Volume 29, Number 4 August 2021
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Exact Solitary Wave Solutions for Wick-type Stochastic (2+1)-dimensional Coupled KdV equations

Hossam A. Ghany[1] and M. Zakarya[2]
[1]Department of Mathematics, Helwan University, Cairo (11282), Egypt.
h.abdelghany@yahoo.com
[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut (71524), Egypt.
mohammed_zakarya@rocketmail.com

Abstract

Variable coefficients and Wick-type stochastic (2+1)-dimensional coupled KdV equations are investigated. By using the F-expansion method , Hermite transform and white noise theory, the white noise functional solutions for Wick-type stochastic (2+1)-dimensional coupled KdV equations are obtained. The exact travelling wave solutions are expressed in terms of Jacobi elliptic (JEF), trigonometric and hyperbolic functions.

Keywords: KdV equations; F-expansion method; Hermite transform; Wick product.
PACS No. : 05.40. ± a, 02.30.Jr.

1 Introduction

In this paper, we shall explore exact solutions for the following variable coefficients (2+1)-dimensional coupled KdV equations.

$$\begin{cases} u_t + \phi_1(t)uv_x + \phi_2(t)vu_x + \phi_3(t)u_{xxx} = 0, \\ \\ u_x + v_y = 0, \end{cases} \tag{1.1}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Random wave is an important subject of stochastic partial differential equations (PDEs). Many authors have studied this subject. Wadati first introduced and studied the stochastic KdV equations and gave the diffusion of soliton of the KdV equation under Gaussian noise in [30, 32] and others [3, 4, 5, 25] also researched stochastic KdV-type

1

equations. Xie first introduced Wick-type stochastic KdV equations on white noise space and showed the auto- Backlund transformation and the exact white noise functional solutions in [37]. Furthermore, Xie [38, 39, 40, 41], Ghany et al. [11, 12, 13, 15, 16, 17, 18, 19, 20] researched some Wick-type stochastic wave equations using white noise analysis.

In this paper we use F-expansion method for finding new periodic wave solutions of nonlinear evolution equations in mathematical physics, and we obtain some new periodic wave solutions for (2+1)-dimensional coupled KdV equations. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations arising in mathematical physics. The effort in finding exact solutions to nonlinear equations is important for the understanding of most nonlinear physical phenomena. For instance, the nonlinear wave phenomena observed in fluid dynamics, plasma, and optical fibers[24]. Many effective methods have been presented, such as tanh-function method [34, 42, 8], variational iteration method [6, 7], exp-function method [22, 23, 36, 43, 44] , homotopy perturbation method [10, 29, 35], homotopy analysis method [1], tanh-coth method [33, 34, 31], Jacobi elliptic function expansion method [27, 28, 9, 26] and F-expansion method [45, 46, 47, 48]. The main objective of this paper is using the F-expansion method to construct white noise functional solutions for wick-type stochastic (2+1)-dimensional coupled KdV equations via hermite transform, wick-type product and white noise analysis. If equation (1.1) is considered in a random environment, we can get stochastic (2+1)-dimensional coupled KdV equations. In order to give the exact solutions of stochastic (2+1)-dimensional coupled KdV equations, we only consider this problem in white noise environment. We shall study the following Wick-type stochastic (2+1)-dimensional coupled KdV equations.

$$\begin{cases} U_t + \Phi_1(t) \diamond U \diamond V_x + \Phi_2(t) \diamond V \diamond U_x + \Phi_3(t) \diamond U_{xxx} = 0, \\ U_x + V_y = 0, \end{cases} \tag{1.2}$$

where " $\diamond$ " is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$ which was defined in [21] and $\Phi_1(t), \Phi_2(t)$ and $\Phi_3(t)$ are $(\mathcal{S})_{-1}$ -valued functions.

2 Description of the F-expansion Method

In order to at the same time obtain more periodic wave solutions expressed by various Jacobi elliptic functions to nonlinear wave equations, we introduce an F-expansion method which can be thought of as a succinctly over-all generalization of Jacobi elliptic function expansion. We briefly show what is F-expansion method and how to use it to obtain various periodic wave solutions to nonlinear wave equations. Suppose a nonlinear wave equation for $u(t,x)$ is given by

$$\theta_1(u, u_t, u_x, u_y, u_{xx}, u_{xxx}, ...) = 0, \tag{2.1}$$

where $u = u(t, x)$ is an unknown function, θ_1 is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of a deformation F-expansion method.

Step 1. Look for traveling wave solution of Eq.(2.1) by taking

$$u(t, x, y) = u(\xi) \ , \xi(t, x, y) = kx + ly + \mu \int_0^t \omega(\tau) d\tau + c, \tag{2.2}$$

Hence, under the transformation (2.2). Eq.(2.1) can be transformed into the following ordinary differential equation (ODE) as following

$$\theta_2(u, \mu\omega u', ku', lu', k^2u'', k^3u''', ...) = 0, \tag{2.3}$$

Step 2. Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$ of the form

$$u(t, x, y) = u(\xi) = \sum_{i=1}^{N} a_i F^i(\xi), \tag{2.4}$$

where $a_0, a_1, ..., a_N$ are constants to be determined later, while $F'(\xi)$ in(2.4) satisfy

$$[F'(\xi)]^2 = PF^4(\xi) + QF^2(\xi) + R, \tag{2.5}$$

and hence holds for $F(\xi)$

$$\begin{cases} F'F'' = 2PF^3F' + QFF', \\ F'' = 2PF^3 + QF, \\ F''' = 6PF^2F' + QF', \\ \cdots \end{cases} \tag{2.6}$$

where $P, Q,$ and R are constants.

Step 3. The positive integer N can be determined by considering the homogeneous balance between the highest derivative term and the nonlinear terms appearing in (2.3). Therefore, we can get the value of N in (2.4).

Step 4. Substituting (2.4) into (2.3) with the condition (2.5), we obtain polynomial in $F^i(\xi)[F'(\xi)]^j$, $(i = 0 \pm 1, \pm 2, ..., j = 0, 1)$. Setting each coefficient of this polynomial to be zero yields a set of algebraic equations for $a_0, a_1, ..., a_N, \mu$ and ω .

Step 5. Solving the algebraic equations with the aid of Maple we have $a_0, a_1, ..., a_N, \mu$ and ω can be expressed by (P, Q, R) . Substituting these results into F-expansion (2.4), then a general form of traveling wave solution of Eq. (2.1) can be obtained.

Step 6. Since the general solutions of (2.4) have been well known for us Choose properly (P, Q and R .) in ODE (2.5) such that the corresponding solution $F(\xi)$ of it is one of Jacobi elliptic functions. (See Appendices A, B and C .)[45, 46, 47]

3

3 New Exact Wave Solutions of Eq. (1.2)

Taking the Hermite transform, white noise theory, and F-expansion method to explore new exact wave solutions for Eq.(1.2). Applying Hermite transform to Eq.(1.2), we get the deterministic equation.

$$\begin{cases} \widetilde{U}_t(t,x,y,z)+\widetilde{\Phi_1}(t,z)\widetilde{U}(t,x,y,z)\widetilde{V}_x(t,x,y,z)+\widetilde{\Phi_2}(t,z)\widetilde{V}(t,x,y,z)\widetilde{U}_x(t,x,y,z) \\ +\widetilde{\Phi_3}(t,z)\widetilde{U}_{xxx}(t,x,y,z)=0, \\ \\ \widetilde{U}_x(t,x,y,z)+\widetilde{V}_y(t,x,y,z)=0, \end{cases} \tag{3.1}$$

where $z=(z_1,z_2,...)\in(\mathbb{C}^{\mathbb{N}})$ is a vector parameter. To look for the travelling wave solution of Eq.(3.1), we make the transformations $\widetilde{\Phi_1}(t,z):=\phi_1(t,z)$, $\widetilde{\Phi_2}(t,z):=\phi_2(t,z)$, $\widetilde{\Phi_3}(t,z):=\phi_3(t,z)$, $\widetilde{U}(t,x,y,z):=u(t,x,y,z)=u(\xi(t,x,y,z))$ and $\widetilde{V}(t,x,y,z):=v(t,x,y,z)=v(\xi(t,x,y,z))$ with

$$\xi(t,x,y,z)=kx+ly+\mu\int_0^t\omega(\tau,z)d\tau+c,$$

where k,μ and c are arbitrary constants which satisfy $k\mu\neq 0$, $\omega(\tau,z)$ is a nonzero function of the indicated variables to be determined later. Hence, Eq.(3.1) can be transformed into the following (ODE).

$$\begin{cases} \mu\omega u'+k\phi_1 uv'+k\phi_2 vu'+k^3\phi_3 u'''=0, \\ ku'+lv'=0, \end{cases} \tag{3.2}$$

where the prime denote to the differential with respect to ξ . In view of F-expansion method, the solution of Eq. (3.1), can be expressed in the form.

$$\begin{cases} u(t,x,y,z)=u(\xi)=\Sigma_{i=1}^{N}a_iF^i(\xi), \\ v(t,x,y,z)=v(\xi)=\Sigma_{i=1}^{M}b_iF^i(\xi), \end{cases} \tag{3.3}$$

where a_i and b_i are constants to be determined later. considering homogeneous balance between the highest order nonlinear terms and the highest order partial derivative of u in (3.2), then we can obtain $N=M=2$ so (3.3) can be rewritten as following

$$\begin{cases} u(t,x,y,z)=a_0+a_1F(\xi)+a_2F^2(\xi), \\ v(t,x,y,z)=b_0+b_1F(\xi)+b_2F^2(\xi), \end{cases} \tag{3.4}$$

where a_0,a_1,a_2,b_0,b_1 and b_2 are constants to be determined later. Substituting (3.4) with the conditions (2.5),(2.6) into (3.2) and collecting all terms with the same power of

4

$F^i(\xi)[F'(\xi)]^j$, $(i = 0 \pm 1, \pm 2, ..., j = 0, 1)$. as following

$$\begin{cases} [\mu\omega a_1 + ka_0b_1\phi_1 + ka_1b_0\phi_2 + k^3a_1\phi_3Q]F' \\ +[2\mu\omega a_2 + 2ka_0b_2\phi_1 + ka_1b_1\phi_1 + 2ka_2b_0\phi_2 + ka_1b_1\phi_2 + 8k^3a_2\phi_3Q]FF' \\ +k[2a_1b_2\phi_1 + a_2b_1\phi_1 + 2a_2b_1\phi_2 + a_1b_1\phi_2 + 6k^2a_1\phi_3P]F^2F' \\ +2ka_2[b_2\phi_1 + b_2\phi_2 + 12k^2\phi_3P]F^2F' = 0, \\ (ka_1 + lb_1)F' + 2[ka_2 + lb_2]FF' = 0. \end{cases} \tag{3.5}$$

Setting each coefficients of $F^i(\xi)[F'(\xi)]^j$ to be zero, we get a system of algebraic equations which can be expressed by.

$$\begin{cases} \mu\omega a_1 + ka_0b_1\phi_1 + ka_1b_0\phi_2 + k^3a_1\phi_3Q = 0, \\ 2\mu\omega a_2 + 2ka_0b_2\phi_1 + ka_1b_1\phi_1 + 2ka_2b_0\phi_2 + ka_1b_1\phi_2 + 8k^3a_2\phi_3Q = 0, \\ k[2a_1b_2\phi_1 + a_2b_1\phi_1 + 2a_2b_1\phi_2 + a_1b_1\phi_2 + 6k^2a_1\phi_3P] = 0, \\ 2ka_2[b_2\phi_1 + b_2\phi_2 + 12k^2\phi_3P] = 0, \\ ka_1 + lb_1 = 0, \\ 2[ka_2 + lb_2] = 0. \end{cases} \tag{3.6}$$

with solving by *Maple* to get the following coefficients

$$\begin{cases} a_2 = b_2 = 0, \;\; a_0 \;, b_0 = \text{arbitrary constant}, \\ a_1 = \frac{6lk\phi_3(t,z)P}{\phi_2(t,z)}, \\ b_1 = -\frac{6k^2\phi_3(t,z)P}{\phi_2(t,z)}, \\ \omega = \frac{k^2a_0\phi_1(t,z) - lk[b_0\phi_2(t,z) + k^2\phi_3(t,z)Q]}{l\mu}. \end{cases} \tag{3.7}$$

Substituting by coefficient (3.7) into (3.4) yields general form solutions of Eq. (1.2).

$$u(t, x, y, z) = a_0 + \frac{6lk\phi_3(t, z)P}{\phi_2(t, z)} \; F(\xi), \tag{3.8}$$

$$v(t, x, y, z) = b_0 - \frac{6k^2\phi_3(t, z)P}{\phi_2(t, z)} \; F(\xi), \tag{3.9}$$

with

$$\xi(t, x, y, z) = kx + ly + \int_0^t \frac{k^2a_0\phi_1(\tau, z) - lk[b_0\phi_2(\tau, z) + k^2\phi_3(\tau, z)Q]}{l} \; d\tau.$$

5

From Appendix A, we give the special cases as following.

Case I:
If we take $P=\frac{1}{4}, Q=\frac{m^2-2}{2}$ and $R=\frac{m^2}{4}$, we have $F(\xi)\rightarrow ns(\xi)\pm ds(\xi)$,

$$u_1(t,x,y,z)=a_0+\frac{3lk\phi_3(t,z)}{2\phi_2(t,z)}\left[ns\left(\xi_1(t,x,y,z)\right)\pm ds\left(\xi_1(t,x,y,z)\right)\right], \tag{3.10}$$

$$v_1(t,x,y,z)=b_0-\frac{3k^2\phi_3(t,z)}{2\phi_2(t,z)}\left[ns\left(\xi_1(t,x,y,z)\right)\pm ds\left(\xi_1(t,x,y,z)\right)\right], \tag{3.11}$$

with

$$\xi_1(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{2k^2a_0\phi_1(\tau,z)-lk[2b_0\phi_2(\tau,z)+k^2\phi_3(\tau,z)(m^2-2)]}{2l}\right\}d\tau.$$

In the limit case when $m\rightarrow o$, we have $ns(\xi)\pm ds(\xi)\rightarrow 2\csc(\xi)$, thus (3.10),(3.11) become.

$$u_2(t,x,y,z)=a_0+\frac{3lk\phi_3(t,z)}{\phi_2(t,z)}\csc\left(\xi_2(t,x,y,z)\right), \tag{3.12}$$

$$v_2(t,x,y,z)=b_0-\frac{3k^2\phi_3(t,z)}{\phi_2(t,z)}\csc\left(\xi_2(t,x,y,z)\right), \tag{3.13}$$

with

$$\xi_2(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{k^2a_0\phi_1(\tau,z)-lk[b_0\phi_2(\tau,z)-k^2\phi_3(\tau,z)]}{l}\right\}d\tau.$$

In the limit case when $m\rightarrow 1$ we have $ns(\xi)\pm ds(\xi\rightarrow \coth(\xi)\pm(\xi)$, thus (3.10).(3.11) become.

$$u_3(t,x,y,z)=a_0+\frac{3lk\phi_3(t,z)}{2\phi_2(t,z)}\left[\coth\,\xi_3(t,x,y,z)\pm\,\left(\xi_3(t,x,y,z)\right)\right], \tag{3.14}$$

$$v_3(t,x,y,z)=b_0-\frac{3k^2\phi_3(t,z)}{2\phi_2(t,z)}\left\{[\coth\,\xi_3(t,x,y,z)\pm\,\left(\xi_3(t,x,y,z)\right)\right], \tag{3.15}$$

with

$$\xi_3(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{2k^2a_0\phi_1(\tau,z)-lk[2b_0\phi_2(\tau,z)-k^2\phi_3(\tau,z)]}{2l}\right\}d\tau.$$

Case II:
If we take $P=1, Q=-(1+m^2)$ and $R=m^2$, then $F(\xi)\to ns(\xi)$,

$$u_4(t,x,y,z)=a_0+\frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ ns\,(\xi_4(t,x,y,z)), \tag{3.16}$$

$$v_4(t,x,y,z)=b_0-\frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ ns\,(\xi_4(t,x,y,z)), \tag{3.17}$$

with

$$\xi_4(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{2k^2a_0\phi_1(\tau,z)-lk[2b_0\phi_2(\tau,z)+k^2\phi_3(\tau,z)(m^2-2)]}{l}\right\}d\tau.$$

In the limit case when $m\to o$ we have $ns(\xi)\pm ds(\xi)\to\csc(\xi)$, thus (3.10),(3.11) become.

$$u_5(t,x,y,z)=a_0+\frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ \csc\,(\xi_2(t,x,y,z)), \tag{3.18}$$

$$v_5(t,x,y,z)=b_0-\frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ \csc\,(\xi_2(t,x,y,z)). \tag{3.19}$$

In the limit case when $m\to 1$ we have $ns(\xi)\to\coth(\xi)$, thus (3.10).(3.11) become.

$$u_6(t,x,y,z)=a_0+\frac{6lk\phi_3(t,z)}{2\phi_2(t,z)}\ \coth\,(\xi_5(t,x,y,z)), \tag{3.20}$$

$$v_6(t,x,y,z)=b_0-\frac{6k^2\phi_3(t,z)}{2\phi_2(t,z)}\ \coth\,(\xi_5(t,x,y,z)), \tag{3.21}$$

with

$$\xi_5(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{k^2a_0\phi_1(\tau,z)-lk[b_0\phi_2(\tau,z)-2k^2\phi_3(\tau,z)]}{l}\right\}d\tau.$$

Case III:
If we take $P=1, Q=(2-m^2)$ and $R=1-m^2$, then $F(\xi)\to cs(\xi)$,

$$u_7(t,x,y,z)=a_0+\frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ cs\,(\xi_6(t,x,y,z)), \tag{3.22}$$

$$v_7(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ cs\,(\xi_6(t,x,y,z)), \tag{3.23}$$

with

$$\xi_6(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0\phi_1(\tau,z) - lk[2b_0\phi_2(\tau,z) + k^2\phi_3(\tau,z)(2-m^2)]}{l} \right\} d\tau.$$

In the limit case when $m \to o$ we have $cs(\xi) \to \cot(\xi)$, thus (3.10),(3.11) become.

$$u_8(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ \cot\,(\xi_7(t,x,y,z)), \tag{3.24}$$

$$v_8(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ \cot\,(\xi_7(t,x,y,z)), \tag{3.25}$$

$$\xi_7(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0\phi_1(\tau,z) - lk[b_0\phi_2(\tau,z) + 2k^2\phi_3(\tau,z)]}{l} \right\} d\tau.$$

In the limit case when $m \to 1$ we have $cs(\xi) \to (\xi)$, thus (3.10).(3.11) become.

$$u_9(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ (\xi_8(t,x,y,z)), \tag{3.26}$$

$$v_9(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ (\xi_8(t,x,y,z)), \tag{3.27}$$

with

$$\xi_8(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0\phi_1(\tau,z) - lk[b_0\phi_2(\tau,z) + k^2\phi_3(\tau,z)]}{l} \right\} d\tau.$$

Obviously, there are another solutions for Eq.(1.2). These solutions come from setting different values for the coefficients P, Q and R . (see Appendix A, B and C.)[46, 47]. The above mentioned cases are just to clarify how far our technique is applicable.

4 White Noise Functional Solutions of Eq.(1.2)

In this section, we employ the results of the Section 3 by using Hermite transform to obtain exact white noise functional solutions for Wick-type stochastic (2+1)-dimensional coupled KdV equations (1.2). The properties of exponential and trigonometric functions yield that there exists a bounded open set $\mathbf{G} \subset \mathbb{R}_+ \times \mathbb{R}^2$, $\rho < \infty$, $\lambda > 0$ such that the solution $u(t,x,y,z)$ of Eq. (3.1) and all its partial derivatives which are involved in Eq. (3.1) are uniformly bounded for $(t,x,y,z) \in \mathbf{G} \times K_\rho(\lambda)$, continuous with respect to $(t,x,y) \in \mathbf{G}$ for all $z \in K_\rho(\lambda)$ and analytic with respect to $z \in K_\rho(\lambda)$, for all $(t,x,y) \in \mathbf{G}$. From Theorem 4.1.1 in [21], there exists $U(t,x,y,z) \in (\mathcal{S})_{-1}$ such that $u(t,x,y,z) = \widetilde{U}(t,x,y)(z)$ for all $(t,x,y,z) \in \mathbf{G} \times K_\rho(\lambda)$ and $U(t,x,y)$ solves Eq.(1.2) in $(\mathcal{S})_{-1}$. Hence, by applying the inverse Hermite transform to the results of Section 3, we get exact white noise functional solutions of Eq. (1.2) as follows.

- **White noise functional solutions of JEF type**:

$$U_1(t,x,y) = a_0 + \frac{3lk\Phi_3(t)}{2\Phi_2(t)} \diamond \Big[ns^\diamond\left(\Xi_1(t,x,y)\right) \pm ds^\diamond\left(\Xi_1(t,x,y)\right)\Big], \tag{4.1}$$

$$V_1(t,x,y) = b_0 - \frac{3k^2\Phi_3(t)}{2\Phi_2(t)} \diamond \Big[ns^\diamond\left(\Xi_1(t,x,y)\right) \pm ds^\diamond\left(\Xi_1(t,x,y)\right)\Big], \tag{4.2}$$

$$U_2(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond ns^\diamond\left(\Xi_2(t,x,y)\right), \tag{4.3}$$

$$V_2(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond ns^\diamond\left(\Xi_2(t,x,y)\right), \tag{4.4}$$

$$U_3(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond cs^\diamond\left(\Xi_3(t,x,y)\right), \tag{4.5}$$

$$V_3(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond cs^\diamond\left(\Xi_3(t,x,y)\right), \tag{4.6}$$

with

$$\Xi_1(t,x,y) = kx + ly + \int_0^t \left\{ \frac{2k^2 a_0 \Phi_1(\tau) - lk[2b_0\Phi_2(\tau) + k^2\phi_3(\tau)(m^2-2)]}{2l} \right\} d\tau,$$

$$\Xi_2(t,x,y) = kx + ly + \int_0^t \left\{ \frac{2k^2 a_0 \Phi_1(\tau) - lk[2b_0\Phi_2(\tau) + k^2\Phi_3(\tau)(m^2-2)]}{l} \right\} d\tau,$$

9

$$\Xi_3(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[2b_0\Phi_2(\tau) + k^2\Phi_3(\tau)(2-m^2)]}{l} \right\} d\tau.$$

- **White noise functional solutions of trigonometric type**:

$$U_4(t,x,y) = a_0 + \frac{3lk\Phi_3(t)}{\Phi_2(t)} \diamond \csc^\diamond(\Xi_4(t,x,y)), \tag{4.7}$$

$$V_4(t,x,y) = b_0 - \frac{3k^2\Phi_3(t)}{\Phi_2(t)} \diamond \csc^\diamond(\Xi_4(t,x,y)), \tag{4.8}$$

$$U_5(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond \csc^\diamond(\Xi_4(t,x,y)), \tag{4.9}$$

$$V_5(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond \csc^\diamond(\Xi_4(t,x,y)), \tag{4.10}$$

$$U_6(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond \cot^\diamond(\Xi_5(t,x,y)), \tag{4.11}$$

$$V_6(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond \cot^\diamond(\Xi_5(t,x,y)), \tag{4.12}$$

with

$$\Xi_4(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[b_0\Phi_2(\tau) - k^2\Phi_3(\tau)]}{l} \right\} d\tau,$$

$$\Xi_5(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[b_0\Phi_2(\tau) + 2k^2\Phi_3(\tau)]}{l} \right\} d\tau.$$

- **White noise functional solutions of hyperbolic type**:

$$U_7(t,x,y) = a_0 + \frac{3lk\Phi_3(t)}{2\Phi_2(t)} \diamond \left[\coth^\diamond(\Xi_6(t,x,y)) \pm^\diamond (\Xi_6(t,x,y)) \right], \tag{4.13}$$

$$V_7(t,x,y) = b_0 - \frac{3k^2\Phi_3(t)}{2\Phi_2(t)} \diamond \left[\coth^\diamond(\Xi_6(t,x,y)) \pm^\diamond (\Xi_6(t,x,y)) \right], \tag{4.14}$$

$$U_8(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{2\Phi_2(t)} \diamond \coth^{\diamond}(\Xi_7(t,x,y)), \tag{4.15}$$

$$V_8(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{2\Phi_2(t)} \diamond \coth^{\diamond}(\Xi_7(t,x,y)), \tag{4.16}$$

$$U_9(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond {}^{\diamond}(\Xi_8(t,x,y)), \tag{4.17}$$

$$V_9(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond {}^{\diamond}(\Xi_8(t,x,y)), \tag{4.18}$$

with

$$\Xi_6(t,x,y) = kx + ly + \int_0^t \left\{ \frac{2k^2a_0\Phi_1(\tau) - lk[2b_0\Phi_2(\tau) - k^2\Phi_3(\tau)]}{2l} \right\} d\tau,$$

$$\Xi_7(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2a_0\Phi_1(\tau) - lk[b_0\Phi_2(\tau) - 2k^2\Phi_3(\tau)]}{l} \right\} d\tau,$$

$$\Xi_8(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2a_0\Phi_1(\tau) - lk[b_0\Phi_2(\tau) + k^2\Phi_3(\tau)]}{l} \right\} d\tau.$$

We observe that, for different forms of Φ_1, Φ_2 and Φ_3, we can get different exact white noise functional solutions of Eq. (1.2) from Eqs. (4.1)-(4.18).

5 Example

It is well known that Wick version of function is usually difficult to evaluate. So, in this section, we give non-Wick version of solutions of Eq. (1.2). Let $W_t = \dot{B}_t$ be the Gaussian white noise, where B_t is the Brownian motion. We have the Hermite transform $\widetilde{W_t}(z) = \sum_{i=1}^{\infty} z_i \int_0^t \eta_i(s)ds$ [21]. Since $\exp^{\diamond}(B_t) = \exp(B_t - \frac{t^2}{2})$, we have $\cot^{\diamond}(B_t) = \cot(B_t - \frac{t^2}{2})$, $\csc^{\diamond}(B_t) = \csc(B_t - \frac{t^2}{2})$, $\coth^{\diamond}(B_t) = \coth(B_t - \frac{t^2}{2})$ and ${}^{\diamond}(B_t) = (B_t - \frac{t^2}{2})$. Suppose that. $\Phi_1(t) = \psi_1\Phi_3(t), \Phi_2(t) = \psi_2\Phi_3(t)$ and $\Phi_3(t) = \Gamma(t) + \psi_3 W_t$ where ψ_1, ψ_2 and ψ_3 are arbitrary constants and $\Gamma(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. Therefore, for $\Phi_1(t)\Phi_2(t)\Phi_3(t) \neq 0$. thus exact white noise functional solutions of Eq. (1.2) are as follows.

$$U_{10}(t,x,y) = a_0 + \frac{3lk}{\psi_2}\ \csc\left(\Omega_1(t,x,y)\right), \tag{5.1}$$

$$V_{10}(t,x,y) = b_0 - \frac{3k^2}{\psi_2}\ \csc\left(\Omega_1(t,x,y)\right), \tag{5.2}$$

$$U_{11}(t,x,y) = a_0 + \frac{6lk}{\psi_2}\ \csc\Omega_1(t,x,y), \tag{5.3}$$

$$V_{11}(t,x,y) = b_0 - \frac{6k^2}{\psi_2}\ \csc\left(\Omega_1(t,x,y)\right), \tag{5.4}$$

$$U_{12}(t,x,y) = a_0 + \frac{6lk}{\psi_2}\ \cot\left(\Omega_2(t,x,y)\right), \tag{5.5}$$

$$V_{12}(t,x,y) = b_0 - \frac{6k^2}{\psi_2}\ \cot\left(\Omega_2(t,x,y)\right), \tag{5.6}$$

with

$$\Omega_1(t,x,y) = kx + ly + \big(\frac{k^2 a_0\psi_1 - lk[b_0\psi_2 - k^2]}{l}\big)\left\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\right\},$$

$$\Omega_2(t,x,y) = kx + ly + \big(\frac{k^2 a_0\psi_1 - lk[b_0\psi_2 + 2k^2]}{l}\big)\left\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\right\},$$

and

$$U_{13}(t,x,y) = a_0 + \frac{3lk}{2\psi_2}\ \Big[\coth\left(\Omega_3(t,x,y)\right) \pm\ \left(\Omega_3(t,x,y)\right)\Big], \tag{5.7}$$

$$V_{13}(t,x,y) = b_0 - \frac{3k^2}{2\psi_2}\ \Big[\coth\left(\Omega_2(t,x,y)\right) \pm\ \left(\Omega_3(t,x,y)\right)\Big], \tag{5.8}$$

$$U_{14}(t,x,y) = a_0 + \frac{6lk}{2\psi_2}\ \coth\left(\Omega_4(t,x,y)\right), \tag{5.9}$$

12

$$V_{14}(t,x,y) = b_0 - \frac{6k^2}{2\psi_2} \ \coth(\Omega_4(t,x,y)), \tag{5.10}$$

$$U_{15}(t,x,y) = a_0 + \frac{6lk}{\psi_2} \ (\Omega_5(t,x,y)), \tag{5.11}$$

$$V_{15}(t,x,y) = b_0 - \frac{6k^2}{\psi_2} \ (\Omega_5(t,x,y)), \tag{5.12}$$

with

$$\Omega_3(t,x,y) = kx + ly + \big(\frac{2k^2a_0\psi_1 - lk[2b_0\psi_2 - k^2]}{2l}\big)\bigg\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\bigg\},$$

$$\Omega_4(t,x,y) = kx + ly + \big(\frac{k^2a_0\psi_1 - lk[b_0\psi_2 - 2k^2]}{l}\big)\bigg\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\bigg\},$$

$$\Omega_5(t,x,y) = kx + ly + \big(\frac{k^2a_0\psi_1 - lk[b_0\psi_2 + k^2]}{l}\big)\bigg\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\bigg\}.$$

6 Conclusion

We have discussed the solutions of (SPDEs) driven by Gaussian white noise. There is a unitary mapping between the Gaussian white noise space and the Poisson white noise space. This connection was given by Benth and Gjerde [2]. By the aid of this connection, we can derive some stochastic exact soliton solutionsfor our problem. In this paper, using Hermite transformation, white noise theory and F-expansion method, we study the white noise functional solutions of the Wick-type stochastic (2+1)-dimensional coupled KdV equations. This paper shows that the F-expansion method is sufficient to solve many stochastic nonlinear equations in mathematical physics. The method which we have proposed in this paper is standard, direct and computerized method, which allows us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other nonlinear (PDEs) in mathematical physics such as modified Hirota-Satsuma coupled KdV, KdV-Burgers, modified KdV Burgers, Sawada-Kotera, Zhiber-Shabat equations and Benjamin-Bona-Mahony equations. Since the equation (1.2) has other solutions if select other values of P, Q and R (see Appendices A, B, C), and there are many other of exact solutions for wick-type stochastic (2+1)-dimensional coupled KdV equations.

Appendix A. The ODE and Jacobi Elliptic Functions

Relation between values of (P , Q , R) and corresponding $F(\xi)$ in ODE.

$$(F')^2(\xi) = PF^4(\xi) + QF^2(\xi) + R,$$

P	Q	R	$F(\xi)$
m^2	$-1-m^2$	1	$\mathrm{sn}\xi$, $\mathrm{cd}\xi = \frac{cn\xi}{dn\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{cn}\xi$
-1	$2-m^2$	m^2-1	$\mathrm{dn}\xi$
1	$-1-m^2$	m^2	$\mathrm{ns}\xi = \frac{1}{\mathrm{sn}\xi}$, $\mathrm{dc}\xi = \frac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{nc}\xi = \frac{1}{\mathrm{cn}\xi}$
m^2-1	$2-m^2$	-1	$\mathrm{nd}\xi = \frac{1}{\mathrm{dn}\xi}$
$1-m^2$	$2-m^2$	1	$\mathrm{sc}\xi = \frac{\mathrm{sn}\xi}{\mathrm{cn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{sd}\xi = \frac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\mathrm{cs}\xi = \frac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{ds}\xi = \frac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$
$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi}$, $\frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\pm\mathrm{dn}\xi}$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}\xi \pm i\mathrm{cn}\xi$, $\frac{\mathrm{dn}\xi}{i\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{cn}\xi}$, $\frac{\mathrm{m\,sn}\xi}{1\pm\mathrm{dn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\xi \pm \mathrm{cs}\xi$, $\frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{dn}\xi}$, $\frac{\mathrm{sn}\xi}{1\pm\mathrm{cn}\xi}$,
$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$\frac{\mathrm{dn}\xi}{1\pm m\mathrm{sn}\xi}$
$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$\mathrm{nc}\xi \pm i\mathrm{sc}\xi$ $\frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$
$\frac{-1}{4}$	$\frac{m^2+1}{2}$	$\frac{-(1-m^2)^2}{4}$	$m\mathrm{cn}\xi \pm \mathrm{dn}\xi$
$\frac{1}{4}$	$\frac{m^2+1}{2}$	$\frac{(1-m^2)^2}{4}$	$\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{ns}\xi \pm \mathrm{ds}\xi$

Appendix B.

the jacobi elliptic functions degenerate into trigonometric functions when $m \to 0$.

$$sn\xi \to \sin\xi, cn\xi \to \cos\xi, dn\xi \to 1, sc\xi \to \tan\xi, sd\xi \to \sin\xi, cd\xi \to \cos\xi,$$
$$ns\xi \to \csc\xi, nc\xi \to \sec\xi, nd\xi \to 1, cs\xi \to \cot\xi, ds\xi \to \csc\xi, dc\xi \to \sec\xi.$$

Appendix C.
the jacobi elliptic functions degenerate into hyperbolic functions when $m \to 1$.

$$sn\xi \to \tan\xi, cn\xi \to \xi, dn\xi \to \xi, sc\xi \to \sinh\xi, sd\xi \to \sinh\xi, cd\xi \to 1,$$
$$ns\xi \to \coth\xi, nc\xi \to \cosh\xi, nd\xi \to \cosh, cs\xi \to \xi, ds\xi \to \xi, dc\xi \to 1.$$

References

[1] S. Abbasbandy. *Chem. Eng. J* , **136** (2008): 144-150.

[2] E. Benth and J. Gjerde. *Potential. Anal.*, **8** (1998): 179-193.

[3] A. de Bouard and A. Debussche. *J. Funct. Anal.*, **154** (1998): 215-251.

[4] A. Debussche and J. Printems. *Physica D*, **134** (1999): 200-226.

[5] A. Debussche and J. Printems. *J. Comput. Anal. Appl.*, **3** (2001): 183-206.

[6] M. Dehghan, J. Manafian and A. Saadatmandi. *Math. Meth. Appl. Sci*, **33** (2010): 1384-1398.

[7] M. Dehghan and M. Tatari. *Chaos Solitons and Fractals*, **36** (2008): 157-166.

[8] E. Fan. *Phys. Lett. A*, **277** (2000): 212-218.

[9] Z. T. Fu, S. K. Liu, S. D. Liu, Q. Zhao. *Phys. Lett. A*, **290** (2001): 72-76.

[10] D. D. Ganji and A. Sadighi. *Int J Non Sci Numer Simul*, **7** (2006): 411-418.

[11] H. A. Ghany. *Chin. J. Phys.*, **49** (2011): 926-940.

[12] H. A. Ghany. *International Journal of pure and applied mathematics*, **78** (2012): 17-27.

[13] H. A. Ghany and A. Hyder. *International Review of Physics*, **6** (2012): 153-157.

[14] H. A. Ghany and A. Hyder. *J. Comput. Anal. Appl.*, **15** (2013): 1332-1343.

[15] H. A. Ghany and A. Hyder. *Kuwait Journal of Science*, **41** (2013): 1-14.

[16] H. A. Ghany and M. S. Mohammed. *Chin. J. Phys.*, **50** (2012): 619-627.

[17] H. A. Ghany, A. S. Okb El Bab, A. M. Zabal and A. Hyder. *Chin. Phys. B*, **22** (2013): 080501-1.

[18] H. A. Ghany and A. Hyder. *Int. Journal of Math. Analysis*, **7** (2013): 2199 - 2208

[19] H. A. Ghany and Hussain E. Hussain. *Int. Journal of Math. Analysis*, **7**: 3019 - 3026

[20] H. A. Ghany and A. Hyder. *Chin. Phys. B*, **23** (2014): 0605031-7.

[21] H. Holden, B. Ø sendal, J. U b ø e and T. Zhang. *Stochastic partial differential equations*, Bihkäuser: Basel, (1996).

[22] J .H. He and X. H. Wu. *Chaos Solitons and Fractals*, **30** (2006):700-708.

[23] J. H. He and M. A. Abdou. *Chaos Solitons and Fractals*, **34** (2007): 1421-1429.

[24] R. S. Johnson, *J. Fluid Mech.*,**42** (1970), 49-60.

[25] V.V. Konotop, L. Vazquez. Nonlinear Random Waves, World Scientific, Singapore, (1994).

[26] S. K. Liu, Z. T. Fu, S. D. Liu, Q. Zhao. *Phys. Lett. A*, **289** (2001): 69-74.

[27] J. Liu, L. Yang, K. Yang *Chaos, Solitons and Fractals*, **20** (2004): 1157-1164.

[28] E. J. Parkes, B. R. Duffy, P. C. Abbott *Phys. Lett. A*, **295** (2002): 280-286.

[29] F. Shakeri and M. Dehghan. *Math. Comput. Model*, **48** (2008): 486-498.

[30] M. Wadati. *J. Phys. Soc. Jpn.*, **52** (1983): 2642-2648.

[31] A. M. Wazwaz. *Chaos Solitons Fractals*, **188** (2007): 1930-1940.

[32] M. Wadati and Y. Akutsu. *J. Phys. Soc. Jpn.*, **53** (1984): 3342-3350.

[33] A. M. Wazwaz. *Appl. Math. Comput*, **177** (2006): 755-760.

[34] A. M. Wazwaz. *Appl. Math. Comput*, **169** (2005): 321-338.

[35] X. Ma, L. Wei, and Z. Guo. *J. Sound Vibration*, **314** (2008): 217-227.

[36] X. H. Wu and J.H. He. *Chaos Solitons and Fractals*, **38** (2008): 903-910.

[37] Y. C. Xie . *Phys. Lett. A*, **310** (2003): 161-167.

[38] Y. C. Xie. *Solitons and Fractals*, **21** (2004): 473-480.

[39] Y. C. Xie. *Solitons and Fractals*, **20** (2004): 337-342.

[40] Y. C. Xie. *J. Phys. A: Math. Gen*, **37** (2004): 5229-5236.

[41] Y. C. Xie. *Phys. Lett. A*, **310** (2003): 161-167.

[42] S. Zhang, T. C. Xia. *Communications in Theoretical Physics*, **46** (2006): 985-990.

[43] S. D. Zhu. *Int J Non Sci Numer Simul*, **8** (2007): 461-464.

[44] S. D. Zhu. *Int J Non Sci Numer Simul*, **8** (2007): 465-468.

[45] Yubin Zhou. Mingliang Wang. Yueming Wang. *Physics Letters A*, **308** (2003): 31-36.

[46] Sheng Zhang. Tiecheng Xia. *Applied Mathematics and Computation*, **189** (2007): 836-843.

[47] Sheng Zhang. TieCheng Xia. *Applied Mathematics and Computation*, **183** (2006): 1190-1200.

[48] Sheng Zhang. Tiecheng Xia. *Communications in Nonlinear Science and Numerical Simulation*, **13** (2008): 1294-1301.

Exact Solutions for Stochastic Fractional Zhiber-Shabat Equations

Hossam A. Ghany[1] **and Ashraf Fathallah** [2]
[1]Department Mathematics, Helwan University, Cairo, Egypt
h.abdelghany@yahoo.com
[2] Department of Mathematics, Misr International University,
Cairo 11341, Egypt.
ashraf.abdhady@miuegypt.edu.eg

Abstract

This paper is devoted to give exact solutions of the variable coefficient fractional Zhiber -Shabat equation with space-time-fractional derivatives. Moreover, by using the Hermite transform and the homogeneous balance principle, the white noise functional solutions for the Wick-type stochastic fractional Zhiber-Shabat equation are explicitly shown. Detailed computations and implemented examples are explicitly provided.

Keywords: Fractional Zhiber-Shabat equations; White noise; Stochastic; Hermite transform.

MSC: 60H30; 60H15; 35R60

1 Introduction

The main task of this paper is to explore exact solutions for the following fractional Zhiber-Shabat equation with variable coefficients:

$$\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}u + p(t)e^{u} + q(t)e^{-u} + r(t)e^{-2u} = 0 \tag{1.1}$$

where $\partial_{x^{\alpha_1}}, \partial_{t^{\alpha_2}}(0 < \alpha_1, \alpha_2 < 0)$ are the modified Riemann-Liouville fractional derivatives defined by Jumarie [6] and $q(t), p(t)$ and $r(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Random waves is an important subject of random fractional partial differential equations. Recently, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equation. In the past decades, an important progress has been made in the research of the exact solutions of nonlinear partial differential equations (PDEs). To seek various exact solutions of multifarious physical models described by nonlinear

PDEs, various methods have been proposed. There are many authers studied this subject. Wadati first introduced and studied the stochastic KdV equation and gave the diffusion of soliton of the KdV equation under Gaussian noise in ([10]-[12]). Xie firstly researched Wick-type stochastic KdV equation on white noise space and showed the auto-Bachlund transformation and the exact white noise functional solutions in [14], furthermore, Chen and Xie ([1]-[3]) and Xie ([15]-[17]) researched some Wick-type stochastic wave equations using white noise analysis method. Recently, Uğurlu and Kaya[9] gave the tanh function method, Wazzan [13] showed the modified tanh-coth method, these methods have been applied to derive nonlinear transformations and exact solutions of nonlinear PDEs in mathematical physics. If Eqn.(1.1) is considered in random environment, we can get random fractional Zhiber-Shabat equation with space-fractional derivatives. In order to give the exact solutions of random fractional Zhiber-Shabat equation with space-fractional derivatives, we only consider this problem in white noise environment. Wick-type stochastic generalized fractional Zhiber-Shabat equations with space-fractional derivatives is the perturbation of Eqn.(1.1) by random force $W(t) \diamond R^{\diamond}(U, U_{xt})$, which represented by:

$$\partial_{x^{\alpha_1}} \partial_{t^{\alpha_2}} U + P(t) \diamond e^{\diamond U} + Q(t) \diamond e^{\diamond(-U)} + R(t) e^{\diamond(-2U)} = W(t) \diamond R^{\diamond}(U, U_{x^{\alpha_1} t^{\alpha_2}}) \tag{1.2}$$

where $W(t)$ is Gaussian white noise, i.e., $W(t) = B^{\cdot}(t)$ and B(t) is a Brownian motion, $R(u, u_{x^{\alpha_1} t^{\alpha_2}}) = -\beta_1 \partial_{x^{\alpha_1}} \partial_{t^{\alpha_2}} u - \beta_2 e^{u} - \beta_3 e^{-u} - \beta_4 e^{-2u}$ is a functional of $u, \partial_{x^{\alpha_1}} \partial_{t^{\alpha_2}} u := \frac{\partial^{\alpha_1+\alpha_2} u}{\partial x^{\alpha_1} \partial x^{\alpha_2}} = u_{x^{\alpha_1} x^{\alpha_2}}$ for some constants $\beta_1, ..., \beta_4$ and $R^{\diamond}$ is the Wick version of the functional R. " $\diamond$ " is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$ and $P(t), Q(t)$ and $R(t)$ are white noise functionals. Eqn.(1.2) can be seen as the perturbation of the coefficients $p(t), q(t)$ and $r(t)$ of Eqn.(1.1) by white noise functionals.

This paper is devoted to give white noise functional solution for Wick-type stochastic generalized fractional Zhiber-Shabat equations with space-fractional derivatives. Moreover, the Hermite transform and the homogenous balance principle are employed to find the exact solution for stochastic fractional Zhiber-Shabat equation with variable coefficient. Finally, implemented examples are explicitly shown.

2 Preliminaries

There are different definitions for fractional derivatives, for more details (see [5, 6]). In our paper we use the modified Riemann-Liouville derivative defined by Jumarie [6]:

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha-1} [f(y) - f(0)] dy, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \int_0^x (x-y)^{-\alpha} [f(y) - f(0)] dy, & 0 < \alpha < 1, \\ \left[f^{(\alpha-n)}(x) \right]^{(n)}, & n \leq \alpha < n+1, \quad n \in \mathbb{N} \end{cases} \tag{2.1}$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. Some properties of the modified Riemann-Liouville derivative were summarized in [5] , three useful formulas of them are

$$\begin{cases} D_x^\alpha x^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > 0, \\ D_x^\alpha (u(x)v(x)) = u(x) D_x^\alpha v(x) + v(x) D_x^\alpha u(x), \\ D_x^\alpha [f(u(x)] = \dfrac{df}{du} D_x^\alpha u(x) = \left(\dfrac{du}{dx}\right)^\alpha D_u^\alpha f(u). \end{cases} \tag{2.2}$$

Now, we outline the main idea of the modified fractional sub-equation method. Many authors considered nonlinear FPDE, say, in two variables

$$F(u, u_x, u_t, D_x^\alpha u, D_t^\alpha u, ...) = 0, \qquad 0 < \alpha \leq 1 \tag{2.3}$$

where F is a nonlinear function with respect to the indicated variables. To determine the solution $u = u(x,t)$ explicitly, we first introduce the following transformation

$$u = u(\xi), \qquad \xi = \xi(x,t) \tag{2.4}$$

which converts Eq.(2.3) into a fractional ordinary differential equation

$$G(u, u', u'', D_\xi^\alpha u, D_\xi^{2\alpha} u, ...) = 0. \tag{2.5}$$

Next we introduce a new variable $Y = Y(\xi)$ which is a solution of the fractional Riccati equation

$$D_\xi^\alpha Y = h_0 + h_1 Y + h_2 Y^2, \qquad 0 < \alpha \leq 1, \tag{2.6}$$

where h_0, h_1 and h_2 are arbitrary constants. Eq.(2.6) is the fractional Riccati differential equation, where α is a parameter describing the order of the fractional derivative. In the case of $\alpha = 1$ Eq.(2.6) is reduced to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems. The feed back gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation which has to be found for the whole time horizon of the control process [18, 19]. Then we propose the following series expansion as a solution of Eq.(2.3)

$$u(x,t) = u(\xi) = \sum_{k=0}^{n} a_k(x,t) Y^k(\xi) + \sum_{k=1}^{n} b_k(x,t) Y^{-k}(\xi), \tag{2.7}$$

where $a_k (k = 0, 1, ..., n), b_k (k = 1, ..., n)$ are functions to be determined later and n is a positive integer which can be determined via the balancing of the highest derivative term with the nonlinear term in equation Eq.(2.5). Inserting Eq.(2.7) into Eq.(2.5) and using Eq.(2.6) will give an algebraic equation in powers of Y . Since all coefficients of Y^k must vanish, this will give a system of algebraic equations with respect to a_k and b_k . With the aid of Mathematica, we can determine a_k and b_k . According to the recent paper by Zhang et al. [19], we can deduce the following set of solutions of Eq.(2.6).

$$\begin{cases} Y_1(\xi) = E_\alpha(\xi) - 1, & h_0 = h_1 = 1, \ h_2 = 0, \\ Y_2(\xi) = \coth_\alpha(\xi) \pm \operatorname{csch}_\alpha(\xi), \ Y_3(\xi) = \tanh_\alpha(\xi) \pm i \operatorname{sech}_\alpha(\xi), & h_0 = -h_2 = \frac{1}{2}, \ h_1 = 0, \\ Y_4(\xi) = \frac{1}{2}\tan_\alpha(2\xi), \ Y_5(\xi) = \frac{1}{2}\cot_\alpha(2\xi), & h_0 = \frac{1}{4}h_2 = 1, \ h_1 = 0, \end{cases} \tag{2.8}$$

with the generalized hyperbolic and trigonometric functions

$$\tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)},\ \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},\ \operatorname{csch}_\alpha(x) = \frac{1}{\sinh_\alpha(x)},\ \operatorname{sech}_\alpha(x) = \frac{1}{\cosh_\alpha(x)},$$

$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2},\ \cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2},\ \tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)},$$

$$\cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)},\ \sin_\alpha(x) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i},\ \cos_\alpha(x) = \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2},$$

defined by the Mittag-Leffler function $E_\alpha(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(1+j\alpha)}$. For more details about the generalized exponential, hyperbolic and trigonometric functions see [8].

3 Exact Solutions of Eqn. (1.2).

Many authors considered nonlinear equations of the form

$$P(u, u_t, u_x, u_{xt}, u_{xx}, u_{xxx}, ...) = 0 \tag{3.1}$$

where P is a nonlinear function with respect to the indicated variables. Introducing the one wave variable $\zeta = x - ct$ carry out the two independent partial differential equation (3.1) into an ODE

$$N(u, u', u'', u''', ...) = 0 \tag{3.2}$$

Equation (3.2) is then integrated as long as all terms contain derivatives. The tanh technique is based on the priori assumption that the travelling wave solutions can be expressed in terms of the tanh function [7]. We therefor introduce a new independent variable

$$Y = tanh(\mu\zeta)$$

that leads to the change of derivatives:

$$\frac{d}{d\zeta} = \mu(1 - Y^2)\frac{d}{dY},$$

$$\frac{d^2}{d\zeta^2} = \mu^2(1 - Y^2)(-2Y\frac{d}{dY} + (1 - Y^2\frac{d^2}{dY^2}))$$

The solution can be proposed by the tanh method as a finite power series in Y in the form:

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{3.3}$$

limiting them to solitary and shock wave profiles. However, the extended tanh method admits the use of the finite expansion

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} a_k Y^{-k}, \tag{3.4}$$

4

where M is a positive integer, in most cases, that will be determined. Expansion (3.4) reduces to the standard tanh method [7] for $a_k = 0, 1 \leqslant k \leqslant M$. Substituting (3.3) or (3.4) into the ODE (3.2) results in an algebraic equation in powers of Y. In this section, we will give exact solutions of Eqn(3.2). Taking the Hermite transform of Eqn.(3.2), we get

$$\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}\widetilde{U}(x,t,z) + \lambda_2(t,z)e^{\widetilde{U}(x,t,z)} + \lambda_2(t,z)e^{-\widetilde{U}(x,t,z)} + \lambda_3(t,z)e^{-2\widetilde{U}(x,t,z)} = 0 \tag{3.5}$$

where $z = (z_1, z_2, ...) \in C^{\mathbb{N}}$ is a parameter. Using the transformation

$$\zeta = \frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)},$$

that will carry out Eqn.(3.5) into

$$\lambda_1\widetilde{U}_{\zeta\zeta} + \lambda_2(t,z)e^{\widetilde{U}(\zeta,z)} + \lambda_3(t,z)e^{-\widetilde{U}(\zeta,z)} + \lambda_4(t,z)e^{-2\widetilde{U}(\zeta,z)} = 0. \tag{3.6}$$

where, $\lambda_1 = \mu\nu$, $\lambda_2 =: \lambda_2(t,z) = \frac{1}{1+\beta_1}\{\widetilde{P}(t,z)+\beta_2\}$, $\lambda_3 =: \lambda_3(t,z) = \frac{1}{1+\beta_1}\{\widetilde{Q}(t,z)+\beta_3\}$ and $\lambda_4 =: \lambda_4(t,z) = \frac{1}{1+\beta_1}\{\widetilde{R}(t,z)+\beta_4\}$. Denote $u(\zeta,z) = \widetilde{U}(\zeta,z)$ and assume that the solutions of (3.6) is the form

$$u(\zeta,z) = \frac{\partial^2 F(\phi(\zeta,z))}{\partial\zeta^2} + V(\zeta,z)$$

Let $v(\zeta,z) = e^{u(\zeta,z)}$, then Eqn.(3.6) becomes

$$\lambda_1\{vv'' - v'^2\} + \lambda_2 v^3 + \lambda_3 v + \lambda_4 = 0; \tag{3.7}$$

Considering the homogeneous balance between vv'' and v^3 in (3.7), gives M=2, hence we set the tanh-coth assumption by

$$v(x,t,z) = S(Y) = a_0(t,z) + a_1(t,z)Y(\zeta) + a_2(t,z)Y^2(\zeta) + b_1(t,z)Y^{-1}(\zeta) + b_2(t,z)Y^{-2}(\zeta) \tag{3.8}$$

where $Y(\zeta)$ satisfies the Riccati equation

$$Y' = c_1 + c_2Y + c_3Y^2, \tag{3.9}$$

and c_1, c_2, c_3 are constant to be prescribed later. By virtue of (3.8) and (3.9) with observation of the linear independence of $Y^n (n = -6, -5, ..., 6)$, Eqn.(3.7) implies the following system of linear equations

$$
\begin{cases}
\lambda_4+\lambda_3 a_0+\lambda_2[a_0(a_0^2+2a_1b_1+2a_2b_2)+a_1(2a_0b_1+2a_1b_2)+a_2(b_1^2+2a_0b_2)+\\
b_1(2a_0a_1+2a_2b_1)+b_2(a_1^2+2a_0a_2)]+\lambda_1[D_0a_0+D_8a_1+D_7a_2+D_1b_1+D_2b_2-(a_1c_1-b_1c_3)^2+\\
(a_1c_2+2a_2c_1)(b_1c_2+2b_2c_3)+(a_1c_3+2a_2c_2)(b_1c_1+2b_2c_2)+4a_2c_3b_2c_1]=0;\\
\lambda_3a_1+\lambda_2[a_0(2a_0a_1+2a_2b_1)+a_1(a_0^2+2a_1b_1+2a_2b_2)+a_2(2a_0b_1+2a_1b_2)+\\
b_1(a_1^2+2a_0a_2)+2a_1a_2b_2]+\lambda_1[D_0a_1+D_1a_0+D_8a_2+D_2b_1+D_3b_2\\
-(a_1c_1-b_1c_3)(a_1c_2+2a_2c_1)+(a_1c_3+2a_2c_2)(b_1c_2+2b_2c_3)]=0;\\
\lambda_3a_2+\lambda_2[a_0(a_1^2+2a_0a_2)+a_1(2a_0a_1+2a_2b_1)+a_2(a_0^2+2a_1b_1+2a_2b_2)+2a_1a_2b_1+a_2^2b_2]+\\
\lambda_1[D_2a_0++D_1a_1+D_3b_1+D_4b_2-(a_1c_1-b_1c_3)(a_1c_3+2a_2c_2)\\
-(a_1c_2+2a_2c_1)^2+2a_2c_3(b_1c_2+2b_2c_3)]=0;\\
\lambda_2[2a_0a_1a_2+a_1(a_2^2+2a_0a_2)+a_2(2a_0a_1+2a_2b_1)+b_1a_2^2]+\\
\lambda_1[D_3a_0+D_2a_1+D_1a_2+D_4b_1-2a_2c_3(a_1c_1-b_1c_3)-(a_1c_2+2a_2c_1)(a_1c_3+2a_2c_2)]=0;\\
\lambda_2[a_0a_2^2+2a_1^2a_2+a_2(a_1^2+2a_0a_2)]+\lambda_1[D_4a_0+D_3a_1+D_2a_2-2a_2c_3(a_1c_2+2a_2c_1)\\
-(a_1c_3+2a_2c_2)^2]=0;\\
\lambda_2[a_1a_2^2+2a_1a_2^2]+\lambda_1[D_4a_1+D_3a_2-2a_2c_3(a_1c_3+2a_2c_2)]=0;\\
\lambda_2[a_2^3]+\lambda_1[D_4a_2-4a_2^2c_3^2]=0;\\
\lambda_3b_2+\lambda_2[a_0(b_1^2+2a_0b_2)+b_1(2a_0b_1+2a_1b_2)+b_2(a_0^2+2a_1b_1+2a_2b_2)+\\
2a_1b_1b_2+a_2b_2^2]+\lambda_1[D_0b_2+D_7a_0+D_6a_1+D_5a_2+D_8b_1-(b_1c_2+2b_2c_3)^2+\\
2b_2c_1(a_1c_2+2a_2c_1)+(a_1c_1-b_1c_3)(b_1c_1+2b_2c_2)]=0;\\
\lambda_3b_1+\lambda_2[a_0(2a_0b_1+2a_1b_2)+b_1(a_0^2+2a_1b_1+2a_2b_2)+b_2(2a_0a_1+2a_2b_1)+\\
a_1(b_1^2+2a_0b_2)+2a_2b_2b_1]+\lambda_1[D_0b_1+D_8a_0+D_7a_1+D_6a_2+\\
D_1b_2+(a_1c_1-b_1c_3)(b_1c_2+2b_2c_3)+(a_1c_2+2a_2c_1)(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[2a_0b_1b_2+b_1(b_1^2+2a_0b_2)+b_2(2a_0b_1+2a_1b_2)+a_1b_2^2]+\lambda_1[D_6a_0+D_5a_1+D_7b_1+D_8b_2\\
+2b_2c_1(a_1c_1-b_1c_3)-(b_1c_2+2b_2c_3)(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[a_0b_2^2+2a_0b_1^2b_2+b_2(b_1^2+2a_0b_2)]+\lambda_1[D_5a_0+D_6b_1+D_7b_2-2b_2c_1(b_1c_2+2b_2c_3)\\
-(b_1c_1+2b_2c_2)^2]=0;\\
\lambda_2[b_1b_2^2+2a_0b_1b_2^2]+\lambda_1[D_5b_1+D_6b_2-2b_2c_1(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[b_2^3]+\lambda_1[D_5b_2-4b_2^2c_1^2]=0.
\end{cases}
$$

where, $D_0 = c_1(a_1c_2+2a_2c_1)+c_3(b_1c_2+2b_2c_3)$, $D_1 = c_2(a_1c_2+2a_2c_1)+2c_1(a_1c_3+2a_2c_2)$, $D_2 = c_3(a_1c_2+2a_2c_1)+2c_2(a_1c_3+2a_2c_2)+6a_2c_3c_1$, $D_3 = 2c_3(a_1c_3+2a_2c_2)+6a_2c_3c_2$, $D_4 = 6a_2c_3^2$, $D_5 = 6a_2b_2c_1^2$, $D_6 = 2c_1(b_1c_1+2b_2c_2)+6b_2c_1c_2$, $D_7 = c_1(b_1c_2+2b_2c_3)+2c_2(b_1c_1+2b_2c_2)+6b_2c_3c_1$, and $D_8 = c_2(b_1c_2+2b_2c_3)+2c_3(b_1c_1+2b_2c_2)$. In the remaining part of this section we will discuss and solve our problem for some special cases for the Riccati equation as follows:

A. $c_1 = c_2 = 1, c_3 = 0$.

This choice for the constants implies that

$$Y_1(\zeta) = exp(\zeta) - 1 \tag{3.10}$$

6

By the aid of Maple 12, the above system of equations can be solve for the following cases:

Case 1: $\lambda_4 = a_1 = a_2 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_0 = \pm i\sqrt{\frac{\lambda_3}{\lambda_2}}$; $b_1 = \frac{3}{\lambda_2 \pm i\sqrt{\lambda_2\lambda_3}}$; $b_2 = -\frac{2\lambda_1}{\lambda_2}$. By virtue of Eqn.(3.8), then Eqn.(3.5) have the solution

$$u_1 = ln\{\pm i\sqrt{\frac{\lambda_3}{\lambda_2}} + \frac{3}{\lambda_2 \pm i\sqrt{\lambda_2\lambda_3}} \times \frac{1}{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1} - \frac{2\lambda_1}{\lambda_2(exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1)^2}\} \tag{3.11}$$

Case 2: For $\lambda_4 = a_1 = a_2 = b_1 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_0 = \pm i\sqrt{\frac{\lambda_3}{\lambda_2}}$; $b_2 = -\frac{2\lambda_1}{\lambda_2}$. Eqn.(3.5) have the solution

$$u_2 = ln\{\pm i\sqrt{\frac{\lambda_3}{\lambda_2}} - \frac{2\lambda_1}{\lambda_2(exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1)^2}\} \tag{3.12}$$

B. $c_1 = -c_3 = 0.5, c_2 = 0$.

This choice for the constants implies that

$$Y_2(\zeta) = coth(\zeta) \pm csch(\zeta) \tag{3.13}$$

or

$$Y_3(\zeta) = tanh(\zeta) \pm isech(\zeta) \tag{3.14}$$

By the aid of Maple 12, the above system of equations can be solve for the following cases:

Case 3: $\lambda_4 = a_0 = a_1 = a_2 = b_1 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $b_2 = -\frac{\lambda_1}{2\lambda_2}$. By virtue of Eqn.(3.8), then Eqn.(3.5) have the solution

$$u_3 = ln\{-\frac{\lambda_1}{2\lambda_2(coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2}\} \tag{3.15}$$

or

$$u_4 = ln\{-\frac{\lambda_1}{2\lambda_2(tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2}\} \tag{3.16}$$

Case 4: For $\lambda_4 = a_0 = a_1 = b_1 = b_2 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_2 = -\frac{\lambda_1}{2\lambda_2}$. Eqn.(3.5) have the solution

$$u_5 = ln\{-\frac{\lambda_1}{2\lambda_2}(coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2\} \tag{3.17}$$

or

$$u_6 = ln\{-\frac{\lambda_1}{2\lambda_2}(tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2\} \tag{3.18}$$

4 White noise functional solutions of (1.2)

In this section, we will use Theorem 2.1 of Xie [17] for $d = 1$ to obtain white noise functional solutions of Eqs.(1.2). The properties of hyperbolic functions yield that there exists a bounded open set $\mathbf{S} \subset \mathbb{R}_+ \times \mathbb{R}, m > 0$ and $n > 0$ such that $u(x,t,z), u_{xt}(x,t,z)$ are uniformally bounded for all $(t,x,z) \in \mathbf{S} \times \mathbb{K}_m(n)$, continuous with respect to $(t,x) \in \mathbf{S}$ for all $z \in \mathbb{K}_m(n)$ and analytic with respect to $z \in \mathbb{K}_m(n)$ for all $(t,x) \in \mathbf{S}$. Using Theorem 2.1 of Xie [17], there exists a stochastic process $U(t,x)$ such that the Hermite transformation of $U(t,x)$ is $u(t,x,z)$ for all $\mathbf{S} \times \mathbb{K}_m(n)$, and $U(t,x)$ is the solution of (1.2). This implies that $U(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$. Hence, for $\Lambda_1\Lambda_2\Lambda_3 \neq 0$ the white noise functional solutions of Eqn.(1.2) as follows:

$$U_1(x,t) = ln^{\diamond}\{\pm i\sqrt{\frac{\Lambda_3(t)}{\Lambda_2(t)}} + \frac{3\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-1}}{\Lambda_2(t) \pm i\sqrt{\Lambda_2(t)\Lambda_3(t)}} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \quad (4.1)$$

$$U_2(x,t) = ln^{\diamond}\{\pm i\sqrt{\frac{\Lambda_3(t)}{\Lambda_2(t)}} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \quad (4.2)$$

$$U_3(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \quad (4.3)$$

$$U_4(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \quad (4.4)$$

$$U_5(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \quad (4.5)$$

$$U_6(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \quad (4.6)$$

We observe that for different form of $\Lambda_2(t)$ and $\Lambda_3(t)$, we can get different solutions of (1.2) from (3.1)-(3.6).

