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Differential equations and inclusions involving mixed fractional derivatives with four-point nonlocal fractional boundary conditions

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Abstract

We study a new class of boundary value problems of mixed fractional differential equations and inclusions involving both left Caputo and right Riemann-Liouville fractional derivatives, and nonlocal four-point fractional boundary conditions. We apply the standard tools of the fixed-point theory to obtain the sufficient criteria for the existence and uniqueness of solutions for the problems at hand. Illustrative examples for the obtained results are also presented.

Keywords: Fractional differential equations; fractional differential inclusions; fractional derivative; boundary value problem; existence; fixed point theorems.
MSC 2000: 34A08, 34B15, 34A60.

1 Introduction

Fractional calculus deals with the study of fractional order integrals and derivatives and their diverse applications [1, 2, 3]. Riemann-Liouville and Caputo are kinds of fractional derivatives. They all generalize the ordinary integral and differential operators. However, the fractional derivatives have fewer properties than the corresponding classical ones. As a result, it makes these derivatives very useful at describing the anomalous phenomena, see [4, 5, 6] and references cited therein.

Some solutions of equations containing left and right fractional derivatives were investigated [7, 8, 9]. The left and the right derivatives found interesting applications in fractional variational principles, fractional control theory as well as in fractional Lagrangian and Hamiltonian dynamics. In [10], the existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative

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was discussed. In [11, 12], the authors studied the existence of solutions for fractional boundary value problems involving both the left Riemann-Liouville and the right Caputo fractional derivatives.

In this paper, we investigate the existence and uniqueness of solutions for a mixed fractional differential equation involving both left Caputo and right Riemann-Liouville types fractional derivatives associated with nonlocal four-point fractional boundary conditions. Precisely, we study the following problems:

$$\begin{cases} {}^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) = f(t, y(t)), \quad t \in J := [0,1], \\ y(0) = 0, \; D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) \in F(t, y(t)), \quad t \in J := [0,1], \\ y(0) = 0, \; D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \xi, \eta < 1, \end{cases} \tag{1.2}$$

where ${}^cD^{\alpha}_{1-}$ and D^{β}_{0+} denote the left Caputo fractional derivative of order $\alpha \in (1,2]$ and the right Riemann-Liouville fractional derivative of order $\beta \in (0,1]$ respectively, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\delta \in \mathbb{R}$ is an appropriate constant. Here we remark that the problem (1.1) with $y'(0) = 0$ in palce of $D^{\beta}_{0+}y(\xi) = 0$, was studied recently in [13].

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and prove a basic result that plays a key role in the forthcoming analysis. Section 3 contains the existence and uniqueness results for the problem (1.1), which rely on fixed point theorems due to Banach, Krasnoselskii and Leray-Schauder nonlinear alternative. In Section 4, we discuss existence results for the problem (1.2), which rely on nonlineqar alternative for Kakutani maps and Covitz and Nadler fixed point theorem. Finally in Section 5 we study illustrative examples for the obtained results.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts [14] that we need in the sequel.

Definition 2.1 *We define the left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ as*

$$I^{\alpha}_{0+}g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds, \tag{2.1}$$

$$I^{\alpha}_{1-}g(t) = \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds, \tag{2.2}$$

provided the right-hand sides are point-wise defined on $(0,\infty)$, *where* Γ *is the Gamma function.*

Definition 2.2 *The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* $g : (0,\infty) \to \mathbb{R}$ *such that* $g \in C^n((0,\infty),\mathbb{R})$ *are respectively given by*

$$D^{\alpha}_{0+}g(t) = \frac{d^n}{dt^n}(I^{n-\alpha}_{0+}g)(t),$$

$$^cD^{\alpha}_{1-}g(t) = (-1)^n I^{n-\alpha}_{1-}g^{(n)}(t),$$

where $n-1 < \alpha < n$.

The following lemma, dealing with a linear variant of the problem (1.1), plays an important role in the forthcoming analysis.

Lemma 2.3 *Let* $h \in C(J,\mathbb{R})$ *and* $P = [(1-\delta\eta^{\beta+1}) - (\beta+1)\xi(1-\delta\eta^{\beta})] \neq 0$. *The function* y *is a solution of the problem*

$$\begin{cases} ^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) = h(t), \quad t \in J := [0,1], \\ y(0) = 0, \ D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \xi, \eta < 1, \end{cases} \tag{2.3}$$

if and only if

$$\begin{aligned} y(t) = {} & I^{\beta}_{0+}I^{\alpha}_{1-}h(t) + \frac{[t^{\beta+1}(1-\delta\eta^{\beta}) - t^{\beta}(1-\delta\eta^{\beta+1})]}{P\Gamma(\beta+1)} I^{\alpha}_{1-}h(t)|_{t=\xi} \\ & + \frac{[t^{\beta+1} - \xi(\beta+1)t^{\beta}]}{P}\left(\delta I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=\eta} - I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=1}\right), \end{aligned} \tag{2.4}$$

where $I^{\alpha}_{1-}y(s)$ *is defined by (2.2).*

Proof. Applying the right fractional integral I^{α}_{1-} to both sides of the equation in the problem (2.3), we get

$$D^{\beta}_{0+}y(t) = I^{\alpha}_{1-}h(t) + c_0 + c_1 t. \tag{2.5}$$

Using the condition $D^{\beta}_{0+}y(\xi) = 0$ in (2.5), we obtain

$$c_0 + c_1\xi = -I^{\alpha}_{1-}h(t)|_{t=\xi}. \tag{2.6}$$

Next we apply the left fractional integral I^{β}_{0+} to the equation (2.5) to get

$$y(t) = I^{\beta}_{0+}I^{\alpha}_{1-}h(t) + c_0\frac{t^{\beta}}{\Gamma(\beta+1)} + c_1\frac{t^{\beta+1}}{\Gamma(\beta+2)} + c_2 t^{\beta-1}. \tag{2.7}$$

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Making use of the conditions $y(0) = 0$ and $y(1) = \delta y(\eta)$ in (2.7) yields $c_2 = 0$ and

$$\frac{(1-\delta\eta^{\beta})}{\Gamma(\beta+1)}c_0 + \frac{(1-\delta\eta^{\beta+1})}{\Gamma(\beta+2)}c_1 = \delta I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=\eta} - I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=1}. \tag{2.8}$$

Solving (2.7) and (2.8) for c_0 and c_1, we find that

$$c_0 = -\frac{\Gamma(\beta+2)}{P}\Big[\frac{(1-\delta\eta^{\beta+1})}{\Gamma(\beta+2)} I_{1-}^{\alpha} h(t)|_{t=\xi} + \xi\left(\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=\eta} - I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=1}\right)\Big],$$

$$c_1 = \frac{\Gamma(\beta+2)}{P}\Big[\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=\eta} - I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=1} + \frac{(1-\delta\eta^{\beta})}{\Gamma(\beta+1)} I_{1-}^{\alpha} h(t)|_{t=\xi}\Big].$$

Substituting the values of c_0 and c_1 in (2.6), we get the solution (2.4). By direct computation, we can obtain the converse of this lemma. This completes the proof. □

Remark 2.4 *Let $\|h\| = \sup_{t\in[0,1]} |h(t)$. Then we have the following estimate:*

$$\|y\| \le \|h\| \max_{t\in[0,1]} \left\{ \frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)| + \frac{[t^{\beta} + (1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)} \right\}, \tag{2.9}$$

where

$$\mu_1(t) = \frac{t^{\beta+1}(1-\delta\eta^{\beta}) - t^{\beta}(1-\delta\eta^{\beta+1})}{P\Gamma(\beta+1)}, \quad \mu_2(t) = \frac{t^{\beta+1} - \xi(\beta+1)t^{\beta}}{P}. \tag{2.10}$$

Indeed, we have

$$\begin{aligned}
|y(t)| &\le \|h\| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)|\|h\| \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + |\mu_2(t)|\|h\| \Bigg[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \\
&\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds\Bigg] \\
&= \|h\| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} ds + |\mu_1(t)|\|h\| \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + |\mu_2(t)|\|h\| \Bigg[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} ds\Bigg] \\
&\le \|h\| \max_{t\in[0,1]} \left\{ \frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)| + \frac{[t^{\beta} + (1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)} \right\},
\end{aligned}$$

where we taken $(1-s)^{\alpha} \le 1$.

For computation convenience, we introduce the notation:

$$\Lambda = \max_{t\in[0,1]} \Big\{ \frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)| + \frac{[t^{\beta} + (1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)} \Big\}. \tag{2.11}$$

3 Existence and uniqueness results for the problem (1.1)

Let $\mathcal{X} = C([0,1], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ equipped with the norm $\|y\| = \sup\{|y(t)| : t \in [0,1]\}$.

In view of Lemma 2.3, we transform the problem (1.1) into a fixed point problem as

$$y = \mathcal{G}y, \tag{3.1}$$

where the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{aligned} \mathcal{G}y(t) \;\; = \;\; & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds + \mu_1(t) \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds \\ & +\mu_2(t)\Big[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds\Big], \end{aligned} \tag{3.2}$$

where μ_1, μ_2 are defined by (2.10).

Our first result deals with the existence and uniqueness of solutions for the problem (1.1).

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that:*

(H_1) $|f(t,y) - f(t,z)| \le L|y - z|$, *for all* $t \in [0,1]$, $y, z \in \mathbb{R}$, $L > 0$.

Then the problem (1.1) has a unique solution on $[0,1]$ *if*

$$L\Lambda < 1, \tag{3.3}$$

where Λ *is defined by (2.11).*

Proof. Let us define $\sup_{t\in[0,1]} |f(t,0)| = M$ and select $r \ge \dfrac{M\Lambda}{1 - L\Lambda}$ to establish that $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{y \in \mathcal{X} : \|y\| \le r\}$ and $\mathcal{G}$ is defined by (3.2). Using the condition (H_1), we have

$$\begin{aligned} |f(t,y)| \;\; & = \;\; |f(t,y) - f(t,0) + f(t,0)| \le |f(t,y) - f(t,0)| + |f(t,0)| \\ & \le \;\; L\|y\| + M \le Lr + M. \end{aligned} \tag{3.4}$$

Then, for $y \in \mathcal{B}_r$, by using Remark 2.4, we obtain

$$\begin{aligned} \|\mathcal{G}y\| \;\; \le \;\; & (Lr + M)\Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\ & +|\mu_2(t)|\Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \end{aligned}$$

$$
\begin{aligned}
& +\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\}\\
= \;& (Lr+M)\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds+|\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\
& +|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds\Bigg]\Bigg\}\\
\leq \;& (Lr+M)\Lambda<r.
\end{aligned}
$$

This show that $\mathcal{G}y \in \mathcal{B}_r$, $y \in \mathcal{B}_r$. Thus $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$. Next we show that $\mathcal{G}$ is a contraction. For that, let $y, z \in \mathcal{X}$. Then, for each $t \in [0,1]$, we have

$$
\begin{aligned}
& \|(\mathcal{G}y)-(\mathcal{G}z)\|\\
\leq \;& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|f(u,y(u))-f(u,z(u))|duds\\
& +|\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))-f(s,z(s))|ds\\
& +|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|f(u,y(u))-f(u,z(u))|duds\\
& +\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|f(u,y(u))-f(u,z(u))|duds\Bigg]\\
\leq \;& L\Lambda\|y-z\|,
\end{aligned}
$$

which, in view of the given condition $L\Lambda < 1$, implies that $\mathcal{G}$ is a contraction. In consequence, it follow by the contraction mapping principle that there exists a unique solution for the problem (1.1) on $[0,1]$. This completes the proof. □

Our next existence result for the problem (1.1) relies on Krasnoselskii's fixed point theorem.

Lemma 3.2 *(Krasnoselskii's fixed point theorem) [15]. Let S be a closed, bounded, convex and nonempty subset of a Banach space X. Let $\mathcal{Y}_1, \mathcal{Y}_2$ be the operators mapping S into X such that (a) $\mathcal{Y}_1 s_1 + \mathcal{Y}_2 s_2 \in S$ whenever $s_1, s_2 \in S$; (b) $\mathcal{Y}_1$ is compact and continuous; (c) $\mathcal{Y}_2$ is a contraction mapping. Then there exists $s_3 \in S$ such that $s_3 = \mathcal{Y}_1 s_3 + \mathcal{Y}_2 s_3$.*

Theorem 3.3 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition (H_1). In addition we assume that:*

(H_2) *$|f(t,y)| \leq m(t)$, for all $(t,y) \in [0,1] \times \mathbb{R}$ and $m \in C([0,1], \mathbb{R}^+)$.*

Then there exists at least one solution for the problem (1.1) on $[0,1]$ *provided that*

$$L \sup_{t\in[0,1]} \left\{ \frac{t^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)| \frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)} \right\} < 1. \tag{3.5}$$

Proof. Setting $\sup_{t\in[0,1]} |m(t)| = \|m\|$, we fix

$$\varrho \geq \|m\|\Lambda, \tag{3.6}$$

where Λ is defined by (2.11), and consider $B_\varrho = \{y \in \mathcal{X} : \|y\| \leq \varrho\}$. Introduce the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on B_ϱ as follows:

$$\begin{aligned} \mathcal{G}_1 y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u,y(u))duds \\ &\quad + \mu_1(t) \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds, \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_2 y(t) &= \mu_2(t) \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u,y(u))duds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u,y(u))duds \Bigg]. \end{aligned}$$

Observe that $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$. Now we verify the hypotheses of Krasnoselskii's fixed point theorem in the following steps.

(i) For $y, z \in B_\varrho$, we have

$$\begin{aligned} &\|\mathcal{G}_1 y + \mathcal{G}_2 z\| = \sup_{t\in[0,1]} |(\mathcal{G}_1 y)(t) + (\mathcal{G}_2 z)(t)| \\ \leq\ & \|m\| \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\ & + |\mu_2(t)| \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \\ & + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \Bigg] \Bigg\} \\ & \leq \|m\|\Lambda \leq \varrho, \end{aligned}$$

where we used (3.6). Thus $\mathcal{G}_1 y + \mathcal{G}_2 z \in B_\varrho$.

(ii) We show that $\mathcal{G}_1$ is a contraction. Indeed, by using the assumption (H_1) together with (3.5) and the fact that $(1-s)^\alpha < 1, (1 < \alpha \leq 2)$ we have

$$\begin{aligned}
|\mathcal{G}_1 y(t) - \mathcal{G}_1 z(t)| &\leq L\|y-z\|\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \\
&\quad +|\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\} \\
&\leq L\|y-z\|\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}ds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\} \\
&\leq L\sup_{t\in[0,1]}\Bigg\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\Bigg\}\|y-z\|,
\end{aligned}$$

which implies that

$$\|\mathcal{G}_1 y - \mathcal{G}_1 z\| \leq L\sup_{t\in[0,1]}\Bigg\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\Bigg\}\|y-z\|.$$

Hence $\mathcal{G}_1$ is a contraction by (3.5).

(iii) Using the continuity of f, it is easy to show that the operator $\mathcal{G}_2$ is continuous. Further, $\mathcal{G}_2$ is uniformly bounded on B_ϱ as

$$\|\mathcal{G}_2 x\| = \sup_{t\in[0,1]}|(\mathcal{G}_2 y)(t)| \leq \frac{\|m\|M_2(\delta\eta^\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}, \quad M_2 = \sup_{t\in[0,1]}|\mu_2(t)|.$$

In order to establish that $\mathcal{G}_2$ is compact, we define $\sup_{(t,y)\in[0,1]\times B_\varrho}|f(t,y)| = \overline{f}$. Thus, for $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
|(\mathcal{G}_2 y)(t_2) - (\mathcal{G}_2 y)(t_1)| &\leq |\mu_2(t_2)-\mu_2(t_1)|\overline{f}\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \\
&\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg] \\
&\leq |\mu_2(t_2)-\mu_2(t_1)|\overline{f}\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \to 0 \text{ as } t_1 \to t_2,
\end{aligned}$$

independent of y. This shows that $\mathcal{G}_2$ is relatively compact on B_ϱ. As all the conditions of the Arzelá-Ascoli theorem are satisfied, so $\mathcal{G}_2$ is compact on B_ϱ. In view of steps *(i)*-*(iii)*, the conclusion of Krasnoselskii's fixed point theorem applies and hence there exists at least one solution for the problem (1.1) on $[0,1]$. The proof is completed. □

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Remark 3.4 *Interchanging the roles of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the foregoing result, we can obtain a second result by requiring the condition:*

$$LM_1 \frac{\delta\eta^\beta + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1, \quad M_1 = \sup_{t\in[0,1]} |\mu_1(t)|,$$

instead of (3.5).

The following existence result is based on Leray-Schauder nonlinear alternative.

Lemma 3.5 *(Nonlinear alternative for single valued maps)[16]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

Theorem 3.6 *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_3) *There exist a function $g \in C([0, 1], \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, y)| \leq g(t)\psi(\|y\|), \quad \forall (t, y) \in [0, 1] \times \mathbb{R}$.*

(H_4) *There exists a constant $K > 0$ such that*

$$\frac{K}{\|g\|\psi(K)\Lambda} > 1.$$

Then the problem (1.1) has at least one solution on $[0, 1]$.

Proof. Consider the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ defined by (3.2). We show that $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{X} = C([0, 1], \mathbb{R})$. For a positive number r, let $\mathcal{B}_r = \{y \in C([0, 1], \mathbb{R}) : \|y\| \leq r\}$ be a bounded set in $C([0, 1], \mathbb{R})$. Then, by using the fact that $(1 - s)^{\alpha-1} \leq 1 \ (1 < \alpha \leq 2)$ we have

$$\begin{aligned}
|\mathcal{G}y(t)| \leq\ & \|g\|\psi(r)\Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + |\mu_2(t)| \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \Bigg] \Bigg\} \\
\leq\ & \|g\|\psi(r)\Lambda,
\end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|\mathcal{G}y\| \leq \|g\|\psi(r)\Lambda.$$

Next we show that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_r$, where $\mathcal{B}_r$ is a bounded set of $C([0,1],\mathbb{R})$. Then, using the fact that $(1-s)^{\alpha-1} \leq 1$ $(1<\alpha\leq 2)$ and the computations for $\mathcal{G}_2$ in previous theorem, we obtain

$$\begin{aligned}
&|\mathcal{G}y(t_2)-\mathcal{G}y(t_1)|\\
\leq\;& \|g\|\psi(r)\Bigg\{\Bigg|\int_0^{t_1}\frac{[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]}{\Gamma(\beta)}ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}ds\Bigg|\\
&+|\mu_1(t_2)-\mu_1(t_1)|\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds+|\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\}\\
\leq\;& \|g\|\psi(r)\Bigg\{\frac{2(t_2-t_1)^\beta+t_2^\beta-t_1^\beta}{\Gamma(\beta+1)}+|\mu_1(t_2)-\mu_1(t_1)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\\
&+|\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\},
\end{aligned}$$

which tends to zero independently of $y \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As $\mathcal{G}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once it is shown that the set of all solutions to the equation $y = \lambda\mathcal{G}y$ is bounded for $\lambda \in [0,1]$. For that, let y be a solution of $y = \lambda\mathcal{G}y$ for $\lambda \in [0,1]$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
|y(t)| = |\lambda\mathcal{G}y(t)| \;\leq\;& \Bigg\{\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds+|\mu_1(t)|\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&+|\mu_2(t)|\Bigg[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\\
&+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\}|g(t)|\psi(\|y\|)\\
\leq\;& \|g\|\psi(\|y\|)\Lambda,
\end{aligned}$$

which implies that

$$\frac{\|y\|}{\|g\|\psi(\|y\|)\Lambda} \leq 1.$$

In view of (H_4), there is no solution y such that $\|y\| \neq K$. Let us set

$$U = \{y \in \mathcal{X} : \|y\| < K\}.$$

The operator $\mathcal{G} : \overline{U} \to \mathcal{X}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda \mathcal{G}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [16, Theorem 5.2], we deduce that $\mathcal{G}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. $\square$

4 Existence results for the problem (1.2)

Before presenting the existence results for the problem (1.2), we outline the necessary concepts on multi-valued maps [17], [18].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex }\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0,1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

For each $y \in \mathcal{X}$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Definition 4.1 *A function $y \in C([0,1],\mathbb{R})$ is said to be a solution of the boundary value problem (1.2) if $y(0) = 0$, $D^{\beta}_{0+}y(\xi) = 0$, $y(1) = \delta y(\eta)$, $0 < \xi, \eta < 1$, and there exists a function $v \in S_{F,y}$ such that $v(t) \in F(t, y(t))$ and*

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-} v(s)ds + \mu_1(t) \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-} v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-} v(s)ds\Big], \quad t \in [0,1]. \end{aligned}$$

4.1 The upper semicontinuous case

In the case when F has convex values we prove an existence result based on nonlinear alternative of Leray-Schauder type.

Definition 4.2 *A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,y)$ *is measurable for each* $y \in \mathbb{R}$;

(ii) $y \longmapsto F(t,y)$ *is upper semicontinuous for almost all* $t \in [0,1]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) *for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1], \mathbb{R}^+)$ such that*

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \le \varphi_\rho(t)$$

for all $y \in \mathbb{R}$ with $\|y\| \le \rho$ and for a.e. $t \in [0,1]$.

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y : y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 4.3 *([17, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4.4 *([19]) Let X be a separable Banach space. Let $F : [0,1] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator*

$$\Theta \circ S_{F,x} : C([0,1], X) \to \mathcal{P}_{cp,c}(C([0,1], X)), \quad y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

For the forthcoming analysis, we need the following lemma.

Lemma 4.5 *(Nonlinear alternative for Kakutani maps)[16]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Theorem 4.6 *Assume that:*

(B_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*

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(B_2) *there exist a function* $\phi \in C([0,1], \mathbb{R}^+)$, *and a nondecreasing function* $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\|F(t,y)\|_{\mathcal{P}} := \sup\{|w| : w \in F(t,y)\} \leq \phi(t)\Omega(\|y\|)$$

for each $(t,y) \in [0,1] \times \mathbb{R}$;

(B_3) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\|\phi\|\Lambda\Omega(M)} > 1,$$

where Λ *is defined by (2.11).*

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. Define an operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$\Omega_F(y) = \{h \in \mathcal{X} : h(t) = N(y)(t)\}$$

where

$$\begin{aligned} N(y)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds + \mu_1(t) \int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds\Big]. \end{aligned}$$

We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that Ω_F *is convex for each* $y \in \mathcal{X}$. This step is obvious since $S_{F,y}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in* $\mathcal{X}$. For a positive number ρ, let $B_\rho = \{y \in \mathcal{X} : \|y\| \leq \rho\}$ be a bounded ball in $\mathcal{X}$. Then, for each $h \in \Omega_F(y), y \in B_\rho$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds + \mu_1(t) \int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Bigg[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds\Bigg]. \end{aligned}$$

Then, by using the fact that $(1-s)^{\alpha-1} \leq 1$ $(1 < \alpha \leq 2)$ we have

$$|h(t)| \leq \|g\|\Omega(r)\Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$
\begin{aligned}
&+|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \\
&+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\} \\
\leq\ & \|\phi\|\Omega(r)\Lambda,
\end{aligned}
$$

which, on taking the norm for $t\in[0,1]$. yields

$$\|h\| \leq \|\phi\|\Omega(r)\Lambda.$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $\mathcal{X}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in B_\rho$. For each $h \in \Omega_F(y)$, using the fact that $(1-s)^{\alpha-1} \leq 1$ $(1 < \alpha \leq 2)$, we obtain

$$
\begin{aligned}
& |h(t_2)-h(t_1)| \\
\leq\ & \|\phi\|\Omega(r)\Bigg\{\Bigg|\int_0^{t_1} \frac{[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]}{\Gamma(\beta)\Gamma(\alpha+1)}ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}ds\Bigg| \\
& +|\mu_1(t_2)-\mu_1(t_1)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds+|\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\} \\
\leq\ & \|\phi\|\Omega(r)\Bigg\{\frac{2(t_2-t_1)^\beta+t_2^\beta-t_1^\beta}{\Gamma(\beta+1)\Gamma(\alpha+1)}+|\mu_1(t_2)-\mu_1(t_1)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)} \\
& +|\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\},
\end{aligned}
$$

which tends to zero independently of $y \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega_F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

In our next step, we show that Ω_F *is upper semicontinuous.* To this end it is sufficient to show that Ω_F *has a closed graph,* by Lemma 4.3. Let $y_n \to y_*$, $h_n \in \Omega_F(y_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(y_*)$. Associated with $h_n \in \Omega_F(y_n)$, there exists $v_n \in S_{F,y_n}$ such that for each $t \in [0,1]$,

$$
\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_n(s)ds+\mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v_n(s)ds \\
&\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_n(s)ds-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_n(s)ds\Big].
\end{aligned}
$$

Thus it suffices to show that there exists $v_* \in S_{F,y_*}$ such that for each $t \in [0,1]$,

$$
h_*(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds+\mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v_*(s)ds
$$

$$+\mu_2(t)\Big[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v_*(s)ds-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v_*(s)ds\Big].$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R})\to\mathcal{X}$ given by

$$\begin{aligned} v\mapsto\Theta(v)(t) &= \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v(s)ds+\mu_1(t)\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\\ &\quad+\mu_2(t)\Big[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v(s)ds-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v(s)ds\Big].\end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t)-h_*(t)\|\\ =\ &\Big\|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}(v_n-v_*)(s)ds+\mu_1(t)\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}(v_n-v_*)(s)ds\\ &+\mu_2(t)\Big[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}(v_n-v_*)(s)ds-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}(v_n-v_*)(s)ds\Big]\Big\|\to 0,\end{aligned}$$

as $n\to\infty$. Thus, it follows by Lemma 4.4 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,y_n})$. Since $y_n\to y_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v_*(s)ds+\mu_1(t)\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v_*(s)ds\\ &\quad+\mu_2(t)\Big[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v_*(s)ds-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}v_*(s)ds\Big].\end{aligned}$$

Finally, we show there exists an open set $U\subseteq\mathcal{X}$ with $y\notin\theta\Omega_F(y)$ for any $\theta\in(0,1)$ and all $y\in\partial U$. Let $\theta\in(0,1)$ and $y\in\theta\Omega_F(y)$. Then there exists $v\in L^1([0,1],\mathbb{R})$ with $v\in S_{F,y}$ such that, for $t\in[0,1]$, we can obtain

$$\begin{aligned} &|y(t)|=|\theta\Omega_F(y)(t)|\\ \le\ &\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|v(u)|duds+|\mu_1(t)|\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds\\ &+|\mu_2(t)|\Bigg[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|v(u)|duds\\ &+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|v(u)|duds\Bigg]\\ \le\ &\|\phi\|\Omega(\|y\|)\Lambda,\end{aligned}$$

which implies that

$$\frac{\|y\|}{\|\phi\|\Omega(\|y\|)\Lambda}\le 1.$$

In view of (B_3), there exists M such that $\|y\| \neq M$. Let us set

$$U = \{y \in \mathcal{X} : \|y\| < M\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(\mathcal{X})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \theta\Omega_F(y)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.5), we deduce that Ω_F has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.2). This completes the proof. □

4.2 The Lipschitz case

We prove in this subsection the existence of solutions for the problem (1.2) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [21].

Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X)\times\mathcal{P}(X) \to \mathbb{R}\cup\{\infty\}$ defined by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [20]).

Definition 4.7 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called* (**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \le \gamma d(x,y)$ for each $x,y \in X$ and* (**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 4.8 *([21]) Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 4.9 *Assume that:*

(A_1) *$F : [0,1]\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, y(t)) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$;*

(A_2) *$H_d(F(t,y), F(t,\bar{y}) \le q(t)|y-\bar{y}|$ for almost all $t \in [0,1]$ and $y, \bar{y} \in \mathbb{R}$ with $q \in C([0,1],\mathbb{R}^+)$ and $d(0, F(t,0)) \le q(t)$ for almost all $t \in [0,1]$.*

Then the problem (1.2) has at least one solution on $[0,1]$ if

$$\|q\|\Lambda < 1, \tag{4.1}$$

where Λ is defined by (2.11).

Proof. Consider the operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined in the beginning of the proof of Theorem 4.6. Observe that the set $S_{F,y}$ is nonempty for each $y \in \mathcal{X}$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [22]). Now we show that the operator Ω_F satisfies the assumptions of Lemma 4.8. To show that $\Omega_F(y) \in \mathcal{P}_{cl}(\mathcal{X})$

for each $y \in \mathcal{X}$, let $\{u_n\}_{n\geq 0} \in \Omega_F(y)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathcal{X}$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds\Big]. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in [0,1]$, we have

$$\begin{aligned} u_n(t) \to u(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds\Big]. \end{aligned}$$

Hence, $u \in \Omega_F(y)$.

Next we show that there exists $\hat{\theta} := \|q\|\Lambda < 1$ such that

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \hat{\theta}\|y-\bar{y}\| \text{ for each } y, \bar{y} \in \mathcal{X}.$$

Let $y, \bar{y} \in \mathcal{X}$ and $h_1 \in \Omega_F(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds\Big]. \end{aligned}$$

By (A_2), we have

$$H_d(F(t,y), F(t,\bar{y}) \leq q(t)|y-\bar{y}|.$$

So, there exists $w \in F(t,\bar{y})$ such that

$$|v_1(t) - w| \leq q(t)|y(t) - \bar{y}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq q(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t,\bar{y})$ is measurable (Proposition III.4 [22]), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t,\bar{y})$. So $v_2(t) \in F(t,\bar{y})$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq q(t)|y(t) - \bar{y}(t)|$. For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_2(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds$$

$$+\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_2(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_2(s)ds\Big].$$

Thus

$$\begin{aligned}
& |h_1(t)-h_2(t)| \\
\leq\ & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha|v_1-v_2|(s)ds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}|v_1-v_2|(s)ds \\
& +|\mu_2(t)|\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha|v_1-v_2|(s)ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha|v_1-v_2|(s)ds\Big] \\
\leq\ & \|q\|\Lambda\|y-\bar{y}\|,
\end{aligned}$$

which yields $\|h_1-h_2\| \leq \|q\|\Lambda\|y-\bar{y}\|$.

Analogously, interchanging the roles of y and $\overline{y}$, we can obtain

$$H_d(\Omega_F(y),\Omega_F(\bar{y})) \leq \|q\|\Lambda\|y-\bar{y}\|.$$

By the condition (4.1), it follows that Ω_F is a contraction and hence it has a fixed point y by Lemma 4.8, which is a solution of the problem (1.2). This completes the proof.□

5 Examples

(a) We construct examples for the illustration of the results obtained in Section 3. For that, we consider the following problem:

$$\begin{cases} D_{1-}^{7/4}D_{0+}^{3/4}y(t) = f(t,y(t)),\ t\in J:=[0,1], \\ y(0)=0,\ D_{0+}^{3/4}y(\xi)=0,\quad y(1)=(5/2)y(2/3), \end{cases} \tag{5.1}$$

Here $\alpha=7/4, \beta=3/4, \xi=1/3, \eta=2/3, \delta=5/2$, and

$$f(t,y) = \frac{1}{2\sqrt{t^2+81}}\Big(\cos y + \frac{|y|}{1+|y|}\Big) + \frac{e^{-2t}}{t+4}. \tag{5.2}$$

With the given data, it is found that

$$\begin{aligned}
& P = [(1-\delta\eta^{\beta+1}) - (\beta+1)\xi(1-\delta\eta^\beta)] \approx 0.262961 \neq 0, \\
& \sup_{t\in[0,1]}\Big\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\Big\} \approx 1.454491,
\end{aligned}$$

and $\Lambda \approx 4.503584$ (Λ is given by (2.11)). Furthermore, $|f(t,y_1)-f(t,y_2)| \leq L|y_1-y_2|$ with $L=1/9$ so that $L\Lambda \approx 0.0.500398 < 1$. Clearly the hypothesis of Theorem 3.1 is satisfied and hence the problem (5.1) has a unique solution by the conclusion of Theorem 3.1.

In order to illustrate Theorem 3.3, we notice that (3.5) is satisfied as

$$L\Big\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}+|\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\Big\}\approx 0.161610<1,$$

and

$$|f(t,y)|\le m(t)=\frac{1}{\sqrt{t^2+81}}+\frac{e^{-2t}}{t+4}.$$

As all the assumptions of Theorem 3.3 hold true, we deduce from the conclusion of Theorem 3.3 that the problem (5.1) has at least one solution on $[0,1]$.

Now we demonstrate the application of Theorem 3.6 by considering the nonlinear function

$$f(t,y)=\frac{e^{-t}}{\sqrt{t+36}}\Big(y+\frac{2}{\pi}\tan^{-1}y+\frac{1}{10}\Big). \tag{5.3}$$

Clearly $|f(t,y)|\le g(t)\psi(\|y\|)$, where $g(t)=\frac{e^{-t}}{\sqrt{t+36}}$, $\psi(\|y\|=(\frac{11}{10}+\|y\|)$. By the condition (H_4), we find that $K>3.310535$. Thus all the conditions of Theorem 3.6 are satisfied and consequently, the problem (5.1) with $f(t,y)$ given by (5.3) has has at least one solution on $[0,1]$.

(b) Here we illustrate the results obtained in Section 4. Let us consider the following fractional differential inclusion involving both left Caputo and right Riemann-Liouville types fractional derivatives equipped with fractional boundary conditions:

$$\begin{cases} D_{1-}^{7/4}D_{0+}^{3/4}y(t)\in F(t,y(t)),\ t\in J:=[0,1],\\ y(0)=0,\ D_{0+}^{3/4}y(\xi)=0,\ \ y(1)=(5/2)y(2/3), \end{cases} \tag{5.4}$$

In order to illustrate Theorem 4.6, we take

$$F(t,y(t))=\left[\frac{1}{\sqrt{t^2+49}}\Big(\frac{|y(t)|}{2(1+|y(t)|)}+|y(t)|+\frac{1}{2}\Big),\ \frac{e^{-t}}{9+t}\Big(\sin y(t)+\frac{1}{80}\Big)\right]. \tag{5.5}$$

Clearly $|F(t,y(t))|\le\phi(t)\Omega(\|y\|)$, where $\phi(t)=\frac{1}{\sqrt{t^2+49}}$ and $\Omega(\|y\|)=\|y\|+1$. Using the condition (B_3), we find that $M>1.804018$. As the hypothesis of Theorem 4.6 is satisfied, the problem (5.4) with $F(t,y(t))$ given by (5.5) has at least one solution on $[0,1]$.

Now we illustrate Theorem 4.9 by considering

$$F(t,x(t))=\left[\frac{1}{\sqrt{100+t^2}},\ \frac{\sin x(t)}{(6+t)}+\frac{1}{50}\right]. \tag{5.6}$$

Obviously $q(t)=1(6+t)$ with $\|q\|=1/6$ and $d(0,F(t,0))\le q(t)$ for almost all $t\in[0,1]$. Moreover, $\|q\|\Lambda\approx 0.750597$. Thus all the assumptions of Theorem 4.9 hold true and consequently its conclusion applies to the problem (5.4) with $F(t,y(t))$ given by (5.6).

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L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces

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Abstract

In this paper, we study the existence of solutions and L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces. We no longer require the Azera-Ascoli theorem to prove the existence of continuous solutions of nonlinear differential systems, but instead we apply the regularity results of general linear differential equations to the case the L^2-primitive process for retarded stochastic neutral functional differential systems with unbounded principal operators, delay terms and local Lipschitz continuity of the nonlinear term. Finally, we give a simple example to which our main result can be applied.

Keywords: stochastic neutral differential equations, retarded system, L^2-primitive process, analytic semigroup, fractional power

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1 Introduction

In this paper, we study the existence of solutions and L^2-primitive process for the following retarded stochastic neutral functional differential equations in Hilbert spaces:

$$\begin{cases} d[x(t)+g(t,x_t)] = [Ax(t)+\int_{-h}^{0} a_1(s)A_1x(t+s)ds + k(t)]dt + f(t,x_t)dW(t), \\ x(0)=\phi^0 \in L^2(\Omega,H), \quad x(s)=\phi^1(s),\ s\in[-h,0). \end{cases} \tag{1.1}$$

where $t>0$, $h>0$, $a_1(\cdot)$ is Hölder continuous, k is a forcing term, $W(t)$ stands for K-valued Brownian motion or Winner process with a finite trace nuclear covariance operator Q, and $g,\ f$, are given functions satisfying some assumptions. Moreover, $A: D(A)\subset H\to H$ is unbounded and $A_1\in B(H)$, where $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$.

This kind of systems arises in many practical mathematical models, such as, population dynamics, physical, biological and engineering problems, etc. (see [6, 11, 23]).

Many authors have studied for the theory of stochastic differential equations in a variety of ways (see [4] [7] and reference therein), impulsive stochastic neutral differential equations [14, 21], approximate controllability of stochastic equations [5, 27, 26].

As for the retarded differential equations, Jeong et al [17, 18], Wang [32], and Sukavanam et al. [28] have discussed the regularity of solutions and controllability of the semilinear retarded systems, and see [8, 15, 16, 24] and references therein for the linear retarded systems.

In [10, 12, 13], the authors have discussed the existence of solutions for mild solutions for the neutral differential systems with state-dependence delay. Most studies about the neutral initial value problems governed by retarded semilinear parabolic equation have been devoted to the control problems.

Recently, second order neutral impulsive integrodifferential systems have been studied in [2, 25], and Stochastic differential systems with impulsive conditions in [1, 3, 29]. Further, as for impulsive neutral stochastic differential inclusions with nonlocal initial conditions have been studied for the existence results by Lin and Hu [22], and controllability results by [19].

Let $(\Omega,\mathcal{F},P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t\in I\}$ satisfying $\mathcal{F}_t\subset\mathcal{F}$. An H valued random variables is an $\mathcal{F}$ -measurable function $x(t):\Omega\to H$. Usually we suppress the dependence on $w\in\Omega$ in the stochastic process $\mathcal{S}=\{x(t,w):\Omega\to H:$

$t \in [0, T]\}$and write $x(t)$ instead of $x(t, w)$ and $x(t) : [0, T] \to H$ in the space of $\mathcal{S}$. Then we have to study on results in connection with solutions of random differential and integral equations in Hilbert spaces. It should be ensured that $x(t, w)$ is a H-valued random variable with finite second moments and L^2-primitive process of (1.1) for all $t \in T$ in order to study stationary random function, Brownian motion, Markov process, and etc. But the papers treating the regularity for second moments of the systems and L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces are not many.

In this paper, we propose a different approach of the earlier works used Azera-Ascoli theorem to prove the existence of the mild solutions of functional differential systems in the Banach space of all continuous functions. Our approach is that regularity results of general differential equations results of the linear cases of Di Blasio et al. [8] and semilinear cases of [17] remain valid under the above formulation of the stochastic neutral differential system (1.1) even though the system (1.1) contains unbounded principal operators, delay term and local Lipschitz continuity of the nonlinear term.

The paper is organized as follows. In Section 2, we construct the strict solution of the semilinear functional differential equations and introduce basic properties. In Section 3, by using properties of the strict solutions in dealt in Section 2, we will obtain the L^2-primitive process of (1.1), and a variation of constant formula of L^2-primitive process of (1.1) on the solution space. Finally, we give a simple example to which our main result can be applied.

2 Preliminaries and Lemmas

The inner product and norm in H are denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively. V is another Hilbert space densely and continuously embedded in H. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. For brevity we may regard that

$$||u||_* \le |u| \le ||u||, \quad u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u, u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0. \tag{2.2}$$

Let A be the operator associated with the sesquilinear form $-a(\cdot, \cdot)$:

$$((c_1 - A)u, v) = -a(u, v), \quad u, \ v \in V.$$

It follows from (2.2) that for every $u \in V$

$$\text{Re}\,(Au, u) \ge c_0||u||^2.$$

Then A is a bounded linear operator from V to V^* according to the Lax-Milgram theorem, and its realization in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

is also denoted by A. Then A generates an analytic semigroup $S(t) = e^{tA}$ in both H and V^* as in Theorem 3.6.1 of [30]. Moreover, there exists a constant C_0 such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}, \tag{2.3}$$

for every $u \in D(A)$, where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [31]).

If X is a Banach space and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into X the p-th powers of norms are integrable.

For the sake of simplicity we assume that the semigroup $S(t)$ generated by A is uniformly bounded, that is, There exists a constant M_0 such that

$$||S(t)||_{B(H)} \leq M_0, \quad ||AS(t)||_{B(H)} \leq \frac{M_0}{t}. \tag{2.4}$$

The following lemma is from [30, Lemma 3.6.2].

Lemma 2.2. *There exists a constant M_0 such that the following inequalities hold:*

$$||S(t)||_{B(H,V)} \leq t^{-1/2} M_0, \tag{2.5}$$
$$||S(t)||_{B(V^*,V)} \leq t^{-1} M_0, \tag{2.6}$$
$$||AS(t)||_{B(H,V)} \leq t^{-3/2} M_0. \tag{2.7}$$

First, consider the following initial value problem for the abstract linear parabolic equation

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + \int_{-h}^{0} a_1(s)A_1x(t+s)ds + k(t), \quad 0 < t \le T, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \ s \in [-h, 0). \end{cases} \tag{2.8}$$

By virtue of Theorem 2.1 of [15] or [8], we have the following result on the corresponding linear equation of (2.8).

Lemma 2.3. *1) For $(\phi^0, \phi^1) \in V \times L^2(-h, 0; D(A))$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (2.8) belonging to*

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||\phi^0|| + ||\phi^1||_{L^2(-h,0;D(A))} + ||k||_{L^2(0,T;H)}), \tag{2.9}$$

where C_1 is a constant depending on T and

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} = \max\{||x||_{L^2(0,T;D(A))}, ||x||_{W^{1,2}(0,T;H)}\}$$

(2) Let $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (2.8) belonging to

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}), \tag{2.10}$$

where C_1 is a constant depending on T.

Let the solution spaces $\mathcal{W}_0(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}_0(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}_0(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

Thus, there exists a constant $c_1 > 0$ such that

$$||x||_{C([0,T];V)} \le c_1||x||_{\mathcal{W}_0(T)}, \quad ||x||_{C([0,T];H)} \le c_1||x||_{\mathcal{W}_1(T)}. \tag{2.11}$$

Lemma 2.4. *Suppose that* $k \in L^2(0,T;H)$ *and* $x(t) = \int_0^t S(t-s)k(s)ds$ *for* $0 \le t \le T$. *Then there exists a constant* C_2 *such that*

$$\begin{aligned} ||x||_{L^2(0,T;D(A))} &\le C_1||k||_{L^2(0,T;H)}, \\ ||x||_{L^2(0,T;H)} &\le C_2T||k||_{L^2(0,T;H)}, \end{aligned} \tag{2.12}$$

and

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.13}$$

Proof. The first assertion is immediately obtained by (2.9). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \\ &\le M_0 \int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M_0 \int_0^T t \int_0^t |k(s)|^2 ds dt \\ &\le M_0 \frac{T^2}{2} \int_0^T |k(s)|^2 ds, \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M_0/2}||k||_{L^2(0,T;H)}. \tag{2.14}$$

From (2.3), (2.12), and (2.14) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M_0/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M_0/2}, C_0\sqrt{C_1}(M_0/2)^{1/4}\},$$

the proof is complete. □

In what follows in this section, we assume $c_1 = 0$ in (2.2) without any loss of generality. So we have that $0 \in \rho(A)$ and the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \ge 0\}$ is contained in the resolvent set of A. In this case, it is possible to define the fractional power A^α for $\alpha > 0$. The subspace $D(A^\alpha)$ is dense in H and the expression

$$||x||_\alpha = ||A^\alpha x||, \quad x \in D(A^\alpha)$$

defines a norm on $D(A^\alpha)$. It is also well known that A^α is a closed operator with its domain dense and $D(A^\alpha) \supset D(A^\beta)$ for $0 < \alpha < \beta$. Due to the well known fact that $A^{-\alpha}$ is a bounded operator, we can assume that there is a constant $C_{-\alpha} > 0$ such that

$$||A^{-\alpha}||_{\mathcal{L}(H)} \le C_{-\alpha}, \quad ||A^{-\alpha}||_{\mathcal{L}(V^*,V)} \le C_{-\alpha}. \tag{2.15}$$

Lemma 2.5. *For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$:*

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \le \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(H,V)} \le \frac{C_\alpha}{t^{3\alpha/2}}. \tag{2.16}$$

Proof. The relation is from the inequalities (2.6) and (2.7) by properties of fractional power of A and the definition of $S(t)$. □

3 Existence of solutions

In this paper $(H, |\cdot|)$ and $(K, |\cdot|_K)$ denote real separable Hilbert spaces. Consider the following retarded semilinear impulsive neutral differential system in Hilbert space H:

$$\begin{cases} d[x(t) + g(t, x_t)] = [Ax(t) + \int_{-h}^0 a_1(s)A_1 x(t+s)ds + k(t)]dt + f(t, x_t)dW(t), \\ x(0) = \phi^0 \in L^2(\Omega, H), \quad x(s) = \phi^1(s), \ s \in [-h, 0). \end{cases} \tag{3.1}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t \in I\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

An H valued random variables is an $\mathcal{F}$ -measurable function $x(t) : \Omega \to H$ and the collection of random variables $\mathcal{S} = \{x(t, w) : \Omega \to H \ : \ t \in [0, T], \ w \in \Omega\}$ is a stochastic process. Generally, we just write $x(t)$ instead of $x(t, w)$ and $x(t) : [0, T] \to H$ in the space of $\mathcal{S}$

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal basis of K, and let $Q \in B(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $\mathrm{Tr}(Q) = \sum_{n=1}^\infty \sqrt{\lambda_n} = \lambda < \infty$ (Tr denotes the trace of the operator), where $\lambda_n \ge 0 (n = 1, 2, \cdots)$, and $B(K, K)$ denotes the space of all bounded linea operators from K into K.

$\{W(t) : t \ge 0\}$ be a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator Q over $(\Omega, \mathcal{F}, P)$, which satisfies that

$$W(t) = \sum_{n=1}^\infty \sqrt{\lambda_n} w_i(t) e_n, \quad t \ge 0,$$

where $\{w_i(t)\}_{i=1}^\infty$ be mutually independent one dimensional standard Wiener processes over $(\Omega, \mathcal{F}, P)$. Then the above K-valued stochastic process $W(t)$ is called a Q-Wiener process.

We assume that $\mathcal{F}_t = \sigma\{W(s) : 0 \le s \le t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\psi \in B(K,H)$ and define

$$|\psi|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} |\sqrt{\lambda_n}\psi e_n|^2.$$

If $|\psi|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. $B_Q(K,H)$ stands for the space of all Q-Hilbert-Schmidt operators. The completion $B_Q(K,H)$ of $B(K,H)$ with respect to the topology induced by the norm $|\psi|_Q$, where $|\psi|_Q^2 = (\psi,\psi)$ is a Hilbert space with the above norm topology.

Let V be a dense subspace of H as mentioned in Section 2. For $T > 0$ we define

$$M^2(-h,T;V) = \{x : [-h,T] \to V : E(\int_{-h}^{T} ||x(s)||^2 ds) < \infty\}$$

with norm defined by

$$||x||_{M^2(0,T;V)} = \big[E(\int_{-h}^{T} ||x(s)||^2 ds)\big]^{1/2}.$$

The spaces $M^2(-h,0;V)$, $M^2(0,T;V)$, and $M^2(0,T;V^*)$ are also defined as the same way and the basic theory of the class of all nonanticipative functions can be founded in [9]. For $h > 0$, we assume that $\phi^1 : [-h,0) \to V$ is a given initial value satisfying

$$E(\int_{-h}^{0} ||\phi^1(s)||^2 ds) < \infty,$$

that is, $\phi^1 \in M^2(-h,0;V)$. In this note, a random variable $x(t) : \Omega \to H$ will be called an L^2-primitive process if $x \in M^2(-h,T;V)$.

For each $s \in [0,T]$, we define $x_s : [-h,0] \to H$ as

$$x_s(r) = x(s+r), \quad -h \le r \le 0.$$

We will set

$$\Pi = M^2(-h,0;V).$$

Definition 3.1. *A stochastic process $x : [-h,T] \times \Omega \to H$ is called a solution of (3.1) if*

(i) *$x(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \ge 0$.*

(ii) *$x(t) \in H$ has cádlág paths on $t \in (0,T)$ such that*

$$\begin{aligned} x(t) =& S(t)[\phi^0 + g(0,x_0)] - g(t,x_t) + \int_0^t AS(t-s)g(s,x_s)ds \\ &+ \int_0^t S(t-s)\{\int_{-h}^0 a_1(\tau)A_1 x(s+\tau)d\tau ds + f(s,x_s)dW(s)\} \\ &+ \int_0^t S(t-s)k(s)ds, \\ x(0) =& \phi^0, \quad x(s) = \phi^1(s), \quad s \in [-h,0). \end{aligned} \tag{3.2}$$

(iii) *$x \in M^2(0,T;V)$ i.e., $E(\int_0^T ||x(s)||^2 ds) < \infty$ and $C([0,T];H)$.*

To establish our results, we introduce the following assumptions on system (3.1).
Assumption (A). We assume that $a_1(\cdot)$ is Hölder continuous of order ρ:

$$|a_1(0)| \le H_1, \quad |a_1(s) - a_1(\tau)| \le H_1(s-\tau)^\rho.$$

Assumption (G). Let $g : [0,T] \times \Pi \to H$ be a nonlinear mapping satisfying the following conditions hold:

(i) For any $x \in \Pi$, the mapping $g(\cdot, x)$ is strongly measurable.

(ii) There exist positive constants L_g and $\beta > 2/3$ such that

$$\begin{aligned} &E|A^\beta g(t,x)|^2 \le L_g(||x||_\Pi + 1)^2, \\ &E|A^\beta g(t,x) - A^\beta g(t,\hat{x})|^2 \le L_g||x-\hat{x}||_\Pi^2, \end{aligned}$$

for all $t \in [0,T]$, and $x, \hat{x} \in \Pi$.

Assumption (F). Let $f : \mathbb{R} \times \Pi \to B(K,H)$ be a nonlinear mapping satisfying the following:

(i) For any $x \in \Pi$, the mapping $f(\cdot, x)$ is strongly measurable.

(ii) There exists a function $L_f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$E|f(t,x) - f(t,y)|^2 \le L_f(r)||x-y||_\Pi^2, \quad t \in [0,T]$$

hold for $||x||_\Pi \le r$ and $||y||_\Pi \le r$.

(iii) The inequality

$$E|f(t,x)|^2 \le L_f(r)(||x||_\Pi + 1)^2$$

holds for every $t \in [0,T]$ and $||x||_\Pi \le r$.

Lemma 3.1. *Let $x \in M^2(-h,T;V)$. Then the mapping $s \mapsto x_s$ belongs to $C([0,T];\Pi)$, and for each $0 < t \le T$*

$$\begin{aligned} &||x_t||_\Pi \le ||x||_{M^2(-h,t;V)} = ||\phi^1||_\Pi + ||x||_{M^2(0,t;V)}, \\ &E(||x||^2_{L^2(0,t;V)}) = ||x||^2_{M^2(0,t;V)}, \\ &||x.||_{L^2(0,t;\Pi)} \le \sqrt{t}||x||_{M^2(-h,t;V)}. \end{aligned} \tag{3.3}$$

Proof. The first paragraph is easy to verify. In fact, it is from the following inequality;

$$||x_t||^2_\Pi = E\big(\int_{-h}^0 ||x(t+\tau)||^2 d\tau\big) \le E\big[\int_{-h}^t ||x(\tau)||^2 d\tau\big] \le ||x||^2_{M^2(-h,t;V)},\ t > 0.$$

The second paragraph is immediately obtained by definition. From the inequality (3.3), we have

$$\begin{aligned} \int_0^t ||x_s||^2_\Pi ds &= \int_0^t \big[E\big(\int_{-h}^0 ||x(s+\tau)||^2 d\tau\big)\big]^2 ds \\ &= \int_0^t \big[E\big(\int_{s-h}^s ||x(\tau)||^2 d\tau\big)\big]^2 ds \le t||x||^2_{M^2(-h,t;V)}, \end{aligned}$$

which completes the last paragraph. □

One of the main useful tools in the proof of existence theorems for nonlinear functional equations is the following Sadvoskii's fixed point theorem.

Lemma 3.2. *(Krasnoselski [20]) Suppose that Σ is a closed convex subset of a Banach space X. Assume that K_1 and K_2 are mappings from Σ into X such that the following conditions are satisfied:*

(i) $(K_1 + K_2)(\Sigma) \subset \Sigma$,

(ii) K_1 is a completely continuous mapping,

(iii) K_2 is a contraction mapping.

