\right)\right| < \frac{\pi}{2}\,\delta \qquad (|z|<|z_0|)$$

and

$$\left|\arg\left(P(z_0)\right)\right| = \frac{\pi}{2}\,\delta \qquad (\delta>0).$$

Then

$$\frac{z_0P'(z_0)}{P(z_0)} = \mathrm{i}k\delta, \tag{1.9}$$

where

$$k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \textit{when} \quad \arg\left(P(z_0)\right) = \frac{\pi}{2}\,\delta$$

and

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \textit{when} \quad \arg\left(P(z_0)\right) = -\frac{\pi}{2}\,\delta,$$

and where

$$P(z_0)^{\frac{1}{\delta}} = \pm\mathrm{i}a$$

with $a>0$.

2. Bounds for the Real Parts

In this section, we investigate the bounds for the real parts of normalized analytic functions defined by the Srivastava-Attiya operator $J_{s,b}$.

Theorem 1. *Let $f \in \mathcal{A}$ and*

$$\Re\left\{\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)}\right\} > \alpha \qquad (z \in \mathbb{D}), \tag{2.1}$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$ and $b \geqq -\alpha$. Then

$$\Re\left\{\frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)}\right\} > -b + (b+1)\left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (z \in \mathbb{D}). \tag{2.2}$$

This result is sharp.

Proof. Let us define a function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)}.$$

Then p is analytic in $\mathbb{D}$ with $p(0) = 1$. Thus, from the recurrence relation (1.3), we have

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} = p(z) + \frac{zp'(z)}{p(z)+b}. \tag{2.3}$$

From (2.1), the above relation shows that

$$p(z) + \frac{zp'(z)}{p(z)+b} \prec \frac{1+(1-2\alpha)z}{1-z}. \tag{2.4}$$

Also, from Lemma 2 with $A = 1-2\alpha$, $B = -1$, $\beta = 1$ and $\gamma = b$, we find that

$$p(z) \prec \frac{1}{Q(z)} - b \qquad (z \in \mathbb{D}), \tag{2.5}$$

where Q is defined by

$$Q(z) = \int_0^1 t^b \left(\frac{1-zt}{1-z}\right)^{-2(1-\alpha)} \mathrm{d}t.$$

By applying Lemma 3, we have

$$Q(z) = \frac{\Gamma(b+1)}{\Gamma(b+2)}\, {}_2F_1\left(2-2\alpha, 1; b+2; \frac{z}{z-1}\right).$$

Moreover, the function Q is represented as follows:

$$Q(z) = \int_0^1 g(t,z)\mathrm{d}\mu(t),$$

where

$$g(t,z) = \frac{1-z}{1-(1-t)z}$$

and

$$\mathrm{d}\mu(t) = \frac{\Gamma(b+1)}{\Gamma(2-2\alpha)\Gamma(b+2\alpha)} t^{1-2\alpha}(1-t)^{b+2\alpha-1}\mathrm{d}t$$

with $t \in [0,1]$ and $z \in \mathbb{D}$. We note that $\mathrm{d}\mu(t)$ is a positive measure on $[0,1]$. We can easily verify that the assertions hold true:

(i) $g(\cdot,t)$ is analytic in $\mathbb{D}$ for each $t \in [0,1]$;
(ii) $g(z,\cdot)$ is integrable with respect to μ on $[0,1]$;
(iii) $\mathfrak{R}\{g(z,t)\} > 0$ for all $z \in \mathbb{D}$ and $t \in [0,1]$;
(iv) $g(-r,t)$ is real for all r and for $t \in [0,1]$.

Indeed, we have

$$\mathfrak{R}\left\{\frac{1}{g(z,t)}\right\} = \mathfrak{R}\left\{1+\frac{zt}{1-z}\right\} \geqq 1-\frac{tr}{1+r} = \frac{1}{g(-r,t)},$$

for $|z| \leqq r < 1$ and $t \in [0,1]$. Therefore, by applying Lemma 4, we obtain

$$\mathfrak{R}\left\{\frac{1}{Q(z)}\right\} \geqq \frac{\Gamma(b+2)}{\Gamma(b+1)}\left[{}_2F_1\left(2-2\alpha,1;b+2;\frac{r}{1+r}\right)\right]^{-1} \qquad (|z| \leqq r < 1). \tag{2.6}$$

Letting $r \to 1-$ in (2.6) we conclude that the inequality (2.2) holds true from the relation (2.5). The sharpness of this result follows from the fact that the function Q is the best dominant of (2.4). □

Theorem 2. *Let $f \in \mathcal{A}$ and suppose that*

$$\mathfrak{R}\left\{\frac{J_{s,b}(f)(z)}{z}\right\} > \alpha \qquad (z \in \mathbb{D}), \tag{2.7}$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$ and $b > -1$. Then

$$\mathfrak{R}\left\{\frac{J_{s+1,b}(f)(z)}{z}\right\} > 1-\frac{(1-\alpha)(b+1)}{b+2}\,{}_2F_1\left(1,1;b+3;\frac{1}{2}\right) \qquad (z \in \mathbb{D}). \tag{2.8}$$

This result is sharp.

Proof. Let us define a function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{J_{s+1,b}(f)(z)}{z} \qquad (z \in \mathbb{D}).$$

Then we have

$$\frac{J_{s,b}(f)(z)}{z} = p(z) + \frac{1}{b+1}zp'(z).$$

From (2.7), we see that

$$p(z) + \frac{zp'(z)}{b+1} \prec h(z) \qquad (z \in \mathbb{D}),$$

where

$$h(z) = \frac{1+(1-2\alpha)z}{1-z} \qquad (z \in \mathbb{D}).$$

Thus, by applying Lemma 1 with $\gamma = b+1$ and h given above, we have $p(z) \prec q(z)$ in $\mathbb{D}$, where q is a convex function in $\mathbb{D}$ defined by

$$q(z) = \frac{b+1}{z^{b+1}}\int_0^z \frac{1+(1-2\alpha)t}{1-t}t^b \mathrm{d}t,$$

which, in view of Lemma 3, yields

$$q(z) = 1 + \frac{2(1-\alpha)(b+1)z}{(b+2)(1-z)}\ {}_2F_1\left(1, 1; b+3; \frac{z}{z-1}\right).$$

Since the function q is convex with real coefficients, by the subordination relation:

$$p(z) \prec q(z) \qquad (z \in \mathbb{D}),$$

we obtain the inequality (2.8) by letting $z \to -1+$. The sharpness of this result follows from the fact that the function q is the best dominant of the differential subordination given by

$$p(z) + \frac{zp'(z)}{b+1} \prec h(z) \qquad (z \in \mathbb{D}).$$

□

We recall the following special case due to Prajapat and Bulboacă [9, Corollary 2.10].

Theorem 3. *Let $f \in \mathcal{A}$ and suppose that*

$$\Re\left\{\frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)}\right\} > \alpha \qquad (z \in \mathbb{D}),$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$ and $b \geqq -\alpha$. Then

$$\Re\left\{\frac{J_{s+1,b}(f)(z)}{J_{s+2,b}(f)(z)}\right\} > \left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (z \in \mathbb{D}). \tag{2.9}$$

This result is sharp.

3. Bounds for the Arguments

For given $\alpha \in [0, 1)$, let the parameters β_1, β_2 and β_3 be real numbers defined by

$$\beta_1 = \beta_1(\alpha, b) := -b + (b+1)\left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (b \geqq -\alpha), \tag{3.1}$$

$$\beta_2 = \beta_2(\alpha, b) := 1 - \frac{(1-\alpha)(b+1)}{b+2}\ {}_2F_1\left(1, 1; b+3; \frac{1}{2}\right) \qquad (b > -1) \tag{3.2}$$

and

$$\beta_3 = \beta_3(\alpha, b) := \left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (b \geqq -\alpha). \tag{3.3}$$

We note that $\beta_j < 1$ $(j = 1, 2, 3)$. We also note that

$$\beta_j \geqq \alpha \qquad (j = 1, 2, 3).$$

These inequalities are immediate consequences of Lemma 1 or 2 with

$$h(z) = \frac{1+Az}{1+Bz} = \frac{1+(1-2\alpha)z}{1-z} \qquad (A = 1-2\alpha;\ B = -1)$$

such that

$$\Re\{h(z)\} > \alpha \qquad (z \in \mathbb{D}).$$

In this section, we investigate the bounds for the arguments of normalized analytic functions defined by the Srivastava-Attiya operator $J_{s,b}$. In order to get our results, we need the following propositions.

Proposition 1. *Let $w_1, w_2, w_3 \in \mathbb{C}$ satisfy the following conditions*:

(i) $\Re\{w_1\} > 0$ *and* $\Im\{w_1\} < 0$;
(ii) $0 < \arg(w_3) \leqq \arg(w_2) < \frac{\pi}{2}$;
(iii) $|w_3| \leqq |w_2|$.

Then the inequality:

$$\arg(w_1 + w_3) \leqq \arg(w_1 + w_2) \tag{3.4}$$

holds true.

Proof. First of all, we consider a case for which $\arg(w_3) = \arg(w_2)$. In this case, we let

$$w_1 = x + \mathrm{i}y, \quad w_2 = R_2 \mathrm{e}^{\mathrm{i}\theta} \quad \text{and} \quad w_3 = R_3 \mathrm{e}^{\mathrm{i}\theta},$$

where $x > 0$, $y < 0$ and $R_2 \geqq R_3$. Then the inequality (3.4) is equivalent to

$$\frac{y + R_2 \sin\theta}{x + R_2 \cos\theta} \geqq \frac{y + R_3 \sin\theta}{x + R_3 \cos\theta}.$$

Furthermore, since $x > 0$ and $\theta \in (0, \pi/2)$, the above inequality is equivalent to

$$(R_2 - R_3)(x \sin\theta - y \cos\theta) \geqq 0.$$

Therefore, it follows from $x > 0$ and $y < 0$ that the above inequality holds true.

To complete the proof of Proposition 1, let $\Omega \subset \mathbb{C}$ be defined by

$$\Omega = \left\{ R\mathrm{e}^{\mathrm{i}\psi} \in \mathbb{C} : 0 < R \leqq R_2 \quad \text{and} \quad 0 < \psi \leqq \arg(w_2) \right\}.$$

Letting $w_3 \in \Omega$, we suppose that ℓ_1 be a straight line through the points $-w_1$ and w_2 and ℓ_2 be a straight line through the points $-w_1$ and w_3. From Condition (ii) of Proposition 1, we can take the unique intersection point denoted by $\widetilde{w}_3 \in \Omega$ of ℓ_1 and ℓ_2. For this point, we have

$$\arg\big(w_3 - (-w_1)\big) = \arg\big(\widetilde{w}_3 - (-w_1)\big) \geqq \arg\big(w_2 - (-w_1)\big),$$

which completes the proof of Proposition 1. □

The demonstration of Proposition 2 below is fairly straightforward.

Proposition 2. *Let w_1 and w_2 be in $\mathbb{C} \setminus \{0\}$. Then*

$$\arg(w_1 + w_2) \geqq \min\{\arg(w_1), \arg(w_2)\}.$$

Theorem 4. *Let $\beta \in \mathbb{R}$ be the parameter β_1 given by (3.1). Suppose also that $f \in \mathcal{A}$ and*

$$\left| \arg\left(\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} - \alpha \right) \right| < \frac{\pi}{2}\,\gamma \qquad (z \in \mathbb{D}), \tag{3.5}$$

where $s \in \mathbb{R}$, $b \geqq -\beta$, $0 \leqq \alpha < 1$ and $0 < \gamma < 1$. Then

$$\left| \arg\left(\frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)} - \beta \right) \right| < \frac{\pi}{2}\,\delta \qquad (z \in \mathbb{D};\ 0 < \delta < 1),$$

where $0 < \delta < 1$ and

$$\gamma = \min\left\{ \delta, \frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta} + x_0^{\delta-1})}{2(\beta-\alpha)[(1-\beta)x_0^{\delta} + \beta + b]} \right) \right\}$$

and $x_0 \in (0,1)$ *is the root of the following equation*:

$$(1-\beta)(x^2-1)x^{\delta} = (\beta+b)[1-\delta-(1+\delta)x^{\delta}]. \tag{3.6}$$

Proof. Let us define the functions p and $P : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)} \quad \text{and} \quad P(z) = \frac{p(z)-\beta}{1-\beta}.$$

Then the functions p and P are analytic in $\mathbb{D}$ with $p(0) = P(0) = 1$. From the recurrence relation (1.3), we have

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} = p(z) + \frac{zp'(z)}{p(z)+b}. \tag{3.7}$$

We now assume that there exists a point $z_0 \in \mathbb{R}$ such that

$$\left|\arg\big(P(z)\big)\right| = \left|\arg\big(p(z)-\beta\big)\right| < \frac{\pi}{2}\,\delta$$

for $|z| < |z_0|$ and

$$\left|\arg\big(P(z_0)\big)\right| = \left|\arg\big(p(z_0)-\beta\big)\right| = \frac{\pi}{2}\,\delta.$$

Consider the case when

$$\arg\big(P(z_0)\big) = \arg\big(p(z_0)-\beta\big) = \frac{\pi}{2}\,\delta.$$

Then, by Lemma 5, we have

$$\frac{z_0P'(z_0)}{P(z_0)} = \frac{z_0p'(z_0)}{p(z_0)-\beta} = \mathrm{i}\delta k, \tag{3.8}$$

where

$$k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \tag{3.9}$$

with $a > 0$. Also, from (3.7) and (3.8), we have

$$\begin{aligned}
&\arg\left(\frac{z_0J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right) \\
&\quad = \arg\left(p(z_0) + \frac{z_0p'(z_0)}{p(z_0)+b} - \alpha\right) \\
&\quad = \arg\big(p(z_0)-\beta\big) + \arg\left(\frac{p(z_0)-\alpha}{p(z_0)-\beta} + \frac{z_0p'(z_0)}{p(z_0)-\beta}\cdot\frac{1}{p(z_0)+b}\right) \\
&\quad = \frac{\pi}{2}\,\delta + \arg\left(1-\beta+(\beta-\alpha)(\mathrm{i}a)^{-\delta} + \frac{\mathrm{i}\delta k(1-\beta)}{(1-\beta)(\mathrm{i}a)^{\delta}+\beta+b}\right).
\end{aligned} \tag{3.10}$$

Let us define w_1, w_2 and w_3 by

$$w_1 = 1-\beta+(\beta-\alpha)(\mathrm{i}a)^{-\delta},$$

$$w_2 = \frac{\mathrm{i}\delta k(1-\beta)}{(1-\beta)(\mathrm{i}a)^{\delta}+\beta+b}$$

and

$$w_3 = \frac{\mathrm{i}\delta k(1-\beta)}{[(1-\beta)a^{\delta}+\beta+b]\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\delta}}.$$

We note that

$$\mathfrak{R}\{w_1\} = 1-\beta+\frac{\beta-\alpha}{(\beta-\alpha)a^\delta}\cos\left(\frac{\pi}{2}\,\delta\right) > 0$$

and

$$\mathfrak{I}\{w_1\} = -\frac{\beta-\alpha}{(\beta-\alpha)a^\delta}\sin\left(\frac{\pi}{2}\,\delta\right) < 0.$$

Also, from the inequality $\beta+b \geqq 0$, we can easily verify that the inequality $\arg(w_2) \geqq \arg(w_3)$ holds true. Furthermore, the inequality $|w_2| \geqq |w_3|$ is true, since

$$\begin{aligned}|w_2|^2 &= \frac{\delta^2k^2(1-\beta)^2}{(1-\beta)^2a^{2\delta}+2(1-\beta)(\beta+b)a^\delta\cos\left(\frac{\pi}{2}\,\delta\right)+(\beta+b)^2}\\ &\geqq \frac{\delta^2k^2(1-\beta)^2}{[(1-\beta)a^\delta+\beta+b]^2}\\ &= |w_3|^2.\end{aligned}$$

Therefore, by applying Proposition 1 with (3.10), we have

$$\begin{aligned}&\arg\left(\frac{z_0J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)}-\alpha\right)\\ &\geqq \frac{\pi}{2}\,\delta+\arg\left(1-\beta+(\beta-\alpha)(\mathrm{i}a)^{-\delta}+\frac{\mathrm{i}\delta k(1-\beta)}{[(1-\beta)a^\delta+\beta+b]\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta}}\right)\\ &= \frac{\pi}{2}\,\delta+\arg\left(\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\,\delta}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta}+\frac{\beta-\alpha}{(1-\beta)a^\delta}+\frac{\mathrm{i}\delta k}{(1-\beta)a^\delta+\beta+b}\right)\right)\\ &\geqq \arg\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta}+\frac{\beta-\alpha}{(1-\beta)a^\delta}+\frac{\mathrm{i}\delta(a+a^{-1})}{2[(1-\beta)a^\delta+\beta+b]}\right).\end{aligned} \tag{3.11}$$

Let us now put

$$w_4 = \frac{\beta-\alpha}{(1-\beta)a^\delta}+\mathrm{i}\frac{\delta(a+a^{-1})}{2[(1-\beta)a^\delta+\beta+b]}.$$

Then

$$\arg(w_4) = \arctan\left(\frac{(1-\beta)\delta}{2(\beta-\alpha)}g(a)\right),$$

where $g:(0,\infty)\to\mathbb{R}$ is a function defined by

$$g(x) = \frac{x+x^{-1}}{1-\beta+(\beta+b)x^{-\delta}}.$$

Differentiating the function g with respect to x, we have

$$x^{\delta+2}[1-\beta+(\beta+b)x^{-\delta}]^2g'(x) = h(x),$$

where the function $h:(0,\infty)\to\mathbb{R}$ is defined by

$$h(x) = (x^2-1)[(1-\beta)x^\delta+\beta+b]+\delta(\beta+b)(x^2+1).$$

Since the function h is continuous on $(0,\infty)$ with

$$h(0) = -(\beta+b)(1-\delta) < 0 \quad \text{and} \quad h(1) = 2\delta(\beta+b) > 0,$$

there exists an $x_0 \in (0,1)$ such that $h(x_0) = 0$, which is equivalent to the equation given by (3.6). Differentiating the function h twice with respect to x, we find that

$$h''(x) = (2+\delta)(1-\beta)(1+\delta)x^{\delta} + \delta(1-\delta)(1-\beta)x^{\delta-2} + 2(1+\delta)(\beta+b) > 0$$

for all $x \in (0,1)$. Since $h(x) > 0$ for $x \in (1,\infty)$, it follows from the convexity of $h(x)$ on $(0,1)$ that the function $g'(x)$ vanishes only at $x_0 \in (0,1)$. Furthermore, we can easily verify that $g(x_0)$ is the minimum value of $g(x)$ on $(0,\infty)$. Therefore, we have

$$\begin{aligned} \arg\,(w_4) &\geqq \arctan\left(\frac{(1-\beta)\delta}{2(\beta-\alpha)}\; g(x_0)\right) \\ &= \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{2(\beta-\alpha)\big[(1-\beta)x_0^{\delta}+\beta+b\big]}\right). \end{aligned} \tag{3.12}$$

Finally, from (3.11), (3.12) and Proposition 2, we have

$$\begin{aligned} &\arg\left(\frac{z_0 J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right) \\ &\geqq \arg\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta} + w_4\right) \\ &\geqq \min\left\{\frac{\pi}{2}\,\delta, \arg(w_4)\right\} \\ &\geqq \min\left\{\frac{\pi}{2}\,\delta, \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{2(\beta-\alpha)[(1-\beta)x_0^{\delta}+\beta+b]}\right)\right\} \\ &= \frac{\pi}{2}\,\gamma, \end{aligned}$$

which leads to a contradiction to the hypothesis (3.5).

For the case when

$$\arg\big(P(z_0)\big) = \arg\big(p(z_0)-\beta\big) = -\frac{\pi}{2}\,\delta,$$

Lemma 5 yields

$$\frac{z_0 P'(z_0)}{P(z_0)} = \frac{z_0 p'(z_0)}{p(z_0)-\beta} = \mathrm{i}\delta k, \tag{3.13}$$

where

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \tag{3.14}$$

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with $a > 0$. We also have

$$\begin{aligned}
&\arg\left(\frac{z_0 J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right)\\
&= \arg\big(p(z_0)-\beta\big) + \arg\left(\frac{p(z_0)-\alpha}{p(z_0)-\beta} + \frac{z_0 p'(z_0)}{p(z_0)-\beta}\cdot\frac{1}{p(z_0)+b}\right)\\
&= -\frac{\pi}{2}\,\delta + \arg\left(\frac{(1-\beta)(-\mathrm{i}a)^{\delta}+\beta-\alpha}{(1-\beta)(-\mathrm{i}a)^{\delta}} + \frac{\mathrm{i}\delta k}{(1-\beta)(-\mathrm{i}a)^{\delta}+\beta+b}\right)\\
&= -\left[\frac{\pi}{2}\,\delta + \arg\left(1-\beta+(\beta-\alpha)(\mathrm{i}a)^{-\delta} + \frac{\mathrm{i}\delta\widetilde{k}(1-\beta)}{(1-\beta)(\mathrm{i}a)^{\delta}+\beta+b}\right)\right],
\end{aligned}$$

where

$$\widetilde{k} := -k > \frac{a+a^{-1}}{2}.$$

Therefore, from the proof of the first case, we have

$$\arg\left(\frac{z_0 J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right) \leqq -\frac{\pi}{2}\,\gamma,$$

which also leads to a contradiction to the hypothesis (3.5). This completes the proof of Theorem 4. □

Theorem 5. *Let $\beta \in \mathbb{R}$ be the parameter β_2 given by (3.2). Let $f \in \mathcal{A}$ and suppose that*

$$\left|\arg\left(\frac{J_{s,b}(f)(z)}{z} - \alpha\right)\right| < \frac{\pi}{2}\,\gamma \qquad (z \in \mathbb{D}), \tag{3.15}$$

where $s \in \mathbb{R}$, $b > -1$, $0 \leqq \alpha < 1$ and $0 < \gamma < 1$. Then

$$\left|\arg\left(\frac{J_{s+1,b}(f)(z)}{z} - \beta\right)\right| < \frac{\pi}{2}\,\delta \qquad (z \in \mathbb{D};\ 0 < \delta < 1),$$

where

$$\gamma = \delta + \frac{2}{\pi}\ \arctan\left\{\frac{-2(b+1)(\beta-\alpha)\sin\left(\frac{\pi}{2}\,\delta\right) + \delta(1-\beta)(x_0^{\delta+1}+x_0^{\delta-1})}{2(b+1)\left[(1-\beta)x_0^{\delta} + (\beta-\alpha)\cos\left(\frac{\pi}{2}\,\delta\right)\right]}\right\},$$

and $x_0 \in (0,1)$ is the unique zero of the function h defined by

$$h(x) = Cx^{\delta}(x^2-1) + AC(\delta+1)x^2 + \delta Bx + AC(\delta-1) \tag{3.16}$$

with

$$A = \frac{\beta-\alpha}{1-\beta}\cos\left(\frac{\pi}{2}\,\delta\right), \quad B = \frac{\beta-\alpha}{1-\beta}\sin\left(\frac{\pi}{2}\,\delta\right) \quad \textit{and} \quad C = \frac{\delta}{2(b+1)}. \tag{3.17}$$

Proof. Let us define the functions p and $P : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{J_{s+1,b}(f)(z)}{z} \quad \text{and} \quad P(z) = \frac{p(z)-\beta}{1-\beta}. \tag{3.18}$$

Then the functions p and P are analytic in $\mathbb{D}$ with $p(0) = P(0) = 1$. We also have

$$\frac{J_{s,b}(f)(z)}{z} = p(z) + \frac{1}{b+1} z p'(z). \tag{3.19}$$

We now assume that there exists a point $z_0 \in \mathbb{R}$ such that

$$\left|\arg\big(P(z)\big)\right| = \left|\arg\big(p(z)-\beta\big)\right| < \frac{\pi}{2}\,\delta,$$

for $|z| < |z_0|$ and

$$\left|\arg\big(P(z_0)\big)\right| = \left|\arg\big(p(z_0)-\beta\big)\right| = \frac{\pi}{2}\,\delta.$$

We consider the case when

$$\arg\big(P(z_0)\big) = \arg\big(p(z_0)-\beta\big) = \frac{\pi}{2}\,\delta.$$

Then, by Lemma 5, we have the relations given by (3.8) and (3.9) with $a > 0$. From (3.18) and (3.19), we have

$$\begin{aligned}
&\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right)\\
&\quad = \arg\big(p(z_0)-\beta\big) + \arg\left(\frac{p(z_0)-\alpha}{p(z_0)-\beta} + \frac{z_0 p'(z_0)}{(b+1)\big(p(z_0)-\beta\big)}\right)\\
&\quad = \frac{\pi}{2}\,\delta + \arg\left(1 + \frac{\beta-\alpha}{(1-\beta)(\mathrm{i}a)^{\delta}} + \frac{\mathrm{i}\delta k}{b+1}\right)\\
&\quad = \frac{\pi}{2}\,\delta + \arctan\left(-\frac{\frac{\beta-\alpha}{(1-\beta)a^{\delta}}\sin\left(\frac{\pi}{2}\,\delta\right) + \frac{\delta k}{b+1}}{1 + \frac{\beta-\alpha}{(1-\beta)a^{\delta}}\cos\left(\frac{\pi}{2}\,\delta\right)}\right)\\
&\quad \geqq \frac{\pi}{2}\,\delta + \arctan\big(g(a)\big),
\end{aligned} \tag{3.20}$$

where

$$g(x) = \frac{-B + C(x^{\delta+1} + x^{\delta-1})}{x^{\delta} + A},$$

and A, B and C are positive constants given by (3.17). Differentiating the function $g(x)$ with respect to x, we have

$$a^{2-\delta}(a^{\delta}+A)^2 g'(x) = h(x),$$

where h is given by (3.16). Simple calculations show that

$$h(0) = -AC(1-\delta) < 0 \quad \text{and} \quad h(x) \geqq h(1) = \delta(2AC+B) > 0 \qquad (x \geqq 1)$$

and

$$h''(x) = C[(\delta+2)(\delta+1)x^{\delta} + \delta(1-\delta)x^{\delta-2}] + 2AC(\delta+1) > 0 \qquad (0 < x < 1).$$

Similar methods as in the proof of Theorem 4 would yield

$$g(x) \geqq g(x_0) \qquad (0 < x < \infty), \tag{3.21}$$

where x_0 is the unique zero of $h(x)$ on $(0,\infty)$. Therefore, by (3.20) and (3.21), we obtain

$$\begin{aligned}
&\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right)\\
&\quad \geqq \frac{\pi}{2}\,\delta + \arctan\left(\frac{-2(b+1)(\beta-\alpha)\sin(\delta\pi/2) + \delta(1-\beta)(x_0^{\delta+1} + x_0^{\delta-1})}{2(b+1)[(1-\beta)x_0^{\delta} + (\beta-\alpha)\cos(\delta\pi/2)]}\right)\\
&\quad = \frac{\pi}{2}\,\gamma,
\end{aligned}$$

which provides a contradiction to the hypothesis (3.15).

For the case when

$$\arg\big(P(z_0)\big) = \arg\big(p(z_0) - \beta\big) = -\frac{\pi}{2}\,\delta,$$

we have the relations given by (3.13) and (3.14) with $a > 0$. Therefore, we have

$$\begin{aligned}
&\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right) \\
&= -\left[\frac{\pi}{2}\,\delta + \arctan\left(-\frac{\frac{\beta-\alpha}{(1-\beta)a^{\delta}}\sin\left(\frac{\pi}{2}\,\delta\right) + \frac{\delta\widetilde{k}}{b+1}}{1 + \frac{\beta-\alpha}{(1-\beta)a^{\delta}}\cos\left(\frac{\pi}{2}\,\delta\right)}\right)\right] \\
&\leqq -\left(\frac{\pi}{2}\,\delta + \arctan\big(g(a)\big)\right), \qquad (3.22)
\end{aligned}$$

where

$$\widetilde{k} := -k > \frac{a + a^{-1}}{2}.$$

Therefore, from (3.22) and (3.21), we have

$$\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right) \leqq -\frac{\pi}{2}\,\gamma,$$

which also provides a contradiction to the hypothesis (3.15). This evidently completes the proof of Theorem 5. □

Next, for given suitable real of the parameters s and b and for $f \in \mathcal{A}$, we define a function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{J_{s+1,b}(f)(z)}{J_{s+2,b}(f)(z)}.$$

Then, by using the recurrence relation (1.3), we obtain

$$\frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)} = p(z) + \frac{zp'(z)}{(b+1)p(z)} \qquad (z \in \mathbb{D}). \tag{3.23}$$

By applying the same methods as in the proof of Theorem 4 to the differential equation (3.23) instead of (3.7), we can establish the following argument property associated with the Srivastava-Attiya operator.

Theorem 6. *Let $\beta \in \mathbb{R}$ be the parameter β_3 given by (3.3). Also let $f \in \mathcal{A}$ and*

$$\left|\arg\left(\frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)} - \alpha\right)\right| < \frac{\pi}{2}\,\gamma \qquad (z \in \mathbb{D}),$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$, $b \geqq -\alpha$ and $0 < \gamma < 1$. Then

$$\left|\arg\left(\frac{J_{s+1,b}(f)(z)}{J_{s+2,b}(f)(z)} - \beta\right)\right| < \frac{\pi}{2}\,\delta \qquad (z \in \mathbb{D}),$$

where $0 < \delta < 1$ and

$$\gamma = \min\left\{\delta, \frac{2}{\pi}\,\arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta} + x_0^{\delta-1})}{2(\beta-\alpha)(b+1)[(1-\beta)x_0^{\delta} + \beta]}\right)\right\}$$

and x_0 is the root in the interval $(0,1)$ of the following equation:

$$[(1-\beta)x^{\delta}+\beta](1-x^2)=\beta\delta(x^2+1). \tag{3.24}$$

4. **Numerical and Computational Analysis**

Let $s=-1$ and $b=0$. Since the following equalities:

$$J_{s,b}(f)(z)=zf'(z) \quad \text{and} \quad J_{s+1,b}(f)(z)=f(z)$$

hold true for $f\in\mathcal{A}$, it follows from Theorem 4 that $f\in\mathcal{C}(\alpha,\gamma_1)$ implies that $f\in\mathcal{S}^*(\beta,\delta)$, where

$$\gamma_1=\min\left\{\delta,\frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{2(\beta-\alpha)[(1-\beta)x_0^{\delta}+\beta]}\right)\right\} \tag{4.1}$$

and $x_0\in(0,1)$ is the root of the following equation:

$$(1-\beta)(x^2-1)x^{\delta}=\beta[1-\delta-(1+\delta)x^{\delta}]).$$

On the other hand, Nunokawa *et al.* [8] showed that $f\in\mathcal{C}(\alpha,\gamma_2)$ implies that $f\in\mathcal{S}^*(\beta,\delta)$, where

$$\gamma_2=\frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{(1-\beta)x_0^{\delta}+\beta}\right), \tag{4.2}$$

and $x_0\in(0,1)$ is the root of the following equation:

$$(1-\beta)(x^2-1)x^{\delta}=\beta(1-\delta-(1+\delta)x^2). \tag{4.3}$$

As it does not seem to be so easy to compare the values γ_1 and γ_2 for the whole ranges of the parameters $\alpha\in(0,1)$ and $\delta\in(0,1)$, we will compare them here in several particular cases of α and δ. Thus, if we fix $\alpha=\frac{1}{2}$, then we have

$$\beta=\frac{1}{2\log 2}.$$

With the aid of *Mathematica*, we can thus obtain Table 1 (see below) which gives the approximate values of $\gamma_1\in(0,1)$ and $\gamma_2\in(0,1)$ defined by (4.1) and (4.2), respectively, when δ is given by

$$\delta=\frac{j}{10} \qquad (j=1,2,\cdots,9).$$

As we see from Table 1, we can verify that the results in this paper would significantly improve the results in the earlier work [8] for the special cases considered above.

Finally, we give another table (Table 2 below) which gives the approximate values of γ defined in Theorem 5 and Theorem 6, respectively, when δ is given by

$$\delta=\frac{j}{10} \qquad (j=1,2,\cdots,9).$$

δ	γ_1	γ_2
0.9	0.44897	0.27427
0.8	0.43647	0.28576
0.7	0.41317	0.28270
0.6	0.38021	0.26598
0.5	0.33781	0.23485
0.4	0.28596	0.18916
0.3	0.22487	0.13310
0.2	0.15544	0.07889
0.1	0.07952	0.03626

Table 1. **The Approximate Values of γ_1 and γ_2**

δ	Theorem 5 (γ)	Theorem 6 (γ)
0.9	0.75302	0.28582
0.8	0.75151	0.27787
0.7	0.67933	0.26303
0.6	0.59106	0.24205
0.5	0.49662	0.21506
0.4	0.39926	0.18205
0.3	0.30036	0.14316
0.2	0.20061	0.09896
0.1	0.10041	0.05062

Table 2. **The Approximate Values of γ in Theorem 5 and Theorem 6**

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SHARP BOUNDS FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS*

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Abstract. In the article, we prove that $\alpha = 3$, $\beta = \log 4/(\pi/2 - \log 4) = 7.51371\cdots$, $\gamma = 1/4$ and $\delta = 1 + \log 2 - \pi/2 = 0.122351\cdots$ are the best possible constants such that the double inequalities

$$\frac{\beta+1}{\beta+r^2}\log\frac{4}{r'} < \mathcal{K}(r) < \frac{\alpha+1}{\alpha+r^2}\log\frac{4}{r'},$$

$$1+\left(\frac{1}{2}\log\frac{4}{r'}-\gamma\right)r'^2 < \mathcal{E}(r) < 1+\left(\frac{1}{2}\log\frac{4}{r'}-\delta\right)r'^2$$

hold for all $r \in (0,1)$, where $r' = \sqrt{1-r^2}$, and $\mathcal{K}(r) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}$ and $\mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}d\theta$ are the complete elliptic integrals of the first and second kinds.