5 Example and Concluding Remarks

Let $\dot{B_t}$ be the Gaussian white noise, where B_t is Brown motion. We have the Hermite transform $\tilde{\dot{B}}(t,z) = \sum_{k=1}^{\infty} z_k \int_0^t \eta_k(s)ds$. Science $\exp^{\diamond}(B_t) = \exp(B_t - t^2/2)$, we have $tanh^{\diamond}(B_t) = tanh(B_t - t^2/2)$, $coth^{\diamond}(B_t) = cot(B_t - t^2/2)$, $sech^{\diamond}(B_t) = sech(B_t - t^2/2)$ and $csch^{\diamond}(B_t) = csch(B_t - t^2/2)$. Suppose $\Lambda_3(t) = \alpha\Lambda_2(t)$ and $\Lambda_2(t) = \lambda_2(t) + \beta\dot{B_t}$, where α, β are arbitrary constants and $\lambda_2(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. The white noise functional solutions of (1.2) are as follows: If $\Lambda_1(t)\Lambda_2(t)\Lambda_3(t) \neq 0$

$$U_7(x,t) = ln\{\pm i\sqrt{\alpha} + \frac{3\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-1}}{\Lambda_2(t)(1+\pm i\sqrt{\alpha})} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \tag{5.1}$$

$$U_8(x,t) = ln\{\pm i\sqrt{\alpha} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \tag{5.2}$$

$$U_9(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{5.3}$$

$$U_{10}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm i\, sech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{5.4}$$

$$U_{11}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{5.5}$$

$$U_{12}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm i\, sech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{5.6}$$

Finally, we remark that for $\alpha_1 = \alpha_2 = 0$, $p(t)$ =1 and $q(t) = r(t) = 0$, Eqn.(1.1) reduces to the Liouville equation. For $\alpha_1 = \alpha_2 = 0$, $r(t)$ =0 and $q(t) = p(t) = 1$, Eqn.(1.1) reduces to the Sinh-Gordon equation. For $\alpha_1 = \alpha_2 = 0$, $p(t) = r(t)$ =1 and $q(t) = 0$, Eqn.(1.1) reduces to the the well known Dodd-Bullough-Mikhailov equation. Moreover, for $\alpha_1 = \alpha_2 = 0$, $p(t)$ =0 , $q(t) = -1$ and $r(t) = 1$, gives Tzitzeica-Dodd-Bullough equation. Hence, our results in this work can be considered as a continuation of our results in our previous papers [4,5], this work gives directly exact solutions for wick-type stochastic form to each one of the above equations. Also, we remark that, since the Riccati equation has other solution if select other values of c_1, c_2 and c_3 , there are many other exact solutions of variable coefficient and wick-type stochastic Zhiber-Shabat equations

References

[1] Chen. B, Xie. Y.C., 2005, Exact solutions for wick-type stochastic coupled Kadomtsev-Petviashili equations, J. Phys. A, 38, 815-22.

[2] Chen. B, Xie. Y.C., 2005, Exact solutions for generalized stochastic Wick-type KdV-mKdV equations, Chaos Solitons Fractals, 23,281-7.

[3] Chen. B, Xie. Y.C., 2007, Periodic-like solutions of variable coefficient and Wick-type stochastic NLS equations, J. Comput. Appl. Math., 203,249-63.

[4] Ghany H. A., 2011, Exact Solutions for Stochastic Generalized Hirota-Satsuma Coupled KdV Equations, Chin. J. Phys., 49,926-940.

[5] Ghany H. A., Mohammed S. A., 2012, White Noise Functional Solutions for the Wick-type Stochastic Fractional KdV-Burgers-Kuramoto Equations, Chin. J. Phys., 50,619-627.

[6] Jumarie G.,2006, Modifed Riemann-Liouville derivative and fractional Taylor series of non differentiable functions further results, Comp. Math. Appl., 51, 1367-1376.

[7] Malfliet W., 1992, Solitary wave solutions of nonlinear wave equations, Am. J. Phys, 60(7),6504.

[8] Podlubny. I., 1999, Fractional Differential Equations, Academic Press, San Diego.

[9] Uğurlu. Y., Kaya. D., 2007, Analytic method for solitary solutions of some partial differential equations, Phys. Lett. A, 370,251-9.

[10] Wadati. M., 1983, Stochastic Korteweg de Vries equation, J. Phys. Soc. Jpn, 52,2642-8.

[11] Wadati. M., 1990, Deformation of solitons in random media, J. Phys. Soc. Jpn, 59,4201-3.

[12] Wadati. M., Akutsu. Y., 1984, Stochastic Korteweg de Vries equation, J. Phys. Soc. Jpn, 53,3342-50.

[13] Wazzan. L., A modified tanh-coth method for solving the KdV and the KdV-Burgers equations, to appear on Commu. Nonlinear Sci. Numer. Simul.

[14] Xie. Y. C., 2003, Exact solutions for stochastic KdV equations, Phys. Lett. A, 310,161-7.

[15] Xie. Y. C., 2004, Positonic solutions for Wick-type stochastic KdV equations, Chaos Solitons Fractals, 20,337-42.

[16] Xie. Y. C., 2004, An auto-Backlund transformation and exact solutions for wick type stochastic generalized KdV equations, J. Phys. A: Math. Gen., 37,5229-36.

[17] Xie. Y. C., 2004, Exact solutions of the Wick-type stochastic Kadomtsev-Peviashvili equations, Chaos Solitons Fractals, 21,473-80.

[18] Zhang. J.L., Ren. D.F. and Wang. M.L., 2003, The periodic wave solutions for the generalized Nizhnik-Novikov-Veselov equation, Chin. Phys, 12(8), 825-830.

[19] Zhang. J.L, Zong. Q.A., Liu. D. and Gao. Q., 2010, A generalized exp-function method for fractional Riccati differential equations, Communication Fractional Calculus, 1, 48-56.

Invariance, solutions, periodicity and asymptotic behavior of a class of fourth order difference equations

Mensah Folly-Gbetoula *

School of Mathematics, University of the Witwatersrand, 2050, Johannesburg, South Africa.

Abstract

We construct Lie symmetry generators of some fourth order difference equations. We use these generators to derive similarity variables that make it possible to obtain exact solutions. In some cases, we study periodicity and asymptotic behavior of the solutions.

2010 Mathematics Subject Classification: 39A11, 39A05.
Key words: Difference equation; symmetry; reduction; group invariant solutions

1 Introduction

Several years back, Sophus Lie studied the invariance property of equations under a group of transformations. The approach used was later known as Lie symmetry method. This method has been used to solve differential equations, and recently it has been applied to difference equations. Although Maeda studied difference equations via Lie symmetry analysis in twentieth century [9, 10], it is Hydon who really rekindled the interest for solving difference equations via symmetry. For Hydon's work, refer to [8].
Most often, difference equations arise as a result of discretizing differential equations, especially in phenomena that depend on time. There are many ways in which a differential equation can be discretized (see [4]). Difference equations have numerous applications. For example, biological systems, population dynamics, economics, physics (see [1, 2]). Although difference equations appear simple, finding their solutions can be incredibly difficult. The symmetry approach to finding solutions of difference equations is recent and the reader can refer to [8] and some recent articles [5–7, 11, 12] for further knowledge on this method.
In this paper, we consider the system of difference equations

$$x_{n+4} = \frac{x_n x_{n+1}}{x_{n+3}(a_n + b_n x_n x_{n+1})} \tag{1}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers. For equation (1), we derive all Lie point symmetries and give formulas for solutions in closed form. We also discuss periodicity and asymptotic behavior of solutions in some cases.

*Mensah.Folly-Gbetoula@wits.ac.za

1.1 Preliminaries

In this section, we give a background on symmetry methods for difference equations. Our definitions and notation come from [3, 8, 13].
Consider the difference equations

$$x_{n+4} = \Omega(x_n, x_{n+1}, x_{n+3}), \tag{2}$$

where n denotes the independent variable; x_n the dependent variable. In this case u_{n+i} denotes the 'i-th shift' of u_n.
Consider the group of transformations

$$(n, x_n) \mapsto (n, \tilde{x}_n = x_n + \varepsilon Q_1(n, x_n) + O(\varepsilon^2)), \tag{3}$$

where Q is the characteristic of the group of point transformations. Let

$$X = Q(n, x_n)\frac{\partial}{\partial x_n} \tag{4}$$

be the corresponding infinitesimal generator. The group of transformations (3) is a symmetry group if and only if

$$Q(n+4, \Omega) - \mathcal{X}(\Omega) = 0, \tag{5}$$

whenever (2) holds. Here,

$$\mathcal{X} = Q(n, x_n)\frac{\partial}{\partial x_n} + Q(n, x_{n+1})\frac{\partial}{\partial x_{n+1}} + Q(n+3, x_{n+3})\frac{\partial}{\partial x_{n+3}}$$

denotes the prolongation of X to all shifts of x_n appearing in the right hand sides of equations in (2). Equation (5), known as the linearized symmetry condition, can be solved for Q by applying the appropriate differential operators. The characteristic, together with the canonical coordinate

$$s = \int \frac{dx_n}{Q(n, x_n)}, \tag{6}$$

are necessary in the reduction of order of (2). The following definition can be used to check if a given function is invariant under a given group of transformations.

Definition 1 *[13] Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \textit{for all} \qquad x \in M.$$

2

2 Main results

2.1 Symmetry and difference invariant

To obtain the the criterion which gives the Lie point symmetries of (1), we force (5) on

$$x_{n+4} = \frac{x_n x_{n+1}}{x_{n+3}(a_n + b_n x_n x_{n+1})}. \tag{7}$$

This leads to

$$\begin{aligned} &Q(n+4, x_{n+4}) + \frac{x_n x_{n+1}(a_n + b_n x_n x_{n+1}) Q(n+3, x_{n+3})}{{x_{n+3}}^2 (a_n + b_n x_n x_{n+1})^2} \\ &- \frac{a_n [x_n Q(n+1, x_{n+1}) + x_{n+1} Q(n, x_n)]}{x_{n+3}(a_n + b_n x_n x_{n+1})^2} = 0. \end{aligned} \tag{8}$$

The methodology of solving these functional equations is given as follows:

- Firstly, apply the differential operator $\frac{\partial}{\partial x_n} + \frac{x_{n+1}}{x_n}\frac{\partial}{\partial x_{n+1}}$ on equation (8). This leads (after simplification) to

$$x_{n+1} Q'(n+1, x_{n+1}) - x_{n+1} Q'(n, x_n) - Q(n+1, x_{n+1}) + \frac{a_n}{x_n} Q(n, x_n) = 0.$$

- Secondly, differentiate with respect to x_n, separate by powers of x_{n+1} and solve the resulting system of over determining equations for Q. This gives

$$Q(n, x_n) = \alpha(n) x_n + \beta(n)$$

for some functions α and β of n.

- Lastly, substitute the latter in (8) to eliminate any dependency among the arbitrary functions that appear in Q. This leads to the constraints

$$\alpha(n) + \alpha(n+1) = 0 \text{ and } \quad \beta(n) = 0. \tag{9}$$

We have omitted the details in the computation. The constraints in (9) are readily solved ($\alpha(n) = (-1)^n$) and we have

$$Q = (-1)^n x_n. \tag{10}$$

Consequently, Equation (1) admits a one dimensional Lie algebra:

$$X = (-1)^n x_n \frac{\partial}{\partial x_n}. \tag{11}$$

The canonical coordinate is given by

$$s_n = \int \frac{dx_n}{(-1)^n x_n} = (-1)^n \ln |x_n| \tag{12}$$

and the difference invariant which is inspired by the form of the final constraints (9) is given by

$$\mathbf{u}_n = (-1)^n s_n + (-1)^{n+1} s_{n+1}. \tag{13}$$

It is not difficult to verify, using Definition 1 together with (11), that (13) is indeed invariant under the group of transformations of (1). For simplicity, we prefer using the compatible variable

$$|u_n| = \exp(-\mathbf{u}_n) \tag{14}$$

which is also invariant. This gives a convenient choice of the change variables which does not require lucky guesses. With this variable u_n, it follows that

$$u_{n+3} = a_n u_n + b_n \tag{15}$$

whose solution is given by

$$u_{3n+j} = u_j \left(\prod_{k_1=0}^{n-1} a_{3k_1+j} \right) + \sum_{l=0}^{n-1} \left(b_{3l+j} \prod_{k_2=l+1}^{n-1} a_{3k_2+j} \right), \quad j = 0, 1, 2. \tag{16}$$

To obtain the solutions of (1), we go up the hierarchy created by the changes of variables. By evaluating (13) as a telescoping series, we have

$$(-1)^n s_n = (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{k_1} \mathbf{u}_{k_1} + (-1)^n s_0 \quad (= \ln|x_n| \quad \text{from} \quad (12)), \tag{17}$$

i.e.

$$x_n = \exp\left\{ (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{k_1} \mathbf{u}_{k_1} + (-1)^n s_0 \right\}, \tag{18}$$

$$= \exp\{ \sum_{k_1=0}^{n-1} (-1)^{n+k_1} \ln u_{k_1} + \ln x_0 \}, \tag{19}$$

where all the u_{k_1}'s are obtained using (16).
Note. Equation (19) gives the closed form solution of (1) in a unified manner. Looking at the form of u_l in (16), we rephrase (19) as follows:

$$x_{6n+j} = \exp\left\{ \sum_{k_1=0}^{6n+j-1} (-1)^{6n+j+k_1} \ln u_{k_1} + \ln x_0 \right\}, \tag{20}$$

$$= x_j \prod_{i=0}^{n-1} \left(\prod_{r=0}^{2} \frac{u_{6i+j+2r}}{u_{6i+j+2r+1}} \right), \tag{21}$$

4

$j = 0, 1, \ldots, 5$. More clearly,

$$
\begin{aligned}
x_{6n} &= x_0 \prod_{i=0}^{n-1} \frac{u_{3(2i)}}{u_{3(2i)+1}} \frac{u_{3(2i)+2}}{u_{3(2i+1)}} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}}, && \text{(22a)}\\
x_{6n+1} &= x_1 \prod_{i=0}^{n-1} \frac{u_{3(2i)+1}}{u_{3(2i)+2}} \frac{u_{3(2i+1)}}{u_{3(2i+1)+1}} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}}, && \text{(22b)}\\
x_{6n+2} &= x_2 \prod_{i=0}^{n-1} \frac{u_{3(2i)+2}}{u_{3(2i+1)}} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}} \frac{u_{3(2i+2)}}{u_{3(2i+2)+1}}, && \text{(22c)}\\
x_{6n+3} &= x_3 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)}}{u_{3(2i+1)+1}} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}} \frac{u_{3(2i+2)+1}}{u_{3(2i+2)+2}}, && \text{(22d)}\\
x_{6n+4} &= x_4 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}} \frac{u_{3(2i+2)}}{u_{3(2i+2)+1}} \frac{u_{3(2i+2)+2}}{u_{3(2i+3)}}, && \text{(22e)}\\
x_{6n+5} &= x_5 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}} \frac{u_{3(2i+2)+1}}{u_{3(2i+2)+2}} \frac{u_{3(2i+3)}}{u_{3(2i+3)+1}}. && \text{(22f)}
\end{aligned}
$$

We then substitute the expressions given in (16) in (22) to get

$$
\begin{aligned}
x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{u_0 \prod\limits_{l_1=0}^{2i-1} a_{3l_1} + \sum\limits_{j=0}^{2i-1} b_{3j} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i-1} a_{3l+1} + \sum\limits_{j=0}^{2i-1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}} \frac{u_2 \prod\limits_{l=0}^{2i-1} a_{3l+2} + \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{kl_2+2}}{u_0 \prod\limits_{l=0}^{2i} a_{3l} + \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{kl_2}}\\
& \frac{u_1 \prod\limits_{l=0}^{2i} a_{3l+1} + \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{kl_2+1}}{u_2 \prod\limits_{l=0}^{2i} a_{3l+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{kl_2+2}}, \qquad \text{(23a)}
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+1} =& x_1 \prod_{i=0}^{n-1} \frac{u_1 \prod\limits_{l_1=0}^{2i-1} a_{3l_1+1} + \sum\limits_{j=0}^{2i-1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}}{u_2 \prod\limits_{l=0}^{2i-1} a_{3l+2} + \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+2}} \frac{u_0 \prod\limits_{l=0}^{2i} a_{3l} + \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i} a_{3l+1} + \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}}\\
& \frac{u_2 \prod\limits_{l=0}^{2i} a_{3l+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod\limits_{l=0}^{2i+1} a_{3l} + \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}}, \qquad \text{(23b)}
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+2} =& x_2 \prod_{i=0}^{n-1} \frac{u_2 \prod\limits_{l_1=0}^{2i-1} a_{3l_1+2} + \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+2}}{u_0 \prod\limits_{l=0}^{2i} a_{3l} + \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{3l_2}} \frac{u_1 \prod\limits_{l=0}^{2i} a_{3l+1} + \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}}{u_2 \prod\limits_{l=0}^{2i} a_{3l+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}} \\
& \frac{u_0 \prod\limits_{l=0}^{2i+1} a_{3l} + \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i+1} a_{3l+1} + \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+1}}, \qquad (23c)
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+3} =& x_3 \prod_{i=0}^{n-1} \frac{u_0 \prod\limits_{l_1=0}^{2i} a_{3l_1} + \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i} a_{3l+1} + \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}} \frac{u_2 \prod\limits_{l=0}^{2i} a_{3l+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod\limits_{l=0}^{2i+1} a_{3l} + \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}} \\
& \frac{u_1 \prod\limits_{l=0}^{2i+1} a_{3l+1} + \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+1}}{u_2 \prod\limits_{l=0}^{2i+1} a_{3l+2} + \sum\limits_{j=0}^{2i+1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+2}}, \qquad (23d)
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+4} =& x_4 \prod_{i=0}^{n-1} \frac{u_1 \prod\limits_{l_1=0}^{2i} a_{3l_1+1} + \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}}{u_2 \prod\limits_{l=0}^{2i} a_{3l+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}} \frac{u_0 \prod\limits_{l=0}^{2i+1} a_{3l} + \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i+1} a_{3l+1} + \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{kl_2+1}} \\
& \frac{u_2 \prod\limits_{l=0}^{2i+1} a_{3l+2} + \sum\limits_{j=0}^{2i+1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+2}}{u_0 \prod\limits_{l=0}^{2i+2} a_{3l} + \sum\limits_{j=0}^{2i+2} b_{3j} \prod\limits_{l_2=j+1}^{2i+2} a_{3l_2}}, \qquad (23e)
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+5} =& x_5 \prod_{i=0}^{n-1} \frac{u_2 \prod\limits_{l_1=0}^{2i} a_{3l_1+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod\limits_{l=0}^{2i+1} a_{3l} + \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}} \frac{u_1 \prod\limits_{l=0}^{2i+1} a_{3l+1} + \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+1}}{u_2 \prod\limits_{l=0}^{2i+1} a_{3l+2} + \sum\limits_{j=0}^{2i+1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+2}} \\
& \frac{u_0 \prod\limits_{l=0}^{2i+2} a_{3l} + \sum\limits_{j=0}^{2i+2} b_{3j} \prod\limits_{l_2=j+1}^{2i+2} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i+2} a_{3l+1} + \sum\limits_{j=0}^{2i+2} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+2} a_{3l_2+1}}. \qquad (23f)
\end{aligned}
$$

6

We rewrite (23) in terms of initial conditions only as follows:

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i-1} a_{3l_1} + x_0x_1\sum\limits_{j=0}^{2i-1} b_{3j}\prod\limits_{l_2=j+1}^{2i-1} a_{3l_2}}{\prod\limits_{l=0}^{2i-1} a_{3l+1} + x_1x_2\sum\limits_{j=0}^{2i-1} b_{3j+1}\prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}} \frac{\prod\limits_{l=0}^{2i-1} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i-1} b_{3j+2}\prod\limits_{l_2=j+1}^{2i-1} a_{kl_2+2}}{\prod\limits_{l=0}^{2i} a_{3l} + x_0x_1\sum\limits_{j=0}^{2i} b_{3j}\prod\limits_{l_2=j+1}^{2i} a_{kl_2}} \frac{\prod\limits_{l=0}^{2i} a_{3l+1} + x_1x_2\sum\limits_{j=0}^{2i} b_{3j+1}\prod\limits_{l_2=j+1}^{2i} a_{kl_2+1}}{\prod\limits_{l=0}^{2i} a_{3l+2}+x_2x_3\sum\limits_{j=0}^{2i} b_{3j+2}\prod\limits_{l_2=j+1}^{2i} a_{kl_2+2}}, \tag{24a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i-1} a_{3l_1+1} + x_1x_2 \sum\limits_{j=0}^{2i-1} b_{3j+1}\prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}}{\prod\limits_{l=0}^{2i-1} a_{3l+2} + x_2x_3\sum\limits_{j=0}^{2i-1} b_{3j+2}\prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+2}} \frac{\prod\limits_{l=0}^{2i} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i} b_{3j}\prod\limits_{l_2=j+1}^{2i} a_{3l_2}}{\prod\limits_{l=0}^{2i} a_{3l+1}+x_1x_2\sum\limits_{j=0}^{2i} b_{3j+1}\prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}} \frac{\prod\limits_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i} b_{3j+2}\prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod\limits_{l=0}^{2i+1} a_{3l} + x_0x_1\sum\limits_{j=0}^{2i+1} b_{3j}\prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}}, \tag{24b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i-1} a_{3l_1+2} + x_2x_3\sum\limits_{j=0}^{2i-1} b_{3j+2}\prod\limits_{l_2=j+1}^{2i-1}a_{3l_2+2}}{\prod\limits_{l=0}^{2i} a_{3l}+x_0x_1\sum\limits_{j=0}^{2i} b_{3j}\prod\limits_{l_2=j+1}^{2i} a_{3l_2}} \frac{\prod\limits_{l=0}^{2i} a_{3l+1}+x_1x_2\sum\limits_{j=0}^{2i}b_{3j+1}\prod\limits_{l_2=j+1}^{2i}a_{3l_2+1}}{\prod\limits_{l=0}^{2i}a_{3l+2}+x_2x_3\sum\limits_{j=0}^{2i}b_{3j+2}\prod\limits_{l_2=j+1}^{2i}a_{3l_2+2}} \frac{\prod\limits_{l=0}^{2i+1}a_{3l}+x_0x_1\sum\limits_{j=0}^{2i+1}b_{3j}\prod\limits_{l_2=j+1}^{2i+1}a_{3l_2}}{\prod\limits_{l=0}^{2i+1}a_{3l+1}+x_1x_2\sum\limits_{j=0}^{2i+1}b_{3j+1}\prod\limits_{l_2=j+1}^{2i+1}a_{3l_2+1}}, \tag{24c}$$

$$x_{6n+3}=x_3\prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i}a_{3l_1}+x_0x_1\sum\limits_{j=0}^{2i}b_{3j}\prod\limits_{l_2=j+1}^{2i}a_{3l_2}}{\prod\limits_{l=0}^{2i}a_{3l+1}+x_1x_2\sum\limits_{j=0}^{2i}b_{3j+1}\prod\limits_{l_2=j+1}^{2i}a_{3l_2+1}} \frac{\prod\limits_{l=0}^{2i}a_{3l+2}+x_2x_3\sum\limits_{j=0}^{2i}b_{3j+2}\prod\limits_{l_2=j+1}^{2i}a_{3l_2+2}}{\prod\limits_{l=0}^{2i+1}a_{3l}+x_0x_1\sum\limits_{j=0}^{2i+1}b_{3j}\prod\limits_{l_2=j+1}^{2i+1}a_{3l_2}} \frac{\prod\limits_{l=0}^{2i+1}a_{3l+1}+x_1x_2\sum\limits_{j=0}^{2i+1}b_{3j+1}\prod\limits_{l_2=j+1}^{2i+1}a_{3l_2+1}}{\prod\limits_{l=0}^{2i+1}a_{3l+2}+x_2x_3\sum\limits_{j=0}^{2i+1}b_{3j+2}\prod\limits_{l_2=j+1}^{2i+1}a_{3l_2+2}}, \tag{24d}$$

$$x_{6n+4}=x_4\prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i}a_{3l_1+1}+x_1x_2\sum\limits_{j=0}^{2i}b_{3j+1}\prod\limits_{l_2=j+1}^{2i}a_{3l_2+1}}{\prod\limits_{l=0}^{2i}a_{3l+2}+x_2x_3\sum\limits_{j=0}^{2i}b_{3j+2}\prod\limits_{l_2=j+1}^{2i}a_{3l_2+2}} \frac{\prod\limits_{l=0}^{2i+1}a_{3l}+x_0x_1\sum\limits_{j=0}^{2i+1}b_{3j}\prod\limits_{l_2=j+1}^{2i+1}a_{3l_2}}{\prod\limits_{l=0}^{2i+1}a_{3l+1}+x_1x_2\sum\limits_{j=0}^{2i+1}b_{3j+1}\prod\limits_{l_2=j+1}^{2i+1}a_{kl_2+1}} \frac{\prod\limits_{l=0}^{2i+1}a_{3l+2}+x_2x_3\sum\limits_{j=0}^{2i+1}b_{3j+2}\prod\limits_{l_2=j+1}^{2i+1}a_{3l_2+2}}{\prod\limits_{l=0}^{2i+2}a_{3l}+x_0x_1\sum\limits_{j=0}^{2i+2}b_{3j}\prod\limits_{l_2=j+1}^{2i+2}a_{3l_2}}, \tag{24e}$$

$$
\begin{aligned}
x_{6n+5} =& x_5 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i} a_{3l_1+2} + x_2x_3 \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod\limits_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}} \frac{\prod\limits_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+1}}{\prod\limits_{l=0}^{2i+1} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i+1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+2}} \\
& \frac{\prod\limits_{l=0}^{2i+2} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i+2} b_{3j} \prod\limits_{l_2=j+1}^{2i+2} a_{3l_2}}{\prod\limits_{l=0}^{2i+2} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i+2} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+2} a_{3l_2+1}},
\end{aligned} \tag{24f}
$$

where $x_4 = x_0x_1/(x_3(a_0 + b_0x_0x_1))$ and $x_5 = x_2x_3(a_0 + b_0x_0x_1)/(x_0(a_1 + b_1x_1x_2))$. In the following subsections, we study some special cases.

2.2 The case where (a_n) and (b_n) are 3 periodic sequences

Let $a_n = \{a_0, a_1, a_2, a_0, a_1, a_2, \dots\}$ and $b_n = \{b_0, b_1, b_2, b_0, b_1, b_2, \dots\}$. Equations in (23) reduce to

$$
\begin{aligned}
x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{a_0^{2i} + b_0x_0x_1 \sum\limits_{j=0}^{2i-1} a_0^j}{a_1^{2i} + b_1x_1x_2 \sum\limits_{j=0}^{2i-1} a_1^j} \frac{a_2^{2i} + b_2x_2x_3 \sum\limits_{j=0}^{2i-1} a_2^j}{a_0^{2i+1} + b_0x_0x_1 \sum\limits_{j=0}^{2i} a_0^j} \frac{a_1^{2i+1} + b_1x_1x_2 \sum\limits_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum\limits_{j=0}^{2i} a_2^j}, \\
x_{6n+1} =& x_1 \prod_{i=0}^{n-1} \frac{a_1^{2i} + b_1x_1x_2 \sum\limits_{j=0}^{2i-1} a_1^j}{a_2^{2i} + b_2x_2x_3 \sum\limits_{j=0}^{2i-1} a_2^j} \frac{a_0^{2i+1} + b_0x_0x_1 \sum\limits_{j=0}^{2i} a_0^j}{a_1^{2i+1} + b_1x_1x_2 \sum\limits_{j=0}^{2i} a_1^j} \frac{a_2^{2i+1} + b_2x_2x_3 \sum\limits_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum\limits_{j=0}^{2i+1} a_0^j}, \\
x_{6n+2} =& x_2 \prod_{i=0}^{n-1} \frac{a_2^{2i} + b_2x_2x_3 \sum\limits_{j=0}^{2i-1} a_2^j}{a_0^{2i+1} + b_0x_0x_1 \sum\limits_{j=0}^{2i} a_0^j} \frac{a_1^{2i+1} + b_1x_1x_2 \sum\limits_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum\limits_{j=0}^{2i} a_2^j} \frac{a_0^{2i+2} + b_0x_0x_1 \sum\limits_{j=0}^{2i+1} a_0^j}{a_1^{2i+2} + b_1x_1x_2 \sum\limits_{j=0}^{2i+1} a_1^j}, \\
x_{6n+3} =& x_3 \prod_{i=0}^{n-1} \frac{a_0^{2i+1} + b_0x_0x_1 \sum\limits_{j=0}^{2i} a_0^j}{a_1^{2i+1} + b_1x_1x_2 \sum\limits_{j=0}^{2i} a_1^j} \frac{u_2a_2^{2i+1} + b_2 \sum\limits_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum\limits_{j=0}^{2i+1} a_0^j} \frac{a_1^{2i+2} + b_1x_1x_2 \sum\limits_{j=0}^{2i+1} a_1^j}{a_2^{2i+2} + b_2x_2x_3 \sum\limits_{j=0}^{2i+1} a_2^j}, \\
x_{6n+4} =& x_4 \prod_{i=0}^{n-1} \frac{a_1^{2i+1} + b_1x_1x_2 \sum\limits_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum\limits_{j=0}^{2i} a_2^j} \frac{a_0^{2i+2} + b_0x_0x_1 \sum\limits_{j=0}^{2i+1} a_0^j}{a_1^{2i+2} + b_1x_1x_2 \sum\limits_{j=0}^{2i+1} a_1^j} \frac{a_2^{2i+2} + b_2x_2x_3 \sum\limits_{j=0}^{2i+1} a_2^j}{a_0^{2i+3} + b_0x_0x_1 \sum\limits_{j=0}^{2i+2} a_0^j},
\end{aligned}
$$

$$
x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{a_2^{2i+1} + b_2x_2x_3 \sum\limits_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum\limits_{j=0}^{2i+1} a_0^j} \frac{a_1^{2i+2} + b_1x_1x_2 \sum\limits_{j=0}^{2i+1} a_1^j}{a_2^{2i+2} + b_2x_2x_3 \sum\limits_{j=0}^{2i+1} a_2^j} \frac{a_0^{2i+3} + b_0x_0x_1 \sum\limits_{j=0}^{2i+2} a_0^j}{a_1^{2i+3} + b_1x_1x_2 \sum\limits_{j=0}^{2i+2} a_1^j}.
$$

8

2.3 The case where (a_n) and (b_n) are real constants

Let $a_n = a$ and $b_n = b$. Equations in (23) give rise to

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_0x_1 \sum_{j=0}^{2i-1} a^j}{a^{2i} + bx_1x_2 \sum_{j=0}^{2i-1} a^j} \frac{a^{2i} + bx_2x_3 \sum_{j=0}^{2i-1} a^j}{a^{2i+1} + bx_0x_1 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + bx_1x_2 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j}, \tag{25a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_1x_2 \sum_{j=0}^{2i-1} a^j}{a^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a^j} \frac{a^{2i+1} + bx_0x_1 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_1x_2 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j}, \tag{25b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_2x_3 \sum_{j=0}^{2i-1} a^j}{a^{2i+1} + bx_0x_1 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j} \frac{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j}, \tag{25c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{a^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a^j}{a^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_2x_3 \sum_{j=0}^{2i+1} a^j}, \tag{25d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{a^{2i+1} + bx_1x_2 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j} \frac{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_2x_3 \sum_{j=0}^{2i+1} a^j}{a^{2i+3} + bx_0x_1 \sum_{j=0}^{2i+2} a^j}, \tag{25e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_2x_3 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+3} + bx_0x_1 \sum_{j=0}^{2i+2} a^j}{a^{2i+3} + bx_1x_2 \sum_{j=0}^{2i+2} a^j}. \tag{25f}$$

2.3.1 The case where $a = 1$

Equations in (25) simplify to

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{1 + 2ibx_0x_1}{1 + 2ibx_1x_2} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_0x_1} \frac{1 + (2i+1)bx_1x_2}{1 + (2i+1)bx_2x_3}, \tag{26a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{1 + 2ibx_1x_2}{1 + 2ibx_2x_3} \frac{1 + (2i+1)bx_0x_1}{1 + (2i+1)bx_1x_2} \frac{1 + (2i+1)bx_2x_3}{1 + (2i+2)bx_0x_1}, \tag{26b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_0x_1} \frac{1 + (2i+1)bx_1x_2}{1 + (2i+1)bx_2x_3} \frac{1 + (2i+2)bx_0x_1}{1 + (2i+2)bx_1x_2}, \tag{26c}$$

9

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_0x_1}{1+(2i+1)bx_1x_2} \frac{1+(2i+1)bx_2x_3}{1+(2i+2)bx_0x_1} \frac{1+(2i+2)bx_1x_2}{1+(2i+2)bx_2x_3}, \tag{26d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_1x_2}{1+(2i+1)bx_2x_3} \frac{1+(2i+2)bx_0x_1}{1+(2i+2)bx_1x_2} \frac{1+(2i+2)bx_2x_3}{1+(2i+3)bx_0x_1}, \tag{26e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_2x_3}{1+(2i+2)bx_0x_1} \frac{1+(2i+2)bx_1x_2}{1+(2i+2)bx_2x_3} \frac{1+(2i+3)bx_0x_1}{1+(2i+3)bx_1x_2}. \tag{26f}$$

2.3.2 The case where $a = -1$

Let $a_n = -1$ and $b_n = b$. Equations in (23) result in

$$x_{6n} = \frac{x_0(x_1x_2b-1)^n}{(x_0x_1b-1)^n(x_2x_3b-1)^n}, x_{6n+1} = \frac{x_1(x_0x_1b-1)^n(x_2x_3b-1)^n}{(x_1x_2b-1)^n},$$

$$x_{6n+2} = \frac{x_2(x_1x_2b-1)^n}{(x_0x_1b-1)^n(x_2x_3b-1)^n}, x_{6n+3} = \frac{x_3(x_0x_1b-1)^n(x_2x_3b-1)^n}{(x_1x_2b-1)^n},$$

$$x_{6n+4} = \frac{x_0x_1(x_1x_2b-1)^n}{x_3(x_0x_1b-1)^{n+1}(x_2x_3b-1)^n}, x_{6n+5} = \frac{x_2x_3(x_0x_1b-1)^{n+1}(x_2x_3b-1)^n}{x_0(x_1x_2b-1)^{n+1}}.$$

2.4 Existence of six periodic solutions

From (26), if $a = 1$ and $b = 0$, then the solution of (1) is periodic with period six as long as $u_0 \neq x_2$ or $x_1 \neq x_3$. It should also be noted that the solutions are periodic with period two when $x_0 = x_2$ and $x_1 = x_3$.
The graphs below are cases where the solutions are six periodic.

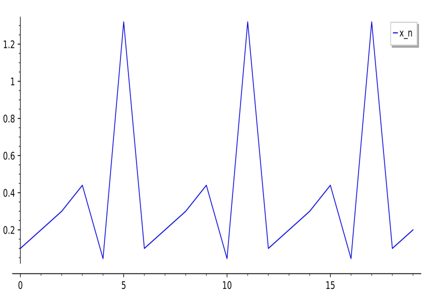

Figure 1: $a = 1, b = 0, x_0 = 0.1, x_1 = 0.2, x_2 = 0.3, x_3 = 0.44$.

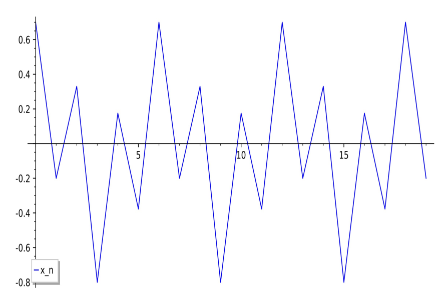

Figure 2: $a = 1, b = 0, x_0 = 0.7, x_1 = -0.2, x_2 = 0.33, x_3 = -0.8$.

10

2.5 Existence of 12-periodic solutions

Using (27), we have that if $a = -1$ and $b = 0$, then the solution of (1) is periodic with period twelve.
The graphs below are cases where the solutions are twelve periodic.

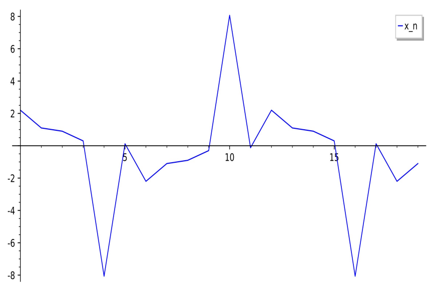

Figure 3: $a = -1, b = 0, x_0 = 2.2, x_1 = 1.1, x_2 = 0.9, x_3 = 0.3$.

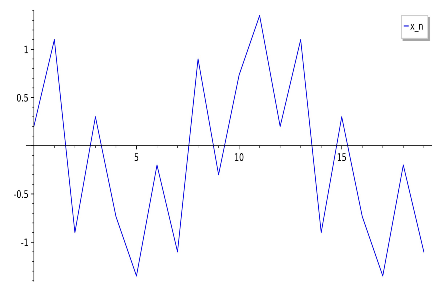

Figure 4: $a = -1, b = 0, x_0 = 0.2, x_1 = 1.1, x_2 = -0.9, x_3 = 0.3$.

3 Asymptotic behavior of the solutions for constant coefficients

Theorem 1 *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be the solution to the sequence in* (1) *where* $a_n = 1$ *for all* $n \geq 0$ *and* $b_n = b \neq 0$. *Then*

$$\lim_{n\to\infty} x_n = 0.$$

Proof 1 *Using* (26), *we have that*

$$\begin{aligned} x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{1+2ibx_0x_1}{1+2ibx_1x_2} \frac{1+2ibx_2x_3}{1+(2i+1)bx_0x_1} \frac{1+(2i+1)bx_1x_2}{1+(2i+1)bx_2x_3} \\ =& x_0 \prod_{i=0}^{n-1} \frac{1+2ibx_0x_1}{1+(2i+1)bx_0x_1} \frac{1+2ibx_2x_3}{1+(2i+1)bx_2x_3} \frac{1+(2i+1)bx_1x_2}{1+(2i)bx_1x_2} \\ =& x_0 \prod_{i=0}^{n-1} \left(1+\frac{bx_0x_1}{1+2ibx_0x_1}\right)^{-1} \left(1+\frac{bx_2x_3}{1+2ibx_2x_3}\right)^{-1} \left(1+\frac{bx_1x_2}{1+2ibx_1x_2}\right) \end{aligned}$$

We know that $1 + 2ix_kx_{k+1} \to \infty$ *as* $i \to \infty$. *Hence, there is a sufficiently large integer* t *such that for* $i \geq t$, *we have*

$$1 + 2ix_kx_{k+1} \sim 2ix_kx_{k+1}.$$

Thus

$$\begin{aligned} x_{6n} =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\left(1+\frac{1}{2i}\right)^{-1}\left(1+\frac{1}{2i}\right)^{-1}\left(1+\frac{1}{2i}\right) \\ =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\exp\left[\ln\left(1+\frac{1}{2i}\right)^{-1}+\ln\left(1+\frac{1}{2i}\right)^{-1}+\ln\left(1+\frac{1}{2i}\right)\right], \end{aligned}$$

where

$$\Gamma(t)=\prod_{i=0}^{t}\left(1+\frac{bx_0x_1}{1+2ibx_0x_1}\right)^{-1}\left(1+\frac{bx_2x_3}{1+2ibx_2x_3}\right)^{-1}\left(1+\frac{bx_1x_2}{1+2ibx_1x_2}\right).$$

Utilizing the expansion $\ln(1+x)=x+O(x^2),(1+x)^{-1}=1-x+O(x^2)$, *for* $x\to 0$, *we obtain*

$$\begin{aligned} x_{6n} =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\exp\left[-\frac{1}{2i}+O\left(\frac{1}{i^2}\right)\right] \\ =& x_0\Gamma(t)\exp\left[-\sum_{i=t+1}^{n-1}\left(\frac{1}{2i}\right)\right]\prod_{i=t+1}^{n-1}\exp\left[O\left(\frac{1}{i^2}\right)\right]. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}x_{6n}=0 \quad as \quad n\to\infty.$$

Similarly,

$$\lim_{n\to\infty}x_{6n+j}=0 \quad as \quad n\to\infty,$$

for $j=1,2,3,4,5$.

References

[1] R.P. Agarwal, Difference Equations and Inequalities, Dekker, New York (1992).

[2] L. Berezansky and E. Braverman, On impulsive BevertonHolt difference equations and their applications, J. Difference Equ. Appl. 10:9 (2004), 851–868.

[3] G. Bluman and S. Anco, Symmetry and Integration Methods for Differential Equations, Springer, New York (2002).

[4] V. A. Dorodnitsyn, R. Kozlov and P. Winternitz,Lie group classiffcation of second-order ordinary difference equations, J. Math. Phys. 41 (2000), 480–504.

[5] M. Folly-Gbetoula and A.H. Kara, Symmetries, conservation laws, and integrability of difference equations, *Advances in Difference Equations*, **2014:224** (2014).

[6] Folly-Gbetoula M, Ndlovu L, Kara A H and A Love , Symmetries, Associated First Integrals, and Double Reduction of Difference Equations, *Abstract and Applied Analysis* **2014**, Article ID 490165, (2014) 6 pages.

[7] M. Folly-Gbetoula and D. Nyirenda, On some sixth-order rational recursive sequences, Journal of computational analysis and applications, **27:6** (2019) 1057–1069.

[8] P. E. Hydon, Difference Equations by Differential Equation Methods, *Cambridge University Press*, Cambridge, (2014).

[9] S. Maeda, Canonical structure and symmetries for discrete systems, Math. Japonica 25 (1980), 405–420.

[10] S. Maeda, The similarity method for difference equations, IMA J. Appl. Math.38 (1987), 129–134.

[11] N. Mnguni, M. Foly-Gbetoula, Invariance Analysis of a Third Order Diference Equation with Variable Coefficients, *Dynamics of Continuous, Discrete and Impulsive Systems* **25** (2018), 63-73.

[12] M. Mnguni, D. Nyirenda and M. Folly-Gbetoula, On solutions of some fifth-order difference equations, *Far East Journal of Mathematical Sciences*, **102:12** (2017) 3053-3065.

[13] P. J. Olver, Applications of Lie Groups to Differential Equations, Second Edition, Springer, New York (1993).

GENERALIZED ZWEIER $\mathcal{I}$-CONVERGENT SEQUENCE SPACES OF FUZZY NUMBERS

KAVITA SAINI AND KULDIP RAJ

ABSTRACT. In the present paper we introduce Zweier ideal convergent sequences spaces of fuzzy numbers by using lacunary sequence, infinite matrix and generalized difference matrix operator A_i^p. We study some topological and algebraic properties of these sequence spaces. Some inclusion relations related to these spaces are also establish.

1. Introduction and Preliminaries

Initially the idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al.[10]. Gurdal [7] studied the ideal convergence sequences in 2-normed spaces. Later on, it was further studied by Savas [21], Savas and Hazarika [8], Tripathy and Dutta [25], Tripathy and Hazarika [26], Raj et al.[17]. Let X be a non-empty set, then a family of sets $\mathcal{I} \subset 2^X$ is called an ideal iff for each $X_1, X_2 \in \mathcal{I}$, we have $X_1 \cup X_2 \in \mathcal{I}$ and for each $X_1 \in \mathcal{I}$ and each $X_2 \subset X_1$, we have $X_2 \in \mathcal{I}$. A non-empty family of sets $U \subset 2^X$ is a filter on X iff $\phi \notin U$, for each $X_1, X_2 \in U$, we have $X_1 \cap X_2 \in U$ and each $X_1 \in U$ and each $X_1 \subset X_2$, we have $X_2 \in U$. An ideal $\mathcal{I}$ is said to be non-trivial ideal if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$. Clearly, $\mathcal{I} \subset 2^X$ is a non-trivial ideal iff $U = U(\mathcal{I}) = \{X - X_1 : X_1 \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset 2^X$ is said to be admissible iff $\{x : x \in X\} \subset \mathcal{I}$. A non-trivial ideal is called maximal if there cannot exists any non-trivial ideal $\mathcal{J} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset.

A sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to a real number x_0 if

$$\{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\} \in \mathcal{I},$$

for every $\epsilon > 0$ (see [10]). We denote it by $\mathcal{I} - \lim x_k = x_0$.

Kızmaz [9] introduced the notion of difference sequence spaces and studied $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. Further this notion generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\Delta^i)$, $c(\Delta^i)$ and $c_0(\Delta^i)$. The new type of generalization of the difference sequence spaces was introduced by Tripathy and Esi [27] who studied the spaces $l_\infty(\Delta_v^i)$, $c(\Delta_v^i)$ and $c_0(\Delta_v^i)$. Let i, v be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_v^i) = \{x = (x_k) \in w : (\Delta_v^i x_k) \in Z\},$$

where $\Delta_v^i x = (\Delta_v^i x_k) = (\Delta_v^{i-1} x_k - \Delta_v^{i-1} x_{k+1})$ and $\Delta_v^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^i x_k = \sum_{n=0}^{i} (-1)^n \begin{pmatrix} i \\ n \end{pmatrix} x_{k+vn}.$$

Başar and Atlay [2] introduced and studied the generalized difference matrix $A(m,n) =$

2010 *Mathematics Subject Classification.* 40A05, 40A30.

Key words and phrases. Musielak-Orlicz function, ideal convergence, generalized difference matrix operator, fuzzy real number.

$(a_{rs}(m,n))$ which is a generalization of $\Delta^1_{(1)}$-difference operator as follows:

$$a_{rs}(m,n) = \begin{cases} m, & (s=r); \\ n, & (s=r-1); \\ 0, & 0 \leq s \leq r-1 \ \text{ or } s > r. \end{cases}$$

for all $r, s \in \mathbb{N}$ and $m, n \in \mathbb{R} - \{0\}$.
Başarir and Kayikçi [3] introduced the generalized difference matrix A^p of order p and the binomial representation of this operator is

$$A^p(x_k) = \sum_{v=0}^{p} \binom{p}{v} m^{p-v} n^v x_{k-v},$$

where $m, n \in \mathbb{R} - \{0\}$ and $r \in \mathbb{N}$.
Recently, Başarir et al.[4] studied the following generalized difference sequence spaces

$$Z(A^p_i) = \{x = (x_k) \in w : (A^p_i x_k) \in Z\},$$

for $Z = l_\infty,\ \bar{c},\ \bar{c_0}$, where $\bar{c}, \bar{c_0}$ are the sets of statistically convergent and statistically null convergent respectively and the binomial representation of operator A^p_i is as follows:

$$A^p_i(x_k) = \sum_{v=0}^{p} \binom{p}{v} m^{p-v} n^v x_{k-iv}.$$

Şengönül [22] defined the sequence $y = (y_k)$ which is frequently used as the $Z-$transformation of the sequence $x = (x_k)$ that is,

$$y_k = \beta x_k + (1-\beta) x_{k-1},$$

where $x_{-1} = 0,\ k \neq 0,\ 1 < k < \infty$ and Z denotes the matrix $Z = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} \beta, & (i=k); \\ 1-\beta, & (i-1=k)(i,k \in \mathbb{N}); \\ 0, & \text{otherwise}. \end{cases}$$

Şengönül [22] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows:

$$\mathcal{Z} = \{x = (x_k) \in w : Z(x) \in c\}$$

and

$$\mathcal{Z}_0 = \{x = (x_k) \in w : Z(x) \in c_0\}.$$

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex, continuous and non-decreasing function which also satisfy $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is called as an Orlicz sequence space. An Orlicz function is said to satisfy Δ_2-condition if for a constant R, $M(Qx) \leq RQM(x)$ for all values of $x \geq 0$ and for $Q > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called as Musielak-Orlicz function.To know more about sequence spaces see ([1], [15], [16], [24], [18], [19] and [28]) and references therein.
An increasing non-negative integer sequence $\theta = (k_r)$ with $k_0 = 0$ and $k_r - k_{r-1} \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and q_r denotes the ratio $\frac{k_r}{k_{r-1}}$. The space of

lacunary strongly convergent sequence was defined by Freedman et al. [6] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0, \text{ for some } L\Big\}.$$

The space N_θ is a $BK-$ space with the norm

$$\|x\| = \sup\Big(\frac{1}{h_r}\sum_{k\in I_r} |x_k|\Big).$$

Let $\lambda = (\lambda_{nk})$ be an infinite matrix of real or complex numbers λ_{nk}, where $n, k \in$ N. Then a matrix transformation of $x = (x_k)$ is denoted as λx and $\lambda x = \lambda_n(x))$ if $\lambda_n(x) = \sum_{k=1}^{\infty} \lambda_{nk}x_k$ converges for each $n \in$ N.

The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [29] in 1965. Subsequently many authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. The theory of sequences of fuzzy numbers was first studied by Matloka [12]. He studied some of their properties and showed that every convergent sequences of fuzzy numbers is bounded. Later on Nanda [13] introduced sequences of fuzzy numbers and studied that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Further, the theory of sequences of fuzzy numbers have been discussed by Savas and Mursaleen [20], Tripathy and Nanda [23], Hazarika and Savas [8] and many more.

