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A Numerical Technique for Solving Fuzzy Fractional Optimal Control Problems†

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Abstract In this paper, the fuzzy fractional optimal control problem with both fixed and free final state conditions has been considered. Our problem is defined in the sense of Riemann-Liouville fractional derivative based on Hukuhara difference, and the dynamic constraint is described by a fractional differential equation of order less than 1. Using fuzzy variational approach, a necessary conditions of our problem has been derived. A numerical technique based on Grünwald-Letnikov definition of fractional derivative and the relation between right Riemann-Liouville fractional derivative and right Caputo fractional derivative is proposed. Finally, some numerical examples are given to illustrate our main results.
Keywords: Fuzzy fractional calculus;Grünwald-Letnikov fractional derivative;Fuzzy fractional optimal control problem;Fixed final state problem;Free final state problem;Fuzzy variational approach;Necessary conditions.

1. Introduction

Optimal control is the standard method for solving dynamic optimization problems, which deal with finding a control law for a given system such that a certain optimality criterion is achieved. It's playing an increasingly important role in modern system design, and considered to be a powerful mathematical tool that can be used to make decisions in real life. On the other hand, accurate modeling of some real problems in scientific fields and engineering, sometimes lead to a set of fractional differential and integral equations. Fractional optimal control problem is an optimal control problem whose dynamic system is described by fractional differential equations. We can define the fractional optimal control problem in sense of different definitions of fractional derivative, for example Riemann-Liouville fractional derivative, Caputo fractional derivative and so on.

Due to, uncertainty in the input, output and manner of many dynamical systems, meanwhile, fuzziness is a way to express an uncertain phenomena in real world. Thus, importing fuzziness in the optimal control theory, give a better display of the problems with control parameters in real world such as physical models and dynamical systems.

In the last decade, fuzzy fractional optimal control problems have attracted a great deal of attention and the interest in the filed of fuzzy fractional optimal control problems has increased. In [1], Fard and Soolaki, prove the necessary optimality conditions of pontryagin type for a class of fuzzy fractional optimal control problems with the fuzzy fractional derivative described in the Caputo sense. In [2], Fard and Salehi studied the constrained and unconstrained fuzzy fractional variational problems containing the Caputo-type fractional derivatives using the approach of the generalized differentiability. In [3], Karimyar and Fakharzadeh introduced the solution of fuzzy fractional optimal control problems by using Mittag-Leffler function.

In this paper, we will study a fixed and free final state fuzzy fractional optimal control problems with the fuzzy fractional derivative described in Riemann-Liouville type in sense of Hukuhara difference.

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Then, we derive the necessary conditions of that problems based on fuzzy variational approach. A numerical algorithm is proposed to solve the necessary conditions to find the optimal fuzzy control and optimal fuzzy state as a solutions of our problems. The definitions of a strong and weak solutions of our problems are given, to guarantee the optimal solutions are a fuzzy functions.

This paper is organized as follows. In Section 2 we introduce and generalize some basic concepts and notations that are key to our discussion. In Section 3 we present basic elements of fuzzy fractional calculus and fuzzy calculus of variations. In Section 4 we establish our main results, Theorem(4.1), that provides the necessary conditions of fuzzy fractional optimal control problems with both fixed and free final state conditions. In Section 5 we propose a numerical technique to solve the necessary conditions. Finally, we discuss the applicability of the main theorem and the numerical algorithm through an examples.

2. Definitions and preliminaries

Here, we start with basic definitions and lemmas needed in the other sections for a better understanding of this work. The details of this concepts are clearly found in [7, 9, 10, 11, 12, 17].

Definition 2.1 A fuzzy set $\tilde{A} : R \to [0,1]$ is called a fuzzy number if $\tilde{A}$ is normal, convex fuzzy set, upper semi-continuous and supp$A = \overline{\{x \in R | \tilde{A}(x) > 0\}}$ is compact, where $\overline{M}$ denotes the closure of M. In the rest of this paper we use E^1 to denote the fuzzy number space.

Where it is $\alpha-$level set $\tilde{a}[\alpha] = \{x \in R : \tilde{a}(x) \geq \alpha\} = [a^l(\alpha), a^r(\alpha)], \forall \alpha \in (0,1]$, and $0-$level set $\tilde{a}[0]$ is defined as $\overline{\{x \in R | \tilde{a}(x) > 0\}}$. Obviously, the α-level set $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ is bounded closed interval in R for all $\alpha \in [0,1]$, where $a^l(\alpha)$ and $a^r(\alpha)$ denote the left-hand and right-hand end points of $\tilde{a}[\alpha]$, respectively. $\tilde{a}$ is a crisp number with value k if its membership function is defined by,

$$\tilde{a}(x) = \begin{cases} 1 & , x = k \\ 0 & , x \neq k \end{cases}$$

Thus,

$$\tilde{0}(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x \neq 0. \end{cases}$$

Let $\tilde{u}, \tilde{v} \in E^1, k \in R$, we can define the addition and scalar multiplication by using α-level set respectively as

$$(\tilde{a} + \tilde{b})[\alpha] = \tilde{a}[\alpha] + \tilde{b}[\alpha],\ (k\tilde{a})[\alpha] = k\tilde{a}[\alpha],$$

where $\tilde{a}[\alpha] + \tilde{b}[\alpha]$ means the usual addition of two intervals of R, and $k\tilde{a}[\alpha]$ means the usual product between a scalar and interval of R. Furthermore, the opposite of the fuzzy number $\tilde{a}$ is $-\tilde{a}$, i.e., $-\tilde{a}(x) = \tilde{a}(-x)$, it means, $-\tilde{a}[\alpha] = [-a^r(\alpha), -a^l(\alpha)]$.

The binary operation "." in R can be extended to the binary operation "$\odot$" of two fuzzy numbers by using the extension principle. Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers, then

$$(\tilde{a} \odot \tilde{b})(z) = \sup_{x \cdot y = z} \min\{\tilde{a}(x), \tilde{b}(x)\}.$$

Using α-level set the product $(\tilde{a} \odot \tilde{b})$ is defined by

$$\begin{aligned}(\tilde{a} \odot \tilde{b})[\alpha] \quad = \quad & \Big[\min\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}, \\ & \max\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}\Big].\end{aligned}$$

The metric structure is given by the Hausdorff distance $\mathbb{D} : E^1 \times E^1 \times R \to R_+ \cup \{0\}$,

$$\mathbb{D}(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{| a^l(\alpha) - b^l(\alpha) |, | a^r(\alpha) - b^r(\alpha) |\}.$$

A special class of fuzzy numbers is the class of triangular fuzzy numbers. For $a_1 < a_2 < a_3$ and $a_1, a_2, a_3 \in R$, the triangular fuzzy number $\tilde{a}$ is generally denoted by $\tilde{a} = (a_1, a_2, a_3)$ is determined by a_1, a_2, a_3 such that $a^l(\alpha) = a_1 + (a_2 - a_1)\alpha$ and $a^r(\alpha) = a_3 - (a_3 - a_2)\alpha$, when $\alpha = 0$ then $\tilde{a}[0] = [a_1, a_3]$ and when $\alpha = 1$ then $\tilde{a}[1] = [a_2, a_2] = a_2$.

We know that, we can identify a fuzzy number $\tilde{a} \in E^1$ by the left and right hand functions of its $\alpha-$level set, the following lemma introduce the properties of this functions.

Lemma 2.1 Suppose that $a^l : [0,1] \to R$ and $a^r : [0,1] \to R$ satisfy the conditions:

C1: a^l is bounded increasing function,

C2: a^r is bounded decreasing function,

C3: $a^l(1) \leq a^r(1)$,

C4: $\lim\limits_{\alpha\to k^-} a^l(\alpha) = a^l(k)$ and $\lim\limits_{\alpha\to k^-} a^r(\alpha) = a^r(k)$, for all $0 < k \leq 1$,

C5: $\lim\limits_{\alpha\to 0^+} a^l(\alpha) = a^l(0)$ and $\lim\limits_{\alpha\to 0^+} a^r(\alpha) = a^r(0)$.

Then $\tilde{a} : R \to [0,1]$ defined by $\tilde{a}(x) = \sup\{\alpha | a^l(\alpha) \leq x \leq a^r(\alpha)\}$ is a fuzzy number with $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$. Moreover, if $\tilde{a} : R \to [0,1]$ is a fuzzy number with $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$, then the functions $a^l(\alpha)$ and $a^r(\alpha)$ satisfy conditions **C1- C5**.

Definition 2.2 (H-difference). Let $\tilde{a}, \tilde{b} \in E^1$, where $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$ for all $\alpha \in [0,1]$, the H-difference is defined by

$$\tilde{a} \ominus \tilde{b} = \tilde{c} \iff \tilde{a} = \tilde{b} + \tilde{c}.$$

Obviously, $\tilde{a} \ominus \tilde{a} = \tilde{0}$, and the α-level set of H-difference is

$$(\tilde{a} \ominus \tilde{b})[\alpha] = [a^l(\alpha) - b^l(\alpha), a^r(\alpha) - b^r(\alpha)], \forall \alpha \in [0,1].$$

Definition 2.3 (Partial ordering). Let $\tilde{a}, \tilde{b} \in E^1$, we write $\tilde{a} \preceq \tilde{b}$, if $a^l(\alpha) \leq b^l(\alpha)$ and $a^r(\alpha) \leq b^r(\alpha)$ for all $\alpha \in [0,1]$. We also write $\tilde{a} \prec \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and there exists $\alpha_0 \in [0,1]$ such that $a^l(\alpha_0) < b^l(\alpha_0)$ or $a^r(\alpha_0) < b^r(\alpha_0)$. Furthermore, $\tilde{a} = \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$. In other words, $\tilde{a} = \tilde{b}$, if $\tilde{a}[\alpha] = \tilde{b}[\alpha]$ for all $\alpha \in [0,1]$.

In the sequel, we say that $\tilde{a}, \tilde{b} \in E^1$ are *comparable* if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$, and *non-comparable* otherwise.

From now we consider S as a subset of R.

Definition 2.4 (Fuzzy valued function). The function $\tilde{f} : S \to E^1$ is called a fuzzy-valued function if $\tilde{f}(t)$ is assign a fuzzy number for any $e \in S$. We also denote $\tilde{f}(t)[\alpha] = [f^l(t,\alpha), f^r(t,\alpha)]$, where $f^l(t,\alpha) = (\tilde{f}(t))^l(\alpha) = \min\{\tilde{f}(t)[\alpha]\}$ and $f^r(t,\alpha) = (\tilde{f}(t))^r(\alpha) = \max\{\tilde{f}(t)[\alpha]\}$. Therefore any fuzzy-valued function $\tilde{f}$ may be understood by $f^l(t,\alpha)$ and $f^r(t,\alpha)$ being respectively a bounded increasing function of α and a bounded decreasing function of α for $\alpha \in [0,1]$. And also it holds $f^l(t,\alpha) \leq f^r(t,\alpha)$ for any $\alpha \in [0,1]$.

Definition 2.5 (Continuity of a fuzzy valued function). We say that $\tilde{f} : S \to E^1$ is continuous at $t \in S$, if both $f^l(t,\alpha)$ and $f^r(t,\alpha)$ are continuous functions at $t \in S$ for all $\alpha \in [0,1]$.

If $\tilde{f}(t)$ is continuous in the metric $\mathbb{D}$, then its definite integral exists and defined by

$$\int_a^b \tilde{f}(t)[\alpha]dt = \left[\int_a^b f^l(t,\alpha)dt, \int_a^b f^r(t,\alpha)dt\right].$$

Definition 2.6 (Distance measure between fuzzy valued functions). Suppose that $\tilde{f}, \tilde{g} : S \to E^1$ are two fuzzy functions. We define the distance measure between $\tilde{f}$ and $\tilde{g}$ by

$$\begin{aligned} \mathbb{D}_{E^1}(\tilde{f}(x), \tilde{g}(x)) &= \sup_{0\leq\alpha\leq 1} \mathbb{H}(\tilde{f}(x)[\alpha], \tilde{g}(x)[\alpha]) \\ &= \max\{\sup_{z\in\tilde{f}(x)[\alpha]} d(z, \tilde{g}(x)[\alpha]), \sup_{y\in\tilde{g}(x)[\alpha]} d(\tilde{f}(x)[\alpha], y)\}, \ \forall x \in S. \end{aligned}$$

Where $\mathbb{H}$ is the Hausdorff metric on the family of all nonempty compact subsets of R, and

$$d(a,B)=\inf_{b\in B} d(a,b).$$

Moreover, we can define

$$\| \tilde{f}(x) \|_{E^1}^2 = \mathbb{D}_{E^1}(\tilde{f}(x),\tilde{f}(x)),\ \ \forall x\in S,$$

for any $\tilde{f}:S\to E^1$.

3. Elements of fuzzy fractional calculus and fuzzy calculus of variations

Several definitions of a fractional derivative have been studied, such as Riemann-Liouville, Grünwald-Letnikov, Caputo and so on. In this paper, we deal with the problems defined by Riemann-Liouville fractional derivative. In this section, we first introduce the definition of fuzzy Riemann-Liouville integrals and derivatives in sense of Hukuhara difference.

Definition 3.1(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$ and $0<\beta\le 1$, then the fuzzy Riemann-Liouville integral of $\tilde{f}(x)$ of order β is defined by

$${}_aI_x^\beta \tilde{f}(x)=\frac{1}{\Gamma(\beta)}\int_a^x \tilde{f}(t)(x-t)^{\beta-1}dt,$$

where $\Gamma(\beta)$ is the Gamma function and $x>a$.

Theorem 3.1(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$. The fuzzy Riemann-Liouville integral of $\tilde{f}(x)$ can be expressed as follows

$${}_aI_x^\beta \tilde{f}(x)\,[\alpha]=\left[{}_aI_x^\beta f^l(x,\alpha),{}_a I_x^\beta f^r(x,\alpha)\right],\ \ 0\le\alpha\le 1,$$

where

$$\begin{aligned}{}_aI_x^\beta f^l(x,\alpha)&=\frac{1}{\Gamma(\beta)}\int_a^x f^l(t,\alpha)(x-t)^{\beta-1}dt,\\ {}_aI_x^\beta f^r(x,\alpha)&=\frac{1}{\Gamma(\beta)}\int_a^x f^r(t,\alpha)(x-t)^{\beta-1}dt.\end{aligned}$$

In the next definition, we define the fuzzy Riemann-Liouville fractional derivative of order $0<\beta<1$ of a fuzzy valued function $\tilde{f}(x)$.

Definition 3.2(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$. $x_0\in(a,b)$ and then: $G(x)=\frac{1}{\Gamma(1-\beta)}\int_a^x\frac{\tilde{f}(t)dt}{(x-t)^\beta}$. We say that $\tilde{f}$ is Riemann-Liouville H-differentiable of order $0<\beta<1$ at x_0, if there exist an element ${}_aD_x^\beta\tilde{f}(x_0)\in E^1$ such that for $h>0$ sufficiently small

(1) ${}_aD_x^\beta\tilde{f}(x_0)=\lim\limits_{h\to0^+}\frac{G(x_0+h)\ominus G(x_0)}{h}=\lim\limits_{h\to0^+}\frac{G(x_0)\ominus G(x_0-h)}{h}$,

or

(2) ${}_aD_x^\beta\tilde{f}(x_0)=\lim\limits_{h\to0^+}\frac{G(x_0)\ominus G(x_0+h)}{-h}=\lim\limits_{h\to0^+}\frac{G(x_0-h)\ominus G(x_0)}{-h}$,

or

(3) ${}_aD_x^\beta\tilde{f}(x_0)=\lim\limits_{h\to0^+}\frac{G(x_0+h)\ominus G(x_0)}{h}=\lim\limits_{h\to0^+}\frac{G(x_0-h)\ominus G(x_0)}{-h}$,

or

(4) ${}_aD_x^\beta\tilde{f}(x_0)=\lim\limits_{h\to0^+}\frac{G(x_0)\ominus G(x_0+h)}{-h}=\lim\limits_{h\to0^+}\frac{G(x_0)\ominus G(x_0-h)}{h}$.

For sake of simplicity, we say that the fuzzy valued function $\tilde{f}(x)$ is Riemann-Liouville $[(i)-\beta]-$differentiable if it is differentiable as in the Definition(3.2) case(i), $i = 1, 2, 3, 4$ respectively.

Theorem 3.2(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a, b] \in R$ and $\tilde{f}(x)[\alpha] = [f^l(x, \alpha), f^r(x, \alpha)]$,then for $\alpha \in [0, 1]$, $x \in (a, b)$ and $\beta \in (0, 1)$

(i) Let us consider $\tilde{f}$ is Riemann-Liouville $[(1) - \beta]-$differentiable fuzzy-valued function, then:

$$ {}_aD_x^\beta \tilde{f}(x_0)[\alpha] = \left[{}_aD_x^\beta f^l(x_0, \alpha), {}_a D_x^\beta f^r(x_0, \alpha) \right]. $$

(ii) Let us consider $\tilde{f}$ is Riemann-Liouville $[(2) - \beta]-$differentiable fuzzy-valued function, then:

$$ {}_aD_x^\beta \tilde{f}(x_0)[\alpha] = \left[{}_aD_x^\beta f^r(x_0, \alpha), {}_a D_x^\beta f^l(x_0, \alpha) \right]. $$

Where

$$ \begin{aligned} {}_aD_x^\beta f^l(x_0, \alpha) &= \left[\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{f^l(t, \alpha) dt}{(x-t)^\beta} \right] \Bigg|_{x=x_0}, \\ {}_aD_x^\beta f^r(x_0, \alpha) &= \left[\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{f^r(t, \alpha) dt}{(x-t)^\beta} \right] \Bigg|_{x=x_0}. \end{aligned} $$

Theorem 3.3(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a, b]$ is a Riemann-Liouville H-differentiable of order $0 < \beta < 1$ on each point $x \in (a, b)$ in the sense of Definition(3.2) case(3) or case(4), then ${}_aD_x^\beta \tilde{f}(x) \in R$ for all $x \in (a, b)$.

Now we state some elements of fuzzy calculus of variations.

Definition 3.3(Fuzzy increment[10]). Suppose that $\tilde{x}(.)$ and $\tilde{x}(.) + \delta\tilde{x}(.)$ are fuzzy functions for which the fuzzy functional $\tilde{J}$ is defined. The increment of $\tilde{J}$, denoted by $\Delta\tilde{J}$, is

$$ \Delta\tilde{J} := \tilde{J}(\tilde{x} + \delta\tilde{x}) \ominus \tilde{J}(x), \tag{3.1} $$

Where $\delta\tilde{x}(.)$ is the variation of $\tilde{x}(.)$.

Because the increment $\Delta\tilde{J}$ depends on the fuzzy functions $\tilde{x}$ and $\delta\tilde{x}$, we denote $\Delta\tilde{J}$ by $\Delta\tilde{J}(\tilde{x}, \delta\tilde{x})$.

Definition 3.4(Differentiability of a fuzzy functional[10, 15]). Suppose that $\Delta\tilde{J}$ can be written as

$$ \Delta\tilde{J}(\tilde{x}, \delta\tilde{x}) := \delta\tilde{J}(\tilde{x}, \delta\tilde{x}) + \tilde{j}(\tilde{x}, \delta\tilde{x}) \cdot \parallel \delta\tilde{x} \parallel_{E^1}, \tag{3.2} $$

Where $\delta\tilde{J}$ is linear in $\delta\tilde{x}$. We say that $\tilde{J}$ is differentiable with respect to $\tilde{x}$ if for any $\epsilon > 0$,

$$ D_{E^1}(\tilde{j}(\tilde{x}, \delta\tilde{x}), 0) < \epsilon, \;\; as \parallel \delta\tilde{x}(.) \parallel_{E^1} \to 0. $$

From now $\tilde{C}[t_0, t_1]$ represent the class of all fuzzy continuous functions on $[t_0, t_1]$.

Definition 3.5(Fuzzy relative minimum[10]) A fuzzy functional $\tilde{J}$ with domain $\tilde{C}[t_0, t_1]$, has a fuzzy relative minimizer $\tilde{x}^* = \tilde{x}^*(t)$, if

$$ \tilde{J}(\tilde{x}) \succeq \tilde{J}(\tilde{x}^*), \tag{3.3} $$

for all fuzzy functions $\tilde{x} \in \tilde{C}[t_0, t_1]$.

It is clear that the inequality (3.3) holds iff

$$ J^l(\tilde{x}, \alpha) \geq J^l(\tilde{x}^*, \alpha), \;\text{ and }\; J^r(\tilde{x}, \alpha) \geq J^r(\tilde{x}^*, \alpha), \tag{3.4} $$

for all $\alpha \in [0, 1]$ and all $\tilde{x} \in \tilde{C}[t_0, t_1]$.

The following theorem is the fundamental theorem of the calculus of variations in fuzzy environment.

Theorem 3.4 Let $\tilde{x}, \delta\tilde{x} \in \tilde{C}[t_0, t_1]$ be two fuzzy functions of $t \in [t_0, t_1]$, and $\tilde{J}(\tilde{x})$ differentiable fuzzy functional of $\tilde{x}$. If $\tilde{x}^*$ is a fuzzy minimizer of $\tilde{J}$, then the variation of $\tilde{J}$ regardless of any boundary conditions must vanish on $\tilde{x}^*$, that is,

$$ \delta\tilde{J}(\tilde{x}^*, \delta\tilde{x}) = 0, \tag{3.5} $$

for all admissible $\delta\tilde{x}$ having the property $\tilde{x}+\delta\tilde{x} \in \tilde{C}[t_0, t_1]$.

It is obviously that the equality (3.5) holds if and only if

$$\delta J^l(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) = 0, \tag{3.6}$$

$$\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) = 0, \tag{3.7}$$

for all $\alpha \in [0,1], t \in [t_0, t_1]$ and all admissible $\delta\tilde{x}$ where,

$$\delta\tilde{x}(t)[\alpha] = [\delta x^l(t,\alpha), \delta x^r(t,\alpha)].$$

Proof. See [10]

4. Fuzzy fractional optimal control problem

In this section, we first define fuzzy fractional optimal control problem with fixed and free final state conditions, and then we derive necessary conditions for optimality by applying fuzzy variational approaches to our problem.

We define fuzzy fractional optimal control problem as:

$$\begin{aligned} &\min_{\tilde{u}} \tilde{J}(\tilde{u}) = \tilde{\phi}(\tilde{x}(t_1), t_1) + \int_{t_0}^{t_1} \tilde{f}(\tilde{x}(t), \tilde{u}(t), t)dt, \\ &\text{subject to:} \quad {}_{t_0}D_t^{\beta}\tilde{x} = \tilde{g}(\tilde{x}(t), \tilde{u}(t), t) \\ &\qquad\qquad\quad \tilde{x}(t_0) = \tilde{x}_0. \end{aligned} \tag{4.1}$$

For fixed final state problem we have additional condition $\tilde{x}(t_1) = \tilde{x}_1$. Where $\tilde{f}, \tilde{g} : E^1 \times E^1 \times R \to E^1$ are assumed to be continuous first and second partial derivatives on $t \in I = [t_0, t_1] \subseteq R$ with respect to all their arguments and Riemann integrable, the fuzzy state $\tilde{x}(t)$ and the fuzzy control $\tilde{u}(t)$ are functions of $t \in I$, and the fuzzy state function $\tilde{x}(t)$ is Riemann-Liouville $[(1)-\beta]-$differentiable fuzzy-valued function and satisfies appropriate boundary conditions, and $\beta \in (0,1)$.

Definition 4.1 We say that an admissible fuzzy curve $(\tilde{x}^*, \tilde{u}^*)$ is solution of (4.1), if for all admissible fuzzy curve $(\tilde{x}, \tilde{u})$ of (4.1),

$$\tilde{J}(\tilde{x}^*, \tilde{u}^*) \preceq \tilde{J}(\tilde{x}, \tilde{u}).$$

Note that, we consider an admissible fuzzy control $\tilde{u}$ is not bounded.

Remark 4.1 If we choose $\beta = 1$, problem (4.1) is reduced to classical fuzzy optimal control problem.

Definition 4.2(Fuzzy Hamiltonian Function). We define fuzzy Hamiltonian function as,

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t) = \tilde{f}(\tilde{x}(t), \tilde{u}(t), t) + \tilde{\lambda}(t)\tilde{g}(\tilde{x}(t), \tilde{u}(t), t). \tag{4.2}$$

It means that,

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)[\alpha] = [H^l(x^l, u^l, \lambda^l, t, \alpha), H^r(x^r, u^r, \lambda^r, t, \alpha)]. \tag{4.3}$$

for any $\alpha \in [0,1]$, and where $H^l(x^l, u^l, \lambda^l, t, \alpha)$ and $H^r(x^r, u^r, \lambda^r, t, \alpha)$ are classical Hamiltonian functions.

Remark 4.2 In the following theorem, we assume that $J^l(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)$ (or $J^r(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)$) is stated in terms containing only $x^l(t,\alpha)$, $u^l(t,\alpha)$ and $\lambda^l(t,\alpha)$ (or only $x^r(t,\alpha)$, $u^r(t,\alpha)$ and $\lambda^r(t,\alpha)$) in order to simplify the result presentations.

4.1 Derivation of Necessary Conditions

Now we are in the position to state a fundamental result of this work in the following theorem.
Theorem 4.1(Necessary Conditions) Assume that $\tilde{x}^*(t)$ be an admissible fuzzy state and $\tilde{u}^*(t)$ be an admissible fuzzy control. Then the necessary conditions for $\tilde{u}^*$ to be an optimal control for (4.1) and for all $\alpha \in [0,1]$, $t \in [t_0, t_1]$ are:

$$ {}_{t_0}D_t^{\beta} x^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.4} $$

$$ {}_{t_0}D_t^{\beta} x^{*^r}(t,\alpha) = \frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.5} $$

$$ {}_t^C D_{t_1}^{\beta} \lambda^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.6} $$

$$ {}_t^C D_{t_1}^{\beta} \lambda^{*^r}(t,\alpha) = \frac{\partial H^r}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.7} $$

$$ \frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha) = 0, \tag{4.8} $$

$$ \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha) = 0. \tag{4.9} $$

with

$$ \lambda^l(t_1,\alpha) = \left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=t_1}, \tag{4.10} $$

$$ \lambda^r(t_1,\alpha) = \left.\frac{\partial \phi^r}{\partial x^r}\right|_{t=t_1}. \tag{4.11} $$

for free final state problems.

Proof. First we adopt fuzzy lagrange multiplier to form an augmented functional incorporating the constraints, then we modify the performance index as,

$$ \tilde{J}_a(\tilde{u}) = \int_{t_0}^{t_1} \left[\tilde{f}(\tilde{x}(t), \tilde{u}(t), t) + \frac{d\tilde{\phi}}{dt} + \tilde{\lambda}\left(\tilde{g}(\tilde{x}(t), \tilde{u}(t), t) \ominus_{t_0} D_t^{\beta} \tilde{x} \right) \right] dt, \tag{4.12} $$

It means that,

$$ \left[J_a^l(u^l,\alpha), J_a^r(u^r,\alpha) \right] = \left[\int_{t_0}^{t_1} \left[f^l(x^l,u^l,t,\alpha) + \frac{d\phi^l}{dt} + \lambda^l(t,\alpha)\left(g^l(x^l,u^l,t,\alpha) - {}_{t_0}D_t^{\beta} x^l \right) \right] dt, \right. $$
$$ \left. \int_{t_0}^{t_1} \left[f^r(x^r,u^r,t,\alpha) + \frac{d\phi^r}{dt} + \lambda^r\left(g^r(x^r,u^r,t,\alpha) - {}_{t_0}D_t^{\beta} x^r \right) \right] dt \right]. $$

In the remaining of the proof we will ignore the similar arguments and only we consider the left hand of all functions of its α-level set.

$$ J_a^l(u^l,\alpha) = \int_{t_0}^{t_1} \left[f^l(x^l(t), u^l(t), t, \alpha) + \lambda^l(t,\alpha) g^l(x^l(t), u^l(t), t, \alpha) - \lambda^l(t,\alpha)_{t_0} D_t^{\beta} x^l(t,\alpha) + \frac{d\phi^l}{dt} \right] dt. \tag{4.13} $$

Using the definition of fuzzy Hamiltonian function, then we can rewrite equation (4.13) as,

$$J_a^l(u^l,\alpha) = \int_{t_0}^{t_1} \left[H^l(x^l(t), u^l(t), \lambda^l(t), t, \alpha) + \frac{d\phi^l}{dt} - \lambda^l(t,\alpha)_{t_0}D_t^\beta x^l(t,\alpha) \right]. \tag{4.14}$$

Taking variation of equation (4.14), we obtain

$$\delta J_a^l(u^l,\alpha) = \int_{t_0}^{t_1} \frac{\partial H^l}{\partial x^l}\delta x^l + \frac{\partial H^l}{\partial u^l}\delta u^l + \frac{\partial H^l}{\partial \lambda^l}\delta\lambda^l + \frac{\partial \phi^l}{\partial x^l}\delta x^l - \delta\lambda^l{}_{t_0}D_t^\beta x^l - \lambda^l \delta_{t_0}D_t^\beta x^l, \tag{4.15}$$

where δx^l, $\delta\lambda^l$ and δu^l are the variations of x^l, λ^l and u^l respectively.

Using the formula for fractional integration by parts, integrate the last term on the RHS of (4.15), then we obtain

$$\delta J_a^l(u^l,\alpha) = \int_{t_0}^{t_1} \left(\frac{\partial H^l}{\partial x^l} -{}_t^C D_{t_1}^\beta \lambda^l \right)\delta x^l + \frac{\partial H^l}{\partial u^l}\delta u^l + \left(\frac{\partial H^l}{\partial \lambda^l} - {}_{t_0}D_t^\beta x^l \right)\delta\lambda^l dt + \left.\left(\frac{\partial \phi^l}{\partial x^l} - \lambda^l \right)\right|_{t=t_1} \delta x^l(t_1). \tag{4.16}$$

where ${}_t^C D_{t_1}^\beta$ represent the classical right Caputo fractional derivative.

u^{*^l} is an extremal if the variation of J_a^l is zero, that is, for all $\alpha \in [0,1]$ we require

$$\int_{t_0}^{t_1} \left(\frac{\partial H^l}{\partial x^l} -{}_t^C D_{t_1}^\beta \lambda^l \right)\delta x^l + \frac{\partial H^l}{\partial u^l}\delta u^l + \left(\frac{\partial H^l}{\partial \lambda^l} - {}_{t_0}D_t^\beta x^l \right)\delta\lambda^l dt + \left.\left(\frac{\partial \phi^l}{\partial x^l} - \lambda^l \right)\right|_{t=t_1} \delta x^l(t_1) = 0. \tag{4.17}$$

It is convenient to choose the coefficients of $\delta x^l, \delta u^l$, and $\delta\lambda^l$ in (4.17) to be zero. This leads to

$${}_{t_0}D_t^\beta x^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.18}$$

$${}_t^C D_{t_1}^\beta \lambda^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.19}$$

$$\frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha) = 0, \tag{4.20}$$

Finally, we have

$$\left.\left(\frac{\partial \phi^l}{\partial x^l} - \lambda^l \right)\right|_{t=t_1} \delta x^l(t_1) = 0, \tag{4.21}$$

1. For the fixed final state problem
$$\delta x^l(t_1) = 0, \tag{4.22}$$

2. For the free final state problem
$$\left.\left(\frac{\partial \phi^l}{\partial x^l} - \lambda^l \right)\right|_{t=t_1} = 0. \tag{4.23}$$

Equations $(4.18)-(4.20)$ represents the necessary conditions for u^{*^l} to be an optimal with the condition (4.22) for the fixed final state problem and (4.23) for the free final state problem.

By following the same steps(using the right hand of all functions of its α-level set) for $\delta J_a^r(u^{*^r},\alpha) = 0$, for all $\alpha \in [0,1]$ and $t \in [0,1]$, we will obtain

$${}_{t_0}D_t^\beta x^{*^l}(t,\alpha) = \frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.24}$$

$$ {}_t^C D_{t_1}^{\beta} \lambda^{*^r}(t,\alpha) = \frac{\partial H^l}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.25} $$

$$ \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha) = 0. \tag{4.26} $$

Equations $(4.24) - (4.26)$ represents the necessary conditions for u^{*^r} to be an extremal with the conditions $\delta x^r(t_1) = 0$ for the fixed final state problem and $\left(\frac{\partial \phi^r}{\partial x^r} - \lambda^l\right)\Big|_{t=t_1} = 0$ for the free final state problem.

The above equations form a set of necessary conditions that the left and right hand functions of its $\alpha-$level set of the fuzzy optimal control $\tilde{u}^*$ and fuzzy optimal state $\tilde{x}^*$ must satisfy. □

We know that, $\tilde{u}^*(t)$ and $\tilde{x}^*(t)$ are a fuzzy numbers with $\tilde{u}^*(t)[\alpha] = \left[u^{*^l}(t,\alpha), u^{*^r}(t,\alpha)\right]$ and $\tilde{x}^*(t)[\alpha] = \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right]$ if $u^{*^l}(t,\alpha), u^{*^r}(t,\alpha), x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ satisfy are related properties in **C1-C5** of Lemma(2.1). In the following definition, based on the conditions **C1** and **C2** of Lemma(2.1), we introduce the definition of strong and weak solutions of our problem.

Definition 4.3(Strong and Weak Solutions).

1. (Strong Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are strong solutions of (4.1) if $u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)$,$x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ obtained from $(4.4) - (4.11)$ satisfy the conditions **C1-C2** of Lemma(2.1), for all $t \in [t_0, t_1]$ and $\alpha \in [0,1]$.

2. (Weak Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are weak solutions of (4.1) if $u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)$,$x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ obtained from $(4.4) - (4.11)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we define $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ as:

   $\tilde{u}^*(t)[\alpha] =$

$$ \begin{cases} [2u^{r^*}(t,1) - u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)], \text{ if } u^{l^*}, u^{r^*} \text{are decreasing functions of } \alpha, \\ [u^{l^*}(t,\alpha), 2u^{l^*}(t,1) - u^{r^*}(t,\alpha)], \text{ if } u^{l^*}, u^{r^*} \text{are increasing functions of } \alpha, \\ [u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)], \text{ if } u^{l^*} \text{is decreasing and } u^{r^*} \text{is increasing of } \alpha \end{cases} $$

   and,

   $\tilde{x}^*(t)[\alpha] =$

$$ \begin{cases} [2x^{r^*}(t,1) - x^{l^*}(t,\alpha), x^{r^*}(t,\alpha)], \text{ if } x^{l^*}, x^{r^*} \text{are decreasing functions of } \alpha, \\ [x^{l^*}(t,\alpha), 2x^{l^*}(t,1) - x^{r^*}(t,\alpha)], \text{ if } x^{l^*}, x^{r^*} \text{are increasing functions of } \alpha, \\ [x^{r^*}(t,\alpha), x^{l^*}(t,\alpha)], \text{ if } x^{l^*} \text{is decreasing and } x^{r^*} \text{is increasing of } \alpha \end{cases} $$

for all $t \in [t_0, t_1]$ and $\alpha \in [0,1]$.

Now, we consider fixed and free final state problems with a quadratic performance index.

4.2 Fixed Final State Problem

We can define fuzzy fractional optimal control problem with fixed final state as

$$ \begin{aligned} &\min_{\tilde{u}} \tilde{J}(\tilde{u}) = \frac{1}{2}\int_{t_0}^{t_1} \left[q(t)\tilde{x}^2 + r(t)\tilde{u}^2\right] dt, \\ &\text{subject to:} \quad {}_0D_t^{\beta}\tilde{x} = a(t)\tilde{x} + b(t)\tilde{u}, \\ &\qquad \tilde{x}(t_0) = \tilde{x}_0, \quad \tilde{x}(t_1) = \tilde{x}_1. \end{aligned} \tag{4.27} $$

where $q(t) \geq 0$ and $r(t) > 0$.

Theorem(4.1), give the necessary conditions for u^{*^l} to be an optimal as

$$ {}_{t_0}D_t^{\beta} x^l = a(t)x^l + b(t)u^l, \tag{4.28} $$

$$ {}_t^C D_{t_1}^{\beta} \lambda^l = q(t)x^l + a(t)\lambda^l, \tag{4.29} $$

$$ r(t)u^l + b(t)\lambda^l = 0. \tag{4.30} $$

Equations (4.28) and (4.30) gives

$$ {}_{t_0}D_t^{\beta} x^l = a(t)x^l - r^{-1}(t)b^2(t)\lambda^l. \tag{4.31} $$

We will obtain $x^l(t,\alpha)$ and $u^l(t,\alpha)$ by solving Equations (4.29) – (4.31) with the boundary conditions $x^l(t_0) = x_0^l$ and $x^l(t_1) = x_1^l$.

Similarly Theorem(4.1), give the necessary conditions for u^{*^r} to be an optimal as

$$ {}_{t_0}D_t^{\beta} x^r = a(t)x^r + b(t)u^r, \tag{4.32} $$

$$ {}_t^C D_{t_1}^{\beta} \lambda^r = q(t)x^r + a(t)\lambda^r, \tag{4.33} $$

$$ r(t)u^r + b(t)\lambda^r = 0. \tag{4.34} $$

Equations (4.32) and (4.34) gives

$$ {}_{t_0}D_t^{\beta} x^r = a(t)x^r - r^{-1}(t)b^2(t)\lambda^r. \tag{4.35} $$

We will obtain $x^r(t,\alpha)$ and $u^r(t,\alpha)$ by solving Equations (4.33) – (4.35) with the boundary conditions $x^r(t_0) = x_0^r$ and $x^r(t_1) = x_1^r$.

4.3 Free Final State Problem

We can define fuzzy fractional optimal control problem with free final state as

$$ \begin{aligned} &\min_{\tilde{u}} \tilde{J}(\tilde{u}) = \tilde{\phi}(\tilde{x}(t_1), t_1) + \frac{1}{2}\int_{t_0}^{t_1} \left[q(t)\tilde{x}^2 + r(t)\tilde{u}^2\right] dt, \\ &\text{subject to:} \quad {}_{t_0}D_t^{\beta}\tilde{x} = a(t)\tilde{x} + b(t)\tilde{u}, \\ &\qquad\qquad\quad \tilde{x}(t_0) = \tilde{x}_0. \end{aligned} \tag{4.36} $$

where $q(t) \geq 0$ and $r(t) > 0$.

Following the same steps, we will obtain $x^l(t,\alpha)$ and $u^l(t,\alpha)$ by solving Equations (4.29) – (4.31) with respect to the conditions

$$ x^l(t_0) = x_0^l \quad \text{and} \quad \lambda^l(t_1,\alpha) = \left.\left(\frac{\partial \phi^l}{\partial x^l}\right)\right|_{t=t_1}. \tag{4.37} $$

Also we will obtain $x^r(t,\alpha)$ and $u^r(t,\alpha)$ by solving Equations (4.33) – (4.35) with respect to the conditions

$$ x^r(t_0) = x_0^r \quad \text{and} \quad \lambda^r(t_1,\alpha) = \left.\left(\frac{\partial \phi^r}{\partial x^r}\right)\right|_{t=t_1}. \tag{4.38} $$

In the next section we propose an algorithm used to find the solution of both cases numerically, the details of this algorithm in [4, 5].

5. Numerical technique

Considering the both cases of fixed and free final state problems defined above, in order to find the solution of our problems, we use the Grünwald-Letnikov(GL-for short) approximation of the left Riemann-Liouville fractional derivative and using the relation between right Riemann-Liouville fractional derivative and right Caputo fractional derivative and then use GL-approximation, we can approximate (4.31) and (4.29) as

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} x_{m-j}^l = a(mh)x_m^l - r^{-1}(mh)b^2(mh)\lambda_m^l, \tag{5.1}$$

for $m = 1, 2, ..., N$, and

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} \lambda_{m+j}^l = q(mh)x_m^l + a(mh)\lambda_m^l + \frac{\lambda_N^l (t_1 - mh)^{-\beta}}{\gamma(1-\beta)}, \tag{5.2}$$

for $m = N-1, N-2, ..., 0$, respectively. Where N is the number of equal divisions of the interval $[0, t_1]$, the nodes are labeled as $0, 1, ..., N$. The size of each division is given as $h = \frac{t_1}{N}$, and $t_j = jh$ represent the time at node j. The coefficients are defined as

$$w_j^{\beta} = (-1)^j \binom{\beta}{j}. \tag{5.3}$$

Where x_i^l and λ_i^l represent the numerical approximations of $x^l(t, \alpha)$ and $\lambda^l(t, \alpha)$ at node i.

Similarly, we can approximate (4.35) and (4.33) as

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} x_{m-j}^r = a(mh)x_m^r - r^{-1}(mh)b^2(mh)\lambda_m^r, \tag{5.4}$$

for $m = 1, 2, ..., N$, and

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} \lambda_{m+j}^r = q(mh)x_m^r + a(mh)\lambda_m^l + \frac{\lambda_N^r (t_1 - mh)^{-\beta}}{\gamma(1-\beta)}, \tag{5.5}$$

for $m = N-1, N-2, ..., 0$, respectively.

Also x_i^r and λ_i^r represent the numerical approximations of $x^r(t, \alpha)$ and $\lambda^r(t, \alpha)$ at node i. In general, Equations (5.1) and (5.2) or Equations (5.4) and (5.5) give a set of $2N$ equations in terms of $2N$ variables, i.e., $Ax = b$, it means that, we can use any linear equation solver to find the solution. Regardless the left and right bounds of the fuzzy numbers $\tilde{x}$ and $\tilde{\lambda}$, the vector x is constructed as follows

- For fixed final state problem

$$x = [x_1 \;\; x_2 \;\; ... \;\; x_{N-1} \;\; \lambda_0 \;\; \lambda_1 \;\; ... \;\; \lambda_N]^T .$$

- For free final state problem

$$x = [x_1 \;\; x_2 \;\; ... \;\; x_N \;\; \lambda_0 \;\; \lambda_1 \;\; ... \;\; \lambda_{N-1}]^T .$$

In the next section, we will give four examples can serve to illustrate our main results.

6. Numerical examples

Example 6.1 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_0^1 \left[\tilde{x}^2 + \tilde{u}^2\right] dt$$

subject to:

$$\begin{array}{rcl} {}_0D_t^\beta \tilde{x} & = & t\tilde{x} + \tilde{u}, \\ \tilde{x}(0) & = & (0,1,2), \quad \tilde{x}(1) = (-2,-1,1). \end{array}$$

Solution. We have,

$$q(t) = r(t) = b(t) = t_1 = 1, \text{ and } a(t) = t,$$

Then for the left bound of state and control Theorem(4.1) gives,

$$ {}_0D_t^\beta x^l = tx^l - \lambda^l, \tag{6.1}$$

$$ {}_t^CD_1^\beta \lambda^l = x^l + t\lambda^l, \tag{6.2}$$

$$ u^l + \lambda^l = 0. \tag{6.3}$$

and the boundary conditions

$$\begin{array}{rcl} x^l(0,\alpha) & = & \alpha, \\ x^l(1,\alpha) & = & -2+\alpha. \end{array}$$

For the right bound of state and control, Theorem(4.1) gives,

$$ {}_0D_t^\beta x^r = tx^r - \lambda^r, \tag{6.4}$$

$$ {}_t^CD_1^\beta \lambda^r = x^r + t\lambda^r, \tag{6.5}$$

$$ u^r + \lambda^r = 0. \tag{6.6}$$

and the boundary conditions

$$\begin{array}{rcl} x^r(0,\alpha) & = & 2-\alpha, \\ x^r(1,\alpha) & = & 1-2\alpha. \end{array}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(1(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(1(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is an increasing function of α, $u^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1), furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a strong fuzzy solution of this problem.

Example 6.2 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_1^2 \tilde{u}^2 dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= (2t-1)\tilde{x} \ominus \sin(t)\tilde{u}, \\ \tilde{x}(1) &= (0,1,2), \quad \tilde{x}(2) = (-2,-1,1). \end{aligned}$$

Solution. We have, $q(t) = 0$, $r(t) = t_0 = 1$, $b(t) = -\sin(t)$, and $a(t) = (2t-1)$, then for the left bound of the state and control, Theorem(4.1) gives,

$${}_1D_t^{\beta}x^l = (2t-1)x^l - \sin^2(t)\lambda^l, \tag{6.7}$$

$${}_t^CD_2^{\beta}\lambda^l = (2t-1)\lambda^l, \tag{6.8}$$

$$u^l - \sin(t)\lambda^l = 0. \tag{6.9}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= \alpha, \\ x^l(1,\alpha) &= -2+\alpha. \end{aligned}$$

For the right bound of state and control Theorem(4.1) gives,

$${}_1D_t^{\beta}x^r = (2t-1)x^r - \sin^2(t)\lambda^r, \tag{6.10}$$

$${}_t^CD_2^{\beta}\lambda^r = (2t-1)\lambda^r, \tag{6.11}$$

$$u^r - \sin(t)\lambda^r = 0. \tag{6.12}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 2-\alpha, \\ x^r(1,\alpha) &= 1-2\alpha. \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(2(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(2(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

Example 6.3 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_0^1 \left[\tilde{x}^2 + \tilde{u}^2\right] dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= -(0,1,3)\tilde{x} + \tilde{u}, \\ \tilde{x}(0) &= (1,1,1), \quad \tilde{x}(1) = (0,0,0). \end{aligned}$$

Solution. We know that,

$$\left[{}_0D_t^{\beta}x^l,{}_0D_t^{\beta}x^r\right]=\left[-(3-2\alpha)x^l+u^l,-\alpha x^r+u^r\right],$$

then we have,

$$q(t)=r(t)=b(t)=x_0=t_1=1,$$

$a(t)=-(3-2\alpha)$ and $a(t)=-\alpha$ for the left and right derivatives respectively, then for the left bound of the state and control Theorem(4.1) gives,

$${}_0D_t^{\beta}x^l=-(3-2\alpha)x^l-\lambda^l, \tag{6.13}$$

$${}_t^CD_1^{\beta}\lambda^l=x^l-(3-2\alpha)\lambda^l, \tag{6.14}$$

$$u^l+\lambda^l=0. \tag{6.15}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= 1,\\ x^l(1,\alpha) &= 0.\end{aligned}$$

For the right bound of the state and control Theorem(4.1) gives,

$${}_1D_t^{\beta}x^r=-\alpha x^r-\lambda^r, \tag{6.16}$$

$${}_t^CD_2^{\beta}\lambda^r=x^r-\alpha\lambda^r, \tag{6.17}$$

$$u^r+\lambda^r=0. \tag{6.18}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 1,\\ x^r(1,\alpha) &= 0.\end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(3(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1)=x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(3(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1)=x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha]=\left[u^{r^*}(t,\alpha),u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

Example 6.4 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t))=\frac{1}{2}\tilde{x}^2(1)+\frac{1}{2}\int_0^1\left[\tilde{x}^2+\tilde{u}^2\right]dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= -(0,1,3)\tilde{x}+\tilde{u},\\ \tilde{x}(0) &= (1,1,1).\end{aligned}$$

Solution. We have,

$$q(t) = r(t) = b(t) = x_0 = t_1 = 1,$$

$a(t) = -(3-2\alpha)$ and $a(t) = -\alpha$ for the left and right derivatives respectively, then Theorem(4.1) gives,

$${}_{t_0}D_t^{\beta} x^l = -(3-2\alpha)x^l - \lambda^l, \tag{6.19}$$

$${}_t^C D_{t_1}^{\beta} \lambda^l = x^l - (3-2\alpha)\lambda^l, \tag{6.20}$$

$$u^l + \lambda^l = 0. \tag{6.21}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= 1, \\ \lambda^l(0,\alpha) &= x^l(1,\alpha). \end{aligned}$$

For the right bound of the state and control Theorem(4.1) gives,

$${}_1D_t^{\beta} x^r = -\alpha x^r - \lambda^r, \tag{6.22}$$

$${}_t^C D_2^{\beta} \lambda^r = x^r - \alpha\lambda^r, \tag{6.23}$$

$$u^r + \lambda^r = 0. \tag{6.24}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 1, \\ \lambda^r(0,\alpha) &= x^r(1,\alpha). \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(4(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(4(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1**-**C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

7. Conclusion

In this paper, the necessary conditions of fuzzy fractional optimal control problem with both fixed and free final state conditions at the final time has been derived using fuzzy variational approach. Our problems is defined in the sense of Riemann-Liouville fractional derivative based on Hukuhara difference. A numerical technique is proposed based on Grünwald-Letnikov definition of fractional derivative. The concepts of strong and weak solutions of our problems are given. lastly, four examples are provided to show the effectiveness of Theorem(4.1) and the numerical algorithm.

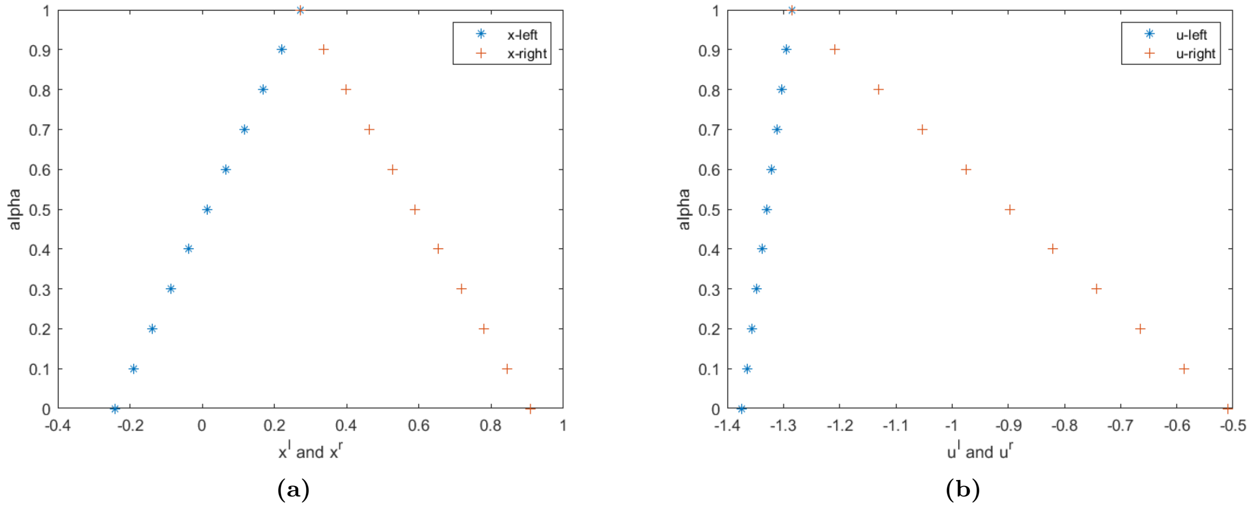


Figure 1: Example(6.1) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.

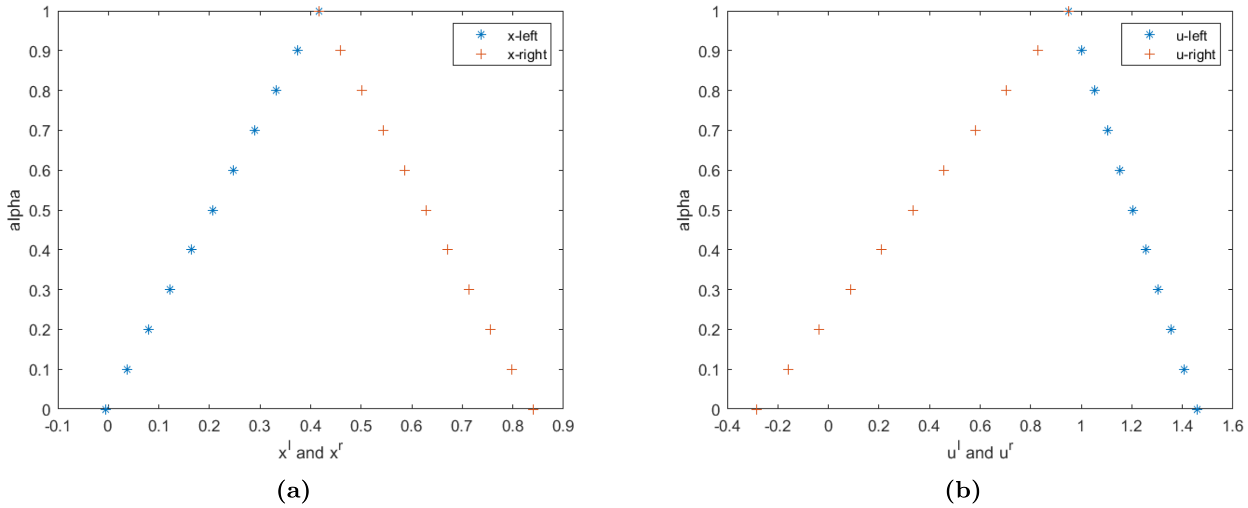
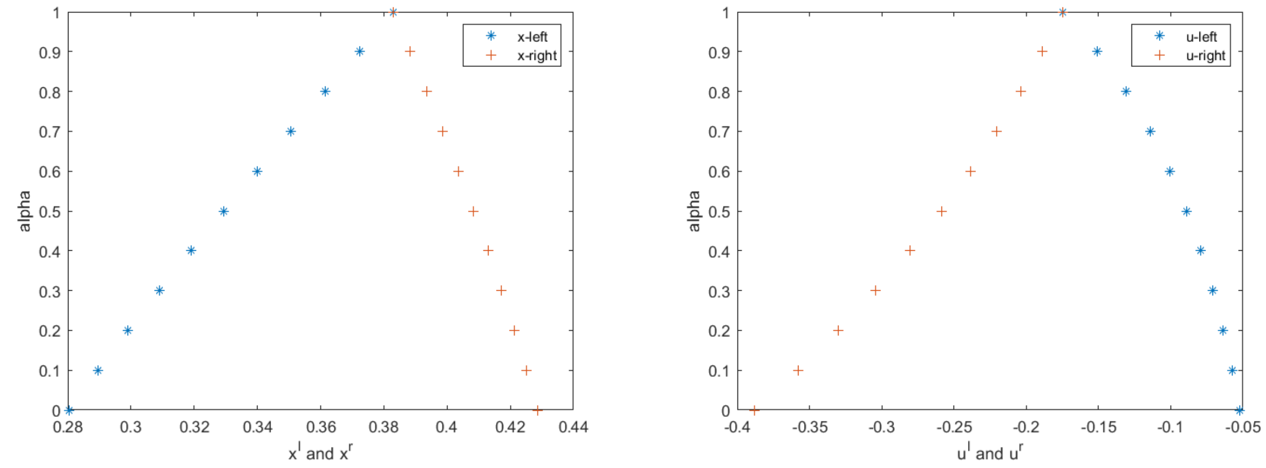
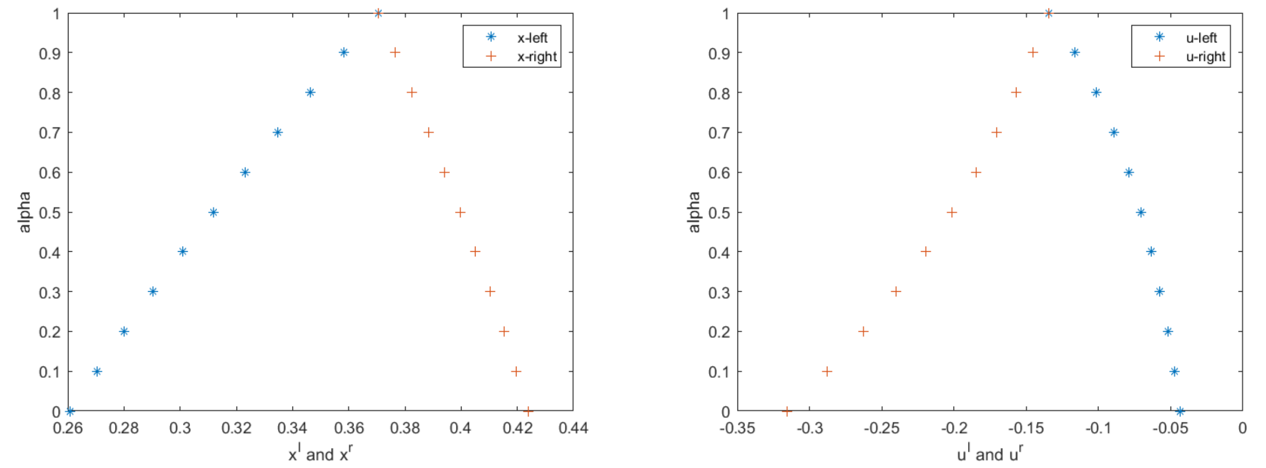


Figure 2: Example(6.2) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.