1. Introduction

The complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1-5] of the first and the second kinds are respectively defined by

$$\begin{cases} \mathcal{K}(r) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}, \\ \mathcal{K}(0) = \frac{\pi}{2}, \qquad \mathcal{K}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}d\theta, \\ \mathcal{E}(0) = \frac{\pi}{2}, \qquad \mathcal{E}(1) = 1. \end{cases}$$

It is well known that the function $r \to \mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$ and the function $r \to \mathcal{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1,\pi/2)$. The complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the particular cases of the Gaussian hypergeometric function [6-15]

$$F(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad (-1<x<1),$$

where $(a)_0 = 1$ for $a \neq 0$, $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x) = \int_0^{\infty}t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma

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function [16-21]. Indeed,

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n},$$

$$\mathcal{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}.$$

The complete elliptic integrals play a very important role in the study of geometric function theory and they have numerous applications in various problems of physics and engineering. In particular, Many remarkable inequalities and elementary approximations for the complete elliptic integrals can be found in the literature [22-34].

In the sequel, we will use the symbols $\mathcal{K}$ and $\mathcal{E}$ for $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively. Throughout this paper we let $r'=\sqrt{1-r^2}$ for $0<r<1$. Then we use the symbols $\mathcal{K}'$ and $\mathcal{E}'$ for $\mathcal{K}(r')$ and $\mathcal{E}(r')$, respectively.

Carlson and Gustafson [35] proved that the double inequality

$$1<\frac{\mathcal{K}(r)}{\log(4/r')}<\frac{4}{3+r^2}$$

holds for all $0<r<1$.

Kühnau [36] proved the inequality

$$\frac{9}{8+r^2}<\frac{\mathcal{K}(r)}{\log(4/r')} \tag{1.1}$$

for all $0<r<1$.

It is well known that the double inequality

$$\frac{\pi}{2}M_{3/2}(1,r')<\mathcal{E}(r)<\frac{\pi}{2}M_2(1,r')$$

holds for all $0<r<1$ (see [37, 19.9.4]), where

$$M_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}\quad (p\neq 0),\qquad M_0(a,b)=\sqrt{ab}$$

is the pth power mean [38-51]

It is the aim of this paper to refine the inequality (1.1) for the complete elliptic integral of the first kind, and to obtain sharp upper and lower bounds for the complete elliptic integral of the second kind. Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1. *The double inequality*

$$\frac{\beta+1}{\beta+r^2}<\frac{\mathcal{K}(r)}{\log(4/r')}<\frac{\alpha+1}{\alpha+r^2} \tag{1.2}$$

holds for all $r\in(0,1)$ *with the best possible constants* $\alpha=3$ *and* $\beta=(\log 4)/(\pi/2-\log 4)=7.51371\cdots$.

Theorem 1.2. *The double inequality*

$$1+r'^2\left(\frac{1}{2}\log\frac{4}{r'}-\gamma\right)<\mathcal{E}(r)<1+r'^2\left(\frac{1}{2}\log\frac{4}{r'}-\delta\right) \tag{1.3}$$

holds for all $r\in(0,1)$ *with the best possible constants* $\gamma=1/4$ *and* $\delta=1+\log 2-\pi/2=0.122351\cdots$.

2. Proof of Theorems 1.1 and 1.2

In order to prove our main results we need to establish some monotonicity properties for the functions defined by the complete elliptic integrals. The following derivative formula can be found in the literature [52]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$

$$\frac{d}{dr}(\mathcal{E} - r'^2\mathcal{K}) = r\mathcal{K}, \quad \frac{d}{dr}(\mathcal{K} - \mathcal{E}) = \frac{r\mathcal{E}}{r'^2}.$$

Theorem 2.1. *The function*

$$F(r) = (3 + r^2)\mathcal{K} - 4\log(4/r')$$

is strictly increasing from $(0, 1)$ *onto* $(-a, 0)$ *with* $a = 4\log 4 - 3\pi/2 = 0.832788\cdots$. *In particular, the double inequality*

$$\frac{4\log(4/r') - a}{3 + r^2} < \mathcal{K} < \frac{4\log(4/r')}{3 + r^2} \tag{2.1}$$

holds for all $r \in (0, 1)$.

Proof. Let the functions g and h be defined by

$$g(r) = (3 + r^2)\mathcal{E} - (r^4 - 4r^2 + 3)\mathcal{K} - 4r^2,$$

$$h(r) = 3r'^2\mathcal{K} + 4\mathcal{E} - 8.$$

Then differentiation gives

$$rr'^2\frac{d}{dr}F(r) = g(r),$$

$$\frac{1}{r}\frac{dg(r)}{dr} = h(r).$$

It follows from [52, Theorem 3.21(7)] that the function h is strictly decreasing from $(0, 1)$ onto $(-4, 7\pi/2 - 8)$ and there exists $r_0 \in (0, 1)$ such that h is positive on $(0, r_0)$ and negative on $(r_0, 1)$. We conclude that g is strictly increasing on $(0, r_0)$ and strictly decreasing on $(r_0, 1)$. From $g(0) = 0 = g(1)$ we clearly see that $g(r) > 0$ for $r \in (0, 1)$ and F is strictly increasing on $(0, 1)$. It is easy to see that the limiting value $F(0) = -a$, and by [53, 112.01] $F(1^-) = 0$. □

Theorem 2.2. *Let* $\beta = \log 4/(\pi/2 - \log 4) = 7.51371\cdots$. *Then there exists* $s_0 \in (0, 1)$ *such that the function*

$$G(r) = (\beta + r^2)\mathcal{K} - (\beta + 1)\log(4/r')$$

is strictly increasing on $(0, s_0)$ *and strictly decreasing on* $(s_0, 1)$ *with the limiting values* $G(0^+) = 0 = G(1^-)$. *In particular, the inequality*

$$\frac{(\beta + 1)\log(4/r')}{\beta + r^2} < \mathcal{K} \tag{2.2}$$

holds for all $r \in (0, 1)$.

Proof. Let the functions g, h, l and p be defined by

$$g(r) = (\beta + r^2)\mathcal{E} - (\beta - r^2)r'^2\mathcal{K} - (\beta + 1)r^2,$$
$$h(r) = 4\mathcal{E} + (\beta - 3r^2)\mathcal{K} - 2(\beta + 1),$$
$$l(r) = (\beta + 4 - 7r^2)\mathcal{E} - (\beta + 4 + 3r^2)r'^2\mathcal{K},$$
$$p(r) = (21r^2 + 8 + \beta)\mathcal{K} - 24\mathcal{E}.$$

Then differentiation leads to

$$rr'^2\frac{d}{dr}G(r) = g(r),$$
$$\frac{1}{r}\frac{d}{dr}g(r) = h(r),$$
$$rr'^2\frac{d}{dr}h(r) = l(r),$$
$$\frac{1}{r}\frac{d}{dr}l(r) = p(r).$$

It is easy to see that the function p is strictly increasing from $(0, 1)$ onto $((8 + \beta - 24)\pi/2, \infty)$. It follows from $(8 + \beta - 24)\pi/2 = -13.3302 \cdots < 0$ that there exists $r_1 \in (0, 1)$ such that p is negative on $(0, r_1)$ and positive on $(r_1, 1)$. Hence the function l is strictly decreasing on $(0, r_1)$ and strictly increasing on $(r_1, 1)$. From the limiting values $l(0^+) = 0$ and $l(1^-) = \beta - 3 = 4.51371 \cdots > 0$ we clearly see that there exists $r_2 \in (0, 1)$ such that l is negative on $(0, r_2)$ and positive on $(r_2, 1)$. We conclude that h is strictly decreasing on $(0, r_2)$ and strictly increasing on $(r_2, 1)$. This together with the values $h(0^+) = 2\pi - 2 + (\pi/2 - 1)\beta = 1.05827 \cdots$, $h(0.8) = -0.760875 \cdots$ and $h(1^-) = \infty$ implies that there exists $0 < r_3 < r_4 < 1$ such that h is positive on $(0, r_3) \cup (r_4, 1)$ and negative on (r_3, r_4). Hence g is strictly increasing on $(0, r_3)$ and $(r_4, 1)$, and strictly decreasing on (r_3, r_4). Since $g(0^+) = 0 = g(1^-)$, we conclude that there exists $s_0 \in (0, 1)$ such that g is positive on $(0, s_0)$ and negative on $(s_0, 1)$. Therefore, the function G is strictly increasing on $(0, s_0)$ and strictly decreasing on $(s_0, 1)$. It is easy to see that $G(0^+) = 0$ and

$$\lim_{r\to 1^-} G(r) = \lim_{r\to 1^-} (a + 1)(\mathcal{K} - \log(4/r')) - r'^2\mathcal{K} = 0.$$

□

Proof of Theorem 1.1. Inequality (1.2) follows from inequality (2.2) and the right-hand side inequality of (2.1) immediately.

Lemma 2.3. *The function*

$$u(r) = (1 + r^2)\mathcal{E} - r'^2\mathcal{K} - \frac{5}{2}r^2 + \frac{1}{2}r^4$$

is negative on $(0, 1)$.

Proof. Let the functions f and g be defined by

$$f(r) = 3\mathcal{E} + 2r^2 - 5,$$
$$g(r) = \frac{3(\mathcal{E} - \mathcal{K})}{r^2} + 4.$$

Then Applying the derivative formulas we get

$$\frac{d}{dr}u(r) = rf(r),$$

$$\frac{d}{dr} f(r) = rg(r).$$

Since the function $r \mapsto (\mathcal{E}-\mathcal{K})/r^2$ is strictly decreasing from $(0,1)$ onto $(-\infty, -\pi/4)$ (see [52, 3.43(11)]), the function g is strictly decreasing from $(0,1)$ onto $(-\infty, (16-3\pi)/4)$. Then from $(16-3\pi)/4) > 0$ we know that there exists $r_0 \in (0,1)$ such that $rg(r)$ is positive on $(0, r_0)$ and negative on $(r_0, 1)$. Hence f is strictly increasing on $(0, r_0)$ and strictly decreasing on $(r_0, 1)$. It is easy to see that $f(0^+) = 3\pi/2 - 5 < 0$ and $f(1^-) = 0$. We conclude that there exists $r_1 \in (0,1)$ such that $rf(r)$ is negative on $(0, r_1)$ and positive on $(r_1, 1)$. Therefore, the function u is strictly decreasing on $(0, r_1)$ and strictly increasing on $(r_1, 1)$. Then from the facts that $u(0^+) = 0 = u(1^-)$ we get $u(x) < 0$ for all $r \in (0,1)$. □

Theorem 2.4. *The function*

$$H(r) = \frac{\mathcal{E}-1}{r'^2} - \frac{1}{2}\log\frac{4}{r'}$$

is strictly decreasing from $(0,1)$ *onto* $(-1/4, -\delta)$ *with* $\delta = 1 + \log 2 - \pi/2 = 0.122351\cdots$.

Proof. Differentiation yields

$$rr'^4 \frac{d}{dr} H(r) = (1+r^2)\mathcal{E} - r'^2\mathcal{K} - \frac{5}{2}r^2 + \frac{1}{2}r^4 = u(r) < 0$$

by Lemma 2.3. Hence, the function H is strictly decreasing on $(0,1)$.

We clearly see that

$$H(0^+) = \pi/2 - 1 - \log 2 = -\delta.$$

Let

$$h_1(r) = \mathcal{E} - 1 - \frac{1}{2}r'^2 \log\frac{4}{r'}, \qquad h_2(r) = r'^2.$$

Then $h_1(1^-) = 0 = h_2(1^-)$, and by l'Hospital's rule one has

$$H(1^-) = \lim_{r\to 1^-} \frac{h_1'(r)}{h_2'(r)} = \lim_{r\to 1^-} \frac{1}{2}\left(\frac{r'^2\mathcal{K}}{r^2} + \mathcal{K} - \log\frac{4}{r'}\right) - \frac{\mathcal{E}}{2r^2} + \frac{1}{4} = -\frac{1}{4},$$

where the last equality follows from the facts (see [52, 3.21(7) and (3.25)] or [53, 112.01] that

$$\lim_{r\to 1^-} {r'}^2\mathcal{K} = 0, \qquad \lim_{r\to 1^-}\left(\mathcal{K} - \log\frac{4}{r'}\right) = 0.$$

□

Proof of Theorem 1.2. Inequality (1.3) follows easily from Theorem 2.4 immediately.

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Symmetric identities for the second kind q-Bernoulli polynomials

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Abstract : In [9], we studied the second kind q-Bernoulli numbers and polynomials. By using these numbers and polynomials, we investigated the zeros of the second kind q-Bernoulli polynomials. In this paper, by applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second kind q-Bernoulli polynomials and the sums of powers of consecutive q-odd integers.

Key words : Symmetric properties, the sums of powers of consecutive q-odd integers, the second kind Bernoulli numbers and polynomials, the second kind q-Bernoulli numbers and polynomials.

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1. Introduction

Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind Bernoulli number, the second kind Bernoulli polynomials, Euler numbers, Euler polynomials, Genocchi numbers, Genocchi polynomials, tangent numbers, tangent polynomials, and Bell polynomials were studied by many authors (see for details [1-11]). Bernoulli numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [8], by using the second kind Euler numbers E_j and polynomials $E_j(x)$, we investigated the alternating sums of powers of consecutive odd integers. Let k be a positive integer. Then we obtain

$$T_j(k-1)=\sum_{n=0}^{k-1}(-1)^n(2n+1)^j=\frac{(-1)^{k+1}E_j(2k)+E_j}{2}.$$

In [9], we introduced the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$. By using computer, we observed an interesting phenomenon of 'scattering' of the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ in complex plane. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind q-Bernoulli polynomials $B_{n,q}(x)$. The outline of this paper is as follows. We introduce the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$. In Section 2, we obtain the sums of powers of consecutive q-odd integers. Finally, we give recurrence identities the second kind q-Bernoulli polynomials and the sums of powers of consecutive q-odd integers.

Throughout this paper, we always make use of the following notations: $\mathbb{N}=\{1,2,3,\cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p=p^{-\nu_p(p)}=p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q\in\mathbb{C}$, or p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}$ one normally assume that $|q|<1$. If $q\in\mathbb{C}_p$, we normally assume that $|q-1|_p<p^{-\frac{1}{p-1}}$ so that $q^x=\exp(x\log q)$ for $|x|_p\le 1$. For

$$g\in UD(\mathbb{Z}_p)=\{g|g:\mathbb{Z}_p\to\mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by [1, 2, 3, 4, 6]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} g(x)q^x.$$

The bosonic integral was considered from a physical point of view to the bosonic limit $q \to 1$, as follows:

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x) \text{ (see [1, 2, 3, 5])}. \tag{1.1}$$

By (1.1), we easily see that

$$I_1(g_1) = I_1(g) + g'(0), \text{ cf. [1, 2, 3, 4, 6, 7]}, \tag{1.2}$$

where $g_1(x) = g(x+1)$ and $g'(0) = \frac{dg(x)}{dx}|_{x=0}$.

First, we introduce the second kind Bernoulli numbers B_n and polynomials $B_n(x)$. The second kind Bernoulli numbers B_n and polynomials $B_n(x)$ are defined by means of the following generating functions (see [7]):

$$F(t) = \frac{2te^t}{e^{2t}-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

and

$$F(x,t) = \frac{2te^t}{e^{2t}-1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!},$$

respectively.

The second kind q-Bernoulli polynomials, $B_{n,q}(x)$ are defined by means of the generating function:

$$F_q(x,t) = \frac{(\log q + 2t)e^t}{qe^{2t}-1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \tag{1.3}$$

The second kind q-Bernoulli numbers $B_{n,q}$ are defined by means of the generating function:

$$F_q(t) = \frac{(\log q + 2t)e^t}{qe^{2t}-1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \tag{1.4}$$

In (1.2), if we take $g(x) = q^x e^{(2x+1)t}$, then we have

$$\int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x) = \frac{(\log q + 2t)e^t}{q^h e^{2t}-1}. \tag{1.5}$$

for $|t| \le p^{-\frac{1}{p-1}}$. In (1.2), if we take $g(x) = e^{2nxt}$, then we also have

$$\int_{\mathbb{Z}_p} e^{2nxt} d\mu_1(x) = \frac{2nt}{e^{2nt}-1}. \tag{1.6}$$

It will be more convenient to write (1.2) as the equivalent bosonic integral form

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_1(x) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) + g'(0), \quad \text{(see [1,2,3,4,6])}. \tag{1.7}$$

For $n \in \mathbb{N}$, we also derive the following bosonic integral form by (1.7),

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_1(x) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) + \sum_{k=0}^{n-1} g'(k), \text{ where } g'(k) = \frac{dg(x)}{dx}|_{x=k}. \tag{1.8}$$

In [9], we introduced the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$ and investigate their properties. The following elementary properties of the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4). We, therefore, choose to omit details involved.

Theorem 1(Witt formula). For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have

$$\int_{\mathbb{Z}_p} q^x (2x+1)^n d\mu_1(x) = B_{n,q},$$

$$\int_{\mathbb{Z}_p} q^y (x+2y+1)^n d\mu_1(y) = B_{n,q}(x).$$

Theorem 2. For any positive integer n, we have

$$B_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{k,q} x^{n-k}.$$

Theorem 3(Distribution Relation). For any positive integer m, we obtain

$$B_{n,q}(x) = m^{n-1} \sum_{i=0}^{m-1} q^i B_{n,q^m}\left(\frac{2i+x+1-m}{m}\right) \text{ for } n \geq 0.$$

2. Symmetry identities for the second kind q-Bernoulli polynomials

In this section, we assume that $q \in \mathbb{C}_p$. In [2], Kim investigated interesting properties of symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli polynomials and Bernoulli polynomials. By using same method of [3], expect for obvious modifications, we obtain recurrence identities the second kind q-Bernoulli polynomials. By (1.7), we obtain

$$\begin{aligned} \frac{1}{h\log q + 2t}&\left(\int_{\mathbb{Z}_p} q^x q^n e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x)\right) \\ &= \frac{n\int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x)}{\int_{\mathbb{Z}_p} q^{nx} e^{2ntx} d\mu_1(x)}. \end{aligned} \tag{2.1}$$

By (1.8), we obtain

$$\begin{aligned} \frac{1}{h\log q + 2t}&\left(\int_{\mathbb{Z}_p} q^x q^n e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x)\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{n-1} q^i (2i+1)^k\right)\frac{t^k}{k!}. \end{aligned} \tag{2.2}$$

For each integer $k \geq 0$, let

$$O_{k,q}(n) = 1^k + q3^k + q^2 5^k + q^3 7^k + \cdots + q^n (2n+1)^k.$$

The above sum $O_{k,q}(n)$ is called the sums of powers of consecutive q-odd integers.

From the above and (2.2), we obtain

$$\frac{1}{\log q + 2t}\left(\int_{\mathbb{Z}_p} q^x q^n e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x)\right)\frac{t^k}{k!} = \sum_{k=0}^{\infty} O_{k,q}(n-1)\frac{t^k}{k!}. \tag{2.3}$$

Thus, we have

$$\sum_{k=0}^{\infty}\left(q^n\int_{\mathbb{Z}_p} q^x(2x+2n+1)^k d\mu_1(x)-\int_{\mathbb{Z}_p} q^x(2x+1)^k d\mu_1(x)\right)\frac{t^k}{k!}=\sum_{k=0}^{\infty}(\log q+2t)O_{k,q}(n-1)\frac{t^k}{k!}.$$

By comparing coefficients $\frac{t^k}{k!}$ in the above equation, we arrive at the following theorem:

Theorem 4. Let k be a positive integer. Then we obtain

$$q^nB_{n,q}(2n)-B_{n,q}=\log qO_{k,q}(n-1)+2kO_{k-1,q}(n-1). \tag{2.4}$$

Remark 5. For the sums of powers of consecutive odd integers, we have

$$\lim_{q\to 1}\left(\log qO_{k,q}(n-1)+2kO_{k-1,q}(n-1)\right)=2k\sum_{i=0}^{n-1}(2i+1)^{k-1}=B_k(2n)-B_k \text{ for } k\in\mathbb{N}.$$

By using (2.1) and (2.3), we arrive at the following theorem:

Theorem 6. Let n be positive integer. Then we have

$$\frac{n\int_{\mathbb{Z}_p} q^x e^{(2x+1)t}d\mu_1(x)}{\int_{\mathbb{Z}_p} q^{nx}e^{2ntx}d\mu_1(x)}=\sum_{m=0}^{\infty}\left(O_{m,q}(n-1)\right)\frac{t^m}{m!}. \tag{2.5}$$

Let w_1 and w_2 be positive integers. By using (1.5) and (1.6), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^{(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p} q^{w_1w_2x}e^{2w_1w_2xt}d\mu_1(x)}\\&=\frac{(\log q+2t)e^{w_1t}e^{w_2t}e^{w_1w_2xt}(q^{w_1w_2}e^{2w_1w_2t}-1)}{(q^{w_1}e^{2w_1t}-1)(q^{w_2}e^{2w_2t}-1)}.\end{aligned} \tag{2.6}$$

By using (2.4) and (2.6), after elementary calculations, we obtain

$$\begin{aligned}a&=\left(\frac{1}{w_1}\int_{\mathbb{Z}_p} q^{w_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p} q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p} q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}\sum_{m=0}^{\infty}B_{m,q^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_2}}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.7}$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}. \tag{2.8}$$

Again, by using the symmetry in (2.7), we have

$$\begin{aligned}a&=\left(\frac{1}{w_2}\int_{\mathbb{Z}_p} q^{w_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p} q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p} q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}\sum_{m=0}^{\infty}B_{m,q^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_2}}(w_1x)w_2^{j-1}O_{m-j,q^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}. \tag{2.9}$$

By comparing coefficients $\frac{t^m}{m!}$ on the both sides of (2.8) and (2.9), we arrive at the following theorem:

Theorem 7. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}(w_1-1)w_2^{m-j}\\&=\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_2}}(w_1x)w_2^{j-1}O_{m-j,q^{w_1}}(w_2-1)w_1^{m-j},\end{aligned}$$

where $B_{k,q}(x)$ and $O_{m,q}(k)$ denote the second kind q-Bernoulli polynomials and the sums of powers of consecutive q-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be positive integers. Then we have

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j-1}x^{j-k}B_{k,q^{w_2}}O_{m-j,q^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j-1}w_2^{m-k}x^{j-k}B_{k,q^{w_1}}O_{m-j,q^{w_2}}(w_1-1).\end{aligned}$$

By using (2.6), we have

$$\begin{aligned}a&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\sum_{j=0}^{w_1-1}q^{w_2j}e^{(2j+1)(w_2t)}\right)\\&=\sum_{j=0}^{w_1-1}q^{w_2j}\int_{\mathbb{Z}_p}q^{w_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_1(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}q^{w_2j}B_{n,q^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.10}$$

Again, by using the symmetry property in (2.10), we also have

$$\begin{aligned}a&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\sum_{j=0}^{w_2-1}q^{w_1j}e^{(2j+1)(w_1t)}\right)\\&=\sum_{j=0}^{w_2-1}q^{w_1j}\int_{\mathbb{Z}_p}q^{w_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_1(x_2)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}q^{w_1j}B_{n,q^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.11}$$

By comparing coefficients $\frac{t^n}{n!}$ on the both sides of (2.10) and (2.11), we have the following theorem.

Theorem 9. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1} q^{w_2 j} B_{n,q^{w_1}}\left(w_2 x+(2j+1)\frac{w_2}{w_1}\right) w_1^{n-1}\\ =&\sum_{j=0}^{w_2-1} q^{w_1 j} B_{n,q^{w_2}}\left(w_1 x+(2j+1)\frac{w_1}{w_2}\right) w_2^{n-1}.\end{aligned} \tag{2.12}$$

Substituting $w_1 = 1$ into (2.12), we arrive at the following corollary.

Corollary 10. Let w_2 be positive integer. Then we obtain

$$B_{n,q}(x) = w_2^{n-1}\sum_{j=0}^{w_2-1} q^j B_{n,q^{w_2}}\left(\frac{x-w_2+(2j+1)}{w_2}\right).$$

This last result(Corollary 10) is shown to yield the known Distribution Relation of the second kind q-Bernoulli polynomials(Theorem 3).

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On some finite difference methods on the Shishkin mesh for the singularly perturbed problem *

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Abstract: This paper studies the convergence behavior of three finite difference schemes on the Shishkin mesh to solve the singularly perturbed two-point boundary value problem. Three new error estimates are proved for the hybrid finite difference scheme that combines the midpoint upwind scheme on the coarse part with the central difference scheme on the fine part, the midpoint upwind scheme and the simple upwind scheme, respectively. Finally, numerical experiments illustrate that these error estimates are sharp and the convergence is uniform with respect to the perturbation parameter.

Keywords: Singularly perturbed boundary value problem; Finite difference scheme; Piecewise equidistant mesh; Error estimate; Uniform convergence

1 Introduction

Consider the singularly perturbed two-point boundary value problem:

$$\begin{cases} Lu(x) := -\varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x),\ x \in (0,1), \\ u(0) = A,\ u(1) = B, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, A and B are given constants, and the functions $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth satisfying $0 < \beta < b(x) < \beta^*$ and $0 \leq c(x) < \gamma^*$, where β, β^* and γ^* are constants. Under these conditions, the singularly perturbed problem (1) has a unique solution that possesses a boundary layer at $x = 1$ (see [1–4]).

Among various numerical methods to solve singularly perturbed problems, finite difference schemes on layer-adapted meshes for the singularly perturbed two-point boundary value problem have been widely studied in the literature, see [1–10]. The simple upwind scheme was proved to have the error estimate $O(N^{-1})$ on the coarse part and $O(N^{-1}\ln N)$ on the fine part on the Shishkin mesh and the error estimate $O(N^{-1})$ on the whole interval on the Bakhvalov-Shishkin mesh, see, e.g., [5, 6]. The central difference scheme on the Shishkin mesh was proved

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to have the convergence $O(N^{-2}\ln^2 N)$ on the whole nodes by discrete Green's functions, although the computed solution had small oscillations on the coarse part, see [3, 7]. In order to avoid oscillation, the midpoint upwind scheme on the Shishkin mesh was constructed and the convergence $O(\max\{N^{-2}, N^{-5+4i/N}\ln N\})$ were proved for $i = 1, \cdots, N$ in [8]. The midpoint upwind scheme on the Bakhvalov-Shishkin mesh was shown to be of order $O(N^{-2})$ on the coarse part and $O(N^{-1})$ on the fine part in [9]. To improve the convergence behaviour of boundary layer, the hybrid finite difference scheme was proposed and the convergence $O(N^{-2})$ on the coarse part and $O(N^{-2}\ln^2 N)$ on the fine part were proved for (1.1) with $c(x) \equiv 0$ in [8] and with geeneral $c(x) \geq 0$ in [4].

In this paper, we construct the hybrid finite difference scheme on the Shishkin mesh to slove (1) with $c(x) \geq 0$, not only give the suitable conditions especially for the $c(x)$ to guarantee an associated M-matrix and the discrete maximum principle, but also obtain a better error estimate. Furthermore, new error estimates for the midpoint upwind scheme and the simple upwind scheme are also obtained. Finally, the convergence behaviours according to these new error estimates for these schemes are confirmed by numerical experiments.

Note: Throughout the paper, the nontrivial case $\varepsilon \leq CN^{-1}$ is considered, C denotes a generic positive constant that is independent of both perturbation parameter ε and mesh parameter N, and C can take different values at each occurrence, even in the same argument.

2 Error estimates on the Shishkin mesh

Lemma 1 (see [1–3] The solution $u(x)$ of (1) can be decomposed as $u(x) = S(x) + E(x)$ on $[0, 1]$, where the smooth part S satisfies

$$LS(x) = f(x) \text{ and } \mid S^{(i)}(x) \mid \leq C,\ 0 \leq i \leq q,$$

while the layer part E satisfies

$$LE(x) = 0 \text{ and } \mid E^{(i)}(x) \mid \leq C\varepsilon^{-i} \exp\left(-\frac{\beta(1-x)}{\varepsilon}\right),\ 0 \leq i \leq q,$$

where the maximal order q depends on the smoothness of the data. □

Let N be a positive even integer and $\tau = \min\left\{\frac{1}{2}, \frac{4\varepsilon}{\beta}\ln N\right\}$. Since the singularly perturbed problem is considered, we generally take $\varepsilon \leq CN^{-1}$ and $\tau = \frac{4\varepsilon}{\beta}\ln N$. Choose $1-\tau$ be the transition point. Divide $[0, 1-\tau]$ and $[1-\tau, 1]$ uniformly into $N/2$ subintervals, respectively. Then the Shishkin mesh is:

$$x_i = \begin{cases} \dfrac{2(1-\tau)}{N} i,\ 0 \leq i \leq \dfrac{N}{2}, \\ 1 - 2\tau\left(1 - \dfrac{i}{N}\right),\ \dfrac{N}{2} \leq i \leq N. \end{cases} \tag{2}$$

Lemma 2 Denote $h_i = x_i - x_{i-1}$ for (2), then $N^{-1} \le h_i < 2N^{-1}$ and $h_{N/2+i} = \dfrac{8\varepsilon}{\beta} N^{-1} \ln N$ for $i = 1, 2, \cdots, N/2$. $\square$

We construct the following hybrid finite difference scheme:

$$L^N u_i^N := \begin{cases} -\varepsilon D^+ D^- u_i^N + b_{i-1/2} D^- u_i^N + c_{i-1/2} u_{i-1/2}^N = f_{i-1/2}, \ 0 < i \le N/2, \\ -\varepsilon D^+ D^- u_i^N + b_i D^0 u_i^N + c_i u_i^N = f_i, \ N/2 < i < N, \end{cases} \tag{3}$$
$$u_0^N = A, \ u_N^N = B,$$

where define L^N as a discrete operator, $D^+ u_i^N = \dfrac{u_{i+1}^N - u_i^N}{h_{i+1}}$, $D^- u_i^N = \dfrac{u_i^N - u_{i-1}^N}{h_i}$, $D^+ D^- u_i^N = \dfrac{2\left(D^+ u_i^N - D^- u_i^N\right)}{h_{i+1} + h_i}$, $D^0 u_i^N = \dfrac{u_{i+1}^N - u_{i-1}^N}{h_{i+1} + h_i}$, $u_{i-1/2}^N = \dfrac{u_{i-1}^N + u_i^N}{2}$, $b_{i-1/2} = b\left(x_{i-1/2}\right)$, $b_i = b\left(x_i\right)$, $f_{i-1/2} = f\left(x_{i-1/2}\right)$ and so on.

The scheme (3) is slightly different from the scheme (2.86) in [4] in the discretization of cu at $x_{i-1/2}$. The scheme (3) gives the following expression:

$$L^N u_i^N = \begin{cases} -\frac{2\varepsilon}{h_{i+1}(h_i+h_{i+1})} u_{i+1}^N + (\frac{2\varepsilon}{h_i h_{i+1}} + \frac{b_{i-1/2}}{h_i} + \frac{c_{i-1/2}}{2}) u_i^N - (\frac{2\varepsilon}{h_i(h_i+h_{i+1})} + \frac{b_{i-1/2}}{h_i} - \frac{c_{i-1/2}}{2}) u_{i-1}^N, \\ -(\frac{2\varepsilon}{h_{i+1}(h_i+h_{i+1})} - \frac{b_i}{h_i+h_{i+1}}) u_{i+1}^N + (\frac{2\varepsilon}{h_i h_{i+1}} + c_i) u_i^N - (\frac{2\varepsilon}{h_i(h_i+h_{i+1})} + \frac{b_i}{h_i+h_{i+1}}) u_{i-1}^N. \end{cases}$$

Lemma 3 (Discrete comparison principle) If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then the operator L^N defined by (3) on (2) satisfies the discrete comparison principle, i.e., let $\{v_i\}$ and $\{w_i\}$ are mesh functions, if $v_0 \le w_0$, $v_N \le w_N$ and $L^N v_i \le L^N w_i$ for $i = 1, 2, \cdots, N-1$, then $v_i \le w_i$ for all i.
Proof. Under the conditions of Lemma 3, the coefficient matrix associated with L^N by the above expression is clearly an $(N-1) \times (N-1)$ strictly diagonally dominant matrix, and has positive diagonal entries and non-positive off diagonal entries. So it is an irreducible M-matrix. Hence, the operator satisfies the discrete comparison principle. $\square$

So, the scheme (3) on (2) has a unique solution and the function w_i is defined as a barrier function for v_i by Lemma 3.