Let B denote the set of all closed bounded intervals $U = [u_1, u_2]$ on the real line $\mathbb{R}$. For $U, V \in B$, we define $U \leq V$ iff $u_1 \leq v_1$ and $u_2 \leq v_2$ and we define

$$d(U, V) = \max\{|u_1 - v_1|, |u_2, v_2|\}.$$

It is well known that d defines a metric on B and (B, d) is a complete metric space (see [14]).

A fuzzy number is a function $U : \mathbb{R} \to [0, 1]$, which satisfy the following conditions:
(i) U is normal i.e there exits an x_0 such that $U(x_0) = 1$,
(ii) U is convex i.e for $x, y \in \mathbb{R}$ and $0 \leq \tau \leq 1$,

$$U(\tau x + (1 - \tau)y) \geq \min\{U(x), U(y)\},$$

(iii) U is upper semi-continuous,
(iv) the closure of the set supp(U) is compact, where supp$(U) = \{x \in \mathbb{R} : U(x) > 0\}$ and it is denoted by $[U]^0$.

The set of all fuzzy numbers are denoted by $\mathbb{R}_\mathbb{F}$. Let $[U]^0 = \overline{x \in \mathbb{R} : u(x) > 0}$ and the r-level set is $[U]^r = \{x \in \mathbb{R} : u(x) \geq r\}, (0 \leq r \leq 1)$. The set $[U]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $U, V \in \mathbb{R}_\mathbb{F}$ and $\lambda \in \mathbb{R}$, it is positive to define uniquely the sum $U \oplus V$ and the product $U \odot V$ as follows:

$$[U \oplus V]^r = [U]^r + [V]^r \text{ and } [\lambda \odot U]^r = \lambda[U]^r.$$

Now, denote the interval $[U]^r$ by $[u_1^{(r)}, u_2^{(r)}]$, where $u_1^{(r)} \leq u_2^{(r)}$ and $u_1^{(r)}, u_2^{(r)} \in \mathbb{R}$, for $r \in [0, 1]$.

Now, define $\hat{d} : \mathbb{R}_\mathbb{F} \times \mathbb{R}_\mathbb{F} \to \mathbb{R}$ by

$$\hat{d}(U, V) = \sup_{r\in[0,1]} d([U]^r, [V]^r).$$

Definition 1.1. A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to a fuzzy number x_0 if for every $\epsilon > 0$ there exist a positive integer n_0 such that

$$\hat{d}(x_k, x_0) < \epsilon, \text{ for } k > n_0.$$

Definition 1.2. A sequence $x = (x_k)$ of fuzzy numbers is said to be $\mathcal{I}$- convergent to a fuzzy number x_0 if for every $\epsilon > 0$ such that

$$\{k \in \mathbb{N} : \hat{d}(x_k, x_0) \geq \epsilon\} \in \mathcal{I}.$$

Throughout the article, we denote Zweier fuzzy number sequence $Z(x)$ by x' for $x \in \omega^F$.

Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ be a bounded sequence of positive real numbers, $\lambda = (\lambda_{nk})$ be an infinite matrix, θ be a lacunary sequence and ω^F is the set of all sequences of fuzzy real numbers. In the present paper we define lacunary Zweier $\mathcal{I}-$convergent, lacunary Zweier $\mathcal{I}-$null and lacunary Zweier $\mathcal{I}-$bounded sequence spaces of fuzzy numbers as follows:

$$\mathcal{Z}^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Big\{x = (x_k) \in \omega^F : \Big\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big]^{q_k} \geq \epsilon\Big\} \in \mathcal{I}$$
$$\text{for some } \rho > 0 \text{ and } x_0 \in \mathbb{R}_{\mathbb{F}}\Big\},$$

$$\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Big\{x = (x_k) \in \omega^F : \Big\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\Big)\Big]^{q_k} \geq \epsilon\Big\} \in \mathcal{I}$$
$$\text{for some } \rho > 0\Big\}$$

and

$$\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Big\{x = (x_k) \in \omega^F : \exists K > 0 \text{ s.t. } \Big\{n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\Big)\Big]^{q_k} \geq K\Big\} \in \mathcal{I}$$
$$\text{for some } \rho > 0\Big\},$$

where,

$$\bar{0}(t) = \begin{cases} 1, & if \ \ t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

If $0 < q_k \leq \sup q_k = D, C = \max(1, 2^{D-1})$. Then

$$|c_k + d_k|^{q_k} \leq C(|c_k|^{q_k} + |d_k|^{q_k}), \tag{1.1}$$

for all $c_k, d_k \in \mathbb{R}$ and for all $k \in \mathbb{N}$.

The main purpose of this paper is to study some classes of lacunary Zweier sequences of fuzzy numbers defined by means of generalized difference matrix operator, Musielak-Orlicz function and infinite matrix. We shall make an effort to study some interesting algebraic and topological properties of concerning sequence spaces. Also, we examine some interrelations between these sequence spaces.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ be a bounded sequence of positive real numbers and θ be a lacunary sequence. Then the sequence spaces $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$, $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ and $\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ are closed under addition and scalar multiplication.*

Proof. Consider $x = (x_k), y = (y_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ and α, β are scalars. Then there exist positive numbers $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho_1}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \in \mathcal{I}$$

and

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y'_k, y_0)}{\rho_2}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \in \mathcal{I}.$$

Since A_i^p is linear and by using the continuity of Musielak-Orlicz function $\mathcal{M}$, we have the following inequality:

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p(\alpha(x'_k) + \beta(y'_k)))}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\right]^{q_k} \\
&\quad\leq\ D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho_1}\right)\right]^{q_k} \\
&\quad+\ D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\left(\frac{\hat{d}(A_i^p y'_k, y_0)}{\rho_2}\right)\right]^{q_k} \\
&\quad\leq\ DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho_1}\right)\right]^{q_k} \\
&\quad+\ DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y'_k, y_0)}{\rho_2}\right)\right]^{q_k},
\end{aligned}$$

where $K = \max\left\{1, \frac{|\alpha|\rho_1}{|\alpha|\rho_1+|\beta|\rho_2}, \frac{|\beta|\rho_2}{|\alpha|\rho_1+|\beta|\rho_2}\right\}$.

Thus, we have

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p(\alpha(x'_k) + \beta(y'_k)))}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\right]^{q_k} \geq \epsilon\right\} \\
&\quad\subseteq\ \left\{n \in \mathbb{N} : DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho_1}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \\
&\quad\cup\ \left\{n \in \mathbb{N} : DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y'_k, y_0)}{\rho_2}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\}.
\end{aligned}$$

Since the sets on right hand side of above relation belong to $\mathcal{I}$. Thus, the sequence space $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ is closed under addition and scalar multiplication . Similarly, we can prove others. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ and $v = (v_k)$ be two bounded sequences of positive real numbers with $0 < q_k \leq v_k$ for each k and $(\frac{v_k}{q_k})$ be bounded. Then*

(i) $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$,
(ii) $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$,
(iii) $\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$.

Proof. The proof of the theorem is straightforward, so we omit it. □

Theorem 2.3. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and $q=(q_k)$ be a bounded sequence of positive numbers. Then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q],\subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}$ $[A_i^P,\theta,\lambda,\mathcal{M},q]$.*

Proof. We know that the first inclusion is obvious. Next, we show that $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$. Let $(x_k)\in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]$. Then we have

$$\frac{1}{h_r}\sum_{k\in I_r}\lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x'_k,\bar{0})}{\rho}\right)\right]^{q_k}$$
$$\begin{aligned}
&\le \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x'_k,x_0)}{\rho}\right)\right]^{q_k}\\
&+ \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\left[M_k\left(\frac{\hat{d}(x_0,\bar{0})}{\rho}\right)\right]^{q_k}\\
&\le \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x'_k,x_0)}{\rho}\right)\right]^{q_k}\\
&+ C\max\left\{1,\sup\left(\lambda_{nk}\left[M_k\left(\frac{\hat{d}(x_0,\bar{0})}{\rho}\right)\right]\right)^D\right\},
\end{aligned}$$

where $\sup q_k=D$ and $C=\max(1,2^{D-1})$. Therefore, $(x_k)\in \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]$. This completes the proof of the theorem. □

Theorem 2.4. *Let $\mathcal{M}=(M_k)$ and $\mathcal{M}'=(M'_k)$ be two Musielak-Orlicz functions. Then the folowing inclusions holds:*
(i) $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$,
(ii) $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$,
(iii) $\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$.

Proof. Suppose $(x_k)\in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]$. Then, we have

$$\lambda_{nk}\left[(M_k+M'_k)\left(\frac{\hat{d}(A_i^p x'_k,\bar{0})}{\rho}\right)\right]^{q_k}$$
$$\le C\left[\lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x'_k,\bar{0})}{\rho}\right)\right]^{q_k}+C\left[\lambda_{nk}\left[M'_k\left(\frac{\hat{d}(A_i^p x'_k,\bar{0})}{\rho}\right)\right]^{q_k},$$

which consequently implies that

$$\frac{1}{h_r}\sum_{k\in I_r}\lambda_{nk}\left[(M_k+M'_k)\left(\frac{\hat{d}(A_i^p x'_k,\bar{0})}{\rho}\right)\right]^{q_k}$$
$$\begin{aligned}
&\le \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x'_k,\bar{0})}{\rho}\right)\right]^{q_k}\\
&+ \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\left[M'_k\left(\frac{\hat{d}(A_i^p x'_k,\bar{0})}{\rho}\right)\right]^{q_k}.
\end{aligned}$$

This implies $(x_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M} + \mathcal{M}', q]$. We can prove the other cases in the same way. □

Theorem 2.5. *Let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M_k')$ be two Musielak-Orlicz functions. Then the folowing inclusion holds:*

$$\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}', q] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}.\mathcal{M}', q].$$

Proof. For given $\epsilon > 0$ and choose ϵ_0 such that $\sup_n \left(\sum_{k \in I_r} \lambda_{nk} \right) \max\{\epsilon_0^h, \epsilon_0^D\} < \epsilon$. Choose $0 < \varphi < 1$ such that $M_k(t) < \epsilon_0$, for all $k \in \mathbb{N}$. Let $x = (x_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}', q]$. Then for some $\rho > 0$, we have

$$B_1 = \left\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right) \right]^{q_k} \geq \varphi^D \right\} \in \mathcal{I}.$$

If $n \notin B_1$, then we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right) \right]^{q_k} < \varphi^D.$$

This implies

$$\left[M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right) \right]^{q_k} < \varphi^D \quad \text{for all } k \in \mathbb{N}.$$

Hence,

$$M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right)^{q_k} < \varphi \quad \text{for all } k \in \mathbb{N}.$$

Therefore,

$$M_k\left(M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right) \right)^{q_k} < \epsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Thus, we get

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right) \right) \right]^{q_k} < \sup_n \left(\sum_{k \in I_r} \lambda_{nk} \right) \max\{\epsilon_0^h, \epsilon_0^D\} < \epsilon.$$

Now, we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right) \right) \right]^{q_k} < \epsilon.$$

This implies

$$\left\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(M_k'\left(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \right) \right) \right]^{q_k} \geq \epsilon \right\} \subset B_1 \in \mathcal{I}.$$

This completes the proof. □

Theorem 2.6. *If $\lim q_k > 0$ and $x = (x_k) \to x_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$, then x_0 is unique.*

Proof. Let $\lim q_k = u_0$. Consider that $(x_k) \to x_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$ and $(x_k) \to y_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$. So, there exist $\rho_1, \rho_2 > 0$, such that

$$X_1 = \left\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \in \mathcal{I} \tag{2.1}$$

and

$$X_2 = \left\{ n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', y_0)}{\rho_2} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \in \mathcal{I}. \tag{2.2}$$

Define $\rho = \max\{2\rho_1, 2\rho_2\}$. Then we have

$$\begin{aligned}
&\sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k} \\
&\quad \leq \quad D \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho} \right) \right]^{q_k} + D \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', y_0)}{\rho} \right) \right]^{q_k}.
\end{aligned}$$

Then from (2.1) and (2.2), we have

$$\begin{aligned}
&\left\{ n \in \mathbb{N} : \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k} \geq \epsilon \right\} \\
&\quad \subseteq \quad \left\{ n \in \mathbb{N} : D \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \\
&\quad \cup \quad \left\{ n \in \mathbb{N} : D \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', y_0)}{\rho_2} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \\
&\quad \subseteq \quad X_1 \cup X_2 \in \mathcal{I}.
\end{aligned}$$

Also,

$$\left[M_k\left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k} \to \left[M_k\left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{u_0} \quad \text{as} \;\; k \to \infty.$$

Then, we have

$$\lim_{k\to\infty} \left[M_k\left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k} = \left[M_k\left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{u_0} = 0.$$

Thus, $x_0 = y_0$. □

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $q = (q_k)$ be a bounded sequence of positive real numbers,*
(a) If $0 < \inf q_k \leq q_k \leq 1$ for all k, then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$ and $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q] \subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$.
(b) If $1 \leq q_k \leq \sup q_k = D < \infty$ for all k, then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$ and $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}] \subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$.

Proof. (a) Suppose $(x_k) \in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$. Since $0 < \inf q_k \leq q_k \leq 1$, then we have

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho} \right) \right] \\
&\quad \leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho} \right) \right]^{q_k}.
\end{aligned}$$

Thus,

$$\Big\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big] \geq \epsilon\Big\}$$

$$\subseteq \quad \Big\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big]^{q_k} \geq \epsilon\Big\} \in \mathcal{I}.$$

The other part can be proved in the same way.

(ii) Suppose $(x_k) \in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$. Since $1 \leq q_k \leq \sup q_k = D < \infty$. Then for each $0 < \epsilon < 1$, there exists a positive integer m_0 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big] \leq \epsilon < 1,$$

for all $n \geq m_0$. This implies

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big]^{q_k}$$

$$\leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big].$$

Thus,

$$\Big\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big]^{q_k} \geq \epsilon\Big\}$$

$$\subseteq \quad \Big\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Big)\Big] \geq \epsilon\Big\} \in \mathcal{I}.$$

The other part can be proved in the same way. $\square$

References

[1] R. Anand, C. Sharma and K. Raj, *Seminormed double sequence spaces of four dimensional matrix and Musielak-Orlicz function*, J. Inequal. Appl., (2018), **2018**:285.

[2] E. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J., **55** (2003), 136-147.

[3] M. Başarir and M. Kayikçi, *On the generalized B^m-Riesz difference sequence spaces and β-property*, J. Inequal. Appl., **2009** (2009), Article ID 385029, 18 pages.

[4] M. Başarir, S. Kayikci and E. E. Kara, *Some generalized difference statistically convergent sequence spaces in 2-normed space*, J. Inequal. Appl., **37** (2013), 2013: 177.

[5] M. Et and R. Çolak, *On some generalized difference sequence spaces and related matrix transformations*, Hokkaido Math. J., **26** (1997), 483-492.

[6] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesaro-type summability spaces*, Proc. London Math. Soc., **37** (1978), 508-520.

[7] M. Gurdal, *On ideal convergent sequences in 2-normed spaces*, Thai J. Math., **4** (2006), 85-91.

[8] B. Hazarika and E. Savas, *Some I-convergent lambda-summable difference sequences spaces of fuzzy real numbers defined by a sequence of Orlicz functions*, Math. Comput. Modelling, **54** (2011), 2986-2998.

[9] H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull., **24** (1981), 169-176.

[10] P. Kostyrko, T. Salat and W. Wilczynski, *I-convergence*, Real Anal. Exchange, **26** (2000-2001), 669-686.

[11] J. Lindenstrauss and L. Tzafriri, *An Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379-390.

[12] M. Matloka, *Sequences of fuzzy numbers*, J. Math. Anal. Appl., **28** (1986), 28-37.

[13] S. Nanda., *On sequences of fuzzy number*, Fuzzy Sets Syst., **33** (1989), 123-126.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

[14] M. L. Puri and D. A. Ralescu, *Differentials of fuzzy functions*, J. Math. Anal. Appl., **91** (1983), 552-558.

[15] K. Raj, A. Choudhary and C. Sharma, *Almost strongly Orlicz double sequence spaces of regular matrices and their applications to statistical convergence*, Asian-Eur. J. Math., **11** 1850073 (2018), doi.org/10.1142/S1793557118500730.

[16] K. Raj, C. Sharma and A. Choudhary, *Applications of Tauberian theorem in Orlicz spaces of double difference sequences of fuzzy numbers*, J. Intell. Fuzzy Systems, **35** (2018), 2513-2524.

[17] K. Raj, A. Abzhapborav and A. Khassymkan, *Some generalized difference sequences of ideal convergence and Orlicz functions*, J. Comput. Anal. Appl., **22** (2017), 52-63.

[18] K. Raj, and R. Anand, *Double difference spaces of almost null and almost convergent sequences for Orlicz function*, J. Comput. Anal. Appl., **24** (2018), 773-783.

[19] A. Choudhary and K. Raj, *Applications of double difference fractional order operators to originate some spaces of sequences*, J. Comput. Anal. Appl., **28** (2020), 94-103.

[20] E. Savas and M. Mursaleen, *On statistically convergent double sequence of fuzzy numbers*, Inform. Sci., **162** (2004), 183-192.

[21] E. Savas, *A sequence spaces in 2-normed space defined by ideal convergence and an Orlicz function*, Abst. Appl. Anal., (2011), Article ID 741382, 1-8.

[22] M. Şengönül, *On The Zweier Sequence Space*, Demonstratio Math., **40** (2007), 181-196.

[23] B. C. Tripathy and S. Nanda, *Absolute value of fuzzy real number and fuzzy sequence spaces*, Jour. Fuzzy Math., **8** (2000), 883-892

[24] B. C. Tripathy, S. Debnath and S. Saha, *On some difference sequence spaces of interval numbers*, Proyecciones, **37** (2018), 603-612.

[25] B. C. Tripathy and A. J. Dutta, *On I-acceleration convergence of sequences of fuzzy numbers*, Math. Modell. Analysis, **17** (2012), 549-557.

[26] B. C. Tripathy and B. Hazarika, *Some I-convergent sequence spaces defined by Orlicz functions*, Acta. Math. Appl. Sinica, **27** (2011), 149-154.

[27] B. C. Tripathy and A. Esi, *A new type of difference sequences spaces*, Int. J. Sci. and Tech., **1** (2006), 11-14.

[28] B. C. Tripathy and R. Goswami, *Statistically convergent multiple sequences in probalistic normed spaces*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **78** (2016), 83-94.

[29] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338-353.

School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
E-mail address: kavitasainitg3@gmail.com
E-mail address: kuldipraj68@gmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Some convergence results using K^* iteration process in $CAT(0)$ spaces

Kifayat Ullah, Dong Yun Shin, Choonkil Park and Bakhat Ayaz Khan

Abstract. In this paper, some strong and Δ-convergence results for Suzuki generalized nonexpansive mappings in the setting of complete $CAT(0)$ spaces are proved. We are using newly introduced K^* iteration process for approximation of fixed point. We also give an example to show the efficiency of the K^* iteration process. Our results are extension, improvement and generalization of many well known results in the literature of fixed point theory in $CAT(0)$ spaces.

Mathematics Subject Classification (2010). Primary 47H09, 47H10.

Keywords. Suzuki generalized nonexpansive mapping; $CAT(0)$ space; K^* iterative process; Δ-convergence; strong convergence.

1. Introduction

It is well-known that several mathematics problems are naturally formulated as fixed point problem $Tx = x$, where T is some suitable mapping, may be nonlinear. For example, for given functions $\zeta : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ and $\xi : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$, the solution of following nonlinear integral equation

$$x(c) = \zeta(c) + \int_a^b \xi(c, r, x(r))dr,$$

where $x \in C[a,b]$ (the set of all continuous real-valued functions defined on $[a,b] \subseteq \mathbb{R}$), is equivalently to fixed point problems for the following mapping $T : C[a,b] \to C[a,b]$ defined by

[0]*Corresponding author: Dong Yun Shin (email: dyshin@uos.ac.kr).
This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937). We would like to thank Prof. Balwant Singh Thakur for technical assistance.

$$(Tx)(c) = \zeta(c) + \int_a^b \xi(c, r, x(r))dr$$

for all $x \in C[a, b]$.

The well-known Banach contraction theorem uses the Picard iteration process for approximation of fixed point. Many iterative processes have been developed to approximate fixed points of contraction type of mapping in $CAT(0)$ type spaces of ground spaces. Some of the other well-known iterative processes are those of Mann [17], Ishikawa [10], Noor [8], Abbas [1], Agarwal [2], Phuengrattana and Suantai [19], Karahan and Ozdemir [11], Chugh, Kumar and Kumar [6], Sahu and Petrusel [20], Khan [14], Gursoy and Karakaya [9], Thakur, Thakur and Postolache [22] and so on. See also [13, 23, 25] for more information on $CAT(0)$ spaces and applications. Recently, Ullah and Arshad [24] introduced a new three steps iteration process as the K^* iteration process and proved that it is strong and converges fast as compared to all above mentioned iteration processes. They use uniformly convex Banach space as a ground space.

Motivated by above, in this paper, first we develop an example of Suzuki generalized nonexpansive mappings is given which is not nonexpansive. We compare the speed of convergence of the K^* iteration process with the leading two steps S-iteration process and leading three steps Picard-S-iteration process for Suzuki generalized nonexpansive mappings, and graphic representation is also given.

Finally, we prove some strong and Δ-convergence theorems for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ spaces.

2. Preliminaries

Let (X, d) be a metric space. A geodesic from x to y in X is a mapping c from closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t^{'})) = |t - t^{'}|$ for all $t, t^{'} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. The space (X, d) is said to be a geodesic space if every two points of X is joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, $y \in X$, which we denote by $[x, y]$, called the segment joining x to y.

A geodesic triangle $\Delta(x_{1,}x_{2,}x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the triangle $\Delta(x_{1,}x_{2,}x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_{1,}x_{2,}x_3) := \Delta(\bar{x}_{1,}\bar{x}_{2,}\bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

$CAT(0)$: Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{E^2}(\bar{x}, \bar{y}).$$

If $x, y_{1,} y_2$ are points in $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2. \tag{CN}$$

This is the (CN) inequality of Burhat and Tits [5].

We recall the following result from Dhompongsa and Panyanak [8].

Lemma 2.1. ([8]) *For $x, y \in X$ and $\alpha \in [0, 1]$, there exists a unique point z $\in [x,\, y]$ such that*

$$d(x, z) = \alpha d(x, y) \text{ and } d(y, z) = (1 - \alpha) d(x, y). \tag{2.1}$$

The notation $((1 - \alpha)x \oplus \alpha y)$ is used for the unique point z satisfying (2.1).

$CAT(0)$ space may be regarded as a metric version of Hilbert space. For example, in $CAT(0)$ space we have the following extended version of parallelogram law:

$$d(z, \alpha x \oplus (1 - \alpha)y)^2 = \alpha d(x, z)^2 + (1 - \alpha) d(z, y)^2 - \alpha(1 - \alpha) d(x, y)^2 \tag{2.2}$$

for any $\alpha \in [0, 1]$, $x, y \in X$.

If $\alpha = \frac{1}{2}$, then the inequality (2.2) becomes the (CN) inequality.

In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (cf. [5]). Complete $CAT(0)$ spaces are often called Hadmard spaces. For more on these spaces, please refer to [3, 4].

Lemma 2.2. ([14, Lemma 2.4]) *For $x, y, z \in X$ and $\alpha \in [0, 1]$, we have*

$$d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha) d(z, y).$$

Let C be a nonempty closed convex subset of a $CAT(0)$ space X let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x_n, x).$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well known that, in a complete $CAT(0)$ space, $A(C, \{x_n\})$ consists of exactly one point.

We now recall the definition of Δ-convergence in $CAT(0)$ space.

Definition 2.3. A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$.

In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-lim of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$.

Since in a $CAT(0)$ space every regular sequence Δ-converges, we see that every bounded sequence in X has a Δ-convergent subsequence.

A $CAT(0)$ space X is said to satisfy the Opial$'$s property [17] if for each sequence $\{x_n\}$ in X, Δ-converges to $x \in X$, we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$$

for all $y \in X$ such that $y \neq x$.

Definition 2.4. A point p is called a fixed point of a mapping T if $T(p) = p$ and $F(T)$ represents the set of all fixed points of the mapping T.

Definition 2.5. Let C be a nonempty subset of a $CAT(0)$ space X.

(i) A mapping $T : C \to C$ is called a contraction if there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in C$.

(ii) A mapping $T : C \to C$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$.

(iii) A mapping is a quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have

$$d(Tx, p) \leq d(x, p).$$

In 2008, Suzuki [21] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C). A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y).$$

Suzuki [21] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness. The mapping satisfying condition (C) is called a Suzuki generalized nonexpansive mapping.

Suzuki [21] obtained fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mapping. In 2011, Phuengrattana [18] proved convergence theorems for Suzuki generalized nonexpansive mappings using the Ishikawa iteration in uniformly convex Banach spaces and $CAT(0)$ spaces. Recently, fixed point theorems for Suzuki generalized nonexpansive mapping have been studied by a number of authors, see, e.g., [22] and references therein.

The following are some basic properties of Suzuki generalized nonexpansive mappings whose proofs in the setup of $CAT(0)$ spaces follow the same lines as those of [12, Propostions 11, 14, 19] and therefore we omit them.

Proposition 2.6. *Let C be a nonempty subset of a $CAT(0)$ space X and $T : C \to C$ be any mapping.*

(i) [21, Proposition 1] *If T is nonexpansive, then T is a Suzuki generalized nonexpansive mapping.*

(ii) [21, Proposition 2] *If T is a Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(iii) [21, Lemma 7] *If T is a Suzuki generalized nonexpansive mapping, then*

$$d(x, Ty) \leq 3d(Tx, x) + d(x, y)$$

for all $x, y \in C$.

Lemma 2.7. [21, Theorem 5] *Let C be a weakly compact convex subset of a $CAT(0)$ space X. Let T be a mapping on C. Assume that T is a Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.8. [16, Lemma 2.9] *Suppose that X is a complete $CAT(0)$ space and $x \in X$. If $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that for some $r \geq 0$, we have*

$$\begin{aligned} \lim_{n\to\infty} \sup d(x_n, x) &\leq r, \\ \lim_{n\to\infty} \sup d(y_n, x) &\leq r, \\ \lim_{n\to\infty} \sup d(t_n x_n + (1 - t_n)y_n, x) &= r, \end{aligned}$$

then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.9. [7, Proposition 2.1] *If C is a closed comvex subset of a complete $CAT(0)$ space X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.10. [15] *Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 2.11. [15, Proposition 3.7] *Let C is a closed comvex subset of a complete $CAT(0)$ space X and $T : C \to X$ be a Suzuki generalized nonexpansive mapping. Then the conditions $\{x_n\}$ Δ-converges to x and $d(Tx_n, x_n) \to 0$ imply $x \in C$ and $Tx = x$.*

The following is an example of Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 1. Define a mapping $T : [0, 1] \to [0, 1]$ by

$$Tx = \begin{cases} 1 - x \text{ if } x \in \left[0, \frac{1}{6}\right) \\ \frac{x+5}{6} \text{ if } x \in \left[\frac{1}{6}, 1\right]. \end{cases}$$

We need to prove that T is a Suzuki generalized nonexpansive but not nonexpansive.

If $x=\frac{15}{96}$ and $y=\frac{1}{6}$, then we have

$$\begin{aligned} d(Tx,Ty) &= |Tx-Ty| \\ &= \left|1-\frac{15}{96}-\frac{31}{36}\right| \\ &= \frac{5}{288} \\ &> \frac{1}{96} \\ &= d(x,y). \end{aligned}$$

Hence T is not a nonexpansive mapping.

To verify that T is a Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x\in\left[0,\frac{1}{6}\right)$. Then $\frac{1}{2}d(\,x,Tx)=\frac{1-2x}{2}\in\left(\frac{1}{3},\frac{1}{2}\right]$. For $\frac{1}{2}d(x,Tx)\leq d(x,y)$, we have $\frac{1-2x}{2}\leq y-x$, i.e., $\frac{1}{2}\leq y$ and hence $y\in\left[\frac{1}{2},1\right]$. We have

$$d(Tx,Ty)=\left|\frac{y+5}{6}-(1-x)\right|=\left|\frac{y+6x-1}{6}\right|<\frac{1}{6}$$

and

$$d(x,y)=|x-y|>\left|\frac{1}{6}-\frac{1}{2}\right|=\frac{2}{6}.$$

Hence $\frac{1}{2}d(x,Tx)\leq d(x,y)\Longrightarrow d(Tx,Ty)\leq d(x,y)$.

Case II: Let $x\in\left[\frac{1}{6},1\right]$. Then $\frac{1}{2}d(\,x,Tx)=\frac{1}{2}\left|\frac{x+5}{6}-x\right|=\frac{5-5x}{12}\in\left[0,\frac{25}{72}\right]$. For $\frac{1}{2}d(x,Tx)\leq d(x,y)$, we have $\frac{5-5x}{12}\leq|y-x|$, which gives two possibilities:

(a) Let $x<y$. Then $\frac{5-5x}{12}\leq y-x\Longrightarrow y\geq\frac{5+7x}{12}\Longrightarrow y\in\left[\frac{37}{72},1\right]\subset\left[\frac{1}{6},1\right]$. So

$$d(Tx,Ty)=\left|\frac{x+5}{6}-\frac{y+5}{6}\right|=\frac{1}{6}d(x,y)\leq d(x,y).$$

Hence $\frac{1}{2}d(x,Tx)\leq d(x,y)\Longrightarrow d(Tx,Ty)\leq d(x,y)$.

(b) Let $x>y$. Then $\frac{5-5x}{12}\leq x-y\Longrightarrow y\leq x-\frac{5-5x}{12}=\frac{17x-5}{12}\Longrightarrow y\in\left[-\frac{13}{72},1\right]$. Since $y\in[0,1]$, $y\leq\frac{17x-5}{12}\Longrightarrow x\in\left[\frac{5}{12},1\right]$. So the case is $x\in\left[\frac{5}{12},1\right]$ and $y\in[0,1]$.

Now the case that $x\in\left[\frac{5}{12},1\right]$ and $y\in\left[\frac{1}{6},1\right]$ is the same case as that of (a). So let $x\in\left[\frac{5}{12},1\right]$ and $y\in\left[0,\frac{1}{6}\right)$. Then

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{x+5}{6}-(1-y)\right| \\ &= \left|\frac{x+6y-1}{6}\right|. \end{aligned}$$

For convenience, first we consider $x\in\left[\frac{5}{12},\frac{1}{2}\right]$ and $y\in\left[0,\frac{1}{6}\right)$. Then $d(Tx,Ty)\leq\frac{1}{12}$ and $d(x,y)\geq\frac{3}{12}$. Hence $d(Tx,Ty)\leq d(x,y)$.

TABLE 1. Some values produced by S, Picard-S and K^* IP

	K^*	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.99809713998382	0.99722222222222	0.98333333333333
x_2	0.99997729192914	0.99993300629392	0.99758822658104
x_3	0.99999985210113	0.99999849779947	0.99967552468466
x_4	0.99999999971662	0.99999996779523	0.99995826261755
x_5	1	0.99999999933035	0.99999479283092
x_6	1	0.99999999998638	0.99999936458953
x_7	1	0.99999999999973	0.99999992375668
x_8	1	0.99999999999999	0.99999999097156
x_9	1	1	0.99999999894221
x_{10}	1	1	0.99999999987715

TABLE 2. Some values produced by S, Picard-S and K^* IP

	K^*	Picard-S	S
x_0	0.5	0.5	0.5
x_1	0.99048569991909	0.99722222222222	0.98333333333333
x_2	0.99988645964572	0.99993300629392	0.99758822658104
x_3	0.99999926050565	0.99999926050566	0.99967552468466
x_4	0.99999999858311	0.99999996779523	0.99995826261755
x_5	1	0.99999999933035	0.99999479283092
x_6	1	0.99999999998638	0.99999936458953
x_7	1	0.99999999999973	0.99999992375668
x_8	1	0.99999999999999	0.99999999097156
x_9	1	1	0.99999999894221
x_{10}	1	1	0.99999999987715

Next consider $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left[0, \frac{1}{6}\right)$. Then $d(Tx, Ty) \leq \frac{1}{6}$ and $d(x, y) \geq \frac{2}{6}$. Hence $d(Tx, Ty) \leq d(x, y)$. So

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y).$$

Hence T is a Suzuki generalized nonexpansive mapping.

In order to show the efficiency of K^* iteration process, we use Example 1 with $x_0 = 0.9, x_0 = 0.5$ and get the above Tables 1 and 2. Graphic representation is given in Figure 1.

Let $n \geq 0$ and $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$. Ullah and Arshad [24] introduced a new iteration process known as the K^* iteration process

$$\begin{cases} x_0 \in C \\ z_n = (1 - \beta_n)x_n + \beta_n T x_n \\ y_n = T((1 - \alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = T y_n. \end{cases}$$

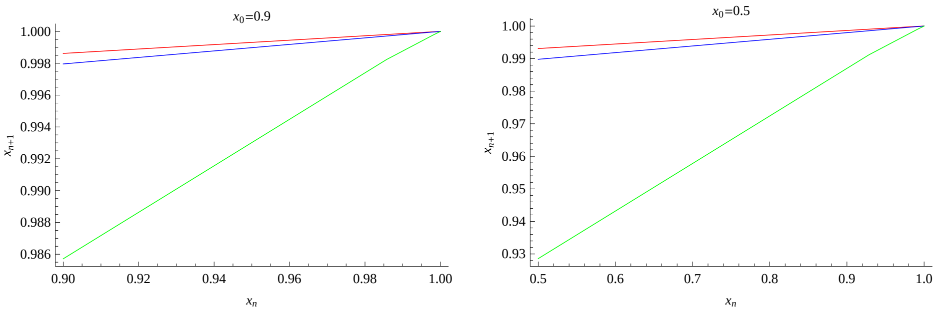

FIGURE 1. Convergence of iterative sequences generated by K^* (red line), Picard-S (blue line) and S (green line) iteration process to the fixed point 1 of the mapping T defined in Example 1.

They also proved that the K^* iteration process is faster than the Picard-S iteration and S-iteration processes with the help of a numerical example.

3. Convergence results for Suzuki generalized nonexpansive mappings

In this section, we prove some strong and Δ-convergence theorems of a sequence generated by a K^* iteration process for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ space. The K^* iteration process in the language of $CAT(0)$ space is given by

$$\begin{array}{c} x_0 \in C \\ z_n = (1-\beta_n)x_n \oplus \beta_n T x_n \\ y_n = T((1-\alpha_n)z_n \oplus \alpha_n T z_n) \\ x_{n+1} = T y_n \end{array} . \tag{3.1}$$

Lemma 3.1. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrarily chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1). Then $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T is a Suzuki generalized nonexpansive mapping,

$$\frac{1}{2}d(p, Tp) = 0 \leq d(p, z) \text{ implies that } d(Tp, Tz) \leq d(p, z).$$

By Proposition 2.6 (ii), we have

$$\begin{aligned} d(z_n,p) &= d(((1-\beta_n)x_n \oplus \beta_n T x_n),p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(Tx_n,p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned} d(y_n,p) &= d((T(1-\alpha_n)z_n \oplus \alpha_n T z_n),p) \\ &\leq d(((1-\alpha_n)z_n \oplus \alpha_n T z_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.3}$$

Similarly by using (3.3), we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &\leq d(x_n,p). \end{aligned}$$

This implies that $\{d(x_n,p)\}$ is bounded and nonincreasing for all $p \in F(T)$. Hence $\lim_{n\to\infty} d(x_n,p)$ exists, as required. $\square$

Theorem 3.2. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a,b]$ for some a,b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Theorem 3.2, $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n,p) = r. \tag{3.4}$$

From (3.2) and (3.4), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{3.5}$$

By Proposition 2.6 (ii) we have

$$\limsup_{n\to\infty} d(y_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{3.6}$$

On the other hand, by using (3.2), we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &= d((T(1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq d(((1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p) \\ &= d(x_n,p)-\alpha_n d(x_n,p)+\alpha_n d(z_n,p). \end{aligned}$$

This implies that

$$\frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p).$$

So

$$d(x_{n+1},p)-d(x_n,p) \leq \frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p),$$

which implies that

$$d(x_{n+1},p) \leq d(z_n,p).$$

Therefore,

$$r \leq \liminf_{n\to\infty} d(z_n,p). \tag{3.7}$$

By (3.5) and (3.7), we get

$$\begin{aligned} r &= \lim_{n\to\infty} d(z_n,p) \\ &= \lim_{n\to\infty} d(((1-\beta_n)x_n+\beta_n Tx_n),p) \\ &= \lim_{n\to\infty} d(\beta_n(Tx_n,p)+(1-\beta_n)(x_n,p)). \end{aligned} \tag{3.8}$$

From (3.4), (3.6), (3.8) and Lemma 2.8, we have that $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n)=0$. Let $p \in A(C,\{x_n\})$. By Proposition 2.6 (iii), we have

$$\begin{aligned} r(Tp,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,Tp) \\ &\leq \limsup_{n\to\infty}(3d(Tx_n,x_n)+d(x_n,p)) \\ &\leq \limsup_{n\to\infty} d(x_n,p) \\ &= r(p,\{x_n\}). \end{aligned}$$

This implies that $Tp \in A(C,\{x_n\})$. Since X is uniformly convex, $A(C,\{x_n\})$ is a singleton and hence we have $Tp=p$. So $F(T) \neq \emptyset$. □

Now we are in the position to prove Δ-convergence theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0,1]$ such that $\{t_n\} \in [a,b]$ and $\{s_n\} \in [0,b]$ or $\{t_n\} \in [a,1]$ and $\{s_n\} \in [a,b$ for some a,b with $0 < a \leq b < 1$. For an arbitrary element $x_1 \in C$, $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$, by Theorem 3.3, we have that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. We now let $w_w\{x_n\} := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $w_w\{x_n\} \subset F(T)$. Let $u \in w_w\{x_n\}$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.9 and 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n$ $\{v_n\} = v \in C$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, $v \in F(T)$ by Lemma 2.11. We claim that $u = v$. Suppose not. Since T is a Suzuki generalized nonexpansive mapping and $v \in F(T)$, $\lim_n d(x_n, v)$ exists by Theorem 3.2. Then by uniqueness of asymptotic centers,

$$
\begin{aligned}
\lim_{n\to\infty} supd(v_n, v) &< \lim_{n\to\infty} supd(v_n, u) \\
&\leq \lim_{n\to\infty} supd(u_n, u) \\
&< \lim_{n\to\infty} supd(u_n, v) \\
&= \lim_{n\to\infty} supd(x_n, v)) \\
&= \lim_{n\to\infty} supd(v_n, v),
\end{aligned}
$$

which is a contradiction and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufices to show that $w_w\{x_n\}$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.9 and 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n$ $\{v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $c \in F(T)$. We can complete the proof by showing that $x = v$. Suppose not. Since $\{d(x_n, v)\}$ is convergent, by the uniqueness of asymptotic centers,

$$
\begin{aligned}
\lim_{n\to\infty} supd(v_n, v) &< \lim_{n\to\infty} supd(v_n, x) \\
&\leq \lim_{n\to\infty} supd(x_n, x) \\
&< \lim_{n\to\infty} supd(x_n, v) \\
&= \lim_{n\to\infty} supd(v_n, v),
\end{aligned}
$$

which is a contradiction and hence the conclusion follows. □

Next we prove the strong convergence theorem.

Theorem 3.4. *Let C be a nonempty compact convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a,b]$ for some a,b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.7, we have that $F(T) \neq \emptyset$ and so by Theorem 3.2 we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.6 (iii), we have

$$d(x_{n_k}, Tp) \leq 3d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p), \text{ for all } n \geq 1.$$

Letting $k \to \infty$, we get $Tp = p$, *i.e.*, $p \in F(T)$. By Theorem 3.2, $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$ and so $\{x_n\}$ converges strongly to p. □

Senter and Dotson [22] introduced the notion of a mappings satisfying condition (I) as follows.

A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$.

Now we prove the strong convergence theorem using condition (I).

Theorem 3.5. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (3.1) *for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we see that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \geq 0$. If $r = 0$, then the result follows. Suppose $r > 0$. Then from the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq d(Tx_n, x_n). \tag{3.9}$$

Since $F(T) \neq \emptyset$, by Theorem 3.3, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. So (3.9) implies that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{3.10}$$

Since f is a nondecreasing function, from (3.10), we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\}$, $y_k \in F(T)$, such that

$$d(x_{n_k}, y_k) < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

So using (3.4), we get

$$d(x_{n_{k+1}}, y_k) \leq d(x_{n_k}, y_k) < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(y_{k+1}, y_k) &\leq d(y_{k+1}, x_{k+1}) + d(x_{k+1}, y_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p \in F(T)$. Hence the proof is complete. □

References

[1] M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mat. Vesn. **66** (2014), 223–234.

[2] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal. **8** (2007), 61–79.

[3] M. Bridson and A. Heaflinger, *Metric Space of Non-positive Curvature*, Springer-Verlag, Berlin, 1999.

[4] D. Burago, Y. Burago and S. Inavo, *A Course in Metric Geometry, in: Graduate Studies in Mathematics*, Vol. 33, Americal Mathematical Society, Providence, 2001.

[5] F. Burhat and J. Tits, *Groups reductifs sur un curps local*, Inst. Hautes Etudes Sci. Publ. Math. **41** (1972), 5–251.

[6] R. Chugh, V. Kumar and S. Kumar, *Strong convergence of a new three step iterative scheme in Banach spaces*, Am. J. Comput. Math. **2** (2012), 345–357.

[7] S. Dhompongsa, W. A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 35–45.

[8] S. Dhompongsa and B. Panyanak, *On* Δ*-convergence theorem in* $CAT(0)$ *spaces*, Comput. Math. Appl. *56* (2008), 2572–2579.

[9] F. Gursoy and V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, arXiv:1403.2546v2 (2014).

[10] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Am. Math. Soc. **44** (1974), 147–150.

[11] I. Karahan and M. Ozdemir, *A general iterative method for approximation of fixed points and their applications*, Adv. Fixed Point Theory **3** (2013), 520–526.

[12] E. Karapinar and K. Tas, *Generalized (C)-conditions and related fixed point theorems*, Comput. Math. Appl. **61** (2011), 3370–3380.

[13] H. Khatibzadeh and S. Ranjbar, *A variational inequality in complete* $CAT(0)$ *spaces*, J. Fixed Point Theory Appl. **17** (2015), 557–574.

[14] S.H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory Appl. **2013**, Article No. 69 (2013).

[15] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spacces*, Nonlinear Anal.–TMA **68** (2008), 3689–3696.

[16] W. Lawaong and B. Panyanak, *Approximating fixed points of nonexpansive nonself mappings in $CAT(0)$ spaces*, Fixed Point Theory Appl. **2010**, Art. ID 367274 (2010).

[17] W.R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506–510.

[18] W. Phuengrattana, *Approximating fixed points of Suzuki-generalized nonexpansive mappings*, Nonlinear Anal. Hybrid Syst. **5** (2011), 583–590.

[19] W. Phuengrattana and S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comput. Appl. Math. **235** (2011), 3006–3014.

[20] D. R. Sahu and A. Petrusel, *Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces*, Nonlinear Anal.–TMA **74** (2011), 6012–6023.

[21] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340** (2008), 1088–1095.

[22] B.S Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput. **275** (2016), 147–155.

[23] G. C. Ugwunnadi, A. R. Khan and M. Abbas, *A hybrid proximal point algorithm for finding minimizers and fixed points in $CAT(0)$ spaces*, J. Fixed Point Theory Appl. **20** (2018), 20:82.

[24] K. Ullah and M. Arshad, *New three step iteration process and fixed point approximation in Banach spaces*, preprint.

[25] Z. Yang and Y.J. Pu, *Generalized Browder-type fixed point theorem with strongly geodesic convexity on Hadamard manifolds with applications*, Indian J. Pure Appl. Math. **43** (2012), 129–144.

Kifayat Ullah
Department of Mathematics, University of Science and Technology, Township Campus, 44000, Bannu, Pakistan
e-mail: `kifayatmath@yahoo.com`

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 02504, Korea
e-mail: `dyshin@uos.ac.kr`

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: `baak@hanyang.ac.kr`

Bakhat Ayaz Khan
Department of Mathematics, University of Science and Technology, Township Campus, 44000, Bannu, Pakistan
e-mail: `bakhtayazkhan580@gmail.com`

Nonlinear Discrete Inequalities Method for the Ulam Stability of First Order Nonlinear Difference Equations

R.Dhanasekaran[1], E.Thandapani[2] and J.M.Rassias[3]

[1] Department of Mathematics,
Vel Tech Rangarajan Dr.Sagunthala,
R and D Institute of Science and Technology,
Chennai - 600 062, India.
e-mail: drrdsekar@yahoo.com

[2] Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Chennai - 600 005, India.
e-mail: ethandapani@yahoo.co.in

[3] Pedagogical Department E.E.,Section of Mathematics and Informatics,
National and Capodistrian University of Athens,
Athens 15342,GREECE.
e-mail:jrassias@primedu.uoa.gr

Abstract

In this paper, first we derive some nonlinear discrete inequalities, and then as an application, we study the Ulam stability of the first order nonlinear difference equation

$$\Delta y(n) = f(n, y(n)),\ n \geq n_0,$$

where f is a given function. The obtained result on Ulam stability is new to the literature in the sense that our approach does not require the explicit form of solutions of the investigated equations.

2010 Mathematics Subject Classification: 39A30,39B82

Keywords and Phrases: Ulam stability, discrete inequality, nonlinear difference equation.

1. Introduction

In the passed years, the Ulam stability of functional equations received a great attention.In general, we say that an equation is stable in the sense of Ulam if for every approximate solution of that equation there exists an exact solution of the equation near it. For more details on Ulam stability, one can refer to [13].

The problem of the Ulam stability of difference equations is related to the notion of the perturbation of discrete dynamical systems. In [2–5, 7–9, 11, 12, 14, 17], the authors studied Ulam stability of linear difference equations and in [16], the authors obtained some results on Ulam stability for some second order linear difference equations. In all these papers, the authors studied the Ulam stability of first and second order linear difference equations and it seems that no results dealing with Ulam stability for the nonlinear difference equations are available in the literature.

Therefore the purpose of this paper is to study that Ulam stability of the following first order nonlinear difference equation

$$\Delta y(n) = f(n, y(n)),\ n \geq \mathbb{N}, \tag{1.1}$$

where $f \in C(\mathbb{N}, \mathbb{R})$ and $\mathbb{N}$ denotes the set of all non-negative integers, without using the explicit form of the solutions.

Next, we present the definition of the Ulam stability for difference equations.

Definition 1.1. *The equation* (1.1) *is called stable in Ulam sense if there exists a constant $L \geq 0$ such that for every $\epsilon > 0$ and every $\{y(n)\}$ in $\mathbb{R}$ satisfying*

$$|\Delta y(n) - f(n, y(n))| \leq \epsilon,\ n \geq 0 \tag{1.2}$$

there exists a sequence $\{x(n)\}$ in $\mathbb{R}$ with the properties

$$\Delta x(n) = f(n, x(n)),\ n \geq 0 \tag{1.3}$$

and

$$|y(n) - x(n)| \leq L\epsilon,\ n \geq 0. \tag{1.4}$$

A sequence $\{y(n)\}$ which satisfies (1.2) for some $\epsilon > 0$ is called an approximate solution of the nonlinear difference equation (1.1), and we reformulate the above definition as: the equation (1.1) is called Ulam stable if for every approximate solution of it there exists an exact solutions close to it. If in Definition 1.1, the number ϵ is replaced by a sequence of positive numbers $\{\epsilon(n)\}$ and $L\epsilon$ from (1.4)by a sequence of positive numbers $\{\eta(n)\}$ the equation (1.1) is called genelized stable in the Ulam sense.

In this paper first we derive some nonlinear discrete inequalities, and as an application we investigate the Ulam stability of equations (1.1).

2. Nonlinear Discrete Inequalities

In this section, we present some nonlinear discrete inequalities which provide us a powerful tool for investigating the Ulam stability of a nonlinear first order difference equations.

We begin with the following results which can be found in: [[6], Theorem 41, pp.39].

Lemma 2.1. *If* $a > 0$ *and* $0 < \alpha \leq 1$, *then*

$$a^{\alpha} \leq \alpha a + (1-\alpha)$$

and the equality holds if $\alpha = 1$.

Theorem 2.2. *Let* $\{u(n)\}$, $\{f(n)\}$, $\{g(n)\}$ *and* $\{h(n)\}$ *be nonnegative real sequences defined for all* $n \in \mathbb{N}$, *and*

$$u(n) \leq f(n) + g(n)\sum_{s=0}^{n-1} h(s)u^{\alpha}(s), \tag{2.1}$$

where $0 < \alpha \leq 1$. *Then*

$$u(n) \leq f(n) + g(n)\sum_{s=0}^{n-1} h(s)(\alpha f(s) + (1-\alpha)) \exp\left(\sum_{t=s+1}^{n-1} \alpha f(t)g(t)\right). \tag{2.2}$$

Proof. Defining a sequence $R(n)$ by

$$R(n) = \sum_{s=0}^{n-1} h(s)u^{\alpha}(s),$$

then $R(0) = 0$ and $u(n) \leq f(n) + g(n)R(n)$. Now using Lemma 2.1, one can obtain

$$\begin{aligned} \Delta R(n) &= h(n)u^{\alpha}(n) \leq h(n)(f(n) + g(n)R(n))^{\alpha} \\ &\leq (\alpha h(n)f(n) + (a-\alpha)h(n)) + \alpha h(n)g(n)R(n) \end{aligned}$$

or

$$R(n+1) - (1+\alpha h(n)g(n))R(n) \leq h(n)(\alpha f(n) + (1-\alpha)). \tag{2.3}$$

Multiplying (2.3) by $\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}$, we have

$$\Delta\left(R(n)\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}\right) \leq h(n)(\alpha f(n)+(1-\alpha))\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}.$$

Summing up the last inequality from 0 to $n-1$, we obtain

$$\begin{aligned} R(n) &\leq \sum_{s=0}^{n-1} h(s)(\alpha f(s)+(1-\alpha)) \prod_{t=s+1}^{n-1}(1+\alpha h(t)g(t)) \\ &\leq \sum_{s=0}^{n-1} h(s)(\alpha f(s)+(1-\alpha))\left(\exp\sum_{t=s+1}^{n-1}\alpha h(t)g(t)\right). \end{aligned} \tag{2.4}$$

Using (2.4) in $u(n) \leq f(n) + g(n)R(n)$, we have the desired inequality (2.2). This completes the proof. ■

Corollary 2.3. *Let $u(n)$ and $p(n)$ be non-negative real sequences defined for all $n \in \mathbb{N}$ such that*

$$u(n) \leq c + \sum_{s=0}^{n-1} p(s)u^{\alpha}(s) \tag{2.5}$$

where $c \geq 0$ and $0 < \alpha \leq 1$. Then

$$u(n) \leq \left(\frac{c\alpha+(1-\alpha)}{\alpha}\right)\exp\left(\sum_{s=0}^{n-1}\alpha p(s)\right). \tag{2.6}$$

Proof. Let $f(n) = c \geq 0$, $g(n) = 1$ and $h(n) = p(n)$ in (2.2), we have

$$\begin{aligned} u(n) &\leq c + \sum_{s=0}^{n-1} p(s)(\alpha c + 1 - \alpha)\prod_{t=s+1}^{n-1}(1+\alpha p(t)) \\ &= c + \frac{(\alpha c+(1-\alpha))}{\alpha}\sum_{s=0}^{n-1}\alpha p(s)\prod_{t=s+1}^{n-1}(1+\alpha p(t)) \\ &= c + \frac{(\alpha c+(1-\alpha))}{\alpha}\left(\prod_{s=0}^{n-1}(1+\alpha p(s)) - 1\right) \\ &\leq \left(\frac{\alpha c+(1-\alpha)}{\alpha}\right)\exp\left(\sum_{s=0}^{n-1}\alpha p(s)\right). \end{aligned}$$

The proof is now complete. ■

Theorem 2.4. *Let $u(n), p(n)$ and $h(n)$ be non-negative real sequences for all $n \in \mathbb{N}$ and*

$$u(n) \leq c + \sum_{s=0}^{n-1} p(s)u(s) + \sum_{s=0}^{n-1} h(s)u^{\alpha}(s), \tag{2.7}$$

where $c \geq 0$ and $0 < \alpha \leq 1$. Then

$$u(n) \leq \left(c + (1-\alpha) \sum_{s=0}^{n-1} h(s) \right) \exp \left(\sum_{s=0}^{n-1} (p(s) + \alpha h(s)) \right). \tag{2.8}$$

Proof. Let $R(n)$ be the right hand side of (2.7). Then $R(0) = c$ and $u(n) \leq R(n)$ and

$$\begin{aligned} \Delta R(n) &= p(n)u(n) + h(n)u^{\alpha}(n) \\ &\leq p(n)R(n) + h(n)R^{\alpha}(n) \\ &\leq p(n)R(n) + h(n)(\alpha R(n) + (1-\alpha)) \\ &= (p(n) + \alpha h(n))R(n) + (1-\alpha)h(n) \end{aligned} \tag{2.9}$$

where we have used Lemma 2.1. Now from (2.9), we have

$$R(n+1) - (1 + p(n) + \alpha h(n))R(n) \leq (1-\alpha)h(n).$$

Arguing as in the proof of Theorem 2.2, one can easily obtain the desired result and hence the details are omitted. ∎

Remark 2.1. *(a) If $\alpha = 1$ in Theorem 2.2, then it reduced to the well-known Pachpatte inequality [10], in 2002. For $0 < \alpha < 1$ the estimate (2.2) of Theorem 2.2 is new to the literature.*
(b) If $\alpha = 1$ and $g(n) \equiv 1$, then Theorem 2.2 reduced to a well-known result due to Sugiyama [15], in 1969.

Remark 2.2. *If $\alpha = 1$ in Corollary 2.3, then it reduced to the discrete analogue of the well-known Gronwall-Bellman inequality [1].*

Remark 2.3. *The result obtained in Theorem 2.4 is different from that one by Willet and Wong [18] for the case $0 < \alpha < 1$.*

3. Ulam Stability

As an application of the discrete inequalities established in Section 2, we investigate the Ulam stability of equation (1.1).

Theorem 3.1. *Let $p(n)$ be a positive real sequence for all $n \in \mathbb{N}$ such that*

$$|f(n,u) - f(n,v)| \le p(n)|u-v|^{\alpha} \tag{3.1}$$

where $0 < \alpha \le 1$, and

$$\sum_{n=0}^{\infty} p(n) < \infty. \tag{3.2}$$

If for a positive real sequence $\phi(n)$ such that $\sum_{n=0}^{\infty} \phi(n) < \infty$, and

$$|\Delta y(n) - f(n, y(n))| \le \phi(n) \tag{3.3}$$

then there exists a real sequence $x(n)$ and a constant $k > 0$ satisfying

$$\Delta x(n) = f(n, x(n)) \tag{3.4}$$

such that $|y(n) - x(n)| \le k$; that is, equation (1.1) has the Ulam stability.

Proof. From the inequality (3.3), we have

$$y(n) \le y(0) + \sum_{s=0}^{n-1} f(s, y(s)) + \sum_{s=0}^{n-1} \phi(s) \tag{3.5}$$

and from the equation (3.4), we obtain

$$x(n) = x(0) + \sum_{s=0}^{n-1} f(s, x(s)). \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$|y(n) - x(n)| \le |y(0) - x(0)| + \sum_{s=0}^{n-1} |f(s, y(s)) - f(s, x(s))| + \sum_{s=0}^{n-1} \phi(s).$$

Using the condition (3.1) in the above inequality, we have

$$|y(n) - x(n)| \le M_1 + \sum_{s=0}^{n-1} p(s)|y(s) - x(s)|^{\alpha} + M_2 \tag{3.7}$$

where $M_1 = |y(0) - x(0)|$ and $\sum_{n=0}^{\infty} \phi(n) \leq M_2$ by hypothesis. Now applying Corollary 2.3 in (3.7), we obtain

$$|y(n) - x(n)| \leq \frac{((M_1 + M_2)\alpha + (1-\alpha))}{\alpha} \exp\left(\sum_{s=0}^{n-1} \alpha p(s)\right). \tag{3.8}$$

It follows from (3.2) that there is a constant $M_3 > 0$ such that $\sum_{n=0}^{\infty} p(n) \leq M_3$, and using this in (3.8), one obtains

$$|y(n) - x(n)| \leq k$$

where $k = \frac{((M_1+M_2)\alpha+(1-\alpha))}{\alpha} \exp(\alpha M_3)$. This completes the proof. ■

4. Conclusion

In this paper, first we have obtained some new nonlinear discrete inequalities and then as an application we investigate the Ulam stability of a nonlinear first order difference equation. In this approach, we do not need to require the explicit form of the solution of the studied equation, where as in [3,4,7-9,11,12,14] the authors used the explicit form of the solutions to prove their established results.