(a) (b)

Figure 3: Example(6.3) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.



(a) (b)

Figure 4: Example(6.4) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control $t = 0.1, \beta = 0.77$.

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Differential Transform Method for Solving Fuzzy Fractional Wave Equation†

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Abstract: In this letter, the differential transform method (DTM) is applied to solve fuzzy fractional wave equation. The elemental properties of this method are investigated based on the two-dimensional differential transform method (DTM), generalized Taylor's formula and fuzzy Coputo's derivative. The proposed method is also illustrated by using some examples. The results reveal that DTM is a highly effective scheme for obtaining analytical solutions of the fuzzy fractional wave equation.

Mathematics Subject Classification. 65L05, 26E50

Keyword: Fuzzy numbers; Fuzzy fractional wave equation; Differential transform method; Fuzzy Caputo's derivative; Generalized Taylor formula.

1 Introduction

In 1965, the fuzzy sets were introduced for the first time by Zadeh in [28]. hundreds of examples have been supplied where the nature of uncertainty in the behavior of given system processes are fuzzy rather than stochastic nature. In the last few years, many authors have interested in the study of the theoretical framework of fuzzy initial value problems. Chang and Zadeh in [6] have introduced the concept of fuzzy derivative. Kandel and Byatt in [12] have initially presented the concept of the fuzzy differential equation. Bede and Gal in [4] have studied the concept of strongly generalized differentiable of fuzzy valued functions, which enlarged the class of differentiable fuzzy valued functions.

In 1695, the fractional calculus was first studied. The subject of fractional calculus has gained importance during the past three decades due mainly to its demonstrated applications in different area of physics and engineering in [16]. Fuzzy fractional differential equations (FFDE) play an important role in modelling of science and engineering problems. Padmapriya and Kaliyappan in [22] established analytical and numerical methods to solve fuzzy fractional differential equations. the concept of differential of fuzzy function with two variables and fuzzy wave equations studied in [26]. In the last years many authors have developed and introduced some variant methods for solving fuzzy wave equation. Kermani in [15] used finite difference method to solve the fuzzy wave equation numerically. Also, Martin and Radek in [25] used f-transforms to solve the fuzzy wave equation.

Zhou in [29] has presented the concept of the differential transform method (DTM), this method constructs an analytical solution inform of a polynomial, which is different from the tradition higher order Taylor formula method. Recently some researchers used differential transform method (DTM) to solve fuzzy fractional differential equations and fuzzy differential equations in [9, 23, 1, 19, 20].

This paper is structured as follows. In Section 2, we call some definitions on fuzzy numbers, fuzzy functions and fuzzy Caputo's derivative. In Section 3, The generalization of Taylor's formula is presented. In Section 4, the generalized two-dimensional differential transform method (DTM) for

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the solution of the fuzzy wave equation with space and time-fractional derivatives are developed and derived. Examples are shown in Section 5. Finely, conclusion is given in section 6.

2 Basic concepts

The results about fuzzy numbers space E^1, we recall that $E^1 = \{\tilde{u} : R \to [0,1] : u$ satisfies (1)(4) below } (refer to [6])

1. $\tilde{u}$ is normal, i.e., there exists $x_0 \in R$ such that $\tilde{u}(x_0) = 1$;
2. $\tilde{u}$ is convex, i.e., for all and $\lambda \in [0,1], x, y \in R$,
$$\tilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\},$$
holds;
3. $\tilde{u}$ is upper semicontinuous, i.e., for any $x_0 \in R$,
$$\tilde{u}(x_0) \geq \lim_{x \longrightarrow x_0^{\pm}} \tilde{u}(x);$$
4. supp $\tilde{u} = \{x \in R | \tilde{u}(x) > 0\}$ is the support of $\tilde{u}$, and its closure cl (supp $\tilde{u}$) is compact.

For $0 < r \leq 1$, denote $[\tilde{u}]_r = \{x : \tilde{u}(x) \geq r\}$. Then from (1)-(4), follows that the r-level set $[\tilde{u}]_r$ is a closed and bounded interval for all $r \in [0,1]$.

For $\tilde{u}, \tilde{v} \in E^1,\ k \in R$, the addition and scalar multiplication are defined using the equations

$$[\tilde{u} + \tilde{v}]_r = [\tilde{u}]_r + [\tilde{v}]_r,$$

$$[k\tilde{u}]_r = k[\tilde{u}]_r,$$

respectively.

Define $D : E^1 \times E^1 \to R^+ \cup \{0\}$ using the equation

$$D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} d([\tilde{u}]_r [\tilde{v}]_r),$$

where d is Hausdorff metric space as

$$\begin{aligned} d([\tilde{u}]_r, [\tilde{v}]_r) &= \inf\{\varepsilon : [\tilde{u}]_r \subset N([\tilde{v}]_r, \varepsilon), [\tilde{v}]_r \subset N([\tilde{u}]_r, \varepsilon)\} \\ &= \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}, \end{aligned}$$

where $N([\tilde{u}]_r, \varepsilon), N([\tilde{v}]_r, \varepsilon)$ is the ε-neighborhood of $[\tilde{u}]_r, [\tilde{v}]_r$, respectively, and $\underline{u}_r, \underline{v}_r, \overline{u}_r, \overline{v}_r$ are endpoints of $[\tilde{u}]_r, [\tilde{v}]_r$, respectively.
By using the results of [13], we see that

- (E^1, D) is complete metric space,
- $D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$ for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^1$,
- $D(k\tilde{u}, k\tilde{v}) = |k| D(\tilde{u}, \tilde{v})$.

In addition, we can introduce a partial order in E^1 by $\tilde{u} \leq \tilde{v}$ if and only if $[\tilde{u}]_r \leq [\tilde{v}]_r,\ r \in [0,1]$ if and only if $\underline{u}_r \leq \underline{v}_r, \overline{u}_r \leq \overline{v}_r, r \in [0,1]$. For applications of the partial order on E^1 (refer to [27]).

As the fuzzy number is resolved by using the interval $\tilde{u}_r = [\underline{u}_r, \overline{u}_r]$, see [8] defined another statements, parametrically, of fuzzy numbers as in following.

Definition 2.1.[31, 32] For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1, \tilde{u} = [\underline{u}_r, \overline{u}_r], \tilde{v} = [\underline{v}_r, \overline{v}_r]$, the quantity $D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}$ is the distance between $\tilde{u}$ and $\overline{v}$ and also the following properties hold:

- (E^1, D) is a complete metric space,
- $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v}, \tilde{w} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e}), \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{0}) \leq D(\tilde{u}, \tilde{0}) + D(\tilde{v}, \tilde{0}), \forall \tilde{u}, \tilde{v} \in E^1$,
- $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v} \in E^1, k \in R$,
- $D(k_1 \odot \tilde{u}, k_2 \odot \tilde{u}) = |k_1 - k_2| D(\tilde{u}, \tilde{0}), \forall \tilde{u} \in E^1, k_1, k_2 \in R$, with $k_1 \cdot k_2 \geq 0$.

Let us recall the definition of the Hukuhara difference (H-difference) in [33]. Suppose that $\tilde{u}, \tilde{v} \in E^1$. The Hukuhara H-difference has been presented as a set $\tilde{w}$ for which $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \tilde{u} = \tilde{v} \oplus \tilde{w}$. The H-difference is unique, but it does not always exist (a necessary condition for $\tilde{u} \ominus_{gH} \tilde{v}$ to exist is that $\tilde{u}$ contains a translate $\{c\} \oplus \tilde{v}$ of $\tilde{v}$). A generalization of the Hukuhara difference aims to overcome this situation.

Definition 2.2.[33, 31] The generalized Hukuhara difference between two fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$ is defined as following:

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \begin{cases} \text{(i) } \tilde{u} = \tilde{v} \oplus \tilde{w}, \\ or \text{ (ii) } \tilde{v} = \tilde{u} \oplus (-\tilde{w}). \end{cases} \tag{2.1}$$

In terms of the $r-$levels, we get $[\tilde{u} \ominus_{gH} \tilde{v}] = [\min\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}, \max\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}]$ and if the H-difference exists, then $\tilde{u} \ominus \tilde{v} = \tilde{u} \ominus_{gH} \tilde{v}$; the conditions for existence of $\tilde{w} = \tilde{u} \ominus_{gH} \tilde{v} \in E^1$ are

$$\text{Case (i)} \quad \begin{cases} \underline{w}_r = \underline{u}_r - \underline{v}_r \text{ and } \overline{w}_r = \overline{u}_r - \overline{v}_r, \forall_r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.2}$$

$$\text{Case (ii)} \quad \begin{cases} \underline{w}_r = \overline{u}_r - \overline{v}_r \text{ and } \overline{w}_r = \underline{u}_r - \underline{v}_r, \forall_r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.3}$$

It is easy to show that (i) and (ii) are both valid if and only if $\tilde{w}$ is a crisp number. In the case, it is possible that the gH-difference of two fuzzy numbers does not exist. To address this shortcoming, a new difference between fuzzy numbers was introduced in [33].

Lemma 2.1.[10, 24] A fuzzy number $\tilde{u}$ in parametric form is a pair $[\underline{u}_r, \overline{u}_r]$ of function $\underline{u}_r$ and $\overline{u}_r$ for any $r \in [0,1]$, which satisfies the following requirements.

- $\underline{u}_r$ is a bounded non-decreasing left continuous function in (0,1];
- $\overline{u}_r$ is a bounded non-increasing left continuous function in (0,1];
- $\underline{u}_r \leq \overline{u}_r$.

Some the author of the classified fuzzy numbers into several types of fuzzy membership function. To the deepest of our study, triangular fuzzy membership function or also often referred to as triangular fuzzy numbers are the most widely used membership function.

In order to describe the fuzzy numbers and real numbers clearly, in convenience, the fuzzy numbers and fuzzy-valued functions in the whole paper are added with a tilde sign at the top, while the real-value function and interval-value functions are written directly.

A fuzzy valued function $\tilde{f}$ of two variables is a rule that assigns to each ordered pair of real numbers, (x, t), in a set D, a unique fuzzy numbers denoted by $\tilde{f}(x, t)$. The set D is the domain of $\tilde{f}$ and its range is the set of values taken by f, i.e., $\{\tilde{f}(x, t) | (x, t) \in D\}$.

The parametric representation of the fuzzy valued function $f : D \to E^1$ is expressed by $f(x, t)(r) = [\underline{f}(x, t)(r), \overline{f}(x, t)(r)]$, for all $(x, t) \in D$ and $r \in [0, 1]$.

Suppose $f : D \to E^1$ be a fuzzy valued function of two variable. Then, we say that the fuzzy limit of $f(x, t)$ as (x, t) approaches to (a, b) is $L \in E^1$, and we write $\lim_{(x,t)\to(a,b)} f(x, t) = L$ if for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that if $(x,t) \in D, \| (x,t) - (a,b) \| < \delta \Rightarrow D(f(x,t), L) < \varepsilon$, where $\| \cdot \|$ denotes the Euclidean norm in R^n (ref. to [3])

A fuzzy valued function $f : D \to E^1$ is said to be fuzzy continuous at $(x_0, t_0) \in D$ if $\lim_{(x,t)\to(x_0,t_0)} f(x,t) = f(x_0,t_0)$. We say that f is fuzzy continuous on D if f is fuzzy continuous at every point (x_0, t_0) in D (ref. to [3, 30]).

Definition 2.3.[11] Suppose that $\tilde{u}(x,t) : D \to E^1$ and $(x_0,t) \in D$. We say that $\tilde{u}$ is strongly generalized differentiable on (x_0,t) if there exists an element $\frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)} \in E^1$ such that

i. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0+h,t) \ominus_{gH} \tilde{u}(x_0,t), \tilde{u}(x_0,t) \ominus_{gH} \tilde{u}(x_0-h,t)$ and the limits (in the metric D)

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0+h,t) \ominus_{gH} \tilde{u}(x_0,t)}{h} = \lim_{h\to 0+} = \frac{(x_0,t) \ominus_{gH} \tilde{u}(x_0-h,t)}{h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

ii. for all $h > 0$ sufficiently small, $\exists_{gH} \tilde{u}(x_0,t) \ominus_{gH} \tilde{u}(x_0+h,t), \tilde{u}(x_0-h,t) \ominus_{gH} \tilde{u}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0,t) \ominus_{gH} \tilde{u}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0-h,t) \ominus_{gH} \tilde{u}(x_0,t)}{-h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

iii. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0+h,t) \ominus_{gH} \tilde{u}(x_0,t), \tilde{u}(x_0-h,t) \ominus_{gH} \tilde{u}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0+h,t) \ominus_{gH} \tilde{u}(x_0,t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0-h,t) \ominus_{gH} \tilde{u}(x_0,t)}{-h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

iv. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0,t) \ominus_{gH} \tilde{u}(x_0+h,t), \tilde{u}(x_0,t) \ominus_{gH} \tilde{u}(x_0-h,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0,t) \ominus_{gH} \tilde{u}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0,t) \ominus_{gH} \tilde{u}(x_0-h,t)}{h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)}.$$

Definition 2.4.[4] Suppose that $\tilde{u}(x,t) : D \to E^1$ and $(x_0,t) \in D$. We define the n th-order derivative of $\tilde{u}$ as follows: we say that $\tilde{u}$ is strongly generalized differentiable of the n th-order at (x_0,t) if there exists an element $\frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)} \in E^1$, $\forall s = 1, 2, \cdots, n$ such that

i. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t), \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)$ and the limits (in the metric D)

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

ii. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t), \tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

iii. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t), \tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

iv. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t), \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)}.$$

2.1 Fuzzy Coputo's derivative

We denote $C^F[a,b]$ as a space of all fuzzy valued functions which are continuous on $[a,b]$, and the space of all Kaleva integrable fuzzy-valued functions on the bounded interval $[a,b] \subset \mathbb{R}$ by $K^F[a,b]$, we denote the space of fuzzy value functions $\tilde{f}(x)$ which have continuous H-derivative up to order $n-1$ on $[a,b]$ such that $\tilde{f}^{(n-1)}(x) \in AC^F([a,b])$ by $AC^{(n)F}([a,b])$, where $AC^F([a,b])$ denote the set of all fuzzy-valued functions which are absolutely continuous (ref. to [13, 9]).

Definition 2.5.[2] Suppose $\tilde{f}(x) \in C^F[a,b] \cap K^F[a,b]$, the fuzzy Riemann Liouville integral of fuzzy valued function $\tilde{f}$ is defined as following:

$$(I^\alpha_{a+}\tilde{f})(x,r) = [(I^\alpha_{a+}\underline{f})(x,r), (I^\alpha_{a+}\overline{f})(x,r)],$$

where $0 \le r \le 1$

$$(I^\alpha_{a+}\underline{f})(x,r) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\underline{f}(t)(r)\mathrm{d}t}{(x-t)^{1-\alpha}},\ 0 \le r \le 1,$$

$$(I^\alpha_{a+}\overline{f})(x,r) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\overline{f}(t)(r)\mathrm{d}t}{(x-t)^{1-\alpha}},\ 0 \le r \le 1.$$

Suppose $\tilde{f}(x) \in C^F((0,a]) \cap K^F(0,a)$, be a given function such that $\tilde{f}(t,r) = [\underline{f}(t,r), \overline{f}(t,r)]$ for all $t \in (0,a]$ and $0 \le r \le 1$. We define $D^\alpha_{*a}\tilde{f}(t;r)$ the fuzzy fractional Riemann-Liouville derivative of order $0 < \alpha < 1$ of $\tilde{f}$ in the parametric from,

$$D^\alpha_{*a}\tilde{f}(t;r) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_0^t (t-s)^{-\alpha}\underline{f}(s,r)ds, \frac{d}{dt}\int_0^t (t-s)^{-\alpha}\overline{f}(s,r)ds\right],$$

provided that equation defines a fuzzy number $D^\alpha_{*a}\tilde{f}(t) \in E^1$. In fact,

$$D^\alpha_{*a}\tilde{f}(t,r) = [D^\alpha_{*a}\underline{f}(t,r), D^\alpha_{*a}\overline{f}(t,r)].$$

Obviously, $D^\alpha_{*a}\tilde{f}(t) = \frac{d}{dt}I^{1-\alpha}\tilde{f}(t)\ for\ t \in (0,a]$.

3 Generalized Taylor's formula

In this section, we present the generalized Taylor's formula that involves Caputo fractional derivative.

Theorem 3.1.[21] Let that $(D^\alpha_{*a})^j f(x) \in C(a,b]$ for $j = 0, 1, \cdots, n+1$, where $0 < \alpha \le 1$, that we get

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D^\alpha_{*a})^i f)(a+) + \frac{((D^\alpha_{*a})^{n+1}f)(\zeta)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}, \tag{3.4}$$

with $a \leq \zeta \leq x$, $\forall x \in (a, b]$ and D^{α}_{*a} is the Caputo fractional derivative of order α, where $(D^{\alpha}_{*a})^j = D^{\alpha}_{*a} D^{\alpha}_{*a} \cdots D^{\alpha}_{*a}$. In case of $\alpha = 1$, the generalized Taylor's formula (3.4) reduces to the classical Taylor's formula.

Theorem 3.2.[17] Let that $(D^{\alpha}_{*a})^j f(x) \in C(a, b]$ for $j = 0, 1, \cdots\cdots, N + 1$, where $0 < \alpha \leq 1$. If $x \in [a, b]$, then

$$f(x) \simeq \sum_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha + 1)} ((D^{\alpha}_{*a})^i f)(a+). \tag{3.5}$$

Furthermore, there is a value ζ with $a \leq \zeta \leq x$ so that the error term $R^{\alpha}_N(x)$ has the from

$$R^{\alpha}_N(x) = \frac{((D^{\alpha}_{*a})^{N+1} f)(\zeta)}{\Gamma((N+1)\alpha + 1)} (x-a)^{(N+1)\alpha}. \tag{3.6}$$

The accuracy of $R^{\alpha}_N(x)$ increases when we choose large N and decreases as value of x moves away from the center a. Hence, we must choose N large enough so that the error does not exceed a specified bound. In the following theorem, we find precise condition under which the exponents hold for arbitrary fractional operators.

Theorem 3.3.[18] Let that $f(x) = x^{\lambda^*} g(x)$, where $\lambda^* > -1$ and $g(x)$ has the generalized power series expansion $g(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n\alpha}$ with radius of convergence $R > 0, where \;\; 0 < \alpha \leq 1$. Then

$$D^{\gamma}_{*a} D^{\beta}_{*a} f(x) = D^{\gamma+\beta}_{*a} f(x) \tag{3.7}$$

for all $x \in (0, R)$ if one of the following conditions is satisfied:

1. $\beta < \lambda^* + 1$, and γ arbitrary,
2. $\beta \geq \lambda^* + 1, \gamma$ arbitrary,, and $a_j = 0$ for $j = 0, 1, \cdots\cdots, m-1$, where $m - 1 < \beta \leq m$.

4 Differential transform method and fuzzy fractional wave equation

4.1 Generalized two-dimensional differential transform method

In this section, we will derive the generalized two-dimensional differential transform method (DTM) that we get developed for the solution of the wave equation with space and time-fractional derivatives. The proposed method is based on Taylor's formula. Consider a function of two variables $u(x, t)$, and Let that it can be represented as a product of two single variable functions, $u(x, t) = f(x)g(t)$. Based on the properties of generalized two dimensional differential transform method, function $u(x, t)$ can be represented as.

$$u(x,t) = \sum_{j=0}^{\infty} F_{\alpha}(j) \cdot (x-x_0)^{j\alpha} \sum_{h=0}^{\infty} G_{\beta}(h) \cdot (t-t_0)^{h\beta} = \sum_{j=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h)(x-x_0)^{j\alpha}(t-t_0)^{h\beta}, \tag{4.8}$$

where $0 < \alpha, \beta \leq 1, U_{\alpha,\beta}(j, h) = F_{\alpha}(j)G_{\beta}(h)$ is called the spectrum of $u(x, t)$. If function $u(x, t)$ is analytical and differentiated continuously with respect to time t^* in the domain of interest, then we define the generalized two-dimensional differential transform method (DTM) of the function $u(x, t)$ as follows:

$$U_{\alpha,\beta}(j,h) = \frac{1}{\Gamma(\alpha j + 1)\Gamma(\beta h + 1)} [(D^{\alpha}_{x_0})^j (D^{\beta}_{t_0})^h u(x,t)]_{(x_0,t_0)}, \tag{4.9}$$

where $(D^{\alpha}_{x_0})^j = D^{\alpha}_{x_0} \cdot D^{\alpha}_{x_0} \cdots\cdot D^{\alpha}_{x_0}$. In this work, the lowercase $u(x, t)$ represents the original function while the uppercase $U_{\alpha,\beta}(j, h)$ stands for the transformed function. The generalized differential transform method (DTM) inverse of $U_{\alpha,\beta}(j, h)$ is defined as follows

$$u(x,t) = \sum_{j=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h) \cdot (x-x_0)^{j\alpha}(t-t_0)^{h\beta} \tag{4.10}$$

In case of $\alpha = 1$ and $\beta = 1$. then generalized two-dimensional differential transform (DTM) (4.9) reduces to the classical two-dimensional DTM [5]. From equation (4.9) and (4.10), some basic properties of the generalized two-dimensional differential transform (DTM) are introduced below (ref. to [17]).

Theorem 4.1 If $u(x,t) = v(x,t) \pm w(x,t),$ *then* $U_{\alpha,\beta}(j,h) = V_{\alpha,\beta}(j,h) \pm W_{\alpha,\beta}(j,h)$.

Theorem 4.2 If $u(x,t) = cv(x,t),$ *then* $U_{\alpha,\beta}(j,h) = cV_{\alpha,\beta}(j,h)$.

Theorem 4.3 If $u(x,t) = v(x,t)w(x,t)$, then

$$U_{\alpha,\beta}(j,h) = \sum_{r=0}^{j}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(j-r,s). \tag{4.11}$$

Theorem 4.4 If $u(x,t) = D_{x_0}^{\alpha} v(x,t)$ and $0 < \alpha \leq 1$, then we get

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)}{\Gamma(\alpha j+1)} V_{\alpha,\beta}(j+1,h). \tag{4.12}$$

Theorem 4.5 If $u(x,t) = D_{x_0}^{\alpha} D_{t_0}^{\beta} v(x,t)$ *and* $0 < \alpha, \beta \leq 1$, then we get

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)\Gamma(\beta(h+1)+1)}{\Gamma(\alpha j+1)\Gamma(\beta h+1)} V_{\alpha,\beta}(j+1,h+1). \tag{4.13}$$

Theorem 4.6 If $u(x,t) = (x-x_0)^{n\alpha}(t-t_0)^{m\alpha},$ then $U_{\alpha,\beta}(j,h) = \delta(j-n)(h-m)$.

Theorem 4.7 If $u(x,t) = D_{x_0}^{\gamma} v(x,t), m-1 < \gamma \leq m$ *and* $v(x,t) = f(x)g(t)$, where $f(x)$ satisfies the conditions in Theorem 3.3, then

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)} U_{\alpha,\beta}(j+\gamma/\alpha,h). \tag{4.14}$$

Theorem 4.8 If $u(x,t) = D_{x_0}^{\gamma} D_{t_0}^{\eta} v(x,t)$, where $m-1 < \gamma \leq m, n-1 < \eta \leq n$ and $v(x,t) = f(x)g(t)$, where the functions $f(x)$ and $g(x)$ satisfy the conditions given in Theorem 3.3, then

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)} \frac{\Gamma(\beta h+\eta+1)}{\Gamma(\beta h+1)} U_{\alpha,\beta}(j+\gamma/\alpha,h+\eta/\beta). \tag{4.15}$$

4.2 Fuzzy fractional wave equation

Consider the fuzzy fractional wave equation with the indicated initial conditions and boundary conditions.

$$\frac{\partial^{\alpha}\tilde{u}}{\partial t^{\alpha}} = c^2 \odot \frac{\partial^2 \tilde{u}}{\partial x^2}, \qquad 0 < \alpha \leq 2, \qquad 0 < x < L, \qquad t > 0, \tag{4.16}$$

subject to the boundary conditions

$$\tilde{u}(0,t) = 0, \quad and \quad \tilde{u}(L,t) = 0, \tag{4.17}$$

and initial conditions.

$$\tilde{u}(x,0) = \tilde{f}(x), \quad and \quad \tilde{u}_t(x,0) = \tilde{g}(x). \tag{4.18}$$

We note that the case (i) of Definition 2.3 is coincident with the Hukuhara derivative [14]. We say that a function is (i) differentiable if it is differentiable as in (i) of Definition 2.3, a function is (ii)

differentiable if it is differentiable as in (ii) of Definition 2.3. In this paper we consider the two cases (i) and (ii). In Ref. [4] the authors consider four cases: the case (i) in [14] is coincident with (i); the case (iii) of Definition 2.1 is equivalent to (ii); in the other cases, the derivative is trivial because it is reduced to crisp element. For details see Theorem 7 in [4]. Thus, we only consider the cases (i) and (ii).

Lemma 4.2. [7]. Let $\tilde{u}(x,t) : D \to E^1$. Then the following statements hold.
(i) If $\tilde{u}(x,t)$ is (i)-partial differentiable for x (i.e. $\tilde{u}$ is partial differentiable for x under the meaning of Definition 2.1 (i), similarly to t), then

$$\left[\frac{\partial \tilde{u}}{\partial x}\right]_r = \left[\frac{\partial \underline{u}(x,t)(r)}{\partial x}, \frac{\partial \bar{u}(x,t)(r)}{\partial x}\right]; \tag{4.19}$$

(ii) If $\tilde{u}(x,t)$ is (ii)-partial differentiable for x (i.e. $\tilde{u}$ is partial differentiable for x under the meaning of Definition 2.1 (ii), similarly to t), then

$$\left[\frac{\partial \tilde{u}}{\partial x}\right]_r = \left[\frac{\partial \bar{u}(x,t)(r)}{\partial x}, \frac{\partial \underline{u}(x,t)(r)}{\partial x}\right]. \tag{4.20}$$

Remark 4.1. For $\tilde{u}(x,t) : D \to E^1$, the following results hold.

$$\left[\frac{\partial^2 \tilde{u}}{\partial x^2}\right]_r = \left[\frac{\partial^2 \underline{u}(x,t)(r)}{\partial x^2}, \frac{\partial^2 \bar{u}(x,t)(r)}{\partial x^2}\right], \tag{4.21}$$

in cases for that $(i,i), (ii,ii)$-$\frac{\partial^2 \tilde{u}}{\partial x^2}$ exist;

$$\left[\frac{\partial^2 \tilde{u}}{\partial x^2}\right]_r = \left[\frac{\partial^2 \bar{u}(x,t)(r)}{\partial x^2}, \frac{\partial^2 \underline{u}(x,t)(r)}{\partial x^2}\right]. \tag{4.22}$$

in cases for that $(i,ii), (ii,i)$-$\frac{\partial^2 \tilde{u}}{\partial t^2}$ exist.

Remark 4.2. In this paper, we only consider that the cases of $(i-ii)^n$-$\frac{\partial^n \tilde{u}}{\partial t^n}$ such that

$$\left[\frac{\partial^n \tilde{u}}{\partial x^n}\right]_r = \left[\frac{\partial^n \underline{u}(x,t)(r)}{\partial x^n}, \frac{\partial^n \bar{u}(x,t)(r)}{\partial x^n}\right], \tag{4.23}$$

where $(i-ii)^n$-$\frac{\partial^n \tilde{u}}{\partial t^n}$ stands for n time derivative in the cases (i) or (ii).

5 Examples

Example 5.1. Consider the following fuzzy fractional wave equation

(A)

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = 4 \odot \frac{\partial^2 \tilde{u}}{\partial x^2} \qquad 0 \le x \le 1, \quad 0 < t, \tag{5.24}$$

subject to the boundary conditions

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \qquad 0 < t, \tag{5.25}$$

and initial conditions

$$\begin{aligned} \tilde{u}(x,0) &= \tilde{f}(x) = \tilde{k}^n \odot \sin(\pi x), \quad && 0 \le x \le 1, \\ \frac{\partial \tilde{u}(x,0)}{\partial t} &= \tilde{g}(x) = 0, && 0 \le x \le 1. \end{aligned} \tag{5.26}$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} s, & s \in [0,1], \\ 2-s & s \in (1,2], \\ 0 & s \notin [0,2], \end{cases} \tag{5.27}$$

and $[\underline{\tilde{k}^n}](r) = r^n, [\overline{\tilde{k}^n}](r) = (2-r)^n$.
The parametric form of (5.24) is

$$\frac{\partial^2 \underline{u}}{\partial t^2} = 4\frac{\partial^2 \underline{u}}{\partial x^2} \qquad 0 \le x \le 1, \qquad 0 < t, \tag{5.28}$$

$$\frac{\partial^2 \overline{u}}{\partial t^2} = 4\frac{\partial^2 \overline{u}}{\partial x^2}, \qquad 0 \le x \le 1, \qquad 0 < t, \tag{5.29}$$

for $r \in [0,1]$, and where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, similar to $\overline{u}$.
Taking the differential transform of equations (5.28) and (5.29), we get

$$(j+2)(j+1)\underline{U}(i,j+2)(r) = 4(i+2)(i+1)\underline{U}(i+2,j)(r), \tag{5.30}$$

$$(j+2)(j+1)\overline{U}(i,j+2)(r) = 4(i+2)(i+1)\overline{U}(i+2,j)(r). \tag{5.31}$$

From the initial given by equation (5.26), we get

$$\underline{u}(x,0)(r) = \sum_{i=0}^{\infty} \underline{U}(i,0)(r)x^i = \underline{k}(r)\sin(\pi x) = r^n \sum_{i=1,3,\dots.}^{\infty} \frac{(-1)^{\frac{(i-1)}{2}}}{i!}\pi^i x^i, \tag{5.32}$$

$$\overline{u}(x,0)(r) = \sum_{i=0}^{\infty} \overline{U}(i,0)(r)x^i = \overline{k}(r)\sin(\pi x) = (2-r)^n \sum_{i=1,3,\dots.}^{\infty} \frac{(-1)^{\frac{(i-1)}{2}}}{i!}\pi^i x^i. \tag{5.33}$$

The corresponding spectra can be obtained as follows,

$$\underline{U}(i,0)(r) = \begin{cases} 0, & \text{for i is even,} \\ \frac{(-1)^{\frac{(i-1)}{2}}}{i!} r^n \pi^i, & \text{for i is odd} \end{cases} \tag{5.34}$$

$$\overline{U}(i,0)(r) = \begin{cases} 0, & \text{for i is even,} \\ \frac{(-1)^{\frac{(i-1)}{2}}}{i!} (2-r)^n \pi^i, & \text{for i is odd} \end{cases} \tag{5.35}$$

and from equation (5.26) it can be obtained that,

$$\frac{\partial \underline{u}(x,0)(r)}{\partial t} = \sum_{i=0}^{\infty} \underline{U}(i,1)(r)x^i = 0, \tag{5.36}$$

$$\frac{\partial \overline{u}(x,0)(r)}{\partial t} = \sum_{i=0}^{\infty} \overline{U}(i,1)(r)x^i = 0. \tag{5.37}$$

Hence,

$$\underline{u}(i,1)(r) = 0, \tag{5.38}$$

$$\overline{u}(i,1)(r) = 0. \tag{5.39}$$

Substituting equations (5.34) -(5.39) to equations (5.30) and (5.31), all spectra can be found as,

$$\underline{U}(i,j)(r) = \begin{cases} 0, & \text{for } i \text{ is even } \textit{or } j \text{ is odd} \\ \dfrac{2^j(-1)^{\frac{(i+j-1)}{2}}}{i!j!}\, r^n\pi^{i+j}, & \text{for } i \text{ is odd } \textit{or } j \text{ is even} \end{cases} \tag{5.40}$$

$$\overline{U}(i,j)(r) = \begin{cases} 0, & \text{for } i \text{ is even } \textit{or } j \text{ is odd} \\ \dfrac{2^j(-1)^{\frac{(i+j-1)}{2}}}{i!j!}\, (2-r)^n\pi^{i+j}, & \text{for } i \text{ is odd } \textit{or } j \text{ is even} \end{cases} \tag{5.41}$$

So, the closed from of the solution can be easily written as

$$\begin{aligned} \underline{u}(x,t)(r) &= \underline{k}^n \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \underline{U}(i,j)(r)x^it^j = r^n \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j \\ &= r^n\left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= r^n \sin(\pi x)\cos(2\pi t), \end{aligned} \tag{5.42}$$

$$\begin{aligned} \overline{u}(x,t)(r) &= \overline{k}^n \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{U}(i,j)(r)x^it^j = (2-r)^n \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j \\ &= (2-r)^n\left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= (2-r)^n \sin(\pi x)\cos(2\pi t). \end{aligned} \tag{5.43}$$

(B) Consider the following fuzzy fractional wave equation (5.24) with the boundary conditions:

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \qquad 0 < t, \tag{5.44}$$

and initial conditions

$$\begin{aligned} \tilde{u}(x,0) &= \tilde{f}(x) = \tilde{k}^n \oplus \sin(\pi x), && 0 \le x \le 1, \\ \frac{\partial \tilde{u}(x,0)}{\partial t} &= \tilde{g}(x) = 0, && 0 \le x \le 1. \end{aligned} \tag{5.45}$$

By following the same steps, we will find that the solution. So, the closed from of the solution can be easily written as

$$\begin{aligned} \underline{u}(x,t)(r) &= \underline{k}^n + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \underline{U}(i,j)(r)x^it^j = r^n + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j \\ &= r^n + \left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= r^n + (\sin(\pi x)\cos(2\pi t)), \end{aligned} \tag{5.46}$$

$$\begin{aligned} \overline{u}(x,t)(r) &= \overline{k}^n + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{U}(i,j)(r)x^it^j = (2-r)^n + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j \\ &= (2-r)^n + \left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= (2-r)^n + (\sin(\pi x)\cos(2\pi t)). \end{aligned} \tag{5.47}$$

(C) Consider the following fuzzy fractional wave equation (5.24) with the boundary conditions:

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \qquad 0 < t, \tag{5.48}$$

and initial conditions

$$\begin{aligned} \tilde{u}(x,0) &= \tilde{f}(x) = \tilde{k}^n \ominus_{gH} \sin(\pi x), \quad && 0 \le x \le 1, \\ \frac{\partial \tilde{u}(x,0)}{\partial t} &= \tilde{g}(x) = 0, && 0 \le x \le 1. \end{aligned} \tag{5.49}$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... , fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} 2(s-0.5), & s \in [0.5,1], \\ 2(1.5-s), & s \in (1,1.5], \\ 0 & s \notin [0.5,1.5], \end{cases} \tag{5.50}$$

and$\{\underline{\tilde{k}^n}\}(r) = (0.5+0.5r)^n$, $\{\overline{\tilde{k}^n}\}(r) = (1.5-0.5r)^n$.
By following the same steps, we will find that the solution. So, the closed from of the solution can be easily written as

$$\begin{aligned} \underline{u}(x,t)(r) &= \underline{k}^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \underline{U}(i,j)(r) x^i t^j = (0.5+0.5r)^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}} \pi^{i+j} x^i t^j \\ &= (0.5+0.5r)^n - \left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= (0.5+0.5r)^n - (\sin(\pi x)\cos(2\pi t)), \end{aligned} \tag{5.51}$$

$$\begin{aligned} \overline{u}(x,t)(r) &= \overline{k}^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{U}(i,j)(r) x^i t^j = (1.5-0.5r)^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}} \pi^{i+j} x^i t^j \\ &= (1.5-0.5r)^n - \left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= (1.5-0.5r)^n - (\sin(\pi x)\cos(2\pi t)). \end{aligned} \tag{5.52}$$

Example 5.2. Consider the following fuzzy time-fractional wave equation.

(A)

$$\frac{\partial^{1.5}\tilde{u}}{\partial t^{1.5}} = \frac{\partial^2 \tilde{u}}{\partial x^2}, \qquad t > 0, \tag{5.53}$$

subject to the initial conditions

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \odot \sin(x), \quad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \odot (-\sin(x)). \tag{5.54}$$

where $\tilde{k}^n \in E^1$, n=1,2,3,..., fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} 2(s-0.5), & s \in [0.5,1], \\ 2(1.5-s), & s \in (1,1.5], \\ 0 & s \notin [0.5,1.5], \end{cases} \tag{5.55}$$

and$\{\underline{\tilde{k}^n}\}(r) = (0.5 + 0.5r)^n, \{\overline{\tilde{k}^n}\}(r) = (1.5 - 0.5r)^n$.
The parametric form of (5.53) is

$$\frac{\partial^{1.5}\underline{u}}{\partial t^{1.5}} = \frac{\partial^2\underline{u}}{\partial x^2}, \qquad t > 0, \tag{5.56}$$

$$\frac{\partial^{1.5}\overline{u}}{\partial t^{1.5}} = \frac{\partial^2\overline{u}}{\partial x^2}, \qquad t > 0. \tag{5.57}$$

for $r \in [0, 1]$, and where $\underline{u}$ stands for $\underline{u}(x, t)(r)$, similar to $\overline{u}$.

Let the solution $u(x, t) = f(x)g(t)$ where the function $g(t)$ satisfies the conditions given in Theorem 3.3. Then selecting $\alpha = 0.5, \beta = 1$ and applying the generalized two-dimensional differential transform method (DTM) to both sides of equations (5.56) and (5.57) by Theorem 4.7, equations (5.56) and (5.57) Transforms to

$$\underline{U}_{0.5,1}(j, h + 3)(r) = \frac{(j + 1)(j + 2)\Gamma(\frac{h}{2} + 1)}{\Gamma(\frac{h}{2} + \frac{5}{2})}\underline{U}_{0.5,1}(j + 2, h)(r), \tag{5.58}$$

$$\overline{U}_{0.5,1}(j, h + 3)(r) = \frac{(j + 1)(j + 2)\Gamma(\frac{h}{2} + 1)}{\Gamma(\frac{h}{2} + \frac{5}{2})}\overline{U}_{0.5,1}(j + 2, h)(r). \tag{5.59}$$

The generalized two-dimensional differential transform of the initial conditions (5.54) are given by

$$\underline{U}_{0.5,1}(j, 0)(r) = (0.5 + 0.5r)^n \frac{1}{j!} \sin(\frac{\pi j}{2}), \tag{5.60}$$

$$\underline{U}_{0.5,1}(j, 1)(r) = 0, \tag{5.61}$$

$$\underline{U}_{0.5,1}(j, 2)(r) = (0.5 + 0.5r)^n \frac{-1}{j!} \sin(\frac{\pi j}{2}), \tag{5.62}$$

$$\overline{U}_{0.5,1}(j, 0)(r) = (1.5 - 0.5r)^n \frac{1}{j!} \sin(\frac{\pi j}{2}), \tag{5.63}$$

$$\overline{U}_{0.5,1}(j, 1)(r) = 0, \tag{5.64}$$

$$\overline{U}_{0.5,1}(j, 2)(r) = (1.5 - 0.5r)^n \frac{-1}{j!} \sin(\frac{\pi j}{2}). \tag{5.65}$$

Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j, h)$ can be calculated.
So, the solution $u(x, t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x, t)(r) &= (0.5 + 0.5r)^n \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right) x \\
&+ (0.5 + 0.5r)^n \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right) x^3 \\
&+ (0.5 + 0.5r)^n \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right) x^5 \\
\underline{u}(x, t)(r) &= (0.5 + 0.5r)^n \left(\sum_{j=0}^{\infty} \frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2} + 1)} \sin(x) - \sum_{j=0}^{\infty} \frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2} + 2)} \sin(x)\right), \\
&= (0.5 + 0.5r)^n \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}}) \sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}}) \sin(x)\right),
\end{aligned} \tag{5.66}$$

$$
\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n\left(1-t-\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{\Gamma(4)}t^3+.....\right)\cdot x\\
&+(1.5-0.5r)^n\left(-\frac{1}{3!}+\frac{1}{3!}t+\frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}-\frac{1}{3!\Gamma(4)}t^3+....\right)\cdot x^3\\
&+(1.5-0.5r)^n\left(\frac{1}{5!}-\frac{1}{5!}t-\frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{5!\Gamma(4)}t^3-...\right)\cdot x^5\\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n\left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x)-\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (1.5-0.5r)^n\left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x)-tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.67)
\end{aligned}
$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$
E_{\alpha,\beta}(z)=\tilde{k}^n\odot\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.68)
$$

(B) Consider the following fuzzy time-fractional wave equation (5.53) with the initial conditions:

$$
\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\oplus\sin(x),\quad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\oplus(-\sin(x)). \qquad (5.69)
$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j,h)$ can be calculated.
So, the solution $u(x,t)$ of equations (5.56) and (5.57) is obtained

$$
\begin{aligned}
\underline{u}(x,t)(r) &= (0.5+0.5r)^n+\left(1-t-\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{\Gamma(4)}t^3+.....\right)x\\
&+(0.5+0.5r)^n+\left(-\frac{1}{3!}+\frac{1}{3!}t+\frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}-\frac{1}{3!\Gamma(4)}t^3+....\right)x^3\\
&+(0.5+0.5r)^n+\left(\frac{1}{5!}-\frac{1}{5!}t-\frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{5!\Gamma(4)}t^3-...\right)x^5\\
\underline{u}(x,t)(r) &= (0.5+0.5r)^n+\left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x)-\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (0.5+0.5r)^n+\left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x)-tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right), \qquad (5.70)
\end{aligned}
$$

$$\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n + \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right)\cdot x \\
&+ (1.5-0.5r)^n + \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right)\cdot x^3 \\
&+ (1.5-0.5r)^n + \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right)\cdot x^5 \\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n + \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (1.5-0.5r)^n + \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right).
\end{aligned} \tag{5.71}$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$E_{\alpha,\beta}(z) = \tilde{k}^n \oplus \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \tag{5.72}$$

(C) Consider the following fuzzy time fractional wave equation (5.53) with initial conditions:

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \ominus_{gH} \sin(x), \quad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \ominus_{gH} (-\sin(x)). \tag{5.73}$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j,h)$ can be calculated.
So, the solution $u(x,t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x,t)(r) &= (0.5+0.5r)^n - \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right) x \\
&+ (0.5+0.5r)^n - \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right) x^3 \\
&+ (0.5+0.5r)^n - \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right) x^5 \\
\underline{u}(x,t)(r) &= (0.5+0.5r)^n - \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (0.5+0.5r)^n - \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right),
\end{aligned} \tag{5.74}$$

$$\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n - \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right)\cdot x \\
&+ (1.5-0.5r)^n - \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right)\cdot x^3 \\
&+ (1.5-0.5r)^n - \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right)\cdot x^5 \\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n - \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (1.5-0.5r)^n - \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.75)
\end{aligned}$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$E_{\alpha,\beta}(z) = \tilde{k}^n \ominus_{gH} \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.76)$$

Example 5.3. Consider the following fuzzy linear space time fractional wave equation

(A)

$$\frac{\partial^{1.5}\tilde{u}}{\partial t^{1.5}} = \frac{1}{2}x^2 \odot \frac{\partial^{1.25}\tilde{u}}{\partial x^{1.25}} \quad x>0, \quad t>0, \qquad (5.77)$$

subject to the initial conditions

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \odot \sum_{n=0}^{\infty} a_n x^n, \qquad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \odot \sum_{n=0}^{\infty} b_n x^n. \qquad (5.78)$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} s, & s\in[0,1], \\ 2-s & s\in(1,2], \\ 0 & s\notin[0,2], \end{cases} \qquad (5.79)$$

and $[\underline{\tilde{k}^n}](r) = r^n, [\overline{\tilde{k}^n}](r) = (2-r)^n$.
The parametric form of (5.77) is

$$\frac{\partial^{1.5}\underline{u}}{\partial t^{1.5}} = \frac{1}{2}x^2\frac{\partial^{1.25}\underline{u}}{\partial x^{1.25}} \quad x>0, \quad t>0 \qquad (5.80)$$

$$\frac{\partial^{1.5}\overline{u}}{\partial t^{1.5}} = \frac{1}{2}x^2\frac{\partial^{1.25}\overline{u}}{\partial x^{1.25}} \quad x>0, \quad t>0 \qquad (5.81)$$

for $r\in[0,1]$, and where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, similar to $\overline{u}$.

Let the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = f(x)g(t)$ where the functions $f(x)$ and $g(t)$ satisfy the conditions given in Theorem 3.3. Selecting $\alpha = 0.5, \beta = 0.25$ and applying the generalized two-dimensional differential transform to both sides of equations (5.80) and (5.81), the fuzzy linear space-time fractional wave equations (5.80) and (5.81) transform to

$$\underline{U}_{1/2,1/4}(j,h+3)(r)=\begin{cases}\frac{1}{2}\frac{\Gamma(h/2+1)\Gamma(j/4+7/4)}{\Gamma(h/2+5/2)\Gamma(j/4+2/4)}\underline{U}_{1/2,1/4}(j+3,h)(r), & j\geq 2\\ 0, & j<2.\end{cases} \tag{5.82}$$

$$\overline{U}_{1/2,1/4}(j,h+3)(r)=\begin{cases}\frac{1}{2}\frac{\Gamma(h/2+1)\Gamma(j/4+7/4)}{\Gamma(h/2+5/2)\Gamma(j/4+2/4)}\overline{U}_{1/2,1/4}(j+3,h)(r), & j\geq 2\\ 0, & j<2.\end{cases} \tag{5.83}$$

The generalized two-dimensional transforms of the initial conditions (5.78) are given by

$$\underline{U}_{1/2,1/4}(j,0)(r)=r^n a_j, \tag{5.84}$$
$$\underline{U}_{1/2,1/4}(j,1)(r)=0, \tag{5.85}$$
$$\underline{U}_{1/2,1/4}(j,2)(r)=r^n b_j, \tag{5.86}$$
$$\overline{U}_{1/2,1/4}(j,0)(r)=(2-r)^n a_j, \tag{5.87}$$
$$\overline{U}_{1/2,1/4}(j,1)(r)=0, \tag{5.88}$$
$$\overline{U}_{1/2,1/4}(j,2)(r)=(2-r)^n b_j. \tag{5.89}$$

Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}\underline{u}(x,t)(r)=&\,r^n\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+r^n\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+r^n\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+r^n\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+r^n\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots,\end{aligned} \tag{5.90}$$

$$\begin{aligned}\overline{u}(x,t)(r)=&\,(2-r)^n\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+(2-r)^n\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+(2-r)^n\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+(2-r)^n\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+(2-r)^n\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots.\end{aligned} \tag{5.91}$$

(B) Consider the following fuzzy linear-space-time-fractional wave equation (5.77) with the initial conditions:

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \oplus \sum_{n=0}^{\infty} a_n x^n, \qquad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \oplus \sum_{n=0}^{\infty} b_n x^n. \tag{5.92}$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}
\underline{u}(x,t)(r) = r^n &+ \left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right) \\
+ r^n &+ \left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4} \\
+ r^n &+ \left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4} \\
+ r^n &+ \left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4} \\
+ r^n &+ \left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots,
\end{aligned} \tag{5.93}$$

$$\begin{aligned}
\overline{u}(x,t)(r) = (2-r)^n &+ \left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right) \\
+ (2-r)^n &+ \left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4} \\
+ (2-r)^n &+ \left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4} \\
+ (2-r)^n &+ \left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4} \\
+ (2-r)^n &+ \left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots.
\end{aligned} \tag{5.94}$$

(C) Consider the following fuzzy linear space-time-fractional wave equation (5.77) with the initial conditions:

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \ominus_{gH} \sum_{n=0}^{\infty} a_n x^n, \qquad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \ominus_{gH} \sum_{n=0}^{\infty} b_n x^n. \tag{5.95}$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$
\begin{aligned}
\underline{u}(x,t)(r) = r^n - &\left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right) \\
+ r^n - &\left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4} \\
+ r^n - &\left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4} \\
+ r^n - &\left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4} \\
+ r^n - &\left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots, \qquad (5.96)
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}(x,t)(r) = (2-r)^n - &\left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right) \\
+ (2-r)^n - &\left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4} \\
+ (2-r)^n - &\left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4} \\
+ (2-r)^n - &\left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4} \\
+ (2-r)^n - &\left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots. \qquad (5.97)
\end{aligned}
$$

6 Conclusions

In this paper, the differential transform method (DTM) has been successfully applied for solving fuzzy fractional wave equation. The proposed method is also illustrated by three examples. The new method is investigated based on the two-dimensional differential transform method, generalized Taylor's formula and fuzzy Caputo,s derivative. The results reveal that DTM is a highly effective scheme for obtaining analytical solutions of the fuzzy fractional wave equation.

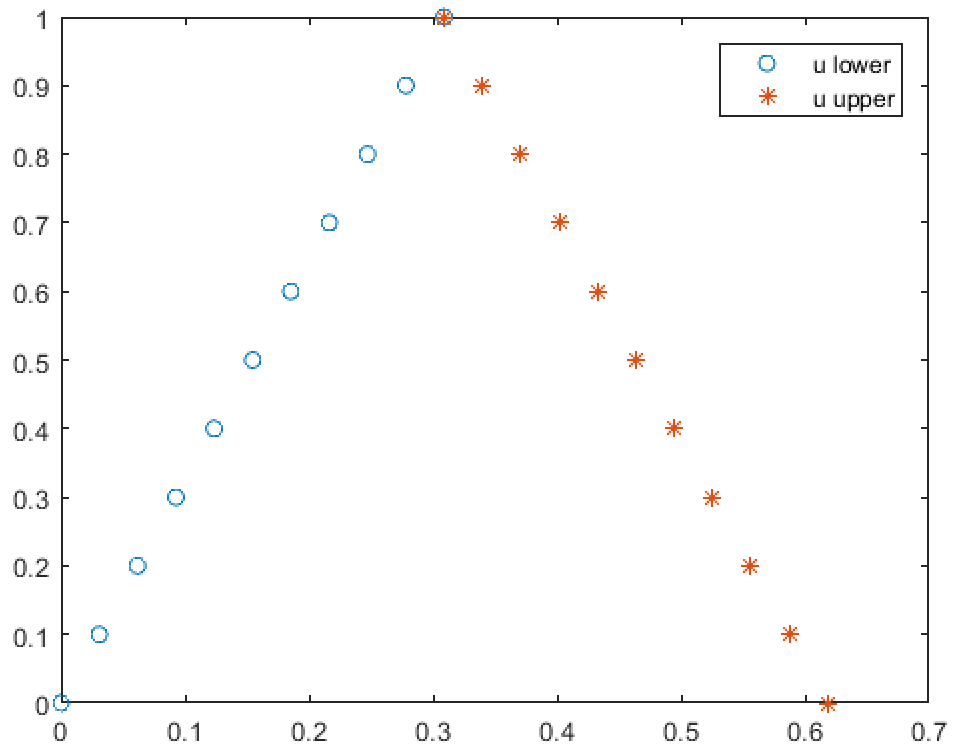

Figure 1: Example (5.1), Case (A), $t = 0.000001, x = 0.1, n = 1$.