Lemma 4 Set $Z_0 = 1$, define the mesh function $Z_i = \prod_{j=1}^{i} \left(1 + \dfrac{\beta h_j}{2\varepsilon}\right)$ for $i = 1, 2, \cdots, N$. Then the operator L^N of (3) satisfies

$$L^N Z_i \ge \frac{C}{\max\{\varepsilon, h_i\}} Z_i \text{ for } i = 1, 2, \cdots, N-1.$$

Proof. Clearly

$$D^+ Z_i = \frac{\beta}{2\varepsilon} Z_i \text{ and } D^- Z_i = \frac{\beta}{2\varepsilon + \beta h_i} Z_i.$$

Hence

$$-\varepsilon D^+ D^- Z_i = -\frac{2\varepsilon}{h_{i+1} + h_i}\left(D^+ Z_i - D^- Z_i\right) = -\frac{\beta^2 h_i}{\left(h_{i+1} + h_i\right)\left(2\varepsilon + \beta h_i\right)} Z_i,$$

and

$$D^0 Z_i = \frac{h_{i+1}D^+Z_i + h_iD^-Z_i}{h_{i+1}+h_i} = \left(\frac{\beta}{2\varepsilon+\beta h_i} + \frac{\beta^2 h_{i+1}h_i}{2\varepsilon\,(h_{i+1}+h_i)\,(2\varepsilon+\beta h_i)}\right) Z_i.$$

Thus, from (3), by using $c(x) \geq 0$ and $b(x) > \beta > 0$, we have

$$\begin{aligned}
L^N Z_i &\geq -\frac{\beta^2 h_i}{(h_{i+1}+h_i)\,(2\varepsilon+\beta h_i)} Z_i + b_{i-1/2}\cdot\frac{\beta}{2\varepsilon+\beta h_i} Z_i \\
&= \frac{\beta}{2\varepsilon+\beta h_i}\left(b_{i-1/2} - \frac{\beta h_i}{h_{i+1}+h_i}\right) Z_i, \quad i = 1, 2, \cdots, N/2, \\
L^N Z_i &\geq -\frac{\beta^2 h_i}{(h_{i+1}+h_i)\,(2\varepsilon+\beta h_i)} Z_i + b_i\left(\frac{\beta}{2\varepsilon+\beta h_i} + \frac{\beta^2 h_{i+1}h_i}{2\varepsilon\,(h_{i+1}+h_i)\,(2\varepsilon+\beta h_i)}\right) Z_i \\
&\geq \frac{\beta}{2\varepsilon+\beta h_i}\left(b_i - \frac{\beta h_i}{h_{i+1}+h_i}\right) Z_i, \quad i = N/2+1, \cdots, N-1,
\end{aligned}$$

and obtain the result. □

Lemma 5 For the Shishkin mesh (2), there exists a constant C such that

$$\prod_{j=i+1}^{N}\left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1} \leq CN^{-4(1-i/N)} \text{ for } N/2 \leq i < N.$$

Proof. By Lemma 4.1(b) in [1] and noting $h_j = h$ for $j = N/2+1, \cdots, N$, we have

$$\prod_{j=i+1}^{N}\left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1} = \prod_{j=i+1}^{N}\left(1+\frac{\beta h}{2\varepsilon}\right)^{-1} \leq \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon+\beta h)} = \mathrm{e}^{-4(1-i/N)\ln N/\left(1+4N^{-1}\ln N\right)}.$$

Then as the proof of Lemma 3.2 in [8], the result is proved. □

Lemma 6 Assuming that $u(x)$ be sufficiently smooth function defined on $[0,1]$, for the truncation error of (3) on the Shishkin mesh to solve (1), there exists a constant C such that

$$\left|L^N(u_i) - (Lu)(x_{i-1/2})\right| \leq C\left[\int_{x_{i-1}}^{x_{i+1}} \varepsilon|u'''(t)|\mathrm{d}t + h_i\int_{x_{i-1}}^{x_i}(|u'''(t)|+|u''(t)|)\mathrm{d}t\right],\ i = 1, \cdots, N/2,$$

$$\left|L^N(u_i) - (Lu)\,(x_i)\right| \leq Ch_i\int_{x_{i-1}}^{x_{i+1}}\left(\varepsilon\left|u^{(4)}(t)\right| + \left|u'''(t)\right|\right)\mathrm{d}t,\ i = N/2+1, \cdots, N-1.$$

Proof. The results follow by noting that $c(x)u(x)$ contributes

$$\left|c_{i-1/2}\right|\left|(u(x_{i-1})+u(x_i))/2 - u\left(x_{i-1/2}\right)\right| \leq Ch_i\int_{x_{i-1}}^{x_i}\left|u''(t)\right|\mathrm{d}t$$

for $i = 1, 2, \cdots, N/2$ to the truncation error in the Lemma 2.4 in [8] and zero for $i = N/2+1, \cdots, N-1$ to the truncation error in Theorem 3.2 in [8], respectively. □

Similarly, the numerical solution can also be split into the smooth part and the layer part by $u_i^N = S_i^N + E_i^N$, where S_i^N satisfies $L^N S_i^N = f_{i-1/2}$, $i = 1, 2, \cdots, N/2$, $L^N S_i^N = f_i$, $i = N/2+1, \cdots, N-1$, $S_0^N = S_0$ and $S_N^N = S_N$, and E_i^N satisfies $L^N E_i^N = 0$, $i = 1, 2, \cdots, N-1$, $E_0^N = E_0$ and $E_N^N = E_N$, therefore

$$\left|u_i - u_i^N\right| \leq \left|S_i - S_i^N\right| + \left|E_i - E_i^N\right|. \tag{4}$$

Lemma 7 If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then for the smooth part of the solutions of (1) and (3) on the Shishkin mesh, there exists a constant C such that

$$\left|S_i - S_i^N\right| \leq CN^{-2} \text{ for all } i.$$

Proof. By Lemma 1 and Lemma 6, we have

$$\left|L^N(S_i - S_i^N)\right| = \begin{cases} \left|L^N(S_i) - (LS)(x_{i-1/2})\right| \leq C(h_i + h_{i+1})(\varepsilon + h_i), \ i = 1, \cdots, N/2, \\ \left|L^N(S_i) - (LS)\left(x_i\right)\right| \leq Ch_i(h_i + h_{i+1})(\varepsilon + 1), \ i = N/2 + 1, \cdots, N-1. \end{cases}$$

Set $w_i = C_0 N^{-1}\left(\varepsilon + N^{-1}\right) x_i$ for all i, where constant C_0 is chosen sufficiently large. Then

$$L^N w_i = \begin{cases} b_{i-1/2} C_0 N^{-1}(\varepsilon + N^{-1}) + c_{i-1/2}\left(w_{i-1} + w_i\right)/2 \geq CN^{-1}(\varepsilon + N^{-1}), \ i = 1, \cdots, N/2, \\ b_i C_0 N^{-1}(\varepsilon + N^{-1}) + c_i w_i \geq CN^{-1}(\varepsilon + N^{-1}), \ i = N/2 + 1, \cdots, N-1. \end{cases}$$

Therefore, $L^N w_i \geq \left|L^N\left(S_i - S_i^N\right)\right|$ for $i = 1, \cdots, N-1$. Clearly, $w_0 = 0 = \left|S_0 - S_0^N\right|$ and $w_N = C_0 N^{-1}\left(\varepsilon + N^{-1}\right) \geq 0 = \left|S_N - S_N^N\right|$. By Lemma 3, w_i is a barrier function for $\left|S_i - S_i^N\right|$ and then the proof is completed. $\square$

Lemma 8 If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then for the layer part of the solutions of (1) and (3) on the Shishkin mesh, there exists a constant C such that

$$\left|E_i - E_i^N\right| \leq CN^{-2} \text{ for } i = 0, 1, \cdots, N/2.$$

Proof. For $i = 0, 1, \cdots, N/2$, from Lemma 1, we have

$$|E_i| \leq C\mathrm{e}^{-\dfrac{\beta(1-x_i)}{\varepsilon}} \leq C\mathrm{e}^{-\dfrac{\beta(1-x_i)}{2\varepsilon}} \leq C\mathrm{e}^{-\dfrac{\beta(1-x_{N/2})}{2\varepsilon}} = CN^{-2}. \tag{5}$$

Recall the function Z_i in Lemma 4. Now $\mathrm{e}^t \geq 1 + t$ for all $t \geq 0$. So,

$$\frac{Z_i}{Z_N} = \prod_{j=i+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \geq \prod_{j=i+1}^{N} \mathrm{e}^{-\beta h_j/(2\varepsilon)} = \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon)}. \tag{6}$$

Let $Y_i = C_0 \dfrac{Z_i}{Z_N}$ for all i, where constant C_0 is chosen sufficiently large. From Lemma 4, we have $L^N Y_i = C_0/Z_N \cdot L^N Z_i \geq 0 = \left|L^N E_i^N\right|$ for $i = 1, \cdots, N-1$. By (6) and Lemma 1, $Y_0 = C_0 Z_0/Z_N \geq C_0 \mathrm{e}^{-\frac{\beta}{2\varepsilon}} \geq C_0 \mathrm{e}^{-\frac{\beta}{\varepsilon}} \geq |E\left(0\right)| = \left|E_0^N\right|$ and $Y_N = C_0 \geq |E\left(1\right)| = \left|E_N^N\right|$. Thus, by Lemma 3, we have

$$\left|E_i^N\right| \leq Y_i = C\prod_{j=i+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \text{ for all } i. \tag{7}$$

By Lemma 5, we have

$$\left|E_i^N\right| \leq C\prod_{j=N/2+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \leq CN^{-2} \text{ for } i = 0, 1, \cdots, N/2.$$

Consequently, combining this inequality with (5), the proof is completed. □

Lemma 9 If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then for the layer part of the solutions of (1) and (3) on the Shishkin mesh, there exists a constant C such that

$$\left|E_i - E_i^N\right| \le C \max\left\{N^{-2}, N^{-6+4i/N} \ln^2 N\right\} \text{ for } i = N/2+1, \cdots, N.$$

Proof. By Lemmas 6, 1 and 2, we have

$$\begin{aligned}
\left|L^N\left(E_i - E_i^N\right)\right| &= \left|L^N\left(E_i\right) - (LE)\left(x_i\right)\right| \\
&\le C h_i \int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon\left|E^{(4)}(t)\right| + \left|E'''(t)\right|\right) \mathrm{d}t \\
&\le C h_i \int_{x_{i-1}}^{x_{i+1}} \varepsilon^{-3} \exp\left(-\frac{\beta(1-t)}{\varepsilon}\right) \mathrm{d}t \\
&= C\varepsilon^{-3} h_i \cdot \frac{\varepsilon}{\beta} \sinh \frac{\beta h_i}{\varepsilon} \cdot \mathrm{e}^{-\beta(1-x_i)/\varepsilon} \\
&\le C\varepsilon^{-3} h_i^2 \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon)} \\
&\le C\varepsilon^{-1} N^{-2} \ln^2 N \prod_{j=i+1}^{N}\left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1}, \text{ since } \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon)} \le \prod_{j=i+1}^{N}\left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1} \\
&= C\varepsilon^{-1} N^{-2} \ln^2 N \cdot Z_i / Z_N.
\end{aligned}$$

Set $\phi_i = C_0\left\{N^{-2} + N^{-2}\ln^2 N \cdot Z_i/Z_N\right\}$ for $i = N/2, \cdots, N$, where constant C_0 is chosen sufficiently large. By Lemma 4, we have $L^N\phi_i \ge CN^{-2}\ln^2 N/Z_N \cdot L^N Z_i \ge \left|L^N\left(E_i - E_i^N\right)\right|$ for $i = N/2+1, \cdots, N-1$. Clearly, $\phi_{N/2} \ge C_0 N^{-2} \ge \left|E_{N/2} - E_{N/2}^N\right|$ by Lemma 8 and $\phi_N \ge 0 = \left|E_N - E_N^N\right|$. Thus, ϕ_i is a barrier function for $\left|E_i - E_i^N\right|$ by Lemma 3. And by Lemma 5, the result follows. □

Theorem 1 Assuming that $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, the hybrid finite difference scheme (3) on the Shishkin mesh (2) for (1) satisfies:

$$\left|u_i - u_i^N\right| \le C \max\left\{N^{-2}, N^{-6+4i/N} \ln^2 N\right\} \text{ for } i = 1, \cdots, N. \tag{8}$$

Furthermore,

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-2},\ 0 \le i \le p_h N, \\ CN^{-2}\ln^2 N,\ p_h N < i \le N, \end{cases} \tag{9}$$

where $p_h = 1 - \dfrac{1}{2\mathrm{e}} \approx 0.8161$.

Proof. From (4) and Lemmas 7, 8 and 9, we have the error estimate (8).

Furthermore, since $N^{-6+4i/N}\ln^2 N = N^{-2}N^{-4+4i/N}\ln^2 N$, we consider the function

$$f(x) = x^{-4+4p_h} \ln^2 x,\ x > 1.$$

From $f'(x) = x^{-4+4p_h-1} \ln x \left[(-4+4p_h)\ln x + 2\right] = 0$, we have $x = \mathrm{e}^{1/(2-2p_h)}$. So,

$$\max_{x>1}\{f(x)\} = \frac{1}{4(1-p_h)^2\mathrm{e}^2},$$

then

$$N^{-4+4p_h}\ln^2 N \le \frac{1}{4(1-p_h)^2\mathrm{e}^2} = 1.$$

Therefore, (9) is proved. □

Theorem 2 Assuming that $N > \frac{\gamma^*}{\beta}$, the midpoint upwind scheme on the Shishkin mesh (2) for (1) satisfies:

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-2}, \ 0 \le i \le p_m N, \\ CN^{-1}\ln N, \ p_m N < i \le N, \end{cases} \tag{10}$$

where $p_m = \frac{3}{4} - \frac{1}{4\mathrm{e}} \approx 0.6580$.

Proof. Under the hypothesis of Theorem 2, the matrix associated with the midpoint upwind scheme is an M-matrix.

In [8], it is shown that $\left|E_i - E_i^N\right| \le C\max\left\{N^{-2}, N^{-5+4i/N}\ln N\right\}$ for $i = N/2, \cdots, N$. Combining this with (3.1) and (3.2) in [8] yields the result:

$$\left|u_i - u_i^N\right| \le C\max\left\{N^{-2}, N^{-5+4i/N}\ln N\right\} \text{ for all } i. \tag{11}$$

Further, as the proof of Theorem 1, Theorem 2 follows. □

Theorem 3 The simple upwind scheme on the Shishkin mesh (2) for (1) satisfies:

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-1}, \ 0 \le i \le p_s N, \\ CN^{-1}\ln N, \ p_s N < i \le N, \end{cases} \tag{12}$$

where $p_s = 1 - \frac{1}{2\mathrm{e}} \approx 0.8161$.

Proof. The matrix associated with the simple upwind scheme is an M-matrix. From Lemma 2.95 in [3], we know $\left|L^N(E_i - E_i^N)\right| \le C\varepsilon^{-1}N^{-1}\mathrm{e}^{-\beta(1-x_i)/\varepsilon}$.

Set a new $\phi_i = C_0\left\{N^{-1} + N^{-1}\ln N \cdot Z_i/Z_N\right\}$ for $i = N/2, \cdots, N$, where constant C_0 is chosen sufficiently large. It is easy to verify that $\left|E_i - E_i^N\right| \le \phi_i$ for $i = N/2, \cdots, N$, by the discrete comparison principle.

Note that $\tau = \frac{2\varepsilon}{\beta}\ln N$ and $h_i = \frac{4\varepsilon}{\beta}N^{-1}\ln N$ for $i = N/2+1, \cdots, N$. As the proof of Lemma 5, we have $\prod_{j=i+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \le CN^{-2(1-i/N)}$ for $N/2 \le i < N$. Thus $\left|E_i - E_i^N\right| \le C\max\left\{N^{-1}, N^{-3+2i/N}\ln N\right\}$ for $i = N/2, \cdots, N$. Combining this inequality with Lemma 2.86 and Corollary 2.95 in [3], we have

$$\left|u_i - u_i^N\right| \le C\max\left\{N^{-1}, N^{-3+2i/N}\ln N\right\} \text{ for all } i. \tag{13}$$

Consequently, as the proof of Theorem 1, the proof is completed. □

Remark. In this paper, for the hybrid scheme and the midpoint upwind scheme, the condition $N > \frac{\gamma^*}{\beta}$, not in [4] and [8], is added in Theorems 1 and 2 for $c(x) \geq 0$ in (1). Moreover, the constants p_h, p_m and p_s are much larger than $\frac{1}{2}$, and the factor C of new error estimates are uniform to ε and N.

3 Numerical results

Example 1 (see [10]). Consider the singularly perturbed problem

$$\begin{cases} -\varepsilon y'' + \dfrac{1}{x+1}y' + \dfrac{1}{x+2}y = f(x),\ 0 < x < 1, \\ y(0) = 1 + 2^{-\frac{1}{\varepsilon}},\ y(1) = \mathrm{e} + 2, \end{cases}$$

where $f(x)$ is chosen such that $y(x) = \mathrm{e}^x + 2^{-\frac{1}{\varepsilon}}(x+1)^{1+\frac{1}{\varepsilon}}$ is the exact solution.

The numerical results of Example 1 by the hybrid scheme (3) on (2) are shown in Table 1, where the numerical convergence order is computed by $\log_2 \frac{\max_{0<i\leq p_h N}\left|u_i - u_i^N\right|}{\max_{0<i\leq p_h N}\left|u_i - u_i^{2N}\right|}$, and the numerical convergence constant is computed by $\max_{0<i\leq p_h N}\left|u_i - u_i^N\right|/N^{-2}$, with the corresponding formulas for $p_h N < i < N$, and for the midpoint upwind scheme and the simple upwind scheme.

Table 1. The numerical results of the hybrid scheme (3) on the Shishkin mesh (2)

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \leq p_h N$	order	const	$i > p_h N$	order	const	$i \leq p_h N$	order	const	$i > p_h N$	order	const
16	0.0178	——	4.553	0.1711	——	5.698	0.0178	——	4.553	0.1711	——	5.698
32	0.0031	2.522	3.212	0.0639	1.421	5.446	0.0031	2.522	3.212	0.0639	1.421	5.446
64	5.1389e-4	2.593	2.105	0.0211	1.599	4.988	5.1395e-4	2.593	2.105	0.0211	1.599	4.988
128	8.4857e-5	2.599	1.390	0.0071	1.571	4.922	8.4882e-5	2.599	1.391	0.0071	1.572	4.922
256	1.5179e-5	2.483	0.995	0.0023	1.626	4.874	1.5190e-5	2.482	0.996	0.0023	1.626	4.874
512	3.1035e-6	2.290	0.814	7.2139e-4	1.673	4.859	3.1083e-6	2.289	0.815	7.2139e-4	1.673	4.859
1024	6.8326e-7	2.183	0.717	2.2239e-4	1.698	4.854	6.8554e-7	2.181	0.719	2.2279e-4	1.695	4.862
2048	1.5924e-7	2.101	0.668	6.7248e-5	1.726	4.852	1.6034e-7	2.096	0.673	6.7628e-5	1.720	4.879

The third and ninth columns in Table 1 show second-order convergence and agree with $(9)_1$ on $[0, x_{[p_h N]}]$. The sixth and twelfth columns show almost second-order convergence and agree with $(9)_2$ on $(x_{[p_h N]}, 1]$. Moreover, the columns of orders and constants in Table 1 show that the convergence is uniform to the different perturbation parameters.

Table 2. The numerical results of the midpoint upwind scheme on the Shishkin mesh

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \leq p_m N$	order	const	$i > p_m N$	order	const	$i \leq p_m N$	order	const	$i > p_m N$	order	const
16	0.0058	——	1.476	0.3578	——	2.065	0.0058	——	1.476	0.3578	——	2.065
32	5.6856e-4	3.351	0.582	0.2550	0.489	2.355	5.6849e-4	3.351	0.582	0.2550	0.489	2.355
64	1.5345e-4	1.890	0.629	0.1735	0.556	2.670	1.5350e-4	1.889	0.629	0.1735	0.556	2.670
128	3.8769e-5	1.985	0.635	0.1091	0.670	2.877	3.8792e-5	1.984	0.636	0.1091	0.670	2.878
256	9.6954e-6	2.000	0.635	0.0655	0.736	3.023	9.7056e-6	1.999	0.636	0.0655	0.736	3.023
512	2.4218e-6	2.001	0.635	0.0381	0.782	3.127	2.4265e-6	2.000	0.636	0.0381	0.782	3.127
1024	6.0438e-7	2.003	0.634	0.0216	0.819	3.191	6.0664e-7	2.000	0.636	0.0216	0.819	3.191
2048	1.5056e-7	2.005	0.632	0.0120	0.848	3.225	1.5166e-7	2.000	0.636	0.0120	0.848	3.225

Table 2 shows the uniform convergence of second-order on $[0, x_{[p_m N]}]$ and almost first-order on $(x_{[p_m N]}, 1]$, and agrees with Theorem 2.

Table 3. The numerical results of the simple upwind scheme on the Shishkin mesh

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \le p_s N$	order	const	$i > p_s N$	order	const	$i \le p_s N$	order	const	$i > p_s N$	order	const
16	0.2228	——	3.565	0.2409	——	1.390	0.2228	——	3.565	0.2409	——	1.390
32	0.1123	0.988	3.594	0.1540	0.646	1.422	0.1123	0.988	3.594	0.1540	0.646	1.422
64	0.0516	1.122	3.303	0.0968	0.670	1.490	0.0516	1.122	3.303	0.0968	0.670	1.490
128	0.0226	1.191	2.888	0.0599	0.693	1.581	0.0226	1.191	2.888	0.0599	0.693	1.581
256	0.0097	1.220	2.484	0.0356	0.751	1.644	0.0097	1.220	2.484	0.0356	0.751	1.644
512	0.0043	1.174	2.187	0.0205	0.796	1.686	0.0043	1.174	2.187	0.0205	0.796	1.686
1024	0.0019	1.178	1.942	0.0116	0.822	1.709	0.0019	1.178	1.942	0.0116	0.822	1.709
2048	8.5786e-4	1.147	1.757	0.0064	0.858	1.719	8.5793e-4	1.147	1.757	0.0064	0.858	1.719

Table 3 shows the uniform convergence of first-order on $[0, x_{[p_s N]}]$ and almost first-order on $(x_{[p_s N]}, 1]$, and verifies Theorem 3.

The log2-log2 graphs of errors to illustrate the convergence orders for the hybrid scheme on $[0, 1-\tau]$, $(1-\tau, x_{[p_h N]}]$ and $(x_{[p_h N]}, 1]$ on the Shishkin mesh are shown in Fig. 1 (a), those for the midpoint upwind scheme and the simple upwind scheme are in Figs. 1 (b) and (c).

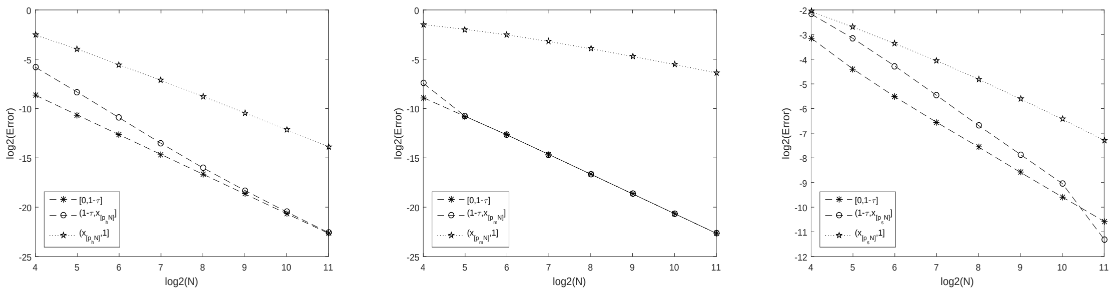

Fig. 1 The log2-log2 graphs of errors on the Shishkin mesh for: (a) the hybrid scheme, (b) the midpoint upwind scheme, (c) the simple upwind scheme.

4 Conclusions

In this paper, the hybrid finite difference scheme is constructed, which is slightly different from the schemes in [4] and [8]. The new estimates on the Shishkin mesh, which are $O(N^{-2})$ for $1 \le i \le p_h N$ and $O(N^{-2} \ln^2 N)$ for $p_h N < i < N$ with $p_h = 1 - \frac{1}{2\mathrm{e}}$ for the hybrid finite difference scheme, $O(N^{-2})$ for $1 \le i \le p_m N$ and $O(N^{-1} \ln N)$ for $p_m N < i < N$ with $p_m = \frac{3}{4} - \frac{1}{4\mathrm{e}}$ for the midpoint upwind scheme, and $O(N^{-1})$ for $1 \le i \le p_s N$ and $O(N^{-1} \ln N)$ for $p_s N < i < N$ with $p_s = 1 - \frac{1}{2\mathrm{e}}$ for the simple upwind scheme, are better than those in [4–6, 8]. The numerical example strongly support our results.

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FEKETE SZEGÖ PROBLEM RELATED TO SIMPLE LOGISTIC ACTIVATION FUNCTION

C. RAMACHANDRAN AND D. KAVITHA

Abstract. In this present paper, we introduce the new subclass of analytic univalent functions associated with quasi-subordination in the field of sigmoid functions. We obtained the coefficient bounds and Fekete-Szego inequality belongs to the defined class. Also, we extracted the new subclasses from the dened class of analytic functions.

Mathematics Subject Classification: Primary:30C45; Secondary:30C50,33E99
Keywords: Univalent functions, Sigmoid function, Subordination, Quasi-subordination, Fekete-Szegö Inequality.

1. Introduction and preliminaries

Sigmoid function playing an important role in the branch of special functions which is the part of logistic activation function developed in eighteenth century. The theory of special functions has been developed by C. F.Gauss, C. G. J. Jacobi, F. Klein and many others in nineteenth century. However, in the twentieth century , from the perspective of fundamental science sigmoid functions are of special interest in abstract areas such as approximation theory, functional analysis, topology, differential equations and probability theory and so on.

A typical applications of the sigmoid function includes neural networks, image processing, artificial networks, biomathematics, chemistry, geoscience, probability theory, economics etc., We can find the similar kind of functions called gompertz function and ogee function which are used in modelling systems to saturate at more values of time period. The evaluation process of sigmoid function in many ways especially by truncated series expansion method was seen in [4, 10].

Recently Ramachandran et al. [13] discssed the problem of Hankel determinant for the subclass of analytic and univalent functions. The sigmoid function is of the form

$$h(z) = \frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

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- Bound output real numbers between 0 and 1, it leads to the probability theory.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is an injective function.
- It increases monotonically.

The above properties permit us to use sigmoid function in the univalent function theory.

In computational networks, this sigmoid function leads the output as digital numbers 1 for ON and 0 for OFF. Kannan et al.[6] brought out contrast enhancement using modified sigmoid function provides the highest measure of contrast and can be effectively used for further analysis of sports color images.

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open disk $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$ is of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.2}$$

and normalized by $f(0) = f'(0) - 1 = 0$ and let $\mathcal{S}$ be a class of all functions in $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.

If $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write $f(z) \prec g(z)$, $z \in \mathbb{U}$ if there exits a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$, then We have that

$$f \prec g \quad or \quad f(z) \prec g(z), z \in \mathbb{U}$$

if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$ defined by[11].

In the year 1970, Robertson [15] introduced the concept of quasi-subordination. For, two analytic functions $f(z)$ and $g(z)$, the function $f(z)$ is quasi-subordinate to $g(z)$ in the open unit disc $\mathbb{U}$, written by

$$f(z) \prec_q g(z).$$

If there exist an analytic function φ and ω, with $|\varphi(z)| \leq 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$f(z) = \varphi(z) g(\omega(z)), \quad (z \in \mathbb{U}).$$

Observe that if $\varphi(z) \equiv 1$, then $f(z) = g(\omega(z))$, so that $f(z) \prec g(z)$ in $\mathbb{U}$. Furthermore, if $\omega(z) = z$, then $f(z) = \varphi(z)g(z)$, said to be that $f(z)$ is majorized by $g(z)$ and symbolically written as $f(z) \ll g(z)$ in $\mathbb{U}$. Hence it is obvious that the quasi-subordination is a generalization of subordination as well as majorization [2, 8, 16].

Haji Mohd and Darus [5] introduced the concepts of q-starlike and q-convex functions as follows:

Definition 1. *Let the class $\mathcal{S}_q^*(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfies the quasi-subordination*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}). \tag{1.3}$$

Example 1. *A function $f \in \mathbb{U} \to \mathbb{C}$ defined by*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) = z(\varphi(z) - 1) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}).$$

belongs to the class $\mathcal{S}_q^*(\varphi)$.

Definition 2. *Let the class $\mathcal{C}_q(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfies the quasi-subordination*

$$\left(\frac{zf''(z)}{f'(z)}\right) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}). \tag{1.4}$$

Example 2. *A function $f \in \mathbb{U} \to \mathbb{C}$ defined by*

$$\left(\frac{zf''(z)}{f'(z)}\right) = z(\varphi(z) - 1) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}).$$

belongs to the class $\mathcal{C}_q(\varphi)$.

To prove our main results, we need the following lemmas:

Lemma 1. [7] *Let ω be the analytic function in $\mathbb{D}$, with $\omega(0) = 0, |\omega(z)| < 1$ and $\omega(z) = \omega_1 z + \omega_2 z^2 + \dots$, then $|\omega_2 - \nu\omega_1^2| \leq max[1; |\nu|]$, where $\nu \in \mathbb{C}$. The result is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.*

Lemma 2. [3] *Let ω be the analytic function in $\mathbb{D}$, with $\omega(0) = 0, |\omega(z)| < 1$ and $\omega(z) = \omega_1 z + \omega_2 z^2 + \dots$, then*

$$|\omega_n| \leq \begin{cases} 1, & n = 1 \\ 1 - |\omega_1|^2, & n \geq 2. \end{cases}$$

The result is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.

Lemma 3. [11] *Let φ be an analytic function with positive real part in $\mathbb{D}$, with $|\varphi(z)| < 1$ and let $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \dots$ Then $|c_0| \leq 1$ and $|c_n| \leq 1 - |c_0|^2 \leq 1$, for $n > 0$.*

Lemma 4. [4] *Let h be the sigmoid function defined in (1.1) and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m, \tag{1.5}$$

then $\Phi(z) \in P, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 5. [4] *Let*

$$\Phi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m$$

then $|\Phi_{n,m}(z)| < 2$.

Lemma 6. [4] *If $\Phi(z) \in P$ and it is starlike, then f is a normalized univalent function of the form (1.2). Taking m = 1, Joseph et al [4] remarked the following:*

Remark 1. *Let*

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \dfrac{(-1)^{n+1}}{2n!}$ then $|c_n| \leq 2, n = 1, 2, 3 \dots$ this result is sharp for each n see [4].

Motivated by the earlier works of Ramachandran et al. [14], we define the class of function involving quasi-subordination in terms of sigmoid functions.

Definition 3. *A function $f \in \mathcal{A}$ is in the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ if*

$$\left[\frac{zf'(z)}{f(z)}\right]^{\alpha} \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} - 1 \prec_q \Phi(z) - 1 \tag{1.6}$$

here $0 < \beta \leq 1, \quad 0 \leq \alpha \leq 1, \quad 0 \leq \lambda \leq 1$.

With various choices of the parameters, the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduces to the following new classes,

(1) $M_q^{0,0,1}(\Phi_{n,m}) \equiv \mathcal{S}_q^*(\Phi_{n,m})$,
(2) $M_q^{0,1,1}(\Phi_{n,m}) \equiv \mathcal{C}_q(\Phi_{n,m})$,
(3) $M_q^{0,\lambda,1}(\Phi_{n,m}) \equiv M_q^{\lambda}(\Phi_{n,m})$.

In this present paper, we determine the coefficient estimates including a Fekete-Szegö inequality of functions belonging to the above defined class and the class

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involving majorization. This result can assist us to represent various geometric interpretation as well as behaviours of the functions in complex domain.

Let $f(z)$ be of the form (1.2), $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \ldots.$ and $\omega(z) = \omega_1 z + \omega_2 z^2 + \ldots$, throughout this paper unless otherwise mentioned.

2. Fekete-Szegö Inequality

In this section we obtain the first two coefficient estimates and the Fekete-Szegö Inequality for the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$.

Theorem 1. *If $f(z) \in M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$, then*

$$|a_2| \leq \frac{1}{2[\alpha + \beta(1+\lambda)]},$$

$$|a_3| \leq \frac{1}{4[\alpha + \beta(1+2\lambda)]} max \left\{1, \left|\frac{\alpha(\alpha-3) + \beta(\beta-1)(1+\lambda)^2 + 2\alpha\beta(1+\lambda) - 2\beta(1+3\lambda)}{4[\alpha+\beta(1+\lambda)]^2}\right|\right\},$$

and for any complex number μ, we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{4[\alpha + \beta(1+2\lambda)]} max \left\{1, \left|\Lambda + \frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right|\right\},$$

where

$$\Lambda = \frac{\alpha(\alpha-3) + \beta(\beta-1)(1+\lambda)^2 + 2\alpha\beta(1+\lambda) - 2\beta(1+3\lambda)}{4[\alpha+\beta(1+\lambda)]^2}. \tag{2.1}$$

Proof. Since $f \in \mathcal{A}$ belongs to the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$, then from (1.6) we have

$$\left[\frac{zf'(z)}{f(z)}\right]^\alpha \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^\beta - 1 = \varphi(z)(\Phi(z)-1), z \in \mathbb{U}. \tag{2.2}$$

The modified sigmoid function $\Phi(z)$ can be expressed as

$$\Phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 + \ldots$$

since $\varphi(z)$ as defined earlier, now we obtain

$$\begin{aligned}\varphi(z)(\Phi(\omega(z))-1) &= (c_0 + c_1 z + c_2 z^2 + \ldots.)\left[\frac{\omega_1}{2}z + \frac{\omega_2}{2}z^2 + \left(\frac{\omega_3}{2} - \frac{\omega_1^3}{24}\right)z^3 + \ldots\right] \\ &= \frac{c_0\omega_1}{2}z + \left(\frac{c_0\omega_2}{2} + \frac{c_1\omega_1}{2}\right)z^2 + \ldots\end{aligned} \tag{2.3}$$

by replacing the equivalent expressions of $f(z)$, $\dfrac{f'(z)}{f(z)}$ and $\dfrac{zf''(z)}{f'(z)}$ in (2.2) and the simple calculation yields the following,

$$\left[\frac{zf'(z)}{f(z)}\right]^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}-1=[\alpha+\beta(1+\lambda)]a_2 z$$
$$+\left\{2[\alpha+\beta(1+2\lambda)]a_3+\left[\frac{\alpha(\alpha-3)}{2}+\frac{\beta(\beta-1)}{2}(1+\lambda)^2+\alpha\beta(1+\lambda)-\beta(1+3\lambda)\right]a_2^2\right\}z^2+\dots \tag{2.4}$$

Equating right hand side part of (2.3) and (2.4), we get,

$$a_2=\frac{c_0\omega_1}{2[\alpha+\beta(1+\lambda)]}, \tag{2.5}$$

and

$$\begin{aligned} a_3=&\frac{1}{4[\alpha+\beta(1+2\lambda)]}\left\{c_1\omega_1+c_0\left[\omega_2\right.\right.\\ &\left.\left.-\left(\frac{\alpha(\alpha-3)+\beta(\beta-1)(1+\lambda)^2+2\alpha\beta(1+\lambda)-2\beta(1+3\lambda)}{4[\alpha+\beta(1+\lambda)]^2}\right)\omega_1^2c_0\right]\right\}. \end{aligned} \tag{2.6}$$

Using the hypothesis of Lemma 3 and the well-known inequality of Lemma 2, for $n>0$

$$|c_n|\le 1-|c_0|^2\le 1.$$

and

$$|\omega_1|\le 1$$

we have,

$$|a_2|\le\frac{1}{2[\alpha+\beta(1+\lambda)]},$$

and for any $\mu\in\mathbb{C}$, we obtain from (2.5) and (2.6)

$$a_3-\mu a_2^2=\frac{1}{4[\alpha+\beta(1+2\lambda)]}\left\{c_1\omega_1+c_0\left[\omega_2-\left(\Lambda+\frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right)\omega_1^2c_0\right]\right\}.$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{U}$, using [11], for some $y, |y|\le 1$:

$$|c_0|\le 1 \qquad and \qquad c_1=(1-c_0^2)y.$$

Now, replacing the value of c_1 as defined above, we get

$$a_3-\mu a_2^2=\frac{1}{4[\alpha+\beta(1+2\lambda)]}\left\{y\omega_1+c_0\omega_2-\left[\left(\Lambda+\frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right)\omega_1^2+y\omega_1\right]c_0^2\right\}. \tag{2.7}$$

If $c_0 = 0$ then

$$|a_3 - \mu a_2^2| \leq \frac{|\omega_1||y|}{4[\alpha+\beta(1+2\lambda)]} = \frac{1}{4[\alpha+\beta(1+2\lambda)]}.$$

If $c_0 \neq 0$ then

$$|a_3 - \mu a_2^2| \leq \frac{1}{4[\alpha+\beta(1+2\lambda)]} max\left\{1, \left|\Lambda + \frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right|\right\} \tag{2.8}$$

and the result is sharp. □

Further setting $\mu = 0$ in (2.8) we get the bound on $|a_3|$. This completes the proof of the Theorem 1.