Then the operator $K_1 + K_2$ has a fixed point in Σ.

From now on, we establish the following results on the solvability of the equation (3.1).

Theorem 3.1. *Let Assumptions (A), (G) and (F) be satisfied. Assume that* $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ *and* $k \in M^2(0,T;V^*)$ *for* $T > 0$*. Then, there exists a solution* x *of the system (3.1) such that*

$$x \in M^2(0,T;V) \cap C([0,T];H).$$

Moreover, there is a constant C_3 *independent of the initial data* (ϕ^0, ϕ^1) *and the forcing term* k *such that*

$$||x||_{M^2(-h,T;V)} \leq C_3(1 + E(|\phi^0|^2) + ||\phi^1||_\Pi + ||k||_{M^2(0,T;V^*)}). \tag{3.4}$$

Proof. Let

$$r := 2\big[C_1 C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) + \sqrt{3}C_1\big(E(|\phi^0|^2) + ||\phi^1||_\Pi^2 + ||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\big],$$

and

$$\begin{aligned} N :=& \sqrt{3}C_{-\alpha}\sqrt{L_g}\big(||\phi^1||_\Pi + r + 1\big) \\ &+ (3\beta - 2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}(||\phi^1||_\Pi + r + 1) \\ &+ C_2 \mathrm{Tr}(Q)\sqrt{L_f(r)}(||\phi^1||_\Pi + r + 1), \end{aligned}$$

where $\beta > 2/3$, C_1 and C_2 is constants in Lemma 2.3 and Lemma 2.4, respectively. Let

$$T_1^\gamma := \max\{T_1^{1/2}, T_1^{3\beta/2}\}$$

and choose $0 < T_1 < T$ such that

$$T_1^\gamma N \leq \frac{r}{2} = C_1 C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) + \sqrt{3}C_1\big(\sqrt{E(|\phi^0|^2)} + ||\phi^1||_\Pi + ||k||_{M^2(0,T_1;V^*)}, \tag{3.5}$$

and

$$\hat{N} := T_1^\gamma\big\{\sqrt{3}C_{-\alpha}\sqrt{L_g} + (3\beta - 1)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g} + C_2\mathrm{Tr}(Q)\sqrt{L_f(r)}\big\} < 1. \tag{3.6}$$

Let J be the operator on $M^2(0,T_1;V)$ defined by

$$\begin{aligned}(Jx)(t) =&S(t)[\phi^0+g(0,\phi^1)]-g(t,x_t)+\int_0^t AS(t-s)g(s,x_s)ds\\&+\int_0^t S(t-s)\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau ds+f(s,x_s)dW(s)\}\\&+\int_0^t S(t-s)k(s)ds.\end{aligned}$$

It is easily seen that J is continuous from $C([0,T_1];H)$ into itself. Let

$$\Sigma=\{x\in M^2(-h,T;V):x(0)=\phi^0,\ \text{and } x(s)=\phi^1(s)(s\in[-h,0))\}.$$

and

$$\Sigma_r=\{x\in\Sigma:||x||_{M^2(0,T_1;V)}\le r\},$$

which is a bounded closed subset of $M^2(0,T_1;V)$. Now, we give the proof of Theorem 3.1 in the following several steps:

Now, in order to show that the operator J has a fixed point in $\Sigma_r\subset M^2(0,T_1;V)$, we take the following steps.

Step 1. J maps Σ_r into Σ_r.

By (2.10), (2.15) and Assumption (G), and noting $x_0=\phi^1$, we know

$$\begin{aligned}E\big[\int_0^{T_1}\|S(t)g(0,x_0)\|^2dt\big]&=E\big[C_1^2|g(0,\phi^1)|^2\big]\\&=E\big[C_1^2\big(||A^{-\beta}||_{B(H)}|A^\beta g(0,\phi^1)|\big)^2\big]\\&\le(C_1C_{-\alpha})^2L_g(||\phi^1||_\Pi+1)^2.\end{aligned}\tag{3.7}$$

From (2.10) of Lemma 2.3 it follows

$$\begin{aligned}&E\big[\int_0^{T_1}\big\|S(t)\phi^0+\int_0^t S(t-s)\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau+k(s)\}ds\big\|^2dt\big]\\&\le E\big[C_1^2\{|\phi^0|+||\phi^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T_1;V^*)}\}^2\big]\\&\le 3C_1^2\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big).\end{aligned}\tag{3.8}$$

By using Assumption (G) and Lemma 3.1, we have

$$\begin{aligned}&||g(\cdot,x.)||^2_{M^2(0,T_1;V)}=E\big(\int_0^{T_1}\big\|A^{-\beta}A^\beta g(t,x_t)\big\|^2dt\big)\\&\le C_{-\alpha}^2E\big(\int_0^{T_1}\big|A^\beta g(t,x_t)\big|^2dt\big)\le C_{-\alpha}^2L_gT_1\big(||x_t||_\Pi+1\big)^2\\&\le 3C_{-\alpha}^2L_gT_1\big(||\phi^1||_\Pi^2+||x||_{M^2(0,T_1;V)}+1\big)\end{aligned}\tag{3.9}$$

Define $H_1 : M^2(0,T_1;V) \to M^2(0,T_1;V)$ by

$$(H_1x)(t) = \int_0^t AS(t-s)g(s,x_s)ds.$$

Then from Lemma 2.5 and Assumption (G) we have

$$\begin{aligned}||AS(t-s)g(s,x_s)|| &= ||A^{1-\beta}S(t-s)||_{B(H,V)}|A^{\beta}g(s,x_s)|\\ &\le \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^{\beta}g(s,x_s)|,\end{aligned}$$

and hence, by using Hólder inequality and Assumption (G),

$$\begin{aligned}&||H_1x||^2_{M^2(0,T_1;V)} = E\big[\int_0^{T_1} \big\|\int_0^t AS(t-s)g(s,x_s)ds\big\|^2 dt\big] \qquad (3.10)\\ &\le E\big[\int_0^{T_1}\big(\int_0^t \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^{\beta}g(s,x_s)|ds\big)^2 dt\big]\\ &\le E\big[\int_0^{T_1} C^2_{1-\beta}(3\beta-2)^{-1}t^{3\beta-2}\int_0^t |A^{\beta}g(s,x_s)|^2 dsdt\big]\\ &\le (3\beta-2)^{-1}C^2_{1-\beta}L_g(||x_s||_{\Pi}+1)^2\int_0^{T_1} t^{3\beta-1}dt\\ &\le (3\beta-2)^{-1}(3\beta)^{-1}C^2_{1-\beta}L_gT_1^{3\beta}(||x||_{M^2(-h,T_1;V)}+1)^2\\ &= (3\beta-2)^{-1}(3\beta)^{-1}C^2_{1-\beta}L_gT_1^{3\beta}(||\phi^1||_{\Pi}+||x||_{M^2(0,T_1;V)}+1)^2\end{aligned}$$

Let

$$(H_2x)(t) = \int_0^t S(t-s)f(s,x_s)dW(s).$$

For (2.13) of Lemma 2.4 it follows

$$\begin{aligned}||H_2x||^2_{M^2(0,T_1;V)} &= E\big[\int_0^{T_1}\big|\int_0^t S(t-s)f(s,x_s)dW(s)\big|^2 dt\big] \qquad (3.11)\\ &\le E[C_2^2\mathrm{Tr}(Q)^2T_1||f(s,x_s)||^2_{L^2(0,T;V^*)}]\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1||f(s,x_s)||^2_{M^2(0,T;V^*)}\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1L_f(r)(||x_s||_{\Pi}+1)^2\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1L_f(r)(||\phi^1||_{\Pi}+||x||_{M^2(0,T_1;V)}+1)^2\end{aligned}$$

Therefore, from (3.7)-(3.11) it follows that

$$
\begin{aligned}
||Jx||_{M^2(0,T_1;V)} \le & C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\
&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\
&+\sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1\big)\\
&+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\
&+C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\
\le & C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\
&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}+T_1^\gamma N\le r,
\end{aligned}
$$

and so, J maps Σ_r into Σ_r.

Define mapping K_1+K_2 on $L^2(0,T_1;V)$ by the formula

$$
\begin{aligned}
&(Jx)(t)=(K_1x)(t)+(K_2x)(t),\\
&(K_1x)(t)=\int_0^t S(t-s)\int_0^s a_1(\tau-s)A_1x(\tau)d\tau ds,
\end{aligned}
$$

and

$$
\begin{aligned}
(K_2x)(t)=&S(t)[\phi^0+g(0,x_0)]-g(t,x_t)+\int_0^t AS(t-s)g(s,x_s)ds\\
&+\int_0^t S(t-s)\{\int_{s-h}^0 a_1(\tau-s)A_1\phi^1(\tau)d\tau ds+f(s,x_s)dW(s)\}\\
&+\int_0^t S(t-s)k(s)ds.
\end{aligned}
$$

Step 2. K_1 is a completely continuous mapping.

We can now employ Lemma 3.2 with Σ_r. Assume that a sequence $\{x_n\}$ of $M^2(0,T_1;V)$ converges weakly to an element $x_\infty\in M^2(0,T_1;V)$, i.e., $w-\lim_{n\to\infty}x_n=x_\infty$. Then we will show that

$$
\lim_{n\to\infty}||K_1x_n-K_1x_\infty||_{M^2(0,T_1;V)}=0, \tag{3.12}
$$

which is equivalent to the completely continuity of K_1 since $M^2(0,T_1;V)$ is reflexive. For a fixed $t\in[0,T_1]$, let $x_t^*(x)=(K_1x)(t)$ for every $x\in M^2(0,T_1;V)$. Then $x_t^*\in M^2(0,T_1;V^*)$ and we have $\lim_{n\to\infty}x_t^*(x_n)=x_t^*(x_\infty)$ since $w-\lim_{n\to\infty}x_n=x_\infty$. Hence,

$$
\lim_{n\to\infty}(K_1x_n)(t)=(K_1x_\infty)(t),\quad t\in[0,T_1].
$$

Set

$$h(s) = \int_0^s a_1(\tau - s)A_1 x(\tau)d\tau.$$

Then by using Hólder inequality we obtain the following inequality

$$\begin{aligned} |h(s)| &\le \Big| \int_0^s (a_1(\tau - s) - a_1(0))A_1 x(\tau)d\tau\Big| \\ &\quad + \Big| \int_0^s a_1(0)A_1 x(\tau)d\tau\Big| \\ &\le \big\{\big((2\rho+1)^{-1}s^{2\rho+1}\big)^{1/2} + \sqrt{s}\big\}H_1||A_1||_{B(H)}\big(\int_0^s ||x(\tau)||^2 d\tau\big)^{1/2}. \end{aligned} \tag{3.13}$$

Thus, by (2.5) and (3.13) it holds

$$\begin{aligned} &||(K_1x)(t)||^2 = \Big\| \int_0^t S(t-s)h(s)ds\Big\|^2 \\ &\le (H_1||A_1||_{B(H)})^2\big(\int_0^t ||x(\tau)||^2 d\tau\big)\Big\| \int_0^t \frac{1}{(t-s)^{1/2}}\big\{((2\rho+1)^{-1}s^{(2\rho+1)/2} + \sqrt{s}\big\}ds\Big\|^2 \\ &\le (H_1||A_1||_{B(H)})^2||x||^2_{L^2(0,t;V)}\big\{(2\rho+1)^{-1}B(1/2,(2\rho+3)/2)t^{\rho+1} + B(1/2,3/2)t\big\}^2. \\ &:= c_2||x||^2_{L^2(0,t;V)}, \end{aligned}$$

where c_2 is a constant and $B(\cdot,\cdot)$ is the Beta function. Here we used

$$B(1/2,(2\rho+3)/2)t^{\rho+1} = \int_0^t (t-s)^{-1/2}s^{(2\rho+1)/2}ds.$$

And we know

$$\sup_{0\le t\le T_1} ||E[(K_1x)(t)]^2|| \le c_2||x||^2_{M^2(0,T_1;V)} \le \infty.$$

Therefore, by Lebesgue's dominated convergence theorem it holds

$$\lim_{n\to\infty} E\big(\int_0^{T_1} ||(K_1x_n)(t)||^2 dt\big) = E\big(\int_0^{T_1} ||(K_1x_\infty)(t)||^2 dt\big),$$

i.e., $\lim_{n\to\infty} ||K_1x_n||_{M^2(0,T_1;V)} = ||K_1x_\infty||_{M^2(0,T_1;V)}$. Since $M^2(0,T_1;V)$ is a reflexive space, it holds (3.12).

Step 3. K_2 is a contraction mapping.

For every x_1 and $x_2 \in \Sigma_r$, we have

$$\begin{aligned}(K_2x_1)(t)-(K_2x_2)(t)=&g(t,x_{2_t})-g(t,x_{1_t})\\&+\int_0^t AS(t-s)\big(g(t,x_{1_s})-g(t,x_{2_s})\big)ds\\&+\int_0^t S(t-s)\{f(s,x_1(s))-f(s,x_2(s))\}dW(s).\end{aligned}$$

In a similar way to (3.9)-(3.11) and Proposition 2.3, we have

$$\begin{aligned}||K_2x_1-K_2x_2||_{M^2(0,T_1;V)}\le&\big\{C_{-\alpha}\sqrt{T_1L_g}+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}T_1^{3\beta/2}\sqrt{L_g}\\&+C_2\sqrt{T_1L_f(r)}\mathrm{Tr}(Q)\big\}||x_1-x_2||_{M^2(0,T_1;V)}\\\le&\hat{N}||x_1-x_2||_{M^2(0,T_1;V)}.\end{aligned}$$

So by virtue of the condition (3.6) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $M^2(0,T_1;V)$. This has proved the local existence and uniqueness of the solution of (3.1).

Step 4. We drive a priori estimate of the solution.

To prove the global existence, we establish a variation of constant formula (3.4) of solution of (3.1). Let x be a solution of (3.1) and $\phi^0 \in H$. Then we have that from (3.7)-(3.11) it follows that

$$\begin{aligned}||x||_{M^2(0,T_1;V)}\le&C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\&+\sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1\big)\\&+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\&+C_2\mathrm{Tr}(Q)T_1\sqrt{T_1L_f(r)}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\\le&\hat{N}||x||_{L^2(0,T_1;V)}+\hat{N}_1,\end{aligned}$$

where

$$\begin{aligned}\hat{N}_1=&C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\&+\sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi+1\big)\\&+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi+1)\\&+C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi+1).\end{aligned}$$

Taking into account (3.6) there exists a constant C_3 such that

$$\begin{aligned}||x||_{L^2(0,T_1;V)} \leq & (1-\hat{N})^{-1}\hat{N}_1 \\ \leq & C_3(1+E(|\phi^0|^2)+||\phi^1||_\Pi+||k||_{M^2(0,T_1;V^*)}),\end{aligned}$$

which obtain the inequality (3.4).

Now we will prove that $E[x(T_1)^2]<\infty$ in order that the solution can be extended to the interval $[T_1, 2T_1]$.

Define a mapping $H_3 : L^2(0,T_1;V) \to L^2(0,T_1;V)$ as

$$(H_3x)(t)=S(t)[\phi^0+g(0,x_0)]+\int_0^t S(t-s)\Big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau+k(s)\Big\}ds$$

The from (2.11) and Lemma 2.3 it follows that

$$\begin{aligned}E|(H_3x)(T_1)|^2 &\leq c_1E||H_3x||^2_{\mathcal{W}_1} \\ &\leq 3c_1C_1E\big\{|\phi^0+g(0,\phi^1)|+||\phi^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T_1;V^*)}\big\}^2 \\ &\leq c_1C_1\big\{E|\phi^0+g(0,\phi^1)|^2+||\phi^1||^2_{M^2(-h,0;V)}+||k||^2_{M^2(0,T_1;V^*)}\big\} := I,\end{aligned} \tag{3.14}$$

and from (2.4) and Assumption (F),

$$\begin{aligned}E|(H_2x)(T_1)|^2 =& E\Big|\int_0^{T_1} S(T_1-s)f(s,x_s)dW(s)\Big|^2 \\ \leq & M_0^2\mathrm{Tr}(Q)^2T_1L_f(r)(||x_s||_\Pi+1)^2 \\ \leq & M_0^2\mathrm{Tr}(Q)^2T_1L_f(r)(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)^2 := II.\end{aligned} \tag{3.15}$$

Moreover, by using Assumption (G) we have

$$\begin{aligned}E|g(T_1,x_{T_1})|^2 &\leq E\big\|A^{-\beta}A^\beta g(t,x_{T_1})\big\|^2, \\ &\leq C\_ALPHA^2L_g\big(||x_{T_1}||_\Pi+1\big)^2 \\ &\leq C\_ALPHA^2L_g\big(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1\big)^2 := III,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}E|(H_1x)(T_1)|^2 &= E|\int_0^{T_1} AS(T_1-s)g(s,x_s)ds|^2 \\ &= E|\int_0^{T_1} A^{1-\beta}W(T_1-s)A^\beta g(s,x_s)ds|^2 \\ &\leq E\Big[\int_0^{T_1}\frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^\beta(g(s,x_s)|ds\Big]^2 \\ &\leq E\big[C^2_{1-\beta}(3\beta-2)^{-1}T_1^{3\beta-2}\int_0^t|A^\beta(g(s,x_s)|^2ds\big]^2 \\ &= C^2_{1-\beta}(3\beta-2)^{-1}T_1^{3\beta-1}L_g(||x||_{M^2(0,T_1;V)}+||\phi^1||_{M^2(-h,0;V)}+1)^2 := IV.\end{aligned} \tag{3.17}$$

Thus,by (3.14)-(3.17) we have

$$\begin{aligned}E|x(T_1)|^2 &= E\big|(H_3x)(T_1) - g(T_1, x_{T_1}) + \int_0^{T_1} AS(T_1 - s)g(s, x_s)ds \\ &\quad + \int_0^{T_1} S(T_1 - s)f(s, x_s)dW(s)\big| \\ &\le I + II + III + IV < \infty.\end{aligned}$$

Hence we can solve the equation in $[T_1, 2T_1]$ with the initial $(x(T_1), x_{T_1})$ and an analogous estimate to (3.4). Since the condition (3.6) is independent of initial values, the solution can be extended to the interval $[0, nT_1]$ for any natural number n, and so the proof is complete. $\square$

Remark 3.1. *Thanks for Lemma 2.3, we note that the solution of (3.1) with the conditions of Theorem 3.1 satisfies also that*

$$E(\int_{-h}^{T} ||x'(s)||_*^2 ds) < \infty.$$

Here we note that by a simple calculation using the properties of analytic semigroup, it is immediately seen that $x \in M^2(-h, T; H)$.

Now, we obtain that the solution mapping is continuous in the following result, which is useful for the control problem and physical applications of the given equation.

Theorem 3.2. *Let Assumptions (A), (G) and (F) be satisfied. Assuming that the initial data $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ and the forcing term $k \in M^2(0, T; V^*)$. Then the solution x of the equation (3.1) belongs to $x \in M^2(0, T; V)$ and the mapping*

$$L^2(\Omega, H) \times \Pi \times M^2(0, T; V^*) \ni (\phi^0, \phi^1, k) \mapsto x \in M^2(0, T; V) \tag{3.18}$$

is continuous.

Proof. From Theorem 3.1, it follows that if $(\phi^0, \phi^1, k) \in L^2(\Omega, H) \times \Pi \times M^2(0, T; V^*)$ then x belongs to $M^2(0, T; V)$. Let $(\phi_i^0, \phi_i^1, k_i)$ and x^i be the solution of (3.1) with

$(\phi_i^0, \phi_i^1, k_i)$ in place of (ϕ^0, ϕ^1, k) for $i = 1,\ 2$. Let $x_i (i = 1, 2) \in \Sigma_r$. Then it holds

$$\begin{aligned}
&x^1(t) - x^2(t) = S(t)[(\phi_1^0 - \phi_2^0) + (g(0, x_0^1) - g(0, x_0^2))] \\
&- (g(t, x_t^1) - g(t, x_t^2)) + \int_0^t AS(t-s)(g(s, x_s^1) - g(t, x_s^2))ds \\
&+ \int_0^t S(t-s)\{ \int_{-h}^0 a_1(\tau)A_1(x^1(s+\tau) - x^2(s+\tau))d\tau ds \\
&+ \int_0^t S(t-s)\{((Fx^1)(s) - (Fx^2)(s)) + (k_1(s) - k_2(s))\}ds. \\
&+ \int_0^t S(t-s)(k_1(s) - k_2(s))ds
\end{aligned}$$

Hence, by applying the same argument as in the proof of Theorem 3.1, we have

$$||x_1 - x_2||_{M^2(0,T_1;V)} \leq \hat{N}||x_1 - x_2||_{L^2(0,T_1;V)} + \hat{N}_2,$$

where

$$\begin{aligned}
\hat{N}_2 =& C_1 C_ALPHA\sqrt{L_g}(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ \sqrt{3}C_1\big(E[|\phi_1^0 - \phi_1^0|^2] + ||\phi_1^1 - \phi_2^1||_\Pi^2 + ||k_1 - k_2||^2_{M^2(0,T_1;V^*)}\big)^{1/2} \\
&+ \sqrt{3}C_ALPHA\sqrt{T_1 L_g}\big(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ (3\beta - 2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{(3\beta+1)/2}(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ C_2 \mathrm{Tr}(Q)\sqrt{T_1 L_f(r)}(||\phi_1^1 - \phi_2^1||_\Pi),
\end{aligned}$$

which implies

$$||x||_{M^2(0,T_1;V)} \leq \hat{N}_2(1 - \hat{N})^{-1}.$$

Therefore, it implies the inequality (3.18). □

4 Example

Let

$$H = L^2(0, \pi),\ V = H_0^1(0, \pi),\ V^* = H^{-1}(0, \pi).$$

Consider the following retarded neutral stochastic differential system in Hilbert space H:

$$\begin{cases} d[x(t,y)+g(t,x_t(t,y))] = [Ax(t,y)+\int_{-h}^{0} a_1(s)A_1 x(t+s,y)ds + k(t,y)]dt \\ \qquad\qquad\qquad\qquad\qquad +F(t,x(t,y))dW(t), \quad (t,y)\in[0,T]\times[0,\pi], \\ x(0,y)=\phi^0(y)\in L^2(\Omega,H), \quad x(s,y)=\phi^1(s,y), \quad (s,y)\in[-h,0)\times[0,\pi], \end{cases} \tag{3.19}$$

where $h>0$, $a_1(\cdot)$ is Hölder continuous, $A_1\in B(H)$, and $W(t)$ stands for a standard cylindrical Winner process in H defined on a stochastic basis $(\Omega,\mathcal{F},P)$. Let

$$a(u,v)=\int_0^\pi \frac{du(y)}{dy}\overline{\frac{dv(y)}{dy}}dy.$$

Then

$$A=\partial^2/\partial y^2 \quad \text{with} \quad D(A)=\{x\in H^2(0,\pi): x(0)=x(\pi)=0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n=-n^2$ and $z_n(y)=(2/\pi)^{1/2}\sin ny$, respectively. Moreover,

(a1) $\{z_n : n\in N\}$ is an orthogonal basis of H and

$$S(t)x=\sum_{n=1}^{\infty} e^{n^2 t}(x,z_n)z_n, \quad \forall x\in H,\ t>0.$$

Moreover, there exists a constant M_0 such that $||S(t)||_{B(H)}\le M_0$.

(a2) Let $0<\alpha<1$. Then the fractional power $A^\alpha : D(A^\alpha)\subset H\to H$ of A is given by

$$A^\alpha x=\sum_{n=1}^{\infty} n^{2\alpha}(x,z_n)z_n,\ D(A^\alpha):=\{x : A^\alpha x\in H\}.$$

In particular,

$$A^{-1/2}x=\sum_{n=1}^{\infty}\frac{1}{n}(x,z_n)z_n, \ \text{ and } \ ||A^{-1/2}||=1.$$

The nonlinear mapping f is a real valued function belong to $C^2([0,\infty))$ which satisfies the conditions

(f1) $f(0)=0$, $f(r)\ge 0$ for $r>0$,

(f2) $|f'(r) \leq c(r+1)$ and $|f''(r)| \leq c$ for $r \geq 0$ and $c > 0$.

If we present

$$F(t, x(t, y)) = f'(|x(t, y)|^2)x(t, y),$$

Then it is well known that F is a locally Lipschitz continuous mapping from the whole V into H by Sobolev's imbedding theorem (see [30, Theorem 6.1.6]). As an example of q in the above, we can choose $q(r) = \mu^2 r + \eta^2 r^2/2$ (μ and η is constants).

Define $g : [0, T] \times \Pi \to H$ as

$$g(t, x_t) = \sum_{n=1}^{\infty} \int_0^t e^{n^2 t}(\int_{-h}^0 a_2(s)x(t+s)ds, z_n)z_n, \quad , t > 0.$$

Then it can be checked that Assumption (G) is satisfied. Indeed, for $x \in \Pi$, we know

$$Ag(t, x_t) = (S(t) - I)\int_{-h}^0 a_2(s)x(t+s)ds,$$

where I is the identity operator form H to itself and

$$|a_2(0)| \leq H_2, \quad |a_2(s) - a_2(\tau)| \leq H_2(s-\tau)^\kappa, \quad s,\ \tau \in [-h, 0]$$

for a constant $\kappa > 0$. Hence we have

$$\begin{aligned} E|Ag(t, x_t)|^2 \leq & (M_0+1)^2\big\{\big|\int_{-h}^0 (a_2(s) - a_2(0))x(t+s)d\tau\big|^2 \\ & + \big|\int_{-h}^0 a_2(0)x(t+s)d\tau\big|^2\big\} \\ \leq & (M_0+1)^2 H_2^2\big\{(2\kappa+1)^{-1}h^{2\rho+1} + h\big\}||x_t||_\Pi^2. \end{aligned}$$

It is immediately seen that Assumption (G) has been satisfied. Thus, all the conditions stated in Theorem 3.1 have been satisfied for the equation (3.19), and so there exists a solution x of the equation (3.19) such that

$$E(\int_{-h}^T ||x(s)||^2 ds) < \infty, \text{ and } E(\int_{-h}^T ||x'(s)||_*^2 ds) < \infty.$$

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FUZZY STABILITY OF CUBIC FUNCTIONAL EQUATIONS WITH EXTRA TERMS

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ABSTRACT. In this paper, we consider the generalized Hyers-Ulam stability for the following cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) + G_f(x,y) = 0.$$

with an extra term G_f which is a functional operator of f.

1. INTRODUCTION AND PRELIMINARIES

In 1940, Ulam proposed the following stability problem (cf. [20]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [8] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [18] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians ([3], [4], [5], [7], and [16]).

Katsaras [11] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a vector space in different points of view. In particular, Bag and Samanta [2] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. In this paper, we use the definition of fuzzy normed spaces given in [2],[14], [15].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space*.

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Let (X, N) be a fuzzy normed space. (i) A sequence $\{x_n\}$ in X is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$. (ii) A sequence $\{x_n\}$ in X is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$ and any $t > 0$, there exists an $m \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $n \geq m$ and all positive integer p.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space.*

For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+||x||}, & \text{if } t > 0 \end{cases}$$

is a fuzzy norm on X.

In 1996, Isac and Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 1.2. [6] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

In 2001, Rassias [19] introduced the following cubic functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) = 0 \tag{1.1}$$

and the following cubic functional equations were investigated

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.2}$$

in ([10]). Every solution of a cubic functional equation is called *a cubic mapping* and Kim and Han [12] investigated the following cubic functional equation

$$\begin{aligned} &f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) \\ &+ k[f(mx + y) + f(mx - y) - m[f(x + y) + f(x - y)] - 2(m^3 - m)f(x)] = 0 \end{aligned}$$

for some rational number m and some real number k and proved the stability for it in fuzzy normed spaces.

In this paper, we investigate the following functional equation which is added a term by G_f to (1.1)

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y) = 0,$$

where G_f is a functional operator depending on functions f. The definition of G_f is given in section 2 and prove the stability for it in fuzzy normed spaces.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

2. Cubic functional equations with extra terms

For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. It is clear that $\sigma_i(0, 0) = 0$.

Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

Now consider the functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y) = 0 \tag{2.1}$$

with the functional operator G_f.

Theorem 2.1. *Suppose that the mapping $f : X \longrightarrow Y$ is a solution of (2.1) with $f(0) = 0$. Then f is cubic if and only if $f(2x) = 8f(x)$ and $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$.*

Proof. Suppose that $f(2x) = 8f(x)$ and $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$. Interchanging x and y in (2.1) , we have

$$f(2x + y) - 3f(x + y) + 3f(y) - f(y - x) - 6f(x) + G_f(y, x) = 0. \tag{2.2}$$

for all $x, y \in X$ and letting $x = -x$ in (2.2) , we have

$$f(-2x + y) - 3f(-x + y) + 3f(y) - f(x + y) - 6f(-x) + G_f(y, -x) = 0. \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$f(2x + y) - f(-2x + y) - 2f(x + y) + 2f(y - x) - 6f(x) + 6f(-x) = 0. \tag{2.4}$$

for all $x, y \in X$, because $G_f(y, x) = G_f(y, -x)$. Letting $y = x$ in (2.4), we have

$$f(3x) - 22f(x) + 5f(-x) = 0. \tag{2.5}$$

for all $x \in X$ and letting $y = 2x$ in (2.4), by (2.5), we have

$$f(4x) - 2f(3x) - 4f(x) + 6(-x) = 16f(x) + 16f(-x) = 0.$$

for all $x \in X$, because $f(2x) = 8f(x)$. Hence f is odd and by (2.2) and (2.3), f satisfies (1.2). Thus f is a cubic mapping. The converse is trivial. $\square$

3. The Generalized Hyers-Ulam stability for (2.1)

In this section, we prove the generalized Hyers-Ulam stability of (2.1) in fuzzy normed spaces. For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x, y) = f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y)$$

for all $x, y \in X$.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow Z$ be a function such that there is a real number L satisfying $0 < L < 1$ and*

$$N'(\phi(2x, 2y), t) \geq N'(8L\phi(x, y), t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x, y), t) \geq N'(\phi(x, y), t) \tag{3.2}$$

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for all $x, y \in X$ *and all* $t > 0$ *and*

$$N(f(2x) - 8f(x), t) \ge \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\} \tag{3.3}$$

for all $x \in X$*, all* $t > 0$ *and some nonnegative real numbers* a, b, c*. Further, assume that if* g *satisfies (2.1), then* g *is a cubic mapping. Then there exists an unique cubic mapping* $C : X \longrightarrow Y$ *such that*

$$\begin{aligned} &N\Big(f(x) - C(x), \frac{1}{8(1-L)}t\Big) \\ &\ge \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\} \end{aligned} \tag{3.4}$$

for all $x \in X$ *and all* $t > 0$*.*

Proof. Let $\psi(x,t) = \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\}$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \ \ge \ \psi(x,t), \ \forall x \in X, \ \forall t > 0\}.$$

Then (S, d) is a complete metric space(see [17]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 2^{-3}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (3.1), we have

$$N(Jg(x) - Jh(x), cLt) \ge N(2^{-3}(g(2x) - h(2x)), cLt) \ge \psi(x,t)$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \le Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (3.3), $d(f, Jf) \le \frac{1}{8} < \infty$ and by Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. Moreover, $C(x) = N - \lim_{n\to\infty} 2^{-3n} f(2^n x)$ for all $x \in X$ and $d(f, C) \le \frac{1}{8(1-L)}$ and hence we have (3.4).

Replacing x, y, and t by $2^n x$, $2^n y$, and $2^{3n} t$ in (3.2), respectively, we have

$$N(D_f(2^n x, 2^n y), 2^{3n} t) \ge N'(\phi(2^n x, 2^n y), 2^{3n} t) \ge N'(L^n \phi(x,y), t)$$

for all $x, y \in X$ and all $t > 0$. Letting $n \longrightarrow \infty$ in the last inequality, we have

$$C(x + 2y) - 3C(x + y) + 3C(x) - C(x - y) - 6C(y) + G_C(x, y) = 0$$

for all $x, y \in X$ and thus C is a cubic mapping.

Now, we show the uniqueness of C. Let $C_0 : X \longrightarrow Y$ be another cubic mapping with (3.4). Then C_0 is a fixed ponit of J in S and by (3.4), we get

$$d(Jf, C_0) \le d(Jf, JC) \le Ld(f, C_0) \le \frac{L}{8(1-L)} < \infty$$

and by (3) of Theorem 1.2, we have $C = C_0$. □

Similar to Theorem 3.1, we can also have the following theorem.

Theorem 3.2. *Let* $\phi : X^2 \longrightarrow Z$ *be a function such that there is a real number* L *satisfying* $0 < L < 1$ *and*

$$N'(\phi(x,y),t) \ge N'\Big(\frac{L}{8}\phi(2x,2y),t\Big) \tag{3.5}$$

for all $x, y \in X$ *and all* $t > 0$*. Let* $f : X \longrightarrow Y$ *be a mapping satisfying* $f(0) = 0$*, (3.2), and (3.3). Further, assume that if* g *satisfies (2.1), then* g *is a cubic mapping.*

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Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that the inequality

$$\begin{aligned} &N\Big(f(x) - C(x), \frac{L}{8(1-L)}t\Big) \\ &\geq \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\} \end{aligned} \tag{3.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let $\psi(x,t) = \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\}$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \ \geq \ \psi(x,t), \ \forall x \in X, \ \forall t > 0\}.$$

Then (S,d) is a complete metric space(see [17]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 8g(2^{-1}x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (3.2) and (3.5), we have

$$N(Jg(x) - Jh(x), cLt) \geq N(8(g(2^{-1}x) - h(2^{-1}x)), cLt) \geq \psi(x,t)$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (3.3), we get

$$N\Big(f(x) - 8f(2^{-1}x), \frac{L}{8}t\Big) \geq \psi\Big(2^{-1}x, \frac{L}{8}t\Big) \geq \psi(x,t) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. Hence $d(f, Jf) \leq \frac{L}{8} < \infty$ and by Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. Moreover, $C(x) = N - \lim_{n\to\infty} 2^{3n} f(2^{-n}x)$ for all $x \in X$ and $d(f,C) \leq \frac{L}{8(1-L)}$ and hence we have (3.6). The rest of the proof is similar to that of Theorem 3.1. □

Using Theorem 3.1 and Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let $\phi : X^2 \longrightarrow Z$ be a function with (3.1). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and (3.2). Further, assume that if g satisfies (2.1), then g is a cubic mapping and that*

$$\begin{aligned} &N(G_f(0,x),t) \geq \min\{N'(a_1\phi(x,0),t), N'(a_2\phi(0,x),t), N'(a_3\phi(x,-x),t)\}, \\ &N(G_f(x,-x),t) \geq \min\{N'(b_1\phi(x,0),t), N'(b_2\phi(0,x),t), N'(b_3\phi(x,-x),t)\} \end{aligned} \tag{3.8}$$

for all $x \in X$, all $t > 0$ and for some nonnegative real numbers $a_i, b_i (i = 1,2,3)$. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$\begin{aligned} &N\Big(f(x) - C(x), \frac{7}{24(1-L)}t\Big) \\ &\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\} \end{aligned} \tag{3.9}$$

for all $x \in X$ and all $t > 0$, where $c_1 = \max\{a_1, b_1\}$, $c_2 = \max\{1, a_2, b_2\}$, and $c_3 = \max\{1, a_3, b_3\}$.

Proof. Setting $x = 0$ and $y = x$ in (3.2), we have

$$N(f(2x) - 9f(x) - f(-x) + G_f(0,x), t) \geq N'(\phi(0,x),t) \tag{3.10}$$

for all $x \in X$ and all $t > 0$. Setting $y = -x$ in (3.2), we have

$$N(3f(x) - 5f(-x) - f(2x) + G_f(x,-x), t) \geq N'(\phi(x,-x),t) \tag{3.11}$$

for all $x \in X$ and all $t > 0$. Hence by (3.10) and (3.11), we get

$$\begin{aligned}
&N(6f(x) + 6f(-x) - G_f(0,x) - G_f(x,-x), 2t) \\
&\geq \min\{N'(\phi(0,x),t), N'(\phi(x,-x),t)\}
\end{aligned} \tag{3.12}$$

for all $x \in X$ and all $t > 0$. Thus by (3.8), (3.10), and (3.12), we get

$$\begin{aligned}
&N\Big(f(2x) - 8f(x), \frac{7}{3}t\Big) \\
&= \min\Big\{N(f(2x) - 9f(x) - f(-x) + G_f(0,x), t), N\Big(\frac{5}{6}G_f(0,x), \frac{5}{6}t\Big), \\
&\quad N\Big(f(x) + f(-x) - \frac{1}{6}G_f(0,x) - \frac{1}{6}G_f(x,-x), \frac{1}{3}t\Big), N\Big(\frac{1}{6}G_f(x,-x), \frac{1}{6}t\Big)\Big\} \\
&\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. By Theorem 3.1, there exists an unique cubic mapping $C : X \longrightarrow Y$ with (3.9). □

Corollary 3.4. *Let $\phi : X^2 \longrightarrow Z$ be a function with (3.5). Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Further, assume that if g satisfies (2.1), then g is a cubic mapping and that (3.8) hold. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that the inequality*

$$\begin{aligned}
&N\Big(f(x) - C(x), \frac{L}{8(1-L)}t\Big) \\
&\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\}
\end{aligned} \tag{3.13}$$

holds for all $x \in X$ and all $t > 0$, where $c_1 = \max\{a_1, b_1\}$, $c_2 = \max\{1, a_2, b_2\}$, and $c_3 = \max\{1, a_3, b_3\}$.

Proof. By (**??**), we get

$$\begin{aligned}
&N\Big(f(x) - 8f(2^{-1}x), \frac{7L}{24}t\Big) \geq \psi\Big(2^{-1}x, \frac{L}{8}t\Big) \\
&\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. By Theorem 3.2, there exists an unique cubic mapping $C : X \longrightarrow Y$ with (3.13). □

From now on, we consider the following functional equation

$$\begin{aligned}
&f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) \\
&+ k[f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)] = 0
\end{aligned} \tag{3.14}$$

for some positive real number k.

Lemma 3.5. [12] *A mapping $f : X \longrightarrow Y$ satisfies (3.14) if and only if f is a cubic mapping.*

Using Theorem 2.1, Theorem 3.1, and Theorem 3.2, we have the following example.

Example 3.6. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\begin{aligned}
&N(f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) + k[f(2x+y) + f(2x-y) \\
&- 2f(x+y) - 2f(x-y) - 12f(x)], t) \geq \frac{t}{t + \|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p}
\end{aligned} \tag{3.15}$$

for all $x, y \in X$, all $t > 0$ and some positive real numbers k, p with $p \neq \frac{3}{2}$. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{2k|8 - 2^{2p}|t}{2k|8 - 2^{2p}|t + \|x\|^{2p}} \tag{3.16}$$

for all $x \in X$.

Proof. Let $G_f(x, y) = k[f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)]$ and $\phi(x, y) = \|x\|^{2p} + \|y\|^{2p} + \|x\|^{2p}\|y\|^p$. Then $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$ and f satisfies (3.2). Letting $y = 0$ in (3.15), we have

$$N(f(2x) - 8f(x), t) \geq N'\Big(\frac{1}{2k}\phi(x, 0), t\Big)$$

for all $x \in X$ and all $t > 0$, where

$$N'(r, t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+|r|}, & \text{if } t > 0 \end{cases}$$

for all $r \in \mathbb{R}$. By Theorem 3.1, and Theorem 3.2, there exists an unique mapping $C : X \longrightarrow Y$ with (2.1) and (3.16). Since $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$, $G_C(y, x) = G_C(y, -x)$ for all $x, y \in X$ and letting $y = 0$ in $D_C(x, y) = 0$, we have $C(2x) = 8C(x)$ for all $x \in X$. By Theorem 2.1, we have the result. □

We can use Corollary 3.3 and Corollary 3.4 to get a classical result in the framework of normed spaces. As an example of $\phi(x, y)$ in Corollary 3.3 and Corollary 3.4, we can take $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then we can formulate the following example.

Example 3.7. Let X be a normed space and Y a Banach space. Suppose that if g satisfies (2.1), then g is a cubic mapping. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|Df(x, y)\| \leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \tag{3.17}$$

for all $x, y \in X$ and a fixed positive real numbers p, ϵ with $p \neq \frac{3}{2}$. Suppose that

$$\|G_f(0, x)\| \leq \epsilon \max\{a_1, a_2, a_3\}\|x\|^{2p}, \ \ \|G_f(x, -x)\| \leq \epsilon \max\{b_1, b_2, b_3\}\|x\|^{2p}$$

for all $x \in X$, all $t > 0$ and for some nonnegative real numbers $a_i, b_i (i = 1, 2, 3)$. Then there is an unique cubic mapping $C : X \longrightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{7\epsilon}{3|8 - 2^{2p}|} \max\{3, a_1, a_2, 3a_3, b_1, b_2, 3b_3\}\|x\|^{2p}$$

for all $x \in X$.

Proof. Define a fuzzy norm N' on $\mathbb{R}$ by

$$N_{\mathbb{R}}(x, t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

for all $x \in \mathbb{R}$ and all $t > 0$. Similary we can define a fuzzy norm N_Y on Y. Then (Y, N_Y) is a fuzzy Banach space. Let $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then by definitions N_Y and N', the following inequality holds :

$$N_Y(Df(x, y), t) \geq N_{\mathbb{R}}(\phi(x, y), t)$$

for all $x, y \in X$ and all $t > 0$. By Corollary 3.3 and Corollary 3.4, we have the result. $\square$

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DRYGAS FUNCTIONAL EQUATIONS WITH EXTRA TERMS AND ITS STABILITY

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Abstract. In this paper, we consider the generalized Hyers-Ulam stability for the following functional equation with an extra term G_f

$$f(x+y)+f(x-y)+G_f(x,y)=2f(x)+f(y)+f(-y),$$

where G_f is a functional operator of f.

1. Introduction and preliminaries

In 1940, Ulam [12] proposed the following stability problem :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941, Hyers [6] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [11] generalized the result of Hyers. Rassias [11] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p)$$

for some $\epsilon \geq 0$ and p with $p < 1$ and for all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. The paper of Rassias [11] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassis approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [10] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability for the quadratic functional equation and Park [9] proved the generalized Hyers-Ulam stability of the quadratic functional eqution in Banach modules over a C^*-algebra.

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In this paper, we are interested in what kind of terms can be added to the Drygas functional equation [4]

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)$$

while the generalized Hyers-Ulam stability still holds for the new functional equation. We denote the added term by $G_f(x,y)$ which can be regarded as a functional operator depending on the variables x, y, and functions f. Then the new functional equation can be written as

$$f(x+y)+f(x-y)+G_f(x,y)=2f(x)+f(y)+f(-y). \tag{1.2}$$

In fact, the functional operator $G_f(x,y)$ was introduced and considered in the cases of additive, quadratic functional equations with somewhat different point of view by the authors([7], [8]).

2. Solutions of 1.2 as additive-quadratic mappings

Let X and Y be normed spacese. For given $l\in\mathbb{N}$ and any $i\in\{1,2,\cdots,l\}$, let $\sigma_i : X\times X\longrightarrow X$ be a binary operation such that

$$\sigma_i(rx,ry)=r\sigma_i(x,y)$$

for all $x,y\in X$ and all $r\in\mathbb{R}$. It is clear that $\sigma_i(0,0)=0$. Also let $F:Y^l\longrightarrow Y$ be a linear, continuous function. For a map $f:X\longrightarrow Y$, define

$$G_f(x,y)=F(f(\sigma_1(x,y)),f(\sigma_2(x,y)),\cdots,f(\sigma_l(x,y))).$$

From now on, for any mapping $f:X\longrightarrow Y$, we deonte

$$f_o(x)=\frac{f(x)-f(-x)}{2},\ \ f_e(x)=\frac{f(x)+f(-x)}{2}$$

First, we consider the following functional equation

$$\begin{aligned}&af(x+y)+bf(x-y)-cf(y-x)\\&=(a+b)f(x)-cf(-x)+(a-c)f(y)+bf(-y)\end{aligned} \tag{2.1}$$

for fixed real numbers a,b,c with $a=b-c$ and $a\neq 0$. We can easily show the following lemma.

Lemma 2.1. *Let $f:X\longrightarrow Y$ be a mapping. Then f satisfies (2.1) if and only if f is an additive-quadratic mapping.*

Definition 2.2. The functional operator G is called *additive-quadratic* if whenever $G_h(x,y)=0$ for all $x,y\in X$, h is an additive-quadratic mapping.

Lemma 2.3. *Let $f:X\longrightarrow Y$ be a mapping satisfying (1.2) and G additve-quadratic. Then the following are equivalent :*
(1) f is additive-quadratic,
(2) the following equality

$$G_f(x,y)=-G_f(y,x) \tag{2.2}$$

holds for all x, $y\in X$, and
(3) there exist real numbers b,c such that $b\neq c$ and

$$bG_f(x,y)=cG_f(y,x) \tag{2.3}$$

holds for all x, $y\in X$.

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Proof. (1) ($\Rightarrow$) (2) ($\Rightarrow$) (3) are trivial.

(3) ($\Rightarrow$) (1) By (2.2), we have $f(0) = 0$ and by (1.2), we have

$$\begin{aligned} G_f(x, y) &= 2f(x) + f(y) + f(-y) - f(x + y) - f(x - y), \text{ and} \\ G_f(y, x) &= 2f(y) + f(x) + f(-x) - f(x + y) - f(y - x) \end{aligned}$$

for all x, $y \in X$. Hence by (2.3), we have

$$(b+c)f(x+y)+bf(x-y)-cf(y-x) = (2b+c)f(x)+cf(-x)+(b+2c)f(y)+bf(-y)$$

for all x, $y \in X$ and by Lemma 2.1, we have that f is additive-quadratic. $\square$

3. The generalized Hyers-Ulam stability of (1.2)

In this section, we deal with the generalized Hyers-Ulam stability of (1.2). Throughout this paper, assume that G is additive-quadratic and the following inequalities hold

$$\begin{aligned} &\|G_h(x, x)\| \le \|G_h(0, x)\| + \sum_{i=1}^{t} |b_i|\|G_h(\delta_i x, 0)\| \;\; if \;\; h : odd, \\ &\|G_h(x, x)\| \le \sum_{i=1}^{r} |p_i|\|G_h(0, \alpha_i x)\| + \sum_{i=1}^{s} |a_i|\|G_h(\lambda_i x, 0)\| \;\; if \;\; h : even \end{aligned} \tag{3.1}$$

for some $r, s, t \in \mathbb{N} \cup \{0\}$, some real numbers $p_i, a_i, b_i, \alpha_i, \lambda_i$, and δ_i and for all $x \in X$.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty \tag{3.2}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an odd mapping such that

$$\|f(x + y) + f(x - y) + G_f(x, y) - 2f(x)\| \le \phi(x, y). \tag{3.3}$$

for all $x, y \in X$. Then there exists an odd mapping $A : X \longrightarrow X$ such that A satisfies (1.2) and

$$\|A(x) - f(x)\| \le \sum_{n=0}^{\infty} 2^{-n-1}\Big[\phi(2^n x, 2^n x) + \phi(0, 2^n x) + \sum_{i=1}^{t} |b_i|\phi(2^n \delta_i x, 0)\Big]. \tag{3.4}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $A : X \longrightarrow X$ is an unique additive mapping with (3.4).

Proof. By (3.3), we have

$$\|G_f(x, 0)\| \le \phi(x, 0), \;\; \|G_f(0, x)\| \le \phi(0, x)$$

for all $x, y \in X$. Setting $y = x$ in (3.3), we have

$$\|f(2x) + G_f(x, x) - 2f(x)\| \le \phi(x, x) \tag{3.5}$$

for all $x \in X$. Hence by (3.1) and (3.5), we have

$$\|f(x) - 2^{-1}f(2x)\| \le 2^{-1}\Big[\phi(x, x) + \phi(0, x) + \sum_{i=1}^{t} |b_i|\phi(\delta_i x, 0)\Big] \tag{3.6}$$

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for all $x \in X$. By (3.6), we have

$$\begin{aligned}&\|f(x) - 2^{-n} f(2^n x)\| \\ &\leq \sum_{k=0}^{n-1} 2^{-k-1}\Big[\phi(2^k x, 2^k x) + \phi(0, 2^k x) + \sum_{i=1}^{t} |b_i|\phi(2^k \delta_i x, 0)\Big]\end{aligned}$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \leq m < n$,

$$\begin{aligned}&\|2^{-m} f(2^m x) - 2^{-n} f(2^n x)\| \\ &= 2^{-m}\|f(2^m x) - 2^{-(n-m)} f(2^{n-m}(2^m x))\| \\ &\leq \sum_{k=m}^{n-1} 2^{-k-1}\Big[\phi(2^k x, 2^k x) + \phi(0, 2^k x) + \sum_{i=1}^{t} |b_i|\phi(2^k \delta_i x, 0)\Big]\end{aligned} \tag{3.7}$$

for all $x \in X$. By (3.2) and (3.7), $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ for all $x \in X$. By (3.7), we have (3.4).

Replacing x and y by $2^n x$ and $2^n y$ in (3.3), respectively and deviding (3.3) by 2^n, we have

$$\|2^{-n}[f(2^n(x + y)) + f(2^n(x - y)) + G_f(2^n x, 2^n y) - 2f(2^n x)]\| \leq 2^{-n}\phi(2^n x, 2^n y)$$

for all $x, y \in X$ and letting $n \to \infty$, we can show that A satisfies (1.2). Since f is odd, A is odd.

Suppose that G_f satisfies (2.2). Then clearly, we can show that G_A satisfies (2.2) and hence by Lemma 2.3, A is an additive-quadratic mapping. Since A is odd, A is an additive mapping.

Now, we show the uniqueness of A. Let $E : X \longrightarrow Y$ be an additive mapping with (3.4). Since A and E are additive,

$$\begin{aligned}\|A(x) - E(x)\| &= \|A(2^n x) - E(2^n x)\| \\ &\leq 2^{-k}\sum_{n=0}^{\infty} 2^{-n}\Big[\phi(2^n x, 2^n x) + \phi(0, 2^n x) + \sum_{i=1}^{t} |b_i|\phi(2^n \delta_i x, 0)\Big]\end{aligned}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Hence, letting $k \to \infty$, by (3.2), we have $A = E$. □

Similar to Theorem 3.1, we have the following theorem.

