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A new techniques applied to Volterra–Fredholm integral equations with discontinuous kernel

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Abstract

The purpose of this paper is to establish the general solution of a Volterra–Fredholm integral equation with discontinuous kernel in a Banach space. Banach’s fixed point theorem is used to prove the existence and uniqueness of the solution. By using separation of variables method, the problem is reduced to a Volterra integral equations of the second kind with continuous kernel. Normality and continuity of the integral operator are also discussed.

Mathematics Subject Classification(2010): 45L05; 46B45; 65R20.

Key–Words: Banach space, Volterra–Fredholm integral equation, Separation of variables method.

1 Introduction

It is well-known that the integral equations govern many mathematical models of various phenomena in physics, economy, biology, engineering, even in mathematics and other fields of science. The illustrative examples of such models can be found in the literature, (see, e.g., [5, 6, 9, 11, 12, 14, 18, 20]). Many problems of mathematical physics, applied mathematics, and engineering are reduced to Volterra–Fredholm integral equations, see [1, 2].

Analytical solutions of integral equations, either do not exist or it’s hard to compute. Eventual an exact solution is computable, the required calculations may be tedious, or the resulting solution may be difficult to interpret. Due to this, it is required to obtain an efficient numerical solution. There are numerous studies in literature concerning the numerical solution of integral equations such as [4, 8, 10, 13, 16, 17, 21].

In this present paper, the existence and uniqueness solution of the Eq. (1) are discussed and proved in the space $L_2(\Omega) \times C[0, T]$, $0 \leq T < 1$. Moreover, the normality and continuity of the

integral operator are obtained. A numerical method is used to translate the Volterra–Fredholm integral equation (1) to a Volterra integral equations of the second kind with continuous kernel,

The outline of the paper is as follows: Sect. 1 is the introduction; In Sect. 2, the existence of a unique solution of the Volterra–Fredholm integral equation is discussed and proved using Picard’s method and Banach’s fixed point method. Sect. 3, include the general solution of the Volterra–Fredholm integral equation by applying the method of separation of variables. A brief conclusion is presented in Sect. 4.

Consider the following linear Volterra–Fredholm integral equation:

$$\mu\psi(x, t) - \lambda \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)\psi(y, \tau)dyd\tau - \lambda \int_0^t F(t, \tau)\psi(x, \tau)d\tau = g(x, t), \quad (1)$$

$$(x = \bar{x}(x_1, x_1, \dots, x_n), \quad y = \bar{y}(y_1, y_1, \dots, y_n)),$$

where μ is a constant, defined the kind of integral equation, λ is constant, may be complex and has many physical meaning. The function $\psi(x, t)$ is unknown in the Banach space $L_2(\Omega) \times C[0, T]$, $0 \leq T < 1$, where Ω is the domain of integration with respect to position and the time $t \in [0, T]$ and it called the potential function of the Volterra–Fredholm integral equation. The kernels of time $\Phi(t, \tau)$, $F(t, \tau)$ are continuous in $C[0, T]$ and the known function $g(x, t)$ is continuous in the space $L_2(\Omega) \times C[0, T]$, $0 \leq t \leq T$. In addition the kernel of position $k(|x - y|)$ is discontinuous function.

2 The existence of a unique solution of the Volterra–Fredholm integral equation

In this paper, for discussing the existence and uniqueness of the solution of Eq. (1), we assume the following conditions:

(i) The kernel of position $k(|x - y|) \in L_2([\Omega] \times [\Omega])$, $x, y \in [\Omega]$ satisfies the discontinuity condition:

$$\left\{ \int_{\Omega} \int_{\Omega} k^2(|x - y|)dxdy \right\}^{\frac{1}{2}} = k^*, \quad k^* \text{ is constant.}$$

(ii) The kernels of time $\Phi(t, \tau)$, $F(t, \tau) \in C[0, T]$ and satisfies $|\Phi(t, \tau)| \leq M_1$, $|F(t, \tau)| \leq M_2$, s.t M_1 , M_2 are constants, $\forall t, \tau \in [0, T]$.

(iii) The given function $g(x, t)$ with its partial derivatives with respect to the position and time is continuous in the space $L_2(\Omega) \times C[0, T]$, $0 \leq \tau \leq T < 1$ and its norm is defined as,

$$\|g(x, t)\| = \max_{0 \leq t \leq T} \int_0^t \left(\int_{\Omega} g^2(x, \tau)dx \right)^{\frac{1}{2}} d\tau = N, \quad N \text{ is a constant.}$$

Theorem 1. *If the conditions (i)–(iii) are satisfied, then Eq. (1) has a unique solution $\psi(x, t)$ in the Banach space $L_2(\Omega) \times C[0, T]$, $0 \leq T < 1$, under the condition,*

$$|\lambda| < \frac{|\mu|}{M_1 k^* + M_2 T}.$$

Proof. To prove the existence of a unique solution of Eq. (1) we use the successive approximations method (*Picard’s method*), or we can use Banach’s fixed point theorem.

2.1 Picard’s method

We assume the solution of Eq. (1) takes the form:

$$\psi(x, t) = \lim_{n \rightarrow \infty} \psi_n(x, t),$$

where

$$\psi_n(x, t) = \sum_{i=0}^n H_i(x, t), \quad t \in [0, T], \quad n = 1, 2, \dots$$

where the functions $G_i(x, t)$, $i = 0, 1, \dots, n$ are continuous functions of the form:

$$\left. \begin{aligned} H_n(x, t) &= \psi_n(x, t) - \psi_{n-1}(x, t), \\ H_0(x, t) &= g(x, t) \end{aligned} \right\}. \tag{2}$$

□

Now we should prove the following lemmas:

Lemma 1. *The series $\sum_{i=0}^n H_i(x, t)$ is uniformly convergent to a continuous solution function $\psi(x, t)$.*

Proof. We construct the sequences,

$$\begin{aligned} \mu \psi_n(x, t) &= g(x, t) + \lambda \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) \psi_{n-1}(y, \tau) dy d\tau + \lambda \int_0^t F(t, \tau) \psi_{n-1}(x, \tau) d\tau, \\ \psi_0(x, t) &= g(x, t). \end{aligned}$$

Then, we get

$$\begin{aligned} \psi_n(x, t) - \psi_{n-1}(x, t) &= \frac{\lambda}{\mu} \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) (\psi_{n-1}(y, \tau) - \psi_{n-2}(y, \tau)) dy d\tau \\ &\quad + \frac{\lambda}{\mu} \int_0^t F(t, \tau) (\psi_{n-1}(x, \tau) - \psi_{n-2}(x, \tau)) d\tau. \end{aligned}$$

From Eq. (2), then, we have

$$H_n(x, t) = |\gamma| \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)H_{n-1}(y, \tau)dyd\tau + |\gamma| \int_0^t F(t, \tau)H_{n-1}(x, \tau)d\tau; \quad \gamma = \frac{\lambda}{\mu},$$

using the properties of the norm, we obtain

$$\|H_n(x, t)\| \leq |\gamma| \left\| \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)H_{n-1}(y, \tau)dyd\tau \right\| + |\gamma| \left\| \int_0^t F(t, \tau)H_{n-1}(x, \tau)d\tau \right\|. \quad (3)$$

For $n = 1$, the formula (3) yields

$$\|H_1(x, t)\| \leq |\gamma| \left\| \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)H_0(y, \tau)dyd\tau \right\| + |\gamma| \left\| \int_0^t F(t, \tau)H_0(x, \tau)d\tau \right\|,$$

by applying Cauchy–Schwarz inequality and using the condition (ii) we get

$$\begin{aligned} \|H_1(x, t)\| \leq |\gamma|M_1 \left\| \left(\int_{\Omega} |k(|x - y|)|^2 dy \right)^{\frac{1}{2}} \cdot \max_{0 \leq t \leq T} \left\| \int_0^t \left(\int_{\Omega} |H_0(y, \tau)|^2 dy \right)^{\frac{1}{2}} d\tau \right\| \right\| \\ + |\gamma|M_2 \int_0^t \|H_0(x, \tau)\| d\tau, \end{aligned}$$

using the conditions (i) and (iii), we have

$$\|H_1(x, t)\| \leq |\gamma|M_1k^*N + |\gamma|M_2N\|t\|, \quad (4)$$

where $\max_{0 \leq t \leq T} |t| = T$, so that formula (4) becomes

$$\|H_1(x, t)\| \leq |\gamma|N(M_1k^* + M_2T),$$

by induction, we get

$$\|H_n(x, t)\| \leq \beta^n N; \quad \beta = |\gamma|(M_1k^* + M_2T) < 1; \quad n = 1, 2, \dots$$

Since

$$|\lambda| < \frac{|\mu|}{M_1k^* + M_2T},$$

this leads us to say that the sequence $\psi_n(x, t)$ has a convergent solution. So that, for $n \rightarrow \infty$, we have

$$\psi(x, t) = \sum_{i=0}^{\infty} H_i(x, t). \quad (5)$$

The above formula represents an infinite convergence series. □

Lemma 2. *The function $\psi(x, t)$ of the series (5) represents an unique solution of Eq. (1).*

Proof. To show that $\psi(x, t)$ is the only solution of Eq. (1), we assume the existence of another solution $\varphi(x, t)$ of Eq. (1), then we obtain

$$\begin{aligned} \mu[\psi(x, t) - \varphi(x, t)] = & \lambda \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)[\psi(y, \tau) - \varphi(y, \tau)]dyd\tau \\ & + \lambda \int_0^t F(t, \tau)[\psi(x, \tau) - \varphi(x, \tau)]d\tau, \end{aligned}$$

which leads us to the following

$$\begin{aligned} \|\psi(x, t) - \varphi(x, t)\| = & |\gamma| \left\| \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)(\psi(y, \tau) - \varphi(y, \tau))dyd\tau \right\| \\ & + |\gamma| \left\| \int_0^t F(t, \tau)(\psi(x, \tau) - \varphi(x, \tau))d\tau \right\|, \end{aligned}$$

by applying the Cauchy–Schwarz inequality and using the conditions (i) and (ii), we get

$$\begin{aligned} \|\psi(x, t) - \varphi(x, t)\| \leq & |\gamma|M_1k^* \int_0^t \int_{\Omega} \|\psi(y, \tau) - \varphi(y, \tau)\|dyd\tau \\ & + |\gamma|M_2 \int_0^t \|\psi(x, \tau) - \varphi(x, \tau)\|d\tau, \tag{6} \\ \leq & \beta\|\psi(x, t) - \varphi(x, t)\|, \quad \beta = |\gamma|M_1k^* + |\gamma|M_2T < 1. \end{aligned}$$

The formula (6) can be adapted as,

$$(1 - \beta)\|\psi(x, t) - \varphi(x, t)\| \leq 0.$$

Since $\beta < 1$, so that $\psi(x, t) = \varphi(x, t)$, that is the solution is unique. □

2.2 Banach’s fixed point theorem

When the Picard’s method fails to prove the existence of a unique solution for the homogeneous integral equations or for the integral equations of the first kind, we must use Banach’s fixed point theorem. For this, we write the formula (1) in the integral operator form:

$$\begin{aligned} (\bar{U}\psi)(x, t) &= \frac{1}{\mu}g(x, t) + (U\psi)(x, t), \\ (U\psi)(x, t) &= \frac{\lambda}{\mu} \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)\psi(y, \tau)dyd\tau + \frac{\lambda}{\mu} \int_0^t F(t, \tau)\psi(x, \tau)d\tau. \end{aligned} \tag{7}$$

To prove the existence of a unique solution of Eq. (1), using Banach’s fixed point theorem, we must prove the normality and continuity of the integral operator (7).

(a) For the normality, we use Eq. (7) to get

$$\|(U\psi)(x, t)\| = \left| \frac{\lambda}{\mu} \right| \left\| \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)\psi(y, \tau)dyd\tau \right\| + \left| \frac{\lambda}{\mu} \right| \left\| \int_0^t F(t, \tau)\psi(x, \tau)d\tau \right\|; \quad \mu \neq 0.$$

Using the condition (ii), then applying Cauchy–Schwarz inequality, we get

$$\begin{aligned} \|(U\psi)(x, t)\| \leq & \left| \frac{\lambda}{\mu} \right| M_1 \left\| \left(\int_{\Omega} |k(|x - y|)|^2 dy \right)^{\frac{1}{2}} \cdot \max_{0 \leq t \leq T} \left| \int_0^t \left(\int_{\Omega} |H_0(y, \tau)|^2 dy \right)^{\frac{1}{2}} d\tau \right\| \right. \\ & \left. + \left| \frac{\lambda}{\mu} \right| M_2 \left\| \int_0^t \|H_0(x, \tau)\| d\tau \right\|, \end{aligned}$$

using the condition (i), we obtain

$$\|(U\psi)(x, t)\| \leq \left| \frac{\lambda}{\mu} \right| (M_1 k^* + M_2 T) \|\psi(x, t)\|,$$

since

$$\|(U\psi)(x, t)\| \leq \beta \|\psi(x, t)\|; \quad \beta = \left| \frac{\lambda}{\mu} \right| (M_1 k^* + M_2 T) < 1,$$

where

$$|\lambda| < \frac{|\mu|}{M_1 k^* + M_2 T}.$$

Therefore, the integral operator U has a normality, which leads immediately after using the condition (iii) to the normality of the operator \bar{U} .

(b) For the continuity, we suppose the two potential functions $\psi_1(x, t)$ and $\psi_2(x, t)$ in the space $L_2(\Omega) \times C[0, T]$ are satisfied Eq. (7), then

$$\begin{aligned} (\bar{U}\psi_1)(x, t) &= \frac{1}{\mu} g(x, t) + \frac{\lambda}{\mu} \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) \psi_1(y, \tau) dy d\tau + \frac{\lambda}{\mu} \int_0^t F(t, \tau) \psi_1(x, \tau) d\tau, \\ (\bar{U}\psi_2)(x, t) &= \frac{1}{\mu} g(x, t) + \frac{\lambda}{\mu} \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) \psi_2(y, \tau) dy d\tau + \frac{\lambda}{\mu} \int_0^t F(t, \tau) \psi_2(x, \tau) d\tau, \end{aligned} \tag{8}$$

Using equations (8), we get

$$\begin{aligned} \bar{U}[\psi_1(x, t) - \psi_2(x, t)] &= \frac{\lambda}{\mu} \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) [\psi_1(y, \tau) - \psi_2(y, \tau)] dy d\tau \\ &+ \frac{\lambda}{\mu} \int_0^t F(t, \tau) [\psi_1(x, \tau) - \psi_2(x, \tau)] d\tau. \end{aligned}$$

using the condition (ii) and applying the Cauchy–Schwarz inequality we get ,

$$\begin{aligned} \|\bar{U}[\psi_1(x, t) - \psi_2(x, t)]\| \leq & \left| \frac{\lambda}{\mu} \right| M_1 \left\| \left(\int_{\Omega} |k(|x - y|)|^2 dy \right)^{\frac{1}{2}} \right. \\ & \cdot \max_{0 \leq t \leq T} \left\| \int_0^t \left(\int_{\Omega} |\psi_1(y, \tau) - \psi_2(y, \tau)|^2 dy \right)^{\frac{1}{2}} d\tau \right\| + \left| \frac{\lambda}{\mu} \right| M_2 \left\| \int_0^t |\psi_1(x, \tau) - \psi_2(x, \tau)| d\tau \right\|. \end{aligned}$$

By using the condition (i), the last inequality becomes,

$$\|\bar{U}[\psi_1(x, t) - \psi_2(x, t)]\| \leq \left| \frac{\lambda}{\mu} \right| (M_1 k^* + M_2 T) \|\psi_1(x, t) - \psi_2(x, t)\|,$$

hence, we have

$$\|\bar{U}[\psi_1(x, t) - \psi_2(x, t)]\| \leq \beta \|\psi_1(x, t) - \psi_2(x, t)\|; \quad \beta = \left| \frac{\lambda}{\mu} \right| (M_1 k^* + M_2 T) < 1, \quad (9)$$

with

$$|\lambda| < \frac{|\mu|}{(M_1 k^* + M_2 T)}.$$

Inequality (9) leads us to the continuity of the integral operator \bar{U} . So that, \bar{U} is a contraction operator. Therefore by Banach's fixed point theorem, there is an unique fixed point $\psi(x, t)$, which is the solution of the linear mixed integral equation (1).

3 Separation of variables method

To obtain the general solution of Eq. (1), we do the following:

For $t = 0$, the formula (1) becomes

$$\mu\psi(x, 0) = g(x, 0). \quad (10)$$

Then, seek the solution of equation (1) in the form:

$$\psi(x, t) = \sum_{n=1}^{\infty} c_n(t)\psi_n(x),$$

in this aim, we write

$$\psi(x, t) = \psi_0(x, t) + \psi_1(x, t), \quad (11)$$

where $\psi_0(x, t)$, $\psi_1(x, t)$ are called, respectively, the complementary and particularly solution of (1). Using Eq. (11) in Eq. (1), we get

$$\mu\psi_k(x, t) - \lambda \int_0^t \int_{\Omega} \Phi(t, \tau)k(|x - y|)\psi_k(y, \tau)dyd\tau - \lambda \int_0^t F(t, \tau)\psi_k(x, \tau)d\tau = \delta_k g(x, t); \quad k = 0, 1, \quad (12)$$

also, for Eq. (10), we have

$$\mu\psi_k(x, 0) = \delta_k g(x, 0), \quad (13)$$

where,

$$\delta_k = \begin{cases} 0; & k = 0 \\ 1; & k = 1 \end{cases}.$$

From the two Eqs. (12), (13), we get

$$\begin{aligned} & \mu[\psi_k(x, t) - \psi_k(x, 0)] - \lambda \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) \psi_k(y, \tau) dy d\tau \\ & - \lambda \int_0^t F(t, \tau) \psi_k(x, \tau) d\tau = \delta_k [g(x, t) - g(x, 0)]. \end{aligned} \tag{14}$$

Now, we can represent the solution of (11) in the series form

$$\psi_k(x, t) = \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(t) \psi_{2n}(x) + c_{2n-1}^{(k)}(t) \psi_{2n-1}(x) \right), \tag{15}$$

where $\psi_{2n}(x)$, $\psi_{2n-1}(x)$ are the even and odd functions respectively.

Using Eq. (15) in Eq. (14), we obtain

$$\begin{aligned} & \mu \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(t) - c_{2n}^{(k)}(0) \right) \psi_{2n}(x) + \mu \sum_{n=1}^{\infty} \left(c_{2n-1}^{(k)}(t) - c_{2n-1}^{(k)}(0) \right) \psi_{2n-1}(x) \\ & - \lambda \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(\tau) \psi_{2n}(y) + c_{2n-1}^{(k)}(\tau) \psi_{2n-1}(y) \right) dy d\tau \\ & - \lambda \int_0^t F(t, \tau) \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(\tau) \psi_{2n}(x) + c_{2n-1}^{(k)}(\tau) \psi_{2n-1}(x) \right) d\tau = \delta_k [g(x, t) - g(x, 0)]. \end{aligned} \tag{16}$$

Taking $k = 0$, in Eq. (14), yields

$$\begin{aligned} & \mu \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(t) - c_{2n}^{(0)}(0) \right) \psi_{2n}(x) + \mu \sum_{n=1}^{\infty} \left(c_{2n-1}^{(0)}(t) - c_{2n-1}^{(0)}(0) \right) \psi_{2n-1}(x) \\ & - \lambda \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(\tau) \psi_{2n}(y) + c_{2n-1}^{(0)}(\tau) \psi_{2n-1}(y) \right) dy d\tau \\ & - \lambda \int_0^t F(t, \tau) \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(\tau) \psi_{2n}(x) + c_{2n-1}^{(0)}(\tau) \psi_{2n-1}(x) \right) dy d\tau = 0. \end{aligned} \tag{17}$$

Theorem 2. (see [3, 19]). For a symmetric and positive kernel of Fredholm integral term of Eq. (1), the integral operator,

$$(K\psi_n)(x) = \int_{\Omega} k(|x - y|) \psi_n(y) dy,$$

through the time interval $0 \leq t \leq T < 1$ is compact and self-adjoint operator. So, we may write $(K\psi_n)(x) = \alpha_n \psi_n(x)$, where α_n and $\psi_n(x)$ are the eigenvalues and the eigenfunctions of the integral operator, respectively.