Corollary 1. *Let $\alpha = 0$, $\lambda = 0$ and $\beta = 1$ the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduced to $\mathcal{S}_q^*(\Phi_{n,m})$ then we have,*

$$a_2 = \frac{c_0\omega_1}{2},$$

and

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} max\left\{1, \left|\frac{2\mu-1}{2}\right|\right\}.$$

Corollary 2. *Let $\alpha = 0$, $\lambda = 1$ and $\beta = 1$ the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduced to $\mathcal{C}_q(\Phi_{n,m})$ then we have,*

$$a_2 = \frac{c_0\omega_1}{4},$$

and

$$|a_3 - \mu a_2^2| \leq \frac{1}{12} max\left\{1, \left|\frac{3\mu-2}{4}\right|\right\}.$$

Corollary 3. *Let $\alpha = 0$ and $\beta = 1$ the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduced to $M_q^{\lambda}(\Phi_{n,m})$ then we have,*

$$a_2 = \frac{c_0\omega_1}{2(1+\lambda)},$$

and

$$|a_3 - \mu a_2^2| \leq \frac{1}{4(1+2\lambda)} max\left\{1, \left|\frac{2\mu(1+2\lambda)-(1+3\lambda)}{2(1+\lambda)^2}\right|\right\}.$$

Theorem 2. *If $f \in \mathcal{A}$, such that the function*

$$\left[\frac{zf'(z)}{f(z)}\right]^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta} - 1 \ll \Phi(z) - 1, \quad z \in \mathbb{U}$$

then,

$$|a_2| \leq \frac{1}{2|\alpha+\beta(1+\lambda)|},$$

and for any complex number μ

$$|a_3 - \mu a_2^2| \leq \frac{1}{4[\alpha+\beta(1+2\lambda)]} max \left\{1, \left|\Lambda + \frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right|\right\}.$$

Proof. Taking $\omega(z) = z$ in the proof of Theorem 1, we get the desired result. □

3. Conclusion

Finding the estimates for various subclasses of analytic functions with normalization is the most important role of geometric function theory. These estimates characterise the behaviours of functions in complex domain. This characterisation provides a tool using the sigmoid function in wide range of fields like image processing, digital communications, neural sciences etc.,

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On the second kind twisted q-Euler numbers and polynomials of higher order

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Abstract : In this paper, we introduce the second kind twisted q-Euler polynomials $E^{(k)}_{n,\omega,q}(x)$ of order k. We also get interesting properties related to the second kind twisted q-Euler numbers and polynomials. Finally, we construct twisted q-zeta function of order which interpolates the second kind twisted q-Euler numbers of higher order at negative integer.

Key words : Euler numbers, Euler polynomials, the second kind Euler numbers and polynomials, q-zeta function, twisted q-Euler numbers and polynomials, twisted q-Euler numbers and polynomials of higher order, twisted q-zeta function.

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1. Introduction

Recently, mathematicians have studied Euler numbers, Euler polynomials, the second kind Euler numbers and the second kind Euler polynomials (see [1-9]). These numbers and polynomials possess many interesting properties and arising in many areas of mathematics, applied mathematics, and physics. In this paper, we introduce the second kind twisted q-Euler numbers $E^{(k)}_{n,\omega,q}$ and polynomials $E^{(k)}_{n,\omega,q}(x)$ of higher order. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ of the function $g \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [1, 3]}. \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} g(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2g(0). \tag{1.2}$$

First, we introduce the second kind q-Euler numbers $E^{(k)}_{n,q}$ of higher order k. The second kind q-Euler numbers $E^{(k)}_{n,q}$ of higher order k are defined by the generating function:

$$\left(\frac{2e^t}{qe^{2t}+1}\right)^k = \sum_{n=0}^{\infty} E^{(k)}_{n,q} \frac{t^n}{n!}, \quad (|\log q + 2t| < \pi). \tag{1.3}$$

Let $T_p = \cup_{N\ge1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$. We introduce the second kind twisted q-Euler polynomials $E_{n,\omega,q}(x)$ as follows:

$$\frac{2e^t}{\omega q e^{2t}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w,q}(x) \frac{t^n}{n!}. \tag{1.4}$$

In [5], we obtain the second kind twisted q-Euler numbers $E_{n,\omega,q}$ polynomials $E_{n,\omega,q}(x)$ and investigate their properties.

Theorem 1. For positive integers $n, \omega \in T_p$, we have

$$\int_{\mathbb{Z}_p} \phi_\omega(x) q^x (2x+1)^n d\mu_{-1}(x) = E_{n,\omega,q},$$

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^y (x+2y+1)^n d\mu_{-1}(y) = E_{n,\omega,q}(x).$$

2. The second kind twisted q-Euler polynomials of higher order

The main purpose of this section is to study a systemic properties of the second kind twisted q-Euler numbers and polynomials of higher order. In this section, we assume that $q \in \mathbb{C}_p$. We construct the second kind twisted q-Euler numbers $E^{(k)}_{n,\omega,q}$ and polynomials $E^{(k)}_{n,\omega,q}(x)$ of higher order k. We use the notation

$$\sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} = \sum_{k_1 \cdots k_n=0}^{m}.$$

The binomial formulae are known as

$$(1-a)^n = \sum_{i=0}^{n} \binom{n}{i} (-a)^i, \text{ where } \binom{n}{i} = \frac{n(n-1)\dots(n-i+1)}{i!},$$

and

$$\frac{1}{(1-a)^n} = (1-a)^{-n} \sum_{i=0}^{n} \binom{-n}{i} (-a)^i = \sum_{i=0}^{n} \binom{n+i-1}{i} a^i$$

Now, using multiple of p-adic q-integral, we introduce the second kind twisted q-Euler polynomials $E^{(k)}_{n,w,q}(x)$ of higher order : For $k \in \mathbb{N}$, we define

$$\begin{aligned} &\sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}(x) \frac{t^n}{n!} \\ &\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} \omega^{x_1+\cdots+x_k} q^{x_1+x_2+\cdots+x_k} e^{(x+2x_1+2x_2+\cdots+2x_k+k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \end{aligned} \tag{2.1}$$

By using Taylor series of $e^{(x+2x_1+2x_2+\cdots+2x_k+k)t}$ in the above equation, we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1+\cdots+x_k} q^{x_1+\cdots+x_k} (x+2x_1+\cdots+2x_k+k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} E^{(k)}_{n,w,q}(x) \frac{t^n}{n!}. \end{aligned}$$

By comparing coefficients $\dfrac{t^n}{n!}$ on the above equation, we arrive at the following theorem.

Theorem 2. For positive integers n and k, we have

$$E^{(k)}_{n,\omega,q}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1+\cdots+x_k} q^{x_1+\cdots+x_k} (x+2x_1+\cdots+2x_k+k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.2}$$

By (2.1), the second kind twisted q-Euler polynomials of higher order, $E^{(k)}_{n,\omega,q}(x)$ are defined by means of the following generating function

$$F^{(k)}_{\omega,q}(x,t) = \left(\frac{2e^t}{\omega q e^{2t}+1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}(x) \frac{t^n}{n!}. \tag{2.3}$$

Again, by using (2,1), the second kind twisted q-Euler numbers of higher order, $E^{(k)}_{n,\omega,q}$ are defined by the following generating function

$$\left(\frac{2e^t}{\omega q e^{2t}+1}\right)^k = \sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}\frac{t^n}{n!}, \quad |t+\log q| < \frac{\pi}{2}. \tag{2.4}$$

When $k = 1$, above (2.3) and (2.4) will become the corresponding definitions of the second kind twisted q-Euler polynomials $E_{n,\omega,q}(x)$ and the second kind twisted q-Euler numbers $E_{n,\omega,q}$. Observe that for $x = 0$, the equation (2.4) reduces to (2.3). Note that when $k = 1$, then we have (1.4), when $q \to 1$, then we have

$$\left(\frac{2e^t}{\omega e^{2t}+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E^{(k)}_{n,\omega}(x)\frac{t^n}{n!},$$

where $E^{(k)}_{n,\omega}(x)$ denote the second kind twisted Euler polynomials of higher order k. In the case when $x = 0$ in (2.1), we have the following corollary.

Corollary 3. For positive integers n, k, we have

$$E^{(k)}_{n,\omega,q} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \omega^{x_1+\cdots+x_k} q^{x_1+\cdots+x_k}(2x_1+\cdots+2x_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).$$

By using binomial expansion in (2.2), we obtain

$$E^{(k)}_{n,\omega,q}(x) = \sum_{l=0}^{n}\binom{n}{l}x^{n-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \omega^{x_1+\cdots+x_k} q^{x_1+\cdots+x_k}(2x_1+\cdots+2x_k+k)^l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).$$

Again, by Corollary 3, we arrive at the following theorem.

Theorem 4. For positive integers n, k, we have

$$E^{(k)}_{n,\omega,q}(x) = \sum_{l=0}^{n}\binom{n}{l}E^{(k)}_{l,\omega,q}x^{n-l}.$$

We define distribution relation of the second kind twisted q-Euler polynomials of higher order as follows: For $m \in \mathbb{N}$ with $m \equiv 1(\mod 2)$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}(x)\frac{t^n}{n!}\\
&= \left(\frac{2e^t}{\omega q e^{2t}+1}\right)\left(\frac{2e^t}{\omega q e^{2t}+1}\right)\cdots\left(\frac{2e^t}{\omega q e^{2t}+1}\right)e^{xt}\\
&= \left(\frac{2e^{mt}}{\omega^m q^m e^{2mt}+1}\right)^k \sum_{a_1,\cdots,a_k=0}^{m-1} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k} e^{\left(\frac{2a_1+\cdots+2a_k+k+x-mk}{m}\right)(mt)}.
\end{aligned}$$

From the above, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}(x)\frac{t^n}{n!}\\
&= \sum_{a_1,\cdots,a_k=0}^{m-1} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k}\sum_{n=0}^{\infty} E^{(k)}_{n,\omega^m,q^m}\left(\frac{2a_1+\cdots+2a_k+k+x-mk}{m}\right)\frac{(mt)^n}{n!}.
\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 5 (Distribution relation of the second kind twisted q-Euler polynomials of higher order). For $m \in \mathbb{N}$ with $m \equiv 1(\mod 2)$, we have

$$E_{n,\omega,q}^{(k)}(x) = m^n \sum_{a_1,\cdots,a_k=0}^{m-1} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k} E_{n,\omega^m,q^m}^{(k)}\left(\frac{2a_1+\cdots+2a_k+k+x-mk}{m}\right).$$

By (2.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!} &= 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k} e^{(2a_1+\cdots+2a_k+k+x)t} \\ &= 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \omega^m q^m e^{(2m+k+x)t}.\end{aligned} \tag{2.5}$$

From the above, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k}(x+2a_1+\cdots+2a_k+k)^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m (2m+k+x)^n\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 6. For positive integers n and k, we have

$$\begin{aligned}E_{n,\omega,q}^{(k)}(x) &= 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k}(2a_1+\cdots+2a_k+k+x)^n \\ &= 2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m(2m+k+x)^n.\end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}\sum_{l=0}^{\infty} E_{l,\omega,q}^{(k)}(x+y)\frac{t^l}{l!} &= \left(\frac{2e^t}{\omega q e^{2t}+1}\right)^k e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!} y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} E_{n,\omega,q}^{(k)}(x) y^{l-n}\right)\frac{t^l}{l!},\end{aligned}$$

we have the following addition theorem.

Theorem 7. The second kind twisted q-Euler polynomials $E_{n,\omega,q}^{(k)}(x)$ of higher order satisfies the following relation:

$$E_{n,\omega,q}^{(k)}(x+y) = \sum_{l=0}^{n}\binom{n}{l} E_{l,\omega,q}^{(k)}(x) y^{n-l}.$$

3. Multiple twisted q-Euler zeta function

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let ω be the p^N-th root of unity. We define multiple twisted q-Euler zeta function. This function interpolates the second kind twisted q-Euler polynomials of higher order at negative integers.

By using (2.5), we have

$$F_{\omega,q}^{(k)}(x,t) = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k} e^{(2a_1+\cdots+2a_k+k+x)t} = \sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!}.$$

For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we can derive the following Eq. (3.1) form the Mellin transformation of $F_{\omega,q}^{(k)}(x,t)$.

$$\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F_{\omega,q}^{(k)}(x,-t)dt = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k}\omega^{a_1+\cdots+a_k}q^{a_1+\cdots+a_k}}{(2a_1+\cdots+2a_k+k+x)^s} \tag{3.1}$$

For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we define the multiple twisted q-Euler zeta function as follows:

Definition 8. For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we define

$$\zeta_{\omega,q}^{(k)}(s,x) = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k}\omega^{a_1+\cdots+a_k}q^{a_1+\cdots+a_k}}{(2a_1+\cdots+2a_k+k+x)^s}. \tag{3.2}$$

For $s = -l$ in (3.2) and using (2.6), we arrive at the following theorem.

Theorem 9. For positive integer l, we have

$$\zeta_{\omega,q}^{(k)}(-l,x) = E_{l,\omega,q}^{(k)}(x).$$

By (2.4), we have

$$\sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}\frac{t^n}{n!} = \left(\frac{2e^t}{\omega q e^{2t}+1}\right)^k = 2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m e^{(2m+k)t}.$$

By using Taylor series of $e^{(2m+k)t}$ in the above, we have

$$\sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m (2m+k)^n\right)\frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$ in the above equation, we have

$$E_{n,\omega,q}^{(k)} = 2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m (2m+k)^n. \tag{3.3}$$

By using (3.3), we define twisted q-Euler zeta function as follows:

Definition 10. For $s \in \mathbb{C}$, we define

$$\zeta_{\omega,q}^{(k)}(s) = 2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}\frac{(-1)^m\omega^m q^m}{(2m+k)^s}. \tag{3.4}$$

The function $\zeta_{\omega,q}^{(k)}(s)$ interpolates the number $E_{n,\omega,q}^{(k)}$ at negative integers. Substituting $s = -n$ with $n \in \mathbb{Z}_+$ into (3.4), and using (3.3), we obtain the following theorem:

Theorem 11. Let $n \in \mathbb{Z}_+$, We have

$$\zeta_{\omega,q}^{(k)}(-n) = E_{n,\omega,q}^{(k)}.$$

Further, by (3.2) and (3.4), we have

$$\sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k}\omega^{a_1+\cdots+a_k}q^{a_1+\cdots+a_k}}{(2a_1+\cdots+2a_k+k)^s} = \sum_{m=0}^{\infty} \binom{m+k-1}{m} \frac{(-1)^m\omega^m q^m}{(2m+k)^s}.$$

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HERMITE-HADAMARD INEQUALITY AND GREEN'S FUNCTION WITH APPLICATIONS*

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ABSTRACT. In the article, we derive the Hermite-Hadamard inequality by using Green's function, establish some Hermite-Hadamrd type inequalities for the class of monotonic as well as convex functions, and give applications for means, mid-point and trapezoid formulae.

1. INTRODUCTION

Convexity plays an important role in different fields of pure and applied sciences such as statistics, optimization theory, economics and finance etc. The fundamental justification for the significance of convexity is its meaningful relationship with the theory of inequalities. Many useful inequalities have been obtained by using convexity. Among those inequalities, the most extensively and intensively attractive inequality in the last decades is the well known Hermite-Hadamard inequality [1-9], which can be stated as follows: the double inequality

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) \le \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx \le \frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} \tag{1.1}$$

holds if the function $\psi : [\alpha_1, \alpha_2] \to \mathbb{R}$ is a convex function. If ψ is a concave function then (1.1) holds in the reverse direction.

The Hermite-Hadamard inequality gives an upper as well as lower estimations for the integral mean of any convex function defined on closed and bounded interval which involves the the endpoints and midpoint of the domain of the function. Also inequality (1.1) provides the necessary and sufficient condition for the function to be convex. There are several applications of the Hermite-Hadamard inequality in the geometry of Banach spaces [10] and nonlinear analysis [11]. Some peculiar convex functions can be used in (1.1) to obtain classical inequalities for means. For some comprehensive surveys on various generalizations and developments of inequality (1.1) we recommend [12]. Due to the great importance of the convexity and the Hermite-Hadamard inequlity, in the recent years many generalizations, refinements and extensions can be found in the literature [13-37]

In the article, we give a new proof for the Hermite-Hadamard inequality by using Green's function, obtain some refinements of the Hermite-Hadamard inequality

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for monotonic functions as well as convex functions. At the end, we give some applications for means, mid-point and trapezoid formulae.

2. Main Results

In order to obtain our main results we need to establish a lemma, which we present in this section.

Lemma 2.1. *Let $\mathcal{G}$ be the Green's function defined on $[\alpha_1, \alpha_2] \times [\alpha_1, \alpha_2]$ by*

$$\mathcal{G}(\lambda, \mu) = \begin{cases} \alpha_1 - \mu\ , & \alpha_1 \leq \mu \leq \lambda; \\ \alpha_1 - \lambda, & \lambda \leq \mu \leq \alpha_2. \end{cases} \tag{2.1}$$

Then any $\psi \in C^2([\alpha_1, \alpha_2])$ can be expressed as

$$\psi(x) = \psi(\alpha_1) + (x - \alpha_1)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x, \mu)\psi''(\mu)d\mu. \tag{2.2}$$

Proof. By using the techniques of integration by parts in $\int_a^b \mathcal{G}(t, \mu)\psi''(\mu)d\mu$, we can easily obtain (2.2). □

The following Theorem 2.2 give a new proof for the Hermite-Hadamard inequality.

Theorem 2.2. *Let $\psi \in C^2([\alpha_1, \alpha_2])$. Then the double inequality*

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \psi(x)dx \leq \frac{\psi(\alpha_1) + \psi(\alpha_2)}{2} \tag{2.3}$$

holds if ψ is convex on $[\alpha_1, \alpha_2]$.

Proof. Let $x = (\alpha_1 + \alpha_2)/2$. Then (2.2) leads to

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) = \psi(\alpha_1) + \left(\frac{\alpha_1 + \alpha_2}{2} - \alpha_1\right)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}\left(\frac{\alpha_1 + \alpha_2}{2}, \mu\right)\psi''(\mu)d\mu,$$

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) = \psi(\alpha_1) + \left(\frac{\alpha_2 - \alpha_1}{2}\right)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}\left(\frac{\alpha_1 + \alpha_2}{2}, \mu\right)\psi''(\mu)d\mu. \tag{2.4}$$

Taking integral of (2.2) with respect to x and dividing by $\alpha_2 - \alpha_1$, we get

$$\begin{aligned} \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \psi(x)dx &= \psi(\alpha_1) + \frac{1}{\alpha_2 - \alpha_1}\left(\frac{\alpha_2^2 - \alpha_1^2}{2} - \alpha_1(\alpha_2 - \alpha_1)\right)\psi'(\alpha_2) \\ &\quad + \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x, \mu)\psi''(\mu)d\mu dx, \end{aligned}$$

$$\begin{aligned} \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \psi(x)dx &= \psi(\alpha_1) + \left(\frac{\alpha_2 - \alpha_1}{2}\right)\psi'(\alpha_2) \\ &\quad + \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x, \mu)\psi''(\mu)d\mu dx. \end{aligned} \tag{2.5}$$

Subtracting (2.5) from (2.4) we obtain

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \psi(x)dx$$

$$= \int_{\alpha_1}^{\alpha_2} \mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) \psi''(\mu)d\mu - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)\psi''(\mu)d\mu dx$$

$$= \int_{\alpha_1}^{\alpha_2} \left[\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx\right] \psi''(\mu)d\mu. \tag{2.6}$$

Note that

$$\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx = \frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu, \tag{2.7}$$

$$\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) = \begin{cases} \alpha_1 - \mu\ , & \alpha_1 \le \mu \le \frac{\alpha_1+\alpha_2}{2}; \\ \frac{\alpha_1-\alpha_2}{2}, & \frac{\alpha_1+\alpha_2}{2} \le \mu \le \alpha_2. \end{cases}$$

If $\alpha_1 \le \mu \le \frac{\alpha_1+\alpha_2}{2}$, then

$$\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx$$

$$= \alpha_1 - \mu - \frac{1}{\alpha_2-\alpha_1}\left(\frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu\right)$$

$$= \frac{-(\mu-\alpha_1)^2}{2(\alpha_2-\alpha_1)} \le 0. \tag{2.8}$$

If $\frac{\alpha_1+\alpha_2}{2} \le \mu \le \alpha_2$, then

$$\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx$$

$$= \frac{\alpha_1-\alpha_2}{2} - \frac{1}{\alpha_2-\alpha_1}\left(\frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu\right)$$

$$= \frac{-(\alpha_2-\mu)^2}{2(\alpha_2-\alpha_1)} \le 0. \tag{2.9}$$

From the convexity of ψ we know that $\psi''(\mu) \ge 0$. Therefore, the first inequality of (2.3) follows easily from (2.6), (2.8) and (2.9).

Next, we prove second inequality of (2.3).

Let $x = \alpha_2$. Then (2.2) gives

$$\psi(\alpha_2) = \psi(\alpha_1) + (\alpha_2-\alpha_1)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}(\alpha_2,\mu)\psi''(\mu)d\mu,$$

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} = \psi(\alpha_1) + \frac{1}{2}(\alpha_2-\alpha_1)\psi'(\alpha_2) + \frac{1}{2}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(\alpha_2,\mu)\psi''(\mu)d\mu. \tag{2.10}$$

It follows from (2.5) and (2.10) that

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \psi(x)dx$$

$$= \int_{\alpha_1}^{\alpha_2} \left(\frac{1}{2}\mathcal{G}(\alpha_2,\mu) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx\right)\psi''(\mu)d\mu. \tag{2.11}$$

From (2.1) one has

$$\mathcal{G}(\alpha_2,\mu) = \alpha_1 - \mu \tag{2.12}$$

if $\alpha_1 \le \mu \le \alpha_2$.

It follows from (2.7) and (2.12) that

$$\begin{aligned}
&\frac{1}{2}\mathcal{G}(\alpha_2,\mu) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x,\mu)dx \\
&= \frac{\alpha_1-\mu}{2} - \frac{1}{\alpha_2-\alpha_1}\left(\frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu\right) \\
&= \frac{1}{2(\alpha_2-\alpha_1)}\left((\alpha_1-\mu)(\alpha_2-\alpha_1) - \mu^2 + \alpha_1^2 - 2\alpha_1\alpha_2 + 2\alpha_2\mu\right) \\
&= \frac{1}{2(\alpha_2-\alpha_1)}\left((\mu-\alpha_1)(\alpha_2-\mu)\right) \ge 0. \qquad (2.13)
\end{aligned}$$

Therefore,

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx \ge 0$$

follows from (2.11) and (2.13) together with $\psi''(\mu) \ge 0$. □

Next, we give some refinements of the Hermite-Hadamard inequality for the class of monotonic and convex functions.

Theorem 2.3. *Let $\psi \in C^2([\alpha_1,\alpha_2])$. Then the following statements are true:*

(1) If $|\psi''|$ is increasing, then

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right| \le \frac{(\alpha_2-\alpha_1)^2}{48}\left[|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)| + |\psi''(\alpha_2)|\right];$$

(2) If $|\psi''|$ is decreasing, then

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right| \le \frac{(\alpha_2-\alpha_1)^2}{48}\left[|\psi''(\alpha_1)| + |\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)|\right];$$

(3) If $|\psi''|$ is convex, then

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right| \\
&\le \frac{(\alpha_2-\alpha_1)^2}{48}\left[\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|, |\psi''(\alpha_1)|\right\} + \max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|, |\psi''(\alpha_2)|\right\}\right].
\end{aligned}$$

Proof. (1) By using (2.6) we have

$$\begin{aligned}
&\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx \\
&= \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\left[\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2},\mu\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x,\mu)dx\right]\psi''(\mu)d\mu \\
&\quad + \int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\left[\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2},\mu\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x,\mu)dx\right]\psi''(\mu)d\mu \\
&= \frac{-1}{2(\alpha_2-\alpha_1)}\left[\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2\psi''(\mu)d\mu + \int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2\psi''(\mu)d\mu\right].
\end{aligned}$$

Taking absolute and using triangular inequality we obtain

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq \frac{1}{2(\alpha_2-\alpha_1)}\left[\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2\left|\psi''(\mu)\right|d\mu+\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2\left|\psi''(\mu)\right|d\mu\right]$$

$$\leq \frac{1}{2(\alpha_2-\alpha_1)}\left[\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2d\mu+\left|\psi''(\alpha_2)\right|\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2d\mu\right]$$

$$=\frac{1}{2(\alpha_2-\alpha_1)}\left[|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)|\times\frac{1}{24}(\alpha_2-\alpha_1)^3+|\psi''(\alpha_2)|\times\frac{1}{24}(\alpha_2-\alpha_1)^3\right]$$

$$=\frac{(\alpha_2-\alpha_1)^2}{48}\left[|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)|+|\psi''(\alpha_2)|\right]. \tag{2.14}$$

Similarly we can prove part (2).

For part (3), using (2.14) and the fact that every convex function ψ defined on the interval $[\alpha_1,\alpha_2]$ is bounded above by $\max\{\psi(\alpha_1),\psi(\alpha_2)\}$, we obtain

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq \frac{1}{2(\alpha_2-\alpha_1)}\left[\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_1)|\right\}\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2d\mu\right.$$

$$\left.+\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_2)|\right\}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2d\mu\right]$$

$$=\frac{(\alpha_2-\alpha_1)^2}{48}\left[\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_1)|\right\}+\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_2)|\right\}\right].$$

□

Theorem 2.4. *Let $\psi\in C^2([\alpha_1,\alpha_2])$. Then the following statements are true:*

(1) If $|\psi''|$ is increasing, then

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\leq\frac{|\psi''(\alpha_2)|(\alpha_2-\alpha_1)^2}{12};$$

(2) If $|\psi''|$ is decreasing, then

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\leq\frac{|\psi''(\alpha_1)|(\alpha_2-\alpha_1)^2}{12};$$

(3) If $|\psi''|$ is a convex function, then

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\leq\frac{|\max\{|\psi''(\alpha_1)|,|\psi''(\alpha_2)|\}|(\alpha_2-\alpha_1)^2}{12}.$$

Proof. It follows from (2.11) that

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx=\frac{1}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}(\mu-\alpha_1)(\alpha_2-\mu)\psi''(\mu)d\mu.$$

Taking absolute and using triangular inequality one has

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq\frac{1}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}((\mu-\alpha_1)(\alpha_2-\mu))\,|\psi''(\mu)|d\mu. \tag{2.15}$$

Since $(\mu-\alpha_1)(\alpha_2-\mu)\geq 0$ and $|\psi''|$ is increasing, therefore

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq\frac{|\psi''(\alpha_2)|}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}(\mu-\alpha_1)(\alpha_2-\mu)d\mu$$

$$=\frac{|\psi''(\alpha_2)|(\alpha_2-\alpha_1)^2}{12}.$$

Similarly we can prove part (2)

For part (3), using (2.15) and the fact that every convex function f defined on the interval $[\alpha_1,\alpha_2]$ is bounded above by $\max\{f(\alpha_1),f(\alpha_2)\}$, we have

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq\frac{\max\{|\psi''(\alpha_1)|,|\psi''(\alpha_2)|\}}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}((\mu-\alpha_1)(\alpha_2-\mu))\,d\mu.$$

□

3. Applications to Means

A bivariate function $M:(0,\infty)\times(0,\infty)\mapsto(0,\infty)$ is said to be a mean if $\min\{a,b\}\leq M(a,b)\leq\max\{a,b\}$, $M(a,b)=M(b,a)$ and $M(\lambda a,\lambda b)=\lambda M(a,b)$ for all $a,b,\lambda\in(0,\infty)$.

Let $a,b>0$ with $a\neq b$. Then the arithmetic mean $A(a,b)$ [38-43], logarithmic mean $L(a,b)$ [44-48] and (α,r)-th generalized logarithmic mean $L_{(\alpha,r)}(a,b)$ [49-52] are defined by

$$A(a,b)=\frac{a+b}{2},\quad L(a,b)=\frac{b-a}{\log b-\log a},\quad L_{(\alpha,r)}(a,b)=\left[\frac{\alpha(b^{r+\alpha}-a^{r+\alpha})}{(r+\alpha)(b^\alpha-a^\alpha)}\right]^{1/r},$$

respectively. Recently, the bivariate means have been the subject of intensive research [53-67] and many remarkable inequalities for the bivariate means and related special functions can be found in the literature [68-90].

In this section we present several new inequalities the arithmetic, logarithmic and generalized logarithmic means by using our results.

Theorem 3.1. *Let $0 < \alpha_1 < \alpha_2$. Then the following statements are true:*
(1) if $r \geq 2$, then

$$\left|A^r(\alpha_1,\alpha_2) - L_r^r(\alpha_1,\alpha_2)\right| \leq \frac{r(r-1)(\alpha_2-\alpha_1)^2}{48}\left[\left(\frac{\alpha_1+\alpha_2}{2}\right)^{r-2} + \alpha_2^{r-2}\right]; \tag{3.1}$$

(2) if $r < 2$ and $r \neq 0, -1$, then

$$\left|A^r(\alpha_1,\alpha_2) - L_r^r(\alpha_1,\alpha_2)\right| \leq \frac{r(r-1)(\alpha_2-\alpha_1)^2}{48}\left[\left(\frac{\alpha_1+\alpha_2}{2}\right)^{r-2} + \alpha_1^{r-2}\right]. \tag{3.2}$$

Proof. Let $\psi(x) = x^r$ $(x > 0)$ and $r \geq 2$. Then we clearly see that $|\psi''|$ is increasing and inequality (3.1) follows easily from Theorem 2.3(1). Similarly, we can prove inequality (3.2). □

Theorem 3.2. *Let $0 < \alpha_1 < \alpha_2$. Then the following statements are true:*
(1) if $r \geq 2$, then

$$|A(\alpha_1^r,\alpha_2^r) - L_r^r(\alpha_1,\alpha_2)| \leq \frac{r(r-1)(\alpha_2-\alpha_1)^2\alpha_2^{r-2}}{48};$$

(2) if $r < 2$ and $r \neq 0, -1$, then

$$|A(\alpha_1^r,\alpha_2^r) - L_r^r(\alpha_1,\alpha_2)| \leq \frac{r(r-1)(\alpha_2-\alpha_1)^2\alpha_1^{r-2}}{48}.$$

Proof. By using Theorem 2.4 and the same arguments as given in the proof of Theorem 3.1, we can obtain the desired results. □

Theorem 3.3. *The inequalities*

$$\left|A^{-1}(\alpha_1,\alpha_2) - L^{-1}(\alpha_1,\alpha_2)\right| \leq \frac{(\alpha_2-\alpha_1)^2}{24}\left[\frac{8}{(\alpha_1+\alpha_2)^3} + \frac{1}{\alpha_1^3}\right],$$

$$\left|A^{-1}(\alpha_1,\alpha_2) - L^{-1}(\alpha_1,\alpha_2)\right|$$
$$\leq \frac{(\alpha_2-\alpha_1)^2}{24}\left[\max\left\{\frac{8}{(\alpha_1+\alpha_2)^3}, \frac{1}{\alpha_2^3}\right\} + \max\left\{\frac{8}{(\alpha_1+\alpha_2)^3}, \frac{1}{\alpha_1^3}\right\}\right]$$

hold for all $\alpha_1, \alpha_2 \in \mathbb{R}^+$ with $\alpha_1 < \alpha_2$.

Proof. Let $x > 0$ and $\psi(x) = 1/x$. Then we clearly see that $|\psi''|$ is decreasing and convex and Theorem 3.3 follows easily from Theorem 2.3(2) and (3). □

Theorem 3.4. *The inequality*

$$\left|A\left(\alpha_1^{-1},\alpha_2^{-1}\right) - L^{-1}(\alpha_1,\alpha_2)\right| \geq \frac{(\alpha_2-\alpha_1)^2}{6\alpha_1^3}$$

holds for all $\alpha_1, \alpha_2 \in \mathbb{R}^+$ with $\alpha_1 < \alpha_2$.

Proof. Similar proof as in Theorem 3.3 but use Theorem 2.4 instead of Theorem 2.3 □

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4. Applications to Trapezoidal and Mid-Point Formulae

In this section we provide some new error estimations for the trapezoidal and mid-point formulae.

Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b \psi(x)dx = \mathtt{T}(\psi, d) + \mathtt{E}(\psi, d),$$

where

$$\mathtt{T}(\psi, d) = \sum_{i=0}^{n-1} \frac{\psi(x_i) + \psi(x_{i+1})}{2} (x_{i+1} - x_i)$$

for the trapezoidal version and

$$\mathtt{T}(\psi, d) = \sum_{i=0}^{n-1} \psi\Big(\frac{x_i + x_{i+1}}{2}\Big) (x_{i+1} - x_i)$$

for the midpoint version and $\mathtt{E}(\psi, d)$ denotes the associated approximation error.

Theorem 4.1. *Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$, $\psi \in C^2([a, b])$ and $E(\psi, d)$ be the trapezoidal error. Then one has*

$$|E(\psi, d)| \leq \sum_{i=0}^{n-1} \frac{|\psi''(x_{i+1})| (x_{i+1} - x_i)^3}{12} \tag{4.1}$$

if $|\psi''|$ is an increasing function;

$$|E(\psi, d)| \leq \sum_{i=0}^{n-1} \frac{\max\{|\psi''(x_i)|, |\psi''(x_{i+1})|\} (x_{i+1} - x_i)^3}{12}$$

if $|\psi''|$ is a decreasing function;

$$|E(\psi, d)| \leq \sum_{i=0}^{n-1} \frac{|\psi''(x_{i+1})| (x_{i+1} - x_i)^3}{12}$$

if $|\psi''|$ is a convex function.