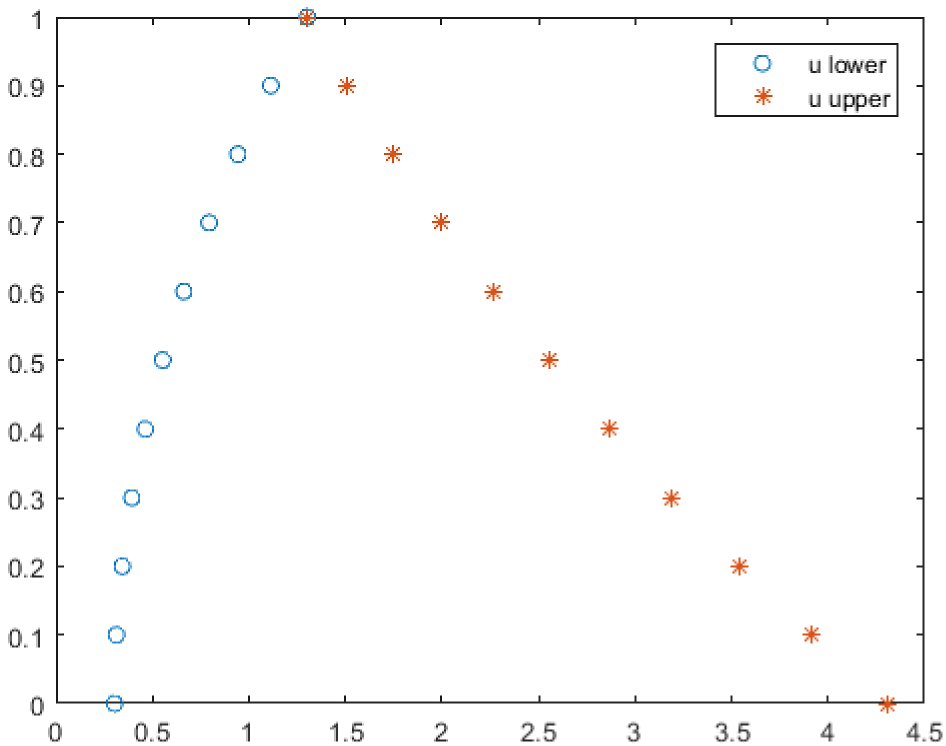

Figure 2: Example (5.1), Case (B), $t = 0.03, x = 0.1, n = 2$.

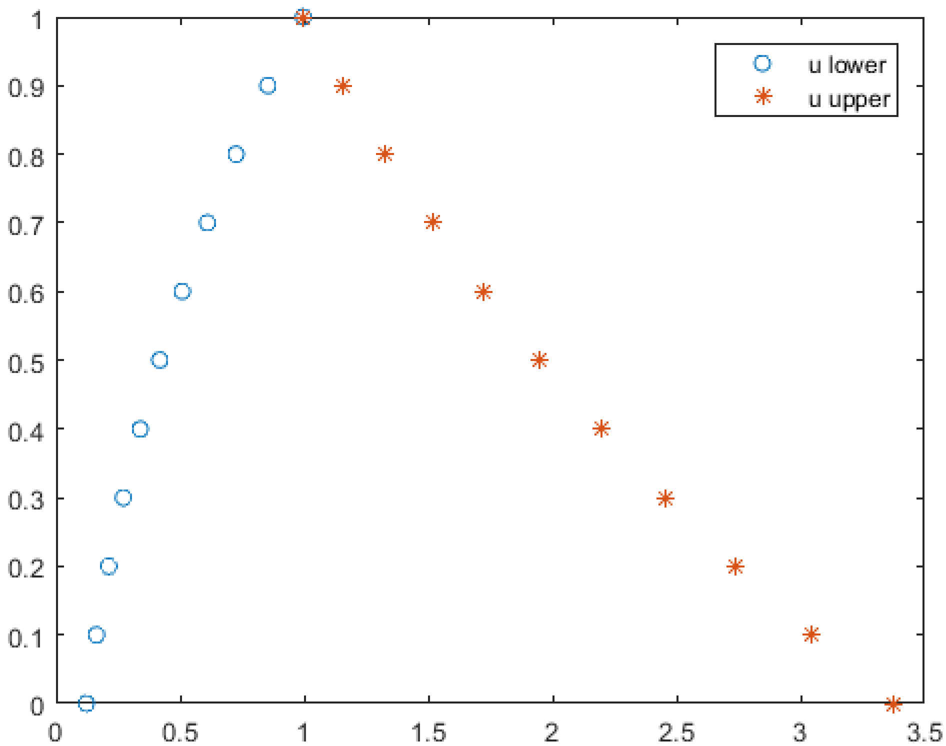

Figure 3: Example (5.1), Case (C), $t = 0.0001, x = 0.001, n = 3$.

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Some Generalized k-Fractional Integral Inequalities for Quasi-Convex Functions

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Abstract

Fractional integral operators generalize the concept of definite integration. Therefore these operators play a vital role in the advancement of subjects of sciences and engineering. The aim of this study is to establish the bounds of a generalized fractional integral operator via quasi-convex functions. These bounds behave as a formula in unified form, and estimations of almost all fractional integrals defined in last two decades can be obtained at once by choosing convenient parameters. Moreover, several related fractional integral inequalities are identified.

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1 Introduction

A function $f : I \to \mathbb{R}$ is said to be *convex* if the following inequality holds:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1.1}$$

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for all $a, b \in I$ and $t \in [0,1]$.

If inequality (1.1) is reversed, then the function f will be the concave on $[a, b]$. Convex functions are very useful in mathematical analysis. A lot of integral inequalities have been established due to convex functions in literature (for details see, [2–6, 10, 11, 18–20]. Quasi-convexity is also class of convex functions which is defined as follows:

Definition 1.1. ([10]) A function $f : I \to \mathbb{R}$ is said to be *quasi-convex* if the following inequality holds:

$$f(ta + (1-t)b) \leq \max\{f(a), f(b)\} \tag{1.2}$$

for all $a, b \in I$ and $t \in [0,1]$.

Example 1.2. ([11, p. 83]) The function $f : [-2, 2] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & x \in [-2, -1] \\ x^2 & x \in (-1, 2] \end{cases}$$

is not a convex function on $[-2, 2]$ but it is quasi-convex function on $[-2, 2]$.

It is noted that class of quasi-convex functions contain the class of finite convex functions defined on finite closed intervals. For some recent citations and utilizations of quasi-convex functions one can see [2, 10, 11, 20] and references therein.

Fractional integral operators play an important role in generalizing the mathematical inequalities. In recent years, authors have proved various interesting mathematical inequalities due to different fractional integral operators, for example see [3–8 11, 15, 20]. The upcoming definitions and remark provide a detailed information of recent and classical fractional integral operators.

Definition 1.3. Let $f \in L_1[a, b]$ with $0 \leq a < b$. Then *Riemann-Liouville fractional integral operators* of order $\mu > 0$ are defined by

$$^{\mu}I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt, \quad x > a \tag{1.3}$$

and

$$^{\mu}I_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)dt, \quad x < b, \tag{1.4}$$

where $\Gamma(\mu)$ is the Gamma function defined by $\Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt$.

Definition 1.4. ([16]) Let $f \in L_1[a, b]$ with $0 \leq a < b$. Then *Riemann-Liouville k-fractional integral operators* of order $\mu, k > 0$ are defined by

$$^{\mu}I^k_{a^+} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)dt, \quad x > a \tag{1.5}$$

and

$$^{\mu}I^k_{b^-} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)dt, \quad x < b, \tag{1.6}$$

where $\Gamma_k(\mu)$ is the k-Gamma function defined as $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$.

Definition 1.5. ([14]) Let $f \in L_1[a,b]$ with $0 \leq a < b$. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). *The left-sided and right-sided fractional integrals of a function* f with respect to another function g on $[a,b]$ of order $\mu > 0$, are defined by

$$ {}_g^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (g(x)-g(t))^{\mu-1}g'(t)f(t)dt, \quad x > a \tag{1.7}$$

and

$$ {}_g^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (g(t)-g(x))^{\mu-1}g'(t)f(t)dt, \quad x < b. \tag{1.8}$$

Definition 1.6. ([15]) Let $f \in L_1[a,b]$ with $0 \leq a < b$. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). *The left-sided and right-sided fractional integrals of a function* f with respect to another function g on $[a,b]$ of order $\mu, k > 0$ are defined by

$$ {}_g^{\mu}I_{a^+}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1}g'(t)f(t)dt, \quad x > a \tag{1.9}$$

and

$$ {}_g^{\mu}I_{b^-}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (g(t)-g(x))^{\frac{\mu}{k}-1}g'(t)f(t)dt, \quad x < b. \tag{1.10}$$

These are compact formulas which give almost all fractional integrals by choosing suitable formations of function g. In this context the following remark is important:

Remark 1.7. Fractional integrals elaborated in (1.9) and (1.10) particularly produce several known fractional integrals corresponding to different settings of k and g.

(i) For $k = 1$ (1.9) and (1.10) fractional integrals coincide with (1.7) and (1.8).

(ii) For taking g as identity function (1.9) and (1.10) fractional integrals coincide with (1.5) and (1.6).

(iii) For $k = 1$, along with g as identity function (1.9) and (1.10) fractional integrals coincide with (1.3) and (1.4).

(iv) For $k = 1$ and $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$, (1.9) and (1.10) produce Katugampola fractional integrals defined by Chen et al. in [1].

(v) For $k = 1$ and $g(x) = \frac{x^{\tau+s}}{\tau+s}$, (1.9) and (1.10) produce generalized conformable fractional integrals defined by Khan et al. in [13].

(vi) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10), then conformable (k,s)-fractional integrals are achieved as defined by Sidra et al. in [9].

(vii) If we take $g(x) = \frac{x^{1+s}}{1+s}$, then conformable fractional integrals are achieved as defined by Sarikaya et al. in [17].

(viii) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10) with $k = 1$, then conformable fractional integrals are achieved as defined by Jarad et al. in [12].

The rest of paper is organized as follows:

In Section 2, the bounds of sum of left-sided and right-sided generalized fractional integrals via quasi-convex function are established. First result provides an upper bound for generalized fractional integrals, and some particular cases are elaborated. Then bounds along with particular cases, in modulus form have been presented. Furthermore, Hadamarad type bounds are formulated. In Section 3, applications of results of Section 2 are given. Moreover concluding remarks are included at the end.

In the next sections the notation $M_a^b(f) = \max\{f(a), f(b)\}$ has been used frequently.

2 Main Results

Firstly, the following theorem set the formula for upper bounds of fractional integrals via quasi-convex functions in a unified form.

Theorem 2.1. *Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be two functions such that g be differentiable and $f \in L[a,b]$ with $a < b$. Also let f be positive, quasi-convex and g be strictly increasing function with $g' \in L[a,b]$. Then for $x \in [a,b]$ and $\mu, \nu \geq k$, the following inequality holds:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) + {}_g^{\nu} I_{b^-}^{k} f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^x(f) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_x^b(f). \tag{2.1}$$

Proof. As f is quasi-convex, therefore for $t \in [a,x]$, $f(t) \leq M_a^x(f)$. Under assumptions on function g, for all $x \in [a,b]$, $t \in [a,x]$ and $\mu \geq k$, the following inequality holds:

$$g'(t)(g(x)-g(t))^{\frac{\mu}{k}-1} \leq g'(t)(g(x)-g(a))^{\frac{\mu}{k}-1}. \tag{2.2}$$

From aforementioned two inequalities, the following integral inequality is yielded:

$$\int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} f(t) g'(t)dt \leq (g(x)-g(a))^{\frac{\mu}{k}-1} M_a^x(f) \int_a^x g'(t)dt. \tag{2.3}$$

By using (1.9) of Definition 1.6, the following bound of fractional integral defined in (1.9) is obtained:

$${}_g^{\mu}I_{a^+}^{k} f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^x(f). \tag{2.4}$$

Again from quasi-convexity of f, for $t \in [x,b]$, $f(t) \leq M_x^b(f)$. Also for $x \in [a,b]$, $t \in [x,b]$ and $\nu \geq k$, the following inequality holds:

$$g'(t)(g(t)-g(x))^{\frac{\nu}{k}-1} \leq g'(t)(g(b)-g(x))^{\frac{\nu}{k}-1}. \tag{2.5}$$

From aforementioned two inequalities, the following integral inequality is yielded:

$$\int_x^b (g(t)-g(x))^{\frac{\nu}{k}-1} f(t) g'(t)dt \leq (g(b)-g(x))^{\frac{\nu}{k}-1} M_x^b(f) \int_x^b g'(t)dt. \tag{2.6}$$

By using (1.10) of Definition 1.6, the following bound of fractional integral defined in (1.10) is obtained:

$${}_g^{\nu}I_{b^-}^{k} f(x) \leq \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_x^b(f). \tag{2.7}$$

From (2.4) and (2.7), the bound of sum of left-sided and right-sided fractional integrals is achieved. □

Special cases of Theorem 2.1, are discussed in the following corollaries.

Corollary 2.2. *If we take $\mu = \nu$ in (2.1), then we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^k f(x) +_g^{\mu} I_{b^-}^k f(x) \leq \frac{1}{k\Gamma_k(\mu)} \left((g(x)-g(a))^{\frac{\mu}{k}} M_a^x(f) + (g(b)-g(x))^{\frac{\mu}{k}} M_x^b(f) \right). \quad (2.8) $$

Corollary 2.3. *If we take $k = 1$ in (2.1), then we get the following generalized (RL) fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+} f(x) +_g^{\nu} I_{b^-} f(x) \leq \frac{(g(x)-g(a))^{\mu}}{\Gamma(\mu)} M_a^x(f) + \frac{(g(b)-g(x))^{\nu}}{\Gamma(\nu)} M_x^b(f). \quad (2.9) $$

Corollary 2.4. *If we take $g(x) = x$ in (2.1), then we get the following (RL) k-fractional integral inequality:*

$$ {}^{\mu}I_{a^+}^k f(x) +^{\nu} I_{b^-}^k f(x) \leq \frac{(x-a)^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^x(f) + \frac{(b-x)^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_x^b(f). \quad (2.10) $$

Corollary 2.5. *If we take $g(x) = x$ and $k = 1$ in (2.1), then we get the following (RL) fractional integral inequality:*

$$ {}^{\mu}I_{a^+} f(x) +^{\nu} I_{b^-} f(x) \leq \frac{(x-a)^{\mu}}{\Gamma(\mu)} M_a^x(f) + \frac{(b-x)^{\nu}}{\Gamma(\nu)} M_x^b(f). \quad (2.11) $$

Corollary 2.6. *Under the assumptions of above theorem if f is increasing on $[a,b]$, then from (2.1), we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^k f(x) +_g^{\nu} I_{b^-}^k f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} f(x) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} f(b). \quad (2.12) $$

Corollary 2.7. *Under the assumptions of above theorem if f is decreasing on $[a,b]$, then from (2.1), we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^k f(x) +_g^{\nu} I_{b^-}^k f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} f(a) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} f(x). \quad (2.13) $$

Next theorem provides the bound of generalized fractional integrals in modulus form.

Theorem 2.8. *Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be two differentiable functions with $a < b$. Also let $|f'|$ be quasi-convex and g be strictly increasing with $g' \in L[a,b]$. Then for $x \in [a,b]$ and $\mu, \nu, k > 0$, the following inequality holds:*

$$ \left| {}_g^{\mu}I_{a^+}^k f(x) +_g^{\nu} I_{b^-}^k f(x) - \left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)} f(a) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)} f(b) \right) \right| \quad (2.14) $$
$$ \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)} M_a^x(|f'|) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)} M_x^b(|f'|). $$

Proof. As $|f'|$ is quasi-convex, therefore for $t \in [a, x]$, we have

$$|f'(t)| \leq M_a^x(|f'|). \tag{2.15}$$

From (2.15), we have

$$f'(t) \leq M_a^x(|f'|). \tag{2.16}$$

Under assumptions of the function g, the following inequality holds:

$$(g(x) - g(t))^{\frac{\mu}{k}} \leq (g(x) - g(a))^{\frac{\mu}{k}} \tag{2.17}$$

for all $x \in [a, b]$, $t \in [a, x]$ and $\mu, k > 0$.

From (2.16) and (2.17), we have

$$\int_a^x (g(x) - g(t))^{\frac{\mu}{k}} f'(t)dt \leq (g(x) - g(a))^{\frac{\mu}{k}} M_a^x(|f'|) \int_a^x dt, \tag{2.18}$$

the left hand side calculate as follows:

$$\begin{aligned}
&\int_a^x (g(x) - g(t))^{\frac{\mu}{k}} f'(t)dt \\
&= f(t)(g(x) - g(t))^{\frac{\mu}{k}}|_a^x + \frac{\mu}{k}\int_a^x (g(x) - g(t))^{\frac{\mu}{k}-1} f(t)g'(t)dt \\
&= -f(a)(g(x) - g(a))^{\frac{\mu}{k}} + \Gamma_k(\mu + k)_g^\mu I_{a^+}^k f(x).
\end{aligned}$$

Using above calculation in (2.18), we get the following inequality:

$${}_g^\mu I_{a^+}^k f(x) - \frac{(g(x) - g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu + k)} f(a) \leq \frac{(g(x) - g(a))^{\frac{\mu}{k}}(x - a)}{\Gamma_k(\mu + k)} M_a^x(|f'|). \tag{2.19}$$

Also from (2.15), we can write

$$f'(t) \geq -M_a^x(|f'|). \tag{2.20}$$

Following the same procedure as we did for (2.16), we also have

$$\frac{(g(x) - g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu + k)} f(a) -{}_g^\mu I_{a^+}^k f(x) \leq \frac{(g(x) - g(a))^{\frac{\mu}{k}}(x - a)}{\Gamma_k(\mu + k)} M_a^x(|f'|). \tag{2.21}$$

From (2.19) and (2.21), we get the following modulus inequality:

$$\left| {}_g^\mu I_{a^+}^k f(x) - \frac{(g(x) - g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu + k)} f(a) \right| \leq \frac{(g(x) - g(a))^{\frac{\mu}{k}}(x - a)}{\Gamma_k(\mu + k)} M_a^x(|f'|). \tag{2.22}$$

Again by using quasi-convexity of $|f'|$, for $t \in [x, b]$, we have

$$|f'(t)| \leq M_x^b(|f'|). \tag{2.23}$$

Now for $x \in [a, b]$, $t \in [x, b]$ and $\nu, k > 0$, the following inequality holds:

$$(g(t) - g(x))^{\frac{\nu}{k}} \leq (g(b) - g(x))^{\frac{\nu}{k}}. \tag{2.24}$$

By adopting the same way as we have done for (2.16), (2.17) and (2.20) one can get from (2.23) and (2.24) the following modulus inequality:

$$\left|{}^{\nu}_{g}I^{k}_{b-}f(x)-\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right|\leq\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}M^{b}_{x}(|f'|). \tag{2.25}$$

From (2.22) and (2.25) via triangular inequality, we get the modulus inequality in (2.14), which is required. □

Special cases of Theorem 2.8, are discussed in the following corollaries.

Corollary 2.9. *If we take $\mu=\nu$ in (2.14), then we get the following fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}_{g}I^{k}_{a+}f(x)+^{\mu}_{g}I^{k}_{b-}f(x)-\frac{1}{\Gamma_k(\mu+k)}\left((g(x)-g(a))^{\frac{\mu}{k}}f(a)+(g(b)-g(x))^{\frac{\mu}{k}}f(b)\right)\right| \qquad (2.26)\\&\leq\frac{1}{\Gamma_k(\mu+k)}\left((g(x)-g(a))^{\frac{\mu}{k}}(x-a)M^{x}_{a}(|f'|)+(g(b)-g(x))^{\frac{\mu}{k}}(b-x)M^{b}_{x}(|f'|)\right).\end{aligned}$$

Corollary 2.10. *If we take $k=1$ in (2.14), then we get the following generalized (RL) fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}_{g}I_{a+}f(x)+^{\nu}_{g}I_{b-}f(x)-\left(\frac{(g(x)-g(a))^{\mu}}{\Gamma(\mu+1)}f(a)+\frac{(g(b)-g(x))^{\nu}}{\Gamma(\nu+1)}f(b)\right)\right| \qquad (2.27)\\&\leq\frac{(g(x)-g(a))^{\mu}(x-a)}{\Gamma(\mu+1)}M^{x}_{a}(|f'|)+\frac{(g(b)-g(x))^{\nu}(b-x)}{\Gamma(\nu+1)}M^{b}_{x}(|f'|).\end{aligned}$$

Corollary 2.11. *If we take $g(x)=x$ in (2.14), then we get the following (RL) k-fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}I^{k}_{a+}f(x)+^{\nu}I^{k}_{b-}f(x)-\left(\frac{(x-a)^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(b-x)^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \qquad (2.28)\\&\leq\frac{(x-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}M^{x}_{a}(|f'|)+\frac{(b-x)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}M^{b}_{x}(|f'|).\end{aligned}$$

Corollary 2.12. *If we take $g(x)=x$ and $k=1$ in (2.14), then we get the following (RL) fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}I_{a+}f(x)+^{\nu}I_{b-}f(x)-\left(\frac{(x-a)^{\mu}}{\Gamma(\mu+1)}f(a)+\frac{(b-x)^{\nu}}{\Gamma(\nu+1)}f(b)\right)\right| \qquad (2.29)\\&\leq\frac{(x-a)^{\mu+1}}{\Gamma(\mu+1)}M^{x}_{a}(|f'|)+\frac{(b-x)^{\nu+1}}{\Gamma(\nu+1)}M^{b}_{x}(|f'|).\end{aligned}$$

Corollary 2.13. *Under the assumptions of above theorem if $|f'|$ is increasing on $[a,b]$, then from (2.14), we get the following fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}_{g}I^{k}_{a+}f(x)+^{\nu}_{g}I^{k}_{b-}f(x)-\left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \qquad (2.30)\\&\leq\frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)}|f'(x)|+\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}|f'(b)|.\end{aligned}$$

Corollary 2.14. *Under the assumptions of above theorem if $|f'|$ is decreasing on $[a,b]$, then from* (2.14)*, we get the following fractional integral inequality:*

$$\left|{}_g^{\mu}I_{a^+}^k f(x) +_g^{\nu} I_{b^-}^k f(x) - \left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \tag{2.31}$$
$$\leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)}|f'(a)| + \frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}|f'(x)|.$$

We need the following lemma in the proof of next result.

Lemma 2.15. *Let $f:[0,\infty)\to R$ be a quasi-convex function. If $f(x) = f(a+b-x)$, then for $x\in[a,b]$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq f(x). \tag{2.32}$$

Proof. We have

$$\frac{a+b}{2} = \frac{1}{2}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right) + \frac{1}{2}\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right). \tag{2.33}$$

As f is quasi-convex, therefore for $x\in[a,b]$, we have

$$f\left(\frac{a+b}{2}\right) \leq \max\{f(x), f(a+b-x)\}. \tag{2.34}$$

Using given condition $f(x) = f(a+b-x)$ in (2.34), then inequality in (2.32) is established. □

Theorem 2.16. *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be two functions such that g be differentiable and $f\in L[a,b]$ with $a<b$. Also let f be positive, quasi-convex, $f(x)=f(a+b-x)$ and g be strictly increasing with $g'\in L[a,b]$. Then for $x\in[a,b]$ and $\mu,\nu,k>0$, the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \tag{2.35}$$
$$\leq {}_g^{\nu+k}I_{b^-}^k f(a) +_g^{\mu+k} I_{a^+}^k f(b)$$
$$\leq \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] M_a^b(f).$$

Proof. As f is quasi-convex, therefore for $x\in[a,b]$, we have

$$f(x) \leq M_a^b(f). \tag{2.36}$$

Under assumptions of the function g, the following inequality holds:

$$g'(x)(g(x)-g(a))^{\frac{\nu}{k}} \leq g'(x)(g(b)-g(a))^{\frac{\nu}{k}} \tag{2.37}$$

for all $x\in[a,b]$ and $\nu,k>0$.

From (2.36) and (2.37), we have

$$\int_a^b (g(x)-g(a))^{\frac{\nu}{k}} f(x)g'(x)dx \le (g(b)-g(a))^{\frac{\nu}{k}} M_a^b(f) \int_a^b g'(x)dx.$$

By using (1.10) of Definition 1.6, we get

$$_g^{\nu+k}I_{b^-}^k f(a) \le \frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{k\Gamma_k(\nu+k)} M_a^b(f). \tag{2.38}$$

Now for $x \in [a,b]$ and $\mu, k > 0$, the following inequality inequality holds:

$$g'(x)(g(b)-g(x))^{\frac{\mu}{k}} \le g'(x)(g(b)-g(a))^{\frac{\mu}{k}}. \tag{2.39}$$

From (2.36) and (2.39), we have

$$\int_a^b (g(b)-g(x))^{\frac{\mu}{k}} f(x)g'(x)dx \le (g(b)-g(a))^{\frac{\mu}{k}} M_a^b(f) \int_a^b g'(x)dx.$$

By using (1.9) of Definition 1.6, we get

$$_g^{\mu+k}I_{a^+}^k f(b) \le \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{k\Gamma_k(\mu+k)} M_a^b(f). \tag{2.40}$$

Adding (2.38) and (2.40), we get the following inequality

$$_g^{\nu+k}I_{b^-}^k f(a) + _g^{\mu+k} I_{a^+}^k f(b) \le \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] M_a^b(f). \tag{2.41}$$

Now on the other hand multiplying (2.32) with $(g(x)-g(a))^{\frac{\nu}{k}} g'(x)$, then integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\int_a^b (g(x)-g(a))^{\frac{\nu}{k}} g'(x)dx \le \int_a^b (g(x)-g(a))^{\frac{\nu}{k}} g'(x)f(x)dx. \tag{2.42}$$

By using (1.10) of Definition 1.6, we get

$$\frac{k(g(b)-g(a))^{\frac{\nu}{k}+1}}{\nu+k} f\left(\frac{a+b}{2}\right) \le k\Gamma_k(\nu+k) {}_g^{\nu+k}I_{b^-}^k f(a). \tag{2.43}$$

Similarly, multiplying (2.32) with $(g(b)-g(x))^{\frac{\mu}{k}} g'(x)$, then integrating over $[a,b]$, we have

$$\frac{k(g(b)-g(a))^{\frac{\mu}{k}+1}}{\mu+k} f\left(\frac{a+b}{2}\right) \le k\Gamma_k(\mu+k) {}_g^{\mu+k}I_{a^+}^k f(b). \tag{2.44}$$

Adding (2.43) and (2.44), we get the following inequality

$$f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \le_g^{\nu+k} I_{b^-}^k f(a) + _g^{\mu+k} I_{a^+}^k f(b). \tag{2.45}$$

From (2.41) and (2.45), we get the inequalities in (2.35), which is required. □

Special cases of Theorem 2.16, are discussed in the following corollaries.

Corollary 2.17. *If we take $\mu=\nu$ in (2.35), then we get the following fractional integral inequality:*

$$2f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \leq {}_g^{\mu+k}I_{b^-}^k f(a) + {}_g^{\mu+k} I_{a^+}^k f(b) \leq \frac{2}{k}\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] M_a^b(f).$$

Corollary 2.18. *If we take $k=1$ in (2.35), then we get the following generalized (RL) fractional integral inequality:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\nu+1}}{\Gamma(\nu+2)}+\frac{(g(b)-g(a))^{\mu+1}}{\Gamma(\mu+2)}\right] \\ \leq\ & {}_g^{\nu+1}I_{b^-} f(a) + {}_g^{\mu+1} I_{a^+} f(b) \\ \leq\ & \left[\frac{(g(b)-g(a))^{\nu+1}}{\Gamma(\nu+1)}+\frac{(g(b)-g(a))^{\mu+1}}{\Gamma(\mu+1)}\right] M_a^b(f). \end{aligned} \tag{2.46}$$

Corollary 2.19. *If we take $g(x)=x$ in (2.35), then we get the following (RL) k-fractional integral inequality:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{(b-a)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(b-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \\ \leq\ & {}^{\nu+k}I_{b^-}^k f(a) + {}^{\mu+k} I_{a^+}^k f(b) \\ \leq\ & \frac{1}{k}\left[\frac{(b-a)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(b-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] M_a^b(f). \end{aligned} \tag{2.47}$$

Corollary 2.20. *If we take $g(x)=x$ and $k=1$ in (2.35), then we get the following (RL) fractional integral inequality:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)}+\frac{(b-a)^{\mu+1}}{\Gamma(\mu+2)}\right] \\ \leq\ & {}^{\nu+1}I_{b^-} f(a) + {}^{\mu+1} I_{a^+} f(b) \\ \leq\ & \left[\frac{(b-a)^{\nu+1}}{\Gamma(\nu+1)}+\frac{(b-a)^{\mu+1}}{\Gamma(\mu+1)}\right] M_a^b(f). \end{aligned} \tag{2.48}$$

Corollary 2.21. *Under the assumptions of above theorem if f is increasing on $[a,b]$, then from (2.35), we get the following fractional integral inequality:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \\ \leq\ & {}_g^{\nu+k}I_{b^-}^k f(a) + {}_g^{\mu+k} I_{a^+}^k f(b) \\ \leq\ & \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] f(b). \end{aligned} \tag{2.49}$$

Corollary 2.22. *Under the assumptions of above theorem if f is decreasing on $[a, b]$, then from* (2.35), *we get the following fractional integral inequality:*

$$f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \tag{2.50}$$
$$\leq {}^{\nu+k}_{g}I^k_{b^-}f(a)+{}^{\mu+k}_{g}I^k_{a^+}f(b)$$
$$\leq \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]f(a).$$

3 Applications

In this section we give applications of the results proved in the previous section. First we apply Theorem 2.1 and get the following result.

Theorem 3.1. *Under the assumptions of Theorem 2.1, we have the following fractional integral inequality:*

$${}^{\mu}_{g}I^k_{a^+}f(b)+{}^{\nu}_{g}I^k_{b^-}f(a)\leq\frac{1}{k}\left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu)}+\frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu)}\right)M^b_a(f). \tag{3.1}$$

Proof. If we put $x = a$ in (2.1), then we have

$${}^{\nu}_{g}I^k_{b^-}f(a)\leq\frac{((g(b)-g(a))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}M^b_a(f). \tag{3.2}$$

If we put $x = b$ in (2.1), then we have

$${}^{\mu}_{g}I^k_{a^+}f(b)\leq\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}M^b_a(f). \tag{3.3}$$

Adding inequalities (3.2) and (3.3), we get (3.1). □

Special cases of Theorem 3.1, are discussed in the following corollaries.

Corollary 3.2. *If we take $\mu = \nu$ in* (3.1), *then we have the following fractional integral inequality:*

$${}^{\mu}_{g}I^k_{a^+}f(b)+{}^{\mu}_{g}I^k_{b^-}f(a)\leq\frac{2(g(b)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}M^b_a(f). \tag{3.4}$$

Corollary 3.3. ([2]) *If we take $\mu = k = 1$ and $g(x) = x$ in* (3.4), *then we get the following inequality:*

$$\frac{1}{b-a}\int_a^b f(t)dt\leq M^b_a(f). \tag{3.5}$$

Next we apply Theorem 2.8 to obtain required results.

Theorem 3.4. *Under the assumptions of Theorem 2.8, we have the following fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}_{g}I^{k}_{a^+}f(b)+{}^{\nu}_{g}I^{k}_{b^-}f(a)-\left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \\ &\leq\left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}+\frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}\right)(b-a)M_a^b(|f'|).\end{aligned} \tag{3.6}$$

Proof. If we put $x=a$ in (2.14), then we have

$$\left|{}^{\nu}_{g}I^{k}_{b^-}f(a)-\frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right|\leq\frac{(g(b)-g(a))^{\frac{\nu}{k}}(b-a)}{\Gamma_k(\nu+k)}M_a^b(|f'|). \tag{3.7}$$

If we put $x=b$ in (2.14), then we have

$$\left|{}^{\mu}_{g}I^{k}_{a^+}f(b)-\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)\right|\leq\frac{(g(b)-g(a))^{\frac{\mu}{k}}(b-a)}{\Gamma_k(\mu+k)}M_a^b(|f'|). \tag{3.8}$$

Adding inequalities (3.7) and (3.8), we get (3.6). □

Special cases of Theorem 3.4, are discussed in the following corollaries.

Corollary 3.5. *If we take $\mu=\nu$ in* (3.6), *then we have the following fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}_{g}I^{k}_{a^+}f(b)+{}^{\mu}_{g}I^{k}_{b^-}f(a)-\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}\left(f(a)+f(b)\right)\right| \\ &\leq\frac{2(g(b)-g(a))^{\frac{\mu}{k}}(b-a)}{\Gamma_k(\mu+k)}M_a^b(|f'|).\end{aligned} \tag{3.9}$$

Corollary 3.6. *If we take $\mu=k=1$ and $g(x)=x$ in* (3.9), *then we get the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(a)+f(b)}{2}\right|\leq(b-a)M_a^b(|f'|). \tag{3.10}$$

By applying Theorem 2.16 similar relations can be established we leave it for the reader.

4 Concluding Remarks

The aim of this study is to explore bounds of fractional integrals in a compact form by using the concept of quasi-convexity. The authors are succeeded in the formulation of bounds of generalized fractional integrals (1.9) and (1.10). Theorem 2.1 provides upper bounds, Theorem 2.8 gives bounds in modulus form while Theorem 2.16 formulates bounds of Hadamard type. Section 3 consists of the applications of these bounds. Also some particular case of all these results are shown. Remark 1.7 includes all possible fractional integrals associated with generalized fractional integrals (1.9) and (1.10). The readers can obtain bounds for desired fractional integrals by putting the corresponding function g from Remark 1.7.

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Asymptotically Almost Automorphic Mild Solutions for Second Order Nonautonomous Semilinear Evolution Equations

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Abstract

The aim of this paper is to study the existence of asymptotically almost automorphic mild solution to some classes of second order semilinear evolution equation via the techniques of measure of noncompactness. The investigation is based on a new fixed point result which is a generalization of the well known Darbo's fixed point theorem. Finally examples are given to illustrate the analytical findings.

Key words: Asymptotically almost automorphic, second order nonautonomous differential equations, mild solution, evolution system, Kuratowski measures of noncompactness, fixed point.
AMS Subject Classification : 34G20

1 Introduction

This work is mainly concerned with the existence of asymptotically almost automorphic mild solution for second differential equations. More

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precisely, we will consider the following problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in \mathbb{R}^+ := [0, +\infty), \tag{1}$$

$$y(0) = y_0,\ y'(0) = y_1, \tag{2}$$

where $\{A(t)\}_{t\in\mathbb{R}^+}$ is a family of linear closed operators from E into E that generate an evolution system of linear bounded operators $\{\mathcal{U}(t,s)\}_{(t,s)\in\mathbb{R}^+\times\mathbb{R}^+}$ for $0 \le s \le t < +\infty$, $f : \mathbb{R}^+ \times E \to E$ is a Carathéodory function, and $(E, |\cdot|)$ is a real Banach space.

Evolution equations arise in many areas of applied mathematics [2, 37]. This type of equations has received much attention in recent years [1]. There are many results concerning the second-order differential equations, see for example [8, 11, 12, 20, 28, 35]. In recent years there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in [0, T], \tag{3}$$

$$y(0) = y_0,\ y'(0) = y_1. \tag{4}$$

The reader is referred to [10, 19, 22, 36] and the references therein. In the above mentioned works, the existence of solutions to the problem (3)-(4) is related to the existence of an evolution operator $\mathcal{U}(t; s)$ for the homogeneous equation

$$y''(t) = A(t)y(t), \quad \text{for} \quad t \ge 0.$$

For this purpose there are many techniques to show the existence of $\mathcal{U}(t, s)$ which has been developed by Kozak [25].
On the other hand, since Bochner [13] introduced the concept of almost automorphy, the automorphic functions have been applied to many areas including ordinary as well as partial differential equations, abstract differential equations, functional differential equations, integral equations, etc.; see [16, 21, 18, 27, 7]. We also refer the reader to the monographs by N'Guérékata [30, 31] for the basic theory of almost automorphic functions and applications. The concept of asymptotically almost automorphy was introduced by N'Guérékata [29]. Since then, these functions have generated lot of developments and applications, see [39, 14, 24, 17] and the references therein. In the previous works, people have established the existence of asymptotically almost automorphic mild solution of differential equations under the conditions that f satisfies or not the Lipschitz condition.

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In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Aissani and Benchohra [3], Akhmerov *et al.* [4], Alv́ares [5], Banaś and Goebel [9], Olszowy and Wędrychowicz [33], Olszowy [34], and the references therein.

Inspired by the above works,, in this work, using the properties of the analytic semigroups, Kuratowski measure of noncompactness, fixed point theorem, we obtain an existence result without assuming that the nonlinearity f satisfies a Lipschitz type condition.

This work is organized of as follows. In Section 2, we recall some fundamental properties of asymptotically almost automorphic and facts about evolution systems. Section 3 is devoted to establishing some criteria the existence of asymptotically almost automorphic mild solutions to the problem (1)-(2). Furthermore, appropriate examples are provided in section 4 to show the feasibility of our results.

2 Preliminaries and basic results

In this section we recall certain definitions and lemmas to be used subsequently in this paper.
Throughout this paper, we denote by E a Banach space with the norm $|\cdot|$. Let $BC(\mathbb{R}^+, \mathbb{E})$ be the Banach space of all bounded and continuous functions y mapping $\mathbb{R}^+$ into E endowed with the usual supremum norm

$$\|y\|_\infty = \sup_{t\in\mathbb{R}^+} |y(t)|.$$

In what follows, let $\{A(t),\ t \in \mathbb{R}^+\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ which is dense in E and independent of t.

In this work the existence of solution the problem (1)-(2) is related to the existence of an evolution operator $\mathcal{U}(t,s)$ for the following homogeneous problem

$$y''(t) = A(t)y(t) \qquad t \in \mathbb{R}^+. \tag{5}$$

This concept of evolution operator has been developed by Kozak [25] and recently used by Henríquez *et al.* [22].

Definition 2.1 *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ : s \leq t\}$, is called an evolution operator of the equation (5) if de following conditions hold:*

(e_1) *For any $x \in E$ the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is continuously differentiable and*

- **(a)** *for each $t \in \mathbb{R}$, $\mathcal{U}(t,t)x = 0, \forall x \in E$,*
- **(b)** *for all $(t,s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x|_{t=s} = -x$.*

(e_2) *For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of class C^2 and*

- **(a)** $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x = A(t)\mathcal{U}(t,s)x,$
- **(b)** $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x = \mathcal{U}(t,s)A(s)x,$
- **(c)** $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x|_{t=s} = 0.$

(e_3) *For all $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and*

- **(a)** $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x = A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x.$
  Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$ is continuous,
- **(b)** $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x = \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x.$

Throughout this paper, we will use the following definition of the concept of Kuratowski measure of noncompactness [9].

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Definition 2.2 *The Kuratowski measure of noncompactness α is defined by*

$$\alpha(D) = \inf\{r > 0 : D \text{ has a finite cover by sets of diameter} \leq r\},$$

for a bounded set D in any Banach space E.

Let us recall the basic properties of Kuratowski measure of noncompactness.

Lemma 2.3 *[9] Let E be a Banach space and $C, D \subset E$ be bounded, then the following properties hold:*

- (i_1) $\alpha(D) = 0$ *if only if D is relatively compact,*
- (i_2) $\alpha(\overline{D}) = \alpha(D)$ *; $\overline{D}$ the closure of D,*
- (i_3) $\alpha(C) \leq \alpha(D)$ *when $C \subset D$,*
- (i_4) $\alpha(C+D) \leq \alpha(C)+\alpha(D)$ *where* $C+D = \{x \mid x = y + z; y \in C; z \in D\}$,
- (i_5) $\alpha(aD) = |a|\alpha(D)$ *for any $a \in \mathbb{R}$,*
- (i_6) $\alpha(ConvD) = \alpha(D)$, *where $ConvD$ is the convex hull of D,*
- (i_7) $\mu(C \cup D) = \max(\alpha(C), \alpha(D))$,
- (i_8) $\alpha(C \cup \{x\}) = \alpha(C)$ *for any $x \in E$.*

Denote by $\omega^T(y, \varepsilon)$ the modulus of continuity of y on the interval $[0, T]$ i.e.

$$\omega^T(y, \varepsilon) = \sup\{|y(t) - y(s)|\,; t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Moreover, let us put

$$\omega^T(D, \varepsilon) = \sup\left\{\omega^T(y, \varepsilon); y \in D\right\},$$

$$\omega_0^T(D) = \lim_{\varepsilon \to 0} \omega^T(D, \varepsilon).$$

Lemma 2.4 *[15] Let E be a Banach space, $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that*

$$\alpha(D) \leq 2\alpha(D_0).$$

Lemma 2.5 *[23] Let $D = \{y_n\}_{n=0}^{+\infty} \subset C(\mathbb{R}^+, E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $\mathbb{R}^+$, and*

$$\alpha\left\{\int_0^t y_n(s))ds\right\}_{n=0}^{\infty} \leq 2\int_0^t \alpha(D(s))ds, \qquad t \in \mathbb{R}^+.$$

Now, we recall some basic definitions and results on almost automorphic functions and asymptotically almost automorphic functions (for more details, see [13, 31, 38]).

Definition 2.6 *A continuous function $f : \mathbb{R} \to E$ is said to be almost automorphic if for every sequence of real numbers $\{\tau_n'\}$, there exists a subsequence $\{\tau_n\}$ such that*

$$g(t) = \lim_{n\to\infty} f(t + \tau_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \tau_n) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

Denote by $AA(\mathbb{R}, E)$ the set of all such functions.

Lemma 2.7 *[30] $AA(\mathbb{R}, E)$ is a Banach space with the supremum norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|.$$

Definition 2.8 *A continuous function $f : \mathbb{R} \times E \to E$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $y \in E$ if for every sequence of real numbers $\{\tau_n'\}$, there exists a subsequence $\{\tau_n\}$ such that*

$$\lim_{n\to\infty} f(t + \tau_n, y) = g(t, y)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \tau_n, y) = f(t, y)$$

for each $t \in \mathbb{R}$ and each $y \in E$. The collection of those functions is denoted by $AA(\mathbb{R} \times E, E)$.

Example 2.9 *[40] The function $f : \mathbb{R} \times E \to E$ given by*

$$f(t, y) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)\cos y$$

is almost automorphic in $t \in \mathbb{R}$ for each $y \in E$, where $E = L^2([0, 1])$.

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The space of all continuous functions $h : \mathbb{R}^+ \to E$ such that $\lim\limits_{t\to\infty} h(t) = 0$ is denoted by $C_0(\mathbb{R}^+, E)$. Moreover, we denote $C_0(\mathbb{R}^+ \times E, E)$; the space of all continuous functions from $\mathbb{R} \times E$ to E satisfying $\lim\limits_{t\to\infty} h(t, y) = 0$ in t and uniformly in $y \in E$.

Remark 2.10 *Note that if $\nu(t) \in C_0(\mathbb{R}^+, E)$, then*

$$\int_0^t e^{-(t-s)}\nu(s)ds \in C_0(\mathbb{R}^+, E).$$

Definition 2.11 *A continuous function $f : \mathbb{R}^+ \to E$ is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t) = g(t) + h(t),$$

where

$$g(t) \in AA(\mathbb{R}, E),\ \ h(t) \in C_0(\mathbb{R}^+, E).$$

Denote by $AAA(\mathbb{R}^+, E)$ the set of all such functions.

Example 2.12 *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(t) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right) + e^{-t}$$

is an asymptotically almost automorphic function with

$$g(t) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right) \in AA(\mathbb{R}, \mathbb{R}), \quad h(t) = e^{-t} \in C_0(\mathbb{R}^+, \mathbb{R}).$$

Lemma 2.13 *[31],[32]. $AAA(\mathbb{R}^+, E)$ is also a Banach space with the norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}^+} |f(t)|.$$

Definition 2.14 *A continuous function $f : \mathbb{R}^+ \times E \to E$ is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t, y) = g(t, y) + h(t, y),$$

where

$$g(t, y) \in AA(\mathbb{R} \times E, E), \quad h(t, y) \in C_0(\mathbb{R}^+ \times E, E).$$

Denote by $AAA(\mathbb{R}^+ \times E, E)$ the set of all such functions.

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Example 2.15 *The function $f : \mathbb{R}^+ \times E \to E$ given by*

$$f(t,x) = \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right)\cos y + e^{-t}|y|$$

is asymptotically almost automorphic in $t \in \mathbb{R}^+$ for each $y \in E$, where $E = L^2([0,1])$.

$$g(t,y) = \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right)\cos y \in AA(\mathbb{R}\times E, E),$$
$$h(t,y) = e^{-t}|y| \in C_0(\mathbb{R}^+ \times E, E).$$

.

Lemma 2.16 *[26] $f : \mathbb{R}\times E \to E$ is almost automorphic, and assume that $f(t,\cdot)$ is uniformly continuous on each bounded subset $K \subset E$ uniformly for $t \in \mathbb{R}$, that is for any $\varepsilon > 0$, there exists $\varrho > 0$ such that $y, z \in K$ and $|y(t) - z(t)| < \varrho$ imply that $|f(t,y) - f(t,z)| < \varepsilon$ for all $t \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to E$ be almost automorphic. Then the function $F : \mathbb{R} \to E$ defined by $F(t) = f(t,\varphi(t))$ is almost automorphic.*

Theorem 2.17 *[6] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, and let $\Gamma : \Omega \to \Omega$ be a continuous operator satisfying the inequality*

$$\alpha(\Gamma(D)) \le \Psi(\alpha(D))$$

for any nonempty subset D of Ω, where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function such that

$$\lim_{n\to+\infty} \Psi^n(t) = 0 \text{ for each } t \ge 0.$$

Then Γ has at least one fixed point in the set Ω.

3 Main results

Definition 3.1 *A function $y \in BC(\mathbb{R}^+, E)$ is said to be a mild solution to the problem (1)-(2) if y satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds.$$

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For the proof of our main theorem, we need the following hypotheses:

(H_1) (a) There exists a constant $M \geq 1$ and $\delta > 0$, such that

$$\|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} \leq Me^{-\delta(t-s)} \quad \text{for any } (t,s) \in \Delta$$

and for any sequence of real numbers $\{\tau_n'\}$, we can extract a subsequence $\{\tau_n\}$ and for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|U(t+\tau_n, s+\tau_n) - U(t,s)\|_{\mathcal{B}(E)} \leq \varepsilon e^{-\delta(t-s)},$$

$$\|U(t-\tau_n, s-\tau_n) - U(t,s)\|_{\mathcal{B}(E)} \leq \varepsilon e^{-\delta(t-s)}$$

for each $t, s \in \mathbb{R}$. for all $n > N$, for each $t, s \in \mathbb{R}$, $t \geq s$.

(H_2) There exist a constant $\widetilde{M} \geq 0$ and $\delta > 0$, such that:

$$\left\| \frac{\partial}{\partial s}\mathcal{U}(t,s) \right\|_{\mathcal{B}(E)} \leq \widetilde{M}e^{-\delta(t-s)}, (t,s) \in \Delta.$$

(H_3) The function $f : \mathbb{R}^+ \times E \to E$ is Carathéodory and asymptotically almost automorphic i.e., $f(t,y) = g(t,y) + h(t,y)$ with

$$g(t,y) \in AA(\mathbb{R} \times E, E), \quad h(t,y) \in C_0(\mathbb{R}^+ \times E, E),$$

and $g(t,y)$ is uniformly continuous on any bounded subset $K \subset E$ uniformly for $t \in \mathbb{R}$.
Moreover,

(a) There exist $p \in L^q(\mathbb{R}, \mathbb{R}^+)$, $q \in [1, \infty)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that for all $t \in \mathbb{R}^+$ and $y \in E$,

$$|g(t,y)| \leq p(t)\psi(|y|) \quad \text{and} \quad \lim_{|y| \to +\infty} \inf \frac{\psi(|y|)}{|y|} = \rho_1.$$

(b) There exist a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}^+$ and $y \in E$ with $|y| \leq R$,

$$|h(t,y)| \leq \beta(t)\phi(|y|) \quad \text{and} \quad \lim_{R \to +\infty} \inf \frac{\phi(R)}{R} = \rho_2.$$

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(H_4) There exist a locally integrable function $\eta : \mathbb{R} \to \mathbb{R}^+$ and a continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any nonempty bounded set $D \subset E$ we have :

$$\alpha(f(t,D)) \leq \eta(t)\varphi(\alpha(D)) \text{ for a.e } t \in \mathbb{R}^+.$$

Additionally we assume that $\lim\limits_{n\to+\infty}(\psi+\phi)^n(t) = 0$ for a.e $t \in \mathbb{R}^+$. Let $\beta(t)$ be the function involved in the assumption (H_3), then

$$\int_0^t e^{-(t-s)}\beta(s)ds \in C_0(\mathbb{R}^+,\mathbb{R}^+).$$

Put

$$\rho = \sup_{t\in\mathbb{R}^+}\int_0^t e^{-(t-s)}\beta(s)ds.$$

We need the following technical lemma.

Lemma 3.2 *Assume that (H_1) hold. If $\varphi(t) \in AA(\mathbb{R},E)$, then*

$$\Lambda(t) := \int_{-\infty}^t U(t,s)\varphi(s)\mathrm{ds},\ \mathrm{t} \in \mathbb{R},$$

belongs to $AA(\mathbb{R},\mathbb{E})$.

Proof. From (H_1) it is clear that $\Lambda(t)$ is well-defined and continuous on $\mathbb{R}$. Since $\varphi(t) \in AA(\mathbb{R},E)$, it follows that for every sequence of real numbers $\{\tau_n'\}$, we can extract a subsequence $\{\tau_n\}$ such that

(c_1) $\lim\limits_{n\to\infty}\varphi(t+\tau_n) - \widetilde{\varphi}(t) = 0$ for each $t \in \mathbb{R}$ and,

(c_2) $\lim\limits_{n\to\infty}\widetilde{\varphi}(t-\tau_n) - \varphi(t) = 0$ for each $t \in \mathbb{R}$.

Notes that $\widetilde{\varphi}$ is also bounded on $\mathbb{R}$, and measurable. Define

$$\widetilde{\Lambda}(t) = \int_{-\infty}^t \mathcal{U}(t,s)\widetilde{\varphi}(s)\mathrm{ds}, \quad \mathrm{t} \in \mathbb{R}.$$

For $t \in \mathbb{R}$, Since $\widetilde{\varphi}$ is measurable, $\widetilde{\Lambda}$ is well-defined.
For $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\Lambda y)(t+\tau_n) - (\widetilde{\Lambda}y)(t)\right| \\
&= \left|\int_{-\infty}^{t+\tau_n} \mathcal{U}(t+\tau_n, s)\varphi(s)ds - \int_{-\infty}^{t} \mathcal{U}(t,s)\widetilde{\varphi}(s)ds\right| \\
&= \left|\int_{-\infty}^{t} \mathcal{U}(t+\tau_n, s+\tau_n)\varphi(s+\tau_n)ds - \int_{-\infty}^{t} \mathcal{U}(t,s)\widetilde{\varphi}(s)ds\right| \\
&\leq \int_{-\infty}^{t} \|\mathcal{U}(t+\tau_n, s+\tau_n)\|_{\mathcal{B}(E)} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| ds \\
&+ \int_{-\infty}^{t} \|\mathcal{U}(t+\tau_n, s+\tau_n) - \mathcal{U}(t,s))\|_{\mathcal{B}(E)}\widetilde{\varphi}(s)ds \\
&\leq \int_{-\infty}^{t} Me^{-\delta(t-s)} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| ds \\
&+ \int_{-\infty}^{t} \varepsilon e^{-\delta(t-s)} \left|\widetilde{\varphi}(s)\right| ds \\
&\leq M \int_{-\infty}^{t} e^{-\delta(t-s)} ds \sup_{s\in\mathbb{R}} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| \\
&+\varepsilon \int_{-\infty}^{t} e^{-\delta(t-s)} ds \sup_{s\in\mathbb{R}} \left|\widetilde{\varphi}(s)\right| \\
&\leq \frac{M}{\delta} \sup_{s\in\mathbb{R}} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| + \frac{\varepsilon}{\delta} \sup_{s\in\mathbb{R}} \left|\widetilde{\varphi}(s)\right|.
\end{aligned}$$

Using (c_1), we obtain that for $n \to \infty$,

$$\Lambda(t+\tau_n) \to \widetilde{\Lambda}(t).$$

Analogously, one can prove that,

$$\widetilde{\Lambda}(t-\tau_n) \to \Lambda(t) \text{ for each } t \in \mathbb{R} \text{ as } n \to \infty.$$

This we show that

$$\Lambda \in AA(\mathbb{R}, E).$$

Theorem 3.3 *Assume that the hypotheses $(H_1) - (H_4)$ are satisfied. If*

$$M\rho_1\|p\|_{L^q} + M\delta^{-1}\rho\rho_2 < 1, \tag{6}$$

and

$$M\max(4\|\eta\|_{L^1}, \|p\|_{L^q}\delta^{-1+\frac{1}{q}}) < 1, \tag{7}$$

then the problem (1)-(2) has a asymptotically almost automorphic mild solution.