In view of theorem 2, and Eq. (17), we arrive to the following

$$\begin{aligned} & \mu \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(t) - c_{2n}^{(0)}(0) \right) \psi_{2n}(x) + \mu \sum_{n=1}^{\infty} \left(c_{2n-1}^{(0)}(t) - c_{2n-1}^{(0)}(0) \right) \psi_{2n-1}(x) \\ & - \lambda \int_0^t \Phi(t, \tau) \sum_{n=1}^{\infty} \left(\alpha_{2n} c_{2n}^{(0)}(\tau) \psi_{2n}(x) + \alpha_{2n-1} c_{2n-1}^{(0)}(\tau) \psi_{2n-1}(x) \right) d\tau \\ & - \lambda \int_0^t F(t, \tau) \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(\tau) \psi_{2n}(x) + c_{2n-1}^{(0)}(\tau) \psi_{2n-1}(x) \right) dyd\tau = 0. \end{aligned}$$

Separating the odd and even terms, we obtain

$$c_{2n}^{(0)}(t) - \gamma \int_0^t (\alpha_{2n} \Phi(t, \tau) + F(t, \tau)) c_{2n}^{(0)}(\tau) d\tau = c_{2n}^{(0)}(0); \quad \gamma = \frac{\lambda}{\mu}, \tag{18}$$

and,

$$c_{2n-1}^{(0)}(t) - \gamma \int_0^t (\alpha_{2n-1} \Phi(t, \tau) + F(t, \tau)) c_{2n-1}^{(0)}(\tau) d\tau = c_{2n-1}^{(0)}(0), \tag{19}$$

the two Eqs. (18) and (19) give the same results for even and odd functions, so it is suffice to study the following equation,

$$c_n^{(0)}(t) - \gamma \int_0^t (\alpha_n \Phi(t, \tau) + F(t, \tau)) c_n^{(0)}(\tau) d\tau = c_n^{(0)}(0); \quad \gamma = \frac{\lambda}{\mu}, \tag{20}$$

where $c_n^{(0)}(0)$ is constant will be determined.

Also, taking $k = 1$ in formula (16), we obtain

$$\begin{aligned} & \mu \sum_{n=1}^{\infty} \left(c_{2n}^{(1)}(t) - c_{2n}^{(1)}(0) \right) \psi_{2n}(x) + \mu \sum_{n=1}^{\infty} \left(c_{2n-1}^{(1)}(t) - c_{2n-1}^{(1)}(0) \right) \psi_{2n-1}(x) \\ & - \lambda \int_0^t \int_{\Omega} \Phi(t, \tau) k(|x - y|) \sum_{n=1}^{\infty} \left(c_{2n}^{(1)}(\tau) \psi_{2n}(y) + c_{2n-1}^{(1)}(\tau) \psi_{2n-1}(y) \right) dyd\tau \\ & - \lambda \int_0^t F(t, \tau) \sum_{n=1}^{\infty} \left(c_{2n}^{(1)}(\tau) \psi_{2n}(x) + c_{2n-1}^{(1)}(\tau) \psi_{2n-1}(x) \right) dyd\tau = [g(x, t) - g(x, 0)]. \end{aligned} \tag{21}$$

Using theorem 2 in Eq. (21), to have

$$\begin{aligned} & \mu \sum_{n=1}^{\infty} \left(c_{2n}^{(1)}(t) - c_{2n}^{(1)}(0) \right) \psi_{2n}(x) + \mu \sum_{n=1}^{\infty} \left(c_{2n-1}^{(1)}(t) - c_{2n-1}^{(1)}(0) \right) \psi_{2n-1}(x) \\ & - \lambda \int_0^t \Phi(t, \tau) \sum_{n=1}^{\infty} \left(\alpha_{2n} c_{2n}^{(1)}(\tau) \psi_{2n}(x) + \alpha_{2n-1} c_{2n-1}^{(1)}(\tau) \psi_{2n-1}(x) \right) d\tau \\ & - \lambda \int_0^t F(t, \tau) \sum_{n=1}^{\infty} \left(c_{2n}^{(1)}(\tau) \psi_{2n}(x) + c_{2n-1}^{(1)}(\tau) \psi_{2n-1}(x) \right) dyd\tau \\ & = \sum_{n=1}^{\infty} a_{2n} \psi_{2n}(x) [g(x, t) - g(x, 0)], \end{aligned} \tag{22}$$

where,

$$1 = \sum_{n=1}^{\infty} a_{2n} \psi_{2n}(x),$$

the formula (22) can be separated to the following equations

$$\begin{aligned} c_{2n}^{(1)}(t) - \gamma \int_0^t (\alpha_{2n} \Phi(t, \tau) + F(t, \tau)) c_{2n}^{(1)}(\tau) d\tau &= \frac{1}{\mu} a_{2n} [g(x, t) - g(x, 0)] + c_{2n}^{(1)}(0); \quad \gamma = \frac{\lambda}{\mu}, \\ c_{2n-1}^{(1)}(t) - \gamma \int_0^t (\alpha_{2n-1} \Phi(t, \tau) + F(t, \tau)) c_{2n-1}^{(1)}(\tau) d\tau &= c_{2n-1}^{(1)}(0). \end{aligned} \quad (23)$$

Eqs. (20) and (23) represent Volterra integral equations of the second kind that have the same continuous kernel $\Phi(t, \tau) \in C([0, T] \times [0, T])$, and each of them has a unique solution in the class $C[0, T]$ the books edited by Linz [15] and Burton [7] contain many different methods to solve the integral equations (20) and (23).

The values of $c_n^{(0)}(0)$, $c_{2n}^{(1)}(0)$ and $c_{2n-1}^{(1)}(0)$ can be obtained, we return to the equation (10), and we seek the solution of this equation in the form,

$$\psi(x, 0) = \sum_{n=1}^{\infty} c_n(0) \psi_n(x).$$

Hence, in this respect, we write

$$\psi(x, t) = \psi_0(x, t) + \psi_1(x, t), \quad (24)$$

where $\psi_0(x, t)$ is a complementary solution while $\psi_1(x, t)$ is a particularly solution. So, from Eq. (10) we write

$$\mu \psi_k(x, 0) = \delta_k g(x, 0), \quad (25)$$

and expand the solution of equation (24) in the form

$$\psi_k(x, 0) = \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(0) \psi_{2n}(x) + c_{2n-1}^{(k)}(0) \psi_{2n-1}(x) \right), \quad (26)$$

using Eq. (26) in Eq. (25), we obtain

$$\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(0) \psi_{2n}(x) + c_{2n-1}^{(k)}(0) \psi_{2n-1}(x) \right) = \delta_k g(x, 0). \quad (27)$$

If we take $k = 0$ in Eq. (27), we obtain

$$\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(0) \psi_{2n}(x) + c_{2n-1}^{(0)}(0) \psi_{2n-1}(x) \right) = 0,$$

equating the odd and even terms in both sides, we get

$$c_{2n}^{(0)}(0) = 0, \quad c_{2n-1}^{(0)}(0) = 0,$$

then , we have

$$c_n^{(0)}(0) = 0.$$

Taking $k = 1$ in Eq. (27), we have

$$\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(1)}(0)\psi_{2n}(x) + c_{2n-1}^{(1)}(0)\psi_{2n-1}(x) \right) = g(x, 0).$$

Equating both sides of the last equation, we get

$$c_{2n}^{(1)}(0) = 0, \quad c_{2n-1}^{(1)}(0) = 0,$$

so, the last two formulas give us

$$c_n^{(1)}(0) = 0.$$

In view of Eqs. (20) and (23), the general solution of (1) can be adapted in the form

$$\psi_N(x, t) = \sum_{n=1}^N \left(c_n^{(0)}(t) + c_n^{(1)}(t) \right) \psi_n(x), \quad (28)$$

where $c_n^{(0)}(t)$ and $c_n^{(1)}(t)$ must satisfy the inequality

$$\sum_{n=1}^N |c_n^{(0)}(t) + c_n^{(1)}(t)| < \epsilon, \quad (N \rightarrow \infty, \epsilon \ll 1, 0 \leq t \leq T < 1). \quad (29)$$

Theorem 3. *If, for $t \in [0, T]$, the inequality (29) holds, the series (28) is uniformly convergent in the space $\ell_2(\Omega) \times C[0, T]$, $N \rightarrow \infty$. Hence the solution of the Volterra–Fredholm integral equation (1) can be obtained in a series form of (28).*

Theorem 4. *For the given functions $g(x, t) \in L_2(\Omega) \times C[0, T]$, $\Phi(t, \tau) \in C([0, T] \times [0, T])$, $k(x, y) \in C([\Omega] \times [\Omega])$, and under the condition (29), we have*

$$\|\psi(x, t) - \psi_N(x, t)\| \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

where $\psi(x, t)$ represents the unique solution of Eq. (1) and the error takes the form:

$$E_N = \|\psi(x, t) - \psi_N(x, t)\|,$$

where

$$E_N \rightarrow 0 \quad \text{as } N \rightarrow 1.$$

4 Conclusion and remarks

From the above results and discussion, the following may be concluded:

1. Equation (1) has a unique solution $\psi(x, t)$ in the space $L_2(\Omega) \times C[0, T]$, under some conditions.
2. The Volterra–Fredholm integral equation of the second kind, in time and position, after using separation of variables method leads to a Volterra integral equations of the second kind with continuous kernel.
3. Solutions of the Volterra integral equations can be obtained by numerical methods.

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References

- [1] M. A. Abdou, M. E. Nasr and M. A. Abdel–Aty, A study of normality and continuity for mixed integral equations, *J. of Fixed Point Theory Appl.*, 20(1) (2018). <https://doi.org/10.1007/s11784-018-0490-0>
- [2] M. A. Abdou, M. E. Nasr and M. A. Abdel–Aty, Study of the Normality and Continuity for the Mixed Integral Equations with Phase-Lag Term, *Inter. J. of Math. Analysis*, 11 (2017), 787–799. <https://doi.org/10.12988/ijma.2017.7798>
- [3] M. A. Abdou, Fredholm–Volterra integral equation of the first kind and contact problem, *Appl. Math. Comput.*, 125(2–3) (2002), 177–193. [https://doi.org/10.1016/S0096-3003\(00\)00118-1](https://doi.org/10.1016/S0096-3003(00)00118-1)
- [4] M. AL–Jawary, G. Radhi and J. Ravnik, Two efficient methods for solving Schlömilch’s integral equation, *International Journal of Intelligent Computing and Cybernetics*, 10(3) (2017), 287–309.
- [5] S. Bazm, Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations, *J. Comput. Appl. Math.*, 275 (2015), 44–60.

- [6] F. Brauer, C. Castillo–Chvez, *Mathematical Models in Population Biology and Epidemiology*, Springer, New York, (2012).
- [7] T. A. Burton, *Volterra Integral and Differential Equations*, Elsevier, London, New York, (2005).
- [8] C. Cattani, A. Kudreyko, Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind, *Appl. Math. Comput.*, 215(12) (2010), 4164–4171. <https://doi.org/10.1016/j.amc.2009.12.037>
- [9] C. Craciun, M. A. Serban, A nonlinear integral equation via picard operators, *Fixed Point Theory*, 12(1) (2011), 57–70.
- [10] Z. Gu, X. Guo, D. Sun, Series expansion method for weakly singular Volterra integral equations, *Applied Numerical Mathematics*, 105 (2016), 112–123. <https://doi.org/10.1016/j.apnum.2016.03.001>
- [11] A. A. Hamoud, K. P. Ghadle, Approximate solutions of fourth-order fractional integro–differential equations, *Acta Universitatis Apulensis*, 55 (2018), 49–61.
- [12] A. A. Hamoud, K. P. Ghadle, On the numerical solution of nonlinear Volterra–Fredholm integral equations by variational iteration method, *Int. J. Adv. Sci. Tech. Research*, 3 (2016), 45–51.
- [13] E. Hashemizadeh, M. Rostami, Numerical solution of Hammerstein integral equations of mixed type using the Sinc–collocation method, *J. Comput. Appl. Math.*, 279 (2015), 31–39. <https://doi.org/10.1016/j.cam.2014.10.019>
- [14] V. Ilea, D. Otrocol, Some properties of solutions of a functional–differential equation of second order with delay, *The Scientific World Journal*, (2014). <http://dx.doi.org/10.1155/2014/878395>
- [15] P. Linz, *Analytic and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, (1985).
- [16] K. Maleknejad, H. Almasieh, M. Roodaki, Triangular functions (TF) method for the solution of nonlinear Volterra–Fredholm integral equations, *Commun. Nonlinear Sci. Numer. Simul.*, 15(11) (2010), 3293–3298. <https://doi.org/10.1016/j.cnsns.2009.12.015>
- [17] M. E. Nasr, M. A. Abdel–Aty, Analytical discussion for the mixed integral equations, *J. of Fixed Point Theory Appl.*, 20(3) (2018). <https://doi.org/10.1007/s11784-018-0589-3>

- [18] D. Otrocol, V. Ilea, Ulam stability for a delay differential equation, *Central European Journal of Mathematics*, 11(7) (2013), 1296–1303.
- [19] G. Ya. Popov, *Contact Problems for a Linearly Deformable Base*, Kiev, Odessa, (1982).
- [20] I. A. Rus, Results and problems in Ulam stability of operatorial equations and inclusions, *Handbook of functional equations*, Springer Optim. Appl., New York, (2014).
<https://doi.org/10.1007/978-1-4939-1286-5-15>
- [21] V. S. Sizikov, D. N. Sidorov, Generalized quadrature for solving singular integral equations of Abel type in application to infrared tomography, *Applied Numerical Mathematics*, 106 (2016), 69–78. <https://doi.org/10.1016/j.apnum.2016.03.004>

A COMPARATIVE STUDY OF THREE FORMS OF ENTROPY ON TRADE VALUES BETWEEN KOREA AND FOUR COUNTRIES

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ABSTRACT. Recently Wood-Jang[21] studied some applications of the Choquet integral in the trading relationship that Korea shares with selected trading partners. In this study, we consider the fuzzy entropy and the Shannon entropy, in addition we also define the Choquet entropy on a fuzzy set and develop four fuzzy sets which are related to that of the Choquet expected utility $CEU(u(a))$ for the trade values of utility u from an act a on S . Using this data set, we calculate three forms of entropy on four fuzzy sets as in the Choquet expected utility for the trade values that exist between Korea and four trading partner countries. Furthermore, we provide comparisons with three forms of entropy on four fuzzy sets which are representative of the four trading partner countries analyzed in this study.

1. INTRODUCTION

Many researchers have studied the Choquet integrals with respect to a fuzzy measure of fuzzy sets or interval-valued fuzzy sets and their applications in [2,4-13,19,20,26]. There are several examples of such analysis, these include student evaluations, similarity measures, the examination of the Choquet expected utility, and various other forms of inequalities. Recently, by using Choquet integrals with respect to a fuzzy measure, Wood-Jang[20,21] studied applications of them. These include some applications of the Choquet integral by firstly examining the imprecise market premium functions, and then more recently the trade relationship that Korea shares with selected trading partners. By using fuzzy sets and Choquet integrals in [17], studies utilized the concept of Choquet integral expected utility and its related areas(see[12,13,19,21,26]). Note that Biswas [2] investigated a student's evaluation on the space of fuzzy sets which include data information from the students' respective classes.

Our first motivation was to build on our previous efforts by considering three forms of fuzzy entropy in [1,3,14,15,18,24,25], the Shannon entropy in [3,21,24,25], and the Choquet entropy which we define in this study. From this point of view, we provide a comparative study of three

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forms of entropy on the level of trade that exists between Korea and four trading partners using data obtained from the WTO [23]. Our second motivation for conducting this study, was to provide a unique analysis of international trade flows using the Shannon entropy, fuzzy entropy, and Choquet entropy techniques.

In this study, we consider the fuzzy entropy and the Shannon entropy, in addition we provide a definition of the Choquet entropy on a fuzzy set and also develop four fuzzy sets which are related to the Choquet expected utility $CEU(u(a))$ for the trade values of utility u from an act a on S for the specified 2-digit HS product codes (01 – 05) for animal product exports between Korea and selected trading partners for years 2010-2013 using data obtained from the WTO regional trade database[23]. Using this data set, we calculate three forms of entropy on four fuzzy sets as in the Choquet expected utility for the trade values that exist between Korea and four trading partner countries (New Zealand, USA, India, Turkey). Furthermore, we provide comparisons for three forms of entropy on four fuzzy sets for the four trading partner countries.

2. THREE FORMS OF ENTROPY ON FUZZY SETS

Let X be a finite set of states of nature and $F(X)$ be the set of all fuzzy sets $A = \{(x, m_A(x)) \mid x \in X, m_A \rightarrow [0, 1] \text{ is a function}\}$. Recall that m_A is called a membership function of A .

Definition 2.1. ([2,4-13,19,20,26])

(1) A real-valued function μ on X is called a fuzzy measure if it satisfies

$$\begin{aligned} \text{(i)} \quad & \mu(\emptyset) = 0, \mu(X) = 1, \\ \text{(ii)} \quad & A \subset B \Rightarrow \mu(A) \leq \mu(B), \end{aligned} \tag{1}$$

where A, B are subsets of X .

(2) The Choquet integrals with respect to a fuzzy measure μ of $A \in F(X)$ is defined by

$$(C) \int m_A d\mu = \int_0^1 \mu(\{x \in X \mid m_A(x) \geq \alpha\}) d\alpha. \tag{2}$$

Remark 2.1. Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set. It is well known that the discrete Choquet integral with respect to a fuzzy measure μ is followings.

$$(C) \int m_A d\mu = \sum_{i=1}^n m_A(x_{(i)}) [\mu(E^{(i)}) - \mu(E^{(i+1)})], \tag{3}$$

where (\cdot) indicates a permutation on $\{1, 2, \dots, n\}$ such that

$$m_A(x_{(1)}) \leq m_A(x_{(2)}) \leq \dots \leq m_A(x_{(n)}), \tag{4}$$

$E^{(i)} = \{x \in X \mid m_A(x) \geq m_A(x_{(i)})\}$ for $i = 1, 2, \dots, n$ and let $E^{n+1} = \emptyset$ (see [4-7,9,10,11,13]).

By using the Choquet integral, we consider the Choquet expected utility $CEU(u(a))$ of utility $u(a)$ from an act a as follows. Note that in economics, the utility function is an important concept that measures preferences over a set of goods and services. Utility is measured in units called utils, which represent the welfare or satisfaction of a consumer from consuming a certain number of goods. Here, we assume that an act is a function from S to X , where S is a finite set of states of nature.

Definition 2.2. ([5]) Let $u : X \rightarrow [0, 1]$ be a utility and a be an act from S to X . The full version of the Choquet expected utility is mentioned above so we can use the CEU abbreviation here. with respect to a fuzzy measure μ of utility u from act a is defined by

$$CEU(u(a)) = (C) \int u(a(s))d\mu(s). \tag{5}$$

Now, we introduce three forms of entropy on a fuzzy set as follows. Firstly, the Shannon entropy which was first developed by Shannon [24]. The mathematical communication theory has been utilized to measure the fuzziness in a fuzzy set or system [25]. According to Shannon, the information source is a person or a device that produces messages, using the average minimum amount of information.

Definition 2.3. ([3,22,24,25]) Let $A \in F(X)$ and $X = \{x_1, x_2, \dots, x_n\}$ be finite set. The Shannon entropy on A is defined by

$$E_S(A) = - \sum_{i=1}^n m_A(x_i) \log m_A(x_i). \tag{6}$$

Secondly, the fuzzy entropy on a fuzzy set is used to express the mathematical values of the fuzziness of fuzzy sets. The concept of entropy, the basic subject of information theory and telecommunication, is a measure of fuzziness in fuzzy sets. Luca-Termini [18] note that Fuzzy entropy $D(A)$ can be represented by the Shannon function as follows

$$D(A) = k \sum_{i=1}^n s(m_A(x_i)),$$

where $s(x) = -x \log x - (1 - x) \log(1 - x)$ is the Shannon function. When we put $X = \{x_1, x_2, \dots, x_n\}$, we consider the following fuzzy entropy $E_F(A)$ which is the fuzzy entropy $D(A)$ with $k = -\frac{1}{n}$.

Definition 2.4. ([1,3,14,15,18,24,25]) Let $A \in F(X)$ and $X = \{x_1, x_2, \dots, x_n\}$ be finite set. The fuzzy entropy on A is defined by

$$E_F(A) = -\frac{1}{n} \sum_{i=1}^n [m_A(x_i) \log m_A(x_i) + (1 - m_A(x_i)) \log(1 - m_A(x_i))]. \tag{7}$$

Thirdly, we define the Choquet entropy on a fuzzy set and compare the Choquet entropy to another two forms of entropy, this helps to demonstrate the role of the Choquet entropy through the trading relationship that exists between Korea and four of its trading partners in the next section.

Definition 2.5. Let $A \in F(X)$ and X be a set. The fuzzy entropy on A is defined by

$$E_C(A) = 1 - (C) \int m_A(x)d\mu(x). \tag{8}$$

Note that if $X = \{x_1, x_2, \dots, x_n\}$ is finite set, then we get

$$E_C(A) = 1 - \sum_{i=1}^n m_A(x_{(i)})[\mu(A_{(i)}) - \mu(A_{(i+1)})], \tag{9}$$

where (\cdot) indicates a permutation on $\{1, 2, \dots, n\}$ such that

$$m_A(x_{(1)}) \leq m_A(x_{(2)}) \leq \dots \leq m_A(x_{(n)}) \tag{10}$$

and $A_{(i)} = i, 2, \dots, n$ and $A_{(n+1)} = \emptyset$.