Proof. Applying Theorem 2.4 on each subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, 2, \cdots, n-1)$ of the division d, we have

$$\left|\frac{\psi(x_i) + \psi(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \psi(x)dx\right| \leq \frac{|\psi''(x_{i+1})|(x_{i+1} - x_i)^2}{12}.$$

Multiplying both sides by $x_{i+1} - x_i$ and taking summation we obtain

$$\left|\int_a^b \psi(x)dx - \mathtt{T}(\psi, d)\right| \leq \sum_{i=0}^{n-1} \left\{\frac{|\psi''(x_{i+1})|}{12} (x_{i+1} - x_i)^3\right\} \leq \sum_{i=0}^{n-1} \left|\left\{\frac{|\psi''(x_{i+1})|}{12} (x_{i+1} - x_i)^3\right\}\right|,$$

which is equivalent to (4.1). Similarly we can prove other parts. □

Theorem 4.2. *Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$, $\psi \in C^2([a, b])$ and $E(\psi, d)$ be the mid-point error. Then one has*

$$|E(\psi, d)| \leq \frac{1}{48} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[\left|\psi''\Big(\frac{x_{i+1} + x_i}{2}\Big)\right| + |\psi''(x_{i+1})|\right]$$

if $|\psi''|$ is an increasing function;

$$|E(\psi, d)| \leq \frac{1}{48} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[\left| \psi'' \Big(\frac{x_{i+1} + x_i}{2} \Big) \right| + |\psi''(x_i)| \right]$$

if $|\psi''|$ is a decreasing function;

$$|E(\psi, d)| \leq \frac{1}{48} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[\max\left\{ \left| \psi'' \Big(\frac{x_{i+1} + x_i}{2} \Big) \right|, |\psi''(x_i)| \right\} + \max\left\{ \left| \psi'' \Big(\frac{x_{i+1} + x_i}{2} \Big) \right|, |\psi''(x_{i+1})| \right\} \right]$$

if $|\psi''|$ is convex function.

Proof. The proof is analogous to the proof of Theorem 4.1. □

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On the Applications of the Girard-Waring Identities

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Abstract

We present here some applications of Girard-Waring identities. Many various identities for things like elementary mathematics and other advanced mathematics come from those identities.

AMS Subject Classification: 05A15, 05A05, 15B36, 15A06, 05A19, 11B83.

Key Words and Phrases: Girard-Waring identities, Recursive sequences, Fibonacci numbers and polynomials, Lucas numbers, Chebyshev polynomials of the first kind, Chebysheve polynomials of the second kind.

1 Introduction

In this paper, we are concerned with the applications of the following Girard-Waring identities:

$$x^n + y^n = \sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(x+y)^{n-2k}(xy)^k \tag{1}$$

and

$$\frac{x^{n+1}-y^{n+1}}{x-y} = \sum_{0\leq k\leq [n/2]} (-1)^k \binom{n-k}{k}(x+y)^{n-2k}(xy)^k. \tag{2}$$

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Albert Girard published these identities in Amsterdam in 1629 and Edward Waring published similar material in Cambridge in 1762-1782. These may be derived from the earlier work of Sir Isaac Newton. It worth noting that $(-1)^k \frac{n}{n-k}\binom{n-k}{k}$ is an integer because

$$\begin{aligned}&\frac{n}{n-k}\binom{n-k}{k}=\binom{n-k}{k}+\binom{n-k-1}{k-1}\\&=2\binom{n-k}{k}-\binom{n-k-1}{k}.\end{aligned}$$

The proofs of formulas (1) and (2) can be seen in Comtet [3] (P. 198) and the survey paper by Gould [6]. Recently, Shapiro and one the authors [8] gave a different proof of (2) by using Riordan arrays.

There are some alternative forms of formula (1). As an example, we give the following one. If $x+y+z=0$, then (1) gives

$$\begin{aligned}x^n+y^n &= \sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(-z)^{n-2k}(xy)^k\\ &= (-1)^n z^n + \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k,\end{aligned}$$

which implies

$$x^n+y^n-(-1)^n z^n = \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k.$$

Thus, when n is even, we have formula

$$x^n+y^n-z^n = \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k, \tag{3}$$

while for odd n we have

$$x^n+y^n+z^n = \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k, \tag{4}$$

where $x+y+z=0$. Particularly, if $n=3$, then

$$x^3+y^3+z^3=3xyz, \tag{5}$$

which will be shown in Corollary **??** and applied in the following examples. The formulas (3) and (4) can be considered as analogies of the results for the case of $xy+yz+zx=0$ shown in Ma [11]. Draim and Bicknell [4] use sums and products of two roots of a quadratic equation to derive a class of Girard-Waring identities. Using Girard-Waring formulas to derive combinatorial identities is also an attractive topic. For instance, Filipponi [5] uses Girard-Waring formula (1) to derive some unusual binomial Fibonacci identities. Furthermore, some well-known identities can be re-derived by using Girard-Waring formulas. As an example, we substitute $x = u + \sqrt{u^2-4}$, $y = u - \sqrt{u^2-4}$, and $z = -x-y$ into (1) and obtain the following identity shown on page 57 of Riordan [13]:

$$\sum_{k=0}^{m}(-1)^k \frac{n}{n-k}\binom{n-k}{k}u^{n-2k} = 2^{-n}\left[(u+\sqrt{u^2-4})^n + (u-\sqrt{u^2-4})^n\right] \tag{6}$$

for $n = 1, 2, \ldots$, where $m = [n/2]$. In particular, if $u = 2$, above identity (6) reduces to

$$\sum_{k=0}^{m}(-1)^k \frac{n}{n-k}\binom{n-k}{k}2^{n-2k} = 2$$

for $n = 1, 2, \ldots$. It worth mentioning that Vasil'ev and Zelevinskii [18] denoted the function shown on the right-hand side of (6) by Q_n and obtained (see (4') on Page 57 of [18])

$$Q_n(x) = \Pi_{1\le k\le n}\left(x - 2\cos\frac{(2k-1)\pi}{2n}\right),$$

which implies

$$\Pi_{1\le k\le m}\cos\frac{(2k-1)\pi}{4m} = \frac{\sqrt{2}}{2^m}$$

for $m \ge 1$ (see (d) on Page 58 of [18]).

In the next section, we present some applications of Girard-Waring identities to the trigonometric identities. In section 3, some applications of Girard-Waring identities to the linear recurrence relations of order 2 will be given.

2 Girard-Waring identities and trigonometric identities

Girard-Waring identities can be applied to construct many interesting trigonometric identities related to the roots of some quadratic equations.

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Our idea of the first application of Girard-Waring identities can be presented as follows: In formulas (1) and (2), there are two terms $x+y$, and xy. If we consider x and y the two roots r_1 and r_2 of a given quadratic equation, $ax^2+bx+c=0$, then we have the sums of, and differences of, n-th powers of the roots of the quadratic equation. Therefore we have $r_1+r_2=-\frac{b}{a}=:p$ and $r_1\, r_2=\frac{c}{a}=:q$. Thus formula (1) and (2) give:

$$r_1^n+r_2^n=\sum_{0\leq k\leq[n/2]}(-1)^k\frac{n}{n-k}\binom{n-k}{k}p^{n-2k}q^k \tag{7}$$

and

$$\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}=\sum_{0\leq k\leq[n/2]}(-1)^k\binom{n-k}{k}p^{n-2k}q^k. \tag{8}$$

We first consider a simple quadratic equation $x^2+c=0$. Then two roots, r_1 and r_2, of the equation satisfy

$$r_1+r_2=0 \quad \text{and} \quad r_1r_2=c.$$

From (7) we have the identity

$$r_1^n+r_2^n=\sum_{0\leq k\leq[n/2]}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(r_1+r_2)^{n-2k}(r_1r_2)^k,$$

which implies

$$r_1^{2\ell}+r_2^{2\ell}=2(-c)^\ell \tag{9}$$

and $r_1^{2\ell+1}+r_2^{2\ell+1}=0$. For instance, if $c=-3$, then $r_1=2\cos(\pi/6)$ and $r_2=2\cos(5\pi/6)$. From (9) we obtain

$$\cos^{2\ell}\left(\frac{\pi}{6}\right)+\cos^{2\ell}\left(\frac{5\pi}{6}\right)=2\left(\frac{3}{4}\right)^\ell.$$

When $\ell=1$ and 2, when $\cos^2\left(\frac{\pi}{6}\right)+\cos^2\left(\frac{5\pi}{6}\right)=1.5$ and $\cos^4\left(\frac{\pi}{6}\right)+\cos^4\left(\frac{5\pi}{6}\right)=9/8$, respectively.

Consider a quadratic equation $ax^2+bx+c=0$, we have $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If $b^2-4ac<0$, then

$$x=A\pm Bi=\rho(\cos\theta\pm i\sin\theta),$$

where $\theta=tan^{-1}\frac{B}{A}$.

Then two roots r_1 and r_2 are:

$$r_1 = \rho(\cos\theta + i\sin\theta) \quad \text{and} \quad r_2 = \rho(\cos\theta - i\sin\theta),$$

which implies $p = r_1 + r_2 = 2\rho\cos\theta$ and $q = r_1 r_2 = \rho$. Thus, equation (7) gives

$$r_1^n + r_2^n = \rho^n \sum_{0\le k\le[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2\cos\theta)^{n-2k},$$

which implies

$$2\cos n\theta = \sum_{0\le k\le[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2\cos\theta)^{n-2k}.$$

Note that

$$\frac{n}{n-k}\binom{n-k}{k} = \frac{n}{k}\binom{n-k-1}{k-1}, \quad k\ge 1.$$

Thus

$$\cos n\theta = \frac{1}{2}\Big\{(2\cos\theta)^n - \frac{n}{1}(2\cos\theta)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos\theta)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos\theta)^{n-6} + \cdots\Big\}.$$

Similarly, from (8) we have

$$\sin(n+1)\theta = \sin\theta \sum_{0\le k\le[n/2]} (-1)^k \binom{n-k}{k}(2\cos\theta)^{n-2k}.$$

Example 2.1 On Page 50 of Comtet [3], it can be seen that

$$\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta),$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind. Thus,

$$U_n(\cos\theta) = \sum_{0\le k\le[n/2]} (-1)^k \binom{n-k}{k}(2\cos\theta)^{n-2k}.$$

On Page 88 of Comtet [3], we also find that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} = \begin{vmatrix} 2\cos\theta & 1 & 0 & 0 & \dots \\ 1 & 2\cos\theta & 1 & 0 & \dots \\ 0 & 1 & 2\cos\theta & 1 & \dots \\ 0 & 0 & 1 & 2\cos\theta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Hence, the determinant of the tridiangonal matrix on the rightmost side of the above equation is equal to

$$\begin{vmatrix} 2\cos\theta & 1 & 0 & 0 & \dots \\ 1 & 2\cos\theta & 1 & 0 & \dots \\ 0 & 1 & 2\cos\theta & 1 & \dots \\ 0 & 0 & 1 & 2\cos\theta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \sum_{0\le k\le[n/2]} (-1)^k \binom{n-k}{k} (2\cos\theta)^{n-2k}.$$

Recall that the Chebyshev polynomials of the first kind $T_n(x)$ are defined by $T_n(x) = \cos(n\cos^{-1} x)$. Thus,

$$\begin{aligned} & T_n(x) = \cos(n\cos^{-1} x) \\ = \ & \frac{1}{2}\left\{(2x)^n - \frac{n}{1}(2x)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2x)^{n-4} - \cdots\right\} \end{aligned}$$

From Page 88 of [3],

$$T_n(\cos\theta) = \cos n\theta = \begin{vmatrix} \cos\theta & 1 & 0 & 0 & \dots \\ 1 & 2\cos\theta & 1 & 0 & \dots \\ 0 & 1 & 2\cos\theta & 1 & \dots \\ 0 & 0 & 1 & 2\cos\theta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Thus,

$$\begin{vmatrix} x & 1 & 0 & 0 & \dots \\ 1 & 2x & 1 & 0 & \dots \\ 0 & 1 & 2x & 1 & \dots \\ 0 & 0 & 1 & 2x & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \frac{1}{2}\left\{(2x)^n - \frac{n}{1}(2x)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2x)^{n-4} - \cdots\right\}.$$

From [19] (see Page 696),

$$T_n(x) = 2^{n-1}\Pi_{k=1}^n \left\{x - \cos\left(\frac{(2k-1)\pi}{2n}\right)\right\}.$$

Since $T_n(\cos\theta) = \cos n\theta$, the above formula implies that

$$\cos n\theta = 2^{n-1}\Pi_{k=1}^n \left\{\cos\theta - \cos\left(\frac{(2k-1)\pi}{2n}\right)\right\}. \tag{10}$$

Remark 2.1 It is well known (see, for example, [19]) that

$$U_n(x) = 2^n\Pi_{k=1}^n \left\{x - \cos\left(\frac{k\pi}{n+1}\right)\right\}.$$

Thus,

$$\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta) = 2^n \Pi_{k=1}^n \left\{\cos\theta - \cos\left(\frac{k\pi}{n+1}\right)\right\}. \tag{11}$$

Substituting different values of θ into (11), we may obtain a type of trigonometric identities. For instance, let $\theta = \pi/2$. Then (11) yields

$$\sin(n+1)\frac{\pi}{2} = -2^n \Pi_{k=1}^n \cos\left(\frac{k\pi}{n+1}\right).$$

If $n = 2m$, $m = 0, 1, , 2\ldots$, because of $\sin(2m+1)\pi/2 = (-1)^m$, the last equation implies

$$(-1)^m = 2^{2m}\Pi_{k=1}^{2m} \cos\left(\frac{k\pi}{2m+1}\right) = 4^m(-1)^m \Pi_{k=1}^m \cos^2\left(\frac{k\pi}{2m+1}\right),$$

where in the last step we use the fact

$$\cos\left(\frac{k\pi}{2m+1}\right) = -\cos\left(\pi - \frac{k\pi}{2m+1}\right) = -\cos\left(\frac{(2m-k+1)\pi}{2m+1}\right)$$

for $k = m+1, m+2, \ldots, 2m$. Thus we obtain the identity

$$\Pi_{k=1}^m \cos^2\left(\frac{k\pi}{2m+1}\right) = \frac{1}{4^m}.$$

Other identities can be obtained by substituting $\theta = \pi/6$, $\pi/4$, $\pi/3$, etc.

Recall also that

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Let $r_1 = e^x$ and $r_2 = e^{-x}$. Then (7) gives

$$\cosh(nx) = \frac{1}{2} \sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2\cosh x)^{n-2k}$$

and

$$\sinh(nx) = \sinh x \sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2\cosh x)^{n-2k}.$$

Other applications of Girard-Waring identities to the product expansions of trigonometric functions similar to the results shown in [2] will be presented in the author's further work.

3 Girard-Waring Identities and linear recurrence relations of order 2

Girard-Waring identities can be applied to construct the expressions of the linear recursive sequences of order 2.

Note that $x^n + y^n = (x+y)\left(x^{n-1} + y^{n-1}\right) - xy\left(x^{n-2} + y^{n-2}\right)$ $(n \geq 2)$. Let $w_n(x,y) = x^n + y^n$. Then

$$w_n(x,y) = (x+y)w_{n-1}(x,y) - xyw_{n-2}(x,y), n \geq 2, \tag{12}$$

with the initial conditions $w_0(x,y) = 2$ and $w_1(x,y) = x+y$. The characteristic equation of the above recurrence relation is $t^2 - (x+y)t + xy = 0$. Thus $t = x, y$

Proposition 3.1. *Let $a_n(x,y) = p(x,y)a_{n-1}(x,y) + q(x,y)a_{n-2}(x,y)$, $n \geq 2$, with given $a_0(x,y)$ and $a_1(x,y)$. Then*

$$\begin{aligned} & a_n(x,y) \\ = \; & \frac{a_1(x,y) - \beta(x,y)a_0(x,y)}{\alpha(x,y) - \beta(x,y)}\alpha^n(x,y) - \frac{a_1(xy) - \alpha(x,y)a_0(x,y)}{\alpha(x,y) - \beta(x,y0}\beta^n(x,y), \end{aligned}$$

where $\alpha(x,y) \neq \beta(x,y)$ are the roots of the characteristic equation $t^2 - p(x,y)t - q(x,y) = 0$.

By using this proposition 3.1, the solution of (12) is $w_n(x,y) = x^n + y^n$.
Example 3.1 The generalized Lucas polynomials (Lucas 1891, see Swamy [16]) $V_n(x,y)$ are defined by

$$V_n(x,y) = xV_{n-1}(x,y) + yV_{n-2}(x,y), \quad V_0(x,y) = 2, \; V_1(x,y) = x.$$

The characteristic equation is $t^2 - xt - y = 0$. Thus

$$t = \frac{x \pm \sqrt{x^2 + 4y}}{2}.$$

By Proposition 3.1 and the Girard-Waring identity (1)

$$V_n(x,y) = \alpha^n(x,y) + \beta^n(x,y) = \sum_{0 \leq k \leq [n/2]} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k} y^k.$$

Example 3.2 Dickson polynomials of the first kind of degree n (Dickson 1897, see Lidl, Mullen, and Turnwald [10]) are defined by

$$D_n(x,a) = xD_{n-1}(x,a) - aD_{n-2}(x,a), \quad D_0(x,a) = 2, \; D_1(x,a) = x.$$

Thus from Proposition 3.1 and the Girard-Waring identity (1),

$$\begin{aligned} & V_n(x,-a) = D_n(x,a) \\ = & \sum_{0\leq k\leq [n/2]} \frac{n}{n-k}\binom{n-k}{k}(-a)^k x^{n-2k}. \end{aligned}$$

Example 3.3 For the Lucas polynomials (see Bicknell [1]) $\{\mathcal{L}_n(x) = V_n(x,1)\}$, i.e., let $y=1$ in $\{V_n(x,y)\}$, we have

$$\mathcal{L}_n(x) = \sum_{0\leq k\leq [n/2]} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}.$$

Note that $D_n(x,-1) = \mathcal{L}_n(x)$. For the Lucas numbers $\mathcal{L}_n = \mathcal{L}_n(1)$, we have

$$\mathcal{L}_n = \sum_{0\leq k\leq [n/2]} \frac{n}{n-k}\binom{n-k}{k}.$$

Example 3.4 The Chebysheve polynomials of the first kind (Chehysher 1821-1894, see Rivlin [14] and Zwillinger [19]) are defined by

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \qquad n\geq 2,$$

with the initial conditions $T_0(x)=1$ and $T_1(x)=x$. Thus, from Proposition 3.1 and the Girard-Waring identity (1), we have

$$T_n(x) = \frac{1}{2}\sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2x)^{n-2k}.$$

Note that from (2), we have

$$\frac{x^{n+1}-y^{n+1}}{x-y} = (x+y)\frac{x^n-y^n}{x-y} - xy\frac{x^{n-1}-y^{n-1}}{x-y}, \quad n\geq 2.$$

Let $W_{n+1}(x,y) = (x^{n+1}-y^{n+1})/(x-y)$. Then

$$W_n(x,y) = (x+y)W_{n-1}(x,y) - xyW_{n-2}(x,y), \quad n\geq 2, \tag{13}$$

with the initial conditions $W_0(x,y)=0$ and $W_1(x,y)=1$. Thus

$$W_n(x,y) = \frac{x^n-y^n}{x-y} = \sum_{0\leq k\leq [(n-1)/2]} (-1)^k \binom{n-1-k}{k}(x+y)^{n-1-k}(xy)^k.$$

Remark 3.1 From the expression of $W_n(x,y)$ and noting the initial condition $W_0(x,y)=0$, we know $\{W_n(x,y)\}$ is a linear divisibility sequence.

More precisely, from authors' recent work [9], if x and y be distinct real (or complex) numbers, then sequence $(W_n(x,y))$ is a second order linear homogenous recursive sequence with $W_0 = 0$ and $W_1 = 1$ and a linear divisibility sequence of order 2. For instance, when $r \neq 1$ and $a_1 = 1$, the geometric sequence $\{s_n = a_1(1-r^n)/(1-r) = (1-r^n)/(1-r)\}_{n\geq 1}$ is a linear divisibility sequence because $s_n = W_n(1,r)$.

Example 3.5 The Generalized Fibonacci polynomials $F_n(x,y)$ (see Swammy [17]) are defined by $F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y)$ $(n \geq 2)$ with the initial conditions $F_0(x,y) = 0$ and $F_1(x,y) = 1$. From Proposition 3.1 and the Girard-Waring identity (2), we have

$$F_n(x,y) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k} x^{n-1-k}y^k.$$

Thus, for the Fibonacci polynomials

$$F_n(x) = F_n(x,1) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k} x^{n-1-k}.$$

For Fibonacci sequence $\{F_n\}$

$$F_n = F_n(1) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k}.$$

For the Pell sequence $\{P_n\}$

$$P_n = F_n(2) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k} 2^{n-1-k}.$$

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Local Fractional Taylor Formula

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Abstract

Here we derive an appropiate local fractional Taylor formula. We provide a complete description of the formula.

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1 Introduction

In [3], [4] was first introduced the local fractional derivative and presented an incomplete local fractional Taylor formula, all done by the use of Riemann-Liouville fractional derivative. Similar work was done in [1], but again with some gaps. The author is greatly motivated by the pioneering work of [1]-[4] and presents a local fractional Taylor formula in a complete suitable form and without any gaps.

2 Main Results

We mention

Definition 1 *([5], pp. 68, 89) Let $x, x' \in [a,b]$, $f \in C([a,b])$. The Riemann-Lioville fractional derivative of a function f of order q $(0 < q < 1)$ is defined as*

$$D_x^q f(x') = \left\{ \begin{array}{l} D_{x+}^q f(x'), \quad x' > x, \\ D_{x-}^q f(x'), \quad x' < x \end{array} \right\} =$$

$$\frac{1}{\Gamma(1-q)} \left\{ \begin{array}{l} \frac{d}{dx'} \int_x^{x'} (x'-t)^{-q} f(t)\, dt, \quad x' > x, \\ -\frac{d}{dx'} \int_{x'}^{x} (t-x')^{-q} f(t)\, dt, \quad x' < x. \end{array} \right. \tag{1}$$

1

We need

Definition 2 *([3]) The local fractional derivative of order q $(0 < q < 1)$ of a function $f \in C([a,b])$ is defined as*

$$\mathcal{D}^q f(x) = \lim_{x' \to x} D_x^q \left(f(x') - f(x)\right). \tag{2}$$

More generally we define

Definition 3 *(see also [1]) Let $N \in \mathbb{Z}_+$, $0 < q < 1$, the local fractional derivative of order $(N+q)$ of a function $f \in C^N([a,b])$ is defined by*

$$\mathcal{D}^{N+q} f(x) = \lim_{x' \to x} D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \right). \tag{3}$$

If $N = 0$, then Definition 3 collapses to Definition 2.
We need

Definition 4 *(related to Definition 3) Let $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. Set*

$$F(x, x'-x; q, N) := D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \right). \tag{4}$$

Let $x' - x := t$, then $x' = x + t$, and

$$F(x, t; q, N) = D_x^q \left(f(x+t) - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} t^n \right). \tag{5}$$

We make

Remark 5 *Here $x', x \in [a,b]$, and $a \le x+t \le b$, equivalently $a - x \le t \le b - x$. From $a \le x \le b$, we get $a - x \le 0 \le b - x$.*

We assume here that $F(x, \cdot; q, N) \in C^1([a-x, b-x])$. Clearly, then it holds

$$\mathcal{D}^{N+q} f(x) = F(x, 0; q, N), \tag{6}$$

and $\mathcal{D}^{N+q} f(x)$ exists in $\mathbb{R}$.

We make

Remark 6 *We observe that:*

I) Let $x' > x$ $(x' - x > 0)$ then

$$f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \overset{([2])}{=}$$

$$D_x^{-q}\left[D_x^q\left(f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n\right)\right] =$$

$$D_x^{-q}\left(F(x,x'-x;q,N)\right) = \frac{1}{\Gamma(q)}\int_x^{x'}(x'-z)^{q-1}F(x,z-x;q,N)\,dz =$$

$$\frac{1}{\Gamma(q)}\int_0^{x'-x}\frac{F(x,t;q,N)}{(x'-x-t)^{-q+1}}dt =$$

(integration by parts)

$$\frac{1}{\Gamma(q)}\left[F(x,t;q,N)\int(x'-x-t)^{q-1}dt\right]_0^{x'-x} + \tag{7}$$

$$\frac{1}{\Gamma(q)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}\frac{(x'-x-t)^q}{q}dt.$$

Thus,

$$f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n = \frac{\mathcal{D}^{N+q}f(x)}{\Gamma(q+1)}(x'-x)^q +$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(x'-x-t)^q\,dt, \quad \text{for } x' > x, \tag{8}$$

$N \in \mathbb{Z}_+$.

II) Let $x' < x$ $(x'-x<0)$: We have similarly,

$$f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n \overset{([2])}{=}$$

$$D_x^{-q}\left[D_x^q\left(f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n\right)\right] =$$

$$D_x^{-q}\left(F(x,x'-x;q,N)\right) = \frac{1}{\Gamma(q)}\int_{x'}^{x}(z-x')^{q-1}F(x,z-x;q,N)\,dz = \tag{9}$$

$$\frac{1}{\Gamma(q)}\int_{x'-x}^{0}(x-x'+t)^{q-1}F(x,t;q,N)\,dt =$$

(integration by parts)

$$\frac{1}{\Gamma(q)}\left[F(x,t;q,N)\int(t+x-x')^{q-1}dt\right]_{x'-x}^{0} -$$

$$\frac{1}{\Gamma(q)}\int_{x'-x}^{0}\frac{dF(x,t;q,N)}{dt}\frac{(t+x-x')^q}{q}dt = \tag{10}$$

3

$$\frac{1}{\Gamma(q)}\left[F(x,0;q,N)\frac{(x-x')^q}{q}\right]+\frac{1}{\Gamma(q)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}\frac{(t+x-x')^q}{q}dt=$$

$$\frac{1}{\Gamma(q+1)}\mathcal{D}^{N+q}f(x)(x-x')^q+\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(t-x'+x)^q\,dt. \tag{11}$$

Conclusion:

We have proved that ($N\in\mathbb{Z}_+$)

I)

$$f(x')=\sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n+\frac{\mathcal{D}^{N+q}f(x)}{\Gamma(q+1)}(x'-x)^q+$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(x'-x-t)^q\,dt,\ \ \textit{when } x'>x, \tag{12}$$

and

II)

$$f(x')=\sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n+\frac{\mathcal{D}^{N+q}f(x)(x-x')^q}{\Gamma(q+1)}+$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(t-x'+x)^q\,dt,\ \ \textit{when } x'<x. \tag{13}$$

We have derived

Theorem 7 *Let $f\in C^N([a,b])$, $N\in\mathbb{Z}_+$. Here $x,x'\in[a,b]$, and $F(x,\cdot;q,N)\in C^1([a-x,b-x])$. Then*

$$f(x')=\sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n+\frac{\mathcal{D}^{N+q}f(x)}{\Gamma(q+1)}|x'-x|^q+ \tag{14}$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}|(x'-x)-t|^q\,dt.$$

In particular we get

Corollary 8 *(to Theorem 7, $N=0$) Let $f\in C([a,b])$; $x,x'\in[a,b]$, and $F(x,\cdot;q,0)\in C^1([a-x,b-x])$. Then*

$$f(x')=f(x)+\frac{\mathcal{D}^{q}f(x)}{\Gamma(q+1)}|x'-x|^q+ \tag{15}$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,0)}{dt}|(x'-x)-t|^q\,dt.$$

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ON VORONOVSKAJA TYPE ESTIMATES OF BERNSTEIN-STANCU OPERATORS

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Abstract. In the present paper, we obtain the Voronovaskaja-type results of approximation by a type of Bernstein-Stancu operators with shifted knots.

1. Introduction and The Main Results

For any $f(x) \in C_{[0,1]}$, the corresponding Bernstein operators $B_n(f,x)$ are defined as follows:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, k = 0, 1, \cdots, n$. The approximation properties of Bernstein operators for continuous functions or functions of smoothness have been investigated extensively. Among them, many authors have studied the Voronovaskaja-type asymptotical estimates (see [5]-[7], [13]).

Stancu ([11]) generalized the Bernstein operators to the following so called Bernstein-Stancu operators:

$$B_{n,\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) p_{n,k}(x). \tag{1.1}$$

It was showed that $B_{n,\alpha,\beta}(f;x)$ converges to continuous function $f(x)$ uniformly in $[0,1]$ for α,β satisfying $0 \le \alpha \le \beta$.

Recently, Gadjiev and Ghorbanalizadeh ([4]) further generalized Bernstein-Stancu operators by using shifted knots as follows:

$$S_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^{n} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) q_{n,k}(x), \tag{1.2}$$

where $x \in \left[\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$, $q_{n,k}(x) = \binom{n}{k}\left(x - \frac{\alpha_2}{n+\beta_2}\right)^k \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-k}$, $k = 0, 1, \cdots, n$, and $\alpha_k, \beta_k, k = 1, 2$ are positive real numbers satisfying $0 \le \alpha_1 \le \beta_1$, $0 \le \alpha_2 \le \beta_2$. They estimated the approximation rate of approxiamtion by $S_{n,\alpha,\beta}(f,x)$ for continuous functions in A_n. In fact, they established the following:

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Theorem 1.1. *([4]) Let f be continuous function on $[0,1]$. Then the following inequalities hold:*

$$|S_{n,\alpha,\beta}(f,x)-f(x)| \le \begin{cases} \frac{3}{2}\omega\left(f,\sqrt{\frac{4(\beta_2-\beta_1)^2\left(\frac{n+\alpha_2}{n+\beta_2}\right)^2+n}{(n+\beta_1)^2}}\right), & if\ (\beta_2-\beta_1)\ge(\alpha_2-\alpha_1), \\ \frac{3}{2}\omega\left(f,\sqrt{\frac{4(\alpha_2-\alpha_1)^2+n}{(n+\beta_1)^2}}\right), & if\ (\beta_2-\beta_1)\le(\alpha_2-\alpha_1). \end{cases}$$

In Theorem 1.1, the approximation properties of $S_{n,\alpha,\beta}(f,x)$ in $A_n := \left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$ are considered. As we know, $S_{n,\alpha,\beta}$ is positive and linear in the set A_n. Although, $S_{n,\alpha,\beta}$ is still definable on $[0,1]\backslash A_n$, but it is not positive in this case. Then, a natural problem is whether $S_{n,\alpha,\beta}(f,x)$ can be used to approximate the continuous functions on the whole interval $[0,1]$. Wang, Yu and Zhou ([14]) give a positive answer by establishing the following:

Theorem 1.2. *Let f be a continuous function on $[0,1]$, $\lambda\in[0,1]$ be a fixed positive number. Then there exists a positive constant C only depending on $\lambda,\alpha_1,\alpha_2,\beta_1$ and β_2 such that*

$$|S_{n,\alpha,\beta}(f,x)-f(x)| \le C\omega_{\varphi^\lambda}\left(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{1.3}$$

where $\varphi(x)=\sqrt{x(1-x)}$, $\delta_n(x):=\varphi(x)+\frac{1}{\sqrt{n}}$, and

$$\omega_{\varphi^\lambda}(f,t) := \sup_{0<h\le t}\ \sup_{x\pm\frac{h\phi^\lambda(x)}{2}\in[0,1]}\left|f\left(x+\frac{h\varphi^\lambda(x)}{2}\right)-f\left(x-\frac{h\varphi^\lambda(x)}{2}\right)\right|.$$

Many authors have generalized $S_{n,\alpha,\beta}(f,x)$ in many ways (see [1], [3], [8]-[10], [12]).

Our purpose of the paper is to give the Voronovskaja type estimates of approximation by $S_{n,\alpha,\beta}(f,x)$ on A_n.

Theorem 1.3. *Let $f\in C^2(A_n)$, $\lambda\in[0,1]$ be a fixed positive number. Then there exists a positive constant C only depending on $\lambda,\alpha_1,\alpha_2,\beta_1$ and β_2 such that*

$$\left|S_{n,\alpha,\beta}(f,x)-f(\theta_n(x))-\frac{1}{2}f''(x)M_n(x)\right| \le C\frac{\delta_n^2(x)}{n}\omega_{\phi^\lambda}\left(f'',\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}, \tag{1.4}$$

where $\delta_n(x)=\phi(x)+\frac{1}{\sqrt{n}}$, $\phi(x)=\sqrt{\left(x-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)}$,

$$\begin{aligned} M_n(x) \ &:= \ -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2+\left(\frac{(n+\beta_2)(1+2\alpha_2)}{(n+\beta_1)^2}-\frac{2\alpha_1}{n+\beta_2}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right) \\ &\quad +\frac{\alpha_1^2}{(n+\beta_1)^2}-\frac{\alpha_1^2}{(n+\beta_2)^2}, \end{aligned}$$

$$\theta_n(x) := S_{n,\alpha,\beta}(t,x) = \left(\frac{n+\beta_2}{n+\beta_1}\right)x-\left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right).$$

When $\alpha_1=\alpha_2=\beta_1=\beta_2=0$, we get the results of [7] for Bernstein operators. Noting that $|\theta(x)-x|\le\left|\frac{(\beta_2-\beta_1)x-\alpha_2+\alpha_1}{n+\beta_1}\right|$, we have

Corollary 1. *Let $f \in C^2(A_n)$, $\lambda \in [0,1]$ be a fixed positive number. Then there exists a positive constants C_1 and C_2 only depending on $\lambda, \alpha_1, \alpha_2, \beta_1$ and β_2 such that*

$$\left|S_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right| \leq C_1\left(\frac{\delta_n^2(x)}{n}\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n} + \omega\left(f, \frac{(\beta_2-\beta_1)x - \alpha_2 + \alpha_1}{n+\beta_1}\right)_{A_n}\right), \quad (1.5)$$

where $\omega(f,t)_{A_n}$ is the usual modulus of continuity of f on A_n.

Throughout the paper, C denotes either a positive absolute constant or a positive constant that may depend on some parameters but not on f, x and n, their values may be different at different occurrences. The symbol $x \sim y$ means that there exists a positive constant C such that $C^{-1} \leq x \leq Cy$.