Proof. Consider the operator $N : AAA(\mathbb{R}^+, E) \to AAA(\mathbb{R}^+, E)$ defined by

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds, \qquad (8)$$

where $y \in AAA(\mathbb{R}^+, E)$ with $y = \gamma + \zeta$, γ is the principal term and ζ the corrective term of y. We need to prove that N is weel- defined, that is $N(AAA(\mathbb{R}^+, E)) \subset AAA(\mathbb{R}^+, E)$. Let

$$\sigma(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1,$$

then

$$\begin{aligned} |\sigma(t)| &= |-\tfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1| \\ &\leq |\tfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0| + |\mathcal{U}(t,0)y_1| \\ &\leq \widetilde{M}e^{-\delta t}|y_0| + Me^{-\delta t}|y_1|. \end{aligned}$$

Since $\delta > 0$, we get $\lim\limits_{t\to+\infty} |(\sigma(t)| = 0$. that is

$$\sigma \in C_0(\mathbb{R}^+, E). \qquad (9)$$

By assumption $f = g + h$ where g is the principal term and h the corrective term. So we can write

$$\begin{aligned} f(t,y(t)) &= g(t,\gamma(t)) + f(t,y(t)) - f(t,\gamma(t)) + h(t,\gamma(t)) \\ &= g(t,\gamma(t)) + H(t,y(t)), \qquad (10) \end{aligned}$$

In view of (10), we have

$$\begin{aligned} W(t) &= \int_0^t \mathcal{U}(t,s)f(s,y(s))ds \\ &= \int_0^t \mathcal{U}(t,s)g(s,\gamma(s))ds + \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &= \int_{-\infty}^t \mathcal{U}(t,s)g(s,\gamma(s))ds - \int_{-\infty}^0 \mathcal{U}(t,s)g(s,\gamma(s))ds \\ &+ \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &= (I_1y)(t) + (I_2y)(t), \end{aligned}$$

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where

$$(I_1y)(t) = \int_{-\infty}^{t} \mathcal{U}(t,s)g(s,\gamma(s))\mathrm{ds},$$

$$\begin{aligned}(I_2y)(t) &= \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &- \int_{-\infty}^{0} \mathcal{U}(t,s)g(s,y(s))ds \\ &= (J_1y)(t) + (J_2y)(t),\end{aligned}$$

where

$$(J_1y)(t) = \int_0^t \mathcal{U}(t,s)H(s,y(s))ds,$$

$$(J_2y)(t) = \int_{-\infty}^{t} \mathcal{U}(t,s)g(s,\gamma(s))\mathrm{ds}.$$

Using (H_3) and Lemma 2.16 , we deduce that $s \to g(s,\gamma(s))$ is in $AA(\mathbb{R},E)$. Thus, by Lemma 3.2 we obtain

$$(I_1y)(t) \in AA(\mathbb{R},E). \tag{11}$$

Let's prove that $J_1 \in C_0(\mathbb{R}^+,E), J_2 \in C_0(\mathbb{R}^+,E)$.
Ideed by definition $H \in C_0(\mathbb{R}^+,E)$, that means given $\varepsilon > 0$, there exists $T > 0$ such that if $t \geq T$, we have $|H(t,y)| \leq \varepsilon$. Therefore if $t \geq T$, we get

$$\begin{aligned}\int_T^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)}|H(s,y(s))|ds &\leq M\varepsilon \int_T^t e^{-\delta(t-s)}ds \\ &\leq \frac{M}{\delta}\varepsilon,\end{aligned}$$

then

$$|(J_1y)(t)| \leq \frac{M}{\delta}\varepsilon \qquad \text{if } t \geq T.$$

So,

$$J_1 \in C_0(\mathbb{R}^+,E). \tag{12}$$

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Next, let us show that $J_2 \in C_0(\mathbb{R}^+, E)$.

$$\begin{aligned}|(J_2y)(t)| &\leq \int_{-\infty}^{0} \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} |g(s,y(s))| ds \\ &\leq M \sup_{t\in\mathbb{R}} |g(t,y(t))| \int_0^T e^{-\delta(t-s)} ds \\ &+ M\|g\|_\infty \frac{e^{-\delta(t}}{\delta} \qquad \to 0 \text{ as } \to \infty.\end{aligned}$$

So,

$$J_2 \in C_0(\mathbb{R}^+, E). \tag{13}$$

Finaly combining (9),(11), (12) and (13) proves our claim that $N \in AAA(\mathbb{R}^+, E)$. Next, we will prove that the operator N satisfies all the assumptions of Theorem 2.17. We will break the proof into several steps.
Let

$$B_R = \left\{ y \in AAA(\mathbb{R}^+, E) : \|y\|_\infty \leq R \right\},$$

where R be any positive constant. Then B_R is a bounded, closed and convex subset of $AAA(\mathbb{R}^+, E)$.

Step 1: $N(y) \in B_R$ for any $y \in B_R$.
In fact, if we assume that the assertion is false, then $R < |(Ny)(t)|$. This yields that

$$\begin{aligned}R < |(Ny)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |g(s,y(s)| ds \\ &+ \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} |h(s,y(s)| ds \\ &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}\, p(s)\psi(|y(s)|) ds \\ &+ \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} \beta(s)\phi(|y(s)|) ds \\ &\leq M\psi(R) \int_0^t e^{-\delta(t-s)} p(s) ds \\ &+ M\ \phi(R) \int_0^t e^{-\delta(t-s)} \beta(s) ds.\end{aligned}$$

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For $t \geq 0$, it follows from the Hölder inequality that

$$R < |(Ny)(t)| \quad \leq \quad M\psi(R)\|p\|_{L^q} + M\rho_2\phi(R).$$

Dividing both sides by R and taking the $\liminf$ as $R \to +\infty$, we have

$$M\rho_1\|p\|_{L^q} + M\delta^{-1}\rho\rho_2 > 1,$$

which contradicts (6). Hence, the operator N transforms the set B_R into itself.

Step 2. N is continuous.

Let $(y_n)_{n\in N}$ be a sequence in B_R such that $y_n \to y$ in B_R.

Case 1. If $t \in [0,T];\ T > 0$, then, we have

$$|(Ny_n)(t) - (Ny)(t)| \leq M\int_0^t |f(s, y_n(s)) - f(s, y(s))|\, ds.$$

Since the functions f is Carathéodory, the Lebesgue dominated convergence theorem implies that

$$\|Ny_n - Ny\|_\infty \to 0 \quad \text{ as } \ n \to +\infty.$$

Case 2. Since the functions f is Carathéodory, we can see that

$$|f(s, y_n(s)) - f(s, y(s))| \leq \frac{\delta\varepsilon}{M} \quad \text{ for } t \geq T. \tag{14}$$

If $t \in (T, \infty)$, $T > 0$, then (14) and the hypotheses give us that

$$\begin{aligned} |Ny_n(t) - Ny(t)| \ &\leq \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)}\Big|f(s, y_n(s)) - f(s, y(s))\Big|ds \\ &\leq M\frac{\delta\varepsilon}{M}\int_0^t e^{-\delta(t-s)}ds \\ &\leq \frac{M}{\delta}\frac{\delta\varepsilon}{M} \\ &\leq \varepsilon. \end{aligned} \tag{15}$$

Then the inequality (15) reduces to

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Now, we conclude that N is continuous from B_R to B_R.

Step 3: $N(B_R)$ is equicontinuous.

Let $t_1, t_2 \in [0,T]$ with $t_2 > t_1$ and $y \in B_R$. Then, we have

$$\begin{aligned}
&|(N_1y)(t_2) - (N_1y(t_1)| \\
&= \left|\int_0^{t_1} (\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s))g(s,y(s)) \right. \\
&\left. + \int_{t_1}^{t_2} \mathcal{U}(t_2,s)g(s,y(s))ds\right| \\
&+ \left|\int_0^{t_1} (\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s))h(s,y(s)) \right. \\
&\left. + \int_{t_1}^{t_2} \mathcal{U}(t_2,s)h(s,y(s))ds\right| \\
&\leq \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{B(E)}\ p(s)\psi(|y(s)|)ds \\
&+M\int_{t_1}^{t_2} e^{-\delta(t-s)}p(s)\psi(|y(s)|)ds. \\
&+ \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{B(E)}\ \beta(s)\phi(|y(s)|)ds \\
&+M\int_{t_1}^{t_2} e^{-\delta(t-s)}\beta(s)\phi(|y(s)|)ds.
\end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned}
&|(N_1y)(t_2) - (N_1y(t_1)| \\
&\leq \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{\mathcal{B}(E)}\ p(s)\psi(|y(s)|)ds \\
&+\frac{M\|p\|_{L^q}\psi(R)}{\delta^{1-\frac{1}{q}}}\left(e^{-\frac{q\delta}{q-1}(t-t_2)} - e^{-\frac{q\delta}{q-1}(t-t_2)}\right)^{1-\frac{1}{q}} \\
&+ \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{B(E)}\ \beta(s)\phi(|y(s)|)ds \\
&+\frac{M\phi(R)\sup\limits_{t\in\mathbb{R}}\beta(t)}{\delta}(e^{-\delta(t-t_2)} - e^{-\delta(t-t_1)}).
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$, which implies that $N(B_R)$ is equicontinuous.
Consider the measure of noncompacteness $\mu(B)$ defined on the family of bounded subsets of the space $AAA(\mathbb{R}^+, E)$ (see [33]) by

$$\mu(B) = \omega_0^T(B) + \sup_{t\in J}\alpha(B(t)) + \lim_{T\to+\infty}\sup\{|y(t)| : t \geq T, y \in E\}.$$

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Step 4: $\mu(N(B)) \leq M \max(4\|\eta\|_{L^1}, \|p\|_{L^q}\delta^{-1+\frac{1}{q}})(\varphi+\psi)(\mu(B))$ for all $B \subset B_R$. For all $B \subset B_R$, $N(B)$ is bounded. Hence, by Lemma 2.4, there exists a countable set $B_1 = \{y\}_{n=1}^{\infty} \subset B$, such that

$$(N(B)) \leq 2\alpha(N(B_1)). \tag{16}$$

Using the properties of α, Lemma 2.4, Lemma 2.5 and assumptions (H_1) and (H_4), we get

$$\begin{array}{rcl}\alpha(NB_1(t)) & \leq & \alpha\left(\left\{\int_0^t \mathcal{U}(t,s)f(s,y_n(s))ds\right\}_{n=0}^{\infty}\right) \\ & \leq & 2M\int_0^t \{\alpha\left(f(s,y_n(s))ds)\right)\}_{n=0}^{\infty} ds \\ & \leq & 2M\int_0^t \eta(s)\varphi\left(\{(\alpha(y_n(s))\}_{n=0}^{\infty})\right)) ds \\ & \leq & 2M\int_0^t \eta(s)\varphi(\alpha(B(s)))ds.\end{array}$$

Form inequality (16), it follows that

$$\alpha(NB(t)) \leq 4M\int_0^t \eta(s)\varphi(\alpha(B(s)))ds,$$

then

$$\alpha(N(B(t)) \leq 4M\|\eta\|_{L^1}\varphi(\sup_{t\in\mathbb{R}^+}\alpha(B(t))).$$

Since

$$\sup_{t\in\mathbb{R}^+}\alpha(B(t)) \leq \sup_{t\in\mathbb{R}^+}\alpha(B(t)) + \lim_{t\to+\infty}\sup\{|y(t)| : t \geq T, y \in E\}),$$

then

$$\alpha(N(B(t)) \leq 4M\|\eta\|_{L^1}\varphi(\sup_{t\in\mathbb{R}^+}\alpha(B(t)) + \lim_{t\to+\infty}\sup\{|y(t)| : t \geq T, y \in E\}). \tag{17}$$

On the other hand, we have

$$\begin{aligned}
|(Ny)(t)| &\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ M\int_{-\infty}^{t} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds + |(I_2y)(t)| \\
&+ M\int_{-\infty}^{T} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds. \\
&+ M\int_{T}^{t} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds + |I_2(t)|. \\
&\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ M\int_{-\infty}^{T} e^{-\delta(t-s)}p(s)ds\psi(\sup_{s\in\mathbb{R}}|\gamma(s)|) \\
&+ M\int_{T}^{t} e^{-\delta(t-s)}p(s)ds\psi(\sup\{|\gamma(t)| : t \geq T, y \in E\}) \\
&+ \sup\{|(I_2y)(t)| : t \geq T, y \in E\}).
\end{aligned}$$

Next, applying the Hölder inequality we derive

$$\begin{aligned}
|(Ny)(t)| &\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}e^{-\delta(t-T)}\psi(\|y\|_\infty). \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}(1-e^{-\frac{q\delta}{q-1}t})^{1-\frac{1}{q}}\psi(\sup\{|y(t)| : t \geq T, y \in E\}) \\
&+ \sup\{|(I_2y)(t)| : t \geq T, y \in E\}).
\end{aligned}$$

Then

$$\begin{aligned}
|(Ny)(t)| &\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}e^{-\delta T}\psi(\|y\|_\infty). \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}\psi(\sup\{|y(t)| : t \geq T, y \in E\}) \\
&+ \sup\{|(I_2y)(t)| : t \geq T, y \in E\}).
\end{aligned}$$

Since $\delta \geq 0$, $I_2 \in C_0(\mathbb{R}^+, E)$ and

$$\lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\} \leq \sup_{t\in\mathbb{R}} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\},$$

then

$$\begin{aligned} &\lim_{T\to+\infty} \sup\{|(Ny)(t) : t \geq T, y \in E\}) \\ &\leq \tfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}} \psi(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}). \end{aligned} \tag{18}$$

Further, combining (17) and (18), we get

$$\begin{aligned} &\sup_{t\in J} \alpha((NB)(t)) + \lim_{T\to+\infty} \sup\{|(Ny)(t) : t \geq T, y \in E\}) \\ &\leq 4M\|\eta\|_{L^1} \varphi(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}) \\ &+ \tfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}} \psi(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\} \\ &\leq M \max(4\|\eta\|_{L^1}, \tfrac{\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}})(\varphi + \psi)(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}). \end{aligned} \tag{19}$$

From **Step 3** and inequality (19), we conclude that

$$\mu(N(B)) \leq M \max\left(4\|\eta\|_{L^1}, \frac{\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}\right)(\varphi + \psi)(\mu(B)).$$

It follows from Lemma 2.17 that N has at least one fixed point $y \in B_R$, which is just a asymptotically almost automorphic mild solution of problem (1)-(2) on $\mathbb{R}^+$.

4 An Example

Consider the second order differential equation of the form;

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} z(t,\tau) &= \dfrac{\partial^2}{\partial \tau^2} z(t,\tau) + 2\sin\left(\dfrac{1}{2+\cos t + \cos\sqrt{2}t}\right)\dfrac{\partial}{\partial t} z(t,\tau) \\ &+ \dfrac{\sin^2 t}{12\sqrt{1+t^2}} \sin\left(\dfrac{1}{2+\cos t+\cos\sqrt{2}t}\right)(|z(t,\tau)| + \ln(1+|z(t,\tau)|)) \\ &+ \dfrac{\sin^2 t}{15\sqrt{1+t^2}} \dfrac{\sin \pi z(t,\tau)}{(1+|z(t,\tau)|)}, \qquad t \in \mathbb{R}^+, \quad \tau \in [0,\pi], \\ z(t,0) = & z(t,\pi) = 0, \qquad t \in \mathbb{R}^+, \\ \dfrac{\partial}{\partial t} z(0,\tau) &= \psi(\tau), \qquad \tau \in [0,\pi]. \end{cases} \tag{20}$$

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Let $E = L^2([0,\pi],\mathbb{R}^+)$ be the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}^+$, and let $H^2([0,\pi],\mathbb{R}^+)$ be the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}^+$, such that $x'' \in L^2([0,\pi],\mathbb{R}^+)$. We consider the operator $A_1 z(\tau) = z''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R}^+,\mathbb{C})$, which is the infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, A_1 has discrete spectrum, the spectrum of A_1 consists of eigenvalues n^2 for $n \in \mathbb{Z}$, with associated eigenvector

$$\omega_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}, n \in \mathbb{Z},$$

the set $\{\omega_n \in \mathbb{Z}\}$ is an orthonormal basis of E. In particular,

$$A_1 x = -\sum_{n=1}^{\infty} n^2 \langle x, w_n\rangle w_n \text{ for } x \in D(A).$$

The cosine function $C(t)$ is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt)\langle x, w_n\rangle w_n \text{ for } x \in D(A), t \in \mathbb{R}^+,$$

form a cosine function on H, with associated sine function

$$S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle x, w_n\rangle w_n \text{ for } x \in D(A), t \in \mathbb{R}^+.$$

From [35], for all $x \in H^2([0,\pi],\mathbb{R}^+), t \in \mathbb{R}^+$, $\|C(t)\|_{B(E)} \le e^{-t}$ and $\|S(t)\|_{B(E)} \le e^{-t}$.
Now, we define an operator $A(t) : D(A) \subset H \to H$ by

$$\begin{cases} D(A(t)) = D(A) \\ A(t) = A_1 + b(t,\tau). \end{cases}$$

where $b(t,\tau) = 2\sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right)$
Note that $A(t)$ generates an evolutionary process $\mathcal{U}(t,s)$ of the form

$$\mathcal{U}(t,s) = S(t-s)e^{\int_s^t b(t,s)ds}$$

Since $b(t,\tau) = 2\sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \le 2$, we have

$$\mathcal{U}(t,s) = S(t-s)e^{-2(t-s)} \tag{21}$$

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and

$$\|\mathcal{U}\|_{\mathcal{B}(E)} \leq \|S\|_{\mathcal{B}(E)} e^{-2(t-s)} \leq e^{-3(t-s)}$$

We conclude that $\mathcal{U}(t,s)$ is a evolutionary process exponentially stable with $M = 1$ and $\delta = 3$.
It follows from the estimate (21) that $\mathcal{U}(t,s) : E \to E$ is well defined and satisfies the conditions of Definition 2.1.
Hence conditions (H_1) and (H_2) are satisfied.
Now, let

$$z(t)(\tau) = w(t)(\tau), \ t \geq 0, \ \tau \in [0,\pi],$$

$$g(t,z)(\tau) \quad = \quad \frac{\sin^2 t}{12\sqrt{1+t^2}} \sin\Big(\frac{1}{2+\cos t + \cos\sqrt{2}t}\Big)(|z(t,\tau)| + \ln(1+|z(t,\tau)|)),$$

$$h(t,z)(\tau) = \frac{\sin^2 t \sin \pi z(t,\tau)}{15\sqrt{1+t^2}(1+|z(t,\tau)|)}.$$

Then it is easy to verify that $g : \mathbb{R} \times E \times E$ is continuous and

$$g \in AA(\mathbb{R} \times E; E).$$

We can estimate for the functions g:

$$g(t,z)(\tau) \leq \frac{\sin^2 t}{12\sqrt{1+t^2}}(|z(t,\tau)| + \ln(1+|z(t,\tau)|)).$$

Hence conditions $(H_3)(a)$ is satisfied with

$$p(t) = \frac{\sin^2 t}{3\sqrt{1+t^2}}, \quad \psi(t) = \frac{1}{4}(t + \ln(1+t)).$$

Then it is easy to verify that $p \in L^2(\mathbb{R})$ and $\rho_1 = \dfrac{1}{4}$.
On the other hand, it is clear that $h : \mathbb{R}^+ \times E \times E$ is continuous and

$$h \in C_0(\mathbb{R}^+ \times E; E).$$

We can also estimate for the functions h:

$$h(t,z)(\tau) \leq \frac{\pi}{15\sqrt{1+t^2}}|z(t,\tau)|.$$

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Hence conditions $(H_3)(b)$ is satisfied with

$$\beta(t) = \frac{\pi}{15\sqrt{1+t^2}}, \quad \phi(R) = R.$$

Then it is easy to verify that $\beta \in C_0(\mathbb{R}^+, \mathbb{R})$, $\rho_2 = 1$ and $\rho \leq \frac{\pi}{15}$.
Furthermore:

$$f(t; z) = g(t; z) + h(t; z) \in AA(\mathbb{R}^+ \times E; E).$$

We can also estimate for the functions f:

$$f(t, z)(\tau) \leq \frac{2\sin^2 t}{\sqrt{1+t^2}} |z(t, \tau)|. \tag{22}$$

By (22), for every $t \in J$, and $B \in D \subset E$, we have

$$\alpha(f(t, D) \leq \frac{\sin^2 t}{12\sqrt{1+t^2}} \alpha(D),$$

Hence conditions $(H4)$ is satisfied with

$$\eta(t) = \frac{1}{6\sqrt{1+t^2}}, \quad \varphi(t) = \frac{\sin^2 t}{2}.$$

Moreover, we have

$$(\psi + \varphi)(t) = \frac{\sin^2 t}{2} + \frac{1}{4}(t + \ln(1+t)) \leq t.$$

We conclude that (see Lemma 2.1. [6])

$$\lim_{n \to +\infty} (\psi + \phi)^n(t) = 0 \text{ for a.e } t \in \mathbb{R}^+.$$

Consequently, can be written in the abstract form (1)-(2) with $A(t)$ and f as defined above. Thus, Theorem 3.3 yields that equation (20) has a asymptotically almost automorphic mild solution.

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Applying hybrid coupled fixed point theory to the nonlinear hybrid system of second order differential equations

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Abstract

In this work, we apply hybrid fixed point theory method to prove the existence of solution of systems of second order ordinary nonlinear hybrid differential equations with periodic boundaries.

Key words and phrases. Second order differential equation; hybrid systems; boundary value problems; coupled fixed point.

AMS Mathematics subject Classification. 34A12, 34A38.

1 Introduction

Let $X \neq \phi$, and $F : X \times X \to X$ is a mapping. A point $(x, y) \in X \times X$ is said to a coupled fixed point of F in $X \times X$ if $F(x, y) = x$ and $F(y, x) = y$. The notation of coupled fixed point was introduced in 2006 by Baskar and Lshmikantham [1]. It will known that the existence of the fixed point play an important role for showing the existence of solutions of nonlinear integral [2, 3] , differential equations [4, 5] and iterative process [6].

In 1964, Krasnoselskii [7] initiated the idea of study the hybrid fixed point theory for the function which can be written as the sum of two other functions.In 2013, Dhage [2] obtained hybrid fixed point theorems for the operator which can be written as the sum of two other operators using Krasnoselskii fixed point theorem techniques and developed a Krasnoselskii fixed point technique helpful to analyze the existence of solution of nonlinear Volterra fractional integral equations under some conditions.

Recently, in 2015, Dhage and Dhage [8] proved the existence of solutions of the boundary value problems of second order ordinary nonlinear differential equations using the hybrid fixed point theorem which was obtained by themselves [9] .

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More recently, in 2017, Yang et al. [10] introduced the notation of hybrid coupled fixed point theorems and applied this ideas to prove the existence of system of fractional differential equations of order $\alpha, 0 < \alpha < 1$.

In this paper, we apply the hybrid fixed point theorem to study and prove the existence of solution of a certain system of boundary -value problems with periodic boundaries(for short BVPPB) of second order ordinary nonlinear hybrid differential equations.

To this end, the remainder of the article is organized as follows. Section 2, is given some preliminaries and basic definitions. Section3, is established the existence results of coupled nonlinear system of second order differential equations.

2 Preliminaries

Throughout this paper, let E be a nonempty set and $(E, \leq, \|.\|)$ be a partially ordered normed linear space. If $Q : E \to E$ is a mapping. Then Q is said to be monotone nondecreasing if $a \leq b$ the $Q(a) \leq Q(b)$, for all $a, b \in E$. Two elements $a, b \in E$ are said to be comparable if $a \leq b$ or $b \leq a$. If C is nonempty subset of E,C is said to be chain if each two elements $a, b \in C$ are comparable.

Definition 1 *[10] Let $Q : E \to E$ be a mapping. Q is called partially compact if for each chain C subset of E, $Q(C)$ is reltively compact subset of E.*

Definition 2 *[2, 10] Let $Q : E \to E$ be a mapping. Given an element $a \in E$. Define orbit $\Gamma(a; Q)$ as:*

$$\Gamma(a; Q) = \{a, Qa, Q^2a, Q^3a, ..., Q^na, ...\}.$$

If for any sequence $\{a_n\} \in \Gamma(a; Q)$ such that: $a_n \to a^*$ as $n \to \infty$ then $Qa_n \to Qa^*$, for each $a \in E$, then Q is called $\Gamma-$ orbitally continuous in E. Furthermore, $(E, \leq, \|.\|)$ is said to be $\Gamma-$ orbitally complete if each sequence $\{a_n\} \in \Gamma(a; Q)$ converges to an element $a^* \in E$.

Definition 3 *[2, 10] A mapping $\phi : E \to E$ is said to be $D-$ function if it is upper semi continuous and monotone nondecreasing such that: $\phi(0) = 0$.*

Definition 4 *[2, 10] A mapping $\Upsilon : E \to E$ is called partially nonlinear $D-$ contraction in E , if for each comparable elements $a, b \in E$, there exist a $D-$ function $\phi : \Re^+ \to \Re^+$ such that the following conditions are satisfied:*

(i) $\|\Upsilon a - \Upsilon b\| \leq \phi(\|a - b\|)$, *and*

(ii) $\phi(r) < r$ *for all* $r > 0$.

Definition 5 *[10] Let $(E, \leq, \|.\|)$ be a $\Gamma-$ orbitally complete linear space. The positive cone $K \in E$ is defined as: $K = \{x \in E : x \geq 0\}$*

The following theorem will be used as tool to prove the main results.

Theorem 6 *[10] Let E is a partially ordered $\Gamma-$ orbitally complete normed linear space and K is the positive cone of E. Let K is normal and C be a nonempty closed subset of E. Consider $P, Q : C \to C$ are two monotone nondecreasing mappings such that the following are satisfied:* (1) *P is $\Gamma-$ orbitally continuous and a partially nonlinear $D-$ contraction,*

(2) *Q is $\Gamma-$ orbitally continuous and a partially compact,*

(3) *there exist an element $a \in C$ such that: $a \leq Pa + Qy$ for all $y \in C$ and*

(4) *every pair of elements in C has an upper and lower bounded.*

Then $F(x, y) = Px + Qy$ has a coupled fixed point in $E \times E$.

Now, we give some notations and definitions about the problem which we will study the existence of its solution.

Definition 7 *Consider $I = [0, b] \in \Re$ where $b > 0$. The periodic boundary value problem of second-order nonlinear differential equation can be written as:*

$$\frac{d^2x(t)}{dt^2} = f(t, x(t) + h(t, x(t)), \qquad x(0) = x(b), x'(0) = x'(b) \tag{1}$$

for all $t \in I$, where $f, h : I \times \Re \to \Re$ are continuous function. The solution of the differential equation (1) is the function $x \in C^2(I, \Re)$ that satisfies equation (1), where by $x \in C^2(I, \Re)$ we mean the space of twice continuously differentiable real-valued functions on I.

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Definition 8 *the space $C(I,\Re)$ is the space of all continuous real-value function defined on I. It is easy to prove that: $C(I,\Re)$ is a Banach space with the norm:*

$$\|x\| = \sup_{t\in I} |x(t)|. \tag{2}$$

Therefore, it is also clear that $C(I,\Re)$ is partially ordered with respect to the partially order relation:

$$x \leq y \ \text{ if and only if } \quad x(t) \leq y(t) \ \text{ for all } \ t \in I. \tag{3}$$

Also $C(I,\Re)$ is a partially ordered $\Gamma-$ orbitally complete linear space with normal cone $K_C = \{x \in C(I,\Re) : x(t) \geq 0, t \in I\}$ [10].

Lemma 9 *[11] Consider $\sigma, g \in L^1(I,\Re)$, then x is a solution of the differential equation:*

$$x''(t) + \sigma(t)x(t) = g(t), \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

if and only if x is a solution of the integral equation:

$$x(t) = \int_0^b G_\sigma(t,s)g(s)ds,$$

where $G_\sigma(t,s)$ is a Green's function associated with the differential equation:

$$x''(t) + \sigma(t)x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

Remark 10 *The Green function G_σ is continuous an nonnegative on $I \times I$ and there exist the number :*

$$M_\sigma = \max\{|G_\sigma(t,s)| : t,s \in [0,b]\}$$

for all $\sigma \in L^1(I,\Re^+)$.

Lemma 11 *[8] Consider the differential equations (1). Let $F(t,x) = f(t,x) + \lambda x$, such that:*

$(P1)$ The function $f : I \times \Re \to \Re$ is continuous and there exist a constant $k_1 > 0$ such that$|\ F(t,x)| \leq k_1$ for all $t \in I$,

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$(P2)$ The function $h : I \times \Re \to \Re$ is continuous and exist a constant $k_2 > 0$ such that$|\ h(t,x)| \leq k_2$ for all $t \in I$.

Then a function $v \in C(I, \Re)$ is a solution of the differential equation:

$$\frac{d^2x(t)}{dt^2} + \lambda x(t) = F(t, x(t) + h(t, x(t)), \qquad x(0) = x(b), x'(0) = x'(b) \tag{4}$$

if and only if v is the solution of the integral equation:

$$x(t) = \int_0^b G(t,s)F(s, x(s))ds + \int_0^b G(t,s)h(s, x(s))ds, \tag{5}$$

for all $t \in I$, where $G(t,s)$ is a Green's function associated with the differential equation:

$$x''(t) + \lambda x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b)\ \ ,\ \ x'(0) = x'(b),$$

3 Main results

Now , consider the system of nonlinear differential equations:

$$\begin{aligned}
&\tfrac{d^2x(t)}{dt^2} = f(t, x(t) + h(t, y(t)), \quad t \in I,\\
&\tfrac{d^2y(t)}{dt^2} = f(t, y(t) + h(t, x(t)), \quad t \in I,\\
&x(0) = x(b) \quad , \quad x'(0) = x'(b),\\
&y(0) = y(b) \quad , \quad y'(0) = y'(b).
\end{aligned} \tag{6}$$

Then the hybrid system (6) can be written as:

$$\begin{aligned}
&\tfrac{d^2x(t)}{dt^2} + \lambda x(t) = F(t, x(t) + h(t, y(t)), \quad t \in I,\\
&\tfrac{d^2y(t)}{dt^2} + \lambda y(t) = F(t, y(t) + h(t, x(t)), \quad t \in I,\\
&x(0) = x(b) \quad , \quad x'(0) = x'(b),\\
&y(0) = y(b) \quad , \quad y'(0) = y'(b),
\end{aligned} \tag{7}$$

where $F(t, x(t)) = f(t, x(t)) + \lambda x(t)$. By applying Lemma 11, we have the following Lemma .

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Lemma 12 *Consider the hybrid system of differential equations (7), such that the conditions (P1) and (P2) hold.*

Then a function $(v_1, v_2) \in C(I, \Re) \times C(I, \Re)$ *is a solution of the differential equation the system (7) :*

if and only if (v_1, v_2) *is the solution of the system of nonlinear integral equation:*

$$\begin{aligned} x(t) &= \int_0^b G(t,s)F(s,x(s))ds + \int_0^b G(t,s)h(s,y(s))ds, \\ y(t) &= \int_0^b G(t,s)F(s,y(s))ds + \int_0^b G(t,s)h(s,x(s))ds \end{aligned} \tag{8}$$

for all $t \in I$*, where* $G(t,s)$ *is a Green's function associated with the differential equation:*

$$x''(t) + \lambda x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

By using the continuity of the integrals, we can define the two mappings: $T_1 : C(I, \Re) \to C(I, \Re)$ and $T_2 : C(I, \Re) \to C(I, \Re)$ as:

$$T_1(x(t)) = \int_0^b G(t,s)F(s,x(s))ds,$$

and

$$T_2(x(t)) = \int_0^b G(t,s)h(s,x(s))ds.$$

Then we can define the following :

$$T(x(t), y(t)) = T_1 x(t) + T_2 y(t)$$

The coupled fixed point of the operator T is the solution of the system (7).

With the two conditions (P1) and (P2) consider the following set of assumptions:

$(P3)$ There exists $\lambda > 0$ and $\varepsilon > 0$ such that:

$$0 \leq [f(t,x) + \lambda x] - [f(t,y) + \lambda y] \leq \varepsilon(x - y),$$

for all $t \in I$ and $x, y \in \Re$ such that: $x \geq y$.

$(P4)$ The function $h(t,x)$ nondecreasing.

$(P5)$ There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) \leq f(t, v(t)) + h(t, y(t)),$$

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$$v(0) \leq v(b) \quad , \quad v'(0) \leq v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$.

Lemma 13 *Assume that (P1) and (P3) holds. Let $\varepsilon Mb < 1$. Then the operator T_1 is $\Gamma-$ orbitally continuous, nondecreasing and a partially nonlinear $D-$ contraction.*

Proof. Let $E = C(I, \Re)$. The proof will be done in 3 steps.

Step 1: To prove that T_1 is nondecreasing.

Let $x, y \in E$, such that: $x \geq y$ Then by (P3), we have that:

$$T_1x(t) = \int_0^b G(t,s)F(s,x(s))ds \geq \int_0^b G(t,s))F(s,y(s))ds \geq T_1y(t),$$

for all $t \in I$. Thus T_1 is nondecreasing operator.

Step 2: To prove that T_1 is $\Gamma-$ orbitally continuous.

Take a sequence $\{x_n\} \in \Gamma(x; T_1)$ for any $x \in E$ with $x_n \to x^*$ as $n \to \infty$. Since T_1 is continues by (P1) again, then we have that:

$$\begin{aligned} \lim_{n\to\infty}(T_1x_n)(t) &= \textstyle\int_0^b G(t,s)\lim_{n\to\infty} F(s,x_n(s))ds \\ &= \textstyle\int_0^b G(t,s)F(s,x^*(s))ds \\ &= (T_1x^*)(t), \end{aligned} \tag{9}$$

for all $t > 0$. Hence , T_1 is $\Gamma-$ orbitally continuous.

Step 3: To prove that T_1 is partially nonlinear $D-$ contraction.

For any comparable element $x, y \in E$ such that:$x \geq y$. For $t \in I$, we have that:

$$\begin{aligned} \mid (T_1x)(t) - (T_1y)(t) \mid &= \mid \textstyle\int_0^b G(t,s)[F(s,x(s)) - F(s,y(s))]ds \mid \\ &\leq \textstyle\int_0^b G(t,s) \mid [F(s,x(s)) - F(s,y(s))] \mid ds \\ &\leq M\varepsilon b\|x-y\|, \end{aligned} \tag{10}$$

Let $\tau = M\varepsilon b$. New define the mapping $\Theta(l) = \tau l$, for each $l \in \Re^+$. Thus $\Theta(l) < l$ and $\Theta(0) = 0$. Therefore, we proved that:$\|T_1x - T_1y\| \leq \Theta(\|x-y\|)$. Hence T_1 is partially nonlinear $D-$ contraction.

7

Lemma 14 *Assume that (P2) and (P4) holds. Then the operator T_2 is $\Gamma-$ orbitally continuous, nondecreasing and a partially compact in $E = C(I, \Re$.*

Proof. The proof is also done in 3 steps.

Step 1: To prove that: T_2 is $\Gamma-$ orbitally continuous.

Consider for any $x \in E$ with $x_n \to x^*$ as $n \to \infty$. Since T_2 is continues by (P2), then we have that:

$$\begin{aligned} \lim_{n\to\infty}(T_2x_n)(t) &= \int_0^b G(t,s)\lim_{n\to\infty} h(s,x_n(s))ds \\ &= \int_0^b G(t,s)h(s,x^*(s))ds \\ &= (T_2x^*)(t), \end{aligned} \tag{11}$$

for all $t > 0$. Hence , T_2 is $\Gamma-$ orbitally continuous.

Step 2: To prove that: T_2 is nondecreasing.

Let $x, y \in E$, such that: $x \geq y$ Then by (P2), we have that:

$$T_2x(t) = \int_0^b G(t,s)h(s,x(s))ds \geq \int_0^b G(t,s))h(s,y(s))ds \geq T_2y(t),$$

for all $t \in I$. Thus T_2 is nondecreasing operator.

Step 3: To prove that T_2 is partially compact in $E = C(I, \Re)$. Let C be an arbitrary chain in E. We show $T_2(C)$ is uniformly bounded and equicontinuous set in E. Consider $x \in C$ be arbitrary, then we get that:

$$\begin{aligned} \mid T_2x(t) \mid &= \mid \int_0^b G(t,s)h(s,x(s))ds \mid \\ &\leq \int_0^b G(t,s) \mid h(s,x(s)) \mid ds \\ &\leq Mk_2b, \end{aligned} \tag{12}$$

Thus , we have that: $\|T_2x\| \leq \tau$, where $\tau = Mk_2b$. Hence, $T_2(C)$ is uniformly bounded subset in E.

To prove $T_2(C)$ is an equicontinuous : let $t_1, t_2 \in I$ such that: $t_1 - t_2 < 0$. We have that:

$$\begin{aligned} \mid T_2x(t_2) - T_2x(t_1) \mid &= \mid \int_0^b [G(t_1,s) - G(t_2,s)]h(s,x(s))ds \mid \\ &\leq \int_0^b \mid G(t_1,s) - G(t_2,s) \mid h(s,x(s))ds \\ &\leq \int_0^b \mid G(t_1,s) - G(t_2,s) \mid k_2 ds \to 0, \end{aligned} \tag{13}$$

as $t_1 \to t_2$. Hence $T_2(C)$ is a compact subset of E . Therefore, T_2 is partially compact in $E = C(I, \Re)$.

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Theorem 15 *Let the hypotheses (P1), (P2), (P3), (P4) , (P5) and* $\varepsilon Mb < 1$ *hold. Then the hybrid system (6) has a coupled solution on* I.

Proof. By (P5) There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) \leq f(t, v(t) + h(t, y(t)),$$

$$v(0) \leq v(b) \quad , \quad v'(0) \leq v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$. Then , we get that: There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) + \lambda v(t) \leq F(t, v(t)) + h(t, y(t)),$$

$$v(0) \leq v(b) \quad , \quad v'(0) \leq v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$. Applying Theorem 6, we get that : the mapping $T(x, y) = T_1x + T_2y$ has a coupled fixed point. This coupled fixed point is the solution of the hybrid system (6).

Corollary 16 *Let the hypotheses (P1), (P2), (P3), (P4) and* $\varepsilon Mb < 1$ *hold. Consider there exists an element* $v \in C^2(I, \Re)$ *such that:*

$$v''(t) \geq f(t, v(t) + h(t, y(t)),$$

$$v(0) \geq v(b) \quad , \quad v'(0) \geq v'(b),$$

for all $t \in I$ *and* $y \in C(I, \Re)$.

Then the hybrid system (6) has a coupled solution on I.

Example 17 *Let* $I = [0, 1]$. *Suppose we have the following system of differential equations:*

$$\begin{aligned} &x''(t) = tan^{-1}x(t) - x(t) + h(t, y(t)), \quad t \in I, \\ &y''(t) = tan^{-1}y(t) - y(t) + h(t, x(t)), \quad t \in I, \\ &x(0) = x(1) \quad , \quad x'(0) = x'(1), \\ &y(0) = y(1) \quad , \quad y'(0) = y'(1), \end{aligned} \tag{14}$$

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and $t \in [0,1]$. *Consider* $g : I \times \Re \to \Re$ *such that:*

$$g(t,x) == \begin{cases} 1, & \text{if } x \le 1 \\ \frac{2x}{1+x}, & \text{if } x > 1. \end{cases} \tag{15}$$

It is clear that $f(t,x) = \tan^{-1} x(t) - x(t)$ *and* f *and* g *are continues functions.Define* $F(t,x) = \tan^{1} x(t) - x(t)$. *Then we get that:* $|F(t,x)| < \frac{1}{\pi} = k_1$. *Therefore ,* $f(t,x)$ *is satisfied condition (P1) . Also, since we have that:*

$$0 \le \tan^{1} x - \tan^{1} y \le \frac{1}{1+\kappa^2}(x-y) \quad \forall x, y \in \Re, \ \ x > y, \ \ x > \kappa > y.$$

Thus f *is satisfied condition (P3) with* $\lambda = 1$. *It is also clear that:* $\lambda > \varepsilon = \frac{1}{1+\kappa^2}$ *, where* $x > \kappa > y$. *Also,* $g(t,x)$ *is nondecreasing in* x *for all* $t \in I$ *and bounded . Thus* g *is satisfied the conditions (P2) and (P4). Finally , the function :*

$$v(t) = -2\int_0^1 G(t,s)ds + \int_0^1 G(t,s))ds,$$

is satisfied the condition (P5). Then the hybrid system has (14) a coupled solution.

4 Declaration of conflicting interests

The author declared no potential conflicts on interest with respect to the research, authorship, and/or publication of this article.

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CO-ORDINATED CONVEX FUNCTIONS OF THREE VARIABLES AND SOME ANALOGOUS INEQUALITIES WITH APPLICATIONS

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Abstract: Co-ordinated convex function of two variables and corresponding inequalities have been studied before by many researchers. This paper deals with Co-ordinated convex function of three variables. In the present paper the idea of co-ordinated convex function of two variables in a rectangle from the plane R^2 is extended to that of three variables in a rectangle from space R^3. Moreover corresponding extended right handed Hermite-Hadamard type inequalities are also incorporated. At the end some applications of resulting inequalities to special means are also given.

Key Words: Hermite-Hadamard's inequality, co-ordinated convex function, Hölder's integral inequality, power mean inequality, arithmetic mean, logarithmic mean.

1. Introduction

It is said that the notion *convex* was pioneered by Archimedes. While estimating value of π, he noticed that the perimeter of a convex figure is smaller than that of any other convex figure, encompassing it. As per J. L. Jensen, idea of convex function is as primeval as an increasing function or a positive function. A documented result spontaneously identified with convex function is Hermite-Hadamard (HH) inequality

$$g\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(u)du \leq \frac{g(\alpha)+g(\beta)}{2}, \tag{1.1}$$

where g is a real valued convex function on the real interval I and $\alpha, \beta \in I$ with $\alpha < \beta$. The idea of co-ordinated convex (CC) function on a rectangle from the plane was coined by Dragomir and Pearce in 2000, see [6]. Co-ordinated convexity is more general than convexity. It only requires a function to be convex on each

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coordinate individually and independently, not simultaneously as in case of convexity. Thereafter Dragomir and many other mathematicians worked in this field, see [1, 5, 8, 9, 11, 13, 14, 16–18, 20–22]. The idea of co-ordinated convexity was joined with other classes of convex functions like s-convex, m-convex, (α, m)-convex, h-convex, log-convex, $((s, m), QC)$-convex functions and many interesting results were obtained, see [2–4, 10, 12, 15, 19] and references there in.

Dragomir and Pearce [6] defined CC function of two variables as follows. The principal outcomes of the present article are actuated by this definition.

Definition 1.1. *A function $g : \triangle \longrightarrow R$ is called CC on $\triangle$ if the partial functions $g_u : [\gamma, \delta] \longrightarrow R$ and $g_v : [\alpha, \beta] \longrightarrow R$, defined as $g_u(y) = g(u, y)$ and $g_v(x) = g(x, v)$ respectively, are convex. Here $\triangle$ is bi-dimensional interval given by $\triangle = [\alpha, \beta] \times [\gamma, \delta]$.*

2. Main Results

In this section we mainly establish results on the basis of definition of CC function of three variables on a rectangle from the space R^3. We derive HH type inequality for CC functions of three variables and then establish inequalities related to the algebraic combination of middle term and right side terms of this inequality, to be called right HH type inequalities. Let us first define CC function of three variables. Motivated by Definition 1.1, CC function of three variables is defined as:

Definition 2.1. *A function $g : \Delta \longrightarrow R$ is called CC on Δ if the partial functions $g_{u,v} : [\zeta, \eta] \longrightarrow R$, $g_{u,w} : [\gamma, \delta] \longrightarrow R$ and $g_{v,w} : [\alpha, \beta] \longrightarrow R$, defined as $g_{u,v}(z) = g(u, v, z), g_{u,w}(y) = g(u, y, w)$ and $g_{v,w}(x) = g(x, v, w)$ respectively, are convex. Here and on wards Δ is tri-dimensional interval given by $\Delta = [\alpha, \beta] \times [\gamma, \delta] \times [\zeta, \eta]$.*

Lemma 2.2. *If a function $g : \Delta \longrightarrow R$ is CC function on Δ, then the subsequent inequality is induced*

$$\begin{aligned}
&g\left(lc + (1-l)d, me + (1-m)h, nr + (1-n)s\right) \\
&\leq lmng(c, e, r) + lm(1-n)g(c, e, s) + l(1-m)ng(c, h, r) \\
&\quad + l(1-m)(1-n)g(c, h, s) + (1-l)mng(d, e, r) + (1-l)m(1-n)g(d, e, s) \\
&\quad + (1-l)(1-m)ng(d, h, r) + (1-l)(1-m)(1-n)g(d, h, s),
\end{aligned}$$

for all $l, m, n \in [0, 1]$ and $(c, e, r), (c, e, s), (c, h, r)$ e.t.c. are in Δ.

Proof. The Ineq. (**??**) can be proved simply by applying Definition 2.1, to the given function g. □

Lemma 2.3. *Every convex function is CC. The converse in general is not true.*

Proof. It can be proved by using Definition 2.1 of CC function. Moreover the function $g : [0, 1]^3 \longrightarrow [0, \infty)$ given by $g(u, v, w) = uvw$, for all $(u, v, w) \in [0, 1]^3$ is CC function but is not convex on $[0, 1]^3$. □

Motivated by Dragomir [7], we extend HH Ineq. (1.1) for CC functions of three variables as follows:

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Theorem 2.4. *Suppose that $g : \Delta \subset R^3 \longrightarrow R$ is CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
& g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right) \\
& \leq \frac{1}{3}\left[\frac{1}{\beta-\alpha} \int_\alpha^\beta g\left(u, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right) du+\frac{1}{\delta-\gamma} \int_\gamma^\delta g\left(\frac{\alpha+\beta}{2}, v, \frac{\zeta+\eta}{2}\right) dv\right. \\
& \left.\quad+\frac{1}{\eta-\zeta} \int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, w\right) dw\right] \qquad (2.1)\\
& \leq \frac{1}{3}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)} \int_\gamma^\delta \int_\alpha^\beta g\left(u, v, \frac{\zeta+\eta}{2}\right) dudv+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\right. \\
& \left.\quad \int_\gamma^\delta \int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, v, w\right) dvdw+\frac{1}{(\beta-\alpha)(\eta-\zeta)} \int_\alpha^\beta \int_\zeta^\eta g\left(u, \frac{\gamma+\delta}{2}, w\right) dudw\right] \qquad (2.2)\\
& \leq \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)} \int_\alpha^\beta \int_\gamma^\delta \int_\zeta^\eta g(u, v, w) dudvdw \qquad (2.3)\\
& \leq \frac{1}{6}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)} \int_\alpha^\beta \int_\gamma^\delta\{g(u, v, \zeta)+g(u, v, \eta)\} dudv\right. \\
& \quad+\frac{1}{(\delta-\gamma)(\eta-\zeta)} \int_\gamma^\delta \int_\zeta^\eta\{g(\alpha, v, w)+g(\beta, v, w)\} dvdw \\
& \left.\quad+\frac{1}{(\beta-\alpha)(\eta-\zeta)} \int_\alpha^\beta \int_\zeta^\eta\{g(u, \gamma, w)+g(u, \delta, w)\} dudw\right] \qquad (2.4)\\
& \leq \frac{1}{12}\left[\frac{1}{\beta-\alpha} \int_\alpha^\beta\{g(u, \gamma, \zeta)+g(u, \gamma, \eta)+g(u, \delta, \zeta)+g(u, \delta, \eta)\} du\right. \\
& \quad+\frac{1}{\delta-\gamma} \int_\gamma^\delta\{g(\alpha, v, \zeta)+g(\alpha, v, \eta)+g(\beta, v, \zeta)+g(\beta, v, \eta)\} dv \\
& \left.\quad+\frac{1}{\eta-\zeta} \int_\zeta^\eta\{g(\alpha, \gamma, w)+g(\alpha, \delta, w)+g(\beta, \gamma, w)+g(\beta, \delta, w)\} dw\right] \qquad (2.5)\\
& \leq \frac{1}{8}[g(\alpha, \gamma, \zeta)+g(\alpha, \gamma, \eta)+g(\alpha, \delta, \zeta)+g(\alpha, \delta, \eta)+g(\beta, \gamma, \zeta)+g(\beta, \gamma, \eta) \\
& \quad+g(\beta, \delta, \zeta)+g(\beta, \delta, \eta)] . \qquad (2.6)
\end{aligned}
$$

Proof. Since $g : \Delta \longrightarrow R$ is CC function, it follows that the function $g_{u,v} : [\zeta, \eta] \longrightarrow R$, defined by $g_{u,v}(z) = g(u, v, z)$ is convex on $[\zeta, \eta]$. Then from Inequality (1.1), we have

$$
\begin{aligned}
& g_{\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}}\left(\frac{\zeta+\eta}{2}\right) \leq \frac{1}{\eta-\zeta} \int_\zeta^\eta g_{\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}}(w) dw, \quad \text{where } w \in[\zeta, \eta] \\
& g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right) \leq \frac{1}{\eta-\zeta} \int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, w\right) dw .
\end{aligned}
$$

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Similar arguments applied to the functions $g_{u,w}$ and $g_{v,w}$ respectively, we get

$$g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\le\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv$$

and

$$g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\le\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g\left(u,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du,$$

adding above three inequalities, we get Ineq. (2.1) as follows

$$\begin{aligned}&g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\\&\le\frac{1}{3}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g\left(u,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\right.\\&\quad\left.+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},w\right)dw\right].\end{aligned}$$

Now consider the following inequality

$$\begin{aligned}&g_{\frac{\alpha+\beta}{2},v}\left(\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{\frac{\alpha+\beta}{2},v}(w)dw,\qquad\text{where } w\in[\zeta,\eta],\\&g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},v,z\right)dw,\\&\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\le\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},v,w\right)dvdw,\end{aligned}$$

adding all such inequalities, we get Ineq. (2.2) as follows

$$\begin{aligned}&\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f\left(x,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}f\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\\&+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}f\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},w\right)dw\le\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}f\left(u,v,\frac{\zeta+\eta}{2}\right)dudv\\&+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}f\left(\frac{\alpha+\beta}{2},v,w\right)dvdw\\&+\frac{1}{(\eta-\zeta)(\beta-\alpha)}\int_{\alpha}^{\beta}\int_{\zeta}^{\eta}f\left(u,\frac{\gamma+\delta}{2},w\right)dwdu.\end{aligned}$$

Considering

$$g_{u,v}\left(\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{u,v}(w)dw\le\frac{g_{u,v}(m)+g_{u,v}(n)}{2},\ \text{ where } w\in[\zeta,\eta],$$

gives

$$g\left(u,v,\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g(u,v,w)dw\le\frac{g(u,v,\zeta)+g(u,v,\eta)}{2},\tag{2.7}$$

integrating Ineq. (2.7) w.r.t u and v over the intervals $[\alpha, \beta]$ and $[\gamma, \delta]$ respectively, then multiplying by $\frac{1}{(\beta-\alpha)(\delta-\gamma)}$, we have

$$\begin{aligned}
&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g\left(u, v, \frac{\zeta+\eta}{2}\right) dudv \\
&\leq \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta} g(u, v, w)dudvdw \\
&\leq \frac{1}{2(\beta-\alpha)(\delta-\gamma)}\left[\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g(u, v, \zeta)dudv + \int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g(u, v, \eta)dudv\right],
\end{aligned}$$

by the same argument applied on the functions $g_{v,w}$ and $g_{u,w}$, we tend to get two additional such inequalities. Then by adding these inequalities, we have Ineqs. (2.3) and (2.4) as follows

$$\begin{aligned}
&\frac{1}{3}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g\left(u, v, \frac{\zeta+\eta}{2}\right) dudv + \frac{1}{(\delta-\gamma)(\eta-\zeta)}\right. \\
&\left.\int_{\gamma}^{\delta}\int_{\zeta}^{\eta} g\left(\frac{\alpha+\beta}{2}, v, w\right) dvdw + \frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_{\zeta}^{\eta}\int_{\alpha}^{\beta} g\left(u, \frac{\gamma+\delta}{2}, w\right) dudw\right] \\
&\leq \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta} g(u, v, w)dudvdw \\
&\leq \frac{1}{6}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g(u, v, \zeta)dudv + \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g(u, v, \eta)dudv\right. \\
&\quad + \frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta} g(\alpha, v, w)dvdw + \frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta} g(\beta, v, w)dvdw \\
&\quad \left. + \frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_{\eta}^{\zeta}\int_{\alpha}^{\beta} g(u, v, \zeta)dudw + \frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_{\eta}^{\zeta}\int_{\alpha}^{\beta} g(u, v, \eta)dudw\right].
\end{aligned}$$

By convexity of $g_{\alpha,v}$, we have

$$\begin{aligned}
\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} g_{\alpha,v}(w)dw &\leq \frac{g_{\alpha,v}(\zeta)+g_{\alpha,v}(\eta)}{2}, \quad \text{where } w \in [\zeta, \eta] \\
\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} g(\alpha, v, w)dw &\leq \frac{g(\alpha, v, \zeta)+g(\alpha, v, \eta)}{2},
\end{aligned}$$

integrating this inequality w.r.t. v over the interval $[\gamma, \delta]$, we have

$$\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta} g(\alpha, v, w)dvdw \leq \frac{1}{2(\delta-\gamma)}\int_{\gamma}^{\delta}\{g(\alpha, v, \zeta)+g(\alpha, v, \eta)\}dv,$$

similarly by convexity of $g_{\beta,v}$ we have

$$\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta} g(\beta, v, w)dvdw \leq \frac{1}{2(\delta-\gamma)}\int_{\gamma}^{\delta}\{g(\beta, v, \zeta)+g(\beta, v, \eta)\}dv,$$

after further calculation we have six such inequalities, adding these we get Ineq. (2.5) as follows

$$\begin{aligned}
&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\{g(u,v,\zeta)+g(u,v,\eta)\}\,dudv\\
&+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta\{g(\alpha,v,w)+g(\beta,v,w)\}\,dvdw\\
&+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\alpha^\beta\int_\zeta^\eta\{g(u,v,\zeta)+g(u,v,\eta)\}\,dudw\\
&\le\frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}\,du\right.\\
&+\frac{1}{\delta-\gamma}\int_\gamma^\delta\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv\\
&\left.+\frac{1}{\eta-\zeta}\int_\zeta^\eta\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}\,dw\right].
\end{aligned}$$

Now to get last inequality we use the convexity of $g_{\alpha,\gamma}$, which implies

$$\begin{aligned}
&\frac{1}{\eta-\zeta}\int_\zeta^\eta g_{\alpha,\gamma}(w)dw\le\frac{g_{\alpha,\gamma}(\zeta)+g_{\alpha,\gamma}(\eta)}{2},\ \text{where } w\in[\zeta,\eta],\\
&\frac{1}{\eta-\zeta}\int_\zeta^\eta g(\alpha,\gamma,w)dw\le\frac{g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)}{2},
\end{aligned}$$

adding such inequalities, we get Ineq. (2.6) as follows

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\left[\int_\alpha^\beta g(u,\gamma,\zeta)du+\int_\alpha^\beta g(u,\gamma,\eta)du+\int_\alpha^\beta g(u,\delta,\zeta)du+\int_\alpha^\beta g(u,\delta,\eta)du\right]\\
&+\frac{1}{\delta-\gamma}\left[\int_\gamma^\delta g(\alpha,v,\zeta)dv+\int_\gamma^\delta g(\alpha,v,\eta)dv+\int_\gamma^\delta g(\beta,v,\zeta)dv+\int_\gamma^\delta g(\beta,v,\eta)dv\right]\\
&+\frac{1}{\eta-\zeta}\left[\int_\zeta^\eta g(\alpha,\gamma,w)dw+\int_\zeta^\eta g(\alpha,\delta,w)dw+\int_\zeta^\eta g(\beta,\gamma,w)dw+\int_\zeta^\eta g(\beta,\delta,w)dw\right]\\
&\le\frac{3}{2}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)\right.\\
&\left.+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)\right].
\end{aligned}$$

Hence proved. □

Motivated by notion given in [22], we now present right handed HH type inequalities related to inequality given in Theorem 2.4, for differentiable CC functions on rectangle from the space R^3. In order to establish further results we need the following lemma.