Theorem 3.2. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^n \phi(2^{-n} x, 2^{-n} y) < \infty \tag{3.8}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an odd mapping satisfying (3.3). Then there exists an odd mapping $A : X \longrightarrow X$ such that A satisfies (1.2) and

$$\|A(x) - f(x)\| \leq \sum_{n=0}^{\infty} 2^{n-1}\Big[\phi(2^{-n} x, 2^{-n} x) + \phi(0, 2^{-n} x) + \sum_{i=1}^{t} |b_i|\phi(2^{-n} \delta_i x, 0)\Big] \tag{3.9}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $A : X \longrightarrow X$ is an unique additive mapping with (3.9)

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Proof. By (3.3), we have

$$\|G_f(x,0)\| \leq \phi(x,0), \ \ \|G_f(0,x)\| \leq \phi(0,x)$$

for all $x, y \in X$. Setting $y = x = \frac{x}{2}$ in (3.5), we have

$$\Big\|f(x) + G_f\Big(\frac{x}{2},\frac{x}{2}\Big) - 2f\Big(\frac{x}{2}\Big)\Big\| \leq \phi\Big(\frac{x}{2},\frac{x}{2}\Big) \tag{3.10}$$

for all $x \in X$. Hence by (3.1), (3.3), and (3.10), we have

$$\Big\|f(x) - 2f\Big(\frac{x}{2}\Big)\Big\| \leq \phi(x,x) + \phi(0,x) + \sum_{i=1}^{t}|b_i|\phi(\delta_i x,0) \tag{3.11}$$

for all $x \in X$. By (3.11), we have

$$\|f(x) - 2^n f(2^{-n}x)\| \leq \sum_{k=0}^{n-1} 2^k\Big[\phi(2^{-k}x,2^{-k}x) + \phi(0,2^{-k}x) + \sum_{i=1}^{t}|b_i|\phi(2^{-k}\delta_i x,0)\Big]$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N}\cup\{0\}$ with $0 \leq m < n$,

$$\begin{aligned}&\|2^m f(2^{-m}x) - 2^n f(2^{-n}x)\|\\ &= 2^m\|f(2^{-m}x) - 2^{(n-m)}f(2^{-(n-m)}(2^{-m}x))\|\\ &\leq \sum_{k=m}^{n-1} 2^k\Big[\phi(2^{-k}x,2^{-k}x) + \phi(0,2^{-k}x) + \sum_{i=1}^{t}|b_i|\phi(2^{-k}\delta_i x,0)\Big]\end{aligned} \tag{3.12}$$

for all $x \in X$. By (3.12), $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence in Y. The rest of proof is similar to Theorem 3.1. □

Theorem 3.3. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{-2n}\phi(2^n x, 2^n y) < \infty \tag{3.13}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an even mapping such that

$$\|f(x+y) + f(x-y) + G_f(x,y) - 2f(x) - 2f(y)\| \leq \phi(x,y). \tag{3.14}$$

for all $x, y \in X$. Then there exists an even mapping $Q : X \longrightarrow X$ such that

$$\|Q(x) - f(x)\| \leq \sum_{n=0}^{\infty} 2^{-2n-2}\Big[\phi(2^n x,2^n x) + \sum_{i=1}^{r}|p_i|\phi(0,2^n a_i x) + \sum_{i=1}^{s}|a_i|\phi(2^n\lambda_i x,0)\Big] \tag{3.15}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $Q : X \longrightarrow Y$ is an unique quadratic mapping with (3.15)

Proof. Setting $y = x$ in (3.14), we have

$$\|2^2 f(x) - f(2x) + G_f(x,x)\| \leq \phi(x,x)$$

for all $x \in X$ and by (3.14), we have

$$\|G_f(x,0)\| \leq \phi(x,0), \ \ \|G_f(0,x)\| \leq \phi(0,x)$$

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for all $x \in X$. Since f is even, letting $y = x$ in (3.14), by (3.1), we have

$$\begin{aligned}
&\|f(x) - 2^{-2} f(2x)\| \\
&\le 2^{-2}\Big[\phi(x,x) + \|G_f(x,x)\|\Big] \\
&\le 2^{-2}\Big[\phi(x,x) + \sum_{i=1}^{r} |p_i| \|G_f(0, \alpha_i x)\| + \sum_{i=1}^{s} |a_i| \|G_f(\lambda_i x, 0)\|\Big] \\
&\le 2^{-2}\Big[\phi(x,x) + \sum_{i=1}^{r} |p_i| \phi(0, \alpha_i x) + \sum_{i=1}^{s} |a_i| \phi(\lambda_i x, 0)\Big]
\end{aligned}$$

for all $x \in X$. Hence we have

$$\begin{aligned}
&\|f(x) - 2^{-2n} f(2^n x)\| \\
&\le \sum_{k=0}^{n-1} 2^{-2k-2}\Big[\phi(2^k x, 2^k x) + \sum_{i=1}^{r} |p_i| \phi(0, 2^k a_i x) + \sum_{i=1}^{s} |a_i| \phi(2^k \lambda_i x, 0)\Big]
\end{aligned} \tag{3.16}$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \le m < n$, by (3.16)

$$\begin{aligned}
&\|2^{-2m} f(2^m x) - 2^{-2n} f(2^n x)\| \\
&= 2^{-2m} \|f(2^m x) - 2^{-2(n-m)} f(2^{n-m}(2^m x))\| \\
&\le \sum_{k=m}^{n-1} 2^{-2k-2}\Big[\phi(2^k x, 2^k x) + \sum_{i=1}^{r} |p_i| \phi(0, 2^k a_i x) + \sum_{i=1}^{s} |a_i| \phi(2^k \lambda_i x, 0)\Big]
\end{aligned} \tag{3.17}$$

for all $x \in X$. By (3.17), $\{2^{-2n} f(2^n x)\}$ is a Cauchy sequence in Y. The rest of proof is similar to Theorem 3.1. □

Theorem 3.4. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{2n} \phi(2^{-n} x, 2^{-n} y) < \infty \tag{3.18}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an even mapping satisfying (3.14). Then there exists an even mapping $Q : X \longrightarrow X$ such that

$$\|Q(x) - f(x)\| \le \sum_{n=0}^{\infty} 2^{2n}\Big[\phi(2^{-n} x, 2^{-n} x) + \sum_{i=1}^{r} |p_i| \phi(0, 2^{-n} a_i x) + \sum_{i=1}^{s} |a_i| \phi(2^{-n} \lambda_i x, 0)\Big] \tag{3.19}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $Q : X \longrightarrow Y$ is an unique quadratic mapping with (3.19)

Proof. Setting $y = x = \frac{x}{2}$ in (3.14), we have

$$\Big\|2^2 f\Big(\frac{x}{2}\Big) - f(x) + G_f\Big(\frac{x}{2}, \frac{x}{2}\Big)\Big\| \le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big)$$

for all $x \in X$. By (3.14), we have

$$\|G_f(x, 0)\| \le \phi(x, 0), \quad \|G_f(0, x)\| \le \phi(0, x)$$

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for all $x \in X$ and so, we have

$$\begin{aligned}\Big\|2^2 f\Big(\frac{x}{2}\Big) - f(x)\Big\| &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \Big\|G_f\Big(\frac{x}{2}, \frac{x}{2}\Big)\Big\| \\ &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \sum_{i=1}^{r} |p_i| \Big\|G_f\Big(0, \alpha_i \frac{x}{2}\Big)\Big\| + \sum_{i=1}^{s} |a_i| \Big\|G_f\Big(\lambda_i \frac{x}{2}, 0\Big)\Big\| \\ &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \sum_{i=1}^{r} |p_i| \phi\Big(0, \alpha_i \frac{x}{2}\Big) + \sum_{i=1}^{s} |a_i| \phi\Big(\lambda_i \frac{x}{2}, 0\Big)\end{aligned}$$

for all $x \in X$. Similar to Theorem 3.1, we have the result. □

Theorem 3.5. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (3.2). Let $f : X \longrightarrow Y$ be a mapping with (3.3). Then there exists a mapping $F : X \longrightarrow X$ such that F satisfies (1.2) and*

$$\begin{aligned}&\|F(x) - f(x)\| \\ &\le \sum_{n=0}^{\infty} 2^{-2n-2} \Big[\phi_1(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i| \phi_1(0, 2^n x) + \sum_{i=1}^{s} |a_i| \phi_1(\lambda_i 2^n x, 0)\Big] \\ &\quad + \sum_{n=0}^{\infty} 2^{-n-1} \Big[\phi_1(2^n x, 2^n x) + \phi_1(0, 2^n x) + \sum_{i=1}^{t} |b_i| \phi_1(\delta_i 2^n x, 0)\Big]\end{aligned} \tag{3.20}$$

for all $x \in X$, where $\phi_1(x, y) = \frac{1}{2}\Big[\phi(x, y) + \phi(-x, -y)\Big]$. Further, if G_f satisfies (2.2), then $F : X \longrightarrow X$ is an unique additive-quadratic mapping with (3.20)

Proof. By (3.3), we have

$$\|f_e(x + y) + f_e(x - y) + G_{f_e}(x, y) - 2f_e(x) - 2f_e(y)\| \le \phi_1(x, y) \tag{3.21}$$

for all $x, y \in X$. By Theorem 3.3, there exists an even mapping $Q : X \longrightarrow Y$ such that $Q(x) = \lim_{n\longrightarrow\infty} 2^{-2n} f_e(2^n x)$ for all $x \in X$,

$$Q(x + y) + Q(x - y) + G_Q(x, y) = 2Q(x) + 2Q(y) \tag{3.22}$$

for all $x, y \in X$, and

$$\begin{aligned}&\|Q(x) - f_e(x)\| \\ &\le \sum_{n=0}^{\infty} 2^{-2n-2} \Big[\phi_1(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i| \phi_1(0, 2^n a_i x) + \sum_{i=1}^{s} |a_i| \phi_1(2^n \lambda_i x, 0)\Big]\end{aligned} \tag{3.23}$$

for all $x \in X$. Similarly, there exists an odd mapping $A : X \longrightarrow Y$ such that $A(x) = \lim_{n\longrightarrow\infty} 2^{-n} f_o(2^n x)$ for all $x \in X$,

$$A(x + y) + A(x - y) + G_A(x, y) - 2A(x) = 0 \tag{3.24}$$

for all $x, y \in X$, and

$$\|A(x) - f_o(x)\| \le \sum_{n=0}^{\infty} 2^{-n-1} \Big[\phi_1(2^n x, 2^n x) + \phi_1(0, 2^n x) + \sum_{i=1}^{t} |b_i| \phi_1(2^n \delta_i x, 0)\Big] \tag{3.25}$$

for all $x \in X$.

Let $F = Q + A$. Since Q is even and A is odd, $2Q(y) = F(y) + F(-y)$ and by (3.22) and (3.24), F satisfies (1.2). Since $\|F(x) - f(x)\| \le \|Q(x) - f_e(x)\| + \|A(x) - f_o(x)\|$, by (3.23) and (3.25), we have (3.20).

Suppose that G_f satisfies (2.2). Then clearly, we can show that G_F satisfies (2.2) and hence by Lemma 2.3, F is an additive-quadratic mapping. The proof of the uniqueness of F is similar to Theorem 3.1. □

Theorem 3.6. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^n \phi(2^{-n}x, 2^{-n}y) < \infty$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping with (3.3). Then there exist a mapping $F : X \longrightarrow X$ such that

(3.26)

$$\begin{aligned}
&\|F(x) - f(x)\| \\
&\le \sum_{n=0}^{\infty} 2^{2n-2}\Big[\phi_1(2^{-n}x, 2^{-n}x) + \sum_{i=1}^{r}|p_i|\phi_1(0, 2^{-n}x) + \sum_{i=1}^{s}|a_i|\phi_1(\lambda_i 2^{-n}x, 0)\Big] \\
&\quad + \sum_{n=0}^{\infty} 2^{n-1}\Big[\phi_1(2^{-n}x, 2^{-n}x) + \phi_1(0, 2^{-n}x) + \sum_{i=1}^{t}|b_i|\phi_1(\delta_i 2^{-n}x, 0)\Big]
\end{aligned}$$

for all $x \in X$, where $\phi_1(x, y) = \frac{1}{2}\Big[\phi(x, y) + \phi(-x, -y)\Big]$. Further, if G_f satisfies (2.2), then $F : X \longrightarrow X$ is an unique additive-quadratic mapping with (3.26).

4. Applicaions

In this section, we illustrate how the theorems in section 3 work well for the generalized Hyers-Ulam stability of various additive-quadratic functional equations.

As examples of $\phi(x, y)$ in Theorem 3.5 and Theorem 3.6, we can take $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then we can formulate the following theorem :

Theorem 4.1. *Assume that all of the conditions in Theorem 3.1 hold and G_f satisfies (2.2). Let p be a real number with $0 < p < \frac{1}{2}, 1 < p$. Let $f : X \longrightarrow Y$ be a mapping such that*

(4.1)

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)+G_f(x,y)\| \le \epsilon(\|x\|^p\|y\|^p+\|x\|^{2p}+\|x\|^{2p})$$

for all $x, y \in X$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \begin{cases} \Psi_1(x), & \text{if } 0 < p < \frac{1}{2} \\ \Psi_2(x), & \text{if } 1 < p \end{cases}$$

for all $x \in X$, where

$$\Psi_1(x) = \Big[3 + \sum_{i=1}^{r}|p_i| + \sum_{i=1}^{s}|a_i||\lambda_i|^{2p}\Big]\frac{\epsilon}{4 - 4^p}\|x\|^{2p} + \Big[4 + \sum_{i=1}^{t}|b_i||\delta_i|^{2p}\Big]\frac{\epsilon}{2 - 4^p}\|x\|^{2p}$$

and

$$\Psi_2(x) = \Big[3 + \sum_{i=1}^{r}|p_i| + \sum_{i=1}^{s}|a_i||\lambda_i|^{2p}\Big]\frac{4^{p-1}\epsilon}{4^p - 4}\|x\|^{2p} + \Big[4 + \sum_{i=1}^{t}|b_i||\delta_i|^{2p}\Big]\frac{2^{2p-1}\epsilon}{4^p - 2}\|x\|^{2p}$$

Lemma 4.2. *Let G be the operator defined by*

$$G_f(x, y) = f(2x + y) - f(x + 2y) + f(x - y) - f(y - x) - 3f(x) + 3f(y)$$

for all mapping $f : X \longrightarrow Y$. Then G is additive-quadratic.

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Proof. Suppose that $G_f(x,y) = 0$ for all $x, y \in X$. Then we have

$$f(2x+y) - f(x+2y) + f(x-y) - f(y-x) - 3f(x) + 3f(y) = 0. \tag{4.2}$$

and so we have

$$f_e(2x+y) - f_e(x+2y) - 3f_e(x) + 3f_e(y) = 0 \tag{4.3}$$

for all $x, y \in X$ and letting $y = y - x$ in (4.3), we have

$$f_e(x+y) - f_e(x-2y) - 3f_e(x) + 3f_e(x-y) = 0 \tag{4.4}$$

for all $x, y \in X$. Letting $y = -y$ in (4.4), we have

$$f_e(x-y) - f_e(x+2y) - 3f_e(x) + 3f_e(x+y) = 0 \tag{4.5}$$

for all $x, y \in X$. By (4.4) and (4.5), we have

$$f_e(x+2y)+f_e(x-2y)-2f_e(x)-8f_e(y)-4[f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)] = 0$$

for all $x, y \in X$ and so f_e is quadratic.

Since f_o is an odd mapping, by (4.2), we have

$$f_o(2x+y) - f_o(x+2y) + 2f_o(x-y) - 3f_o(x) + 3f_o(y) = 0 \tag{4.6}$$

for all $x, y \in X$ and letting $y = -x - y$ in (4.6), we have

$$f_o(x-y) + f_o(x+2y) + 2f_o(2x+y) - 3f_o(x) - 3f_o(x+y) = 0 \tag{4.7}$$

for all $x, y \in X$. By (4.6) and (4.7), we have

$$f_o(2x+y) + f_o(x-y) - 2f_o(x) + f_o(y) - f_o(x+y) = 0 \tag{4.8}$$

for all $x, y \in X$ and letting $y = -y$ in (4.10), we have

$$f_o(2x-y) + f_o(x+y) - 2f_o(x) - f_o(y) - f_o(x-y) = 0 \tag{4.9}$$

for all $x, y \in X$. By (4.10) and (4.9), we have

$$f_o(2x+y) + f_o(2x-y) - 4f_o(x) = 0 \tag{4.10}$$

for all $x, y \in X$ and hence f_o is additive. Thus f is an additive-quadratic mapping. □

By Lemma 2.3, Theorem 4.1, and Lemma 4.2, we have the following theorem :

Theorem 4.3. *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\|f(x+2y) - f(2x+y) + f(x+y) + f(y-x) + f(x) - 4f(y) - f(-y)\|$$
$$\le \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|x\|^{2p})$$

for all $x, y \in X$ and some a real number p with $0 < p < \frac{1}{2}, 1 < p$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \begin{cases} \left[\frac{3}{4-4^p} + \frac{4}{2-4^p}\right]\epsilon\|x\|^{2p}, & \text{if } 0 < p < \frac{1}{2} \\ \\ \left[\frac{3\times 4^{p-1}}{4^p-4} + \frac{2\times 4^p}{4^p-2}\right]\epsilon\|x\|^{2p}, & \text{if } 1 < p \end{cases}$$

for all $x \in X$.

Proof. For a mapping $h : X \longrightarrow Y$, let $G_h(x,y) = h(2x+y) - h(x+2y) + h(x-y) - h(y-x) - 3h(x) + 3h(y)$. By Lemma 4.2, G is additive-quadratic and f, G satisfiy (4.1). Since G_f satisfies (2.2) in Lemma 2.3, by Theorem 4.1, we have the result. □

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Theoretical and Numerical Discussion for the Mixed Integro–Differential Equations

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Abstract

In this paper, we tend to apply the proposed modified Laplace Adomian decomposition method that is the coupling of Laplace transform and Adomian decomposition method. The modified Laplace Adomian decomposition method is applied to solve the Fredholm–Volterra integro–differential equations of the second kind in the space $L_2[a, b]$. The nonlinear term will simply be handled with the help of Adomian polynomials. The Laplace decomposition technique is found to be fast and correct. Several examples are tested and also the results of the study are discussed. The obtained results expressly reveal the complete reliability, efficiency, and accuracy of the proposed algorithmic rule for solving the Fredholm–Volterra integro–differential equations and therefore will be extended to other problems of numerous nature.

Mathematics Subject Classification: 41A10, 45J05, 65R20.

Key-Words: Fredholm-Volterra Integro-Differential Equations; Adomian Decomposition Method; Laplace Transform Method; Laplace Adomian Decomposition Method.

1. Introduction

Mathematical modeling of real-life problems usually results in functional equations, such as differential, integral, and integro-differential equations. Many mathematical formulations of physical phenomena reduced to integro-differential equations, like fluid dynamics, biological models, chemical mechanics and contact problems, see [6, 14, 19].

Many problems from physics and engineering and alternative disciplines cause linear and nonlinear integral equations. Now, for the solution of those equations several analytical and numerical methods are introduced, however numerical methods are easier than analytical methods and most of the time numerical methods are used to solve these equations we refer to [1, 2, 18].

Laplace Adomians decomposition method was first introduced by Suheil A. Khuri [16,17] and has been with successfully used to find the solution of differential equations [20]. This method generates a solution in the form of a series whose terms are determined by a recursive relation using the Adomian polynomials.

Most of the nonlinear integro-differential equations don't have an exact analytic solution, therefore approximation and numerical technique should be used, there are only a number of techniques for the solution of integro-differential equations, since it's relatively a new subject in arithmetic.

The modified laplace decomposition technique has applied for solving some nonlinear ordinary, partial differential equations. Recently, the authors have used many methods for the numerical or the analytical solution of linear and nonlinear Fredholm and Volterra integral and integrodifferential equations of the second kind [8, 9, 11, 12, 21].

In this paper, we consider the Fredholm–Volterra integro–differential equations of the second kind with continuous kernels with respect to position. We applied Laplace transform and Adomian polynomials to solve nonlinear Fredholm–Volterra integro–differential equations. Al–Towaiq and Kasasbeh [7] have applied the modification of Laplace decomposition method to solve linear interval Fredholm integro–differential equations of the form :

$$u'(x) = f(x) + \int_a^b k(x,t)u(t)dt; \qquad u(a) = \alpha.$$

But in this paper, we will study the modification of Laplace Adomian decomposition method to solve the nonlinear interval Fredholm–Volterra integro–differential equation of the form:

$$\phi(u+q) = p(u) + \lambda \int_a^b k(u,v)\mu(v,\phi(v))dv + \lambda \int_0^u \psi(u,v)\nu(v,\phi(v))dv; \quad (q << 1), \tag{1}$$

where q is the Phase-Lag is positive, very small and assumed to be intrinsic properties of the medium. The constant parameter λ may be complex and has many physical meanings, the function $\phi(u)$ is unknown in the Banach space and continuous with their derivative with respect to time in the space $L_2[a,b]$, where $[a,b]$ is the domain of integration with respect to the position and it's called the potential function of the mixed integral equation. The kernels $k(u,v), \psi(u,v)$ are positive and continuous in $L_2[a,b]$ and the known function $p(u)$ is continuous and its derivatives with respect to position.

Using Taylor Expansion after neglecting the second derivative in the equation (1) we get,

$$\phi(u) + q\frac{d\phi(u)}{du} = p(u) + \lambda \int_a^b k(u,v)\mu(v,\phi(v))dv + \lambda \int_0^u \psi(u,v)\nu(v,\phi(v))dv; \quad (q << 1), \tag{2}$$

with initial condition,

$$\phi(a) = \alpha. \tag{3}$$

The equation (2) with initial condition (3) is called Integro-Differential Equation for the Phase-Lag. The Integro-Differential Equation is a kind of functional equation that has associate integral and derivatives of unknown function. These equations were named after the leading mathematicians who have first studied them such as Fredholm, Volterra. Fredholm and Volterra equations are the most encountered types, see [10]. There is, formally only one difference between them, in the Fredholm equation the region of integration is fixed where in the Volterra equation the region is variable. Integro-Differential Equations (IDEs) are given as a combination of differential and integral equations.

2. Preliminaries

In this section, we give some definitions and properties of the Adomian polynomials and Laplace transform.

2.1 Laplace transform

Definition 1. The Laplace transform of a function $\phi(u); u > 0$ is defined as

$$L[\phi(u)] = \Phi(s) = \int_0^{+\infty} e^{-su}\phi(u)du, \tag{4}$$

where s can be either real or complex.

Definition 2. Given two functions ϕ and ψ, we define, for any $u > 0$,

$$(\phi * \psi)(u) = \int_0^u \phi(v)\psi(u-v)dv, \tag{5}$$

the function $\phi * \psi$ is called the convolution of ϕ and ψ.

Theorem 1. *The convolution theorem*

$$L[\phi * \psi](u) = L[\phi(u)] * L[\psi(u)]. \tag{6}$$

Lemma 1. *Laplace Transform of an Integral: If $\Phi(s) = L[\phi(u)]$ then*

$$L\left[\int_0^u \phi(v)dv\right] = \frac{\Phi(s)}{s}. \tag{7}$$

Theorem 2. *The Laplace transform $L[\phi(u)]$ of the derivatives are defined by*

$$L[\phi^{(n)}(u)] = s^n L[\phi(u)] - s^{n-1}\phi(0) - s^{n-2}\phi'(0) - \cdots - \phi^{(n-1)}(0). \tag{8}$$

2.2 Adomians Decomposition method

Consider the general functional equation:

$$\phi = p + N_1\phi + N_2\phi, \tag{9}$$

where N_1, N_2 are a nonlinear operators, p is a known function, and we are seeking the solution ϕ satisfying (9). We assume that for every p , Eq. (9) has one and only one solution.

The Adomians technique consists of approximating the solution of (9) as an infinite series

$$\phi = \sum_{n=0}^{\infty} \phi_n, \tag{10}$$

and decomposing the nonlinear operators N_1, N_2 as respectively

$$N_1\phi = \sum_{n=0}^{\infty} A_n, \qquad N_2\phi = \sum_{n=0}^{\infty} B_n, \tag{11}$$

where A_n, B_n are polynomials (called Adomian polynomials)of $\{\phi_0, \phi_1, \ldots, \phi_n\}$ [4,5] given by

$$\begin{aligned} A_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_1\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0, 1, 2, \ldots \\ B_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_2\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0, 1, 2, \ldots \end{aligned}$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty}\phi_n, \sum_{n=0}^{\infty} A_n$ and $\sum_{n=0}^{\infty} B_n$ are given in [3,13]. Substituting (10) and (11) into (9) yields, we get

$$\sum_{n=0}^{\infty}\phi_n = p + \sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n.$$

Thus, we can identify

$$\begin{aligned} \phi_0 =& p, \\ \phi_{n+1} =& A_n(\phi_0, \phi_1, \ldots, \phi_n) + B_n(\phi_0, \phi_1, \ldots, \phi_n); \quad n = 0, 1, 2, \ldots \end{aligned}$$

Thus all components of ϕ can be calculated once the A_n, B_n are given. We then define the n-terms approximate to the solution ϕ by

$$\Psi_n[\phi] = \sum_{i=0}^{n-1}\phi_i \quad , with \quad \lim_{n\to\infty}\Psi_n[\phi] = \phi.$$

3. Description of the Method

The purpose of this section is to discuss the use of modified Laplace decomposition algorithm for the Fredholm–Volterra integro–differential equation. Applying the Laplace transform (denoted by L) on the both sides of the equation yield (2), we have

$$\begin{aligned} L[\phi(u)] + qL\left[\frac{d\phi(u)}{du}\right] =& L[p(u)] + \lambda L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ &+ \lambda L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right], \end{aligned} \tag{12}$$

using the differentiation property of Laplace transform (8) we get

$$\begin{aligned} L[\phi(u)] + qsL[\phi(u)] - q\phi(0) =& L[p(u)] + \lambda L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ &+ \lambda L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right]. \end{aligned} \tag{13}$$

Thus, the given equation is equivalent to

$$\begin{aligned} L[\phi(u)] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)} L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ + \frac{\lambda}{(1+qs)} L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right]. \end{aligned} \tag{14}$$

The Adomian decomposition method and the Adomian polynomials can be used to handle (14) and to address the nonlinear terms $\mu(v,\phi(v)), \nu(v,\phi(v))$. We first represent the linear term $\phi(u)$ at the left side by an infinite series of components given by

$$\phi(u) = \sum_{n=0}^{\infty} \phi_n(u), \tag{15}$$

where the components $\phi_n; n \geq 0$ will be determined recursively. However, the nonlinear terms $\mu(v,\phi(v)), \nu(v,\phi(v))$ at the right side of Eq. (14) will be represented by an infinite series of the Adomian polynomials A_n, B_n respectively in the form

$$\mu(v,\phi(v)) = \sum_{n=0}^{\infty} A_n(v), \qquad \nu(v,\phi(v)) = \sum_{n=0}^{\infty} B_n(v), \tag{16}$$

where $A_n, B_n; n \geq 0$ are defined by

$$\begin{aligned} A_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mu\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0,1,2,\ldots \\ B_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\nu\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0,1,2,\ldots \end{aligned}$$

where the so-called Adomian polynomials A_n, B_n can be evaluated for all forms of nonlinearity [22]. In other words, assuming that the nonlinear function is $\mu(v,\phi(v)), \nu(v,\phi(v))$, therefore the Adomian polynomials are given by

$$\begin{aligned} A_0 &= \mu(\phi_0), & B_0 &= \nu(\phi_0), \\ A_1 &= \phi_1\mu'(\phi_0), & B_1 &= \phi_1\nu'(\phi_0), \\ A_2 &= \phi_2\mu'(\phi_0) + \frac{1}{2}\phi_1^2\mu''(\phi_0), & B_2 &= \phi_2\nu'(\phi_0) + \frac{1}{2}\phi_1^2\nu''(\phi_0). \end{aligned}$$

Substituting (15) and (16) into (14), we will get

$$
\begin{aligned}
L\left[\sum_{0}^{\infty}\phi_n(u)\right] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)\sum_0^\infty A_n(v)dv\right] \\
+\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)\sum_0^\infty B_n(v)dv\right].
\end{aligned}
\tag{17}
$$

The Adomian decomposition method presents the recursive relation

$$
L[\phi_0(u)] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)}, \tag{18}
$$

$$
L[\phi_1(u)] = \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_0(v)dv\right] + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_0(v)dv\right], \tag{19}
$$

$$
L[\phi_2(u)] = \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_1(v)dv\right] + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_1(v)dv\right]. \tag{20}
$$

In general, the recursive relation is given by

$$
\begin{aligned}
L[\phi_{n+1}(u)] =& \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right] \\
&+\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right], \quad n=0,1,2,\ldots
\end{aligned}
\tag{21}
$$

A necessary condition for Eq. (21) to work is that

$$
\lim_{s\to\infty}\frac{\lambda}{(1+qs)} = 0.
$$

Applying inverse Laplace transform to Eqs. (18)–(21), so our required recursive relation

$$
\phi_0(u) = G(u), \tag{22}
$$

and

$$
\begin{aligned}
\phi_{n+1}(u) =& L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right]\right] \\
&+L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right]\right],
\end{aligned}
\tag{23}
$$

where $G(u)$ may be a function that arises from the source term and also the prescribed initial conditions, the initial solution is very important, the choice of (22) as the initial solution always leads to noise oscillation during the iteration procedure, the modified laplace decomposition method [15] suggests that the operate $G(u)$ defined above in (18) be rotten into two parts:

$$
G(u) = G_1(u) + G_2(u).
$$

Instead of iteration procedure (22) and (23), we suggest the following modification

$$
\begin{aligned}
\phi_0(u) &= G_1(u), \\
\phi_1(u) &= G_2(u) + L^{-1}\left[\frac{\lambda}{(1+qs)} L\left[\int_a^b k(u,v)A_0(v)dv\right]\right] \\
&\quad + L^{-1}\left[\frac{\lambda}{(1+qs)} L\left[\int_0^u \psi(u,v)B_0(v)dv\right]\right], \\
\phi_{n+1}(u) &= L^{-1}\left[\frac{\lambda}{(1+qs)} L\left[\int_a^b k(u,v)A_n(v)dv\right]\right] \\
&\quad + L^{-1}\left[\frac{\lambda}{(1+qs)} L\left[\int_0^u \psi(u,v)B_n(v)dv\right]\right], \qquad n = 0, 1, 2, \ldots
\end{aligned}
$$

We then define the n-terms approximate to the solution $\phi(u)$ by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1} \phi_i(u), \quad with \quad \lim_{n\to\infty} \Psi_n[\phi(u)] = \phi(u).$$

In this paper, the obtained series solution converges to the exact solution.

3.1 A Test of Convergence

In fact, on every interval the inequality $\|\phi_{i+1}\|_2 < \beta\|\phi_i\|_2$ is required to hold for $i = 0, 1, \ldots, n$, wherever $0 < \beta < 1$ may be a constant and n is that the maximum order of the approximate used in the computation. Of course, this is often only a necessary condition for convergence, as a result of it might be necessary to compute $\|\phi_i\|_2$ for each $i = 0, 1, \ldots, n$ so as to conclude that the series is convergent.

4. Application of the Laplace transform–Adomian decomposition method

In this section, the Laplace transform–Adomian decomposition method for solving Fredholm–Volterra integro–differential equation is illustrated in the two examples given below. To show the high accuracy of the solution results from applying the present method to our problem (2) compared with the exact solution, the maximum error is defined as:

$$R_n = \|\phi_{Exact}(u) - \Psi_n[\phi(u)]\|_\infty,$$

where $n = 1, 2, \ldots$ represents the number of iterations.

Example 1

Consider the nonlinear Fredholm–Volterra integro–differential equation

$$\phi(u + 0.2) = p(u) + \frac{1}{4}\int_0^1 cos(u)\phi^2(v)dv + \int_0^u \phi^3(v)dv, \tag{24}$$

where

$$p(u) = \frac{1}{12}(-3 - u^3 cos(u)).$$

Using Taylor Expansion after neglecting the second derivative in the equation (24) we get,

$$\phi(u) + 0.2\frac{d\phi(u)}{du} = p(u) + \frac{1}{4}\int_0^1 cos(u)\phi^2(v)dv + \int_0^u \phi^3(v)dv; \quad \phi(0) = 0. \tag{25}$$

The exact solution for this problem is

$$\phi(u) = cos(u) - sin(u).$$

First, we apply the Laplace transform to both sides of (25)

$$L[\phi(u)] + 0.2L\left[\frac{d\phi(u)}{du}\right] = L[p(u)] + \frac{1}{4}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + L\left[\int_0^u \phi^3(v)dv\right], \tag{26}$$

Using the property of Laplace transform and the initial conditions, we get

$$L[\phi(u)] + 0.2sL[\phi(u)] = L[p(u)] + \frac{1}{4}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + L\left[\int_0^u \phi^3(v)dv\right], \tag{27}$$

or equivalently

$$L[\phi(u)] = \frac{L[p(u)]}{1+0.2s} + \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u \phi^3(v)dv\right]. \tag{28}$$

Substituting the series assumption for $\phi(u)$ and the Adomian polynomials for $\phi^2(u), \phi^3(u)$ as given above in (15) and (16) respectively into Eq. (28) we obtain

$$\begin{aligned} L\left[\sum_{n=0}^{\infty}\phi_n(u)\right] &= \frac{L[p(u)]}{1+0.2s} + \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)\sum_{n=0}^{\infty}A_n(v)dv\right] \\ &\quad + \frac{1}{1+0.2s}L\left[\int_0^u \sum_{n=0}^{\infty}B_n(v)dv\right]. \end{aligned} \tag{29}$$

The recursive relation is given below

$$\begin{aligned} L[\phi_0(u)] &= \frac{L[p(u)]}{1+0.2s}, \\ L[\phi_1(u)] &= \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)A_0(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u B_0(v)dv\right], \\ L[\phi_{n+1}(u)] &= \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)A_n(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u B_n(v)dv\right], \end{aligned} \tag{30}$$

where A_n, B_n are the Adomian polynomials for the nonlinear terms $\phi^2(u), \phi^3(u)$ respectively. The Adomian polynomials for $\mu(v, \phi(v)) = \phi^2(u), \nu(v, \phi(v)) = \phi^3(u)$ are given by

$$\begin{aligned} A_0 &= \phi_0^2, & B_0 &= \phi_0^3, \\ A_1 &= 2\phi_0\phi_1, & B_1 &= 3\phi_0^2\phi_1, \\ A_2 &= 2\phi_0\phi_2 + \phi_1^2, & B_2 &= 3\phi_0^2\phi_2 + 3\phi_0\phi_1^2, \\ A_3 &= 2\phi_0\phi_3 + 2\phi_1\phi_2, & B_3 &= 3\phi_0^2\phi_3 + 6\phi_0\phi_1\phi_2 + \phi_1^3. \end{aligned}$$

Taking the inverse Laplace transform of both sides of the first part of (30), and using the recursive relation (30) gives

$$\begin{aligned} \phi_0(u) &= 1 - u - u^2 + \frac{1}{2}u^3 + \frac{1}{12}u^4 - \dots \\ \phi_1(u) &= \frac{1}{2}u^2 - \frac{1}{3}u^3 - \frac{1}{8}u^4 + \frac{1}{6}u^5 + \dots \\ \phi_2(u) &= \frac{1}{12}u^4 - \frac{1}{12}u^5 + \dots \end{aligned} \tag{31}$$

Thus the series solution is given by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1} \phi_i(u) = \left(1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \dots\right) - \left(u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \dots\right) \quad n = 1, 2, \dots$$

$$\phi(u) = \lim_{n\to\infty} \Psi_n[\phi(u)] = \lim_{n\to\infty} \left[\left(1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \dots\right) - \left(u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \dots\right)\right],$$

that converges to the exact solution

$$\phi(u) = cos(u) - sin(u).$$

Example 2

Consider the nonlinear Fredholm–Volterra integro–differential equation

$$\phi(u + 0.01) = p(u) + \int_0^1 \phi(v)dv + \int_0^u e^{-u}\phi^2(v)dv, \tag{32}$$

where

$$p(u) = 1 - \frac{1}{4}e^{-u} + 0.0100502e^u.$$

Using Taylor Expansion after neglecting the second derivative in the equation (32) we get,

$$\phi(u) + 0.01\frac{d\phi(u)}{du} = p(u) + \int_0^1 \phi(v)dv + \int_0^u e^{-u}\phi^2(v)dv; \quad \phi(0) = 1. \tag{33}$$

The exact solution for this problem is

$$\phi(u) = e^u.$$

First, we apply the Laplace transform to both sides of (33)

$$L[\phi(u)] + 0.01L\left[\frac{d\phi(u)}{du}\right] = L[p(u)] + L\left[\int_0^1 \phi(v)dv\right] + L\left[\int_0^u e^{-u}\phi^2(v)dv\right], \tag{34}$$

using the property of Laplace transform and the initial conditions, we get

$$L[\phi(u)] + 0.01sL[\phi(u)] - 0.01 = L[p(u)] + L\left[\int_0^1 \phi(v)dv\right] + L\left[\int_0^u e^{-u}\phi^2(v)dv\right], \tag{35}$$

or equivalently

$$\begin{aligned} L[\phi(u)] = \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s} + \frac{1}{1+0.01s}L\left[\int_0^1 \phi(v)dv\right] \\ + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}\phi^2(v)dv\right]. \end{aligned} \tag{36}$$

Substituting the series assumption for $\phi(u)$ and the Adomian polynomials for $\phi^2(u)$ as given above in (15) and (16) respectively into above equation, we obtain

$$\begin{aligned} L\left[\sum_{n=0}^{\infty}\phi_n(u)\right] = \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s} + \frac{1}{1+0.01s}L\left[\int_0^1 \sum_{n=0}^{\infty}\phi_n(v)dv\right] \\ + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}\sum_{n=0}^{\infty}A_n(v)dv\right], \end{aligned} \tag{37}$$

the recursive relation is given below

$$\begin{aligned} L[\phi_0(u)] &= \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s}, \\ L[\phi_1(u)] &= \frac{1}{1+0.01s}L\left[\int_0^1 \phi_0(v)dv\right] + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}A_0(v)dv\right], \\ L[\phi_{n+1}(u)] &= \frac{1}{1+0.01s}L\left[\int_0^1 \phi_n(v)dv\right] + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}A_n(v)dv\right], \end{aligned} \tag{38}$$

where A_n are the Adomian polynomials for the nonlinear terms $\phi^2(u)$. The Adomian polynomials for $\mu(v, \phi(v)) = \phi^2(u)$ is given by

$$\begin{aligned} A_0 &= \phi_0^2, \\ A_1 &= 2\phi_0\phi_1, \\ A_2 &= 2\phi_0\phi_2 + \phi_1^2, \\ A_3 &= 2\phi_0\phi_3 + 2\phi_1\phi_2, \end{aligned}$$

Taking the inverse Laplace transform of both sides of the first part of (38), and using the recursive relation (38) gives

$$\begin{aligned}
\phi_0(u) &= 1 + u - \frac{1}{2}u^3 - \frac{1}{2}u^4 - \frac{13}{40}u^5 + \dots \\
\phi_1(u) &= \frac{1}{2}u^2 + \frac{2}{3}u^3 + \frac{5}{12}u^4 + \frac{7}{120}u^5 + \dots \\
\phi_2(u) &= \frac{1}{8}u^4 + \frac{11}{40}u^5 + \dots
\end{aligned} \tag{39}$$

Thus the series solution is given by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1} \phi_i(u) = \left(1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \frac{1}{5!}u^5 + \dots\right) \quad n = 1, 2, \dots$$

$$\phi(u) = \lim_{n\to\infty} \Psi_n[\phi(u)] = \lim_{n\to\infty} \left[\left(1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \frac{1}{5!}u^5 + \dots\right)\right],$$

that converges to the exact solution

$$\phi(u) = e^u.$$

5. Conclusions

In this work, the Laplace decomposition technique has been successfully applied to finding the approximate solution of the nonlinear Fredholm–Volterra integro–differential equation. The method is extremely powerful and efficient find analytical moreover as numerical solutions for wide classes of nonlinear Fredholm–Volterra integro–differential equations. It provides a lot of realistic series solutions that converge very rapidly in real physical issues.

The main advantage of this technique is that the fact that it provides the analytical solution. Some examples are given and therefore the results reveal that the method is extremely effective. some of the nonlinear equations are examined by the modified technique to Illustrate the effectiveness and convenience of this technique, and in all cases, the modified technique performed excellently. The results reveal that the proposed technique is extremely effective and easy.

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Representation of the Matrix for Conversion between Triangular Bézier Patches and Rectangular Bézier Patches

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Abstract

In this paper we studied Bézier surfaces that are very famous techniques and widely used in Computer Aided Geometric Design. Mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches. In this paper we will give a representation for the conversion matrix which converts one type to another.

1 Introduction

The theory of Bézier curves has an important role and they are numerically the most stable among all polynomial bases currently used in CAD systems. On the other hand in these days Bézier surfaces are very famous techniques and widely used in Computer Aided Geometric Design [1]-[13]. Mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches and they are defined in terms of the univariate Bernstein polynomials $B_i^n(s) = \binom{n}{i} s^i (1-s)^{n-i}$ and the bivariate Bernstein polynomial $B_{i,j,k}^n(u,v,w) = \binom{n}{i,j,k} u^i v^j w^k$ where $u+v+w=1$. A triangular Bézier patch of degree n with control points $T_{i,j,k}$ is defined by

$$T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k} B_{i,j,k}^n(u,v,w),\ u,v,w \geq 0,\ u+v+w=1.$$

and a rectangular Bézier patch of degree $n \times m$ with control points $P_{i,j}$ is represented by

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{ij} B_i^n(s) B_j^n(t) \quad 0 \leq s,t \leq 1, \text{(see [3])}$$

Since the two patches have different geometric properties it is not easy to use both of them in the same CAD system and conversion of one type to another is needed.

2 Construction of the Conversion Matrices

The following theorem gives the conversion of degree n triangular Bézier patch to degenerate rectangular Bézier patch of degree $n \times n$.

Definition 1 *For all nonnegative integers x the falling factorial is defined by*

$$(x)_n = x(x-1)...(x-n+1) = \prod_{k=1}^{n} (x-(k-1))$$

Theorem 2 *A degree n triangular Bézier patch $T(u,v,w)$ can be represented as a degenerate Bézier patch of degree $n \times n$:*

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{ij} B_i^n(s) B_j^n(t), \qquad 0 \leq s,t \leq 1$$

where the control points P_{ij} are determined by

$$\begin{pmatrix} P_{i0} \\ P_{i1} \\ \vdots \\ P_{in} \end{pmatrix} = A_1 A_2 ... A_i \begin{pmatrix} T_{i0} \\ T_{i1} \\ \vdots \\ T_{i,n-i} \end{pmatrix}, \qquad i = 0,1,2,...,n.$$

and $A_i (i = 0,1,...,n)$ are degree elevation operators in the form

$$A_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \cdots & 0 & 0 \\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{n-k}{n+1-k} & \frac{1}{n+1-k} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(n-k+2)\times(n-k+1)}$$

Until now no one has studied the generalization of the product $A_1 A_2 ... A_k$ mentioned in the above theorem and indeed the product of these matrices is not easy to calculate for different values of n and k. Here we will give the generalization of this product which will make all the computations easier.

Theorem 3 *The following formula is true*

$$A_1 A_2 ... A_k = \bar{A}_k = \left[\bar{a}_{i,j}^{(k)}\right]_{(n+1)\times(n-k+1)},$$

where

$$\bar{a}_{i,j}^{(k)} = \frac{\binom{i-1}{j-1} (k)_{i-j} (n-k)_{j-1}}{(n)_{i-1}},$$

$$(k)_n = k(k-1)...(k-n+1) = \prod_{j=1}^{n} (k-(j-1)) \quad \textit{and}$$

$$A_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \cdots & 0 & 0 \\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{n-k}{n+1-k} & \frac{1}{n+1-k} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(n-k+2)\times(n-k+1)}.$$

Proof. For $k = 1$,

$$\bar{A}_1 = A_1.$$

Suppose it is true for k, that is

$$A_1 A_2 ... A_k = \bar{A}_k.$$

We will show that it also true for $k+1$, i.e

$$\bar{A}_k A_{k+1} = \bar{A}_{k+1}.$$

Let $c_{i,j}$ be the element at the i^{th} row, j^{th} column of the matrix $\bar{A}_k A_{k+1}$:

$$\begin{aligned} c_{i,j} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,j}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{i-1}{m-1} (k)_{i-m} (n-k)_{m-1}}{(n)_{i-1}} a_{m,j}^{(k+1)}, \end{aligned}$$

where $i = \{1, 2, ..., n+1\}$ and $j = \{1, 2, ..., n-k\}$.

For $j = 1$ (first column)

$$\begin{aligned} c_{i,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{i-1}{m-1} (k)_{i-m} (n-k)_{m-1}}{(n)_{i-1}} a_{m,1}^{(k+1)}. \end{aligned}$$

For $i = 1$ and $j = 1$

$$\begin{aligned} c_{1,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_{m-1}}{(n)_0} a_{m,1}^{(k+1)} \\ &= a_{1,1}^{(k+1)} = 1 = \bar{a}_{1,1}^{(k+1)}. \end{aligned}$$

For $i = 2$ and $j = 1$,

$$\begin{aligned} c_{2,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{2,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1} (k)_{2-m} (n-k)_{m-1}}{(n)_1} a_{m,1}^{(k+1)} \\ &= \frac{k+1}{n} = \frac{(k+1)_1}{(n)_1}. \end{aligned}$$

For $i = n+1$ and $j = 1$,

$$\begin{aligned} c_{n+1,1} &= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1} (k)_{n+1-m} (n-k)_{m-1}}{(n)_n} a_{m,1}^{(k+1)} \\ &= \frac{(k+1)k(k-1)...(k-n+2)}{n(n-1)(n-2)...1} \\ &= \frac{(k+1)_n}{(n)_n}. \end{aligned}$$

For $j = 2$ (second solumn), for $i = 1$ and $j = 2$

$$\begin{aligned} c_{1,2} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,2}^{(k+1)} \\ c_{1,2} &= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_0}{(n)_0} a_{m,2}^{(k+1)} \\ c_{1,2} &= a_{1,2}^{(k+1)} = 0 = \bar{a}_{1,2}^{(k+1)}. \end{aligned}$$

For $i=2$ and $j=2$

$$\begin{aligned}
c_{2,2} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,2}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1} (k)_{2-m} (n-k)_{m-1}}{(n)_1} a_{m,2}^{(k+1)} \\
&= \frac{n-k-1}{n} = \bar{a}_{2,2}^{(k+1)}.
\end{aligned}$$

For $i=n+1$ and $j=2$

$$\begin{aligned}
c_{n+1,2} &= \sum_{m=1}^{n-k+1} \bar{a}_{n+1,m}^{(k)} a_{m,n+1}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1} (k)_{n+1-m} (n-k)_{m-1}}{(n)_n} a_{m,2}^{(k+1)} \\
&= \frac{n(k+1)k(k-1)...(k-n+3)(n-k-1)}{n(n-1)(n-2)...1} \\
&= \frac{n(k+1)_{n-1}(n-k-1)_1}{(n)_n} = \bar{a}_{n+1,2}^{(k+1)}.
\end{aligned}$$

For $j=n-k$ (last column), for $i=1$ and $j=n-k$

$$\begin{aligned}
c_{1,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,n-k}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_{m-1}}{(n)_0} a_{m,n-k}^{(k+1)} \\
&= a_{1,n-k}^{(k+1)} = 0 = \bar{a}_{1,n-k}^{(k+1)}.
\end{aligned}$$

For $i=2$ and $j=n-k$

$$\begin{aligned}
c_{2,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}_{2,m}^{(k)} a_{m,j}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1} (k)_{2-m} (n-k)_{m-1}}{(n)_1} a_{m,n-k}^{(k+1)} \\
&= 0 = \bar{a}_{2,n-k}^{(k+1)}.
\end{aligned}$$

For $i=n+1$ and $j=n-k$

$$\begin{aligned}
c_{n+1,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}_{n+1,m}^{(k)} a_{m,n-k}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1} (k)_{n+1-m} (n-k)_{m-1}}{(n)_n} a_{m,n-k}^{(k+1)} \\
&= 1 = \bar{a}_{n+1,n-k}^{(k+1)}.
\end{aligned}$$

Hence, $\bar{A}_k A_{k+1} = [c_{i,j}]_{(n+1)\times(n-k)} = \left[\bar{a}_{i,j}^{(k+1)}\right]_{(n+1)\times(n-k)}$, where $\bar{a}_{i,j}^{(k+1)} = \frac{\binom{i-1}{j-1}(k+1)_{i-j}(n-k-1)_{j-1}}{(n)_{i-1}}$. ■

Remark 4 *Sum of the elements in each row of the matrix $\bar{A}_k$ is equal to* 1.

Now in the following theorem we consider the inverse process

Theorem 5 *A rectangular Bézier patch $P(s,t)$ of degree $n \times n$ can be represented as a Triangular Bézier patch $T(u,v,w)$ of degree n :*

$$T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k}\ B^n_{i,j,k}(u,v,w), \qquad u,v,w \geq 0, u+v+w=1.$$

where the control points $T_{i,j,k}$ are determined by

$$\begin{pmatrix} T_{i0} \\ T_{i1} \\ \vdots \\ T_{i,n-i} \end{pmatrix} = B_i B_{i-1} ... B_1 \begin{pmatrix} P_{i0} \\ P_{i1} \\ \vdots \\ P_{in} \end{pmatrix} \qquad i = 0,1,2,...,n.$$

and $B_i (i = 0,1,...,n)$ are degree elevation operators in the form

$$B_k = \begin{bmatrix} 1-t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-t & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-t & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-t & t \end{bmatrix}_{(n-k+1)\times(n-k+2)}$$

Proof. Indeed

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j} B_i^n(s) B_j^n(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j} B_i^n(s) \left\{ t B_{j-1}^{n-1}(t) + (1-t) B_j^{n-1}(t) \right\} \\ &= \sum_{i=0}^{n} B_i^n(s) \left\{ t \sum_{j=0}^{n} P_{i,j} B_{j-1}^{n-1}(t) + (1-t) \sum_{j=0}^{n} P_{i,j} B_j^{n-1}(t) \right\} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-r} P_{i,j}^r B_i^n(s) B_j^{n-r}(t), \end{aligned}$$

where

$$\begin{aligned} &P_{i,j}^0(t) \equiv P_{i,j}^0 = P_{i,j} \\ &P_{i,j}^r(t) = t P_{i,j+1}^{r-1} + (1-t) P_{i,j}^{r-1}. \end{aligned}$$

Let $r = i$,

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^i B_i^n(s) B_j^{n-i}(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^i \binom{n}{i} \binom{n-i}{j} s^i (1-s)^{n-i} t^j (1-t)^{n-i-j} \end{aligned}$$

if we use the following reparametrization

$$\begin{cases} s = u \\ t = \frac{v}{1-u} = \frac{v}{v+w} \end{cases}$$

we get

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i} \binom{n}{i} \binom{n-i}{j} u^{i}(1-u)^{n-i} \left(\frac{v}{v+w}\right)^{j} (1-\frac{v}{v+w})^{n-i-j}.$$

Now if $i+j+k=n$

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i} \binom{n}{i} \binom{n-i}{j} u^{i}(1-u)^{n-i} \left(\frac{v}{v+w}\right)^{j} (1-\frac{v}{v+w})^{k} \\ &= \sum_{i+j+k=n} T_{i,j,k} B_{i,j,k}^{n}(u,v,w) \end{aligned}$$

$$T_{i,j,k} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i}(t).$$

For each value of i, we obtain $(n-i+1)\times(n-i+2)$ matrix B_i. ■

Theorem 6 *The product of the matrices in the above theorem $B_k B_{k-1} ... B_1$ can be generalized as follows*

$$Z^{k} = B_k B_{k-1} ... B_1 = \begin{bmatrix} b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} \end{bmatrix}$$

where $b_{k,j} = \binom{k}{j} t^{j}(1-t)^{k-j}$ and Z^k is $(n-k+1)\times(n+1)$ matrix.

Proof. For $k=1$,

$$Z^1 = B_1,$$

suppose it is true for k, that is

$$B_k B_{k-1} ... B_1 = Z^k,$$

we will show that it also true for $k+1$, i.e

$$B_{k+1} B_k B_{k-1} ... B_1 = Z^{k+1}.$$

Let $z_{i,j}$ be the element at the i^{th} row, j^{th} column of the matrix $B_{k+1}B_k B_{k-1}...B_1$,

$$z_{i,j} = \sum_{m=1}^{n-k+1} b_{i,m}^{(k+1)} b_{m,j}^{(k)}$$

where $b_{i,m}^{(k+1)}$ is the element at the i^{th} row, m^{th} column of the matrix B_{k+1}, $b_{m,j}^{(k)}$ is the element at the m^{th} row, j^{th} column of the matrix $B_k B_{k-1}...B_1$, $i = \{1,2,...,n-k\}$ and $j = \{1,2,...,n+1\}$.