2. Auxiliary Lemmas

Lemma 2.1. *([3], Lemma 3) For any given $\gamma \geq 0$, we have*

$$\Delta_{n,\gamma}(x) := \sum_{k=0}^{n}\left|\frac{k+\alpha_1}{n+\beta_1} - x\right|^\gamma q_{n,k}(x) \leq C\frac{\delta_n^\gamma(x)}{n^{\frac{\gamma}{2}}}, \quad x \in A_n. \quad (2.1)$$

Lemma 2.2. *If $g \in D_\lambda := \left\{g : g' \in AC_{loc}, \|\phi^\lambda g'\| < \infty, \|g'\| < \infty\right\}$, then for any $x \in A_n = [\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$, we have*

$$\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u)-g(x))du, x\right)\right| \leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right). \quad (2.2)$$

Proof. We need the following inequality:

$$\int_x^t \frac{1}{\phi^\lambda(u)}du \leq C\frac{|t-x|}{\phi^\lambda(x)}, \text{ for any } x, t \in A_n. \quad (2.3)$$

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In fact, when $\lambda = 1$, (2.3) is obvious. When $0 < \lambda \leq 1$, we have (by using Hölder's inequality for $0 < \lambda < 1$)

$$\begin{aligned}
\int_x^t \frac{1}{\phi^\lambda(u)}du &\leq \left|\int_x^t \frac{1}{\phi(u)}du\right|^\lambda \left|\int_x^t du\right|^{1-\lambda} \\
&\leq C|t-x|^{1-\lambda}\left|\int_x^t \left(\frac{1}{\sqrt{u-\frac{\alpha_2}{n+\beta_2}}}+\frac{1}{\sqrt{\frac{n+\alpha_2}{n+\beta_2}-u}}\right)du\right|^\lambda \\
&\leq C|t-x|^{1-\lambda}\left(\left|\sqrt{t-\frac{\alpha_2}{n+\beta_2}}-\sqrt{x-\frac{\alpha_2}{n+\beta_2}}\right|+\left|\sqrt{\frac{n+\alpha_2}{n+\beta_2}-t}-\sqrt{\frac{n+\alpha_2}{n+\beta_2}-x}\right|\right)^\lambda \\
&\leq C|t-x|\left(\frac{1}{\sqrt{t-\frac{\alpha_2}{n+\beta_2}}+\sqrt{x-\frac{\alpha_2}{n+\beta_2}}}+\frac{1}{\sqrt{\frac{n+\alpha_2}{n+\beta_2}-t}+\sqrt{\frac{n+\alpha_2}{n+\beta_2}-x}}\right)^\lambda \\
&\leq C|t-x|\left(\frac{1}{\sqrt{x-\frac{\alpha_2}{n+\beta_2}}}+\frac{1}{\sqrt{\frac{n+\alpha_2}{n+\beta_2}-x}}\right)^\lambda \\
&\leq C\frac{|t-x|}{\phi^\lambda(x)}, \qquad (2.4)
\end{aligned}$$

which proves (2.3).

Now, we prove (2.2) by considering the following two difference cases: $x \in B_n = \left[\frac{\alpha_2+1}{n+\beta_2}, \frac{n+\alpha_2-1}{n+\beta_2}\right]$ and $x \in B_n^c = [\frac{\alpha_2}{n+\beta_2}, \frac{\alpha_2+1}{n+\beta_2}] \cup [\frac{n+\alpha_2-1}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$, respectively.

When $x \in B_n = \left[\frac{\alpha_2+1}{n+\beta_2}, \frac{n+\alpha_2-1}{n+\beta_2}\right]$, we have

$$\phi(x) \geq \min\left(\phi\left(\frac{\alpha_2+1}{n+\beta_2}\right), \phi\left(\frac{n+\alpha_2-1}{n+\beta_2}\right)\right) \geq \frac{C}{\sqrt{n}},$$

which means that

$$\delta_n(x) \sim \phi(x), \quad x \in B_n. \qquad (2.5)$$

Then, by Lemma 2.1, we have

$$\begin{aligned}
&\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u)-g(x))du, x\right)\right| \\
\leq\ & \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)\left(\int_x^u \frac{\phi^\lambda(s)g'(s)}{\phi^\lambda(s)}ds\right)du, x\right)\right| \\
\leq\ & C\|\phi^\lambda g'\|\left|S_{n,\alpha,\beta}\left(\int_x^t |t-u|\frac{|x-u|}{\phi^\lambda(x)}du, x\right)\right| \\
\leq\ & C\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)}\left|S_{n,\alpha,\beta}\left(|t-x|^3, x\right)\right| \\
=\ & C\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)}\left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^n \left|\frac{k+\alpha_1}{n+\beta_1}-x\right|^3 |q_{n,k}(x)| \\
\leq\ & C\frac{\delta_n^2(x)}{n}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\|. \qquad (2.6)
\end{aligned}$$

When $x \in B_n^c = [\frac{\alpha_2}{n+\beta_2}, \frac{\alpha_2+1}{n+\beta_2}] \cup [\frac{n+\alpha_2-1}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$, we have $\delta_n(x) \sim \frac{1}{\sqrt{n}}$. Then,

$$\begin{aligned}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\delta_n^\lambda g'\| &\leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\right)^\lambda\|g'\|\right)\\ &\leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right). \qquad (2.7)\end{aligned}$$

By Lemma 2.1 again, we get

$$\begin{aligned}&\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u)-g(x))du, x\right)\right|\\ \leq\ & C\|\delta_n^\lambda g'\|\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)\left(\int_x^u \frac{1}{\delta_n^\lambda(s)}ds\right)du, x\right)\right|\\ \leq\ & C\|\delta_n^\lambda g'\|\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)^2\left(\frac{1}{\delta_n^\lambda(u)} + \frac{1}{\delta_n^\lambda(s)}\right)du, x\right)\right|\\ \leq\ & C\|\delta_n^\lambda g'\|\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)^2\frac{1}{\delta_n^\lambda(u)}du, x\right)\right|\\ \leq\ & C\|\delta_n^\lambda g'\|\sum_{k=0}^n\left(\frac{1}{\delta_n^\lambda(x)} + \frac{1}{\delta_n^\lambda\left(\frac{k+\alpha_1}{n+\beta_1}\right)}\right)\left|x - \frac{k+\alpha_1}{n+\beta_1}\right|^3 q_{n,k}(x)\\ \leq\ & C\frac{\|\delta_n^\lambda g'\|}{\delta_n^\lambda(x)}\sum_{k=0}^n\left|x - \frac{k+\alpha_1}{n+\beta_1}\right|^3 q_{n,k}(x)\\ \leq\ & C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right). \qquad (2.8)\end{aligned}$$

We prove Lemma 2.2 by combining (2.6), (2.7) and (2.8). □

Lemma 2.3. *Under the conditions of Lemma 2.2, we have for $x \in A_n = [\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$ that*

$$\left|\int_x^{S_{n,\alpha,\beta}(t,x)} (S_{n,\alpha,\beta}(t,x)-u)(g(u)-g(x))du\right| \leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right). \qquad (2.9)$$

Proof. If $x \in B_n = [\frac{\alpha_2+1}{n+\beta_2}, \frac{n+\alpha_2-1}{n+\beta_2}]$, by (2.3) and the fact (see, [4])

$$S_{n,\alpha,\beta}(t,x) = \left(\frac{n+\beta_2}{n+\beta_1}\right)x - \left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right). \qquad (2.10)$$

we have for any $\gamma \geq 0$ that

$$|S_{n,\alpha,\beta}(t,x) - x|^\gamma \leq \frac{C}{n^\gamma}. \qquad (2.11)$$

By (2.3) and (2.11), we get

$$\begin{aligned}
&\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)(g(u)-g(x))du\right| \\
\leq\ & \left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)\left(\int_x^u g'(s)ds\right)du\right| \\
\leq\ & C\|\phi^\lambda g'\|\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)\left(\int_x^u \frac{1}{\phi^\lambda(s)}ds\right)du\right| \\
\leq\ & C\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)}|S_{n,\alpha,\beta}(t,x)-x|^3 \\
\leq\ & C\frac{1}{n^3}\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)} \\
\leq\ & C\frac{\delta_n^2(x)}{n}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\|. \qquad (2.12)
\end{aligned}$$

If $x\in B_n^c=\left[\frac{\alpha_2}{n+\beta_2},\frac{\alpha_2+1}{n+\beta_2}\right]\cup\left[\frac{n+\alpha_2-1}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$, we have

$$S_{n,\alpha,\beta}(t,x)=\left(\frac{n+\beta_2}{n+\beta_1}\right)x-\left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right)\in\left[\frac{\alpha_1}{n+\beta_2},\frac{\alpha_1+1}{n+\beta_2}\right]\cup\left[\frac{n+\alpha_1-1}{n+\beta_2},\frac{n+\alpha_1}{n+\beta_2}\right].$$

It is easy to observe that

$$\begin{aligned}
\delta_n(S_{n,\alpha,\beta}(t,x)) &= \delta_n\left(\frac{n+\beta_2}{n+\beta_1}x-\frac{\alpha_2-\alpha_1}{n+\beta_1}\right) \\
&= \sqrt{\left[\frac{n+\beta_2}{n+\beta_1}x-\frac{\alpha_2-\alpha_1}{n+\beta_1}-\frac{\alpha_2}{n+\beta_1}\right]\left[\frac{n+\alpha_2}{n+\beta_1}-\frac{n+\beta_2}{n+\beta_1}x+\frac{\alpha_2-\alpha_1}{n+\beta_1}\right]}+\frac{1}{\sqrt{n}} \\
&= \sqrt{\left[\frac{n+\beta_2}{n+\beta_1}x-\frac{2\alpha_2-\alpha_1}{n+\beta_1}\right]\left[\frac{n+2\alpha_2-\alpha_1}{n+\beta_1}-\frac{n+\beta_2}{n+\beta_1}x\right]}+\frac{1}{\sqrt{n}} \\
&\sim \delta_n(x)\sim\frac{1}{\sqrt{n}}.
\end{aligned}$$

Therefore, by (2.11) again, we get

$$\begin{aligned}
&\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)(g(u)-g(x))du\right| \\
\leq\ & C\|\delta_n^\lambda g'\|\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)^2\left(\frac{1}{\delta_n^\lambda(x)}+\frac{1}{\delta_n^\lambda(u)}\right)du\right| \\
\leq\ & C\|\delta_n^\lambda g'\|\left||S_{n,\alpha,\beta}(t,x)-x|^3\cdot\left(\frac{1}{\delta_n^\lambda(x)}+\frac{1}{\delta_n^\lambda(S_{n,\alpha,\beta}(t,x))}\right)\right| \\
\leq\ & C\frac{\delta_n^2(x)}{n}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\delta_n^\lambda g'\|. \qquad (2.13)
\end{aligned}$$

We get Lemma 2.3 by combining (2.7), (2.12) and (2.13). □

3. Proof of Results

Define the auxiliary operators $\overline{S}_{n,\alpha,\beta}(f,x)$ as follows:

$$\overline{S}_{n,\alpha,\beta}(f,x) = S_{n,\alpha,\beta}(f,x) + L_{n,\alpha,\beta}(f,x), \tag{3.1}$$

where $L_{n,\alpha,\beta}(f,x) = f(x) - f(\theta_n(x))$, and $\theta_n(x) = S_{n,\alpha,\beta}(t,x)$.

It follows from the facts $S_{n,\alpha,\beta}(1,x) = 1$ and (2.10) that

$$\overline{S}_{n,\alpha,\beta}(1,x) = 1, \ \ \overline{S}_{n,\alpha,\beta}((t-x),x) = 0. \tag{3.2}$$

For any $f(x) \in C(A_n)$, $0 \le \lambda \le 1$, define the K-functional:

$$K_{\varphi^\lambda}(f,t) := \inf_{g \in A.C_{loc}} \left\{ \|f-g\| + t\left\|\phi^\lambda g'\right\| + t^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\| \right\},$$

Then ([2])

$$K_{\phi^\lambda}(f,t) \sim \omega_{\phi^\lambda}(f,t).$$

Then, by taking $t = \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}$, there is a $g \in AC_{loc}$ such that

$$\left\|f''-g\right\| \le C\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.3}$$

$$\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\phi^\lambda g'\right\| \le C\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.4}$$

$$\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\| \le C\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.5}$$

By (3.1), we have

$$\left|S_{n,\alpha,\beta}(f,x) - f(\theta_n(x)) - \frac{1}{2}f''(x)M_n(x)\right| = \left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right|.$$

Hence, we only need to prove the following inequality:

$$\left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right| \le C\frac{\delta_n^2(x)}{n}\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.6}$$

It follows from (3.1) that

$$\begin{aligned}
\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)du, x\right) &= S_{n,\alpha,\beta}\left(\int_x^t (t-u)du, x\right) - \int_x^{S_{n,\alpha,\beta}(t,x)} (S_{n,\alpha,\beta}(t,x)-u)\,du \\
&= \frac{1}{2}\left[S_{n,\alpha,\beta}\left((t-x)^2,x\right) - (S_{n,\alpha,\beta}(t,x)-x)^2\right] \\
&= \frac{1}{2}\left[S_{n,\alpha,\beta}(t^2,x) - S^2_{n,\alpha,\beta}(t,x)\right]
\end{aligned}$$

It was proved in ([4]) that

$$\begin{aligned}
S_{n,\alpha,\beta}(t^2,x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2 - \frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2 \\
&+ \left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{1}{n+\beta_1}\left(x-\frac{\alpha_2}{n+\beta_2}\right) + \left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{2\alpha_2}{n+\beta_1}\left(x-\frac{\alpha_2}{n+\beta_2}\right) + \frac{\alpha_1^2}{(n+\beta_1)^2}.
\end{aligned}$$

By (2.10), we can rewrite $S_{n,\alpha,\beta}(t,x)$ as follows:

$$\begin{aligned} S_{n,\alpha,\beta}(t,x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)x - \left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right) = \frac{n+\beta_2}{n+\beta_1}\left(x - \frac{\alpha_2-\alpha_1}{n+\beta_2}\right) \\ &= \frac{n+\beta_2}{n+\beta_1}\left(x - \frac{\alpha_2}{n+\beta_2} + \frac{\alpha_1}{n+\beta_2}\right), \end{aligned}$$

which means that

$$\begin{aligned} (S_{n,\alpha,\beta}(t,x))^2 &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2} + \frac{\alpha_1}{n+\beta_2}\right)^2 \\ &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 + \left(\frac{\alpha_1}{n+\beta_2}\right)^2 + \frac{2\alpha_1}{n+\beta_2}\cdot\left(x - \frac{\alpha_2}{n+\beta_2}\right). \end{aligned}$$

Then

$$\begin{aligned} S_{n,\alpha,\beta}(t^2,x) - S^2_{n,\alpha,\beta}(t,x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 - \frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 \\ &\quad + \left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{1}{n+\beta_1}\left(x - \frac{\alpha_2}{n+\beta_2}\right) + \left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{2\alpha_1}{n+\beta_1}\left(x - \frac{\alpha_2}{n+\beta_2}\right) \\ &\quad + \frac{\alpha_1^2}{(n+\beta_1)^2} - \left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 - \left(\frac{\alpha_1}{n+\beta_2}\right)^2 \\ &\quad - \frac{2\alpha_1}{n+\beta_2}\cdot\left(\frac{\alpha_2}{n+\beta_2}\right) \\ &= -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 + \left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{1}{n+\beta_1}\left(x - \frac{\alpha_2}{n+\beta_2}\right) \\ &\quad + \frac{n+\beta_2}{n+\beta_1}\cdot\frac{2\alpha_1}{n+\beta_1}\left(x - \frac{\alpha_2}{n+\beta_2}\right) + \frac{\alpha_1^2}{(n+\beta_1)^2} \\ &\quad - \frac{\alpha_1^2}{(n+\beta_2)} - \frac{2\alpha_1}{n+\beta_2}\left(x - \frac{\alpha_2}{n+\beta_2}\right) \\ &= -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 + \left[\frac{n+\beta_2}{(n+\beta_1)^2} + \frac{2\alpha_1(n+\beta_2)}{(n+\beta_1)^2} - \frac{2\alpha_1}{n+\beta_2}\right] \\ &\cdot \quad \left(x - \frac{\alpha_2}{n+\beta_2}\right) + \frac{\alpha_1^2}{(n+\beta_1)^2} - \frac{\alpha_1^2}{(n+\beta_2)^2} \\ &= -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 + \left(\frac{(n+\beta_2)(1+2\alpha_1)}{(n+\beta_1)^2} - \frac{2\alpha_1}{n+\beta_2}\right) \\ &\quad \times \left(x - \frac{\alpha_2}{n+\beta_2}\right) + \frac{\alpha_1^2}{(n+\beta_1)^2} - \frac{\alpha_1^2}{(n+\beta_2)^2} \\ &=: M_n(x) \end{aligned}$$

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By Taylor's formula: $f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du$, we have

$$\begin{aligned}
& \left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right| \\
\leq & \left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - f''(x)\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)du, x\right)\right| \\
\leq & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)f''(u)du, x\right) - \overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)f''(x)du, x\right)\right| \\
\leq & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)(f''(u)-f''(x))du, x\right)\right| \\
\leq & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)|f''(u)-g(u)|du, x\right)\right| + \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)|f''(x)-g(x)|du, x\right)\right| \\
+ & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)|g(u)-g(x)|du, x\right)\right| \\
=: & \ I_1 + I_2 + I_2.
\end{aligned} \tag{3.7}$$

For I_1, by (2.11), (3.1) and Lemma 2.1, we have

$$\begin{aligned}
I_1 \leq & \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(f''(u)-g(u))du, x\right)\right| \\
+ & \left|\int_x^{S_{n,\alpha,\beta}(t,x)} (S_{n,\alpha,\beta}(t,x)-u)|f''(u)-g(u)|du\right| \\
\leq & \ C\|f''-g\|\left(\frac{\delta_n^2(x)}{n} + \frac{1}{n^2}\right) \\
\leq & \ C\frac{\delta_n^2(x)}{n}\|f''-g\|.
\end{aligned} \tag{3.8}$$

Similarly, we also have

$$I_2 \leq C\frac{\delta_n^2(x)}{n}\|f''-g\|. \tag{3.9}$$

For I_3, by Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
I_3 \leq & \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u)-g(x))du, x\right)\right| \\
+ & \left|\int_x^{S_{n,\alpha,\beta}(t,x)} (S_{n,\alpha,\beta}(t,x)-u)(g(u)-g(x))du\right| \\
\leq & \ C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right).
\end{aligned} \tag{3.10}$$

We finish the proof of (3.6) by combining (3.3)-(3.5), (3.7)-(3.10) .

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Double-sided Inequalities of Ostrowski's Type and Some Applications

Waseem Ghazi Alshanti and Gradimir V. Milovanović

Abstract

We construct a new general Ostrowski type inequality for differentiable mappings whose first derivatives are bounded in terms of pre-assigned continuous functions. Applications to composite quadrature rules are also given.

1 Introduction

In 1938, A. Ostrowski [14] introduced the following interesting and useful integral inequality for differentiable mappings with bounded derivatives:

Theorem 1.1 *Let $f : [a,b] \to \mathbb{R}$ be continuous mapping on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup\limits_{t\in[a,b]} |f'(t)| < \infty$, then for all $x \in [a,b]$*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty . \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

Ostrowski's inequality is one of the most famous inequalities in the integral calculus. It measures the deviation of a function from its integral mean. Also, an estimation of approximating area under the curve of a function by a rectangle can be obtained in this case.

In 1975, Milovanović [10] (see also [12, pp. 26–29]) proposed a generalization of (1.1) for a function f of several variables as follows:

Theorem 1.2 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function defined on $\overline{D}$ and let $\left|\frac{\partial f}{\partial x_i}\right| \le M_i$ in D, where $M_i > 0$ for each $i = 1, \ldots, m$. Then, for every $X = (x_1, \ldots, x_m) \in \overline{D}$, we have*

$$\begin{aligned} \Bigg| f(x_1,\ldots,x_m) \;&-\; \frac{1}{\prod\limits_{i=1}^{n}(b_i-a_i)} \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} f(y_1,\ldots,y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m \Bigg| \\ &\le \sum_{i=1}^{m}\left[\frac{1}{4} + \frac{\left(x_i - \frac{a_i+b_i}{2}\right)^2}{(b_i-a_i)^2}\right](b_i-a_i)M_i. \end{aligned}$$

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One year later, in 1976, Milovanović and Pečarić [11] presented the following generalization when $\left|f^{(n)}(x)\right| \leq M$ $(\forall x \in (a,b))$, and $n > 1$:

Theorem 1.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be $n\,(>1)$ times differentiable function such that $\left|f^{(n)}(x)\right| \leq M$ $\big(\forall x \in (a,b)\big)$. Then, for every $x \in [a,b]$*

$$\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k\right)-\frac{1}{b-a}\int\limits_a^b f(y)\mathrm{d}y\right| \leq \frac{M}{n(n+1)!}\cdot\frac{(x-a)^{n+1}+(b-x)^{n+1}}{b-a},$$

where F_k is defined by

$$F_k \equiv F_k(f;n;x;a;b) \equiv \frac{n-k}{k!}\cdot\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}.$$

For $n = 2$, Theorem 1.3 gives

$$\left|\frac{1}{2}\left(f(x) \quad + \quad \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right)-\frac{1}{b-a}\int\limits_a^b f(y)\mathrm{d}y\right|$$
$$\leq \quad \frac{M\,(b-a)^2}{4}\left[\frac{1}{12}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right].$$

2 Preliminaries

Associated with differentiable mappings, there has been extensive research in the literature on related results. Over the past few decades, many studies on obtaining sharp bounds of Ostrowski's tpye inequalities have been conducted. Most of the calculations within these sharp bounds depend mainly on the magnitudes of Lebesgue norms of derivatives of given functions.

In [5]–[8], Dragomir and Wang obtained the following bounds on the deviation of an absolutely continuous mapping f, defined over the interval $[a,b]$, from its integral mean

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^x f(t)\mathrm{d}t\right| \leq \begin{cases} \left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]\frac{\|f'\|_\infty}{b-a}, & f' \in L_\infty[a,b]; \\ \frac{1}{q+1}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{1/q}\frac{\|f'\|_p}{b-a}, & f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\frac{\|f'\|_1}{b-a}, & f' \in L_1[a,b]. \end{cases}$$

In [9], Masjed-Jamei and Dragomir provided the following analogues of the Ostrowski's inequality for a differentiable function f whose first derivative f' is bounded, bounded from below, and bounded from above in terms of two functions $\alpha, \beta \in C[a,b]$ as follows:

Theorem 2.1 *Let $f : I \to \mathbb{R}$, where I is an interval, be a function differentiable in the interior $\mathring{I}$ of I, and let $[a,b] \subset \mathring{I}$. For any $\alpha, \beta \in C[a,b]$ and $x \in [a,b]$, we have the following three cases:*

1° *If* $\alpha(x) \le f'(x) \le \beta(x)$, *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\mathrm{d}t + \int_x^b (t-b)\,\beta(t)\mathrm{d}t\right) \le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\mathrm{d}t + \int_x^b (t-b)\,\alpha(t)\mathrm{d}t\right); \tag{2.1}$$

2° *If* $\alpha(x) \le f'(x)$, *then*

$$\begin{aligned}\frac{1}{b-a}\Bigg[\int_a^x (t-a)\alpha(t)\mathrm{d}t &+ \int_x^b (t-b)\alpha(t)\mathrm{d}t \\ &- \max\{x-a, b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right)\Bigg] \\ &\le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\le \frac{1}{b-a}\Bigg[\int_a^x (t-a)\alpha(t)\mathrm{d}t + \int_x^b (t-b)\alpha(t)\mathrm{d}t \\ &+ \max\{x-a, b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right)\Bigg],\end{aligned} \tag{2.2}$$

3° *If* $f'(x) \le \beta(x)$, *then*

$$\begin{aligned}\frac{1}{b-a}\Bigg[\int_a^x (t-a)\beta(t)\mathrm{d}t &+ \int_x^b (t-b)\beta(t)\mathrm{d}t \\ &- \max\{x-a, b-x\}\left(\int_a^b \beta(t)\mathrm{d}t - f(b)+f(a)\right)\Bigg] \\ &\le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\le \frac{1}{b-a}\Bigg[\int_a^x (t-a)\beta(t)\mathrm{d}t + \int_x^b (t-b)\beta(t)\mathrm{d}t \\ &+ \max\{x-a, b-x\}\left(\int_a^b \beta(t)\mathrm{d}t - f(b)+f(a)\right)\Bigg],\end{aligned} \tag{2.3}$$

The listed inequalities in Theorem 2.1 are significant as they improve all previous results in which the Lebesgue norms of f' come into play when handling the bounds calculations. In this case, the required computations in bounds are just in terms of pre-assigned functions. For other related general results, the reader may be refer to [3], [15], [16], [18], [1], and [2].

In this paper, motivated by [9], new integral inequalities of Ostrowski type are obtained. Namely, under certain conditions on f', we give the lower and upper bounds for the difference

$$E(f;h) = \frac{h}{2}[f(a) + f(b)] + (1-h)f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \tag{2.4}$$

where $h \in [0,1]$ and $x \in [a+h\frac{b-a}{2}, b-h\frac{b-a}{2}]$. Our results provides range of estimates including those given by [9] and [5]–[8]. Utilizing general Peano kernel, we recapture the three inequalities (2.1)–(2.3) obtained by [9]. Some special cases of our result and applications to numerical quadrature rules are also given.

3 Main Results

In order to formulate our main results, we need a kernel $K(t;\cdot) : [a,b] \to \mathbb{R}$ defined by

$$K(t;x) = \begin{cases} t - \left(a + h\frac{b-a}{2}\right), & t \in [a,x], \\ t - \left(b - h\frac{b-a}{2}\right), & t \in (x,b], \end{cases} \tag{3.1}$$

for all $h \in [0,1]$ and $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$. Also, for two functions $\alpha, \beta \in C[a,b]$, such that $\alpha(t) \le \beta(t)$ for each $t \in [a,b]$, we define the functions $A(t;\cdot) : [a,b] \to \mathbb{R}$ and $B(t;\cdot) : [a,b] \to \mathbb{R}$ by

$$A(t;x) = \frac{1}{2}\Big\{\big[1 - \operatorname{sgn} K(t;x)\big]\beta(t) + \big[1 + \operatorname{sgn} K(t;x)\big]\alpha(t)\Big\} \tag{3.2}$$

and

$$B(t;x) = \frac{1}{2}\Big\{\big[1 - \operatorname{sgn} K(t;x)\big]\alpha(t) + \big[1 + \operatorname{sgn} K(t;x)\big]\beta(t)\Big\}, \tag{3.3}$$

respectively. We note that

$$\operatorname{sgn} K(t;x) = \begin{cases} -1, & t \in \left[a, a + h\frac{b-a}{2}\right), \\ 1, & t \in \left(a + h\frac{b-a}{2}, x\right], \\ -1, & t \in \left(x, b - h\frac{b-a}{2}\right), \\ 1, & t \in \left(b - h\frac{b-a}{2}, b\right], \end{cases} \tag{3.4}$$

and equal to zero at $t = a + h\frac{b-a}{2}$ and $t = b - h\frac{b-a}{2}$.

Obviously, (2.4) provides range of estimates including those introduced by [9] and [5]–[8]. For

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instance, when $h = 0$, $h = 1/2$, and $h = 1$, (2.4) can be, respectively, reduced to

$$E(f;0) = f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad (x \in [a,b]), \tag{3.5}$$

$$E(f;1/2) = \frac{1}{4}\left[f(a) + f(b) + 2f(x)\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]\right), \tag{3.6}$$

$$E(f;1) = \frac{1}{2}[f\,(a) + f\,(b)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{3.7}$$

Theorem 3.1 *Let $f : I \to \mathbb{R}$, where I is an interval, be a function differentiable in the interior $\overset{\circ}{I}$ of I, and let $[a,b] \subset \overset{\circ}{I}$. Also, let $E(f;h)$, $K(t;x)$, $A(t;x)$, and $B(t;x)$ be given by* (2.4), (3.1), (3.2), *and* (3.3), *respectively.*

For any $\alpha, \beta \in C[a,b]$, $h \in [0,1]$, and $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$, we have the following three cases:

1° *If $\alpha(x) \le f'(x) \le \beta(x)$, then*

$$\frac{1}{b-a}\int_a^b K(t;x)A(t;x)\mathrm{d}t \le E(f;h) \le \frac{1}{b-a}\int_a^b K(t;x)B(t;x)\mathrm{d}t; \tag{3.8}$$

2° *If $\alpha(x) \le f'(x)$, then*

$$\begin{aligned}\frac{1}{b-a}\left\{\int_a^b K(t;x)\alpha(t)\mathrm{d}t - L(x,h)\left(f(b) - f(a) - \int_a^b \alpha(t)\mathrm{d}t\right)\right\} &\le E(f;h)\\ \le \frac{1}{b-a}\left\{\int_a^b K(t;x)\alpha(t)\mathrm{d}t + L(x,h)\left(f(b) - f(a) - \int_a^b \alpha(t)\mathrm{d}t\right)\right\}&,\end{aligned} \tag{3.9}$$

where

$$L(x,h) = \max_{t\in[a,b]}\left|K(t;x)\right| = \max\left\{x - a - h\frac{b-a}{2}, b - x - h\frac{b-a}{2}, h\frac{b-a}{2}\right\}; \tag{3.10}$$

3° *If $f'(x) \le \beta(x)$, then*

$$\begin{aligned}\frac{1}{b-a}\left\{\int_a^b K(t;x)\beta(t)\mathrm{d}t - L(x,h)\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\right\} &\le E(f;h)\\ \le \frac{1}{b-a}\left\{\int_a^b K(t;x)\beta(t)\mathrm{d}t + L(x,h)\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\right\}&,\end{aligned} \tag{3.11}$$

where $L(x,h)$ is defined by (3.10).

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Proof. By considering the kernel $K(t;x)$ in (3.1), we have

$$\begin{aligned}\int_a^b K(t;x)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)\mathrm{d}t &= \int_a^b K(t;x)f'(t)\mathrm{d}t-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t\\ &= (b-a)E(f;h)-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t, \qquad (3.12)\end{aligned}$$

because of

$$\begin{aligned}\int_a^b K(t;x)f'(t)\mathrm{d}t &= \int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]f'(x)\mathrm{d}t+\int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]f'(x)\mathrm{d}t\\ &= \int_a^b tf'(t)\mathrm{d}t-\left(a+h\frac{b-a}{2}\right)(f(x)-f(a))-\left(b-h\frac{b-a}{2}\right)(f(b)-f(x))\\ &= (b-a)\left[\frac{h}{2}[f(a)+f(b)]+(1-h)f(x)\right]-\int_a^b f(t)\mathrm{d}t\\ &= (b-a)E(f;h).\end{aligned}$$

Now, for the first inequality (3.8), the given assumption $\alpha(x)\le f'(x)\le\beta(x)$ yields

$$\left|f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\le\frac{\beta(t)-\alpha(t)}{2}. \qquad (3.13)$$

Therefore, from (3.12) and (3.13), we get

$$\begin{aligned}\left|(b-a)E(f;h)-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t\right| &= \left|\int_a^b K(t;x)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)\mathrm{d}t\right|\\ &= \frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t,\end{aligned}$$

i.e.,

$$\begin{aligned}-\frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t+\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t &\le (b-a)E(f;h)\\ \le\frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t+\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t.\end{aligned}$$

Since $\big|K(t;x)\big|=K(t;x)\,\mathrm{sgn}\,K(t;x)$, and $A(t;x)$ and $B(t;x)$ are defined by (3.2) and (3.3), respectively, the previous inequalities reduce to (3.8).

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For the second case, when $\alpha(x) \le f'(x)$, we have

$$\begin{aligned}\int\limits_a^b K(t;x)\left(f'(t)-\alpha(t)\right)\mathrm{d}t &= \int\limits_a^b K(t;x)f'(t) - \int\limits_a^b K(t;x)\alpha(t)\mathrm{d}t \\ &= (b-a)E(f;h) - \int\limits_a^b K(t;x)\alpha(t)\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}\left|(b-a)E(f;h) - \int\limits_a^b K(t;x)\alpha(t)\mathrm{d}t\right| &\le \left|\int\limits_a^b K(x,t)\left(f'(t)-\alpha(t)\right)\mathrm{d}t\right| \\ &\le \int_a^x \big|K(t;x)\big|\big(f'(t)-\alpha(t)\big)\mathrm{d}t \\ &\le \left(\max_{t\in[a,b]}|K(t;x)|\right)\int_a^b \big(f'(t)-\alpha(t)\big)\mathrm{d}t \\ &= L(x,h)\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right), \qquad (3.14)\end{aligned}$$

where $L(x,h)$ is defined by (3.10). Then, (3.14) gives (3.9).

Finally, for the third case, when $f'(x) \le \beta(x)$, we have

$$\begin{aligned}\int\limits_a^b K(t;x)\left(f'(t)-\beta(t)\right)\mathrm{d}t &= \int\limits_a^b K(t;x)f'(t) - \int\limits_a^b K(t;x)\beta(t)\mathrm{d}t \\ &= (b-a)E(f;h) - \int\limits_a^b K(t;x)\beta(t)\mathrm{d}t,\end{aligned}$$

from which, as before, we obtain

$$\begin{aligned}\left|(b-a)E(f;h) - \int\limits_a^b K(t;x)\beta(t)\mathrm{d}t\right| &\le \left|\int\limits_a^b K(x,t)\left(f'(t)-\beta(t)\right)\mathrm{d}t\right| \\ &\le \int_a^x \big|K(t;x)\big|\big(\beta(t)-f'(t)\big)\mathrm{d}t \\ &\le \left(\max_{t\in[a,b]}|K(t;x)|\right)\int_a^b \big(\beta(t)-f'(t)\big)\mathrm{d}t \\ &= L(x,h)\Big(\int_a^b \beta(t)\mathrm{d}t - f(b)+f(a)\Big),\end{aligned}$$

i.e., (3.11).