Note: From here onwards we use '$\mathcal{A}$' to represent the following algebraic combination of middle and the right sided terms of HH type inequality given in Theorem

2.4.

$$\frac{1}{8}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)+g(\beta,\delta,\zeta)\right.$$
$$\left.+g(\beta,\delta,\eta)\right]-\frac{1}{4}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}\,du\right.$$
$$+\frac{1}{\delta-\gamma}\int_\gamma^\delta\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv$$
$$\left.+\frac{1}{\eta-\zeta}\int_\zeta^\eta\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}\,dw\right]+\frac{1}{2}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\right.$$
$$\int_\alpha^\beta\int_\gamma^\delta\{g(u,v,\zeta)+g(u,v,\eta)\}dudv+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta\{g(\alpha,v,w)+g(\beta,v,w)\}dvdw$$
$$\left.+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\alpha^\beta\int_\zeta^\eta\{g(u,\gamma,w)+g(u,\delta,w)\}\,dudw\right]$$
$$-\frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_\alpha^\beta\int_\gamma^\delta\int_\zeta^\eta g(u,v,w)dudvdw.$$

Lemma 2.5. *Let $g:\Delta\subset R^3\longrightarrow R$ be partial differentiable function on Δ. If $g_{lmn}\in L(\Delta)$, then the subsequent identity is induced*

$$\mathcal{A}=\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1(1-2l)(1-2m)(1-2n)$$
$$g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn.$$

Proof. Considering the following triple integral and integrating it by parts w.r.t. l,m and n respectively, we get

$$\int_0^1\int_0^1\int_0^1(1-2l)(1-2m)(1-2n)$$
$$g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\,dldmdn$$
$$=\int_0^1\int_0^1(1-2m)(1-2n)\left[\frac{(1-2l)}{\alpha-\beta}g_{mn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\Big|_0^1\right.$$
$$\left.-\int_0^1\frac{(-2)}{\alpha-\beta}g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dl\right]dmdn$$
$$=\int_0^1\int_0^1\frac{(1-2m)(1-2n)}{\beta-\alpha}[g_{mn}(\alpha,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)+g_{mn}(\beta,m\gamma+(1-m)\delta,$$
$$n\zeta+(1-n)\eta)-2\int_0^1g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dl]dmdn$$
$$=\frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\{[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)$$
$$+g(\beta,\gamma,\eta)+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)]-2\left[\int_0^1\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)\right.$$

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$$+g(u,\delta,\eta)\} \, du + \int_0^1 \{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\} \, dv$$
$$+ \int_0^1 \Big\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\Big\}dw\Big]$$
$$+4\Big[\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\zeta)+g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\eta)\}dldm$$
$$+\int_0^1\int_0^1\{g(\alpha,m\gamma+(1-m)\delta,\zeta n+(1-n)\eta)+g(\beta,m\gamma+(1-m)\delta,\zeta n+(1-n)\eta)\} \, dmdn$$
$$+\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,\gamma,\zeta n+(1-n)\eta)+g(l\alpha+(1-l)\beta,\delta,\zeta n+(1-n)\eta)\} \, dldn\Big]$$
$$-8\int_0^1\int_0^1\int_0^1 g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn\Big\}, \tag{2.8}$$

on multiplying both sides Eq. (2.8) by $\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}$, we get

$$\frac{1}{8}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)\right.$$
$$\left.+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)\right]-\frac{1}{4}\Big[\int_0^1\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\} \, du$$
$$+\int_0^1\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\} \, dv$$
$$+\int_0^1\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\} \, dw\Big]$$
$$+\frac{1}{2}\Big[\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\zeta)+g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\eta)\}dldm$$
$$+\int_0^1\int_0^1\{g(\alpha,m\gamma+(1-m)\delta,n\zeta+(1-\zeta n)\eta)+g(\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\}dmdn$$
$$+\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,\gamma,n\zeta+(1-n)\eta)+g(l\alpha+(1-l)\beta,\delta,n\zeta+(1-n)\eta)\} \, dldn\Big]$$
$$-\int_0^1\int_0^1\int_0^1 g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn$$
$$=\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1[(1-2l)(1-2m)(1-2n)$$
$$g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)]dldmdn. \tag{2.9}$$

By changing variable on left hand side of Eq. (2.9), we get the required result. □

Theorem 2.6. *Let $g:\Delta\subset R^3\longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|$ is a CC function on Δ, then the subsequent inequality is induced*

$$|\mathcal{A}|\le\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{512}\left[|g_{lmn}(\alpha,\gamma,\zeta)|+|g_{lmn}(\alpha,\gamma,\eta)+|g_{lmn}(\alpha,\delta,\zeta)\right.$$
$$\left.+|g_{lmn}(\alpha,\delta,\eta)|+|g_{lmn}(\beta,\gamma,\zeta)|+|g_{lmn}(\beta,\gamma,\eta)|+|g_{lmn}(\beta,\delta,\zeta)|+|g_{lmn}(\beta,\delta,\eta)|\right].$$

Proof. From Lemma 2.5, properties of modulus, co-ordinated convexity of $|g_{lmn}|$ and integrating by parts w.r.t. l, m and n respectively, we have

$$
\begin{aligned}
&|\mathcal{A}| \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |1-2l||1-2m||1-2n|\,[l|g_{lmn}(\alpha, m\gamma+(1-m)\delta, \\
&\quad \zeta n+(1-n)\eta)| + (1-l)|g_{lmn}(\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|]\,dldmdn \\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8\times 4}\int_0^1\int_0^1 |1-2m||1-2n|\,[|g_{mn}(\alpha, m\gamma+(1-m)\delta, \\
&\quad n\zeta+(1-n)\eta)| + |g_{mn}(\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|]\,dmdn \\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8\times 4\times 4}\int_0^1 |1-2n|\,[|g_n(\alpha,\gamma,\zeta n+(1-n)\eta)| \\
&\quad +|g_n(\alpha,\delta,n\zeta+(1-n)\eta)|+|g_n(\beta,\gamma,n\zeta+(1-n)\eta)||g_n(\beta,\delta,n\zeta+(1-n)\eta)|]dn \\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{512}\,[|g_{lmn}(\alpha,\gamma,\zeta)| + |g_{lmn}(\alpha,\gamma,\eta) + |g_{lmn}(\alpha,\delta,\zeta) \\
&\quad +|g_{lmn}(\alpha,\delta,\eta)| + |g_{lmn}(\beta,\gamma,\zeta)| + |g_{lmn}(\beta,\gamma,\eta)| + |g_{lmn}(\beta,\delta,\zeta)| + |g_{lmn}(\beta,\delta,\eta)|].
\end{aligned}
$$

Hence proved. □

Theorem 2.7. *Let $g : \Delta \subset R^3 \longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|^q$ with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, is a CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
|\mathcal{A}| \leq\; & \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8(p+1)^{\frac{3}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}} \\
& [|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q + |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q \\
& + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q + |g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2.5, properties of modulus, Hölder's integral inequality and Lemma 2.2, we have

$$
\begin{aligned}
&|\mathcal{A}| \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)| \\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|dldmdn \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|^p dldmdn\right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1\int_0^1\int_0^1 |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|^q dldmdn\right)^{\frac{1}{q}} \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |1-2l|^p\,|1-2m|^p\,|1-2n|^p dldmdn\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&(\int_0^1\int_0^1\int_0^1 (lmn|g_{lmn}(\alpha,\gamma,\zeta)|^q + lm(1-n)|g_{lmn}(\alpha,\gamma,\eta)|^q \\
&+ l(1-m)n|g_{lmn}(\alpha,\delta,\zeta)|^q + l(1-m)(1-n)|g_{lmn}(\alpha,\delta,\eta)|^q \\
&+ (1-l)mn|g_{lmn}(\beta,\gamma,\zeta)|^q + (1-l)m(1-n)|g_{lmn}(\beta,\gamma,\eta)|^q \\
&+(1-l)(1-m)n|g_{lmn}(\beta,\delta,\zeta)|^q + (1-l)(1-m)(1-n)|g_{lmn}(\beta,\delta,\eta)|^q)\, dldmdn)^{\frac{1}{q}} \\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{(p+1)^3}\right)^{\frac{1}{p}}\left(\frac{1}{8}\right)^{\frac{1}{q}}(|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q \\
&+ |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q \\
&+|g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q)^{\frac{1}{q}}\,.
\end{aligned}
$$

Hence proved. □

Theorem 2.8. *Let $g : \Delta \subset R^3 \longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|^q$, $q \geq 1$ is a CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
|\mathcal{A}| \leq\ & \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{64}\left(\frac{1}{8}\right)^{\frac{1}{q}} \\
& [|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q + |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q \\
& + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q + |g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2.5, properties of modulus, power mean inequality and Lemma 2.2, we have

$$
\begin{aligned}
&|\mathcal{A}| \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)| \\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|dldmdn \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|dldmdn\right)^{1-\frac{1}{q}} \\
&\quad (\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)| \\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|^q dldmdn)^{\frac{1}{q}} \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}(\int_0^1\int_0^1\int_0^1 |1-2l||1-2m||1-2n| \\
&\quad (lmn|g_{lmn}(\alpha,\gamma,\zeta)|^q + lm(1-n)|g_{lmn}(\alpha,\gamma,\eta)|^q + l(1-m)n|g_{lmn}(\alpha,\delta,\zeta)|^q \\
&\quad + l(1-m)(1-n)|g_{lmn}(\alpha,\delta,\eta)|^q + (1-l)mn|g_{lmn}(\beta,\gamma,\zeta)|^q + \\
&\quad (1-l)m(1-n)|g_{lmn}(\beta,\gamma,\eta)|^q + (1-l)(1-m)n|g_{lmn}(\beta,\delta,\zeta)|^q \\
&\quad + (1-l)(1-m)(1-n)|g_{lmn}(\beta,\delta,\eta)|^q)dldmdn)^{\frac{1}{q}}
\end{aligned}
$$

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$$= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\frac{1}{64}\right)^{\frac{1}{q}}$$
$$(|g_{lmn}(\alpha,\gamma,\zeta)|^q+|g_{lmn}(\alpha,\gamma,\eta)|^q+|g_{lmn}(\alpha,\delta,\zeta)|^q+|g_{lmn}(\alpha,\delta,\eta)|^q$$
$$+|g_{lmn}(\beta,\gamma,\zeta)|^q+|g_{lmn}(\beta,\gamma,\eta)|^q+|g_{lmn}(\beta,\delta,\zeta)|^q+|g_{lmn}(\beta,\delta,\eta)|^q)^{\frac{1}{q}},$$

after simplification we get the required result. □

Remark. *Since* $\frac{1}{8} < \frac{1}{(p+1)^{\frac{3}{p}}}$ *for* $p > 1$*, therefore estimation given in Theorem 2.8 with an improved and simplified constant is even better than that of Theorem 2.7.*

3. Some Applications to special Means

In Section 2, we established some inequalities based on CC functions of three variables. Now we apply these inequalities to get estimates for absolute values of different patterns of some special means. For this let us first have a look at following special means of positive real numbers ϵ, κ ($\epsilon \neq \kappa$).

Arithmetic mean

$$A(\epsilon,\kappa) = \frac{\epsilon+\kappa}{2}.$$

Harmonic Mean

$$H(\epsilon,\kappa) = \frac{2\epsilon\kappa}{\epsilon+\kappa}.$$

Logarithmic mean

$$L(\epsilon,\kappa) = \frac{\kappa-\epsilon}{ln(\kappa)-ln(\epsilon)}.$$

Genralised Log-mean

$$L_i(\epsilon,\kappa) = \left[\frac{\kappa^{i+1}-\epsilon^{i+1}}{(i+1)(\kappa-\epsilon)}\right]^{\frac{1}{i}},\ i \in Z \setminus \{-1,0\}.$$

Proposition 3.1. *Let* $\alpha,\beta,\gamma,\delta,\zeta,\eta \in R^+$ *such that* $\alpha<\beta, \gamma<\delta, \zeta<\eta$*, we have*

$$\left|A_iA_jA_k-\left(L_i^iA_jA_k+A_iL_j^jA_k+A_iA_jL_k^k\right)+\left(L_i^iL_j^jA_k+L_i^iA_jL_k^k+A_iL_j^jL_k^k\right)-L_i^iL_j^jL_k^k\right|$$
$$\leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}A_{i-1}A_{j-1}A_{k-1}. \tag{3.1}$$

$$\left|A_iA_jA_k-\left(L_i^iA_jA_k+A_iL_j^jA_k+A_iA_jL_k^k\right)+\left(L_i^iL_j^jA_k+L_i^iA_jL_k^k+A_iL_j^jL_k^k\right)-L_i^iL_j^jL_k^k\right|$$
$$\leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{8(p+1)^{\frac{3}{p}}}\left(A_{q(i-1)}\,A_{q(j-1)}\,A_{q(k-1)}\right)^{\frac{1}{q}}. \tag{3.2}$$

$$\left|A_iA_jA_k-\left(L_i^iA_jA_k+A_iL_j^jA_k+A_iA_jL_k^k\right)+\left(L_i^iL_j^jA_k+L_i^iA_jL_k^k+A_iL_j^jL_k^k\right)-L_i^iL_j^jL_k^k\right|$$
$$\leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}\left(A_{q(i-1)}\,A_{q(j-1)}\,A_{q(k-1)}\right)^{\frac{1}{q}}. \tag{3.3}$$

Where $A_i = A\left(\alpha^i,\beta^i\right)$, $A_j = A\left(\gamma^j,\delta^j\right)$, $A_k = A\left(\zeta^k,\eta^k\right)$, $L_i^i = L_i^i(\alpha,\beta)$, $L_j^j = L_j^j(\gamma,\delta)$, $L_k^k = L_k^k(\zeta,\eta)$, $A_{i-1} = A\left(\alpha^{i-1},\beta^{i-1}\right)$, $A_{q(i-1)} = A\left(\alpha^{q(i-1)},\beta^{q(i-1)}\right)$ *etc.*

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Proof. The assertions in Proposition 3.1 follow by taking $g(u,v,w) = u^i v^j w^k$ as the CC function in Theorem 2.6, Theorem 2.7 and Theorem 2.8 respectively. Where $u, v, w \in R^+$ and $i, j, k \in Z^+$. $\square$

Proposition 3.2. *Let $\alpha, \beta, \gamma, \delta, \zeta, \eta \in R^+$ such that $\alpha < \beta, \gamma < \delta, \zeta < \eta$, we have*

$$\Big|H_i H_j H_k - \left(L^i_{-i} H_j H_k + H_i L^j_{-j} H_k + H_i H_j L^k_{-k}\right) + \left(L^i_{-i} L^j_{-j} H_k + L^i_{-i} H_j L^k_{-k} + H_i L^j_{-j} L^k_{-k}\right) - L^i_{-i} L^j_{-j} L^k_{-k}\Big| \leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}(H_i H_j H_k)\left(H^{-1}_{i+1} H^{-1}_{j+1} H^{-1}_{k+1}\right)\left(L^i_{-i} L^j_{-j} L^k_{-k}\right).$$

$$\Big|H_i H_j H_k - \left(L^i_{-i} H_j H_k + H_i L^j_{-j} H_k + H_i H_j L^k_{-k}\right) + \left(L^i_{-i} L^j_{-j} H_k + L^i_{-i} H_j L^k_{-k} + H_i L^j_{-j} L^k_{-k}\right) - L^i_{-i} L^j_{-j} L^k_{-k}\Big| \leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{8(p+1)^{\frac{3}{p}}}(H_i H_j H_k)\left(H^{-1}_{q(i+1)} H^{-1}_{q(j+1)} H^{-1}_{q(k+1)}\right)^{\frac{1}{q}}\left(L^i_{-i} L^j_{-j} L^k_{-k}\right).$$

$$\Big|H_i H_j H_k - \left(L^i_{-i} H_j H_k + H_i L^j_{-j} H_k + H_i H_j L^k_{-k}\right) + \left(L^i_{-i} L^j_{-j} H_k + L^i_{-i} H_j L^k_{-k} + H_i L^j_{-j} L^k_{-k}\right) - L^i_{-i} L^j_{-j} L^k_{-k}\Big| \leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}(H_i H_j H_k)\left(H^{-1}_{q(i+1)} H^{-1}_{q(j+1)} H^{-1}_{q(k+1)}\right)^{\frac{1}{q}}\left(L^i_{-i} L^j_{-j} L^k_{-k}\right).$$

Where $H_i = H\left(\alpha^i, \beta^i\right), H_j = H\left(\gamma^j, \delta^j\right), H_k = H\left(\zeta^k, \eta^k\right), L^i_i = L^i_{-i}(\alpha, \beta)$, $L^j_{-j} = L^j_j(\gamma, \delta), L^k_{-k} = L^k_k(\zeta, \eta), H^{-1}_{i+1} = H^{-1}\left(\alpha^{i+1}, \beta^{i+1}\right)$, $H^{-1}_{q(i+1)} = H^{-1}\left(\alpha^{q(i+1)}, \beta^{q(i+1)}\right)$ etc.

Proof. The assertions in Proposition 3.2 follow by taking $g(u,v,w) = \frac{1}{u^i v^j w^k}$ as CC function in Theorem 2.6, Theorem 2.7 and Theorem 2.8 respectively. Where $u, v, w \in R^+$ and $i, j, k \in Z^+$. $\square$

4. Conclusion

In this paper the idea of CC function of two variables in a rectangle from the plane R^2 is extended to that of three variables in a rectangle from the space R^3. Then HH type inequality for CC function of three variables is established. Consequently by using this inequality the analogous extended right handed HH type inequalities for CC functions of three variables are obtained. Thus obtained right handed inequalities are utilized to give bounds for algebraic combinations of some special means.

Motivated by these results one can also find extensions of existing left handed HH type inequalities. Furthermore it is asserted that in the same way idea of CC function of three variables in a rectangle from the space R^3 can be further extended to CC functions of n variables in a rectangle from n-dimensional Euclidean space R^n.

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Studies on the Higher Order Difference Equation $x_{n+1} = \beta x_{n-l} + \alpha x_{n-k} + \dfrac{ax_{n-t}}{bx_{n-t}+c}$

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ABSTRACT

The main objective of this paper is to study the local and the global stability of the solutions, the periodic character and the boundedness of the difference equation

$$x_{n+1} = \beta x_{n-l} + \alpha x_{n-k} + \frac{ax_{n-t}}{bx_{n-t}+c}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1}, ...,\ x_{-1}$, x_0 are positive real numbers where $s = max\{l,\ k,\ t\}$.

Keywords: Difference equations, Stability, Global stability, Boundedness, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

The higher-order difference equations are of paramount importance in applications. Such equations also seem naturally as discrete analogues and as numerical solutions of differential which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics and so on [1-9]. The theory of difference equations gets a central position in applicable analysis. That is, the theory of difference equations will continue to play an important role in mathematics as a whole. Hence, it is very interesting to study the behavior of solutions of a difference equations and to discuss the local and global asymptotic stability of their equilibrium points [10-15]. In recent years, the behavior of solutions of various difference equations has been one of the main topics in the theory of difference equations [16-34].

Abo-Zeid [35] obtained the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B+Cx_nx_{n-1}x_{n-2}}, \quad n = 0, 1, ...,$$

where $A,\ B,\ C$ are positive real numbers and the initial conditions $x_{-2},\ x_{-1},\ x_0$ are real numbers.

Abu-Saris et al. [36] studied the globally asymptotically stability of the equilibrium solution of the rational difference equation

$$x_{n+1} = \frac{a+x_nx_{n-k}}{x_n+x_{n-k}}, \quad n = 0, 1, ...,$$

where k is a nonnegative integer, $a \geq 0$, and $x_{-k},\ ...,\ x_0 > 0$.

You-Hui et al. [37] investigated the global attractivity of the nonlinear difference equation

$$y_{n+1} = \frac{p+qy_n}{1+y_n+ry_{n-k}}, \quad n = 0, 1, ...,$$

where $p,\ q,\ r \in [0,\ \infty)$, $k \geq 1$ is a positive integer and the initial conditions $y_{-k},\ \dots,\ y_{-1}$ are nonnegative real numbers and y_0 is a positive real number.

Zayed et al. [38] investigated the boundedness character, the periodicity character, the convergence and the global stability of positive solutions of the difference equation,

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}}, \qquad n = 0, 1, ...,$$

where the coefficients $\alpha_i,\ \beta_i \in (0,\ \infty)$ for $i = 0,\ 1,\ 2,$ and $l,\ k$ are positive integers such that $l < k$. The initial conditions $x_{-k},\ ,\ ...,\ x_{-l},\ ...,\ x_{-2},\ x_{-1}, x_0$ are arbitrary positive real numbers.

El-Dessoky [39] investigated some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{dx_{n-s} - e}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $a,\ b,\ c,\ d$ and e are positive real numbers and the initial conditions $x_{-t},\ x_{-t+1}, ...,\ x_{-1},$ x_0 are positive real numbers where $t = max\{l,\ k,\ s\}$.

Our goal is to obtain some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = \beta x_{n-l} + \alpha x_{n-k} + \frac{ax_{n-t}}{bx_{n-t} + c}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1}, ...,\ x_{-1},$ x_0 are positive real numbers where $s = max\{l,\ k,\ t\}$.

2. LOCAL STABILITY

In this section, we study the local stability of the equilibrium point of equation (1).

The equilibrium points of Eq. (1) are given by

$$\overline{x} = \beta\overline{x} + \alpha\overline{x} + \frac{a\overline{x}}{b\overline{x} + c},$$

$$b(1 - \alpha - \beta)\overline{x}^2 + c(1 - \alpha - \beta)\overline{x} = a\overline{x}.$$

So, $\overline{x}_0 = 0$ is forever an equilibrium point of the difference equation (1). If $\alpha + \beta < 1$, then the positive equilibrium point of the Eq. (1) is given by

$$\overline{x}_1 = \frac{a}{b(1 - \alpha - \beta)} - \frac{c}{b}.$$

Let $f : (0,\ \infty)^3 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$f(u,\ v,\ w) = \beta u + \alpha v + \frac{aw}{bw + c}.$$

Therefore, it follows that

$$\frac{\partial f(u,\ v,\ w)}{\partial u} = \beta,\ \frac{\partial f(u,\ v,\ w)}{\partial v} = \alpha,\ \frac{\partial f(u,\ v,\ w)}{\partial w} = \frac{ac}{(bw + c)^2}. \tag{2}$$

THEOREM 2.1. *The zero equilibrium $\overline{x}_0$ of the difference equation (1) is locally asymptotically stable if*

$$c\,(\alpha + \beta) + a < c. \tag{3}$$

Proof: So, we can write Eq. (2) at zero equilibrium point $\overline{x}_0 = 0$ in the form

$$\frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial u} = \beta = p_1,\ \frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial v} = \alpha = p_2 \text{ and } \frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial w} = \frac{a}{c} = p_3.$$

Then the linearized equation of Eq. (1) about $\overline{x}_0$ is

$$y_{n+1} - p_1 y_{n-k} - p_2 y_{n-l} - p_3 y_{n-s} = 0,$$

According to Theorem 1.6 page 7 in [1], then Eq. (1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|\beta| + |\alpha| + \left|\frac{a}{c}\right| < 1,$$

and so

$$c\,(\alpha+\beta) + a < c.$$

The proof is complete.

Example 1. Consider $l = 2$, $k = 1$, $t = 3$, $\beta = 0.3$, $\alpha = 0.2$, $a = 0.5$, $b = 0.7$ and $c = 6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the zero solution of the difference equation (1) is local stability (see Fig. 1).

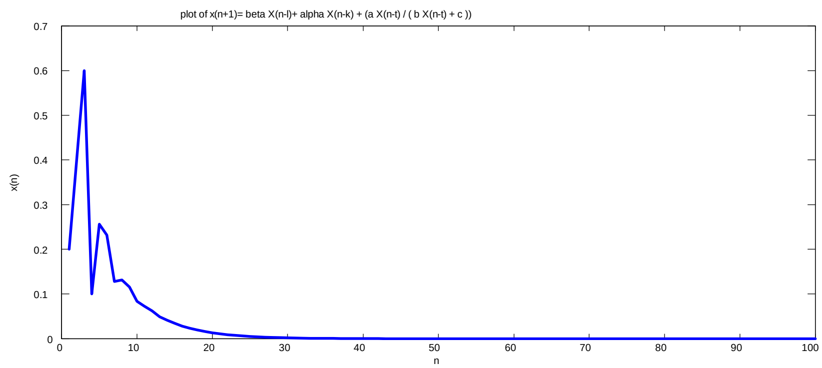

Figure 1. Plot the behavior of zero solution of Eq. (1) is local stable.

THEOREM 2.2. *The positive equilibrium $\overline{x}_1$ of the difference equation (1) is locally asymptotically stable if*

$$c\,(1-\alpha-\beta) < a. \tag{4}$$

Proof: So, we can write Eq. (2) at the positive equilibrium point $\overline{x}_1 = \frac{a}{b(1-\alpha-\beta)} - \frac{c}{b}$

$$\frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial u} = \beta = p_1,\ \frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial v} = \alpha = p_2 \text{ and } \frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial w} = \frac{c(1-\alpha-\beta)^2}{a} = p_3.$$

Then the linearized equation of Eq. (1) about $\overline{x}_1$ is

$$y_{n+1} - p_1 y_{n-k} - p_2 y_{n-l} - p_3 y_{n-s} = 0,$$

According to Theorem 1.6 page 7 in [1], then Eq. (1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|\beta| + |\alpha| + \left|\frac{c(1-\alpha-\beta)^2}{a}\right| < 1,$$

and so

$$\frac{c(1-\alpha-\beta)^2}{a} < 1 - a - b,$$

if $\alpha+\beta < 1$, then

$$c\,(1-\alpha-\beta) < a.$$

The proof is complete.

Example 2. Figure (2) shows the solution of the difference equation (1) is local stability if $l = 2$, $k = 1$, $t = 3$, $\beta = 0.3$, $\alpha = 0.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$.

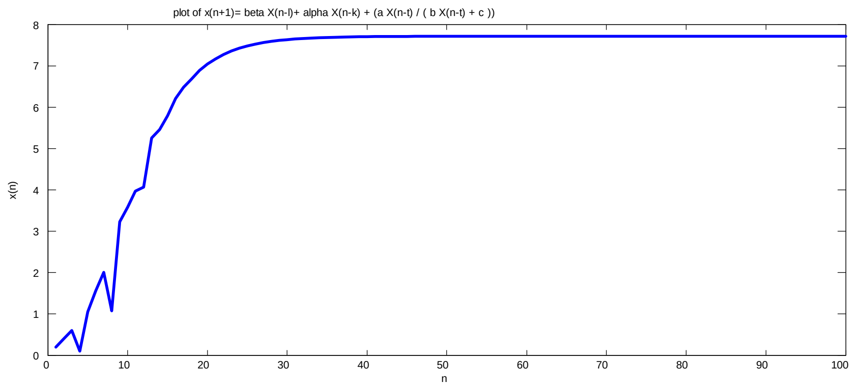

Figure 2. Draw the behavior of the positve solution of Eq. (1) is local stable.

Example 3. The solution of the difference equation (1) is unstable if $l = 2$, $k = 1$, $t = 3$, $\beta = 0.9$, $\alpha = 0.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 3).

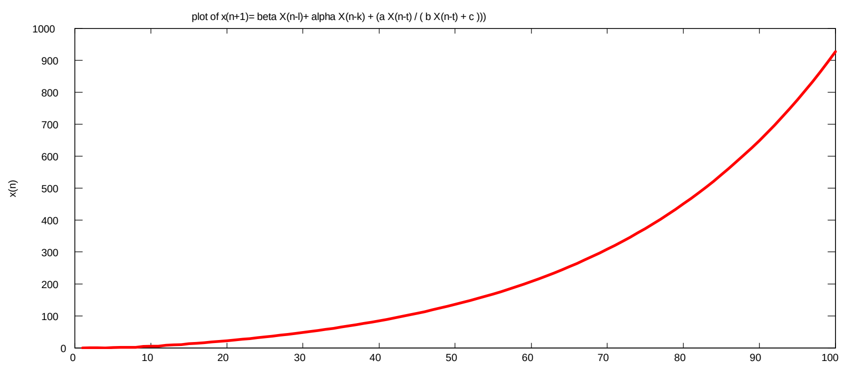

Figure 3. Sketch the behavior of the solution of Eq. (1) is unstable.

3. GLOBAL STABILITY

In this section, the global asymptotic stability of equation (1) is studied.

THEOREM 3.1. *The equilibrium point $\overline{x}_0$ is a global attractor of difference equation (1) if*

$$\alpha + \beta + \frac{a}{c} < 1. \tag{5}$$

Proof: Suppose that ζ and η are real numbers and assume that $F : [\zeta, \eta]^3 \longrightarrow [\zeta, \eta]$ is a function defined by

$$F(x,\ y,\ z) = \beta x + \alpha y + \frac{az}{bz+c}.$$

Then

$$\frac{\partial F(x,\ y,\ z)}{\partial x} = \beta,\ \frac{\partial F(x,\ y,\ z)}{\partial y} = \alpha \text{ and } \frac{\partial F(x,\ y,\ z)}{\partial z} = \frac{ac}{(bz+c)^2}.$$

Now, we can see that the function $F(x,\ y,\ z)$ increasing in $x,\ y$ and z. Then

$$\begin{aligned} & \left[\beta x + \alpha x + \frac{ax}{bx+c} - x\right](x - \overline{x}_0) \\ \leq \quad & \left[-(1-\alpha-\beta)\, x + \frac{ax}{bx} + \frac{ax}{c}\right](x-0) \leq -\left(1-\alpha-\beta-\frac{a}{c}\right)x^2 < 0 \end{aligned}$$

If $\alpha+\beta+\frac{a}{c}<1$, then $F(x,\ y,\ z)$ satisfies the negative feedback property

$$[F(x,\ x,\ x)-x](x-\overline{x}_0)<0, \quad \text{for } \overline{x}_0 = 0.$$

According to Theorem 1.10 page 15 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

Example 4. Consider $l=2$, $k=1$, $t=3$, $\beta=0.03$, $\alpha=0.02$, $a=0.5$, $b=0.7$ and $c=4$ and the initial conditions $x_{-3}=0.5$, $x_{-2}=0.7$, $x_{-1}=0.6$ and $x_0=1.1$, the zero solution of the difference equation (1) is global stability (see Fig. 4).

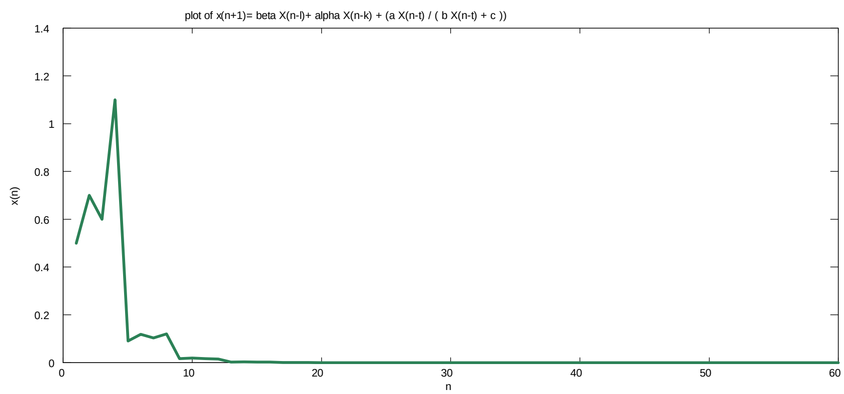

Figure 4. Plot the behavior of the zero solution of Eq. (1) is global stability.

THEOREM 3.2. *The equilibrium point $\overline{x}_1$ is a global attractor of difference equation (1) if*

$$\beta+\alpha<1. \tag{6}$$

Proof: Suppose that ζ and η are real numbers and assume that $g:[\zeta,\eta]^3 \longrightarrow [\zeta,\eta]$ is a function defined by

$$g(u,\ v,\ w)=\beta u+\alpha v+\tfrac{aw}{bw+c}.$$

Then

$$\tfrac{\partial g(u,\ v,\ w)}{\partial u}=\beta,\ \tfrac{\partial g(u,\ v,\ w)}{\partial v}=\alpha \text{ and } \tfrac{\partial g(u,\ v,\ w)}{\partial w}=\tfrac{ac}{(bw+c)^2}.$$

Now, we can see that the function $g(u,\ v,\ w)$ increasing in $u,\ v$ and w.

Let (m,M) be a solution of the system $M=g(M,\ M,\ M)$ and $m=g(m,\ m,\ m)$. Then from Eq. (1), we see that

$$M=\beta M+\alpha M+\frac{aM}{bM+c},\ m=\beta m+\alpha m+\frac{am}{bm+c},$$

thus

$$\begin{aligned} b(1\alpha-\beta)M^2-c(1-\alpha-\beta)M &= aM, \\ b(1\alpha-\beta)m^2-c(1-\alpha-\beta)m &= am. \end{aligned}$$

Subtracting we obtain

$$\begin{aligned} b(1-\alpha-\beta)(M^2-m^2)-(a+c(1-\alpha-\beta))(M-m) &= 0, \\ (M-m)\{b(1-\alpha-\beta)(M+m)-a-c(1-\alpha-\beta)\} &= 0 \end{aligned}$$

under the condition $0 \neq b(1-\alpha-\beta)$, we see that

$$M = m.$$

According to Theorem 1.15 page 18 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

Example 5. The solution of the difference equation (1) is global stability when $l = 2$, $k = 1$, $t = 3$, $\beta = 0.1$, $\alpha = 0.2$, $a = 2$, $b = 1$ and $c = 0.01$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 5).

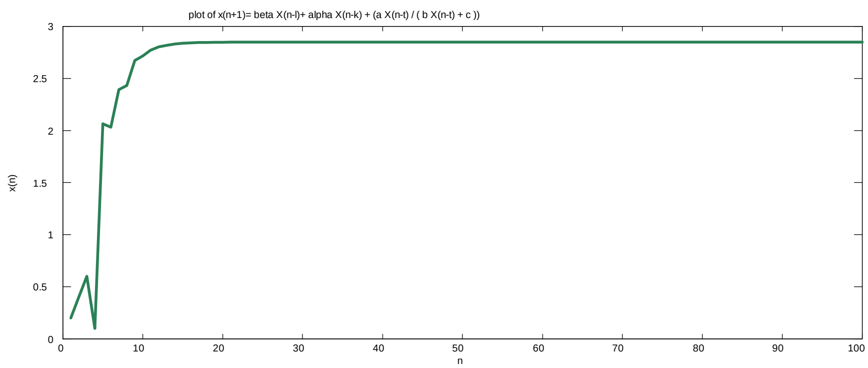

Figure 5. Plot the behavior of the solution of Eq. (1) is global stability.

4. PERIODIC SOLUTIONS

THEOREM 4.1. *Let l, k and t are both odd positive integers then for all β, α, a, b and c are positive real numbers, then Eq. (1) has a prime period two solution if*

$$\alpha+\beta<1 \text{ and } c(1-\alpha-\beta)<a. \tag{7}$$

Proof: First, suppose that there exists distinct nonnegative solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l, k and t are both odd, then $x_{n+1} = x_{n-l} = x_{n-k} = x_{n-t} = P$. It follows Eq. (1) that

$$P = \beta P + \alpha P + \tfrac{aP}{bP+c} \text{ and } Q = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c}.$$

Therefore,

$$b(1-\alpha-\beta)P^2 + (c(1-\alpha-\beta)-a)P = 0, \tag{8}$$

$$b(1-\alpha-\beta)Q^2 + (c(1-\alpha-\beta)-a)Q = 0, \tag{9}$$

Subtracting (9) from (8) gives

$$P+Q = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} \tag{10}$$

Again, adding (8) and (9) yields

$$PQ = 0. \tag{11}$$

where $a > c(1-\alpha-\beta)$ and $1 > \alpha+\beta$. Let P and Q are the two distinct nonnegative real roots of the quadratic

$$b(1-\alpha-\beta)t^2 - (a - c(1-\alpha-\beta))t = 0, \tag{12}$$

and so

$$a - c(1-\alpha-\beta) > 0 \text{ and } 1-\alpha-\beta > 0, \tag{13}$$

from Inequality (13), we obtain Inequality (7).

Second suppose that Inequality (7) is true. We will show that Eq. (1) has a prime period two solution.

Therefore P and Q are distinct nonnegative real numbers.

Set

$$x_{-l} = P,\ x_{-k} = P,\ x_{-t} = P,\ ,...,\ x_{-3} = P,\ x_{-2} = Q,\ x_{-1} = P,\ x_0 = Q.$$

We would like to show that

$$x_1 = x_{-1} = P = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} \quad \text{and} \quad x_2 = x_0 = Q = 0.$$

It follows from Eq. (1) that

$$\begin{aligned} x_1 &= \beta P + \alpha P + \tfrac{aP}{bP+c} = (\beta+\alpha)P + \frac{a\left(\frac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right)}{b\left(\frac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right)+c}, \\ &= (\beta+\alpha)\left(\tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right) + \tfrac{a(a-c(1-\alpha-\beta))}{b(a-c(1-\alpha-\beta))+cb(1-\alpha-\beta)} = (\beta+\alpha)\left(\tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right) + \tfrac{a(a-c(1-\alpha-\beta))}{ab}, \\ &= \tfrac{(\beta+\alpha)(a-c(1-\alpha-\beta))+(1-\alpha-\beta)(a-c(1-\alpha-\beta))}{b(1-\alpha-\beta)} = \tfrac{(a-c(1-\alpha-\beta))(\beta+\alpha+1-\alpha-\beta)}{b(1-\alpha-\beta)} = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} = P. \end{aligned}$$

and

$$x_2 = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c} = 0 = Q,$$

Then by induction we get

$$x_{2n} = Q \quad \text{and} \quad x_{2n+1} = P \quad \text{for all} \quad n \geq -2.$$

Thus Eq. (1) has the prime period two solution

$$...,\ P,\ Q,\ P,\ Q,\ ...,$$

where P and Q are the distinct nonnegative roots of the quadratic Eq. (12) and the proof is complete.

THEOREM 4.2. *Let l, k and t are both even positive integers then for all β, α, a, b and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Let that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l, k and t are both even, then $x_{n+1} = P$ and $x_{n-l} = x_{n-k} = x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c} \text{ and } Q = \beta P + \alpha P + \tfrac{aP}{bP+c}.$$

Therefore,

$$bPQ + cP = b(\beta+\alpha)Q^2 + (a + c(\beta+\alpha))Q, \tag{14}$$

$$bPQ + cQ = b(\beta+\alpha)P^2 + (a + c(\beta+\alpha))P, \tag{15}$$

Subtracting (15) from (14) gives

$$P + Q = -\tfrac{a+c(1+\beta+\alpha)}{b(\beta+\alpha)} \tag{16}$$

Again, adding (14) and (15) yields

$$PQ = \tfrac{c(a+c(1+\beta+\alpha))}{b^2(\beta+\alpha)(1+\beta+\alpha)}. \tag{17}$$

From (16) and (17), we have

$$(P+Q)\,PQ = -\tfrac{c(a+c(1+\beta+\alpha)+c)^2}{b^3(\beta+\alpha)^2(1+\beta+\alpha)} < 0$$

This contradicts the hypothesis that both P and Q are positive. Thus, the proof is now completed.

THEOREM 4.3. *Let $l,\ k$ are even and t is odd positive integers then for all $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Let that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $l,\ k$ are even and t is odd, then $x_{n+1} = x_{n-t} = P$ and $x_{n-l} = x_{n-k} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha Q + \tfrac{aP}{bP+c} \text{ and } Q = \beta P + \alpha P + \tfrac{aQ}{bQ+c}.$$

Therefore,

$$bP^2 + cP = b\,(\alpha+\beta)\,PQ + c\,(\alpha+\beta)\,Q + aP, \tag{18}$$

$$bQ^2 + cQ = b\,(\alpha+\beta)\,PQ + c\,(\alpha+\beta)\,P + aQ, \tag{19}$$

By subtracting (18) from (19), we deduce

$$P + Q = \tfrac{a-c(1+\alpha+\beta)}{b} \tag{20}$$

Again, by adding (18) and (19), we get

$$PQ = -\left(\tfrac{c(\alpha+\beta)(a-c(1+\alpha+\beta))}{b^2(\alpha+\beta+1)}\right). \tag{21}$$

If $a > c\,(1+\alpha+\beta)$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Therefor, the proof is completed.

THEOREM 4.4. *If $l,\ t$ are even and k is odd positive integers then Eq. (1) has no positive prime period two solution.*

Proof: Let there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $l,\ t$ are even and k is odd, then $x_{n+1}=x_{n-k}=P$ and $x_{n-l}=x_{n-t}=Q$. It follows Eq. (1) that

$$P=\beta Q+\alpha P+\tfrac{aQ}{bQ+c} \text{ and } Q=\beta P+\alpha Q+\tfrac{aP}{bP+c}.$$

Therefore,

$$b\left(1-\alpha\right)PQ+c\left(1-\alpha\right)P=b\beta Q^{2}+\left(c\beta+a\right)Q, \tag{22}$$

$$b\left(1-\alpha\right)PQ+c\left(1-\alpha\right)Q=b\beta P^{2}+\left(c\beta+a\right)P, \tag{23}$$

By subtracting (23) from (22), we get

$$P+Q=-\left(\tfrac{a+c(1-\alpha+\beta)}{b\beta}\right) \tag{24}$$

While, by adding (22) and (23), we deduce

$$PQ=\tfrac{c(1-\alpha)(a+c(1-\alpha+\beta))}{b^{2}\beta(1-\alpha+\beta)}. \tag{25}$$

If $\alpha<1$ and $\alpha<1+\beta$ then from (24) and (25), we have

$$PQ\left(P+Q\right)=-\tfrac{c(1-\alpha)(a+c(1-\alpha+\beta))^{2}}{b^{3}\beta^{2}(1-\alpha+\beta)}<0$$

This contradicts the hypothesis that both P, Q are positive. Thus, the proof is now completed.

Theorem 4.5. *Suppose that $k,\ t$ are even and l is odd positive integers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q,...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $k,\ t$ are even and l is odd, then $x_{n+1}=x_{n-l}=P$ and $x_{n-k}=x_{n-t}=Q$. It follows Eq. (1) that

$$P=\beta P+\alpha Q+\tfrac{aQ}{bQ+c} \text{ and } Q=\beta Q+\alpha P+\tfrac{aP}{bP+c}.$$

Therefore,

$$b\left(1-\beta\right)PQ+c\left(1-\beta\right)P=b\alpha Q^{2}+\left(c\alpha+a\right)Q, \tag{26}$$

$$b\left(1-\beta\right)PQ+c\left(1-\beta\right)Q=b\alpha P^{2}+\left(c\alpha+a\right)P, \tag{27}$$

By subtracting (27) from (26), we have

$$P+Q=-\left(\tfrac{a+c(1+\alpha-\beta)}{b\alpha}\right) \tag{28}$$

Again, by adding (26) and (27),we deduce

$$PQ=\tfrac{c(1-\beta)(a+c(1+\alpha-\beta))}{b^{2}\alpha(1+\alpha-\beta)}. \tag{29}$$

If $\beta < 1$ and $\beta < 1+\alpha$ then from (28) and (29), we have

$$PQ\left(P+Q\right) = -\frac{c(1-\beta)(a+c(1+\alpha-\beta))^2}{b^2\alpha^2(1+\alpha-\beta)} < 0$$

This contradicts the hypothesis that both P, Q are positive. Thus, the proof is now completed.

THEOREM 4.6. *Let k is even and $l,\ t$ are odd positive integers then for all $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q,...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when k is even and $l,\ t$ are odd, then $x_{n+1} = x_{n-l} = x_{n-t} = P$ and $x_{n-k} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha Q + \frac{aP}{bP+c} \text{ and } Q = \beta Q + \alpha P + \frac{aQ}{bQ+c}.$$

Therefore,

$$b\left(1-\beta\right)P^2 + c\left(1-\beta\right)P = b\alpha PQ + c\alpha Q + aP, \tag{30}$$

$$b\left(1-\beta\right)Q^2 + c\left(1-\beta\right)Q = b\alpha PQ + c\alpha P + aQ, \tag{31}$$

By subtracting (31) from (30), we get

$$P+Q = \frac{a-c(1+\alpha-\beta)}{b(1-\beta)} \tag{32}$$

Again, by adding (30) and (31), we have

$$PQ = -\frac{c\alpha(a-c(1+\alpha-\beta))}{b^2(1-\beta)(1+\alpha-\beta)}. \tag{33}$$

where $\beta < 1$, $\beta < 1+\alpha$ and $c\left(1+\alpha-\beta\right) < a$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Therefor, the proof is completed.

THEOREM 4.7. *If l is even and $k,\ t$ are odd positive integers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q,...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l is even and $k,\ t$ are odd, then $x_{n+1} = x_{n-k} = x_{n-t} = P$ and $x_{n-l} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha P + \frac{aP}{bP+c} \text{ and } Q = \beta P + \alpha Q + \frac{aQ}{bQ+c}.$$

Therefore,

$$b\left(1-\alpha\right)P^2 + c\left(1-\alpha\right)P = b\beta PQ + c\beta Q + aP, \tag{31}$$

$$b\left(1-\alpha\right)Q^2 + c\left(1-\alpha\right)Q = b\beta PQ + c\beta P + aQ, \tag{32}$$

Subtracting (32) from (31) gives

$$P+Q = \frac{a-c(1+\beta-\alpha)}{b(1-\alpha)} \tag{33}$$

Again, adding (31) and (32) yields

$$PQ = -\frac{c\beta(a-c(1+\beta-\alpha))}{b^2(1-\alpha)(1+\beta-\alpha)}. \tag{34}$$

If $\alpha < 1$, $\alpha < 1+\beta$ and $c(1+\beta-\alpha) < a$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Thus, the proof is completed.

Theorem 4.8. *Let t is even and l, k are odd positive integers. If*

$$c(1-\alpha-\beta)+a \neq 0,$$

then Eq. (1) has no prime period two solution.

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when t is even and l, k are odd, then $x_{n+1} = x_{n-k} = x_{n-l} = P$ and $x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha P + \frac{aQ}{bQ+c} \text{ and } Q = \beta Q + \alpha Q + \frac{aP}{bP+c}.$$

Therefore,

$$b(1-\alpha-\beta)PQ + c(1-\alpha-\beta)P = aQ, \tag{35}$$

$$b(1-\alpha-\beta)PQ + c(1-\alpha-\beta)Q = aP, \tag{36}$$

Subtracting (47) from (46) gives

$$(c(1-\alpha-\beta)+a)(P-Q) = 0$$

Since $c(1-\alpha-\beta)+a \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is completed.

Example 6. Figure (6) shows the Eq. (1) has a period two solution when $l = 1$, $k = 3$, $t = 5$, $\beta = 0.1$, $\alpha = 0.2$, $a = 0.5$, $b = 0.07$ and $c = 0.05$ and the initial conditions $x_{-5} = P$, $x_{-4} = Q$, $x_{-3} = P$, $x_{-2} = Q$, $x_{-1} = P$ and $x_0 = Q$ where $P = \frac{a-c(1+\beta-\alpha)}{b(1-\alpha)}$ and $Q = 0$.

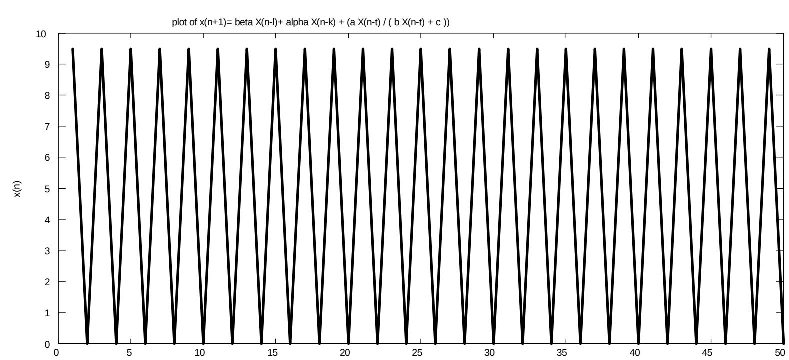

Figure 6. Sketch the solution of Eq. (1) has a period two solution.

Example 7. Consider $l = 5$, $k = 2$, $t = 4$, $\beta = 0.6$, $\alpha = 0.2$, $a = 0.4$, $b = 0.7$ and $c = 0.5$ and the initial conditions $x_{-5} = 1.2$, $x_{-4} = 1.4$, $x_{-3} = 0.6$, $x_{-2} = 1.1$, $x_{-1} = 0.3$ and $x_0 = 0.8$ the solution of Eq. (1) has no period two solution (See Fig. 7).

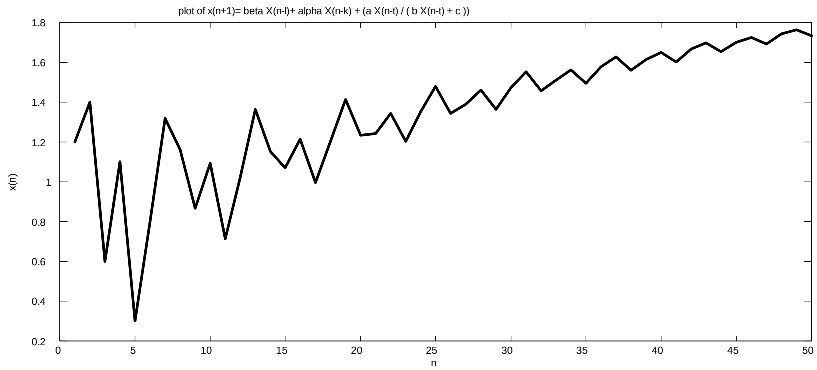

Figure 7. Draw the solution of Eq. (1) has no periodic.

5. BOUNDEDNESS OF THE SOLUTIONS

In this section, we investigate the boundedness nature of the positive solutions of equation (1).

THEOREM 5.1. *Every solution of difference equation (1) is bounded if* $\beta+\alpha<1$.

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}+c}, \\ &\leqslant \beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{a}{b} \quad \text{for all } n\geq 1. \end{aligned}$$

By using a comparison, we can right hand side as follows

$$t_{n+1}=\beta t_{n-l}+\alpha t_{n-k}+\tfrac{a}{b}.$$

and this equation is locally asymptotically stable if $\beta+\alpha<1$, and converges to the equilibrium point $\overline{t}=\frac{a}{b(1-\beta-\alpha)}$.Therefore

$$\lim_{n\to\infty}\sup x_n\leqslant \tfrac{a}{b(1-\beta-\alpha)}.$$

Thus the solution is bounded.

THEOREM 5.2. *Every solution of difference equation (1) is unbounded if* $\beta>1 or\ \alpha>1$.

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1).Then from Equation (1) we see that

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}+c}>\beta x_{n-l} \quad for \text{ all } \ n\geq 1.$$

We see that the right hand side can be written as follows

$$t_{n+1}=\beta t_{n-l}.$$

then

$$t_{ln+i}=\beta^n t_{l+i}+const., \qquad i=0,1,...,l,$$

and this equation is unstable because $\beta>1$, and $\lim_{n\to\infty}t_n=\infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above.

Similarly from Equation (1) we see that

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}+c}>\alpha x_{n-k} \ for\ all\ n\geq 1.$$

We see that the right hand side can be written as follows

$$t_{n+1}=\alpha t_{n-k}.$$

then

$$t_{kn+i} = \alpha^n t_{k+i} + const., \qquad i = 0, 1, ..., k,$$

and this equation is unstable because $\alpha > 1$, and $\lim_{n\to\infty} t_n = \infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above. Thus, the proof is now completed.

Example 8. We assume $l = 2$, $k = 1$, $t = 3$, $\beta = 0.4$, $\alpha = 1.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the solution of the difference equation (1) is unbounded (see Fig. 8).

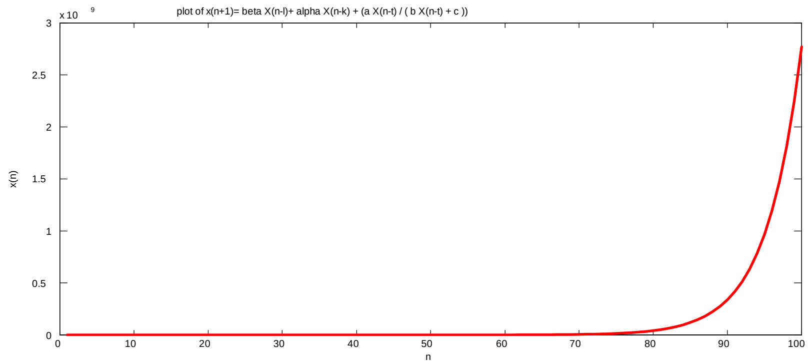

Figure 8. Plot the behavior of the solution of Eq. (1) is unbounded.

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Hermite-Hadamard inequality for Sugeno integral based on harmonically convex functions

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Abstract. For the classical Hermite-Hadamard inequality of harmonically convex functions, i.e.,

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

an upper bound is proved in the framework of the Sugeno integral.

Keywords: the Sugeno integral; the Hermite-Hadamard inequality; harmonically convex function.

2010 Mathematics Subject Classification: 26D15, 28A25, 28E15, 39B62.

1. Introduction

One of the most important integral inequalities which is related to harmonically convex functions is classical Hermite-Hadamard integral inequality. Double inequality

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

is known as Hermite-Hadamard integral inequality for harmonically convex functions, where $f \in L([a,b])$ [7, 5]. When we are trying to obtain these inequalities in the spirit of monotone measures and non-additive integrals, we get different results than the classic form.

The concept of the fuzzy integral was introduced and initially examined by Sugeno [17]. Further theoretical investigations of the integral and its generalizations have been pursued by many researchers [14, 15, 12, 2, 8, 1]. The study of inequalities for the Sugeno integral was initiated by Román-Flores and Chalco-Cano [13]. In this article, at the first we prove some Hermite-Hadamard type inequalities for harmonically convex functions in the case of non-additive integrals. Consequently, upper bound for these functions are established. In fact, the main purpose of this article is to obtain an approximation for non-solvable integral of this type.

This paper is organized as follows. Some necessary preliminaries are presented in Section 2. We address the essential problems in Sections 3 and upper bound for the Sugeno integral based on a harmonically convex function is presented. Finally, a conclusion is drawn and a problem for further investigations is given in Section 4.