3. CONSTRUCTIONS OF FOUR FUZZY SETS

In this section, we create four fuzzy sets, the USA-fuzzy set U , the NZ-fuzzy set N , IN-fuzzy set I , TR-fuzzy set T for the CEU of the trade values that exist between Korea and four countries. Now, we consider the CEU of a utility on a set of trade values (in USD) that represent the trading relationship that Korea shares with selected trading partners(i.e. Korea-USA, Korea-New Zealand, Korea-India, and Korea-Turkey). In [21], we examined these respective trading relationships by incorporating a clearly defined set of Harmonized System (HS) product code categories (i.e. HS Codes $i = 1, 2, 3, 4, 5$) for each individual year that is under review (i.e. 2010, 2011, 2012, 2013). We note that the product code definitions have been provided by the UN Comtrade’s online database and the relevant categories are defined as follows(see [21]):

1. Live animals; animal products.
2. Meat and edible meat offal.
3. Fish and crustaceans, mollusks and other aquatic invertebrates.
4. Dairy produce; birds’ eggs; natural honey; edible products of animal origin, not elsewhere specified or included.
5. Products of animal origin, not elsewhere specified or included.

Denote that HSPC=HS Product Code, s =Year, $a(s)$ =Trade Value, $u(a(s))$ =the utility of $a(s)$, $CEU(u(a(s)))$ =the Choquet Expected Utility of $u(a)$ from a . By using the trade values in tables A_1 through to A_4 in the Appendix, we calculate the Choquet expected utility $CEU(u(a))$ for the set of trade values (in USD) that represent Korea’s trading relationship with a particular country for years 2010, 2011, 2012, and 2013. Let $s_1 = 2010, s_2 = 2011, s_3 = 2012, s_4 = 2013$. Let $X = \{1, 2, 3, 4, 5\}$. Note that (\cdot) indicates a permutation on $\{1, 2, 3, 4, 5\}$, such that

$$CEU(u(a_{(1)})) \leq CEU(u(a_{(2)})) \leq \dots \leq CEU(u(a_{(5)})). \tag{11}$$

Then we denote that $a_{(i)} = a(s_{(i)})$ for $i = 1, 2, 3, 4, 5$ satisfy the equation (11). We define a fuzzy measure μ on X as follows.

$$\begin{aligned} \mu(E^{(4)}) &= \mu_1(\{a_{(4)}\}) = 0.1, \quad \mu(E^{(3)}) = \mu_1(\{a_{(3)}, a_{(4)}\}) = 0.3, \\ \mu(E^{(2)}) &= \mu_1(\{a_{(2)}, a_{(3)}, a_{(4)}\}) = 0.6, \quad \mu(E^{(1)}) = \mu_1(\{a_{(4)}, a_{(3)}, a_{(2)}, a_{(1)}\}) = 1, \end{aligned} \tag{12}$$

and if $a(s)$ is the trade value of s and $u(a) = \sqrt{\frac{a}{100141401}}$, then by using Definition 2.3, we obtain the following $CEU(u(a))$ as follows:

$$\begin{aligned} CEU(u(a)) &= \sum_{i=1}^4 u(a(s^{(i)})) \left(\mu(E^{(i)}) - (\mu(E^{(i+1)})) \right) \\ &= 0.4u(a(s^{(1)})) + 0.3u(a(s^{(2)})) + 0.2u(a(s^{(3)})) + 0.1u(a(s^{(4)})). \end{aligned} \tag{13}$$

By using (5), we obtained the four tables $A_1 \sim A_4$ (see [17]). If we take $m_Y(i) = CEU(u(a(s_i)))$, then we develop four fuzzy sets $Y : \{1, 2, 3, 4, 5\} \rightarrow [0, 1]$ by $Y = \{(i, m_X(i)) | i = 1, 2, 3, 4, 5\}$. Here, Y is one of USA-fuzzy set U , NZ-fuzzy set N , IN-fuzzy set I , and TR-fuzzy set T defined by

$$U = \{(1, 0.05664), (2, 0.04483), (3, 0.93879), (4, 0.20821), (5, 0.04858)\} \tag{14}$$

$$N = \{(1, 0.00533), (2, 0.00000), (3, 0.78873), (4, 0.15976), (5, 0.01557)\} \tag{15}$$

$$I = \{(1, 0.00154), (2, 0.00000), (3, 0.04570), (4, 0.00000), (5, 0.00000)\} \tag{16}$$

$$T = \{(1, 0.00264), (2, 0.00887), (3, 0.00368), (4, 0.00470), (5, 0.00000)\} \tag{17}$$

4. CALCULATE THREE FORMS OF ENTROPY ON FOUR FUZZY SETS

In this section, we calculate three forms of entropy on four fuzzy sets U, N, I, T . Note that $0 \log 0 = 1$. Firstly, from (6) and (14)-(17), we get Shannon entropies $E_S(U), E_S(N), E_S(T)$ on four fuzzy sets as follows.

$$E_S(U) = -(0.05664 \log 0.05664 + 0.04483 \log 0.04483 + 0.93879 \log 0.93879 + 0.20821 \log 0.20821 + 0.04858 \log 0.04858) = 0.83476, \tag{18}$$

$$E_S(N) = -(0.00533 \log 0.00533 + 0.00000 \log 0.00000 + 0.78873 \log 0.78873 + 0.15978 \log 0.15978 + 0.01557 \log 0.01557) = 0.57291, \tag{19}$$

$$E_S(I) = -(0.00154 \log 0.00154 + 0.00000 \log 0.00000 + 0.04570 \log 0.04570 + 0.00000 \log 0.00000 + 0.00000 \log 0.00000) = 0.15099, \tag{20}$$

$$E_S(T) = -(0.00264 \log 0.00264 + 0.00887 \log 0.00887 + 0.00368 \log 0.00368 + 0.00470 \log 0.00470 + 0.00000 \log 0.00000) = 0.10340. \tag{21}$$

Secondly, from (7) and (14)-(17), we get fuzzy entropies $E_F(U), E_F(N), E_F(I), E_F(T)$ on four fuzzy sets as follows.

$$E_F(U) = -\frac{1}{5}(0.05664 \log 0.05664 + (1 - 0.05664) \log(1 - 0.05664) + 0.04483 \log 0.04483 + (1 - 0.04483) \log(1 - 0.04483) + 0.93879 \log 0.93879 + (1 - 0.93879) \log(1 - 0.93879) + 0.20821 \log 0.20821 + (1 - 0.20821) \log(1 - 0.20821) + 0.04858 \log 0.04858 + (1 - 0.04858) \log(1 - 0.04858)) = 0.26736, \tag{22}$$

$$E_F(N) = -\frac{1}{5}(0.00533 \log 0.00533 + (1 - 0.00533) \log(1 - 0.00533) + 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000) + 0.78873 \log 0.78873 + (1 - 0.78873) \log(1 - 0.78873) + 0.15978 \log 0.15978 + (1 - 0.15978) \log(1 - 0.15978) + 0.01557 \log 0.01557 + (1 - 0.01557) \log(1 - 0.01557)) = 0.21368, \tag{23}$$

$$E_F(I) = -\frac{1}{5}(0.00154 \log 0.00154 + (1 - 0.00154) \log(1 - 0.00154) + 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000) + 0.04570 \log 0.04570 + (1 - 0.04570) \log(1 - 0.04570) + 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000) + 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000)) = 0.03834, \tag{24}$$

$$E_F(T) = -\frac{1}{5}(0.00264 \log 0.00264 + (1 - 0.00264) \log(1 - 0.00264) + 0.00887 \log 0.00887 + (1 - 0.00887) \log(1 - 0.00887) + 0.00368 \log 0.00368 + (1 - 0.00368) \log(1 - 0.00368) + 0.00470 \log 0.00470 + (1 - 0.00470) \log(1 - 0.00470) + 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000)) = 0.01447. \tag{25}$$

Thirdly, from (9) and (14)-(17), we get Choquet entropys $E_C(U), E_C(N), E_C(I), E_C(T)$ on four fuzzy sets as follows.

$$E_C(U) = 1 - (0.04483 \times 0.1 + 0.04858 \times 0.1 + 0.05664 \times 0.6 + 0.20821 \times 0.1 + 0.93879 \times 0.1) = 0.536119, \tag{26}$$

$$E_C(N) = 1 - (0.00000 \times 0.1 + 0.00533 \times 0.1 + 0.01557 \times 0.6 + 0.15976 \times 0.1 + 0.78873 \times 0.1) = 0.895276, \tag{27}$$

$$E_C(I) = 1 - (0.00154 \log 0.00154 + 0.00000 \log 0.00000 + 0.04570 \log 0.04570 + 0.00000 \log 0.00000 + 0.00000 \log 0.00000) = 0.15099, \tag{28}$$

$$E_C(T) = 1 - (0.00264 \log 0.00264 + 0.00887 \log 0.00887 + 0.00368 \log 0.00368 + 0.00470 \log 0.00470 + 0.00000 \log 0.00000) = 0.10340. \tag{29}$$

Through investigations (18)-(29), we can compare the three forms of entropy on the four fuzzy sets U, N, I, T as follows.

(1) In this study, we found that the Shannon entropy had a range $[0, 1]$ and the fuzzy entropy had a range $[0, 0.5]$ and the Choquet entropy had a range $[0.5, 1]$.

(2) The Shannon entropy E_S and the fuzzy entropy E_F have the same order of four countries in the trading relationship that Korea shares with selected trading partner as follows.

$$E_S(U) \geq E_S(N) \geq E_S(I) \geq E_S(T), \tag{30}$$

and

$$E_F(U) \geq E_F(N) \geq E_F(I) \geq E_F(T). \tag{31}$$

(3) The Choquet entropy has the following order of the trading relationship that Korea shares with selected trading partner.

$$E_C(U) \leq E_C(N) \leq E_C(I) \leq E_C(T). \tag{32}$$

From (2) and (3), we observe the order in which Choquet entropy was investigated is the opposite of the order in the other two forms.

5. CONCLUSIONS

According to the data analyzed in experiments (27) - (29), we can see that the Shannon entropy and the Fuzzy entropy contain the same order of results. However, the results for the Choquet entropy were very different to that of the Fuzzy and Shannon entropies, as the opposite order of results being obtained. This finding also suggests that the results for the Choquet entropy exhibit a much higher level of ambiguity across the four countries (New Zealand, India, the US, and Turkey) analyzed, particular those countries that have a smaller trading relationship with Korea.

From an economic perspective, the Shannon and Fuzzy entropies have provided scholars with a means of better understanding the scope and magnitude of the potential relationship that exists between particular entities, in this case the trading relationship between Korea and four trading partners. In an era where the development of stronger bilateral economic ties through trade, such an analysis provides a unique but timely portrayal.

Furthermore, the ambiguities present in the Choquet entropy findings highlight the important need to carry out additional research. Such efforts would help to establish a clearer understanding on the types of trading relationships present between Korea and the four countries selected for this study.

REFERENCES

- [1] S. Al-sharhan, F. Karray, W. Gveaieb, O. Bosir, *Fuzzy entropy; A brief surbey, 10th IEEE International conference an fuzzy systems*, Melbourne, Australia, 1135-1139.
- [2] R. Biswas, *An application of fuzzy sets in students' evaluation*, *Fuzzy Sets Syst.* **74** (1995) 187-194.
- [3] P. Burillo, *Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets*, *Fuzzy Sets Syst.* **78** (1996) 305-316.
- [4] G. Choquet, *Theory of capacities*, *Ann. Inst. Fourier* **5** (1953) 131-295.
- [5] I. Gilboa, D. Schmeidler, *Maximin expected utility with non-unique prior*, *J. Math. Econ.* **18** (1989) 141-153.
- [6] L.C. Jang, B.M. Kil, Y.K. Kim, J.S. Kwon, *Some properties of Choquet integrals of set-valued functions*, *Fuzzy Sets Syst.*, **91** (1997) 61-67.
- [7] L.C. Jang, J.S. Kwon, *On the representation of Choquet integrals of set-valued functions and null sets*, *Fuzzy Sets Syst.*, **112** (2000) 233-239.
- [8] L.C. Jang, *A note on the interval-valued fuzzy integral by means of an interval-representable pseudo-multiplication and their convergence properties*, *Fuzzy Sets Syst.* **137** (2003) 11-26.
- [9] L.C. Jang, *The application of interval-valued Choquet integrals in multicriteria decision aid*, *J. Appl. Math. computing*, **20(1-2)** (2006) 549-556.
- [10] L.C. Jang, *A note on Jensen type inequality for Choquet integrals*, *Int. J. Fuzzy logic and Intell. syst.*, **9(2)** (2009) 71-75.
- [11] L.C. Jang, *On properties of the Choquet integral of interval-valued functions*, *Journal of Applied Mathematics*, **2011** (2011) Article ID 492149, 10 pages.
- [12] L.C. Jang, J. Wood, *The application of the Choquet integral expected utility in international trade*, *Advan. Stud. Contem. Math.*, **27(2)** (2017) 159-173.
- [13] L.C. Jang, *A note on convergence properties of interval-valued capacity functionals and Choquet integrals*, *Information Sciences* **183** (2012) 151-158.
- [14] B. Kasko, *Fuzzy entropy and conditioning*, *Information Sciences* **40** (1986) 165-174.
- [15] A.D. Luca, S. Termini, *A definition of a nonprobabilistic entropy in setting of fuzzy sets theory*, *Information Control* **20** (1972) 301-312.
- [16] L. Mangelsdorff, M. Weber, *Testing Choquet expected utility* *J. Econ. Behavior and Organization* **25** (1994) 437-457.
- [17] T. Murofushi, M. Sugeno, *Some quantities represented by Choquet integral* *Fuzzy Sets Syst.* **56** (1993) 229-235.
- [18] E. Pasha, R. Farnoosh, A. Fatemi, *Fuzzy entropy as cost function in image processing* *Proceedings of the 2nd IMT-GT Regionsl Conference on Mathematics, Statistics and Applications* Universiti Sains Malaysia, Penang, June 13-15, 2006.
- [19] D. Schmeidler, *Subjective probability and expected utility without additivity*, *Econometrica* **57(3)** (1989) 571-587.
- [20] J. Wood, L.C. Jang, *A note on Choquet integrals and imprecise market premium functionals*, *Proc. Jangjeon Math.Soc.*, **18(4)** (2015) 601-608.
- [21] J. Wood, L.C. Jang, *A study on the Choquet integral with respect to a capacity and its applications*, *Global J. Pure Applied Math.*, **12(2)** (2016) 1593-1599.
- [22] C.E. Shannon, *Amathematical theory of communication*, *The Best System Technical Journal*, **27** (1948) 379-423.
- [23] WTO (2016). WTO Regional Trade Database.
- [24] L. Xuechang, *Entropy, distance maesure and similarity measure of fuzzy sets and their relations*, *Fuzzy Sets Syst.*, **52** (1992) 201-227.
- [25] W. Zeng, H. Li, *Relationship between similarity maesure and entropy of interval-valued fuzzy sets* , *Fuzzy Sets Syst.*, **57** (2006) 1477-1484.
- [26] D. Zhang, *Subjectiv ambiguity, expected utility and Choquet expected utility*, *Econ. Theory* **20** (2002) 159-181.

6. APPENDIX

Table A1: The CEU for animal product exports between Korea and the USA for years 2010-2013

HSPC	s	$a(s)$ (USD)	$u(a(s))$	$CEU_{(i,USA)}(u(a))$
1	s_1	$286892 = a(s^{(1)})$	0.05352	0.05664
	s_2	$330299 = a(s^{(2)})$	0.05743	
	s_3	$358496 = a(s^{(3)})$	0.05983	
	s_4	$364918 = a(s^{(4)})$	0.06037	
2	s_1	$997539 = a(s^{(4)})$	0.09981	0.04483
	s_2	$376805 = a(s^{(3)})$	0.06034	
	s_3	$30005 = a(s^{(1)})$	0.01731	
	s_4	$272884 = a(s^{(2)})$	0.05220	
3	s_1	$74866073 = a(s^{(1)})$	0.86464	0.93879
	s_2	$95654573 = a(s^{(2)})$	0.97734	
	s_3	$100141401 = a(s^{(4)})$	1.00000	
	s_4	$99871717 = a(s^{(3)})$	0.99865	
4	s_1	$3722326 = a(s^{(1)})$	0.19280	0.20821
	s_2	$4323214 = a(s^{(2)})$	0.20778	
	s_3	$5016833 = a(s^{(4)})$	0.22382	
	s_4	$4910771 = a(s^{(3)})$	0.22145	
5	s_1	$235669 = a(s^{(2)})$	0.04851	0.04858
	s_2	$359747 = a(s^{(3)})$	0.05994	
	s_3	$101795 = a(s^{(1)})$	0.05994	
	s_4	$863858 = a(s^{(4)})$	0.09088	

Table A2: The CEU for animal product exports between Korea and New Zealand for years 2010-2013

HSPC	s	$a(s)$ (USD)	$u(a(s))$	$CEU_{(i,NZ)}(u(a))$
1	s_1	$6650 = a(s^{(4)})$	0.00815	0.00533
	s_2	$4497 = a(s^{(3)})$	0.00670	
	s_3	$1589 = a(s^{(1)})$	0.00398	
	s_4	$2779 = a(s^{(2)})$	0.00527	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$70759196 = a(s^{(2)})$	0.84059	0.78873
	s_2	$91263506 = a(s^{(4)})$	0.95464	
	s_3	$70763937 = a(s^{(3)})$	0.84062	
	s_4	$46632301 = a(s^{(1)})$	0.68240	
4	s_1	$165773 = a(s^{(3)})$	0.04069	0.15976
	s_2	$113751 = a(s^{(1)})$	0.03370	
	s_3	$148756 = a(s^{(2)})$	0.03854	
	s_4	$277350 = a(s^{(4)})$	0.05263	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.01557
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$218022 = a(s^{(3)})$	0.04666	
	s_4	$393025 = a(s^{(4)})$	0.00265	

Table A3: the CEU for Animal product export between Korea and India for years 2010-2013

HSPC	s	$a(s)$ (USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$1050 = a(s^{(3)})$	0.00324	0.00264
	s_2	$1300 = a(s^{(4)})$	0.00360	
	s_3	$450 = a(s^{(1)})$	0.00212	
	s_4	$700 = a(s^{(2)})$	0.00264	
2	s_1	$35432 = a(s^{(3)})$	0.01881	0.00887
	s_2	$50639 = a(s^{(4)})$	0.02249	
	s_3	$2656 = a(s^{(1)})$	0.00515	
	s_4	$8230 = a(s^{(2)})$	0.00907	
3	s_1	$8695 = a(s^{(4)})$	0.009318	0.00368
	s_2	$5247 = a(s^{(3)})$	0.00724	
	s_3	$0 = a(s^{(1)})$	0.00000	
	s_4	$1865 = a(s^{(2)})$	0.00432	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00470
	s_2	$21614 = a(s^{(3)})$	0.01469	
	s_3	$30938 = a(s^{(4)})$	0.01758	
	s_4	$0 = a(s^{(2)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

Table A4: The CEU for animal product exports between Korea and Turkey for years 2010-2013

HSPC	s	$a(s)$ (USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$0 = a(s^{(1)})$	0.00000	0.00154
	s_2	$6900 = a(s^{(4)})$	0.00830	
	s_3	$150 = a(s^{(2)})$	0.00122	
	s_4	$300 = a(s^{(3)})$	0.00173	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$0 = a(s^{(1)})$	0.00000	0.04570
	s_2	$672952 = a(s^{(3)})$	0.08198	
	s_3	$2532837 = a(s^{(4)})$	0.15904	
	s_4	$199874 = a(s^{(2)})$	0.04468	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

QUADRATIC FUNCTIONAL INEQUALITY IN MODULAR SPACES AND ITS STABILITY

CHANG IL KIM AND GILJUN HAN*

ABSTRACT. In this paper, we prove the generalized Hyers-Ulam stability for the following functional inequality

$$\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \geq \rho(k[f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y)])$$

in modular spaces without Δ_2 -conditions.

1. INTRODUCTION AND PRELIMINARIES

In 1940, Ulam proposed the following stability problem (cf. [16]):

“Let G_1 be a group and G_2 a metric group with the metric d . Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \rightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \rightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?”

In the next year, Hyers [4] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ theorem was generalized by Aoki [2] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. A generalization of the Rassias’ theorem was obtained by Găvruta [3] by replasing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

A problem that mathematicians has dealt with is ”how to generalize the classical function space L^p ”. A first attempt was made by Birnbaum and Orlicz in 1931. The more abstract generalization was given by Nakano [11] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*([1], [6], [7], [8], [9], [12], [15], [18]). This idea was refined and generalized by Musielak and Orlicz [10] in 1959.

Recently, Sadeghi [14] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the Δ_2 -condition and Wongkum, Chaipunya, and Kumam [17] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the Δ_2 -condition.

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In this paper, we prove the generalized Hyers-Ulam stability for the following quadratic functional equation

$$(1.1) \quad \begin{aligned} & \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \\ & \geq \rho(k[f(ax+by) + f(ax-by) - 2a^2f(x) - 2b^2f(y)]) \end{aligned}$$

in modular spaces without Δ_2 -conditions by using a fixed point theorem.

Definition 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}, \mathbb{C}, \text{ or } \mathbb{N})$.

- (1) A generalized functional $\rho : X \rightarrow [0, \infty]$ is called a *modular* if
 - (M1) $\rho(x) = 0$ if and only if $x = 0$,
 - (M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
 - (M3) $\rho(z) \leq \rho(x) + \rho(y)$ whenever z is a convex combination of x and y .
- (2) If (M3) is replaced by
 - (M4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$

for all $x, y \in V$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$, then we say that ρ is *convex*.