References

[1] R.P.Agarwal, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.

[2] A.R.Baias, F.Blaga and D.Popa, Best Ulam constant for a linear difference equations, Carpathian J. Math., 35(2019), 13-21.

[3] J.Brzdek, D.Popa and B.Xu, The Hyers-Ulam stabiility of nonlinear recurrences, J. Math. Anal. Appl., 335(2007), 443-449.

[4] J.Brzdek, D.Popa and B.Xu, Remarks on stability of linear recurrence of higher order, Appl. Math.Lett., 23(2010), 1459-1463.

[5] J.Brzdek, D.Popa and B.Xu, On nonstability of the linear recurrence of order one, J. Math. Anal. Appl., 367(2010), 146-153.

[6] G.H.Hardy, J.E.Littlewood and G.Polya, Inequalities, Cambridge Univ. Press, Cambridge, 1934.

[7] S.M.Jung, Hyers-Ulam stability of the first order matrix differene equations, Adv. Diff. Equ., 2015(2015), Art. ID.2015:170, 13 pages.

[8] S.M.Jung and Y.W.Nam, On the Hyers-Ulam stability of the first order difference equation, J. Funct. Spaces, Vol.2016, Art. ID:6078298, 6 pages.

[9] M.Onitsuka, Hyers-Ulam stability of first order nonhomogeneous difference equations with a constant stepsize, Appl. Math. Comput., 330(2018), 143-151.

[10] B.G.Pachpatte, Inequalities For Finite Difference Equations, Marcel Dekker, New York, 2002.

[11] D.Popa, Hyers-Ulam stability of the linear recurrence with constant coefficients, Adv. Diff. Equ., 2(2005), 101-107.

[12] D.Popa, Hyers-Ulam-Rassias stability of a linear recurrence, J. Math. Anal. Appl., 309(2005), 591-597.

[13] J.M.Rassias, E.Thandapani, K.Ravi and B.V.Senthil kumar,Book: Functional Equations and Inequalities: Solutions and Stability Results,Series on Concrete and Applicable Mathematics:Vol.21, World Sci. Pub.Comp., Singapore, 2017,Pages 396.

[14] Y.Shen and Y.Li, The Z-transform method for the Ulam stability of linear difference equations with constant coefficients, Adv. Diff. Equ., (2018), 2018:396, 15 pages.

[15] S.Sugiyama, On the stability of difference equations, Bull.Sci.Engr.Research Lab., Waseda Univ., 45(1969), 140-144.

[16] A.K.Tripathy, Hyers-Ulam stability of second order linear difference equation, Inter. J. Diff. Equ. Appl., 16(2017), 53-65.

[17] A.K.Tripathy and P.Senapati, Hyers-Ulam stability of first order linear difference operators on Banach spaces, J. Adv. Math., 14(2018), 7475-7485.

[18] D.Willett and J.S.W.Wong, On the discrete analogues of some generalizations

of Gronwall's inequality, Monatsh. Math., 69(1965), 362-367.

Algebras and Smarandache Types

Jung Mi Ko[1] **and Sun Shin Ahn**[2,*]

[1]*Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea*

[2]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. In this paper we introduce the notion of a BQ-algebra and show that is is equivalent to an abelian group. For deep investigations of several algebraic structures, we introduce the notions of a Smarandache V-algebra-type U-algebra and a Smarandache V-algebra-trans-type U-algebra, and apply the notions to several algebras.

1. Introduction

W. B. Vasantha Kandasamy ([8]) studied the concept of Smarandache groupoids, ideals of groupoids, Smarandache Bol groupoids and strong Bol groupoids, and obtained many interesting results about them. Smarandace semigroups are very important for the study of congruences, and it was studied by R. Padilla ([18]). It will be very interesting to study the Smarandache structure in general algebraic structures. Kim et al. ([11]) defined the concept of a Smarndache d-algebra and investigated some related properties of it. Seo et al. ([19]) introduced the concept of a Smarndache fuzzy BCI-algebra and investigated some related properties of it. Neggers et al. ([17]) defined the notion of a B-algebra and investigated some related properties of it. Some properties of B-algebra are studied in ([3, 12, 13]).

In this paper, we introduce the notion of a BQ-algebras and show that it is equivalent to an abelian group. Moreover, we introduce the notions of a Smarandache V-algebra-type U-algebra and a Smarandache V-algebra-trans-type U-algebra, and apply the notions to several algebras.

2. Preliminaries

A *B-algebra* ([17]) is a non-empty set X with a selected point 0 and a binary operation "$*$" satisfying the following axioms: (i) $x*x = 0$, (ii) $x*0 = x$, (iii) $(x*y)*z = x*(z*(0*y))$ for any $x, y, z \in X$. A B-algebra $(X, *, 0)$ is said to be *0-commutative* ([2]) if $x*(0*y) = y*(0*x)$ for any $x, y \in X$. Let $(X, *, 0)$ be a B-algebra and let $g \in X$. We define $g^{[0]} := 0, g^{[1]} := g^{[0]}*(0*g) = 0*(0*g) = g$ and $g^{[n]} := g^{[n-1]}*(0*g)$ where $n \geq 1$.

[0]**2010 Mathematics Subject Classification**: 06F35; 03G25.

[0]**Keywords**: Smarandache algebra, point algebra, p-derived algebra.

* The corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409

[0]**E-mail**: jmko@gwnu.ac.kr (J. M. Ko); sunshine@dongguk.edu (S. S. Ahn)

Theorem 2.1. *Let $(X, *, 0)$ be a B-algebra and let $g \in X$. Then*

$$g^{[m]} * g^{[n]} = \begin{cases} g^{[m-n]} & \text{if } m \geq n, \\ 0 * g^{[n-m]} & \text{otherwise.} \end{cases}$$

Theorem 2.2. ([10]) *Every 0-commutative B-algebra is a BCI-algebra.*

Theorem 2.3. ([10]) *The following are equivalent:*

(i) *X is an abelian group,*
(ii) *X is a p-semisimple BCI-algebra,*
(iii) *X is a 0-commutative B-algebra.*

Let $(X, *, 0)$ be a B-algebra. Given $x, y \in X$, we define $x *^{\langle 1 \rangle} y := x * y, x *^{\langle 2 \rangle} y := (x * y) * y, x *^{\langle n \rangle} y := (x *^{\langle n-1 \rangle} y) * y$ where $n \geq 3$. For general references for BCK/BCI-algebras, we refer to [5, 6, 14].

3. Several algebras

Let $(X, *)$ be a groupoid (or a binary system, an algebra), i.e., X is a set and "$*$" is a binary operation on X. If we take an element p in X which plays an important role in $(X, *)$, then we say that p is a *selected point* and we write it by $(X, *, p)$. Such an algebra $(X, *, p)$ is said to be a *pointed algebra*.

Example 3.1. *Let $(X, *)$ be a group with identity e. The identity element e plays an important role in $(X, *)$ and hence we may write it by $(X, *, e)$ and e becomes a selected point in $(X, *)$.*

We regard all algebras below as pointed algebras without loss of generality. For simplicity's sake, we shall write $p = 0$, not intending 0 to have the usual meaning. Thus, in Example 3.1, $(X, *, e)$ becomes $(X, *, 0)$ unless it is important to distinguish the algebra $(X, *, 0)$ which contains the subalgebra (not necessary a subgroup) $(Y, *)$ with its selected point p to produce $(Y, *, p)$.

Example 3.2. Consider $X := \{a, b, c, d\}$ with the following table:

$*$	a	b	c	d
a	a	a	a	a
b	a	a	a	b
c	a	b	c	c
d	a	b	c	d

Then $(X, *, d)$ is an pointed algebra and the selected point d is the right identity. Consider $Y := \{a, c\}$ and $Z := \{a, d\}$ with the following tables:

*	a	c
a	a	a
c	a	c

*	a	d
a	a	a
d	a	d

Then $(Y, *, c)$ is a pointed algebra with a selected point c is the right identity, and $(Z, *, d)$ is also a pointed algebra with a special point d is the left identity.

Definition 3.3. Let $(X, *, p)$ be a pointed algebra. Define a binary operation "$\bullet$" on X by

$$x \bullet y := x * (p * y)$$

for any $x, y \in X$. Then the algebra $(X, \bullet, p)$ is called a *p-derived algebra from* $(X, *, p)$.

Example 3.4. (i) Let $(X, *, e)$ be a group with identity e. If $(X, \bullet, e)$ is an e-derived algebra of $(X, *, e)$, then $(X, \bullet) = (X, *)$, since e is the identity, we have $x \bullet y = x * (e * y) = x * y$ for all $x, y \in X$.
(ii) Let $(X, *, p)$ be a left-zero-semigroup with a selected point p. If $(X, \bullet, p)$ is a p-derived algebra of $(X, *, p)$, then $(X, *) = (X, \bullet)$.

Let X be a d-algebra and $x \in X$. Define $x * X := \{x * a | a \in X\}$. X is said to be *edge* ([16]) if for any $x \in X$, $x * X = \{x, 0\}$.

Lemma 3.5. ([16]) *Let X be an edge d-algebra. Then*

(i) $x * 0 = x$ *for all* $x \in X$.
(ii) $(x * (x * y)) * y = 0$ *for all* $x, y \in X$.

Example 3.6. (i) Let $(X, *, 0)$ be an edge d-algebra. If $(X, \bullet, 0)$ is an e-derived algebra of $(X, *, 0)$, then $(X, \bullet)$ is a left-zero-semigroup.
(ii) Let $(X, *, 0)$ be a BCK-algebra. If $(X, \bullet, 0)$ is an e-derived algebra of $(X, *, 0)$, then $(X, \bullet)$ is a left-zero-semigroup. In fact, $x \bullet y = x * (0 * y) = x * 0 = x$ for all $x, y \in X$.

In terms of list of axioms to be used to describe the various algebra types we note the following section of axioms:

(1) $x * x = 0$ for all $x \in X$.
(2) $x * 0 = x$ for all $x \in X$.
(3) $0 * x = x$ for all $x \in X$.
(4) $x * y = y * x$ for all $x, y \in X$.
(5) $x * y = y * x = 0 \Leftrightarrow x = y$ for all $x, y \in X$.
(6) $x * y = y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.
(7) $x * y = y * x \Rightarrow x = y$ for all $x, y \in X$.
(8) $0 * x = 0$ for all $x \in X$.

(9) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.
(10) $(x * y) * z = x * (z * y)$ for all $x, y, z \in X$.
(11) $(x * y) * z = x * (z * (0 * y))$ for all $x, y, z \in X$.
(12) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.
(13) $(x * y) * (0 * y) = x$ for all $x, y \in X$.
(14) $x * (y * z) = (x * y) * z$ for all $x, y, z \in X$.
(15) $(x * (x * y)) * y = 0$ for all $x, y \in X$.
(16) $((x * y) * (x * z)) * (x * y) = 0$ for all $x, y, z \in X$.
(17) for any $x \in X$, there exists $y \in X$ with $x * y = 0$.
(18) for any $x \in X$, there exists $y \in X$ with $y * x = 0$.

An algebra $(X, *, 0)$ is called a *group* if it satisfies (2),(3), (14), (17), and (18). An algebra $(X, *)$ is called a *semigroup* if it satisfies (14). An algebra $(X, *, 0)$ is called a *semigroup with identity* if it satisfies (2),(3), and (14). An algebra $(X, *, 0)$ is called a *B-algebra* ([17]) if it satisfies (1),(2), and (11). An algebra $(X, *, 0)$ is called a *BG-algebra* ([9]) if it satisfies (1),(2), and (13). An algebra $(X, *, 0)$ is called a *BH-algebra* ([7]) if it satisfies (1), (2), and (6). An algebra $(X, *, 0)$ is called a *Q-algebra* ([15]) if it satisfies (1), (2), and (9). An algebra $(X, *, 0)$ is called a *d-algebra* ([16]) if it satisfies (1), (5), and (8). An algebra $(X, *, 0)$ is called a *BCK-algebra* ([14]) if it satisfies (1), (5), (8), (15), and (16). An algebra $(X, *, 0)$ is called a *$gBCK$-algebra* ([4]) if it satisfies (1), (2), (9) and (12). An algebra $(X, *, 0)$ is called an *abelian group* if it satisfies (2), (3), (4), (14), (17), and (18). An algebra $(X, *, 0)$ is called a *commutative semigroup* if it satisfies (4) and (14).

4. BQ-algebras

In this section, we introduce the notion of a BQ-algebra and we show that it is equivalent to an abelian group. An algebra $(X, *, 0)$ is said to be a *BQ-algebra* if it satisfies the conditions (1), (2), (9) and (11).

Theorem 4.1. *Let $(X, *, 0)$ be a BQ-algebra. If we define $x \bullet y := x * (0 * y)$ for any $x, y \in X$, then $(X, \bullet, 0)$ is an abelian group.*

Proof. Since $(X, *, 0)$ is a BQ-algebra, it is both a B-algebra and a Q-algebra. It was proved that if $(X, *, 0)$ is a B-algebra, then $(X, \bullet, 0)$ is a group ([1]). By (9) we obtain $(x * (0 * y)) * (0 * z) = (x * (0 * z)) * (0 * y)$ for any $x, y, z \in X$. It follows that $(x \bullet y) \bullet z = (x \bullet z) \bullet y$ for any $x, y, z \in X$. If we take $x := 0$, then $(0 \bullet y) \bullet z = (0 \bullet z) \bullet y$. Since $(X, *, 0)$ is a B-algebra, we have $0 \bullet y = 0 * (0 * y) = y$ for any $y \in X$. Hence we obtain $y \bullet z = z \bullet y$ for any $y, z \in X$. This proves that $(X, \bullet, 0)$ is an abelian group. □

Theorem 4.2. *Let $(X, \bullet, 0)$ be an abelian group. If we define $x * y := x \bullet y^{-1}$ for any $x, y \in X$, then $(X, *, 0)$ is a BQ-algebra.*

Proof. (1) For any $x \in X$, we have $x*x = x \bullet x^{-1} = 0$. (2) For any $x \in X$, $x*0 = x \bullet 0^{-1} = x \bullet 0 = x$. (9) Given $x, y, z \in X$, since $(X, \bullet, 0)$ is a group, we obtain

$$\begin{aligned}(x * y) * z &= (x \bullet y^{-1}) \bullet z^{-1} \\ &= (x \bullet z^{-1}) \bullet y^{-1} \\ &= (x * z) * y.\end{aligned}$$

(11) Given $x, y, z \in X$, since $(X, \bullet, 0)$ is a group, we have

$$\begin{aligned}x * (z * (0 * y)) &= x \bullet [z \bullet [(y^{-1})^{-1} \bullet 0^{-1}]]^{-1} \\ &= x \bullet [z \bullet (y \bullet 0)]^{-1} \\ &= x \bullet (z \bullet y)^{-1} \\ &= x \bullet (y^{-1} \bullet z^{-1}).\end{aligned}$$

Similarly, we prove that $(x * y) * z = (x \bullet y^{-1}) \bullet z^{-1}$. Since $(X, \bullet, 0)$ is a group, we obtain $(x * y) * z = x * (z * (0 * y))$. Hence $(X, *, 0)$ is a BQ-algebra. □

By Theorems 4.1 and 4.2, we conclude that *the class of all BQ-algebras is equivalent to the class of all abelian groups.*

The interesting fact to note is that we are able to take advantage of the relationship $x \bullet y = x * (0 * y)$ to understand better what the meaning of the class of BQ-algebra is. Other such questions around in this setting as well as others. E.q., what class of B-algebras corresponds to the class of solvable groups ? Can it be considered to be of the form: B"V"-algebras corresponds to solvable groups where "V"-algebras is some nicely identifiable class, as the same as the class for BQ-algebras ?

5. Smarandache types

Let $(X, *)$ be an U-algebra. Then $(X, *)$ is said to be a *Smarandache V-algebra-type U-algebra* if there exists $Y \subseteq X$ such that $(Y, *)$ is a non-trivial subalgebra of $(X, *)$ and $|Y| \geq 2$, and $(Y, *)$ is a V-algebra. For example, a B-algebra $(X, *, 0)$ is said to be a *Smarandache Q-algebra-type B-algebra* it it contains a non-trivial sub-B-algebra $(Y, *, 0)$ of $(X, *, 0)$ and $|Y| \geq 2$, and $(Y, *, 0)$ is a Q-algebra. Similarly, a Q-algebra $(X, *, 0)$ is called a *Smarandache group-type Q-algebra* if it contains a non-trivial sub-Q-algebra $(Y, *, 0)$ of $(X, *, 0)$, and $(Y, *, 0)$ is a group where $|Y| \geq 2$.

Theorem 5.1. *There is no Smarandache d-algebra-type commutative groupoid.*

Proof. Assume that there is a Smarandache d-algebra-type commutative groupoid $(X, *, 0)$. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial subgroupoid of a commutative groupoid $(X, *, 0)$, $|Y| \geq 2$ and $(Y, *, 0)$ is a d-algebra. It follows that $0 * y = 0$ for all $y \in Y$. Since $(X, *, 0)$

is a commutative groupoid and $Y \subseteq X$, we obtain $0 * y = y * 0 = 0$ for all $y \in Y$. Since $(X, *, 0)$ is a d-algebra and $Y \subseteq X$, we obtain $y = 0$, i.e., $|Y| = 1$, a contradiction. □

Theorem 5.2. *There is no Smarandache semigroup-type d-algebra.*

Proof. Assume that there is a Smarandache semigroup-type d-algebra $(X, *, 0)$. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial subalgebra of a d-algebra $(X, *, 0)$, $|Y| \geq 2$ and $(Y, *, 0)$ is a semigroup. It follows that $0 * (y * 0) = 0$ for any $y \in Y$, since $Y \subset X$ and $(X, *, 0)$ is a d-algebra. Hence

$$\begin{aligned} y * 0 &= y * (0 * (y * 0)) \\ &= (y * 0) * (y * 0) \\ &= 0. \end{aligned}$$

Since $Y \subseteq X$ and $(X, *, 0)$ is a d-algebra, we obtain $0 * y = 0$ for all $y \in Y$. By (6), we have $y = 0$, i.e., $|Y| = 1$, a contradiction. □

Theorem 5.3. *A Smarandache group-type B-algebra is equal to a Smarandache Boolean-group-type B-algebra.*

Proof. Since every Boolean group is a group, it is enough to show that every Smarandache group-type B-algebra is a Smarandache Boolean-group-type B-algebra. Assume $(X, *, 0)$ is a Smarandache group-type B-algebra. Then there exists $Y \subseteq X$ such that $|Y| \geq 2$, $(Y, *, 0)$ is a non-trivial subalgebra of a B-algebra and $(Y, *, 0)$ is a group. For any $y \in Y$, since $Y \subseteq X$ and $(X, *, 0)$ is a B-algebra, we obtain $y * y = 0$. Since $(Y, *)$ is a group, the order of y is 2 in the group $(Y, *)$ for any $y \neq 0$ in Y and hence $(Y, *, 0)$ is a Boolean group, proving the theorem. □

Corollary 5.4. *A Smarandache group-type Q-algebra is equal to a Smarandache Boolean-group-type Q-algebra.*

Proof. Every Q-algebra has also the condition (1), and the proof is similar to the proof of Theorem 5.3. □

Theorem 5.5. *Every Smarandache B-algebra-type group is a Smarandache Boolean-group-type group.*

Proof. Let $(X, *, 0)$ be a Smarandache B-algebra-type group. Then there exists $Y \subseteq X$ such that $|Y| \geq 2$, $(Y, *, 0)$ is a non-trivial subgroup of a group $(X, *, 0)$ and $(Y, *, 0)$ is a B-algebra. It follows that $y * y = 0$ for all $y \in Y$. Since $Y \subseteq X$ and $(X, *, 0)$ is a group, we obtain $y = y^{-1}$ in the group. Hence $x * y^{-1} = x * y \in Y$, which shows that $(Y, *)$ is a subgroup of $(X, *)$ and the order of y is 2. Thus $(Y, *)$ is a Boolean group. This proves that $(X, *, 0)$ is a Smarandache Boolean-group-type group. □

Theorem 5.6. *Let $(X, *, 0)$ be a Smarandache L-algebra-type M-algebra. If every L-algebra is an N-algebra, then $(X, *, 0)$ is a Smarandache N-algebra-type M-algebra.*

Proof. It is easy and omit the proof. □

Theorem 5.7. *Let $(X, *, 0)$ be a Smarandache 0-commutative-B-algebra-type M-algebra. Then $(X, *, 0)$ is a Smarandache BCI-algebra-type M-algebra, where M-algebra is any algebra.*

Proof. By applying Theorems 2.2 and 5.6, we prove the theorem. □

Theorem 5.8. *Let $(X, *, 0)$ be an M-algebra. Then the following are equivalent:*

(i) *X is a Smarandache abelian-group-type M-algebra*
(ii) *X is a Smarandache p-semisimple BCI-algebra-type M-algebra,*
(iii) *X is a Smarandache 0-commutative B-algebra-type M-algebra.*

Proof. It follows immediately from Theorems 2.3 and 5.6. □

Proposition 5.9. *If $(X, *, 0)$ is a Smarandache Q-algebra-type group, then it is a Smarandache Boolean-group-type group.*

Proof. Let $(X, *, 0)$ be a Smarandache Q-algebra-type group. Then there exists $Y \subseteq X$ such that $|Y| \geq 2$, $(Y, *, 0)$ is a non-trivial subgroup of a group $(X, *, 0)$ and $(Y, *, 0)$ is a Q-algebra. Since Y is a Q-algebra, we have $y * y = 0$ for any $y \in Y$. This means the order of y is 2 in the group $(Y, *)$, i.e., $y = y^{-1}$, which shows that $(Y, *, 0)$ is a Boolean-group. Hence $(X, *, 0)$ is a Smarandache Boolean-group-type group. □

Theorem 5.10. *Any non-trivial d-algebra cannot be a Smarandache group-type d-algebra.*

Proof. Assume there exists a Smarandache group-type d-algebra $(X, *, 0)$. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial sub-d-algebra of $(X, *, 0)$ and $(Y, *, 0)$ is a group where $|Y| \geq 2$. Since $(Y, *, 0)$ is a group and $(X, *, 0)$ is a d-algebra, we have $y = 0 * y = 0$ for all $y \in Y$. It follows that $|Y| = 1$, a contradiction. □

Theorem 5.11. *Any non-trivial group cannot be a Smarandache d-algebra-type group.*

Proof. Assume that there exists a Smarandache d-algebra-type group $(X, *, 0)$. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial subgroup of a group $(X, *, 0)$, and $(Y, *, 0)$ is a d-algebra and $|Y| \geq 2$. Then $0 * x = 0$ for all $x \in Y$. Since $(Y, *, 0)$ is a group, we obtain $x = 0$ for all $x \in Y$, proving that $|Y| = 1$, a contradiction. □

Theorem 5.12. *Any non-trivial $gBCK$-algebra cannot be a Smarandache group-type $gBCK$-algebra.*

Proof. Let $(X, *, 0)$ be a Smarandache group-type $gBCK$-algebra. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial sub-$gBCK$-algebra of $(X, *, 0)$, and $(Y, *, 0)$ is a group and $|Y| \geq 2$. Since $Y \subseteq X$ and $(X, *, 0)$ is a $gBCK$-algebra, we obtain $y * y = 0$ for all $y \in Y$. It follows from

$(Y, *, 0)$ is a group that the order of y is 2, i.e., $(Y, *, 0)$ is a Boolean group. Now, since $(Y, *, 0)$ is a $gBCK$-algebra, we have $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$. It follows that $(x * x) * x = (x * x) * (x * x)$ for all $x \in X$. Since $(X, *, 0)$ is a group, we obtain $x = 0$ for all $x \in X$, proving that $|X| = 1$, a contradiction. □

Corollary 5.13. *Any non-trivial group cannot be a Smarandache $gBCK$-algebra-type group.*

Proof. The proof is similar to Theorem 5.12, and we omit it. □

Definition 5.14. Let $(X, *, p)$ be an L-algebra and let $(Y, *, p)$ be both a sub-L-algebra of $(X, *, p)$ and an M-algebra. $(X, *, p)$ is said to be a *Smarandache N-algebra-trans-type L-algebra* if $(Y, *, p)$ is isomorphic with an N-algebra $(Y, \odot, q)$.

$$\begin{tikzcd}
(X, *, p) \arrow[d, no head] & \\
(Y, *, p) \arrow[r, "\cong"] & (Y, \odot, q)
\end{tikzcd}$$

where $L-, M-, N-$ algebras are arbitrary algebras.

Theorem 5.15. *If $(X, *, 0)$ is a Smarandache B-algebra-type Q-algebra, then it is a Smarandache abelian-group-trans-type Q-algebra.*

Proof. Let $(X, *, 0)$ be a Smarandache B-algebra-type Q-algebra. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial sub-Q-algebra of a Q-algebra $(X, *, 0)$, $|Y| \geq 2$ and $(Y, *, 0)$ is a B-algebra. Define $x \bullet y := x * (0 * y)$ for any $x, y \in Y$. Then $(Y, \bullet, 0)$ is an abelian group. In fact, since Y is both a Q-algebra and B-algebra, $(Y, *, 0)$ is a BQ-algebra. By Theorem 4.1, $(Y, \bullet, 0)$ is an abelian group. By Theorems 4.1 and 4.2, $(Y, *, 0) \cong (Y, \bullet, 0)$. This shows that $(X, *, 0)$ is a Smarandache abelian-group-trans-type Q-algebra. □

Corollary 5.16. *If $(X, *, 0)$ is a Smaradache Q-algebra-type B-algebra, then it is a Smarandache abelian-group-trans-type B-algebra.*

Proof. It is similar to Theorem 5.15. □

Proposition 5.17. *Every B-algebra is a Smarandache BQ-algebra-trans-type B-algebra.*

Proof. Let $(X, *, 0)$ be a B-algebra. Define $x \bullet y := x * (0 * y)$ for all $x, y \in X$. Then $(X, \bullet, 0)$ is a group. Let $x \in X$ such that $x \neq 0$. Let $\langle x \rangle$ be a cyclic group generated by x. Then $\langle x \rangle$ is a non-trivial abelian subgroup of $(X, \bullet, 0)$. If we let $Y_x := \{x *^{\langle n \rangle} (0 * x) \mid n \in \mathbf{Z}\}$, then $Y_x \cong \langle x \rangle$. By Theorems 4.1 and 4.2, Y_x is a non-trivial BQ-algebra. This shows that X is a Smarandache BQ-algebra-trans-type B-algebra. □

6. Conclusion

We introduced the notion of a BQ-algebra and proved that it is equivalent to an abelian group. For detailed investigations among several algebraic structures, we introduced the notions of a Smarandache V-type U-algebra and a Smarandache V-trans-type U-algebra, and applied this notions to several algebras. For further investigations, we will apply the notions of a hyper structure theory and several fuzzy related algebras to the notions of a Smarandache V-type U-algebra and a Smarandache V-trans-type U-algebra.

References

[1] P. J. Allen, J. Neggers and H. S. Kim, *B-algebras and groups*, Sci. Math. Japo. **59**(2004), 23-29.

[2] J. R. Cho and H. S. Kim, *On B-algebras and quasigroups*, Quasigroups and Related Systems **8**(2001), 1-6.

[3] J. S. Han and S. S. Ahn, *Quotient B-algebras induced by an int-soft normal subalgebras*, J. Comput. Anal. Appl. **26**(2019), no. 5, 791-801.

[4] S. M. Hong, Y. B. Jun and M. A. Ozturk, *Generalizations of BCK-algebras*, Sci. Math. Japo. **58**(2003), 603-611.

[5] Y. Huang, BCI-algebras, Science Press, Beijing, 2006.

[6] A. Iorgulescu, Algebras of logic as BCK-algebras, Editura ASE, Bucharest, 2008.

[7] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematicae **1**(1998), 347-354.

[8] W. B. V. Kandasamy, *Smarandache groupoids*, http://www.gallwp.unm.edu/∼smarandache/Groupoids.pdf.

[9] C. B. Kim and H. S. Kim, *On BG-algebras*, Demonstratio Math. **41**(2008), 497-505.

[10] H. S. Kim and H. G. Park, *On 0-commutative B-algebras*, Sci. Math. Japo. **62**(2005), 31-36.

[11] Y. H. Kim, Y. H. Kim and S. S. Ahn, *Smarandache d-algebras*, Honam Math. J. **40**(2018), no.3, 539-548.

[12] J. M. Ko and S. S. Ahn, *On fuzzy B-algebras over t-norm*, J. Comput. Anal. Appl. **19**(2015), no. 6, 975-983.

[13] J. M. Ko and S. S. Ahn, *Hesitant fuzzy normal subalgebras in B-algebras*, J. Comput. Anal. Appl. **26**(2019), no. 6, 1084-1094.

[14] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa, Seoul, 1994.

[15] J. Neggers, S. S. Ahn and H. S. Kim, *On Q-algebras*, Int. J. Math. & Math. Sci. **27**(2001), 749-757.

[16] J. Neggers, and H. S. Kim, *On d-algebras*, Math. Slovaca **49**(1999), 19-26.

[17] J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54**(2002), 21-29.

[18] R. Padilla, Smarandache algebric structures, Bull. Pure Appl. Sci., **1998**, 17E, 119-121.

[19] Y. J. Seo and S. S. Ahn, *Smarndache fuzzy BCI-algebras*, J. Comput. Anal. Appl. **24**(2018), no. 4, 619-627.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Nonlinear differential equations associated with degenerate (h, q)-tangent numbers

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : In this paper, we study nonlinear differential equations arising from the generating functions of degenerate (h, q)-tangent numbers. We give explicit identities for the degenerate (h, q)-tangent numbers.

Key words : Nonlinear differential equations, (h, q)-tangent numbers and polynomials, degenerate tangent numbers, degenerate (h, q)-tangent numbers, higher-order degenerate tangent numbers.

AMS Mathematics Subject Classification : 05A19, 11B83, 34A30, 65L99.

1. Introduction

Recently, many mathematicians have studied in the area of the degenerate Euler numbers and polynomials, degenerate Bernoulli numbers and polynomials, degenerate Genocchi numbers and polynomials, and degenerate tangent numbers and polynomials(see [1, 2, 3, 4, 5, 6, 7]). In [1], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.*[2] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. The degenerate (h, q)-tangent numbers $\mathcal{T}_{n,q}^{(h)}(\lambda)$ are defined by the generating function:

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(h)}(\lambda)\frac{t^n}{n!} = \frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}. \tag{1.1}$$

The degenerate (h, q)-tangent numbers of higher order, $\mathcal{T}_{n,\lambda,q}^{(k,h)}$ are defined by means of the following generating function

$$\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^k = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k,h)}(\lambda)\frac{t^n}{n!}. \tag{1.2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.5}$$

Many mathematicians have studied in the area of the linear and nonlinear differential equations arising from the generating functions of special numbers and polynomials in order to give explicit identities for special polynomials. In this paper, we study nonlinear differential equations arising from the generating functions of degenerate (h,q)-tangent numbers . We give explicit identities for the degenerate (h,q)-tangent numbers .

2. Nonlinear differential equations associated with degenerate (h,q)-tangent numbers

In this section, we study nonlinear differential equations arising from the generating functions of degenerate twisted (h,q)-tangent numbers. Let

$$F = F(t,\lambda,q,h) = \frac{2}{q^h(1+\lambda t)^{2/\lambda}+1} = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(h)}(\lambda)\frac{t^n}{n!}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} F^{(1)} &= \frac{\partial}{\partial t}F(t,\lambda,q,h) = \frac{\partial}{\partial t}\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right) \\ &= \frac{1}{1+\lambda t}\left(\frac{-4}{q^h(1+\lambda t)^{2/\lambda}+1}\right) + \frac{1}{1+\lambda t}\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^2 \\ &= \frac{-2F+F^2}{1+\lambda t}. \end{aligned} \tag{2.2}$$

By (2.2), we have

$$F^2 = 2F + (1+\lambda t)F^{(1)}. \tag{2.3}$$

Taking the derivative with respect to t in (2.3), we obtain

$$\begin{aligned} 2FF^{(1)} &= 2F^{(1)} + \lambda F^{(1)} + (1+\lambda t)F^{(2)} \\ &= (\lambda+2)F^{(1)} + (1+\lambda t)F^{(2)}. \end{aligned} \tag{2.4}$$

From (2.2), (2.3), and (2.4), we have

$$2F^3 = 4F + (1+\lambda)(1+\lambda t)F^{(1)} + (1+\lambda t)^2F^{(2)}.$$

Continuing this process, we can guess that

$$N!F^{N+1} = \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}, \quad (N=0,1,2,\ldots), \tag{2.5}$$

where $F^{(i)} = \left(\frac{\partial}{\partial t}\right)^i F(t,\lambda,q,h)$. Differentiating (2.5) with respect to t, we have

$$(N+1)!F^NF^{(1)} = \sum_{i=0}^{N} i\lambda a_i(N,\lambda,q,h)(1+\lambda t)^{i-1}F^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i+1)} \tag{2.6}$$

and

$$(N+1)!F^NF^{(1)} = (N+1)!F^N\left(\frac{-2F+F^2}{1+\lambda t}\right) = (N+1)!\left(\frac{F^{N+2}-2F^{N+1}}{1+\lambda t}\right). \tag{2.7}$$

By (2.5), (2.6), and (2.7), we have

$$\begin{aligned}(N+1)!F^{N+2} &= 2(N+1)!F^{N+1} \\ &\quad + \sum_{i=0}^{N} \lambda i a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^{i+1} F^{(i+1)} \\ &= 2(N+1)\sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} \\ &\quad + \sum_{i=0}^{N} \lambda i a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^{i+1} F^{(i+1)} \\ &= \sum_{i=0}^{N} (2(N+1)+\lambda i)\, a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} + \sum_{i=1}^{N+1} a_{i-1}(N,\lambda,q,h)(1+\lambda t)^i F^{(i)}.\end{aligned} \tag{2.8}$$

Now replacing N by $N+1$ in (2.5), we find

$$(N+1)!F^{N+2} = \sum_{i=0}^{N+1} a_i(N+1,\lambda,q,h)(1+\lambda t)^i F^{(i)}. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned}\sum_{i=0}^{N+1} a_i(N+1,\lambda,q,h)(1+\lambda t)^i F^{(i)} &= \sum_{i=0}^{N} (2(N+1)+\lambda i)\, a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} \\ &\quad + \sum_{i=1}^{N+1} a_{i-1}(N,\lambda,q,h)(1+\lambda t)^i F^{(i)}.\end{aligned} \tag{2.10}$$

Comparing the coefficients on both sides of (2.10), we obtain

$$\begin{aligned}2(N+1)a_0(N,\lambda,q,h) &= a_0(N+1,\lambda,q,h), \\ a_{N+1}(N+1,\lambda,q,h) &= a_N(N,\lambda,q,h),\end{aligned} \tag{2.11}$$

and

$$a_i(N+1,\lambda,q,h) = (2(N+1)+\lambda i)\, a_i(N,\lambda,q,h) + a_{i-1}(N,\lambda,q,h), (1 \le i \le N). \tag{2.12}$$

In addition, by (2.5), we have

$$F = a_0(0,\lambda,q,h)F, \tag{2.13}$$

which gives

$$a_0(0,\lambda,q,h) = 1. \tag{2.14}$$

It is not difficult to show that

$$F^2 = a_0(1,\lambda,q,h)F + a_1(1,\lambda,q,h)(1+\lambda t)F^{(1)} = 2F + (1+\lambda t)F^{(1)}. \tag{2.15}$$

Thus, by (2.15), we also find

$$a_0(1,\lambda,q,h) = 2, \quad a_1(1,\lambda,q,h) = 1. \tag{2.16}$$

From (2.11), we note that

$$\begin{aligned}a_0(N+1,\lambda,q,h) &= 2(N+1)a_0(N,\lambda,q,h) = 4(N+1)Na_0(N-1,\lambda,q,h) \\ &= \cdots = 2^{N+1}(N+1)!,\end{aligned} \tag{2.17}$$

and

$$a_{N+1}(N+1,\lambda,q,h)=a_N(N,\lambda,q,h)=\cdots=1. \tag{2.18}$$

For $i=1,2,3$ in (2.11), then we find that

$$\begin{aligned}
a_1(N+1,\lambda,q,h)&=\sum_{k=0}^{N}2^k\left(N+1+\frac{\lambda}{2}\right)_k a_0(N-k,\lambda,q,h),\\
a_2(N+1,\lambda,q,h)&=\sum_{k=0}^{N-1}2^k\left(N+1+\frac{\lambda}{2}\times 2\right)_k a_1(N-k,\lambda,q,h),\\
a_3(N+1,\lambda,q,h)&=\sum_{k=0}^{N-2}2^k\left(N+1+\frac{\lambda}{2}\times 3\right)_k a_2(N-k,\lambda,q,h).
\end{aligned}$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$a_i(N+1,\lambda,q,h)=\sum_{k=0}^{N-i+1}2^k\left(N+1+\frac{\lambda}{2}\times i\right)_k a_{i-1}(N-k,\lambda,q,h). \tag{2.19}$$

Note that, here the matrix $a_i(j,\lambda,q,h)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix}
1 & 2 & 2!2^2 & 3!2^3 & \cdots & (N+1)!2^{N+1}\\
0 & 1 & \cdot & \cdot & \cdots & \cdot\\
0 & 0 & 1 & \cdot & \cdots & \cdot\\
0 & 0 & 0 & 1 & \cdots & \cdot\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,\lambda,q,h)$. By (2.17), (2.18), and (2.19), we have

$$\begin{aligned}
a_1(N+1,\lambda,q,h)&=\sum_{k_1=0}^{N}2^{k_1}\left(N+1+\frac{\lambda}{2}\right)_{k_1} a_0(N-k_1,\lambda,q,h)\\
&=\sum_{k_1=0}^{N}2^{N}(N-k_1)!\left(N+1+\frac{\lambda}{2}\right)_{k_1},
\end{aligned}$$

$$\begin{aligned}
&a_2(N+1,\lambda,q,h)=\sum_{k_2=0}^{N-1}2^{k_2}\left(N+1+\frac{\lambda}{2}\times 2\right)_{k_2} a_1(N-k_2,\lambda,q,h)\\
&=\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}2^{N-1}(N-k_2-k_1-1)!\left(N+1+\frac{\lambda}{2}\times 2\right)_{k_2}\left(N-k_2+\frac{\lambda}{2}\right)_{k_1},
\end{aligned}$$

and

$$\begin{aligned}
&a_3(N+1,\lambda,q,h)=\sum_{k_3=0}^{N-2}2^{k_3}\left(N+1+\frac{\lambda}{2}\times 3\right)_{k_3} a_2(N-k_3,\lambda,q,h)\\
&=\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}2^{N-2}(N-k_3-k_2-k_1-2)!\left(N+1+\frac{\lambda}{2}\times 3\right)_{k_3}\\
&\quad\times\cdots\times\left(N-k_3+\frac{\lambda}{2}\times 2\right)_{k_2}\left(N-k_3-k_2-1+\frac{\lambda}{2}\right)_{k_1}
\end{aligned}$$

Continuing this process, we have

$$\begin{aligned}
a_i(N+1,\lambda,q,h) = &\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i+1} 2^{N-i+1}\\
&\times(N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1)!\\
&\times\left(N+1+\frac{\lambda}{2}\times i\right)_{k_i}\left(N-k_i+\frac{\lambda}{2}\times(i-1)\right)_{k_{i-1}}\\
&\times\left(N-k_i-k_{i-1}-1+\frac{\lambda}{2}\times(i-2)\right)_{k_{i-2}}\\
&\times\left(N-k_i-k_{i-1}-k_{i-2}-2+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots\\
&\times\left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+2+\frac{\lambda}{2}\right)_{k_1}.
\end{aligned} \tag{2.20}$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 1. For $N=0,1,2,\ldots$, the nonlinear functional equation

$$N!F^{N+1}=\sum_{i=0}^{N}a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}$$

has a solution

$$F=F(t,\lambda,q,h)=\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1},$$

where

$$\begin{aligned}
&a_0(N,\lambda,q,h)=2^NN!,\\
&a_N(N,\lambda,q,h)=1,\\
&a_i(N,\lambda,q,h)=\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(2q^h-x)^{N-i}\\
&\times(N-k_i-k_{i-1}-\cdots-k_2-k_1-i)!\\
&\times\left(N+\frac{\lambda}{2}\times i\right)_{k_i}\left(N-k_i-1+\frac{\lambda}{2}\times(i-1)\right)_{k_{i-1}}\\
&\times\left(N-k_i-k_{i-1}-2+\frac{\lambda}{2}\times(i-2)\right)_{k_{i-2}}\\
&\times\left(N-k_i-k_{i-1}--k_{i-2}-3+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots\\
&\times\left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+1+\frac{\lambda}{2}\right)_{k_1}.
\end{aligned}$$

From (1.1) and (1.2), we note that

$$N!F^{N+1}=N!\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^{N+1}=N!\sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(N+1,h)}(\lambda)\frac{t^n}{n!}. \tag{2.21}$$

From (2.5), we note that

$$F^{(i)}=\left(\frac{\partial}{\partial t}\right)^iF(t,\lambda,q,h)=\sum_{l=0}^{\infty}\mathcal{T}_{i+l,q}^{(h)}(\lambda)\frac{t^l}{l!}. \tag{2.22}$$

From Theorem 1, (1.5), (2.21), and (2.22), we can derive the following equation:

$$\begin{aligned}
N!F^N\sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(N+1,h)}(\lambda)\frac{t^n}{n!} &= \sum_{i=0}^{N}a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}\\
&= \sum_{i=0}^{N}a_i(N,\lambda,q,h)\sum_{k=0}^{\infty}(i)_k\lambda^k\frac{t^k}{k!}\sum_{l=0}^{\infty}\mathcal{T}_{i+l,q}^{(h)}(\lambda)\frac{t^l}{l!} \qquad (2.23)\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}a_i(N,\lambda,q,h)(i)_k\lambda^k\mathcal{T}_{n-k+i,q}^{(h)}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides of (2.23), we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \ldots$, we have

$$N!\mathcal{T}_{n,q}^{(N+1,h)}(\lambda) = \sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}a_i(N,\lambda,q,h)(i)_k\lambda^k\mathcal{T}_{n-k+i,q}^{(h)}(\lambda),$$

where

$$\begin{aligned}
&a_0(N,\lambda) = N!2^N, \quad a_N(N,\lambda) = 1,\\
&a_i(N,\lambda) = \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}2^{N-i}\\
&\qquad\times(N-k_i-k_{i-1}-\cdots-k_2-k_1-i)!\\
&\qquad\times\left(N-k_i-k_{i-1}--k_{i-2}-3+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots\\
&\qquad\times\left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+1+\frac{\lambda}{2}\right)_{k_1}.
\end{aligned}$$

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

REFERENCES

1. Carlitz, L.(1979). *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. v.15, pp. 51-88.

2. Qi, F., Dolgy, D.V., Kim, T., Ryoo, C.S.(2015). *On the partially degenerate Bernoulli polynomials of the first kind*, Global Journal of Pure and Applied Mathematics, v.11, pp. 2407-2412.

3. Ryoo, C.S.(2015). *Notes on degenerate tangent polynomials*, Global Journal of Pure and Applied Mathematics v.11, pp. 3631-3637.

4. Ryoo, C.S.(2015). *Note on degenerate tangent polynomials of higher order*, Global Journal of Pure and Applied Mathematics v.11, pp. 4547-4554.

5. Ryoo, C.S.(2014). *A numerical investigation on the zeros of the tangent polynomials*, J. App. Math. & Informatics, v.32, pp. 315-322.

6. Ryoo, C.S.(2011). *On the alternating sums of powers of consecutive odd integers*, Journal of Computational Analysis and Applications, v.13, pp. 1019-1024.

7. Young, P.T.(2008) *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, Journal of Number Theory, v. 128, pp. 738-758

On the symmetries of the second kind (h, q)-Bernoulli polynomials

C. S. RYOO

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : In this paper, by applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second kind (h, q)-Bernoulli polynomials and the sums of powers of consecutive (h, q)-odd integers.

Key words : Bernoulli numbers and polynomials, the second kind Bernoulli numbers and polynomials, the second kind q-Bernoulli numbers and polynomials, the second kind (h, q)-Bernoulli numbers and polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind Bernoulli number and the second kind Bernoulli polynomials were studied by many authors(see [1-8]). Bernoulli numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In [5], by using the second kind Bernoulli numbers B_j and polynomials $B_j(x)$, we investigated the q-analogue of sums of powers of consecutive odd integers(see [6]). Let k be a positive integer. Then we obtain

$$O_k(n-1) = \sum_{i=0}^{n-1}(2i+1)^{k-1} = \frac{B_k(2n) - B_k}{2k}.$$

In [4], we introduced the second kind (h, q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$. By using computer, we observed an interesting phenomenon of 'scattering' of the zeros of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ in complex plane. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$. In this paper, we give recurrence identities the second kind (h, q)-Bernoulli polynomials and the sums of powers of consecutive (h, q)-odd integers.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by [2, 5]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} g(x) q^x.$$

The bosonic integral was considered from a physical point of view to the bosonic limit $q \to 1$, as follows:

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x) \text{ (see [2])}. \tag{1.1}$$

By (1.1), we easily see that

$$I_1(g_1) = I_1(g) + g'(0), \tag{1.2}$$

where $g_1(x) = g(x+1)$ and $g'(0) = \frac{dg(x)}{dx}\Big|_{x=0}$.

First, we introduce the second kind Bernoulli numbers B_n and polynomials $B_n(x)$. The second kind Bernoulli numbers B_n and polynomials $B_n(x)$ are defined by means of the following generating functions (see [3]):

$$\frac{2te^t}{e^{2t}-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

and

$$\left(\frac{2te^t}{e^{2t}-1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

respectively.

The second kind (h, q)-Bernoulli polynomials, $B_{n,q}^{(h)}(x)$ are defined by means of the generating function:

$$\left(\frac{(h\log q + 2t)e^t}{q^h e^{2t}-1}\right) e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{1.3}$$

The second kind (h, q)-Bernoulli numbers $E_{n,q}^{(h)}$ are defined by means of the generating function:

$$\frac{(h\log q + 2t)e^t}{q^h e^{2t}-1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)} \frac{t^n}{n!}. \tag{1.4}$$

In (1.2), if we take $g(x) = q^{hx}e^{(2x+1)t}$, then we have

$$\int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) = \frac{(h\log q + 2t)e^t}{q^h e^{2t}-1}. \tag{1.5}$$

for $|t| \le p^{-\frac{1}{p-1}}, h \in \mathbb{Z}$. In (1.2), if we take $g(x) = e^{2nxt}$, then we also have

$$\int_{\mathbb{Z}_p} e^{2nxt} d\mu_1(x) = \frac{2nt}{e^{2nt}-1}. \tag{1.6}$$

for $|t| \le p^{-\frac{1}{p-1}}$. It will be more convenient to write (1.2) as the equivalent bosonic integral form

$$\int_{\mathbb{Z}_p} g(x+1) d\mu_1(x) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) + g'(0), \quad \text{(see [2])}. \tag{1.7}$$

For $n \in \mathbb{N}$, we also derive the following bosonic integral form by (1.7),

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_1(x) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) + \sum_{k=0}^{n-1} g'(k), \text{ where } g'(k) = \frac{dg(x)}{dx}\Big|_{x=k}. \tag{1.8}$$

In [4], we introduced the second kind (h, q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$ and investigate their properties. The following elementary properties of the second kind (h, q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4). We, therefore, choose to omit details involved.

Theorem 1. For $h \in \mathbb{Z}, q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have

$$\begin{aligned} B_{n,q}^{(h)} &= \int_{\mathbb{Z}_p} q^{hx}(2x+1)^n d\mu_1(x), \\ B_{n,q}^{(h)}(x) &= \int_{\mathbb{Z}_p} q^{hy}(x+2y+1)^n d\mu_1(y). \end{aligned}$$

Theorem 2. For any positive integer n, we have

$$B_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{k,q}^{(h)} x^{n-k}.$$

Theorem 3. For any positive integer m, we obtain

$$B_{n,q}^{(h)}(x) = m^{n-1} \sum_{i=0}^{m-1} q^{hi} B_{n,q^m}^{(h)} \left(\frac{2i+x+1-m}{m} \right) \text{ for } n \geq 0.$$

2. On the symmetries of the second kind (h,q)-Bernoulli polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $h \in \mathbb{Z}$. We investigate interesting properties of symmetry p-adic invariant integral on $\mathbb{Z}_p$ for the second kind (h,q)-Bernoulli polynomials. W also obtain recurrence identities the second kind (h,q)-Bernoulli polynomials.

By (1.7), we obtain

$$\begin{aligned} &\frac{1}{h\log q + 2t} \left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) \right) \\ &\qquad = \frac{n \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x)}{\int_{\mathbb{Z}_p} q^{hnx} e^{2ntx} d\mu_1(x)} \end{aligned} \tag{2.1}$$

By (1.8), we obtain

$$\begin{aligned} &\frac{1}{h\log q + 2t} \left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) \right) \\ &\qquad = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} q^{hi}(2i+1)^k \right) \frac{t^k}{k!}. \end{aligned} \tag{2.2}$$

For each integer $k \geq 0$, let

$$O_{k,q}^{(h)}(n) = 1^k + q^h 3^k + q^{2h} 5^k + q^{3h} 7^k + \cdots + q^{nh}(2n+1)^k.$$

The above sum $O_{k,q}^{(h)}(n)$ is called the sums of powers of consecutive (h,q)-odd integers. From the above and (2.2), we obtain

$$\begin{aligned} &\frac{1}{h\log q + 2t} \left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) \right) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} O_{k,q}^{(h)}(n-1) \frac{t^k}{k!}. \end{aligned} \tag{2.3}$$

Thus, we have

$$\sum_{k=0}^{\infty} \left(q^{hn} \int_{\mathbb{Z}_p} q^{hx}(2x+2n+1)^k d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx}(2x+1)^k d\mu_1(x) \right) \frac{t^k}{k!} = \sum_{k=0}^{\infty} (h\log q + 2t) O_{k,q}^{(h)}(n-1) \frac{t^k}{k!}$$

By comparing coefficients $\frac{t^k}{k!}$ in the above equation, we have

$$\begin{aligned}&(h\log q+2t)O_{k,q}^{(h)}(n-1)\\&=\int_{\mathbb{Z}_p}q^{hx}(2x+2n+1)^k d\mu_1(x)-\int_{\mathbb{Z}_p}q^{hx}(2x+1)^k d\mu_1(x)\end{aligned}.$$

By using the above equation we arrive at the following theorem:

Theorem 4. Let k be a positive integer. Then we obtain

$$q^{hn}B_{n,q}^{(h)}(2n)-B_{n,q}^{(h)}=h\log qO_{k,q}^{(h)}(n-1)+2kO_{k-1,q}^{(h)}(n-1). \tag{2.4}$$

Remark 5. For the alternating sums of powers of consecutive integers, we have

$$\begin{aligned}\lim_{q\to 1}\left(h\log qO_{k,q}^{(h)}(n-1)+2kO_{k-1,q}^{(h)}(n-1)\right)&=\sum_{i=0}^{n-1}(2i+1)^{k-1}\\&=\frac{B_k(2n)-B_k}{2k},\ \text{ for } k\in\mathbb{N}.\end{aligned}$$

By using (2.1) and (2.3), we arrive at the following theorem:

Theorem 6. Let n be positive integer. Then we have

$$\frac{n\int_{\mathbb{Z}_p}q^{hx}e^{(2x+1)t}d\mu_1(x)}{\int_{\mathbb{Z}_p}q^{hnx}e^{2ntx}d\mu_1(x)}=\sum_{m=0}^{\infty}\left(O_{m,q}^{(h)}(n-1)\right)\frac{t^m}{m!}. \tag{2.5}$$

Let w_1 and w_2 be positive integers. By using (1.5) and (1.6), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}q^{h(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2xt}d\mu_1(x)}\\&=\frac{(h\log q+2t)e^{w_1t}e^{w_2t}e^{w_1w_2xt}(q^{hw_1w_2}e^{2w_1w_2t}-1)}{(q^{hw_1}e^{2w_1t}-1)(q^{hw_2}e^{2w_2t}-1)}\end{aligned} \tag{2.6}$$

By using (2.4) and (2.6), after calculations, we obtain

$$\begin{aligned}S&=\left(\frac{1}{w_1}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}\sum_{m=0}^{\infty}B_{m,q^{w_1}}^{(h)}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_2}}^{(h)}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.7}$$

By using Cauchy product in the above, we have

$$S=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}^{(h)}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}^{(h)}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!} \tag{2.8}$$

By using the symmetry in (2.7), we have

$$\begin{aligned}S&=\left(\frac{1}{w_2}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}\sum_{m=0}^{\infty}B_{m,q^{w_2}}^{(h)}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_1}}^{(h)}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$S=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B^{(h)}_{j,q^{w_2}}(w_1x)w_2^{j-1}O^{(h)}_{m-j,q^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!} \tag{2.9}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.8) and (2.9), we arrive at the following theorem:

Theorem 7. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}B^{(h)}_{j,q^{w_1}}(w_2x)w_1^{j-1}O^{(h)}_{m-j,q^{w_2}}(w_1-1)w_2^{m-j}\\&=\sum_{j=0}^{m}\binom{m}{j}B^{(h)}_{j,q^{w_2}}(w_1x)w_2^{j-1}O^{(h)}_{m-j,q^{w_1}}(w_2-1)w_1^{m-j},\end{aligned}$$

where $B^{(h)}_{k,q}(x)$ and $O^{(h)}_{m,q}(k)$ denote the second kind (h,q)-Bernoulli polynomials and the sums of powers of consecutive (h,q)-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be positive integers. Then we have

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j-1}x^{j-k}B^{(h)}_{k,q^{w_2}}O^{(h)}_{m-j,q^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j-1}w_2^{m-k}x^{j-k}B^{(h)}_{k,q^{w_1}}O^{(h)}_{m-j,q^{w_2}}(w_1-1),\end{aligned}$$

By using (2.6), we have

$$\begin{aligned}S&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\sum_{j=0}^{w_1-1}q^{w_2hj}e^{(2j+1)(w_2t)}\right)\\&=\sum_{j=0}^{w_1-1}q^{w_2hj}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_1(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}q^{w_2hj}B^{(h)}_{n,q^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

By using the symmetry property in (2.10), we also have

$$\begin{aligned}S&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\sum_{j=0}^{w_2-1}q^{w_1hj}e^{(2j+1)(w_1t)}\right)\\&=\sum_{j=0}^{w_2-1}q^{w_1hj}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_1(x_2)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}q^{w_1hj}B^{(h)}_{n,q^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.10) and (2.11), we have the following theorem.

Theorem 9. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1} q^{w_2hj} B_{n,q^{w_1}}^{(h)}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\\&=\sum_{j=0}^{w_2-1} q^{w_1hj} B_{n,q^{w_2}}^{(h)}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}.\end{aligned}\tag{2.12}$$

Observe that if $h=1$, then (2.12) reduces to Theorem 5 in [9](see [5, 9]). Substituting $w_1=1$ into (2.12), we arrive at the following corollary.

Corollary 10. Let w_2 be positive integer. Then we obtain

$$B_{n,q}^{(h)}(x)=w_2^{n-1}\sum_{j=0}^{w_2-1} q^{hj} B_{n,q^{w_2}}^{(h)}\left(\frac{x-w_2+2j+1}{w_2}\right).\tag{2.13}$$

The Corollary 10 is shown to yield the known distribution relation of the second kind (h,q)-Bernoulli polynomials(see Theorem 3). Note that if $q\to 1$, then (2.13) reduces to distribution relation of the second kind Bernoulli polynomials(see [8]).