The proof of this theorem is completed. $\square$

Remark 3.1 According to (3.10) and $\max\{u,v\} = \frac{1}{2}(u+v+|u-v|)$, we can see that

$$L(x,h) = \frac{b-a}{2}(1-h) + \left|x - \frac{a+b}{2}\right| \qquad \text{if} \quad h \le \frac{1}{2}.$$

This expression holds also for $h > \frac{1}{2}$, but only when $|x| > 2h-1$. However, for $|x| \le 2h-1$, the

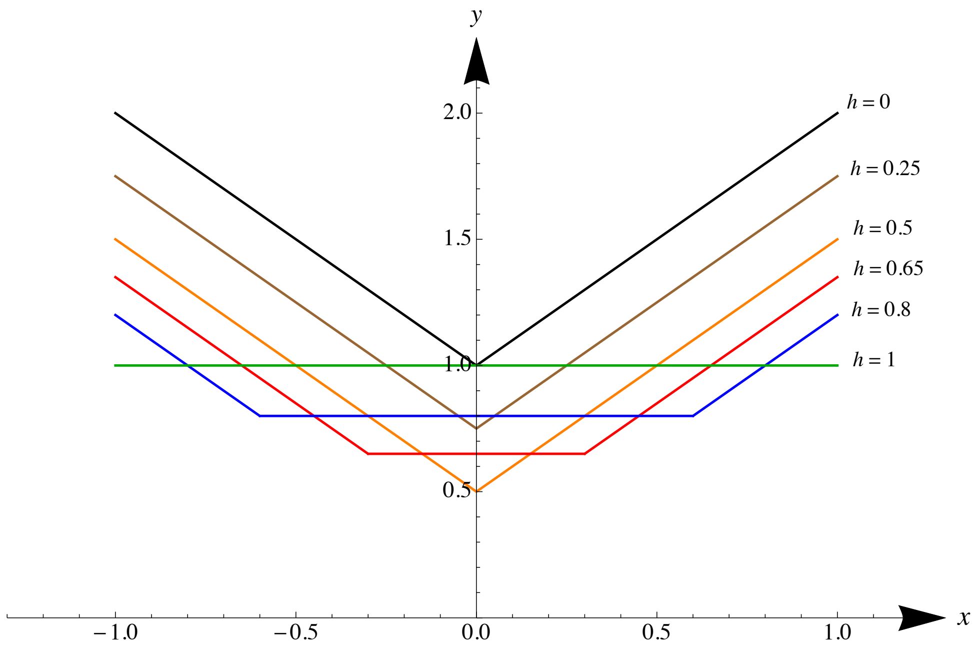

Figure 1: The function $x \mapsto L(x,h)$ for $h = 0$, 0.25, 0.5, 0.65, 0.8, and 1.

function $x \mapsto L(x,h)$ is a constant, i.e.,

$$L(x,h) = \frac{b-a}{2}h.$$

This function on $[a,b] = [-1,1]$ for different value of h is presented in Figure 1.

Now, we consider cases with constant functions α and β, i.e., when $\alpha(x) = \alpha_0$ and $\beta(x) = \beta_0$ on $[a,b]$.

According to (3.2), (3.3), and (3.4), we get

$$\big(A(t;x), B(t;x)\big) = \begin{cases} (\beta_0, \alpha_0), & t \in \left[a, a + h\frac{b-a}{2}\right), \\ (\alpha_0, \beta_0), & t \in \left(a + h\frac{b-a}{2}, x\right], \\ (\beta_0, \alpha_0), & t \in \left(x, b - h\frac{b-a}{2}\right), \\ (\alpha_0, \beta_0), & t \in \left(b - h\frac{b-a}{2}, b\right], \end{cases}$$

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so that the corresponding bounds in (3.8) become

$$\begin{aligned}\underline{B}^{(1)} &= \frac{1}{b-a}\int_a^b K(t;x)A(t;x)\mathrm{d}t \\ &= -\frac{1}{2(b-a)}\big[b^2\beta_0 - a^2\alpha_0 - 2(b\beta_0 - a\alpha_0)x + (\beta_0-\alpha_0)x^2\big] \\ &\quad +\frac{1}{2}\big[a\alpha_0 + b\beta_0 - (\alpha_0+\beta_0)x\big]h - \frac{1}{4}(b-a)(\beta_0-\alpha_0)h^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\overline{B}^{(1)} &= \frac{1}{b-a}\int_a^b K(t;x)B(t;x)\mathrm{d}t \\ &= \frac{1}{2(b-a)}\big[a^2\beta_0 - b^2\alpha_0 + 2(b\alpha_0 - a\beta_0)x + (\beta_0-\alpha_0)x^2\big] \\ &\quad +\frac{1}{2}\big[b\alpha_0 + a\beta_0 - (\alpha_0+\beta_0)x\big]h + \frac{1}{4}(b-a)(\beta_0-\alpha_0)h^2. \end{aligned} \tag{3.16}$$

Also,

$$\begin{aligned}\frac{1}{b-a}\int_a^b K(t;x)\mathrm{d}t &= \frac{1}{b-a}\left\{\int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]\mathrm{d}t + \int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]\mathrm{d}t\right\} \\ &= \frac{1}{2}(1-h)(2x-a-b),\end{aligned}$$

so that we can find the corresponding lower and upper bounds in the inequalities (3.9) and (3.11):

$$\underline{B}^{(2)} = \frac{\alpha_0}{2}(1-h)(2x-a-b) - L(x,h)\left(\frac{f(b)-f(a)}{b-a}-\alpha_0\right), \tag{3.17}$$

$$\overline{B}^{(2)} = \frac{\alpha_0}{2}(1-h)(2x-a-b) + L(x,h)\left(\frac{f(b)-f(a)}{b-a}-\alpha_0\right), \tag{3.18}$$

$$\underline{B}^{(3)} = \frac{\beta_0}{2}(1-h)(2x-a-b) - L(x,h)\left(\beta_0-\frac{f(b)-f(a)}{b-a}\right), \tag{3.19}$$

$$\overline{B}^{(3)} = \frac{\beta_0}{2}(1-h)(2x-a-b) + L(x,h)\left(\beta_0-\frac{f(b)-f(a)}{b-a}\right), \tag{3.20}$$

where $L(x,h)$ is defined by (3.10).

Thus, for constant functions α and β on $[a,b]$, we get the following result:

Corollary 3.1 *Under the assumptions of Theorem* 3.1 *with $\alpha(x)=\alpha_0$ and $\beta(x)=\beta_0$, we have:*

1° *If $\alpha_0 \le f'(x) \le \beta_0$, then $\underline{B}^{(1)} \le E(f;h) \le \overline{B}^{(1)}$;*

2° *If $\alpha_0 \le f'(x)$, then $\underline{B}^{(2)} \le E(f;h) \le \overline{B}^{(2)}$;*

3° *If $f'(x) \le \beta_0$, then $\underline{B}^{(3)} \le E(f;h) \le \overline{B}^{(3)}$,*

where the bounds are given in (3.15)–(3.19).

4 Some Applications in Numerical Integration

Inequalities of Ostrowski's type have attracted considerable interest over the years. Many authors have worked on this subject and proved many extensions and generalizations, including applications in numerical integration (cf. [4]). These inequalities can be considered as error estimates of certain elementary quadrature rules in some classes of functions.

Beside the bounds of (3.5)–(3.7), in this section we consider also ones for $h = 1/3,\ 1/4,\ 2/3$, and $3/4$, i.e.,

$$\begin{aligned}
E(f;1/3) &= \frac{1}{6}\left[f(a)+f(b)+4f(x)\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{5a+b}{6},\frac{a+5b}{6}\right]\right), && (4.1)\\
E(f;1/4) &= \frac{1}{8}[f(a)+f(b)+6f(x)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{7a+b}{8},\frac{a+7b}{8}\right]\right),\\
E(f;2/3) &= \frac{1}{3}[f(a)+f(b)+f(x)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{2a+b}{3},\frac{a+2b}{3}\right]\right),\\
E(f;3/4) &= \frac{1}{8}[3f(a)+3f(b)+2f(x)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{5a+3b}{8},\frac{3a+5b}{8}\right]\right),
\end{aligned}$$

respectively.

For $x = (a+b)/2$, $E(f;1/3)$, given before by (4.1), represents the error in the well-known Simpson formula (cf. [13, pp. 343–350]).

In order to get the corresponding estimates of (2.4), i.e.,

$$E(f;h) = \frac{h}{2}\big[f(a)+f(b)\big] + (1-h)f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[a+h\frac{b-a}{2},\, b-h\frac{b-a}{2}\right]\right),$$

for different values of h, we use here Corollary 3.1 (Case 1°).

Case $h=0$. Here, the value of x can be arbitrary in $[a,b]$. Then, $\underline{B}^{(1)}$ and $\overline{B}^{(1)}$ reduce to

$$\underline{B}^{(1)} = -\frac{1}{2(b-a)}\big[b^2\beta_0 - a^2\alpha_0 - 2(b\beta_0 - a\alpha_0)x + (\beta_0-\alpha_0)x^2\big]$$

and

$$\overline{B}^{(1)} = \frac{1}{2(b-a)}\big[a^2\beta_0 - b^2\alpha_0 + 2(b\alpha_0 - a\beta_0)x + (\beta_0-\alpha_0)x^2\big],$$

so that, under the condition $\alpha_0 \le f'(x) \le \beta_0$, for each $x\in[a,b]$, we have

$$\underline{B}^{(1)} \le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \overline{B}^{(1)}. \tag{4.2}$$

For the symmetric bounds of the first derivative f' ($|f'(x)| \leq \beta_0$), i.e., if $\alpha_0 = -\beta_0$, (4.2) reduces to

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\beta_0,$$

which is, in fact, the original Ostrowski inequality (1.1).

Otherwise, (4.2) for $x = (a+b)/2$ gives the error estimate for the midpoint rule,

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{1}{8}(b-a)(\beta_0 - \alpha_0),$$

while for $x = b$ it gives the error estimate for the so-called endpoint rule

$$\frac{1}{2}(b-a)\alpha_0 \leq f(b) - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \leq \frac{1}{2}(b-a)\beta_0.$$

Case $h = 1$. Here x must be $(b-a)/2$! Taking $h = 1$ in (3.15) and (3.15), for the trapezoidal rule (3.7), we obtain the same bound as for the midpoint rule,

$$\left| \frac{1}{2}\big[f(a) + f(b)\big] - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{1}{8}(b-a)(\beta_0 - \alpha_0).$$

Case $0 < h < 1$. Now we take $x = (a+b)/2$ in (3.15) and (3.15). Since, in that case,

$$-\underline{B}^{(1)} = \overline{B}^{(1)} = \frac{1}{8}(b-a)\big(1 - 2h + 2h^2\big)(\beta_0 - \alpha_0),$$

we get

$$\left| \frac{h}{2}\big[f(a) + f(b)\big] + (1-h) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{b-a}{4}\Big(\frac{1}{2} - h + h^2\Big)(\beta_0 - \alpha_0), \qquad (4.3)$$

provided that $\alpha_0 \leq f'(x) \leq \beta_0$ for $x \in [a, b]$.

For $h = 1/2$, $1/3$, $1/4$, $2/3$, and $3/4$, the inequality (4.3) reduces to

$$\left| \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{b-a}{16}(\beta_0 - \alpha_0),$$

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{5(b-a)}{72}(\beta_0 - \alpha_0),$$

$$\left| \frac{1}{8}\left[f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{5(b-a)}{64}(\beta_0 - \alpha_0),$$

$$\left| \frac{1}{3}\left[f(a) + f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{13(b-a)}{144}(\beta_0 - \alpha_0),$$

and

$$\left|\frac{1}{8}\left[3f(a)+2f\left(\frac{a+b}{2}\right)+3f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le \frac{5(b-a)}{64}(\beta_0-\alpha_0),$$

respectively.

5 Conclusion

Inspired and motivated by the work of Masjed-Jamei and Dragomir [9], new integral inequalities of Ostrowski type are obtained with bounds are just in terms of pre-assigned functions. Our results provides a generalization of error bounds that is independent of Lebesgue norms including those given by [9] and [5]–[8]. We utilize general Peano kernel to recapture the three inequalities (3.8), (3.9), and (3.11), obtained in [9]. Some special cases and applications to numerical quadrature rules are also proposed.

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ITERATES OF CHENEY-SHARMA TYPE OPERATORS ON A TRIANGLE WITH CURVED SIDE

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ABSTRACT. We consider some Cheney–Sharma type operators as well as their product and Boolean sum for a function defined on a triangle with one curved side. Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of these operators.

Keywords: Triangle with curved side, Cheney-Sharma operators, contraction principle, weakly Picard operators.

MSC 2010 Subject Classification: 41A36, 41A25, 39B12, 47H10.

1. Cheney-Sharma type operators

We recall some results regarding Cheney-Sharma type operators on a triangle with one curved side, introduced in [6]. Similar operators were introduced and studied in [3], [4], [5] and [9].

We consider the standard triangle $\widetilde{T}_h$ with vertices $V_1 = (0, h)$, $V_2 = (h, 0)$ and $V_3 = (0, 0)$, with two straight sides Γ_1, Γ_2, along the coordinate axes, and with the third side Γ_3 (opposite to the vertex V_3) defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$, for $h > 0$. Also, we have $f(x) \leq h$ and $g(y) \leq h$, for $x, y \in [0, h]$.

Let F be a real-valued function defined on $\widetilde{T}_h$ and $(0, y)$, $(g(y), y)$, respectively, $(x,\ 0)$, $(x, f(x))$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in \widetilde{T}_h$, intersect the sides Γ_i, $i = 1, 2, 3$. (See Figure 1.)

In [6], we have obtained the following extensions of Cheney-Sharma operator of second kind, to the case of functions defined on $\widetilde{T}_h$:

$$(1.1)\qquad \begin{aligned}(Q_m^x F)(x,y) &= \sum_{i=0}^{m} q_{m,i}(x,y) F\left(i\tfrac{g(y)}{m}, y\right),\\ (Q_n^y F)(x,y) &= \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, j\tfrac{f(x)}{n}\right),\end{aligned}$$

with

$$q_{m,i}(x,y) = \binom{m}{i} \tfrac{1}{(1+m\beta)^{m-1}} \tfrac{x}{g(y)} (\tfrac{x}{g(y)} + i\beta)^{i-1} (1 - \tfrac{x}{g(y)}) [1 - \tfrac{x}{g(y)} + (m-i)\beta]^{m-i-1},$$

$$q_{n,j}(x,y) = \binom{n}{j} \tfrac{1}{(1+nb)^{n-1}} \tfrac{y}{f(x)} (\tfrac{y}{f(x)} + jb)^{j-1} (1 - \tfrac{y}{f(x)}) [1 - \tfrac{y}{f(x)} + (n-j)b]^{n-j-1},$$

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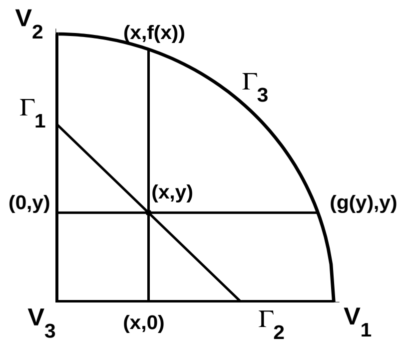

FIGURE 1. Triangle $\tilde{T}_h$.

where

$$\Delta_m^x = \left\{ i\frac{g(y)}{m} \middle| \; i = \overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ j\frac{f(x)}{n} \middle| \; j = \overline{0,n} \right\}$$

are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$ and $m, n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$.

Remark 1.1. *As the Cheney-Sharma operator of second kind interpolates a given function at the endpoints of the interval, we may use the operators Q_m^x and Q_n^y as interpolation operators on $\widetilde{T}_h$.*

Theorem 1.2. [6] *If F is a real-valued function defined on $\widetilde{T}_h$ then the following properties hold:*

(i) $Q_m^x F = F$ on $\Gamma_1 \cup \Gamma_3$;
(ii) $Q_n^y F = F$ on $\Gamma_2 \cup \Gamma_3$;
(iii) $(Q_m^x e_{ij})(x, y) = x^i y^j, \quad i = 0, 1;\ j \in \mathbb{N}$;
(iv) $(Q_n^y e_{ij})(x, y) = x^i y^j, \ \ i \in \mathbb{N};\ j = 0, 1$, where $e_{ij}(x, y) = x^i y^j, \ \ i, j \in \mathbb{N}$.

Let $P_{mn}^1 = Q_m^x Q_n^y$, respectively, $P_{nm}^2 = Q_n^y Q_m^x$ be the products of the operators Q_m^x and Q_n^y. We have

$$\left(P_{mn}^1 F\right)(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} q_{m,i}(x, y)\, q_{n,j}\left(i\frac{g(y)}{m}, y\right) F\left(i\frac{g(y)}{m}, j\frac{f(i\frac{g(y)}{m})}{n}\right), \tag{1.2}$$

respectively,

$$\left(P_{nm}^2 F\right)(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} q_{m,i}\left(x, j\frac{f(x)}{n}\right) q_{n,j}(x, y)\, F\left(i\frac{g(j\frac{f(x)}{n})}{m}, j\frac{f(x)}{n}\right).$$

Theorem 1.3. *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $(P_{mn}^1 F)(V_i) = F(V_i), \qquad i = 1, ..., 3;$
$(P_{mn}^1 F)(\Gamma_3) = F(\Gamma_3),$
(ii) $(P_{nm}^2 F)(V_i) = F(V_i), \qquad i = 1, ..., 3;$
$(P_{nm}^2 F)(\Gamma_3) = F(\Gamma_3).$

We consider the Boolean sums of the operators Q_m^x and Q_n^y,

$$\begin{aligned} S_{mn}^1 &:= Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y, \\ S_{nm}^2 &:= Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x. \end{aligned} \tag{1.3}$$

Theorem 1.4. *If F is a real-valued function defined on $\widetilde{T}_h$, then*

$$S^1_{mn}F\Big|_{\partial\widetilde{T}_h} = F\Big|_{\partial\widetilde{T}_h},$$
$$S^2_{mn}F\Big|_{\partial\widetilde{T}_h} = F\Big|_{\partial\widetilde{T}_h}.$$

2. Weakly Picard operators

We recall some results regarding weakly Picard operators that will be used in the sequel (see, e.g., [21]).

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by

$F_A := \{x \in X \mid A(x) = x\}$-the fixed points set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y,\ Y \neq \emptyset\}$-the family of the nonempty invariant subsets of A;

$A^0 := 1_X,\ A^1 := A,\ \ldots,\ A^{n+1} := A \circ A^n,\quad n \in \mathbb{N}$.

Definition 2.1. *The operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*
(i) $F_A = \{x^*\}$;
(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.2. *The operator A is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *If A is a weakly Picard operator then we consider the operator A^∞, $A^\infty : X \to X$, defined by*

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Theorem 2.4. *An operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A)$, $\forall \lambda \in \Lambda$;
(b) $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator*, $\forall \lambda \in \Lambda$.

3. Iterates of Cheney-Sharma type operators

We study the convergence of the iterates of the Cheney-Sharma type operators (1.1) and of their product and Boolean sum operators, using the weakly Picard operators technique and the contraction principle. The same approach for some other linear and positive operators lead to similar results in [1], [2], [7], [8], [22]-[24].

The limit behavior for the iterates of some classes of positive linear operators were also studied, for example, in [10]-[20]. In the papers [10]-[12] were introduced new methods for the study of the asymptotic behavior of the iterates of positive linear operators. These techniques enlarge the class of operators for which the limit of the iterates can be calculated.

Let F be a real-valued function defined on $\widetilde{T}_h$, $h \in \mathbb{R}_+$. First we study the convergence of the iterates of the Cheney–Sharma type operators given in (1.1).

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Theorem 3.1. *The operators Q_m^x and Q_n^y are weakly Picard operators and*

$$(Q_m^{x,\infty}F)(x,y) = \frac{F(g(y),y) - F(0,y)}{g(y)}x + F(0,y), \tag{3.1}$$

$$(Q_n^{y,\infty}F)(x,y) = \frac{F(x,f(x)) - F(x,0)}{f(x)}y + F(x,0). \tag{3.2}$$

Proof. Taking into account the interpolation properties of Q_m^x and Q_n^y (from Theorem 1.2), let us consider the following sets:

(3.3)
$$\begin{aligned} X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} &= \{F \in C(\widetilde{T}_h) \mid F(0,y) = \varphi|_{\Gamma_1},\ F(g(y),y) = \varphi|_{\Gamma_3}\}, \quad \text{for } y \in [0,h], \\ X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} &= \{F \in C(\widetilde{T}_h) \mid F(x,0) = \psi|_{\Gamma_2},\ F(x,f(x)) = \psi|_{\Gamma_3}\}, \quad \text{for } x \in [0,h], \end{aligned}$$

and for $\varphi, \psi \in C(\widetilde{T}_h)$ we denote by

$$\begin{aligned} F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y) &= \frac{\varphi|_{\Gamma_3} - \varphi|_{\Gamma_1}}{g(y)}x + \varphi|_{\Gamma_1}, \\ F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y) &= \frac{\psi|_{\Gamma_3} - \psi|_{\Gamma_2}}{f(x)}y + \psi|_{\Gamma_2}. \end{aligned}$$

We have the following properties:

(i) $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are closed subsets of $C(\widetilde{T}_h)$;

(ii) $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ is an invariant subset of Q_m^x and $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ is an invariant subset of Q_n^y, for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$;

(iii) $C(\widetilde{T}_h) = \underset{\varphi \in C(\widetilde{T}_h)}{\cup} X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $C(\widetilde{T}_h) = \underset{\psi \in C(\widetilde{T}_h)}{\cup} X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are partitions of $C(\widetilde{T}_h)$;

(iv) $F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \cap F_{Q_m^x}$ and $F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \in X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \cap F_{Q_n^y}$, where $F_{Q_m^x}$ and $F_{Q_n^y}$ denote the fixed points sets of Q_m^x and Q_n^y.

The statements (i) and (iii) are obvious.

(ii) By linearity of Cheney-Sharma operators and Theorem 1.2, it follows that $\forall F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $\forall F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \in X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ we have

$$\begin{aligned} Q_m^x F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y) &= F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y), \\ Q_n^y F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y) &= F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y). \end{aligned}$$

So, $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are invariant subsets of Q_m^x and, respectively, of Q_n^y, for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$;

(iv) We prove that

$$Q_m^x|_{X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}} : X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \to X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \text{ and } Q_n^y|_{X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}} : X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \to X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$$

are contractions for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$.

Let $F,G\in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$. From (1.1) and (3.3) we get

$$\begin{aligned}
&|Q^x_m(F)(x,y)-Q^x_m(G)(x,y)|=\\
&=|Q^x_m(F-G)(x,y)|\leq\\
&\leq|q_{m,0}\,(x;y)\,[F\,(0,0)-G(0,0)]|\\
&\quad+\left|\sum_{i=1}^{m}q_{m,i}\,(x;y)\left[F\left(\tfrac{ig(y)}{m},y\right)-G\left(x,\tfrac{jf(x)}{n}\right)\right]\right|\\
&=\left|\sum_{i=1}^{m}q_{m,i}\,(x;y)\left[F\left(\tfrac{ig(y)}{m},y\right)-G\left(x,\tfrac{jf(x)}{n}\right)\right]\right|\\
&\leq\sum_{i=1}^{m}q_{m,i}\,(x;y)\,\|F-G\|_\infty\\
&=\left[\sum_{i=0}^{m}q_{m,i}\,(x;y)-q_{m,0}\,(x;y)\right]\|F-G\|_\infty\\
&=\left\{1-\left(1-\tfrac{x}{g(y)}\right)\left[1-\tfrac{x}{g(y)(1+m\beta)}\right]^{m-1}\right\}\|F-G\|_\infty\\
&\leq\left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\right]\|F-G\|_\infty\,,
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm.

Hence,

$$\begin{aligned}
&\|Q^x_m(F)(x,y)-Q^x_m(G)(x,y)\|_\infty\leq\\
&\leq\left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\right]\|F-G\|_\infty\,,\ \forall F,G\in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}},
\end{aligned}\tag{3.4}$$

i.e., $Q^x_m|_{X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}}$ is a contraction for $\varphi\in C(\widetilde{T}_h)$.

Analogously, we prove that $Q^y_n|_{X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}}$ is a contraction for $\psi\in C(\widetilde{T}_h)$.

On the other hand, $\frac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}(\cdot)+\varphi|_{\Gamma_1}\in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $\frac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}(\cdot)+\psi|_{\Gamma_2}\in X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are fixed points of Q^x_m and Q^y_n, i.e.,

$$\begin{aligned}
Q^x_m\left(\tfrac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}(\cdot)+\varphi|_{\Gamma_1}\right)&=\tfrac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}(\cdot)+\varphi|_{\Gamma_1}\,,\\
Q^y_n\left(\tfrac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}(\cdot)+\psi|_{\Gamma_2}\right)&=\tfrac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}(\cdot)+\psi|_{\Gamma_2}\,.
\end{aligned}$$

From the contraction principle, $F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y):=\frac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}x+\varphi|_{\Gamma_1}$ is the unique fixed point of Q^x_m in $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $Q^x_m|_{X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}}$ is a Picard operator, with

$$\left(Q^{x,\infty}_m F\right)(x,y)=\tfrac{F(g(y),y)-F(0,y)}{g(y)}x+F\,(0,y)\,,$$

and, similarly, $F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y) := \frac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}y + \psi|_{\Gamma_2}$ is the unique fixed point of Q^y_n in $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ and $Q^y_n|_{X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}}$ is a Picard operator, with

$$(Q^{y,\infty}_n F)(x,y) = \tfrac{F(x,f(x))-F(x,0)}{f(x)}y + F(x,0).$$

Consequently, taking into account (ii), by Theorem 2.4, it follows that the operators Q^x_m and Q^y_n are weakly Picard operators. $\square$

Further we study the convergence of the product and Boolean sum operators given in (1.2) and (1.3).

Theorem 3.2. *The operator P^1_{mn} is a weakly Picard operator and*

$$\left(P^{1,\infty}_{mn}F\right)(x,y) = \frac{F(g(y),y)}{g(y)}x. \tag{3.5}$$

Proof. Let

$$X_\alpha = \{F \in C(\widetilde{T}_h) \mid F(g(y),y) = \alpha\}, \qquad \alpha \in \mathbb{R}$$

and denote by

$$F_\alpha(x,y) := \frac{\alpha}{g(y)}x.$$

We remark that:

(i) X_α is a closed subset of $C(\widetilde{T}_h)$;
(ii) X_α is an invariant subset of P^1_{mn}, for $\alpha \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$;
(iii) $C(\widetilde{T}_h) = \underset{\alpha}{\cup} X_\alpha$ is a partition of $C(\widetilde{T}_h)$;
(iv) $F_\alpha \in X_\alpha \cap F_{P^1_{mn}}$, where $F_{P^1_{mn}}$ denote the fixed points sets of P^1_{mn}.

The statements (i) and (iii) are obvious.

(ii) Similarly with the proof of Theorem 3.1, by linearity of Cheney-Sharma operators and Theorem 1.3, it follows that X_α is an invariant subset of P^1_{mn}, for $\alpha \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$;

(iv) We prove that

$$P^1_{mn}\big|_{X_\alpha} : X_\alpha \to X_\alpha$$

is a contraction for $\alpha \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$. Let $F,G \in X_\alpha$. From [2, Lemma 8] and (3.4), it follows that

$$\begin{aligned}&\left|P^1_{mn}(F)(x,y) - P^1_{mn}(G)(x,y)\right| = \left|P^1_{mn}(F-G)(x,y)\right| \\ &\leq \left[1 - \left(\tfrac{m\beta}{1+m\beta}\right)^{m-1}\left(\tfrac{nb}{1+nb}\right)^{n-1}\right]\|F-G\|_\infty,\end{aligned}$$

so, $P^1_{mn}\big|_{X_\alpha}$ is a contraction for $\alpha \in \mathbb{R}$.

From the contraction principle we have that F_α is the unique fixed point of P^1_{mn} in X_α and $P^1_{mn}\big|_{X_\alpha}$ is a Picard operator, so (3.5) holds. Consequently, taking into account (ii), by Theorem 2.4, it follows that the operators P^1_{mn} is a weakly Picard operator. $\square$

Remark 3.3. *Similar results can be obtained for the operator P^2_{mn}.*

Theorem 3.4. *The operator S^1_{mn} is a weakly Picard operator and*

$$\left(S^{1,\infty}_{mn}F\right)(x,y) = \tfrac{-F(0,y)}{g(y)}x + \tfrac{F(x,f(x))-F(x,0)}{f(x)}y + F(x,0) + F(0,y).$$

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Proof. The proof follows the same steps as the proof of Theorem 3.2, using the following inequality

$$\begin{aligned}&\left\|S_{mn}(F)(x,y)-S_{mn}(G)(x,y)\right\|_{\infty}\\&\leq\left\{1-\left[\left(\tfrac{m\beta}{1+m\beta}\right)^{m-1}+\left(\tfrac{nb}{1+nb}\right)^{n-1}-\left(\tfrac{m\beta}{1+m\beta}\right)^{m-1}\left(\tfrac{nb}{1+nb}\right)^{n-1}\right]\right\}\left\|F-G\right\|_{\infty},\end{aligned}$$

for proving that S^1_{mn} is a contraction. □

Remark 3.5. *We have a similar result for the operator S^2_{nm}.*

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Contemporary Concepts of Neutrosophic Fuzzy Soft BCK-submodules

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Abstract

In this paper, we introduce the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft BCK-submodules and discusse the relation between them. Also, we define the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft BCK-submodules. Finally, we investigate some resultes.

Keywords: BCK-algebras, BCK-modules, soft sets, fuzzy soft sets, neutrosophic sets, neutrosophic soft sets, neutrosophic fuzzy soft BCK-submodules, neutrosophic fuzzy soft translations, neutrosophic fuzzy soft multiplications and neutrosophic fuzzy soft extensions.

1 Introduction

Fuzzy set theory which was developed by Zadeh [23] is an appropriate theory for dealing with vagueness. It is consedered as the one of theories can be handled with uncertainties. Combining fuzzy set models with other mathematical models has attracted the attention of many researchers. Interval-valued fuzzy sets [24], hesitant fuzzy sets [21] , intuitionistic fuzzy sets [3, 4], Intutionistic Fuzzy BCK-submodules [5] and $(\epsilon, \epsilon \vee q)$-fuzzy BCK-submodules [2] are some of the researches that have dealt this subject.

Neutrosophic algebraic structure is a very recent study. It was applied in many fields in order to solve problems related to uncertainties and indeterminacy where they happens to be one of the major factors in almost all real-world problems. Neutrosophic set is a generalizations of the fuzzy set especially of intuitionistic fuzzy set. The intuitionistic fuzzy set has the degree of non-membership as introduced by K. Atanassov [3]. Smarandache in 1998 [19] has introduced the degree of indeterminacy as an independent component and defined the neutrosophic set on three components: truth, indeterminacy and falsity.

The concept of BCK-algebra was first initiated by Imai and Iseki [8]. In 1994, the notion of BCK-modules was introduced by H. Abujable, M. Aslam and A. Thaheem as an action of BCK-algebras on abelian group [1]. BCK-modules theory then was developed by Z. perveen, M. Aslam and A. Thaheem [18]. Bakhshi [6] presented the concept of fuzzy BCK-submodules and investigated their properties. Recently, H. Bashir and Z. Zahid applied the theory of soft sets on BCK-modules in [12].

Translations, multiplications and extensions are very interested mathematical tools. They are types of operations that researchers like to apply with fuzzy set theory. In this paper, we introduce the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft BCK-submodules and discusse the relation between them. Also, we define the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft BCK-submodules. Finally, we investigate some resultes.

2 Preliminaries

In this section, some preliminaries from the soft set theory, neutrosophic soft sets, BCK-algebras and BCK-modules are induced.

Definition 2.1.[17] Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and let A be a nonempty subset of E. A pair $F_A = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping, called the approximate function of the soft set (F, A). It is easy to represent a soft set (F, A) by a set of ordered pairs as follows:

$$(F, A) = \{(x, F(x)) : x \in A\}$$

Definition 2.2.[20] A neutrosophic set A on the universe of discourse U is defined as $A = \{(x, T_A(x), I_A(x), F_A(x)), x \in U\}$ where $T_A : X \to]^-0, 1^+[$ is a truth membership function, $I_A : U \to]^-0, 1^+[$ is an indeterminate membership function, and $F_A : X \to]^-0, 1^+[$ is a false membership function and $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Definition 2.3.[13] Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the neutrosophic soft set (NSS) over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.4.[8, 9] An algebra $(X, *, 0)$ of type $(2, 0)$ is called BCK-algebra if it satisfying the following axioms:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCK-2) $(x * (x * y)) * y = 0$,

(BCK-3) $x * x = 0$,

(BCK-4) $0 * x = 0$,

(BCK-5) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y, z \in X$.

A partial ordering "$\leq$ " is defined on X by $x \leq y \Leftrightarrow x * y = 0$. A BCK-algebra X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$, for all $x \in X$, commutative if it satisfies the identity $x \wedge y = y \wedge x$, where $x \wedge y = y * (y * x)$, for all $x, y \in X$ and implicative if $x * (y * x) = x$, for all $x, y \in X$.

Definition 2.5.[1] Let X be a BCK-algebra. Then by a left X-module (abbreviated X-module), we mean an abelian group M with an operation $X \times M \to M$ with $(x, m) \longmapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$:

(i) $(x \wedge y)m = x(ym)$,

(ii) $x(m + n) = xm + xn$,

(iii) $0m = 0$.

If X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary.

A mapping $\mu : X \to [0, 1]$ is called a fuzzy set in a BCK-algebra X. For any fuzzy set μ in X and any $t \in [0, 1]$, we define set $U(\mu; t) = \mu^t = \{x \in X | \mu(x) \geq t\}$, which is called upper t-level cut of μ.

Definition 2.6.[6] A fuzzy subset μ of M is said to be a fuzzy BCK-submodule if for all m, m_1, $m_2 \in M$ and $x \in X$, the following axioms hold:

(FBCKM1) $\mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2)\}$,

(FBCKM2) $\mu(-m) = \mu(m)$,

(FBCKM3) $\mu(xm) \geq \mu(m)$.

Definition 2.7.[6] Let M, N be modules over a BCK-algebra X. A mapping $f : M \to N$ is called BCK-module homomorphism if

(1) $f(m_1 + m_2) = f(m_1) + f(m_2)$,

(2) $f(xm) = xf(m)$ for all $m, m_1, m_2 \in M$ and $x \in X$.

A BCK-module homomorphism is said to be monomorphism (epimorphism) if it is one to one (onto). If it is both one to one and onto, then we say that it is an isomorphism.