For $i = 1$(first row)

$$z_{1,j} = \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i=1$ and $j=1$

$$\begin{aligned} z_{1,1} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,1}^{(k)} + b_{1,2}^{(k+1)} b_{2,1}^{(k)} + ... + b_{1,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= (1-t)\, b_{k,0} = (1-t)^{k+1} = Z_{(1,1)}^{k+1}. \end{aligned}$$

For $i=1$ and $j=2$,

$$\begin{aligned} z_{1,2} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,2}^{(k)} + b_{1,2}^{(k+1)} b_{2,2}^{(k)} + ... + b_{1,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= (1-t)\, b_{k,1} + t b_{k,0} \\ &= b_{k+1,1} = Z_{(1,2)}^{k+1}. \end{aligned}$$

For $i=1$ and $j=n+1$,

$$\begin{aligned} z_{1,n+1} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{1,2}^{(k+1)} b_{2,n+1}^{(k)} + ... + b_{1,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= 0 = Z_{(1,n+1)}^{k+1}. \end{aligned}$$

For $i=2$ (second row)

$$z_{2,j} = \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i=2$ and $j=1$,

$$\begin{aligned} z_{2,1} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,1}^{(k)} + b_{2,2}^{(k+1)} b_{2,1}^{(k)} + ... + b_{2,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= 0 = Z_{(2,1)}^{k+1}. \end{aligned}$$

For $i=2$ and $j=2$

$$\begin{aligned} z_{2,2} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,2}^{(k)} + b_{2,2}^{(k+1)} b_{2,2}^{(k)} + ... + b_{2,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= (1-t) b_{k,0} \\ &= (1-t)^{k+1} = Z_{2,2}^{k+1}. \end{aligned}$$

For $i=2$ and $j=n+1$,

$$\begin{aligned} z_{2,n+1} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{2,2}^{(k+1)} b_{2,n+1}^{(k)} + ... + b_{2,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= t b_{k,k} = Z_{2,n+1}^{k+1}. \end{aligned}$$

For $i = n-k+1$

$$z_{n-k+1,j} = \sum_{m=1}^{n-k+1} b^{(k+1)}_{n-k+1,m} b^{(k)}_{m,j}.$$

For $i = n-k$ and $j = 1$

$$\begin{aligned} z_{n-k,1} &= \sum_{m=1}^{n-k+1} b^{(k+1)}_{n-k,m} b^{(k)}_{m,1} \\ &= b^{(k+1)}_{n-k,1} b^{(k)}_{1,1} + b^{(k+1)}_{n-k,2} b^{(k)}_{2,1} + ... + b^{(k+1)}_{n-k,n-k+1} b^{(k)}_{n-k+1,1} \\ &= 0 = Z^{k+1}_{n-k,1}. \end{aligned}$$

For $i = n-k$ and $j = 2$

$$\begin{aligned} z_{n-k,2} &= \sum_{m=1}^{n-k+1} b^{(k+1)}_{n-k,m} b^{(k)}_{m,2} \\ &= b^{(k+1)}_{n-k,1} b^{(k)}_{1,2} + b^{(k+1)}_{n-k,2} b^{(k)}_{2,2} + ... + b^{(k+1)}_{n-k,n-k+1} b^{(k)}_{n-k+1,2} \\ &= 0 = Z^{k+1}_{n-k,2}. \end{aligned}$$

For $i = n-k$ and $j = n+1$

$$\begin{aligned} z_{n-k,n+1} &= \sum_{m=1}^{n-k+1} b^{(k+1)}_{n-k,m} b^{(k)}_{m,n+1} \\ &= b^{(k+1)}_{n-k,1} b^{(k)}_{1,n+1} + b^{(k+1)}_{n-k,2} b^{(k)}_{2,n+1} + ... + b^{(k+1)}_{n-k,n-k+1} b^{(k)}_{n-k+1,n+1} \\ &= tb_{k,k} = Z^{k+1}_{n-k,n+1}. \end{aligned}$$

$$\bar{A}_k A_{k+1} = \bar{A}_{k+1} = \left[\begin{array}{cc} 1 & 0 \\ \frac{k+1}{n} & \frac{n-k-1}{n} \\ \frac{(k+1)k}{n(n-1)} & \frac{2(k+1)(n-k-1)}{n(n-1)} \\ \frac{(k+1)(k)(k-1)}{n(n-1)(n-2)} & \frac{3(k+1)k(n-k-1)}{n(n-1)(n-2)} \\ \frac{(k+1)k(k-1)(k-2)}{n(n-1)(n-2)(n-3)} & \frac{4(k+1)k(k-1)(n-k-1)}{n(n-1)(n-2)(n-3)} \\ \frac{(k+1)k(k-1)(k-2)(k-3)}{n(n-1)(n-2)(n-3)} & \frac{5(k+1)k(k-1)(k-2)(n-k-1)}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots \\ \frac{(k+1)k...(k-n+3)}{n(n-1)(n-2)...2} & \frac{(n-1)(k+1)k...(k-n+4)(n-k-1)}{n(n-1)(n-2)...2} \\ \frac{(k+1)k...(k-n+2)}{n(n-1)(n-2)...1} & \frac{n(k+1)k...(k-n+3)(n-k-1)}{n(n-1)(n-2)...1} \end{array} \right.$$

$$\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \frac{(n-k-1)(n-k-2)}{n(n-1)} & 0 \\ \frac{3(k+1)(n-k-1)(n-k-2)}{n(n-1)(n-2)} & \frac{(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)} \\ \frac{6(k+1)k(n-k-1)(n-k-2)}{n(n-1)(n-2)(n-3)} & \frac{4(k+1)(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)} \\ \frac{10(k+1)k(k-1)(n-k-1)(n-k-2)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{10(k+1)k(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{array}$$

$$\left.\begin{matrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{(n-k-1)(n-k-2)(n-k-3)(n-k-4)}{n(n-1)(n-2)(n-3)} & 0 & 0 & \cdots & 0 & 0 \\ \frac{5(k+1)(n-k-1)(n-k-2)(n-k-3)(n-k-4)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{(n-k-1)(n-k-2)(n-k-3)(n-k-4)(n-k-5)}{n(n-1)(n-2)(n-3)(n-4)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ \vdots & \vdots & & & \frac{n-k-1}{n} & \frac{k+1}{n} \\ \vdots & \vdots & & & 0 & 1 \end{matrix}\right] \blacksquare$$

Conclusion 7 *As we mentioned before, mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches. These two types of patches have different geometric properties so it is difficult to use both of them in the same CAD system. One may need to convert one type to another and here in this paper we studied on the conversion matrix to convert triangular Bézier patch to a rectangular Bézier patch and a rectangular Bézier patch to a triangular Bézier patch. We found simple representations for these two matrices which will allow the conversion in one step.*

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Permeable values with applications in BE-algebras

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Abstract. The notions of energetic subsets and (anti) permeable values are introduced, and related properties are investigated. These notions are applied to the theory of BE-algebras. Regarding (anti) fuzzy subalgebras/filters and energetic subsets are investigated.

1. Introduction

As a generalization of a BCK-algebra, the notion of BE-algebras has been introduced by H. S. Kim and Y. H. Kim in [5]. The study of BE-algebras has been continued in papers [1], [2], and [6]. Jun et al. [3] introduced the notions of S-energetic subsets and I-energetic subsets in BCK/BCI-algebras, and investigated several properties. Jun et.al [4] defined the notions of a C-energetic subset and (anti) permeable C-value in BCK-algebras and studied some related properties of them.

In this paper, we introduce the notions of energetic subsets and (anti) permeable values, and investigate some related properties. These notions are applied to the theory of BE-algebras. Regarding (anti) fuzzy subalgebras/filters and energetic subsets are investigated.

2. Preliminaries

We display basic notions on BE-algebras. We refer the reader to the papers [2, 5] for further information regarding BE-algebras.

By a *BE-algebra* [5] we mean a system $(X; *, 1)$ of type $(2, 0)$ which the following axioms hold:
(BE1) $(\forall x \in X)\,(x * x = 1)$,
(BE2) $(\forall x \in X)\,(x * 1 = 1)$,
(BE3) $(\forall x \in X)\,(1 * x = x)$,
(BE4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z))$ (exchange).

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We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general [5].

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

(2.1) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(2.2) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,
(2.3) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$.

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* [5] of X if

(i) $1 \in F$;
(ii) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

The concept of fuzzy sets was introduced by Zadeh [7]. Let X be a set. The mapping $f : X \to [0, 1]$ is called a *fuzzy set* in X. A fuzzy set f in a BE-algebra X is called a *fuzzy subalgebra* of X if it satisfies

(F_0) $(\forall x, y \in X)(f(x * y) \geq \min\{f(x), f(y)\})$.

A fuzzy set f in a BE-algebra X is called a fuzzy filter of X if it satisfies

(F_1) $(\forall x \in X)(f(1) \geq f(x))$;
(F_2) $(\forall x, y \in X)(f(y) \geq \min\{f(x * y), f(x)\})$.

Note that every fuzzy filter f of a BE-algebra X satisfies

$$(\forall x, y \in X)(x \leq y \Rightarrow f(y) \geq f(x)).$$

For a fuzzy set f in X and $t \in [0, 1]$, the (strong) upper (resp. lower) t-level sets are defined as follows:

$$U(f; t) := \{x \in X | f(x) \geq t\}, \quad U^*(f; t) := \{x \in X | f(x) > t\},$$
$$L(f; t) := \{x \in X | f(x) \leq t\}, \quad L^*(f; t) := \{x \in X | f(x) < t\}.$$

3. Energetic subsets

In what follows, let X denote a BE-algebra unless otherwise specified.

Definition 3.1. A nonempty subset A of a BE-algebra X is said to be S-*energetic* if it satisfies

(S) $(\forall a, b \in X)(a * b \in A \Rightarrow \{a, b\} \cap A \neq \emptyset)$.

Definition 3.2. A nonempty subset A of a BE-algebra X is said to be F-*energetic* if it satisfies

(F) $(\forall x, y \in X)(y \in A \Rightarrow \{x * y, x\} \cap A \neq \emptyset)$.

Example 3.3. (1) Let $X := \{1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

It is easy to show that $A := \{b, c\}$ is a S-energetic subset of X. But $B := \{a\}$ is not an S-energetic subset of X since $c * b = a \in B$ and $\{c, b\} \cap B = \emptyset$. It is routine to verify that $C := \{c\}$ is an S-energetic subset of X. But it is not an F-energetic subset of X, since $c \in C$ and $\{b * c, b\} \cap C = \emptyset$.

(2) Let $X := \{1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

It is easy to show that $A := \{a, b\}$ is an F-energetic subset of X.

Theorem 3.4. *For any nonempty subset A of X, if A is a subalgebra of a BE-algebra X, then $X \setminus A$ is an S-energetic a subset of X.*

Proof. Let $a, b \in X$ be such that $a * b \in X \setminus A$. If $\{a, b\} \cap (X \setminus A) = \emptyset$, then $a, b \in A$ and so $a * b \in A$ since A is a subalgebra of X. This is a contradiction. Thus $\{a, b\} \cap (X \setminus A) \neq \emptyset$. Therefore $X \setminus A$ is an S-energetic subset of X. □

Theorem 3.5. *For any nonempty subset A of X, if A is a filter of a BE-algebra X, then $X \setminus A$ is an F-energetic a subset of X.*

Proof. Let $x, y \in X$ be such that $y \in X \setminus A$. If $\{x * y, x\} \cap X \setminus A = \emptyset$, then $x * y, x \in A$ and so $y \in A$, since A is a filter of X. This is a contradiction. Therefore $\{x * y, x\} \cap X \setminus A \neq \emptyset$. Thus $X \setminus A$ is an F-energetic subset of X. □

Theorem 3.6. *Let A be a nonempty subset of a BE-algebra X with $1 \notin A$. If A is F-energetic, then $X \setminus A$ is a filter of X.*

Proof. Obviously, $1 \in X \setminus A$. Let $x, y \in X$ be such that $x * y, x \in X \setminus A$. Assume that $y \in A$. Then $\{x * y, x\} \cap A \neq \emptyset$ by (F). Hence $x * y \in A$ or $x \in A$, which is a contradiction. Therefore $y \in X \setminus A$. This completes the proof. □

Theorem 3.7. *If f is a fuzzy filter of a BE-algebra X, then the nonempty lower t-level set $L(f; t)$ is an F-energetic subset of X for all $t \in [0, 1]$.*

Proof. Assume that $L(f;t) \neq \emptyset$ for $t \in [0,1]$ and let $x, y \in X$ be such that $y \in L(f;t)$. Then $t \geq f(y) \geq \min\{f(x*y), f(x)\}$. Hence $f(x*y) \leq t$ or $f(x) \leq t$, i.e., $x*y \in L(f;t)$ or $x \in L(f;t)$. Thus $\{x*y, x\} \cap L(f;t) \neq \emptyset$. Therefore $L(f;t)$ is an F-energetic subset of X. □

Corollary 3.8. *If f is a fuzzy filter of a BE-algebra X, then the nonempty stronger lower t-level set $L^*(f;t)$ is an F-energetic subset of X.*

Since $L(f;t) \cup U^*(f;t) = X$ and $L(f;t) \cap U^*(f;t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

Corollary 3.9. *If f is a fuzzy filter of a BE-algebra X, then $U^*(f;t)$ is empty set or a filter of X for all $t \in [0,1]$.*

For any $a, b \in X$, we consider sets

$$X_a^b := \{x \in X | a*(b*x) = 1\} \text{ and } A_a^b := X \setminus X_a^b.$$

Obviously, $a, b \notin A_a^b, A_a^b = A_b^a$ and $1 \notin A_a^b$. In the following example, we know that there exist $a, b \in X$ such that A_a^b may not be F-energetic.

Example 3.10. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra [2] with the following Cayley table

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $A_c^d = \{0, b\}$ and it is not F-energetic since $b \in A_c^d$ but $\{a*b, a\} \cap A_c^d = \emptyset$.

We consider conditions for the set A_a^b to be F-energetic.

Theorem 3.11. *If X is a self distributive BE-algebra X, then A_a^b is F-energetic for all $a, b \in X$.*

Proof. Let $y \in A_a^b$ for any $a, b, y \in X$. Assume that $\{x*y, x\} \cap A_a^b = \emptyset$ for any $x \in X$. Then $x*y \notin A_a^b$ and $x \notin A_a^b$ and so $a*(b*(x*y)) = 1$ and $a*(b*x) = 1$. Using (BE3) and the self distributivity of X, we have

$$\begin{aligned} 1 =& a*(b*(x*y)) = a*((b*x)*(b*y)) \\ =& (a*(b*x))*(a*(b*y)) = 1*(a*(b*y)) = a*(b*y) \end{aligned}$$

and so $y \notin A_a^b$. This is a contradiction, and therefore $\{a*b, a\} \cap A_a^b \neq \emptyset$. Hence A_a^b is an F-energetic subset of X for all $a, b \in X$. □

Definition 3.12. A fuzzy set f in a BE-algebra X is called an *anti fuzzy subalgebra* of X if $f(x*y) \leq \max\{f(x), f(y)\}$ for all $x, y \in X$. A fuzzy set f in a BE-algebra X is called an *anti fuzzy filter* of X if it satisfies

(AF_1) $(\forall x \in X)(f(1) \leq f(x))$;
(AF_2) $(\forall x, y \in X)(f(y) \leq \max\{f(x*y), f(x)\})$.

Proposition 3.13. *For any anti fuzzy filter of a BE-algebra X, then following are valid.*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow f(y) \leq f(x))$;
(ii) $(\forall x, y, z \in X)(f(x*z) \leq \max\{f(x*(y*z)), f(y)\})$;
(iii) $(\forall a, x \in X)(f((a*x)*x) \leq f(a))$.

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x*y = 1$. It follows from Definition 3.12 that $f(y) \leq \max\{f(x*y), f(x)\} = \max\{f(1), f(x)\} = f(x)$.
(ii) Using (AF_2) and (BE4), we have $f(x*z) \leq \max\{f(y*(x*z)), f(y)\} = \max\{f(x*(y*z)), f(y)\}$ for any $x, y, z \in X$.
(iii) Taking $y := (a*x)*x$ and $x := a$ in (AF_2), we have $f((a*x)*x) \leq \max\{f(a*((a*x)*x)), f(a)\} = \max\{f((a*x)*(a*x)), f(a)\} = \max\{f(1), f(a)\} = f(a)$ for any $a, x \in X$. □

Theorem 3.14. *Any fuzzy set of a BE-algebra X satisfying (AF_1) and Proposition 3.13 (ii) is an anti fuzzy filter of X.*

Proof. Taking $x := 1$ in Proposition 3.13 (ii) and (BE3), we have $f(z) = f(1*z) \leq \max\{f(1*(y*z)), f(y)\} = \max\{f(y*z), f(y)\}$ for all $y, z \in X$. Hence f is an anti fuzzy filter of X. □

Corollary 3.15. *For any fuzzy set f of a BE-algebra X, f is an anti fuzzy filter of X if and only if it satisfies (AF_1) and Proposition 3.13 (ii).*

Theorem 3.16. *Any fuzzy set f of a BE-algebra X is an anti fuzzy filter of X if and only if it satisfies the following conditions:*

(i) $(\forall x, y \in X)(f(y*x) \leq f(x))$;
(ii) $(\forall x, a, b \in X)(f((a*(b*x))*x) \leq \max\{f(a), f(b)\})$.

Proof. Assume that f is an anti fuzzy filter of X. It follows from Definition 3.12 that $f(y*x) \leq \max\{f(x*(y*x)), f(x)\} = \max\{f(1), f(x)\} = f(x)$ for all $x, y \in X$. Using Proposition 3.13, we have $f((a*(b*x))*x) \leq \max\{f((a*(b*x))*(b*x)), f(b)\} \leq \max\{f(a), f(b)\}$ for any $a, b, x \in X$.

Conversely, let f be a fuzzy set satisfying conditions (i) and (ii). Setting $y := x$ in (i), we have $f(x*x) = f(1) \leq f(x)$ for all $x \in X$. Using (ii), we obtain $f(y) = f(1*y) = f((x*y)*(x*y))*y) \leq \max\{f(x*y), f(y)\}$ for all $x, y \in X$. Hence f is an anti fuzzy filter of X. □

Proposition 3.17. *For any fuzzy set of a BE-algebra X, then f is an anti fuzzy filter of X if and only if*

$(*)$ $(\forall x, y, z \in X)(z \leq x*y \Rightarrow f(y) \leq \max\{f(x), f(z)\})$.

Proof. Assume that f is an anti fuzzy filter of X. Let $x, y, z \in X$ be such that $z \leq x * y$. By Proposition 3.13, we have $f(y) \leq \max\{f(x*y), f(x)\} \leq \max\{f(z), f(x)\}$.

Conversely, suppose that f satisfies $(*)$. By (BE2), we have $x \leq x * 1 = 1$. Using $(*)$, we have $f(1) \leq f(x)$ for all $x \in X$. It follows from (BE1) and (BE4) that $x \leq (x*y)*y$ for all $x, y \in X$. Using $(*)$, we have $f(y) \leq \max\{f(x*y), f(x)\}$. Therefore f is an anti fuzzy filter of X. □

4. Permeable values in BE-algebras

Definition 4.1. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0,1]$ is called a *permeable S-value* for f if $U(f;t) \neq \emptyset$ and the following assertion is valid.

(4.1) $(\forall a, b \in X)(f(a*b) \geq t \Rightarrow \max\{f(a), f(b)\} \geq t)$.

Example 4.2. Consider a BE-algebra $X = \{1, a, b, c\}$ as in Example 3.3 (1). Let f be a fuzzy set of X defined by $f(1) = 0.2, f(a) = 0.3$, and $f(b) = f(c) = 0.6$. Take $t \in (0.3, 0.6]$. Then $U(f;t) = \{b, c\}$. It is easy to check that t is a permeable S-value for f.

Theorem 4.3. *Let f be a fuzzy subalgebra of a BE-algebra X. If $t \in [0,1]$ is a permeable S-value for f, then the nonempty upper t-level set $U(f;t)$ is an S-energetic subset of X.*

Proof. Let $a, b \in X$ be such that $a * b \in U(f;t)$. Then $f(a*b) \geq t$ and so $\max\{f(a), f(b)\} \geq t$. Therefore $f(a) \geq t$ or $f(b) \geq t$, i.e., $a \in U(f;t)$ or $b \in U(f;t)$. Hence $\{a, b\} \cap U(f;t) \neq \emptyset$. Thus $U(f;t)$ is an S-energetic subset of X. □

Since $U(f;t) \cup L^*(f;t) = X$ and $U(f;t) \cap L^*(f;t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

Corollary 4.4. *Let f be a fuzzy subalgebra of a BE-algebra X. If $t \in [0,1]$ is a permeable S-value for f, then $L^*(f;t)$ is empty or a subalgebra of X.*

Definition 4.5. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0,1]$ is called an *anti permeable S-value* for f if $L(f;t) \neq \emptyset$ and the following assertion is valid.

(4.2) $(\forall a, b \in X)(f(a*b) \leq t \Rightarrow \min\{f(a), f(b)\} \leq t)$.

Example 4.6. Consider a BE-algebra $X = \{1, a, b, c\}$ as in Example 3.3 (1). Let f be a fuzzy set of X defined by $f(1) = 0.4, f(a) = f(b) = 0.5$, and $f(c) = 0.3$. Take $t \in [0.3, 0.4)$. Then $L(f;t) = \{c\}$. It is easy to check that t is an anti permeable S-value for f.

Theorem 4.7. *Let f be an anti fuzzy subalgebra of a BE-algebra X. For any anti permeable S-value $t \in [0,1]$ for f, we have $L(f;t) \neq \emptyset \Rightarrow L(f;t)$ is an S-energetic subset of X.*

Proof. Let $a, b \in X$ be such that $a * b \in L(f;t)$. Then $f(a*b) \leq t$ and so $\min\{f(a), f(b)\} \leq t$. Thus $f(a) \leq t$ or $f(b) \leq t$, i.e., $a \in L(f;t)$ or $b \in L(f;t)$. Hence $\{a, b\} \cap L(f;t) \neq \emptyset$. Therefore $L(f;t)$ is an S-energetic subset of X. □

Theorem 4.8. *Let f be a fuzzy subalgebra of a BE-algebra X and let $t \in [0,1]$ be such that $L(f;t) \neq \emptyset$. Then t is an anti permeable S-value for f.*

Proof. Let $a, b \in X$ be such that $f(a{*}b) \leq t$ for all $t \in [0,1]$. Then $\min\{f(a), f(b)\} \leq f(a{*}b) \leq t$. Therefore t is an anti permeable S-value for f. □

Definition 4.9. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0,1]$ is called a *permeable F-value* for f if $U(f;t) \neq \emptyset$ and the following assertion is valid.

(4.3) $(\forall x, y \in X)(f(y) \geq t \Rightarrow \max\{f(x * y), f(x)\} \geq t)$.

Example 4.10. Consider the BE-algebra $X = \{1, a, b, c, d, 0\}$ as in Example 3.10. Let f be a fuzzy set in X defined by $f(1) = 0.2, f(a) = f(b) = 0.4$, and $f(c) = f(d) = f(0) = 0.7$. If $t \in (0.4, 0.7]$, then $U(f;t) = \{0, c, d\}$ and it is easy to check that t is a permeable F-value for f.

Theorem 4.11. *Let f be a fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is a permeable F-value for f, then the nonempty upper t-level set $U(f;t)$ is an F-energetic subset of X.*

Proof. Assume that $U(f;t) \neq \emptyset$ for $t \in [0,1]$. Let $y \in X$ be such that $y \in U(f;t)$. Then $t \leq f(y)$. It follows from (4.3) that $t \leq \max\{f(x*y), f(x)\}$ for all $x \in X$. Hence $f(x*y) \geq t$ or $f(x) \geq t$, i.e., $x * y \in U(f;t)$ or $x \in U(f;t)$. Hence $\{x*y, x\} \cap U(f;t) \neq \emptyset$. Therefore $U(f;t)$ is an F-energetic subset of X. □

Since $U(f;t) \cup L^*(f;t) = X$ and $U(f;t) \cap L^*(f;t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

Corollary 4.12. *Let f be a fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is a permeable F-value for f, then $L^*(f;t)$ is empty or a filter of X.*

Theorem 4.13. *For a fuzzy set f in a BE-algebra X, if there exists a subset K of $[0,1]$ such that $\{U(f;t), L^*(f;t)\}$ is a partition of X and $L^*(f;t)$ is a filter of X for all $t \in K$, then t is a permeable F-value for f.*

Proof. Assume that $f(y) \geq t$ for any $y \in X$. Then $y \in U(f;t)$ and so $\{x*y, x\} \cap U(f;t) \neq \emptyset$ for any $x \in X$, since $U(f;t)$ is an F-energetic subset of X. Hence $x*y \in U(f;t)$ or $x \in U(f;t)$ and so $\max\{f(x*y), f(x)\} \geq t$. Therefore t is a permeable F-value for f. □

Theorem 4.14. *Let f be a fuzzy set in a BE-algebra X with $U(f;t) \neq \emptyset$ for $t \in [0,1]$. If f is an anti fuzzy filter of X, then t is a permeable F-value for f.*

Proof. Let $y \in X$ be such that $f(y) \geq t$. Then $t \leq f(y) \leq \max\{f(x*y), f(x)\}$ for all $x \in X$. Hence t is a permeable F-value for f. □

Theorem 4.15. *Let f be an anti fuzzy filter of a BE-algebra X. Then the following assertion is valid.*

$$(\forall t \in [0,1])(U(f;t) \neq \emptyset \Rightarrow U(f;t) \text{ is an } F\text{-energetic subset of } X).$$

Proof. Let $y \in X$ be such that $y \in U(f;t)$. Then $f(y) \geq t$. By (AF_2), we have $t \leq f(y) \leq \max\{f(x*y), f(x)\}$ for all $x \in X$. Hence $f(x*y) \geq t$ or $f(x) \geq t$, i.e., $x*y \in U(f;t)$ or $x \in U(f;t)$. Therefore $\{x*y, x\} \cap U(f;t) \neq \emptyset$. Thus $U(f;t)$ is an F-energetic subset of X. □

Definition 4.16. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0,1]$ is called an *anti permeable F-value* for f if $L(f;t) \neq \emptyset$ and the following assertion is valid.

(4.4) $(\forall x, y \in X)(f(y) \leq t \Rightarrow \min\{f(x*y), f(x)\} \leq t)$.

Theorem 4.17. *Let f be a fuzzy set in a BE-algebra X with $L(f;t) \neq \emptyset$ for $t \in [0,1]$. If f is a fuzzy filter of X, then t is an anti permeable F-value for f.*

Proof. Let $y \in X$ be such that $f(y) \leq t$. Then $\min\{f(x*y), f(x)\} \leq f(y) \leq t$ for all $x \in X$. Hence t is an anti permeable F-value for f. □

Theorem 4.18. *Let f be an anti fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is an anti permeable F-value for f, then the lower t-level set $L(f;t)$ is an F-energetic subset of X.*

Proof. Let $y \in X$ be such that $y \in L(f;t)$. Then $f(y) \leq t$. It follows from (4.4) that $\min\{f(x*y), f(x)\} \leq t$ for all $x \in X$. Hence $x*y \in L(f;t)$ or $x \in L(f;t)$ and so $\{x*y, x\} \cap L(f;t) \neq \emptyset$. Therefore $L(f;t)$ is an F-energetic subset of X. □

Corollary 4.19. *Let f be an anti fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is an anti permeable F-value for f, then $U^*(f;t)$ is empty or a filter of X.*

Theorem 4.20. *For a fuzzy set f in a BE-algebra X, if there exists a subset K of $[0,1]$ such that $\{U^*(f;t), L(f;t)\}$ is a partition of X and $U^*(f;t)$ is a filter of X for all $t \in K$, then t is an anti permeable F-value for f.*

Proof. Assume that $f(y) \leq t$ for any $y \in X$. Then $y \in L(f;t)$ and so $\{x*y, x\} \cap L(f;t)\} \neq \emptyset$ for all $x \in X$, since $L(f;t)$ is an F-energetic subset of X. Hence $f(x*y) \leq t$ or $f(x) \leq t$ and so $\min\{f(x*y), f(x)\} \leq t$. Therefore t is anti permeable F-value for f. □

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Rapid gradient penalty schemes and convergence for solving constrained convex optimization problem in Hilbert spaces

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Abstract

The purposes of this paper are to establish and study the convergence of a new gradient scheme with penalization terms called rapid gradient penalty algorithm (**RGPA**) for minimizing a convex differentiable function over the set of minimizers of a convex differentiable constrained function. Under the observation of some appropriate choices for the available properties of the considered functions and scalars, we can generate a suitable algorithm that weakly converges to a minimal solution of the considered constraint minimization problem. Further, we also provide a numerical example to compare the rapid gradient penalty algorithm (**RGPA**) and the algorithm introduced by Peypouquet [20].

Keywords: Rapid gradient penalty algorithm, penalization, constraint minimization, fenchel conjugate

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with the norm and inner product given by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $f:\mathcal{H}\to\mathbb{R}$ and $g:\mathcal{H}\to\mathbb{R}$ be convex and (Fréchet) differentiable functions on the space $\mathcal{H}$ and the gradients ∇f and ∇g are Lipschitz continuous operators with constants L_f and L_g, respectively. We consider the following constrained convex optimization problem

$$\min_{x\in\arg\min g} f(x). \tag{1.1}$$

Throughout the paper, we also assume that the solution set $\mathcal{S}:=\arg\min\{f(x):x\in\arg\min g\}$ is a nonempty set. Further, without loss of generality, we may assume that $\min g=0$.

Due to the interesting applications of (1.1) in many branches of mathematics and sciences, many researchers have paid attention to solve the problem (1.1) which can be mentioned briefly as follows: In 2010, Attouch and Czarnecki [1] initially presented and studied a numerical algorithm called the *multiscale asymptotic gradient* (**MAG**) for solving general constrained convex optimization problem. They proved that every sequence generated by (**MAG**) converges weakly to a solution of their

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considered problem. It seems that their representation is the starting point for the development of numerical algorithms in the context of solving type of constrained convex optimization problem (see, for instance [2–4, 7–10, 18]) and the references therein. Inspired by Attouch and Czarnecki [1], in 2012 Peypouquet [20] proposed and analyzed an algorithm called *diagonal gradient scheme* (**DGS**) via gradient method and exterior penalization scheme for constrained minimization of convex functions. He also provided a weak convergence to find a solution of the considered constrained minimization of convex functions. Several applications are provided such as relaxed feasibility, mathematical programming with convex inequality constraints, and Stokes equation and signal reconstruction, etc. In 2013, Shehu et al. [21] studied the problem (1.1) in the case when the constrained set is simple enough and also proposed an algorithm for solving (1.1). In the last two decades, intensive research efforts dedicated to algorithms of inertial type and their convergence behavior can be noticed (see [6, 11, 13–17, 19]). In 2017, Bot et al. [9] considered the problem of minimizing a smooth convex objective function subject to the set of minima of another differentiable convex function. They proposed a new algorithm called *gradient-type penalty with inertial effects method* (**GPIM**) for solving the problem (1.1). They also illustrated the usability of their method via a numerical experiment for image classification via support vector machines.

In the remaining part of this section, we recall some elements of convex analysis. For a function $h : \mathcal{H} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ we denote by $\operatorname{dom} h = \{x \in \mathcal{H} : h(x) < +\infty\}$ its effective domain and say that h is proper, if $\operatorname{dom} h \neq \emptyset$ and $h(x) \neq -\infty$ for all $x \in \mathcal{H}$. The *Fenchel conjugate* of h is $h^* : \mathcal{H} \to \overline{\mathbb{R}}$, which is defined by

$$h^*(z) = \sup_{x \in \mathcal{H}} \{\langle z, x\rangle - h(x)\} \text{ for all } z \in \mathcal{H}.$$

The subdifferential of h at $x \in \mathcal{H}$, with $h(x) \in \mathbb{R}$, is the set

$$\partial h(x) := \{v \in \mathcal{H} : h(y) - h(x) \geq \langle v, y - x\rangle \ \forall y \in \mathcal{H}\}.$$

We take by convention $\partial h(x) := \emptyset$, if $h(x) \in \{\pm\infty\}$.

The convex and differentiable function $T : \mathcal{H} \to \mathbb{R}$ has a Lipschitz continuous gradient with Lipschitz constant $L_T > 0$, if $\|\nabla T(x) - \nabla T(y)\| \leq L_T\|x - y\|$ for all $x, y \in \mathcal{H}$.

Let $\mathcal{C} \subset \mathcal{H}$ be a nonempty closed convex set. The *indicator function* is defined as:

$$\delta_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

The *support function* of $\mathcal{C}$ is defined as: $\sigma_{\mathcal{C}}(x) := \sup_{c \in \mathcal{C}}\langle x, c\rangle$ for all $x \in \mathcal{H}$. The *normal cone* $\mathcal{C}$ at a point x is

$$N_{\mathcal{C}}(x) := \begin{cases} \{\overline{x} \in \mathcal{H} : \langle \overline{x}, c - x\rangle \leq 0 \text{ for all } c \in \mathcal{C}\}, & \text{if } x \in \mathcal{C} \\ \emptyset, & \text{otherwise.} \end{cases}$$

We denote by $\mathbf{Ran}(N_{\mathcal{C}})$ for the range of $N_{\mathcal{C}}$. Notice that $\delta^*_{\mathcal{C}} = \sigma_{\mathcal{C}}$. Moreover, it holds that $\overline{x} \in N_{\mathcal{C}}(x)$ if and only if $\sigma_{\mathcal{C}}(\overline{x}) = \langle \overline{x}, x\rangle$.

Inspired by the research works in this direction, we are interested in the development and improvement of the method for finding solutions of the considered problem, that is, we wish to establish the algorithm called rapid gradient penalty algorithm (**RGPA**) for solving (1.1) which is generated by a controlling sequence of scalars together with the gradient of objective and feasibility gap functions as follows:

$$(\mathbf{RGPA}) \quad \begin{cases} x_1 \in \mathcal{H}; \\ y_n = x_n - \lambda_n \nabla f(x_n) - \lambda_n \beta_n \nabla g(x_n); \\ x_{n+1} = y_n + \alpha_n(y_n - x_n) \qquad \text{for all } n \geq 1, \end{cases}$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ are sequences of positive parameters and $\{\alpha_n\} \subseteq (0,1)$.

For $n \geq 1$, we write $\Omega_n := f + \beta_n g$, which is also (Fréchet) differentiable function. Therefore, $\nabla\Omega_n$ is Lipschitz continuous with constant $L_n := L_f + \beta_n L_g$. In particular, if we setting $\alpha_n = 0$ for all $n \geq 1$, the algorithm (**RGPA**) can be reduced to (**DGS**) in Peypouquet [20].

In order to support our ideas, we also provide a numerical example to simulate an event for solving problem (1.1). We also compare the time and the iteration between two algorithms including (**RGPA**) and (**DGS**).

2. The Hypotheses

In this section, we will carry out the main assumptions to prove the convergence results for rapid gradient penalty algorithm (**RGPA**). In order to prove the convergence results, the following assumptions will be proposed.

Assumption A

(I) The function f is bounded from below;

(II) There exists a positive $K > 0$ such that $\beta_{n+1} - \beta_n \leq K\lambda_{n+1}\beta_{n+1}$, $\frac{L_n}{2} - \frac{1}{2\lambda_n} \leq -K$ and $\frac{\alpha_n^2 - 1}{2\lambda_n} + (1+\alpha_n)^2 K < 0$ for all $n \geq 1$;

(III) $\{\alpha_n\} \in l^2 \setminus l^1$, $\sum_{n=1}^{\infty} \lambda_n = +\infty$ and $\liminf_{n\to\infty} \lambda_n\beta_n > 0$;

(IV) For each $p \in \mathbf{Ran}(N_{\arg\min g})$, we have $\sum_{n=1}^{\infty} \lambda_n\beta_n \left[g^*\left(\frac{p}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{p}{\beta_n}\right)\right] < +\infty$.

Remark 2.1. The conditions in **Assumption A** sparsely extend the hypotheses in [20]. The differences are given by the second and third inequality in (II), which here involves a sequence $\{\alpha_n\}$ which controls the inertial terms, and by $\{\alpha_n\} \in l^2 \setminus l^1$.

In the following remark, we present some situations where **Assumption A** is verified.

Remark 2.2. Let $K > 0$, $q \in (0,1)$, $\delta > 0$ and $\gamma \in (0, \frac{1}{3L_g})$ be any given. Then we set $\alpha_n := \frac{1}{n+1}$ for all $n \geq 1$, which implies that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$ and $\alpha_n \leq \frac{1}{2}$ for all $n \geq 1$. We also set

$$\beta_n := \frac{3\gamma[L_f + 2(K+\delta)]}{1 - 3\gamma L_g} + \gamma K n^q \quad \text{and} \quad \lambda_n := \frac{\gamma}{\beta_n} \quad \text{for all } n \geq 1.$$

Since $\beta_n \geq \frac{3\gamma[L_f+2(K+\delta)]}{1-3\gamma L_g}$, we have for each $n \geq 1$

$$\beta_n(1 - 3\gamma L_g) \geq 3\gamma[L_f + 2(K+\delta)].$$

It follows that

$$\frac{1}{3\lambda_n} - \beta_n L_g \geq L_f + 2(K_+\delta) \quad \text{for all } n \geq 1,$$

which implies that

$$-(K+\delta) \geq \frac{L_n}{2} - \frac{1}{6\lambda_n} \quad \text{for all } n \geq 1. \tag{2.1}$$

According to (2.1), we obtain that

$$-K \geq \frac{L_n}{2} - \frac{1}{2\lambda_n} \quad \text{and} \quad \frac{1}{3} > 2\lambda_n K \quad \text{for all } n \geq 1.$$

Let us consider, for each $n \geq 1$

$$\frac{\alpha_n^2 - 1}{2\lambda_n} + (1+\alpha_n)^2 K \leq \frac{-\frac{3}{4} + \frac{9}{4} 2\lambda_n K}{2\lambda_n} < \frac{-\frac{3}{4} + \frac{3}{4}}{2\lambda_n} = 0.$$

On the other hand,

$$\beta_{n+1} - \beta_n = \gamma K[(n+1)^q - n^q] \leq \gamma K = K\lambda_{n+1}\beta_{n+1}.$$

Hence, we can conclude that **Assumption A** (II) holds.

Since $q \in (0,1)$, we obtain that $\sum_{n=1}^{\infty} \frac{1}{\beta_n} = +\infty$, so $\sum_{n=1}^{\infty} \lambda_n = +\infty$. Notice that $\lambda_n \beta_n = \gamma$ for all $n \geq 1$. It follows that $\liminf_{n\to\infty} \lambda_n \beta_n = \liminf_{n\to\infty} \gamma > 0$. Thus **Assumption A** (III) holds.

Finally, since $g^* - \sigma_{\arg\min g} \geq 0$. If $g(x) \geq \frac{k}{2}\operatorname{dist}^2(x, \arg\min g)$ where $k > 0$, then $g^*(x) - \sigma_{\arg\min g}(x) \leq \frac{1}{2k}\|x\|^2$ for all $x \in \mathcal{H}$.

Therefore, for each $p \in \mathbf{Ran}(N_{\arg\min g})$, we obtain that

$$\lambda_n \beta_n \left[g^*\left(\frac{p}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{p}{\beta_n}\right)\right] \leq \frac{\lambda_n}{2k\beta_n}\|p\|^2.$$

Thus, $\sum_{n=1}^{\infty} \lambda_n \beta_n \left[g^*\left(\frac{p}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{p}{\beta_n}\right)\right]$ converges, if $\sum_{n=1}^{\infty} \frac{\lambda_n}{\beta_n}$ converges, which is equivalently to $\sum_{n=1}^{\infty} \frac{1}{\beta_n^2}$ converges. This holds for the above choices of $\{\beta_n\}$ and $\{\lambda_n\}$ when $q \in (\frac{1}{2}, 1)$.

3. Convergence analysis for convexity

In this section, we will prove the convergence of the sequence of $\{x_n\}$ generated by (**RGPA**) and of the sequence of objective values $\{f(x_n)\}$.

We start the convergence analysis of this section with three technical lemmas.

Lemma 3.1. *Let x^* be an arbitrary element in $\mathcal{S}$ and set $p^* := -\nabla f(x^*)$. Then for each $n \geq 1$*

$$\begin{aligned} \|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + (1+\alpha_n)\lambda_n\beta_n g(x_n) &\leq (1+\alpha_n)^2\|x_n - y_n\|^2 \\ &+ (1+\alpha_n)\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right]. \end{aligned} \tag{3.1}$$

Proof. Applying to the first-order optimality condition, we have

$$0 \in \nabla f(x^*) + N_{\arg\min g}(x^*).$$

It follows that

$$p^* = -\nabla f(x^*) \in N_{\arg\min g}(x^*).$$

Note that for each $n \geq 1$, $\frac{x_n - y_n}{\lambda_n} - \beta_n \nabla g(x_n) = \nabla f(x_n)$.

By monotonicity of ∇f, we obtain that

$$\begin{aligned} \left\langle \frac{x_n - y_n}{\lambda_n} - \beta_n \nabla g(x_n) + p^*, x_n - x^* \right\rangle &= \langle \nabla f(x_n) - \nabla f(x^*), x_n - x^* \rangle \\ &\geq 0 \quad , \forall n \geq 1, \end{aligned}$$

and hence, for each $n \geq 1$

$$2\langle x_n - y_n, x_n - x^*\rangle \geq 2\lambda_n\beta_n \langle \nabla g(x_n), x_n - x^*\rangle - 2\lambda_n \langle p^*, x_n - x^*\rangle. \tag{3.2}$$

Since g is convex and differentiable, we have for each $n \geq 1$

$$\langle \nabla g(x_n), x^* - x_n \rangle + g(x_n) \leq g(x^*) = 0,$$

whence

$$2\lambda_n \beta_n g(x_n) \leq 2\lambda_n \beta_n \langle \nabla g(x_n), x_n - x^* \rangle. \tag{3.3}$$

On the other hand,

$$2\langle x_n - y_n, x_n - x^* \rangle = \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|y_n - x^*\|^2. \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we get that

$$\|y_n - x^*\|^2 \leq \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - 2\lambda_n \beta_n g(x_n) + 2\lambda_n \langle p^*, x_n - x^* \rangle. \tag{3.5}$$

Since $x^* \in \mathcal{S}$ and $p^* \in N_{\arg\min g}(x^*)$, we have

$$\sigma_{\arg\min g}(p^*) = \langle p^*, x^* \rangle.$$

In (3.5), we observe that

$$\begin{aligned}
2\lambda_n \langle p^*, x_n - x^* \rangle - \lambda_n \beta_n g(x_n) &= 2\lambda_n \langle p^*, x_n \rangle - \lambda_n \beta_n g(x_n) - 2\lambda_n \langle p^*, x^* \rangle \\
&= \lambda_n \beta_n \left[\left\langle \frac{2p^*}{\beta_n}, x_n \right\rangle - g(x_n) - \left\langle \frac{2p^*}{\beta_n}, x^* \right\rangle \right] \\
&\leq \lambda_n \beta_n \left[g^* \left(\frac{2p^*}{\beta_n} \right) - \sigma_{\arg\min g} \left(\frac{2p^*}{\beta_n} \right) \right].
\end{aligned} \tag{3.6}$$

Combining (3.6) and (3.5), we obtain that

$$\|y_n - x^*\|^2 \leq \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \lambda_n \beta_n g(x_n) + \lambda_n \beta_n \left[g^* \left(\frac{2p^*}{\beta_n} \right) - \sigma_{\arg\min g} \left(\frac{2p^*}{\beta_n} \right) \right]. \tag{3.7}$$

On the other hand, we observe that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|y_n + \alpha_n (y_n - x_n) - x^*\|^2 = \|(1 + \alpha_n)(y_n - x^*) + \alpha_n (x^* - x_n)\|^2 \\
&= (1 + \alpha_n)\|y_n - x^*\|^2 - \alpha_n \|x_n - x^*\|^2 + \alpha_n (1 + \alpha_n)\|x_n - y_n\|^2.
\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we obtain the desired result. □

Lemma 3.2. *For all $n \geq 1$, we have*

$$\begin{aligned}
\Omega_{n+1}(x_{n+1}) \leq \Omega_n(x_n) + (\beta_{n+1} - \beta_n) g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n} \|y_n - x_n\|^2 \\
+ \left[\frac{L_n}{2} - \frac{1}{2\lambda_n} \right] \|x_{n+1} - x_n\|^2.
\end{aligned}$$

Proof. Since $\nabla \Omega$ is L_n-Lipschitz continuous and by Descent Lemma (see [5, Theorem 18.15]), we obtain that

$$\Omega_n(x_{n+1}) \leq \Omega_n(x_n) + \langle \nabla \Omega_n(x_n), x_{n+1} - x_n \rangle + \frac{L_n}{2} \|x_{n+1} - x_n\|^2.$$

Recall that $-\frac{y_n - x_n}{\lambda_n} = \nabla \Omega_n(x_n)$.

It follows that

$$\begin{aligned}
&f(x_{n+1}) + \beta_n g(x_{n+1})\\
&\leq f(x_n) + \beta_n g(x_n) - \left\langle \frac{y_n - x_n}{\lambda_n}, x_{n+1} - x_n \right\rangle + \frac{L_n}{2}\|x_{n+1} - x_n\|^2\\
&= f(x_n) + \beta_n g(x_n) - \frac{1}{2\lambda_n}\|y_n - x_n\|^2 - \frac{1}{2\lambda_n}\|x_{n+1} - x_n\|^2 + \frac{1}{2\lambda_n}\|y_n - x_{n+1}\|^2 + \frac{L_n}{2}\|x_{n+1} - x_n\|^2\\
&= f(x_n) + \beta_n g(x_n) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2 + \left[\frac{L_n}{2} - \frac{1}{2\lambda_n}\right]\|x_{n+1} - x_n\|^2.
\end{aligned}$$

Adding $\beta_{n+1}g(x_{n+1})$ to both sides, we have

$$\begin{aligned}
f(x_{n+1}) + \beta_{n+1}g(x_{n+1}) \leq\ & f(x_n) + \beta_n g(x_n) + (\beta_{n+1} - \beta_n)g(x_{n+1})\\
&+ \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2 + \left[\frac{L_n}{2} - \frac{1}{2\lambda_n}\right]\|x_{n+1} - x_n\|^2,
\end{aligned}$$

which means that

$$\Omega_{n+1}(x_{n+1}) \leq \Omega_n(x_n) + (\beta_{n+1} - \beta_n)g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2 + \left[\frac{L_n}{2} - \frac{1}{2\lambda_n}\right]\|x_{n+1} - x_n\|^2.$$

□

For $n \geq 1$ and $x^* \in \mathcal{S}$, we denote by

$$\begin{aligned}
\Lambda_n &:= f(x_n) + (1 - (1 + \alpha_n)K\lambda_n)\beta_n g(x_n) + K\|x_n - x^*\|^2\\
&= \Omega_n(x_n) - (1 + \alpha_n)K\lambda_n\beta_n g(x_n) + K\|x_n - x^*\|^2.
\end{aligned}$$

Lemma 3.3. *Let $x^* \in \mathcal{S}$ and set $p^* := -\nabla f(x^*)$. Then there exists $\theta > 0$ such that for each $n \geq 1$*

$$\Lambda_{n+1} - \Lambda_n + \theta\|y_n - x_n\|^2 \leq (1 + \alpha_n)K\lambda_n\beta_n\left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

Proof. From Lemma 3.2 and **Assumption A** (II), we obtain that

$$\begin{aligned}
\Omega_{n+1}(x_{n+1}) - \Omega_n(x_n) &\leq K\lambda_{n+1}\beta_{n+1}g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2\\
&\leq (1 + \alpha_{n+1})K\lambda_{n+1}\beta_{n+1}g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2. \qquad (3.9)
\end{aligned}$$

On the other hand, multiplying (3.1) by K, we have

$$\begin{aligned}
&K\|x_{n+1} - x^*\|^2 - K\|x_n - x^*\|^2 + (1 + \alpha_n)K\lambda_n\beta_n g(x_n)\\
&\qquad\leq (1 + \alpha_n)^2K\|x_n - y_n\|^2 + (1 + \alpha_n)K\lambda_n\beta_n\left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right]. \qquad (3.10)
\end{aligned}$$

Combining (3.9) and (3.10), we have

$$\Lambda_{n+1} - \Lambda_n \leq \left[\frac{\alpha_n^2 - 1}{2\lambda_n} + (1 + \alpha_n)^2K\right]\|y_n - x_n\|^2 + (1 + \alpha_n)K\lambda_n\beta_n\left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right]. \qquad (3.11)$$

For each $n \geq 1$, $\frac{\alpha_n^2 - 1}{2\lambda_n} + (1 + \alpha_n)^2K < 0$, we have there exists $\theta > 0$ such that

$$\frac{\alpha_n^2 - 1}{2\lambda_n} + (1 + \alpha_n)^2K < -\theta.$$

From (3.11), we have

$$\Lambda_{n+1} - \Lambda_n + \theta\|y_n - x_n\|^2 \leq (1+\alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

This completes the proof. □

The next lemma is an important role in convergence analysis (see in [3, Lemma 2] or [12, Lemma 3.1]).

Lemma 3.4. *Let $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\varepsilon_n\}$ be real sequences. Assume that $\{\gamma_n\}$ is bounded from below, $\{\delta_n\}$ is non-negative and $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ such that*

$$\gamma_{n+1} - \gamma_n + \delta_n \leq \varepsilon_n \quad \textit{for all} \quad n \geq 1.$$

Then $\lim_{n\to\infty} \gamma_n$ exists and $\sum_{n=1}^{\infty} \delta_n < +\infty$.

Lemma 3.5. *Let $x^* \in \mathcal{S}$. Then the following statements hold:*

(i) *The sequence $\{\Lambda_n\}$ is bounded from below and $\lim_{n\to\infty} \Lambda_n$ exists;*

(ii) $\sum_{n=1}^{\infty} \|y_n - x_n\|^2 < +\infty$;

(iii) $\lim_{n\to\infty} \|x_n - x^*\|^2$ *exists and* $\sum_{n=1}^{\infty} \lambda_n\beta_n g(x_n) < +\infty$;

(iv) $\lim_{n\to\infty} \Omega_n(x_n)$ *exists;*

(v) $\lim_{n\to\infty} g(x_n) = 0$ *and every weak cluster point of the sequence $\{x_n\}$ lies in* $\arg\min g$.

Proof. We set $p^* := -\nabla f(x^*)$.

(i). From **Assumption A** (II) implies $1-(1+\alpha_n)K\lambda_n \geq 0$. Since f is convex and differentiable, we have for each $n \geq 1$

$$\begin{aligned}
\Lambda_n &= f(x_n) + (1-(1+\alpha_n)K\lambda_n)\beta_n g(x_n) + K\|x_n - x^*\|^2 \geq f(x_n) + K\|x_n - x^*\|^2 \\
&\geq f(x^*) + \langle \nabla f(x^*), x_n - x^*\rangle + K\|x_n - x^*\|^2 = f(x^*) - \left\langle \frac{p^*}{\sqrt{2K}}, \sqrt{2K}(x_n - x^*)\right\rangle + K\|x_n - x^*\|^2 \\
&\geq f(x^*) - \frac{\|p^*\|^2}{4K} - K\|x_n - x^*\|^2 + K\|x_n - x^*\|^2 = f(x^*) - \frac{\|p^*\|^2}{4K}.
\end{aligned}$$

Therefore, $\{\Lambda_n\}$ is bounded from below.

Next, we set $\gamma_n = \Lambda_n$, $\delta_n = \theta\|y_n - x_n\|^2$ and

$$\varepsilon_n = (1+\alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

Recall that $\min g = 0$. Thus $g \leq \delta_{\arg\min g}$. Therefore $\sigma_{\arg\min g} = (\delta_{\arg\min g})^* \leq g^*$ and hence, $g^* - \sigma_{\arg\min g} \geq 0$. It follows that

$$\varepsilon_n = (1+\alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right] \leq 2K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

By using **Assumption A** (IV) and $p^* \in N_{\arg\min g}(x^*)$, we have $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. Applying Lemma 3.3 and Lemma 3.4, we obtain that $\lim_{n\to\infty} \Lambda_n$ exists.

(ii). Follows immediately from Lemmas 3.3 and 3.4.
(iii). We set $\gamma_n = \|x_n - x^*\|^2$, $\delta_n = (1+\alpha_n)\lambda_n\beta_n g(x_n)$ and

$$\varepsilon_n = (1+\alpha_n)^2\|y_n - x_n\|^2 + (1+\alpha_n)\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

From statement (ii), Lemma 3.4 and Lemma 3.1, we get that

$$\lim_{n\to\infty}\|x_n - x^*\| \text{ exists and } \sum_{n=1}^{\infty}\lambda_n\beta_n g(x_n) < +\infty.$$

For (iv) since for each $n \geq 1$ $\Omega_n(x_n) = \Lambda_n + (1+\alpha_n)K\lambda_n\beta_n g(x_n) - K\|x_n - x^*\|^2$, by using (i), (iii) and $\lim_{n\to\infty}\alpha_n = 0$, we have $\lim_{n\to\infty}\Omega_n(x_n)$ exists.