Corollary 11. Let w_2 be positive integer. Then we have

$$B_n(x)=w_2^{n-1}\sum_{j=0}^{w_2-1} B_n\left(\frac{x-w_2+2j+1}{w_2}\right).$$

Acknowledgement: This work was supported by 2020 Hannam University Research Fund.

REFERENCES

1. Adelberg, A.(1992). *On the degrees of irreducible factors of higher order Bernoulli polynomials*, Acta Arith., v.62, pp. 329-342.

2. Kim, T.(2002). *q-Volkenborn integration,* Russ. J. Math. phys., v.9, pp. 288-299.

3. Ryoo, C.S.(2011). *Distribution of the roots of the second kind Bernoulli polynomials*, Journal of Computational Analysis and Applications, v.13, pp. 971-976.

4. Ryoo, C.S.(2012). *Zeros of the second kind (h,q)-Bernoulli polynomials*, Applied Mathematical Sciences, v.6, pp. 5869-5875.

5. Ryoo, C.S.(2020). *Symmetric identities for the second kind q-Bernoulli polynomials*, Journal of Computational Analysis and Applications, v.28, pp. 654-659.

6. Ryoo, C.S.(2011). *On the alternating sums of powers of consecutive odd integers*, Journal of Computational Analysis and Applications, v.13, pp. 1019-1024.

7. Ryoo, C.S.(2015). *Symmetric identities for Carlitzs twisted q-Bernoulli numbers and polynomials associated with p-adic invariant integral on Z_p*, Global Journal of Pure and Applied Mathematics, v.11, pp. 2413-2417.

8. Ryoo, C.S.(2010). *A note on the second kind Bernoulli polynomials*, Far East J. Math. Sci., v.42, pp. 109-115.

9. Ryoo, C.S.(2011). *A note on q-extension of the second kind Bernoulli numbers and polynomials*, Far East J. Math. Sci., v.58, pp. 75-82.

SOME NEW FUZZY BEST PROXIMITY POINT THEOREMS IN NON-ARCHIMEDEAN FUZZY METRIC SPACES

MÜZEYYEN SANGURLU SEZEN [1], HÜSEYIN IŞIK [2,†]

Abstract. In this paper, we define fuzzy weak P-property. Then we prove a fuzzy best proximity point theorems for γ-contractions with condition fuzzy weak P-property. Later, we give definition of fuzzy isometric distance between two functions in non-Archimedean fuzzy metric spaces. Also, we introduce γ-proximal contraction type-1 and type-2 contraction respectively via functions preserving fuzzy isometric distance and providing fuzzy isometry. Then, we obtain some fuzzy best proximity results for γ-proximal contractions types in non-Archimedean fuzzy metric spaces. Finally, we present some examples to illustrate the validity of the definitions and results obtained in the paper.

1. Introduction and Preliminaries

The Banach contraction principle found by Banach has an important resonance in mathematics as well as in other fields [1]. Later, the subject of fixed point theory attracted the attention of many aouthors and caused this subject to be discussed in different areas of mathematics and different topological spaces. Then, authors intensively introduced many works regarding the fixed point theory. On the other hand, the concept of fuzzy metric space was introduced in different ways by some authors (see [2, 7]). Importantly, Gregori and Sapena [5] introduced the notion of fuzzy contractive mapping and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. At the same time, there are presented by many authors by expanding the Banach's result in the literature (see [9–11, 14, 16, 20, 21]).

In this work, we prove some fuzzy best proximity point results for mappings providing γ-proximal contractions. Then, we give some examples are supplied in order to support the useability of our results. Also, we show that our main results are more general than known results in the existing literature.

2010 *Mathematics Subject Classification.* 47H10,54H25.

Key words and phrases. γ-proximal contraction, fuzzy best proximity point, non-Archimedean fuzzy metric space.

[†]Corresponding author.

1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Definition 1. *[12] A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangular norm (in short, continuous* $t-$*norm) if it satisfies the following conditions:*

(TN-1) $*$ *is commutative and associative,*
(TN-2) $*$ *is continuous,*
(TN-3) $*(a,1) = a$ *for every* $a \in [0,1]$,
(TN-4) $*(a,b) \le *(c,d)$ *whenever* $a \le c$, $b \le d$ *and* $a,b,c,d \in [0,1]$.

An arbitrary $t-$norm $*$ can be extended (by associativity) in a unique way to an nary operator taking for $(x_1, x_2, ..., x_n) \in [0,1]^n$, $n \in N$, the value $*(x_1, x_2, ..., x_n)$ is defined, in [4], by $*_{I=1}^{0} x_i = 1$, $*_{I=1}^{n} x_i = *(*_{I=1}^{n-1} x_i, x_n) = *(x_1, x_2, ..., x_n)$.

Definition 2. *[3] A fuzzy metric space is an ordered triple* $(X, M, *)$ *such that* X *is a nonempty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$, *satisfying the following conditions, for all* $x,y,z \in X,\ s,t > 0:$

(FM-1) $M(x,y,t) > 0$,
(FM-2) $M(x,y,t) = 1$ *iff* $x = y$,
(FM-3) $M(x,y,t) = M(y,x,t)$,
(FM-4) $M(x,z,t+s) \ge M(x,y,t) * M(y,z,s)$,
(FM-5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous.*

If, in the above definition, the triangular inequality (FM-4) is replaced by
(NA) $M(x,z,max\{t,s\}) \ge M(x,y,t) * M(y,z,s)$ for all $x,y,z \in X,\ s,t > 0$, or equivalently,
$M(x,z,t) \ge M(x,y,t) * M(y,z,t)$
then the triple $(X, M, *)$ is called a non-Archimedean fuzzy metric space [6].

Definition 3. *Let* $(X, M, *)$ *be a fuzzy metric space (or non-Archimedean fuzzy metric space). Then*

(i) *A sequence* $\{x_n\}$ *in* X *is said to converge to* x *in* X, *denoted by* $x_n \to x$, *if and only if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$, *i.e. for each* $r \in (0,1)$ *and* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - r$ *for all* $n \ge n_0$ *[7, 13].*

(ii) *A sequence* $\{x_n\}$ *is a M-Cauchy sequence if and only if for all* $\varepsilon \in (0,1)$ *and* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) \ge 1 - \varepsilon$ *for all* $m > n \ge n_0$ *[3, 13]. A sequence* $\{x_n\}$ *is a G-Cauchy sequence if and only if* $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ *for any* $p > 0$ *and* $t > 0$ *[4, 5, 15].*

(iii) *The fuzzy metric space* $(X, M, *)$ *is called M-complete (G-complete) if every M-Cauchy (G-Cauchy)sequence is convergent.*

Definition 4. *[18,19] Let A, B be a non-empty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. The mapping $g : A \to A$ is said to be a fuzzy isometric if*

$$M(gx_1, gx_2, t) = M(x_1, x_2, t)$$

for all $x_1, x_2 \in A$.

Definition 5. *[17] For $t > 0$, a non-empty subset A of a fuzzy metric space $(X, M, *)$ is said to be t-approximatively compact if for each x in X and each sequence y_n in A with $M(y_n, x, t) \longrightarrow M(A, x, t)$, there exists a subsequence y_{n_k} of y_n converging to an element y_0 in A.*

Definition 6. *[22] Let $\gamma : [0, 1) \to \mathbb{R}$ be a strictly increasing, continuous mapping and for each sequence $\{a_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} a_n = 1$ if and only if $\lim_{n\to\infty} \gamma(a_n) = +\infty$. Let Γ is the family of all γ functions.*
A mapping $T : X \to X$ is said to be a γ-contraction if there exists a $\delta \in (0, 1)$ such that

$$M(Tx, Ty, t) < 1 \Rightarrow \gamma(M(Tx, Ty, t)) \geq \gamma(M(x, y, t)) + \delta \tag{1.1}$$

for all $x, y \in X$ and $\gamma \in \Gamma$.

2. Main Results

In this section, we present some definitions and deduce some best proximity point results in non-Archimedean fuzzy metric spaces.

Let $A_0(t)$ and $B_0(t)$ two nonempty subsets of a fuzzy metric space $(X, M, *)$. We will use the following notations:

$$M(A, B, t) = \sup \{M(x, y, t) : x \in A, y \in B\};$$
$$A_0(t) = \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\};$$
$$B_0(t) = \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}.$$

Now, let us state our main results.

Definition 7. *Let (A, B) be a pair of nonempty subsets of a non-Archimedean fuzzy metric space X with $A_0 \neq 0$. Then the pair (A, B) is said to have the fuzzy weak P-property if and ony if*

$$\left\{ \begin{array}{l} M(x_1, y_1, t) = M(A, B, t) \\ M(x_2, y_2, t) = M(A, B, t) \end{array} \right. \Longrightarrow M(x_1, x_2, t) \geq M(y_1, y_2, t)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 8. *Let $X = R{\times}R$ and $M : X \times X \times (0,\infty) \to (0,1]$ be the non-Archimedean fuzzy metric given by*

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

for all $t > 0$, where $d : X \times X \to [0,\infty)$ is the standart metric $d(x,y) = |x-y|$ for all $x \in X$. Let $A = \{(0,0)\}$, $B = \{(1,0),(-1,0)\}$. Then here, $d(A,B) = 1$ and $M(A,B,t) = \frac{t}{t+1}$. Let us consider

$$\begin{aligned} M(u_1,x_1,t) &= M(A,B,t) \\ M(u_2,x_2,t) &= M(A,B,t). \end{aligned}$$

Herefrom, we have
$(u_1,x_1) = ((0,0),(1,0))$ and $(u_2,x_2) = ((0,0),(-1,0))$

$$M(u_1,u_2,t) = M((0,0),(0,0),t) = 1 > \frac{t}{t+2} = M(x_1,x_2,t).$$

Then it is easy to see that (A,B) is said to have the fuzzy weak P-property.

Definition 9. *Let A, B be a nonempty subset of a non-Archimedean fuzzy metric space $(X,M,*)$. Given $T : A \to B$ and a fuzzy isometry $g : A \to A$, the mapping T is said to preserve fuzzy isometric distance with respect to g if*

$$M(Tgx_1, Tgx_2, t) = M(Tx_1, Tx_2, t)$$

for all $x_1, x_2 \in A$.

Example 10. *Let $X = R\times [0,1]$ and $M : X{\times}X{\times}(0,\infty) \to (0,1]$ be the non-Archimedean fuzzy metric given by*

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

for all $t > 0$, where $d : X \times X \to [0,\infty)$ is the standart metric $d(x,y) = |x-y|$ for all $x \in X$. Let $A = \{(x,0) : \text{for all } x \in R\}$. Define $g : A \to A$ by $g(x,0) = (-x,0)$. Then $M(x,y,t) = \frac{t}{t+d(x,y)} = M(gx,gy,t)$, where $x = (x_1,0)$ and $y = (y_1,0) \in A$. Therefore, g is a fuzzy isometry.

Theorem 11. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X,M,*)$ such that $A_0(t)$ is nonempty. Let $T : A \to B$ be γ-contraction such that $T(A_0(t)) \subseteq B_0(t)$. Suppose that the pair (A,B) has the fuzzy P-property. Then, there exists a unique x^* in A such that $M(x^*,Tx^*,t) = M(A,B,t)$.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Proof. Let we choose an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$, we can find $x_1 \in A_0(t)$ such that $M(x_1, Tx_0, t) = M(A, B, t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$, it follows that there is an element x_2 in $A_0(t)$ such that $M(x_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \quad \text{for all } n \in N. \tag{2.1}$$

(A, B) satisfies the fuzzy weak P-property, therefore from (2.1) we obtain

$$M(x_n, x_{n+1}, t) \geq M(Tx_{n-1}, Tx_n, t), \quad \text{for all } n \in N. \tag{2.2}$$

Now we will prove that the sequence $\{x_n\}$ is convergent in $A_0(t)$. If there exists $n_0 \in N$ such that $M(Tx_{n_0-1}, Tx_{n_0}, t) = 1$, then by (2.2) we get $M(x_{n_0}, x_{n_0+1}, t) = 1$ which implies $x_{n_0} = x_{n_0+1}$. Therefore, we get

$$Tx_{n_0} = Tx_{n_0+1} \Longrightarrow M(Tx_{n_0}, Tx_{n_0+1}, t) = 1. \tag{2.3}$$

From (2.2) and (2.3), we have that

$$M(x_{n_0+2}, x_{n_0+1}, t) \geq M(Tx_{n_0+1}, Tx_{n_0}, t) = 1 \Longrightarrow x_{n_0+2} = x_{n_0+1}.$$

Therefore, $x_n = x_{n_0}$, for all $n \geq n_0$ and $\{x_n\}$ is convergent in $A_0(t)$. Also, we obtain

$$M(x_{n_0}, Tx_{n_0}, t) = M(x_{n_0+1}, Tx_{n_0}, t) = M(A, B, t).$$

This shows that x_{n_0} is a fuzzy best proximity point of T and the proof is completed. Due to this reason, we suppose that $M(Tx_{n-1}, Tx_n, t) \neq 1$, for all $n \in N$. In view of (1.1) and by (2.2), we get

$$\begin{aligned} \gamma(M(x_n, x_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\ &\geq \gamma(M(x_{n-2}, x_{n-1}, t)) + 2\delta \\ &\quad \ldots \\ &\geq \gamma(M(x_0, x_1, t)) + n\delta. \end{aligned} \tag{2.4}$$

Letting $n \to \infty$, from (2.4) we get

$$\lim_{n\to\infty} \gamma(M(x_n, Tx_{n+1}, t)) = +\infty.$$

Then, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{2.5}$$

Now, we want to show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all

$k \in N$ there exist $n(k), m(k) \in N$ with $n(k) > m(k) > k$ and

$$M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \varepsilon\ . \tag{2.6}$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the inequality (2.6). Then, we have

$$M(x_{m(k)-1}, x_{n(k)}, t_0) > 1 - \varepsilon$$

and so, for all $k \in N$, we get

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{n(k)}, x_{m(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) * M(x_{m(k)-1}, x_{n(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) * (1 - \varepsilon\ ). \end{aligned} \tag{2.7}$$

By taking $k \to \infty$ in (2.7) and using (2.5), we obtain

$$\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, t_0) = 1 - \varepsilon. \tag{2.8}$$

From (FM-4), we get

$$\begin{aligned} M(x_{m(k)+1}, x_{n(k)+1}, t_0) &\geq M(x_{m(k)+1}, x_{m(k)}, t_0) \\ &\quad * M(x_{m(k)}, x_{n(k)}, t_0) \\ &\quad * M(x_{n(k)1}, x_{n(k)+1}, t_0). \end{aligned} \tag{2.9}$$

Taking the limit as $k \to \infty$ in (2.9), we obtain

$$\lim_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, t_0) = 1 - \varepsilon\ . \tag{2.10}$$

By applying the inequality (1.1) with $x = x_{m(k)}$ and $y = x_{n(k)}$

$$\gamma(M(x_{n(k)+1}, x_{m(k)+1}, t)) \geq \gamma(M(x_{n(k)}, x_{m(k)}, t)) + \delta. \tag{2.11}$$

Taking the limit as $k \to \infty$ in (2.11), applying (1.1), from (2.8), (2.10) and continuitiy of γ, we obtain

$$\gamma(1 - \varepsilon) \geq \gamma(1 - \varepsilon) + \delta$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X. Since $A_0(t)$ is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $x^* \in A_0(t)$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Since T is continuous, we obtain $Tx_n \to Tx^*$. Also, from continuity of the fuzzy metric function M, we have $M(x_{n+1}, Tx_n, t) = M(x^*, Tx^*, t)$. From (2.1), $M(x^*, Tx^*, t) = M(A, B, t)$. So, we prove that x^* is a fuzzy best proximity point of T. The uniqueness of the best proximity point of T. From the condition that T is γ-contraction, we get

$$x_1, x_2 \in A \text{ such that } x_1 \neq x_2 \text{ and } M(x_1, Tx_1, t) = M(x_2, Tx_2, t) = M(A, B, t).$$

Then by the fuzzy weak P-property of (A, B), we have $M(x_1, x_2, t) \geq M(Tx_1, Tx_2, t)$. Also

$$x_1 \neq x_2 \Longrightarrow M(x_1, x_2, t) \neq 1.$$

Hence,

$$\gamma(M(x_1, x_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) \geq \gamma(M(x_1, x_2, t)) + \delta > \gamma(M(x_1, x_2, t))$$

which is a contradiction. Therefore the fuzzy best proximity point is unique. □

Corollary 12. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space and $A_0(t)$ a nonempty closed subsets of X. Let $T : A \to A$ be a γ-contraction. Then, there exists a unique x^* in A.*

Example 13. *Let $X = [0, 1] \times R$ and $M : X \times X \times (0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by as in Example 10. Let $A = \{(0, x) :$ for all $x \in R\}$, $B = \{(1, y) :$ for all $y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{1-x}$ for all $x \in X$. Now, define $T : A \to B$ by $T(0, x) = (1, \frac{x}{6})$. Then, we get $T(A_0(t)) = B_0(t)$. Let us consider*

$$\begin{aligned} M(u_1, Tx_1, t) &= M(A, B, t) \\ M(u_2, Tx_2, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have $(u_1, x_1) = ((0, -\frac{z_1}{6}), (0, -z_1))$ *or* $(u_2, x_2) = ((0, -\frac{z_2}{6}), (0, -z_2))$. *Then from (1.1), we obtain,*

$$\begin{aligned}\gamma(M(u_1, u_2, t)) &= \gamma(M((0, \tfrac{-z_1}{6}), (0, \tfrac{-z_2}{6}), t)) = \gamma(\frac{t}{t + \frac{|z_1 - z_2|}{6}}) \\ &= \frac{1}{1 - \frac{t}{t + \frac{|z_1 - z_2|}{6}}} > \frac{1}{1 - \frac{t}{t + |z_1 - z_2|}} = \gamma(\frac{t}{t + |z_1 - z_2|}) \\ &= \gamma(M(x_1, x_2, t)).\end{aligned}$$

That is,

$$\gamma(M(u_1, u_2, t)) > \gamma(M(x_1, x_2, t)).$$

Therefore, there exixts a $\delta \in (0, 1)$ *such that*

$$\gamma(M(u_1, u_2, t)) \geq \gamma(M(x_1, x_2, t)) + \delta$$

Then it is easy to see that T *is a* γ*-contraction and* $(0, 0)$ *is a unique fuzzy best proximity point of* T.

Definition 14. *(* γ*-proximal contraction of Type-1) Let* A *and* B *be two nonempty subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that a mapping* $T : A \to B$ *is said to be a* γ*-proximal contraction if there exists a* $\delta \in (0, 1)$ *for all* $u_1, u_2, x_1, x_2 \in X$ *such that*

$$\begin{cases} M(u_1, Tx_1, t) = M(A, B, t) \\ M(u_2, Tx_2, t) = M(A, B, t) \\ M(u_1, u_2, t), M(x_1, x_2, t) < 1 \end{cases} \Longrightarrow \gamma(M(u_1, u_2, t)) \geq \gamma(M(x_1, x_2, t)) + \delta. \quad (2.12)$$

Definition 15. *(*γ*-proximal contraction of Type-2) Let* A *and* B *be two nonempty subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that a mapping* $T : A \to B$ *is said to be a* γ*-proximal contraction if there exists a* $\delta \in (0, 1)$ *for all* $u_1, u_2, x_1, x_2 \in X$ *such that*

$$\begin{cases} M(u_1, Tx_1, t) = M(A, B, t) \\ M(u_2, Tx_2, t) = M(A, B, t) \\ M(Tu_1, Tu_2, t), M(Tx_1, Tx_2, t) < 1 \end{cases} \Longrightarrow \gamma(M(Tu_1, Tu_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) + \delta. \quad (2.13)$$

Theorem 16. *Let* A *and* B *be two nonempty, closed subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that* $T : A \to B$ *and* $g : A \to A$ *satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,
(ii) $T : A \to B$ *is a continuous* γ*-proximal contraction of type-1,*
(iii) g *is a fuzzy isometry,*
(iv) $A_0(t) \subseteq g(A_0(t))$.

Then, there exists a unique element x *in* A *such that* $M(gx, Tx, t) = M(A, B, t)$.

Proof. Let we choose an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$. Further, since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t), \quad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

Now we will prove that the sequence $\{x_n\}$ is convergent in $A_0(t)$. If there exists $n_0 \in N$ such that $M(gx_{n_0}, Tx_{n_0+1}, t) = 1$, then it is clear that sequence $\{x_n\}$ is convergent. Hence, let $M(gx_{n_0}, gx_{n_0+1}, t) \neq 1$, for all $n \in N$. From T is a γ-proximal contraction of type-1 and (2.14), we have

$$\begin{aligned}
\gamma(M(gx_n, gx_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\
\Rightarrow \gamma(M(x_n, x_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\
&\;\;\ldots \\
&\geq \gamma(M(x_0, x_1, t)) + n\delta.
\end{aligned} \tag{2.15}$$

Letting $n \to \infty$, from (2.15) we get

$$\lim_{n\to\infty} \gamma(M(x_n, Tx_{n+1}, t)) = +\infty.$$

Then, if we similarly continue as the process in the proof of Theorem 11, we have $\{x_n\}$ is a Cauchy sequence.

Since is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $x \in A_0(t)$ such that $\lim_{n\to\infty} x_n = x$.

Since T,g and M are continuous, passing to the limit $n \to \infty$, we have

$$M(gx, Tx, t) = M(A, B, t).$$

Let x^* be in $A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Now, we will show that $x = x^*$. Suppose to the contrary, let $x \neq x^*$. Therefore, $M(x, x^*, t) \neq 1$. Since T is a γ-proximal contraction of type-1 and g is an isometry, we have

$$\gamma(M(x, x^*, t)) = \gamma(M(gx, gx^*, t)) \geq \gamma(M(x, x^*, t)) + \delta > \gamma(M(x, x^*, t))$$

which is a contradiction. Hence, $x = x^*$. Therefore, the proof of Theorem 16 is completed. □

If we take g is the identity mapping, we obtain the following result.

Corollary 17. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Assume that A is approximatively compact with respect to B. Also, suppose that $T : A \to B$ satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,

(ii) $T : A \to B$ is a continuous γ-proximal contraction of type-1,

Then, T has a unique fuzzy best proximity point in A.

Example 18. *Let $X = R\times[-2, 2]$ and $M : X \times X \times (0,\infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0,\infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(x,-2) : \text{for all } x \in R\}$, $B = \{(y, 2) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 4$ and $M(A, B, t) = \frac{t}{t+4}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{1-x^2}$ for all $x \in X$. Now, define $T : A \to B$ and $g : A \to A$ by

$$T(x,-2) = (\frac{x}{2}, 2) \text{ and } g(x,-2) = (-x,-2)$$

Clearly, g is fuzzy isometry. Then, we have, we get $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Let us consider

$$\begin{aligned} M(gu_1, Tx, t) &= M(A, B, t) \\ M(gu_2, Tx, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have $(u_1, x_1) = ((-\frac{z_1}{2},-2), (z_1,-2))$ or $(u_2, x_2) = ((-\frac{z_2}{2},-2), (z_2,-2))$. We claim that T is a γ-proximal contraction type-1. Now, putting $u_1 = (-\frac{z_1}{2},-2), x_1 = (z_1,-2)$, $u_2 = (-\frac{z_2}{2},-2)$ and $x_2 = (z_2,-2)$ in (2.12), we have

$$\begin{aligned} \gamma(M(gu_1, gu_2, t)) &= \gamma(M((\tfrac{z_1}{2},-2), (\tfrac{z_2}{2},-2), t) = \gamma(\frac{t}{t + \frac{|z_1-z_2|}{2}}) \\ &= \frac{1}{1 - (\frac{t}{t+\frac{|z_1-z_2|}{2}})^2} > \frac{1}{1 - (\frac{t}{t+|z_1-z_2|})^2} = \gamma(\frac{t}{t + |z_1 - z_2|}) \\ &= \gamma(M(x_1, x_2, t)). \end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

That is, we have

$$\gamma(M(u_1, u_2, t)) > \gamma(M(x_1, x_2, t)).$$

Therefore, there exixts a $\delta \in (0, 1)$ such that

$$\gamma(M(u_1, u_2, t) \geq \gamma(M(x_1, x_2, t)) + \delta.$$

Then it is easy to see that T is a γ-proximal contraction type-1. It now follows from Theorem 16 that $(0, -2)$ is a unique fuzzy best proximity point of T.

Theorem 19. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Assume that A is approximatively compact with respect to B. Also, suppose that $T : A \to B$ and $g : A \to A$satisfy the following conditions:*

(*i*) $T(A_0(t)) \subseteq B_0(t)$,
(*ii*) $T : A \to B$ *is a continuous γ-proximal contraction of type-2,*
(*iii*) g *is a fuzzy isometry,*
(*iv*) $A_0(t) \subseteq g(A_0(t))$,
(*v*) T *preserves fuzzy isometric distance with respect to g.*

Then, there exists an element x in A such that $M(gx, Tx, t) = M(A, B, t)$. Moreover, if x^* is another element of A such that $M(gx^*, Tx^*, t) = M(A, B, t)$.

Proof. Let we choose an element Tx_0 in $T(A_0(t))$. Since $Tx_0 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$ and and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t), \quad \text{for all } n \in N. \tag{2.16}$$

Now we will prove that the sequence $\{Tx_n\}$ is convergent in B. If there exists $n_0 \in N$ such that $M(Tgx_{n_0}, Tgx_{n_0+1}, t) = 1$, then it is clear that sequence $\{Tx_n\}$ is convergent. Hence, let $M(Tgx_{n_0}, Tgx_{n_0+1}, t) \neq 1$, for all $n \in N$. From T is a γ-proximal contraction of type-2, T preserves fuzzy isometric distance with respect to g and (2.16), we have

$$\begin{aligned}
\gamma(M(Tgx_n, Tgx_{n+1}, t)) &\geq \gamma(M(Tx_{n-1}, Tx_n, t)) + \delta \\
\Rightarrow \gamma(M(Tx_n, Tx_{n+1}, t)) &\geq \gamma(M(Tx_{n-1}, Tx_n, t)) + \delta \\
&\cdots \\
&\geq \gamma(M(Tx_0, Tx_1, t)) + n\delta. \qquad (2.17)
\end{aligned}$$

Letting $n \to \infty$, from (2.17) we get

$$\lim_{n\to\infty} \gamma(M(Tx_n, Tx_{n+1}, t)) = +\infty.$$

Then, if we similarly continue as the process in the proof of Theorem 11, we have $\{Tx_n\}$ is a Cauchy sequence in B.

Since B is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $y \in B$ such that $\lim_{n\to\infty} Tx_n = y$. From the triangular inequality, we obtain

$$\begin{aligned} M(y, A, t) \geq M(y, gx_n, t) \geq M(y, Tx_{n-1}, t) * M(Tx_{n-1}, gx_n, t) \\ = M(y, Tx_{n-1}, t) * M(A, B, t) \\ \geq M(y, Tx_{n-1}, t) * M(y, A, t). \end{aligned} \tag{2.18}$$

Passing to the limit as $n \to \infty$ in (2.18), we have

$$\lim_{n\to\infty} M(y, gx_n, t) = M(y, A, t).$$

Since $A_0(t)$ is approximatively compact with respect to B, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that converges to some z in $A_0(t)$. Therefore, we have

$$M(z, y, t) = \lim_{k\to\infty} M(gx_{n_k}, Tgx_{n_k-1}, t) = M(y, A, t).$$

Hence, it implies that $z \in A_0(t)$. Since $A_0(t) \subseteq g(A_0(t))$, there exists $x \in A_0(t)$ such that $z = gx$. Taking to the limit as $\lim_{k\to\infty} gx_{n_k} = gx$ and g is a fuzzy isometry, we obtain

$$\lim_{k\to\infty} x_{n_k} = x.$$

Since T is continuous and $\{Tx_n\}$ is convergent to y, we have

$$\lim_{k\to\infty} Tx_{n_k} = Tx = y.$$

Hence, it follows that

$$M(gx, Tx, t) = \lim_{k\to\infty} M(gx_{n_k}, Tgx_{n_k}, t) = M(A, B, t).$$

Let x^* be in $A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Now, we will show that $Tx = Tx^*$. Suppose to the contrary, let $Tx \neq Tx^*$. Therefore, $M(x, Tx^*, t) \neq 1$. Since T is a γ-proximal contraction of type-2 and T preserves fuzzy isometric distance with respect to g, we have

$$\gamma(M(Tx, Tx^*, t)) = \gamma(M(Tgx, Tgx^*, t)) \geq \gamma(M(x, x^*, t)) + \delta > \gamma(M(x, x^*, t))$$

which is a contradiction. Hence, $Tx = Tx^*$. Therefore, the proof of Theorem 19 is completed. $\square$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

If we take g is the identity mapping, we obtain the following result.

Corollary 20. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Assume that A is approximatively compact with respect to B. Also, suppose that $T : A \to B$ satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,

(ii) $T : A \to B$ is a continuous γ-proximal contraction of type-2,

Then, T has a unique fuzzy best proximity point in A. Moreover, if x^* is another fuzzy best proximity point T, then $Tx = Tx^*$.

Example 21. *Let $X = [0, 1]\times R$ and $M : X\times X\times(0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0, \infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(0, x) : \text{for all } x \in R\}$, $B = \{(1, y) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{\sqrt{1-x}}$ for all $x \in X$. Now, define $T : A \to B$ and $g : A \to A$ by

$$T(0, x) = (1, \frac{x}{3}) \text{ and } g(0, x) = (0, -x)$$

Clearly, g is a fuzzy isometry. Then, we have, we get $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Let us consider

$$\begin{aligned} M(gu_1, Tx_1, t) &= M(A, B, t) \\ M(gu_2, Tx_2, t) &= M(A, B, t). \end{aligned}$$

. Clearly, T is preserve isometric distance with respect to g. That is $M(Tgx_1, Tgx_2, t) = M(Tx_1, Tx_2, t)$. We claim that T is a $\gamma-$proximal contraction type-2. Now, putting $u_1 = (0, -\frac{z_1}{3}), x_1 = (0, z_1)$, $u_2 = (0, -\frac{z_2}{3},)$ and $x_2 = (0, z_2)$ in (2.13), we have

$$\begin{aligned} \gamma(M(Tgu_1, Tgu_2, t) &= \gamma(M(1, \tfrac{z_1}{9}), (1, \tfrac{z_2}{9}), t) = \gamma(\frac{t}{t + \frac{|z_1-z_2|}{9}}) \\ &= \frac{1}{\sqrt{1 - \frac{t}{t+\frac{|z_1-z_2|}{9}}}} > \frac{1}{\sqrt{1 - \frac{t}{t+\frac{|z_1-z_2|}{3}}}} = \gamma(\frac{t}{t + \frac{|z_1-z_2|}{3}}) \\ &= \gamma(M(Tx_1, Tx_2, t)). \end{aligned}$$

Since, T preserves isometric distance with respect to g, we have

$$\gamma(M(Tu_1, Tu_2, t)) > \gamma(M(Tx_1, Tx_2, t)).$$

Therefore, there exixts a $\delta \in (0,1)$ *such that*

$$\gamma(M(Tu_1, Tu_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) + \delta.$$

Then it is easy to see that T *is a* γ*-proximal contraction type-2. It now follows from Theorem 19 that* $(0,0)$ *is a unique fuzzy best proximity point of* T.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales*, Fund. Math., *3 (1922), 133-181.*

[2] *Z. Deng,* Fuzzy pseudometric spaces*, J. Math. Anal. Appl., 86 (1922), 74-95.*

[3] *A. George and P. Veeramani,* On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395-399.*

[4] *M. Grabiec,* Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems 27 (1988), 385-389.*

[5] *V. Gregori and A. Sapena,* Sn fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems 125 (2001), 245-253.*

[6] *V. Istrăţescu,* An Introduction to Theory of Probabilistic Metric Spaces, with Applications, Ed, Tehnică, Bucureşti*, in Romanian, (1974).*

[7] *O. Kramosil, and J. Michalek,* Fuzzy metric and statistical metric spaces*, Kybernetika, 11 (1975), 336-344.*

[8] *D. Mihet,* Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces*, Fuzzy Sets and Systems, 159 (2008), 739 -744.*

[9] *D. Mihet,* A class of contractions in fuzzy metric spaces*, Fuzzy Sets Syst., 161 (2010), 1131-1137.*

[10] *P. Salimi, C. Vetro and P. Vetro,* Some new fixed point results in non-Archimedean fuzzy metric spaces*, Nonlinear Analysis: Modelling and Control, 18 2013, 3, 344-358.*

[11] *M. Sangurlu and D. Turkoglu,* Fixed point theorems for $(\psi \circ \varphi)$-contractions in a fuzzy metric spaces*, J. Nonlinear Sci. Appl., 8 (2015), 687-694.*

[12] *B. Schweizer and A. Sklar,* Statistical metric spaces*, Pacific Journal of Mathematics, 10 (1960) 385-389.*

[13] *B. Schweizer, and A. Sklar,* Probabilistic Metric Spaces*, North-Holland, Amsterdam, 1983.*

[14] *D. Turkoglu and M. Sangurlu,* Fixed point theorems for fuzzy (ψ)-contractive mappings in fuzzy metric spaces*, Journal of Intelligent and Fuzzy Systems, 26(2014), 1, 137-142.*

[15] *R. Vasuki and P. Veeramani,* Fixed point theorems and Cauchy sequences in fuzzy metric spaces*, Fuzzy Sets and Systems 135 (2003), 3, 409-413.*

[16] *C. Vetro,* Fixed points in weak non-Archimedean fuzzy metric spaces*, Fuzzy Sets and Systems, 162 (2011), 84-90.*

[17] *Z. Razaa, N. Saleem, M. Abbas,* Optimal coincidence points of proximal quasi-contraction mappings in non-Archimedean fuzzy metric spaces*, Journal of Nonlinear Science and Appl., 9 (2016), 3787-3801.*

[18] *M. Abbas, N. Saleem, M. De la Sen,* Optimal coincidence point results in partially ordered non-Archimedean fuzzy metric spaces*, Fixed Point Theory Appl., 2016 (2016), 44 pages.*

[19] *N. Saleem, M. Abbas, Z. Raza,* Optimal coincidence best approximation solution in non-Archimedean Fuzzy Metric Spaces, *Iranian J. Fuzzy Sys., 13 (2016), 113-124.*

[20] *C. Vetro, P. Salimi,* Best proximity point results in non-Archimedean fuzzy metric spaces, *Fuzzy Information and Engineering, 5 (2013), 417-4291.*

[21] *V. S. Raj,* A Best proximity point theorem for weakly contractive non-self mappings, *Nonlinear Anal., 74 (2011), 449-455.*

[22] *M. Sangurlu Sezen,* Fixed Point Theorems for New Type Contractive Mappings, *Journal of Function Spaces, 2019 (2019), Article ID 2153563, 6 pages.*

[1] Department of Mathematics, Faculty of Science and Arts, University of Giresun, Güre, Giresun, Turkey.

E-mail address: `muzeyyen.sezen@giresun.edu.tr`

[2] Department of Mathematics, Faculty of Science and Arts, Muş Alparslan University, Muş 49250, Turkey.

E-mail address: `isikhuseyin76@gmail.com`

Exact solutions of conformable fractional Harry Dym equation

Asma ALHabees

Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan

*Corresponding author e-mail address: a.habees@ju.edu.jo

Abstract: The aim of this paper is to find exact solutions for the conformable fractional Harry Dym Equation. In this work we deal with three different forms of conformable fractional Harry Dym Equation and for each form a suitable wave variable substitution is found. Each substitution transform its corresponding problem to an ordinary differential equation, What is more, the resulted ordinary differential equations in the three cases are the same. General solutions are obtained by applying the direct integration method on the resulted ordinary differential equation. These obtained solutions are found for some particular choices for the constants values. The behavior of every solution is discussed and illustrated in graphs. The tedious integrals and difficult computations associated with calculations in this paper are performed and simplified by using Mathematica 9.0.

Keywords: Conformable fractional derivative, Harry Dym Equation, Conformable Harry Dym Equation, Exact solutions.

1. Introduction

Recently, differential equations with fractional derivatives attracted the interest of many researchers; since such equations describe effectively many phenomena in applied sciences such as physics, biology, technology, and engineering [3, 7, 14].

Harry Dym equation (HD) was so named related to the name of its discoverer Harry Dym in his unpublished paper 1973-1974, although it appeared to first time in Kruskal and Moser [9]. HD equation represents a system which gathers non-linearity and dispersion, also it is a completely integrable nonlinear evolution equation which obeys an infinite number of conservation laws, but it does not have the Painleve property. More properties for HD equation discussed in details can be found in the reference [4]. Moreover HD equation can be connected to the Korteweg-ge Vries equation which has many applications in hydrodynamics [4, 15].

Many efforts have been done to find exact and approximate solutions for both HD equation and fractional HD equation like algebraic geometric solution of the HD equation[13], solitions solutions of the (2+1) dimensional HD equation via Darboux transformation [2], explicit solutions for HD equation [1], exact solution of the HD equation [12], an efficient approach for fractional HD equation by using sumudu transform [10], symmetries and exact solutions of the time fractional HD equation with Rieman-Liouville derivative [5], and a fractional model of HD equation and its approximate solution [11].

Fractional derivatives have many definitions [14] but the most used of these definitions are Riemann-Liouville derivative and Caputo derivative. They were defined as follows:

(i) Riemann - Liouville Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

(ii) Caputo Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Recently, a new definition called conformable fractional derivative was introduced by authors in [6], Since then the interest of it keeps growing and many equations were solved using such definition [8]. In this paper we intend to find exact solutions for fractional HD equation in the sense of this definition rather than Rieman-Liouville definition or Caputo definition. The rest of the paper is organized as follows: Basics of conformable fractional derivative are stated in section 2, in section 3 solutions for conformable fractional HD equation are found, in section 4 some examples are discussed.

2. Basic results on conformable fractional derivatives.

Now, Let us summarize the basic properties of the conformable fractional derivative definition.

Definition [6]: Given a function$f: [0, \infty) \to \mathbb{R}$. And $t > 0, \alpha \in (0, 1]$, then the conformable fractional derivative of order α is defined as

$T_\alpha (f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$,

T_α is called the conformable fractional derivative of f of order α .

Let $f^\alpha(t)$ stands for $T_\alpha (f)(t) = \frac{d^\alpha f}{dt^\alpha}$.

If f is α-differentiable in some$(0, b), b > 0$, and $\lim_{t \to 0^+} f^\alpha(t)$ exists, then by definition:

$f^\alpha(0) = \lim_{t \to 0^+} f^\alpha(t)$

Theorem 1 [6]: Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then

1. $T_\alpha (af + bg) = a\, T_\alpha (f) + b\, T_\alpha (g),\ for\ all\ a, b \in \mathbb{R}$.
2. $T_\alpha (t^p) = pt^{p-\alpha}\ for\ all\ p \in \mathbb{R}$.
3. $T_\alpha (\lambda) = 0\ for\ all\ constants\ functions\ f(t) = \lambda$.
4. $T_\alpha (fg) = f\, T_\alpha (g) + g\, T_\alpha (f)$.
5. $T_\alpha \left(\frac{f}{g}\right) = \frac{g\, T_\alpha (f) - f\, T_\alpha (g)}{g^2}$.
6. If, in addition, f is differentiable, then $T_\alpha (f)(t) = t^{1-\alpha} \frac{df}{dt}$.

Theorem 2 [8]: let f be an α-differentiable function in conformable sense and differentiable and suppose that g is also differentiable and defined in the range of f. Then

$$T_\alpha\,(fog)\,(t) = t^{1-\alpha}\, g^{'}(t) f^{'}\big(g(t)\big).$$

More properties, definitions and theorems as Roll's Theorem and Mean Value Theorem for conformable fractional derivative are expressed in the work [6],

3. Fractional Harry Dym Equation.

The classical HD equation is:

$$u_t = u^3 u_{xxx} \qquad (*)$$

Where $u(x,t)$ is a function of two real variables $x\ and\ t$.

Let us write:

$$u_t^\alpha = T_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha}\ , \qquad u_x^\alpha = T_x^\alpha u = \frac{\partial^\alpha u}{\partial x^\alpha}\ , \qquad u_x^{(3\alpha)} = T_x^{(3\alpha)} u = T_x^\alpha\, T_x^\alpha\, T_x^\alpha u\,.$$

Now we will solve three fractional forms of$(*)$:

(i) $u_t^\alpha = u^3 u_{xxx}$. (1)

(ii) $u_t = u^3 u_x^{(3\alpha)}$. (2)

(iii) $u_t^\alpha = u^3 u_x^{(3\alpha)}$. (3)

Where $\alpha \in (0,1]$.

Using suitable wave variable substitution in each form will transform the equation to an ordinary differential equation as follows:

1. For form (i) let the wave variable substitution $\eta = x + \frac{c}{\alpha} t^\alpha$ and $u(x,t) = v(\eta)$. So one can write $u = v \circ \eta$, now apply Theorem 2 to find u_t^α . You will get that $u_t^\alpha = t^{1-\alpha}\, \eta^{'}(t) v^{'}\big(\eta(t)\big) = cv^{'}, also\ u^3 = v^3 and\ u_{xxx} = v^{'''}$. Hence equation (1) is transformed to:

$$cv^{'} = v^3 v^{'''} \qquad (4)$$

2. For form (ii) let the wave variable substitution $\eta = \frac{1}{\alpha} x^\alpha + ct$ and $u(x,t) = v(\eta) = v \circ \eta.$ so

$u_t^\alpha = cv^{'}, u^3 = v^3 and\ u_x^{(3\alpha)} = v^{'''}$. Then equation (2) is transformed to:

$$cv^{'} = v^3 v^{'''} \qquad (4)$$

3. For form (iii) let the wave variable substitution $\eta=\frac{1}{\alpha}x^{\alpha}+\frac{c}{\alpha}t^{\alpha}$ and $u(x,t)=v(\eta)$. so $u_t^{\alpha}=cv', u^3=v^3 and\ u_x^{(3\alpha)}=v'''$. Then equation (3) is transformed to:

$$cv'=v^3v''' \qquad (4)$$

Now to solve the resulted ordinary differential equation (4), rewrite it as:

$$v'''+\left(\frac{c}{2v^2}\right)'=0 \qquad (5)$$

Integrate (5) with respect to η, gets

$$v''+\frac{c}{2v^2}=\frac{c_1}{2} \qquad (6)$$

Multiply (6) by v' then integrate with respect to η yields

$$(v')^2=\frac{c}{v}+c_1v+c_2 \qquad (7)$$

Using the separation of variables changes (7) to

$$d\eta=\pm\sqrt{\frac{v}{c_1v^2+c_2v+c}}\ dv \qquad (8)$$

Integrate both sides of (8) using Mathematica 9.0 you will obtain

$$\eta=\pm\int\sqrt{\frac{v}{c_1v^2+c_2v+c}}dv+c_3 \qquad (9)$$

$$\eta=\pm i\ \frac{ABCD}{c_1G}[Elliptic\ E(i\ sinh^{-1}(G),K)-Elliptic\ F(i\ sinh^{-1}(G),K)]+c_3 \qquad (10)$$

Where: $A=\sqrt{\frac{v}{c_1v^2+c_2v+c}}$, $B=-c_2+\sqrt{-4cc_1+c_2{}^2}$, $C=\sqrt{1+\frac{2c_1v}{c_2-\sqrt{-4cc_1+c_2{}^2}}}$

$D = \sqrt{1 + \frac{2c_1 v}{c_2 + \sqrt{-4cc_1 + c_2{}^2}}}$, $G = \sqrt{\frac{2c_1 v}{c_2 + \sqrt{-4cc_1 + c_2{}^2}}}$ and $K = \frac{c_2 + \sqrt{-4cc_1 + c_2{}^2}}{c_2 - \sqrt{-4cc_1 + c_2{}^2}}$.

$Elliptic\ F$ and $Elliptic\ E$ are elliptic integrals of the first and second kind respectively.

For some particular choices to the constants c, c_1 and c_2 in equation (9) one can get simpler solutions as follows:

- Let $c_1 = c_2 = 0$, then $\eta = \pm \frac{2}{3}\, v \sqrt{\frac{v}{c}} + c_3$, hence

$$v = (c_3 \pm \frac{3}{2}\sqrt{c}\ \eta)^{\frac{2}{3}} \qquad (11)$$

- Let $c_1 = 0, c_2 \neq 0$, then $\eta = \pm\left(\frac{\sqrt{cv + c_2 v^2}}{c_2} - \frac{c}{c_2^{\frac{3}{2}}} \log(\, 2c_2\sqrt{v} + 2\sqrt{vc_2{}^2 + c_2 c}\ )\right) + c_3$

Other suggested constants are:

1. Let $c_2 = 2\sqrt{cc_1}$.
2. Let $c_2 = -2\sqrt{cc_1}$.

You can easily using Mathematica 9.0 to perform the integration of equation (9) to get formula of η after you determine the suggested constants, however the difficulty that faces is how to get v with respect to η explicitly , except the formula in (11), this what was discussed in [12], Hence it seems that formula (11) is the only explicit solution for equations (1), (2) and (3). So results can be summarized as follows:

· The solution of equation (1) is $u(x,t) = (c_3 \pm \frac{3}{2}\sqrt{c}\ \left(x + \frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$.

· The solution of equation (2) is $u(x,t) = (c_3 \pm \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha}x^{\alpha} + ct)\ )^{\frac{2}{3}}$.

· The solution of equation (3) is $u(x,t) = (c_3 \pm \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha}x^{\alpha} + \frac{c}{\alpha}t^{\alpha})\ )^{\frac{2}{3}}$.

Remarks:

1. The same ordinary differential equation is obtained from the three different forms of conformable fractional Harry Dym- Equation after using special wave variable for each form.

2. A function could be α-differentiable at a point but not differentiable, illustrating example was discussed in [6].

4. Examples.

Example 1: Let $\alpha = 0.7$, for the graph of equation (1) solution $u(x,t) = (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\,\left(x + \frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4$ and $c = 1$ see Figure 1.

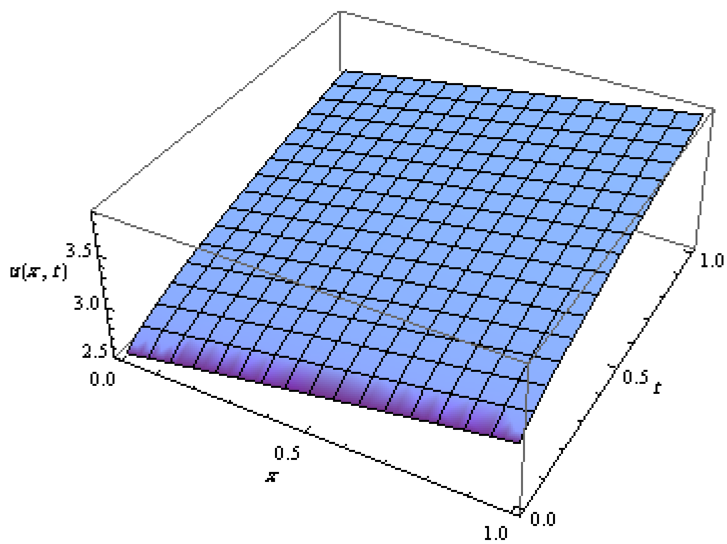

Fig. 1 The graph of $u(x,t) = (4 + \frac{3}{2}\left(x + \frac{1}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ at $\alpha = 0.7$ for example 1

Example 2: The graph of equation (1) solution $u(x,t) = (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\,\left(x + \frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ versus x at $t = 1$, $c_3 = 4\ and\ c = 1$ for different values of α is in Figure 2.

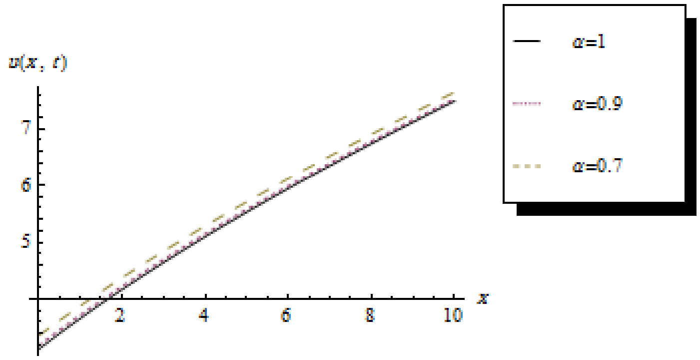

Fig. 2 The graph of $u(x,t) = (4+\frac{3}{2}\left(x+\frac{1}{\alpha}\right))^{\frac{2}{3}}$ versus x at $t = 1$ *at* $\alpha = 1, 0.9$ *and* 0.7 for example 2

Example 3: Let $\alpha = 0.9$, for the graph of equation (2) solution $u(x,t) = \ (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha}x^{\alpha} + ct)\ )^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4$ *and* $c = 1$ see Figure 3.

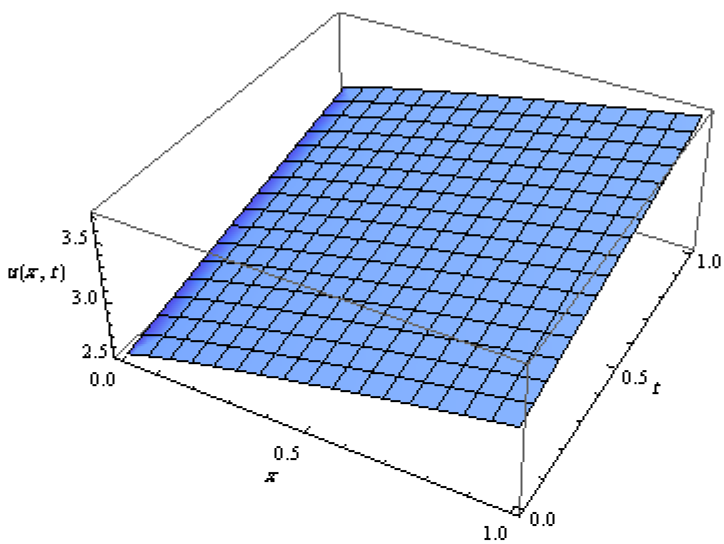

Fig. 3 The graph of $u(x,t) = \ (4+\frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha}+t\right))^{\frac{2}{3}}$ at $\alpha = 0.9$ for example 3

Example 4: The graph of equation (2) solution $u(x,t) = \left(4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + t\right)\right)^{\frac{2}{3}}$ versus x at $t = 0, c_3 = 4\ and\ c = 1$ for different values of α is in Figure 4.

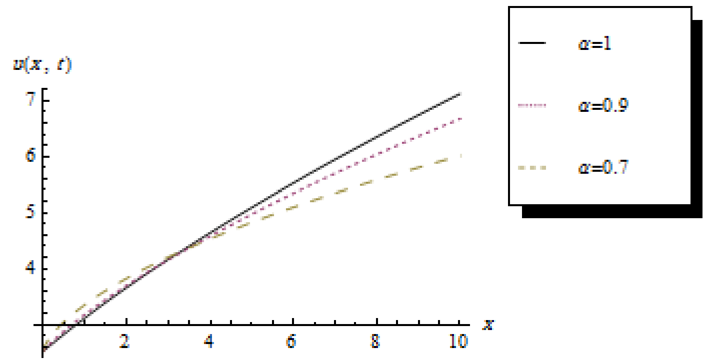

Fig. 4 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}\ x^{\alpha} + t\right))^{\frac{2}{3}}$ versus x at $\mathrm{t} = 0\ at\ \alpha = 1, 0.9\ and\ 0.7$ for example 4

Example 5: Let $\alpha = 0.9$, for the graph of equation (3) solution $u(x,t) = (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha}x^{\alpha} + \frac{c}{\alpha}t^{\alpha})\ )^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4\ and\ c = 1$ see Figure 5.

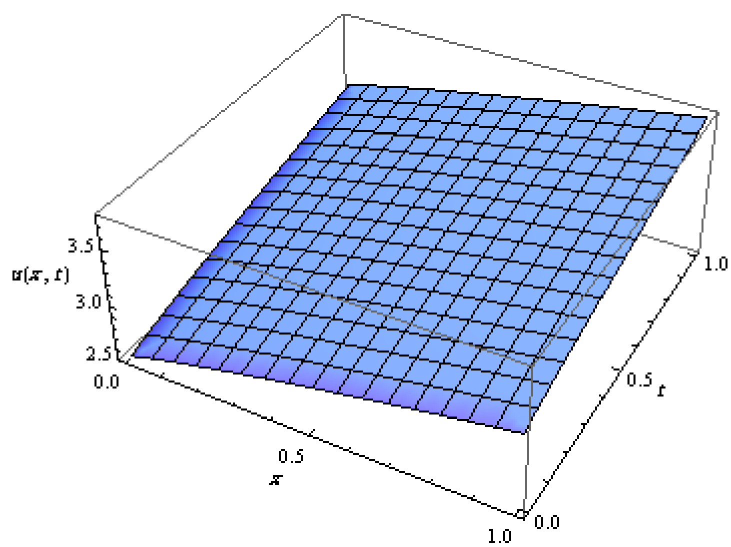

Fig. 5 The graph of $u(x,t) = \ (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + \frac{1}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ at $\alpha = 0.9$ for example 5

Example 6: The graph of equation (3) solution $u(x,t) = (c_3 + \frac{3}{2}\sqrt{c}\,(\frac{1}{\alpha}x^\alpha + \frac{c}{\alpha}t^\alpha))^{\frac{2}{3}}$ versus x at $t = 1$, $c_3 = 4$ and $c = 1$ for different values of α is in Figure 6.

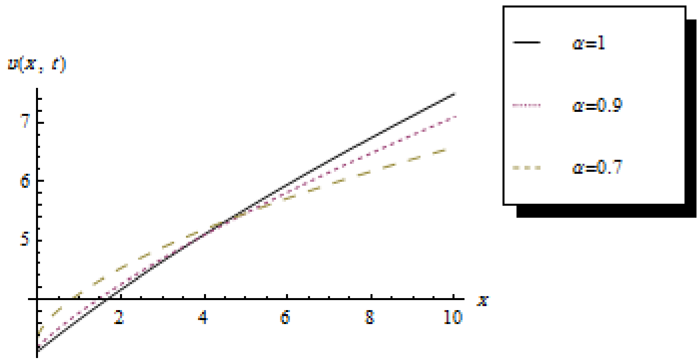

Fig. 6 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^\alpha + \frac{1}{\alpha}t^\alpha\right))^{\frac{2}{3}}$ versus x at $\mathrm{t} = 1$ at $\alpha = 1, 0.9$ and 0.7 for example 6

References

[1] B. Fuchssteinert , T. Schulzet, S. Carllot, Explicit solutions for Harry Dym equation, J Phys A: Math Gen.25(1992) 223-30

[2] A.A. Halim, Solition solutions of the (2+1) dimentional Harry Dym equation via Darboux transformation, Chaos Solitions Fractals 36 (2008) 646-53.

[3] J.H. He, Some applications of non linear fractional differential equations and their approximations, Bull. Sci. Technol. 15 (2) (1999) 86--90.

[4] W.Hereman, P.P. Banerjee, M.R. Chatterjee, On the nonlocal equations and nonlocal charges associated with the Harry-Dym hierarchy Korteweg-de Vries equation, J. Phys. A: Math. 22 (1989) 241-252.

[5] Q. Huang , R. Zhdanov, Symmetries and exact solutions of the time fractional Harry- Dym equation with Rieman-Liouville derivative, Physica A. 409 (2014) 110-118.

[6] R. Khalil, M. Al horani, A. Yousef , M. Sababheh, Anew definition of fractional derivative, Journal of Computational Applied Mathematics, 264 (2014) 65-70.

[7] A.A. Kilbas, H.M. Srivastava, J.J Trujillo,Theory and Applications of Fractional Differential Equations. North-Holland Math. Stud. 204 (2006).

[8] A. Korkmaz, Exact Solutions to Some Conformable Time Fractional Equations in Benjamin-Bona-Mohany Family. https://arxiv.org/abs/1611.07086, 2007 (accessed 3 December 2007).

[9] M.D.Kruskal , J.Moser, Dynamical system: Theory and Applications (Lecture Notes in Physics 38), Springer, Berlin (1975).

[10] D. Kumar , J. Singh, A. Kılıçman, An Efficient Approach for Fractional Harry Dym Equation by Using Sumudu Transform, Abstract and Applied Analysis. Article ID 608943(2013) 8 pages .