For any modular ρ on X , the modular space X_ρ is defined by

$$X_\rho = \{x \in X \mid \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}$$

and the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf \left\{ \lambda > 0 \mid \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

Let X_ρ be a modular space and $\{x_n\}$ a sequence in X_ρ . Then (i) $\{x_n\}$ is called ρ -Cauchy if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, (ii) $\{x_n\}$ is called ρ -convergent to a point $x \in X_\rho$ if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, and (iii) a subset K of X_ρ is called ρ -complete if each ρ -Cauchy sequence is ρ -convergent to a point in K .

A modular space X_ρ is said to *satisfy the Δ_2 -condition* if there exists $k \geq 2$ such that $X_\rho(2x) \leq kX_\rho(x)$ for all $x \in X$.

Example 1.2. ([9], [11], [12]) A convex function ζ defined on the interval $[0, \infty)$, nondecreasing and continuous, such that $\zeta(0) = 0, \zeta(\alpha) > 0$ for $\alpha > 0$, $\zeta(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$, is called an Orlicz function. Let (Ω, Σ, μ) be a measure space and $L^0(\mu)$ the set of all measurable real valued (or complex valued) functions on Ω . Define the Orlicz modular ρ_ζ on $L^0(\mu)$ by the formula $\rho_\zeta(f) = \int_\Omega \zeta(|f|)d\mu$. The associated modular space with respect to this modular is called an Orlicz space, and will be denoted by (L^ζ, Ω, μ) or briefly L^ζ . In other words,

$$L^\zeta = \{f \in L^0(\mu) \mid \rho_\zeta(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

It is known that the Orlicz space L^ζ is ρ_ζ -complete. Moreover, $(L^\zeta, \|\cdot\|_{\rho_\zeta})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\zeta}$ is defined as follows

$$\|f\|_{\rho_\zeta} = \inf \left\{ \lambda > 0 \mid \int_\Omega \zeta\left(\frac{|f|}{\lambda}\right)d\mu \leq 1 \right\}.$$

Further, if μ is the Lebesgue measure on \mathbb{R} and $\zeta(t) = e^t - 1$, then ρ_ζ does not satisfy the Δ_2 -condition.

For a modular space X_ρ , a nonempty subset C of X_ρ , and a mapping $T : C \rightarrow C$, the orbit of T at $x \in C$ is the set

$$\mathbb{O}(x) = \{x, Tx, T^2x, \dots\}.$$

If $\delta_\rho(x) = \sup\{\rho(u - v) \mid u, v \in \mathbb{O}(x)\} < \infty$, then one says that T has a bounded orbit at x .

Lemma 1.3. [5] *Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ -complete subset. If $T : C \rightarrow C$ is a ρ -contraction, that is, there is a constant $L \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \quad \forall x, y \in C$$

and T has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is ρ -convergent to a point $w \in C$.

For any modular ρ on X and any linear space V , we define a set \mathbb{M}

$$\mathbb{M} := \{g : V \rightarrow X_\rho \mid g(0) = 0\}$$

and the generalized function $\tilde{\rho}$ on \mathbb{M} by for each $g \in \mathbb{M}$,

$$\tilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\psi(x, 0), \quad \forall x \in V\},$$

where $\psi : V^2 \rightarrow [0, \infty)$ is a mapping. The proof of the following lemma is similar to the proof of Lemma 10 in [17].

Lemma 1.4. *Let V be a linear space, X_ρ a ρ -complete modular space where ρ is convex lower semi-continuous and $f : V \rightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\psi : V^2 \rightarrow [0, \infty)$ be a mapping such that*

$$(1.2) \quad \psi(ax, ax) \leq a^2 L\psi(x, x)$$

for all $x, y \in V$ and some a and L with $a \geq 2$ and $0 \leq L < 1$. Then we have the following :

- (1) \mathbb{M} is a linear space,
- (2) $\tilde{\rho}$ is a convex modular, and
- (3) $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$ and $\mathbb{M}_{\tilde{\rho}}$ is $\tilde{\rho}$ -complete, and
- (4) $\tilde{\rho}$ is lower semi-continuous.

2. SOLUTIONS OF (1.1)

In this section, we consider solutions of (1.1).

For any $f : V \rightarrow X_\rho$, let

$$A_f(x, y) = k[f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y)]$$

and

$$B_f(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y).$$

Lemma 2.1. *Let ρ be a convex modular on X and $f : V \rightarrow X_\rho$ an even mapping with $f(0) = 0$. Suppose that $ka^2 \geq 1$ and $b^2 > a^2$. Then f is a quadratic mapping if and only if f is a solution of (1.1).*

Proof. Since $k \neq 0$ and f is even, we have

$$(2.1) \quad f(ax) = a^2 f(x), \quad f(bx) = b^2 f(x)$$

for all $x \in V$. Putting $y = ay$ in (1.1), by (2.1), we have

$$(2.2) \quad \begin{aligned} & \rho(f(x + ay) + f(x - ay) - 2f(x) - 2a^2 f(y)) \\ & \geq \rho(ka^2[f(x + by) + f(x - by) - 2f(x) - 2b^2 f(y)]) \end{aligned}$$

for all $x, y \in V$ and letting $y = \frac{y}{a}$ in (2.2), by (2.1), we have

$$(2.3) \quad \rho(B_f(x, y)) \geq \rho(ka^2[f(x + py) + f(x - py) - 2f(x) - 2p^2 f(y)])$$

for all $x, y \in V$, where $p = \frac{b}{a}$. Since ρ is convex and $ka^2 \geq 1$, by (2.3),

$$(2.4) \quad \rho(B_f(x, y)) \geq ka^2 \rho(f(x + py) + f(x - py) - 2f(x) - 2p^2 f(y))$$

for all $x, y \in V$. Letting $x = py$ in (2.3), by (2.1), we have

$$(2.5) \quad \begin{aligned} & \rho(f(px + y) + f(px - y) - 2p^2 f(x) - 2f(y)) \\ & \geq kb^2 \rho(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \end{aligned}$$

for all $x, y \in V$, because ρ is convex and $b^2 > a^2$. Interchanging x and y in (2.5), we have

$$(2.6) \quad \rho(f(x + py) + f(x - py) - 2f(x) - 2p^2 f(y)) \geq kb^2 \rho(B_f(x, y))$$

for all $x, y \in V$. By (M4), (2.4), and (2.6), we have

$$(2.7) \quad \begin{aligned} & \rho(f(x + py) + f(x - py) - 2f(x) - 2p^2 f(y)) \\ & \geq k^2 a^2 b^2 \rho(f(x + py) + f(x - py) - 2f(x) - 2p^2 f(y)) \end{aligned}$$

for all $x, y \in V$. Since $k^2 a^2 b^2 > 1$, by (2.7) and (M1), we get

$$f(x + py) + f(x - py) - 2f(x) - 2p^2 f(y) = 0$$

for all $x, y \in V$ and hence f is a quadratic mapping. The converse is trivial. \square

Theorem 2.2. *Let ρ be a convex modular on X and $f : V \rightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $ka^2 \geq 2$ and $b^2 > a^2$. Then f is a quadratic mapping if and only if f is a solution of (1.1).*

Proof. By (1.1), we have

$$(2.8) \quad \begin{aligned} \rho(A_{f_o}(x, y)) & \leq \frac{1}{2} \rho(A_f(x, y)) + \frac{1}{2} \rho(A_f(-x, -y)) \\ & \leq \frac{1}{2} \rho(B_f(x, y)) + \frac{1}{2} \rho(B_f(-x, -y)) \\ & \leq \frac{1}{2} \rho(2B_{f_o}(x, y)) + \frac{1}{2} \rho(2B_{f_e}(x, y)) \end{aligned}$$

for all $x, y \in V$ and similarly, we have

$$(2.9) \quad \rho(A_{f_e}(x, y)) \leq \frac{1}{2} \rho(2B_{f_o}(x, y)) + \frac{1}{2} \rho(2B_{f_e}(x, y))$$

for all $x, y \in V$. Letting $x = 0$ in (2.8), by (M4), we have

$$(2.10) \quad \frac{1}{2} \rho(4f_o(y)) \geq \rho(2kb^2 f_o(y)) \geq \frac{kb^2}{2} \rho(4f_o(y))$$

for all $y \in V$, because ρ is convex and $kb^2 > 2$. Since $kb^2 > 1$, by (2.10) and (M1), we have $f_o(y) = 0$ for all $y \in V$ and hence by (2.9), we have

$$(2.11) \quad \rho(A_{f_e}(x, y)) \leq \rho(2B_{f_e}(x, y))$$

for all $x, y \in V$. Since $ka^2 \geq 2$ and $b^2 > a^2$, by Lemma 2.1 and (2.11), $2f_e$ is a quadratic mapping and since $f_o(x) = 0$ for all $x \in X$, f is a quadratic mapping. \square

For $k = 1$ in Theorem 2.2, we have the following corollary:

Corollary 2.3. *Let ρ be a convex modular on X and $f : V \rightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $b^2 > a^2 \geq 2$. The f is quadratic if and only if*

$$\rho(B_f(x, y)) \geq \rho(f(ax + by) + f(ax - by) - 2a^2f(x) - 2b^2f(y))$$

for all $x, y \in V$.

Corollary 2.4. *Let ρ be a convex modular on X and $f : V \rightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $ka^2 \geq 2$ and $b^2 > a^2$. Then the following are equivalent*

- (1) f is quadratic,
- (2) f satisfies (1.1), and
- (3) f satisfies the following

$$\rho(rB_f(x, y)) \geq \rho(rA_f(x, y))$$

for all $x, y \in V$ and all real number r .

3. THE GENERALIZED HYERS-ULAM STABILITY FOR (1.1) IN MODULAR SPACES

Throughout this section, we assume that every modular is convex and lower semi-continuous. In this section, we will prove the generalized Hyers-Ulam stability for (1.1).

Lemma 3.1. *Let ρ be a convex modular on X and t a real number with $2 \leq t$. Then*

$$\rho\left(\frac{1}{t}x + \frac{1}{t}y\right) \leq \frac{1}{t}\rho(x) + \frac{1}{t}\rho(y)$$

for all $x, y \in X$.

Proof. Since ρ is a convex modular on X , we have

$$\rho\left(\frac{1}{t}x + \frac{1}{t}y\right) \leq \frac{1}{t}\rho(x) + \left(1 - \frac{1}{t}\right)\rho\left(\frac{1}{t-1}y\right) \leq \frac{1}{t}\rho(x) + \frac{1}{t}\rho(y)$$

for all $x, y \in X$, because $2 \leq t$. \square

Theorem 3.2. *Let ρ be a modular on X , V a linear space, X_ρ a ρ -complete modular space and $f : V \rightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $a \geq 2$, $k \geq a^2$, and $b^2 > a^2$. Let $\phi : V^2 \rightarrow [0, \infty)$ be a mapping such that*

$$(3.1) \quad \phi(ax, ay) \leq a^2L\phi(x, y)$$

for all $x, y \in V$ and some L with $0 < L < 1$ and

$$(3.2) \quad \rho(rA_f(x, y)) \leq \rho(rB_f(x, y)) + |r|\phi(x, y)$$

for all $x, y \in V$ and all real number r . Then there exists a unique quadratic mapping $Q : V \rightarrow X_\rho$ such that

$$(3.3) \quad \rho\left(Q(x) - \frac{1}{a^2}f(x)\right) \leq \frac{1}{ka^4(1-L)}\phi(x, 0)$$

for all $x \in V$.

Proof. By Lemma 1.4, $\tilde{\rho}$ is a lower semi-continuous convex modular on $\mathbb{M}_{\tilde{\rho}}$, $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$, and $\mathbb{M}_{\tilde{\rho}}$ is $\tilde{\rho}$ -complete. Define $T : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$ by $Tg(x) = \frac{1}{a^2}g(ax)$ for all $g \in \mathbb{M}_{\tilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\tilde{\rho}}$. Suppose that $\tilde{\rho}(g - h) \leq c$ for some nonnegative real number c . Then by (3.1), we have

$$\rho(Tg(x) - Th(x)) \leq \frac{1}{a^2}\rho(g(ax) - h(ax)) \leq Lc\phi(x, 0)$$

for all $x \in V$ and so $\tilde{\rho}(Tg - Th) \leq L\tilde{\rho}(g - h)$. Hence T is a $\tilde{\rho}$ -contraction. Since $2k > 1$, by (3.2), for $r = 1$, we get

$$(3.4) \quad \rho\left(f(ax) - a^2f(x)\right) \leq \frac{1}{2k}\rho(2kf(ax) - 2ka^2f(x)) \leq \frac{1}{2k}\phi(x, 0)$$

for all $x \in X$. Since $a \geq 2$, by (3.4),

$$(3.5) \quad \rho(Tf(x) - f(x)) = \rho\left(\frac{1}{a^2}f(ax) - f(x)\right) \leq \frac{1}{a^2}\rho(f(ax) - a^2f(x)) \leq \frac{1}{2ka^2}\phi(x, 0)$$

for all $x \in X$.

Now, we claim that T has a bounded orbit at $\frac{1}{a^2}f$. By Lemma 3.1 and (3.5), for any nonnegative integer n , we obtain

$$\begin{aligned} \rho\left(\frac{1}{a}T^n f(x) - \frac{1}{a}f(x)\right) &\leq \frac{1}{a}\rho\left(T^n f(x) - \frac{1}{a^2}f(ax)\right) + \frac{1}{a}\rho\left(\frac{1}{a^2}f(ax) - f(x)\right) \\ &\leq \frac{1}{a^2}\rho\left(\frac{1}{a}T^{n-1}f(ax) - \frac{1}{a}f(ax)\right) + \frac{1}{2ka^3}\phi(x, 0) \end{aligned}$$

for all $x \in V$ and by (3.1), we have

$$(3.6) \quad \rho\left(\frac{1}{a}T^n f(x) - \frac{1}{a}f(x)\right) \leq \frac{1}{2ka^3} \sum_{i=0}^{n-1} L^i \phi(x, 0) \leq \frac{1}{2ka^3(1-L)}\phi(x, 0)$$

for all $x \in V$ and all $n \in \mathbb{N}$. By Lemma 3.1 and (3.6), we get

$$(3.7) \quad \rho\left(\frac{1}{a^2}T^n f(x) - \frac{1}{a^2}T^m f(x)\right) = \rho\left(\frac{1}{a^2}T^n f(x) - \frac{1}{a^2}T^m f(x)\right) \leq \frac{1}{ka^4(1-L)}\phi(x, 0)$$

for all $x \in V$ and all nonnegative integers n, m . Hence T has a bounded orbit at $\frac{1}{a^2}f$.

By Lemma 1.3, there is a $Q \in \mathbb{M}_{\tilde{\rho}}$ such that $\{T^n \frac{1}{a^2}f\}$ is $\tilde{\rho}$ -convergent to Q . Since $\tilde{\rho}$ is lower semi-continuous, we get

$$0 \leq \tilde{\rho}(TQ - Q) \leq \liminf_{n \rightarrow \infty} \tilde{\rho}\left(TQ - T^{n+1}\frac{1}{a^2}f\right) \leq \liminf_{n \rightarrow \infty} L\tilde{\rho}\left(Q - T^n\frac{1}{a^2}f\right) = 0$$

and hence Q is a fixed point of T in $\mathbb{M}_{\tilde{\rho}}$. Since $a \geq 2$, there is a natural number t with $k < a^{t-6}$ and $2kb^2 < a^{t-3}$ and hence we have

$$\begin{aligned} &\rho\left(\frac{1}{a^t}\left[A_Q(x, y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\right]\right) \\ &\leq \frac{k}{a^t}\rho\left(Q(ax + by) - \frac{1}{a^{2n+2}}f(a^{n+1}x + a^n by)\right) + \frac{2k}{a^{t-2}}\rho\left(Q(x) - \frac{1}{a^{2n+2}}f(a^n x)\right) \\ &\quad + \frac{k}{a^t}\rho\left(Q(ax - by) - \frac{1}{a^{2n+2}}f(a^{n+1}x - a^n by)\right) + \frac{2kb^2}{a^t}\rho\left(Q(y) - \frac{1}{a^{2n+2}}f(a^n y)\right) \end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ in the above inequality, we get

$$(3.8) \quad \lim_{n \rightarrow \infty} \rho\left(\frac{1}{a^t}\left[A_Q(x, y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\right]\right) = 0$$

for all $x, y \in V$, because $\{\frac{1}{a^{2n+2}}f\}$ is $\tilde{\rho}$ -convergent to Q . Similarly, we have

$$(3.9) \quad \lim_{n \rightarrow \infty} \rho\left(\frac{1}{a^t} \left[B_Q(x, y) - \frac{1}{a^{2n+2}} B_f(a^n x, a^n y) \right]\right) = 0$$

for all $x, y \in V$. Since $a^2 \leq k$, by (3.2), we have

$$\begin{aligned} & \rho\left(\frac{1}{ka^{t+1}} A_Q(x, y)\right) \\ & \leq \frac{1}{a} \rho\left(\frac{1}{ka^t} \left[A_Q(x, y) - \frac{1}{a^{2n+2}} A_f(a^n x, a^n y) \right]\right) + \frac{1}{a} \rho\left(\frac{1}{a^{2n+t+4}} A_f(a^n x, a^n y)\right) \\ & \leq \frac{1}{a^3} \rho\left(\frac{1}{a^t} \left[A_Q(x, y) - \frac{1}{a^{2n+2}} A_f(a^n x, a^n y) \right]\right) + \frac{1}{a} \rho\left(\frac{1}{a^{2n+t+4}} B_f(a^n x, a^n y)\right) \\ & \quad + \frac{1}{a^{2n+t+5}} \phi(a^n x, a^n y) \\ & \leq \frac{1}{a^3} \rho\left(\frac{1}{a^t} \left[A_Q(x, y) - \frac{1}{a^{2n+2}} A_f(a^n x, a^n y) \right]\right) + \frac{1}{a^2} \rho\left(\frac{1}{a^{t+1}} B_Q(x, y)\right) \\ & \quad + \frac{1}{a^3} \rho\left(\frac{1}{a^t} \left[\frac{1}{a^{2n+2}} B_f(a^n x, a^n y) - B_Q(x, y) \right]\right) + \frac{L^n}{a^{t+5}} \phi(x, y) \end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ in the last inequality, by (3.8) and (3.9), we get

$$\rho\left(\frac{1}{ka^{t+1}} A_Q(x, y)\right) \leq \rho\left(\frac{1}{a^{t+1}} B_Q(x, y)\right)$$

for all $x, y \in V$. By Corollary 2.3, Q is a quadratic mapping. Moreover, since ρ is lower semi-continuous, by (3.7), we have (3.3). \square

Corollary 3.3. *Let X and Y be normed spaces and ϵ, θ , and p real numbers with $\epsilon \geq 0, \theta \geq 0$, and $0 < p < 1$. Suppose that $a \geq 2, k \geq a^2$, and $b^2 > a^2$. Let $f : X \rightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|A_f(x, y)\| \leq \|B_f(x, y)\| + \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p \|y\|^p)$$

for all $x, y \in X$. Then there is a quadratic mapping $Q : X \rightarrow Y$ such that

$$\|Q(x) - f(x)\| \leq \frac{1}{k(a^2 - a^{2p})} (\epsilon + \theta \|x\|^{2p})$$

for all $x \in X$.

Proof. Let $\rho(z) = \|z\|$ for all $y \in Y$ and $\phi(x, y) = \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p \|y\|^p)$ for all $x, y \in V$. Then ρ is a convex modular on a normed space $Y, Y = Y_\rho$, and $\phi(ax, ay) \leq a^{2p} \phi(x, y)$ for all $x, y \in V$. By Theorem 3.2, we have the results. \square

Using Example 1.1, we get the following example.

Example 3.4. Let θ , and p be real numbers with $\theta \geq 0$ and $0 < p < 1$. Suppose that $a \geq 2, k \geq a^2$, and $b^2 > a^2$. Let ζ be an Orlicz function and L^ζ the Orlicz space. Let $f : V \rightarrow L^\zeta$ be a mapping such that $f(0) = 0$ and

$$\int_{\Omega} \zeta(r A_f(x, y)) d\mu \leq \int_{\Omega} \zeta(r B_f(x, y)) d\mu + |r| \theta (\|x\|^{2p} + \|y\|^{2p} + \|x\|^p \|y\|^p)$$

for all $x, y \in X$ and all real number r . Then there is a quadratic mapping $Q : X \rightarrow Y$ such that

$$\int_{\Omega} \zeta\left(\left|Q(x) - \frac{1}{a^2} f(x)\right|\right) d\mu \leq \frac{\theta}{ka^2(a^2 - a^{2p})} \|x\|^{2p}$$

for all $x \in X$.