(v). It follows from **Assumption A** (III) that $\liminf_{n\to\infty}\lambda_n\beta_n > 0$. According to statement (iii) implies $\lim_{n\to\infty} g(x_n) = 0$. Let $\overline{x}$ be any weak cluster point of the sequence $\{x_n\}$. Therefore, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to $\overline{x}$ as $k \to \infty$. By the weak lower semicontinuity of g, we get that

$$g(\overline{x}) \leq \liminf_{k\to\infty} g(x_{n_k}) \leq \lim_{n\to\infty} g(x_n) = 0,$$

which means that $\overline{x} \in \arg\min g$. This completes the proof. □

Lemma 3.6. *Let $x^* \in \mathcal{S}$. Then*

$$\sum_{n=1}^{\infty}\lambda_n\left[\Omega_n(x_n) - f(x^*)\right] < +\infty.$$

Proof. Since f is differentiable and convex function, we obtain that for each $n \geq 1$

$$f(x^*) \geq f(x_n) + \langle \nabla f(x_n), x^* - x_n\rangle.$$

Since g is differentiable, convex function and $\min g = 0$, we obtain that for each $n \geq 1$

$$0 = g(x^*) \geq g(x_n) + \langle \nabla g(x_n), x^* - x_n\rangle,$$

which implies that

$$0 \geq \beta_n g(x_n) + \langle \beta_n\nabla g(x_n), x^* - x_n\rangle, \quad \text{for all } n \geq 1.$$

Therefore, we can conclude that

$$f(x^*) \geq \Omega_n(x_n) + \langle \nabla\Omega_n(x_n), x^* - x_n\rangle = \Omega_n(x_n) + \left\langle \frac{x_n - y_n}{\lambda_n}, x^* - x_n\right\rangle. \tag{3.12}$$

From (3.12), we obtain that

$$2\lambda_n\left[\Omega_n(x_n) - f(x^*)\right] \leq 2\langle x_n - y_n, x_n - x^*\rangle = \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|y_n - x^*\|^2. \tag{3.13}$$

On the other hand, for each $n \geq 1$

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|x_{n+1} - \alpha_n(y_n - x_n) - x^*\|^2 = \|x_{n+1} - x^*\|^2 + \alpha_n^2\|y_n - x_n\|^2 - 2\langle \alpha_n(x_{n+1} - x^*), y_n - x_n\rangle \\
&= \|x_{n+1} - x^*\|^2 + \alpha_n^2\|y_n - x_n\|^2 - \alpha_n^2\|x_{n+1} - x^*\|^2 - \|y_n - x_n\|^2 + \|\alpha_n(x_{n+1} - x^*) - (y_n - x_n)\|^2 \\
&\geq \|x_{n+1} - x^*\|^2 + \alpha_n^2\|y_n - x_n\|^2 - \alpha_n^2\|x_{n+1} - x^*\|^2 - \|y_n - x_n\|^2,
\end{aligned}$$

which implies that

$$-\|y_n - x^*\|^2 \leq -\|x_{n+1} - x^*\|^2 - \alpha_n^2\|y_n - x_n\|^2 + \alpha_n^2\|x_{n+1} - x^*\|^2 + \|y_n - x_n\|^2. \tag{3.14}$$

Combining (3.13) and (3.14), we have for all $n \geq 1$

$$\begin{aligned} 2\lambda_n\left[\Omega_n(x_n) - f(x^*)\right] &\leq (2-\alpha_n^2)\|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n^2\|x_{n+1} - x^*\|^2 \\ &\leq 2\|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n^2\|x_{n+1} - x^*\|^2. \end{aligned}$$

Therefore, according to Lemma 3.5 (iii), we get that the sequence $\{\|x_n - x^*\|\}$ is bounded, which means that there exists $M > 0$ such that $\|x_n - x^*\| \leq M$ for all $n \geq 1$.

By **Assumption A** (III) and Lemma 3.5, we obtain that

$$2\sum_{n=1}^{\infty}\lambda_n[\Omega_n(x_n) - f(x^*)] \leq 2\sum_{n=1}^{\infty}\|y_n - x_n\|^2 + \|x_1 - x^*\|^2 + M^2\sum_{n=1}^{\infty}\alpha_n^2 < +\infty.$$

□

The following proposition will play an important role in convergence analysis, which is the main result of this paper.

Proposition 3.7 ([5, Opial Lemma]). *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{C} \subseteq \mathcal{H}$ be nonempty set and $\{x_n\}$ be any given sequence such that:*

(i) *For every $z \in \mathcal{C}$, $\lim_{n\to\infty}\|x_n - z\|$ exists;*

(ii) *Every weak cluster point of the sequence $\{x_n\}$ lies in $\mathcal{C}$.*

Then the sequence $\{x_n\}$ converges weakly to a point in $\mathcal{C}$.

Let $\{x_n\}$ be define by (**RGPA**). Then $\{x_n\}$ converges weakly to a point in $\mathcal{S}$. Moreover, the sequence $\{f(x_n)\}$ converges to the optimal objective value of the optimization problem (1.1).

Proof. From Lemma 3.5 (iii), $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in \mathcal{S}$. Let $\overline{x}$ be any weak cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\overline{x}$ as $k \to \infty$. According to Lemma 3.5 (v) implies $\overline{x} \in \arg\min g$. It suffices to show that $f(\overline{x}) \leq f(x)$ for all $x \in \arg\min g$. Since $\sum_{n=1}^{\infty}\lambda_n = +\infty$, and by Lemma 3.6 and Lemma 3.5 (iv), we have

$$\lim_{n\to\infty}\Omega_n(x_n) - f(x^*) \leq 0 \quad \text{for all } x^* \in \mathcal{S}.$$

Therefore, $f(\overline{x}) \leq \liminf_{k\to\infty} f(x_{n_k}) \leq \lim_{n\to\infty}\Omega_n(x_n) \leq f(x^*)$, $\forall x^* \in \mathcal{S}$, which implies that $\overline{x} \in \mathcal{S}$. Applying Opial Lemma, we obtain that $\{x_n\}$ converges weakly to a point in $\mathcal{S}$. The last statement follows immediately from the above. □

4. Numerical experiments

In this section, we present the convergence of the algorithm proposed (**RGPA**) in this paper by the minimization problem with linear equality constraints. Firstly, we are given a linear system of the form

$$\mathbf{A}x = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. In addition, we assume that $n > m$. In this section, the system has many solutions. This leads us to the question of which solution should be considered. As a result, we may consider the following problem, say, the minimization problem with linear equality constraints.

$$\begin{aligned} &\text{minimize } \frac{1}{2}\|x\|^2 \\ &\text{subject to } \mathbf{A}x = \mathbf{b}, \end{aligned}$$

Table 1: Comparison of the convergence of (**RGPA**) and (**DGS**) for the parameters $K = 0.001$ and $q \in (\frac{1}{2}, 1)$.

q	(RGPA)		(DGS)	
	Time (sec)	#(Iters)	Time (sec)	#(Iters)
0.6	2.38	566	10.23	2221
0.7	2.31	568	107.78	25336
0.8	2.46	581	384.00	90636
0.9	44.96	11458	447.11	103487

or , equivalently,

$$\begin{aligned} &\text{minimize } \frac{1}{2}\|x\|^2 \\ &\text{subject to } x \in \arg\min \frac{1}{2}\|\mathbf{A}(\cdot) - \mathbf{b}\|^2. \end{aligned}$$

The above problem can be written in the form of the problem (1.1) as

$$\begin{aligned} &\text{minimize } f(x) := \frac{1}{2}\|x\|^2 \\ &\text{subject to } g(x) := \frac{1}{2}\|\mathbf{A}(x) - \mathbf{b}\|^2. \end{aligned}$$

In this setting, we have $\nabla f(x) = x$ and notice that ∇f is 1-Lipschitz continuous.

Furthermore, we get that $\nabla g(x) = \mathbf{A}^\top(\mathbf{A}x - \mathbf{b})$ and notice that ∇g is $\|\mathbf{A}\|^2$-Lipschitz continuous.

All the numerical experiments were performed under MATLAB (R2015b). We begin with the problem by random matrix $\mathbf{A}$ in $\mathbb{R}^{m\times n}$, vector $x_1 \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$ with $m = 1000$ and $n = 4000$ generated by using MATLAB command **randi**, which the entries of $\mathbf{A}$, x_1 and $\mathbf{b}$ are integer in [-10,10]. Next, we are going to compare a performance of the algorithms (**RGPA**) and (**DGS**). The choice of the parameters for the computational experiment is based on Remark 2.2. We chooses $\gamma = \frac{1}{4\|\mathbf{A}\|^2}$ and $\delta = 1$. We consider different choices of the parameters $K \in (0, 1]$ and $q \in (\frac{1}{2}, 1)$. We will terminate the algorithms (**RGPA**) and (**DGS**) when the errors become small, i.e.,

$$\|x_n - x^*\| \leq 10^{-6},$$

where $x^* = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{b}$.

In Table 1 we present a comparison of the convergence between two algorithms including (**RGPA**) and (**DGS**) for the parameters $K = 0.001$ and different choices for the parameters $q \in (\frac{1}{2}, 1)$. We observe that when $q = 0.6$ leads to lowest computation time for (**RGPA**) and (**DGS**) with 2.38 second and 10.23 second, respectively. Furthermore, we also observe that (**DGS**) hit a big number of iterations than (**RGPA**) for all choices of parameter q.

In Table 2 we present a comparison of the convergence of (**RGPA**) and (**DGS**) for the parameters $q = 0.6$ and $K \in (0, 1]$. We observe that the number of iterations and computation time for (**RGPA**) smaller than the number of iterations for (**DGS**) for each choice of parameters K. Furthermore, (**RGPA**) needs tiny computation time to reach the optimality tolerance than (**DGS**) for each choice of parameter K.

We observe that our algorithm (**RGPA**) performs an advantage behavior when comparing with algorithm (**DGS**) for all different choices of parameters. Note that the number of iterations for (**RGPA**) smaller than the number of iterations for (**DGS**). Furthermore, (**RGPA**) needs tiny computation time to reach optimality tolerance than (**DGS**) for each different choice of parameters.

5. Conclusions

We have presented a new gradient penalty scheme, say, *rapid gradient penalty algorithm* **(RGPA)**. We provide sufficient conditions to guarantee the convergences of **(RGPA)** for the considered con-

Table 2: Comparison of the convergence of (**RGPA**) and (**DGS**) for the parameters $q = 0.6$ and $K \in (0, 1]$.

K	(**RGPA**)		(**DGS**)	
	Time (sec)	#(Iters)	Time (sec)	#(Iters)
0.001	2.38	566	10.23	2221
0.005	2.40	585	171.46	40888
0.01	6.63	1612	254.93	64469
0.05	83.22	20480	288.39	65722
0.1	107.41	26257	212.02	52464
0.5	79.95	18606	100.33	24419
1	51.46	13414	67.20	16616

strained convex optimization problem (1.1). We also provide a numerical example to compare the performance of the algorithms (**RGPA**) and (**DGS**). As a result, we observe that our algorithm (**RGPA**) performs an advantage behavior when comparing with (**DGS**) for all different choices of parameters.

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Disclosure statement

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Approximation by modified Lupaş operators based on (p, q)–integers

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Abstract

The purpose of this paper is to introduce a new modification of Lupaş operators in the frame of post quantum setting and to investigate their approximation properties. First using the relations between q-calculus and post quantum calculus, the post quantum analogue of operators constructed will be linear and positive but will not follow Korovkin's theorem. Hence a new modification of q-Lupaş operators is constructed which will preserve test functions. Based on these modification of operators, approximation properties have been investigated. Further, the rate of convergence of operators by mean of modulus of continuity and functions belonging to the Lipschitz class as well as Peetre's K-functional are studied.

Keywords and phrases: Lupaş operators; Post quantum analogue; q analogue; Peetre's K-functional; Korovkin's type theorem; Convergence theorems.
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1. Introduction and preliminaries

A. Lupaş [17] introduced the linear positive operators at the International Dortmund Meeting held in Witten (Germany, March, 1995) as follows:

$$L_m(f;u) = (1-a)^{-mu} \sum_{l=0}^{\infty} \frac{(mu)_l a^l}{l!} f\left(\frac{l}{m}\right),\ u \geq 0, \tag{1.1}$$

with $f : [0, \infty) \to \mathbb{R}$. If we impose $L_m(u) = u$, we get $a = \frac{1}{2}$. Thus operators (1.1) becomes

$$L_m(f;u) = 2^{-mu} \sum_{l=0}^{\infty} \frac{(mu)_l}{l!2^l} f\left(\frac{l}{m}\right),\ u \geq 0, \tag{1.2}$$

where $(mu)_l$ is the rising factorial defined as:

$$(mu)_0 = 1,\ (mu)_l = mu(mu+1)(mu+2)\cdots(mu+l-1),\ l \geq 0.$$

The q-analogue of Lupaş operators (1.2) is defined in [26] as:

$$L_m^{p,q}(f;u) = 2^{-[m]_q u} \sum_{l=0}^{\infty} \frac{([m]_q u)_l}{[l]!2^l} f\left(\frac{[l]_q}{[m]_q}\right),\ u \geq 0. \tag{1.3}$$

1

2. Construction of new operators and auxiliary results

Let us recall certain notations and definitions of (p,q)-calculus. Let $p>0,\ q>0$. For each non negative integer $l,\, m,\, m\geq l\geq 0$, the (p,q)-integer, (p,q)-binomial are defined, as

$$[j]_{p,q}=p^{j-1}+p^{j-2}q+p^{j-3}q^2+...+pq^{j-2}+q^{j-1}=\begin{cases}\frac{p^j-q^j}{p-q}, & \text{when } p\neq q\neq 1,\\ j\ p^{j-1}, & \text{when } p=q\neq 1,\\ [j]_q, & \text{when } p=1,\\ j, & \text{when } p=q=1.\end{cases}$$

where $[j]_q$ denotes the q-integers and $m=0,1,2,\cdots$.

The formula for (p,q)-binomial expansion is as follows:

$$(au+bv)^m_{p,q}:=\sum_{l=0}^{m}p^{\frac{(m-l)(m-l-1)}{2}}q^{\frac{l(l-1)}{2}}\left[\begin{array}{c}m\\ l\end{array}\right]_{p,q}a^{m-l}b^lu^{m-l}v^l,$$

$$(u+v)^m_{p,q}=(u+v)(pu+qv)(p^2u+q^2v)\cdots(p^{m-1}u+q^{m-1}v),$$

$$(1-u)^m_{p,q}=(1-u)(p-qu)(p^2-q^2u)\cdots(p^{m-1}-q^{m-1}u),$$

where (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c}m\\ l\end{array}\right]_{p,q}=\frac{[m]_{p,q}!}{[l]_{p,q}![m-l]_{p,q}!}.$$

Details on (p,q)-calculus can be found in [9, 11, 21].

In the case of $p=1$, the above notations reduce to q-analogues and one can easily see that $[m]_{p,q}=p^{m-1}[m]_{q/p}$. Mursaleen et al. [21] introduced (p,q)-calculus in approximation theory and constructed post quantum analogue of Bernstein operators. On the other hand Khalid and Lobiyal defined the (p,q)- analogue of Lupaş Bernstein operators in [12] and have shown its application in computer aided geometric design for construction of Beizer curves and surfaces. For another applications of extra parameters p in the field of approximation on compact disk, one can refer [4]. For related literature, one can refer [1, 2, 9, 3, 13, 14, 18, 19, 20, 22, 23, 25, 24] papers based on q and (p,q) integers in approximation theory and CAGD. Motivated by the above mentioned work, we introduce a new analogue of Lupaş operators. The post quantum analogue of (1.3) are as follows:

Definition 2.1. Let $f:[0,\infty)\to\mathbb{R}$, $0<q<p\leq 1$ and for any $m\in\mathbb{N}$. we define the (p,q)-analogue of Lupaş operators as

$$L_m^{p,q}(f;u)=2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}f\left(\frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}}\right),\ u\geq 0. \tag{2.1}$$

The operators (2.1) are linear and positive. For $p=1$, the operators (2.1) turn out to be q-Lupaş operators defined in (1.3). Next, we prove some auxiliary results for (2.1).

Lemma 2.2. *Let $0<q<p\leq 1$ and $m\in\mathbb{N}$. We have*

(i) $L_m^{p,q}(1;u)=1$,

(ii) $L_m^{p,q}(t;u)=\frac{u}{p^{m-1}(2-p)^{[m]_{p,q}u+1}}$,

(iii) $L_m^{p,q}(t^2;u)=\frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}}+\frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}}+\frac{qu}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}$.

Proof. we have

(i)

$$L_m^{p,q}(1;u) = 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} = 1.$$

(ii)

$$\begin{aligned}
L_m^{p,q}(t;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}} \\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}} \\
&= 2^{-[m]_{p,q}u}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}}p^{l+1-m} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{m-1}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l p^l}{[l]_{p,q}!2^l} \\
&= \frac{u}{(p^{m-1})(2-p)^{([m]_{p,q}u+1)}}.
\end{aligned}$$

(iii)

$$\begin{aligned}
L_m^{p,q}(t^2;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{p^{2l-2m}[l]_{p,q}^2}{[m]_{p,q}^2} \\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!a^l}\frac{p^{2l-2m}[l]_{p,q}^2}{[m]_{p,q}^2} \\
&= 2^{-[m]_{p,q}u}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}}\frac{p^{2l+2-2m}[l+1]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{p^{2l}[l+1]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{p^{2l}[p^l+q[l]_{p,q}]}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{p^{3l}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{p^{2l}q[l]_{p,q}}{[m]_{p,q}} \\
&= I_1 + I_2(say),
\end{aligned}$$

□

we find that I_1 and I_2 are

$$\begin{aligned}
I_1 &= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{3l}}{[m]_{p,q}} \\
&= \frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}}. \\
I_2 &= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}q[l]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}([m]_{p,q}u+1)}{p^{2m-2}} qu \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+2)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l} \frac{p^{2l}[l]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}([m]_{p,q}u+1)}{[m]_{p,q}p^{2m-2}} qu \sum_{l=1}^{\infty} \frac{([m]_{p,q}u+2)_{l-1}p^{2l}}{[l-1]_{p,q}!2^l} \\
&= \frac{2^{-[m]_{p,q}u-2}([m]_{p,q}u+1)}{[m]_{p,q}p^{2m-4}} qu \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+2)_l p^{2l}}{[l]_{p,q}!2^l} \\
&= \frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}} + \frac{qu}{p^{2n-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}.
\end{aligned}$$

On adding $I_1$and I_2, we get

$$L_m^{p,q}(t^2;u) = \frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}} + \frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}} + \frac{qu}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}.$$

The sequence of (p,q)-Lupaş operators constructed in (2.1) however do not preserve the test functions t and t^2. Hence one can not guarantee approximation via these operators. Therefore, we construct the modified (p,q)- Lupaş operators as follows:

Lemma 2.3. *Let $0 < q < p \leq 1$ and $m \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$, we define the (p,q)-analogue of Lupaş operators as:*

$$\widetilde{L}_m^{p,q}(f;u) = 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!a^l} f\left(\frac{[l]_{p,q}}{[m]_{p,q}}\right), \ u \geq 0. \tag{2.2}$$

The operators (2.2) are linear and positive. For $p = 1$, the operators (2.2) turn out to be q-Lupaş operator defined in (1.3).
We shall investigate approximation properties of the operators (2.2). We obtain rate of convergence of the operators via modulus of continuities. We also obtain approximation behaviors of the operators for functions belonging to Lipschitz spaces.

Lemma 2.4. *Let $0 < q < p \leq 1$ and $m \in \mathbb{N}$. We have*

(i) $\widetilde{L}_m^{p,q}(1;u) = 1$,
(ii) $\widetilde{L}_m^{p,q}(t;u) = u$,
(iii) $\widetilde{L}_m^{p,q}(t^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2$.

Proof. We have

(i)

$$\widetilde{L}_m^{p,q}(1;u) = 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} = 1.$$

(ii)

$$\begin{aligned}
\widetilde{L}_m^{p,q}(t;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{[l]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{[l]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u-1}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\\
&= u.
\end{aligned}$$

(iii)

$$\begin{aligned}
\widetilde{L}_m^{p,q}(t^2;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{[l]_{p,q}^2}{[m]_{p,q}^2}\\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{[l]_{p,q}^2}{[m]_{p,q}^2}\\
&= 2^{-[m]_{p,q}u}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}}\frac{[l+1]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u-1}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{[p^l+q[l]_{p,q}]}{[m]_{p,q}}\\
&= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}p^l\\
&+ \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l[l]_{p,q}}{[l]_{p,q}!2^l}\\
&= I_1+I_2(Say).
\end{aligned}$$

After solving I_1 and I_2, we get

$$\begin{aligned}
I_1 &= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}p^l\\
&= \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}.\\
I_2 &= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l[l]_{p,q}}{[l]_{p,q}!2^l}\\
&= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=1}^{\infty}\frac{([m]_{p,q}u+1)([m]_{p,q}u+2)_{l-1}[l]_{p,q}}{[l]_{p,q}[l-1]_{p,q}!2^l}\\
&= \frac{2^{-[m]_{p,q}u-2}([m]_{p,q}u+1)qu}{[n]_{p,q}}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+2)_l}{[l]_{p,q}!2^l}\\
&= \frac{qu}{[m]_{p,q}}+qu^2.
\end{aligned}$$

On adding $I_1$and I_2, we get

$$\widetilde{L}_m^{p,q}(t^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}+\frac{qu}{[m]_{p,q}}+qu^2.$$

□

Corollary 2.5. *Using Lemma 2.4, we get the following central moments.*

$\widetilde{L}_n^{p,q}(t-u;u)=0$
$\widetilde{L}_n^{p,q}((t-u)^2;u)=\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}+\frac{qu}{[m]_{p,q}}+qu^2-u^2=\delta_m(u)$ *(say).*

Remark 2.6. One can observe that

$$\lim_{m\to\infty}[m]_{p,q}=\begin{cases}0, & \text{p,q} \in (0,1),\\ \frac{1}{1-q}, & p=1 \text{ and } q\in(0,1).\end{cases}$$

Thus for approximation processes, one need to choose convergent sequences (p_m) and (q_m) such that for each n, $0<q_m<p_m\leq 1$ and $p_m,q_m\to 1$ so that $[m]_{p_m,q_m}\to\infty$ as $m\to\infty$.

Theorem 2.7. *Let $f\in C_B[0,\infty)$ and $q_m\in(0,1)$, $p_m\in(q_m,1]$ such that $q_m\to 1$, $p_m\to 1$, as $m\to\infty$. Then for each $u\in[0,\infty)$ we have*

$$\lim_{n\to\infty}\widetilde{L}_m^{p_m,q_m}(f;u)=f(u).$$

Proof. By Korovkin's theorem it is enough to show that

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(t^m;u)=u^m,\ m=0,1,2.$$

By Lemma 2.4, it is clear that

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(1;u)=1$$

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(1;u)=u$$

and

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(t^2;u)=\lim_{m\to\infty}\left[\frac{u}{(2-p_m)^{([m]_{p_m,q_m}u+1)}[m]_{p_m,q_m}}+\frac{q_mu}{[m]_{p_m,q_m}}+q_mu^2\right]$$

$$=u^2.$$

This completes the proof. ∎

3. Direct results

Let $C_B[0,\infty)$ be the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \|=\sup_{0\leq x<\infty}\mid f(x)\mid .$$

Let us consider the K-functional as:

$$K_2(f,\delta)=\inf_{s\in W^2}\{\| f-s\| +\delta\| s^{''}\|\},$$

where $\delta>0$ and $W^2=\{s\in C_B[0,\infty): s^{'},s^{''}\in C_B[0,\infty)\}$.
Then as in ([4], p. 177, Theorem 2.4), there euists an absolute constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}). \tag{3.1}$$

Second order modulus of smoothness of $f \in C_B[0,\infty)$ is as follows

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{u\in[0,\infty)} \mid f(u+2h) - 2f(u+h) + f(u) \mid .$$

The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{u\in[0,\infty)} \mid f(u+h) - f(u) \mid .$$

Theorem 3.1. *Let $f \in C_B[0,\infty)$, $p,q \in (0,1)$ such that $0 < q < p \leq 1$. Then for every $u \in [0,\infty)$ we have*

$$\mid \widetilde{L}_m^{p,q}(f;u) - f(u) \mid \leq C\omega_2(f;\delta_m(u)),$$

where

$$\delta_m^2(u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2.$$

Proof. Let $s \in \mathcal{W}^2$. Then from Taylor's expansion, we get

$$s(t) = s(u) + s'(u)(t-u) + \int_u^t (t-u)s''(u)\mathrm{d}u,\ t \in [0,\mathcal{A}],\ \mathcal{A} > 0.$$

Now by Corollary 2.5, we have

$$\widetilde{L}_m^{p,q}(s;u) = s(u) + \widetilde{L}_m^{p,q}\left(\int_u^t (t-u)s''(u)\mathrm{d}u;u\right).$$

$$\begin{aligned} |\widetilde{L}_m^{p,q}(s;u)(s;u) - s(u)| &\leq \widetilde{L}_m^{p,q}\left(\left|\int_u^t \mid (t-u) \mid \mid s''(u) \mid \mathrm{d}u;u\right|\right) \\ &\leq \widetilde{L}_m^{p,q}\left((t-u)^2;u\right) \parallel s'' \parallel, \end{aligned}$$

hence we get

$$|\widetilde{L}_m^{p,q}(s;u(s;u) - s(u)| \leq \parallel s'' \parallel \left(\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2\right).$$

By Lemma 2.3, we have

$$|\widetilde{L}_m^{p,q}(f;u)| \leq 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\left|f\left(\frac{[l]_{p,q}}{[m]_{p,q}}\right)\right| \leq \parallel f \parallel .$$

Thus, we have

$$|\widetilde{L}_m^{p,q}(f;u)| \ \leq \mid \widetilde{L}_m^{p,q}((f-s);u) - (f-s)(u) \mid + |\widetilde{L}_m^{p,q}(s;u) - s(u)|.$$

After substituting all values, we get

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \ \leq \parallel f - s \parallel + \parallel s'' \parallel \left(\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2\right).$$

By taking the infimum on the right hand side over all $s \in \mathcal{W}^2$, we get

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \ \leq \ \mathcal{C}K_2\left(f,\delta_m^2(u)\right).$$

By using the property of K-functional, we have

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \ \leq \ \mathcal{C}\omega_2\left(f,\delta_m(u)\right).$$

This completes the proof. $\square$

4. Pointwise estimates

Theorem 4.1. *Let $0 < \alpha \leq 1$ and Ȩ be any bounded subset of the interval $[0, \infty)$. If $f \in C_B[0, \infty)$, is locally Lip(α), i.e., the condition*

$$|f(v) - f(u)| \ \leq \text{Ł}|v - u|^{\alpha}, \ v \in \text{Ȩ} \text{ and } u \in [0, \infty) \tag{4.1}$$

holds, then, for each $u \in [0, \infty)$, we have

$$|\widetilde{L}_m^{p,q}(f; u) - f(u)| \ \leq \text{Ł}\left\{\delta_m(u)^{\frac{\alpha}{2}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\}, \ u \in [0, \infty)$$

where L is a constant depending on α and f and $d(u; \text{Ȩ})$ is the distance between u and Ȩ defined by

$$d(u, \text{Ȩ}) = inf\,\{|t - u|; t \in E\} \text{ and } \delta_m(u) = \widetilde{L}_m^{p,q}((t - u)^2; u).$$

Proof. Let $\overline{\text{Ȩ}}$ be the closure of Ȩ in $[0, 1)$. Then, there exists a point $t_0 \in \overline{\text{Ȩ}}$ such that $d(u, \text{Ȩ}) = |u - t_0|$.
Using the triangle inequality, we have

$$|f(t) - f(u)| \ \leq |f(t) - f(t_0)| + |f(t_0) - f(u)|.$$

By using (4.1) we get,

$$\begin{aligned}
|\widetilde{L}_m^{p,q}(f; u) - f(u)| &\leq \widetilde{L}_m^{p,q}\,(|f(t) - f(t_0)|; u) + \widetilde{L}_m^{p,q}\,(|f(u) - f(t_0)|; u) \\
&\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\,(|t - t_0|^{\alpha}; u) + (|u - t_0|^{\alpha}; u) + |u - t_0|^{\alpha}\right\} \\
&\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\,(|t - u|^{\alpha}; u) + 2|u - t_0|^{\alpha}\right\}.
\end{aligned}$$

By choosing $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get $\frac{1}{p} + \frac{1}{q} = 1$. Then by using Hölder's inequality we get

$$\begin{aligned}
|\widetilde{L}_m^{p,q}(f; u) - f(u)| &\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\,(|t - u|^{\alpha p}; u)^{\frac{1}{p}}\,[\widetilde{L}_m^{p,q}(1^q; u)]^{\frac{1}{q}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\} \\
&\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\left(((t - u)^2; u)\right)^{\frac{\alpha}{2}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\} \\
&\leq \text{Ł}\left\{\delta_m(u)^{\frac{\alpha}{2}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\}.
\end{aligned}$$

Hence the proof is completed. □

Now, we recall local approximation in terms of α order Lipschitz-type maximal function given by Lenze [16] as

$$\widetilde{\omega}_{\alpha}(f; u) = \sup_{t \neq u, t \in (0, \infty)} \frac{|f(t) - f(u)|}{|t - u|^{\alpha}}, \ u \in [0, \infty) \text{ and } \alpha \in (0, 1]. \tag{4.2}$$

Then we get the next result

Theorem 4.2. *Let $f \in C_B[0, \infty)$ and $\alpha \in (0, 1]$. Then, for all $u \in [0, \infty)$, we have*

$$|\widetilde{L}_m^{p_m,q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{\omega}_{\alpha}(f; u)\Big(\delta_m(u)\Big)^{\frac{\alpha}{2}},$$

where $\delta_m(u)$ is defined in Corollary 2.5.

Proof. We know that

$$|\widetilde{L}_m^{p_m,q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{L}_m^{p_m,q_m}(|f(t) - f(u)|; u).$$

From equation (4.2), we have

$$|\widetilde{L}_m^{p_m,q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{\omega}_{\alpha}(f; u)\widetilde{L}_m^{p_m,q_m}(|t - u|^{\alpha}; u).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$|\widetilde{L}_m^{p_m,q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{\omega}_{\alpha}(f; u)\big(\widetilde{L}_m^{p_m,q_m}(|t - u|^2; u)\big)^{\frac{\alpha}{2}},$$

which proves the desired result. $\square$

5. Weighted approximation by $\widetilde{L}_m^{p,q}$

In this section we shall discuss weighted approximation theorems for the operators $\widetilde{L}_m^{p,q}$ on the interval $[0,\infty)$.

Theorem 5.1 (cf. [5, 15]). *Let (T_m) be the sequence of linear positive operators from $C_{u^2}[0,\infty)$ to $B_{u^2}[0,\infty)$ satisfy*

$$\lim_{m\to\infty} \| T_m\kappa_i - \kappa_i \|_{u^2} = 0, \qquad i = 0,1,2.$$

Then for any function $f \in C^*_{u^2}[0,\infty)$

$$\lim_{m\to\infty} \| T_m f - f \|_{u^2} = 0.$$

Theorem 5.2 (cf. [6, 7]). *Let (q_m) and (p_m) be two sequences such that $0 < q_m < p_m \leq 1$, for all n and both converge to 1. Then for each function $f \in C^*_{u^2}[0,\infty)$, we get*

$$\lim_{m\to\infty} \| L_m^{p_m,q_m} f - f \|_{u^2} = 0.$$

Proof. By Theorem 5.1, it is enough to show

$$\lim_{m\to\infty} \| \widetilde{L}_m^{p_m,q_m}\kappa_i - \kappa_i \|_{u^2} = 0, \qquad i = 0,1,2. \tag{5.1}$$

By Lemma 2.4 (i) and (ii), it is clear that

$$\begin{aligned} \| L_m^{p_m,q_m}(1;u) - 1 \|_{u^2} &= 0 \\ \| \widetilde{L}_m^{p_m,q_m}(t;u) - u \|_{u^2} &= 0 \end{aligned}$$

and by Lemma 2.4 (iii), we have

$$\begin{aligned} \| \widetilde{L}_m^{p_m,q_m}(t^2;u) - u^2 \|_2 &= \sup_{u\in[0,\infty)} \frac{\left(\frac{1}{(2-p_m)^{([m]_{p_m,q_m}u+1)}[m]_{p_m,q_m}} + \frac{q_m}{[m]_{p_m,q_m}}\right)u + (q_m-1)u^2}{1+u^2} \\ &\leq \left(\frac{1}{(2-p_m)[m]_{p_m,q_m}} + \frac{q_m}{[m]_{p_m,q_m}}\right) + (q_m-1). \end{aligned}$$

Last inequality means that (5.1) holds for $i = 2$. By Theorem 5.1, the proof is completed. $\square$

Theorem 5.3. *Let $q_m \in (0,1)$, $p_m \in (q,1]$ such that $q_m \to 1$, $p_m \to 1$ as $m \to \infty$. Let $f \in C^*_{u^2}[0,\infty)$, and its modulus of continuity $\omega_{d+1}(f;\delta)$ be defined on $[0,d+1] \subset [0,\infty)$. Then, we have*

$$|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)\|_{C[0,d]} \leq 6M_f(1+d^2)\delta_m(d) + 2\omega_{d+1}(f;\sqrt{\delta_m(d)}),$$

where $\delta_m(d) = \widetilde{L}_m^{p,q}((t-u)^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2.$

Proof. From ([10] p. 378), for $u \in [0,d]$ and $t \in [0,\infty)$, we have

$$|f(t) - f(u)| \leq 6M_f(1+d^2)(t-u)^2 + \left(1 + \frac{|t-u|}{\delta}\right)\omega_{d+1}(f;\delta).$$

Applying $\widetilde{L}_m^{p,q}$ both the sides, we have

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \leq 6M_f(1+d^2)\widetilde{L}_m^{p,q}((t-u)^2;u) + \left(1 + \frac{\widetilde{L}_m^{p,q}(|t-u|;u)}{\delta}\right)\omega_{d+1}(f;\delta).$$

Applying Cauchy-Schwarz inequality,for $u \in [0,d]$ and $t \geq 0$, we get

$$\begin{aligned} |\widetilde{L}_m^{p,q}(f;u) - f(u)| &\leq \widetilde{L}_m^{p,q}\left(|(f;u) - f(u)|;u\right) \\ &\leq 6M_f(1+d^2)\widetilde{L}_m^{p,q}((t-u)^2;u) \\ &+ \omega_{d+1}(f;\delta)\left(1 + \frac{1}{\delta}\widetilde{L}_m^{p,q}\left((t-u)^2;u\right)^{\frac{1}{2}}\right). \end{aligned}$$

Thus, from Lemma 2.4, for $u \in [0,d]$, we get

$$|\widetilde{L}_m^{p;q}(f;u) - f(u)| \leq 6M_f(1+d^2)\delta_m(d) + \omega_{d+1}(f;\delta)\left(1 + \frac{\sqrt{\delta_m(d)}}{\delta}\right).$$

By Choosing $\delta = \sqrt{\delta_m(d)}$, we get the required result. □

Now, we prove a theorem to approximate all functions in $C^*_{u^2}$ Such type of results are given in [8] for locally integrable functions.

Theorem 5.4. *Let $0 < q_m < p_m \leq 1$ such that $q_m \to 1$, $p_m \to 1$ as $m \to \infty$. Then for each function $f \in C^*_{u^2}[0,\infty)$, and $\alpha > 1$*

$$\lim_{m\to\infty} \sup_{u\in[0,\infty)} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} = 0.$$

Proof. Let for any fixed $u_0 > 0$,

$$\begin{aligned}
\sup_{u\in[0,\infty)} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} &\leq \sup_{u\leq u_0} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} + \sup_{u\geq u_0} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} \\
&\leq \| L_m^{p_m,q_m}(f) - f \|_{[c_0,u_0]} + \| f \|_{u^2} \sup_{u\leq u_0} \frac{|\widetilde{L}_m^{p_m,q_m}(1+t^2;u)|}{(1+u^2)^\alpha} \\
&+ \sup_{u\geq u_0} \frac{|f(u)|}{(1+u^2)^\alpha}. \qquad (5.2)
\end{aligned}$$

Since, $|f(u)| \leq M_f(1+u^2)$ we have,

$$\sup_{u\geq u_0} \frac{|f(u)|}{(1+u^2)^\alpha} \leq \sup_{u\geq u_0} \frac{M_f}{(1+u^2)^{\alpha-1}} \leq \frac{M_f}{(1+u^2)^{\alpha-1}}.$$

Let $\epsilon > 0$, and let us choose u_0 large then we have

$$\frac{M_f}{(1+{u_0}^2)^{\alpha-1}} < \frac{\epsilon}{3} \qquad (5.3)$$

and in view of (2.4), we get,

$$\begin{aligned}
\| f \|_{u^2} \lim_{m\to\infty} \frac{|\widetilde{L}_m^{p_m,q_m}(1+t^2;u)|}{(1+u^2)^\alpha} &= \| f \|_{u^2} \frac{1+u^2}{(1+u^2)^\alpha} \\
&\leq \frac{\| f \|_{u^2}}{(1+u^2)^{\alpha-1}} \\
&\leq \frac{\| f \|_{u^2}}{(1+{u_0}^2)^{\alpha-1}} \\
&\leq \frac{\epsilon}{3}. \qquad (5.4)
\end{aligned}$$

By using Theorem 5.3, the first term of inequality (5.2) becomes

$$\| \widetilde{L}_m^{p_m,q_m}(f) - f \|_{[c_0,u_0]} < \frac{\epsilon}{3}, \text{as } m \to \infty. \qquad (5.5)$$

Hence we get the required proof by combining (5.3)-(5.5)

$$\sup_{u\in[0,\infty)} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} < \epsilon.$$

□

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ADDITIVE-QUADRATIC FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES AND STABILITY

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Abstract. In this paper, we investigate the following functional inequality

$$N(f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)]$$
$$-f(-x)-f(-y)-f(-z),t) \geq N(f(x+y+z),t)$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction and preliminaries

The concept of a fuzzy norm on a linear space was introduced by Katsaras [11] in 1984. Later, Cheng and Mordeson [3] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for all $x, y \in X$ and all $c, s, t \in \mathbb{R}$,

(N1) $N(x,\ t) = 0$ for all $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space.*

Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X,N) *and one denotes it by* $N - \lim_{n\to\infty} x_n = x$. *A sequence* $\{x_n\}$ *in* (X,N) *is said to be Cauchy if for any* $\epsilon > 0$, *there is an* $m \in N$ *such that for any* $n \geq m$ *and any positive integer* p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ *for all* $t > 0$. *A fuzzy normed space is said to be complete if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called a fuzzy Banach space.*

In 1940, Ulam proposed the following stability problem (cf.[21]*):*

"Let G_1 *be a group and* G_2 *a metric group with the metric* d. *Given a constant* $\delta > 0$, *does there exist a constant* $c > 0$ *such that if a mapping* $f : G_1 \longrightarrow G_2$ *satisfies* $d(f(xy), f(x)f(y)) < c$ *for all* $x, y \in G_1$, *then there exists an unique homomorphism* $h : G_1 \longrightarrow G_2$ *with* $d(f(x), h(x)) < \delta$ *for all* $x \in G_1$*?"*

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In the next year, Hyers [10] *gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki* [1] *for additive mappings, and by Rassias* [17] *for linear mappings, to consider the stability problem with unbounded Cauchy differences. A generalization of the Rassias' theorem was obtained by Găvruta* [7] *by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach. In 2008, for the first time, Mirmostafaee and Moslehian* [14], [15] *used the definition of a fuzzy norm in* [2] *to obtain a fuzzy version of stability for the Cauchy functional equation*

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.2}$$

Glányi [8] *and Rätz* [18] *showed that if a mapping $f : X \longrightarrow Y$ satisfies the following functional inequality*

$$\|2f(x)+2f(y)-f(xy^{-1})\| \le \|f(xy)\|, \tag{1.3}$$

then f satisfies the following Jordan-Von Neumann functional equation

$$2f(x)+2f(y)-f(xy^{-1})=f(xy).$$

for an abelian group X divisible by 2 into an inner product space Y. Glányi [9] *and Fechner* [6] *proved the Hyers-Ulam stability of (1.3). The stability problems of several functional equations and inequalities have been extensively investigated by a number of authors and there are many interesting results concerning the stability of various functional equations and inequalities.*

Now, we consider the following fixed point theorem on generalized metric spaces.

Definition 1.2. Let X be a non-empty set. Then a mapping $d : X^2 \longrightarrow [0,\infty]$ is called *a generalized metric on X* if d satisfies the following conditions:
(D1) $d(x,y)=0$ if and only if $x=y$,
(D2) $d(x,y)=d(y,x)$, and
(D3) $d(x,y)\le d(x,z)+d(z,y)$.
In case, (X,d) is called *a generalized metric space.*

Theorem 1.3. [4] *Let (X,d) be a complete generalized metric space and let $J : X \longrightarrow X$ a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x)=\infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y=\{y\in X \mid d(J^{n_0}x,y)<\infty\}$* and
(4) *$d(y,y^*) \le \dfrac{1}{1-L}\, d(y,Jy)$ for all $y\in Y$.*

The following function equation $f : X \longrightarrow Y$ is called the Drygas functional equation :

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)$$

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for all $x, y \in X$. The Drygas functional equation has been studied by Szabo [20] *and Ebanks, Făiziev and Sahoo* [5]. *The solutions of the Drygas functional equation in abelian group are obtained by H. Stetkær in* [19].

In this paper, we investigate the following functional inequality which is related with the Drygas type functional equation

$$\begin{aligned} N(f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)] \\ -f(-x)-f(-y)-f(-z),t) \geq N(f(x+y+z),t) \end{aligned} \tag{1.4}$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and $(\mathbb{R}, N')$ is a fuzzy normed space.

2. Solutions and the stability for (1.4)

In this section, we investigate the functional equation (1.4) and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces. For any mapping $f : X \longrightarrow Y$, let

$$f_o(x) = \frac{f(x)-f(-x)}{2}, \quad f_e(x) = \frac{f(x)+f(-x)}{2}.$$

In [12], *the authors proved the following theorem:*

Lemma 2.1. [12] *Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. Then f is quadratic if and only if f satisfies the following functional equation*

$$f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y) = k[f(x+y)+f(x-y)-2f(x)-2f(y)]$$

for all $x, y \in X$, a fixed nonzero rational number a and fixed real numbers b, k with $a^2 \neq b^2$.

Using this, we have the following theorem:

Theorem 2.2. *If a mapping $f : X \longrightarrow Y$ saisfies (1.4), then f is an additive-quadratic mapping.*

Proof. Suppose that f satisfies (1.4). Setting $x = y = z = 0$ in (1.4), by (N3), we have

$$N(f(0),t) \leq N(6f(0),t) = N\Big(f(0), \frac{t}{6}\Big)$$

for all $t > 0$ and by (N5), $N(f(0), \frac{t}{6}) \leq N(f(0), t)$ for all $t > 0$. Hence we have

$$N(f(0),t) = N(f(0),6t)$$

for all $t > 0$. By induction, we get

$$N(f(0),t) = N(f(0),6^n t)$$

for all $t > 0$ and all $n \in \mathbb{N}$. By (N5), we get

$$N(f(0),t) = \lim_{n\to\infty} N(f(0),6^n t) = 1$$

for all $t > 0$ and hence by (N2), $f(0) = 0$. Letting $z = -x-y$ in (1.4), we have

$$\begin{aligned} &N(f(x-y)+f(x+2y)+f(-2x-y)-2f(x)-2f(y)-2f(-x-y) \\ &-f(-x)-f(-y)-f(x+y),t) \geq N(0,t) = 1 \end{aligned}$$

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for all $x, y \in X$ and all $t > 0$ and so by (N2), we get

$$\begin{aligned} &f(x-y)+f(x+2y)+f(-2x-y) \\ &= 2f(x)+2f(y)+2f(-x-y)+f(x+y)+f(-x)+f(-y) \end{aligned} \tag{2.1}$$

for all $x, y \in X$. By (2.1), we have

$$f_o(x-y)+f_o(x+2y)-f_o(2x+y) = -f_o(x+y)+f_o(x)+f_o(y) \tag{2.2}$$

for all $x, y \in X$ and interchanging x and y in (2.2), we have

$$-f_o(x-y)+f_o(2x+y)-f_o(x+2y) = -f_o(x+y)+f_o(x)+f_o(y) \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$f_o(x+y) = f_o(x)+f_o(y)$$

for all $x, y \in X$ and hence f_o is an additive mapping. By (2.1), we have

$$f_e(x-y)+f_e(2x+y)+f_e(x+2y) = 3f_e(x+y)+3f_e(x)+3f_e(y) \tag{2.4}$$

for all $x, y \in X$ and letting $y = -y$ in (2.4), we have

$$f_e(x+y)+f_e(2x-y)+f_e(x-2y) = 3f_e(x-y)+3f_e(x)+3f_e(y) \tag{2.5}$$

for all $x, y \in X$. By (2.4) and (2.5), we have

$$\begin{aligned} &f_e(2x+y)+f_e(2x-y)+f_e(x+2y)+f_e(x-2y) \\ &= 2f_e(x+y)+2f_e(x-y)+6f_e(x)+6f_e(y) \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Letting $y = 0$ in (2.6), we get

$$f_e(2x) = 4f_e(x) \tag{2.7}$$

for all $x \in X$ and letting $y = 2y$ in (2.6), by (2.7), we have

$$\begin{aligned} &4f_e(x+y)+4f_e(x-y)+f_e(x+4y)+f_e(x-4y) \\ &= 2f_e(x+2y)+2f_e(x-2y)+6f_e(x)+24f_e(y) \end{aligned} \tag{2.8}$$

for all $x, y \in X$. By (2.6) and (2.8), we have

$$2f_e(2x+y)+2f_e(2x-y)+f_e(x+4y)+f_e(x-4y) = 18f_e(x)+36f_e(y) \tag{2.9}$$

for all $x, y \in X$. Letting $y = 2y$ in (2.9), by (2.7), we have

$$f_e(x+8y)+f_e(x-8y)+8f_e(x+y)+8f_e(x-y) = 18f_e(x)+144f_e(y)$$

for all $x, y \in X$. By Lemma 2.1, f_e is a quadratic mapping. Thus f is an additive-quadratic mapping. $\square$

Now, we will prove the generalized Hyers-Ulam stability of (1.4) in fuzzy normed spaces. For any mapping $f : X \longrightarrow Y$, let

$$Df(x,y,z) = f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)]-f(-x)-f(-y)-f(-z).$$

Theorem 2.3. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big), t\Big) \geq N'\Big(\frac{L}{4}\phi(x,y,z), t\Big) \tag{2.10}$$

for all $x, y, z \in X$, $t > 0$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x,y,z), t) \geq \min\{N(f(x+y+z), t), N'(\phi(x,y,z), t)\} \tag{2.11}$$

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for all $x, y, z \in X$ and all $t > 0$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{L}{4(1-L)}t\Big) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \tag{2.12}$$

for all $x \in X$ and all $t > 0$. Further, we have

$$F(x) = N - \lim_{n\to\infty} \Big[\frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)\Big] \tag{2.13}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = \inf\{c \in [0,\infty)|\ & N(g(x) - h(x), ct) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\}, \\ & \forall x \in X, \forall t > 0\}. \end{aligned}$$

Then (S, d) is a complete metric space([16]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 3g(\frac{x}{2}) + g(-\frac{x}{2})$ for all $g \in S$ and all $x \in X$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0,\infty)$. Then by (2.10), we have

$$\begin{aligned} & N(Jg(x) - Jh(x), cLt) \\ &= N\Big(3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big) - 3h\Big(\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), cLt\Big) \\ &\geq \min\Big\{N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), \frac{1}{4}cLt\Big), N\Big(g\Big(-\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), \frac{1}{4}cLt\Big)\Big\} \\ &\geq \min\Big\{N'\Big(\phi\Big(\frac{x}{2}, -\frac{x}{2}, 0\Big), \frac{1}{4}Lt\Big), N'\Big(\phi\Big(-\frac{x}{2}, \frac{x}{2}, 0\Big), \frac{1}{4}Lt\Big)\Big\} \\ &\geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = -x$ and $z = 0$ in (2.11), we get

$$N(f(2x) - 3f(x) - f(-x), t) \geq N'(\phi(x, -x, 0), t) \tag{2.14}$$

for all $x \in X$, $t > 0$ and hence

$$\begin{aligned} N\Big(f(x) - Jf(x), \frac{L}{4}t\Big) &= N\Big(f(x) - 3f\Big(\frac{x}{2}\Big) - f\Big(-\frac{x}{2}\Big), \frac{L}{4}t\Big) \\ \geq N'\Big(\phi\Big(\frac{x}{2}, -\frac{x}{2}, 0\Big), \frac{L}{4}t\Big) &\geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{L}{4} < \infty$. By Theorem 1.3, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. By induction, we have

$$J^n f(x) = \frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence we have (2.13).

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Replacing x, y, z, and t by $\frac{x}{2^n}$, $\frac{y}{2^n}$, $\frac{z}{2^n}$, and $\frac{t}{2^{2n}}$ in (2.11), respectively, by (2.11), we have

$$\begin{aligned}
&N\Big(Df_e\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big)\\
&\geq \min\Big\{N\Big(Df\Big(\frac{x}{2^n},\frac{y}{2^n},-\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big),N\Big(Df\Big(-\frac{x}{2^n},-\frac{y}{2^n},\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big)\Big\}\\
&\geq \min\{N'(L^n\phi(x,y,z),t),N'(L^n\phi(-x,-y,z),t)\}
\end{aligned}\tag{2.15}$$

for all $x, y, z \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.15), we have

$$DF_e(x, y, z) = 0$$

for all $x, y, z \in X$ and by Lemma 2.1, F_e is an quadratic mapping. Similarly, F_o is an additive mapping and thus F is an additive-quadratic mapping. Since $d(f, Jf) \leq \frac{L}{4}$, by Theorem 1.3, we have (2.12).

Now, we show the uniqueness of F. Let G be another additive-quadratic mapping with (2.12). Then clearly, G is a fixed point of J and

$$d(Jf, G) = d(Jf, JG) \leq Ld(f, G) \leq \frac{L^2}{4(1-L)} < \infty \tag{2.16}$$

and hence by (3) in Theorem 1.3, $F = G$. □

Similar to Theorem 2.3, we have the following threoem:

Theorem 2.4. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function such that*

$$N'(\phi(2x, 2y, 2z), t) \geq N'(2L\phi(x, y, z), t) \tag{2.17}$$

for all $x, y, z \in X$, $t > 0$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (2.11). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{1}{2(1-L)}t\Big) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \tag{2.18}$$

for all $x \in X$ and all $t > 0$. Further, we have

$$F(x) = \lim_{n\to\infty}\Big[\frac{2^n+1}{2^{2n+1}}f(2^n x) - \frac{2^n-1}{2^{2n+1}}f(-2^n x)\Big]$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned}
d(g,h) = \inf\{c\in[0,\infty)|\ &N(g(x)-h(x),ct)\geq\min\{N'(\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\},\\
&\forall x\in X,\forall t>0\}.
\end{aligned}$$

Then (S, d) is a complete metric space([16]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$ for all $g \in S$ and all $x \in X$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (2.10), we have

$$\begin{aligned}
&N(Jg(x)-Jh(x),cLt)\\
&= N\Big(\frac{3}{8}g(2x)-\frac{1}{8}g(-2x)-\frac{3}{8}h(2x)+\frac{1}{8}h(-2x),cLt\Big)\\
&\geq \min\{N(g(2x)-h(2x),2cLt),N(g(-2x)-h(-2x),2cLt)\}\\
&\geq \min\{N'(\phi(2x,-2x,0),2Lt),N'(\phi(-2x,2x,0),2Lt)\}\\
&\geq \min\{N'\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\}
\end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \le Ld(g, h)$ for any $g, h \in S$ and since $0 < L < 1$, J is a strictly contractive mapping. By (2.14), we get

$$\begin{aligned}
&N\Big(f(x) - Jf(x), \frac{t}{2}\Big) \\
&= N\Big(\frac{3}{8}[f(2x) - 3f(x) - f(-x)] - \frac{1}{8}[f(-2x) - 3f(-x) - f(x)], \frac{t}{2}\Big) \\
&\ge \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Jf) \le \frac{1}{2} < \infty$. The rest of proof the proof is similar to Theorem 2.3. □

As examples of $\phi(x, y, z)$ and $N'(x, t)$ in Theorem 2.3 and Theorem 2.4, we can take $\phi(x, y, z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ and

$$N'(x, t) = \begin{cases} \frac{t}{t+k|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \le 0 \end{cases}$$

for all $x \in \mathbb{R}$, $t > 0$, and for some $\epsilon > 0$, where $k = 1, 2$. Then we can formulate the following corollary:

Corollary 2.5. *Let X be a normed space and (Y, N) a fuzzy Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(Df(x, y, z), t) \ge \min\Big\{N(f(x + y + z), t), \frac{t}{t + k\epsilon(\|x\|^p + \|y\|^p + \|z\|^p)}\Big\}$$

for all $x, y, z \in X$, $t > 0$, a fixed real number p with $0 < p < 1$ or $2 < p$. Then there is an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \ge \begin{cases} \dfrac{(2^p - 4)t}{(2^p - 4)t + 2k\epsilon\|x\|^p}, & \text{if } 2 < p \\[2ex] \dfrac{(2 - 2^p)t}{(2 - 2^p)t + 2k\epsilon\|x\|^p}, & \text{if } 0 < p < 1 \end{cases} \tag{2.19}$$

for all $x \in X$ and all $t > 0$.