[11] S. Kumar, M. P. Tripathi , O. P. Singh, A fractional model of Harry Dym equation and its approximate solution, Ain Shams Engineering Journa l .4 (2013),111-115.

[12] R. Mokhtari, Exact solutions of the Harry- Dym equation, Commun. Theor. Phys. 55 (2).(2011) 204-208.

[13] D.P. Novikov, Alalgebraic geometric solution of the Harry Dym equation, Siberian Math J. 40 (1) (1999). 136-140.

[14] I. Podlubny, Fractional differential equations. An introduction to fractional derivatives fractional differential equations some methods of their Solution and some of their applications, Academic Press, San Diego, 1999.

[15] G.L. Vasconcelos, L.P. Kadanoff, Stationary solutions for the Saffman-Taylor problem with surface tention, . Phys Re A. 44 (10). (1991) 6490-6495.

.

.

Some Properties of the q-Exponential Functions

Mahmoud J. S. Belaghi

Bahçeşehir University, Istanbul, Turkey
mahmoud.belaghi@eng.bau.edu.tr

Abstract. This paper aims to investigate some striking properties of the q-exponential functions more profoundly. To achieve this, at first, the Gauss q-binomial formula is generalized and based on the formula, important properties of the q-exponential functions are established.

Keywords. q-Exponential function, q-Binomial formula.
Mathematics Subject Classification. 11B65, 05A30.

1 Introduction

The q-analogue of any real number t is defined as $[t]_q = \frac{1-q^t}{1-q}$ and the q-factorial, denoted by $[n]_q!$, is defined [1, 2] as

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q \times [n-1]_q \times \cdots \times [1]_q & \text{if } n = 1, 2, \ldots. \end{cases} \tag{1}$$

The q-analogue of $(a+x)^n$, denoted by $(a+x)_q^n$, is defined [3] as

$$(a+x)_q^n = \begin{cases} 1 & n = 0, \\ \prod_{m=0}^{n-1}(a+q^m x) & n = 1, 2, \ldots. \end{cases} \tag{2}$$

It is also defined for any complex number α as

$$(a+x)_q^\alpha = \frac{(a+x)_q^\infty}{(a+q^\alpha x)_q^\infty}, \tag{3}$$

where $(a+x)_q^\infty := \lim_{n\to\infty} \prod_{m=0}^{n}(a+q^m x)$, and the principal value of q^α is considered, $0 < q < 1$. Yet, the q-Maclaurin series expansion of $(a+x)_q^n$ is

$$(a+x)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k} x^k q^{\binom{k}{2}} \tag{4}$$

where $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ are called q-binomial coefficients. Expression (4) is called Gauss q-binomial formula (see [3], p. 15). In the q-binomial coefficients, if $|q| < 1$ and n tends to infinity (see [3], p. 30) we obtain $\lim_{n\to\infty} \binom{n}{k}_q = \frac{1}{(1-q)_q^k}$. More details about the identities involving q-binomial coefficients can be found in reference [4].
One can also recall definitions of the q-functions [2, 5, 6] as follows:

$$e_q^x = \frac{1}{(1-(1-q)x)_q^\infty} = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n, \qquad |x| < 1, \tag{5}$$

$$E_q^x = (1+(1-q)x)_q^\infty = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n q^{\binom{n}{2}}, \qquad x \in \mathbb{C}. \tag{6}$$

It can be seen that $e_q^x E_q^{-x} = 1$ and $e_{q^{-1}}^x = E_q^x$. The product of the two functions are investigated in a more detailed way in [6, 7, 8]. The contribution of the corresponding references can be summarized in the following theorem:

Theorem 1. *For all $x, y \in \mathbb{C}$ the following equation holds*

$$e_q^x\, E_q^y = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} (x+y)_q^n = e_q^{(x+y)_q} \tag{7}$$

where $(x+y)_q^n$ is defined in (4).

In the light of aforementioned preliminaries, this paper aims at studying about the q-exponential functions more closely. At first, the Gauss q-binomial formula is generalized and based on the formula, some properties of the q-exponential functions are established.

2 q-Exponential Functions

First, let us generalize the q-binomial formula given in (4). The generalization of the q-binomial can then be carried out as follows.

Theorem 2. *For any $x, y, z \in \mathbb{C}$ and positive integer n, the following identity holds:*

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k}. \tag{8}$$

Proof. The induction is used to prove the theorem. Equation (8) is valid for $n = 1$. Assuming that (8) holds for any n and we show that it holds for $n+1$. Then

$$\begin{aligned}
(x+y)_q^{n+1} &= (x+y)_q^n \left(q^k(z+q^{n-k}y) + (x-q^k z)\right) \\
&= \sum_{k=0}^{n} \binom{n}{k}_q q^k (x-z)_q^k (z+y)_q^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^{k+1} (z+y)_q^{n-k} \\
&= (z+y)_q^{n+1} + (x-z)_q^{n+1} + \sum_{k=1}^{n} \binom{n}{k}_q q^k (x-z)_q^k (z+y)_q^{n+1-k} \\
&\qquad + \sum_{k=1}^{n} \binom{n}{k-1}_q (x-z)_q^k (z+y)_q^{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k}_q (x-z)_q^k (z+y)_q^{n+1-k}.
\end{aligned}$$

Thus, the proof is complete. □

It is realized that the identity in Theorem 2 can be re-written as

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^{n-k} (z+y)_q^{k}. \tag{9}$$

Its proof can be readily derived form the proof of Theorem 2.

Theorem 2 and its re-expression (9) allow one to conclude the striking identities given as follows:

- For $y = 0$ and $z = 1$, the q-Taylor expansion of x^n about $x = 1$, (see [3], p. 23) becomes

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-1)_q^k.$$

- For $x = 1$, $y = -ab$ and $z = a$, the following identity (see [2], p. 25) is obtained

$$(1-ab)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k} (1-a)_q^k (1-b)_q^{n-k}.$$

- For $y = -x$, the identity

$$\sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z-x)_q^{n-k} = 0.$$

  is found.

- For the case of $z = 0$ in (9), the q-binomial formula in (4) is reached.

- For $x = 1$, $y = -ab$ and $z = b$ in (9); the identity (see [2], p. 25)

$$(1-ab)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q b^k (1-a)_q^k (1-b)_q^{n-k}$$

  is stated.

Theorem 3. *For $x, y, z \in \mathbb{C}$, the following equations hold*

$$\frac{(x+y)_q^\infty}{(z+y)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(x-z)_q^k}{(1-q)^k} \frac{1}{z^k} = e_q^{\frac{(x-z)_q}{(1-q)z}}, \tag{10}$$

and

$$\frac{(x+y)_q^\infty}{(x-z)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(z+y)_q^k}{(1-q)^k} \frac{1}{x^k} = e_q^{\frac{(z+y)_q}{(1-q)x}}. \tag{11}$$

Proof. As $n \to \infty$ in equation (8), it is arrived at

$$\begin{aligned}
(x+y)_q^\infty &= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k \, (z+y)_q^{n-k} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k \frac{(z+y)_q^n}{(z+yq^{n-k})_q^k} \\
&= \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{1}{(1-q)^k} (x-z)_q^k \frac{(z+y)_q^\infty}{z^k}.
\end{aligned}$$

Dividing both sides of the last equation by $(z+y)_q^\infty$ gives

$$\frac{(x+y)_q^\infty}{(z+y)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(x-z)_q^k}{(1-q)^k} \frac{1}{z^k}.$$

By using Theorem 1, the right hand side of the previous equation can be re-written as $e_q^{\frac{(x-z)_q}{(1-q)z}}$ which completes the proof of equation (10). In a similar manner, the latter can be proven. $\square$

Example 1. *If we take $x = 1$ and $y = -az$ in equation (11), we will get (see [2], p. 8)*

$$\frac{(1-az)_q^\infty}{(1-z)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(z-az)_q^k}{(1-q)^k} = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} z^k = {}_1\phi_0\left(a; -; q, z\right).$$

The function on the right hand side of the above equation is called *basic hypergeometric series* and more details about it can be found in [2].

Now we concentrate about the q-exponential functions. At first, product of the q-exponential functions is given in the next theorem and then some properties of the q-exponential functions are derived.

Remark 1. *For $|x| < 1$ and $|q| < 1$, the following identity holds*

$$\frac{(1-y)_q^\infty}{(1-x)_q^\infty} = \sum_{k=0}^{\infty} \frac{(x-y)_q^k}{(1-q)_q^k}. \tag{12}$$

Theorem 4. *For $x, y, z \in \mathbb{C}$, the following identity holds*

$$e_q^{(x+y)_q} = e_q^{(x-z)_q} \; e_q^{(z+y)_q}. \tag{13}$$

Proof. The identity (7) is taken to expand the q-exponential functions on the right hand side of (13), and thus

$$\begin{aligned}
e_q^{(x-z)_q} \; e_q^{(z+y)_q} &= \Big(\sum_{n=0}^{\infty} \frac{1}{[n]_q!} (x-z)_q^n\Big)\Big(\sum_{n=0}^{\infty} \frac{1}{[n]_q!} (z+y)_q^n\Big) \\
&= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} (x+y)_q^n = e_q^{(x+y)_q}.
\end{aligned}$$

$\square$

Corollary 1. *For $x, z, \in \mathbb{C}$, the following identity holds*

$$e_q^{-(x+z)_q} = \frac{1}{e_q^{(z+x)_q}}.$$

Proof. By taking $y := -x$ and $z := -z$ in Theorem 4, the requirement can be easily carried out. □

Theorem 5. *For $x \in \mathbb{C}$ and $m, n \in \mathbb{Z}$, the following identity*

$$e_q^{(m-n)_q x} = \begin{cases} \prod_{j=n}^{m-1} e_q^{((j+1)-j)_q x} & if\ m > n \\ \prod_{j=m}^{n-1} e_q^{(j-(j+1))_q x} & if\ m < n \end{cases}$$

holds.

Proof. First, consider the case of $m > n$. The theorem is proven by induction. For the basis step, $m = n + 1$, the theorem is valid. Take the case $m = k$, $k > n$. Then it needs to be proven that it holds for the case $m = k + 1$. By using identity (13) and the induction, it can be reached

$$e_q^{((k+1)-n)_q x} = e_q^{((k+1)-k)_q x} e_q^{(k-n)_q x} = e_q^{((k+1)-k)_q x} \prod_{j=n}^{k-1} e_q^{((j+1)-j)_q x} = \prod_{j=n}^{k} e_q^{((j+1)-j)_q x}$$

which completes the proof of the first part.
For the case of $m < n$, Corollary 1 is used. Then the result of the first part is applied to get

$$e_q^{(m-n)_q x} = \frac{1}{e_q^{(n-m)_q (x)}} = \frac{1}{\prod_{j=m}^{n-1} e_q^{((j+1)-j)_q x}} = \prod_{j=m}^{n-1} e_q^{(j-(j+1))_q x}$$

which completes the proof. □

Corollary 2. *For $x \in \mathbb{C}$, and positive integers m and n, the following identities hold:*

$$e_q^{mx} = \prod_{j=0}^{m-1} e_q^{((j+1)-j)_q x}, \tag{14}$$

$$E_q^{-nx} = \prod_{j=0}^{n-1} e_q^{(j-(j+1))_q x} \tag{15}$$

Proof. Consideration of (7) with $n = 0$ and m any positive integer in Theorem 5 leads to the complete proof of the first identity. Replacing m and n values between each other in the first identity gives the proof of the second one. □

Now then, the n-*th* q-derivative of the q-exponential functions is found in the next theorem.

Theorem 6. *For $\alpha, \beta, x \in \mathbb{C}$ and positive integer n,*

$$D_q^n e_q^{(\alpha+\beta)_q x} = (\alpha+\beta)_q^n \ e_q^{(\alpha+q^n\beta)_q x}. \tag{16}$$

Proof. We use the induction to prove the theorem. For the case of $n = 1$, we need to get the q-derivative of $e_q^{(\alpha+\beta)_q x}$. So we use equation (7) and then take the q-derivative to obtain

$$D_q e_q^{(\alpha+\beta)_q x} = D_q\Big(\sum_{k=0}^{\infty} \frac{1}{[k]_q!}(\alpha+\beta)_q^k x^k\Big) = (\alpha+\beta)\sum_{k=0}^{\infty} \frac{1}{[k]_q!}(\alpha+q\beta)_q^k x^k = (\alpha+\beta)\ e_q^{(\alpha+q\beta)_q x}.$$

Assuming that (16) holds for a given k and to prove that it holds for $k + 1$, we need to obtain the q-derivative of $D_q^k e_q^{(\alpha+\beta)_q x}$. Hence

$$D_q^{k+1} e_q^{(\alpha+\beta)_q x} = D_q\Big(D_q^k e_q^{(\alpha+\beta)_q x}\Big) = (\alpha+\beta)_q^k\ D_q\Big(e_q^{(\alpha+q^k\beta)_q x}\Big) = (\alpha+\beta)_q^{k+1}\ e_q^{(\alpha+q^{k+1}\beta)_q x}.$$

Thus the proof is complete. □

Theorem 7. *For $|x| < 1$, $|q| < 1$ and any arbitrary α, the following identity holds*

$$e_q^{(1-q^\alpha)_q x} = \frac{1}{(1-(1-q)x)_q^\alpha} \tag{17}$$

Proof. To prove the theorem, we use equations (3), (5), (6) and (7). Then we have

$$e_q^{(1-q^\alpha)_q x} = e_q^x \; E_q^{-q^\alpha x} = \frac{1}{(1-(1-q)x)_q^\infty} \; (1-(1-q)q^\alpha x)_q^\infty = \frac{1}{(1-(1-q)x)_q^\alpha}$$

which completes the proof. □

Remark 2. *Equation (17) can be rewritten as* $e_q^{(q^\alpha-1)_q x} = (1-(1-q)x)_q^\alpha$.

In the next example, we show that the q-binomial theorem (see: [1] P. 247 or [9] P. 488) can be proven shortly by using Theorem 1.

Example 2. *For $|x| < 1$ and $|q| < 1$,*

$$\sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} x^k = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{[k]_q!} \Big(\frac{x}{1-q}\Big)^k = e_q^{\frac{(1-a)_q x}{(1-q)}} = e_q^{(\frac{x}{1-q})} E_q^{(\frac{-ax}{1-q})} = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty}.$$

Note that to reach this result; (7) in the second and third equations, and (5) and (6) in the last equation have been considered.

3 Conclusions and Recommendation

Some striking properties of the q-exponential functions have been analyzed in detail. In doing so, the Gauss q-binomial identity has generalized and based on it, remarkable properties of the q-exponential have been established. For further studies, similar discussion can be carried out for q-trigonometric functions.

4 Acknowledgment

I am thankful to Dr. M. Sari of Yildiz Technical University for patiently helping, advising me and also spending his valuable time to revise the paper.

References

[1] Ernst, T., *A Comprehensive Treatment of Q-calculus*, Springer, 2012.

[2] Gasper, G., Rahman, M., *Basic hypergeometric series*, Vol. 96 Cambridge university press, 2004.

[3] Kac, V., Cheung, P., *Quantum Calculus*, Springer, 2002.

[4] Gould, H. W., The q-Series Generalization of a Formula of Sparre Andersen. *Mathematica Scandinavica*, Vol. 9, pp. 90–94, 1961.

[5] Exton, H., *q-Hypergeometric Functions and ApplicationsEllis*, Horwood, Chichester, 1983.

[6] Jackson, F. H., A basic-sine and cosine with symbolical solutions of certain differential equations. *Proceedings of the Edinburgh Mathematical Society*, Vol. 22, pp. 28–39, 1904.

[7] Hahn, W., Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der LaplaceTransformation, *Mathematische Nachrichten*, Vol. 2, no. 6, pp. 340–379, 1949.

[8] Jackson, F. H., On basic double hypergeometric functions, *The Quarterly Journal of Mathematics*, Vol. 1, pp. 69–82, 1942.

[9] Andrews, G. E., Askey, R., Roy, R., *Special functions*, Vol. 71, Cambridge university press, 1999.

BCI-implicative ideals of BCI-algebras using neutrosophic quadruple structure

Young Bae Jun[1] , Seok-Zun Song[2,*] and and G. Muhiuddin [3]

[1]*Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*
[2] *Department of Mathematics, Jeju National University, Jeju 63243, Korea*
[3] *Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia*

Abstract. Neutrosophic quadruple structure is used to study BCI-implicative ideal in BCI-algebra. The conceot of neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra is introduced, and their related properties are investigated. Relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal are consulted. Conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal are provided. A characterization of a neutrosophic quadruple BCI-implicative ideal is displayed, and the extension property of neutrosophic quadruple BCI-implicative ideal is established.

1. Introduction

In [14], Smarandche has introduced the neutrosophic quadruple numbers for the first time. Using the notion of Smarandache's neutrosophic quadruple numbers, Akinleye et al. [2] presented the notion of neutrosophic quadruple algebraic structures. In particular, they studied neutrosophic quadruple rings. Agboola et al. [1] studied neutrosophic quadruple algebraic hyperstructures, in particular, they developed neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings. Using BCK/BCI-algebras, Jun et al. [7] have established neutrosophic quadruple BCK/BCI-algebra, and have studied neutrosophic quadruple (positive implicative) ideal in neutrosophic quadruple BCK-algebra and neutrosophic quadruple closed ideal in neutrosophic quadruple BCI-algebra. Muhiuddin et al. [13] have studied neutrosophic quadruple q-ideal and (regular) neutrosophic quadruple ideal in neutrosophic quadruple BCI-algebra. Muhiuddin et al. [12] also have studied implicative neutrosophic quadruple ideal in neutrosophic quadruple BCK-algebra.

In this article, we study BCI-implicative ideal in BCI-algebra using neutrosophic quadruple structure. We define neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra, and investigate their related properties. We consult relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal. We provide conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal. We discuss a characterization of an neutrosophic quadruple BCI-implicative ideal, and establish the extension property of neutrosophic quadruple BCI-implicative ideal.

[0]**2010 Mathematics Subject Classification**: 06F35, 03G25, 08A72.

[0]**Keywords**: neutrosophic quadruple BCK/BCI-algebra; neutrosophic quadruple BCI-implicative ideal; neutrosophic quadruple BCI-positive implicative ideal; neutrosophic quadruple BCI-commutative ideal.

* The corresponding author.

[0]**E-mail** : skywine@gmail.com (Y. B. Jun); szsong@jejunu.ac.kr (S. Z. Song); chishtygm@gmail.com (G. Muhiuddin)

2. **Preliminaries**

A BCK/BCI-algebra, which is an important class of logical algebras, is introduced by K. Iséki (see [4, 5]) and it is being studied by many researchers.

A *BCI-algebra* is a set X with a binary operation "$\cdot$" and a special element "0" that satisfies the following conditions:

(I) $(\forall x, y, z \in X)$ $(((x \cdot y) \cdot (x \cdot z)) \cdot (z \cdot y) = 0)$,

(II) $(\forall x, y \in X)$ $((x \cdot (x \cdot y)) \cdot y = 0)$,

(III) $(\forall x \in X)$ $(x \cdot x = 0)$,

(IV) $(\forall x, y \in X)$ $(x \cdot y = 0,\, y \cdot x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 \cdot x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x \cdot 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \le y \ \Rightarrow\ x \cdot z \le y \cdot z,\ z \cdot y \le z \cdot x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x \cdot y) \cdot z = (x \cdot z) \cdot y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x \cdot z) \cdot (y \cdot z) \le x \cdot y) \tag{2.4}$$

where $x \le y$ if and only if $x \cdot y = 0$.

Any BCI-algebra X satisfies the following conditions (see [3]):

$$(\forall x, y \in X)(x \cdot (x \cdot (x \cdot y)) = x \cdot y), \tag{2.5}$$

$$(\forall x, y \in X)(0 \cdot (x \cdot y) = (0 \cdot x) \cdot (0 \cdot y)), \tag{2.6}$$

$$(\forall x, y \in X)(0 \cdot (0 \cdot (x \cdot y)) = (0 \cdot y) \cdot (0 \cdot x)). \tag{2.7}$$

An element a in a BCI-algebra X is said to be *minimal* (see [3]) if the following assertion is valid.

$$(\forall x \in X)(x \le a \ \Rightarrow\ x = a). \tag{2.8}$$

Note that the zero element 0 in a BCI-algebra X is minimal (see [3]).

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x \cdot y \in S$ for all $x, y \in S$. A subset G of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in G, \tag{2.9}$$

$$(\forall x \in X)\,(\forall y \in G)\,(x \cdot y \in G \ \Rightarrow\ x \in G)\,. \tag{2.10}$$

A subset G of a BCI-algebra X is called

- a *closed ideal* of X (see [3]) if it is an ideal of X which satisfies:

$$(\forall x \in X)(x \in G \ \Rightarrow\ 0 \cdot x \in G), \tag{2.11}$$

- a *BCI-positive implicative ideal* of X (see [8, 9]) if it satisfies (2.9) and

$$(\forall x, y, z \in X)\,(((x \cdot z) \cdot z) \cdot (y \cdot z) \in G,\, y \in G \ \Rightarrow\ x \cdot z \in G)\,, \tag{2.12}$$

- a *BCI-commutative ideal* of X (see [10]) if it satisfies (2.9) and
$$\begin{aligned}&(x\cdot y)\cdot z\in G,\ z\in G\\ &\Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in G\end{aligned}\tag{2.13}$$
for all $x,y,z\in X$,
- a *BCI-implicative ideal* of X (see [8]) if it satisfies (2.9) and
$$\begin{aligned}&(((x\cdot y)\cdot y)\cdot(0\cdot y))\cdot z\in G,\ z\in G\\ &\Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in G\end{aligned}\tag{2.14}$$
for all $x,y,z\in X$.

Note that every BCI-implicative ideal is an ideal, but the converse is not true (see [8]).

Lemma 2.1 ([8]). *A subset K of X is a BCI-implicative ideal of a BCI-algebra X if and only if it is an ideal of X that satisfies the following condition.*
$$((x\cdot y)\cdot y)\cdot(0\cdot y)\in K\ \Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K\tag{2.15}$$
for all $x,y\in X$.

Lemma 2.2 ([10]). *An ideal K of X is a BCI-commutative ideal of X if and only if it satisfies:*
$$x\cdot y\in K\ \Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K\tag{2.16}$$
for all $x,y,z\in X$.

Lemma 2.3 ([9]). *An ideal K of X is a BCI-positive implicative ideal of X if and only if it satisfies:*
$$((x\cdot y)\cdot y)\cdot(0\cdot y)\in K\ \Rightarrow\ x\cdot y\in K\tag{2.17}$$
for all $x,y,z\in X$.

We refer the reader to the books [3, 11] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let X be a set. A *neutrosophic quadruple X-number* is an ordered quadruple (a,xT,yI,zF) where $a,x,y,z\in X$ and T, I, F have their usual neutrosophic logic meanings (see [7]).

The set of all neutrosophic quadruple X-numbers is denoted by $N_q(X)$, that is,
$$N_q(X):=\{(a,xT,yI,zF)\mid a,x,y,z\in X\},$$
and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $N_q(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let X be a BCK/BCI-algebra. We define a binary operation $\boxdot$ on $N_q(X)$ by
$$(a,xT,yI,zF)\boxdot(b,uT,vI,wF)=(a\cdot b,(x\cdot u)T,(y\cdot v)I,(z\cdot w)F)$$
for all (a,xT,yI,zF), $(b,uT,vI,wF)\in N_q(X)$. Given $a_1,a_2,a_3,a_4\in X$, the neutrosophic quadruple BCK/BCI-number (a_1,a_2T,a_3I,a_4F) is denoted by $\tilde{a}$, that is,
$$\tilde{a}=(a_1,a_2T,a_3I,a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

Then $(N_q(X); \boxdot, \tilde{0})$ is a BCK/BCI-algebra (see [7]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $N_q(X)$.

We define an order relation "$\ll$" and the equality "$=$" on $N_q(X)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in N_q(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $N_q(X)$.

Let X be a BCK/BCI-algebra. Given nonempty subsets K and J of X, consider the set

$$N_q(K, J) := \{(a, xT, yI, zF) \in N_q(X) \mid a, x \in K \ \& \ y, z \in J\},$$

which is called the neutrosophic quadruple set based on K and J.

The set $N_q(K, K)$ is denoted by $N_q(K)$, and it is called the neutrosophic quadruple set based on K.

3. Neutrosophic quadruple BCI-implicative ideals

In what follows, let X denote a BCI-algebra unless otherwise specified.

Definition 3.1. Let K and J be nonempty subsets of X. Then the neutrosophic quadruple set based on K and J is called a *neutrosophic quadruple BCI-implicative ideal* (briefly, *NQ-BCI-implicative ideal*) over (X, K, J) if it is a BCI-implicative ideal of $N_q(X)$. If $K = J$, then we say that it is an *NQ-BCI-implicative ideal* over (X, K).

Example 3.2. Consider a BCI-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the binary operation $\cdot$, which is given in Table 1.

Table 1. Cayley table for the binary operation "$\cdot$"

$\cdot$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 6^4 elements. If we take $K = \{0, 1, 2\}$, then the neutrosophic quadruple set based on K has 81-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\zeta}_i \mid i = 1, 2, \cdots, 80\},$$

and it is an NQ-BCI-implicative ideal over (X, K) where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\zeta}_1 = (0, 0T, 0I, 1F)$, $\tilde{\zeta}_2 = (0, 0T, 0I, 2F)$,
$\tilde{\zeta}_3 = (0, 0T, 1I, 0F)$, $\tilde{\zeta}_4 = (0, 0T, 1I, 1F)$, $\tilde{\zeta}_5 = (0, 0T, 1I, 2F)$,
$\tilde{\zeta}_6 = (0, 0T, 2I, 0F)$, $\tilde{\zeta}_7 = (0, 0T, 2I, 1F)$, $\tilde{\zeta}_8 = (0, 0T, 2I, 2F)$,
$\tilde{\zeta}_9 = (0, 1T, 0I, 0F)$, $\tilde{\zeta}_{10} = (0, 1T, 0I, 1F)$, $\tilde{\zeta}_{11} = (0, 1T, 0I, 2F)$,

$\tilde{\zeta}_{12} = (0, 1T, 1I, 0F)$, $\tilde{\zeta}_{13} = (0, 1T, 1I, 1F)$, $\tilde{\zeta}_{14} = (0, 1T, 1I, 2F)$,
$\tilde{\zeta}_{15} = (0, 1T, 2I, 0F)$, $\tilde{\zeta}_{16} = (0, 1T, 2I, 1F)$, $\tilde{\zeta}_{17} = (0, 1T, 2I, 2F)$,
$\tilde{\zeta}_{18} = (0, 2T, 0I, 0F)$, $\tilde{\zeta}_{19} = (0, 2T, 0I, 1F)$, $\tilde{\zeta}_{20} = (0, 2T, 0I, 2F)$,
$\tilde{\zeta}_{21} = (0, 2T, 1I, 0F)$, $\tilde{\zeta}_{22} = (0, 2T, 1I, 1F)$, $\tilde{\zeta}_{23} = (0, 2T, 1I, 2F)$,
$\tilde{\zeta}_{24} = (0, 2T, 2I, 0F)$, $\tilde{\zeta}_{25} = (0, 2T, 2I, 1F)$, $\tilde{\zeta}_{26} = (0, 2T, 2I, 2F)$,
$\tilde{\zeta}_{27} = (1, 0T, 0I, 0F)$, $\tilde{\zeta}_{28} = (1, 0T, 0I, 1F)$, $\tilde{\zeta}_{29} = (1, 0T, 0I, 2F)$,
$\tilde{\zeta}_{30} = (1, 0T, 1I, 0F)$, $\tilde{\zeta}_{31} = (1, 0T, 1I, 1F)$, $\tilde{\zeta}_{32} = (1, 0T, 1I, 2F)$,
$\tilde{\zeta}_{33} = (1, 0T, 2I, 0F)$, $\tilde{\zeta}_{34} = (1, 0T, 2I, 1F)$, $\tilde{\zeta}_{35} = (1, 0T, 2I, 2F)$,
$\tilde{\zeta}_{36} = (1, 1T, 0I, 0F)$, $\tilde{\zeta}_{37} = (1, 1T, 0I, 1F)$, $\tilde{\zeta}_{38} = (1, 1T, 0I, 2F)$,
$\tilde{\zeta}_{39} = (1, 1T, 1I, 0F)$, $\tilde{\zeta}_{40} = (1, 1T, 1I, 1F)$, $\tilde{\zeta}_{41} = (1, 1T, 1I, 2F)$,
$\tilde{\zeta}_{42} = (1, 1T, 2I, 0F)$, $\tilde{\zeta}_{43} = (1, 1T, 2I, 1F)$, $\tilde{\zeta}_{44} = (1, 1T, 2I, 2F)$,
$\tilde{\zeta}_{45} = (1, 2T, 0I, 0F)$, $\tilde{\zeta}_{46} = (1, 2T, 0I, 1F)$, $\tilde{\zeta}_{47} = (1, 2T, 0I, 2F)$,
$\tilde{\zeta}_{48} = (1, 2T, 1I, 0F)$, $\tilde{\zeta}_{49} = (1, 2T, 1I, 1F)$, $\tilde{\zeta}_{50} = (1, 2T, 1I, 2F)$,
$\tilde{\zeta}_{51} = (1, 2T, 2I, 0F)$, $\tilde{\zeta}_{52} = (1, 2T, 2I, 1F)$, $\tilde{\zeta}_{53} = (1, 2T, 2I, 2F)$,
$\tilde{\zeta}_{54} = (2, 0T, 0I, 0F)$, $\tilde{\zeta}_{55} = (2, 0T, 0I, 1F)$, $\tilde{\zeta}_{56} = (2, 0T, 0I, 2F)$,
$\tilde{\zeta}_{57} = (2, 0T, 1I, 0F)$, $\tilde{\zeta}_{58} = (2, 0T, 1I, 1F)$, $\tilde{\zeta}_{59} = (2, 0T, 1I, 2F)$,
$\tilde{\zeta}_{60} = (2, 0T, 2I, 0F)$, $\tilde{\zeta}_{61} = (2, 0T, 2I, 1F)$, $\tilde{\zeta}_{62} = (2, 0T, 2I, 2F)$,
$\tilde{\zeta}_{63} = (2, 1T, 0I, 0F)$, $\tilde{\zeta}_{64} = (2, 1T, 0I, 1F)$, $\tilde{\zeta}_{65} = (2, 1T, 0I, 2F)$,
$\tilde{\zeta}_{66} = (2, 1T, 1I, 0F)$, $\tilde{\zeta}_{67} = (2, 1T, 1I, 1F)$, $\tilde{\zeta}_{68} = (2, 1T, 1I, 2F)$,
$\tilde{\zeta}_{69} = (2, 1T, 2I, 0F)$, $\tilde{\zeta}_{70} = (2, 1T, 2I, 1F)$, $\tilde{\zeta}_{71} = (2, 1T, 2I, 2F)$,
$\tilde{\zeta}_{72} = (2, 2T, 0I, 0F)$, $\tilde{\zeta}_{73} = (2, 2T, 0I, 1F)$, $\tilde{\zeta}_{74} = (2, 2T, 0I, 2F)$,
$\tilde{\zeta}_{75} = (2, 2T, 1I, 0F)$, $\tilde{\zeta}_{76} = (2, 2T, 1I, 1F)$, $\tilde{\zeta}_{77} = (2, 2T, 1I, 2F)$,
$\tilde{\zeta}_{78} = (2, 2T, 2I, 0F)$, $\tilde{\zeta}_{79} = (2, 2T, 2I, 1F)$, $\tilde{\zeta}_{80} = (2, 2T, 2I, 2F)$.

Theorem 3.3. *Every NQ-BCI-implicative ideal is a neutrosophic quadruple ideal.*

Proof. It is straightforward since every BCI-implicative ideal is an ideal in BCI-algebras. □

The converse of Theorem 3.3 is not true in general as seen in the following example.

Example 3.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $\cdot$, which is given in Table 2.

TABLE 2. Cayley table for the binary operation "·"

$\cdot$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	0	4
2	2	2	0	0	4
3	3	3	2	0	4
4	4	4	4	4	0

Then X is a BCI-algebra (see [8]), and the neutrosophic quadruple BCI-algebra $N_q(X)$ has 625 elements. If we take $K = \{0, 1\}$, then the neutrosophic quadruple set based on K has 16-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\zeta}_i \mid i = 1, 2, \cdots, 15\},$$

and it is a neutrosophic quadruple ideal over (X, K) where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\zeta}_1 = (0, 0T, 0I, 1F)$,
$\tilde{\zeta}_2 = (0, 0T, 1I, 0F)$, $\tilde{\zeta}_3 = (0, 0T, 1I, 1F)$,
$\tilde{\zeta}_4 = (0, 1T, 0I, 0F)$, $\tilde{\zeta}_5 = (0, 1T, 0I, 1F)$,
$\tilde{\zeta}_6 = (0, 1T, 1I, 0F)$, $\tilde{\zeta}_7 = (0, 1T, 1I, 1F)$,
$\tilde{\zeta}_8 = (1, 0T, 0I, 0F)$, $\tilde{\zeta}_9 = (1, 0T, 0I, 1F)$,
$\tilde{\zeta}_{10} = (1, 0T, 1I, 0F)$, $\tilde{\zeta}_{11} = (1, 0T, 1I, 1F)$,
$\tilde{\zeta}_{12} = (1, 1T, 0I, 0F)$, $\tilde{\zeta}_{13} = (1, 1T, 0I, 1F)$,
$\tilde{\zeta}_{14} = (1, 1T, 1I, 0F)$, $\tilde{\zeta}_{15} = (1, 1T, 1I, 1F)$.

If we take $\tilde{x} = (2, 2T, 2I, 2F)$ and $\tilde{y} = (3, 3T, 3I, 3F)$ in $N_q(X)$, then

$$\begin{aligned}
&(((\tilde{y} \boxdot \tilde{x}) \boxdot \tilde{x}) \boxdot (\tilde{0} \boxdot \tilde{x})) \boxdot \tilde{0} \\
&= ((((3, 3T, 3I, 3F) \boxdot (2, 2T, 2I, 2F)) \boxdot (2, 2T, 2I, 2F)) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot (2, 2T, 2I, 2F))) \boxdot (0, 0T, 0I, 0F) \\
&= (0, 0T, 0I, 0F) \in N_q(K).
\end{aligned}$$

But

$$\begin{aligned}
&\tilde{y} \boxdot ((\tilde{x} \boxdot (\tilde{x} \boxdot \tilde{y})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{y} \boxdot \tilde{x})))) \\
&= (3, 3T, 3I, 3F) \boxdot (((2, 2T, 2I, 2F) \boxdot ((2, 2T, 2I, 2F) \boxdot (3, 3T, 3I, 3F))) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot ((3, 3T, 3I, 3F) \boxdot (2, 2T, 2I, 2F))))) \\
&= (2, 2T, 2I, 2F) \notin N_q(K).
\end{aligned}$$

Hence $N_q(K)$ is not a BCI-implicative ideal of $N_q(X)$, and so it is not an NQ-BCI-implicative ideal over (X, K).

Lemma 3.5 ([7]). *If K and J are ideals of X, then the neutrosophic quadruple set based on K and J is a neutrosophic quadruple ideal over (X, K, J).*

Theorem 3.6. *The neutrosophic quadruple set based on BCI-implicative ideals K and J of X is an NQ-BCI-implicative ideal over (X, K, J).*

Proof. Let K and J be BCI-implicative ideals of X. Since $0 \in K \cap J$, we get $\tilde{0} \in N_q(K, J)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $N_q(X)$ such that

$$(((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot \tilde{z} \in N_q(K, J)$$

and $\tilde{z} \in N_q(K, J)$. Then $\tilde{z} = (z_1,\, z_2T,\, z_3I,\, z_4F) \in N_q(K, J)$ and

$$\begin{aligned}
&(((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot \tilde{z}\\
&= ((((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot\\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F))) \boxdot (z_1, z_2T, z_3I, z_4F)\\
&= (((((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1)) \cdot z_1), ((((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2)) \cdot z_2)T,\\
&((((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3)) \cdot z_3)I, ((((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4)) \cdot z_4)F)\\
&\in N_q(K, J).
\end{aligned}$$

Hence $z_i \in K$ and $(((x_i \cdot y_i) \cdot y_i) \cdot (0 \cdot y_i)) \cdot z_i \in K$ for $i = 1, 2$; and $z_j \in J$ and $(((x_j \cdot y_j) \cdot y_j) \cdot (0 \cdot y_j)) \cdot z_j \in K$ for $j = 3, 4$. Since K and J are BCI-implicative ideals of X, it follows that $x_i \cdot ((y_i \cdot (y_i \cdot x_i)) \cdot (0 \cdot (0 \cdot (x_i \cdot y_i)))) \in K$ and $x_j \cdot ((y_j \cdot (y_j \cdot x_j)) \cdot (0 \cdot (0 \cdot (x_j \cdot y_j)))) \in J$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\begin{aligned}
&\tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \cdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y}))))\\
&= (x_1, x_2T, x_3I, x_4F) \cdot (((y_1, y_2T, y_3I, y_4F) \cdot ((y_1, y_2T, y_3I, y_4F) \cdot\\
&(x_1, x_2T, x_3I, x_4F))) \cdot ((0, 0T, 0I, 0F) \cdot ((0, 0T, 0I, 0F) \cdot\\
&((x_1, x_2T, x_3I, x_4F) \cdot (y_1, y_2T, y_3I, y_4F)))))\\
&= (x_1 \cdot ((y_1 \cdot (y_1 \cdot x_1)) \cdot (0 \cdot (0 \cdot (x_1 \cdot y_1)))),\\
&(x_2 \cdot ((y_2 \cdot (y_2 \cdot x_2)) \cdot (0 \cdot (0 \cdot (x_2 \cdot y_2)))))T,\\
&(x_3 \cdot ((y_3 \cdot (y_3 \cdot x_3)) \cdot (0 \cdot (0 \cdot (x_3 \cdot y_3)))))I,\\
&(x_4 \cdot ((y_4 \cdot (y_4 \cdot x_4)) \cdot (0 \cdot (0 \cdot (x_4 \cdot y_4)))))F)\\
&\in N_q(K, J).
\end{aligned}$$

Hence $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J). □

Corollary 3.7. *The neutrosophic quadruple set based on a BCI-implicative ideal K of X is an NQ-BCI-implicative ideal over (X, K).*

Proposition 3.8. *Every neutrosophic quadruple set based on BCI-implicative ideals K and J of X satisfies the following condition.*

$$\begin{aligned}
&((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J)\\
&\Rightarrow \tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y})))) \in N_q(K, J).
\end{aligned} \tag{3.1}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$.

Proof. Let $((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J)$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$. Then

$$
\begin{aligned}
&((((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1)) \cdot 0, ((((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2)) \cdot 0)T, \\
&((((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3)) \cdot 0)I, ((((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4)) \cdot 0)F) \\
&= (((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1), (((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2))T, \\
&(((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3))I, (((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4))F) \\
&= (((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F)) \\
&= ((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J),
\end{aligned}
$$

and so $(((x_i \cdot y_i) \cdot y_i) \cdot (0 \cdot y_i)) \cdot 0 \in K$ for $i = 1, 2$ and $(((x_j \cdot y_j) \cdot y_j) \cdot (0 \cdot y_j)) \cdot 0 \in J$ for $j = 3, 4$. Since $0 \in K \cap J$, and since K and J are BCI-implicative ideals of X, it follows that $x_i \cdot ((y_i \cdot (y_i \cdot x_i)) \cdot (0 \cdot (0 \cdot (x_i \cdot y_i)))) \in K$ for $i = 1, 2$, and $x_j \cdot ((y_j \cdot (y_j \cdot x_j)) \cdot (0 \cdot (0 \cdot (x_j \cdot y_j)))) \in J$ for $j = 3, 4$. Hence we have

$$
\begin{aligned}
&\tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y})))) \\
&= (x_1 \cdot ((y_1 \cdot (y_1 \cdot x_1)) \cdot (0 \cdot (0 \cdot (x_1 \cdot y_1)))), \\
&(x_2 \cdot ((y_2 \cdot (y_2 \cdot x_2)) \cdot (0 \cdot (0 \cdot (x_2 \cdot y_2)))))T, \\
&(x_3 \cdot ((y_3 \cdot (y_3 \cdot x_3)) \cdot (0 \cdot (0 \cdot (x_3 \cdot y_3)))))I, \\
&(x_4 \cdot ((y_4 \cdot (y_4 \cdot x_4)) \cdot (0 \cdot (0 \cdot (x_4 \cdot y_4)))))F) \\
&\in N_q(K, J).
\end{aligned}
$$

This completes the proof. □

We provide conditions for a neutrosophic quadruple set to be an NQ-BCI-implicative ideal.

Theorem 3.9. *Let K and J be ideals of X such that*

$$
\begin{aligned}
&((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J) \\
&\Rightarrow\ x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J)
\end{aligned}
\tag{3.2}
$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Assume that $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z \in K$ (resp., J) for all $x, y \in X$ and $z \in K$ (resp., J). Then $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) since K and J are ideals of X. It follows from the condition (3.2) that $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J). Hence K and J are BCI-implicative ideals of X, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 3.10. *Let K be an ideal of X such that*

$$
\begin{aligned}
&((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \\
&\Rightarrow\ x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K
\end{aligned}
\tag{3.3}
$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X, K).

Theorem 3.11. *Let K and J be ideals of X such that*

$$0 \cdot x \in K \text{ (resp., } J), \tag{3.4}$$

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J) \;\Rightarrow\; x \cdot (y \cdot (y \cdot x)) \in K \text{ (resp., } J) \tag{3.5}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Assume that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) for all $x, y \in X$. Then $x \cdot (y \cdot (y \cdot x)) \in K$ (resp., J) by (3.5). Using (I), (II), (2.3), (2.5), (2.6) and (3.4), we have

$$\begin{aligned}
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y))))) \cdot (x \cdot (y \cdot (y \cdot x))) \\
&\leq (y \cdot (y \cdot x)) \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \\
&\leq 0 \cdot (0 \cdot (x \cdot y)) \\
&= 0 \cdot ((0 \cdot x) \cdot (0 \cdot y)) \\
&= 0 \cdot ((((0 \cdot y) \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y)) \\
&= 0 \cdot ((((0 \cdot (0 \cdot (0 \cdot y))) \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y)) \\
&= 0 \cdot ((((0 \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y)) \cdot (0 \cdot (0 \cdot y))) \\
&= 0 \cdot (((0 \cdot (x \cdot y)) \cdot (0 \cdot y)) \cdot (0 \cdot (0 \cdot y))) \\
&= 0 \cdot (0 \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y))) \\
&\in K \text{ (resp., } J).
\end{aligned}$$

It follows that $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J). Hence K and J are BCI-implicative ideals of X by Lemma 2.1. Therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 3.12. *Let K be an ideal of X such that*

$$0 \cdot x \in K, \tag{3.6}$$

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \;\Rightarrow\; x \cdot (y \cdot (y \cdot x)) \in K \tag{3.7}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X, K).

Theorem 3.13. *Let X be a BCI-algebra satisfying the conditions:*

$$(\forall x, y \in X)(x \cdot y = ((x \cdot y) \cdot y) \cdot (0 \cdot y)), \tag{3.8}$$

$$(\forall x, y \in X)(x \cdot (x \cdot y) = y \cdot (y \cdot (x \cdot (x \cdot y)))). \tag{3.9}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Let K and J be closed ideals of X. Assume that $((x\cdot y)\cdot y)\cdot(0\cdot y) \in K$ (resp., J). Then $0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)) \in K$ (resp., J). Using the conditions (3.8), (3.9), (2.3), (2.5), (I) and (III), we have

$$\begin{aligned}
&(x\cdot(y\cdot(y\cdot x)))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&=(x\cdot(y\cdot(y\cdot x)))\cdot(x\cdot y)\\
&=(x\cdot(x\cdot y))\cdot(y\cdot(y\cdot x))\\
&=(y\cdot(y\cdot(x\cdot(x\cdot y))))\cdot(y\cdot(y\cdot x))\\
&=(y\cdot(y\cdot(y\cdot x)))\cdot(y\cdot(x\cdot(x\cdot y)))\\
&=(y\cdot x)\cdot(y\cdot(x\cdot(x\cdot y)))\\
&\leq(x\cdot(x\cdot y))\cdot x\\
&=0\cdot(x\cdot y)\\
&=0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&\in K \text{ (resp., } J).
\end{aligned} \tag{3.10}$$

It follows that $x\cdot(y\cdot(y\cdot x)) \in K$ (resp., J), and so that

$$x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y)))) \in K \text{ (resp., } J)$$

in the proof of Theorem 3.18. Thus K and J are BCI-implicative ideals of X by Lemma 2.1, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J) by Theorem 3.6. □

Corollary 3.14. *Let X be a BCI-algebra satisfying the conditions* (3.8) *and* (3.9)*. If K is a closed ideal of X, then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X,K).*

Corollary 3.15. *Let X be a BCI-algebra satisfying the condition:*

$$(\forall x,y\in X)((x\cdot(x\cdot y))\cdot(y\cdot x)=y\cdot(y\cdot x)). \tag{3.11}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J).

Proof. If X satisfies the condition (3.11), then it satisfies two conditions (3.8) and (3.9) (see [**?**, **?**]). Hence the result is induced from Theorem 3.13. □

Corollary 3.16. *Let X be a BCI-algebra satisfying the condition* (3.11)*. If K is a closed ideal of X, then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X,K).*

Theorem 3.17. *Let X be a BCI-algebra satisfying the condition* (3.9) *and*

$$(\forall x,y\in X)((x\cdot(y\cdot x))\cdot(0\cdot(y\cdot x))=x). \tag{3.12}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J).

Proof. Let K and J be closed ideals of X. The conditions (3.12) and (III) lead to the following fact.

$$(z\cdot y)\cdot(((z\cdot y)\cdot(z\cdot(z\cdot y)))\cdot(0\cdot(z\cdot(z\cdot y))))=0. \tag{3.13}$$

It follows from (2.1), (I), (2.2), (2.3) and (III) that

$$\begin{aligned}
&(z \cdot y) \cdot (((z \cdot y) \cdot y) \cdot (0 \cdot y)) = ((z \cdot y) \cdot (((z \cdot y) \cdot y) \cdot (0 \cdot y))) \cdot 0 \\
&= ((z \cdot y) \cdot (((z \cdot y) \cdot y) \cdot (0 \cdot y))) \cdot ((z \cdot y) \cdot (((z \cdot y) \cdot (z \cdot (z \cdot y))) \cdot \\
&(0 \cdot (z \cdot (z \cdot y))))) \\
&\leq (((z \cdot y) \cdot (z \cdot (z \cdot y))) \cdot (0 \cdot (z \cdot (z \cdot y)))) \cdot (((z \cdot y) \cdot y) \cdot (0 \cdot y)) \\
&\leq (((z \cdot y) \cdot y) \cdot (0 \cdot (z \cdot (z \cdot y)))) \cdot (((z \cdot y) \cdot y) \cdot (0 \cdot y)) \\
&\leq (0 \cdot y) \cdot (0 \cdot (z \cdot (z \cdot y))) \\
&\leq (z \cdot (z \cdot y)) \cdot y \\
&= (z \cdot y) \cdot (z \cdot y) = 0.
\end{aligned} \tag{3.14}$$

Hence $(z \cdot y) \cdot (((z \cdot y) \cdot y) \cdot (0 \cdot y)) = 0$ since 0 is a minimal element of X, that is,

$$z \cdot y \leq ((z \cdot y) \cdot y) \cdot (0 \cdot y). \tag{3.15}$$

On the other hand, we get

$$\begin{aligned}
&(((z \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot (z \cdot y) = (((z \cdot y) \cdot y) \cdot (z \cdot y)) \cdot (0 \cdot y) \\
&= (((z \cdot y) \cdot (z \cdot y)) \cdot y) \cdot (0 \cdot y) = (0 \cdot y) \cdot (0 \cdot y) = 0
\end{aligned}$$

by (2.3) and (III), that is,

$$((z \cdot y) \cdot y) \cdot (0 \cdot y) \leq z \cdot y. \tag{3.16}$$

Conditions (3.15) and (3.16) induce

$$z \cdot y = ((z \cdot y) \cdot y) \cdot (0 \cdot y).$$

Therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.13. □

We now consider extension property of NQ-BCI-implicative ideal.

Theorem 3.18. *For any nonempty subsets K and J of X, let A and B be closed ideals of X such that $K \subseteq A$ and $J \subseteq B$. If K and J are BCI-implicative ideals of X, then the neutrosophic quadruple set based on A and B is an NQ-BCI-implicative ideal over (X, A, B), which is larger than the NQ-BCI-implicative ideal over (X, K, J).*

Proof. Assume that K and J are BCI-implicative ideals of X. It is clear that $N_q(K, J) \subseteq N_q(A, B)$. Let $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in A$ (resp., B) for all $x, y \in X$. Then $0 \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y)) \in A$ (resp., B) since A and B are closed ideals of X. Using (2.3) and (III) induce

$$\begin{aligned}
&(((x \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y))) \cdot y) \cdot y) \cdot (0 \cdot y) \\
&= (((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y)) \\
&= 0 \in K \text{ (resp., } J),
\end{aligned} \tag{3.17}$$

which implies from Lemma 2.1 that

$$\begin{aligned}
&(x \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y))) \cdot ((y \cdot (y \cdot (x \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y))))) \cdot \\
&(0 \cdot (0 \cdot ((x \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y))) \cdot y)))) \\
&\in K \subseteq A \text{ (resp., } J \subseteq B).
\end{aligned} \tag{3.18}$$

Since

$$\begin{aligned}
0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)) &= ((0\cdot(x\cdot y))\cdot(0\cdot y))\cdot(0\cdot(0\cdot y)) \\
&= (((0\cdot x)\cdot(0\cdot y))\cdot(0\cdot y))\cdot(0\cdot(0\cdot y)) \\
&= (((0\cdot(0\cdot(0\cdot y)))\cdot x)\cdot(0\cdot y))\cdot(0\cdot y) \\
&= (((0\cdot y)\cdot x)\cdot(0\cdot y))\cdot(0\cdot y) \\
&= (0\cdot x)\cdot(0\cdot y) \\
&= 0\cdot(x\cdot y)
\end{aligned} \tag{3.19}$$

by (2.6), (2.3), (2.5) and (III), we have

$$\begin{aligned}
&0\cdot(0\cdot((x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot y)) \\
&= 0\cdot(0\cdot((x\cdot y)\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))) \\
&= 0\cdot((0\cdot(x\cdot y))\cdot(0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))) \\
&= 0\cdot((0\cdot(x\cdot y))\cdot(0\cdot(x\cdot y))) \\
&= 0.
\end{aligned} \tag{3.20}$$

Combining (3.18) and (3.20) implies that

$$\begin{aligned}
&(x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)) \\
&= (x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))) \\
&\in A \text{ (resp., } B).
\end{aligned} \tag{3.21}$$

Since A and B are ideals of X, it follows that

$$x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\in A \text{ (resp., } B). \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}
&(x\cdot(y\cdot(y\cdot x)))\cdot(x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))) \\
&\leq (y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\cdot(y\cdot(y\cdot x)) \\
&\leq (y\cdot x)\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))) \\
&\leq (x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot x \\
&= 0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)) \\
&\in A \text{ (resp., } B).
\end{aligned} \tag{3.23}$$

By (3.22) and (3.23), we get $x\cdot(y\cdot(y\cdot x))\in A$ (resp., B). Using (3.19), (I), (II) we get

$$\begin{aligned}
&(x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y)))))\cdot(x\cdot(y\cdot(y\cdot x))) \\
&\leq (y\cdot(y\cdot x))\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y)))) \\
&\leq 0\cdot(0\cdot(x\cdot y)) \\
&= 0\cdot(0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\in A \text{ (resp., } B).
\end{aligned} \tag{3.24}$$

It follows that $x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in A$ (resp., B). Hence A and B are BCI-implicative ideals of X by Lemma 2.1. Therefore the neutrosophic quadruple set based on A and B is an NQ-BCI-implicative ideal over (X, A, B) by Theorem 3.6. □

Corollary 3.19. *For any nonempty subset K of X, let A be a closed ideal of X such that $K \subseteq A$. If K is a BCI-implicative ideals of X, then the neutrosophic quadruple set based on A is an NQ-BCI-implicative ideal over (X, A), which is larger than the NQ-BCI-implicative ideal over (X, K).*

4. Relations between NQ-BCI-commutative ideal, NQ-BCI-positive implicative ideal and NQ-BCI-implicative ideal

Theorem 4.1. *For any nonempty subsets K and J of X, every NQ-BCI-implicative ideal over (X, K, J) is an NQ-BCI-commutative ideal over (X, K, J).*

Proof. Let K and J be nonempty subsets of X such that the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J). Let $x, y, z \in X$ be such that $z \in K$ (resp., J) and $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z \in K$ (resp., J). Then $(z, zT, zI, zF) \in N_q(K, J)$ and

$$\begin{aligned}
&((((x, xT, xI, xF) \boxdot (y, yT, yI, yF)) \boxdot (y, yT, yI, yF))\boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y, yT, yI, yF))) \boxdot (z, zT, zI, zF) \\
&= ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z, ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)T, \\
&((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)I, ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)F) \\
&\in N_q(K, J)
\end{aligned}$$

Since $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$, it follows that

$$\begin{aligned}
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))), (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))T, \\
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))I, (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))F) \\
&= (x, xT, xI, xF) \boxdot (((y, yT, yI, yF) \boxdot ((y, yT, yI, yF) \boxdot (x, xT, xI, xF)))\boxdot \\
&((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot ((x, xT, xI, xF) \boxdot (y, yT, yI, yF))))) \\
&\in N_q(K, J).
\end{aligned}$$

Hence $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J), and so K and J are BCI-implicative ideals of X. Thus K and J are ideals of X. Assume that $x \cdot y \in K$ (resp., J) for all $x, y \in X$. Then

$$(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot (x \cdot y) = (0 \cdot y) \cdot (0 \cdot y) = 0 \in K \text{ (resp., } J)$$

by using (2.3) and (III), which implies that

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J).$$

Hence $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot 0 \in K$ (resp., J) and $0 \in K$ (resp., J). Since K (resp., J) is a BCI-implicative ideal of X, it follows that

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J).$$

Therefore K (resp., J) is a BCI-commutative ideal of X by Lemma 2.2, and consequently the neutrosophic quadruple set based on K and J is an NQ-BCI-commutative ideal over (X, K, J). □

The converse of Theorem 4.1 is not true in general. In fact, $N_q(K)$ in Example 3.4 is not a BCI-implicative ideal of $N_q(X)$. But it is routine to verify that $N_q(K)$ is a BCI-commutative ideal of $N_q(X)$.

Lemma 4.2 ([6]). *If K and J are BCI-positive implicative ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-positive implicative ideal over (X, K, J).*

Theorem 4.3. *For any nonempty subsets K and J of X, every NQ-BCI-implicative ideal over (X, K, J) is an NQ-BCI-positive implicative ideal over (X, K, J).*

Proof. Let K and J be nonempty subsets of X such that $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$. Then K and J are ideals of X (see the proof of Theorem 4.1). Let $x, y \in X$ be such that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J). Then

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J)$$

by Lemma 2.1. Note that

$$\begin{aligned}
&(x \cdot y) \cdot (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y))))) \\
&\leq ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \cdot y \\
&= (0 \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y))) \\
&= (0 \cdot (x \cdot y)) \cdot (y \cdot x) \\
&= ((0 \cdot x) \cdot (0 \cdot y)) \cdot (y \cdot x) \\
&= (0 \cdot (0 \cdot x)) \cdot x \\
&= 0 \in K \text{ (resp., } J).
\end{aligned}$$

It follows that $x \cdot y \in K$ (resp., J). Hence K and J are BCI-positive implicative ideals of X by Lemma 2.3, and therefore $N_q(K, J)$ is a BCI-positive implicative ideal of $N_q(X)$ by Lemma 4.2. □

In the following example, we can see that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation "$\cdot$", which is given in Table 3.