REFERENCES

- [1] I. Amemiya, On the representation of complemented modular lattices, *J. Math. Soc. Japan.* **9**(1957), 263-279.
- [2] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan* **2**(1950), 64-66.
- [3] P. Găvruta, A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184**(1994), 431-436.
- [4] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.* **27**(1941), 222-224.
- [5] M. A. Khamsi, Quasicontraction mappings in modular spaces without Δ_2 -condition, *Fixed Point Theory and Applications*, **2008**(2008), 1-6.
- [6] S. Koshi and T. Shimogaki, On F-norms of quasi-modular spaces, *J. Fac. Sci., Hokkaido Univ., Ser. 1* **15**(1961), 202-218.
- [7] W. A. Luxemburg, Banach function spaces. Ph.D. Thesis, Technische Hogeschool te Delft, Netherlands, 1955.
- [8] B. Mazur, Modular curves and the Eisenstein ideal. *Publ. Math. IHS* **47**(1977), 33-186.
- [9] J. Musielak, *Orlicz Spaces and Modular Spaces*. Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1983.
- [10] J. Musielak and W. Orlicz, On modular spaces, *Studia Mathematica*, **18**(1959), 49-65.
- [11] H. Nakano, *Modular semi-ordered spaces*, Tokyo, Japan, 1950.
- [12] W. Orlicz, *Collected Papers*, vols. I, II. PWN, Warszawa 1988.
- [13] Th. M. Rassias, On the stability of the linear mapping in Banach sapces, *Proc. Amer. Math. Sco.* **72**(1978), 297-300.
- [14] G. Sadeghi, A fixed point approach to stability of functional equations in modular spaces, *Bulletin of the Malaysian Mathematical Sciences Society*, **37**(2014), 333-344.
- [15] Ph. Turpin, Fubini inequalities and bounded multiplier property in generalized modular spaces, *Comment. Math.* **1**(1978), 331-353.
- [16] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1964.
- [17] K. Wongkum, P. Chaipunya, and P. Kumam, On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without Δ_2 -conditions, **2015**(2015), 1-6.
- [18] S. Yamamuro, On conjugate spaces of Nakano spaces, *Trans. Am. Math. Soc.* **90**(1959), 291-311.

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Complex Multivariate Taylor's formula

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Abstract

We derive here a Taylor's formula with integral remainder in the several complex variables and we estimate its remainder.

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1 Main Results

We need the following vector Taylor's formula:

Theorem 1 (Shilov, [3], pp. 93-94) *Let $n \in \mathbb{N}$ and $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Then*

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx. \quad (1)$$

The remainder here is the Riemann X -valued integral (defined similar to numerical one) given by

$$Q_{n-1} = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx, \quad (2)$$

with the property:

$$\|Q_{n-1}\| \leq \max_{a \leq x \leq b} \|f^{(n)}(x)\| \frac{(b-a)^n}{n!}. \quad (3)$$

The derivatives above are defined similar to the numerical ones. We make

Remark 2 *Here Q is an open convex subset of \mathbb{C}^k , $k \geq 2$; $z := (z_1, \dots, z_k)$, $x_0 := (x_{01}, \dots, x_{0k}) \in Q$. Let $f : Q \rightarrow \mathbb{C}$ be a coordinate-wise holomorphic*

function. Then, by the famous Hartog's fundamental theorem ([1], [2]) f is jointly holomorphic and jointly continuous on Q . Let $n \in \mathbb{N}$. Each n^{th} order complex partial derivative is denoted by $f_\alpha := \frac{\partial^n f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, \dots, k$ and $|\alpha| := \sum_{i=1}^k \alpha_i = n$.

Consider

$$g_z(t) := f(x_0 + t(z - x_0)), \quad 0 \leq t \leq 1. \quad (4)$$

Clearly it holds that $x_0 + t(z - x_0) \in Q$ and $g_z(t) \in \mathbb{C}$, $\forall t \in [0, 1]$.

Then we derive

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), \dots, x_{0k} + t(z_k - x_{0k})), \quad (5)$$

for all $j = 0, 1, \dots, n$.

Notice here that any mixed partials commute. We remind that $(\mathbb{C}, |\cdot|)$ is a Banach space. By Shilov's Theorem 1, about the Taylor's formula for Banach space valued functions, we obtain

Theorem 3 *It holds*

$$f(z_1, \dots, z_k) = g_z(1) = \sum_{j=0}^{n-1} \frac{g_z^{(j)}(0)}{j!} + R_n(z, 0), \quad (6)$$

where

$$R_n(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1-\theta)^{n-1} g_z^{(n)}(\theta) d\theta, \quad (7)$$

and notice that $g_z(0) = f(x_0)$.

We make

Remark 4 *Notice that (by (7)) we get*

$$|R_n(z, 0)| \leq \left(\max_{0 \leq \theta \leq 1} |g_z^{(n)}(\theta)| \right) \frac{1}{n!}. \quad (8)$$

We also have for $j = 0, 1, \dots, n$:

$$g_z^{(j)}(0) = \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \quad (9)$$

Furthermore it holds

$$g_z^{(n)}(\theta) = \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)), \quad (10)$$

$$0 \leq \theta \leq 1.$$

Another version of (6) is

$$f(z_1, \dots, z_k) = g_z(1) = \sum_{j=0}^n \frac{g_z^{(j)}(0)}{j!} + \overline{R}_n(z, 0), \quad (11)$$

where

$$\overline{R}_n(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1-\theta)^{n-1} \left(g_z^{(n)}(\theta) - g_z^{(n)}(0) \right) d\theta. \quad (12)$$

Identities (6) and (11) are the multivariate complex Taylor's formula with integral remainders.

We give

Example 5 Let $n = k = 2$. Then

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in [0, 1],$$

and

$$g'_z(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)). \quad (13)$$

In addition,

$$\begin{aligned} g''_z(t) &= (z_1 - x_{01}) \left(\frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) \right)' + (z_2 - x_{02}) \left(\frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)) \right)' \\ &= (z_1 - x_{01}) \left\{ (z_1 - x_{01}) \frac{\partial^2 f}{\partial x_1^2}(\ast) + (z_2 - x_{02}) \frac{\partial^2 f}{\partial x_2 \partial x_1}(\ast) \right\} + \\ &\quad (z_2 - x_{02}) \left\{ (z_1 - x_{01}) \frac{\partial^2 f}{\partial x_1 \partial x_2}(\ast) + (z_2 - x_{02}) \frac{\partial^2 f}{\partial x_2^2}(\ast) \right\}. \end{aligned} \quad (14)$$

Hence,

$$\begin{aligned} g''_z(t) &= (z_1 - x_{01})^2 \frac{\partial^2 f}{\partial x_1^2}(\ast) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial^2 f}{\partial x_2 \partial x_1}(\ast) + \\ &\quad (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial^2 f}{\partial x_1 \partial x_2}(\ast) + (z_2 - x_{02})^2 \frac{\partial^2 f}{\partial x_2^2}(\ast), \end{aligned} \quad (15)$$

where $\ast := x_0 + t(z - x_0)$.

Notice that $g_z(t), g'_z(t), g''_z(t) \in \mathbb{C}$.

We make

Remark 6 We define

$$\|f\|_{p, \overline{zx_0}} := \left(\int_0^1 |f(x_0 + \theta(z - x_0))|^p d\theta \right)^{\frac{1}{p}}, \quad p \geq 1, \quad (16)$$

where $\overline{zx_0}$ denotes the segment $\overline{zx_0} \subset Q$.

We also define

$$\|f\|_{\infty, \overline{zx_0}} := \max_{\theta \in [0,1]} |f(x_0 + \theta(z - x_0))|. \quad (17)$$

By (10) we obtain

$$|g_z^{(n)}(\theta)| \leq \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i} \right) |f_\alpha(x_0 + \theta(z - x_0))|, \quad (18)$$

$\forall \theta \in [0, 1]$.

Therefore, by norm properties for $1 \leq p \leq \infty$, it holds

$$\begin{aligned} \|g_z^{(n)}\|_{p, \overline{zx_0}} &\leq \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i} \right) \|f_\alpha\|_{p, \overline{zx_0}} \quad (19) \\ &\leq \left(\sum_{i=1}^k |z_i - x_{0i}| \right)^n \|f_\alpha^*\|_{p, \overline{zx_0}}, \end{aligned}$$

where

$$\|f_\alpha^*\|_{p, \overline{zx_0}} := \max_{|\alpha|=n} \|f_\alpha\|_{p, \overline{zx_0}}, \quad (20)$$

for all $1 \leq p \leq \infty$.

That is

$$\|g_z^{(n)}\|_{p, \overline{zx_0}} \leq (\|z - x_0\|_{l_1})^n \|f_\alpha^*\|_{p, \overline{zx_0}}, \quad (21)$$

for all $1 \leq p \leq \infty$.

Therefore by (8) we obtain

$$|R_n(z, 0)| \leq \frac{(\|z - x_0\|_{l_1})^n \|f_\alpha^*\|_{\infty, \overline{zx_0}}}{n!}. \quad (22)$$

Next, we put things together and we further estimate $R_n(z, 0)$.

Theorem 7 Here $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. It holds

$$|R_n(z, 0)| \leq \min \left\{ \begin{array}{l} \frac{\|g_z^{(n)}\|_{\infty, \overline{z x_0}}}{n!}, \\ \frac{\|g_z^{(n)}\|_{1, \overline{z x_0}}}{(n-1)!}, \\ \frac{\|g_z^{(n)}\|_{p, \overline{z x_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}} \end{array} \right\} \leq \quad (23)$$

$$(\|z - x_0\|_{l_1})^n \min \left\{ \begin{array}{l} \frac{\|f_\alpha^*\|_{\infty, \overline{z x_0}}}{n!}, \\ \frac{\|f_\alpha^*\|_{1, \overline{z x_0}}}{(n-1)!}, \\ \frac{\|f_\alpha^*\|_{p, \overline{z x_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}}. \end{array} \right. \quad (24)$$

Proof. Based on (7), Hölder's inequality and (21). ■

References

- [1] C. Caratheodory, *Theory of Functions of a complex variable*, Volume Two, Chelsea publishing Company, New York, 1954.
- [2] S.G. Krantz, *Function theory of several complex variables*, second edition, AMS Chelsea publishing Providence, Rhode Island, 2001.
- [3] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications Inc., New York, 1996.

On the Barnes-type multiple twisted q -Euler zeta function of the second kind

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Abstract :

In this paper we introduce the Barnes-type multiple twisted q -Euler numbers and polynomials of the second kind, by using fermionic p -adic invariant integral on \mathbb{Z}_p . Using these numbers and polynomials, we construct the Barnes-type multiple twisted q -Euler zeta function of the second kind. Finally, we obtain the relations between these numbers and polynomials and Barnes-type multiple twisted q -Euler zeta function.

Key words : p -adic invariant integral on \mathbb{Z}_p , Euler numbers and polynomials of the second kind, q -Euler numbers and polynomials of the second kind, Barnes-type multiple twisted q -Euler numbers and polynomials of the second kind, Barnes-type multiple twisted q -Euler zeta function.

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1. Introduction

Recently, Bernoulli numbers, Bernoulli polynomials, q -Bernoulli numbers, q -Bernoulli polynomials, the second kind Bernoulli number, the second kind Bernoulli polynomials, Euler numbers of the second kind, Euler polynomials of the second kind, Genocchi numbers, Genocchi polynomials, tangent numbers, tangent polynomials, and Bell polynomials were studied by many authors(see [1, 2, 3, 4, 9]). Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [5], by using Euler numbers E_j and polynomials $E_j(x)$ of the second kind, we investigated the alternating sums of powers of consecutive odd integers. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind Euler polynomials $E_n(x)$ (see [6]). The outline of this paper is as follows. We introduce the Barnes-type multiple twisted q -Euler numbers and polynomials of the second kind, by using fermionic p -adic invariant integral on \mathbb{Z}_p . In Section 2, we construct the Barnes-type multiple twisted q -Euler zeta function of the second kind. Finally, we obtain the relations between these numbers and polynomials and Barnes-type multiple twisted q -Euler zeta function.

Throughout this paper we use the following notations. By \mathbb{Z}_p we denote the ring of p -adic rational integers, \mathbb{Q}_p denotes the field of rational numbers, \mathbb{N} denotes the set of natural numbers, \mathbb{C} denotes the complex number field, and \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p . Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p -adic invariant integral on \mathbb{Z}_p of the function $g \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [1, 2, 4].} \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2g(0). \tag{1.2}$$

First, we introduced the second kind Euler numbers E_n . The second kind Euler numbers E_n are defined by the generating function:

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$\frac{2e^t}{e^{2t} + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

In [5, 6], we studied the second kind Euler numbers E_n and polynomials $E_n(x)$ and investigate their properties.

2. Barnes-type multiple twisted q -Euler numbers and polynomials of the second kind

In this section, we assume that $w_1, \dots, w_k \in \mathbb{Z}_p$ and $a_1, \dots, a_k \in \mathbb{Z}$. Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \rightarrow \infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N . For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ the locally constant function $x \mapsto \omega^x$.

We construct the Barnes-type multiple twisted q -Euler polynomials of the second kind,

$$E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x).$$

For $k \in \mathbb{N}$, we define Barnes-type multiple twisted q -Euler polynomials of the second kind as follows:

$$\begin{aligned} & \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \omega^{x_1 + \dots + x_k} q^{a_1 x_1 + \dots + a_k x_k} e^{(x + 2w_1 x_1 + \dots + 2w_k x_k + k)t} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k) \\ &= \frac{2^k e^{kt}}{(\omega q^{a_1} e^{2w_1 t} + 1)(\omega q^{a_2} e^{2w_2 t} + 1) \dots (\omega q^{a_k} e^{2w_k t} + 1)} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

In the special case, $x = 0$, $E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | 0) = E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k)$ are called the n -th Barnes-type multiple twisted q -Euler numbers of the second kind. By (2.1) and Taylor expansion of $e^{(x + 2w_1 x_1 + \dots + 2w_k x_k + k)t}$, we have the following theorem.

Theorem 1. For positive integers n and k , we have

$$\begin{aligned} & E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \\ &= \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \omega^{x_1 + \dots + x_k} q^{a_1 x_1 + \dots + a_k x_k} (x + 2w_1 x_1 + \dots + 2w_k x_k + k)^n d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k). \end{aligned}$$

In the case when $x = 0$ in (2.1), we have the following corollary.

Corollary 2. For positive integers n , we have

$$\begin{aligned} & E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k) \\ &= \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \omega^{\sum_{i=1}^k x_i} q^{\sum_{i=1}^k a_i x_i} (2w_1 x_1 + \dots + 2w_k x_k + k)^n d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k). \end{aligned} \tag{2.2}$$

By Theorem 1 and (2.2), we obtain

$$E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) = \sum_{l=0}^n \binom{n}{l} E_{l,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k) x^{n-l}, \quad (2.3)$$

where $\binom{n}{k}$ is a binomial coefficient.

We define distribution relation of Barnes-type multiple twisted q -Euler polynomials of the second kind as follows: For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \frac{t^n}{n!} \\ &= \frac{2^k e^{kmt}}{(\omega^m q^{a_1 m} e^{2w_1 mt} + 1)(\omega^m q^{a_2 m} e^{2w_2 mt} + 1) \dots (\omega^m q^{a_k m} e^{2w_k mt} + 1)} \\ & \quad \times \sum_{l_1, \dots, l_k=0}^{m-1} (-1)^{l_1 + \dots + l_k} \omega^{\sum_{i=1}^k l_i} q^{\sum_{i=1}^k a_i l_i} e^{\left(\frac{x + 2w_1 l_1 + \dots + 2w_k l_k + k - mk}{m}\right) (mt)}. \end{aligned}$$

From the above equation, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} m^n \sum_{l_1, \dots, l_k=0}^{m-1} (-1)^{l_1 + \dots + l_k} \omega^{\sum_{i=1}^k l_i} q^{\sum_{i=1}^k a_i l_i} \\ & \quad \times E_{n,\omega^m,q^m} \left(w_1, \dots, w_k; a_1, \dots, a_k \mid \frac{x + 2w_1 l_1 + \dots + 2w_k l_k + k - mk}{m} \right) \frac{t^n}{n!}. \end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 3 (Distribution relation). For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we have

$$\begin{aligned} & E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \\ &= m^n \sum_{l_1, \dots, l_k=0}^{m-1} (-1)^{l_1 + \dots + l_k} \omega^{\sum_{i=1}^k l_i} q^{\sum_{i=1}^k a_i l_i} \\ & \quad \times E_{n,\omega^m,q^m} \left(w_1, \dots, w_k; a_1, \dots, a_k \mid \frac{x + 2w_1 l_1 + \dots + 2w_k l_k + k - mk}{m} \right). \end{aligned}$$

From (2.1), we derive

$$\begin{aligned} & \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \omega^{x_1 + \dots + x_k} q^{a_1 x_1 + \dots + a_k x_k} e^{(x + 2w_1 x_1 + \dots + 2w_k x_k + k)t} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k) \\ &= 2^k \sum_{m_1, \dots, m_k=0}^{\infty} (-1)^{m_1 + \dots + m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i} e^{(x + 2w_1 m_1 + \dots + 2w_k m_k + k)t}. \end{aligned} \quad (2.4)$$

From (2.2) and (2.4), we note that

$$\begin{aligned} & E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \\ &= 2^k \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1 + \dots + m_r} q^{\sum_{i=1}^k a_i m_i} (x + 2w_1 m_1 + \dots + 2w_k m_k + k)^n. \end{aligned} \quad (2.5)$$

By using binomial expansion and (2.1), we have the following addition theorem.

Theorem 4(Addition theorem). Barnes-type multiple twisted q -Euler polynomials of the second kind $E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x)$ satisfies the following relation:

$$E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x + y) = \sum_{l=0}^n \binom{n}{l} E_{l,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) y^{n-l}.$$

3. Barnes-type multiple twisted q -Euler zeta function of the second kind

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$ and the parameters w_1, \dots, w_k are positive. Let ω be the p^N -th root of unity. By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the generating function of Barnes-type multiple twisted q -Euler polynomials of the second kind, $E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x)$, we define Barnes-type multiple twisted q -Euler zeta function of the second kind. This function interpolates the Barnes-type multiple twisted q -Euler polynomials of the second kind at negative integers.

By (2.1), we obtain

$$\begin{aligned} F_{\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x, t) &= \frac{2^k e^{kt}}{(\omega q^{a_1} e^{2w_1 t} + 1) \dots (\omega q^{a_k} e^{2w_k t} + 1)} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

Hence, by (3.1), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \frac{t^n}{n!} \\ &= 2^k \sum_{m_1, \dots, m_k=0}^{\infty} (-1)^{m_1+\dots+m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i} e^{(x+2w_1 m_1+\dots+2w_k m_k+k)t}. \end{aligned}$$

By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the above equation, we have

$$\begin{aligned} &E_{n,\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | x) \\ &= 2^k \sum_{m_1, \dots, m_k=0}^{\infty} (-1)^{m_1+\dots+m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i} (x + 2w_1 m_1 + \dots + 2w_k m_k + k)^n. \end{aligned} \tag{3.2}$$

By (3.2), we define the Barnes-type multiple twisted q -Euler zeta function of the second kind $\zeta_{\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | s, x)$ as follows:

Definition 5. For $s, x \in \mathbb{C}$ with $Re(x) > 0$, $a_1, \dots, a_k \in \mathbb{C}$, we define

$$\zeta_{\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | s, x) = 2^k \sum_{m_1, \dots, m_k=0}^{\infty} \frac{(-1)^{m_1+\dots+m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i}}{(x + 2w_1 m_1 + \dots + 2w_k m_k + k)^s}, \tag{3.3}$$

For $s = -l$ in (3.3) and using (3.2), we arrive at the following theorem.

Theorem 6. For positive integer l , we have

$$\zeta_{\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | -l, x) = E_l(w_1, \dots, w_k; a_1, \dots, a_k | x).$$

By (2.6), we define multiple twisted q -Euler zeta function of the second kind $\zeta_{\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | s)$ as follows:

Definition 7. For $s \in \mathbb{C}$, we define

$$\zeta_{\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | s) = 2^k \sum_{m_1, \dots, m_k=0}^{\infty} \frac{(-1)^{m_1+\dots+m_k} \omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i}}{(2w_1 m_1 + \dots + 2w_k m_k + k)^s}, \quad (3.4)$$

For $s = -l$ in (3.4) and using (2.5), we arrive at the following theorem.