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2. Preliminaries

In this section, we are going to review some well known results from the theory of non-additive measures.

Definition 2.1. [8, 18] Let Σ be a Σ-algebra of subsets of X and let $\mu : \Sigma \to [0, \infty)$ be a non-negative, extended real-valued set function, we say that μ is a monotone measure (or fuzzy measure) iff:

(FM1): $\mu(\emptyset) = 0$;

(FM2): $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);

(FM3): $(E_n) \subseteq \Sigma, \quad E_1 \subseteq E_2 \subseteq \dots$ imply $\lim_{n\to+\infty} \mu(E_n) = \mu(\bigcup\limits_{i=1}^{\infty} E_i)$ (continuity from below);

(FM4): $(E_n) \subseteq \Sigma, \quad E_1 \supseteq E_2 \supseteq \dots, \mu(E_1) < \infty$ imply $\lim_{n\to+\infty} \mu(E_n) = \mu(\bigcap\limits_{i=1}^{\infty} E_i)$ (continuity from above).

Let (X, Σ, μ) be a monotone measure space and f is a non-negative real-valued function on X. We denote the set of all non-negative measurable functions f by $\mathcal{F}_+$ and F_α denote the set $\{x \in X \mid f(x) \geq \alpha\}$, the α-level of f, for $\alpha \geq 0$. $F_0 = \{x \in X \mid f(x) > 0\} = supp(f)$ is the support of f. We know that: $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

Definition 2.2. [17, 8, 18] Let μ be a monotone measure (or fuzzy measure) on (X, Σ). If $f \in \mathcal{F}_+$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the monotone measure μ is defined by

$$\fint_A f d\mu := \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$, $\wedge$ denotes the operation sup and inf on $[0, \infty)$ respectively. In particular if $A = X$, then

$$\fint_X f d\mu := \fint f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

The following properties of the Sugeno integral are well known and can be found in [18, 19].

Proposition 2.3. Let (X, Σ, μ) be a fuzzy measure space, with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. We have

1. $\fint_A f d\mu \leq \mu(A)$;
2. $\fint_A k d\mu \leq k \wedge \mu(A)$, for k non-negative constant;
3. if $f \leq g$ on A, then $\fint_A f d\mu \leq \fint_A g d\mu$;
4. if $A \subset B$, then $\fint_A f d\mu \leq \fint_B f d\mu$;
5. if $\mu(A) < \infty$, then $\fint_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$;
6. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow \fint_A f d\mu \leq \alpha$;
7. $\fint_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;
8. $\fint_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.4. Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap F_\alpha)$. Then, due to (5) and (6) of Proposition 2.3, we have that

$$F(\alpha) = \alpha \Rightarrow \fint_A f d\mu = \alpha.$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

The following proposition shows how to transform a Sugeno integral $⨍_A f \mathrm{d}\mu$, which is defined on a monotone measure space (X, Σ, μ), into another Sugeno integral $⨍ g\mathrm{d}m$ defined on the Lebesgue measure space $([0,\infty), \overline{B_+}, m)$, where $\overline{B_+}$ is the class of all Borel sets in $[0,\infty)$ and m is the Lebesgue measure.

Proposition 2.5. [18] For any $A \in \Sigma$

$$⨍_A f \mathrm{d}\mu = ⨍ \mu(A \cap F_\alpha)\mathrm{d}m,$$

where $F_\alpha = \{x \in X \mid f(x) \geq \alpha\}$ and m is the Lebesgue measure.

Definition 2.6. [16] A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

- **(T_1):** $T(x,1) = T(1,x) = x$ for any $x \in [0,1]$;
- **(T_2):** For any $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, $T(x_1, y_1) \leq T(x_2, y_2)$;
- **(T_3):** $T(x,y) = T(y,x)$ for any $x, y \in [0,1]$;
- **(T_4):** $T(T(x,y),z) = T(x, T(y,z))$ for any $x, y, z \in [0,1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm [9] if there is a t-norm T such that $S(x,y) = 1 - T(1-x, 1-y)$.

Example 2.7. The following functions are t-norms:

- **1:** $T_M(x,y) = x \wedge y$.
- **2:** $T_P(x,y) = x.y$.
- **3:** $T_L(x,y) = (x + y - 1) \vee 0$.

Hereafter, we assume that (X, Σ, μ) is a monotone measure space. To simplify the calculation of the Sugeno integral, for a given $f \in \mathcal{F}_+(X)$ and $A \in \Sigma$, we write

$$\Gamma = \{\alpha : \alpha \geq 0, \ \ \mu(A \cap F_\alpha) > \mu(A \cap F_\beta) \ \ for \ \ any \ \ \beta > \alpha\}.$$

It is easy to see that

$$⨍_A f \mathrm{d}\mu = \bigvee_{\alpha \in \Gamma} (\alpha \wedge \mu(A \cap F_\alpha)).$$

Remark 2.8. A binary operator T on $[0,1]$ is called a t-seminorm[16] if it satisfies the above condition (T_1) and (T_2). Notice that if T is a t-seminorm, for any $x, y \in [0,1]$, we have $T(x,y) \leq T(x,1) = x$ and $T(x,y) \leq T(1,y) = y$, and consequently, $T(x,y) \leq T_M(x,y)$.

By using the concept of t-seminorm, García and Álvarez [16] proposed the following family of fuzzy integral.

Definition 2.9. Let T be a t-seminorm. Then the seminormed Sugeno's fuzzy integral of a function $f \in \mathcal{F}_+$ over $A \in \Sigma$ with respect to T and the fuzzy measure μ is defined by

$$\int_{T,A} f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(A \cap F_\alpha)).$$

Notice that the Sugeno integral of $f \in \mathcal{F}_+$ over $A \in \Sigma$ is the seminormed Sugeno's fuzzy integral of f over $A \in \Sigma$ with respect to the t-seminorm T_M.

Proposition 2.10. (García and Álvarez [16])Let (X, Σ, μ) be a monotone measure space and T be a t-seminorm. Then,

1: For any $A \in \Sigma$ and $f, g \in \mathcal{F}_+$ with $f \leq g$, we have
$$\int_{T,A} f \mathrm{d}\mu \leq \int_{T,A} g \mathrm{d}\mu.$$
2: For $A, B \in \Sigma$ with $A \subset B$ and any $f \in \mathcal{F}_+$,
$$\int_{T,A} f \mathrm{d}\mu \leq \int_{T,B} f \mathrm{d}\mu.$$

Definition 2.11. [7] Let $I \subset \mathbb{R} - \{0\}$ is a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex on I if the inequality
$$f\left(\frac{ab}{ta + (1-t)b}\right) \leq tf(b) + (1-t)f(a) \tag{2.1}$$
holds, for all $a, b \in I$ and $t \in [0, 1]$. If the inequality (2.1) is reversed, then f is said to be harmonically concave. We note that for $t = \frac{1}{2}$, we have the definition of Jensen type of harmonic convex functions, that is
$$f\left(\frac{2ab}{a+b}\right) \leq \frac{f(a) + f(b)}{2}, \ \forall a, b \in I.$$

Proposition 2.12. [7] Let $I \subset \mathbb{R} - \{0\}$ be a real interval and $f : I \to \mathbb{R}$ is function, then:

1: if $I \subset (0, +\infty)$ and f is convex and nondecreasing, then f is harmonically convex.
2: if $I \subset (0, +\infty)$ and f is harmonically convex and nonincreasing, then f is convex.
3: if $I \subset (-\infty, 0)$ and f is harmonically convex and nondecreasing, then f is convex.
4: if $I \subset (-\infty, 0)$ and f is convex and nonincreasing, then f is harmonically convex.

Proposition 2.13. [4] If $[a, b] \subset I \subseteq (0, \infty)$ and we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(t) = f(\frac{1}{t})$, then f is harmonically convex on $[a, b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Proposition 2.14. [6] A function $f : (0, \infty) \to \mathbb{R}$ is harmonically convex if and only if $xf(x)$ is convex.

Theorem 2.15. Let $f : [a, b] \subseteq (0, \infty) \to [0, +\infty)$ be a convex function with $f(a) \neq f(b)$.Then
$$⨍_a^b f \mathrm{d}\mu \leq \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu\left([a, b] \cap \left\{x \geq \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}\right\}\right)\right)$$
where $\Gamma = [f(a), f(b))$ for $f(b) > f(a)$ and $\Gamma = [f(b), f(a))$ for $f(a) > f(b)$.

Proof. As f is convex function, for $x \in [a, b]$ we have,
$$f(x) = f\left((1 - \frac{x-a}{b-a})a + \frac{x-a}{b-a}b\right) \leq (1 - \frac{x-a}{b-a})f(a) + \frac{x-a}{b-a}f(b)$$

and so by (3) of Proposition 2.3

$$\fint_a^b f\mathrm{d}\mu \leq \fint_a^b \left((1-\frac{x-a}{b-a})f(a)+\frac{x-a}{b-a}f(b)\right)\mathrm{d}\mu = \fint_a^b g(x)\mathrm{d}\mu.$$

In order to calculate the integral in the right hard part of the last inequality, we consider the distribution function $F(\alpha)$ given by

$$F(\alpha)=\mu([a,b]\cap\{g\geq\alpha\})=\mu\left([a,b]\cap\left\{\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)\geq\alpha\right\}\right).$$

If $f(a)<f(b)$, then

$$F(\alpha)=\mu\left([a,b]\cap\left\{x\geq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)=\mu\left([\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)},b]\right).$$

Thus $\Gamma=[f(a),f(b))$ and we only consider $\alpha\in[f(a),f(b))$.

If $f(a)>f(b)$, then

$$F(\alpha)=\mu\left([a,b]\cap\left\{x\leq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)=\mu\left([a,\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\right).$$

Thus $\Gamma=[f(b),f(a))$ and only need $\alpha\in[f(b),f(a))$.

This completes the proof. □

Remark 2.16. In the case $f(a)=f(b)$ in Theorem 2.15, we have $g(x)=f(x)$ and so

$$\fint_a^b f\mathrm{d}\mu \leq \fint_a^b g\mathrm{d}\mu = \fint_a^b f(a)\mathrm{d}\mu = f(a)\wedge\mu([a,b]).$$

Corollary 2.17. Let $f:[a,b]\subseteq(0,\infty)\to(0,\infty)$ be a convex function and Σ be the Borel field and μ be the Lebesgue measure on $X=\mathbb{R}$, then

$$\fint_a^b f\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge(b-\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)})\right) & ,f(a)<f(b)\\ f(a)\wedge(b-a) & ,f(a)=f(b)\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge(\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}-a)\right) & ,f(a)>f(b)\end{cases}$$

So

$$\fint_a^b f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}\wedge(b-a) & ,f(a)<f(b)\\ f(a)\wedge(b-a) & ,f(a)=f(b)\\ \frac{(b-a)f(a)}{f(a)-f(b)+(b-a)}\wedge(b-a) & ,f(a)>f(b).\end{cases}$$

Proof. In the case where $f(a)<f(b)$, we have

$$\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge(b-\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)})\right)=\frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}.$$

In fact, $\alpha = \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}$ is as the solution of the equation $F(\alpha) = \alpha$, where F is the distribution function. So taking into account (1) of Proposition 2.3 ($\fint_a^b f\mathrm{d}\mu \le \mu([a,b]) = b-a$) and Remark 2.4 we have

$$\fint_a^b f\mathrm{d}\mu \le \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)} \wedge (b-a).$$

Proofs the other cases is analogous. □

Note that Corollary 2.17 is the same as the Sadarangani Theorem [3].

3. Main Results

Let $I \subset \mathbb{R} - \{0\}$ be a harmonically convex function and $a, b \in I$ with $a < b$ and $f \in L([a,b])$. The following inequalities

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}. \tag{3.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for harmonically convex functions.

Unfortunately, as we will see in the following example, in general, the Hermite-Hadamard inequality is not valid in the fuzzy context.

Example 3.1. Let μ be the usual Lebesgue measure on $\mathbb{R}$ and the function $f(x) = \frac{3}{7}x^2$ on $X = [\frac{1}{2}, 1]$. Obviously, this function is convex and nondecreasing as a result f is harmonically convex function on $[\frac{1}{2}, 1]$. With the above inequality we have

$$\fint_{\frac{1}{2}}^1 \frac{f(x)}{x^2}dx = \fint_{\frac{1}{2}}^1 \frac{3}{7}dx = \frac{3}{7} \wedge \mu([\frac{1}{2}, 1]) = \frac{3}{7} \simeq 0.42.$$

on the other hand, $\frac{f(\frac{1}{2})+f(1)}{2} = \frac{15}{56} \simeq 0.26$.

This proves that the right-hand side of inequality (3.1) is not satisfied for the Sugeno integrals.

The aim of this work is to show a the Hermite-Hadamard type inequality for the Sugeno integral in the case where f is a harmonically convex function.

Lemma 3.2. Let $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave, then

$$\fint_a^b f\mathrm{d}\mu \le \begin{cases} \bigvee_{\alpha\in[f(a),f(b))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}, b]\right) & , f(a) < f(b) \\ f(a) \wedge \mu([a,b]) & , f(a) = f(b) \\ \bigvee_{\alpha\in[f(b),f(a))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\right) & , f(a) > f(b). \end{cases}$$

Proof. Since $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ is harmonically convex function on the interval $[a,b]$, then by Proposition 2.13 the function $g : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$, $g(s) = f(\frac{1}{s})$ is convex on $[\frac{1}{b}, \frac{1}{a}]$. Obviously for any $x \in [a,b]$, $f(x) = g(\frac{1}{x})$, and therefor applying Theorem 2.15 to g, we have

$$\fint_a^b f(x)\mathrm{d}\mu = \fint_a^b g(\frac{1}{x})\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[g(\frac{1}{a}),g(\frac{1}{b}))} \Big(\alpha \wedge \mu[\frac{\alpha(b-a)+ag(\frac{1}{b})-bg(\frac{1}{a})}{g(\frac{1}{b})-g(\frac{1}{a})}, b]\Big) & , g(\frac{1}{a}) < g(\frac{1}{b}) \\ g(\frac{1}{a}) \wedge \mu([a,b]) & , g(\frac{1}{a}) = g(\frac{1}{b}) \\ \bigvee_{\alpha\in[g(\frac{1}{b}),g(\frac{1}{a}))} \Big(\alpha \wedge \mu[a, \frac{\alpha(b-a)+ag(\frac{1}{b})-bg(\frac{1}{a})}{g(\frac{1}{b})-g(\frac{1}{a})}]\Big) & , g(\frac{1}{a}) > g(\frac{1}{b}) \end{cases}$$

$$= \begin{cases} \bigvee_{\alpha\in[f(a),f(b))} \Big(\alpha \wedge \mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}, b]\Big) & , f(a) < f(b) \\ f(a) \wedge \mu([a,b]) & , f(a) = f(b) \\ \bigvee_{\alpha\in[f(b),f(a))} \Big(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\Big) & , f(a) > f(b). \end{cases}$$

□

Corollary 3.3. Let $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave, Σ be the Borel field and μ be the Lebesgue measure on $X = \mathbb{R}$, then

$$\fint_a^b f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+b-a} \wedge (b-a) & , f(a) < f(b) \\ f(a) \wedge (b-a) & , f(a) = f(b) \\ \frac{(b-a)f(a)}{f(a)-f(b)+b-a} \wedge (b-a) & , f(a) > f(b). \end{cases}$$

Remark 3.4. If $[a,b] \subseteq (0,\infty)$ and f is harmonically convex and nonincreasing, then taking into account (2) of Proposition 2.12 the function f is convex and hance the upper bound for the Sugeno integral of f mentioned in article "Hermite-Hadamard inequality for fuzzy integral", were written by K. sadarangani is established.

Remark 3.5. If $[a,b] \subseteq (-\infty,0)$ and f is harmonically convex and nondecreasing, then taking into account (3) of Proposition 2.12 the function f is convex and hance the upper bound for the Sugeno integral of f is established.

Example 3.6. Let μ be a Lebesgue measure and consider function $f(x) = e^{-\frac{1}{x}}$ on $[\frac{1}{3}, \frac{3}{4}]$. Obviously, this function is non-negative and harmonically convex but neither convex, nor concave. we have,

$$\begin{aligned}
⨍_{\frac{1}{3}}^{\frac{3}{4}} f\mathrm{d}\mu &= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([\frac{1}{3},\frac{3}{4}]\cap\left\{e^{-\frac{1}{x}}\geq\alpha\right\}\right)\right)\\
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([\frac{1}{3},\frac{3}{4}]\cap\left\{-\frac{1}{x}\geq\ln\alpha\right\}\right)\right)\\
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([\frac{1}{3},\frac{3}{4}]\cap\left\{-1\geq xln\alpha\right\}\right)\right)\\
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([\frac{1}{3},\frac{3}{4}]\cap\left\{x\geq\frac{-1}{ln\alpha}\right\}\right)\right).
\end{aligned}$$

As result with the solution of the equation

$$\frac{1}{ln\alpha}+\frac{3}{4}=\alpha$$

we conclude that $\alpha \simeq 0/175$. Then $⨍_{\frac{1}{3}}^{\frac{3}{4}} f\mathrm{d}\mu \simeq 0/175$.

On the other hand, since $f(\frac{3}{4}) = \frac{1}{e^{\frac{4}{3}}}$ and $f(\frac{1}{3}) = \frac{1}{e^3}$. By Corollary 3.3, we have

$$⨍_{\frac{1}{3}}^{\frac{3}{4}} f\mathrm{d}\mu \leq \frac{f(\frac{3}{4})(\frac{3}{4}-\frac{1}{3})}{f(\frac{3}{4})-f(\frac{1}{3})+(\frac{3}{4}-\frac{1}{3})}\wedge(\frac{3}{4}-\frac{1}{3})$$

$$\simeq 0/234\wedge\frac{5}{12} = 0/234\wedge 0/416 = 0/234$$

that is a logical inequality.

Example 3.7. The function $f(x) = x - \ln(x+1)$ is nondecreasing and harmonic convex function on $[\frac{1}{2}, 1]$. $f(1) = 1 - \ln 2$ and $f(\frac{1}{2}) = \frac{1}{2} - \ln(\frac{3}{2})$. As $f(1) > f(\frac{1}{2})$, Corollary 3.3 gives us,

$$⨍_{\frac{1}{2}}^{1} f\mathrm{d}\mu \leq \frac{(1-\frac{1}{2})f(1)}{f(1)-f(\frac{1}{2})+\frac{1}{2}}\wedge(\frac{1}{2})\simeq 0.718\wedge\frac{1}{2}=\frac{1}{2}.$$

Thus, we find an upper bound for the Sugeno integral of this function on $[\frac{1}{2}, 1]$.

Example 3.8. The function $f(x) = e^{x^2+x}$ is nondecreasing and harmonic convex function on $[1, 2]$ and $f(1) = e^2$ and $f(2) = e^5$. As follows we find an upper bound for the Sugeno integral of this function,

$$⨍_{1}^{2} e^{x^2+x}\mathrm{d}\mu \leq \frac{e^5}{e^5-e^2+1}\wedge(1)\simeq 1.0449\wedge 1 = 1.$$

Remark 3.9. $f(x) = log(x)$ is a harmonically convex function but not convex, that is why in the Corollary 3.3, does not apply because it is concave. For concave function, we use the Sadarangani paper.

Corollary 3.10. Let $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically convex function which is not concave and $g : \mathbb{R} \to \mathbb{R}$ is a linear function, then $f \circ g$ is harmonically convex[10] and so,

$$\fint_a^b (f \circ g)\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[f(g(a)),f(g(b)))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(g(b))-bf(g(a))}{f(g(b))-f(g(a))}, b]\right) & , f(g(a)) < f(g(b)) \\ f(g(a)) \wedge \mu([a, b]) & , f(g(a)) = f(g(b)) \\ \bigvee_{\alpha\in[f(g(b)),f(g(a)))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(g(b))-bf(g(a))}{af(g(b))-bf(g(a))}]\right) & , f(g(a)) > f(g(b)). \end{cases}$$

Corollary 3.11. Let $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically convex function which is not concave and $g : \mathbb{R} \to \mathbb{R}$ is a linear function, Σ be the Borel field and μ be the Lebesgue measure on $X = \mathbb{R}$, then $f \circ g$ is harmonic convex function[10] and so,

$$\fint_a^b (f \circ g)\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(g(b))}{f(g(b))-f(g(a))+b-a} \wedge (b-a) & , f(g(a)) < f(g(b)) \\ f(g(a)) \wedge (b-a) & , f(g(a)) = f(g(b)) \\ \frac{(b-a)f(g(a))}{f(g(a))-f(g(b))+b-a} \wedge (b-a) & , f(g(a)) > f(g(b)). \end{cases}$$

Remark 3.12. In the case g be harmonic convex function and f be relative convex function, we know that $f \circ g$ is harmonically convex function [11]. Thus similar results of Corollary 3.10 and Corollary 3.11 hold.

Corollary 3.13. Let $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ be a harmonically convex function which is not concave function, Σ be the Borel field and μ be the Lebesgue measure on $X = \mathbb{R}$, then

$$\int_{T_P,[a,b]} f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)^2 f(b)}{f(b)-f(a)+b-a} & , f(a) < f(b) \\ (b-a)f(a) & , f(a) = f(b) \\ \frac{(b-a)^2 f(a)}{f(a)-f(b)+b-a} & , f(a) > f(b). \end{cases}$$

Proof. For harmonically convex function $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ with $f(a) \neq f(b)$ according to Proposition 2.10 and Corollary 3.3 with t-norm T_p, we have

$$\int_{T_P,[a,b]} f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+b-a}.(b-a) & , f(a) < f(b) \\ f(a).(b-a) & , f(a) = f(b) \\ \frac{(b-a)f(a)}{f(a)-f(b)+b-a}.(b-a) & , f(a) > f(b) \end{cases}$$

$$= \begin{cases} \frac{(b-a)^2 f(b)}{f(b)-f(a)+b-a} & , f(a) < f(b) \\ (b-a)f(a) & , f(a) = f(b) \\ \frac{(b-a)^2 f(a)}{f(a)-f(b)+b-a} & , f(a) > f(b). \end{cases}$$

□

Example 3.14. Let μ be the Lebesgue measure on $\mathbb{R}$. Consider the function $f(x) = \frac{1}{x^2}$ on $X = [1,3]$. Obviously, this function is harmonically convex and positive on $X = [1,3]$. As $f(1) = 1$ and $f(3) = \frac{1}{9}$, using Corollary 3.13, we can get the following estimate:

$$\int_{T_P,[1,3]} \frac{1}{x^2} \mathrm{d}\mu \leq \frac{(3-1)^2 f(1)}{f(1) - f(3) + (3-1)} = \frac{18}{13}.$$

Now, let's introduce the most important theorem of this article. With the help of it, an upper bound in the framework of the Sugeno integral for Hermite-Hadamard inequality of harmonically convex functions can be established.

Theorem 3.15. Let $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave, then

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu \leq \fint_a^b f \mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha \in [f(a),f(b))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}, b]\right) & , f(a) < f(b) \\ f(a) \wedge \mu([a,b]) & , f(a) = f(b) \\ \bigvee_{\alpha \in [f(b),f(a))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\right) & , f(a) > f(b) \end{cases}$$

where $m_0 = min\{a^2, b^2\}$.

Proof. Let f be a harmonically convex function which is not concave and $m_0 = min\{a^2, b^2\}$. By Proposition 2.5 we have,

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu = \fint_a^b \mu([a,b] \cap F_\alpha) \mathrm{d}m \tag{3.2}$$

where m is the Lebesgue measure and

$$F_\alpha = \{x \in X : m_0 \frac{f(x)}{x^2} \geq \alpha\}.$$

Obviously,

$$\left([a,b] \cap \left\{f(x) \geq \frac{x^2}{m_0}\alpha\right\}\right) \subseteq ([a,b] \cap \{f(x) \geq \alpha\}).$$

By monotonicity μ, we deduce

$$\mu\left([a,b] \cap \left\{f(x) \geq \frac{x^2}{m_0}\alpha\right\}\right) \leq \mu([a,b] \cap \{f(x) \geq \alpha\}).$$

Now, by Proposition 2.3 and Proposition 2.5, we obtain

$$\fint_a^b \mu\left([a,b] \cap \{f \geq \frac{x^2}{m_0}\alpha\}\right) \mathrm{d}m \leq \fint_a^b \mu([a,b] \cap \{f \geq \alpha\}) \mathrm{d}m = \fint_a^b f \mathrm{d}\mu. \tag{3.3}$$

Combining (3.2 , 3.3), we have

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu \leq \fint_a^b f \mathrm{d}\mu.$$

The last inequality follows from Lemma 3.2. □

Corollary 3.16. If $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave then,

$$\fint_a^b xf(x)\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[af(a),bf(b))}\left(\alpha\wedge\mu[\frac{\alpha(b-a)+abf(b)-baf(a)}{bf(b)-af(a)},b]\right) & ,af(a)<bf(b) \\ af(a)\wedge\mu([a,b]) & ,af(a)=bf(b) \\ \bigvee_{\alpha\in[bf(b),af(a))}\left(\alpha\wedge\mu[a,\frac{\alpha(b-a)+abf(b)-baf(a)}{bf(b)-af(a)}]\right) & ,af(a)>bf(b). \end{cases}$$

Proof. f is harmonically convex function.Therefore, according to the Proposition 2.14 $xf(x)$ is convex. Finally, the proof is complete by using Theorem 2.15. □

Corollary 3.17. If $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave, Σ be Borel field and μ be a Lebesgue measure on $X = \mathbb{R}$, then

$$\fint_a^b xf(x)\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)bf(b)}{bf(b)-af(a)+b-a}\wedge(b-a) & ,af(a)<bf(b) \\ af(a)\wedge(b-a) & ,af(a)=bf(b) \\ \frac{(b-a)af(a)}{af(a)-bf(b)+b-a}\wedge(b-a) & ,af(a)>bf(b). \end{cases}$$

Example 3.18. Let μ be the usual Lebesgue measure on X and the function $f(x) = \frac{3}{5}x^2$ on $X = [1,2]$. Obviously, this function is convex and nondecreasing. So by (1) of Proposition 2.12 f is harmonically convex on $[1,2]$. With use the Corollary 3.17 we have

$$\fint_1^2 xf(x)dx \leq \frac{(2-1)2f(2)}{2f(2)-f(1)+(2-1)}\wedge(2-1) \simeq 0.923.$$

On the other hand, $\fint_1^2 xf(x)dx \simeq 0.87$. This show that the Corollary 3.17 is valid.

4. **Conclusion**

In this paper, we have researched the Hermite-Hadamard inequality for the Sugeno integral based on harmonically convex functions. For further investigations we propose to consider the Hermite-Hadamard inequality for the Choquet integral, and also for some other non-additive integrals. In the future research, we will continue to explore other integral inequalities for non-additive measures and integrals based on harmonically convex function.

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On the Order, Type and Zeros of Meromorphic Functions and Analytic Functions of $[p, q]$-Order in the Unit Disc

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Abstract

In this paper, the authors investigate the $[p, q]$-order and $[p, q]$-type of $f_1 + f_2$, f_1f_2, f_1/f_2, where f_1, f_2 are meromorphic functions or analytic functions with the same $[p, q]$-order and different $[p, q]$-type in the unit disc, and the authors also study the $[p, q]$-order and $[p, q]$-type of f and its derivative. At the end, the authors investigate the relationship between two different $[p, q]$-convergence exponents of f. The obtained results are the improvements and supplements to many previous results.

Key words:meromorphic function; analytic function; unit disc; $[p, q]$-order; $[p, q]$-type
AMS Subject Classification(2000): 30D35, 30D15

1. Notations and Results

We use $\mathbb{C}$ to denote the complex plane and $\Delta = \{z : |z| < 1\}$ to denote the unit disc. By a meromorphic function f, we mean a meromorphic function in the complex plane or a meromorphic function in the unit disc. We shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory in the complex plane or in the unit disc (*see* $[4, 10, 14-17, 19, 20]$). Firstly for $r \in (0, +\infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r), i \in \mathbb{N}$, for all r sufficiently large in $(0, +\infty)$, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r), i \in \mathbb{N}$, we also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Moreover, we denote the logarithmic measure of a set $E \subset [0, 1)$ by $m_l E = \int_E \frac{dt}{1-t}$. Throughout this paper, we use p, q to denote positive integers satisfying $1 \leq q \leq p$. Secondly, we recall some notations about meromorphic functions and analytic functions.

Definition 1.1 (*see* $[4, 17, 19, 20]$). The order $\sigma(f)$ and lower order $\mu(f)$ of a meromorphic function f in the complex plane are respectively defined by

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}, \qquad \mu(f) = \underline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r},$$

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where $T(r,f)$ is the characteristic function of a meromorphic function f in the complex plane or in the unit disc.

Definition 1.2 (*see* [4, 19, 20]). Let f be a meromorphic function in the complex plane or an entire function satisfying $0<\sigma(f)<\infty$, then the type of f is respectively defined by

$$\tau(f)=\overline{\lim_{r\to\infty}}\frac{T(r,f)}{r^{\sigma(f)}},\qquad \tau_M(f)=\overline{\lim_{r\to\infty}}\frac{\log M(r,f)}{r^{\sigma(f)}}.$$

Definition 1.3 (*see* [8, 9, 11, 13]). The $[p,q]$-order of a meromorphic function f in the complex plane is defined by

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q r}.$$

If f is a transcendental entire function, the $[p,q]$-order of f is defined by (*see* [11, 13])

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q r}=\overline{\lim_{r\to\infty}}\frac{\log_{p+1} M(r,f)}{\log_q r}.$$

If f is a polynomial, then $\sigma_{[p,q]}(f)=0$ for any $p\geq q\geq 1$. From Definition 1.3, if $q=1$, we denote $\sigma_{[1,1]}=\sigma_1(f)=\sigma(f)$, and $\sigma_{[p,1]}=\sigma_p(f)$. Similar with Definition 1.2, we can also give the definitions of $\tau_p(f)$ and $\tau_{M,p}(f)$ when $p>1$. In order to keep accordance with Definition 1.1, we give Definition 1.3 by making a small change to the original definition of entire functions of $[p,q]$-order (*see* [8, 9]).

Definition 1.4 (*see* [3, 7]). The iterated p-order of a meromorphic function f in Δ is defined by

$$\sigma_p(f)=\overline{\lim_{r\to 1^-}}\frac{\log_p T(r,f)}{-\log(1-r)}\qquad (p\in\mathbb{N}).$$

For an analytic function f in Δ, we also define

$$\sigma_{M,p}(f)=\overline{\lim_{r\to 1^-}}\frac{\log_{p+1} M(r,f)}{-\log(1-r)}.$$

Remark 1.1. If $p=1$, then we denote $\sigma_1(f)=\sigma(f)$ and $\sigma_{M,1}(f)=\sigma_M(f)$, and we have $\sigma(f)\leq\sigma_M(f)\leq\sigma(f)+1$ (*see* [6, 12, 16, 17]) and $\sigma_{M,p}(f)=\sigma_p(f)$ $(p\geq 2)$ (*see* [3, 7]).

Definition 1.5 (*see* [2]). Let f be a meromorphic function in Δ, then the $[p,q]$-order and lower $[p,q]$-order of f are respectively defined by

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to 1^-}}\frac{\log_p T(r,f)}{\log_q\left(\frac{1}{1-r}\right)},\qquad \mu_{[p,q]}(f)=\underline{\lim_{r\to 1^-}}\frac{\log_p T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Definition 1.6 (*see* [2]). Let f be an analytic function in Δ, then the $[p,q]$-order and lower $[p,q]$-order about maximum modulus of f are respectively defined by

$$\sigma_{M,[p,q]}(f)=\overline{\lim_{r\to 1^-}}\frac{\log_{p+1}M(r,f)}{\log_q\left(\frac{1}{1-r}\right)},\qquad \mu_{M,[p,q]}(f)=\underline{\lim_{r\to 1^-}}\frac{\log_{p+1}M(r,f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Definition 1.7 (*see* [2]). The $[p,q]$-type of a meromorphic function f of $[p,q]$-order in Δ with $0<\sigma_{[p,q]}(f)=\sigma_1<\infty$ is defined by

$$\tau_{[p,q]}(f)=\overline{\lim_{r\to 1^-}}\frac{\log_{p-1}T(r,f)}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_1}}.$$

For an analytic function f in Δ, and the $[p,q]$-type about maximum modulus of f of $[p,q]$-order with $0<\sigma_{M,[p,q]}(f)=\sigma_2<\infty$ is defined by

$$\tau_{M,[p,q]}(f)=\overline{\lim_{r\to 1^-}}\frac{\log_{p}M(r,f)}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_2}}.$$

Definition 1.8 The lower $[p,q]$-type of a meromorphic function f of lower $[p,q]$-order in Δ with $0<\mu_{[p,q]}(f)=\mu_1<\infty$ is defined by

$$\underline{\tau}_{[p,q]}(f)=\underline{\lim_{r\to \infty}}\frac{\log_{p-1}T(r,f)}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_1}}.$$

Similarly for an analytic function f in Δ, and the lower $[p,q]$-type about maximum modulus of f of lower $[p,q]$-order with $0<\mu_{M,[p,q]}(f)=\mu_2<\infty$ is defined by

$$\underline{\tau}_{M,[p,q]}(f)=\underline{\lim_{r\to \infty}}\frac{\log_{p}M(r,f)}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_2}}.$$

Remark 1.2. From Definitions 1.7 and 1.8, it is easy to see that $\tau_{[p,q]}(f)\le\tau_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f)\le\underline{\tau}_{M,[p,q]}(f)$.

Definition 1.9. For any $a\in\mathbb{C}\cup\{\infty\}$, we use $n\left(r,\frac{1}{f-a}\right)$ to denote the unintegrated counting function for the sequence of a-point of a meromorphic function f in Δ. Then the $[p,q]$-exponents of convergence of a-point of f about $n\left(r,\frac{1}{f-a}\right)$ is defined by

$$\lambda^n_{[p,q]}(f,a)=\overline{\lim_{r\to 1^-}}\frac{\log_p n\left(r,\frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Definition 1.10. Let $N\left(r, \frac{1}{f-a}\right)$ be the integrated counting function for the sequence of a-point of a meromorphic function f in Δ. Then the $[p,q]$-exponents of convergence of a-point of f about $N\left(r, \frac{1}{f-a}\right)$ is defined by

$$\lambda^N_{[p,q]}(f,a) = \overline{\lim_{r\to 1^-}} \frac{\log_p N\left(r, \frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Remark 1.3. Similar with Definitions 1.9 and 1.10, we can also give the definitions of the $[p,q]$-exponents of convergence of distinct a-point of f about $n\left(r, \frac{1}{f-a}\right)$ and $N\left(r, \frac{1}{f-a}\right)$, i.e., $\overline{\lambda}^n_{[p,q]}(f,a)$ and $\overline{\lambda}^N_{[p,q]}(f,a)$.

The order and type are two important indicators in revealing the growth of the entire functions or meromorphic functions, many authors have investigated the growth of entire functions or meromorphic functions in the complex plane or in the unit disc $(e.g., see[4, 8-10, 14-20])$ since the first half of the twentieth century. In the following, we list some classic results in the complex plane.

Theorem A(*see* $[4, 10, 19, 20]$). If f_1 and f_2 are meromorphic functions of finite order with $\sigma(f_1) = \sigma_3$ and $\sigma(f_2) = \sigma_4$, then $\sigma(f_1 + f_2) \le \max\{\sigma_3, \sigma_4\}, \sigma(f_1 f_2) \le \max\{\sigma_3, \sigma_4\}, \sigma(f_1/f_2)$ $\le \max\{\sigma_3, \sigma_4\}$; if $\sigma_3 < \sigma_4$, then $\sigma(f_1 + f_2) = \sigma(f_1 f_2) = \sigma(f_1/f_2) = \sigma_4$.

Theorem B (*see* [20]). If f_1 and f_2 are meromorphic functions of finite order, then $\mu(f_1+f_2) \le \min\{\max\{\sigma(f_1), \mu(f_2)\}, \max\{\mu(f_1), \sigma(f_2)\}\}, \mu(f_1 f_2) \le \min\{\max\{\sigma(f_1), \mu(f_2)\}, \max\{\mu(f_1), \sigma(f_2)\}\}$. Furthermore, if $\sigma(f_1) < \mu(f_2)$, then $\mu(f_1 + f_2) = \mu(f_1 f_2) = \mu(f_2)$; or if $\sigma(f_2) < \mu(f_1)$, then $\mu(f_1 + f_2) = \mu(f_1 f_2) = \mu(f_1)$.

Theorem C (*see* [10]). If f_1 and f_2 are entire functions of finite order satisfying $\sigma(f_1) = \sigma(f_2) = \sigma_5$, then the following two statements hold:
(i) If $\tau_M(f_1) = 0$ and $0 < \tau_M(f_2) < \infty$, then $\sigma(f_1 f_2) = \sigma_5, \tau_M(f_1 f_2) = \tau_M(f_2)$.
(ii) If $\tau_M(f_1) < \infty$ and $\tau_M(f_2) = \infty$, then $\sigma(f_1 f_2) = \sigma_5, \tau_M(f_1 f_2) = \infty$.

Theorem D (*see* [18]). Let $f_1(z)$ and $f_2(z)$ be entire functions satisfying $0 < \sigma_p(f_1) = \sigma_p(f_2) = \sigma_6 < \infty,\ 0 \le \tau_{M,p}(f_1) < \tau_{M,p}(f_2) \le \infty$. Then the following statements hold:
(i) If $p \ge 1$, then $\sigma_p(f_1 + f_2) = \sigma_6,\ \tau_{M,p}(f_1 + f_2) = \tau_{M,p}(f_2)$;
(ii) If $p > 1$, then $\sigma_p(f_1 f_2) = \sigma_6,\ \tau_{M,p}(f_1 f_2) = \tau_{M,p}(f_2)$.

Theorem E (*see* [18]). Let $p \ge 1,\ f(z)$ be an entire function or a meromorphic function in the complex plane satisfying $0 < \sigma_p(f) < \infty$. If $p \ge 1$, then $\sigma_p(f) = \sigma_p(f'), \tau_{M,p}(f') = \tau_{M,p}(f)$; if $p > 1$, then $\sigma_p(f) = \sigma_p(f'), \tau_p(f') = \tau_p(f)$.

From Theorems A-E, we can easily obtain the following similar propositions in the unit disc.

Propositions (i) If f_1 and f_2 are meromorphic functions satisfying $\sigma_{[p,q]}(f_1)=\sigma_6$ and $\sigma_{[p,q]}(f_2)=\sigma_7$ in Δ, then $\sigma_{[p,q]}(f_1\pm f_2)\leq\max\{\sigma_6,\ \sigma_7\},\sigma_{[p,q]}(f_1f_2)\leq\max\{\sigma_6,\sigma_7\}$ and $\sigma_{[p,q]}(f_1/f_2)\leq\max\{\sigma_6,\sigma_7\}$.

(ii) If $\sigma_6\neq\sigma_7$ in Proposition (i), then $\sigma_{[p,q]}(f_1\pm f_2)=\sigma_{[p,q]}(f_1f_2)=\sigma_{[p,q]}(f_1/f_2)=\max\{\sigma_6,\sigma_7\}$.

(iii) If f_1 and f_2 are meromorphic functions in Δ, then $\mu_{[p,q]}(f_1+f_2)\leq\max\{\sigma_{[p,q]}(f_1),\mu_{[p,q]}(f_2)\}$ or $\mu_{[p,q]}(f_1+f_2)\leq\max\{\mu_{[p,q]}(f_1),\sigma_{[p,q]}(f_2)\}$ and $\mu_{[p,q]}(f_1f_2)\leq\max\{\sigma_{[p,q]}(f_1),\mu_{[p,q]}(f_2)\}$ or $\mu_{[p,q]}(f_1f_2)\leq\max\{\mu_{[p,q]}(f_1),\sigma_{[p,q]}(f_2)\}$.

(iv) If f_1 and f_2 are meromorphic functions in Δ satisfying $\sigma_{[p,q]}(f_1)<\mu_{[p,q]}(f_2)\leq\infty$, then $\mu_{[p,q]}(f_1+f_2)=\mu_{[p,q]}(f_1f_2)=\mu_{[p,q]}(f_1/f_2)=\mu_{[p,q]}(f_2)$.

(v) If f_1 and f_2 are analytic functions in Δ satisfying $\sigma_{M,[p,q]}(f_1)=\sigma_8$ and $\sigma_{M,[p,q]}(f_2)=\sigma_9$, then $\sigma_{M,[p,q]}(f_1\pm f_2)\leq\max\{\sigma_8,\sigma_9\}$ and $\sigma_{M,[p,q]}(f_1f_2)\leq\max\{\sigma_8,\sigma_9\}$. If $\sigma_8\neq\sigma_9$, then $\sigma_{M,[p,q]}(f_1\pm f_2)=\max\{\sigma_8,\sigma_9\}$.

(vi) If f_1 and f_2 are analytic functions in Δ, then $\max\{\mu_{M,[p,q]}(f_1\pm f_2),\mu_{M,[p,q]}(f_1f_2)\}\leq\max\{\sigma_{M,[p,q]}(f_1),\mu_{M,[p,q]}(f_2)\}$ or $\max\{\mu_{M,[p,q]}(f_1\pm f_2),\mu_{M,[p,q]}(f_1f_2)\}\leq\max\{\mu_{M,[p,q]}(f_1),\sigma_{M,[p,q]}(f_2)\}$.

(vii) If f_1 and f_2 are analytic functions of $[p,q]$-order in Δ, for any $r\in[0,1)$, by the inequality $T(r,f)\leq\log^+M(r,f)\leq\frac{4}{1-r}T\left(\frac{1+r}{2},f\right)$ $(see$ $[4,17])$, we easily obtain that if $p=q\geq2$ and $\sigma_{[p,q]}(f)>1$, or $p>q\geq1$, then $\sigma_{[p,q]}(f)=\sigma_{M,[p,q]}(f)$ and $\tau_{[p,q]}(f)=\tau_{M,[p,q]}(f)$. Similarly, we have $\mu_{[p,q]}(f)=\mu_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f)=\underline{\tau}_{M,[p,q]}(f)$ if $p=q\geq2$ and $\mu_{[p,q]}(f)>1$, or $p>q\geq1$.

Combining Theorems D and E, a natural question is: Can we get the similar results with Theorems D, E for meromorphic functions or analytic functions of $[p,q]$ order in Δ? In fact, we obtain the following results:

Theorem 1.1. Let f_1 and f_2 be meromorphic functions in Δ satisfying $0<\sigma_{[p,q]}(f_1)=\sigma_{[p,q]}(f_2)=\sigma_{10}<\infty$ and $0\leq\tau_1=\tau_{[p,q]}(f_1)<\tau_{[p,q]}(f_2)=\tau_2\leq\infty$. Then $\sigma_{[p,q]}(f_1+f_2)=\sigma_{[p,q]}(f_1f_2)=\sigma_{[p,q]}(f_1/f_2)=\sigma_{10}$, and the following two statements hold:

(i) If $p>1$ and $p\geq q\geq1$, then $\tau_{[p,q]}(f_1+f_2)=\tau_{[p,q]}(f_1f_2)=\tau_{[p,q]}(f_1/f_2)=\tau_{[p,q]}(f_2)$.
(ii) If $p=q=1$, then $\tau_2-\tau_1\leq\max\{\tau(f_1+f_2),\tau(f_1f_2),\tau(f_1/f_2)\}\leq\tau_2+\tau_1$.

Theorem 1.2. Let f_1 and f_2 be meromorphic functions in Δ satisfying $0<\sigma_{[p,q]}(f_1)=\mu_{[p,q]}(f_2)<\infty$ and $0\leq\tau_{[p,q]}(f_1)<\underline{\tau}_{[p,q]}(f_2)\leq\infty$, then $\mu_{[p,q]}(f_1+f_2)=\mu_{[p,q]}(f_1f_2)=\mu_{[p,q]}(f_1/f_2)=\mu_{[p,q]}(f_2)$. And if $p>1$ and $p\geq q\geq1$, we have $\underline{\tau}_{[p,q]}(f_1+f_2)=\underline{\tau}_{[p,q]}(f_1f_2)=\underline{\tau}_{[p,q]}(f_1/f_2)=\underline{\tau}_{[p,q]}(f_2)$.

In the following, when f_1 and f_2 are analytic functions of $[p,q]$-order in the unit disc we have the similar results.

Theorem 1.3. Let f_1 and f_2 be analytic functions in Δ satisfying $0 < \sigma_{M,[p,q]}(f_1) = \sigma_{M,[p,q]}(f_2) = \sigma_{11} < \infty$ and $0 \le \tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) \le \infty$, then $\sigma_{M,[p,q]}(f_1+f_2) = \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) = \tau_{M,[p,q]}(f_2)$.

Remark 1.4. By Proposition (vii), we know that Theorem 1.3 is of the same with Theorem 1.1 for $p > q \ge 1$ and $p = q \ge 2, \sigma_{[p,q]}(f) > 1$. For the case $p = q = 1$, the result of Theorem 1.3 is better than that of Theorem 1.1.

Corollary 1.1. Let f_1 and f_2 be analytic functions in Δ satisfying $0 < \sigma_{M,[p,q]}(f_1) = \mu_{M,[p,q]}(f_2) < \infty$ and $0 \le \tau_{M,[p,q]}(f_1) < \underline{\tau}_{M,[p,q]}(f_2) \le \infty$, then $\mu_{M,[p,q]}(f_1 + f_2) = \mu_{M,[p,q]}(f_2)$ and $\underline{\tau}_{M,[p,q]}(f_1 + f_2) = \underline{\tau}_{M,[p,q]}(f_2)$.

Theorem 1.4. Let f be an analytic function of $[p,q]$-order in Δ, then $\sigma_{M,[p,q]}(f) = \sigma_{M,[p,q]}(f')$, $\mu_{M,[p,q]}(f) = \mu_{M,[p,q]}(f')$. If $0 < \sigma_{M,[p,q]}(f) < \infty$ or $0 < \mu_{M,[p,q]}(f) < \infty$, then $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$, $\underline{\tau}_{M,[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f')$.

Theorem 1.5. Let f be a meromorphic function of $[p,q]$-order in Δ, then
(i) If $p \ge q \ge 2$ and $p > q = 1$, then $\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f'), \mu_{[p,q]}(f) = \mu_{[p,q]}(f')$ and $\tau_{[p,q]}(f) = \tau_{[p,q]}(f')$, $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{[p,q]}(f')$ for $0 < \sigma_{[p,q]}(f) < \infty$ or $0 < \mu_{[p,q]}(f) < \infty$.
(ii) If $p = q = 1$, then $\sigma(f) = \sigma(f'), \mu(f) = \mu(f')$ and $\tau_{[1,1]}(f') \le 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \le 2\underline{\tau}_{[1,1]}(f)$.

Theorem 1.6. Let f be a meromorphic function of $[p,q]$-order in Δ, $a \in \mathbb{C} \cup \{\infty\}$. Then the following statements hold:

(i) If $p > q \ge 1$, then $\lambda^N_{[p,q]}(f,a) = \lambda^n_{[p,q]}(f,a)$.
(ii) If $p = q = 1$, then $\lambda^N(f,a) \le \lambda^n(f,a) \le \lambda^N(f,a) + 1$ (see [12]).
(iii) If $p = q \ge 2$, then $\lambda^N_{[p,p]}(f,a) \le \lambda^n_{[p,p]}(f,a) \le \max\left\{\lambda^N_{[p,p]}(f,a), 1\right\}$. Furthermore, we have $\lambda^N_{[p,p]}(f,a) = \lambda^n_{[p,p]}(f,a)$ if $\lambda^N_{[p,p]}(f,a) \ge 1$, and if $\lambda^N_{[p,p]}(f,a) < 1$ then $\lambda^N_{[p,p]}(f,a) \le \lambda^n_{[p,p]}(f,a) \le 1$.

Remark 1.4. The conclusions of Theorem 1.6 also hold between $\overline{\lambda}^n_{[p,q]}(f,a)$ and $\overline{\lambda}^N_{[p,q]}(f,a)$.

2. Preliminary Lemmas

Lemma 2.1 (see [4, 19, 20]). Let $f_1,\ f_2,\ \cdots,\ f_m(z)$ be meromorphic functions in Δ, where $m \ge 2$ is a positive integer. Then

(i) $T(r, f_1 f_2 \cdots f_m) \le \sum\limits_{i=1}^{m} T(r, f_i)$,

(ii) $T(r, f_1 + f_2 + \cdots + f_m) \le \sum\limits_{i=1}^{m} T(r, f_i) + \log m$.

Lemma 2.2 (see [6, 17]). Let f be a meromorphic function in Δ, and let $k \ge 1$ be an integer.

Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\left\{\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right\}$, possibly outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dt}{1-t} < \infty$.

Lemma 2.3 (*see* [1]). Let $g : (0,1) \to R$ and $h : (0,1) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E_2 \subset [0,1)$ for which $\int_{E_2} \frac{dt}{1-t} < \infty$. Then there exisits a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.

Lemma 2.4 (*see* [5, 15]) Suppose that f is meromorphic in Δ with $f(0) = 0$. Then

$$m(r,f) \leq \left[1 + \varphi(\frac{r}{R})\right] T(R,f') + N(R,f'), \tag{3.13}$$

where $0 < r < R < 1, \varphi(t) = \frac{1}{\pi} \log \frac{1+t}{1-t}$.

3. Proofs of Theorems 1.1 -1.6

Proof of Theorem 1.1. Assume that $0 \leq \tau_1 = \tau_{[p,q]}(f_1) < \tau_{[p,q]}(f_2) = \tau_2 < \infty$, by Definition 1.7, it is easy to see that for any given $\varepsilon > 0$ and $r \to 1^-$, we have

$$T(r,f_1) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}, \tag{3.1}$$

$$T(r,f_2) \leq \exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}. \tag{3.2}$$

By using (3.1)-(3.2) and Lemma 2.1, we have

$$T(r, f_1 + f_2) \leq T(r,f_1) + T(r,f_2) + \log 2$$

$$\leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\} + \exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\} + \log 2$$

$$\leq 2\exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}.$$

Hence $\sigma_{[p,q]}(f_1 + f_2) \leq \sigma_{10}$. In addition, if $p = q = 1$, we can get $\tau_{[p,q]}(f_1 + f_2) \leq \tau_1 + \tau_2$, if $p > 1$, then $\tau_{[p,q]}(f_1 + f_2) \leq \tau_2$ for any $p \geq q \geq 1$. On the other hand, for any given $\varepsilon > 0$, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying

$$T(r_n,f_1) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}, \tag{3.3}$$

$$T(r_n, f_2) \geq \exp_{p-1}\left\{(\tau_2 - \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}. \tag{3.4}$$

By (3.3)-(3.4) and Lemma 2.1, we obtain

$$T(r_n, f_1 + f_2) \geq T(r_n, f_2) - T(r_n, f_1) - \log 2$$

$$\geq \exp_{p-1}\left\{(\tau_2 - \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\} - \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\} - \log 2. \tag{3.5}$$

By (3.5) we have $\sigma_{[p,q]}(f_1 + f_2) \geq \sigma_{10}$. Furthermore, if $p = q = 1$, then $\tau_{[p,q]}(f_1 + f_2) \geq \tau_2 - \tau_1$ and $\tau_{[p,q]}(f_1 + f_2) \geq \tau_2$ for $p > 1$ and $p \geq q \geq 1$.

Therefore, we have $\sigma_{[p,q]}(f_1 + f_2) = \sigma_{10}$, and if $p > 1$, then $\tau_{[p,q]}(f_1 + f_2) = \tau_{[p,q]}(f_2)$, if $p = q = 1$, then $\tau_2 - \tau_1 \leq \tau_{[p,q]}(f_1 + f_2) \leq \tau_2 + \tau_1$. Since $T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2)$, $T(r, f_1 f_2) \geq T(r, f_2) - T(r, f_1) - O(1)$ and $T\left(r, \frac{1}{f_2}\right) = T(r, f_2) + O(1)$, by the above proof, $\sigma_{[p,q]}(f_1 f_2) = \sigma_{[p,q]}(f_1/f_2) = \sigma_{10}$, $\tau_{[p,q]}(f_1 f_2) = \tau_{[p,q]}(f_1/f_2) = \tau_{[p,q]}(f_2)$ for $p > 1$ and $\tau_2 - \tau_1 \leq \max\{\tau(f_1 f_2), \tau(f_1/f_2)\} \leq \tau_2 + \tau_1$ if $p = q = 1$ also can hold. Moreover, Theorem 1.1 also holds for $\tau_{[p,q]}(f_2) = \tau_2 = \infty$.

Proof of Theorem 1.2. Without loss of generality, we suppose that $0 \leq \tau_3 = \tau_{[p,q]}(f_1) < \underline{\tau}_{[p,q]}(f_2) = \tau_4 < \infty$. Assume that $\sigma_{[p,q]}(f_1) = \mu_{[p,q]}(f_2) = \mu_3$, and by Definition 1.8, it is easy to see that for any given $\varepsilon > 0$, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying

$$T(r_n, f_1) < \exp_{p-1}\left\{(\tau_3 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}, \tag{3.6}$$

$$T(r_n, f_2) < \exp_{p-1}\left\{(\tau_4 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}. \tag{3.7}$$

By (3.6)-(3.7) and Lemma 2.1, we have

$$T(r_n, f_1 + f_2) \leq T(r_n, f_1) + T(r_n, f_2) + \log 2$$

$$\leq \exp_{p-1}\left\{(\tau_3 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\} + \exp_{p-1}\left\{(\tau_4 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\} + \log 2$$

$$\leq 2\exp_{p-1}\left\{(\tau_4 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}.$$

Hence $\mu_{[p,q]}(f_1 + f_2) \leq \mu_3$. In addition, if $p > 1$, then $\underline{\tau}_{[p,q]}(f_1 + f_2) \leq \tau_4$ for $p \geq q \geq 1$. On the other hand, for any given $\varepsilon > 0$ and $r \to 1^-$, we have

$$T(r, f_1) \leq \exp_{p-1}\left\{(\tau_3 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\}, \tag{3.8}$$

$$T(r, f_2) \geq \exp_{p-1}\left\{(\tau_4 - \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\}. \tag{3.9}$$

By (3.8)-(3.9) and Lemma 2.1, we obtain

$$T(r, f_1+f_2) \geq T(r, f_2) - T(r, f_1) - \log 2$$

$$\geq \exp_{p-1}\left\{(\tau_4-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\} - \exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\} - \log 2. \quad (3.10)$$

By (3.10) we have $\mu_{[p,q]}(f_1+f_2) \geq \mu_3$ and $\underline{\tau}_{[p,q]}(f_1+f_2) \geq \tau_4$ for $p > 1$ and $p \geq q \geq 1$. Thus we have $\mu_{[p,q]}(f_1+f_2) = \mu(f_2)$ and if $p > 1$ and $p \geq q \geq 1$, then $\underline{\tau}_{[p,q]}(f_1+f_2) = \underline{\tau}_{[p,q]}(f_2)$. Since $T(r, f_1f_2) \leq T(r, f_1) + T(r, f_2)$, $T(r, f_1f_2) \geq T(r, f_2) - T(r, f_1) - O(1)$ and $T\left(r, \frac{1}{f_2}\right) = T(r, f_2) + O(1)$, by the above proof, $\mu_{[p,q]}(f_1f_2) = \mu_{[p,q]}(f_1/f_2) = \mu_{[p,q]}(f_2)$ and $\underline{\tau}_{[p,q]}(f_1f_2) = \underline{\tau}_{[p,q]}(f_1/f_2) = \underline{\tau}_{[p,q]}(f_2)$ also hold if $p > 1$ and $p \geq q \geq 1$.