Table 3. Cayley table for the binary operation "$\cdot$"

$\cdot$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	1	0	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then X is a BCI-algebra (see [8]), and the neutrosophic quadruple BCI-algebra $N_q(X)$ has 625 elements. If we take $K = \{0, 2\}$, then the neutrosophic quadruple set based on K has 16-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\rho}_i \mid i = 1, 2, \cdots, 15\},$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\rho}_1 = (0, 0T, 0I, 2F)$, $\tilde{\rho}_2 = (0, 0T, 2I, 0F)$,
$\tilde{\rho}_3 = (0, 0T, 2I, 1F)$, $\tilde{\rho}_4 = (0, 2T, 0I, 0F)$, $\tilde{\rho}_5 = (0, 2T, 0I, 2F)$,

$\tilde{\rho}_6 = (0, 2T, 2I, 0F)$, $\tilde{\rho}_7 = (0, 2T, 2I, 2F)$, $\tilde{\rho}_8 = (2, 0T, 0I, 0F)$,
$\tilde{\rho}_9 = (2, 0T, 0I, 2F)$, $\tilde{\rho}_{10} = (2, 0T, 2I, 0F)$, $\tilde{\rho}_{11} = (2, 0T, 2I, 2F)$,
$\tilde{\rho}_{12} = (2, 2T, 0I, 0F)$, $\tilde{\rho}_{13} = (2, 2T, 0I, 2F)$, $\tilde{\rho}_{14} = (2, 2T, 2I, 0F)$,
$\tilde{\rho}_{15} = (2, 2T, 2I, 2F)$.

It is routine to verify that $N_q(K)$ is an NQ-BCI-positive implicative ideal over (X, K). If we take $\tilde{\alpha}_1 = (1, 1T, 1I, 1F)$ and $\tilde{\alpha}_3 = (3, 3T, 3I, 3F)$ in $N_q(X)$, then $\tilde{0} \in N_q(K)$ and

$$(((\tilde{\alpha}_1 \boxdot \tilde{\alpha}_3) \boxdot \tilde{\alpha}_3) \boxdot (\tilde{0} \boxdot \tilde{\alpha}_3)) \boxdot \tilde{0} = \tilde{0} \in N_q(K).$$

But,

$$\tilde{\alpha}_1 \boxdot ((\tilde{\alpha}_3 \boxdot (\tilde{\alpha}_3 \boxdot \tilde{\alpha}_1)) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{\alpha}_1 \boxdot \tilde{\alpha}_3)))) = \tilde{\alpha}_1 \boxdot (\tilde{0} \boxdot \tilde{0}) = \tilde{\alpha}_1 \notin N_q(K).$$

Hence $N_q(K)$ is not an NQ-BCI-implicative ideal over (X, K).

We display a characterization of an NQ-BCI-implicative ideal.

Theorem 4.5. *For any nonempty subsets K and J of X, the neutrosophic quadruple set based on K and J is both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K, J) if and only if it is an NQ-BCI-implicative ideal over (X, K, J).*

Proof. For the sufficiency, see Theorems 4.1 and 4.3. Let $N_q(K, J)$ be both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K, J). Then K and J are both a BCI-commutative ideal and a BCI-positive implicative ideal of X. Assume that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) for all $x, y \in X$. Then $x \cdot y \in K$ (resp., J) by Lemma 2.3, and so

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K(\text{resp., } J)$$

by Lemma 2.2. It follows from Lemma 2.1 that K and J are BCI-implicative ideals of X. Therefore the neutrosophic quadruple set based on K and J is an NQ-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 4.6. *For any nonempty subset K of X, the neutrosophic quadruple set based on K is both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K) if and only if it is an NQ-BCI-implicative ideal over (X, K).*

5. Conclusions

Smarandache introduced the notion of neutrosophic quadruple numbers by considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where a, b, c and d are real or complex numbers and T, I, F have their usual neutrosophic logic meanings. Jun et al. made up neutrosophic quadruple BCK/BCI-algebras and (positive) implicative neutrosophic quadruple BCK-algebras using neutrosophic quadruple numbers based on BCK/BCI-algebras (instead of real or complex numbers). In this article, we have studied BCI-implicative ideal in BCI-algebra using neutrosophic quadruple structure. We have introduced neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra, and have investigated their related properties. We have consulted relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal. We have provided conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal. We have discussed a characterization of an NQ-BCI-implicative ideal, and have established the extension property of neutrosophic quadruple BCI-implicative ideal. Based on the contents and ideas of this manuscript, we will study neutrosophic quadruple structure for various algebraic sub-structures in the future.

Acknowledgement The second author, Seok-Zun Song, was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812). The last author, G. Muhiuddin, is partially supported by the research grant S-0064-1439, Deanship of Scientific Research, University of Tabuk, Tabuk-71491, Saudi Arabia.

References

[1] A.A.A. Agboola, B. Davvaz and F. Smarandache, Neutrosophic quadruple algebraic hyperstructures, Ann Fuzzy Math. Inform. **14** (2017), no. 1, 29–42.

[2] S.A. Akinleye, F. Smarandache and A.A.A. Agboola, On neutrosophic quadruple algebraic structures, Neutrosophic Sets and Systems **12** (2016), 122–126.

[3] Y. Huang, BCI-algebra, Science Press, Beijing, 2006.

[4] K. Iséki, On BCI-algebras, Math. Seminar Notes **8** (1980), 125–130.

[5] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. **23** (1978), 1–26.

[6] Y.B. Jun, S.Z. Song and S.J. Kim, Neutrosophic quadruple BCI-positive implicative ideals, Mathematics 2019, 7, 385; doi:10.3390/math7050385

[7] Y.B. Jun, S.Z. Song, F. Smarandache and H. Bordbar, Neutrosophic quadruple BCK/BCI-algebras, Axioms 2018, 7, 41; doi:10.3390/axioms7020041

[8] Y.L. Liu, Y. Xu, and J. Meng, BCI-implicative ideals of BCI-algebras, Inform. Sci. **177** (2007), 4987–4996.

[9] Y.L. Liu and X.H. Zhang, Characterization of weakly positive implicative BCI-algebras, J. Hanzhong Teachers College (Natural) (1) (1994), 4–8.

[10] J. Meng, An ideal characterization of commutative BCI-algebras, Pusan Kyongnam Math. J. **9** (1993), no. 1, 1–6.

[11] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa Co. Seoul, Korea 1994.

[12] G. Muhiuddin, A.N. Al-Kenani, E.H. Roh and Y.B. Jun, Implicative neutrosophic quadruple BCK-algebras and ideals, Symmetry 2019, 11, 277; doi:10.3390/sym11020277.

[13] G. Muhiuddin, F. Smarandache and Y.B. Jun, Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras, Neutrosophic Sets and Systems **25** (2019), 161–173.

[14] F. Smarandache, Neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, and the multiplication of neutrosophic quadruple numbers, Neutrosophic Sets and Systems, **10** (2015), 96–98.

Isolation numbers of matrices over nonbinary Boolean semiring

LeRoy B. Beasley[1] , Madad Khan[2] and Seok-Zun Song[3,*]

[1]***Department of Mathematics and Statistics, Utah State University, Logan, UT84322-3900, USA***

[2]***Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Pakistan***

[3] ***Department of Mathematics, Jeju National University, Jeju 63243, Korea***

Abstract. Let $\mathbb{B}_k$ be the nonbinary Boolean semiring and A be a $m \times n$ Boolean matrix over $\mathbb{B}_k$. The Boolean rank of a Boolean matrix A is the smallest k such that A can be factored as an $m \times k$ Boolean matrix times a $k \times n$ Boolean matrix. The isolation number of A is the maximum number of nonzero entries in A such that no two are in any row or any column, and no two are in a 2×2 submatrix of all nonzero entries. We have that the isolation number of A is a lower bound on the Boolean rank of A. We also compare the isolation number with the binary Boolean rank of the support of A, and determine the equal cases of them.

1. Introduction

There are many papers on the study of rank of matrices over several semirings containing binary Boolean algebra, fuzzy semiring, semiring of nonegative integers, and so on ([2], [3], [6], and [7]). But there are few papers on isolation numbers of matrices. Gregory et al ([7]) introduced set of isolated entries and compared binary Boolean rank with biclique covering number. Recently Beasley ([2]) introduced isolation number of Boolean matrix and compare it with binary Boolean rank.

In this paper, we investigate the possible isolation number of Boolean matrix and compare it with Boolean rank of Boolean matrix and the binary Boolean rank of the support of the Boolean matrix.

2. Preliminaries

Definition 2.1. A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ with two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (the identity is denoted by 0);
- $\mathcal{S}$ is a monoid under multiplication (the identity is denoted by 1, $1 \neq 0$);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

Definition 2.2. A semiring $\mathcal{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

[0]**2010 Mathematics Subject Classification**: 15A23; 15A03; 15B15.

[0]**Keywords**: Boolean rank; nonbinary Boolean semiring; binary Boolean algebra; isolation number.

* The corresponding author.

[0]**E-mail** : leroy.b.beasley@aggiemail.usu.edu (L. B. Beasley); szsong@jejunu.ac.kr (S. Z. Song); madadmath@yahoo.com (M. Khan)

Definition 2.3. A semiring $\mathcal{S}$ is called a *Boolean semiring* if $\mathcal{S}$ is equivalent to a set of subsets of a given set $\mathbb{X}$, the sum of two subsets is their union, and the product is their intersection. The zero element 0 is the empty set and the identity element 1 is the whole set $\mathbb{X}$.

Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k. Then $\mathcal{P}(S_k)$ is the Boolean semiring of all subsets of S_k with operations in above definition. Let $\mathbb{B}_k$ be a Boolean semiring of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, that is a subset of $\mathcal{P}(S_k)$. It is straightforward to see that a Boolean semiring $\mathbb{B}_k$ is a commutative and antinegative semiring. Moreover, all of its elements, except 0 and 1, are zero-divisors. If $\mathbb{B}_k$ consists of only 0 (the empty subset) and 1 (the whole set S_k) then it is called a *binary Boolean semiring*, which is denoted as $\mathbb{B}_1$. If $\mathbb{B}_k$ is not a binary Boolean semiring then it is called a *nonbinary Boolean semiring*.

Throughout the paper, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes a nonbinary Boolean semiring, which contains at least 3 elements. Let $\mathcal{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from a Boolean semiring $\mathbb{B}_k$.

Let $\mathcal{M}_n(\mathbb{B}_k) = \mathcal{M}_{m,n}(\mathbb{B}_k)$ if $m = n$, let I_m denote the $m \times m$ identity matrix, $O_{m,n}$ denote the zero matrix in $\mathcal{M}_{m,n}(\mathbb{B}_k)$, $J_{m,n}$ denote the matrix of all ones in $\mathcal{M}_{m,n}(\mathbb{B}_k)$. The subscripts are usually omitted if the order is obvious, and we write I, O, J.

Definition 2.4. The matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ is said to be of *Boolean rank r* if there exist matrices $B \in \mathcal{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathcal{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer such that such a factorization exists. We denote $b(A) = r$.

By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

Now let $\mathcal{M}_{m,n}(\mathbb{B}_1)$ denote the set of all $m \times n$ binary Boolean matrices with entries in $\mathbb{B}_1$. The *binary Boolean rank* of $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ is the Boolean rank over $\mathbb{B}_1$ and denoted $b_1(A)$.

Definition 2.5. For two (binary) Boolean matrices A and B, A *dominates* B if $a_{i,j} = 0$ implies $b_{i,j} = 0$.

Given a matrix $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, we let $\mathbf{x}^{(j)}$ denote the j^{th} column of X and $\mathbf{x}_{(i)}$ denote the i^{th} row. Now if $b(A) = r$ and $A = BC$ is a factorization of $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}$. Since each of the terms $\mathbf{b}^{(i)}\mathbf{c}_{(i)}$ is a Boolean rank one matrix, the Boolean rank of A is also the minimum number of Boolean rank one matrices whose sum is A.

The binary Boolean rank has many applications in combinatorics, especially graph theory, for example, if $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ is the adjacency matrix of the bipartite graph G with bipartition (X, Y), the binary Boolean rank of A is the minimum number of bicliques that cover the edges of G, called the *biclique covering number*.

Definition 2.6. Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, a set of *isolated entries* ([7]) is a set of entries, usually written as $E = \{a_{i,j}\}$ such that $a_{i,j} \neq 0$, no two entries in E are in the same row, no two entries in E are in the same column, and, if $a_{i,j}, a_{k,l} \in E$ then, $a_{i,l} = 0$ or $a_{k,j} = 0$. That is, isolated entries are independent entries and any two isolated entries $a_{i,j}$ and $a_{k,l}$ do not lie in a submatrix of A of the form $\begin{bmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{bmatrix}$ with all entries nonzero. The *isolation number of* A, $\iota(A)$, is the maximum size of a set of isolated entries.

Note that $\iota(A) = 0$ if and only if $A = O$.

Example 2.7. Let $\sigma \in \mathbb{B}_k$ be neither 0 nor 1 and

$$A = \begin{bmatrix} 1 & 1 & \sigma & 0 & 0 \\ \sigma & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \sigma \\ 0 & \sigma & 0 & 1 & 1 \\ 0 & 0 & 1 & \sigma & 1 \end{bmatrix}$$

be a Boolean matrix over $\mathbb{B}_k$ and E_1 is the set of $\sigma's$ which are located at the positions $\{a_{1,3}, a_{2,1}, a_{3,5}, a_{4,2}, a_{5,4}\}$ of A. The entries $\sigma's$ of A are isolated entries and hence $\iota(A) = 5$. But the entries of A in the position in $E_2 = \{a_{1,1}, a_{2,2}, a_{3,5}, a_{4,4}, a_{5,3}\}$ are not isolated.

Suppose that $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ and $b(A) = r$. Then there are r Boolean rank one matrices A_i such that

$$A = A_1 + A_2 + \cdots + A_r. \tag{2.1}$$

Because each Boolean rank one matrix can be permuted to a matrix of the form $\begin{bmatrix} N & O \\ O & O \end{bmatrix}$ with all nonzero entries in N, it is easily seen that the matrix consisting of two isolated entries of A cannot be dominated by any one A_i among the Boolean rank one summand of A in (2.1). Thus

$$i(A) \leq b(A). \tag{2.2}$$

Many functions, sets and relations concerning matrices do not depend upon the magnitude or nature of the individual entries of a matrix, but rather only on whether the entry is zero or nonzero. These combinatorially significant matrices have become increasingly important in recent years. Of primary interest is the binary Boolean rank. Finding the binary Boolean rank of a $(0, 1)$-matrix is an NP-Complete problem ([8]), and consequently finding bounds on the binary Boolean rank of a matrix is of interest to those researchers that would use binary Boolean rank in their work. If the $(0, 1)$-matrix is the reduced adjacency matrix of a bipartite graph, the isolation number of the matrix is the maximum size of a non-competitive matching in the bipartite graph. This is related to the study of such combinatorial problems as the patient hospital problem, the stable marriage problem, etc. An additional reason for studying the isolation number is that it is a lower bound on the Boolean rank of a Boolean matrix over $\mathbb{B}_k$. While finding the isolation number as well as finding the Boolean rank of a Boolean matrix is an NP-Complete problem ([1]), for some matrices finding the isolation number can be easier than finding the Boolean rank especially if the matrix is sparse:

Example 2.8. Let $\sigma \in \mathbb{B}_k$ and

$$F = \begin{bmatrix} 1 & 1 & 1 & \sigma & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \sigma & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & \sigma & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a Boolean matrix in $\mathcal{M}_9(\mathbb{B}_k)$.

Then we can easily see $b(F) \leqq 6$ from first 3 rows and columns, however to find that Boolean rank is not 5, requires much calculation if the isolation number is not considered. However, the isolation number is easily seen to be 6, both computationally and visually, the σ's in this matrix represent a set of isolated entries. Thus we have $b(F) = 6$ by (2.2).

Note that if any of the 1's in F are replaced by zeros, the resulting matrix still has Boolean rank 6 as well as isolation number 6.

Terms not specifically defined here can be found in Brualdi and Ryser [5] for matrix terms, or Bondy and Murty [4] for graph theoretic terms.

For our use in the next section, we define the support matrix of a Boolean matrix. If $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then the *support* of A is the binary Boolean matrix $\overline{A} = (\overline{a_{i,j}}) \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ such that $\overline{a_{i,j}} = 1$ if $a_{i,j} \neq 0$ and $\overline{a_{i,j}} = 0$ if $a_{i,j} = 0$.

3. Comparisons between isolation numbers and Boolean ranks over $\mathcal{M}_{m,n}(\mathbb{B}_k)$

In this section, we compare the isolation number with Boolean rank of a Boolean matrix, and also we compare the isolation number with binary Boolean rank of the support of a Boolean matrix.

Lemma 3.1. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $b(A + B) \leq b(A) + b(B)$. And for $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_1)$, $b_1(A + B) \leq b_1(A) + b_1(B)$.*

Proof. It follows from the definition of Boolean rank and equation (2.1). ■

Lemma 3.2. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $\overline{A + B} = \overline{A} + \overline{B}$ in $\mathcal{M}_{m,n}(\mathbb{B}_1)$.*

Proof. It follows from the facts that $\mathbb{B}_k$ is an antinegative semiring and $1 + 1 = 1$ in $\mathbb{B}_1$. ■

Lemma 3.3. *For $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $b_1(\overline{A}) \leq b(A)$.*

Proof. If $b(A) = r$, then A has a Boolean rank one factorization such that $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}$ with $B = [\mathbf{b}^{(1)}\mathbf{b}^{(2)} \cdots \mathbf{b}^{(r)}] \in \mathcal{M}_{m,k}(\mathbb{B}_k)$ and $C = [\mathbf{c}_{(1)}\mathbf{c}_{(2)} \cdots \mathbf{c}_{(k)}]^t \in \mathcal{M}_{k,n}(\mathbb{B}_k)$ from (2.1). Therefore
$b_1(\overline{A}) = b_1(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}}) = b_1(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)}} + \overline{\mathbf{b}^{(2)}\mathbf{c}_{(2)}} + \cdots + \overline{\mathbf{b}^{(r)}\mathbf{c}_{(r)}}) \leq r$, from Lemma 3.2. Hence $b_1(\overline{A}) \leq b(A)$. ■

We may have strict inequality in Lemma 3.3 as we see in the following example.

Example 3.4. Let $S_3 = \{x, y, z\}$ and $\mathbb{B}_3 = \{0, \{x\}, \{x, y\}, 1\}$ with $1 = \{x, y, z\}$. Consider $X = \begin{bmatrix} 1 & \{x\} \\ \{x, y\} & \{x, y\} \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & \{x\} \\ \{x, y\} & \{x\} \end{bmatrix}$ in $\mathcal{M}_2(\mathbb{B}_3)$. Then $b(X) = 2$ but $b_1(\overline{X}) = b_1(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = 1$. Hence $b_1(\overline{X}) < b(X)$. But $b(Y) = b_1(\overline{Y}) = 1$ since $Y = \begin{bmatrix} 1 \\ \{x, y\} \end{bmatrix} \begin{bmatrix} 1 & \{x\} \end{bmatrix}$ over $\mathbb{B}_3$.

Lemma 3.5. *For $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $\iota(A) = \iota(\overline{A})$.*

Proof. If $a_{i,j}$ and $a_{k,l}$ are any isolated entries in A, then $i \neq k$ and $j \neq l$, and that $a_{i,l} = 0$ or $a_{k,j} = 0$. Hence $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are isolated entries in $\overline{A}$, so we have $\iota(A) \leq \iota(\overline{A})$.

Conversely, if $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are any isolated entries in $\overline{A}$, then $a_{i,j} \neq 0$ and $a_{k,l} \neq 0$ and that $a_{i,l} = \overline{a_{i,l}} = 0$ or $a_{k,j} = \overline{a_{k,j}} = 0$. Hence $a_{i,j}$ and $a_{k,l}$ are isolated entries in A, so we have $\iota(\overline{A}) \leq \iota(A)$. ■

Theorem 3.6. *If $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then $\iota(A) = 1$ if and only if $b_1(\overline{A}) = 1$.*

Proof. Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. If $b_1(\overline{A}) = 1$ then $A \neq O$ so that $\iota(A) \neq 0$ and since $\iota(A) = \iota(\overline{A}) \leq b_1(\overline{A})$ by (2.2), we have $\iota(A) = 1$.

Conversely, suppose on the contrary that there exists a matrix $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ such that $\iota(A) = 1$, $b_1(\overline{A}) > 1$. Then, there exists two non-equal and nonzero rows of $\overline{A}$, say ith and jth. Hence, without loss of generality, there exists a k such that $\overline{a_{i,k}} = 1$ and $\overline{a_{j,k}} = 0$. Then, $\overline{a_{i,k}}$ and any unit entry in jth row of $\overline{B}$ constitute a set of two isolated entries. Thus, $\iota(A) = \iota(\overline{A}) > 1$, a contradiction. ■

It follows that the subset of $\mathcal{M}_{m,n}(\mathbb{B}_k)$ of matrices with isolation number 1 is the same as the set of matrices whose support has Boolean rank 1.

For $A = A_1 + A_2 + \cdots + A_r$ with $b(A) = r$, let $\mathcal{R}_i$ denote the indices of the nonzero rows of A_i and $\mathcal{C}_j$ denote the indices of the nonzero columns of A_j, $i, j = 1.\cdots, k$. Let $r_i = |\mathcal{R}_i|$, the number of nonzero rows of A_i and $c_j = |\mathcal{C}_j|$, the number of nonzero columns of A_j.

Lemma 3.7. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then if $b(A) \geq b_1(\overline{A}) = 2$ then $\iota(A) = 2$, and if $\iota(A) = 2$ then $b_1(\overline{A}) \neq 3$.*

Proof. If $b_1(\overline{A}) = 2$, then $\iota(A) > 1$ by Theorem 3.6. Since $\iota(A) = \iota(\overline{A}) \leq b_1(\overline{A})$ from Lemma 3.5 and (2.2), we have that $\iota(A) = \iota(\overline{A}) = 2$.

Now, suppose that $\iota(A) = 2$ and that $b_1(\overline{A}) = 3$. Then, we have a factorization of $\overline{A}$ as $\overline{A} = C \times D$ with $C \in \mathcal{M}_{m,3}(\mathbb{B}_1)$ and $D \in \mathcal{M}_{3,n}(\mathbb{B}_1)$. Then, the three rows of D generate all the rows of $\overline{A}$. Since $b_1(\overline{A}) = 3$, D cannot have binary Boolean rank 2 or less. Thus, we have $b_1(D) = 3$. Therefore, we have a factorization of D as $D = E \times F$ with $E \in \mathcal{M}_{3,3}(\mathbb{B}_1)$ and $F \in \mathcal{M}_{3,n}(\mathbb{B}_1)$. Then, the three column of E generate all the columns of D and $b_1(E) = 3$. Therefore, it is sufficient to consider 3×3 matrices of binary Boolean rank 3. However, there are only 10 following 3×3 matrices of binary Boolean rank 3 up to permutations:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ B_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_6 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ B_7 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ B_8 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B_9 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_{10} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since B_5 can be permuted to B_2 and B_7 can be permuted to B_4, and B_9 can be permuted to B_6 with transposing. Therefore, there are only seven non-equivalent 3×3 matrices of binary Boolean rank 3. However, these matrices have three isolation entries on the main diagonal. Thus, we have a contradiction to the conditions that $\iota(B) = 2$ and $r_{\mathbb{B}_1}(\overline{B}) = 3$. Thus, if $\iota(A) = 2$ then $b_1(\overline{A}) \neq 3$. ■

Theorem 3.8. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then, $\iota(A) = 2$ if and only if $b_1(\overline{A}) = 2$.*

Proof. From Lemma 3.7, we have the sufficiency. So we only need show the necessity.

Suppose there exists $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ with $\iota(A) = \iota(\overline{A}) = 2$ and $b_1(\overline{A}) > 2$. By Lemma 3.7, $b_1(\overline{A}) \neq 3$, and hence $b_1(\overline{A}) \geq 4$. Thus we choose A such that if $b_1(\overline{A}) > b_1(\overline{C}) > 2$ then $\iota(C) > 2$. Suppose that $\overline{A} = \overline{A_1} + \overline{A_2} + \cdots + \overline{A_r}$ for $r = b_1(\overline{A})$ where each $\overline{A_i}$ is binary Boolean rank 1, i.e., r is the minimum r such that $b_1(\overline{A}) = r$ and $\iota(A) = 2$. Suppose that $\overline{A_1}$ has the fewest number of nonzero rows of the $\overline{A_i}$'s. As in the proof of the above lemma 3.7, permute the rows of $\overline{A}$ so that $\overline{A_1}$ has nonzero rows $1, 2, \cdots, r_1$. For $j = 1, \cdots, r_1$, let $\overline{B_j}$ be the matrix whose first j rows are the first j rows of $\overline{A}$ and whose last $m-j$ rows are all zero. Let $\overline{C_j}$ be the matrix whose first j rows are all zero and whose last $m-j$ rows are the last $m-j$ rows of $\overline{A}$. Then $\overline{A} = \overline{B_j} + \overline{C_j}$. Further any set of isolated entries of $\overline{C_j}$ is a set of isolated entries for $\overline{A}$. Now, from $b_1(\overline{A}) \leq b_1(\overline{B_j}) + b_1(\overline{C_j})$, and the fact that $b_1(\overline{C_j}) = b_1(\overline{C_{j-1}})$ or $b_1(\overline{C_j}) = b_1(\overline{C_{j-1}}) - 1$, there is some t such that $b_1(\overline{C_t}) = b_1(\overline{A}) - 1$. Since $b_1(\overline{C_t}) < r$ by the choice of $\overline{A}$, for this t, we have that $\iota(\overline{C_t}) > 2$ since $b_1(\overline{C_t}) \geq 3$. That is, $\iota(A) = \iota(\overline{A}) > 2$, which is impossible since $\iota(A) = 2$. Therefore $b_1(\overline{A}) = 2$. ■

Now, as we can see in the following example, there is a Boolean matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ such that $\iota(\overline{A}) = 3$ and $b_1(\overline{A})$ is relative large, depending on m and n.

Example 3.9. For $n \geq 3$, let $\overline{D_n} = J \setminus I \in \mathcal{M}_n(\mathbb{B}_1)$. Then, it is easily shown that $\iota(\overline{D_n}) = 3$ while $b_1(\overline{D_n}) = r$ where $r = min\left\{h : n \leq \left(\begin{array}{c} h \\ \frac{h}{2} \end{array} \right)\right\}$, see [6](Corollary 2). So, $\iota(\overline{D_{20}}) = 3$ while $b_1(\overline{D_{20}}) = 6$.

Definition 3.10. A *tournament matrix* $[T] \in \mathcal{M}_n(\mathbb{B}_k)$ is the adjacency matrix of a directed graph called a tournament, T. It is characterized by $[T] \circ [T]^t = O$ and $[T] + [T]^t = J - I$, where $\circ$ denotes entrywise multiplication of two matrices.

Now, for each $r = 1, 2, \cdots, \min\{m, n\}$, can we characterize the matrices in $\mathcal{M}_{m,n}(\mathbb{B}_k)$ for which $\iota(A) = b_1(\overline{A})$? Of course it is done if $r = 1$ or $r = 2$ in the above theorems, but only in those cases. For $r = m$ we can also find a characterization:

Theorem 3.11. *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then, $\iota(A) = b_1(\overline{A}) = m$ if and only if there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B}_1)$ and $Q \in \mathcal{M}_n(\mathbb{B}_1)$ such that $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T} \in \mathcal{M}_m(\mathbb{B}_1)$ where $\overline{T} \in \mathcal{M}_m(\mathbb{B}_1)$ is dominated by a tournament matrix. (There are no restrictions on C.)*

Proof. Suppose that $\iota(A) = m$. Then we permute A by permutation matrices P and Q so that the set of isolated entries are in the (d, d) positions, $d = 1, \cdots, m$. That is, if $X = PAQ$ then $I = \{x_{1,1}, x_{2,2}, \cdots, x_{m,m}\}$ is the set of isolated entries in X. Therefore $X = [B|C]$, with $\overline{b_{i,i}} = \overline{x_{i,i}} = 1$ and $\overline{b_{i,j}} \cdot \overline{b_{j,i}} = 0$ for every i and $j \neq i$ from the definition of the isolated entries. Thus, $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix. Thus, $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T}$ and clearly there are no conditions on C.

Conversely, if $PAQ = [B|C]$ and $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix, then the diagonal entries of B form a set of isolated entries for PAQ and hence A has a set of m isolated entries. Thus $\iota(A) = b_1(\overline{A}) = m$. ∎

Corollary 3.12. *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. If there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B}_1)$ and $Q \in \mathcal{M}_n(\mathbb{B}_1)$ such that $PAQ = [B|C]$ where $B \in \mathcal{M}_m(\mathbb{B}_k)$ is a diagonal matrix or a triangular matrix with nonzero diagonal entries, then $\iota(A) = b_1(\overline{A}) = m$.*

4. Conclusions

In this paper, we investigated the nonbinary Boolean rank of a matrix A and the rank of its support for the given isolation number k over nonbinary Boolean semirings. Thus, we proved that the isolation number of A is the same as the Boolean rank of the support of it if the isolation numbers are 1 and 2. If the isolation number were greater than 2, then we showed by example that binary Boolean rank of the support of the given nonbinary Boolean matrix may be strictly greater than the isolation number of the matrix. In addition, in some special cases involving tournament matrices, we obtained that the isolation number of the given matrix and the Boolean rank of its support of the nonbinary Boolean matrix are the same.

Acknowledgement The third author, Seok-Zun Song, was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

References

[1] K. Akiko, Complexity of the sex-equal stable marriage problem (English summary), Japan J. Indust. Appl. Math., **10**(1993), 1-19.

[2] L. B. Beasley, Isolation number versus Boolean rank, Linear Algebra Appl., **436**(2012), 3469-3474.

[3] L. B. Beasley and N. J. Pullman, Nonnegative rank-preserving operators, Linear Algebra Appl., **65**(1985), 207-223.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate texts in Mathematics 244, Springer, New York, 2008.

[5] R. Brualdi and H. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991.

[6] D. de Caen, D.A. Gregory,and N. J. Pullman, The Boolean rank of zero-one matrices, Proceedings of the Third Caribbean Conference on Combinatorics and Computing (Bridgetown), 169-173, Univ. West Indies, Cave Hill Campus, Barbados, 1981

[7] D. Gregory, N. J. Pullman, K. F. Jones and J. R. Lundgren, Biclique coverings of regular bigraphs and minimum semiring ranks of regular matrices. J. Combin. Theory Ser. B, **51**(1991), 73-89.

[8] G. Markowsky, Ordering D-classes and computing the Schein rank is hard, Semigroup Forum, **44**(1992), 373-375.

ORTHOGONALLY EULER-LAGRANGE TYPE CUBIC FUNCTIONAL EQUATIONS IN ORTHOGONALITY NORMED SPACES

CHANG IL KIM AND GILJUN HAN*

Abstract. In this paper, we investigate the orthogonally Euler-Lagrange type cubic functional equation

$$\begin{aligned} &f(ax+by)+f(ax-by)-ab^2[f(x+y)+f(x-y)]-2a(a^2-b^2)f(x)\\ &+c[f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y)]=0,\ \ x\perp y \end{aligned}$$

for fixed non-zero rational numbers a, b and a fixed non-zero real number c with $a^2 \neq b^2$ and $a \neq \pm 1$ and prove the generalized Hyers-Ulam stability for it by using the fixed point method,

1. Introduction

Assume that X is a real inner product space and $f : X \longrightarrow \mathbb{R}$ is a solution of the orthogonally Cauchy functional equation $f(x+y) = f(x)+f(y)$, $< x, y >= 0$. By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus, orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

The orthogonally Cauchy functional equation

$$f(x+y) = f(x)+f(y),\ \ x\perp y$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [5]. Rätz [16] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [17] investigated the problem in a rather more general framework.

Definition 1.1. [17] Let X be a real vector space with $\dim X \geq 2$ and $\perp$ a binary relation on X with the following properties:

(O1) totality for zero: $x\perp 0$ and $0\perp x$ for all $x \in X$;

(O2) independence: if $x, y \in X - \{0\}$, $x\perp y$, then x, y are linearly independent;

(O3) homogeneity: if $x, y \in X$, $x\perp y$, then $\alpha x\perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and a non-negative real number k, then there exists an $y \in P$ such that $x\perp y$ and $x + y\perp kx - y$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure.

2010 *Mathematics Subject Classification.* 39B55, 47H10, 39B52, 46H25.

Key words and phrases. Hyers-Ulam stability, fixed point theorem, orthogonally cubic functional equation, orthogonality space.

* Corresponding author.

Remark 1.2. (i) The trivial orthogonality on a vector space X defined by (O1) and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, < \cdot, \cdot >)$ given by $x \perp y$ if and only if $< x, y >= 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, \| \cdot \|)$ defined by $x \perp y$ if and only if $\|x + ky\| \geq \|x\|$ for all $k \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Then clearly examples (i) and (ii) are symmetric but example (iii) is not. However, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

In 1940, S. M. Ulam proposed the following stability problem (cf. [19]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive mappings. In 1978, Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy difference. The result of Rassias has provided a lot of influence in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam stability of functional equations. Ger and Sikorska [4] investigated the orthogonal stability of the Cauchy functional equation

$$f(x + y) = f(x) + f(y), \ \ x \perp y \tag{1.1}$$

and Vajzović [20] investigated the orthogonally additive-quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \ \ x \perp y \tag{1.2}$$

when X is a Hilbert space, Y is a scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, many mathematicians have investigated the orthogonal stability of functional equations ([3], [9], [10], [11], [12], [13], and [18]).

In 2001, Rassias [15] introduced the following cubic functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) = 0 \tag{1.3}$$

and every solution of the cubic functional equation is called *a cubic mapping* and Jun, Kim, and Chang [8] introduced the Euler-Lagrange cubic functional equation.

In this paper, we consider the following orthogonally Euler-Lagrange type cubic functional equation

$$\begin{aligned} &f(ax + by) + f(ax - by) - ab^2[f(x + y) + f(x - y)] - 2a(a^2 - b^2)f(x) \\ &+ c[f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y)] = 0, \ \ x \perp y. \end{aligned} \tag{1.4}$$

for fixed non-zero rational numbers a, b and a fixed non-zero real numbers c with $a^2 \neq b^2$ and $a \neq \pm 1$ and prove the generalized Hyers-Ulam stability for it. Every solution of (1.4) is called *an orthogonally Euler-Lagrange type cubic mapping.*

Throughtout this paper, $(X, \perp)$ is an orthogonality normed space with the norm $\| \cdot \|_X$ and $(Y, \| \cdot \|)$ is a Banach space.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

2. Solutions of (1.4)

In this section, we investigate solutiuons of (1.4). We will show that a mapping f satisfying (1.4) is an orthogonally cubic mapping.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. If f satisfies (1.4) and $c \neq 0$, then f is an orthogonally cubic mapping.*

Proof. Suppose that f satisfies (1.4). Setting $y = 0$ in (1.4), we have

$$f(ax) = a^3 f(x) \tag{2.1}$$

for all $x \in X$ and setting $x = 0$ and $y = x$ in (1.4), we have

$$f(bx) + f(-bx) = (ab^2 + 9c)f(x) + (ab^2 + c)f(-x) - cf(2x) \tag{2.2}$$

for all $x \in X$. Replacing x by $-x$ in (2.2), we have

$$f(bx) + f(-bx) = (ab^2 + 9c)f(-x) + (ab^2 + c)f(x) - cf(-2x) \tag{2.3}$$

for all $x \in X$. Since $c \neq 0$, by (2.2) and (2.3), we have

$$f(2x) - f(-2x) = 8[f(x) - f(-x)] \tag{2.4}$$

for all $x \in X$. Relpacing y by ay in (1.4), by (2.2), we have

$$\begin{aligned} &a^3[f(x + by) + f(x - by)] - (ab^2 + 3c)f(x + ay) - (ab^2 + c)f(x - ay) \\ &+ cf(x + 2ay) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(ay) = 0 \end{aligned} \tag{2.5}$$

for all $x, y \in X$ with $x \perp y$ and letting $y = \frac{y}{b}$ in (2.5), we have

$$\begin{aligned} &a^3[f(x + y) + f(x - y)] - (ab^2 + 3c)f(x + py) - (ab^2 + c)f(x - py) \\ &+ cf(x + 2py) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(py) = 0 \end{aligned} \tag{2.6}$$

for all $x, y \in X$ with $x \perp y$, where $p = \frac{a}{b}$. Letting $y = -y$ in (2.6), we have

$$\begin{aligned} &a^3[f(x - y) + f(x + y)] - (ab^2 + 3c)f(x - py) - (ab^2 + c)f(x + py) \\ &+ cf(x - 2py) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(-py) = 0 \end{aligned} \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. By (2.6) and (2.7), we have

$$\begin{aligned} &c[f(x + 2py) - f(x - 2py)] - 2c[f(x + py) - f(x - py)] \\ &- 6c[f(py) - f(-py)] = 0 \end{aligned} \tag{2.8}$$

for all $x, y \in X$ with $x \perp y$. Letting $y = \frac{1}{p}y$ in (2.8), we have

$$[f(x + 2y) - f(x - 2y)] - 2[f(x + y) - f(x - y)] - 6[f(y) - f(-y)] = 0 \tag{2.9}$$

for all $x, y \in X$ with $x \perp y$.

Let $f_o(x) = \frac{f(x) - f(-x)}{2}$. Then f_o satisfies (2.9). Letting $x = 0$ in (2.9), we have

$$f_o(2y) = 8f_o(y) \tag{2.10}$$

for all $y \in X$. Letting $x = 2x$ in (2.9), by (2.10), we have

$$4[f_o(x + y) - f_o(x - y)] = f_o(2x + y) - f_o(2x - y) + 6f_o(y) \tag{2.11}$$

for all $x, y \in X$ with $x \perp y$. Interchanging x and y in (2.11), we have

$$4[f_o(x + y) + f_o(x - y)] = f_o(x + 2y) + f_o(x - 2y) + 6f_o(x) \tag{2.12}$$

for all $x, y \in X$ with $x \perp y$. By (2.9) and (2.12), we have

$$f_o(x + 2y) - 3f_o(x + y) + 3f_o(x) - f_o(x - y) - 6f_o(y) = 0$$

for all $x, y \in X$ with $x \perp y$ and hence f_0 is an orthogonally cubic mapping.

Let $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_e satisfies (2.9) and so we have

$$f_e(x+2y) - f_e(x-2y) - 2[f_e(x+y) - f_e(x-y)] = 0 \tag{2.13}$$

for all $x, y \in X$ with $x \perp y$. Letting $y = x$ in (2.13), we have

$$f_e(3x) = 2f_e(2x) + f_e(x)$$

for all $x \in X$ and letting $y = 2x$ in (2.13), we have

$$f_e(4x) = 2f_e(3x) - 2f_e(x)$$

for all $x \in X$. Hence we have $f_e(4x) = 4f_e(2x)$ for all $x \in X$ and so

$$f_e(2x) = 4f_e(x), \ \ f_e(3x) = 9f_e(x), \ \ f_e(4x) = 16f_e(x)$$

for all $x \in X$. By induction on n, we have

$$f_e(nx) = n^2 f_e(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence

$$f_e(rx) = r^2 f_e(x)$$

for all $x \in X$ and all rational number r. By (2.1), since a is a non-zero rational number with $a \neq 1$, $f(x) = 0$ for all $x \in X$. Hence $f = f_o + f_e = f_o$ is an orthogonally cubic mapping. □

3. The Generalized Hyers-Ulam stability for (1.4)

In this section, we prove the generalized Hyers-Ulam stability for the orthogonally cubic functional equation (1.4) by using the fixed point method.

In 1996, Isac and Rassias [7] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 3.1. [1], [2] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$\begin{aligned} Df(x,y) = f(ax+by) + f(ax-by) - ab^2[f(x+y) + f(x-y)] - 2a(a^2-b^2)f(x) \\ + c[f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y)] \end{aligned}$$

for all $x, y \in X$.

Theorem 3.2. *Assume that $\phi : X^2 \longrightarrow [0, \infty)$ is a function such that*

$$\phi(x, y) \leq \frac{L}{|a|^3}\phi(ax, ay) \tag{3.1}$$

for all $x, y \in X$ and some real number L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|Df(x, y)\| \leq \phi(x, y) \tag{3.2}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \frac{L}{2|a|^3(1 - L)}\phi(x, 0) \tag{3.3}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g, h) = \inf\{c \in [0, \infty) \mid \|g(x) - h(x)\| \leq c\ \phi(x, 0), \forall x \in X\}.$$

Then (S, d) is a complete metric space([9]). Define a mapping $T : S \longrightarrow S$ by $Tg(x) = a^3 g(\frac{x}{a})$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (3.1), we have

$$\|Tg(x) - Th(x)\| = |a|^3\Big\|g\Big(\frac{x}{a}\Big) - h\Big(\frac{x}{a}\Big)\Big\| \leq cL\phi(x, 0)$$

for all $x \in X$. Hence we have $d(Tg, Th) \leq Ld(g, h)$ for all $g, h \in S$ and so T is a strictly contractive mapping. Putting $y = 0$ in (3.2), we get

$$\|2f(ax) - 2a^3 f(x)\| \leq \phi(x, 0)$$

for all $x \in X$ and hence

$$\Big\|f(x) - a^3 f\Big(\frac{x}{a}\Big)\Big\| \leq \frac{L}{2|a|^3}\phi(x, 0)$$

for all $x \in X$ and hence $d(f, Tf) \leq \frac{L}{2|a|^3} < \infty$. By Theorem 3.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of T such that $d(T^n f, F) \to 0$ as $n \to \infty$ and

$$\|F(x) - f(x)\| \leq \frac{L}{2|a|^3(1 - L)}\phi(x, 0)$$

for all $x \in X$. Replacing x, y by $\frac{x}{a^n}$, $\frac{y}{a^n}$ in (3.2), respectively, and multiplying (3.2) by $|a|^{3n}$, by (O3), we have

$$\Big\|a^{3n} Df\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big)\Big\| \leq L^n \phi(x, y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DF(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$ and by Theorem 2.1, F is an orthogonally cubic mapping.

Now, we will show the uniqueness of F. Let $G : X \longrightarrow Y$ be another orthogonally cubic mapping with (3.3). Since F and G are fixed points of T, by (3.3), we get

$$\begin{aligned}\|G(x) - F(x)\| &= \|T^nG(x) - T^nF(x)\|\\ &\le \|T^nG(x) - T^nf(x)\| + \|T^nF(x) - T^nf(x)\|\\ &\le \frac{L^{n+1}}{|a|^3(1-L)}\phi(x,0)\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Since $0 < L < 1$, letting $n \to \infty$ in the above inequality, we have $F = G$. □

Related with Theorem 3.2, we can also have the following theorem. And the proof is similar to that of Theorem 3.2.

Theorem 3.3. *Assume that $\phi : X^2 \longrightarrow [0,\infty)$ is a function such that*

$$\phi(ax, ay) \le |a|^3L\phi(x,y) \tag{3.4}$$

for all $x, y \in X$ and some real number L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that satisfying (3.2). Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{1}{2|a|^3(1-L)}\phi(x,0) \tag{3.5}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g,h) = \inf\{c \in [0,\infty) \mid \|g(x) - h(x)\| \le c\ \phi(x,0), \forall x \in X\}.$$

Then (S,d) is a complete metric space([9]). Define a mapping $T : S \longrightarrow S$ by $Tg(x) = \frac{1}{a^3}g(ax)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (3.4), we have

$$\|Tg(x) - Th(x)\| = \frac{1}{|a|^3}\|g(ax) - h(ax)\| \le cL\phi(x,0)$$

for all $x \in X$. Hence we have $d(Tg,Th) \le Ld(g,h)$ for all $g, h \in S$ and so T is a strictly contractive mapping. Putting $y = 0$ in (3.2), we get

$$\|2f(ax) - 2a^3f(x)\| \le \phi(x,0)$$

for all $x \in X$ and hence

$$\left\|f(x) - \frac{1}{a^3}f(ax)\right\| \le \frac{1}{2|a|^3}\phi(x,0)$$

for all $x \in X$ and hence $d(f,Tf) \le \frac{1}{2|a|^3} < \infty$. By Theorem 3.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of T such that $d(T^nf,F) \to 0$ as $n \to \infty$ and

$$\|F(x) - f(x)\| \le \frac{1}{2|a|^3(1-L)}\phi(x,0)$$

for all $x \in X$. Replacing x, y by a^nx, a^ny in (3.2), respectively, and multiplying (3.2) by $|a|^{-3n}$, by (O3), we have

$$\left\|a^{-3n}Df(a^nx, a^ny)\right\| \le L^n\phi(x,y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DF(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$ and by Theorem 2.1, F is an orthogonally cubic mapping.

Now, we will show the uniqueness of F. Let $G : X \longrightarrow Y$ be another orthogonally cubic mapping with (3.3). Since F and G are fixed points of T, by (3.3), we get

$$\begin{aligned}\|G(x) - F(x)\| &= \|T^n G(x) - T^n F(x)\| \\ &\le \|T^n G(x) - T^n f(x)\| + \|T^n F(x) - T^n f(x)\| \\ &\le \frac{L^n}{|a|^3(1-L)}\phi(x, 0)\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Since $0 < L < 1$, letting $n \to \infty$ in the above inequality, we have $F = G$. □

As an example of $\phi(x, y)$ in Theorem 3.2 and Theorem 3.3, we can take $\phi(x, y) = \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p})$ for some positive real numbers ϵ and p. Then we can formulate the following corollary :

Corollary 3.4. *Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\|Df(x, y)\| \le \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p}) \tag{3.6}$$

for all $x, y \in X$ with $x \perp y$ and a fixed positive number p with $p \neq \frac{3}{2}$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{1}{2\Big||a|^{2p} - |a|^3\Big|}\|x\|^{2p}$$

for all $x \in X$.

By Theorem 2.1, if $c = -\frac{1}{3}ab^2$, then we have the following orthogonally Euler-Lagrange type cubic functional equation :

$$\begin{aligned}&f(ax + by) + f(ax - by) - \frac{2}{3}ab^2 f(x - y) - \frac{1}{3}ab^2 f(x + 2y) \\ &- a(2a^2 - b^2)f(x) + 2ab^2 f(y) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. By Corollary 3.6, we have the following exmaple.

Example 3.5. Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that

$$\begin{aligned}&\|f(ax + by) + f(ax - by) - \frac{2}{3}ab^2 f(x - y) - \frac{1}{3}ab^2 f(x + 2y) \\ &- a(2a^2 - b^2)f(x) + 2ab^2 f(y)\| \le \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p})\end{aligned}$$

for all $x, y \in X$ with $x \perp y$ and a fixed positive number p with $p \neq \frac{3}{2}$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{1}{2\Big||a|^{2p} - |a|^3\Big|}\|x\|^{2p}$$

for all $x \in X$.

It should be remarked that if a functional inequality can be deformed into the type of (3.2), then a solution of the original functional equation is cubic. In the following theorems, we give a simple example.

Theorem 3.6. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\phi(x,y) \leq \frac{1}{8}L\phi(2x,2y) \tag{3.7}$$

for all $x, y \in X$, some real number L with $0 < L < 1$ and $f : X \longrightarrow Y$ a mapping such that $f(0) = 0$ and

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \leq \phi(x,y) \tag{3.8}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \frac{L}{16(1-L)}[3\phi(x,0) + 8\phi(0,x)]$$

for all $x \in X$.

Proof. Letting $x = 0$ in (3.8), we have

$$\|f(y) + f(-y)\| \leq \phi(0,y) \tag{3.9}$$

for all $y \in X$ and letting $y = 0$ in (3.8), we have

$$\|f(2x) - 8f(x)\| \leq \frac{1}{2}\phi(x,0) \tag{3.10}$$

for all $y \in X$. Letting $y = 2y$ in (3.8), by (3.10), we have

$$\begin{aligned} &\|8f(x+y) + 8f(x-y) - 2f(x+2y) - 2f(x-2y) - 12f(x)\| \\ \leq\ &\frac{1}{2}\phi(x+y,0) + \frac{1}{2}\phi(x-y,0) + \phi(x,2y) \end{aligned} \tag{3.11}$$

for all $x, y \in X$ with $x \perp y$. Interchang x and y in (3.8), by (3.9), we get

$$\begin{aligned} &\|f(x+2y) - f(x-2y) - 2f(x+y) + 2f(x-y) - 12f(y)\| \\ \leq\ &\phi(y,x) + \phi(0,x-2y) + 2\phi(0,x-y) \end{aligned} \tag{3.12}$$

for all $x, y \in X$ with $x \perp y$. Putting $a = 2$, $b = 1$, and $c = -4$ in $Df(x,y)$, by (3.8), (3.11), and (3.12), we have

$$\|Df(x,y)\| \leq \psi(x,y)$$

for all $x, y \in X$, where

$$\begin{aligned} \psi(x,y) = \phi(x,y) + 2\phi(y,x) + \frac{1}{2}\phi(x+y,0) + \frac{1}{2}\phi(x-y,0) + \phi(x,2y) \\ + 2\phi(0,x-2y) + 4\phi(0,x-y) \end{aligned}$$

Since ψ satisfies (3.1), by Theorem 3.2, we get the result. □

Similar to Theorem 3.6, we have the following theorem :

Theorem 3.7. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\phi(2x,2y) \leq 8L\phi(2x,2y) \tag{3.13}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

for all $x, y \in X$, some real number L with $0 < L < 1$ and $f : X \longrightarrow Y$ a mapping satisfying $f(0) = 0$ (3.8). Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \frac{1}{16(1-L)}[3\phi(x,0) + 8\phi(0,x)]$$

for all $x \in X$.

By Theorem 3.6 and Theorem 3.7, we have the following corollary :

Corollary 3.8. *Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \leq \|x\|^p\|y\|^p+\|x\|^{2p}+\|y\|^{2p}.$$

for all $x, y \in X$ and a fixed positive real number p with $p \neq \frac{3}{2}$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \frac{11}{2|8 - 2^{2p}|}\|x\|^{2p}$$

for all $x \in X$.

References

[1] L. Cădariu and V. Radu, *Fixed points and the stability of Jensens functional equation*, J Inequal Pure Appl. Math. **4**(2003), 1-7.

[2] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74**(1968), 305-309.

[3] M. Fochi, *Functional equations in A-orthogonal vectors*, Aequationes Math. **38**(1989), 28-40.

[4] R. Ger and J. Sikorska, *Stability of the orthogonal additivity*, Bull. Polish. Acad. Sci. Math. **43**(1995), 143-151.

[5] S. Gudder and D. Strawther, *Orthogonally additive and orthogonally increasing functions on vector spaces*, Pacific. J. Math. **58**(1975), 427-436.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27**(1941), 222-224.

[7] G. Isac and Th. M. Rassias, *Stability of ψ-additive mappings: applications to nonlinear analysis*, Intern. J. Math. Math. Sci. **19**(1996), 219-228.

[8] K. Jun, H. Kim and I. Chang, *On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation*, J. Comput. Anal. Appl., **7** (2005) 21-33 .

[9] D. Mihe and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343**(2008), 567-572.

[10] M. S. Moslehian, *On the orthogonal stability of the Pexiderized quadratic equation*, J. Differ. Equat. Appl. **11**(2005), 999-1004.

[11] M. S. Moslehian, *On the stability of the orthogonal Pexiderized Cauchy equation*, J. Math. Anal. Appl. **318**(2006), 211-223.

[12] M. S. Moslehian and Th. M. Rassias, *Orthogonal stability of additive type equations*, Aequationes Math. **73**(2007), 249-259.

[13] C. Park, *Orthogonal Stability of an Additive-Quadratic Functional Equation*, Fixed Point Theory and Applications **2011**(2011), 1-11.

[14] Th. M. Rassias, *On the stability of the linear mapping in Banach sapces*, Proc. Amer. Math. Sco. **72**(1978), 297-300.

[15] J. M. Rassias, *Solution of the Ulam stability problem for cubic mappings*, Glasnik Matematički, 36(2001), 63-72.

[16] J. Rätz, *On orthogonally additive mappings*, Aequationes Math. **28**(1985), 35-49 .

[17] J. Rätz and G. Y. Szabó, *On orthogonally additive mappings IV*, Aequationes Math. **38**(1989), 73-85.

[18] G. Y. Szabó, *Sesquilinear-orthogonally quadratic mappings*, Aequationes Math. **40**(1990), 190-200.

[19] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960, Chapter VI.

[20] F. Vajzović, *ber das Funktional H mit der Eigenschaft:* $(x, y) = 0 \Rightarrow H(x + y) + H(x - y) = 2H(x) + 2H(y)$, Glasnik Mat. Ser III. **2**(1967), 73-81.

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 16890, Korea
E-mail address: kci206@hanmail.net

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 16890, Korea
E-mail address: gilhan@dankook.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

CERTAIN SUBCLASS OF HARMONIC MULTIVALENT FUNCTIONS DEFINED BY DERIVATIVE OPERATOR

ADRIANA CĂTAŞ[1*], ROXANA ŞENDRUŢIU[2] AND LOREDANA-FLORENTINA IAMBOR[3]

Abstract. In the present paper, we investigate new properties of a new subclass of multivalent harmonic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, under certain conditions involving a new generalized differential operator. Furthermore, a representation theorem, an integral property and convolution conditions for the subclass denoted by $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ are also obtained. Finally, we will give an application of neighborhood.

Keywords: differential operator, harmonic function, extreme points, convolution, neighborhood.
2000 Mathematical Subject Classification: 30C45.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simple connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. (See also Clunie and Sheil-Small [5] for more details.)

Denote by $S_{\mathcal{H}}(p, n), (p, n \in \mathbb{N} = \{1, 2, \ldots\})$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Then for $f = h + \bar{g} \in S_{\mathcal{H}}(p, n)$ we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p+n-1}^{\infty} b_k z^k, \quad |b_{p+n-1}| < 1. \tag{1.1}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Let $\tilde{S}_{\mathcal{H}}(p,n,m),(p,n\in\mathbb{N},m\in\mathbb{N}_0\cup\{0\})$ denote the family of functions $f_m=h+\bar{g}_m$ that are harmonic in D with the normalization
(1.2)

$$h(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k,\quad g_m(z)=(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|z^k,\quad |b_{p+n-1}|<1.$$

Definition 1.1. [4] Let $H(U)$ denote the class of analytic functions in the open unit disc $U=\{z\in\mathbb{C}:\ |z|<1\}$ and let $\mathcal{A}(p)$ be the subclass of the functions belonging to $H(U)$ of the form

$$h(z)=z^p+\sum_{k=p+n}^{\infty}a_kz^k.$$

For $m\in\mathbb{N}_0$, $\lambda\geq 0$, $\delta\in\mathbb{N}_0$, $l\geq 0$ we define the generalized differential operator $I^m_{\lambda,\delta}(p,l)$ on $\mathcal{A}(p)$ by the following infinite series

$$I^m_{\lambda,\delta}(p,l)h(z)=(p+l)^mz^p+\sum_{k=p+n}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)a_kz^k, \tag{1.3}$$

where

$$C(\delta,k)=\begin{pmatrix} k+\delta-1\\ \delta\end{pmatrix}=\frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(\delta+1)}. \tag{1.4}$$

Remark 1.2. When $\lambda=1$, $p=1$, $l=0$, $\delta=0$ we get Sălăgean differential operator [13]; $p=1$, $m=0$ gives Ruscheweyh operator [12]; $p=1$, $l=0$, $\delta=0$ implies Al-Oboudi differential operator of order m (see [1]); $\lambda=1$, $p=1$, $l=0$ operator (1.3) reduces to Al-Shaqsi and Darus differential operator [2] and when $p=1$, $l=0$ we reobtain the operator introduced by Darus and Ibrahim in [6].

Definition 1.3. [4] Let $f\in S_{\mathcal{H}}(p,n)$, $p\in\mathbb{N}$. Using the operator (1.3) for $f=h+\bar{g}$ given by (1.1) we define the differential operator of f as

$$I^m_{\lambda,\delta}(p,l)f(z)=I^m_{\lambda,\delta}(p,l)h(z)+(-1)^m\overline{I^m_{\lambda,\delta}(p,l)g(z)} \tag{1.5}$$

where

$$I^m_{\lambda,\delta}(p,l)h(z)=(p+l)^mz^p+\sum_{k=p+n}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)a_kz^k \tag{1.6}$$

and

$$I^m_{\lambda,\delta}(p,l)g(z)=\sum_{k=p+n-1}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)b_kz^k. \tag{1.7}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Remark 1.4. When $\lambda = 1$, $l = 0$, $\delta = 0$ the operator (1.5) reduces to the operator introduced earlier in [8] by Jahangiri et al.