Theorem 8. For positive integer l , we have

$$\zeta_{\omega,q}(w_1, \dots, w_k; a_1, \dots, a_k | -l) = E_l(w_1, \dots, w_k; a_1, \dots, a_k).$$

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REFERENCES

1. Kim, T.(2008). *Euler numbers and polynomials associated with zeta function*, *Abstract and Applied Analysis*, Art. ID 581582.
2. Kim, T.(2008) *Symmetry p-adic invariant integral on \mathbb{Z}_p for Bernoulli and Euler polynomials*, *Journal of Difference Equations and Applications*, v.12, pp. 1267-1277.
3. Liu, G.(2006). *Congruences for higher-order Euler numbers*, *Proc. Japan Acad.*, v.82 A, pp. 30-33.
4. Ryoo, C.S., Kim, T., Jang, L. C.(2007). *Some relationships between the analogs of Euler numbers and polynomials*, *Journal of Inequalities and Applications*, v.2007, ID 86052, pp. 1-22.
5. Ryoo, C.S.(2011). *On the alternating sums of powers of consecutive odd integers*, *Journal of Computational Analysis and Applications*, v.13, pp. 1019-1024.
6. Ryoo, C.S.(2010). *Calculating zeros of the second kind Euler polynomials*, *Journal of Computational Analysis and Applications*, v.12, pp. 828-833.
7. Ryoo, C.S.(2014). *Note on the second kind Barnes' type multiple q-Euler polynomials*, *Journal of Computational Analysis and Applications*, v.16, pp. 246-250.
8. Ryoo, C.S.(2015). *On the second kind Barnes-type multiple twisted zeta function and twisted Euler polynomials*, *Journal of Computational Analysis and Applications*, v.18, pp. 423-429.
9. Ryoo, C.S.(2020). *Symmetric identities for the second kind q-Bernoulli polynomials*, *Journal of Computational Analysis and Applications*, v.28, pp. 654-659.
10. Ryoo, C.S.(2020). *On the second kind twisted q-Euler numbers and polynomials of higher order*, *Journal of Computational Analysis and Applications*, v.28, pp. 679-684.
11. Ryoo, C.S.(2020). *Symmetric identities for Dirichlet-type multiple twisted (h, q) -l-function and higher-order generalized twisted (h, q) -Euler polynomials*, *Journal of Computational Analysis and Applications*, v.28, pp. 537-542.

Some Approximation Results of Kantorovich type operators

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In this manuscript, we investigate a variant of the operators define by Lupaş. We compute the rate of convergence for different class of functions. In section 3, the weighted approximation results are established. At the end, stated the problems for further research.

Keyword: Positive linear operators; Rate Convergence; Weighted approximation

2000 Mathematics Subject Classification: primary 41A25, 41A30, 41A36.

1 Introduction

In [1], Lupaş proposed to study the following sequence of linear and positive operators

$$P_n^{[0]}(f, x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0, \quad f : [0, \infty) \rightarrow \mathbb{R}, \quad (1.1)$$

where $(nx)_0 = 1$ and $(nx)_k = nx(nx+1)(nx+2)\dots(nx+k-1)$, $k \geq 1$.

We can consider that $P_n^{[0]}$, $n \geq 1$, are defined on E where $E = \bigcup_{a>0} E_a$ and

E_a is the subspace of all real valued continuous functions f on $[0, \infty)$ such as $\sup_{x \geq 0} (\exp(-ax)|f(x)|) < \infty$. The space E_a is endowed with the norm $\|f\|_a = \sup_{x \geq 0} (\exp(-ax)|f(x)|)$ with respect to which it becomes a Banach space.

In recent year, Patel and Mishra [2] generalized Jain operators type variant of the Lupaş operators defined as

$$P_n^{[\beta]}(f, x) = \sum_{k=0}^{\infty} \frac{(nx+k\beta)_k}{2^k k!} 2^{-(nx+k\beta)} f\left(\frac{k}{n}\right), \quad x \geq 0, \quad f : [0, \infty) \rightarrow \mathbb{R}, \quad (1.2)$$

where $(nx+k\beta)_0 = 1$, $(nx+k\beta)_1 = nx$ and $(nx+k\beta)_k = nx(nx+k\beta+1)(nx+k\beta+2)\dots(nx+k\beta+k-1)$, $k \geq 2$.

We mention that $\beta = 0$, the operators $P_n^{[0]}$ reduce to Lupaş operators (1.1). In

[2], the authors have used following Lagrange's formula to define the operators (1.2):

$$\phi(z) = \phi(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[\frac{d^{k-1}}{dz^{k-1}} ((f(z)^k) \phi'(z)) \right]_{z=0} \left(\frac{z}{f(z)} \right)^k. \quad (1.3)$$

But, if we use following Lagrange's formula then the generalization of the operators (1.1) is written better way:

$$\phi(z) \left[1 - \frac{z}{f(z)} \frac{df(z)}{dz} \right]^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k}{dz^k} ((f(z)^k) \phi(z)) \right]_{z=0} \left(\frac{z}{f(z)} \right)^k.$$

By choosing $\phi(z) = (1 - z)^{-\alpha}$ and $f(z) = (1 - z)^\beta$, we have

$$\begin{aligned} & (1 - z)^{-\alpha} [1 - z\beta(1 - z)^{-1}]^{-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + k\beta)(\alpha + k\beta + 1) \dots (\alpha + k\beta + k - 1) \left(\frac{z}{(1 - z)^{-\beta}} \right)^k. \end{aligned}$$

Taking $z = \frac{1}{2}$, we get

$$1 = (1 - \beta) \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\alpha + \beta k)_k 2^{-(\alpha + \beta k)}.$$

Now, we may define the operators as

$$P_n^{[\beta]}(f, x) = \sum_{k=0}^{\infty} \mathbf{p}_\beta(k, nx) f\left(\frac{k}{n}\right) \quad (1.4)$$

where $\mathbf{p}_\beta(k, nx) = (1 - \beta) \sum_{k=0}^{\infty} \frac{1}{2^k k!} (nx + \beta k)_k 2^{-(nx + \beta k)}$. where $(nx + \beta k)_0 = 1$ and $(nx + \beta k)_k = (nx + \beta k)(nx + \beta k + 2) \dots (nx + \beta k + k - 1)$, $k \geq 1$ and $0 \leq \frac{\beta + 1}{2} < 1$

The parameter β may depend on the natural number n . It is easy to see that for $\beta = 0$, the operators $P_n^{[\beta]}(f, x)$ reduces to Lupaş operator (1.1). We mention that, the operators (1.2) and (1.4) has no much difference as their moments are same. To calculate the moments of (1.4), we follows techniques developed in [2]

and along the lines of [2], we have

$$\begin{aligned}
 P_n^{[\beta]}(1, x) &= 1 \\
 P_n^{[\beta]}(t, x) &= \frac{x}{1-\beta} + \frac{2\beta}{n(1-\beta)^2} \\
 P_n^{[\beta]}(t^2, x) &= \frac{x^2}{(1-\beta)^2} + \frac{2x(1+2\beta)}{n(1-\beta)^3} + \frac{6(\beta+\beta^2)}{n^2(1-\beta)^4} \\
 P_n^{[\beta]}(t^3, x) &= \frac{x^3}{(1-\beta)^3} + \frac{6x^2(1+\beta)}{n(1-\beta)^4} + \frac{6x(1+6\beta+3\beta^2)}{n^2(1-\beta)^5} \\
 &\quad + \frac{2(13\beta+34\beta^2+13\beta^3)}{n^3(1-\beta)^6} \\
 P_n^{[\beta]}(t^4, x) &= \frac{x^4}{(1-\beta)^4} + \frac{4x^3(3+2\beta)}{n(1-\beta)^5} + \frac{36x^2(1+3\beta+\beta^2)}{n^2(1-\beta)^6} \\
 &\quad + \frac{2x(13+146\beta+209\beta^2+52\beta^3)}{n^3(1-\beta)^7} + \frac{30(5\beta+23\beta^2+23\beta^3+5\beta^4)}{n^4(1-\beta)^8}.
 \end{aligned}$$

In the present paper, we modify the operators defined by (1.4) into integral form in Kantorovich sense, see also G.G. Lorentz [3, Ch.II, p.30]. Actually, we replace $f\left(\frac{k}{n}\right)$ by an integral mean of $f(x)$ over a small interval around the point $\frac{k}{n}$ as follows

$$K_n^{[\beta]}(f, x) = n \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} f(t) dt, \tag{1.5}$$

where $\mathbf{p}_{\beta}(k, nx)$ was as defined in (1.4) and f belongs to the class of local integrable functions defined on $[0, \infty)$.

The focus of the paper is to investigate these linear and positive operators. Section 2, provided results in connection with the rate of convergence for $K_n^{[\beta]}$ under different assumptions of the function f .

2 Approximation properties

For any integer $s \geq 0$, we denote by e_s the test function, $e_s(x) = x^s, x \geq 0$, and we also introduce the s -th order central moment of the operator $K_n^{[\beta]}$, that is

$$\Omega_{n,s}(x) = K_n^{[\beta]}(\psi_{x,s}, x), \text{ where } \psi_{x,s}(t) = (t-x)^s, x \geq 0, t \geq 0.$$

Lemma 1 *The operators $K_n^{[\beta]}$, $n \in \mathbf{N}$ defined by (1.5), verify*

1. $K_n^{[\beta]}(1, x) = 1$;
2. $K_n^{[\beta]}(t, x) = \frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}$;

$$\begin{aligned}
 3. K_n^{[\beta]}(t^2, x) &= \frac{x^2}{(1-\beta)^2} + \frac{x(3+2\beta+\beta^2)}{n(1-\beta)^3} + \frac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}; \\
 4. K_n^{[\beta]}(t^3, x) &= \frac{x^3}{(1-\beta)^3} + \frac{3x^2(5+2\beta+\beta^2)}{2n(1-\beta)^4} + \frac{x(10+32\beta+15\beta^2+2\beta^3+\beta^4)}{n^2(1-\beta)^5} \\
 &\quad + \frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6} \\
 5. K_n^{[\beta]}(t^4, x) &= \frac{x^4}{(1-\beta)^4} + \frac{2x^3(7+2\beta+\beta^2)}{n(1-\beta)^5} \\
 &\quad + \frac{2x^2(25+44\beta+18\beta^2+2\beta^3+\beta^4)}{n^2(1-\beta)^6} \\
 &\quad + \frac{x(43+326\beta+329\beta^2+116\beta^3+23\beta^4+2\beta^5+\beta^6)}{n^3(1-\beta)^7} \\
 &\quad + \frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}.
 \end{aligned}$$

Proof: Observe that $K_n^{[\beta]}(1, x) = P_n^{[\beta]}(1, x) = 1$.
 Now,

$$\begin{aligned}
 K_n^{[\beta]}(t, x) &= n \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} t \, dt \\
 &= \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \frac{1+2k}{2n} = \frac{1}{2n} P_n^{[\beta]}(1, x) + P_n^{[\beta]}(t, x) \\
 &= \frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 K_n^{[\beta]}(t^2, x) &= n \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} t^2 \, dt \\
 &= \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \frac{1+3k+3k^2}{3n^2} \\
 &= \frac{1}{3n^2} P_n^{[\beta]}(1, x) + \frac{1}{n} P_n^{[\beta]}(t, x) + P_n^{[\beta]}(t^2, x) \\
 &= \frac{x^2}{(1-\beta)^2} + \frac{x(3+2\beta+\beta^2)}{n(1-\beta)^3} + \frac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}.
 \end{aligned}$$

$$\begin{aligned}
 K_n^{[\beta]}(t^3, x) &= n \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} t^3 dt \\
 &= \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \left(\frac{1 + 4k + 6k^2 + 4k^3}{4n^3} \right) \\
 &= \frac{1}{4n^3} P_n^{[\beta]}(1, x) + \frac{1}{n^2} P_n^{[\beta]}(t, x) + \frac{6}{4n} P_n^{[\beta]}(t^2, x) + P_n^{[\beta]}(t^3, x) \\
 &= \frac{x^3}{(1-\beta)^3} + \frac{3x^2(5+2\beta+\beta^2)}{2n(1-\beta)^4} + \frac{x(10+32\beta+15\beta^2+2\beta^3+\beta^4)}{n^2(1-\beta)^5} \\
 &\quad + \frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6}
 \end{aligned}$$

and

$$\begin{aligned}
 K_n^{[\beta]}(t^4, x) &= n \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} t^4 dt \\
 &= \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \left(\frac{1}{5n^4} + \frac{k}{n^4} + \frac{2k^2}{n^4} + \frac{2k^3}{n^4} + \frac{k^4}{n^4} \right) \\
 &= \frac{1}{5n^4} P_n^{[\beta]}(1, x) + \frac{1}{n^3} P_n^{[\beta]}(t, x) + \frac{2}{n^2} P_n^{[\beta]}(t^2, x) \\
 &\quad + \frac{2}{n} P_n^{[\beta]}(t^3, x) + P_n^{[\beta]}(t^4, x) \\
 &= \frac{x^4}{(1-\beta)^4} + \frac{2x^3(7+2\beta+\beta^2)}{n(1-\beta)^5} + \frac{2x^2(25+44\beta+18\beta^2+2\beta^3+\beta^4)}{n^2(1-\beta)^6} \\
 &\quad + \frac{x(43+326\beta+329\beta^2+116\beta^3+23\beta^4+2\beta^5+\beta^6)}{n^3(1-\beta)^7} \\
 &\quad + \frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}.
 \end{aligned}$$

Lemma 1 implies the following identities

$$\Omega_{n,0}(x) = 1, \quad \Omega_{n,1}(x) = \frac{x\beta}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}, \tag{2.1}$$

$$\Omega_{n,2}(x) = \frac{x^2\beta^2}{(1-\beta)^2} + \frac{x(2+\beta+2\beta^2+\beta^3)}{n(1-\beta)^3} + \frac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}, \tag{2.2}$$

$$\begin{aligned} \Omega_{n,3}(x) &= \frac{x^3\beta^3}{(1-\beta)^3} + \frac{3x^2\beta(4+\beta+2\beta^2+\beta^3)}{2n(1-\beta)^4} \\ &+ \frac{x(9+13\beta+23\beta^2+12\beta^3+2\beta^4+\beta^5)}{n^2(1-\beta)^5} \\ &+ \frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \Omega_{n,4}(x) &= \frac{x^4\beta^4}{(1-\beta)^4} + \frac{2x^3\beta^2(6+\beta+2\beta^2+\beta^3)}{n(1-\beta)^5} \\ &+ \frac{2x^2(6+18\beta+25\beta^2+26\beta^3+12\beta^4+2\beta^5+\beta^6)}{n^2(1-\beta)^6} \\ &+ \frac{x(42+185\beta+252\beta^2+239\beta^3+100\beta^4+19\beta^5+2\beta^6+\beta^7)}{n^3(1-\beta)^7} \\ &+ \frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}. \end{aligned} \tag{2.4}$$

Remark 1 Since $\beta \in [0, 1)$, $(1-\beta)^2 \leq 1$ and $(1-\beta)^{-2} \leq (1-\beta)^{-3} \leq (1-\beta)^{-4}$, we have

$$\Omega_{n,1}(x) \leq \frac{2xn\beta + (1+\beta)^2}{2n(1-\beta)^2} \tag{2.5}$$

and

$$\begin{aligned} \Omega_{n,2}(x) &\leq \frac{3n^2x^2\beta^2 + 3nx(2+\beta+2\beta^2+\beta^3) + 1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4} \\ &\leq \frac{3n^2x^2\beta^2 + 6nx(1+2\beta) + 1+35\beta}{3n^2(1-\beta)^4}. \end{aligned} \tag{2.6}$$

Also, using $\max\{1, x, x^2, x^3, x^4\} \leq (1+x+x^2+x^3+x^4)$, we have

$$\begin{aligned} \Omega_{n,4}(x) &\leq \left(\frac{\beta^4}{(1-\beta)^8} + \frac{20}{n(1-\beta)^8} + \frac{180}{n^2(1-\beta)^8} \right. \\ &\quad \left. + \frac{840}{n^3(1-\beta)^8} + \frac{8400}{5n^4(1-\beta)^8} \right) (1+x+x^2+x^3+x^4) \\ &\leq B_\beta(n)(1+x+x^2+x^3+x^4), \end{aligned} \tag{2.7}$$

where

$$B_\beta(n) = \frac{\beta^4}{(1-\beta)^8} + \frac{20}{n(1-\beta)^8} + \frac{180}{n^2(1-\beta)^8} + \frac{840}{n^3(1-\beta)^8} + \frac{8400}{5n^4(1-\beta)^8}. \tag{2.8}$$

Theorem 1 Let $K_n^{[\beta_n]}$ be defined by (1.5) and $\beta_n \in [0, 1)$ with $\beta_n \rightarrow 0$. Then for $f \in C[0, \infty)$ one has $\lim_{n \rightarrow \infty} K_n^{[\beta_n]}(f, \cdot) = f$ uniformly on any compact $K \subset [0, \infty)$.

Proof: By making use of Lemma 1, we have

$$\lim_{n \rightarrow \infty} K_n^{[\beta_n]}(t^j, x) = x^j, \text{ with } \beta_n \rightarrow 0$$

$j = 0, 1, 2$, uniformly on any compact $K \subset [0, \infty)$. Consequently, our assertion follows directly from the well-known theorem of Bohman-Korovkin.

Let $C_B[0, \infty)$ denote the space of real valued continuous and bounded functions f on the interval $[0, \infty)$, endowed with the norm

$$\|f\| = \sup_{0 \leq x < \infty} |f(x)|$$

For any $\delta > 0$, Peetre's K -functional is define by

$$K_2(f, \delta) = \inf_{g \in C_B^2[0, \infty)} \{\|f - g\| + \delta \|g''\|\},$$

where $C_B^2[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By DeVore and Lorentz [4, P.177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \tag{2.9}$$

where the second order modulus of smoothness of $g \in C_B[0, \infty)$ is defined as

$$\omega_2(g; \delta) = \sup_{0 < h \leq \delta} \sup_{x \geq 0} |g(x) - 2g(x+h) + g(x+2h)|, \quad \delta > 0,$$

also usual modulus of continuity of $f \in C_B[0, \infty)$ is defined by

$$\omega_1(g; \delta) = \sup_{0 < h \leq \delta} \sup_{x \geq 0} |g(x+h) - g(x)|, \quad \delta > 0.$$

Theorem 2 Let $K_n^{[\beta]}$ be defined by (1.5) and $\beta \in [0, 1)$ then for each $x \geq 0$ the following inequality

$$|K_n^{[\beta]}(f, x) - f(x)| \leq \frac{4}{3} \omega_1 \left(f; \sqrt{\frac{3n^2 x^2 \beta^2 + 6nx(1+2\beta) + 1 + 35\beta}{3n(1-\beta)^2}} \right)$$

holds.

Proof: Since $K_n^{[\beta]}(1, x) = 1$ and $\mathbf{p}_\beta(k, nx) \geq 0$, we can write

$$|K_n^{[\beta]}(f, x) - f(x)| \leq n \sum_{k=0}^{\infty} \mathbf{p}_\beta(k, nx) \int_{k/n}^{(k+1)/n} |f(t) - f(x)| dt. \tag{2.10}$$

On the other hand

$$|f(t) - f(x)| \leq \omega_1(f; |t-x|) \leq (1 + \delta^{-2}(t-x)^2) \omega_1(f; \delta).$$

For $|t - x| < \delta$ the last increase is clear. For $|t - x| \geq \delta$, we use the following properties

$$\omega_1(f; \lambda\delta) \leq (1 + \lambda)\omega_1(f; \delta) \leq (1 + \lambda^2)\omega_1(f; \delta),$$

where one can choose $\lambda = \delta^{-1}|t - x|$. This way the relation (2.6) implies

$$\begin{aligned} |K_n^{[\beta]}(f, x) - f(x)| &\leq n \sum_{k=0}^{\infty} \mathbf{p}_\beta(k, nx) \int_{k/n}^{(k+1)/n} (1 + \delta^{-2}(x - t)^2)\omega_1(f; \delta)dt \\ &= (\Omega_{n,0}(x) + \delta^{-2}\Omega_{n,2}(x)) \omega_1(f; \delta) \\ &= \left(1 + \delta^{-2} \left\{ \frac{3n^2x^2\beta^2 + 6nx(1 + 2\beta) + 1 + 35\beta}{3n^2(1 - \beta)^4} \right\}\right) \omega_1(f; \delta). \end{aligned}$$

Choosing $\delta = \left(\frac{3n^2x^2\beta^2 + 6nx(1 + 2\beta) + 1 + 35\beta}{n^2(1 - \beta)^4}\right)^{1/2}$, we obtain the desired result.

Further, we estimate the rate of convergence for smooth functions.