The conclusions of Theorem 1.2 also hold for $\tau_3 = \tau_{[p,q]}(f_1) < \underline{\tau}_{[p,q]}(f_2) = \tau_4 = \infty$.

Proof of Theorem 1.3. Set $0 \leq \tau_5 = \tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) = \tau_6 < \infty$, by Definition 1.7, for any given ε $(0 < 2\varepsilon < \tau_6 - \tau_5)$, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying

$$M(r_n, f_1) \leq \exp_p\left\{(\tau_5+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}, \quad (3.11)$$

$$M(r_n, f_2) > \exp_p\left\{(\tau_6-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}. \quad (3.12)$$

We can choose a sequence $\{z_n\}_{n=1}^{\infty}$ satisfying $|z_n| = r_n (n = 1, 2, \cdots)$ and $|f_2(z_n)| = M(r_n, f_2)$, by (3.11)-(3.12) we have

$$M(r_n, f_1+f_2) \geq |f_1(z_n)+f_2(z_n)| \geq |f_2(z_n)| - |f_1(z_n)| \geq M(r_n, f_2) - M(r_n, f_1)$$

$$\geq \exp_p\left\{(\tau_6-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\} - \exp_p\left\{(\tau_5+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}$$

$$\geq \frac{1}{2}\exp_p\left\{(\tau_6-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\} \quad (r_n \to 1^-).$$

Hence $\sigma_{M,[p,q]}(f_1+f_2) \geq \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) \geq \tau_6$. On the other hand, we have

$$M(r, f_1+f_2) \leq M(r, f_1) + M(r, f_2)$$

$$\leq \exp_p\left\{(\tau_5+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\} + \exp_p\left\{(\tau_6+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\}$$

$$\leq 2\exp_p\left\{(\tau_6+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\},$$

therefore $\sigma_{M,[p,q]}(f_1+f_2) \leq \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) \leq \tau_6$. Thus we can get $\sigma_{M,[p,q]}(f_1+f_2) = \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) = \tau_{M,[p,q]}(f_2)$. Moreover, Theorem 1.3 also holds for $\tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) = \tau_6 = \infty$.

Proof of Theorem 1.4. Since f is an analytic function in the unit disc, from the formula

$$f(z) = f(0) + \int_0^z f'(\zeta)d\zeta \qquad (|z| = r < 1)$$

where the integral route is a line from 0 to z in the unit disc. We obtain that

$$M(r,f) \leq |f(0)| + |\int_0^z f'(\zeta)d\zeta| \leq |f(0)| + rM(r,f') \leq |f(0)| + M(r,f'),$$

i.e.

$$M(r,f') \geq M(r,f) - |f(0)|. \tag{3.13}$$

By (3.13), we have

$$\sigma_{M,[p,q]}(f') \geq \sigma_{M,[p,q]}(f), \quad \mu_{M,[p,q]}(f') \geq \mu_{M,[p,q]}(f).$$

On the other hand, in the circle $|z| = r \in (0,1)$, we take a point z_0 satisfying $|f'(z_0)| = M(r,f')$. By the Cauchy inequality

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta,$$

where $C = \{\zeta : |\zeta - z_0| = s(r) - r\}$ and $s(r) = 1 - d(1-r)$, $d \in (0,1)$. We deduce that

$$M(r,f') = |f'(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f(\zeta)}{(\zeta - z_0)^2}(s(r) - r)\right| d\theta \leq \frac{M(s(r),f)}{s(r) - r},$$

i.e.

$$M(r,f') \leq \frac{M(s(r),f)}{(1-d)(1-r)}. \tag{3.14}$$

By (3.14), then $\sigma_{M,[p,q]}(f') \leq \sigma_{M,[p,q]}(f)$, $\mu_{M,[p,q]}(f') \leq \mu_{M,[p,q]}(f)$. Hence

$$\sigma_{M,[p,q]}(f) = \sigma_{M,[p,q]}(f'), \quad \mu_{M,[p,q]}(f') = \mu_{M,[p,q]}(f). \tag{3.15}$$

If $0 < \sigma_{M,[p,q]}(f) < \infty$ and by (3.13), (3.15), we can get $\tau_{M,[p,q]}(f') \geq \tau_{M,[p,q]}(f)$. Then by (3.14)-(3.15), if $p \geq q = 1$ we can obtain

$$\frac{\log_p M(r,f')}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f')}} \leq \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}}, \frac{\log_p M(s(r),f)}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}}\right\}$$

$$\leq \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}}, \frac{\log_p M(s(r),f)}{\left[\frac{1}{1-s(r)}\right]^{\sigma_{M,[p,1]}(f)}} \cdot \left(\frac{1}{d}\right)^{\sigma_{M,[p,1]}(f)}\right\},$$

let $d \to 1$, therefore $\tau_{M,[p,q]}(f') \leq \tau_{M,[p,q]}(f)$, then $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$. If $p \geq q \geq 2$, then

$$\frac{\log_p M(r, f')}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f')}} \leq \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f)}}, \frac{\log_p M(s(r), f)}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f)}}\right\},$$

thus we have $\tau_{M,[p,q]}(f') \leq \tau_{M,[p,q]}(f)$ and $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$. If $0 < \mu_{M,[p,q]}(f) < \infty$, we can similarly obtain $\underline{\tau}_{M,[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f')$.

Proof of Theorem 1.5. By Lemma 2.2, we have

$$T(r, f') = m(r, f') + N(r, f') \leq m(r, f) + m\left(r, \frac{f'}{f}\right) + 2N(r, f)$$

$$\leq 2T(r, f) + m\left(r, \frac{f'}{f}\right) \leq (2+\varepsilon)T(r, f) + O\left\{\log\frac{1}{1-r}\right\} \quad (r \notin E_1). \tag{3.16}$$

From (3.16) and Lemma 2.3, we have $\sigma_{[p,q]}(f') \leq \sigma_{[p,q]}(f), \mu_{[p,q]}(f') \leq \mu_{[p,q]}(f)$ for $p \geq q \geq 1$, $\tau_{[p,q]}(f') \leq \tau_{[p,q]}(f)$, $\underline{\tau}_{[p,q]}(f') \leq \underline{\tau}_{[p,q]}(f)$, for $p > 1$ and $\tau_{[1,1]}(f') \leq 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \leq 2\underline{\tau}_{[1,1]}(f)$ for $p = q = 1$. On the other hand, set $R = s(r) = 1 - d(1-r), d \in (0, 1)$ in Lemma 2.4, we have

$$T(r, f) < \left(2 + \frac{1}{\pi}\log\frac{3}{(1-d)(1-r)}\right)T(s(r), f'). \tag{3.17}$$

By (3.17) and by the similar proof in Theorem 1.4, we have $\sigma_{[p,q]}(f) \leq \sigma_{[p,q]}(f')$, $\mu_{[p,q]}(f) \leq \mu_{[p,q]}(f')$ for $p \geq q \geq 1$, $\tau_{[p,q]}(f) \leq \tau_{[p,q]}(f')$, $\underline{\tau}_{[p,q]}(f) \leq \underline{\tau}_{[p,q]}(f')$ for $p \geq q \geq 2$ and $\tau_{[p,q]}(f) \leq (\frac{1}{d})^{\sigma_{[p,q]}(f)}\tau_{[p,q]}(f')$ for $p > q = 1$, letting $d \to 1$, therefore the following statements hold:

If $p \geq q \geq 2$ and $p > q = 1$, then $\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f'), \mu_{[p,q]}(f) = \mu_{[p,q]}(f')$ and $\tau_{[p,q]}(f) = \tau_{[p,q]}(f')$ for $0 < \sigma_{[p,q]}(f) < \infty$, $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{[p,q]}(f')$ for $0 < \mu_{[p,q]}(f) < \infty$.

If $p = q = 1$, then $\sigma(f) = \sigma(f'), \mu(f) = \mu(f')$ and $\tau_{[1,1]}(f') \leq 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \leq 2\underline{\tau}_{[1,1]}(f)$.

Proof of Theorem 1.6. Without loss of generality, assume that $f(a) \neq 0$, by

$$N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n\left(t, \frac{1}{f-a}\right) - n\left(0, \frac{1}{f-a}\right)}{t}dt \quad (0 < r < 1),$$

we have

$$n\left(r, \frac{1}{f-a}\right) \leq \frac{1}{\log\left(1 + \frac{1-r}{2r}\right)}\int_r^{r+\frac{1-r}{2}} \frac{n\left(t, \frac{1}{f-a}\right)}{t}dt \leq \frac{1}{\log\left(1 + \frac{1-r}{2r}\right)}N\left(\frac{1+r}{2}, \frac{1}{f-a}\right), \tag{3.18}$$

where $0 < r < 1, \log(1 + \frac{1-r}{2r}) \sim \frac{1-r}{2r}, r \to 1^-$. By (3.18), we have

$$\overline{\lim_{r\to 1^-}}\frac{\log_p n\left(r, \frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)} \leq \max\left\{\overline{\lim_{r\to 1^-}}\frac{\log_p N\left(\frac{1+r}{2}, \frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)}, \overline{\lim_{r\to 1^-}}\frac{\log_p\left(\frac{2r}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}\right\}. \tag{3.19}$$

By (3.19), we can obtain
(i) if $p > q \geq 1$, then $\lambda^{n}_{[p,q]}(f,a) \leq \lambda^{N}_{[p,q]}(f,a)$;
(ii) if $p = q = 1$, then $\lambda^{n}(f,a) \leq \lambda^{N}(f,a) + 1$;
(iii) if $p = q \geq 2$, then $\lambda^{n}_{[p,p]}(f,a) \leq \max\left\{\lambda^{N}_{[p,p]}(f,a), 1\right\}$.
On the other hand, by

$$N\left(r,\frac{1}{f-a}\right)=\int_{r_0}^{r}\frac{n\left(t,\frac{1}{f-a}\right)}{t}dt+N\left(r_0,\frac{1}{f-a}\right)\leq n\left(r,\frac{1}{f-a}\right)\log\left(\frac{r}{r_0}\right)+O(1), \quad (3.20)$$

where $0 < r_0 < r < 1$. By (3.20), we can get
(i) if $p > q \geq 1$, then $\lambda^{N}_{[p,q]}(f,a) \leq \lambda^{n}_{[p,q]}(f,a)$;
(ii) if $p = q = 1$, then $\lambda^{N}(f,a) \leq \lambda^{n}(f,a)$;
(iii) if $p = q \geq 2$, then $\lambda^{N}_{[p,p]}(f,a) \leq \lambda^{n}_{[p,p]}(f,a)$.

Therefore, the conclusions of Theorem 1.6 hold.

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Reachable sets for semilinear integrodifferential control systems

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Abstract
In this paper, we consider a control system for semilinear integrodifferential equations in Hilbert spaces with Lipschitz continuous nonlinear term. Our method is to find the equivalence of approximate controllability for the given semilinear system and the linear system excluded the nonlinear term, which is based on results on regularity for the mild solution. Finally, we give a simple example to which our main result can be applied.

Keywords: approximate controllability, semilinear control system, lipschtiz continuity, approximate controllability, reachable set

AMS Classification Primary 35B37; Secondary 93C20

1 Introduction

Let H and V be real Hilbert spaces such that V is a dense subspace in H. In this paper, we are concerned with the control results for the following retarded semilinear control system in Hilbert space H:

$$\begin{cases} x'(t) = Ax(t) + g(t, x(t), \int_0^t k(t,s,x(s))ds)) + Bu(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

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where $t > 0$, B is a bounded linear controller, and $u(t)$ is an appropriate control functions. Let A be the operator associated with a bounded sesquilinear form defined in $V \times V$ satisfying Gårding inequality. Then it is well known that $S(t)$ generated by A is an analytic semigroup in both H and V^*, where V^* is the dual space of V, and so the system (1.1) may be considered as an system in both H and V^*. g is is a nonlinear mapping as detailed in Section 2.

Whether the reachable set associated with control space in dense subset of H. This is called an approximate controllability problem. As for linear evolution systems in general Banach, there are many papers and monographs, see [1, 2], Triggiani [3], Curtain and Zwart [4] and references and therein.

The controllability for nonlinear control systems has been studied by many authors, for example, control of nonlinear infinite dimensional systems in [5], controllability for parabolic equations with uniformly bounded nonlinear terms in [6], local controllability of neutral functional differential systems in [7].

Recently, the approximate controllability for semilinear control systems can be founded in [8, 9, 10], their results give sufficient condition on strict assumptions on the control action operator B. Similar considerations of semilinear systems have been dealt with in many references [11, 12, 13, 14].

We investigate the equivalence of approximate controllability for (1.1) such that excluded the nonlinear term and the controller. The solution mapping from the initial space to the solution space is Lipschitz continuous in $[0, T]$. We no longer require the strict range condition on B, and the uniform boundedness in [6] but instead we need the regularity and a variation of solutions of the given equations. For the basis of our study we construct the fundamental solution and establish variations of constant formula of solutions for the linear systems, see [15, 16].

Based on L^2-regularity properties of semilinear integrodifferential equations in Hilbert space and the regularity of solutions discussed in Section 2. We will obtain the relations between the reachable set of the semilinear system and that of its corresponding linear system in Section 3. Finally, a simple example to which our main result can be applied is given.

2 Regularity for retarded semilinear equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norms on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u, u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u, \ v \in V. \tag{2.3}$$

Then A is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. It is well known that A generates an analytic semigroup in both of H and V^*(see [17]).

From the following inequalities

$$\omega_1 ||u||^2 \leq \text{Re}\, a(u, u) + \omega_2 |u|^2 \leq |Au|\,|u| + \omega_2 |u|^2 \leq \max\{1, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C > 0$ such that

$$||u|| \leq C ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.4}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.5}$$

where each space is dense in the next one, which is continuous injection.

Lemma 2.1. *With the notations (2.1), (2.4), and (2.5), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [18]).

Assumption (K). Let $k : \mathbb{R}^+ \times [-h, 0] \times V \to H$ be a nonlinear mapping satisfying the following:

(K1) For any $x \in V$ the mapping $k(\cdot, \cdot, x)$ is measurable;

(K2) There exist positive constants K_0, K_1 such that

$$|k(t,s,x) - k(t,s,y)| \le K_1 ||x - y||,$$
$$|k(t,s,0)| \le K_0$$

for all $(t,s) \in \mathbb{R}^+ \times [-h, 0]$ and $x, y \in V$.

Assumption (G). Let $g : \mathbb{R}^+ \times V \times H \to H$ be a nonlinear mapping satisfying the following:

(G1) For any $x \in V$, $y \in H$ the mapping $g(\cdot, x, y)$ is measurable;

(G2) There exist positive constants L_0, L_1, L_2 such that

$$|g(t,x,y) - g(t,\hat{x},\hat{y})| \le L_1 ||x - \hat{x}|| + L_2 |y - \hat{y}|,$$
$$|g(t,0,0)| \le L_0$$

for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $y, \hat{y} \in H$.

For $x \in L^2(-h, T; V)$, $T > 0$ we set

$$G(t,x) = g(t, x(t), \int_0^t k(t,s,x(s))ds).$$

The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [19]. The mild solution of (1.1) is represented by

$$x(t) = S(t)x_0 + \int_0^t \big\{G(s, x(s)0 + Bu(s)\big\}ds, \quad t \ge 0.$$

Lemma 2.2. *Let $x \in L^2(0,T;V)$, $T > 0$. Then $G(\cdot, x) \in L^2(0,T;H)$ and*

$$||G(\cdot, x)||_{L^2(0,T;H)} \le (L_0 + K_0 L_2)\sqrt{T} + (L_1 + L_2 K_1 T)||x||_{L^2(0,T;V)}.$$

Moreover if $x_1,\ x_2 \in L^2(0,T;V)$, then

$$||G(\cdot, x_1) - G(\cdot, x_2)||_{L^2(0,T;H)} \le (L_1 + L_2 K_1 T)||x_1 - x_2||_{L^2(0,T;V)}. \tag{2.6}$$

Proof. Hence, from (K2), (G2) and the above inequality it is easily seen that

$$\begin{aligned}
||G(\cdot,x)||_{L^2(0,T;H)} &\le ||G(\cdot,0)|| + ||G(\cdot,x) - G(\cdot,0)|| \\
&\le L_0\sqrt{T} + L_1||x||_{L^2(0,T;V)} + L_2||\int_0^{\cdot} k(\cdot,s,x(s))ds||_{L^2(0,T;H)} \\
&\le L_0\sqrt{T} + L_1||x||_{L^2(0,T;V)} + L_2K_1T||x||_{L^2(0,T;V)} + K_0L_2\sqrt{T} \\
&\le (L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)||x||_{L^2(0,T;V)}
\end{aligned}$$

Similarly, we can prove (2.6). □

In view of Lemma 2.2, we can apply the regularity results of Theorem 3.1 of [10] to (1.1), and so we obtain the following results.

Proposition 2.1. *1) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (2.7) belonging to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.7}$$

where C_1 is a constant depending on T.

2) If $x_0 \in H$ and $k \in L^2(0,T;V^*)$, then the mapping

$$H \times L^2(0,T;V^*) \ni (x_0,k) \mapsto x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$$

is Lipschitz continuous.

Here, we note that by using interpolation theory, we have that for $z \in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$, there exists a constant $C_2 > 0$ such that

$$||z||_{C([0,T];H)} \le C_2||z||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}. \tag{2.8}$$

3 Approximately reachable sets

Let U be a Banach space and the controller operator B is bounded linear operator from another Banach space U to X.

Let $S(t)$ be an analytic semigroup generation by A. Then we may assume that there exists a positive constant C_0 such that

$$||S(t)|| \le C_0, \quad ||AS(t)|| \le C_0/t(t>0). \tag{3.1}$$

The solution $x(t) = x(t; x_0, G, u)$ of initial value problem (1,1) is the following form:

$$x(t; x_0, G, u) = S(t)x_0 + \int_0^t S(t-s)\{G(t, x(s)) + Bu(s)\}ds, \quad t > 0,$$

For $T > 0$, $x_0 \in H$ and $u \in L^2(0, T; U)$ we define reachable sets as follows.

$$\begin{aligned} L_T(x_0) &= \{x(T; x_0, 0, u) : u \in L^2(0, T; U)\}, \\ R_T(x_0) &= \{x(T; x_0, G, u) : u \in L^2(0, T; U)\}, \\ L(x_0) &= \bigcup_{T>0} L_T(x_0), \quad R(x_0) = \bigcup_{T>0} R_T(x_0). \end{aligned}$$

Definition 3.1. *(1) System (1.1) is said to be H-approximately controllable for initial value x_0 (resp. in time T) if $\overline{R(x_0)} = H$ (resp. $\overline{R_T(x_0)} = H$).*
(2) The linear system corresponding (1.1) is said to be H-approximately controllable for initial value x_0 (resp. in time T) if $\overline{L(x_0)} = H$ (resp. $\overline{L_T(x_0)} = H$).

Remark 3.1. *Since A generate an analytic semigroup, the following (1)-(4) are equivalent for the linear system (see [2, Theorem 3.10]).*

(1) $\overline{L(x_0)} = H \quad \forall x_0 \in H$.

(2) $\overline{L(0)} = H$.

(3) $\overline{L_T(x_0)} = H \quad \forall x_0 \in H$.

(4) $\overline{L_T(0)} = H$.

Theorem 3.1. *For any $T > 0$ we have*

$$\overline{R_T(0)} \subset \overline{L_T(0)}.$$

Proof. Let $z_0 \notin \overline{L_T(0)}$. Since $\overline{L_T(0)}$ is a balanced closed convex subspace, we have $\alpha z_0 \notin \overline{L_T(0)}$ for every $\alpha \in \mathbb{R}$, and

$$\inf\{||z_0 - z|| : z \in \overline{L_T(0)}\} = d.$$

By the formula (2.7) we have

$$||x(\cdot; 0, G, u)||_{L^2(0,T;V)} \le C_1||B||||u||_{L^2(0,T;U)}, \tag{3.2}$$

where C_1 is the constant in Proposition 2.1. For every $u \in L^2(0,T;U)$, we choose a constant $\alpha > 0$ such that

$$C_0\{(L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)C_1||B||||u||_{L^2(0,T;U)}\} < \alpha d. \tag{3.3}$$

Hence form (3.2), (3.3) and by using Hölder inequality, it follows that

$$\begin{aligned}
&|x(T;0,G,u) - \alpha z_0| \\
&\geq |\int_0^T S(T-s)Bu(s)ds - \alpha z_0| - |\int_0^T S(T-s)G(s,x(s))ds| \\
&\geq \alpha d - C_0\{(L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)||x||_{L^2(0,T;V)}\} \\
&\geq \alpha d - C_0\{(L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)C_1||B||||u||_{L^2(0,T;U)}\} > 0.
\end{aligned}$$

Thus, we have $\alpha z_0 \notin \overline{R_T(0)}$. □

Lemma 3.1. *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \leq t \leq T$. Then there exists a constant C_3 such that*

$$||x||_{L^2(0,T;D(A))} \leq C_1||k||_{L^2(0,T;H)}, \tag{3.4}$$
$$||x||_{L^2(0,T;H)} \leq C_3T||k||_{L^2(0,T;H)}, \tag{3.5}$$

and

$$||x||_{L^2(0,T;V)} \leq C_3\sqrt{T}||k||_{L^2(0,T;H)}. \tag{3.6}$$

Proof. The assertion (3.4) is immediately obtained by (2.7). Since

$$\begin{aligned}
||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2dt \leq C_0\int_0^T(\int_0^t |k(s)|ds)^2dt \\
&\leq C_0\int_0^T t\int_0^t |k(s)|^2dsdt \leq C_0\frac{T^2}{2}\int_0^T |k(s)|^2ds
\end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \leq T\sqrt{C_0/2}||k||_{L^2(0,T;H)}.$$

From (2.4), (3.4), and (3.5) it holds that

$$||x||_{L^2(0,T;V)} \leq C\sqrt{C_1T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_3 > 0$ such that

$$C_3 = \max\{\sqrt{C_0/2}, C\sqrt{C_1}(C_0/2)^{1/4}\},$$

the proof is complete. □

Theorem 3.2. *Under Assumptions (K) and (G), for any $x_0 \in H$ we have*

$$\overline{L_T(x_0)} \subset \overline{R_T(x_0)}.$$

.

Proof. Let $u \in L^2(0,T;U)$ be arbitrary fixed. Then by (2.7) we have

$$||x_u||_{L^2(0,T;V)} \le C_1(|x_0| + ||B||||u||_{L^2(0,T;U)}),$$

where x_u is the solution of (1.1) corresponding to the control u. For any $\epsilon > 0$, we can choose a constant $\delta > 0$ satisfying

$$\begin{aligned}\min\{\sqrt{\delta},\delta\} < \min\big[&\{7C_3(L_1 + L_2K_1T))\}^{-1},\\ &\epsilon\{C_3(L_0 + K_0L_2\sqrt{T})\}^{-1},\\ &\epsilon\{C_3(L_1 + L_2K_1T)(C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} + \epsilon)\}^{-1},\\ &\epsilon\{C_3(C_0||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} + \epsilon)(L_1 + L_2K_1T)\}^{-1},\\ &\epsilon\{(C_3^2(L_0 + K_0L_2)\sqrt{T} + \epsilon)(L_1 + L_2K_1T)\}^{-1}\big]/6.\end{aligned} \tag{3.7}$$

Set

$$\begin{aligned}x_1 := x(T-\delta; x_0, G, u) = S(T-\delta)x_0 +&\\ + \int_0^{T-\delta} S(T-\delta-s)G(s, x_u(s))ds + \int_0^{T-\delta} S(T-\delta-s)Bu(s)ds,&\end{aligned}$$

where $x_u(t) = x(t; x_0, G, u)$ for $0 < t \le T$. Consider the following problem:

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \quad \delta < t \le T, \\ y(T-\delta) = x_1, \quad y(s) = 0 \quad -h \le s \le 0. \end{cases} \tag{3.8}$$

The solution of (3.8) with respect to the control $w \in L^2(T-\delta, T; U)$ is denoted by

$$\begin{aligned} y_w(T) &= S(\delta)x_1 + \int_{T-\delta}^{T} S(T-s)Bw(s)ds \\ &= S(T)x_0 + S(\delta)\int_0^{T-\delta} S(T-\delta-s)G(s, x_u(s))ds \\ &+ S(\delta)\int_0^{T-\delta} S(T-\delta-s)Bu(s)ds + \int_{T-\delta}^{T} S(T-s)Bw(s)ds. \end{aligned} \tag{3.9}$$

Then since $z \in \overline{L_T(x_0)}$, and $\overline{L_T(x_0)} = \overline{L(0)}$ is independent of the time T and initial data x_0(see Remark 2.1), there exists $w_1 \in L^2(T-\delta, T; U)$ such that

$$\sup_{T-\delta \le t \le T} |y_{w_1}(t) - z| < \frac{\epsilon}{6}, \tag{3.10}$$

and hence, by (3.9),

$$|\int_{T-\delta}^{t} S(T-s)Bw_1(s)ds| \le C_0 ||x_u||_{L^2(0,T-\delta;V)} + \frac{\epsilon}{6}, \quad t-\delta \le t \le T. \tag{3.11}$$

Now, we set

$$v(s) = \begin{cases} u & \text{if } \ 0 \le s \le T-\delta, \\ w_1(s) & \text{if } \ T-\delta < s < T. \end{cases}$$

Then $v \in L^2(0, T; U)$. Observing that

$$x_v(t; G, v) = S(t)x_0 + \int_0^t S(t-\tau)\{G(\tau, x_v(\tau)) + Bv(\tau)\}d\tau,$$

from (3.9) and (3.10) we obtain that

$$\begin{aligned} &|x(T; x_0, G, v) - z| \le |y_{w_1}(T) - z| + |x(T; x_0, G, v) - y_{w_1}(T)| \\ &\le |y_{w_1}(T) - z| \\ &\quad + \Big|\int_0^T S(T-s)G(s, x_v(s))ds - S(\delta)\int_0^{T-\delta} S(T-\delta-s)G(s, x_u(s))ds\Big| \\ &\quad + \Big|\int_0^T S(T-s)Bv(s)ds - S(\delta)\int_0^{T-\delta} S(T-\delta-s)Bu(s)ds \\ &\quad - \int_{T-\delta}^T S(T-s)Bw_1(s)ds\Big| \\ &\le \frac{\epsilon}{6} + \Big|\int_{T-\delta}^T S(T-s)G(s, x_{w_1}(s))ds\Big| \\ &\le \frac{\epsilon}{6} + II. \end{aligned} \tag{3.12}$$

Here, we remind that the x_{w_1} is represented by

$$\begin{aligned} x_{w_1}(t) =& S(t)x(T-\delta; x_0, G, u) \\ &+ \int_{T-\delta}^t S(T-s)G(s, x_{w_1}(s))ds + \int_{T-\delta}^t S(T-s)Bw_1(s))ds \end{aligned}$$

for $T-\delta < t \le T$. Here, by (2.7) we have

$$\begin{aligned} ||S(\cdot)x(T-\delta;x_0,G,u)||_{L^2(0,T;V)} &\le C_1|x(T-\delta;x_0,G,u)| \\ &\le C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}. \end{aligned} \tag{3.13}$$

Put

$$p(t)=\int_{T-\delta}^{t} S(t-s)G(s,x_{w_1}(s))ds, \quad T-\delta<t\le T,$$

and

$$q(t):=\int_{t-\delta}^{T} S(t-s)Bw_1(s)ds \quad T-\delta<t\le T.$$

Then with aid of (3.6) of Lemma 3.1 and Lemma 2.2, we have

$$\begin{aligned} &||p||_{L^2(T-\delta,T;V)} \le C_3\sqrt{\delta}||G(\cdot,x_{w_1})||_{L^2(T-\delta,T;V)} \\ &\le C_3\sqrt{\delta}\{(L_0+K_0L_2)\sqrt{T}+(L_1+L_2K_1T)||x_{w_1}||_{L^2(T-\delta,T;V)}\}, \end{aligned} \tag{3.14}$$

and by (3.11),

$$||q||_{L^2(T-\delta,T;V)} \le \sqrt{\delta}(C_0||x_u||_{L^2(0,T-\delta;V)}+\frac{\epsilon}{6}). \tag{3.15}$$

Since $C_3\sqrt{\delta}(L_1+L_2K_1T))<1$ by virtue of (3.7), by (3.13)-(3.15), we get

$$\begin{aligned} ||x_{w_1}||_{L^2(T-\delta,T;V)} \le &\{C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \\ &+\sqrt{\delta}(C_0||x_u||_{L^2(0,T-\delta;V)}+\frac{\epsilon}{6}) \\ &+C_3\sqrt{\delta T}(L_0+K_0L_2)\}\{1-C_3\sqrt{\delta}(L_1+L_2K_1T))\}^{-1}. \end{aligned} \tag{3.16}$$

Hence, with aid of (3.6), (3.7), (3.16), and by using the Hölder inequality, we have

$$\begin{aligned} II =&\left|\int_{T-\delta}^{T} S(T-s)G(s,x_{w_1}(s))ds\right| \\ &\le C_3\sqrt{\delta T}\{(L_0+K_0L_2)+(L_1+L_2K_1T)||x_{w_1}||_{L^2(T-\delta,T;V)}\} \\ &\le C_3\sqrt{\delta T}(L_0+K_0L_2)+C_3\sqrt{\delta}(L_1+L_2K_1T)\big\{C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \\ &+\sqrt{\delta}(C_0||x_u||_{L^2(0,T-\delta;V)}+\frac{\epsilon}{6}) \\ &+C_3\sqrt{\delta T}(L_0+K_0L_2)\big\}\big\{1-C_3\sqrt{\delta}(L_1+L_2K_1T))\big\}^{-1} < \frac{5\epsilon}{6}. \end{aligned} \tag{3.17}$$

Therefore, by (3.12) and (3.17), we have

$$||x(T;x_0,G,v)-z||_H<\epsilon,$$

that is, $z\in\overline{R_T(x_0)}$ and the proof is complete. □

Remark 3.2. *Noting that $H([0,T];U)$ is dense in $L^2(0,T;U)$, we can obtain the same results of Theorem 3.2 corresponding to (1.1) with control space*

$$H([0,T];U) = \{w : [0,T] \to U : |w(t) - w(s)| \le H_0|t-s|^\theta,\ 0 < \theta < 1,\ H_0 > 0\}$$

instead of $L^2(0,T;U)$

From Theorems 3.1-2, we obtain the following control results of (1.1).

Corollary 3.1. *Under Assumptions (K) and (G), for $T > 0$ we have*

$$\overline{L_T(x_0)} = H \iff \overline{R_T(x_0)} = H.$$

Therefore, the approximate controllability of linear system (1.1) with $g = 0$ is equivalent to the condition for the approximate controllability of the nonlinear system (1.1).

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On modified degenerate poly-tangent numbers and polynomials

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Abstract : In this paper we introduce the modified degenerate degenerate poly-tangent polynomials and numbers. We also give some properties, explicit formulas, several identities, a connection with modified degenerate poly-tangent numbers and polynomials, and some integral formulas. Finally, we investigate the zeros of the modified degenerate poly-tangent polynomials by using computer.

Key words : Tangent numbers and polynomials, degenerate poly-tangent numbers and polynomials, Cauchy numbers, Stirling numbers, modified degenerate poly-tangent polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials(see [1-11]). In this paper, we define modified degenerate poly-tangent polynomials and numbers and study some properties of the modified degenerate poly-tangent polynomials and numbers. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Carlitz [1] has defined the degenerate Stirling numbers of the first kind and second kind, $S_1(n,k,\lambda)$ and $S_2(n,k,\lambda)$ by means of

$$\left(\frac{1-(1-t)^{\lambda}}{\lambda}\right)^k = k!\sum_{n=k}^{\infty} S_1(n,k,\lambda)\frac{t^n}{n!}, \tag{1.1}$$

$$\left((1+\lambda t)^{1/\lambda}-1\right)^k = k!\sum_{n=k}^{\infty} S_2(n,k,\lambda)\frac{t^n}{n!}. \tag{1.2}$$

Howard [12] has defined the degenerate weighted Stirling numbers of the first kind and second kind, $S_1(n,k,x,\lambda)$ and $S_2(n,k,x,\lambda)$ by means of

$$(1-t)^{\lambda-x}\left(\frac{1-(1-t)^{\lambda}}{\lambda}\right)^k = k!\sum_{n=k}^{\infty} S_1(n,k,x,\lambda)\frac{t^n}{n!}, \tag{1.3}$$

$$(1+\lambda t)^{x/\lambda}\left((1+\lambda t)^{1/\lambda}-1\right)^k = k!\sum_{n=k}^{\infty} S_2(n,k,x,\lambda)\frac{t^n}{n!}. \tag{1.4}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k).$$

The generalized raising factorial $< x|\lambda >_n$ with increment λ is defined by

$$< x|\lambda >_n = \prod_{k=0}^{n-1}(x+\lambda k).$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}.$$

The degenerate poly-Bernoulli numbers $\mathcal{B}_n^{(k)}(\lambda)$ were introduced by Kaneko [5] by using the following generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-(1+\lambda t)^{-1/\lambda}} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(\lambda)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}), \tag{1.5}$$

where

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k} \tag{1.6}$$

is the kth polylogarithm function.

The degenerate poly-Euler polynomials $\mathcal{E}_n^{(k)}(x,\lambda)$ are defined by generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{1.7}$$

The familiar degenerate tangent polynomials $\mathbf{T}_n(x,\lambda)$ are defined by the generating function([7]):

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{T}_n(x,\lambda)\frac{t^n}{n!}, \quad (|2t| < \pi). \tag{1.8}$$

When $x = 0$, $\mathbf{T}_n(0,\lambda) = \mathbf{T}_n(\lambda)$ are called the degenerate tangent numbers. The degenerate tangent polynomials $\mathbf{T}_n^{(r)}(x,\lambda)$ of order r are defined by

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^r(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{T}_n^{(r)}(x,\lambda)\frac{t^n}{n!}, \quad (|2t| < \pi). \tag{1.9}$$

It is clear that $r = 1$ we recover the degenerate tangent polynomials $\mathbf{T}_n(x,\lambda)$.

The degenerate Bernoulli polynomials $\mathbf{B}_n^{(r)}(x,\lambda)$ of order r are defined by the following generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{B}_n^{(r)}(x,\lambda)\frac{t^n}{n!}, \quad (|t| < 2\pi). \tag{1.10}$$

The degenerate Frobenius-Euler polynomials of order r, denoted by $\mathbf{H}_n^{(r)}(u,x,\lambda)$, are defined as

$$\left(\frac{1-u}{(1+\lambda t)^{1/\lambda}-u}\right)^r(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}. \tag{1.11}$$

The values at $x = 0$ are called degenerate Frobenius-Euler numbers of order r; when $r = 1$, the polynomials or numbers are called ordinary degenerate Frobenius-Euler polynomials or numbers.

The degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are defined by the generating function:

$$\frac{2\mathrm{Li}_k\left(1-e^{-t}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{1.12}$$

When $x = 0$, $\mathcal{T}_n^{(k)}(0,\lambda) = \mathcal{T}_n^{(k)}(x,\lambda)$ are called the degenerate poly-tangent numbers. Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-12]). In the following section, we introduce the modified degenerate poly-tangent polynomials and numbers. After that we will investigate some their properties. We also give some relationships both between these polynomials and modified degenerate poly-tangent polynomials and between these polynomials and cauchy numbers. Finally, we investigate the zeros of the modified degenerate poly-tangent polynomials by using computer.

2. Modified degenerate poly-tangent polynomials

In this section, we define modified degenerate poly-tangent numbers and polynomials and provide some of their relevant properties.

The modified degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are defined by the generating function:

$$\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!},\quad (k\in\mathbb{Z}). \tag{2.1}$$

When $x=0$, $\mathcal{T}_n^{(k)}(0,\lambda)=\mathcal{T}_n^{(k)}(x,\lambda)$ are called the degenerate poly-tangent numbers. Upon setting $k=1$ in (2.1), we have

$$\mathcal{T}_n^{(1)}(x,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\lambda^{n-1}S_1(l,1)\mathbf{T}_{n-l}(x,\lambda) \text{ for } n\geq 1.$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda}\\&=\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

By comparing the coefficients on both sides of (2.2), we have the following theorem.

Theorem 2.1. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(\lambda)(x|\lambda)_{n-l}.$$

The following elementary properties of the degenerate poly-tangent numbers $\mathcal{T}_n^{(k)}(\lambda)$ and polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are readily derived form (2.1). We, therefore, choose to omit details involved.

Theorem 2.2. For $k\in\mathbb{Z}$, we have

$$(1)\quad \mathcal{T}_n^{(k)}(x+y,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)(y|\lambda)_{n-l}.$$

$$(2)\quad \mathcal{T}_n^{(k)}(2-x,\lambda)=\sum_{l=0}^{n}(-1)^l\binom{n}{l}\mathcal{T}_{n-l}^{(k)}(2,\lambda)<x|\lambda>_l.$$

Theorem 2.3 For any positive integer n, we have

$$\begin{aligned}&(1)\quad \mathcal{T}_n^{(k)}(mx,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)((m-1)x|\lambda)_{n-l}.\\&(2)\quad \mathcal{T}_n^{(k)}(x+1,\lambda)-\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{n-1}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)(1|\lambda)_{n-l}.\end{aligned} \tag{2.3}$$

From (1.6), (1.8), and (2.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}=\left(2\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda}\\
&=\sum_{l=0}^{\infty}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{l+1}}{(l+1)^k}\frac{2(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\\
&=\sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\frac{2(1+\lambda t)^{x/\lambda}\left(1+\lambda t\right)^{-i/\lambda}}{(1+\lambda t)^{2/\lambda}+1} \qquad (2.4)\\
&=\sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathbf{T}_j(x,\lambda)(-1)^{n-j}<i|\lambda>_{(n-j)}\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n}\frac{1}{(l+1)^k}\binom{l+1}{i}(-i)^{n+i-j}\binom{n}{j}\mathbf{T}_j(x,\lambda)<i|\lambda>_{(n-j)}\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides of (2.4), we have the following theorem.

Theorem 2.4 For $n\in\mathbb{Z}_+$, we have

$$\begin{aligned}
\mathcal{T}_n^{(k)}(x,\lambda)&=\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n}\frac{(-i)^{n+i-j}}{(l+1)^k}\binom{l+1}{i}\binom{n}{j}\mathbf{T}_j(x,\lambda)<i|\lambda>_{(n-j)}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\frac{(-i)^i}{(l+1)^k}\binom{l+1}{i}\mathbf{T}_n(x-i,\lambda).
\end{aligned}$$

By (2.1), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}=2\sum_{l=0}^{\infty}(-1)^l(1+\lambda t)^{2l/\lambda}\sum_{l=0}^{\infty}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{l+1}}{(l+1)^k}(1+\lambda t)^{x/\lambda}\\
&=2\sum_{l=0}^{\infty}\sum_{i=0}^{l}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{i+1}}{(i+1)^k}(-1)^{l-i}(1+\lambda t)^{(2l-2i)/\lambda}(1+\lambda t)^{x/\lambda}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}}{(i+1)^k}(1+\lambda t)^{(2l-2i+x)/\lambda}\left(1+\lambda t\right)^{-j/\lambda}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\sum_{m=0}^{n}\frac{2(-1)^{l+j-i}\binom{i+1}{j}\binom{n}{m}(2l-2i+x|\lambda)_m<j|\lambda>_{(n-m)}}{(i+1)^k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we have the following theorem.

Theorem 2.5 For $n\in\mathbb{Z}_+$, we have

$$\begin{aligned}
\mathcal{T}_n^{(k)}(x,\lambda)&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\sum_{m=0}^{n}\frac{2(-1)^{l+j-i}\binom{i+1}{j}\binom{n}{m}(2l-2i+x|\lambda)_m<j|\lambda>_{(n-m)}}{(i+1)^k}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}(2l-2i-j+x|\lambda)_m}{(i+1)^k}.
\end{aligned}$$

3. Some identities involving degenerate poly-tangent numbers and polynomials

In this section, we give several combinatorics identities involving degenerate poly-tangent numbers and polynomials in terms of degenerate Stirling numbers, generalized falling factorial functions, generalized raising factorial functions, Beta functions, degenerate Bernoulli polynomials of higher order, and degenerate Frobenius-Euler functions of higher order.

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(1-(1-(1+\lambda t)^{-1/\lambda})\right)^{-x}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-(1+\lambda t)^{-1/\lambda})^l\\
&=\sum_{l=0}^{\infty}<x>_l\frac{((1+\lambda t)^{1/\lambda}-1)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{-l/\lambda}\right)\\
&=\sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(-l,\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}(-l,\lambda)<x>_l\right)\frac{t^n}{n!},
\end{aligned}\tag{3.1}$$

where $<x>_l=x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0=1$.

By comparing the coefficients on both sides of (3.1), we have the following theorem.

Theorem 3.1 For $n\in\mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}(-l,\lambda)<x>_l.$$

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(1-(1-(1+\lambda t)^{-1/\lambda})\right)^{-x}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-(1+\lambda t)^{-1/\lambda})^l\\
&=\sum_{l=0}^{\infty}<x>_l\frac{(1+\lambda t)^{-l/\lambda}((1+\lambda t)^{1/\lambda}-1)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l,-l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,-l,\lambda)\mathcal{T}_{n-i}^{(k)}(\lambda)<x>_l\right)\frac{t^n}{n!},
\end{aligned}\tag{3.2}$$

where $<x>_l=x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0=1$.

By comparing the coefficients on both sides of (3.2), we have the following theorem.

Theorem 3.2 For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i} S_2(i,l,-l,\lambda)\mathcal{T}_{n-i}^{(k)}(\lambda) < x >_l .$$

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!} &= \left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(((1+\lambda t)^{1/\lambda}-1)+1\right)^x \\
&= \frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x}{l}\left((1+\lambda t)^{1/\lambda}-1\right)^l \\
&= \sum_{l=0}^{\infty}(x)_l\frac{\left((1+\lambda t)^{1/\lambda}-1\right)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right) \\
&= \sum_{l=0}^{\infty}(x)_l\sum_{n=0}^{\infty}S_2(n,l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.3}$$

By comparing the coefficients on both sides of (3.3), we have the following theorem.

Theorem 3.3 For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}.$$

By (1.2), (1.10), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!} \\
&= \left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} \\
&= \frac{((1+\lambda t)^{1/\lambda}-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!} \\
&= \frac{((1+\lambda t)^{1/\lambda}-1)^r}{r!}\left(\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\frac{r!}{t^r} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r,\lambda)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x,\lambda)\mathcal{T}_{n-l-i}^{(k)}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.4 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{n}\sum_{i=0}^{n-l}\frac{\binom{n}{l}\binom{n-l}{i}}{\binom{l+r}{r}}S_2(l+r,r)T_{n-l-i}^{(k)}\mathbf{B}_i^{(r)}(x,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&=\frac{((1+\lambda t)^{1/\lambda}-u)^r}{(1-u)^r}\left(\frac{1-u}{(1+\lambda t)^{1/\lambda}-u}\right)^r(1+\lambda t)^{x/\lambda}\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\\
&=\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}\sum_{i=0}^{r}\binom{r}{i}(1+\lambda t)^{i/\lambda}(-u)^{r-i}\frac{1}{(1-u)^r}\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\\
&=\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(i,\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{l=0}^{n}\binom{n}{l}\mathbf{H}_l^{(r)}(u,x,\lambda)\mathcal{T}_{n-l}^{(k)}(i,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.5 For $n\in\mathbb{Z}_+$ and $r\in\mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}_l^{(r)}(u,x,\lambda)\mathcal{T}_{n-l}^{(k)}(i,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\frac{(1+\lambda t)^{1/\lambda}+1}{(1+\lambda t)^{1/\lambda}+1}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}\left(\frac{(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{2/\lambda}+1}+\frac{1}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\left(\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{1}{2}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\mathcal{E}_{n-l}^{(k)}(x,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.6 For $n\in\mathbb{Z}_+$ and $r\in\mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\mathcal{E}_{n-l}^{(k)}(x,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\frac{1-(1+\lambda t)^{-1/\lambda}}{1-(1+\lambda t)^{-1/\lambda}}\\
&=\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{1-(1+\lambda t)^{-1/\lambda}}\left(\frac{2(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{2/\lambda}+1}-\frac{2(1+\lambda t)^{(x-1)/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\left(\sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\mathbf{T}_n(x,\lambda)-\mathbf{T}_n(x-1,\lambda)\right)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(x,\lambda)-\mathbf{T}_n(x-1,\lambda)\right)\mathcal{B}_{n-l}^{(k)}(x,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.7 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(x,\lambda) - \mathbf{T}_n(x-1,\lambda)\right) \mathcal{B}_{n-l}^{(k)}(\lambda).$$

By Theorem 3.6 and Theorem 3.7, we have the following corollary.

Corollary 3.8 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}&\sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(1,\lambda) + \mathbf{T}_n(\lambda)\right) \mathcal{E}_{n-l}^{(k)}(x,\lambda)\\ &= 2\sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(x,\lambda) - \mathbf{T}_n(x-1,\lambda)\right) \mathcal{B}_{n-l}^{(k)}(\lambda).\end{aligned}$$

3. Distribution of zeros of the degenerate poly-tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$. The degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ can be determined explicitly. A few of them are

$$\begin{aligned}
&\mathcal{T}_0^{(k)}(x,\lambda) = 0,\\
&\mathcal{T}_1^{(k)}(x,\lambda) = 1,\\
&\mathcal{T}_2^{(k)}(x,\lambda) = -3 + 2^{1-k} - \lambda + 2x\\
&\mathcal{T}_3^{(k)}(x,\lambda) = 4 - 3\cdot 2^{2-k} + 2\cdot 3^{1-k} + 9\lambda - 3\cdot 2^{1-k}\lambda + 2\lambda^2 - 9x\\
&\qquad + 3\cdot 2^{1-k}x - 6\lambda x + 3x^2,\\
&\mathcal{T}_4^{(k)}(x,\lambda) = 3 + 3^{3-2k} + 7\cdot 2^{1-k} + 3\cdot 2^{3-k} - 8\cdot 3^{1-k} - 4\cdot 3^{2-k} - 24\lambda\\
&\qquad + 3\cdot 2^{3-k}\lambda + 3\cdot 2^{4-k}\lambda - 4\cdot 3^{2-k}\lambda - 33\lambda^2 + 11\cdot 2^{1-k}\lambda^2 - 6\lambda^3\\
&\qquad + 16x - 3\cdot 2^{4-k}x + 8\cdot 3^{1-k}x + 54\lambda x - 3\cdot 2^{2-k}\lambda x - 3\cdot 2^{3-k}\lambda x\\
&\qquad + 22\lambda^2 x - 18x^2 + 3\cdot 2^{2-k}x^2 - 18\lambda x^2 + 4x^3.
\end{aligned}$$

We investigate the beautiful zeros of thedegenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ by using a computer. We plot the zeros of the poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ for $n = 30, k = -5, -1, 1, 5, \lambda = 1/2$, and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 30$ and $k = -5$. In Figure 1(top-right), we choose $n = 30$ and $k = -1$. In Figure 1(bottom-left), we choose $n = 30$ and $k = 1$. In Figure 1(bottom-right), we choose $n = 30$ and $k = 5$. Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$ from a 3-D structure are presented(Figure 2). In Figure 2(left), we choose $k = -5$. In Figure 2(middle), we choose $k = 1$. In Figure 2(right), we choose $k = 5$. Our numerical results for approximate solutions of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda), \lambda = 1/2$ are displayed(Tables 1, 2).

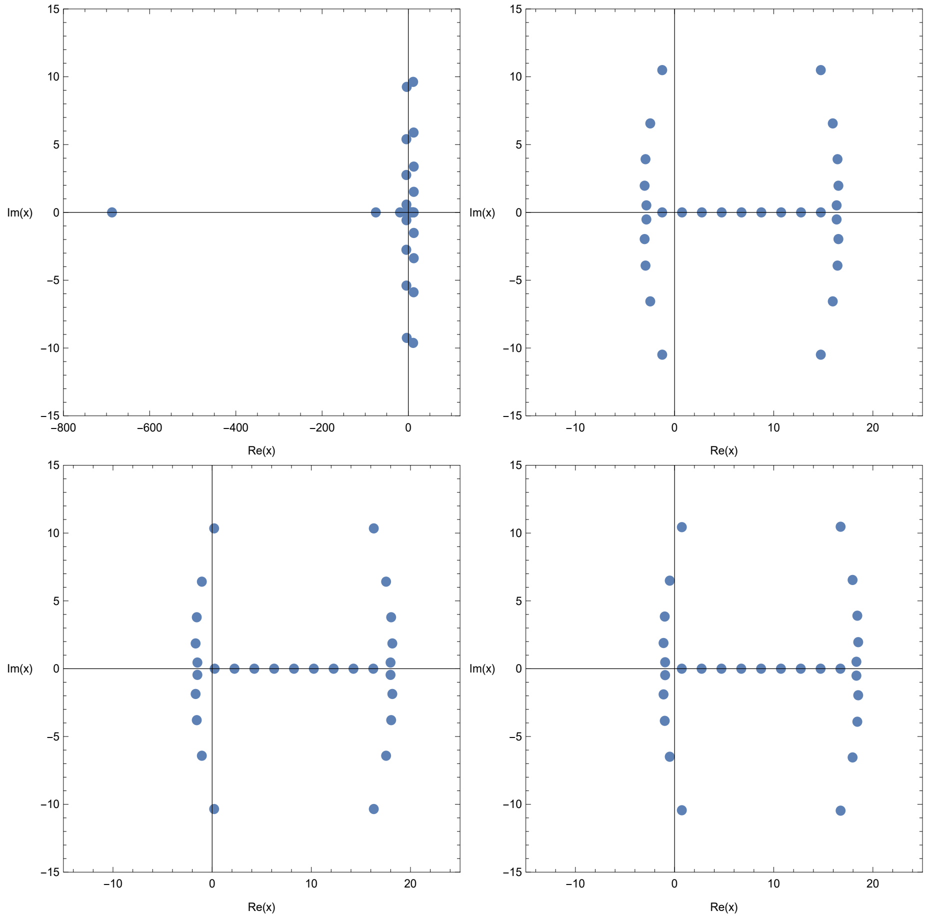

Figure 1: Zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

Table 1. Numbers of real and complex zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

	$k=-10$		$k=1$		$k=10$	
degree n	real	complex zeros	real	complex zeros	real	complex zeros
2	1	0	1	0	1	0
3	2	0	2	0	2	0
4	3	0	3	0	3	0
5	4	0	4	0	4	0
6	5	0	5	0	5	0
7	6	0	2	4	2	4
8	5	2	3	4	3	4
9	6	2	4	4	4	4
10	5	4	5	4	5	4
11	6	4	6	4	6	4
12	7	4	7	4	5	6

The plot of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \leq n \leq 30$ structure are presented(Figure 3). In Figure 3(left), we choose $k=-5$ and $\lambda=1/2$. In Figure 3(middle),we choose $k=1$ and $\lambda=1/2$. In Figure 3(right), we choose $k=5$ and $\lambda=1/2$.

We observe a remarkable regular structure of the complex roots of the degenerate poly-tangent

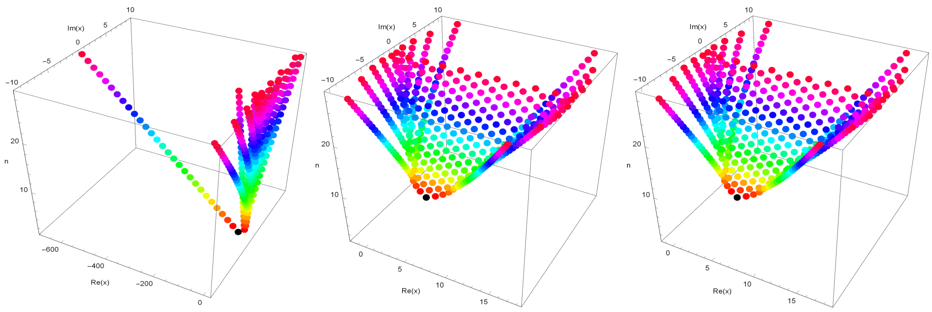

Figure 2: Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$

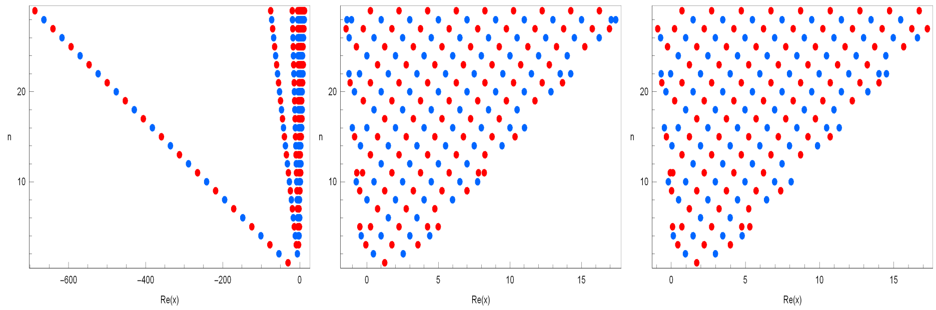

Figure 3: Real zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$

polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$. We also hope to verify a remarkable regular structure of the complex roots of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$(Table 1).