Definition 2.8.[12] Let (F, A) and (G, B) be two soft modules over M and N respectively, $f : M \to N$, $g : A \to B$ be two functions. Then we say that (f, g) is a soft BCK-homomorphism if the following conditions are satisfied:

(1) f is a homomorphism from M onto N,

(2) g is a mapping from A onto B, and

(3) $f(F(x)) = G(g(x))$ for all $x \in A$.

3 Neutrosophic fuzzy soft BCK-submodules

Definition 3.1. A neutrosophic fuzzy soft set (F, A) over a BCK-module M is said to be a neutrosophic fuzzy soft BCK-submodule over M if for all $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$ the following axioms hold :

(NFSS1) $T_{F(\varepsilon)}(m_1 + m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 + m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 + m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$,

(NFSS2) $T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}(m)$,

$I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$,

$F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$,

(NFSS3) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m)$,

$I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m)$,

$F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$.

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a set along with a binary operation $*$ defined in Table 1, then $(X, *, 0)$ forms a commutative BCK-algebra which is not bounded (see [16]). Let $M = \{0, a, b, c\}$ be a subset of X along with an operation $+$ defined by Table 2. Then $(M, +)$ forms a commutative group. Table 3 explains the action of X on M under the operation $xm = x \wedge m$ for all $x \in X$ and $m \in M$. Consequently, M forms an X-module (see [11]).

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	d
d	d	d	d	d	0

Table 1

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 2

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c
d	0	0	0	0

Table 3

Let $A = \{0, a\}$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 4

Consequently, a routine exercise of calculations show that (F, A) forms a neutrosophic fuzzy soft BCK-submodule over M.

(F, A)	0	a	b	c
$T_{F(0)}$	0.9	0.7	0.8	0.7
$I_{F(0)}$	0.8	0.5	0.6	0.5
$F_{F(0)}$	0.1	0.1	0.1	0.1
$T_{F(a)}$	0.5	0.2	0.3	0.2
$I_{F(a)}$	0.3	0.1	0.3	0.1
$F_{F(a)}$	0.1	0.5	0.4	0.5

Table 4

For the sake of simplicity, we shall use the symbols $NFS(M)$ and $NFSS(M)$ for the set of all neutrosophic fuzzy soft sets over M and the set of all neutrosophic fuzzy soft BCK-submodules over M, respectively.

Theorem 3.3. A neutrosophic fuzzy soft set $(F, A) \in NFSS(M)$ if and only if

(i) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m), \quad F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m),$

(ii) $T_{F(\varepsilon)}(m_1 - m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\},$

$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$

$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$

for all $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M then by the definition(3.1) condition (i) is hold.

(ii) $T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(m_1 + (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\},$

$I_{F(\varepsilon)}(m_1 - m_2) = I_{F(\varepsilon)}(m_1 + (-m_2)) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(-m_2)\} = \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$

$F_{F(\varepsilon)}(m_1 - m_2) = F_{F(\varepsilon)}(m_1 + (-m_2)) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(-m_2)\} = \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$

Conversely suppose (F, A) satisfies the conditions (i),(ii). Then we have by (i)

$$T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}((-1)m) \geq T_{F(\varepsilon)}(m),$$

and

$$T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}((-1)(-1)m) \geq T_{F(\varepsilon)}(-m).$$

Thus, $T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}(-m)$. Similarly for $I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$.

$T_{F(\varepsilon)}(m_1 + m_2) = T_{F(\varepsilon)}(m_1 - (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\},$

$I_{F(\varepsilon)}(m_1 + m_2) = I_{F(\varepsilon)}(m_1 - (-m_2)) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(-m_2)\} = \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$

$F_{F(\varepsilon)}(m_1 + m_2) = F_{F(\varepsilon)}(m_1 - (-m_2)) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(-m_2)\{= \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$

Hence (F,A) is a neutrosophic fuzzy soft BCK-submodule over M.

Theorem 3.4. A neutrosophic fuzzy soft set $(F,A) \in NFSS(M)$ if and only if for all $m, m_1, m_2 \in M$, $x, y \in X$ and $\varepsilon \in A$ the following statements hold:

(i) $T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m), \quad F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m),$

(ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\},$

$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$

$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$

Proof. Let $(F,A) \in NFSS(M)$ then by theorem (3.3) and since $0m = 0$ for all $m \in M$,we have

$$\begin{aligned} &\text{(i) } T_{F(\varepsilon)}(0) = T_{F(\varepsilon)}(0m) \geq T_{F(\varepsilon)}(m), \\ &\quad I_{F(\varepsilon)}(0) = I_{F(\varepsilon)}(0m) \geq I_{F(\varepsilon)}(m), \text{ and} \\ &\quad F_{F(\varepsilon)}(0) = F_{F(\varepsilon)}(0m) \leq F_{F(\varepsilon)}(m). \\ &\text{(ii) } T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(xm_1), T_{F(\varepsilon)}(ym_2)\} \\ &\qquad\qquad \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}. \end{aligned}$$

Similarly for

$$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$$

and

$$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Conversely suppose (F,A) satisfies (i),(ii), then we have

$$T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m) and \quad F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m).$$

Then

$$T_{F(\varepsilon)}(xm) = T_{F(\varepsilon)}(x(m-0)) \geq \min\{T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(0)\} = T_{F(\varepsilon)}(m).$$

Similarly for

$$I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) \text{ and } F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m).$$

Also,

$$T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(1m_1 - 1m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}.$$

Similarly for

$$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\} \text{ and } F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Hence (F,A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.5. Let (F, A) be a neutrosophic fuzzy soft set over a BCK-module M and $\alpha \in [0, \perp]$ such that $\perp = 1 - \sup \left\{ F_{F(\varepsilon)}(m) : m \in M, \varepsilon \in A \right\}$.Then $\tilde{T}_\alpha [(F, A)] = (G, A_\alpha^T)$ is called a neutrosophic fuzzy soft α-translation of (F, A) if it satisfies:

$$G(\varepsilon) = \left(\left(T_{F(\varepsilon)}\right)_\alpha^T (m), \left(I_{F(\varepsilon)}\right)_\alpha^T (m), \left(F_{F(\varepsilon)}\right)_\alpha^T (m) \right),$$

for all $\varepsilon \in A, m \in M$ where:

$$\begin{aligned} \left(T_{F(\varepsilon)}\right)_\alpha^T (m) &= T_{F(\varepsilon)}(m) + \alpha, \\ \left(I_{F(\varepsilon)}\right)_\alpha^T (m) &= I_{F(\varepsilon)}(m), \\ \left(F_{F(\varepsilon)}\right)_\alpha^T (m) &= F_{F(\varepsilon)}(m) - \alpha. \end{aligned}$$

Theorem 3.6. A neutrosophic fuzzy soft set (F, A) is said to be a neutrosophic fuzzy soft BCK-submodule over M if and only if the α-translation neutrosophic fuzzy soft set $\tilde{T}_\alpha [(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for all $\alpha \in [0, \perp]$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M and $\alpha \in [0, \perp]$, then by Theorem (3.3)

$$\begin{aligned} \left(T_{F(\varepsilon)}\right)_\alpha^T (xm) &= T_{F(\varepsilon)}(xm) + \alpha \geq T_{F(\varepsilon)}(m) + \alpha = \left(T_{F(\varepsilon)}\right)_\alpha^T (m), \\ \left(F_{F(\varepsilon)}\right)_\alpha^T (xm) &= F_{F(\varepsilon)}(xm) - \alpha \leq F_{F(\varepsilon)}(m) - \alpha = \left(F_{F(\varepsilon)}\right)_\alpha^T (m), \end{aligned}$$

for all $m \in M$, $x \in X$. Also, for all $m_1, m_2 \in M$ we have

$$\begin{aligned} \left(T_{F(\varepsilon)}\right)_\alpha^T (m_1 - m_2) &= T_{F(\varepsilon)}(m_1 - m_2) + \alpha \\ &\geq \min \left\{ T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2) \right\} + \alpha \\ &= \min \left\{ T_{F(\varepsilon)}(m_1) + \alpha, T_{F(\varepsilon)}(m_2) + \alpha \right\} \\ &= \min \left\{ \left(T_{F(\varepsilon)}\right)_\alpha^T (m_1), \left(T_{F(\varepsilon)}\right)_\alpha^T (m_2) \right\}, \end{aligned}$$

and

$$\begin{aligned} \left(F_{F(\varepsilon)}\right)_\alpha^T (m_1 - m_2) &= F_{F(\varepsilon)}(m_1 - m_2) - \alpha \\ &\leq \max \left\{ F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2) \right\} - \alpha \\ &= \max \left\{ F_{F(\varepsilon)}(m_1) - \alpha, F_{F(\varepsilon)}(m_2) - \alpha \right\} \\ &= \max \left\{ \left(F_{F(\varepsilon)}\right)_\alpha^T (m_1), \left(F_{F(\varepsilon)}\right)_\alpha^T (m_2) \right\}. \end{aligned}$$

Hence $\tilde{T}_\alpha [(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M.

Conversely, assume that $\tilde{T}_\alpha [(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for some $\alpha \in [0, \perp]$. Then for all $m \in M$, $x \in X$

$$\begin{aligned} T_{F(\varepsilon)}(xm) + \alpha = \left(T_{F(\varepsilon)}\right)_\alpha^T (xm) &\geq \left(T_{F(\varepsilon)}\right)_\alpha^T (m) = T_{F(\varepsilon)}(m) + \alpha \\ &\Longrightarrow T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m). \end{aligned}$$

Also,

$$F_{F(\varepsilon)}(xm)-\alpha=\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(xm)\leq\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m)=F_{F(\varepsilon)}(m)-\alpha$$
$$\Longrightarrow F_{F(\varepsilon)}(xm)\leq F_{F(\varepsilon)}(m).$$

Now let $m_1, m_2 \in M$, then

$$\begin{aligned}T_{F(\varepsilon)}(m_1-m_2)+\alpha&=\left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1-m_2)\\&\geq\min\left\{\left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1),\left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m_2)\right\}\\&=\min\left\{T_{F(\varepsilon)}(m_1)+\alpha,T_{F(\varepsilon)}(m_2)+\alpha\right\}\\&=\min\left\{T_{F(\varepsilon)}(m_1),T_{F(\varepsilon)}(m_2)\right\}+\alpha\end{aligned}$$

$$\Longrightarrow T_{F(\varepsilon)}(m_1-m_2)\geq\min\left\{T_{F(\varepsilon)}(m_1),T_{F(\varepsilon)}(m_2)\right\},$$

and

$$\begin{aligned}F_{F(\varepsilon)}(m_1-m_2)-\alpha&=\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1-m_2)\\&\leq\max\left\{\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1),\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m_2)\right\}\\&=\max\left\{F_{F(\varepsilon)}(m_1)-\alpha,F_{F(\varepsilon)}(m_2)-\alpha\right\}\\&=\max\left\{F_{F(\varepsilon)}(m_1),F_{F(\varepsilon)}(m_2)\right\}-\alpha\end{aligned}$$

$$\Longrightarrow F_{F(\varepsilon)}(m_1-m_2)\leq\max\left\{F_{F(\varepsilon)}(m_1),F_{F(\varepsilon)}(m_2)\right\}.$$

Hence by Theorem (3.3), (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.7. Let (F, A) and (G, B) be two neutrosophic fuzzy soft sets over a BCK-module M. If $A \subset B$ and $T_{F(\varepsilon)}(m) \leq T_{G(\varepsilon)}(m)$, $I_{F(\varepsilon)}(m) \leq I_{G(\varepsilon)}(m)$, $F_{F(\varepsilon)}(m) \geq F_{G(\varepsilon)}(m)$, $\forall\varepsilon \in A$ and $m \in M$. Then we say that (G, B) is a neutrosophic fuzzy soft extinsion of (F, A).

Definition 3.8. Let (F, A) and (G, B) be two neutrosophic fuzzy soft sets over a BCK-module M. Then (G, B) is a neutrosophic fuzzy soft s-extinsion of (F, A) if the following assertions hold:

(i) (G, B) is a neutrosophic fuzzy soft extinsion of (F, A).

(ii) If (F, A) is a neutrosophic fuzzy soft BCK-submodule over M, then so (G, B).

Theorem 3.9. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M and $\alpha \in [0, \perp]$. Then the neutrosophic fuzzy soft α-translation $\tilde{T}_\alpha[(F, A)]$ is a neutrosophic fuzzy soft s-extinsion of (F, A).

Proof. Since $\tilde{T}_\alpha[(F, A)]$ is an α-translation, we know that $\left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m) \geq T_{F(\varepsilon)}(m)$, $\left(I_{F(\varepsilon)}\right)_{\alpha}^{T}(m) = I_{F(\varepsilon)}(m)$ and $\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m) \leq F_{F(\varepsilon)}(m)$ for all $m \in M, \varepsilon \in A$. Hence $\tilde{T}_\alpha[(F, A)]$ is a neutrosophic fuzzy soft extinsion of (F, A). According to Theorem (3.6), $\tilde{T}_\alpha[(F, A)]$ is a neutrosophic fuzzy soft s-extinsion of (F, A).

The converse of Theorem (3.9) is not true in general as seen in the following example:

Example 3.10. Let $X = \{0, a, b, c\}$ along with a binary operation $*$ defined in Table 5, then $(X, *, 0)$ forms a bounded implicative BCK-algebra (see [16]). Let $M = \{0, a\}$ be a subset of X with a binary operation $+$ defined by $x + y = (x * y) \vee (y * x)$. Then M is a commutative group as shown in table 6. Define scalar multiplication $(X, M) \to M$ by $xm = x \wedge m$ for all $x \in X$ and $m \in M$ that is given in Table 7. Consequently, M forms an X-module (see [11]).

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 5

$+$	0	a
0	0	a
a	a	0

Table 6

$\wedge$	0	a
0	0	0
a	0	a
b	0	0
c	0	a

Table 7

Let $A = M$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 8.

(F, A)	0	a
$T_{F(0)}$	0.9	0.5
$I_{F(0)}$	0.8	0.6
$F_{F(0)}$	0.1	0.3
$T_{F(a)}$	0.3	0.3
$I_{F(a)}$	0.2	0.2
$F_{F(a)}$	0.3	0.5

Table 8

Then (F, A) is a neutrosophic fuzzy soft BCK-submodule over M. Let (G, B) be a neutrosophic fuzzy soft set over M given by Table 9.

Then (G, B) is also a neutrosophic fuzzy soft BCK-submodule over M. Since $T_{F(\varepsilon)}(m) \geq T_{G(\varepsilon)}(m)$, $I_{F(\varepsilon)}(m) \geq I_{G(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(m) \leq F_{G(\varepsilon)}(m)$ for all $m \in M$ and $\varepsilon \in A \subset B$, hence (F, A) is a neutrosophic fuzzy soft s-extension of (G, B), but since $I_{F(0)}(0) = 0.8 \neq I_{G(0)}(0) = 0.7$ then (F, A) is not a neutrosophic fuzzy soft α-translation of (G, B) for all $\alpha \in [0, \perp]$.

(G,B)	0	a
$T_{G(0)}$	0.5	0.3
$I_{G(0)}$	0.7	0.6
$F_{G(0)}$	0.1	0.4
$T_{G(a)}$	0.2	0.2
$I_{G(a)}$	0.1	0.1
$F_{G(a)}$	0.4	0.5

Table 9

Definition 3.11. Let (F,A) be a neutrosophic fuzzy soft set over a BCK-module M and $\upsilon \in [0,1]$. A neutrosophic fuzzy soft υ-multiplication of (F,A) denoted by $\tilde{M}_\upsilon[(F,A)] = (G, m_\upsilon(A))$ is defined as:

$$G(\varepsilon) = \left(m_\upsilon\left(T_{F(\varepsilon)}\right)(m), m_\upsilon\left(I_{F(\varepsilon)}\right)(m), m_\upsilon\left(F_{F(\varepsilon)}\right)(m)\right),$$

where

$$\begin{aligned} m_\upsilon\left(T_{F(\varepsilon)}\right)(m) &= T_{F(\varepsilon)}(m).\upsilon, \\ m_\upsilon\left(I_{F(\varepsilon)}\right)(m) &= I_{F(\varepsilon)}(m), \\ m_\upsilon\left(F_{F(\varepsilon)}\right)(m) &= F_{F(\varepsilon)}(m).\upsilon, \end{aligned}$$

for all $\varepsilon \in A$ and $m \in M$.

Theorem.3.12. If $(F,A) \in NFSS(M)$, then the neutrosophic fuzzy soft υ-multiplication $\tilde{M}_\upsilon[(F,A)] \in NFSS(M)$ for all $\upsilon \in [0,1]$.

Proof. Assume that (F,A) is a neutrosophic fuzzy soft BCK-submodule over M and let $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$. Then

$$\begin{aligned} m_\upsilon\left(T_{F(\varepsilon)}\right)(xm) &= T_{F(\varepsilon)}(xm).\upsilon \geq T_{F(\varepsilon)}(m).\upsilon = m_\upsilon\left(T_{F(\varepsilon)}\right)(m), \\ m_\upsilon\left(I_{F(\varepsilon)}\right)(xm) &= I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) = m_\upsilon\left(I_{F(\varepsilon)}\right)(m), \\ m_\upsilon\left(F_{F(\varepsilon)}\right)(xm) &= F_{F(\varepsilon)}(xm).\upsilon \leq F_{F(\varepsilon)}(m).\upsilon = m_\upsilon\left(F_{F(\varepsilon)}\right)(m). \end{aligned}$$

Moreover,

$$\begin{aligned} m_\upsilon\left(T_{F(\varepsilon)}\right)(m_1 - m_2) &= T_{F(\varepsilon)}(m_1 - m_2).\upsilon \\ &\geq \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\}.\upsilon \\ &= \min\left\{T_{F(\varepsilon)}(m_1).\upsilon, T_{F(\varepsilon)}(m_2).\upsilon\right\} \\ &= \min\left\{m_\upsilon\left(T_{F(\varepsilon)}\right)(m_1), m_\upsilon\left(T_{F(\varepsilon)}\right)(m_2)\right\}, \end{aligned}$$

$$\begin{aligned} m_{\upsilon}\left(I_{F(\varepsilon)}\right)(m_1-m_2) &= I_{F(\varepsilon)}(m_1-m_2) \\ &\geq \min\left\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\right\} \\ &= \min\left\{m_{\upsilon}\left(I_{F(\varepsilon)}\right)(m_1), m_{\upsilon}\left(I_{F(\varepsilon)}\right)(m_2)\right\}, \end{aligned}$$

$$\begin{aligned} m_{\upsilon}\left(F_{F(\varepsilon)}\right)(m_1-m_2) &= F_{F(\varepsilon)}(m_1-m_2).\upsilon \\ &\leq \max\left\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\right\}.\upsilon \\ &= \max\left\{F_{F(\varepsilon)}(m_1).\upsilon, F_{F(\varepsilon)}(m_2).\upsilon\right\} \\ &= \max\left\{m_{\upsilon}\left(F_{F(\varepsilon)}\right)(m_1), m_{\upsilon}\left(F_{F(\varepsilon)}\right)(m_2)\right\}. \end{aligned}$$

Therefore by Theorem (3.3), $\tilde{M}_{\upsilon}[(F,A)]$ is a neutrosophic fuzzy soft BCK-submodule over M.

The converse of Theorem (3.12) is not true in general as seen in the following example:

Example 3.13. Consider a BCK-algebra $X = \{0, a, b, c\}$ and X-module $M = \{0, a\}$ that are defined in Example 3.10. Table 10 defines a neutrosophic fuzzy soft set (F, A) over M

(F,A)	0	a
$T_{F(0)}$	0.3	0.4
$I_{F(0)}$	0.7	0.5
$F_{F(0)}$	0.1	0.5
$T_{F(a)}$	0.1	0.1
$I_{F(a)}$	0.1	0.1
$F_{F(a)}$	0.5	0.6

Table 10

If we take $\upsilon = 0$, then the υ-multiplication is a neutrosophic fuzzy soft BCK-submodule over M since

$$\begin{aligned} m_0\left(T_{F(\varepsilon)}\right)(xm) = 0 = m_0\left(T_{F(\varepsilon)}\right)(m), \\ m_0\left(I_{F(\varepsilon)}\right)(xm) \geq m_0\left(I_{F(\varepsilon)}\right)(m), \\ m_0\left(F_{F(\varepsilon)}\right)(xm) = 0 = m_0\left(F_{F(\varepsilon)}\right)(m), \end{aligned}$$

and

$$\begin{aligned} m_0\left(T_{F(\varepsilon)}\right)(m_1-m_2) = 0 = \min\left\{m_0\left(T_{F(\varepsilon)}\right)(m_1), m_0\left(T_{F(\varepsilon)}\right)(m_2)\right\}, \\ m_0\left(I_{F(\varepsilon)}\right)(m_1-m_2) \geq \min\left\{m_0\left(I_{F(\varepsilon)}\right)(m_1), m_0\left(I_{F(\varepsilon)}\right)(m_2)\right\}, \\ m_0\left(F_{F(\varepsilon)}\right)(m_1-m_2) = 0 = \min\left\{m_0\left(F_{F(\varepsilon)}\right)(m_1), m_0\left(F_{F(\varepsilon)}\right)(m_2)\right\}, \end{aligned}$$

for all $m, m_1, m_2 \in M$ and $x \in X$. But if we take $m_1 = 0, m_2 = a$ and $\varepsilon = 0$ then

$$T_{F(0)}(0 + a) = T_{F(0)}(a) = 0.4 \ngeq \min\left\{T_{F(0)}(0), T_{F(0)}(a)\right\} = 0.3.$$

Hence (F, A) is not a neutrosophic fuzzy soft BCK-submodule over M.

Theorem.3.14. A neutrosophic fuzzy soft set (F, A) is said to be a neutrosophic fuzzy soft BCK-submodule over M if and only if the υ-multiplication neutrosophic fuzzy set $\tilde{M}_\upsilon[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for all $\upsilon \in (0, 1]$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M then by Theorem (3.12) $\tilde{M}_\upsilon[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for all $\upsilon \in (0, 1]$.

Now let $\upsilon \in (0, 1]$ be such that $\tilde{M}_\upsilon[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M and let $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$. Then

$$T_{F(\varepsilon)}(xm).\upsilon = m_\upsilon\left(T_{F(\varepsilon)}\right)(xm) \geq m_\upsilon\left(T_{F(\varepsilon)}\right)(m) = T_{F(\varepsilon)}(m).\upsilon,$$
$$I_{F(\varepsilon)}(xm) = m_\upsilon\left(I_{F(\varepsilon)}\right)(xm) \geq m_\upsilon\left(I_{F(\varepsilon)}\right)(m) = I_{F(\varepsilon)}(m),$$
$$F_{F(\varepsilon)}(xm).\upsilon = m_\upsilon\left(F_{F(\varepsilon)}\right)(xm) \leq m_\upsilon\left(F_{F(\varepsilon)}\right)(m) = F_{F(\varepsilon)}(m).\upsilon,$$

and since $\upsilon \neq 0$, then $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$. Now

$$\begin{aligned} T_{F(\varepsilon)}(m_1 - m_2).\upsilon &= m_\upsilon\left(T_{F(\varepsilon)}\right)(m_1 - m_2) \\ &\geq \min\left\{m_\upsilon\left(T_{F(\varepsilon)}\right)(m_1), m_\upsilon\left(T_{F(\varepsilon)}\right)(m_2)\right\} \\ &= \min\left\{T_{F(\varepsilon)}(m_1).\upsilon, T_{F(\varepsilon)}(m_2).\upsilon\right\} \\ &= \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\}.\upsilon, \end{aligned}$$

which means that

$$T_{F(\varepsilon)}(m_1 - m_2) \geq \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\}.$$

Similarly,

$$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\left\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\right\}.$$

Hence (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

4 Ismorphism Theorem Of Neutrosophic Fuzzy Soft BCK-submodules

Definition 4.1. Let M and N be two BCK-modules over a BCK-algebra X. Let $f : M \longrightarrow N$ be a BCK-submodule homomorphism and let (F, A), (G, B) be two neutrosophic fuzzy soft BCK-submodule over M and N respectively. Then the image of (F, A) is a neutrosophic fuzzy soft set over N defined as follows for all $x \in M$, $y \in N$ and $\varepsilon \in A$.

$$f(F(\varepsilon))(x) = \left(T_{f(F)}(y), I_{f(F)}(y), F_{f(F)}(y)\right) = \left(f(T_F)(y), f(I_F)(y), f(F_F)(y)\right),$$

where

$$f(T_F)(y) = \begin{cases} \sup T_F(x) & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases},$$
$$f(I_F)(y) = \begin{cases} \sup I_F(x) & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases},$$
$$f(F_F)(y) = \begin{cases} \inf F_F(x) & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases},$$

and the preimage of (G, B) is a neutrosophic fuzzy soft set over M defined as

$$f^{-1}(G(\delta))(y) = \left(T_{f^{-1}(G)}(x), I_{f^{-1}(G)}(x), F_{f^{-1}(G)}(x)\right) = \left(T_G(f(x)), I_G(f(x)), F_G(f(x))\right),$$

where $\delta \in B$.

Theorem 4.2. Let $(X, *, 0)$ be a BCK-algebra, M and N are modules of X. A mapping $f : M \longrightarrow N$ is a BCK-submodule homomorphism and $(F, A) \in NFSS(N)$, then the inverse image $\left(f^{-1}(F), A\right) \in NFSS(M)$.

Proof. Since (F, A) is a neutrosophic fuzzy soft BCK-submodule over N. Let $m \in M$, $\varepsilon \in A$ then by Theorem (3.4)

$$T_{f^{-1}(F)}(0) = T_{F(\varepsilon)}(f(0)) = T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(f(m)) = T_{f^{-1}(F)}(m),$$
$$I_{f^{-1}(F)}(0) = I_{F(\varepsilon)}(f(0)) = I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(f(m)) = I_{f^{-1}(F)}(m),$$
$$F_{f^{-1}(F)}(0) = F_{F(\varepsilon)}(f(0)) = F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(f(m)) = F_{f^{-1}(F)}(m).$$

Now let $m_1, m_2 \in M$, $x, y \in X$, and $\varepsilon \in A$, then

$$\begin{aligned} T_{f^{-1}(F)}(xm_1 - ym_2) &= T_{F(\varepsilon)}(f(xm - ym_2)) \\ &= T_{F(\varepsilon)}(xf(m_1) - yf(m_2)) \\ &\geq \min\left\{T_{F(\varepsilon)}(f(m_1)), T_{F(\varepsilon)}(f(m_2))\right\} \\ &= \min\left\{T_{f^{-1}(F)}(m_1), T_{f^{-1}(F)}(m_2)\right\}. \end{aligned}$$

Similarly for

$$I_{f^{-1}(F)}(xm_1 - ym_2) \geq \min\left\{I_{f^{-1}(F)}(m_1), I_{f^{-1}(F)}(m_2)\right\},$$

and

$$F_{f^{-1}(F)}(xm_1 - ym_2) \leq \max\left\{F_{f^{-1}(F)}(m_1), F_{f^{-1}(F)}(m_2)\right\}.$$

Hence $\left(f^{-1}(F), A\right)$ is a neutrosophic fuzzy soft BCK-submodule over M.

Theorem.4.3. Let $(X, *, 0)$ be a BCK-algebra, M and N are modules of X. A mapping $f : M \longrightarrow N$ is a BCK-submodule epimorphism. If (F, A) is a neutrosophic fuzzy soft set over N such that $\left(f^{-1}(F), A\right) \in NFSS(M)$, then $(F, A) \in NFSS(N)$.

Proof. Assume that $\left(f^{-1}(F), A\right)$ is a neutrosophic fuzzy soft BCK-submodule over M. Let $n \in N$ then there exist $m \in M$ such that $f(m) = n$. Then for all $\varepsilon \in A$

$$\begin{aligned} T_{F(\varepsilon)}(n) &= T_{F(\varepsilon)}(f(m)) = T_{f^{-1}(F)}(m) \leq T_{f^{-1}(F)}(0) = T_{F(\varepsilon)}(f(0)) = T_{F(\varepsilon)}(0), \\ I_{F(\varepsilon)}(n) &= I_{F(\varepsilon)}(f(m)) = I_{f^{-1}(F)}(m) \leq I_{f^{-1}(F)}(0) = I_{F(\varepsilon)}(f(0)) = I_{F(\varepsilon)}(0), \\ F_{F(\varepsilon)}(n) &= F_{F(\varepsilon)}(f(m)) = F_{f^{-1}(F)}(m) \geq F_{f^{-1}(F)}(0) = F_{F(\varepsilon)}(f(0)) = F_{F(\varepsilon)}(0). \end{aligned}$$

Let $m, \grave{m} \in M$, $n, \grave{n} \in N$ such that $f(m) = n$ and $f(\grave{m}) = \grave{n}$ and $x, y \in X$ then

$$\begin{aligned} T_{F(\varepsilon)}(xn - y\grave{n}) &= T_{F(\varepsilon)}(xf(m) - yf(\grave{m})) \\ &= T_{F(\varepsilon)}(f(xm - y\grave{m})) \\ &= T_{f^{-1}(F)}(xm - y\grave{m}) \\ &\geq \min\left\{T_{f^{-1}(F)}(m), T_{f^{-1}(F)}(\grave{m})\right\} \\ &= \min\left\{T_{F(\varepsilon)}(f(m)), T_{F(\varepsilon)}(f(\grave{m}))\right\} \\ &= \min\left\{T_{F(\varepsilon)}(n), T_{F(\varepsilon)}(\grave{n})\right\}. \end{aligned}$$

Similarly for

$$I_{F(\varepsilon)}(xn - y\grave{n}) \geq \min\left\{I_{F(\varepsilon)}(n), I_{F(\varepsilon)}(\grave{n})\right\},$$

and

$$F_{F(\varepsilon)}(xn - y\grave{n}) \leq \max\left\{F_{F(\varepsilon)}(n), F_{F(\varepsilon)}(\grave{n})\right\}.$$

Hence according to Theorem (3.4), (F, A) is a neutrosophic fuzzy soft BCK-submodule over N.

Theorem.4.4. Let $(X, *, 0)$ be a BCK-algebra, M and N are modules of X. A mapping $f : M \longrightarrow N$ is a BCK-submodule epimorphism and let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M. Then the homomorphic image $(f(F), A)$ is a neutrosophic fuzzy soft BCK-submodule over N.

Proof. Assume that (F, A) is a neutrosophic fuzzy soft BCK-submodule over M. Let $n \in N$ then there exist $m \in M$ such that $f(m) = n$. Then

$$\begin{aligned} T_{f(F)}(n) &= f(T_F)(n) = \sup T_F(m) \leq \sup T(0) = f(T_F)(0) = T_{f(F)}(0), \\ I_{f(F)}(n) &= f(I_F)(n) = \sup I_F(m) \leq \sup I(0) = f(I_F)(0) = I_{f(F)}(0), \\ F_{f(F)}(n) &= f(F_F)(n) = \inf F_F(m) \geq \inf F(0) = f(F_F)(0) = F_{f(F)}(0). \end{aligned}$$

Let $m_1, m_2 \in M$, $n_1, n_2 \in N$ such that $f(m_1) = n_1$ and $f(m_2) = n_2$ and $x, y \in X$ then

$$\begin{aligned}
T_{f(F)}(xn_1 - yn_2) &= f(T_F)(xn_1 - yn_2) \\
&= \sup T_F(xm_1 - ym_2) \\
&\geq \sup\{\min\{T_F(m_1), T_F(m_2)\}\} \\
&= \min\{\sup T_F(m_1), \sup T_F(m_2)\} \\
&= \min\{f(T_F)(n_1), f(T_F)(n_2)\} \\
&= \min\{T_{f(F)}(n_1), T_{f(F)}(n_2)\}.
\end{aligned}$$

Similarly for

$$I_{f(F)}(xn_1 - yn_2) \geq \min\{I_{f(F)}(n_1), I_{f(F)}(n_2)\},$$

and

$$F_{f(F)}(xn_1 - yn_2) \leq \max\{F_{f(F)}(n_1), F_{f(F)}(n_2)\}.$$

Hence by Theorem (3.4), $(f(F), A)$ is a neutrosophic fuzzy soft BCK-submodule over N.

Corollary 4.5. Let $f : M \longrightarrow N$ be a homomorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft set over N. If (F, A) is a neutrosophic fuzzy soft BCK-submodule, then so is $\left(f^{-1}(F), A_\alpha^T\right)$ for any α-translation $\tilde{T}_\alpha[(F, A)]$ of (F, A) with $\alpha \in [0, \perp]$.

Proof. Directly by Theorem(3.6) and Theorem(4.2).

Joining Theorems (3.6), (4.3) and (4.4) we have the following corollaries:

Corollary 4.6. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft set over N. If the inverse image of a neutrosophic fuzzy soft α-translation of (F, A) is a neutrosophic fuzzy soft BCK-submodule for some $\alpha \in [0, \perp]$, then so is (F, A).

Corollary 4.7. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft BCK-submodule over M, then the homomorphic image of a neutrosophic fuzzy soft α-translation of (F, A) is a neutrosophic fuzzy soft BCK-submodule over N for any $\alpha \in [0, \perp]$.

Using Theorems (3,14), (4.2), (4.3) and (4.4), we deduce the following results:

Corollary 4.8. Let $f : M \longrightarrow N$ be a homomorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft BCK-submodule over N, then the inverse image of a neutrosophic fuzzy soft υ-multiplication of (F, A) is a neutrosophic fuzzy soft BCK-submodule over M for any υ-multiplication of (F, A) with $\upsilon \in [0, 1]$.

Corollary 4.9. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules. If the inverse image of a neutrosophic fuzzy soft υ-multiplication of (F, A) is a neutrosophic fuzzy soft BCK-submodule over M for some $\upsilon \in (0, 1]$, then (F, A) is a neutrosophic fuzzy soft BCK-submodule over N.

Corollary 4.10. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft BCK-submodule over M, then the homomorphic image of a neutrosophic fuzzy soft υ-multiplication of (F, A) is a neutrosophic fuzzy soft BCK-submodule over N for any $\upsilon \in (0, 1]$.

5 Conclusion

Translations, multiplications and extensions are very interested mathematical tools. They are types of operations that researchers like to apply with fuzzy set theory. In this paper, the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft BCK-submodules were introduced and the relation between them were discussed. Also, the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft BCK-submodules was defined. Finally, some results were investigated.

6 Compliance with Ethical Standards

Conflict of Interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical Approval: This artical does not contain any studies with human participants or animals performed by any of the authors.

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