Theorem 3 Let $K_n^{[\beta]}$ be defined by (1.5) and $\beta \in [0, 1)$. Then for $f \in C^1[0, \infty)$ and $a > 0$ one has

$$|K_n^{[\beta]}(f, x) - f(x)| \leq \frac{1}{2n(1 - \beta)^2} \left(b_n \|f'\|_{C[0,a]} + c_n \omega_1\left(f'; \frac{1}{\sqrt{n}}\right) \right),$$

where $b_n = 2an\beta + (1 + \beta)^2$ and

$$c_n = \frac{2\sqrt{n^2a^2\beta^2 + 2na(1 + 2\beta) + 1 + 35\beta}}{\left(1 + (1 - \beta)^{-2}\sqrt{na^2\beta^2 + 2a(1 + 2\beta) + (1 + 35\beta)n^{-1}}\right)}.$$

Proof: We can write

$$f(x) - f(t) = (x - t)f'(x) + (x - t)(f'(\xi) - f'(x)),$$

where $\xi = \xi(t, x)$ is a point of the interval determinate by x and t . If we multiply both members of this inequality by $n\mathbf{p}_\beta(k, nx) \int_{k/n}^{(k+1)/n} dt$ and sum over k , there follows

$$\begin{aligned} |K_n^{[\beta]}(f, x) - f(x)| &\leq |f'(x)|\Omega_{n,1}(x) \\ &\quad + n \sum_{k=0}^{\infty} \mathbf{p}_\beta(k, nx) \int_{k/n}^{(k+1)/n} |x - t| \cdot |f'(\xi) - f'(t)|dt \\ &\leq \left(\frac{2xn\beta + (1 + \beta)^2}{2n(1 - \beta)^2}\right) \max_{x \in [0,a]} |f'(x)| \tag{2.11} \\ &\quad + n \sum_{k=0}^{\infty} \mathbf{p}_\beta(k, nx) \int_{k/n}^{(k+1)/n} |x - t|(1 + \delta^{-1}|t - x|)\omega_1(f'; \delta)dt. \end{aligned}$$

According to Cauchy's inequality, we have

$$\begin{aligned} & n \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} |x-t| dt \\ & \leq \sqrt{n} \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \left\{ \int_{k/n}^{(k+1)/n} (x-t)^2 dt \right\}^{1/2} \\ & \leq \sqrt{n} \left\{ \left[\sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \right] \left[\sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} (x-t)^2 dt \right] \right\}^{1/2}. \end{aligned}$$

Hence,

$$n \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \int_{k/n}^{(k+1)/n} |x-t| dt \leq \sqrt{\Omega_{n,2}(x)}. \tag{2.12}$$

Using inequalities (2.12) in (2.11), we write

$$\begin{aligned} |K_n^{[\beta]}(f, x) - f(x)| & \leq \left(\frac{2an\beta + (1+\beta)^2}{2n(1-\beta)^2} \right) \|f'\|_{C[0,a]} \\ & + \sqrt{\Omega_{n,2}(x)} \left(1 + \delta^{-1} \sqrt{\Omega_{n,2}(x)} \right) \omega_1(f'; \delta). \end{aligned} \tag{2.13}$$

Inserting $\delta = \frac{1}{\sqrt{n}}$ and using $\sqrt{\Omega_{n,2}(x)} \leq \frac{\sqrt{n^2 a^2 \beta^2 + 2na(1+2\beta) + 1 + 35\beta}}{n(1-\beta)^2}$, $x \in [0, a]$, the proof of our theorem is complete.

Theorem 4 *Let $f \in C_B[0, \infty)$. Then for all $x \in [0, \infty)$ there exists a constant $A > 0$ such that*

$$|K_n^{[\beta]}(f, x) - f(x)| \leq A\omega_2(f, \xi_n(x)) + \omega_1\left(f, \frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}\right),$$

where $\xi_n(x) = \frac{3n^2 x^2 \beta^2 + 6nx(1+2\beta) + 1 + 35\beta}{3n^2(1-\beta)^4} + \left(\frac{2xn\beta + (1+\beta)^2}{2n(1-\beta)^2}\right)^2$.

Proof: Consider the following operator

$$\hat{K}_n^{[\beta]}(f, x) = K_n^{[\beta]}(f, x) - f\left(\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}\right) + f(x). \tag{2.14}$$

By the definition of the operators $\hat{K}_n^{[\beta]}$ and Lemma 1, we have

$$\hat{K}_n^{[\beta]}(t-x, x) = 0.$$

Let $g \in C_B^2[0, \infty)$ and $x \in [0, \infty)$. By Taylor's formula of g , we get

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du, \quad t \in [0, \infty).$$

One may write

$$\begin{aligned} \hat{K}_n^{[\beta]}(g, x) - g(x) &= g'(x)\hat{K}_n^{[\beta]}(t-x, x) + \hat{K}_n^{[\beta]} \left(\int_x^t (t-u)g''(u)du, x \right) \\ &= \hat{K}_n^{[\beta]} \left(\int_x^t (t-u)g''(u)du, x \right) \\ &= K_n^{[\beta]} \left(\int_x^t (t-u)g''(u)du, x \right) \\ &\quad - \int_x^{\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}} \left(\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2} - u \right) du. \end{aligned}$$

Now, using the following inequalities

$$\left| \int_x^t (t-u)g''(u)du \right| \leq (t-x)^2 \|g''\| \tag{2.15}$$

and

$$\left| \int_x^{\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}} \left(\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2} - u \right) du \right| \leq \left[\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2} \right]^2 \|g''\|,$$

we reach to

$$\begin{aligned} |\hat{K}_n^{[\beta]}(g, x) - g(x)| &\leq \left\{ K_n^{[\beta]}((t-x)^2, x) + \left[\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2} \right]^2 \right\} \|g''\| \\ &\leq \left\{ \frac{3n^2x^2\beta^2 + 6nx(1+2\beta) + 1 + 35\beta}{3n^2(1-\beta)^4} \right. \\ &\quad \left. + \left(\frac{2xn\beta + (1+\beta)^2}{2n(1-\beta)^2} \right)^2 \right\} \|g''\|. \end{aligned} \tag{2.16}$$

By means of the definitions of the operators $\hat{K}_n^{[\beta]}$ and $K_n^{[\beta]}$, we have

$$\begin{aligned} |K_n^{[\beta]}(f, x) - f(x)| &\leq |\hat{K}_n^{[\beta]}(f-g, x)| + |(f-g)(x)| + |\hat{K}_n^{[\beta]}(g, x) - g(x)| \\ &\quad + \left| f \left(\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2} \right) - f(x) \right| \end{aligned}$$

and

$$\hat{K}_n^{[\beta]}(f, x) \leq |K_n^{[\beta]}(f, x)| + 2\|f\| \leq \|f\|K_n^{[\beta]}(1, x) + 2\|f\| = 3\|f\|.$$

Thus, we may conclude that

$$\begin{aligned} |K_n^{[\beta]}(f, x) - f(x)| &\leq 4\|f-g\| + |\hat{K}_n^{[\beta]}(g, x) - g(x)| \\ &\quad + \left| f \left(\frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2} \right) - f(x) \right|. \end{aligned}$$

In the light of inequality (2.16), one gets

$$\begin{aligned}
 |K_n^{[\beta]}(f, x) - f(x)| &\leq 4\|f - g\| \\
 &+ \left\{ \frac{3n^2x^2\beta^2 + 6nx(1 + 2\beta) + 1 + 35\beta}{3n^2(1 - \beta)^4} \right. \\
 &+ \left. \left(\frac{2xn\beta + (1 + \beta)^2}{2n(1 - \beta)^2} \right)^2 \right\} \|g''\| \\
 &+ \omega_1 \left(f, \frac{x}{1 - \beta} + \frac{(1 + \beta)^2}{2n(1 - \beta)^2} \right).
 \end{aligned}$$

Therefore taking the infimum over all $g \in C_B[0, \infty)$ on the right-hand side of the last inequality and considering (2.9), we find that

$$\begin{aligned}
 |K_n^{[\beta]}(f, x) - f(x)| &\leq 4K_2(f, \xi_n(x)) + \omega_1 \left(f, \frac{x}{1 - \beta} + \frac{(1 + \beta)^2}{2n(1 - \beta)^2} \right) \\
 &\leq 4C\omega_2(f, \xi_n(x)) + \omega_1 \left(f, \frac{x}{1 - \beta} + \frac{(1 + \beta)^2}{2n(1 - \beta)^2} \right) \\
 &\leq A\omega_2(f, \xi_n(x)) + \omega_1 \left(f, \frac{x}{1 - \beta} + \frac{(1 + \beta)^2}{2n(1 - \beta)^2} \right),
 \end{aligned}$$

which completes the proof.

Theorem 5 *Let $0 < \gamma \leq 1$, $\beta \in [0, 1)$ and $f \in C_B[0, \infty)$. Then if $f \in Lip_M(\gamma)$, that is, the inequality $|f(t) - f(x)| \leq M|t - x|^\gamma$, $x, t \in [0, \infty)$ holds, then for each $x \in [0, \infty)$, we have*

$$|K_n^{[\beta]}(f, x) - f(x)| \leq d_n^{\frac{\gamma}{2}}(x),$$

where $d_n(x) = \frac{3n^2x^2\beta^2 + 6nx(1 + 2\beta) + 1 + 35\beta}{3n^2(1 - \beta)^4}$ and $M > 0$ is a constant.

Proof: Let $f \in C_B[0, \infty) \cap Lip_M(\gamma)$. By the linearity and monotonicity of the operators $K_n^{[\beta]}$, we get

$$\begin{aligned}
 |K_n^{[\beta]}(f, x) - f(x)| &\leq K_n^{[\beta]}(|f(t) - f(x)|, x) \\
 &\leq MK_n^{[\beta]}(|t - x|^\gamma, x) \\
 &= Mn \sum_{k=0}^{\infty} \mathbf{p}_\beta(k, nx) \int_{k/n}^{(k+1)/n} |t - x|^\gamma dt.
 \end{aligned}$$

Now, applying the Hölder inequality two times successively with $p = \frac{2}{\gamma}$, $q = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned} |K_n^{[\beta]}(f, x) - f(x)| &\leq M \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) \left(n \int_{k/n}^{(k+1)/n} |t-x|^{\gamma} dt \right)^{\frac{\gamma}{2}} \\ &\leq M (\Omega_{n,2}(x))^{\frac{\gamma}{2}} \\ &\leq M \left(\frac{3n^2 x^2 \beta^2 + 6nx(1+2\beta) + 1 + 35\beta}{3n^2(1-\beta)^4} \right)^{\frac{\gamma}{2}}. \end{aligned}$$

This completes the proof.

3 Weighted approximation properties

Now, we introduce convergence properties of the operators $K_n^{[\beta]}$ via the weighted Korovkin type theorem given by Gadzhiev in [5, 6]. For this purpose, we recall some definitions and notations.

Let $\rho(x) = 1 + x^2$ and $B_{\rho}[0, \infty)$ be the space of all functions having the property

$$|f(x)| \leq M_f \rho(x),$$

where $x \in [0, \infty)$ and M_f is a positive constant depending only on f . The set $B_{\rho}[0, \infty)$ is equipped with the norm

$$\|f\|_{\rho} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2}.$$

$C_{\rho}[0, \infty)$ denotes the space of all continuous functions belonging to $B_{\rho}[0, \infty)$. By $C_{\rho}^0[0, \infty)$, we denote the subspace of all functions $f \in C_{\rho}[0, \infty)$ for which

$$\lim_{x \rightarrow \infty} \frac{|f(x)|}{\rho(x)} < \infty.$$

Theorem 6 ([5, 6]) *Let $\{A_n\}$ be a sequence of positive linear operators acting from $C_{\rho}[0, \infty)$ to $B_{\rho}[0, \infty)$ and satisfying the conditions*

$$\lim_{n \rightarrow \infty} \|A_n(t^v; x) - x^v\|_{\rho} = 0, v = 0, 1, 2.$$

Then for any function $f \in C_{\rho}^0[0, \infty)$,

$$\lim_{n \rightarrow \infty} \|A_n(f; \cdot) - f(\cdot)\|_{\rho} = 0.$$

Note that, a sequence of linear positive operators A_n acts from $C_{\rho}[0, \infty)$ to $B_{\rho}[0, \infty)$ if and only if

$$\|A_n(\rho; x)\| \leq M_{\rho},$$

where M_{ρ} is positive constant. This fact also given in [5, 6].

Theorem 7 Let $\{K_n^{[\beta_n]}\}$ be the sequence of linear positive operators defined by (1.5) and $\beta_n \in [0, 1)$ with $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. Then for each $f \in C_\rho^0[0, \infty)$, we have

$$\lim_{n \rightarrow \infty} \|K_n^{[\beta]}(f; x) - f(x)\|_\rho = 0.$$

Proof: Using Lemma 1, we may write

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|K_n^{[\beta_n]}(\rho, x)|}{1+x^2} &\leq \frac{1}{(1-\beta_n)^2} + \frac{(3+2\beta_n+\beta_n^2)}{n(1-\beta_n)^3} \\ &\quad + \frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4} + 1. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \beta_n = 0$, there exists a positive constant M^* such that

$$\frac{1}{(1-\beta_n)^2} + \frac{(3+2\beta_n+\beta_n^2)}{n(1-\beta_n)^3} + \frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4} \leq M^*$$

for each n . Hence, we get

$$\|K_n^{[\beta_n]}(\rho, x)\|_\rho \leq 1 + M^*,$$

which shows that $\{K_n^{[\beta_n]}\}$ is a sequence of positive linear operators acting from $C_\rho[0, \infty)$ to $B_\rho[0, \infty)$.

In order to complete the proof, it is enough to prove that the conditions of Theorem 6

$$\lim_{n \rightarrow \infty} \|K_n^{[\beta_n]}(t^v; x) - x^v\|_\rho = 0, v = 0, 1, 2$$

are satisfied. It is clear that

$$\lim_{n \rightarrow \infty} \|K_n^{[\beta_n]}(1; x) - 1\|_\rho = 0$$

By Lemma 1, we have

$$\begin{aligned} \|K_n^{[\beta_n]}(t; x) - x\|_\rho &= \sup_{x \in \infty} \left| \left(\frac{1}{1-\beta_n} - 1 \right) \frac{x}{1+x^2} + \frac{(1+\beta_n)^2}{2n(1-\beta_n)^2} \frac{1}{1+x^2} \right| \\ &\leq \left| \frac{\beta_n}{1-\beta_n} + \frac{(1+\beta_n)^2}{2n(1-\beta_n)^2} \right|. \end{aligned}$$

Thus taking into consideration the conditions $\beta_n \rightarrow 0$ as $n \rightarrow \infty$, we can conclude that

$$\lim_{n \rightarrow \infty} \|K_n^{[\beta_n]}(t; x) - x\|_\rho = 0$$

Similarly, one gets

$$\begin{aligned} & \|K_n^{[\beta_n]}(t^2; x) - x^2\|_\rho \\ &= \sup_{x \in \infty} \left| \left(\frac{1}{(1 - \beta_n)^2} - 1 \right) \frac{x^2}{1 + x^2} + \frac{(3 + 2\beta_n + \beta_n^2)}{n(1 - \beta_n)^3} \frac{x}{1 + x^2} \right. \\ & \quad \left. + \frac{1 + 20\beta_n + 12\beta_n^2 + 2\beta_n^3 + \beta_n^4}{3n^2(1 - \beta_n)^4} \frac{1}{1 + x^2} \right| \\ &\leq \sup_{x \in \infty} \left| \left(\frac{1}{(1 - \beta_n)^2} - 1 \right) + \frac{(3 + 2\beta_n + \beta_n^2)}{n(1 - \beta_n)^3} + \frac{1 + 20\beta_n + 12\beta_n^2 + 2\beta_n^3 + \beta_n^4}{3n^2(1 - \beta_n)^4} \right| \\ &\leq \sup_{x \in \infty} \left| \frac{2\beta_n - \beta_n^2}{(1 - \beta_n)^2} + \frac{(3 + 2\beta_n + \beta_n^2)}{n(1 - \beta_n)^3} + \frac{1 + 20\beta_n + 12\beta_n^2 + 2\beta_n^3 + \beta_n^4}{3n^2(1 - \beta_n)^4} \right| \end{aligned}$$

which leads to

$$\lim_{n \rightarrow \infty} \|K_n^{[\beta_n]}(t^2; x) - x^2\|_\rho = 0 \text{ with } \beta_n \rightarrow 0.$$

Thus the proof is completed.

Now, we compute the order of approximation of the operators $K_n^{[\beta]}$ in terms of the weighted modulus of continuity $\Omega_2(f, \delta)$ (see[7]) defined by

$$\Omega_2(f, \delta) = \sup_{x \geq 0, 0 < h \leq \delta} \frac{|f(x + h) - f(x)|}{1 + (x + h)^2}, \quad f \in C_\rho^0[0, \infty)$$

and has the following properties:

- (a) $\Omega_2(f, \delta)$ is monotone increasing function of δ ,
- (b) $\lim_{\delta \rightarrow 0^+} \Omega_2(f, \delta) = 0$,
- (c) for each $\lambda \in [0, \infty)$, $\Omega_2(f, \lambda\delta) \leq (1 + \lambda)\Omega_2(f, \delta)$.

Theorem 8 Let $\{K_n^{[\beta]}\}$ be the sequence of linear positive operators defined by (1.5). Then for each $f \in C_\rho^0[0, \infty)$, we have

$$\sup_{0 \leq x < \infty} \frac{|K_n^{[\beta]}(f; x) - f(x)|}{(1 + x^2)^3} \leq C\Omega_2 \left(f, (B_\beta(n))^{1/4} \right),$$

where C is positive constant and $B_\beta(n)$ is defined in (2.8).

Proof: For $x \geq 0$ and $t \geq 0$, by the definition of $\Omega_2(f, \delta)$ and the property (c), we may write

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + (x + |t - x|^2)) \left(1 + \frac{|t - x|}{\delta_n} \right) \Omega_2(f, \delta_n) \\ &\leq 2(1 + x^2)(1 + (t - x)^2) \left(1 + \frac{|t - x|}{\delta_n} \right) \Omega_2(f, \delta_2). \end{aligned}$$

By using the monotonicity of $K_n^{[\beta]}$ and the following inequality (see [8])

$$(1 + (t - x)^2) \left(1 + \frac{|t - x|}{\delta_n}\right) \leq 2(1 + \delta_n^2) \left(1 + \frac{(t - x)^4}{\delta_n^4}\right),$$

one gets

$$\begin{aligned} |K_n^{[\beta]}(f, x) - f(x)| &\leq 2(1 + x^2)K_n^{[\beta]} \left((1 + (t - x)^2) \left(1 + \frac{|t - x|}{\delta_n}\right), x \right) \Omega_2(f, \delta_n) \\ &\leq 4(1 + x^2)(1 + \delta_n^2)K_n^{[\beta]} \left(\left(1 + \frac{(t - x)^4}{\delta_n^4}\right), x \right) \Omega_2(f, \delta_n) \\ &\leq 4(1 + x^2)(1 + \delta_n^2) \left(1 + \frac{1}{\delta_n^4}K_n^{[\beta]}((t - x)^4, x)\right) \Omega_2(f, \delta_n) \\ &\leq C_1(1 + x^2) \left(1 + \frac{1}{\delta_n^4}K_n^{[\beta]}((t - x)^4, x)\right) \Omega_2(f, \delta_n), \end{aligned}$$

with the help of the inequality (2.7) this inequality leads to

$$|K_n^{[\beta]}(f, x) - f(x)| \leq C_1(1 + x^2) \left(1 + \frac{B_\beta(n)}{\delta_n^4}(1 + x + x^2 + x^3 + x^4)\right) \Omega_2(f, \delta_n),$$

which gives the required result.

Remark 2 In [9], the authors has consider the generalization of the operators (1.1) as

$$P_n^{[0]}(f, a_n, b_n, x) = 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} f\left(\frac{k}{b_n}\right), \quad x \geq 0, \quad f : [0, \infty) \rightarrow \mathbb{R}, \quad (3.1)$$

where $\{a_n\}, \{b_n\}$ are increasing and unbounded sequences of positive numbers such that

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} = 0, \quad \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right).$$

They studied the convergence properties of these operators in weighted spaces of continuous functions on positive semi-axis. Also, A. Erençin and F Taşdelen [10] consider the generalization of the Kantorovich type operators $P_n^{[0]}(f, a_n, b_n, x)$ given by (3.1) as follows:

$$K_n^{[0]}(f, a_n, b_n, x) = b_n 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \int_{k/b_n}^{(k+1)/b_n} f(t) dt, \quad (3.2)$$

where f is an integrable function on $[0, \infty)$ and bounded on every compact subinterval of $[0, \infty)$.

Motivated by the operators (3.1) and (3.2), we generalize the operators $P_n^{[0]}$ and $K_n^{[0]}$ in following way

$$P_n^{[\beta]}(f, a_n, b_n, x) = \sum_{k=0}^{\infty} \frac{2^{-(a_n x + k\beta)} (a_n x + k\beta)_k}{2^k k!} f\left(\frac{k}{b_n}\right), \quad x \geq 0, \quad f : [0, \infty) \rightarrow \mathbb{R}, \quad (3.3)$$

and

$$K_n^{[\beta]}(f, a_n, b_n, x) = b_n \sum_{k=0}^{\infty} \frac{2^{-(a_n x + k\beta)} (a_n x)_k}{2^k k!} \int_{k/b_n}^{(k+1)/b_n} f(t) dt \quad (3.4)$$

and extend the studies of the present article in a similar direction for the operators (3.3) and (3.4). The analysis is different so we may discuss that elsewhere.

References

- [1] A. Lupas, The approximation by some positive linear operators, in: Proceedings of the International Dortmund Meeting on Approximation Theory, Akademie Verlag, Berlin, 1995, pp. 201–229.
- [2] P. Patel, V. N. Mishra, On new class of linear and positive operators, Bollettino dell'Unione Matematica Italiana 8 (2) (2015) 81–96.
- [3] G. G. Lorentz, Bernstein polynomials, no. 8, American Mathematical Soc., 1953.
- [4] R. A. DeVore, G. G. Lorentz, Constructive approximation, Vol. 303, Springer Verlag, 1993.
- [5] A. D. Gadzhiev, The convergence problem for a sequence of positive linear operators on unbounded sets, and theorems analogous to that of PP Korovkin, in: Doklady Akademii Nauk, Vol. 218, Russian Academy of Sciences, 1974, pp. 1001–1004.
- [6] A. D. Gadzhiev, Theorems of the type of PP Korovkin's theorems, Mat. Zametki 20 (5) (1976) 781–786.
- [7] A. J. López-Moreno, Weighted simultaneous approximation with Baskakov type operators, Acta Mathematica Hungarica 104 (1-2) (2004) 143–151.
- [8] N. Ispir, On modified Baskakov operators on weighted spaces, Turkish Journal of Mathematics 25 (3) (2001) 355–365.
- [9] A. Erençin, F. Tasdelen, On a family of linear and positive operators in weighted spaces, Journal of Inequalities in Pure & Applied Mathematics 8 (2) (2007) 6 pages.
- [10] A. Erençin, F. Tasdelen, On certain Kantorovich type operators, Fasc. Math 41 (2009) 65–71.