For any $f : X \longrightarrow Y$, let

$$\begin{aligned} D_1f(x, y) = &f(x - y) + f(x + 2y) + f(-2x - y) \\ &- 2f(x) - 2f(y) - 2f(-x - y) - f(-x) - f(-y) - f(x + y) \end{aligned} \tag{2.20}$$

Using Corollary 2.5, we have the following corollary:

Corollary 2.6. *Let X be a normed space and (Y, N) a fuzzy Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(D_1f(x, y), t) \ge \frac{t}{t + k\epsilon(\|x\|^p + \|y\|^p + \|x + y\|^p)} \tag{2.21}$$

for all $x, y, z \in X$, $t > 0$, a fixed real number p with $0 < p < 1$ or $2 < p$. Then there is an unique additive-quadratic mapping $F : X \longrightarrow Y$ with (2.19).

We remark that the functional inequality (1.4) is not stable for $p = 1$ in Corollary 2.6. The following example shows that the inequality (2.21) is not stable for $p = 1$.

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Example 2.7. Define mappings $t, s : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$t(x) = \begin{cases} x, & \text{if } |x| < 1 \\ -1, & \text{if } x \le -1 \\ 1, & \text{if } 1 \le x, \end{cases}$$

$$s(x) = \begin{cases} x^2, & \text{if } |x| < 1 \\ 1, & \text{ortherwise} \end{cases}$$

and a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \Big[\frac{t(2^n x)}{2^n} + \frac{s(2^n x)}{4^n}\Big]$$

We will show that f satisfies the following inequality

$$|D_1 f(x,y)| \le 112(|x| + |y| + |x+y|) \tag{2.22}$$

for all $x, y \in \mathbb{R}$ and so f satisfies (2.21). But there do not exist an additive-quadratic mapping $F : \mathbb{R} \longrightarrow \mathbb{R}$ and a non-negative constant K such that

$$|F(x) - f(x)| \le K|x| \tag{2.23}$$

for all $x \in \mathbb{R}$.

Proof. Note that $t_o(x) = t(x)$, $s_o(x) = 0$, and $|f_o(x)| \le 2$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \le |x| + |y| + |x+y|$. Then $|D_1 f_o(x,y)| \le 48(|x|+|y|+|x+y|)$. Now suppose that $\frac{1}{4} > |x| + |y| + |x+y|$. Then there is a non-negative integer m such that

$$\frac{1}{2^{m+3}} \le |x| + |y| + |x+y| < \frac{1}{2^{m+2}}$$

and so

$$2^m|x| < \frac{1}{4}, \quad 2^m|y| < \frac{1}{4}, \quad 2^m|x+y| < \frac{1}{4}.$$

Hence we have

$$\{2^m x, 2^m y, 2^m(x-y),\ 2^m(x+y),\ 2^m(x+2y),\ 2^m(2x+y)\} \subseteq (-1,\ 1)$$

and so for any $n = 0, 1, 2, \cdots, m$,

$$|D_1 t_0(2^n x, 2^n y)| = 0,$$

because $t(x) = t_o(x) = x$ on $(-1, 1)$. Thus

$$\begin{aligned} |D_1 f_o(x,y)| &= \Big|\sum_{n=0}^{\infty} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y)\Big| \\ &\le \Big|\sum_{n=0}^{m} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y)\Big| + \Big|\sum_{n=m+1}^{\infty} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y)\Big| \\ &\le \frac{12}{2^{m+1}} \le 48(|x| + |y| + |x+y|), \end{aligned}$$

because $|D_1 t_0(2^n x, 2^n y)| \le 6$.

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Note that $t_e(x) = 0$, $s_e(x) = s(x)$, and $|f_e(x)| \le \frac{4}{3}$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \le |x|+|y|+|x+y|$. Then $|D_1 f_e(x,y)| \le 64(|x|+|y|+|x+y|)$. Now suppose that $\frac{1}{4} > |x|+|y|+|x+y|$. Then there is a non-negative integer k such that

$$\frac{1}{2^{2k+4}} \le |x|+|y|+|x+y| < \frac{1}{2^{2k+2}}$$

and so

$$2^{2k}|x| < \frac{1}{4}, \quad 2^{2k}|y| < \frac{1}{4}, \quad 2^{2k}|x+y| < \frac{1}{4}.$$

Hence we have

$$\{2^k x, 2^k y, 2^k(x-y), 2^k(x+y),\ 2^k(x+2y),\ 2^k(2x+y)\} \subseteq (-1,\ 1)$$

and so for any $n = 0, 1, 2, \cdots, k$,

$$|D_1 s_e(2^n x, 2^n y)| = 0.$$

Thus

$$\begin{aligned}
|D_1 f_e(x,y)| &= \Big|\sum_{n=0}^{\infty} \frac{1}{2^n} D_1 s_e(2^n x, 2^n y)\Big| \\
&\le \Big|\sum_{n=0}^{k} \frac{1}{4^n} D_1 s_e(2^n x, 2^n y)\Big| + \Big|\sum_{n=k+1}^{\infty} \frac{1}{4^n} D_1 s_e(2^n x, 2^n y)\Big| \\
&\le \frac{16}{4^{k+1}} \le 64(|x|+|y|+|x+y|),
\end{aligned}$$

because $|D_1 s_e(2^n x, 2^n y)| \le 12$. Hence we have

$$|D_1 f_o(x,y)| \le 48(|x|+|y|+|x+y|),\ |D_1 f_e(x,y)| \le 64(|x|+|y|+|x+y|)$$

for all $x, y \in X$ and so we have (2.22).

Suppose that there exist an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$, a quadratic mapping $Q : \mathbb{R} \longrightarrow \mathbb{R}$, and a non-negative constant K such that $A+Q$ satisfies (2.23). Since $|f(x)| \le \frac{10}{3}$, by (2.23), we have

$$-\frac{10}{3n^2} - K\frac{|x|}{n} \le \frac{A(x)}{n} + Q(x) \le \frac{10}{3n^2} + K\frac{|x|}{n}$$

for all $x \in X$ and all positive integers n and so $Q(x) = 0$ for all $x \in X$. Since A is additive,

$$-\frac{10}{3n} - K|x| \le A(x) \le \frac{10}{3n} + K|x|$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence $|A(x)| \le K|x|$. By (2.23), we have

$$|f(x)| \le 2K|x| \tag{2.24}$$

for all $x \in X$. Take a positive integer l such that $l > 2K$ and $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Since $x > 0$,

$$f(x) \ge \sum_{n=0}^{\infty} \frac{t(2^n x)}{2^n} \ge \sum_{n=0}^{l-1} \frac{t(2^n x)}{2^n} = lx > 2Kx$$

which contradicts to (2.24). □

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GENERALIZED ADDITIVE-CUBIC FUCNTIONAL EQUATION AND ITS STABILITY

CHANG IL KIM

ABSTRACT. In this paper, we establish some stability results for the following additive-cubic functional equation with an extra term G_f

$$f(2x+y)+f(2x-y)+G_f(x,y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x).$$

in Banach spaces, where G_f is a functional operator of f. Using these, we give new additive-cubic functional equations and prove their stability.

1. INTRODUCTION

In 1940, Ulam [12] raised the following question concerning the stability of group homomorphisms: "Under what conditions does there is an additive mapping near an approximately additive mapping between a group and a metric group ? "
In the next year, Hyers [5] gave a partial solution of Ulam's problem for the case of additive mappings. Hyers 's result, using unbounded Cauchy different, was generalized for additive mappings in [1] and for a linera mapping in [11]. Some stability results for additive, quardartic and mixed additve-cubic functional equations were investigated ([2], [3], [4], [6], [7], [8], [9], [10]).

The generalized Hyers–Ulam stability for the mixed additive-cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x) \tag{1.1}$$

in quasi-Banach spaces has been investigated by Najati and Eskandani [8]. Functional equation (1.1) is called *an additive-cubic functional equation*, since the function $f(x)=ax^3+bx$ is its solution. Every solution of this mixed additive-cubic functional equation is said to be *an additive-cubic mapping*.

In this paper, we are interested in what kind of a term $G_f(x,y)$ can be added to (1.1) while the solution of the new functional equation is also an additive-cubic funtional equation and the generalized Hyers-Ulam stability for it still holds, where $G_f(x,y)$ is a functional operator depending on the variables x,y, and function f. The new functional equation can be written as

$$f(2x+y)+f(2x-y)+G_f(x,y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.2}$$

We give some new functional equations in section 3 as examples of our results and prove the generalized Hyers-Ulam stability for these.

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2. The generalized Hyers-Ulam stability for (1.2)

Let X be a real normed linear space and Y a real Banach space. For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

Throughout this section we always assume that G_f satisfies the following two conditions unless a specific expression for G_f is given.

Condition $\mathbf{P_1}$: Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(2x) = 2f(x)$ and

$$f(2x + y) + f(2x - y) + G_f(x, y) = 2f(x + y) + 2f(x - y) \tag{2.1}$$

for all $x, y \in X$. Then f is an additive mapping.

Condition $\mathbf{P_2}$: Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(2x) = 8f(x)$ and

$$f(2x + y) + f(2x - y) + G_f(x, y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{2.2}$$

for all $x, y \in X$. Then f is a cubic mapping.

For any $f : X \longrightarrow Y$, let

$$f_a(x) = \frac{4}{3}f(x) - \frac{1}{6}f(2x), \;\; f_c(x) = -\frac{1}{3}f(x) + \frac{1}{6}f(2x)$$

Now, we prove the following main theorem.

Theorem 2.1. *Let G_t be a functional operator satisfying* **Condition $\mathbf{P_1}$** *and* **Condition $\mathbf{P_2}$**. *Further, suppose that there is a real number $\lambda (\lambda \neq -1)$ such that*

$$G_t(x, 2x) + 2G_t(x, x) - 2G_t(0, x) = \lambda[t(4x) - 10t(2x) + 16t(x)] \tag{2.3}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty \tag{2.4}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\begin{aligned} &\|f(2x + y) + f(2x - y) + G_f(x, y) \\ &- 2f(x + y) - 2f(x - y) - 2f(2x) + 4f(x)\| \le \phi(x, y) \end{aligned} \tag{2.5}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned} &\|F_a(x) - f_a(x)\| \\ &\le \frac{1}{12|\lambda + 1|}\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\|F_c(x) - f_c(x)\| \\ &\le \frac{1}{48|\lambda+1|}\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]\end{aligned} \tag{2.7}$$

for all $x \in X$.

Proof. By (2.5), we have

$$\|f(x) + f(-x) - G_f(0, x)\| \le \phi(0, x), \tag{2.8}$$

$$\|f(3x) - 4f(2x) + 5f(x) + G_f(x, x)\| \le \phi(x, x), \tag{2.9}$$

and

$$\|f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x) + G_f(x, 2x)\| \le \phi(x, 2x) \tag{2.10}$$

for all $x \in X$. By (2.3), (2.8), (2.9), and (2.10), we have

$$\|2^{-1}f_a(2x) - f_a(x)\| \le \frac{1}{12|\lambda+1|}[\phi(x, 2x) + 2\phi(x, x) + 2\phi(0, x)] \tag{2.11}$$

for all $x \in X$. By (2.11), for $m, n \in \mathbb{N} \cup \{0\}$ with $0 \le m < n$, we have

$$\begin{aligned}&\|2^{-n}f_a(2^n x) - 2^{-m}f_a(2^m x)\| \\ &= 2^{-m}\|2^{-(n-m)}f_a(2^{n-m}\cdot 2^m x) - f_a(2^m x)\| \\ &\le \frac{1}{12|\lambda+1|}\sum_{k=m}^{n-1} 2^{-k}[\phi(2^k x, 2^{k+1}x) + 2\phi(2^k x, 2^k x) + 2\phi(0, 2^k x)]\end{aligned} \tag{2.12}$$

for all $x \in X$. By (2.12), $\{2^{-n}f_a(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that

$$A(x) = \lim_{n\to\infty} 2^{-n}f_a(2^n x)$$

for all $x \in X$. Moreover, by (2.12), we have

$$\begin{aligned}&\|A(x) - f_a(x)\| \\ &\le \frac{1}{12|\lambda+1|}\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]\end{aligned} \tag{2.13}$$

for all $x \in X$. By (2.5), we have

$$\begin{aligned}&\|f_a(2x+y) + f_a(2x-y) + G_{f_a}(x, y) - 2f_a(x+y) - 2f_a(x-y) \\ &- 2f_a(2x) + 4f_a(x)\| \le \frac{4}{3}\phi(x, y) + \frac{1}{6}\phi(2x, 2y)\end{aligned} \tag{2.14}$$

for all $x, y \in X$. Replacing x and y by $2^n x$ and $2^n y$ in (2.14), respectively and deviding (2.5) by 2^n, we have

$$\begin{aligned}&\|2^{-n}f_a(2^n(2x+y)) + 2^{-n}f_a(2^n(2x-y)) + 2^{-n}G_{f_a}(2^n x, 2^n y) \\ &- 2\cdot 2^{-n}f_a(2^n(x+y)) - 2\cdot 2^{-n}f_a(2^n(x-y)) - 2\cdot 2^{-n}f_a(2^{n+1}x) \\ &+ 4\cdot 2^{-n}f_a(2^n x)\| \le \frac{4}{3}\cdot 2^{-n}\phi(2^n x, 2^n y) + \frac{1}{6}\cdot 2^{-n}\phi(2^{n+1}x, 2^{n+1}y)\end{aligned}$$

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for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$\begin{aligned} &A(2x+y) + A(2x-y) + \lim_{n\to\infty} 2^{-n} G_{f_a}(2^n x, 2^n y) \\ &- 2A(x+y) - 2A(x-y) - 2A(2x) + 4A(x) = 0 \end{aligned} \tag{2.15}$$

for all $x, y \in X$ and since F is continuous,

$$\begin{aligned} &\lim_{n\to\infty} 2^{-n} G_{f_a}(2^n x, 2^n y) \\ &= \lim_{n\to\infty} F(2^{-n} f_a(2^n\sigma_1(x,y)), 2^{-n} f_a(2^n\sigma_2(x,y)), \cdots, 2^{-n} f_a(2^n\sigma_l(x,y))) \\ &= G_A(x,y) \end{aligned}$$

for all $x, y \in X$. Hence by (2.15), we have

$$A(2x+y) + A(2x-y) + G_A(x,y) = 2A(x+y) + 2A(x-y) + 2A(2x) - 4A(x) \tag{2.16}$$

for all $x, y \in X$. Relpacing x by $2^n x$ in (2.11) and deviding (2.11) by 2^n, we have

$$\begin{aligned} &\|2^{-n-1} f_a(2^n \cdot 2x) - 2^{-n} f_a(2^n x)\| \\ &\le \frac{2^{-n}}{12|\lambda+1|}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned}$$

for all $x \in X$ and letting $n \to \infty$ in the above inequality, we have

$$A(2x) = 2A(x) \tag{2.17}$$

for all $x, y \in X$. By (2.16) and (2.17), A satisfies (2.1). By **Condition $\mathbf{P_1}$**, A is an additive mapping .

By (2.3), (2.8), (2.9), and (2.10), we have

$$\|8^{-1} f_c(2x) - f_c(x)\| \le \frac{1}{48|\lambda+1|}[\phi(x, 2x) + 2\phi(x,x) + 2\phi(0,x)] \tag{2.18}$$

for all $x \in X$. By (2.18), for $m, n \in \mathbb{N} \cup \{0\}$ with $0 \le m < n$, we have

$$\begin{aligned} &\|2^{-3n} f_c(2^n x) - 2^{-3m} f_c(2^m x)\| \\ &= 2^{-3m}\|2^{-3(n-m)} f_c(2^{n-m} \cdot 2^m x) - f_c(2^m x)\| \\ &\le \frac{1}{48|\lambda+1|}\sum_{k=m}^{n-1} 2^{-3k}[\phi(2^k x, 2^{k+1}x) + 2\phi(2^k x, 2^k x) + 2\phi(0, 2^k x)] \end{aligned} \tag{2.19}$$

for all $x \in X$. By (2.19), $\{2^{-3n} f_c(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $C : X \longrightarrow Y$ such that

$$C(x) = \lim_{n\to\infty} 2^{-3n} h(2^n x)$$

for all $x \in X$. Moreover, by (2.19), we have

$$\begin{aligned} &\|C(x) - f_c(x)\| \\ &\le \frac{1}{48|\lambda+1|}\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned} \tag{2.20}$$

for all $x \in X$. By (2.5), we have

$$\begin{aligned} &\|f_c(2x+y) + f_c(2x-y) + G_{f_c}(x,y) - 2f_c(x+y) - 2f_c(x-y) \\ &- 2f_c(2x) + 4f_c(x)\| \le \frac{1}{3}\phi(x,y) + \frac{1}{6}\phi(2x, 2y) \end{aligned} \tag{2.21}$$

for all $x, y \in X$. Replacing x and y by $2^n x$ and $2^n y$ in (2.21), respectively and deviding (2.21) by 2^{3n}, we have

$$\begin{aligned}
&\|2^{-3n}f_c(2^n(2x+y)) + 2^{-3n}f_c(2^n(2x-y)) + 2^{-3n}G_{f_c}(2^n x, 2^n y) \\
&- 2\cdot 2^{-3n}f_c(2^n(x+y)) - 2\cdot 2^{-3n}f_c(2^n(x-y)) - 2\cdot 2^{-n}f_c(2^{n+1}x) \\
&+ 4\cdot 2^{-3n}f_c(2^n x)\| \le \frac{1}{3}\cdot 2^{-3n}\phi(2^n x, 2^n y) + \frac{1}{6}\cdot 2^{-3n}\phi(2^{n+1}x, 2^{n+1}y)
\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$\begin{aligned}
&C(2x+y) + C(2x-y) + \lim_{n\to\infty} 2^{-3n}G_{f_c}(2^n x, 2^n y) \\
&- 2C(x+y) - 2C(x-y) - 2C(2x) + 4C(x) = 0
\end{aligned} \tag{2.22}$$

for all $x, y \in X$ and since F is continuous,

$$\begin{aligned}
&\lim_{n\to\infty} 2^{-3n}G_{f_c}(2^n x, 2^n y) \\
&= \lim_{n\to\infty} F(2^{-3n}h(2^n\sigma_1(x,y)), 2^{-3n}h(2^n\sigma_2(x,y)), \cdots, 2^{-3n}h(2^n\sigma_l(x,y))) \\
&= G_C(x,y)
\end{aligned}$$

for all $x, y \in X$. Hence by (2.22), we have

$$C(2x+y)+C(2x-y)+G_C(x,y) = 2C(x+y)+2C(x-y)+2C(2x)-4C(x) \tag{2.23}$$

for all $x, y \in X$. Relpacing x by $2^n x$ in (2.18) and deviding (2.18) by 2^{3n}, we have

$$\begin{aligned}
&\|2^{-3}\cdot 2^{-3n}f_c(2^n\cdot 2x) - 2^{-3n}f_c(2^n x)\| \\
&\le \frac{2^{-3n}}{48|\lambda+1|}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]
\end{aligned}$$

for all $x \in X$ and letting $n \to \infty$ in the above inequality, we have

$$C(2x) = 8C(x) \tag{2.24}$$

for all $x, y \in X$. By (2.23) and (2.24), C satisifes (2.2). By **Condition $\mathbf{P_2}$**, C is a cubic mapping.

Let $F = A + C$. Then F is an additive-cubic mapping, $F_a = A$, and $F_c = C$. By (2.13) and (2.20), we have (2.6) and (2.7).

For the uniqueness of F, let H be another additive-cubic mapping with (2.6) and (2.7). Then F_a and H_a are additive mappings and hence

$$\begin{aligned}
&\|F_a(x) - H_a(x)\| = 2^{-k}\|F_a(2^k x) - H_a(2^k x)\| \\
&\le \frac{1}{6|\lambda+1|}\sum_{n=k}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]
\end{aligned}$$

for all $x \in X$. Hence, letting $k \to \infty$ in the above inequality, we have $F_a = H_a$ and similarly, we have $F_c = H_c$. Thus $F = H$. □

Similarly, we have the following theorem:

Theorem 2.2. *Let G_t be a functional operator satisfying* **Condition $\mathbf{P_1}$**, **Condition $\mathbf{P_2}$**, *and*

$$G_t(x, 0) = -G_t(0, x). \tag{2.25}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Further, suppose that there are real numbers $\lambda, \delta(\lambda \neq -1)$ such that

$$\begin{aligned} &G_t(x,2x) + 2G_t(x,x) - 2G_t(0,x) \\ &= \lambda[t(4x) - 10t(2x) + 16t(x)] + \delta[f(x) + f(-x)] \end{aligned} \tag{2.26}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (2.5). Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned} \|F_a(x) - f_a(x)\| \leq \frac{1}{12|\lambda+1|} \sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) \\ + 2\phi(2^n x, 2^n x) + |\delta|\phi(2^n x, 0) + (2+|\delta|)\phi(0, 2^n x)] \end{aligned}$$

and

$$\begin{aligned} \|F_c(x) - f_c(x)\| \leq \frac{1}{48|\lambda+1|} \sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) \\ + 2\phi(2^n x, 2^n x) + |\delta|\phi(2^n x, 0) + (2+|\delta|)\phi(0, 2^n x)] \end{aligned}$$

for all $x \in X$.

Proof. By (2.8) and (2.25), we have

$$\|f(x) + f(-x)\| \leq \phi(x,0) + \phi(0,x)$$

for all $x \in X$, because $\|G_f(x,0)\| \leq \phi(x,0)$ and $G_f(x,0) = -G_f(0,x)$. Similar to the proof of Theorem 2.1, we have

$$\begin{aligned} &\|(1+\lambda)[f(4x) - 10f(2x) + 16f(x)]\| \\ &\leq \phi(x,2x) + 2\phi(x,x) + 2\phi(0,x) + |\delta|\|f(x) + f(-x)\| \\ &\leq \phi(x,2x) + 2\phi(x,x) + |\delta|\phi(x,0) + (2+|\delta|)\phi(0,x) \end{aligned}$$

for all $x \in X$ and so we get

$$\|2^{-1}f_a(2x) - f_a(x)\| \leq \frac{1}{12|\lambda+1|}[\phi(x,2x) + 2\phi(x,x) + |\delta|\phi(x,0) + (2+|\delta|)\phi(0,x)]$$

for all $x \in X$. The rest of this proof is similar to the proof of Theorem 2.1. □

3. Applications

In this section, using Theorem 2.1 and Theorem 2.2, we will prove the generalized Hyers-Ulam stability for some additive-cubic functional equations.

First, we consider the following functional equation :

$$f(2x+y) + f(2x-y) - f(4x) = 2f(x+y) + 2f(x-y) - 8f(2x) + 12f(x). \tag{3.1}$$

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - f(4x) - 2f(x+y) - 2f(x-y) \\ &+ 8f(2x) - 12f(x)\| \leq \phi(x,y) \end{aligned} \tag{3.2}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\|F_a(x) - f_a(x)\| \leq \frac{1}{24} \sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \tag{3.3}$$

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and

$$\|F_c(x) - f_c(x)\| \leq \frac{1}{96}\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \tag{3.4}$$

for all $x \in X$.

Proof. Let $G_f(x,y) = -f(4x) + 10f(2x) - 16f(x)$. Then f satisfies (2.5) and

$$G_t(x, 2x) + 2G_t(x,x) - 2G_t(0,x) = -3[t(4x) - 10t(2x) + 16t(x)]$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. If $t : X \longrightarrow Y$ is a mapping with $t(2x) = 2t(x)$ for all $x \in X$ and (2.1), then $G_t(x,y) = 0$ for all $x, y \in X$ and so t is an additive mapping. Hence G_t satifies **Condition $\mathbf{P_1}$** and similarly G_t satifies **Condition $\mathbf{P_2}$**. By Theroem 2.1, there is an unique additive-cubic mapping $F : X \longrightarrow Y$ with (3.3) and (3.4). □

Using the above theorem, we have the following corollaries:

Corollary 3.2. *Let $f : X \longrightarrow Y$ be a mapping. Then f satisfies (3.1) if and only if f is an additive-cubic mapping.*

Ostadbashi and Kazemzadeh [9] investigated the following additive-cubic functional equation :

$$\begin{aligned} &f(2x+y) + f(2x-y) - f(4x) \\ &= 2f(x+y) + 2f(x-y) - 8f(2x) + 10f(x) - 2f(-x). \end{aligned} \tag{3.5}$$

Corollary 3.3. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - f(4x) - 2f(x+y) - 2f(x-y) \\ &+ 8f(2x) - 10f(x) + 2f(-x)\| \leq \phi(x,y) \end{aligned} \tag{3.6}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\|F_a(x) - f_a(x)\| \leq \frac{1}{24}\sum_{n=0}^{\infty} 2^{-n}[\phi_1(2^n x, 2^{n+1}x) + 2\phi_1(2^n x, 2^n x) + 2\phi_1(0, 2^n x)] \tag{3.7}$$

and

$$\|F_c(x) - f_c(x)\| \leq \frac{1}{96}\sum_{n=0}^{\infty} 2^{-3n}[\phi_1(2^n x, 2^{n+1}x) + 2\phi_1(2^n x, 2^n x) + 2\phi_1(0, 2^n x)] \tag{3.8}$$

for all $x \in X$, where $\phi_1(x,y) = \phi(x,y) + \phi(0,x)$.

Proof. By (3.6), we have

$$\|f(x) + f(-x)\| \leq \phi(0,x)$$

for all $x \in X$ and hence we have

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - f(4x) - 2f(x+y) - 2f(x-y) \\ &+ 8f(2x) - 12f(x)\| \leq \phi(x,y) + \phi(0,x) = \phi_1(x,y) \end{aligned}$$

for all $x, y \in X$. By Theorem 3.3, we have the results. □

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Finally, we consider the following new functional equation :

$$\begin{aligned} f(2x+y)+f(2x-y)-2f(x+y)+3f(x-y)-5f(y-x) \\ -10(x)+14f(y)-2f(2y)=0. \end{aligned} \tag{3.9}$$

Lemma 3.4. *Let G_f be a functional operator such that*

$$G_f(x,y)=-G_f(y,x) \tag{3.10}$$

for all mapping $f : X \longrightarrow Y$ and all $x, y \in X$. Then **Condition $\mathbf{P_1}$** *and* **Condition $\mathbf{P_2}$** *hold.*

Proof. Suppose that $f : X \longrightarrow Y$ is a mapping with $f(2x) = 2f(x)$ and (2.1). Letting $y = 0$ in (2.1), we have

$$G_f(x,0)=0 \tag{3.11}$$

for all $x \in X$ and by (3.10) and (3.11), we get

$$G_f(x,0)=-G_f(0,x)=-[f(x)+f(-x)]=0$$

for all $x \in X$. Hence

$$f(-x)=-f(x) \tag{3.12}$$

for all $x \in X$. Interchaging x and y in (2.1), by (3.12), we have

$$f(x+2y)-f(x-2y)+G_f(y,x)=2f(x+y)-2f(x-y) \tag{3.13}$$

for all $x, y \in X$ and by (2.1), (3.10), and (3.13), we have

$$f(2x+y)+f(2x-y)+f(x+2y)-f(x-2y)=4f(x+y) \tag{3.14}$$

for all $x, y \in X$. Letting $y = -y$ in (3.14), we have

$$f(2x+y)+f(2x-y)+f(x-2y)-f(x+2y)=4f(x-y) \tag{3.15}$$

for all $x, y \in X$. By (3.14) and (3.15), we have

$$f(x+y)+f(x-y)=f(x+2y)+f(x-2y) \tag{3.16}$$

for all $x, y \in X$. Letting $x = x + y$ in (3.16), we get

$$f(x+2y)+f(x)=f(x+3y)+f(x-y) \tag{3.17}$$

for all $x, y \in X$ and letting $x = 2x$ in (3.16), we get

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y) \tag{3.18}$$

for all $x, y \in X$. Letting $y = x + y$ in (3.18), we get

$$f(3x+y)+f(x-y)=2f(2x+y)-2f(y) \tag{3.19}$$

for all $x, y \in X$ and interchaging x and y in (3.19), we have

$$f(x+3y)-f(x-y)=2f(x+2y)-2f(x) \tag{3.20}$$

for all $x, y \in X$. By (3.17) and (3.20), we have

$$f(x+2y)-3f(x)+2f(x-y)=0 \tag{3.21}$$

for all $x, y \in X$. Letting $x = x - y$ in (3.21), we get

$$f(x+y)-3f(x-y)+2f(x-2y)=0 \tag{3.22}$$

for all $x, y \in X$ and letting $y = -y$ in (3.22), we get

$$f(x-y)-3f(x+y)+2f(x+2y)=0 \tag{3.23}$$

for all $x, y \in X$. By (3.21) and (3.23), we have

$$f(x+y) + f(x-y) - 2f(x) = 0$$

for all $x, y \in X$ and hence f is an additive mapping. Thus **Condition $\mathbf{P_1}$** holds.

(2) Suppose that $f : X \longrightarrow Y$ is a mapping with $f(2x) = 8f(x)$ and (2.2). Similar to (1), we have

$$G_f(x, 0) = -G_f(0, x) = 0, \ f(-x) = -f(x)$$

for all $x, y \in X$. Interchaging x and y in (2.2), we have

$$f(x+2y) - f(x-2y) + G_f(y, x) = 2f(x+y) - 2f(x-y) + 12f(y) \tag{3.24}$$

for all $x, y \in X$ and by (2.2), (3.10), and (3.24), we have

$$f(2x+y)+f(2x-y)+f(x+2y)-f(x-2y) = 4f(x+y)+12f(x)+12f(y) \tag{3.25}$$

for all $x, y \in X$. Letting $y = -y$ in (3.25), we have

$$f(2x+y)+f(2x-y)+f(x-2y)-f(x+2y) = 4f(x-y)+12f(x)-12f(y) \tag{3.26}$$

for all $x, y \in X$. By (3.25) and (3.26), we have

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

for all $x, y \in X$ and hence f is a cubic mapping. Thus **Condition $\mathbf{P_2}$** holds. □

Using Lemma 3.4, we investigate solutions and the generalized Hyers-Ulam stability for (3.9).

Theorem 3.5. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - 2f(x+y) + 3f(x-y) - 5f(y-x) \\ &- 10(x) + 14f(y) - 2f(2y)\| \le \phi(x, y) \end{aligned} \tag{3.27}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned} \|F_a(x) - f_a(x)\| \le \frac{1}{12} \sum_{n=0}^{\infty} 2^{-n} [&\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) \\ &+ 5\phi(2^n x, 0) + 7\phi(0, 2^n x)] \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \|F_c(x) - f_c(x)\| \le \frac{1}{48} \sum_{n=0}^{\infty} 2^{-3n} [&\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) \\ &+ 5\phi(2^n x, 0) + 7\phi(0, 2^n x)] \end{aligned} \tag{3.29}$$

for all $x \in X$.

Proof. Let $G_f(x, y) = 5[f(x-y) - f(y-x)] - 14[f(x) - f(y)] + 2[f(2x) - f(2y)]$. Then f satisfies (2.5) and

$$\begin{aligned} &G_t(x, 2x) + 2G_t(x, x) - 2G_t(0, x) \\ &= -2[t(4x) - 10t(2x) + 16t(x)] - 5[f(x) + f(-x)] \end{aligned}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Since G_f satifies (3.10), by Lemma 3.4, **Condition $\mathbf{P_1}$** and **Condition $\mathbf{P_2}$** satisfy. By Theroem 2.2, there is an unique additive-cubic mapping $F : X \longrightarrow Y$ with (3.28) and (3.29). □

Corollary 3.6. *Let $f : X \longrightarrow Y$ be a mapping. Then f satisfies (3.9) if and only if f is an additive-cubic mapping.*

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TOEPLITZ DUALS OF FIBONACCI SEQUENCE SPACES

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Abstract. In this paper we introduce and study some classes of almost strongly convergent difference sequences of Fibonacci numbers defined by a sequence of modulus functions. We also make an effort to study some topological properties and inclusion relations between these classes of sequences. Further, we compute toeplitz duals of theses classes and study matrix transformations on these classes of sequences.

1. Introduction and Preliminaries

Let w be the vector space of all real sequences. We shall write c, c_0 and l_∞ for the sequence spaces of all convergent, null and bounded sequences. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively. Also, we use the conventions that $e = (1, 1, 1, ...)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$.
Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix transformation from X into Y and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{A_n(x)\}$ and the A-transform of x is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (X, Y) we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax \in Y$ for all $x \in X$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

which is a sequence space. By using the matrix domain of a triangle infinite matrix, so many sequence spaces have recently been defined by several authors, (see [1], [2], [15], [25]). In the literature, the matrix domain X_Δ is called the difference sequence space whenever X is a normed or paranormed sequence space, where Δ denotes the backward difference matrix $\Delta = (\Delta_{nk})$ and $\Delta' = (\Delta'_{nk})$ denotes the forward difference matrix (the transpose of the matrix Δ), which are defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n, \\ 0 \quad , & 0 \le k < n-1 \text{ or } k > n \end{cases}$$

$$\Delta'_{nk} = \begin{cases} (-1)^{n-k}, & n \le k \le n+1, \\ 0 \quad , & 0 \le k < n \text{ or } k > n+1 \end{cases}$$

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for all $k, n \in \mathbb{N}$ respectively. The notion of difference sequence spaces was introduced by Kızmaz [16], who defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in X\}$$

for $X = l_\infty, c$ and c_0. The difference space $b\nu_p$, consisting of all sequences (x_k) such that $(x_k - x_{k-1})$ is in the sequence space l_p, was studied in the case $0 < p < 1$ by Altay and Başar [3] and in the case $1 \leq p \leq \infty$ by Başar and Altay [7] and Çolak et al. [9]. Kirişçi and Başar [15] have been introduced and studied the generalized difference sequence spaces

$$\hat{X} = \{x = (x_k) \in w : B(r, s)x \in X\}$$

where X denotes any of the spaces l_∞, l_p, c and c_0, $(1 \leq p < \infty)$ and $B(r, s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R}\backslash\{0\}$. Following Kirişçi and Başar [15], Sönmez [31] have been examined the sequence space $X(B)$ as the set of all sequences whose $B(r, s, t)$-transforms are in the space $X \in \{l_\infty, l_p, c, c_0\}$, where $B(r, s, t)$ denotes the triple band matrix $B(r, s, t) = \{b_{nk}(r, s, t)\}$ defined by

$$b_{nk}(r, s, t) = \begin{cases} r, & n = k \\ s, & n = k + 1 \\ t, & n = k + 2 \\ 0, & \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R}\backslash\{0\}$. Also in ([10-13], [26]) authors studied certain difference sequence spaces.

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. For example, the space $l_p (1 \leq p < \infty)$ is a BK-space with $\|x\|_p = \Big(\sum_{k=0}^{\infty} |x_k|^p\Big)^{\frac{1}{p}}$ and c_0, c and l_∞ are BK-spaces with $\|x\|_\infty = \sup_k |x_k|$. The sequence space X is said to be solid (see [17, p. 48]) if and only if

$$\widetilde{X} = \{(v_k) \in w : \exists (x_k) \in X \text{ such that } |v_k| \leq |x_k| \text{ for all } k \in \mathbb{N}\} \subset X.$$

A sequence (b_n) in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_m \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$.

The following lemma (known as the Toeplitz Theorem) contains necessary and sufficient condition for regularity of a matrix.

Lemma 1.1. *(Wilansky, 1984): Matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is regular if and only if the following three conditions hold:*

(1) There exists $M > 0$ such that for every $n = 1, 2, ...$ the following inequality holds:

$$\sum_{k=1}^{\infty} |a_{nk}| \leq M;$$

(2) $\lim_{n\to\infty} a_{nk} = 0$ for every $k = 1, 2, ...$

(3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

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The sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers is given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2},\ n \geq 2$. Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, in [18] some basic properties of Fibonacci numbers are given as follows:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \phi \text{ (golden ratio)},\\
&\sum_{k=0}^{n} f_k = f_{n+2} - 1 \ (n \in \mathbb{N}),\\
&\sum_k \frac{1}{f_k} \text{ converges},\\
&f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \text{ (Cassini formula)}.
\end{aligned}$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.

Now, let $A = (a_{nk})$ be an infinite matrix and list the following conditions:

$$\sup_{n\in\mathbb{N}} \sum_k \left| a_{nk} \right| < \infty \tag{1.3}$$

$$\lim_{n\to\infty} a_{nk} = 0 \ \text{ for each } k \in \mathbb{N} \tag{1.4}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} a_{nk} = \alpha_k \ \text{ for each } k \in \mathbb{N} \tag{1.5}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0 \tag{1.6}$$

$$\exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} \sum_k a_{nk} = \alpha \tag{1.7}$$

$$\sup_{k\in\mathcal{H}} \sum_n \Big| \sum_{k\in K} a_{nk} \Big| < \infty \tag{1.8}$$

where $\mathbb{C}$ and $\mathcal{H}$ denote the set of all complex numbers and the collection of all finite subsets of $\mathbb{N}$, respectively.
Now, we may give the following lemma on the characterization of the matrix transformations between some classical sequence spaces.

Lemma 1.2. *The following statements hold:*
(a) $A = (a_{nk}) \in (c_0, c_0)$ *if and only if (1.3) and (1.4) hold.*
(b) $A = (a_{nk}) \in (c_0, c)$ *if and only if (1.3) and (1.5) hold.*
(c) $A = (a_{nk}) \in (c, c_0)$ *if and only if (1.3), (1.4) and (1.6) hold.*
(d) $A = (a_{nk}) \in (c, c)$ *if and only if (1.3), (1.5) and (1.7) hold.*
(e) $A = (a_{nk}) \in (c_0, l_\infty) = (c, l_\infty)$ *if and only if condition (1.3) holds.*
(f) $A = (a_{nk}) \in (c_0, l_1) = (c, l_1)$ *if and only if condition (1.8) holds.*

Recently, Kara [19] has defined the sequence spaces $l_p(\hat{F})$ as follows:

$$l_p(\hat{F}) = \{x \in w : \hat{F}x \in l_p\},\ (1 \leq p \leq \infty)$$

where $\hat{F} = (\hat{f}_{nk})$ is the double band matrix defined by the sequence (f_n) of Fibonacci numbers as follows

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1, \\ \frac{f_n}{f_{n+1}}, & k = n, \\ 0, & 0 \le k < n-1 \text{ or } k > n \end{cases} \qquad (k, n \in \mathbb{N}).$$

Also, in [20] Kara et al. have characterized some classes of compact operators on the spaces $l_p(\hat{F})$ and $l_\infty(\hat{F})$, where $1 \le p < \infty$.
The inverse $\hat{F}^{-1} = (g_{nk})$ of the Fibonacci matrix $\hat{F}$ is given by

$$g_{nk} = \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}}, & 0 \le k \le n, \\ 0, & k > n \end{cases} \quad (k, n \in \mathbb{N}).$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ... \\ \frac{4}{1} & \frac{4}{2} & 0 & 0 & 0 & 0 & ... \\ \frac{9}{1} & \frac{9}{2} & \frac{9}{6} & 0 & 0 & 0 & ... \\ \frac{25}{1} & \frac{25}{2} & \frac{25}{6} & \frac{25}{15} & 0 & 0 & ... \\ \frac{64}{1} & \frac{64}{2} & \frac{64}{6} & \frac{64}{15} & \frac{64}{40} & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is obvious that the matrix $\hat{F}$ is a triangular matrix, that is, $f_{nn} \neq 0$ and $f_{nk} = 0$ for $k > n$ $(n = 1, 2, 3...)$. Also, it follows by Lemma 1.1 that the method $\hat{F}$ is regular.
In [8] Başarir et al. introduce the Fibonacci difference sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ as the set of all sequences whose $\hat{F}$-transforms are in the spaces c_0 and c, respectively, i.e.,

$$c_0(\hat{F}) = \left\{ x = (x_n) \in w : \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = 0 \right\},$$

and

$$c(\hat{F}) = \left\{ x = (x_n) \in w : \exists l \in \mathbb{C} \ni \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = l \right\}.$$

Define the sequence $y = (y_n)$ by the $\hat{F}$-transform of a sequence $x = (x_n)$, i.e.,

$$y_n = \hat{F}_n(x) = \begin{cases} x_0, & n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \ge 1 \end{cases} \quad (n \in \mathbb{N}). \tag{1.9}$$

A linear functional L on l_∞ is said to be a Banach limit if it has the following properties:
(1) $L(x) \ge 0$ if $n \ge 0$ (i.e. $x_n \ge 0$ for all n),
(2) $L(e) = 1$, where $e = (1, 1, ...)$,
(3) $L(Dx) = L(x)$,
where the shift operator D is defined by $D(x_n) = \{x_{n+1}\}$ (see [6]).
Let B be the set of all Banach limits on l_∞. A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [22], it was shown that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exits, uniformly in s} \right\}$$

In ([23], [24]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number l such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l| = 0, \text{ uniformly in s.}$$

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Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, pp. 183).
A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x+y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([4], [27], [28], [29], [30]) and references therein.
Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. In this paper we define the following sequence spaces:

$$c_0(\hat{F}, \mathcal{F}, u, p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} = 0 \right\},$$

and

$$c(\hat{F}, \mathcal{F}, u, p) = \left\{ x = (x_k) \in w : \exists l \in \mathbb{C} \ni \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} = l \right\}.$$

If $F_k(x) = x$, for all $k \in \mathbb{N}$. Then above sequence spaces reduces to $c_0(\hat{F}, u, p)$ and $c(\hat{F}, u, p)$.
By taking $p_k = 1$ and $u_k = 1$, for all $k \in \mathbb{N}$, then we get the sequence spaces $c_0(\hat{F}, \mathcal{F})$ and $c(\hat{F}, \mathcal{F})$.
With the notation of (1.2), the sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ can be redefined as follows:

$$c_0(\hat{F}, \mathcal{F}, u, p) = \{c_0(\mathcal{F}, u, p)\}_{\hat{F}} \text{ and } c(\hat{F}, \mathcal{F}, u, p) = \{c(\mathcal{F}, u, p)\}_{\hat{F}}. \tag{1.10}$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.11}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper, we introduce the sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$. We investigate some topological properties of these new sequence spaces and establish some inclusion relations between these spaces. Also we determine the $\alpha-, \beta-$ and $\gamma-$ duals of these spaces and construct the matrix transformation of the spaces $(c_0(\hat{F}, \mathcal{F}, u, p), X)$ and $(c(\hat{F}, \mathcal{F}, u, p), X)$, where X denote the spaces $l_\infty, f, c, f_0, c_0, bs, fs$ and l_1.

2. Some topological properties of the spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$

Theorem 2.1. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are linear spaces over the field $\mathbb{R}$ of real numbers.*

Proof. Let $x=(x_k),\ y=(y_k)\in c_0(\hat{F},\mathcal{F},u,p)$ and $\lambda,\ \mu\in\mathbb{C}$. Then there exist integers M_λ and N_μ such that $|\lambda|\le M_\lambda$ and $|\mu|\le N_\mu$. Using inequality (1.11) and definition of modulus function, we have

$$\tfrac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\lambda\Big(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big)+\mu\Big(\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big)\Big|\right]^{p_k}$$

$$\le\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k|\lambda|\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}+\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k|\mu|\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}$$

$$\le K\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_kM_\lambda\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}+K\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_kN_\mu\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}$$

$$\le KM_\lambda^H\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}+KN_\mu^H\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}$$

$\to 0$ as $n\to\infty$.

Thus $\lambda x+\mu y\in c_0(\hat{F},\mathcal{F},u,p)$. This proves that $c_0(\hat{F},\mathcal{F},u,p)$ is a linear space. Similarly we can prove that $c(\hat{F},\mathcal{F},u,p)$ is a linear space over the real field $\mathbb{R}$. □

Theorem 2.2. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are paranormed space with the paranorm defined by*

$$g(x)=\sup\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}$$

where $0\le p_k\le\sup p_k=H$, $M=\max(1,H)$.

Proof. Since the proof is similar for the space $c(\hat{F},\mathcal{F},u,p)$, we consider only the space $c_0(\hat{F},\mathcal{F},u,p)$. Clearly $g(-x)=g(x)$, for all $x\in c_0(\hat{F},\mathcal{F},u,p)$. It is trivial that $\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}=0$, for $x=0$. Hence we get $g(0)=0$. Since $\frac{p_k}{M}\le 1$, using Minkowski's inequality, we have

$$\left(\tfrac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\Big(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big)+\Big(\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big)\Big|\right]^{p_k}\right)^{\frac{1}{M}}$$

$$\le\left(\tfrac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|+u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}$$

$$\le\left(\tfrac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}+\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}.$$

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Now it follows that $g(x)$ is subadditive. Finally to check the continuity of scalar multiplication let us take any real number ρ. By definition of modulus function F_k, we have

$$\begin{aligned} g(\rho x) &= \sup_k \left(\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \rho \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \right)^{\frac{1}{M}} \\ &\leq C_\rho^{\frac{H}{M}} g(x). \end{aligned}$$

where C_ρ is a positive integer such that $|\rho| \leq C_\rho$. Now, Let $\rho \to 0$ for any fixed x with $g(x) = 0$. By definition for $|\rho| < 1$, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} < \epsilon \text{ for } n > N(\epsilon). \tag{2.1}$$

Also for $1 \leq n < N$, taking ρ small enough. Since F_k is continuous, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} < \epsilon. \tag{2.2}$$

Now from equation (2.1) and (2.2), we have

$$g(\rho x) \to 0 \text{ as } \rho \to 0.$$

This completes the proof. □

Theorem 2.3. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions, $u = (u_k)$ be a sequence of strictly positive real numbers. If $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers with $0 \leq p_k \leq q_k < \infty$ for each k, then $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c(\hat{F}, \mathcal{F}, u, q)$.*

Proof. Let $x \in c_0(\hat{F}, \mathcal{F}, u, p)$. Then

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

This implies that

$$\left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \leq 1,$$

for sufficiently large values of k. Since F_k is increasing and $p_k \leq q_k$ we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{q_k} &\leq \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $x \in c(\hat{F}, \mathcal{F}, u, q)$. This completes the proof. □

Theorem 2.4. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions and $\varrho = \lim_{t \to \infty} \dfrac{F_k(t)}{t} > 0$. Then $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c_0(\hat{F}, u, p)$.*

Proof. In order to prove that $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c_0(\hat{F}, u, p)$. Let $\varrho > 0$. By definition of ϱ, we have $F_k(t) \geq \varrho(t)$, for all $t > 0$. Since $\varrho > 0$, we have $t \leq \frac{1}{\varrho} F_k(t)$ for all $t > 0$.
Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}, u, p)$. Thus, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \leq \frac{1}{\varrho n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k}$$

which implies that $x = (x_k) \in c_0(\hat{F}, u, p)$. This completes the proof. □

Theorem 2.5. *Let $\mathcal{F}' = (F'_k)$ and $\mathcal{F}'' = (F''_k)$ are sequences of modulus functions, then*

$$c_0(\hat{F}, \mathcal{F}', u, p) \cap c_0(\hat{F}, \mathcal{F}'', u, p) \subseteq c_0(\hat{F}, \mathcal{F}' + \mathcal{F}'', u, p).$$

Proof. Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}', u, p) \cap c_0(\hat{F}, \mathcal{F}'', u, p)$. Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_k F''_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

Then we have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[u_k (F'_k + F''_k)\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \\
&\quad\le \quad K\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right\} \\
&\quad + \quad K\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_k F''_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right\} \\
&\quad\longrightarrow \quad 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\frac{1}{n}\sum_{k=1}^{n}\left[u_k (F'_k + F''_k)\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0$ as $n \to \infty$.

Therefore $x = (x_k) \in c_0(\hat{F}, \mathcal{F}' + \mathcal{F}'', u, p)$ and this completes the proof. □

Theorem 2.6. *Let $\mathcal{F} = (F_k)$ and $\mathcal{F}' = (F'_k)$ be two sequences of modulus functions, then*

$$c_0(\hat{F}, \mathcal{F}', u, p) \subseteq c_0(\hat{F}, \mathcal{F}o\mathcal{F}', u, p).$$

Proof. Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}', u, p)$. Then we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} = 0.$$

Let $\epsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $F_k(t) < \epsilon$ for $0 \le t \le \delta$.

Write $y_k = \left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]$ and consider

$$\frac{1}{n}\sum_{k=1}^{n}[F_k(y_k)]^{p_k} = \frac{1}{n}\sum_{1}[F_k(y_k)]^{p_k} + \frac{1}{n}\sum_{2}[F_k(y_k)]^{p_k}$$

where the first summation is over $y_k \le \delta$ and second summation is over $y_k \ge \delta$. Since F_k is continuous, we have

$$\frac{1}{n}\sum_{1}[F_k(y_k)]^{p_k} < \epsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

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By the definition, we have for $y_k > \delta$

$$F_k(y_k) < 2F_k(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{n}\sum_{2}[F_k(y_k)]^{p_k} \leq \max\Big(1, (2F_k(1)\delta^{-1})^H\Big)\frac{1}{n}\sum_{k}[y_k]^{p_k}. \tag{2.4}$$

From equation (2.3) and (2.4), we have

$$c_0(\hat{F}, \mathcal{F}', u, p) \subseteq c_0(\hat{F}, \mathcal{F}o\mathcal{F}', u, p).$$

This completes the proof. □

Theorem 2.7. *The sets $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are BK-spaces with the norm $\|x\|_{c_0(\hat{F},\mathcal{F},u,p)} = \|x\|_{c(\hat{F},\mathcal{F},u,p)} = \|\hat{F}x\|_\infty$.*

Proof. Since (1.10) holds, c_0 and c are the BK-spaces with respect to their natural norms and the matrix $\hat{F}$ is a triangle; Theorem 4.3.12 of Wilansky [33, p.63] gives the fact that the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are BK-spaces with the given norms. This completes the proof. □

Remark 2.8. One can easily check that the absolute property does not hold on the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$, that is, $\|x\|_{c_0(\hat{F},\mathcal{F},u,p)} \neq \||x|\|_{c_0(\hat{F},\mathcal{F},u,p)}$ and $\|x\|_{c(\hat{F},\mathcal{F},u,p)} \neq \||x|\|_{c(\hat{F},\mathcal{F},u,p)}$ for at least one sequence in the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$, and this shows that $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are the sequence spaces of non-absolute type, where $|x| = (|x_k|)$.

Theorem 2.9. *The Fibonacci difference sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ of non-absolute type are linearly isomorphic to the spaces c_0 and c respectively, i.e., $c_0(\hat{F}, \mathcal{F}, p, u) \cong c_0$ and $c(\hat{F}, \mathcal{F}, p, u) \cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0. Consider the transformation T defined with the notation of (1.9), from $c_0(\hat{F}, \mathcal{F}, u, p)$ to c_0 by $x \to y = Tx$. The linearity of T is clear. Further it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.
We assume that $y = (y_k) \in c_0$, for $1 \leq p \leq \infty$ and defined the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j, \quad \text{for all } \ k \in \mathbb{N}.$$

Then we have

$$\lim_{k\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_jf_{j+1}}y_j - \frac{f_{k+1}}{f_k}\sum_{j=0}^{k-1}\frac{f_k^2}{f_jf_{j+1}}y_j\Big|\right]^{p_k}\right\} = \lim_{k\to\infty} y_k = 0$$

which shows that $x \in c_0(\hat{F}, \mathcal{F}, p, u)$. Additionally, we have for every $x \in c_0(\hat{F}, \mathcal{F}, p, u)$ that

$$\begin{aligned}
\|x\|_{c_0(\hat{F},\mathcal{F},p,u)} &= \sup_{k\in\mathbb{N}} \left| \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \right| \\
&= \sup_{k\in\mathbb{N}} \left| \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_k^2}{f_j f_{j+1}} y_j \Big| \right]^{p_k} \right| \\
&= \sup_{k\in\mathbb{N}} \left(|y_k|^{p_k} \right) \\
&= \|y\|_\infty < \infty.
\end{aligned}$$

Consequently, we see from here that T is surjective and norm preserving. Hence, T is a linear bijection which shows that the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0 are linearly isomorphic. It is clear here that if the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0 are respectively replaced by the spaces $c(\hat{F}, \mathcal{F}, u, p)$ and c, then we obtain the fact that $c(\hat{F}, \mathcal{F}, p, u) \cong c$. This concludes the proof. $\square$

Now, we give some inclusion relations concerning with the space $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.