Next, we calculated an approximate solution satisfying poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda) = 0$ for $x \in \mathbb{R}$. The results are given in Table 2 and Table 3.

Table 2. Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda) = 0, \lambda = 1/2, k = -5$

degree n	x
2	30.250
3	-53.896, -6.1044
4	-77.421, -8.8591, -2.9699
5	-100.91, -11.489, -3.9628, -1.6365
6	-124.39, -14.080, -4.7720, -2.3421, -0.66874
7	-147.85, -16.655, -5.4611, -3.0181, -1.0879, 0.076439

Table 3. Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda)=0, \lambda=1/2, k=5$

degree n	x
2	1.7188
3	0.95682, 2.9807
4	0.44597, 2.2234, 3.9869
5	0.13979, 1.4750, 3.4758, 4.7844
6	0.090663, 0.71964, 2.7246, 4.7571, 5.3017
7	1.9752, 3.9751

By numerical computations, we will make a series of the following conjectures:

Conjecture 4.1. Prove that $\mathcal{T}_n^{(k)}(x,\lambda), x \in \mathbb{C}$, has $Im(x,\lambda)=0$ reflection symmetry analytic complex functions. However, $T_n^{(k)}(x,\lambda), k \neq 1$, has not $Re(x,\lambda)=a$ reflection symmetry for $a \in \mathbb{R}$.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n(see Figures 1, 2, 3). We are able to decide if $\mathcal{T}_n^{(k)}(x,\lambda))=0$ has $n-1$ distinct solutions(see Tables 1, 2, 3).

Conjecture 4.2. Prove that $\mathcal{T}_n^{(k)}(x,\lambda))=0$ has $n-1$ distinct solutions.

Since $n-1$ is the degree of the polynomial $\mathcal{T}_n^{(k)}(x,\lambda)$, the number of real zeros $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ lying on the real plane $Im(x,\lambda)=0$ is then $R_{\mathcal{T}_n^{(k)}(x,\lambda)} = n-1-C_{\mathcal{T}_n^{(k)}(x,\lambda)}$, where $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ and $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ which appear in mathematics and physics.

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On the Carlitz's type twisted (p, q)-Euler polynomials and twisted (p, q)-Euler zeta function

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Abstract : In this paper we construct Carlitz's type twisted (p, q)-Euler zeta function. In order to define Carlitz's type twisted (p, q)-Euler zeta function, we introduce the Carlitz's type twisted (p, q)-Euler numbers and polynomials by generalizing the Euler numbers and polynomials, Carlitz's type q-Euler numbers and polynomials. We also give some interesting properties, explicit formulas, a connection with Carlitz's type twisted (p, q)-Euler numbers and polynomials. Finally, we investigate the zeros of the Carlitz's type twisted (p, q)-Euler polynomials by using computer.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, (h, q)-Euler numbers and polynomials, Carlitz's type twisted (p, q)-Euler numbers and polynomials, (p, q)-Euler zeta function, twisted (p, q)-Euler zeta function.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-10]). In this paper, we define Carlitz's type twisted (p, q)-Euler numbers and polynomials and study some properties of the Carlitz's type twisted (p, q)-Euler numbers and polynomials.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by the following generating functions(see [1, 2, 3, 4, 5])

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

and

$$\left(\frac{2}{e^t + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.2}$$

respectively.

The (p, q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2q^{n-3} + pq^{n-2} + q^{n-1}.$$

It is clear that (p, q)-number contains symmetric property, and this number is q-number when $p = 1$. In particular, we can see $\lim_{q\to 1}[n]_{p,q} = n$ with $p = 1$.

By using (p, q)-number, we define the (p, q)-analogue of Euler polynomials and numbers, which generalized the previously known numbers and polynomials, including the Carlitz's type q-Euler numbers and polynomials. We begin by recalling here the Carlitz's type q-Euler numbers and polynomials(see 1, 2, 3, 4, 5]).

Definition 1. The Carlitz's type q-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating function

$$F_q(t,x)=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_q t}. \tag{1.3}$$

and their values at $x=0$ are called the Carlitz's type q-Euler numbers and denoted $E_{n,q}$.

Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-10]). Based on this idea, we generalize the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$. It follows that we define the following (p,q)-analogues of the the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$ (see [6, 7, 9, 10]).

Definition 2. For $0<q<p\le 1$, the Carlitz's type (p,q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_{p,q}t}. \tag{1.4}$$

and

$$F_{p,q}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t}, \tag{1.5}$$

respectively.

In the following section, we define Carlitz's type twisted (p,q)-Euler zeta function. We introduce the Carlitz's type twisted (p,q)-Euler polynomials and numbers. After that we will investigate some their properties. Finally, we investigate the zeros of the Carlitz's type twisted (p,q)-Euler polynomials by using computer.

2. Twisted (p,q)-Euler numbers and polynomials

In this section, we define twisted (p,q)-Euler numbers and polynomials and provide some of their relevant properties. Let r be a positive integer, and let ω be rth root of 1.

Definition 2. For $0<q<p\le 1$, the Carlitz's type twisted (p,q)-Euler numbers $E_{n,p,q,\omega}$ and polynomials $E_{n,p,q,\omega}(x)$ are defined by means of the generating functions

$$F_{p,q,\omega}(t)=\sum_{n=0}^{\infty}E_{n,p,q,\omega}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m\omega^m e^{[m]_{p,q}t}. \tag{2.1}$$

and

$$F_{p,q,\omega}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q,\omega}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m\omega^m e^{[m+x]_{p,q}t}, \tag{2.2}$$

respectively.

Setting $p=1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type twisted q-Euler number $E_{n,q,\omega}$ and q-Euler polynomials $E_{n,q,\omega}(x)$ respectively. Obviously, if we put $\omega=1$, then we have

$$E_{n,p,q,\omega}(x)=E_{n,p,q}(x),\quad E_{n,p,q,\omega}=E_{n,p,q}.$$

Putting $p=1$, we have

$$\lim_{q\to 1}E_{n,p,q,\omega}(x)=E_{n,\omega}(x),\quad \lim_{q\to 1}E_{n,p,q,\omega}=E_{n,\omega}.$$

By using above equation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,p,q,\omega}\frac{t^n}{n!} &= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m]_{p,q}t} \\ &= \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{1}{1+\omega q^{l+1}p^{n-l}}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 3. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q,\omega} = [2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{1}{1+\omega q^{l+1}p^{n-l}}.$$

If we put $p=1$ in the above theorem we obtain

$$E_{n,p,q,\omega} = [2]_q\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{1}{1+\omega q^{l+1}}.$$

By (2.2), we obtain

$$E_{n,p,q,\omega}(x) = [2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x} \frac{1}{1+\omega q^{l+1}p^{n-l}}. \tag{2.4}$$

By using (2.2) and (2.4), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,p,q,\omega}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x} \frac{1}{1+\omega q^{l+1}p^{n-l}}\right)\frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m+x]_{p,q}t}.\end{aligned} \tag{2.5}$$

Since $[x+y]_{p,q} = p^y[x]_{p,q} + q^x[y]_{p,q}$, we see that

$$E_{n,p,q,\omega}(x) = [2]_q \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left(\frac{1}{p-q}\right)^l \frac{1}{1+\omega q^{k+1}p^{n-k}}. \tag{2.6}$$

Next, we introduce Carlitz's type twisted (h,p,q)-Euler polynomials $E_{n,p,q,\omega}^{(h)}(x)$.

Definition 4. The Carlitz's type twisted (h,p,q)-Euler polynomials $E_{n,p,q,\omega}^{(h)}(x)$ are defined by

$$E_{n,p,q}^{(h)}(x) = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m p^{hm}\omega^m [m+x]_{p,q}^n. \tag{2.7}$$

By using (2.7) and (p,q)-number, we have the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q,\omega}^{(h)}(x) = [2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x} \frac{1}{1+\omega q^{l+1}p^{n-l+h}}.$$

By (2.6) and Theorem 2.4, we have

$$E_{n,p,q,\omega}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}E_{l,p,q,\omega}^{(n-l)}$$

The following elementary properties of the (p,q)-analogue of Euler numbers $E_{n,p,q,\omega}$ and polynomials $E_{n,p,q,\omega}(x)$ are readily derived form (2.1) and (2.2). We, therefore, choose to omit details involved.

Theorem 6. (Distribution relation) For any positive integer m(=odd), we have

$$E_{n,p,q,\omega}(x)=\frac{[2]_q}{[2]_{q^m}}[m]_{p,q}^n\sum_{a=0}^{m-1}(-1)^a q^a\omega^a E_{n,p^m,q^m,\omega^m}\left(\frac{a+x}{m}\right), n\in\mathbb{Z}_+.$$

Theorem 7. (Property of complement) For $n\in\mathbb{Z}_+$, we have

$$E_{n,p^{-1},q^{-1},\omega^{-1}}(1-x)=(-1)^n\omega p^n q^n E_{n,p,q,\zeta}(x).$$

Theorem 8. For $n\in\mathbb{Z}_+$, we have

$$\omega q E_{n,p,q,\omega}(1)+E_{n,p,q,\omega}=\begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n\neq 0.\end{cases}$$

By (2.1) and (2.2), we get

$$-[2]_q\sum_{l=0}^{\infty}(-1)^{l+n}q^{l+n}\omega^{l+n}e^{[l+n]_{p,q}t}+[2]_q\sum_{l=0}^{\infty}(-1)^l q^l\omega^l e^{[l]_{p,q}t}=[2]_q\sum_{l=0}^{n-1}(-1)^l q^l\omega^l e^{[l]_{p,q}t}. \tag{2.8}$$

Hence we have

$$(-1)^{n+1}q^n\omega^n\sum_{m=0}^{\infty}E_{m,p,q,\omega}(n)\frac{t^m}{m!}+\sum_{m=0}^{\infty}E_{m,p,q,\omega}\frac{t^m}{m!}=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l q^l\omega^l[l]_{p,q}^m\right)\frac{t^m}{m!}. \tag{2.9}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.9), we have the following theorem.

Theorem 9. For $n\in\mathbb{Z}_+$, we have

$$\sum_{l=0}^{n-1}(-1)^l q^l\omega^l[l]_{p,q}^m=\frac{(-1)^{n+1}q^n\omega^n E_{m,p,q,\omega}(n)+E_{m,p,q,\omega}}{[2]_q}.$$

We investigate the zeros of the twisted (p,q)-Euler polynomials $E_{n,p,q,\omega}(x)$ by using a computer. We plot the zeros of the twisted (p,q)-Euler polynomials $E_{n,p,q,\omega}(x)$ for $x\in\mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n=20, p=1/2, q=1/10$ and $\omega=e^{\frac{2\pi i}{2}}$. In Figure 1(top-right), we choose $n=40, p=1/2, q=1/10$ and $\omega=e^{\frac{2\pi i}{2}}$. In Figure 1(bottom-left), we choose $n=20, p=1/2, q=1/10$ and $\omega=e^{\frac{2\pi i}{4}}$. In Figure 1(bottom-right), we choose $n=40, p=1/2, q=1/10$ and $\omega=e^{\frac{2\pi i}{4}}$.

3. Twisted (p,q)-Euler zeta function

By using twisted (p,q)-Euler numbers and polynomials, (p,q)-Euler zeta function and Hurwitz (p,q)-Euler zeta function is defined. These functions interpolate the twisted (p,q)-Euler numbers $E_{n,p,q,\omega}$, and polynomials $E_{n,p,q,\omega}(x)$, respectively. From (2.1), we note that

$$\begin{aligned}\left.\frac{d^k}{dt^k}F_{p,q,\omega}(t)\right|_{t=0}&=[2]_q\sum_{m=0}^{\infty}(-1)^n q^m\omega^m[m]_{p,q}^k\\&=E_{k,p,q,\omega},(k\in\mathbb{N}).\end{aligned}$$

By using the above equation, we are now ready to define twisted (p,q)-Euler zeta function.

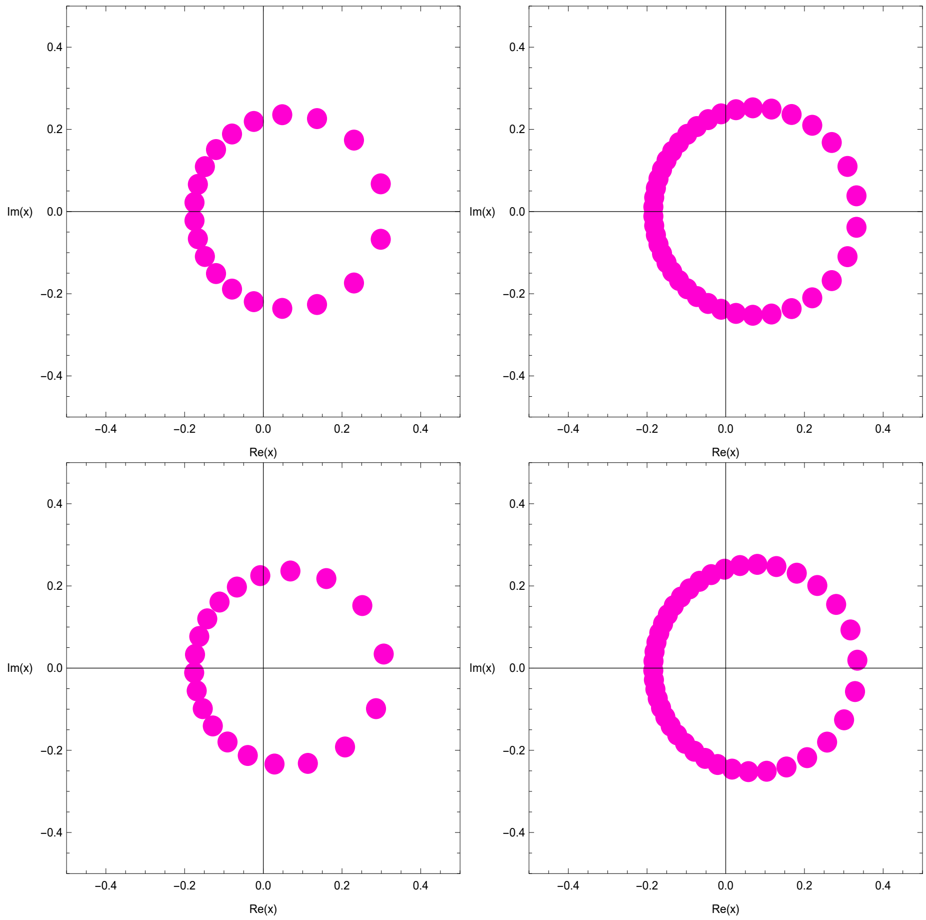

Figure 1: Zeros of $E_{n,p,q,\omega}(x)$

Definition 10. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\zeta_{p,q,\omega}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n \omega^n}{[n]_{p,q}^s}. \tag{3.1}$$

Note that $\zeta_{p,q,\omega}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $p = 1, q \to 1$, then $\zeta_{p,q,\omega}(s) = \zeta_E(s)$ which is the Euler zeta functions(see [4]). Relation between $\zeta_{p,q,\omega}(s)$ and $E_{k,p,q,\omega}$ is given by the following theorem.

Theorem 11. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q,\omega}(-k) = E_{k,p,q,\omega}.$$

Observe that $\zeta_{p,q,\omega}(s)$ function interpolates $E_{k,p,q,\omega}$ numbers at non-negative integers. By using (2.2), we note that

$$\left.\frac{d^k}{dt^k} F_{p,q,\omega}(t,x)\right|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \omega^m [m+x]_{p,q}^k \tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k \left.\left(\sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!}\right)\right|_{t=0} = E_{k,p,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz (p,q)-Euler zeta function.

Definition 12. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$\zeta_{p,q,\omega}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n \omega^n}{[n+x]_{p,q}^s}. \tag{3.4}$$

Note that $\zeta_{p,q,\omega}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $p = 1$ and $q \to 1$, then $\zeta_{p,q,\omega}(s,x) = \zeta_E(s,x)$ which is the Hurwitz Euler zeta functions(see [1, 3, 6]). Relation between $\zeta_{p,q,\omega}(s,x)$ and $E_{k,p,q,\omega}(x)$ is given by the following theorem.

Theorem 13. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q,\omega}(-k,x) = E_{k,p,q,\omega}(x).$$

Observe that $\zeta_{p,q,\omega}(-k,x)$ function interpolates $E_{k,p,q,\omega}(x)$ numbers at non-negative integers.

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STABILITY OF SET-VALUED PEXIDER FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we investigate a set-valued solution of the following Pexider functional equation

$$F(ax + by) = \alpha G(x) + \beta H(y)$$

with three unknown functions F, G and H, where a, b, α, β are positive real scalars.

1. Introduction and preliminaries

Assume that Y is a topological vector space satisfying the T_0 separation axiom. For real numbers s, t and sets $A, B \subset Y$ we put $sA + tB := \{y \in Y; y = sa + tb, a \in A, b \in B\}$. Suppose that the space 2^Y of all subsets of Y is endowed with the Hausdorff topology (see [4]). A set-valued function $F : X \to 2^Y$ is said to be additive if it satisfies the Cauchy functional equation $F(x_1 + x_2) = F(x_1) + F(x_2), x_1, x_2 \in X$. The family of all closed and convex subsets of Y will be denoted by $CC(Y)$, and the sets of all real, rational and positive integer numbers are denoted by $\mathbb{R}, \mathbb{Q}, \mathbb{N}$, respectively.

Lemma 1.1. [1] *Let λ and μ be real numbers. If A and B are nonempty subsets of a real vector space X, then*

$$\lambda(A + B) = \lambda A + \lambda B,$$
$$(\lambda + \mu)A \subseteq \lambda A + \mu B.$$

Moreover, if A is a convex set and $\lambda, \mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$

Lemma 1.2. [3] *Let A, B be subsets of Y and assume that B is closed and convex. If there exists a bounded and nonempty set $C \subset Y$ such that $A + C \subset B + C$, then $A \subset B$.*

Lemma 1.3. *If $(A_n)_{n\in\mathbf{N}}$ and $(B_n)_{n\in\mathbf{N}}$ are decreasing sequences of compact subsets of Y, then $\bigcap_{n\in\mathbb{N}}(A_n + B_n) = \bigcap_{n\in\mathbb{N}} A_n + \bigcap_{n\in\mathbb{N}} B_n$.*

Lemma 1.4. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets of Y, then $A_n \to \bigcap_{n\in\mathbb{N}} A_n$.*

Lemma 1.5. *If A is a bounded subset of Y and $(s_n)_{n\in\mathbb{N}}$ is a real sequence converging to an $s \in \mathbb{R}$, then $s_n A \to sA$.*

Lemma 1.6. *If $A_n \to A$ and $B_n \to B$, then $A_n + B_n \to A + B$.*

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Lemma 1.7. *If $A_n \to A$ and $A_n \to B$, then $clA = clB$.*

Lemma 1.3–1.7 are rather known and can be easily verified. The proofs of them can be found in [1, 2].

2. Set-valued solution of the Pexider functional equation

In the section, we give the solution of the Pexider functional equation.

Theorem 2.1. *Assume that $(X, +)$ is a vector space and Y is a T_0 topological vector space. If set-valued functions $F : X \to CC(Y), G : X \to CC(Y)$ and $H : X \to CC(Y)$ satisfy the functional equation*

$$F(ax + by) = \alpha G(x) + \beta H(y) \tag{2.1}$$

for all $x, y \in X$, where a, b, α and β are positive real numbers, then there exist an additive set-valued function $F_0 : X \to CC(Y)$ and sets $A, B \in CC(Y)$ such that

$$F(x) = \alpha F_0(x) + \alpha A + \beta B, \quad G(x) = F_0(ax) + A \quad \text{and} \quad H(x) = F_0(bx) + B$$

for all $x \in X$.

Proof. First, assume that $0 \in G(0)$ and $0 \in H(0)$. Then, for all $x, y \in X$, we have

$$\begin{aligned}
&F(x+y) = F\left(a\frac{x}{a} + b\frac{y}{b}\right) = \alpha G\left(\frac{x}{a}\right) + \beta H\left(\frac{y}{b}\right)\\
&\subset \alpha G\left(\frac{x}{a}\right) + \beta H(0) + \alpha G(0) + \beta H\left(\frac{y}{b}\right)\\
&= F(x) + F(y).
\end{aligned}$$

Letting $x = y$ in the above equation, we get $F(2x) \subset 2F(x)$, which implies that the sequence $(2^{-n}F(2^n x))_{n\in N}$ is decreasing. Put $F_0(x) := \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x), x \in X$. It is clear that $F_0(x) \in CC(Y)$ for all $x \in X$. Similarly, we get

$$\begin{aligned}
&\alpha G(2x) + \beta H(0) = F(2ax) = F\left(ax + b\left(\frac{ax}{b}\right)\right)\\
&= \alpha G(x) + \beta H\left(\frac{ax}{b}\right) \subset \alpha G(x) + \alpha G(0) + \beta H\left(\frac{ax}{b}\right)\\
&= \alpha G(x) + F(ax) = \alpha G(x) + \alpha G(x) + \beta H(0) = 2\alpha G(x) + \beta H(0).
\end{aligned}$$

In view of Lemma 1.2, we obtain that $G(2x) \subset 2G(x)$, and consequently the sequence $(2^{-n}G(2^n x))_{n\in\mathbb{N}}$ is decreasing. Applying Lemma1.3 and this equality $F(a2^n x) = \alpha G(2^n x) + \beta H(0), n \in \mathbb{N}$, we obtain

$$F_0(ax) = \bigcap_{n\in\mathbb{N}} 2^{-n}F(a2^n x) = \alpha \bigcap_{n\in\mathbb{N}} 2^{-n}G(2^n x) + \beta \bigcap_{n\in\mathbb{N}} 2^{-n}H(0).$$

But $\bigcap_{n\in\mathbb{N}} 2^{-n}H(0) = \{0\}$, since the set $H(0)$ is bounded. Therefore $F_0(ax) = \alpha \bigcap_{n\in\mathbb{N}} 2^{-n}G(2^n x)$ for all $x \in X$. In an analogous way we show that the sequence $(2^{-n}H(2^n x))_{n\in\mathbb{N}}$ is decreasing and $F_0(bx) = \beta \bigcap_{n\in\mathbb{N}} 2^{-n} H(2^n x)$ for all $x \in X$. Hence,using once more Lemma 1.3, we get

$$\begin{aligned} F_0(x_1+x_2) &= \bigcap_{n\in\mathbb{N}} 2^{-n} F(2^n x_1 + 2^n x_2) = \bigcap_{n\in\mathbb{N}} 2^{-n}\left(\alpha G\left(\frac{2^n x_1}{a}\right) + \beta H\left(\frac{2^n x_2}{b}\right)\right) \\ &= \bigcap_{n\in\mathbb{N}} 2^{-n}\alpha G\left(\frac{2^n x_1}{a}\right) + \bigcap_{n\in\mathbb{N}} 2^{-n}\beta H\left(\frac{2^n x_2}{b}\right) \\ &= F_0(x_1) + F_0(x_2), x_1, x_2 \in X, \end{aligned}$$

which means that the set-valued function F_0 is additive.

Now observe that

$$F(nbx) + (n-1)\beta H(0) = F(bx) + (n-1)\beta H(x) \tag{2.2}$$

for all $x \in X$ and $n \in \mathbb{N}$. Indeed, for $n = 1$ the equality is trivial. Assume that it holds for a natural number k. Then, in virtue of (2.1), we obtain

$$\begin{aligned} &F((k+1)bx) + k\beta H(0) = \alpha G\left(\frac{kbx}{a}\right) + \beta H(x) + k\beta H(0) = F(kbx) + \beta H(x) + (k\beta - \beta)H(0) \\ &= F(bx) + (k-1)\beta H(x) + \beta H(x) = F(x) + k\beta H(x). \end{aligned}$$

which proves that (2.2) holds for $n = k+1$. Thus, by induction, it holds for all $n \in \mathbb{N}$. In particular, we have

$$F(2^n x) + (2^n - 1)H(0) = F(x) + (2^n - 1)H\left(\frac{x}{b}\right),$$

and so

$$2^{-n}F(2^n x) + (1 - 2^{-n})H(0) = 2^{-n}F(x) + (1 - 2^{-n})H\left(\frac{x}{b}\right)$$

for all $x \in X$. By Lemma 1.4, $2^{-n}F(2^n x) \to \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x) = F_0(x)$.

On the other hand, by Lemma 1.5, $1-2^{-n}H(0) \to H(0), 2^{-n}F(x) \to \{0\}$ and $(1-2^{-n})H\left(\frac{x}{b}\right) \to H\left(\frac{x}{b}\right)$. Thus, using Lemmas 1.6 and 1.7, we get $cl[F_0(x) + H(0)] = clH\left(\frac{x}{b}\right)$, whence $H\left(\frac{x}{b}\right) = F_0(x) + H(0)$ for all $x \in X$. Similarly, we can obtain $G\left(\frac{x}{a}\right) = F_0(x) + G(0), x \in X$. Let $A := G(0)$ and $B := H(0)$. Then $G(x) = F_0(ax) + A$ and $H(x) = F_0(bx) + B$ for all $x \in X$. Moreover $F(x) = \alpha F_0(x) + \alpha A + \beta B, x \in X$. This finishes our proof in the case that $0 \in G(0)$ and $0 \in H(0)$.

In the opposite case, fix arbitrarily points $a \in G(0)$ and $b \in H(0)$, and consider the set-valued functions $F_1, G-1, H_1 : X \to CC(Y)$ defined by $F_1(x) := F(x) - \alpha a - \beta b, G_1(x) := G(x) - a$ and $H_1 := H(x) - b, x \in X$. These set-valued functions satisfy the equation (2.1) and moreover $0 \in G_1(0)$ and $0 \in H_1(0)$. Therefore, by what we have discussed previously, we can get the same result. This completes the proof. □

In [2], Nikodem proved that a set-valued function $F_0 : [0,\infty) \to CC(Y)$, where Y is a locally convex Hausdorff space, is additive if and only if there exists an additive function $f : [0,\infty) \to Y$ and a set $K \in CC(Y)$ such that $F_0(x) = f(x) + xK, x \in [0,\infty)$. Thus we can get the following.

Theorem 2.2. *Let Y be a locally convex Hausdorff space. The set-valued functions $F : [0,\infty) \to CC(Y), G : [0,\infty) \to CC(Y)$ and $H : [0,\infty) \to CC(Y)$ satisfy the functional equation (2.1) if and only if there exist an additive function $f : [0,\infty) \to Y$ and sets $K, A, B \in CC(Y)$ such that*

$$F(x) = \alpha f(x) + \alpha Kx + \alpha A + \beta B, G(x) = f(ax) + akx + A \quad \text{and} \quad H(x) = f(bx) + bkx + B$$

for all $x \in [0, \infty)$.

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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On symmetries and solutions of certain sixth order difference equations

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Abstract

We use the Lie group analysis method to investigate the invariance properties and the solutions of

$$x_{n+1} = \frac{x_{n-5}x_{n-3}}{x_{n-1}(a_n + b_n x_{n-5}x_{n-3})}.$$

We show that this equation has a two-dimensional Lie algebra and that its solutions can be presented in a unified manner. Besides presenting solutions of the recursive sequence above where a_n and b_n are sequences of real numbers, some specific cases are emphasized.

Key words: Difference equation; symmetry; reduction; group invariant solutions, periodicity
MSC 2010: 39A05, 39A23, 70G65

1 Introduction

Difference equations are important in mathematical modelling, especially where discrete time evolving variables are concerned. They also occur when studying discretization methods for differential equations. Countless results in the subject of difference equations have been recorded. For rational difference equations of order greater than 1, the study can be quite challenging at the same time rewarding. Rewarding in the sense that such a study lays ground for the theory of global properties of difference equations (not necessarily rational) of higher order.
In [4], the author developed an effective symmetry based algorithm to deal with the obtention of solutions of difference equations of any order. However, the calculation one deals with in this application to difference equations of order greater than one can become cumbersome but with great recompense often times. The method consists of finding a group of transformations that maps solutions onto themselves. Symmetry method is a valuable tool and it has been used to solve several difference equations [1–3, 7, 8].
In this paper, our objective is to obtain the symmetry operators of

$$x_{n+1} = \frac{x_{n-5}x_{n-3}}{x_{n-1}(a_n + b_n x_{n-5}x_{n-3})} \tag{1}$$

1

where a_n and b_n are real sequences and to find its solutions by way of symmetries. Without loss of generality, we equivalently study the forward difference equation

$$u_{n+6} = \frac{u_n u_{n+2}}{u_{n+4}(A_n + B_n u_n u_{n+2})}. \tag{2}$$

We refer the interested reader to [4, 9] for a deeper knowledge of Lie analysis.

2 Definitions and Notation

In this section, we briefly present some definitions and notation (largely from Hydon in [4]) indispensable for the understanding of Lie symmetry analysis of difference equations.

Definition 2.1 *Let G be a local group of transformations acting on a manifold M. A subset $\mathcal{S} \subset M$ is called G-invariant, and G is called symmetry group of $\mathcal{S}$, if whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.*

Definition 2.2 *Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\mathcal{V} : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\mathcal{V}) = 0 \qquad \text{for all} \qquad x \in M,$$

and every infinitesimal generator X of G.

Definition 2.3 *A parameterized set of point transformations,*

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon), \tag{3}$$

where $x = x_i$, $i = 1, \ldots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. *Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$*
2. *$\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0*
3. *Each $\hat{x_i}$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore*

$$\hat{x_i}(x : \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, ..., p. \tag{4}$$

Assuming that the sixth-order difference equation has the form

$$u_{n+6} = \Psi(n, u_n, \ldots, u_{n+5}), \quad n \in D \tag{5}$$

for some smooth function Ω and a regular domain $D \subset \mathbb{Z}$. To deduce the symmetry group of (5), we search for a one parameter Lie group of point transformations

$$\Gamma_\varepsilon : (n, u_n) \mapsto (n, u_n + \varepsilon Q(n, u_n)), \tag{6}$$

in which ε is the parameter and Q a continuous function, referred to as characteristic. Let

$$X = Q(n,u_n)\frac{\partial}{\partial u_n} + Q(n+1,u_{n+1})\frac{\partial}{\partial u_{n+1}} + \cdots + Q(n+5,u_{n+5})\frac{\partial}{\partial u_{n+5}} \quad (7)$$

be the corresponding 'prolonged' infinitesimal generator and $S : n \mapsto n+1$ the shift operator. The linearized symmetry condition is given by

$$S^6 Q - X\Psi = 0. \quad (8a)$$

Upon knowledge of the characteristic Q, it is important to introduce the canonical coordinate

$$S_n = \int \frac{du_n}{Q(n,u_n)}, \quad (9)$$

a useful tool which allows one to obtain the invariant $\mathcal{V}$.

3 Main results

As earlier emphasized, our equation under study is

$$u_{n+6} = \Psi = \frac{u_n u_{n+2}}{u_{n+4}(A_n + B_n u_n u_{n+2})}. \quad (10)$$

Appliying the criterion of invariance (8) to (10), we get

$$\begin{aligned} &Q(n+6,u_{n+6}) + \frac{u_n u_{n+2}}{u_{n+4}^2 \left(A_n + B_n u_n u_{n+2}\right)} Q\left(n+4,u_{n+4}\right) \\ &- \frac{A_n u_n}{u_{n+4}\left(\left(A_n + B_n u_n u_{n+2}\right)^2\right.} Q\left(n+2,u_{n+2}\right) \\ &- \frac{A_n u_{n+2}}{u_{n+4}\left(A_n + B_n u_n u_{n+2}\right)^2} Q\left(n,u_n\right) = 0. \end{aligned} \quad (11)$$

In order to eliminate u_{n+3}, we invoke implicit differentiation with respect to u_n (regarding u_{n+4} as a function of u_n, u_{n+2} and u_{n+3}) via the operator

$$L = \frac{\partial}{\partial u_n} - \frac{\Psi_{u_n}}{\Psi_{u_{n+4}}}\frac{\partial}{\partial u_{n+4}}.$$

With some simplification, one gets

$$\begin{aligned} &\left(A_n + B_n u_n u_{n+2}\right) Q'\left(n+4,u_{n+4}\right) - \frac{\left(A_n + B_n u_n u_{n+2}\right)}{u_{n+4}} Q\left(n+4,u_{n+4}\right) \\ &+ B_n u_n Q(n+2,u_{n+2}) - \left(A_n + B_n u_n u_{n+2}\right) Q'(n,u_n) \\ &+ \left(2B_n u_{n+2} + \frac{A_n}{u_n}\right) Q\left(n,u_n\right) = 0. \end{aligned} \quad (12)$$

3

Note that the symbol $'$ stands for the derivative with respect to the continuous variable. After twice differentiating (12) with respect to u_n, keeping u_{n+2} and u_{n+4} fixed, we are led to the equation

$$\begin{aligned}&- B_n u_n u_{n+2} Q'''(n,u_n) - A_n Q'''(n,u_n) + \frac{A_n}{u_n} Q''(n,u_n) - \frac{2A_n}{{u_n}^2} Q'(n,u_n) \\ &+ \frac{2A_n}{{u_n}^3} Q(n,u_n) = 0 \end{aligned} \tag{13}$$

Note that the characteristic in (13) is not a function of u_{n+2} and so we split (13) up with respect to u_{n+2} to get the system

$$1 : Q'''(n,u_n) - \frac{1}{u_n} Q''(n,u_n) + \frac{2}{{u_n}^2} Q'(n,u_n) - \frac{2}{{u_n}^3} Q(n,u_n) = 0 \tag{14a}$$

$$u_{n+2} : Q'''(n,u_n) = 0. \tag{14b}$$

We find that the solution to (14) is

$$Q(n,u_n) = \alpha_n {u_n}^2 + \beta_n u_n \tag{15}$$

for some arbitrary functions α_n and β_n that depend on n. Substituting (16) and its first, second and third shifts in (11), and then replacing the expression of u_{n+3} given in (10) in the resulting equation yields

$$\begin{aligned}& B_n {u_n}^2 {u_{n+2}}^2 {u_{n+4}}^2 \alpha_{n+4} + B_n {u_n}^2 {u_{n+2}}^2 u_{n+4} (\beta_{n+4} + \beta_{n+3}) \\ &- A_n {u_n}^2 u_{n+2} u_{n+4} \alpha_n - A_n u_n {u_{n+2}}^2 u_{n+4} \alpha_{n+2} + A_n u_n u_{n+2} {u_{n+4}}^2 \alpha_{n+4} \\ &+ {u_n}^2 {u_{n+2}}^2 \alpha_{n+2} - A_n (\beta_n + \beta_{n+2} - \beta_{n+4} - \beta_{n+3}) = 0. \end{aligned} \tag{16}$$

Equating all coefficients of all powers of shifts of u_n to zero and simplifying the resulting system, we get its reduced form

$$\alpha_n = 0, \tag{17}$$

$$\beta_n + \beta_{n+2} = 0. \tag{18}$$

The two independent solutions of the linear second-order difference equation above are given by

$$\beta_n = \beta^n \text{ and } \beta_n = \bar{\beta}^n, \tag{19}$$

where $\beta = \exp\{i\pi/2\}$ and $\bar{\beta} = -\exp\{i\pi/2\}$ is its complex conjugate. The characteristic functions are given by

$$Q_1(n,u_n) = \beta^n u_n \quad \text{and} \quad Q_2(n,u_n) = \bar{\beta}^n u_n, \tag{20}$$

and so the Lie algebra of (10) is generated by

$$X_1 = \beta^n u_n \frac{\partial}{\partial_{u_n}} + \beta^{n+2} u_{n+2} \frac{\partial}{\partial_{u_{n+2}}} + \beta^{n+4} u_{n+4} \frac{\partial}{\partial_{u_{n+4}}} \tag{21}$$

$$X_2 = \bar{\beta}^n u_n \frac{\partial}{\partial_{u_n}} + \bar{\beta}^{n+2} u_{n+2} \frac{\partial}{\partial_{u_{n+2}}} + \bar{\beta}^{n+4} u_{n+4} \frac{\partial}{\partial_{u_{n+4}}}. \tag{22}$$

4

Using the canonical coordinate

$$S_n = \int \frac{du_n}{Q_1(n,u_n)} = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n| \tag{23}$$

and (17), we derive the invariant function $\tilde{\mathcal{V}}_n$ as follows:

$$\tilde{\mathcal{V}}_n = S_n \beta^n + S_{n+2} \beta^{n+2}. \tag{24}$$

Actually,

$$X_1(\tilde{\mathcal{V}}_n) = \beta^n + \beta^{n+2} = 0 \tag{25}$$

and

$$X_2(\tilde{\mathcal{V}}_n) = \bar{\beta}^n + \bar{\beta}^{n+2} = 0. \tag{26}$$

For the sake of convenience, we use

$$|\mathcal{V}_n| = \exp\{-\tilde{\mathcal{V}}_n\} \tag{27}$$

instead/ In other words, $\mathcal{V}_n = \pm 1/(u_n u_{n+2})$. Using (10) and (27), one can prove that

$$\mathcal{V}_{n+4} = A_n \mathcal{V}_n \pm B_n. \tag{28}$$

Utilizing the plus sign, the solution of (28) can be written as

$$\mathcal{V}_{4n+j} \quad = \mathcal{V}_j \left(\prod_{k_1=0}^{n-1} A_{4k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{4l+j} \prod_{k_2=l+1}^{n-1} A_{4k_2+j} \right), \tag{29}$$

where $j = 0, 1, 2, 3$. From here, obtaining the solution of (10) is straightforward. We first employ (23) to get

$$|u_n| = \exp\left(\beta_n S_n\right). \tag{30}$$

Secondly, we employ (24) to obtain

$$|u_n| = \exp\left(\beta^n c_1 + \bar{\beta}^n c_2 - \frac{1}{2} \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} \tilde{\mathcal{V}}_{k_1} - \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} \tilde{\mathcal{V}}_{k_2} \right). \tag{31}$$

Lastly, we use (27) to get

$$|u_n| = \exp\left(\beta^n c_1 + \bar{\beta}^n c_2 + \frac{1}{2} \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} \ln |\mathcal{V}_{k_1}| + \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} \ln |\mathcal{V}_{k_2}| \right), \tag{32a}$$

in which V_k is given in (28) with $\gamma(n,k) = \beta^n \bar{\beta}^k$. Note that the constants c_1 and c_2 satisfy

$$c_1 + c_2 = \ln |u_0| \quad \text{and} \quad \beta(c_1 - c_2) = \ln |u_1|. \tag{32b}$$

5

Note. Equations in (32) give the solutions of (2) in a unified manner.
On a further note, $\gamma(n,k)=\beta^n\bar{\beta}^k$ satisfies

$$\begin{aligned}&\gamma(0,1)=\bar{\beta},\gamma(0,2)=-1,\gamma(1,0)=\beta,\gamma(1,2)=-\beta,\gamma(1,3)=-1,\\&\gamma(n,n)=1,\gamma(n+2,k)=-\gamma(n,k),\\&\gamma(n,k+2)=-\gamma(n,k),\gamma(4n,k)=\gamma(0,k),\gamma(n,4k)=\gamma(n,0).\end{aligned}\tag{33}$$

From u_n given in (32a) and equation (33), observe that

$$|u_{4n+j}|=\exp\left(H_j+\sum_{k_1=0}^{4n+j-1}\mathrm{Re}(\gamma(0,k_1))\ln|\mathcal{V}_{k_1}|\right)\tag{34}$$

in which

$$H_j=\beta^j c_1+\bar{\beta}^j c_2.$$

For $j=0$, we have,

$$\begin{aligned}|u_{4n}|&=\exp(H_0+\ln|\mathcal{V}_0|-\ln|\mathcal{V}_2|+\ldots+\ln|\mathcal{V}_{4n-4}|-\ln|\mathcal{V}_{4n-2}|)\\&=\exp(H_0)\prod_{s=0}^{n-1}\left|\frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}}\right|.\end{aligned}\tag{35}$$

It can be shown that there is no need for the absolute values via the utilization of the fact that

$$\mathcal{V}_i=\frac{1}{u_iu_{i+2}}.\tag{36}$$

In order to deduce $\exp(H_0)$, we set $n=0$ in (35) and note that $|u_0|=\exp(H_0)$. Thus

$$u_{4n}=u_0\prod_{s=0}^{n-1}\frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}}.$$

Similarly, replacing n with $4n+j$ for $j=0,1,2,3$, we obtain

$$U_{4n+j}=u_j\prod_{s=0}^{n-1}\frac{\mathcal{V}_{4s+j}}{\mathcal{V}_{4s+j+2}}.\tag{37}$$

Nevertheless, from (28), using the plus sign we are led to

$$\mathcal{V}_{4n+j}\quad=\mathcal{V}_j\left(\prod_{k_1=0}^{n-1}A_{4k_1+j}\right)+\sum_{l=0}^{n-1}\left(B_{4l+j}\prod_{k_2=l+1}^{n-1}A_{4k_2+j}\right),\tag{38}$$

for $j = 0, 1, 2, 3$, where $\mathcal{V}_0 = \frac{1}{u_0 u_2}$. Thus, using (37) with $j = 0$, we get

$$
\begin{aligned}
U_{4n} &= u_0 \prod_{s=0}^{n-1} \frac{V_{4s}}{V_{4s+2}} \\
&= u_0 \prod_{s=0}^{n-1} \frac{V_0 \left(\prod\limits_{k_1=0}^{s-1} A_{4k_1} \right) + \sum\limits_{l=0}^{s-1} \left(B_{4l} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2} \right)}{V_2 \left(\prod\limits_{k_1=0}^{s-1} A_{4k_1+2} \right) + \sum\limits_{l=0}^{s-1} \left(B_{4l+2} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2+2} \right)} \\
&= u_0 \prod_{s=0}^{n-1} \frac{u_4}{u_0} \frac{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1} \right) + u_0 u_2 \sum\limits_{l=0}^{s-1} \left(B_{4l} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2} \right)}{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum\limits_{l=0}^{s-1} \left(B_{4l+2} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2+2} \right)} \\
&= u_0^{1-n} u_4^n \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1} \right) + u_0 u_2 \sum\limits_{l=0}^{s-1} \left(B_{4l} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2} \right)}{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum\limits_{l=0}^{s-1} \left(B_{4l+2} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2+2} \right)}
\end{aligned}
$$

For $j = 1$, we have

$$
\begin{aligned}
U_{4n+1} &= u_1 \prod_{s=0}^{n-1} \frac{V_{4s+1}}{V_{4s+3}} \\
&= u_1^{1-n} u_5^n \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1+1} \right) + u_1 u_3 \sum\limits_{l=0}^{s-1} \left(B_{4l+1} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2+1} \right)}{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1+3} \right) + u_3 u_5 \sum\limits_{l=0}^{s-1} \left(B_{4l+3} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2+3} \right)}
\end{aligned}
$$

For $j = 2$, we have

$$
\begin{aligned}
U_{4n+2} &= u_2 \prod_{s=0}^{n-1} \frac{V_{4s+2}}{V_{4s+4}} \\
&= u_0^n u_4^{-n} u_2 \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum\limits_{l=0}^{s-1} \left(B_{4l+2} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2+2} \right)}{\left(\prod\limits_{k_1=0}^{s} A_{4k_1} \right) + u_0 u_2 \sum\limits_{l=0}^{s} \left(B_{4l} \prod\limits_{k_2=l+1}^{s} A_{4k_2} \right)}
\end{aligned}
$$

For $j=3$, we get

$$\begin{aligned} U_{4n+3} &= u_3 \prod_{s=0}^{n-1} \frac{V_{4s+3}}{V_{4s+5}} \\ &= u_1^n u_5^{-n} u_3 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1+3}\right) + u_3u_5 \sum_{l=0}^{s-1} \left(B_{4l+3} \prod_{k_2=l+1}^{s-1} A_{4k_2+3}\right)}{\left(\prod_{k_1=0}^{s} A_{4k_1+1}\right) + u_1u_3 \sum_{l=0}^{s} \left(B_{4l+1} \prod_{k_2=l+1}^{s} A_{4k_2+1}\right)} \end{aligned}$$

Hence, our solution in terms of x_n $(n > 0)$ is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^{n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1}\right) + x_{-5}x_{-3} \sum_{l=0}^{s-1} \left(b_{4l} \prod_{k_2=l+1}^{s-1} a_{4k_2}\right)}{\left(\prod_{k_1=0}^{s-1} a_{4k_1+2}\right) + x_{-3}x_{-1} \sum_{l=0}^{s-1} \left(b_{4l+2} \prod_{k_2=l+1}^{s-1} a_{4k_2+2}\right)}, \tag{39}$$

$$x_{4n-4} = x_{-4}^{1-n} x_{0}^{n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1+1}\right) + x_{-4}x_{-2} \sum_{l=0}^{s-1} \left(b_{4l+1} \prod_{k_2=l+1}^{s-1} a_{4k_2+1}\right)}{\left(\prod_{k_1=0}^{s-1} a_{4k_1+3}\right) + x_{-2}x_{0} \sum_{l=0}^{s-1} \left(b_{4l+3} \prod_{k_2=l+1}^{s-1} a_{4k_2+3}\right)}, \tag{40}$$

$$x_{4n-3} = x_{-5}^{n} x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1+2}\right) + x_{-3}x_{-1} \sum_{l=0}^{s-1} \left(b_{4l+2} \prod_{k_2=l+1}^{s-1} a_{4k_2+2}\right)}{\left(\prod_{k_1=0}^{s} a_{4k_1}\right) + x_{-5}x_{-3} \sum_{l=0}^{s} \left(b_{4l} \prod_{k_2=l+1}^{s} a_{4k_2}\right)} \tag{41}$$

and

$$x_{4n-2} = x_{-4}^{n} x_{0}^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1+3}\right) + x_{-2}x_{0} \sum_{l=0}^{s-1} \left(b_{4l+3} \prod_{k_2=l+1}^{s-1} a_{4k_2+3}\right)}{\left(\prod_{k_1=0}^{s} a_{4k_1+1}\right) + x_{-4}x_{-2} \sum_{l=0}^{s} \left(b_{4l+1} \prod_{k_2=l+1}^{s} a_{4k_2+1}\right)} \tag{42}$$

In the following sections, we specifically look at some special cases.

4 The case a_n , b_n are 1-periodic

Let $a_n = a$ and $b_n = b$, where a and b are non-zero constants.

4.1 Case: $a \neq 1$

We have

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^{n} \prod_{s=0}^{n-1} \frac{a^s + bx_{-5}x_{-3}\frac{1-a^s}{1-a}}{a^s + bx_{-3}x_{-1}\frac{1-a^s}{1-a}},$$

$$x_{4n-4} = x_{-4}^{1-n} x_{0}^{n} \prod_{s=0}^{n-1} \frac{a^s + bx_{-4}x_{-2}\frac{1-a^s}{1-a}}{a^s + bx_{-2}x_{0}\frac{1-a^s}{1-a}},$$

$$x_{4n-3} = x_{-5}^{n} x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{a^s + bx_{-3}x_{-1}\frac{1-a^s}{1-a}}{a^{s+1} + bx_{-5}x_{-3}\frac{1-a^{s+1}}{1-a}},$$

$$x_{4n-2} = x_{-4}^{n} x_{0}^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{a^s + bx_{-2}x_{0}\frac{1-a^s}{1-a}}{a^{s+1} + bx_{-4}x_{-2}\frac{1-a^{s+1}}{1-a}}$$

as long as any of the denominators does not vanish.

Case: $a = -1$

We have

$$x_{4n-5} = \begin{cases} x_{-5}^{1-n} x_{-1}^{n} \left(\frac{-1+bx_{-5}x_{-3}}{-1+bx_{-3}x_{-1}}\right)^{\lfloor \frac{n-1}{2} \rfloor}, & \text{if } n \text{ is odd;} \\ \\ x_{-5}^{1-n} x_{-1}^{n} \left(\frac{-1+bx_{-5}x_{-3}}{-1+bx_{-3}x_{-1}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}$$

$$x_{4n-4} = \begin{cases} x_{-4}^{1-n} x_{0}^{n} \left(\frac{-1+bx_{-4}x_{-2}}{-1+bx_{-2}x_{0}}\right)^{\lfloor \frac{n-1}{2} \rfloor}, & \text{if } n \text{ is even;} \\ \\ x_{-4}^{1-n} x_{0}^{n} \left(\frac{-1+bx_{-4}x_{-2}}{-1+bx_{-2}x_{0}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \end{cases}$$

$$x_{4n-3} = \begin{cases} \frac{x_{-5}^{n} x_{-1}^{-n} x_{-3}}{-1+bx_{-5}x_{-3}} \left(\frac{-1+bx_{-3}x_{-1}}{-1+bx_{-5}x_{-3}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \\ \\ x_{-5}^{n} x_{-1}^{-n} x_{-3} \left(\frac{-1+bx_{-3}x_{-1}}{-1+bx_{-5}x_{-3}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}$$

and

$$x_{4n-2} = \begin{cases} \frac{x_{-4}^{n} x_{0}^{-n} x_{-2}}{-1+bx_{-4}x_{-2}} \left(\frac{-1+bx_{-2}x_{0}}{-1+bx_{-4}x_{-2}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \\ \\ x_{-4}^{n} x_{0}^{-n} x_{-2} \left(\frac{-1+bx_{-2}x_{0}}{-1+bx_{-4}x_{-2}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}$$

where $bx_{-i}x_{2-i} \neq 1$ for $i = 2, 3, 4, 5$.

4.2 Case: $a = 1$

The solution is given by

$$x_{4n-5} = x_{-5}^{1-n}x_{-1}^{n}\prod_{s=0}^{n-1}\frac{1+bx_{-5}x_{-3}s}{1+bx_{-3}x_{-1}s}, x_{4n-4} = x_{-4}^{1-n}x_{0}^{n}\prod_{s=0}^{n-1}\frac{1+bx_{-4}x_{-2}s}{1+bx_{-2}x_{0}s},$$

$$x_{4n-3} = x_{-5}^{n}x_{-1}^{-n}x_{-3}\prod_{s=0}^{n-1}\frac{1+bx_{-3}x_{-1}s}{1+bx_{-5}x_{-3}(s+1)},$$

$$x_{4n-2} = x_{-4}^{n}x_{0}^{-n}x_{-2}\prod_{s=0}^{n-1}\frac{1+bx_{-2}x_{0}s}{1+bx_{-4}x_{-2}(s+1)}.$$

5 The case a_n, b_n are 2-periodic

In this case, we have $\{a_n\}_{n=0}^{\infty} = a_0, a_1, a_0, a_1, \ldots,$ and similarly $\{b_n\}_{n=0}^{\infty} = b_0, b_1, b_0, b_1, \ldots$ where $a_0 \neq a_1$, and $b_0 \neq b_1$. Then the solution is given by

$$x_{4n-5} = x_{-5}^{1-n}x_{-1}^{n}\prod_{s=0}^{n-1}\frac{a_0^s + b_0x_{-5}x_{-3}\sum\limits_{l=0}^{s-1}a_0^l}{a_0^s + b_0x_{-3}x_{-1}\sum\limits_{l=0}^{s-1}a_0^l},$$

$$x_{4n-4} = x_{-4}^{1-n}x_{0}^{n}\prod_{s=0}^{n-1}\frac{a_1^s + b_1x_{-4}x_{-2}\sum\limits_{l=0}^{s-1}a_1^l}{a_1^s + b_1x_{-2}x_{0}\sum\limits_{l=0}^{s-1}a_1^l},$$

$$x_{4n-3} = x_{-5}^{n}x_{-1}^{-n}x_{-3}\prod_{s=0}^{n-1}\frac{a_0^s + b_0x_{-3}x_{-1}\sum\limits_{l=0}^{s-1}a_0^l}{a_0^{s+1} + b_0x_{-5}x_{-3}\sum\limits_{l=0}^{s}a_0^l}$$

and

$$x_{4n-2} = x_{-4}^{n}x_{0}^{-n}x_{-2}\prod_{s=0}^{n-1}\frac{a_1^s + b_1x_{-2}x_{0}\sum\limits_{l=0}^{s-1}a_1^l}{a_1^{s+1} + b_1x_{-4}x_{-2}\sum\limits_{l=0}^{s}a_1^l}$$

as long as any of the denominators does not vanish.

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6 The case a_n, b_n are 4-periodic

We assume that $\{a_n\} = a_0, a_1, a_2, a_3, a_0, a_1, a_2, a_3, \dots$ and $\{b_n\} = b_0, b_1, b_2, b_3, b_0, b_1, b_2, b_3, \cdots$. The solution is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^{n} \prod_{s=0}^{n-1} \frac{a_0^s + b_0 x_{-5} x_{-3} \sum_{l=0}^{s-1} a_0^l}{a_2^s + b_2 x_{-3} x_{-1} \sum_{l=0}^{s-1} a_2^l}, \tag{43}$$

$$x_{4n-4} = x_{-4}^{1-n} x_{0}^{n} \prod_{s=0}^{n-1} \frac{a_1^s + b_1 x_{-4} x_{-2} \sum_{l=0}^{s-1} a_1^l}{a_3^s + b_3 x_{-2} x_{0} \sum_{l=0}^{s-1} a_3^l}, \tag{44}$$

$$x_{4n-3} = x_{-5}^{n} x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{a_2^s + b_2 x_{-3} x_{-1} \sum_{l=0}^{s-1} a_2^l}{a_0^{s+1} + b_0 x_{-5} x_{-3} \sum_{l=0}^{s} a_0^l} \tag{45}$$

and

$$x_{4n-2} = x_{-4}^{n} x_{0}^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{a_3^s + b_3 x_{-2} x_{0} \sum_{l=0}^{s-1} a_3^l}{a_1^{s+1} + b_1 x_{-4} x_{-2} \sum_{l=0}^{s} a_1^l} \tag{46}$$

as long as any of the denominators does not vanish.

7 Conclusion

In this paper, non-trivial symmetries for difference equations of the form (1) were found. Consequently, the results were used to find formulas for the solutions of the equations. Specific cases of the solutions were also discussed.

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