Definition 1.5. [4] A function $f \in S_{\mathcal{H}}(p,n)$ is said to be in the class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if

$$\frac{1}{p+l}\mathrm{Re}\left\{\frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^{m}_{\lambda,\delta}(p,l)f(z)}\right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{1.8}$$

where $I^m_{\lambda,\delta}f$ is defined by (1.5), for $m \in \mathbb{N}_0$.

Finally, we define the subclass

$$\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \equiv AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \cap \tilde{S}_{\mathcal{H}}(p,n,m). \tag{1.9}$$

Remark 1.6. The class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ includes a variety of well-known subclasses of $S_{\mathcal{H}}(p,n)$. For example, letting $n = 1$ we get $AL_{\mathcal{H}}(1,1,0,\alpha,1,0) \equiv HK(\alpha)$ in [7], for $n = 1$, $AL_{\mathcal{H}}(1,m-1,0,\alpha,1,0) \equiv S_H(t,u,\alpha)$ in [14], $AL_{\mathcal{H}}(p,n+p,0,\alpha,1,0) \equiv SH_p(n,\alpha)$ in [11] and $n = 1$, $AL_{\mathcal{H}}(1,m,\delta,\alpha,1,0) \equiv M_{\mathcal{H}}(m,\delta,\alpha)$ in [3].

Theorem 1.7. [4] *Let $f_m = h + \bar{g_m}$ be given by (1.2). Then $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if and only if*

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k| + \tag{1.10}$$

$$+ \sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \leq 1,$$

where $\lambda n \geq \alpha(p+l)$, $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda \geq 0$ and

$$d_{p,k}(m,\lambda,l) = [p+\lambda(k-p)+l]^m. \tag{1.11}$$

Remark 1.8. The harmonic function

$$f(z) = z^p + \sum_{k=p+n}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}x_k z^k + \tag{1.12}$$

$$+ \sum_{k=p+n-1}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\overline{y_k z^k},$$

where $\sum_{k=p+n}^{\infty}|x_k| + \sum_{k=p+n-1}^{\infty}|y_k| = 1$, $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda n \geq \alpha(p+l)$, $\lambda \geq 0$ and $d_{p,k}(m,\lambda,l)$ is given in (1.11), show that the coefficient bound expressed by (1.10) is sharp.

2. Convex combination and extreme points

In this section, we show that the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination of its members.

For $i=1,2,3,...$, let the functions $f_{m_i}(z)$ be

$$f_{m_i}(z)=z^p-\sum_{k=p+n}^{\infty}|a_{k,i}|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_{k,i}|\bar{z}^k. \tag{2.1}$$

Theorem 2.1. *The class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination.*

Proof. For $i=1,2,3,...$, let $f_{m_i}(z)\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, where the functions $f_{m_i}(z)$ are defined by (2.1). Then by (1.10) we have

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_{k,i}|+ \tag{2.2}$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_{k,i}|\leq 1.$$

For $\sum_{i=1}^{\infty}t_i=1$, $0\leq t_i\leq 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty}t_if_{m_i}(z)=z^p-\sum_{k=p+n}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{k,i}|\right)z^k+(-1)^m\sum_{k=p+n-1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{k,i}|\right)\bar{z}^k.$$

Then by (2.2) one obtains

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\left(\sum_{i=1}^{\infty}t_i|a_{k,i}|\right)+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\left(\sum_{i=1}^{\infty}t_i|b_{k,i}|\right)=$$
$$\sum_{i=1}^{\infty}t_i\cdot\left\{\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_{k,i}|+\right.$$
$$\left.+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_{k,i}|\right\}\leq\sum_{i=1}^{\infty}t_i=1,$$

and therefore $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. □

Further, we will determine a representation theorem for functions in $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ from which we also establish the extreme points of closed convex hulls of $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ denoted by $clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 2.2. *Let $f_m(z)$ given by (1.2). Then $f_m(z) \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if and only if*

$$f_m(z) = X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z), \tag{2.3}$$

where $h_p(z) = z^p$

$$h_k(z) = z^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} z^k, \tag{2.4}$$

$$k = p+n, p+n+1, ...,$$

and

$$g_{m_k}(z) = z^p + (-1)^m \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} \bar{z}^k, \tag{2.5}$$

$$k = p+n-1, p+n, ...,$$

with $X_k \geq 0, Y_k \geq 0, X_p = 1 - \sum_{k=p+n}^{\infty} X_k - \sum_{k=p+n-1}^{\infty} Y_k$.

In particular, the extreme points of $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof. For the functions f_m of the form (2.3), we have

$$f_m(z) = X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z) =$$

$$= z^p - \sum_{k=p+n}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} X_k z^k +$$

$$+(-1)^m \sum_{k=p+n-1}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} Y_k \bar{z}^k.$$

Consequently,

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} a_k +$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}b_k=$$

$$=\sum_{k=p+n}^{\infty}X_k+\sum_{k=p+n-1}^{\infty}Y_k=1-X_p\leq 1,$$

where

$$a_k=\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}X_k$$

$$b_k=\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}Y_k$$

and therefore $f_m\in clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Conversely, suppose that $f_m\in clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.
Setting

$$X_k=\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|, \tag{2.6}$$

$$k=p+n,p+n+1,...,$$

$$Y_k=\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|$$

$$k=p+n-1,p+n,...,$$

and $X_p=1-\sum_{k=p+n}^{\infty}X_k-\sum_{k=p+n-1}^{\infty}Y_k$. We note by Theorem 1.7 that $0\leq Y_k\leq 1, 0\leq X_k\leq 1,\ and\ X_p\geq 0.$

We obtain the required representation since f_m can be written as

$$f_m(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|\bar{z}^k=$$

$$=z^p-\sum_{k=p+n}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)X_k}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}z^k+$$

$$+(-1)^m\sum_{k=p+n-1}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)Y_k}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\bar{z}^k=$$

$$=z^p-\sum_{k=p+n}^{\infty}(z^p-h_k(z))X_k+\sum_{k=p+n-1}^{\infty}(g_{m_k}(z)-z^p)Y_k=$$

$$=\sum_{k=p+n}^{\infty}h_k(z)X_k+\sum_{k=p+n-1}^{\infty}g_{m_k}(z)Y_k+z^p\left(1-\sum_{k=p+n}^{\infty}X_k-\sum_{k=p+n-1}^{\infty}Y_k\right)=$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

$$= X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z),$$

as required. $\square$

3. Integral property and convolution conditions

In this section we will examine the closure properties of the class $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ under the generalized Bernardi-Libera-Livingston integral operator and also convolution properties of the same class.

Now, for $f = h + \bar{g}$ given by (1.1) we define the modified generalized Bernardi-Libera-Livingston integral operator of f as

$$\mathcal{L}_c(f(z)) = \mathcal{L}_c(h(z)) + \overline{\mathcal{L}_c(g(z))}, \quad c > -p, \tag{3.1}$$

where

$$\mathcal{L}_c(h(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} h(t) dt$$

and

$$\mathcal{L}_c(g(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} g(t) dt.$$

Putting $g = 0$ in (3.1), we get the definition of the generalized Bernardi-Libera-Livingston integral operator on analytic functions, (see [9], [10]).

Theorem 3.1. *Let $f \in \widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$. Then $\mathcal{L}_c(f)$ belongs to the class $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$.*

Proof. From the representation of $\mathcal{L}_c(f)$, it follows that

$$\mathcal{L}_c(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} (h(t) + \bar{g}_m(t)) dt =$$

$$= \frac{c+p}{z^c} \left[\int_0^z t^{c-1} \left(t^p - \sum_{k=p+n}^{\infty} |a_k| t^k \right) dt + (-1)^m \overline{\int_0^z t^{c-1} \left(\sum_{k=p+n-1}^{\infty} |b_k| t^k \right) dt} \right] =$$

$$= z^p - \sum_{k=p+n}^{\infty} |A_k| z^k + (-1)^m \sum_{k=p+n-1}^{\infty} |B_k| \bar{z}^k,$$

where

$$A_k = \frac{c+p}{c+k} a_k, \ \ B_k = \frac{c+p}{c+k} b_k.$$

Further, one obtains

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha) + \lambda(k-p)] d_{p,k}(m, \lambda, l) C(\delta, k)}{(p+l)^{m+1}(1-\alpha)} \cdot \frac{c+p}{c+k} |a_k| +$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\frac{c+p}{c+k}|b_k|\leq$$
$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\leq 1.$$

Since $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, by Theorem 1.7 we have $\mathcal{L}_c(f) \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. □

For the harmonic functions

$$f_1(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|\bar{z}^k,\ |b_{p+n-1}|<1, \tag{3.2}$$

and

$$f_2(z)=z^p-\sum_{k=p+n}^{\infty}|A_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k,\ |B_{p+n-1}|<1, \tag{3.3}$$

we define the convolution of f_1 and f_2 as

$$(f_1*f_2)(z)=f_1(z)*f_2(z)=z^p-\sum_{k=p+n}^{\infty}|a_kA_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_kB_k|\bar{z}^k.$$

In the following theorem, we examine the convolution properties of the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 3.2. *For $0\leq\beta\leq\alpha<1$ let $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ and $f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. Then $f_1*f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$.*

Proof. Let $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ and $f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. Obviously, the coefficients of f_1 and f_2 must satisfy similar conditions to the inequality (1.10). Therefore, for the coefficients of f_1*f_2 we can write

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|a_kA_k|+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|b_kB_k|\leq$$
$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|a_k|+$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|b_k|\leq$$
$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)+}|b_k|\leq 1,$$

because $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. In view of Theorem 1.7, it follows that $f_1*f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. $\square$

4. An application of neighborhood

Let us define a generalized (n,η)-neighborhood of a function f given in (1.2) to be the set

$$N_{n,\eta}(f)=\Big\{F_m(z)\in\tilde{S}_{\mathcal{H}}(p,n,m):$$
$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k-A_k|+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k-B_k|\leq\eta\Big\}$$

where $F_m(z)=z^p-\sum_{k=p+n}^{\infty}|A_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k$.

Theorem 4.1. *Let $f_m=h+\bar{g}_m$ be given by (1.2). If the functions f_m satisfy the conditions*

$$\sum_{k=p+n}^{\infty}k\cdot\Bigg[\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$
$$+\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\Bigg]\leq 1-U_{p,\delta}^{\alpha}(m,\lambda,l)\tag{4.1}$$

and

$$\eta\leq\frac{p+n-\alpha-1}{p+n-\alpha}\left(1-U_{p,\delta}^{\alpha}(m,\lambda,l)\right),\tag{4.2}$$

$\lambda n\geq\alpha(p+l)$, where

$$U_{p,\delta}^{\alpha}(m,\lambda,l)=\frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,p+n-1}(m,\lambda,l)C(\delta,p+n-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|$$

then $N_{n,\eta}(f)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Proof. Let f_m satisfy (4.1) and $F_m \in N_{n,\eta}(f)$. We have

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|A_k|+$$

$$+\sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|B_k| \leq$$

$$\leq \eta + \sum_{k=p+n}^{\infty} \left(\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+ \right.$$

$$\left. \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \right) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq$$

$$\eta + \frac{1}{p+n-\alpha} \sum_{k=p+n}^{\infty} k \cdot \left(\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+ \right.$$

$$\left. \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \right) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq$$

$$\leq \eta + \frac{1}{p+n-\alpha}(1-U^{\alpha}_{p,\delta}(m,\lambda,l)) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq 1.$$

Hence, for $\eta \leq \frac{p+n-\alpha-1}{p+n-\alpha}\left(1-U^{\alpha}_{p,\delta}(m,\lambda,l)\right)$ we deduce that $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. □

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Inter. J. of Math. and Mathematical Sci., **27**(2004), 1429-1436.

[2] K. Al-Shaqsi, M. Darus, *An operator defined by convolution involving polylogarithms functions*, Journal of Math. and Statistics, **4**(1)(2008), 46-50.

[3] K. Al-Shaqsi, M. Darus, *On Harmonic Functions Defined by Derivative Operator*, Journal of Inequalities and Applications, vol. 2008, Article ID 263413, doi: 10.1155/2008/263413.

[4] A. Cătaş, R. Şendruţiu *On harmonic multivalent functions defined by a new derivative operator*, Journal of Computational Analysis and Applications, Volume: 28, Issue: 5, Pages: 775-780, 2020.

[5] J. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn, Ser. A I. Math. **9**(1984), 3-25.

[6] M. Darus, R. W. Ibrahim, *On new classes of univalent harmonic functions defined by generalized differential operator*, Acta Universitatis Apulensis, **18**(2009), 61-69.

[7] J. M. Jahangiri, *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklowdowska Sect. A, **52**(1998), 57-66.

[8] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya *Sălăgean type harmonic univalent functions* South. J. Pure Appl. Math., **2**(2002), 77-82.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

[9] R. J. Libera, *Some classes of regular univalent functions*, Proc. Am. Math. Soc. 63(1965), 755-758.

[10] A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Am. Math. Soc. 17(1966), 352-357.

[11] Om P. Ahuja and J. M. Jahangiri, *Multivalent harmonic starlike functions with missing coefficients*, Math. Sci. Res. J., **7**(9)(2003), 347-352.

[12] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[13] Gr. Şt. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, Heidelberg and New York, 1013(1983), 362-372.

[14] Sibel Yalcin, *A new class of Sălăgean-type harmonic univalent functions* Appl. Math. Letters, **18**(2005), 191-198.

[1] Department of Mathematics and Computer Sciences, University of Oradea, Str. Universităţii, No.1, 410087 Oradea, Romania
[*] *Corresponding author: acatas@gmail.com*

[2] Faculty of Environmental Protection, University of Oradea, Str. B-dul Gen. Magheru, No.26, 410048 Oradea, Romania
E-mail address: roxana.sendrutiu@gmail.com

[3] Department of Mathematics and Computer Sciences, University of Oradea, Str. Universităţii, No.1, 410087 Oradea, Romania
E-mail addresses: iambor.loredana@gmail.com

ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS

N. E. CHO, M. K. AOUF, AND A. O. MOSTAFA

Abstract. TThe purpose of the present paper is to investigate some argument properties for certain analytic functions in the open unit disk. The main results presented in here generalize some previous those concerning starlike function of reciprocal of order beta and strongly starlike functions.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C}\ ,\ |z| < 1\}). \tag{1.1}$$

A function $f(z) \in \mathcal{A}$ is said to be in the class $C(\alpha)$ of convex functions of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < 1) \tag{1.2}$$

and is said to be in the class $\mathbb{S}^*(\alpha)$ of starlike functions of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1). \tag{1.3}$$

We note that $\mathcal{C}(0) = \mathcal{C}$ and $\mathbb{S}^*(0) = \mathbb{S}^*$, where $\mathcal{C}$ and $\mathbb{S}^*$ are, respectively, the well-known classes of convex and starlike functions.

The classical result of Marx [5] and Strahhäcker [8] asserts that a convex function is starlike of order $1/2$, that is,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}) \Longrightarrow \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{1.4}$$

If $f(z) \in \mathbb{S}^*$ satisfies the condition

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \beta \quad (0 \leq \beta < 1; z \in \mathbb{U}), \tag{1.5}$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. univalent functions, starlike function of reciprocal of order β, strongly starlike functions, convex functions.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

then $f(z)$ is said to be starlike of reciprocal of order β (see Nunokawa et al. [4]).

In [7] Sakaguchi proved that: If $f(z) \in \mathcal{A}$ and $g(z) \in \mathbb{S}^*$, then

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \quad (z \in \mathbb{U}) \Longrightarrow \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.6}$$

In [6] Pommerenke generalized Sakaguchi's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{C}$ and

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1; z \in \mathbb{U}), \tag{1.7}$$

then

$$\left|\arg \frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)}\right| \leq \frac{\pi}{2}\alpha \quad (|z_1| < 1, |z_2| < 1). \tag{1.8}$$

Recently, Nunokawa et al. [4] generalized Pommerenke's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{C}$, then $g(z)$ is starlike of reciprocal of order β and

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{\pi}{2}\alpha + \tan^{-1} \frac{\alpha\beta}{1+\alpha} \quad (z \in \mathbb{U}; 0 < \alpha \leq 1; 0 \leq \beta < 1), \tag{1.9}$$

then

$$\left|\arg \frac{f(z)}{g(z)}\right| \leq \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{1.10}$$

Also Kanas et al. [1] generalized Sakaguchi's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathbb{S}^*$, then

$$\operatorname{Re}\left\{\left(\frac{f(z)}{g(z)}\right)^{1-\alpha}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\right\} > 0 \ (z \in \mathbb{U}; 0 \leq \alpha \leq 1) \Longrightarrow \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > \alpha \ (z \in \mathbb{U}), \tag{1.11}$$

where the powers in (1.11) are meant as the principal values.

Also Kanas et al. [1] defined the class $\mathcal{H}(\alpha)$ as follows.

$$\mathcal{H}(\alpha) = \left\{f(z) \in \mathcal{A}, g(z) \in \mathbb{S}^* : \operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > 0 \ (0 \leq \alpha \leq 1)\right\}. \tag{1.12}$$

In the present paper, we extend some results obtained by Kanas et al. [1], Liu [2], Nunokawa et al. [4], Pommerenke [6] and Sakaguchi [7] by using Nunowawa's lemma [3].

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

2. Main results

To derive our results, we need the following lemma due to Nunokawa [3].

Lemma 2.1. *[3] Let a function $p(z)$ with $p(0) = 1$ and $p(z) \neq 0$ be analytic in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|\,,\ \alpha > 0),$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha \ \textit{and}\ |\arg p(z_0)| = \frac{\pi}{2}\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \ \textit{when}\ \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \ \textit{when}\ \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia(a > 0).$$

Theorem 2.2. *Let $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β. Suppose that*

$$\left|\arg\left[(1-\lambda)\frac{f(z)}{g(z)} + \lambda\frac{f'(z)}{g'(z)} - \gamma\right]\right| < \frac{\pi}{2}\rho \quad (0 \leq \lambda \leq 1;\ 0 \leq \gamma < 1;\ z \in \mathbb{U}), \tag{2.1}$$

where

$$\rho = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\beta\lambda}{1+\alpha\lambda}\right)(0 < \alpha \leq 1;\ 0 \leq \gamma < 1). \tag{2.2}$$

Then we have

$$\left|\left(\arg\frac{f(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{2.3}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{f(z)}{g(z)} - \gamma\right). \tag{2.4}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $p(z) \neq 0$. It follows from (2.4) that

$$\frac{f'(z)}{g'(z)} = \gamma + (1-\gamma)p(z)\left[1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right]. \tag{2.5}$$

Also, from (2.4) and (2.5), we have

$$(1-\lambda)\frac{f(z)}{g(z)} + \lambda\frac{f'(z)}{g'(z)} - \gamma = (1-\gamma)p(z)\left[1 + \lambda\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right]. \tag{2.6}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1).$$

Then from Lemma 1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k,$$

where

$$k \geq \frac{1}{2}(a + a^{-1}) \geq 1 \text{ when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}(a + a^{-1}) \leq -1 \text{ when} \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where $(p(z_0))^{1/\alpha} = \pm ia (a > 0)$. Since $g(z) \in \mathcal{C}$, from Marx-Strohhäcker's theorem [5,8], we have

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 1/2 \quad (z \in \mathbb{U}),$$

so that $g(z) \in \mathbb{S}^*(1/2)$. Putting $\frac{zg'(z)}{g(z)} = u + iv$, where $u > 1/2$. Then

$$\left|\frac{g(z)}{zg'(z)} - 1\right|^2 = \left|\frac{1 - u - iv}{u + iv}\right|^2 = \frac{1 - 2u + u^2 + v^2}{u^2 + v^2} < 1.$$

Therefore,

$$\left|\frac{g(z)}{zg'(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{2.7}$$

which implies that

$$\left|\mathrm{Im}\left\{\frac{g(z)}{zg'(z)}\right\}\right| < 1 \quad (z \in \mathbb{U}), \tag{2.8}$$

and from the assumption of the theorem, we have

$$\mathrm{Re}\left\{\frac{g(z)}{zg'(z)}\right\} > \beta \quad (0 \leq \beta < 1;\ z \in \mathbb{U}). \tag{2.9}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

For the case $|\arg p(z_0)| = \frac{\pi}{2}\alpha$, from (2.5), (2.6) and (2.8), we have

$$\begin{aligned}
& \arg\left\{(1-\lambda)\frac{f(z_0)}{g(z_0)} + \lambda\frac{f'(z_0)}{g'(z_0)} - \gamma\right\} \\
= \ & \arg p(z_0) + \arg\left\{1 + \lambda\frac{z_0 p'(z_0)}{p(z_0)}\left(\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\} \\
= \ & \frac{\pi}{2}\alpha + \arg\left\{1 + i\alpha k\lambda\left(\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)} + i\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\} \\
= \ & \frac{\pi}{2}\alpha + \arg\left\{1 - \alpha k\lambda\left(\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right) + ik\alpha\lambda\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)}\right\} \\
= \ & \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha k\lambda\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)}}{1 + \alpha k\lambda\left|\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right|}\right\} \\
\geq \ & \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha k\lambda\beta}{1+\alpha k\lambda}\right\} \geq \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha\lambda\beta}{1+\alpha\lambda}\right\}.
\end{aligned}$$

This contradicts the assumption of the theorem, then

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

For the case $|\arg p(z_0)| = -\frac{\pi}{2}\alpha$, applying the same method above, we have a contradiction. This completes the proof of Theorem 2.2. ∎

Remark. Putting $\lambda = 1$ in Theorem 1, we get the result obtained by Liu [2, Theorem 2.1]. Also, from Theorem 1, we have the results obtained by Kanas [1], Nunokawa [4] and Sakaguchi [7].

Theorem 2.3. *Let $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β $(0 \leq \beta < 1)$. Suppose that*

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma}\right| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U}), \tag{2.10}$$

where

$$\rho = (\mu+\gamma)\alpha + \frac{2\gamma}{\pi}\tan^{-1}\left(\frac{\alpha\beta}{1+\alpha}\right) \quad (z \in \mathbb{U}), \tag{2.11}$$

μ and γ are fixed positive real numbers with $0 < \mu + \gamma \leq 1$ and $0 < \alpha \leq 1$. Then

$$\left|\arg\frac{f(z)}{g(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{2.12}$$

Proof. Let us define the function $p(z)$ by (2.4). It follows from (2.4) and (2.5) that

$$\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma}=(p(z))^{\mu+\gamma}\left(1+\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right)^{\gamma}$$

and

$$\begin{aligned}\arg\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma} &= \mu\arg\frac{f(z)}{g(z)}+\gamma\arg\frac{f'(z)}{g'(z)}\\ &= (\mu+\gamma)\arg p(z)+\gamma\arg\left(1+\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right).\end{aligned}\tag{2.13}$$

Suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$|\arg p(z)|<\frac{\pi}{2}\alpha\quad(|z|<|z_0|)\ \textit{and}\ |\arg p(z_0)|=\frac{\pi}{2}\alpha\quad(0<\alpha\leq 1).$$

Then, using Lemma 1, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\beta.$$

For the case $\arg p(z)=\frac{\pi}{2}\alpha$, from (2.7), (2.8) and (2.13), we have

$$\begin{aligned}\arg\left(\frac{f(z_0)}{g(z_0)}\right)^{\mu}\left(\frac{f'(z_0)}{g'(z_0)}\right)^{\gamma} &= (\mu+\gamma)\arg p(z_0)+\gamma\arg\left(1+\frac{z_0p'(z_0)}{p(z_0)}\frac{g(z_0)}{z_0g'(z_0)}\right)\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\arg\left\{1+i\alpha k\left(\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}+i\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)\right\}\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\arg\left\{1-\alpha k\left(\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)+i\alpha k\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}\right\}\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha k\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}}{1+\alpha k\left|\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right|}\right\}\\ &\geq (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha\beta k}{1+\alpha k}\right\}\\ &\geq (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha\beta}{1+\alpha}\right\}.\end{aligned}$$

This contradicts the assumption of the theorem, then we have

$$|\arg p(z)|<\frac{\pi}{2}\alpha\quad(z\in\mathbb{U}).$$

For the case $|\arg p(z_0)|=-\frac{\pi}{2}\alpha$, applying the same method above, we have a contradiction. This completes the proof of Theorem 2.3.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Putting $\mu = 1 - \gamma$ $(\gamma > 0)$ in Theorem 2, we obtain the following corollary.

Corollary 1. Let $f(z) \in \mathcal{A}, g(z) \in \mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β $(0 < \beta \leq 1)$. Suppose that

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)^{1-\gamma}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma}\right| < \frac{\pi}{2}\rho \quad (\gamma > 0;\ z \in \mathbb{U}),$$

$$\rho = \alpha + \frac{2\gamma}{\pi}\tan^{-1}\left(\frac{\alpha\beta}{1+\alpha}\right) \quad (0 < \alpha \leq 1).$$

Then

$$\left|\arg\frac{f(z)}{g(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Remark. Putting $\gamma = 1$ in Corollary 1, we have the result obtained by Nunokawa et al. [[4], Theorem 2.3].

Acknowledgement

The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A01050861).

References

[1] S. Kanas, A. Lecko and A. Moleda, Certain generalization of the Sakaguchi lemma, Zeszyty Nauk. Pol. Rzes. Folia Sci. Univ. Tech. Resov., 38(1987), 35-41.
[2] Jin-Lin Liu, Some argument inequalities for certain analytic functions, Math. Solvaca, 62(2012), 25-28.
[3] M. Nunokawa, On the order of strongly convex functions, Proc. Japan Acad. Ser. A, 69(1993), no. 7, 234-237.
[4] M. Nunokawa, S. Owa, J. Nishiwaki, K. Kuroni and T. Hayanni, Differential subordination and argument property, Comput. Math. Appl., 56(2008), 2733-2736.
[5] A. Marx, Untersuchungen über schlichte Abildung, Math. Ann., 107(1932-1933), 40-67.
[6] Ch. Pommerenke, On close-to-convex analytic functions, Trans. Amer. Math. Soc., 114(1965), no. 1, 176-186.
[7] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan, 11(1959), 72-75.
[8] E. Strohhächer, Beitrage zur theorie der schlichten funktionen, Math. Z., 37(1933), 356-380.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

(N. E. Cho) Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

Email address: necho@pknu.ac.kr

(M. K. Aouf) Department of Mathematics Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Email address: mkaouf127@yahoo.com

(A. O. Mostafa) Department of Mathematics Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Email address: adelaeg254@yahoo.com

Some properties of the second kind degenerate q-Euler polynomials associated with the p-adic integral on $\mathbb{Z}_p$

C. S. RYOO

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : In this paper, we introduce the second kind degenerate q-Euler numbers and polynomials associated with the p-adic integral on $\mathbb{Z}_p$. We also obtain some explicit formulas for the second kind degenerate q-Euler numbers and polynomials.

Key words : Euler numbers and polynomials, the second kind Euler numbers and polynomials, the second kind degenerate Euler numbers and polynomials, the second kind degenerate q-Euler numbers and polynomials, p-adic integral on $\mathbb{Z}_p$.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers. Let p be a fixed odd prime number. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$.

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $g \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_g(x,y) = \frac{g(x)-g(y)}{x-y}$$

have a limit $l = g'(a)$ as $(x,y) \to (a,a)$. For $g \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0\le x<p^N} g(x)(-1)^x, \text{ (see [3]).} \tag{1}$$

If we take $g_1(x) = g(x+1)$ in (1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations(see [6])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{3}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. Note that $(x|\lambda)$ is a homogeneous polynomials in λ and x of degree n, so if $\lambda \neq 0$ then $(x|\lambda)_n = \lambda^n(\lambda^{-1}x|1)_n$. Clearly $(x|0)_n = x^n$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}. \tag{5}$$

For $q \in \mathbb{C}_p$ with $|1-q|_p \le 1$, if we take $g(x) = q^x e^{(2x+1)t}$ in (2), then we easily see that

$$I_{-1}(q^x e^{(2x+1)t}) = \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t}{qe^{2t}+1}.$$

Let us define the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows(see [5]):

$$\int_{\mathbb{Z}_p} q^y e^{(2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \tag{6}$$

$$\int_{\mathbb{Z}_p} q^y e^{(x+2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{7}$$

Recently, many mathematicians have studied in the area of the degenerate Bernoulli umbers and polynomials, degenerate Euler numbers and polynomials, degenerate tangent numbers and polynomials(see [1, 2, 3, 4, 6]). Our aim in this paper is to define the second kind degenerate q-Euler polynomials $\mathcal{E}_{n,q}(x,\lambda)$. We investigate some properties which are related to the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$.

2. Some properties of the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$

In this section, we introduce the second kind degenerate q-Euler numbers and polynomials, and we obtain explicit formulas for them. For $t, \lambda \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$, if we take $g(x) = q^x(1+\lambda t)^{(2x+1)/\lambda}$ in (2), then we easily see that

$$\int_{\mathbb{Z}_p} q^x (1+\lambda t)^{(2x+1)/\lambda} d\mu_{-1}(x) = \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}. \tag{8}$$

Let us define the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$ as follows:

$$\int_{\mathbb{Z}_p} q^y (1+\lambda t)^{(2y+1)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\lambda) \frac{t^n}{n!}, \tag{9}$$

$$\int_{\mathbb{Z}_p} q^y (1+\lambda t)^{(2y+1+x)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda) \frac{t^n}{n!}. \tag{10}$$

Note that $(1+\lambda t)^{1/\lambda}$ tends to e^t as $\lambda \to 0$. From (7) and (10), we note that

$$\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \mathcal{E}_{n,q}(x,\lambda) \frac{t^n}{n!} = \lim_{\lambda\to 0} \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.$$

Thus, we have

$$\lim_{\lambda\to 0} \mathcal{E}_{n,q}(x,\lambda) = E_{n,q}(x), (n \ge 0).$$

From (5) and (9), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda) \frac{t^n}{n!} &= \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\
&= \left(\sum_{m=0}^{\infty} \mathcal{E}_{m,q}(\lambda) \frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} (x|\lambda)_l \frac{t^l}{l!}\right) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_{l,q}(\lambda)(x|\lambda)_{n-l}\right) \frac{t^n}{n!}.
\end{aligned} \tag{11}$$

Therefore, we obtain the following theorem.

Theorem 1. For $n \geq 0$, we have

$$\mathcal{E}_{n,q}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_{l,q}(\lambda)(x|\lambda)_{n-l}.$$

By (8), (9), and (10), we obtain the following Witt's formula.

Theorem 2. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} q^x (2x+1|\lambda)_n d\mu_{-1}(x) = \mathcal{E}_{n,q}(\lambda),$$
$$\int_{\mathbb{Z}_p} q^y (x+2y+1|\lambda)_n d\mu_{-1}(y) = \mathcal{E}_{n,q}(x,\lambda).$$

By (5) and (9), we can derive the following recurrence relation:

$$\begin{aligned} \sum_{n=0}^{\infty} 2(1|\lambda)_n \frac{t^n}{n!} &= 2(1+\lambda t)^{1/\lambda} = (q(1+\lambda t)^{2/\lambda}+1) \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} \\ &= q(1+\lambda t)^{2/\lambda} \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} \\ &= \left(\sum_{l=0}^{\infty} q(2|\lambda)_l \frac{t^l}{l!} \sum_{m=0}^{\infty} \mathcal{E}_{m,q}(\lambda)\frac{t^m}{m!} \right) + \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} q(2|\lambda)_l \mathcal{E}_{n-l,q}(\lambda) + \mathcal{E}_{n,q}(\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{12}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (12), we obtain the following theorem.

Theorem 3. For $n \in \mathbb{Z}_+$, we have

$$q\sum_{l=0}^{n} \binom{n}{l} (2|\lambda)_l \mathcal{E}_{n-l,q}(\lambda) + \mathcal{E}_{n,q}(\lambda) = 2(1|\lambda)_n.$$

By (5), (9), and (10), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} q\mathcal{E}_{n,q}(x+2,\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} \\ &= \frac{2q(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+2)/\lambda} + \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= 2(1+\lambda t)^{(x+1)/\lambda} = 2\sum_{n=0}^{\infty} (x+1|\lambda)_n \frac{t^n}{n!}. \end{aligned} \tag{13}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (13), we have the following theorem.

Theorem 4. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$q\mathcal{E}_{n,q}(x+2,\lambda) + \mathcal{E}_{n,q}(x,\lambda) = 2(x+1|\lambda)_n.$$

By (1) and (5), we have

$$\begin{aligned} &\sum_{m=0}^{\infty} \left(q^n \mathcal{E}_{m,q}(2n,\lambda) + \mathcal{E}_{m,q}(\lambda) \right) \frac{t^m}{m!} \\ &= \int_{\mathbb{Z}_p} q^{x+n}(1+\lambda t)^{(2x+2n+1)/\lambda} d\mu_{-1}(x) + (-1)^n \int_{\mathbb{Z}_p} q^x (1+\lambda t)^{(2x+1)/\lambda} d\mu_{-1}(x) \\ &= 2\sum_{l=0}^{n-1} (-1)^{n-1-l} q^l (1+\lambda t)^{(2l+1)/\lambda} = \sum_{m=0}^{\infty} \left(2\sum_{l=0}^{n-1} (-1)^{n-1-l} q^l (2l+1|\lambda)_m \right) \frac{t^m}{m!}. \end{aligned} \tag{14}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (14), we have the following theorem.

Theorem 5. For $m \in \mathbb{Z}_+$, we have

$$q^n \mathcal{E}_{m,q}(2n,\lambda) + \mathcal{E}_{m,q}(\lambda) = 2\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l(2l+1|\lambda)_m.$$

By (10), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,q^{-1}}(-x,-\lambda)\frac{t^n}{n!} &= \frac{2(1-\lambda t)^{-1/\lambda}}{q^{-1}(1-\lambda t)^{-2/\lambda}+1}(1-\lambda t)^{x/\lambda}\\ &= \frac{2q}{(1-\lambda t)^{2/\lambda}+1}(1-\lambda t)^{(x+1)/\lambda} = \sum_{n=0}^{\infty}(-1)^n q\mathcal{E}_{n,q}(x+1,\lambda)\frac{t^n}{n!}.\end{aligned} \tag{15}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (15), we have the following theorem.

Theorem 6. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q^{-1}}(-x,-\lambda) = (-1)^n q\mathcal{E}_{n,q}(x+1,\lambda), \quad \mathcal{E}_{n,q^{-1}}(-\lambda) = (-1)^n q\mathcal{E}_{n,q}(1|\lambda).$$

For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} &= \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \frac{2(1+\lambda t)^{1/\lambda}}{q^d(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^l q^l(1+\lambda t)^{2l/\lambda}\\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{l=0}^{d-1}(-1)^l q^l \mathcal{E}_{n,q^d}\left(\frac{2l+x+1-d}{d},\frac{\lambda}{d}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\dfrac{t^n}{n!}$ in the above equation, we have the following theorem:

Theorem 7. For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$ and $n \in \mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x,\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l q^l \mathcal{E}_{n,q^d}\left(\frac{2l+x+1-d}{d},\frac{\lambda}{d}\right).$$

In particular,

$$\mathcal{E}_{n,q}(\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l q^l \mathcal{E}_{n,q^d}\left(\frac{2l+1-d}{d},\frac{\lambda}{d}\right).$$

From (10), we derive

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x+y,\lambda)\frac{t^n}{n!} = \frac{2(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda}\\ &= \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}(1+\lambda t)^{y/\lambda}\\ &= \left(\sum_{n=0}^{\infty}\mathcal{E}_{m,q}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(x,\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{16}$$

Therefore, by (16), we have the following theorem.

Theorem 8. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x+y,\lambda) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(x,\lambda)(y|\lambda)_{n-l}.$$

From Theorem 8, we note that $\mathcal{E}_{n,q}(x,\lambda)$ is a Sheffer sequence.

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (10), we obtain

$$\begin{aligned}\frac{2e^t}{qe^{2t}+1}e^{xt}&=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n\frac{1}{n!}=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^m\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,q}(x,\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned}\tag{17}$$

Thus, by (17), we have the following theorem.

Theorem 9. For $n\in\mathbb{Z}_+$, we have

$$E_{m,q}(x)=\sum_{n=0}^{m}\lambda^{m-n}\mathcal{E}_{n,q}(x,\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (7), we have

$$\sum_{n=0}^{\infty}E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!}=\frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}=\sum_{m=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^m}{m!},\tag{18}$$

and

$$\sum_{n=0}^{\infty}E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!}=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,q}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}.\tag{19}$$

Thus, by (18) and (19), we have the following theorem.

Theorem 10. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x,\lambda)=\sum_{n=0}^{m}\lambda^{m-n}E_{n,q}(x)S_1(m,n).$$

Letting $q\to 1$ in Theorem 10 gives the theorem

$$\mathcal{E}_{n}(x,\lambda)=\sum_{n=0}^{m}\lambda^{m-n}E_{n}(x)S_1(m,n).$$

which was proved by Ryoo [4].

Acknowledgement: This work was supported by 2021 Hannam University Research Fund.

REFERENCES

1. Carlitz, L.(1979). *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., v.15, pp. 51-88.

2. Qi, F.; Dolgy, D.V.; Kim, T.; Ryoo, C.S.(2015). *On the partially degenerate Bernoulli polynomials of the first kind*, Global Journal of Pure and Applied Mathematics, v.11, pp. 2407-2412.

3. Kim, T.(2015). *Barnes' type multiple degenerate Bernoulli and Euler polynomials*, Appl. Math. Comput., v. 258, pp. 556-564

4. Ryoo, C.S.(2015). *On the second kind degenerate Euler numbers and polynomials associated with the p-adic integral on $\mathbb{Z}_p$*, Global Journal of Pure and Applied Mathematics, v.12, pp. 5087-5094.

5. Ryoo, C.S.(2012). *A numerical investigation of the structure of the roots of the second kind q-Euler polynomials*, Journal of Computational Analysis and Applications, v.14, pp. 321-327.

6. Young, P.T.(2008). *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, Journal of Number Theory, v. 128, pp. 738-758.

Some symmetric identities for twisted (p, q)-L-function

C. S. RYOO

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : The main of this paper is to obtain some interesting symmetric identities for twisted (p, q)-L-function in complex field. We define the twisted (p, q)-L-function by generalizing the Carlitz's type twisted (p, q)-Euler numbers and polynomials. We give some new symmetric identities for twisted (p, q)-L-function. We also obtain symmetric identities for Carlitz's type twisted (p, q)-Euler numbers and polynomials by using symmetric property for twisted (p, q)-L-function.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, twisted q-Euler numbers and polynomials, twisted (p, q)-Euler numbers and polynomials, q-L-function, twisted (p, q)-L-function, symmetric identities.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many (p, q)-extensions of some special numbers, polynomials, and functions have been studied(see [1, 2, 3, 4, 7]). Luo and Zhou [5] introduced the l-function and q-L-function. Ryoo [6] investigated some identities on the higher-order twisted q-Euler numbers and polynomials. In [8], Ryoo presented the multiple twisted (h, q)-l-function. In this paper, we construct twisted (p, q)-L-function in complex field and Carlitz's type twisted (p, q)-Euler numbers and polynomials. We obtain some new symmetric identities for twisted (p, q)-L-function. We also give symmetric identities for Carlitz's type twisted (p, q)-Euler numbers and polynomials of by using symmetric property for twisted (p, q)-L-function.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. The (p, q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2q^{n-3} + pq^{n-2} + q^{n-1}.$$

Note that this number is q-number when $p = 1$. By substituting q by $\frac{q}{p}$ in the q-number, we can not obtain (p, q)-number. Therefore, much research has been developed in the area of special numbers and polynomials, and functions by using (p, q)-number(see [1, 2, 3, 4, 7]).

By using q-number, Luo and Zhou defined the q-L-function $L_q(s, a)$ and q-l-function $l_q(s)$ (see [5])

$$L_q(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+a}}{[n + a]_q^s}, \quad (Re(s) > 1; a \notin \mathbb{Z}_0^-), \text{ and } l_q(s) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_q^s}, \quad (Re(s) > 1).$$

Inspired by their work, the (p, q)-extension of the twisted q-L-function can be defined as follow: Let ζ be rth root of 1 and $\zeta \neq 1$. For $s, x \in \mathbb{C}$ with $Re(x) > 0$, the twisted (p, q)-L-function $L_{p,q,\zeta}(s, x)$ is define by

$$L_{p,q,\zeta}(s, x) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m \zeta^m}{[m + x]_{p,q}^s}.$$

2. Twisted (p,q)-Euler numbers and polynomials

In this section, we define twisted (p,q)-Euler numbers and polynomials and provide some of their relevant properties. Let r be a positive integer, and let ζ be rth root of 1.

Definition 1. For $0 < q < p \leq 1$, the Carlitz's type twisted (p,q)-Euler numbers $E_{n,p,q,\zeta}$ and polynomials $E_{n,p,q,\zeta}(x)$ are defined by means of the generating functions

$$G_{p,q,\zeta}(t) = \sum_{n=0}^{\infty} E_{n,p,q,\zeta}\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m\zeta^m e^{[m]_{p,q}t}. \tag{2.1}$$

and

$$G_{p,q,\zeta}(t,x) = \sum_{n=0}^{\infty} E_{n,p,q,\zeta}(x)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m\zeta^m e^{[m+x]_{p,q}t}, \tag{2.2}$$

respectively.

Setting $p = 1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type twisted q-Euler number $E_{n,q,\zeta}$ and q-Euler polynomials $E_{n,q,\zeta}(x)$, respectively.

By (2.1), we get

$$\sum_{n=0}^{\infty} E_{n,p,q,\zeta}\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m\zeta^m e^{[m]_{p,q}t} = \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+\zeta q^l p^{n-l}}\right)\frac{t^n}{n!}.$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 2. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q,\zeta} = [2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+\zeta p^{n-l}q^l}.$$

By (2.2), we obtain

$$E_{n,p,q,\zeta}(x) = [2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+\zeta p^{n-l}q^l}. \tag{2.3}$$

Next, we introduce Carlitz's type twisted (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q,\zeta}(x)$.

Definition 3. The Carlitz's type twisted (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q,\zeta}(x)$ are defined by

$$E^{(h)}_{n,p,q,\zeta}(x) = [2]_q \sum_{m=0}^{\infty}(-1)^m p^{hm}\zeta^m[m+x]^n_{p,q}. \tag{2.4}$$

When $x = 0$, $E^{(h)}_{n,p,q,\zeta} = E^{(h)}_{n,p,q,\zeta}(0)$ are called the twisted (h,p,q)-Euler numbers $E^{(h)}_{n,p,q,\zeta}$.

By using (2.4) and (p,q)-number, we have the following theorem.

Theorem 4. For $n \in \mathbb{Z}_+$, we have

$$E^{(h)}_{n,p,q,\zeta}(x) = [2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+\zeta p^{n-l+h}q^l}.$$

By (2.4) and Theorem 2, we have

$$\begin{aligned} E_{n,p,q,\zeta}(x) &= \sum_{l=0}^{n}\binom{n}{l}q^{(n-l)x}E^{(l)}_{n-l,p,q,\zeta}[x]^l_{p,q} \\ E_{n,p,q,\zeta}(x+y) &= \sum_{l=0}^{n}\binom{n}{l}p^{xl}q^{(n-l)y}E^{(l)}_{n-l,p,q,\zeta}(x)[y]^l_{p,q}. \end{aligned} \tag{2.5}$$

By (2.1) and (2.2), we get

$$-[2]_q\sum_{l=0}^{\infty}(-1)^{l+n}\zeta^{l+n}e^{[l+n]_{p,q}t}+[2]_q\sum_{l=0}^{\infty}(-1)^{l}\zeta^{l}e^{[l]_{p,q}t}=[2]_q\sum_{l=0}^{n-1}(-1)^{l}\zeta^{l}e^{[l]_{p,q}t}.$$

Hence we have

$$(-1)^{n+1}\zeta^n\sum_{m=0}^{\infty}E_{m,p,q,\zeta}(n)\frac{t^m}{m!}+\sum_{m=0}^{\infty}E_{m,p,q,\zeta}\frac{t^m}{m!}=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l\zeta^l[l]_{p,q}^m\right)\frac{t^m}{m!}. \tag{2.6}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.6), we have the following theorem.

Theorem 5. For $m \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n-1}(-1)^l\zeta^l[l]_{p,q}^m=\frac{(-1)^{n+1}\zeta^nE_{m,p,q,\zeta}(n)+E_{m,p,q,\zeta}}{[2]_q}.$$

3. Twisted (p,q)-l-function and twisted (p,q)-L-function

By using twisted (p,q)-Euler numbers and polynomials, twisted (p,q)-L-function is defined. These functions interpolate the twisted (p,q)-Euler numbers $E_{n,p,q,\zeta}$, and polynomials $E_{n,p,q,\zeta}(x)$, respectively. From (2.1), we note that

$$\left.\frac{d^k}{dt^k}G_{p,q,\zeta}(t)\right|_{t=0}=[2]_q\sum_{m=0}^{\infty}(-1)^n\zeta^m[m]_{p,q}^k=E_{k,p,q,\zeta},\,(k\in\mathbb{N}).$$

By using the above equation, we are now ready to define twisted (p,q)-l-function.

Definition 6. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$l_{p,q,\zeta}(s)=[2]_q\sum_{n=1}^{\infty}\frac{(-1)^n\zeta^n}{[n]_{p,q}^s}. \tag{3.1}$$

Relation between $l_{p,q,\zeta}(s)$ and $E_{k,p,q,\zeta}$ is given by the following theorem.

Theorem 7. For $k \in \mathbb{N}$, we have

$$l_{p,q,\zeta}(-k)=E_{k,p,q,\zeta}.$$

By using (2.2), we note that

$$\left.\frac{d^k}{dt^k}G_{p,q,\zeta}(t,x)\right|_{t=0}=[2]_q\sum_{m=0}^{\infty}(-1)^m\zeta^m[m+x]_{p,q}^k \tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k\left.\left(\sum_{n=0}^{\infty}E_{n,p,q,\zeta}(x)\frac{t^n}{n!}\right)\right|_{t=0}=E_{k,p,q,\zeta}(x),\ \text{for } k\in\mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the twisted (p,q)-L-function.

Definition 8. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$L_{p,q,\zeta}(s,x)=[2]_q\sum_{n=0}^{\infty}\frac{(-1)^n\zeta^n}{[n+x]_{p,q}^s}. \tag{3.4}$$

Note that $L_{p,q,\zeta}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $L_{p,q,\zeta}(s,x)$ and $E_{k,p,q,\zeta}(x)$ is given by the following theorem.

Theorem 9. For $k \in \mathbb{N}$, we have $L_{p,q,\zeta}(-k,x) = E_{k,p,q,\zeta}(x)$.

Observe that $L_{p,q,\zeta}(-k,x)$ function interpolates $E_{k,p,q,\zeta}(x)$ numbers at non-negative integers.

3. Some symmetric identities for twisted (p,q)-L-function

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain certain symmetric identities for twisted (p,q)-L-function.

Theorem 10. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. Then we obtain

$$\begin{aligned} &[w_2]_{p,q}^s [2]_{q^{w_2}} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} L_{p^{w_1},q^{w_1},\zeta^{w_1}} \left(s, w_2 x + \frac{w_2}{w_1} j\right) \\ &= [w_1]_{p,q}^s [2]_{q^{w_1}} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} L_{p^{w_2},q^{w_2},\zeta^{w_2}} \left(s, w_1 x + \frac{w_1}{w_2} j\right). \end{aligned} \tag{4.1}$$

Proof. Note that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In (3.4), by substitute $w_2 x + \frac{w_2}{w_1} j$ for x in and replace q, p, and ζ by q^{w_1}, p^{w_1} and ζ^{w_1}, respectively, we derive next result

$$\begin{aligned} \frac{1}{[2]_{q^{w_1}}} L_{p^{w_1},q^{w_1},\zeta^{w_1}} \left(s, w_2 x + \frac{w_2}{w_1} j\right) &= \sum_{m=0}^{\infty} \frac{(-1)^m \zeta^{w_1 m}}{\left[m + w_2 x + \frac{w_2}{w_1} j\right]_{p^{w_1},q^{w_1}}^s} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m \zeta^{w_1 m}}{\left[\frac{w_1 m + w_1 w_2 x + w_2 j}{w_1}\right]_{p^{w_1},q^{w_1}}^s} \\ &= [w_1]_{p,q}^s \sum_{m=0}^{\infty} \sum_{i=0}^{w_2-1} \frac{(-1)^{w_2 m + i} \zeta^{w_1(w_2 m + i)}}{[w_1(w_2 m + i) + w_1 w_2 x + w_2 j]_{p,q}^s} \\ &= [w_1]_{p,q}^s \sum_{m=0}^{\infty} \sum_{i=0}^{w_2-1} \frac{(-1)^m (-1)^i \zeta^{w_1 w_2 m} \zeta^{w_1 i}}{[w_1 w_2 (x+m) + w_1 i + w_2 j]_{p,q}^s}. \end{aligned} \tag{4.2}$$

Thus, from (4.2), we can derive the following equation.

$$\begin{aligned} &\frac{[w_2]_{p,q}^s}{[2]_{q^{w_1}}} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} L_{p^{w_1},q^{w_1},\zeta^{w_1}} \left(s, w_2 x + \frac{w_2}{w_1} j\right) \\ &= [w_1]_{p,q}^s [w_2]_{p,q}^s \sum_{m=0}^{\infty} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_1-1} \frac{(-1)^{j+i+m} \zeta^{w_1 w_2 m} \zeta^{w_1 i} \zeta^{w_2 j}}{[w_1 w_2 (x+m) + w_1 i + w_2 j]_{p,q}^s} \end{aligned} \tag{4.3}$$

By using the same method as (4.3), we have

$$\begin{aligned} &\frac{[w_1]_{p,q}^s}{[2]_{q^{w_2}}} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} L_{p^{w_2},q^{w_2},\zeta^{w_2}} \left(s, w_1 x + \frac{w_1}{w_2} j\right) \\ &= [w_1]_{p,q}^s [w_2]_{p,q}^s \sum_{m=0}^{\infty} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \frac{(-1)^{j+i+m} \zeta^{w_1 w_2 m} \zeta^{w_2 i} \zeta^{w_1 j}}{[w_1 w_2 (x+m) + w_1 j + w_2 i]_{p,q}^s} \end{aligned} \tag{4.4}$$

Therefore, by (4.3) and (4.4), we have the following theorem. □

Taking $w_2 = 1$ in Theorem 10, we obtain the following corollary.

Corollary 11. Let $w_1 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$L_{p,q,\zeta}(s, w_1 x) = \frac{[2]_q}{[2]_{q^{w_1}} [w_1]_{p,q}^s} \sum_{j=0}^{w_1-1} (-1)^j \zeta^j L_{p^{w_1},q^{w_1},\zeta^{w_1}} \left(s, x + \frac{j}{w_1}\right).$$

Let us take $s=-n$ in Theorem 10. For $n\in\mathbb{Z}_+$, we obtain certain symmetry identities for twisted (p,q)-Euler polynomials.

Theorem 12. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1 \pmod 2$. For $n\in\mathbb{Z}_+$, we obtain

$$\begin{aligned}&[w_1]_{p,q}^n[2]_{q^{w_2}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=[w_2]_{p,q}^n[2]_{q^{w_1}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}E_{n,p^{w_2},q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right).\end{aligned}$$

Taking $w_2=1$ in Theorem 12, we obtain the following distribution relation.

Corollary 13. Let $w_1\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$. For $n\in\mathbb{Z}_+$, we obtain

$$E_{n,p,q,\zeta}(w_1x)=\frac{[2]_q}{[2]_{q^{w_1}}}[w_1]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j\zeta^jE_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(s,x+\frac{j}{w_1}\right).$$

By (2.5), we have

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}\sum_{i=0}^{n}\binom{n}{i}q^{w_2j(n-i)}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\left[\frac{w_2}{w_1}j\right]^i_{p^{w_1},q^{w_1}}\\&=\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}\sum_{i=0}^{n}\binom{n}{i}q^{w_2j(n-i)}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\left(\frac{[w_2]_{p,q}}{[w_1]_{p,q}}\right)^i[j]^i_{p^{w_2},q^{w_2}}\end{aligned}$$

Hence we have the following theorem.

Theorem 14. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1 \pmod 2$. For $n\in\mathbb{Z}_+$, we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=\sum_{i=0}^{n}\binom{n}{i}[w_2]^i_{p,q}[w_1]^{-i}_{p,q}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2(n-i)j}[j]^i_{p^{w_2},q^{w_2}}.\end{aligned}$$

For each integer $n\geq 0$, let $\mathcal{A}_{n,i,p,q,\zeta}(w)=\sum_{j=0}^{w-1}(-1)^j\zeta^jq^{j(n-i)}[j]^i_{p,q}$. The sum $\mathcal{A}_{n,i,p,q,\zeta}(w)$ is called the alternating twisted (p,q)-power sums.

By Theorem 14, we have

$$\begin{aligned}&[2]_{q^{w_2}}[w_1]^n_{p,q}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=[2]_{q^{w_2}}\sum_{i=0}^{n}\binom{n}{i}[w_2]^i_{p,q}[w_1]^{n-i}_{p,q}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1)\end{aligned}\tag{4.5}$$

By using the same method as in (4.5), we have

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]^n_{p,q}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}E_{n,p^{w_2},q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right)\\&=[2]_{q^{w_1}}\sum_{i=0}^{n}\binom{n}{i}[w_1]^i_{p,q}[w_2]^{n-i}_{p,q}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1x)\mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2)\end{aligned}\tag{4.6}$$

Therefore, by (4.5) and (4.6) and Theorem 12, we have the following theorem.

Theorem 15. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]_{p,q}^{i} [w_2]_{p,q}^{n-i} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1 x) \mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2)\\ &= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]_{p,q}^{i} [w_1]_{p,q}^{n-i} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2 x) \mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1).\end{aligned}$$

By Theorem 15, we obtain the interesting symmetric identity for the twisted (h, p, q)-Euler numbers $E^{(h)}_{n,p,q,\zeta}$ in complex field.

Corollary 16. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]_{p,q}^{i} [w_2]_{p,q}^{n-i} p^{w_1 w_2 x i} \mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2) E^{(i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}\\ &= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]_{p,q}^{i} [w_1]_{p,q}^{n-i} p^{w_1 w_2 x i} \mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1) E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}.\end{aligned}$$

Acknowledgement: This work was supported by 2020 Hannam University Research Fund.

REFERENCES

1. Agarwal, R.P.; Kang, J.Y.; Ryoo, C.S.(2018). *Some properties of (p, q)-tangent polynomials*, Journal of Computational Analysis and Applications, v.24, pp. 1439-1454.

2. Araci, S.; Duran, U.; Acikgoz, M.; Srivastava, H.M.(2016). *A certain (p, q)-derivative operator and associated divided differences*, Journal of Inequalities and Applications, v. 2016:301, DOI 10.1186/s13660-016-1240-8.

3. Duran, U.; Acikgoz, M.; Araci, S.(2016). *On (p, q)-Bernoulli, (p, q)-Euler and (p, q)-Genocchi polynomials*, J. Comput. Theor. Nanosci., v.13, pp. 7833-7846.

4. Hwang, K.W.; Ryoo, C.S.(2019). *Some symmetric identities for degenerate Carlitz-type (p, q)-Euler numbers and polynomials,* Symmetry, v.11, 830; doi:10.3390/sym11060830.

5. Luo, Q.M.; Zhou, Y.(2011). *Extension of the Genocchi polynomials and its q-analogue,* Utilitas Mathematica, v.85, pp. 281 - 297.

6. Ryoo, C.S.(2017). *Some identities on the higher-order twisted q-Euler numbers and polynomials,* J. Computational Analysis and Applications, v.22, pp. 825-830.

7. Ryoo, C.S.(2017). *On the (p, q)-analogue of Euler zeta function*, J. Appl. Math. & Informatics v.35, pp. 303-311.

8. Ryoo, C.S.(2020). *Symmetric identities for Dirichlet-type multiple twisted (h, q)-l-function and higher-order generalized twisted (h, q)-Euler polynomials*, Journal of Computational Analysis and Applications, v.28, pp. 537-542.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO. 4, 2021

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO. 4, 2021

(continues)