Theorem 2.10. *The inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ strictly holds.*

Proof. It is clear that the inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ holds. Further, to show that this inclusion is strict, consider the sequence $x = (x_k) = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j^2}$. Then, we obtain (1.9) for all $k \in \mathbb{N}$ that

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j^2} - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_{k+1}^2}{f_j^2} \Big| \right]^{p_k} = \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big(\frac{f_{k+1}}{f_k} \Big) \right]^{p_k}$$

which shows that $\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big(\frac{f_{k+1}}{f_k} \Big) \right]^{p_k} \to \varphi$, as $k \to \infty$. This is to say that $\hat{F}(x) \in c \backslash c_0$. Thus, the sequence x is in the $c(\hat{F}, \mathcal{F}, u, p)$ but not in $c_0(\hat{F}, \mathcal{F}, u, p)$. Hence, the inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ is strict. $\square$

Theorem 2.11. *The space l_∞ does not include the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.*

Proof. Let us consider the sequence $x = (x_k) = (f_{k+1}^2)$. Since $f_{k+1}^2 \to \infty$ as $k \to \infty$ and $\hat{F}(x) = e^{(0)} = (1, 0, 0, ...)$, the sequence x is in the space $c_0(\hat{F}, \mathcal{F}, u, p)$ but is not in the space l_∞. This shows that the space l_∞ does not include the space $c_0(\hat{F}, \mathcal{F}, u, p)$ and the space $c(\hat{F}, \mathcal{F}, u, p)$, as desired. $\square$

Theorem 2.12. *The inclusions $c_0 \subset c_0(\hat{F}, \mathcal{F}, u, p)$ and $c \subset c(\hat{F}, \mathcal{F}, u, p)$ strictly holds.*

Proof. Let $X = c_0$ or c. Since the matrix $\hat{F} = (f_{nk})$ satisfies the conditions

$$\sup_{n\in\mathbb{N}} \sum_k |f_{nk}| = \sup_{n\in\mathbb{N}} \Big(\frac{f_n}{f_{n+1}} + \frac{f_{n+1}}{f_n} \Big) = 2 + \frac{1}{2} = \frac{5}{2},$$

$$\lim_{n\to\infty} f_{nk} = 0,$$

$$\lim_{n\to\infty} \sum_k f_{nk} = \lim_{n\to\infty} \Big(\frac{f_n}{f_{n+1}} - \frac{f_{n+1}}{f_n} \Big) = \frac{1}{\varphi} - \varphi$$

we conclude by parts (a) and (c) of Lemma 1.2 that $(\hat{F},\mathcal{F},u,p)\in(X,X)$. This leads that $(\hat{F},\mathcal{F},u,p)x\in X$ for any $x\in X$. Thus, $x\in X_{(\hat{F},\mathcal{F},u,p)}$. This shows that $X\subset X_{(\hat{F},\mathcal{F},u,p)}$. Now, let $x=(x_k)=(f_{k+1}^2)$. Then, it is clear that $x\in X_{(\hat{F},\mathcal{F},u,p)}\backslash X$. This says that the inclusion $X\subset X_{(\hat{F},\mathcal{F},u,p)}$ is strict. □

Theorem 2.13. *The spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are not solid.*

Proof. Consider the sequences $r=(r_k)$ and $s=(s_k)$ defined by $r_k=f_{k+1}^2$ and $s_k=(-1)^{k+1}$ for all $k\in\mathbb{N}$. Then, it is clear that $r\in c_0(\hat{F},\mathcal{F},u,p)$ and $s\in l_\infty$. Nevertheless $rs=\{(-1)^{k+1}f_{k+1}^2\}$ is not in the space $c_0(\hat{F},\mathcal{F},u,p)$, since

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}(-1)^{k+1}f_{k+1}^2-\frac{f_{k+1}}{f_k}(-1)^kf_k^2\Big|\right]^{p_k}\\&=\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\big(2(-1)^{k+1}f_kf_{k+1}\big)\right]^{p_k}\quad\text{for all }\ k\in\mathbb{N}.\end{aligned}$$

This shows that the multiplication $l_\infty c_0(\hat{F},\mathcal{F},u,p)$ of the spaces l_∞ and $c_0(\hat{F},\mathcal{F},u,p)$ is not a subset of $c_0(\hat{F},\mathcal{F},u,p)$. Hence, the space $c_0(\hat{F},\mathcal{F},u,p)$ is not solid.
It is clear here that if the spaces $c_0(\hat{F},\mathcal{F},u,p)$ is replaced by the space $c(\hat{F},\mathcal{F},u,p)$, then we obtain the fact $c(\hat{F},\mathcal{F},u,p)$ is not solid. This completes the proof. □

It is known from Theorem 2.3 of Jarrah and Malkowsky [14] that the domain X_T of an infinite matrix $T=(t_{nk})$ in a normed sequence space X has a basis if and only if X has a basis, if T is a triangle. As a direct consequence of this fact, we have

Corollary 2.14. *Define the sequences $c^{(-1)}=\{c_k^{(-1)}\}_{k\in\mathbb{N}}$ and $c^{(n)}=\{c_k^{(n)}\}_{k\in\mathbb{N}}$ for every fixed $n\in\mathbb{N}$ by*

$$c_k^{(-1)}=\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_jf_{j+1}}\quad\textit{and}\quad c_k^{(n)}=\begin{cases}\quad 0 &,\ 0\le k\le n-1\\ \frac{f_{k+1}^2}{f_nf_{n+1}}, & k\ge n\end{cases}$$

Then, the following statements hold:
(a) The sequence $\{c^{(n)}\}_{n=0}^\infty$ is a basis for the space $c_0(\hat{F},\mathcal{F},u,p)$ and every sequence $x\in c_0(\hat{F},\mathcal{F},u,p)$ has a unique representation $x=\sum_n\hat{F}_n(x)c^{(n)}$.
(b) The sequence $\{c^{(n)}\}_{n=-1}^\infty$ is a basis for the space $c(\hat{F},\mathcal{F},u,p)$ and every sequence $z=(z_n)\in c(\hat{F},\mathcal{F},u,p)$ has a unique representation $z=lc^{(-1)}+\sum_n[\hat{F}_n(z)-l]c^{(n)}$, where $l=\lim_{n\to\infty}\hat{F}_n(z)$.

3. The $\alpha-,\beta-$ and $\gamma-$ duals of the spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ and some matrix transformations

The $\alpha-,\beta-$ and $\gamma-$ duals of the sequence space X are respectively defined by

$$X^\alpha=\{a=(a_k)\in w:ax=(a_kx_k)\in l_1\ \text{ for all } x=(x_k)\in X\},$$
$$X^\beta=\{a=(a_k)\in w:ax=(a_kx_k)\in cs\ \text{ for all } x=(x_k)\in X\}$$

and

$$X^\gamma=\{a=(a_k)\in w:ax=(a_kx_k)\in bs\ \text{ for all } x=(x_k)\in X\}$$

In this section, we determine $\alpha-,\beta-$ and $\gamma-$ duals of the sequence spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$, and characterize the classes of infinite matrices from the spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ to the spaces $c_0,c,l_\infty,f,f_0,bs,fs,cs$ and l_1, and from the space f to the

spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.
The following two lemmas are essential for our results.

Lemma 3.1. [8] *Let X be any of the spaces c_0 or c and $a = (a_n) \in w$, and the matrix $B = (b_{nk})$ be defined by $B_n = a_n \hat{F}_n^{-1}$, that is ,*

$$b_{nk} = \begin{cases} a_n g_{nk}, & 0 \leq k \leq n, \\ 0 \quad , & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then $a \in X_{\hat{F}}^{\beta}$ if and only if $B \in (X, l_1)$.

Lemma 3.2 (5, Theorem 3.1)**.** *Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in w$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $Z = (z_{nk})$ by*

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \leq k \leq n, \\ \quad 0 \quad , & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then for any sequence space X,

$$X_Z^{\gamma} = \{a = (a_k) \in w : C \in (X, l_\infty)\},$$

$$X_Z^{\beta} = \{a = (a_k) \in w : C \in (X, c)\}.$$

Combining Lemmas (1.2), (3.1), and (3.2), we have

Corollary 3.3. *Consider the sets d_1, d_2, d_3 and d_4 defined as follows:*

$$d_1 = \left\{ a = (a_k) \in w : \sup_{k \in \mathcal{H}} \sum_n \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{k \in K} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \right| \right]^{p_k} < \infty \right\},$$

$$d_2 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} < \infty \right\},$$

$$d_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} \text{ exists for each } k \in \mathbb{N} \right\},$$

$$d_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} \text{ exists} \right\}.$$

Then the following statements hold:
(a) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\alpha} = \{c(\hat{F}, \mathcal{F}, u, p)\}^{\alpha} = d_1$.
(b) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\beta} = d_2 \cap d_3$ and $\{c(\hat{F}, \mathcal{F}, u, p)\}^{\beta} = d_2 \cap d_3 \cap d_4$.
(c) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\gamma} = \{c(\hat{F}, \mathcal{F}, u, p)\}^{\gamma} = d_2$.

Theorem 3.4. *Let $X = c_0$ or c and Y be an arbitrary subset of w. Then, we have $A = (a_{nk}) \in (X_{\hat{F}}, Y)$ if and only if*

$$D^{(m)} = (d_{nk}^{(m)}) \in (X, c) \text{ for all } n \in \mathbb{N}, \tag{3.1}$$

$$D = (d_{nk}) \in (X, Y), \tag{3.2}$$

where

$$d_{nk}^{(m)} = \begin{cases} \frac{1}{n} \sum_{k=1}^{n} \left(u_k F_k \left| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right| \right)^{p_k}, & 0 \leq k \leq m \\ 0 & , \quad k > m \end{cases}$$

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and

$$d_{nk} = \frac{1}{n}\sum_{k=1}^{n}\left(u_k F_k \left|\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}\right|\right)^{p_k} \text{ for all } k, m, n \in \mathbb{N}.$$

By changing the roles of the spaces $X_{\hat{F}}$ and X with Y in Theorem 3.4, we have

Theorem 3.5. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} = \frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k \left| -\frac{f_{n+1}}{f_n} a_{n-1,k} + \frac{f_n}{f_{n+1}} a_{nk}\right|\right]^{p_k} \tag{3.3}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y, X_{\hat{F}})$ if and only if $B \in (Y, X)$.

Proof. Let $z = (z_k) \in Y$. Then, by taking into account the relation (3.3) one can easily derive the following equality

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{k=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k \left| -\frac{f_{n+1}}{f_n} a_{n-1,k} + \frac{f_n}{f_{n+1}} a_{nk}\right|\right]^{p_k}\right) z_k \text{ for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = [\hat{F}(Az)]_n$. Therefore, we conclude that $Az \in X_{\hat{F}}$ whenever $z \in Y$ if and only if $Bz \in X$ whenever $z \in Y$. This completes the proof. □

By f_0, f and fs we denote the spaces of almost null and almost convergent sequences and series respectively. Now, the following two lemmas characterizing the strongly and almost conservative matrices:

Lemma 3.6. *(see [32]) $A = (a_{nk}) \in (f, c)$ if and only if (1.3), (1.5), and (1.7) hold, and*

$$\lim_{n\to\infty} \sum_k \Delta(a_{nk} - \alpha_k) = 0 \tag{3.4}$$

also holds, where $\Delta(a_{nk} - \alpha_k) = a_{nk} - \alpha_k - (a_{n,k+1} - \alpha_{k+1})$ for all $k, n \in \mathbb{N}$.

Lemma 3.7. *(see [21]) $A = (a_{nk}) \in (c, f)$ if and only if (1.3) holds, and*

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N}, \tag{3.5}$$

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \alpha. \tag{3.6}$$

Now, we list the following conditions:

$$\sup_{m\in\mathbb{N}} \sum_{k=0}^{m} \left|d_{mk}^{(n)}\right| < \infty \tag{3.7}$$

$$\exists d_{nk} \in \mathbb{C} \ni \lim_{m\to\infty} d_{mk}^{(n)} = d_{nk} \text{ for each } k, n \in \mathbb{N} \tag{3.8}$$

$$\sup_{n\in\mathbb{N}} \sum_k |d_{nk}| < \infty \tag{3.9}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} d_{nk} = \alpha_k \text{ for each } k \in \mathbb{N} \tag{3.10}$$

$$\sup_{N,K\in\mathcal{H}}\Big|\sum_{n\in\mathbb{N}}\sum_{k\in K} d_{nk}\Big| < \infty \tag{3.11}$$

$$\exists\beta_n\in\mathbb{C}\ni\lim_{m\to\infty}\sum_{k=0}^{m} d_{mk}^{(n)} = \beta_n \text{ for each } n\in\mathbb{N} \tag{3.12}$$

$$\exists\alpha\in\mathbb{C}\ni\lim_{n\to\infty}\sum_{k} d_{nk} = \alpha \tag{3.13}$$

It is trivial that Theorem 3.4 and Theorem 3.5 have several consequences. Indeed, combining Theorem 3.4, 3.5 and Lemmas 1.1, 3.6 and 3.7 we derive the following results:

Corollary 3.8. *Let $A = (a_{nk})$ be an infinite matrix and $a(n,k) = \sum_{j=0}^{n} a_{jn}$ for all $k, n \in \mathbb{N}$. Then, the following statements hold:*
(a) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), c_0)$ if and only if (3.7), (3.8), (3.9) hold and (3.10) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.
(b) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), cs_0)$ if and only if (3.7), (3.8), (3.9) hold and (3.10) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $a(n,k)$ instead of a_{nk}.
(c) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), c)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold.
(d) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), cs)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $a(n,k)$ instead of a_{nk}.
(e) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), l_\infty)$ if and only if (3.7), (3.8) and (3.9) hold.
(f) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), bs)$ if and only if (3.7), (3.8) and (3.9) hold with $a(n,k)$ instead of a_{nk}.
(g) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), l_1)$ if and only if (3.7), (3.8) and (3.11) hold.
(h) $A = (a_{nk}) \in (c_0(\hat{F},\mathcal{F},u,p), bv_1)$ if and only if (3.7), (3.8) and (3.11) hold with $a_{nk} - a_{n-1,k}$ instead of a_{nk}.

Corollary 3.9. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*
(a) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), l_\infty)$ if and only if (3.7), (3.8), (3.9) and (3.12) hold.
(b) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), bs)$ if and only if (3.7), (3.8), (3.9) and (3.12) hold with $a(n,k)$ instead of a_{nk}.
(c) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), c)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold.
(d) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), cs)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold with $a(n,k)$ instead of a_{nk}.
(e) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), c_0)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$, (3.12) and (3.13) also hold with $\alpha = 0$.
(f) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), cs_0)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$, (3.12) and (3.13) also hold with $\alpha = 0$ with $a(n,k)$ instead of a_{nk}.
(g) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), l_1)$ if and only if (3.7), (3.8), (3.11) and (3.12)hold.
(h) $A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), bv_1)$ if and only if (3.7), (3.8), (3.11) and (3.12) hold with $a_{nk} - a_{n-1,k}$ instead of a_{nk}..

Corollary 3.10. *$A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), f)$ if and only if (3.7), (3.8), (3.12) and (3.13) hold, and (3.9), (3.10) also hold with d_{nk} instead of a_{nk}.*

Corollary 3.11. *$A = (a_{nk}) \in (c(\hat{F},\mathcal{F},u,p), f_0)$ if and only if (3.7), (3.8), (3.12) and (3.13) hold, and (3.9), (3.10) also hold with d_{nk} instead of a_{nk} and $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Corollary 3.12. *$A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), f_s)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold with $a(n,k)$ instead of a_{nk} and (3.9), (3.10) also hold with $d(n,k)$ instead of d_{nk}.*

Corollary 3.13. *$A = (a_{nk}) \in (f, c(\hat{F}, \mathcal{F}, u, p))$ if and only if (1.3), (1.5), (1.7) and (3.8) hold with b_{nk} instead of a_{nk}, where $b(n,k)$ is defined by (3.3) .*

Corollary 3.14. *$A = (a_{nk}) \in (f, c_0(\hat{F}, \mathcal{F}, u, p))$ if and only if (1.3) and (1.7) hold, (1.5) and (3.8) also hold with b_{nk} instead of a_{nk} and $\alpha_k = 0$ for all $k \in \mathbb{N}$, where $b(n,k)$ is defined by (3.3).*

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Completely monotonic functions involving Bateman's $G-$function

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Abstract

In this paper, we prove the complete monotonicity of some functions involving Bateman's $G-$function and show that

$$\frac{1}{2x^2+\alpha} < G(x) - \frac{1}{x} < \frac{1}{2x^2+\beta}, \qquad x > 0$$

where $\alpha = 1$ and $\beta = 0$ are the best possible constants, which is a refinement of a recent result. Then, we give a new proof of Slavić inequality about Wallis ratio W_m and provide a new inequality for W_m. Our new inequality improves some recent related works. We also present two inequalities for the hyperbolic tangent function.

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1 Introduction

A function $H : J \to \mathbb{R}$ is said to be completely monotonic (see [45] and [11]), if $H^{(m)}(x)$ exists on J for all $m \geq 0$ and

$$(-1)^m H^{(m)}(x) \geq 0 \qquad x \in J;\ m \geq 0. \tag{1}$$

For $x > 0$, the necessary and sufficient condition for the function $H(x)$ to be completely monotonic is the convergence of the following integral

$$H(x) = \int_0^\infty e^{-xt} dv(t), \tag{2}$$

where $v(t)$ is a nonnegative measure on $t \geq 0$. The function $H(x)$ is said to be strictly completely monotonic if the inequality (1) is strict for all $x \in J$ and $m \geq 0$. The concept of completely monotonic function is the continuous analogue of the totally monotone sequence presented by Hausdorff in 1921 [15] (see also [45]). These functions find applications in several diverse fields such as in the theory of special functions, asymptotic analysis, probability, physics, and the list continues, see [2], [5], [6], [12] , [13], [32], [34], [35], [38], [44] and the references therein.

The Bateman's $G-$function is defined by (see Erdélyi [10])

$$G(t) = \psi\left(\frac{t}{2} + \frac{1}{2}\right) - \psi\left(\frac{t}{2}\right), \qquad t \neq 0, -1, -2, ... \tag{3}$$

where $\psi(t)$ is the digamma (Psi) function which is defined by

$$\psi(t) = \frac{d}{dt} \ln \Gamma(t)$$

and $\Gamma(z)$ is the classical Euler gamma function which is defined for $Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-w} w^{z-1} dw.$$

For more details on bounds, identities, properties and applications of Bateman's $G-$function, refer to [10], [21]-[25], [31], [39] and the references therein. The following relations hold for the function $G(x)$ [10]:

$$G(x+1) = -G(x) + 2x^{-1}, \tag{4}$$

$$G(x) = \int_0^\infty \frac{2e^{-xv}}{1+e^{-v}} dv, \qquad x > 0 \tag{5}$$

$$G(x) = x^{-1}\, {}_2F_1\left(1, 1; 1+x; \frac{1}{2}\right), \tag{6}$$

where

$${}_lF_m(v_1, ..., v_l; w_1, ..., w_m; z) = \sum_{k=0}^\infty \frac{(v_1)_k ... (v_l)_k}{(w_1)_k ... (w_m)_k} \frac{z^k}{k!}$$

is the generalized hypergeometric function [3] defined for $l, m \in \mathbb{N}$, $v_j, w_j \in \mathbb{C}$, $w_j \neq 0, -1, -2, ...$ and

$$(v)_0 = 1 \quad \text{and} \quad (v)_n = \frac{\Gamma(v+m)}{\Gamma(v)}, \qquad n \in \mathbb{N}.$$

Qiu and Vuorinen [39] established the inequality

$$\frac{(6 - 4\ln 4)}{x^2} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 1/2 \tag{7}$$

and Mortici [25] improved the inequality (7) to the double inequality

$$0 < \psi(x+h) - \psi(x) \leq \psi(h) + \gamma - h + h^{-1}, \qquad x \geq 1;\ h \in (0,1) \tag{8}$$

where γ is the Euler constant. Mahmoud and Agarwal [21] deduced the following asymptotic formula for $x \to \infty$

$$G(x) - \frac{1}{x} \sim \sum_{k=1}^{\infty} \frac{(2^{2k}-1)B_{2k}}{k} x^{-2k}, \tag{9}$$

where $B_m{}'s$ are the Bernoulli numbers [17] and they also presented the following inequality

$$\frac{1}{2x^2 + \frac{3}{2}} < G(x) - x^{-1} < \frac{1}{2x^2}, \qquad x > 0 \tag{10}$$

which improves the lower bound of the inequality (7) for $x > \left(\frac{9-12\ln 2}{16\ln 2 - 11}\right)^{1/2}$. In [22] Mahmoud and Almuashi proved the following inequality

$$\sum_{n=1}^{2m} \frac{(2^{2n}-1)}{n} B_{2n} x^{-2n} < G(x) - x^{-1} < \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)}{n} B_{2n} x^{-2n}, \qquad m \in \mathbb{N} \tag{11}$$

where $\frac{(2^{2n}-1)}{n} B_{2n}$ are the best possible constants. Also, Mahmoud, Talat and Moustafa [23] studied the following family of approximations of Bateman's $G-$function

$$\chi(\rho, x) = \ln\left(1 + \frac{1}{x+\rho}\right) + \frac{2}{x(x+1)}, \qquad 1 \leq \rho \leq 2;\ x > 0$$

which is asymptotically equivalent to the function $G(x)$ for $x \to \infty$.

Recently, Mahmoud and Almuashi [24] presented some identities, functional equations and an asymptotic expansion of the generalized Bateman's $G-$function $G_\sigma(x)$ defined by

$$G_\sigma(x) = \psi\left(\frac{x+\sigma}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq -2r, -2r-\sigma;\ \sigma \in (0,2);\ \text{for}\ \ r = 0, 1, 2, \ldots .$$

Also, they presented the double inequality

$$\ln\left(1 + \frac{\sigma}{x+\phi}\right) < G_\sigma(x) - \frac{2\sigma}{x(x+\sigma)} < \ln\left(1 + \frac{\sigma}{x+\theta}\right), \quad x > 0;\ \sigma \in (0,2)$$

where $\phi = \frac{\sigma}{e^{\gamma + \frac{2}{\sigma} + \psi(\frac{\sigma}{2})} - 1}$ and $\theta = 1$ are the best possible constants.

In this paper, we will study the complete monotonicity of some functions involving the function $G(x)$ and as a consequence, we will deduce a double inequality of it. Also, we will prove that the function

$$q(x) = \frac{1}{G(x) - \frac{1}{x}} - 2x^2, \qquad x > 0$$

is strictly increasing and present a refinement of the lower bound of the inequality (10). We will apply our results to present a new proof of Slavić inequality about Wallis ratio $W_m = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}$ for $m \in N$. We will also present a new inequality of W_m, which improves some recent results. Further, we will present two inequalities involving the hyperbolic tangent function.

2 Main Results

We begin by proving some auxiliary results involving Bernoulli numbers.

Lemma 2.1. *For any positive integer $s \geq 1$, we have*

$$B_{2s} = \frac{1}{2(2^{2s}-1)}\left[1 - \frac{1}{2s+1}\sum_{k=1}^{s-1} 2(2^{2k}-1)\binom{2s+1}{2k} B_{2k}\right] \tag{12}$$

and

$$B_{2s} = \frac{1}{2(2^{2s}-1)}\left[s - \sum_{k=1}^{s-1}(2^{2k}-1)\binom{2s}{2k} B_{2k}\right]. \tag{13}$$

Proof. The identity [30]

$$B_m = \frac{1}{2(1-2^m)}\sum_{j=0}^{m-1} 2^j \binom{m}{j} B_j, \qquad m \in \mathbb{N} \tag{14}$$

can be rewritten as

$$B_m = \frac{1}{2(1-2^m)}[1 - m + \sum_{j=1}^{[\frac{m-1}{2}]} 2^{2j}\binom{m}{2j} B_{2j}], \qquad m \geq 1$$

where $B_{2r+1} = 0$ for $r \in \mathbb{N}$ and hence

$$\sum_{k=1}^{s} 2^{2k}\binom{2s+1}{2k} B_{2k} = 2s, \quad s \geq 1 \tag{15}$$

$$\sum_{k=1}^{s-1} 2^{2k}\binom{2s}{2k} B_{2k} = (2s-1) + 2(1-2^{2s})B_{2s}, \quad s \geq 2. \tag{16}$$

Also, Bernoulli numbers satisfy [4]

$$s - \frac{1}{2} = \sum_{k=1}^{s}\binom{2s+1}{2k} B_{2k}, \quad s \geq 1 \tag{17}$$

$$s - 1 = \sum_{k=1}^{s-1}\binom{2s}{2k} B_{2k}, \quad s \geq 2. \tag{18}$$

From the two identities (15) and (17), we get

$$\sum_{k=1}^{s} 2(2^{2k}-1)\binom{2s+1}{2k} B_{2k} = 2s+1 \qquad s \geq 1 \tag{19}$$

and the two identities (16) and (18) give us

$$2(2^{2s}-1)B_{2s} + \sum_{k=1}^{s-1}(2^{2k}-1)\binom{2s}{2k} B_{2k} = s \qquad s \geq 1. \tag{20}$$

□

Lemma 2.2. *For $v = 2, 3, 4, \cdots$, Bernoulli numbers satisfy*

$$\frac{(2^{2v+2}-1)}{(2^{2v}-1)}\pi^2 < \frac{|B_{2v}|}{|B_{2v+2}|}(2v+1)(2v+2) < \frac{(2^{2v+2}-1)}{(2^{2v}-1)}(\pi^2+1). \tag{21}$$

Proof. The function

$$f(x) = x(8x-(9+3\pi^2))+1, \qquad x \geq \frac{9+3\pi^2+\sqrt{49+54\pi^2+9\pi^4}}{16} \approx 4.80006...$$

is increasing and positive, and hence

$$2^{2v-1}(2^{2v+2}-(9+3\pi^2))+1 > 0, \qquad v \geq 2.$$

Then

$$(\pi^2+1)(2^{2v+2}-1)(2^{2v-1}-1) - \pi^2(2^{2v+1}-1)(2^{2v}-1) > 0, \qquad v \geq 2$$

or

$$\frac{(2^{2v+1}-1)}{(2^{2v-1}-1)}\pi^2 < \frac{(2^{2v+2}-1)}{(2^{2v}-1)}(\pi^2+1), \qquad v \geq 2. \tag{22}$$

From the Qi's result [36]

$$\frac{(2^{2v+2}-1)}{(2^{2v}-1)}\frac{\pi^2}{(2v+1)(2v+2)} < \frac{|B_{2v}|}{|B_{2v+2}|} < \frac{(2^{2v+1}-1)}{(2^{2v-1}-1)}\frac{\pi^2}{(2v+1)(2v+2)}, \qquad v \geq 1 \tag{23}$$

and the inequality (22), we complete the proof. □

Now we will prove the complete monotonicity of some functions involving the function $G(x)$.

Lemma 2.3. *For a positive integer m, the function*

$$F(x) = G(x) - \frac{1}{x} - \sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}}{kx^{2k}}, \qquad x > 0 \tag{24}$$

is strictly completely monotonic.

Proof. Using the formula [1]

$$\frac{1}{x^k} = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-xt}dt, \qquad k \in \mathbb{N} \tag{25}$$

and the integral representation of $G(x)$, we get

$$\begin{aligned} F(x) &= \int_0^\infty \left[e^t - 1 - (1+e^t)\sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}t^{2k-1}}{k(2k-1)!}\right]\frac{e^{-xt}}{1+e^t}dt \\ &= \int_0^\infty \varphi(t)\frac{e^{-xt}}{1+e^t}dt, \end{aligned}$$

where

$$\varphi(t) = e^t - 1 - (1+e^t)\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}. \tag{26}$$

Now

$$\begin{aligned}
\varphi(t) &= \sum_{r=1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{r=0}^{\infty}\frac{t^{r+2k-1}}{r!}\\
&= \sum_{r=1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{\infty}\frac{t^s}{(s-2k+1)!}\\
&= \sum_{r=1}^{4m}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{4m}\frac{t^s}{(s-2k+1)!}\\
&\quad+\sum_{r=4m+1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=4m+1}^{\infty}\frac{t^s}{(s-2k+1)!}.
\end{aligned}$$

Rewrite infinite summations from 0 and split finite summations by even and odd power of t we obtain

$$\begin{aligned}
\varphi(t) &= \sum_{s=1}^{2m}\frac{t^{2s-1}}{(2s-1)!}-\sum_{s=1}^{2m}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m}\frac{t^{2s-1}}{(2s-2k)!}\\
&\quad+\sum_{s=1}^{2m}\frac{t^{2s}}{(2s)!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m}\frac{t^{2s}}{(2s-2k+1)!}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!(s+4m-2k+2)!}t^{s+4m+1},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\varphi(t) &= \sum_{s=1}^{2m}\frac{t^{2s-1}}{(2s-1)!}-\sum_{s=1}^{2m}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1}-\sum_{s=1}^{2m}\frac{1}{(2s)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)(2s!)B_{2k}}{(2k)!(2s-2k)!}t^{2s-1}\\
&\quad+\sum_{s=1}^{2m}\frac{t^{2s}}{(2s)!}-\sum_{s=1}^{2m}\frac{1}{(2s+1)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)((2s+1)!)B_{2k}}{(2k)!(2s-2k+1)!}t^{2s}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{m}\left(\frac{2(2^{4k}-1)B_{4k}}{(4k)!(s+4(m-k)+2)!}+\frac{2(2^{4k-2}-1)B_{4k-2}}{(4k-2)!(s+4(m-k)+4)!}\right)t^{s+4m+1}\\
&= \sum_{s=1}^{2m}\left[2s-4(2^{2s}-1)B_{2s}-\sum_{k=1}^{s-1}2(2^{2k}-1)\binom{2s}{2k}B_{2k}\right]\frac{t^{2s-1}}{(2s)!}\\
&\quad+\sum_{s=1}^{2m}\left[2s+1-\sum_{k=1}^{s}2(2^{2k}-1)\binom{2s+1}{2k}B_{2k}\right]\frac{t^{2s}}{(2s+1)!}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{m}\left[\left(1+\frac{(2^{4k-2}-1)(4k)(4k-1)B_{4k-2}}{(2^{4k}-1)(s+4(m-k)+3)(s+4(m-k)+4)B_{4k}}\right)\right.\\
&\qquad\qquad\left.\frac{2(2^{4k}-1)t^{s+4m+1}B_{4k}}{(4k)!(s+4(m-k)+2)!}\right].
\end{aligned}$$

Using the identities (12) and (13) with the relation

$$(-1)^{r+1}B_{2r} > 0, \qquad r \in \mathbb{N} \tag{27}$$

we obtain

$$\begin{aligned} \varphi(t) &= \sum_{s=0}^{\infty}\sum_{k=1}^{m}\left[\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{(2^{4k}-1)(s+4(m-k)+3)(s+4(m-k)+4)|B_{4k}|}\right)\right. \\ &\quad \left.\frac{2(2^{4k}-1)|B_{4k}|t^{s+4m+1}}{(4k)!(s+4(m-k)+2)!}\right]+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}. \end{aligned}$$

For $s \geq 0$ and $m \geq k \geq 1$, we have

$$(s+4(m-k)+3)(s+4(m-k)+4) \geq (s+3)(s+4) \geq 12$$

and then

$$\begin{aligned} \varphi(t) &\geq \sum_{s=0}^{\infty}\sum_{k=1}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{12(2^{4k}-1)|B_{4k}|}\right)t^{s+4m+1} \\ &\quad +\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!} \\ &\geq \sum_{s=0}^{\infty}\sum_{k=2}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{12(2^{4k}-1)|B_{4k}|}\right)t^{s+4m+1} \\ &\quad +\sum_{s=0}^{\infty}\frac{30|B_4|}{(4!)(s+4m-2)!)}\left(1-\frac{|B_2|}{5|B_4|}\right)t^{s+4m+1}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}. \end{aligned}$$

Using inequality (21) with $v = 2k-1$ for $k \in \mathbb{N}$, we get

$$\varphi(t) > \sum_{s=0}^{\infty}\sum_{k=2}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{\pi^2+1}{12}\right)t^{s+4m+1}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!} > 0,$$

which complete the proof. □

Lemma 2.4. *For a positive integer m, the function*

$$M(x) = \frac{1}{x} - G(x) + \sum_{k=1}^{2m-1}\frac{(2^{2k}-1)B_{2k}}{kx^{2k}}, \qquad x > 0 \tag{28}$$

is strictly completely monotonic.

Proof. Using the formula (25) and the integral representation of $G(x)$, we have

$$\begin{aligned} M(x) &= \int_0^{\infty}\left[(1+e^t)\sum_{k=1}^{2m-1}\frac{(2^{2k}-1)B_{2k}t^{2k-1}}{k(2k-1)!}-(e^t-1)\right]\frac{e^{-xt}}{1+e^t}dt \\ &= \int_0^{\infty}\mu(t)\frac{e^{-xt}}{1+e^t}dt, \end{aligned}$$

where

$$\mu(t)=(1+e^t)\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-(e^t-1).$$

Now

$$\begin{aligned}
\mu(t) &= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}+\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{r=0}^{\infty}\frac{t^{r+2k-1}}{r!}-\sum_{r=1}^{\infty}\frac{t^r}{r!}\mu(t)\\
&= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}+\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{\infty}\frac{t^s}{(s-2k+1)!}-\sum_{r=1}^{\infty}\frac{t^r}{r!}\\
&= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}+\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{4m-2}\frac{t^s}{(s-2k+1)!}-\sum_{r=1}^{4m-2}\frac{t^r}{r!}\\
&\quad+\sum_{s=4m-1}^{\infty}\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^s}{(s-2k+1)!}-\sum_{r=4m-1}^{\infty}\frac{t^r}{r!}.
\end{aligned}$$

Rewrite infinite summations from 0 and split finite summations by even and odd power of t, we obtain

$$\begin{aligned}
\mu(t) &= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}+\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m-1}\frac{t^{2s-1}}{(2s-2k)!}-\sum_{r=1}^{2m-1}\frac{t^{2r-1}}{(2r-1)!}\\
&\quad+\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m-1}\frac{t^{2s}}{(2s-2k+1)!}-\sum_{r=1}^{2m-1}\frac{t^{2r}}{(2r)!}\\
&\quad+\sum_{s=0}^{\infty}\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^{s+4m-1}}{(s+4m-2k)!}-\sum_{r=0}^{\infty}\frac{t^{r+4m-1}}{(r+4m-1)!},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\mu(t) &= \sum_{s=1}^{2m-1}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1}+\sum_{s=1}^{2m-1}\frac{1}{(2s)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)(2s!)B_{2k}}{(2k)!(2s-2k)!}t^{2s-1}-\sum_{s=1}^{2m-1}\frac{t^{2s-1}}{(2s-1)!}\\
&\quad+\sum_{s=1}^{2m-1}\frac{1}{(2s+1)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)(2s+1)!B_{2k}}{(2k)!(2s-2k+1)!}t^{2s}-\sum_{s=1}^{2m-1}\frac{t^{2s}}{(2s)!}\\
&\quad+\sum_{s=0}^{\infty}\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^{s+4m-1}}{(s+4m-2k)!}-\sum_{s=0}^{\infty}\frac{t^{s+4m-1}}{(s+4m-1)!}\\
&= \sum_{s=1}^{2m-1}\left[2(2^{2s}-1)B_{2s}-s+\sum_{k=1}^{s-1}(2^{2k}-1)\binom{2s}{2k}B_{2k}\right]\frac{2t^{2s-1}}{(2s)!}\\
&\quad+\sum_{s=1}^{2m-1}\left[-(2s+1)+\sum_{k=1}^{s}2(2^{2k}-1)\binom{2s+1}{2k}B_{2k}\right]\frac{t^{2s}}{(2s+1)!}\\
&\quad+\sum_{s=0}^{\infty}\left[\frac{1}{2}\frac{s+4m-3}{(s+4m-1)!}+\sum_{k=2}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!(s+4m-2k)!}\right]t^{s+4m-1}.
\end{aligned}$$

Using the identities (12) and (13) with the relation (27), $\mu(t)$ satisfies

$$\begin{aligned}\mu(t) &> \sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(\frac{2(2^{4k}-1)B_{4k}}{(4k)!(s+4(m-k))!}+\frac{2(2^{4k+2}-1)B_{4k+2}}{(4k+2)!(s+4(m-k)-2)!}\right)t^{s+4m-1}\\ &> \left[\sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(1-\frac{(2^{4k}-1)(4k+1)(4k+2)|B_{4k}|}{(2^{4k+2}-1)(s+4(m-k)-1)(s+4(m-k))|B_{4k+2}|}\right)\right.\\ &\qquad\left.\frac{2(2^{4k+2}-1)|B_{4k+2}|t^{s+4m-1}}{(4k+2)!(s+4(m-k)-2)!}\right].\end{aligned}$$

For $s \geq 0$ and $m-k \geq 1$, we have

$$(s+4(m-k)-1)(s+4(m-k)) \geq (s+3)(s+4) \geq 12$$

and then μ satisfies

$$\mu(t) > \sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(1-\frac{\pi^2+1}{12}\right)\frac{2(2^{4k+2}-1)|B_{4k+2}|t^{s+4m-1}}{(4k+2)!(s+4(m-k)-2)!} > 0,$$

which complete the proof. □

From the complete monotonicity of the two functions $F(x)$ and $M(x)$ with the asymptotic expansion (9), we get the following double inequality which posed as a conjecture in [21].

Lemma 2.5. *The following double inequality holds*

$$\sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k} < G(x)-x^{-1} < \sum_{k=1}^{2l-1}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k}, \qquad l,m \in N;\ x>0. \tag{29}$$

From the positivity of the two functions $\varphi(t)$ and $\mu(t)$ in the proofs of Lemmas 2.3 and 2.4, we obtain the following result:

Lemma 2.6. *The following double inequality holds*

$$\sum_{k=1}^{2m}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1} \leq \tanh(x) \leq \sum_{k=1}^{2l-1}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1}, \qquad l,m\in N;\ x\geq 0 \tag{30}$$

and the inequality is reversed if $x \leq 0$. Equality holds if $x=0$.

Remark 1. In the case $|x| < \frac{\pi}{2}$ and l or $m =$ tends to ∞, in the inequality (30) in fact equality holds, since

$$\tanh(x) = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1}, \qquad |x| < \frac{\pi}{2}.$$

Elbert and Laforgia established the following lemma to study the monotonicity of some functions involving gamma function [9] (see also [48]).

Lemma 2.7. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then $K(x) > 0$, if $K(x) > K(x+1)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+1)$ for all $x > a$.*

To present our next result, we can easily prove the following simple modification on Lemma 2.7:

Corollary 2.8. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then for $m \in \mathbb{N}$, $K(x) > 0$, if $K(x) > K(x+m)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+m)$ for all $x > a$.*

Proof. For $m \in \mathbb{N}$, if we have $K(x) > K(x+m)$ and $\lim_{x\to\infty} K(x) = 0$, then

$$K(x) > K(x+m) > ... > K(x+rm) > ... > \lim_{r\to\infty} K(x+rm) = \lim_{y\to\infty} K(y) = 0.$$

The other case is similarly treated. □

Lemma 2.9. *The function*

$$q(x) = \frac{1}{G(x) - \frac{1}{x}} - 2x^2, \qquad x > 0 \tag{31}$$

is strictly increasing.

Proof. For $x > 0$, we have

$$q'(x) = \frac{L(x)}{[G(x) - \frac{1}{x}]^2},$$

where

$$L(x) = -G'(x) - 4xG^2(x) + 8G(x) - \frac{(4x+1)}{x^2}.$$

Now,

$$\begin{aligned} L(x+1) - L(x) &= G'(x) - G'(x+1) + 4x\left[G^2(x) - G^2(x+1)\right] - 4G^2(x+1) \\ &- 8\left[G(x) - G(x+1)\right] + \frac{4x^2+6x+1}{x^2(x+1)^2} \end{aligned}$$

and using equation (4) and its derivative, we get

$$L(x+1) - L(x) = 2G'(x) - 4G^2(x+1) + \frac{6x^2+10x+3}{x^2(x+1)^2} \triangleq L_1(x).$$

Consider the difference

$$\begin{aligned} L_1(x+2) - L_1(x) &= 2\left[G'(x+2) - G'(x)\right] - 4\left[G^2(x+3) - G^2(x+1)\right] \\ &- \frac{4\left(27 + 135x + 220x^2 + 158x^3 + 51x^4 + 6x^5\right)}{x^2(x+1)^2(x+2)^2(x+3)^2} \end{aligned}$$

and using equation (4) and its derivative, we obtain

$$\begin{aligned} L_1(x+2) - L_1(x) &= \frac{16}{(x+1)(x+2)} \left\{ G(x+1) - \frac{4x^5 + 34x^4 + 98x^3 + 99x^2 + 3x - 9}{4x^2(x+1)(x+2)(x+3)^2} \right\} \\ &\triangleq \frac{16}{(x+1)(x+2)} L_2(x). \end{aligned}$$

Using equation (4), the function $L_2(x)$ satisfies

$$L_2(x+2) - L_2(x) = -\frac{3(7x+15)(7x+20)}{2x^2(x+1)(x+2)^2(x+3)^2(x+4)(x+5)^2} < 0.$$

From the asymptotic formula (9) and its derivative

$$G'(x) \sim -\frac{1}{x^2} - \sum_{k=1}^{\infty} \frac{2(2^{2k}-1)B_{2k}}{x^{2k+1}}, \qquad x \to \infty \tag{32}$$

we have

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty} L_1(x) = \lim_{x\to\infty} L_2(x) = 0.$$

Hence, using Corollary 2.8, we get that $L(x) > 0$ for all $x > 0$ which completes the proof. □

As a consequence of the monotonicity of the function $q(x)$ with the asymptotic expansion (9), we obtain the following inequality:

Lemma 2.10. *The following double inequality holds*

$$\frac{1}{2x^2+\alpha} < G(x) - \frac{1}{x} < \frac{1}{2x^2+\beta}, \qquad x > 0 \tag{33}$$

where $\alpha = 1$ and $\beta = 0$ are the best possible constants.

Remark 2. The double inequality (33) is a refinement of the double inequality (10).

Lemma 2.11. *The function*

$$U(x) = G(x) - \frac{1}{x} - \frac{1}{2x^2+1}, \qquad x > 0 \tag{34}$$

is strictly completely monotonic.

Proof. Using the formula (25), the integral representation of $G(x)$ and the Laplace transform of sine function, we have

$$U(x) = \int_0^{\infty} \lambda(t) e^{-xt} dt,$$

where

$$\lambda(t) = \frac{e^t - 1}{e^t + 1} - \frac{1}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right).$$

Since $\sin z < 1$, we get

$$\lambda(t) > \frac{e^t - 1}{e^t + 1} - \frac{1}{\sqrt{2}} > 0, \quad t > \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 1.76275\ .$$

Also, from the generalization of Redheffer-Williams's inequality [40], [41], [42], [46]

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x} \leq \frac{12 - x^2}{12 + x^2}, \qquad 0 < x \leq \pi$$

and the inequality (30) for $m = 4$, we obtain $\lambda(t) > \frac{t^5(2352-240t^2-17t^4)}{40320(24+t^2)} > 0$ for $0 < t < \sqrt{\frac{4\sqrt{3399}-120}{17}} \approx 2.58051$. □

As a consequence of the Lemma 2.11, we get

Lemma 2.12.

1. *For odd positive integer r , we have*

$$G^{(r)}(x) < -\frac{r!}{x^{r+1}} + \frac{r!(\sqrt{2})^r}{(2x^2+1)^{r+1}} \sum_{l=1}^{\frac{r+1}{2}} (-1)^l \binom{r+1}{2l-1} (\sqrt{2}x)^{r-2l+2} \quad x > 0 \tag{35}$$

2. *For even positive integer r , we have*

$$G^{(r)}(x) > \frac{r!}{x^{r+1}} + \frac{r!(\sqrt{2})^r}{(2x^2+1)^{r+1}} \sum_{l=1}^{\frac{r}{2}+1} (-1)^{l+1} \binom{r+1}{2l-1} (\sqrt{2}x)^{r-2l+2} \quad x > 0 \tag{36}$$

Also, as a consequence of the proof of Lemma 2.11, we obtain the following inequality:

Lemma 2.13. *The following double inequality holds*

$$\tanh(x) \geq \frac{1}{\sqrt{2}} \sin(\sqrt{2}x), \qquad x \geq 0. \tag{37}$$

Equality holds iff $x = 0$.

3 Applications: Some inequalities of Wallis ratio

The Wallis ratio

$$W_m = \frac{1.3.5...(2m-1)}{2.4.6...(2m)} = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}, \qquad m \in N \tag{38}$$

plays an important role in mathematics especially in special functions, combinatorics, graph theory and many other branches. For further details about its history and applications, we refer to [7], [16], [18], [20], [26]-[29].

Guo, Xu and Qi [14] deduced the inequality

$$\frac{C_1}{m}\left(1-\frac{1}{2m}\right)^m\sqrt{m-1}<W_m\le\frac{C_2}{m}\left(1-\frac{1}{2m}\right)^m\sqrt{m-1},\quad m\ge 2 \tag{39}$$

with the best possible constants $C_1=\sqrt{\frac{e}{\pi}}$ and $C_2=\frac{4}{3}$.

Recently, Qi and Mortici [37] presented the following improvement of the double inequality (39)

$$\sqrt{\frac{e}{\pi\, m}}\left[1-\frac{1}{2(m+1/3)}\right]^{m+1/3}<W_m<\sqrt{\frac{e}{\pi\, m}}\left[1-\frac{1}{2(m+1/3)}\right]^{m+1/3}e^{\frac{1}{144m^3}},\qquad m\in N. \tag{40}$$

Also, Zhang, Xu and Situ [47] presented the inequality

$$\frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m}}<W_m\le\frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m+16}},\qquad m\in N. \tag{41}$$

Recently, Cristea [8] improved the upper bound of the inequality (41) by

$$W_m\le\frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m}+\frac{1}{48m^2}-\frac{1}{2880m^3}},\qquad m\in N \tag{42}$$

which is better than the upper bound of the inequality (40).

3.1 New proof of Slavić inequality

Slavić [43] presented the following double inequality

$$\frac{1}{\sqrt{x}}\exp\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{k(1-2k)x^{2k-1}}\right)<\frac{\Gamma(x+1/2)}{\Gamma(x+1)}<\frac{1}{\sqrt{x}}\exp\left(\sum_{k=1}^{2m}\frac{(1-2^{-2k})B_{2k}}{k(1-2k)x^{2k-1}}\right), \tag{43}$$

where $x>0$ and $l,m\in N$. In the following sequel, we will present a new proof of Slavić inequality (43). Consider the two functions

$$S_L(x)=\frac{\Gamma(x+1/2)}{\Gamma(x+1)}\sqrt{x}\exp\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right),\qquad l\in N$$

and

$$S_U(x)=\frac{\Gamma(x+1/2)}{\Gamma(x+1)}\sqrt{x}\exp\left(\sum_{k=1}^{2m}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right),\qquad m\in N.$$

Using Lemma 2.5, we obtain

$$\frac{S'_L(x)}{S_L(x)}=G(2x)-\frac{1}{2x}-\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{kx^{2k}}\right)<0,\qquad l\in N$$

and

$$\frac{S_U'(x)}{S_U(x)} = G(2x) - \frac{1}{2x} - \left(\sum_{k=1}^{2m} \frac{(1-2^{-2k})B_{2k}}{kx^{2k}}\right) > 0, \qquad m \in N.$$

Then the function $S_L(x)$ is decreasing and the function $S_U(x)$ is increasing and using the asymptotic expansion of the ratio of two gamma functions [19]

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b}\left[1 + \frac{(a-b)(a+b-1)}{2x} + O(x^{-2})\right], \qquad a, b \geq 0 \tag{44}$$

as $x \to \infty$, we have

$$\lim_{x\to\infty} S_L(x) = \lim_{x\to\infty} S_U(x) = 1.$$

Hence we get

$$S_L(x) > 1 \quad \text{and} \quad S_U(x) < 1,$$

which complete the proof of Slavić inequality (43).

Remark 3. In the case of $l = 1$, $m = 1$ and $x = m$, the inequality (43) will gives

$$\frac{e^{\frac{-1}{8m}}}{\sqrt{\pi m}} < W_m < \frac{e^{\frac{-1}{8m}+\frac{1}{192m^3}}}{\sqrt{\pi m}}, \qquad m \in N \tag{45}$$

which is better than inequality (40) of Qi and Mortici [37].

3.2 New upper bound of W_n

Consider the function

$$M_L(x) = \frac{\Gamma(x+1/2)}{\Gamma(x+1)}\sqrt{x}e^{\frac{-1}{2\sqrt{2}}\left[\tan^{-1}(2\sqrt{2}x)-\frac{\pi}{2}\right]}, \qquad x > 0.$$

Using the inequality (33), we get

$$\frac{M_L'(x)}{M_L(x)} = G(2x) - \frac{1}{2x} - \frac{1}{8x^2+1} > 0$$

and using the expansion (44), we have $\lim_{x\to\infty} M_L(x) = 1$. Then

$$M_L(x) < 1$$

and we obtain the following result:

Lemma 3.1. *The following double inequality holds*

$$\frac{\Gamma(x+1/2)}{\Gamma(x+1)} < \frac{e^{\frac{1}{2\sqrt{2}}\left[\tan^{-1}(2\sqrt{2}x)-\frac{\pi}{2}\right]}}{\sqrt{x}}, \qquad x > 0. \tag{46}$$

Remark 4. In the case of $x = m$ in the inequality (46), we have

$$W_m < \frac{e^{\frac{1}{2\sqrt{2}}\left[\tan^{-1}(2\sqrt{2}m)-\frac{\pi}{2}\right]}}{\sqrt{\pi m}}, \qquad m \in N \tag{47}$$

which is better than inequality (42) of Cristea [8].

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On Ramanujan's asymptotic formula for $n!$

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Abstract

In this paper, we present the following new asymptotic formula of factorial n

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-U(n)}, \qquad n \to \infty$$

where $U(n) = \left(\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\,n} + \frac{1455925}{5021863\,n^2} - \frac{6391301400 29}{92804028240\,n^3} + ...\right)^{-1}$ depending on Ramanujan's approximation formula for $n!$ and we deduce the following upper bound for gamma function $\Gamma(x+1) < \sqrt{\pi}\,(x/e)^x \left[8x^3+4x^2+x+\frac{1}{30}+\frac{1}{\frac{240x}{11}+\frac{9480}{847}}\right]^{1/6}$, $x > 0$.

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Key Words: Factorial, Ramanujan's formula, asymptotic formula, best possible constant, rate of convergence,bounds.

1 Introduction.

In many science branches, we need estimations of big factorials. Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \to \infty$$

is the most well known and used approximation formula for factorial n, which is satisfactory in many branches such as statistical physics and statistics but we need more precise estimates in many pure mathematics studies. For more details about Stirling's formula refinements and its related inequalities, we refer to [2], [12], [22].

Other known formula for estimating $n!$ for large values of n is Ramanujan formula:

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}}, \tag{1}$$

which is a refinement of Stirling's formula and was recorded in the book "The lost notebook and other unpublished papers" as a conjecture of Srinivasa Ramanujan based on some numerical evidence. For more details please refer to [1], [4], [13], [24], [29].

Starting from Ramanujan formula (1), Karatsuba presented the following asymptotic formula [13]

$$\Gamma(x+1) \sim \sqrt{\pi}\,(x/e)^x \left[8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\frac{79}{3360x^2}+\frac{3539}{201600x^3}+...\right]^{1/6}, \quad (2)$$

where $\Gamma(x)=\int_0^\infty e^{-r}r^{x-1}dr$, $x>0$ is the ordinary gamma function and $n!=\Gamma(n+1)$ for $n\in N$. Mortici [23] improve the Ramanujan formula by establishing the following asymptotic formula:

$$\Gamma(x+1) \sim \sqrt{\pi}\,(x/e)^x \left[8x^3+4x^2+x+\frac{1}{30}\right]^{1/6} \exp\left[-\frac{11}{11520x^4}+\frac{13}{3440x^5}+\frac{1}{691200x^6}+...\right], \quad (3)$$

which is faster than formula (2).

Dumitrescu and Mortici [9] introduced the following class of approximations:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\,(x/e)^x \sqrt[6]{1+\frac{1}{2(x-\delta)}+\frac{\alpha}{2(x-\delta)^2}+\frac{\beta}{2(x-\delta)^3}}, \qquad \alpha,\beta,\delta\in R \quad (4)$$

which is a generalization of the Ramanujan's formula (1) at $\delta=0$, $\alpha=1/8$ and $\beta=1/240$.

More various results involving approximations for the gamma function and the factorial can be found in [7], [8], [15], [16], [25], [26], [30] and the references therein.

In sequel, we need the following important Lemma, which is due to Mortici in 2010 and is a very useful tool for constructing asymptotic expansions and measuring the convergence rate of a family of null sequences [19]:

Lemma 1.1. *If $\{\sigma_m\}_{m\in N}$ is a null sequence and there is $s\in R$ and $n>1$ such that*

$$\lim_{m\to\infty} m^n(\sigma_m-\sigma_{m+1})=s, \quad (5)$$

then we have

$$\lim_{m\to\infty} m^{n-1}\sigma_m=\frac{s}{n-1}.$$

From Lemma (1.1), we can conclude that the convergence rate of the sequence $\{\sigma_m\}_{m\in N}$ will increase with the increasing of the value of n in relation (5). Several approximations, formulas and inequalities have been produced using the technique developed by this Lemma. For more details please refer to [5], [6], [11], [14], [17], [20], [21], [28] and the references therein.

In the rest of this paper, we will present a new asymptotic formula of $n!$ depending on Ramanujan's asymptotic formula (1) and we deduce a new upper bound for the ordinary gamma function related to our new asymptotic formula.

2 Main results.

In our first step, we will try to find the best possible constants k_1 and k_2 in the approximation formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{k_1 n+k_2}}, \qquad n\to\infty \tag{6}$$

by defining a sequence A_n satisfies

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{k_1 n+k_2}}\, e^{A_n}, \qquad n\geq 1.$$

Then

$$A_n - A_{n+1} = \left(\frac{1}{12k_1}-\frac{11}{2880}\right)\frac{1}{n^5}+\left(-\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}\right)\frac{1}{n^6}$$
$$+\left(\frac{-9031k_1^3+158200k_1^2+100800k_1k_2+33600k_2^2}{268800k_1^3}\right)\frac{1}{n^7}+O(n^{-8}).$$

If $\left(\frac{1}{12k_1}-\frac{11}{2880}\right)\neq 0$ and $\left(-\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}\right)\neq 0$, then the sequence $A_n - A_{n+1}$ has a rate of worse than n^{-6}. So, we will consider

$$\begin{cases} \frac{1}{12k_1}-\frac{11}{2880}=0 \\ -\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}=0 \end{cases}$$

that is, $k_1=\frac{240}{11}$ and $k_2=\frac{9480}{847}$. Now by Lemma (1.1), we obtain the following result:

Lemma 2.1. *The sequence*

$$A_n = \ln n! - \ln\sqrt{\pi} - n\ln n - n - \frac{1}{6}\ln n\left(8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}}\right) \tag{7}$$

has a rate of convergence equal to n^{-6}, where

$$\lim_{n\to\infty} n^7(A_n - A_{n+1}) = \frac{459733}{124185600}.$$

In our second step, we will try to find the best possible constants T_1, T_2 and T_3 in the approximation formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}+\frac{T_1}{n}+\frac{T_2}{n^2}+\frac{T_3}{n^3}}}, \qquad n\to\infty \tag{8}$$

by defining a sequence B_n satisfies

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}+\frac{T_1}{n}+\frac{T_2}{n^2}+\frac{T_3}{n^3}}}\, e^{B_n}, \qquad n\geq 1.$$

Hence

$$\begin{aligned}
B_n - B_{n+1} &= \frac{(919466 - 65219T_1)}{248371200\ n^7} + \frac{(454457643T_1 - 10043726T_2 - 637955952)}{32784998400\ n^8} \\
&+ \frac{1}{265066712064000\ n^9}(42535179 61T_1^2 - 1277759560770T_1 + 4664306635440T_2 \\
&- 92804028240T_3 + 16394247383595) \\
&+ \frac{1}{54427031543808000\ n^{10}}(-59336575555595T_1^2 + 1965125297982T_1T_2 \\
&+ 7507357980624 81T_1 - 361540539736530T_2 + 1184643420483 60T_3 \\
&- 8420494064916176) \\
&+ \frac{1}{3017434628788715 52000\ n^{11}}(-2774101878984 59T_1^3 + 143136026144382810T_1^2 \\
&- 79155247002714960T_1T_2 + 12105171835569120T_1T_3 \\
&- 105500477122314928 50T_1 + 60525859177845 60T_2^2 + 6180552136457196960T_2 \\
&- 2679997511635567200T_3 + 10139336461783525 5540) \\
&+ O(n^{-12}).
\end{aligned}$$

To obtain the best possible values of the constants T_1, T_2 and T_3, we put

$$\begin{cases} 65219T_1 = 919466 \\ 454457643T_1 - 10043726T_2 = 637955952 \\ 4253517961T_1^2 - 1277759560770T_1 + 4664306635440T_2 - 92804028240T_3 = -16394247383595 \end{cases},$$

that is, $T_1 = \frac{919466}{65219}$, $T_2 = \frac{1455925}{5021863}$ and $T_3 = -\frac{639130140029}{92804028240}$. Hence by Lemma (1.1), we get the following result:

Lemma 2.2. *The sequence*

$$B_n = \ln n! - \ln\sqrt{\pi} - n\ln n - n - \frac{1}{6}\ln n\left(8n^3 + 4n^2 + n + \frac{1}{30}\right.$$
$$\left. - \frac{1}{\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\ n} + \frac{1455925}{5021863\ n^2} - \frac{639130140029}{92804028240\ n^3}}\right) \tag{9}$$

has a rate of convergence equal to n^{-9}, where

$$\lim_{n\to\infty} n^{10}(B_n - B_{n+1}) = \frac{142970656174139}{108854063087616000}.$$

In our third step, we can follow the same technique to get the following result:

Lemma 2.3. *The sequence C_n defined by*

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3 + 4n^2 + n + \frac{1}{30} - V(n)}\ e^{C_n},$$

where

$$V(n) = \frac{1}{\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\ n} + \frac{1455925}{5021863\ n^2} - \frac{639130140029}{92804028240\ n^3} + \frac{T_4}{n^4} + \frac{T_5}{n^5} + \frac{T_6}{n^6}},$$

converges to zero as n^{-12} *with the best possible constants* $T_4 = \frac{142970656174139}{42875461046880}$, $T_5 = \frac{288878734012247231}{22009403337398400}$ *and* $T_6 = -\frac{54220526084844090958 73}{396565429333244371200}$ *since*

$$\lim_{n\to\infty} n^{13}(C_n - C_{n+1}) = -\frac{3843770155487944813 11979}{191419595788599033856 00000}.$$

Hence, we get the asymptotic formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3 + 4n^2 + n + \frac{1}{30} - U(n)}, \qquad n \to \infty \tag{10}$$

where

$$U(n) = \left(\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\, n} + \frac{1455925}{5021863\, n^2} - \frac{639130140029}{92804028240\, n^3} + \frac{142970656174139}{42875461046880\, n^4}\right.$$
$$\left. + \frac{288878734012247231}{22009403337398400\, n^5} - \frac{5422052608484409095873}{396565429333244371200\, n^6} + \ldots\right)^{-1}.$$

3 An inequality of Gamma function.

In this section, we will follow a method presented by Elbert and Laforgia in their paper [10] (see also, [3], [27], [32] and its simple modification in [18]):

Corollary 3.1. *Let* $T(t)$ *be a real-valued function defined on* $t > t_0 \in \mathbb{R}$ *with* $\lim_{t\to\infty} T(t) = 0$. *Then* $T(t) > 0$, *if* $T(t) > T(t+1)$ *for all* $t > t_0$ *and* $T(t) < 0$, *if* $T(t) < T(t+1)$ *for all* $t > t_0$.

Now, Consider the following function

$$F(x) = -\frac{1}{6}\ln\left(8x^3 + 4x^2 + x + \frac{1}{\frac{240x}{11} + \frac{9480}{847}} + \frac{1}{30}\right) + x - x\ln(x) + \ln\Gamma(x+1) - \ln(\sqrt{\pi}), \qquad x > 0$$

which satisfies

$$\lim_{x\to\infty} F(x) = 0.$$

$$\begin{aligned} F(x) - F(x+1) &= \frac{-1}{6}\ln\left(8x^3 + 4x^2 + x + \frac{847}{18480x + 9480} + \frac{1}{30}\right) - x\ln(x) + x\ln(x+1) \\ &+ \frac{1}{6}\ln\left(8(x+1)^3 + 4(x+1)^2 + x + \frac{847}{18480x + 27960} + \frac{31}{30}\right) - 1 \\ &\doteqdot H(x) \end{aligned}$$

The function $H(x)$ satisfies

$$H''(x) = \frac{H_1(x)}{H_2(x)} < 0, \qquad x > 0$$

where

$$\begin{aligned} H_1(x) = & - 1.84724 \times 10^{29}x^{16} - 2.37023 \times 10^{30}x^{15} - 1.39723 \times 10^{31}x^{14} - 5.01631 \times 10^{31}x^{13} \\ & - 1.22596 \times 10^{32}x^{12} - 2.15964 \times 10^{32}x^{11} - 2.83269 \times 10^{32}x^{10} - 2.81806 \times 10^{32}x^{9} \\ & - 2.14586 \times 10^{32}x^{8} - 1.25283 \times 10^{32}x^{7} - 5.57791 \times 10^{31}x^{6} - 1.86841 \times 10^{31}x^{5} \\ & - 4.59618 \times 10^{30}x^{4} - 7.9786 \times 10^{29}x^{3} - 9.149 \times 10^{28}x^{2} - 6.15185 \times 10^{27}x \\ & - 1.83421 \times 10^{26} < 0 \end{aligned}$$

and

$$H_2(x) = 3x(x+1)^2(154x+79)^2(154x+233)^2 \left(147840x^4 + 149760x^3 + 56400x^2 + 10096x + 1163\right)^2$$
$$\left(147840x^4 + 741120x^3 + 1392720x^2 + 1163536x + 365259\right)^2.$$

Then $H(x)$ is strictly concave function satisfies

$$\lim_{x\to 0} H(x) = \frac{1}{6}\left(\log\left(\frac{28855461}{270979}\right) - 6\right) < 0$$

and

$$\lim_{x\to\infty} H(x) = 0.$$

So, $F(x) < 0$ for $x > 0$ and hence we get the following inequality

Lemma 3.2.

$$\Gamma(x+1) < \sqrt{\pi}\,(x/e)^x \left[8x^3 + 4x^2 + x + \frac{1}{30} + \frac{1}{\frac{240x}{11} + \frac{9480}{847}}\right]^{1/6}, \qquad x > 0. \tag{11}$$

Remark 1. In 2018, Yang and Tian [31] presented the inequality

$$\Gamma(x+1) < \left(\frac{x^2 + \frac{6\gamma}{\pi^2 - 12\gamma}}{x + \frac{6\gamma}{\pi^2 - 12\gamma}}\right)^{\frac{6\gamma^2}{\pi^2 - 12\gamma}}, \qquad 0 < x < 1 \tag{12}$$

which is not included in inequality (11).

Remark 2. From the spirit of the previous inequality (11), we can suggest the following inequality:

$$\Gamma(x+1) > \sqrt[6]{\frac{9480}{1163}}\,(x/e)^x \left[8x^3 + 4x^2 + x + \frac{1}{30} + \frac{1}{\frac{240x}{11} + \frac{9480}{847}}\right]^{1/6}, \qquad x > 0$$

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An inertial extragradient subgradient method for solving bilevel equilibrium problems

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Abstract

In this paper, we propose an algorithm by combining an inertial term with the extragradient subgradient method for finding some solutions of bilevel equilibrium problems in a real Hilbert space. Then, we establish a strongly convergent theorem of the proposed algorithm under some sufficient assumptions on the bifunctions involving pseudomonotone and Lipschitz-type conditions. Some numerical experiments are tested to illustrate the advantage performance of our algorithm.

Keywords: Bilevel equilibrium problem; Extragradient subgradient method; Inertial method; Strong convergence

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H, and let f and g be bifunctions from $H \times H$ to $\mathbb{R}$ such that $f(x,x) = 0$ and $g(x,x) = 0$ for all $x \in H$. The equilibrium problem associated with g and C is denoted by $EP(C,g)$: Find $x^* \in C$ such that

$$g(x^*, y) \geq 0 \quad \text{for every} \quad y \in C, \tag{1.1}$$

which was considered by Blum and Oettli [4]. The solution set of problem (1.1) is denoted by Ω.

It can be seen that the equilibrium problem is related to science in various fields and is very important because many problems arise in applied areas such as the fixed point problem, the (generalized) Nash equilibrium problem in game theory, the saddle point problem, the variational inequality problem, the optimization problem and others.

The simple basic method for solving some monotone equilibrium problems is the proximal point method (see [20, 22, 27]). In 2008, Tran et al. [37] proposed the extragradient algorithm for solving the equilibrium problem by using the strongly convex minimization problem to solve at each iteration. Furthermore, Hieu [16] introduced subgradient extragradient methods for pseudomonotone equilibrium problem and the other methods (see the details in [1, 12, 21, 23, 31, 39]).

In this paper, we consider the bilevel equilibrium problems, that is, the equilibrium problem whose constraints are the solution sets of equilibrium problems: Find $x^* \in \Omega$ such that

$$f(x^*, y) \geq 0 \quad \text{for every} \quad y \in \Omega. \tag{1.2}$$

The solution set of problem (1.2) is denoted by Ω^*.

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The bilevel equilibrium problems were introduced by Chadli et al. [7] in 2000. This kind of problems is very important and interesting because it is a generalization class of problems such as optimization problems over equilibrium constraints, variational inequality over equilibrium constraints, hierarchical minimization problems, and complementarity problems. Furthermore, the particular case of the bilevel equilibrium can be applied to a real word model such as the variational inequality over the fixed point set of a firmly nonexpansive mapping applied to the power control problem of CDMA networks which were introduced by Iiduka [18]. For more on the relation of bilevel equilibrium with particular cases, see [10, 19, 30].

Methods for solving bilevel equilibrium problems have been studied extensively by many authors. In 2010, Moudafi [28] introduced a simple proximal method and proved the weak convergence to a solution of problem (1.2). In 2014, Quy [33] introduced the algorithm by combining the proximal method with the Halpern method for solving bilevel monotone equilibrium and fixed point problem. For more details and most recent works on the methods for solving bilevel equilibrium problems, we refer the reader to [2, 8, 36]. The authors considered the method for monotone and pseudoparamonotone equilibrium problem. If a bifunction is more generally monotone, we cannot use the above methods for solving bilevel equilibrium problem, for example, the pseudomonotone property.

In 2018, Yuying et. al [40] proposed a method for finding the solution for bilevel equilibrium problems where f is strongly monotone and g is pseudomonotone and Lipschitz-type continuous. They obtained the convergent sequence by combining an extragradient subgradient method with the Halpern method.

On the other hand, an inertial-type algorithm was first proposed by Polyak [32] as an acceleration process in solving a smooth convex minimisation problem. An inertial-type algorithm is a two-step iterative method in which the next iterate is defined by making use of the previous two iterates. It is well known that incorporating an inertial term in an algorithm speeds up or accelerates the rate of convergence of the sequence generated by the algorithm. Consequently, a lot of research interest is now devoted to the inertial-type algorithm(see e.g. [5, 13, 24] and the references contained in them).

Motivated and inspired by the research work in this direction, in this work, we provide an algorithm which is generated by an inertial term and the extragradient subgradient method for solving bilevel equilibrium problems in a real Hilbert space. Then, the strong convergence theorem of the proposed algorithm are established under some sufficient assumptions on the bifunctions involving pseudomonotone and Lipschitz-type conditions. The numerical experiments are investigated to illustrate the advantage performance together with some improvement of our algorithm.

2. Preliminaries

Throughout this paper, H is a real Hilbert space, C is a nonempty closed convex subset of H. Denote that $x_n \rightharpoonup x$ and $x_n \to x$ are the weak convergence and the strong convergence of a sequence $\{x_n\}$ to x, respectively. For every $x \in H$, there exists a unique element $P_C x$ defined by

$$P_C x = \operatorname{argmin}\{\|x - y\| : y \in C\},$$

which can be found, e.g., in [[6], Sect. 1.2.2, Theorem 1.7], [[11], Theorem 3.4(2)], [[14], Theorem 7.43], [[17], Chap. III, Sect. 3.1] or [[29], Theorem 8.25].

Lemma 2.1 ([15]). *The metric projection P_C has the following basic properties:*

(i) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ *for all* $x \in H$ *and* $y \in C$;
(ii) $\langle x - P_C x, P_C x - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$;
(iii) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ *for all* $x, y \in H$.

We now recall the concept of proximity operator introduced by Moreau [26]. For a proper, convex and lower semicontinuous function $g : H \to (-\infty, \infty]$ and $\gamma > 0$, the Moreau envelope of g of parameter γ is the convex function

$$^{\gamma}g(x) = \inf_{y\in H}\left\{g(y) + \frac{1}{2\gamma}\|y-x\|^2\right\} \quad \forall x \in H.$$

For all $x \in H$, the function $y \mapsto g(y)+\frac{1}{2\gamma}\|y-x\|^2$ is proper, strongly convex and lowe semicontinuous, thus the infimum is attained, i.e. $^{\gamma}g : H \to \mathbb{R}$.

The unique minimum of $y \mapsto g(y) + \frac{1}{2\gamma}\|y-x\|^2$ is called proximal point of g at x and it is denoted by $\operatorname{prox}_g(x)$. The operator

$$\operatorname{prox}_g(x) : H \to H$$

$$x \mapsto \arg\min_{y\in H}\left\{g(y) + \frac{1}{2\gamma}\|y-x\|^2\right\}$$

is well-defined and is said to be the proximity operator of g. When $g = i_C$ (the indicator function of the convex set C), one has

$$\operatorname{prox}_{i_C}(x) = P_C(x)$$

for all $x \in H$.

We also recall that the subdifferential of $g : H \to (-\infty, \infty]$ at $x \in \operatorname{dom}g$ is defined as the set of all subgradient of g at x

$$\partial g(x) := \{w \in H : g(y) - g(x) \geq \langle w, y - x\rangle \, \forall y \in H\}.$$

The function g is called subdifferentiable at x if $\partial g(x) \neq \emptyset$, g is said to be subdifferentiable on a subset $C \subset H$ if it is subdifferentiable at each point $x \in C$, and it is said to be subdifferentiable, if it is subdifferentiable at each point $x \in H$, i.e., if $\operatorname{dom}(\partial g) = H$.

The normal cone of C at $x \in C$ is defined by

$$N_C(x) := \{q \in H : \langle q, y - x\rangle \leq 0, \forall y \in C\}.$$

Definition 2.2 ([34, 35]). A bifunction $\psi : H \times H \to \mathbb{R}$ is called:

(i) β-strongly monotone on C if there exists $\beta > 0$ such that

$$\psi(x, y) + \psi(y, x) \leq -\beta\|x - y\|^2 \quad \forall x, y \in C;$$

(ii) monotone on C if

$$\psi(x, y) + \psi(y, x) \leq 0 \quad \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$\psi(x, y) \geq 0 \Rightarrow \psi(y, x) \leq 0 \quad \forall x, y \in C.$$

(iv) β-strongly pseudomonotone on C if there exists $\beta > 0$ such that

$$\psi(x, y) \geq 0 \Rightarrow \psi(y, x) \leq -\beta\|x - y\|^2 \quad \forall x, y \in C.$$

It is easy to see from the aforementioned definitions that the following implications hold,

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \quad \text{and} \quad \text{(i)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)}$$

The converses in general are not true.

In this paper, we consider the bifunctions f and g under the following conditions.

Condition A

(A1) $f(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
(A2) $f(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
(A3) f is β-strongly monotone on $H \times H$.
(A4) For each $x, y \in H$, there exists $L > 0$ such that

$$\|w - v\| \leq L\|x - y\|, \quad \forall w \in \partial f(x,\cdot)(x), v \in \partial f(y,\cdot)(y).$$

(A5) The function $x \mapsto \partial f(x,\cdot)(x)$ is bounded on the bounded subsets of H.

Condition B

(B1) $g(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
(B2) $g(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
(B3) g is pseudomonotone on C with respect to Ω, i.e.,

$$g(x, x^*) \leq 0, \quad \forall x \in C,\ x^* \in \Omega.$$

(B4) g is Lipschitz-type continuous, i.e., there exist two positive constants L_1, L_2 such that

$$g(x,y) + g(y,z) \geq g(x,z) - L_1\|x - y\|^2 - L_2\|y - z\|^2,\ \forall x, y, z \in H.$$

(B5) g is jointly weakly continuous on $H \times H$ in the sense that, if $x, y \in H$ and $\{x_n\}, \{y_n\} \in H$ converge weakly to x and y, respectively, then $g(x_n, y_n) \to g(x,y)$ as $n \to +\infty$.

Example 2.3 ([40])**.** Let $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,y) = 5y^2 - 7x^2 + 2xy$ and $g(x,y) = 2y^2 - 7x^2 + 5xy$. It follows that f and g satisfy Condition A and Condition B, respectively.

Lemma 2.4 ([3], Propositions 3.1, 3.2)**.** *If the bifunction g satisfies Assumptions (B1), (B2), and (B3), then the solution set Ω is closed and convex.*

Remark 2.5. Let the bifunction f satisfy Condition A and the bifunction g satisfy Condition B. If $\Omega \neq \emptyset$, then the bilevel equilibrium problem (1.2) has a unique solution, see the details in [33].

Lemma 2.6 ([9])**.** *Let $\phi : C \to \mathbb{R}$ be a convex, lower semicontinuous, and subdifferentiable function on C. Then x^* is a solution to the convex optimization problem*

$$\min\{f(x) : x \in C\}$$

if and only if

$$0 \in \partial\phi(x^*) + N_C(x^*).$$

The following lemmas will be used in the proof of the convergence result.

Lemma 2.7 ([38])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$, and $\{\xi_n\}$ be a sequence in $\mathbb{R}$ satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \xi_n, \quad \forall n \geq 0,$$

where $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \xi_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 ([25])**.** *Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that*

$$a_{n_j} < a_{n_j+1} \quad \textit{for all} \quad j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined, for all $n \geq n_0$, by

$$\tau(n) = \max\{k \leq n \mid a_k < a_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$, the following two estimates hold:

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \text{and} \quad a_n \leq a_{\tau(n)+1}.$$

Lemma 2.9 ([40])**.** *Suppose that f is β-strongly monotone on H and satisfies (A4). Let $0 < \alpha < 1$, $0 \leq \eta \leq 1-\alpha$, and $0 < \mu < \frac{2\beta}{L^2}$. For each $x, y \in H$, $w \in \partial f(x,\cdot)(x)$, and $v \in \partial f(y,\cdot)(y)$, we have*

$$\|(1-\eta)x - \alpha\mu w - [(1-\eta)y - \alpha\mu v]\| \leq (1-\eta-\alpha\sigma)\|x-y\|,$$

where $\sigma = 1 - \sqrt{1-\mu(2\beta-\mu L^2)} \in (0,1]$.

3. Main Result

In this section, we propose the algorithm for finding the solution of a bilevel equilibrium problem under the strong monotonicity of f and the pseudomonotonicity and Lipschitztype continuous conditions on g.

Algorithm 3.1. Initialization: Choose $x_0, x_1 \in H$, $0 < \mu < \frac{2\beta}{L^2}$, $\theta \in [0,1)$, the sequences $\{\alpha_n\} \subset (0,1)$, $\{\epsilon_n\} \subset [0,+\infty)$ and $\{\eta_n\}$ are such that

$$\begin{cases} \lim_{n\to\infty}\alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty, \\ 0 \leq \eta_n \leq 1-\alpha_n \ \forall n \geq 0, \ \lim_{n\to\infty}\eta_n = \eta < 1, \\ \sum_{n=0}^{\infty} \epsilon_n < \infty. \end{cases}$$

Select initial $x_0, x_1 \in C$ and set $n \geq 1$.
Step 1.: Given x_{n-1} and x_n $(n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where

$$\theta_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{if otherwise.} \end{cases} \tag{3.1}$$

Choose $\{\lambda_n\}$ such that

$$0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \min\left(\frac{1+\theta_n}{2L_1}, \frac{1+\theta_n}{2L_2}\right).$$

Compute

$$s_n = x_n + \theta_n(x_n - x_{n-1}),$$

$$y_n = \arg\min_{y\in C}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - s_n\|^2\right\},$$

$$z_n = \arg\min_{y\in C}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}.$$

Step 2. Compute $w_n \in \partial f(z_n,\cdot)(z_n)$ and

$$x_{n+1} = \eta_n x_n + (1-\eta_n)z_n - \alpha_n\mu w_n.$$

Set $n := n+1$ and return to **Step 1**.

Remark 3.2. Some remarks on the algorithm are in order now.

(1) Evidently, we have from (3.1) that

$$\sum_{n=0}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty, \tag{3.2}$$

due to $\theta_n \|x_n - x_{n-1}\| \leq \bar{\theta_n} \|x_n - x_{n-1}\| \leq \epsilon_n$.

(2) When $\theta_n = 0$, Algorithm 3.1 reduces to Algorithm 1 of [40].

Theorem 3.3. *Let bifunctions f and g satisfy Condition A and Condition B, respectively. Assume that $\Omega \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution of the bilevel equilibrium problem* (1.2).

Proof. Under assumptions of two bifunctions f and g, we get the unique solution of the bilevel equilibrium problem (1.2), denoted by x^*.

Step 1: Show that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1+\theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1+\theta_n - 2\lambda_n L_2)\|y_n - z_n\|^2 - \theta_n\|x_n - x_{n-1}\|^2. \tag{3.3}$$

The definition of y_n and Lemma 2.6 imply that

$$0 \in \partial \left\{ \lambda_n g(x_n, y) + \frac{1}{2}\|y - s_n\|^2 \right\} (y_n) + N_C(y_n).$$

There are $w \in \partial g(x_n, \cdot)(y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$\lambda_n w + y_n - s_n + \bar{w} = 0. \tag{3.4}$$

Since $\bar{w} \in N_C(y_n)$, we have

$$\langle \bar{w}, y - y_n \rangle \leq 0 \quad \text{for all}\, y \in C. \tag{3.5}$$

By using (3.4) and (3.5), we obtain $\lambda_n \langle w, y - y_n \rangle \geq \langle s_n - y_n, y - y_n \rangle$ for all $y \in C$. Since $z_n \in C$, we have

$$\lambda_n \langle w, z_n - y_n \rangle \geq \langle s_n - y_n, z_n - y_n \rangle. \tag{3.6}$$

It follows from $w \in \partial g(x_n, \cdot)(y_n)$ that

$$g(x_n, y) - g(x_n, y_n) \geq \langle w, y - y_n \rangle \quad \text{for all}\, y \in H. \tag{3.7}$$

By using (3.6) and (3.7), we get

$$\lambda_n \{g(x_n, z_n) - g(x_n, y_n)\} \geq \langle s_n - y_n, z_n - y_n \rangle. \tag{3.8}$$

Similarly, the definition of z_n implies that

$$0 \in \partial \left\{ \lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\} (z_n) + N_C(z_n).$$

There are $u \in \partial g(y_n, \cdot)(z_n)$ and $\bar{u} \in N_C(x)$ such that

$$\lambda_n u + z_n - x_n + \bar{u} = 0. \tag{3.9}$$

Since $\bar{u} \in N_C(z_n)$, we have

$$\langle \bar{u}, y - z_n \rangle \leq 0 \quad \text{for all}\, y \in C. \tag{3.10}$$

By using (3.9) and (3.10), we obtain $\lambda_n\langle u, y - z_n\rangle \geq \langle x_n - z_n, y - z_n\rangle$ for all $y \in C$. Since $x^* \in C$, we have

$$\lambda_n\langle u, x^* - z_n\rangle \geq \langle x_n - z_n, x^* - z_n\rangle \tag{3.11}$$

It follows from $u \in \partial g(y_n, \cdot)(z_n)$ that

$$g(y_n, y) - g(y_n, z_n) \geq \langle u, y - z_n\rangle \quad \text{for all}\, y \in H. \tag{3.12}$$

By using (3.11) and (3.12), we get

$$\lambda_n\{g(y_n, x^*) - g(y_n, z_n)\} \geq \langle x_n - z_n, x^* - z_n\rangle.$$

Since $x^* \in \Omega$, we have $g(x^*, y_n) \geq 0$. If follows from the pseudomonotonicity of g on C with respect to Ω that $g(y_n, x^*) \leq 0$. This implies that

$$\langle x_n - z_n, z_n - x^*\rangle \geq \lambda_n g(y_n, z_n). \tag{3.13}$$

Since g is Lipschitz-type continuous, there exist two positive constants L_1, L_2 such that

$$g(y_n, z_n) \geq g(x_n, z_n) - g(x_n, y_n) - L_1\|x_n - y_n\|^2 - L_2\|y_n - z_n\|^2. \tag{3.14}$$

By using (3.13) and (3.14), we get

$$\langle x_n - z_n, z_n - x^*\rangle \geq \lambda_n\{g(x_n, z_n) - g(x_n, y_n)\} - \lambda_n L_1\|x_n - y_n\|^2 - \lambda_n L_2\|y_n - z_n\|^2.$$

From (3.8) and the above inequality, we obtain

$$2\langle x_n - z_n, z_n - x^*\rangle \geq 2\langle s_n - y_n, z_n - y_n\rangle - 2\lambda_n L_1\|x_n - y_n\|^2 - 2\lambda_n L_2\|y_n - z_n\|^2. \tag{3.15}$$

By the definition of s_n, we have that

$$\begin{aligned} 2\langle s_n - y_n, z_n - y_n\rangle &= 2\langle x_n + \theta_n(x_n - x_{n-1}) - y_n, z_n - y_n\rangle \\ &= -2\langle x_n - y_n, y_n - z_n\rangle + 2\theta_n\langle x_n - x_{n-1}, z_n - y_n\rangle. \end{aligned}$$

We know that

$$\begin{aligned} 2\langle x_n - z_n, z_n - x^*\rangle &= \|x_n - x^*\|^2 - \|z_n - x_n\|^2 - \|z_n - x^*\|^2 \\ -2\langle x_n - y_n, y_n - z_n\rangle &= -\|x_n - z_n\|^2 + \|x_n - y_n\|^2 + \|y_n - z_n\|^2 \\ 2\theta_n\langle x_n - x_{n-1}, z_n - y_n\rangle &= \theta_n(\|x_n - y_n\|^2 - \|x_n - x_{n+1}\|^2 - \|y_n - z_n\|^2). \end{aligned}$$

From (3.15), we can conclude that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1+\theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1+\theta_n - 2\lambda_n L_2)\|y_n - z_n\|^2 - \theta_n\|x_n - x_{n-1}\|^2. \tag{3.16}$$

Step 2: The sequences $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.
Since $0 < \lambda_n < a$, where $a = \min\left\{\frac{1+\theta_n}{2L_1}, \frac{1+\theta_n}{2L_2}\right\}$, we have

$$(1 + \theta_n - 2\lambda_n L_1) > 0 \quad \text{and} \quad (1 + \theta_n - 2\lambda_n L_2) > 0.$$

It follows from (3.3) and the above inequalities that

$$\|z_n - x^*\| \leq \|x_n - x^*\| \quad \text{for all}\, n \in \mathbb{N}. \tag{3.17}$$

By Lemma 2.9 and (3.17), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\eta_n x_n + (1-\eta_n)z_n - \alpha\mu w_n - x^* + \eta_n x^* - \eta_n x^* + \alpha_n\mu v - \alpha_n\mu v\| \\
&= \|(1-\eta_n)z_n - \alpha_n\mu w_n - (1-\eta_n)x^* + \alpha_n\mu v + \eta_n(x_n - x^*) - \alpha_n\mu v\| \\
&\le \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v]\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\
&\le (1-\eta_n-\alpha_n\sigma)\|z_n - x^*\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\
&\le (1-\eta_n-\alpha_n\sigma)\|x_n - x^*\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\
&= (1-\alpha_n\sigma)\|x_n - x^*\| + \alpha_n\tau\left(\frac{\mu\|v\|}{\sigma}\right), \qquad (3.18)
\end{aligned}$$

where $w_n \in \partial f(z_n,\cdot)(z_n)$ and $v \in \partial f(x^*,\cdot)(x^*)$. This implies that

$$\|x_{n+1}-x^*\| \le \max\left\{\|x_n - x^*\|, \frac{\mu\|v\|}{\sigma}\right\}.$$

By induction, we obtain

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\mu\|v\|}{\sigma}\right\}.$$

Thus the sequence $\{x_n\}$ is bounded. By using (3.17), we have $\{z_n\}$, and using Condition (A5), we can conclude that $\{w_n\}$ is also bounded.

Step 3: Show that the sequence $\{x_n\}$ converges strongly to x^*.

Since $x^* \in \Omega^*$, we have $f(x^*, y) \ge 0$ for all $y \in \Omega$. Note that $f(x^*, x^*) = 0$. Thus x^* is a minimum of the convex function $f(x^*,\cdot)$ over Ω. By Lemma 2.6, we obtain $0 \in \partial f(x^*,\cdot)(x^*) + N_\Omega(x^*)$. Then there exists $v \in \partial f(x^*,\cdot)(x^*)$ such that

$$\langle v, z - x^*\rangle \ge 0 \quad \text{for all } z \in \Omega. \qquad (3.19)$$

Note that

$$\|x-y\|^2 \le \|x\|^2 - 2\langle y, x-y\rangle \quad \text{for all } x, y \in H. \qquad (3.20)$$

From Lemma 2.9 and (3.20), we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&= \|\eta_n x_n + (1-\eta_n)z_n - \alpha\mu w_n - x^*\|^2 \\
&= \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*) - \alpha_n\mu v\|^2 \\
&\le \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*)\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\
&\le \left\{\|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*)\|^2\right\} - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\
&\le [(1-\eta_n-\alpha_n\sigma)\|z_n - x^*\| + \eta_n\|x_n - x^*\|]^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\
&\le (1-\eta_n-\alpha_n\sigma)\|z_n - x^*\|^2 + \eta_n\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\
&\le (1-\eta_n-\alpha_n\sigma)\|x_n - x^*\|^2 + \eta_n\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n\sigma)\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \qquad (3.21)
\end{aligned}$$

It follows that

$$\|x_{n+1}-x^*\|^2 \le (1-\alpha_n\sigma)\|x_n - x^*\|^2 + 2\alpha_n\mu\langle v, x^* - x_{n+1}\rangle. \qquad (3.22)$$

Let us consider two cases.

Case 1: There exists n_0 such that $\|x_n - x^*\|$ is decreasing for $n \geq n_0$. Therefore the limit of sequence $\|x_n - x^*\|$ exists. By using (3.17) and (3.22), we obtain

$$\begin{aligned}
0 &\leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2 \\
&\leq -\frac{\alpha_n \sigma}{1-\eta_n}\|z_n - x^*\|^2 - \frac{2\alpha_n \mu}{1-\eta_n}\langle v, x_{n+1} - x^*\rangle \\
&\quad + \frac{1}{1-\eta_n}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2).
\end{aligned}$$

Since $\lim_{n\to\infty} = \eta_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and the limit of $\|x_n - x^*\|$ exists, we have

$$\lim_{n\to\infty}(\|x_n - x^*\|^2 - \|z_n - x^*\|^2) = 0. \tag{3.23}$$

From $0 < \lambda_n < a$ and inequality (3.3), we get

$$(1 + \theta_n - a)\|x_n - y_n\|^2 \leq (1 + \theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2.$$

By using (3.23), we obtain $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Next, we show that

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle \leq 0. \tag{3.24}$$

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle = \limsup_{k\to\infty}\langle v, x^* - x_{n_k}\rangle.$$

Since $\{x_{n_k}\}$ is bounded, we may assume that $\{x_{n_k}\}$ converges weakly to some $\bar{x} \in H$. Therefore

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle = \limsup_{k\to\infty}\langle v, x^* - x_{n_k}\rangle = \langle v, x^* - \bar{x}\rangle. \tag{3.25}$$

Since $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we have $y_{n_k} \rightharpoonup \bar{x}$. Let us consider that

$$\lim_{n\to\infty}\|s_n - y_n\| \leq \lim_{n\to\infty}\|s_n - x_n\| + \lim_{n\to\infty}\|x_n - y_n\|.$$

By the definition of s_n, we have that

$$\begin{aligned}
\lim_{n\to\infty}\|s_n - x_n\| &= \lim_{n\to\infty}\|x_n - \theta_n(x_n - x_{n-1}) - x_n\| \\
&= \lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\|.
\end{aligned}$$

Using the assumption $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < \infty$, it implies that $\lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| = 0$. Thus $\lim_{n\to\infty}\|s_n - x_n\| = 0$. Since $\lim_{n\to\infty}\|s_n - x_n\| = 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we have $s_{n_k} \rightharpoonup \bar{x}$. Since C is closed and convex, it is also weakly closed and thus $\bar{x} \in C$. Next, we show that $\bar{x} \in \Omega$. From the definition of $\{y_n\}$ and Lemma 2.6, we obtain

$$0 \in \partial\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|s_n - y\|^2\right\}(y_n) + N_C(y_n).$$

There exist $\bar{w} \in N_C(y_n)$ and $w \in \partial g(x_n, \cdot)(y_n)$ such that

$$\lambda_n w + y_n - s_n + \bar{w} = 0. \tag{3.26}$$

Since $\bar{w} \in N_C(y_n)$, we have $\langle \bar{w}, y - y_n\rangle \leq 0$ for all $y \in C$. From (3.26), we obtain

$$\lambda_n\langle w, y - y_n\rangle \geq \langle s_n - y_n, y - y_n\rangle \quad \text{for all}\, y \in C. \tag{3.27}$$

Since $w \in \partial g(x_n, \cdot)(y_n)$, we have

$$g(x_n, y) - g(x_n, y_n) \geq \langle w, y - y_n \rangle \quad \text{for all}\, y \in H. \tag{3.28}$$

Combining (3.27) and (3.28), we get

$$\lambda_n\{g(x_n, y) - g(x_n, y_n)\} \geq \langle s_n - y_n, y - y_n \rangle \quad \text{for all}\, y \in C. \tag{3.29}$$

Taking $n = n_k$ and $k \to \infty$ in (3.29), the assumption of λ_n and (B5), we obtain $g(\bar{x}, y) = 0$ for all $y \in C$. This implies that $\bar{x} \in \Omega$. By inequality (3.19), we obtain $\langle v, \bar{x} - x^* \rangle \geq 0$. It follows from (3.25) that

$$\limsup_{n\to\infty} \langle v, x^* - x_{n+1} \rangle \leq 0. \tag{3.30}$$

We can write inequality (3.22) in the following form:

$$\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n \sigma)\|x_n - x^*\|^2 + \alpha_n \sigma \xi_n,$$

where $\xi_n = \frac{2\mu}{\sigma}\langle v, x^* - x_{n+1} \rangle$. It follows from (3.30) that $\limsup_{n\to\infty} \xi_n \leq 0$. By Lemma 2.7, we can conclude that $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$. Hence $x_n \to x^*$ as $n \to \infty$.

Case 2: There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - x^*\| \leq \|x_{n_{j+1}} - x^*\|$ for all $j \in \mathbb{N}$. By Lemma 2.8, there exists a nondecreasing sequence $\{\tau(n)\}$ of N such that $\lim_{n\to\infty} \tau(n) = \infty$, and for each sufficiently large $n \in \mathbb{N}$, we have

$$\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \quad \text{and} \quad \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\|. \tag{3.31}$$

Combining (3.18) and (3.31), we have

$$\begin{aligned}\|x_{\tau(n)} - x^*\| &\leq \|x_{\tau(n)+1} - x^*\| \\ &\leq (1 - \eta_{\tau(n)} - \alpha_{\tau(n)}\sigma)\|z_{\tau(n)} - x^*\| + \eta_{\tau(n)}\|x_{\tau(n)} - x^*\| + \alpha_{\tau(n)}\mu\|v\|. \end{aligned} \tag{3.32}$$

From (3.17) and (3.32), we get

$$0 \leq \|x_{\tau(n)} - x^*\| - \|z_{\tau(n)} - x^*\| \leq -\frac{\alpha_{\tau(n)}\sigma}{1 - \eta_{\tau(n)}}\|z_{\tau(n) - x^*}\| + \frac{\alpha_{\tau(n)}\sigma}{1 - \eta_{\tau(n)}}\|v\|. \tag{3.33}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \eta_n = \eta < 1$, $\{z_n\}$ is bounded, and (3.33), we have $\lim_{n\to\infty}(\|x_{\tau(n)} - x^*\| - \|z_{\tau(n)} - x^*\|) = 0$. It follows from the boundedness of $\{x_n\}$ and $\{z_n\}$ that

$$\lim_{n\to\infty} (\|x_{\tau(n)} - x^*\|^2 - \|z_{\tau(n)} - x^*\|^2) = 0. \tag{3.34}$$

By using the assumption of $\{\lambda_n\}$, we get the following two inequalities:

$$1 + \theta_n - 2\lambda_{\tau(n)}L_1 > 1 + \theta_n - 2aL_1 > 0 \quad \text{and} \quad 1 + \theta_n - 2\lambda_{\tau(n)}L_2 > 1 + \theta_n - 2aL_2 > 0.$$

From (3.3), we obtain

$$\begin{aligned}\|z_{\tau(n)} - x^*\|^2 &\leq \|x_{\tau(n)} - x^*\|^2 - (1 + \theta_n - 2\lambda_{\tau(n)}L_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad - (1 + \theta_n - 2\lambda_{\tau(n)}L_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - x^*\|^2 - (1 + \theta_n - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad - (1 + \theta_n - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2. \end{aligned}$$

This implies that

$$\begin{aligned}0 &< (1 + \theta_n - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 + (1 + \theta_n - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - x^*\|^2 - \|z_{\tau(n)} - x^*\|^2. \end{aligned}$$

It follows from (3.34) and the above inequality that

$$\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{3.35}$$

Note that $\|x_{\tau(n)} - z_{\tau(n)}\| \le \|x_{\tau(n)} - y_{\tau(n)}\| + \|y_{\tau(n)} - z_{\tau(n)}\|$. From (3.35), we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{3.36}$$

By using the definition of x_{n+1} and Lemma 2.9, we obtain

$$\begin{aligned}
\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 &= \|\eta_{\tau(n)}x_{\tau(n)} + (1-\eta_{\tau(n)})z_{\tau(n)} - \alpha_{\tau(n)}\mu t_{\tau(n)} - x_{\tau(n)}\| \\
&= \|(1-\eta_{\tau(n)})z_{\tau(n)} - \alpha_{\tau(n)}\mu t_{\tau(n)} \\
&\quad - [(1-\eta_{\tau(n)})x_{\tau(n)} - \alpha_{\tau(n)}w_{\tau(n)}] - \alpha_{\tau(n)}w_{\tau(n)}\| \\
&\le \|(1-\eta_{\tau(n)})z_{\tau(n)} - \alpha_{\tau(n)}t_{\tau(n)} \\
&\quad - [(1-\eta_{\tau(n)})x_{\tau(n)} - \alpha_{\tau(n)}w_{\tau(n)}]\| + \alpha_{\tau(n)}\|w_{\tau(n)}\| \\
&\le (1-\eta_{\tau(n)} - \alpha_{\tau(n)}\sigma)\|z_{\tau(n)} - x_{\tau(n)}\| + \alpha_{\tau(n)}\|w_{\tau(n)}\| \\
&\le \|z_{\tau(n)} - x_{\tau(n)}\| + \alpha_{\tau(n)}\|w_{\tau(n)}\|,
\end{aligned}$$

where $t_{\tau(n)} \in \partial f(z_{\tau(n)}, \cdot)(z_{\tau(n)})$ and $w_{\tau(n)} \in \partial f(x_{\tau(n)}, \cdot)(x_{\tau(n)})$. Since $\lim_{n\to\infty} \alpha_n = 0$, the boundedness of $\{w_{\tau(n)}\}$ and (3.36), we have $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$. As proved in the first case, we can conclude that

$$\limsup_{n\to\infty} \langle v, x^* - x_{\tau(n)+1}\rangle = \limsup_{n\to\infty} \langle v, x^* - x_{\tau(n)}\rangle \le 0. \tag{3.37}$$

Combining (3.22) and (3.31), we obtain

$$\begin{aligned}
\|x_{\tau(n)+1} - x^*\|^2 &\le (1-\alpha_{n(\tau)}\sigma)\|x_{\tau(n)} - x^*\|^2 + 2\alpha_{n(\tau)}\mu\langle v, x^* - x_{\tau(n)+1}\rangle \\
&\le (1-\alpha_{n(\tau)}\sigma)\|x_{\tau(n)+1} - x^*\|^2 + 2\alpha_{n(\tau)}\mu\langle v, x^* - x_{\tau(n)+1}\rangle.
\end{aligned}$$

By using (3.31) again, we have

$$\|x_n - x^*\|^2 \le \|x_{\tau(n)+1} - x^*\|^2 \le \frac{2\mu}{\sigma}\langle v, x^* - x_{\tau(n)+1}\rangle.$$

From (3.37), we can conclude that $\limsup_{n\to\infty} \|x_n - x^*\|^2 \le 0$. Hence $x_n \to x^*$ as $n \to \infty$. This completes the proof. ∎

4. Numerical example

In this section, we provide a numerical example to test our algorithm. All Matlab colds were performed on a computer with CPU Intel Core i7-7500U, up to 3.5GHz, 4GB of RAM under version MATLAB R2015b. In the following example, we use the standard Euclidean norm and inner product.

Example 4.1. We compare our algorithm with Algorithm 1 proposed in Yuying et al. [40]. Let us consider a problem when $H = \mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n : -5 \le x_i \le 5, \forall i \in \{1, 2, ..., n\}\}$. Let the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$g(x, y) = \langle Px + Qy, y - x\rangle \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$

where P and Q are randomly symmetric positive semidefinite matrices such that $P - Q$ is positive semidefinite. Then g is pseudomonotone on $\mathbb{R}^n$. Next, we obtain that g is Lipschitz-type continuous with $L_1 = L_2 = \frac{1}{2}\|P - Q\|$. Furthermore, we define the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$f(x, y) = \langle Ax + By, y - x\rangle \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$

Table 1: Comparison: proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall i = 1, 2, ..., n\}$.

n	Algorithm 3.1				Yuying et al. Algorithm	
	$\theta = 0.6$		$\theta = 0.9$			
	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)
5	29	1.0015	28	1.0178	34	1.3618
10	43	1.7310	38	1.3645	54	1.9099
50	90	4.3222	88	4.6822	98	5.5028

Table 2: Comparison: proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall i = 1, 2, ..., n\}$.

n	Algorithm 3.1				Yuying et al. Algorithm	
	$\theta = 0.6$		$\theta = 0.9$			
	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)
5	32	1.1074	30	1.0388	37	1.3528
10	50	1.8239	45	1.8472	61	2.3260
50	108	6.6858	105	6.5254	116	6.7247

with A and B being positive definite matices defined by

$$B = N^T N + nI_n \quad \text{and} \quad A = B + M^T M + nI_n, \tag{4.1}$$

where M, N are randomly $n \times n$ matrices and I_n is the identity matrix.
Moreover, $\partial f(x, \cdot)(x) = \{(A+B)x\}$ and $\|(A+B)x - (A+B)y\| \leq \|A+B\|\|x-y\|$ for all $x, y \in \mathbb{R}^n$. Thus the mapping $x \to \partial f(x, \cdot)(x)$ is bounded and $\|A+B\|$-Lipschitz continuous on every bounded subset of H.

It is easy to see that all the conditions of Theorem 3.3 and of Theorem 3.1 in [40] are satisfied. New, we compare the performance of our algorithm and algorithm of Yuying et al. [40], we take $\lambda_k = \dfrac{1}{k+5}$, $\alpha_k = \dfrac{1}{k+4}$, $\eta_k = \dfrac{k+1}{3(k+4)}$, $\mu = \dfrac{2}{\|A+B\|^2}$, the same starting point $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall\, i = 1, 2, ..., n\}$ and $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall\, i = 1, 2, ..., n\}$ for all the algorithms. For Algorithm 3.1, we choose $\epsilon_k = \dfrac{1}{k^{1.1}}$, $\theta \in [0, 1)$ and θ_k such that $0 \leq \theta_k \leq \bar{\theta_k}$, where

$$\theta_k = \begin{cases} \min\left\{\theta, \dfrac{1}{k^{1.1}\,\|x_k - x_{k-1}\|}\right\} & \text{if } x_k \neq x_{k-1}, \\ \theta & \text{if otherwise.} \end{cases}$$

To terminate the algorithm, we used the stopping criteria $\|x_{k+1} - x_k\| < \varepsilon$ with $\varepsilon = 10^{-6}$ is a tolerance. The results are reported in the Table 1 and Table 2, we can see that the number of iterations (No. of Iter.) by Algorithm 3.1 with different inertial parameters ($\theta = 0.6$ and $\theta = 0.9$) is less than that of Yuying et al. Algorithm [40], for two different starting points, we can see that in this example the starting points $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall\, i = 1, 2, ..., n\}$ give better performance than $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall\, i = 1, 2, ..., n\}$. Moreover, Figure 1 and Figure 2 illustrate the numerical behavior of both algorithms. In these figures, the value of errors $\|x_{k+1} - x_k\|$ is represented by the y-axis, number of iterations is represented by the x-axis.

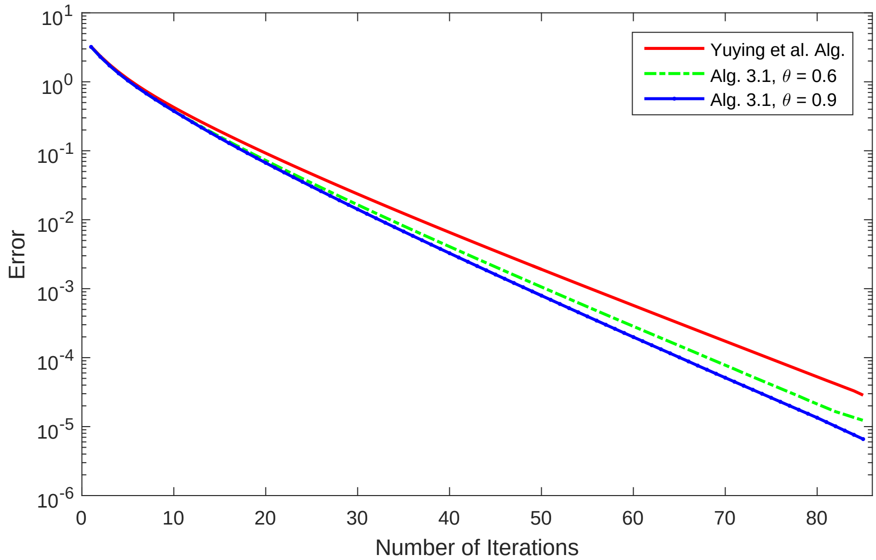

Figure 1: Comparison of proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = (1, 1, ..., 1)^T$ and n=50.

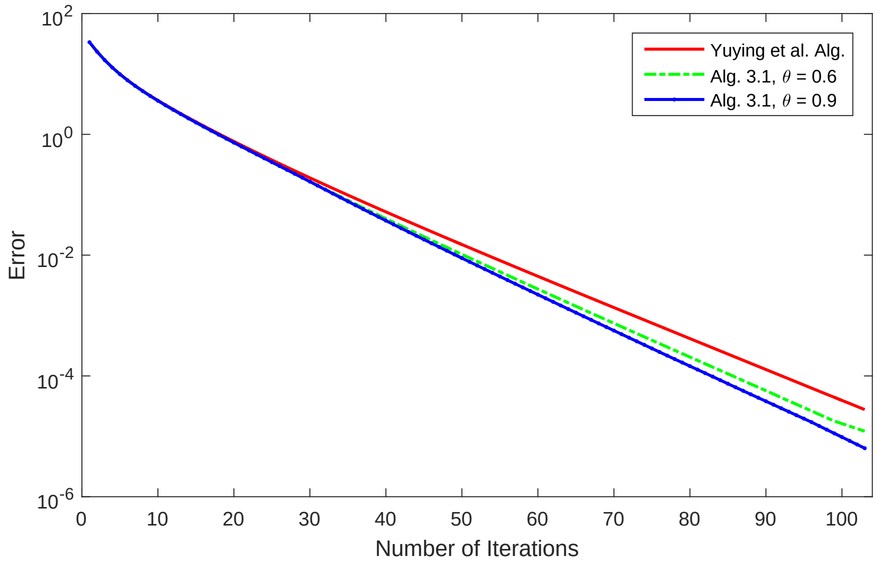

Figure 2: Comparison of proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = (1, 2, ..., 50)^T$ and n=50.

5. Conclusions

In this article, we introduced an iterative algorithm for finding the solution of a bilevel equilibrium problem in real Hilbert space. Under some suitable conditions imposed on parameters, we proved the strong convergence of the algorithm. We showed the efficiency of the proposed algorithm is verified by a numerical experiment and preliminary comparison. These numerical results have also confirmed that the algorithm with inertial effects seems to work better than without inertial effects.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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