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Mittag-Leffler-Hyers-Ulam Stability of Linear Differential Equations using Fourier Transforms

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ABSTRACT. In this paper, we are going to study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the general Linear Differential Equations of Higher order with constant coefficients using Fourier Transforms method. Moreover, the Mittag-Leffler-Hyers-Ulam stability constants of these differential equations are obtained. Some examples are given to illustrate the main results.

AMS Subject Classification: 34K20, 26D10, 34A40, 39B82.

Key Words and Phrases: Mittag-Leffler-Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam-Rassias stability, Linear Differential equation, Fourier Transforms.

1. Introduction

A classical question in the theory of functional equation is the following : “when is it true that a function which approximately satisfies a functional equation (g) must be close to an exact solution of (g)?” If the problem accepts a solution, we say that the equation (g) is stable.

A simulating and famous talk presented by Ulam [40] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talks before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. One of his question was that when is it true that a mapping that approximately satisfies a functional equation must be close to an exact solution of the equation? If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [9] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In the course of time, the Theorem formulated by Hyers was generalized by Rassias Th. [36], Aoki [4], Bourgin [6] and J.M.Rassias [29] for additive mappings. Then a number of authors has studied the Ulam problem for various functional equations by different methods in [2, 21, 30, 31, 32, 33, 34, 35].

A generalization of Ulam’s problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi \left(f, x, x', x'', \dots x^{(n)} \right) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left| \phi \left(f, x, x', x'', \dots x^{(n)} \right) \right| \leq \epsilon,$$

there exists a solution x_a of the differential equation

$$\phi \left(f, x, x', x'', \dots x^{(n)} \right) = 0$$

such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon \rightarrow 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability. Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [25, 26]. Thereafter, In 1998, C. Alsina and R. Ger [3] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved in [3] the following Theorem.

Theorem 1.1. *Assume that a differentiable function $f : I \rightarrow R$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of \mathbb{R} . Then there exists a solution $g : I \rightarrow R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of C. Alsina and R. Ger [3] has been generalized by Takahasi [39]. They proved in [39] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [11, 12, 13, 17, 18, 19, 20]. Using the approach as in [40], Miura, Takahasi and Choda [19], Miura [20], Takahasi, Miura and Miyajima [39] and Miura, Jung and Takahasi are [17] proved that the Hyers-Ulam stability holds true for the differential equation $x' = \lambda x$, while Jung [11] proved a similar result for the differential equation $\phi(t)x'(t) = x$.

In 2006, S.M. Jung [14] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. In 2007, G. Wang, M. Zhou, L. Sun [42] studied the Hyers-Ulam stability of a class of first-order linear differential equations. I. A. Rus [37] discussed four types of Ulam stability: Ulam-Hyers stability, Generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability of the Ordinary Differential Equation $u'(t) = A(u(t)) + f(t, u(t)), t \in [a, b]$. In 2014, Q. H. Alqifiary and S. M. Jung [5] proved the Generalized Hyers-Ulam stability of linear differential equation by using the Laplace Transforms. These days the Hyers-Ulam stability of differential equations are investigated [1, 7, 8, 15, 16, 22, 24, 27, 28, 43] and the investigation is ongoing.

Recently, Vida Kalvandi, N. Eghbali and J.M. Rassias [41] studied the Mittag-Leffler-Hyers-Ulam stability of a fractional differential equation of second order. In this paper, with the help of Fourier Transforms, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation

$$x'(t) + l x(t) = 0 \tag{1.1}$$

and the non-homogeneous linear differential equation

$$x'(t) + l x(t) = r(t) \tag{1.2}$$

where l is a scalar, $x(t)$ and $r(t)$ are the continuously differentiable functions. Also, by using Fourier Transforms, we establish the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order homogeneous linear differential equation

$$x''(t) + l x'(t) + m x(t) = 0 \tag{1.3}$$

and the non-homogeneous second order differential equation

$$x''(t) + l x'(t) + m x(t) = r(t) \tag{1.4}$$

where l and m are scalars, $x(t)$ is a twice continuously differentiable function and $r(t)$ is a continuously differentiable function.

2. Preliminaries

In this section, we introduce some standard notations, Definitions and Theorems, it will be very useful to prove our main results.

Throughout this paper, \mathbb{F} denotes the real field \mathbb{R} or the complex field \mathbb{C} . A function $f : (0, \infty) \rightarrow \mathbb{F}$ of exponential order if there exists a constants $A, B \in \mathbb{R}$ such that $|f(t)| \leq Ae^{tB}$ for all $t > 0$.

For each function $f : (0, \infty) \rightarrow \mathbb{F}$ of exponential order. Let g denote the Fourier Transform of f so that

$$g(u) = \int_{-\infty}^{\infty} f(t) e^{-itu} dt.$$

Then, at points of continuity of f , we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) e^{-ixu} du,$$

this is called the inverse Fourier transforms. The Fourier transform of f is denoted by $\mathcal{F}(\xi)$. We also introduce a notion, the convolution of two functions.

Definition 2.1. (Convolution). Given two functions f and g , both Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t) g(x - t) dt$$

exists. This integral defines a function h on S called the convolution of f and g . We also write $h = f * g$ to denote this function.

Theorem 2.2. *The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of the Fourier transform of $f(x)$ and $g(x)$. That is,*

$$\mathcal{F}\{f(x) * g(x)\} = \mathcal{F}\{f(x)\} \mathcal{F}\{g(x)\} = F(s) G(s)$$

or

$$\mathcal{F} \left\{ \int_{-\infty}^{\infty} f(t) g(x-t) dt \right\} = F(s) G(s),$$

where $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively.

Definition 2.3. [41] The Mittag-Leffler function of one parameter is denoted by $E_{\alpha}(z)$ and defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k$$

where $z, \alpha \in \mathbb{C}$ and $Re(\alpha) > 0$. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + 1)} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Definition 2.4. [41] The generalization of $E_{\alpha}(z)$ is defined as a function

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k$$

where $z, \alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

Now, we give the definition of Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equations (1.1), (1.2), (1.3) and (1.4).

Definition 2.5. The linear differential equation (1.1) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l x(t)| \leq \epsilon E_{\alpha}(t^{\alpha}),$$

where E_{α} is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.1) such that $|x(t) - y(t)| \leq K\epsilon E_{\alpha}(t^{\alpha})$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.1).

Definition 2.6. The linear differential equation (1.2) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l x(t) - r(t)| \leq \epsilon E_{\alpha}(t^{\alpha}),$$

where E_{α} is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.2) such that $|x(t) - y(t)| \leq K\epsilon E_{\alpha}(t^{\alpha})$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.2).

Definition 2.7. The linear differential equation (1.3) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function satisfying

$$|x''(t) + l x'(t) + m x(t)| \leq \epsilon E_{\alpha}(t^{\alpha}),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.3) such that $|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.3).

Definition 2.8. The linear differential equation (1.4) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function satisfying

$$|x''(t) + l x'(t) + m x(t) - r(t)| \leq \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.4) such that $|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.4).

Definition 2.9. We say that the homogeneous linear differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function, if there exists $\phi : (0, \infty) \rightarrow (0, \infty)$ satisfies the inequality

$$|x'(t) + l x(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.1) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.1).

Definition 2.10. We say that the non-homogeneous linear differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function, if there exists $\phi : (0, \infty) \rightarrow (0, \infty)$ satisfies the inequality

$$|x'(t) + l x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.2) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.2).

Definition 2.11. We say that the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function, if there exists $\phi : (0, \infty) \rightarrow (0, \infty)$ satisfies the inequality

$$|x''(t) + l x'(t) + m x(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.3) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.3).

Definition 2.12. We say that the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function, if there exists $\phi : (0, \infty) \rightarrow (0, \infty)$ satisfies the inequality

$$|x''(t) + l x'(t) + m x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfies the differential equation (1.4) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.4).

3. Mittag-Leffler-Hyers-Ulam Stability

In the following theorems, we prove the Mittag-Leffler-Hyers-Ulam stability of the homogeneous and non-homogeneous linear differential equations (1.1), (1.2), (1.3) and (1.4). Firstly, we prove the Mittag-Leffler-Hyers-Ulam stability of first order homogeneous differential equation (1.1).

Theorem 3.1. *The differential equation (1.1) has Mittag-Leffler-Hyers-Ulam stability.*

PROOF. Let l be a constant in \mathbb{F} . For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l x(t)| \leq \epsilon E_\alpha(t^\alpha) \tag{3.1}$$

for all $t > 0$. We will prove that, there exists a solution $y : (0, \infty) \rightarrow \mathbb{F}$ satisfying the differential equation $y'(t) + l y(t) = 0$ such that

$$|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha)$$

for any $t > 0$. Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x'(t) + l x(t)$ for each $t > 0$. In view of (3.1), we have $|p(t)| \leq \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\{x'(t) + l x(t)\} \\ P(\xi) &= \mathcal{F}\{x'(t)\} + l \mathcal{F}\{x(t)\} = -i\xi X(\xi) + l X(\xi) = (l - i\xi)X(\xi) \\ X(\xi) &= \frac{P(\xi)}{(l - i\xi)}. \end{aligned}$$

Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) (l + i\xi)}{l^2 - \xi^2}. \tag{3.2}$$

Taking $Q(\xi) = \frac{1}{(l - i\xi)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l - i\xi)} \Rightarrow q(t) = \mathcal{F}^{-1} \left\{ \frac{1}{(l - i\xi)} \right\}.$$

Now, we set $y(t) = e^{-lt}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt} e^{ist} dt = \int_{-\infty}^0 e^{-lt} e^{ist} dt + \int_0^{\infty} e^{-lt} e^{ist} dt = 0. \tag{3.3}$$

Now,

$$\mathcal{F}\{y'(t) + l y(t)\} = \mathcal{F}\{y'(t)\} + l \mathcal{F}\{y(t)\} = -i\xi Y(\xi) + l Y(\xi) = (l - i\xi)Y(\xi).$$

Then by using (3.3), we have $\mathcal{F}\{y'(t) + l y(t)\} = 0$, since \mathcal{F} is one-to-one operator, thus $y'(t) + l y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.1). Then by using (3.2) and (3.3) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= X(\xi) - Y(\xi) = \frac{P(\xi) (l + i\xi)}{l^2 - \xi^2} = P(\xi) Q(\xi) = \mathcal{F}\{p(t)\} \mathcal{F}\{q(t)\} \\ \Rightarrow \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\}. \end{aligned}$$

Since the operator \mathcal{F} is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t) q(t-s) ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s) ds \right| \leq K \epsilon E_{\alpha}(t^{\alpha}).$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s) ds \right|$ exists for each value of t . Then by virtue of Definition 2.5 the homogeneous linear differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam stability. \square

Now, we are going prove the Mittag-Leffler-Hyers-Ulam stability of the non-homogeneous linear differential equation (1.2) using Fourier transform method.

Theorem 3.2. *The differential equation (1.2) has Mittag-Leffler-Hyers-Ulam stability.*

PROOF. Let l be a constant in \mathbb{F} . For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l x(t) - r(t)| \leq \epsilon E_{\alpha}(t^{\alpha}) \tag{3.4}$$

for all $t > 0$. We have to show that there exists a solution $y : (0, \infty) \rightarrow \mathbb{F}$ satisfying the non-homogeneous differential equation $y'(t) + l y(t) = r(t)$ such that $|x(t) - y(t)| \leq K \epsilon E_{\alpha}(t^{\alpha})$, for any $t > 0$.

Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x'(t) + l x(t) - r(t)$ for each $t > 0$. In view of (3.4), we have $|p(t)| \leq \epsilon E_{\alpha}(t^{\alpha})$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\{x'(t) + l x(t) - r(t)\} \\ P(\xi) &= \mathcal{F}\{x'(t)\} + l \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\} \\ &= -i\xi X(\xi) + l X(\xi) - R(\xi) = (l - i\xi)X(\xi) - R(\xi) \\ X(\xi) &= \frac{P(\xi) + R(\xi)}{(l - i\xi)}. \end{aligned}$$

Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{\{P(\xi) + R(\xi)\} (l + i\xi)}{l^2 - \xi^2}. \tag{3.5}$$

Let us choose $Q(\xi)$ as $\frac{1}{(l - i\xi)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l - i\xi)} \Rightarrow q(t) = \mathcal{F}^{-1} \left\{ \frac{1}{(l - i\xi)} \right\}.$$

Now, we set $y(t) = e^{-lt} + (r(t) * q(t))$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt} e^{ist} dt + \frac{R(\xi)}{(l - i\xi)} = \frac{R(\xi)}{(l - i\xi)} \tag{3.6}$$

Now, $\mathcal{F}\{y'(t) + l y(t)\} = -i\xi Y(\xi) + l Y(\xi) = R(\xi)$. Then by using (3.6), we have

$$\mathcal{F}\{y'(t) + l y(t)\} = \mathcal{F}\{r(t)\},$$

since \mathcal{F} is one-to-one operator, thus $y'(t) + l y(t) = r(t)$, Hence $y(t)$ is a solution of the differential equation (1.2). Then by using (3.5) and (3.6) we have

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= X(\xi) - Y(\xi) = \frac{\{P(\xi) + R(\xi)\} (l + i\xi)}{l^2 - \xi^2} - \frac{R(\xi)}{(l - i\xi)} \\ &= P(\xi) Q(\xi) = \mathcal{F}\{p(t)\} \mathcal{F}\{q(t)\} \\ \Rightarrow \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator \mathcal{F} is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t) q(t - s) ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t - s) ds \right| \leq K \epsilon E_{\alpha}(t^{\alpha}).$$

Where $K = \left| \int_{-\infty}^{\infty} q(t - s) ds \right|$, the integral exists for each value of t . Hence, by the virtue of Definition 2.6 the non-homogeneous differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam stability. \square

Now, we prove the Mittag-Leffler-Hyers-Ulam stability of the homogeneous and non-homogeneous second order linear differential equations (1.3) and (1.4).

Theorem 3.3. *The differential equation (1.3) has Mittag-Leffler-Hyers-Ulam stability.*

PROOF. Let l, m be constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m, \mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ be a twice continuously differentiable function satisfying the inequality

$$|x''(t) + l x'(t) + m x(t)| \leq \epsilon E_{\alpha}(t^{\alpha}) \tag{3.7}$$

for all $t > 0$. We will show that there exists a solution $y : (0, \infty) \rightarrow \mathbb{F}$ satisfying the homogeneous differential equation $y''(t) + l y'(t) + m y(t) = 0$ such that

$$|x(t) - y(t)| \leq K \epsilon E_{\alpha}(t^{\alpha}),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x''(t) + l x'(t) + m x(t)$ for each $t > 0$. In view of (3.7), we have $|p(t)| \leq \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\{x''(t) + l x'(t) + m x(t)\} \\ P(\xi) &= \mathcal{F}\{x''(t)\} + l \mathcal{F}\{x'(t)\} + m \mathcal{F}\{x(t)\} = (\xi^2 - i\xi l + m) X(\xi) \\ X(\xi) &= \frac{P(\xi)}{\xi^2 - i\xi l + m}. \end{aligned}$$

Since l, m are constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)(i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \mu)(i\xi - \nu)}. \tag{3.8}$$

Let $Q(\xi) = \frac{1}{(i\xi - \mu)(i\xi - \nu)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)(i\xi - \nu)} \Rightarrow q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(i\xi - \mu)(i\xi - \nu)}\right\}.$$

Now, setting $y(t)$ as $\frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}$ and taking Fourier transform, we obtain

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} e^{i \xi t} dt = 0. \tag{3.9}$$

Now,

$$\mathcal{F}\{y''(t) + l y'(t) + m y(t)\} = (\xi^2 - i\xi l + m) Y(\xi).$$

Then by using (3.9), we have $\mathcal{F}\{y''(t) + l y'(t) + m y(t)\} = 0$. Since \mathcal{F} is one-to-one operator, then $y''(t) + l y'(t) + m y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.3). Then by using (3.8) and (3.9) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= X(\xi) - Y(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m} = P(\xi) Q(\xi) = \mathcal{F}\{p(t)\} \mathcal{F}\{q(t)\} \\ \Rightarrow \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator \mathcal{F} is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t) q(t-s) ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s) ds \right| \leq K \epsilon E_\alpha(t^\alpha).$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s) ds \right|$, the integral exists for each value of t . Then by virtue of Definition 2.7 the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam stability. \square

Theorem 3.4. *The differential equation (1.4) has Mittag-Leffler-Hyers-Ulam stability.*

PROOF. Let l, m be constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ is a twice continuously differentiable function satisfying the inequality

$$|x''(t) + l x'(t) + m x(t) - r(t)| \leq \epsilon E_\alpha(t^\alpha) \tag{3.10}$$

for all $t > 0$. We have to prove that there exists a solution $y : (0, \infty) \rightarrow \mathbb{F}$ satisfying the non-homogeneous differential equation $y''(t) + l y'(t) + m y(t) = r(t)$ such that

$$|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Assume that $x(t)$ is a continuously differentiable function satisfying the inequality (3.10). Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x''(t) + l x'(t) + m x(t) - r(t)$ for each $t > 0$. In view of (3.10), we have $|p(t)| \leq \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\{x''(t) + l x'(t) + m x(t) - r(t)\} \\ P(\xi) &= \mathcal{F}\{x''(t)\} + l \mathcal{F}\{x'(t)\} + m \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\} = (\xi^2 - i\xi l + m) X(\xi) - R(\xi) \\ X(\xi) &= \frac{P(\xi) + R(\xi)}{\xi^2 - i\xi l + m}. \end{aligned}$$

Since l, m are constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)(i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \mu)(i\xi - \nu)}. \tag{3.11}$$

Taking

$$Q(\xi) = \mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)(i\xi - \nu)},$$

setting

$$y(t) = \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} + (r(t) * q(t))$$

and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} e^{ist} dt + \frac{R(\xi)}{(i\xi - \mu)(i\xi - \nu)} = \frac{R(\xi)}{(i\xi - \mu)(i\xi - \nu)}. \tag{3.12}$$

Now,

$$\begin{aligned} \mathcal{F}\{y''(t) + l y'(t) + m y(t)\} &= \mathcal{F}\{y''(t)\} + l \mathcal{F}\{y'(t)\} + m \mathcal{F}\{y(t)\} \\ &= (\xi^2 - i\xi l + m) Y(\xi) = R(\xi). \end{aligned}$$

Then by using (3.12), we have $\mathcal{F}\{y''(t) + l y'(t) + m y(t)\} = \mathcal{F}\{r(t)\}$, since \mathcal{F} is one-to-one operator, thus $y''(t) + l y'(t) + m y(t) = r(t)$, Hence $y(t)$ is a solution of the differential

equation (1.4). Then by using (3.11) and (3.12) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= X(\xi) - Y(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \mu)(i\xi - \nu)} - \frac{R(\xi)}{(i\xi - \mu)(i\xi - \nu)} \\ &= P(\xi) Q(\xi) = \mathcal{F}\{p(t)\} \mathcal{F}\{q(t)\} \\ \Rightarrow \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator \mathcal{F} is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t) q(t-s) ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s) ds \right| \leq K \epsilon E_{\alpha}(t^{\alpha}).$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s) ds \right|$, the integral exists for each value of t . Then by virtue of Definition 2.8 the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam stability. \square

4. Mittag-Leffler-Hyers-Ulam-Rassias Stability

In the following theorems, we are going to investigate the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equations (1.1), (1.2), (1.3) and (1.4).

Theorem 4.1. *The differential equation (1.1) has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

PROOF. Let l be a constant in \mathbb{F} . For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ be a continuously differentiable function and $\phi : (0, \infty) \rightarrow (0, \infty)$ be an integrable function satisfies

$$|x'(t) + l x(t)| \leq \phi(t) \epsilon E_{\alpha}(t^{\alpha}) \tag{4.1}$$

for all $t > 0$. We will prove that, there exists a solution $y : (0, \infty) \rightarrow \mathbb{F}$ which satisfies the differential equation $y'(t) + l y(t) = 0$ such that

$$|x(t) - y(t)| \leq K \phi(t) \epsilon E_{\alpha}(t^{\alpha})$$

for any $t > 0$. Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x'(t) + l x(t)$ for each $t > 0$. In view of (4.1), we have $|p(t)| \leq \phi(t) \epsilon E_{\alpha}(t^{\alpha})$. Now, taking Fourier transform to $p(t)$, we have

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)(l + i\xi)}{l^2 - \xi^2}. \tag{4.2}$$

Choosing $Q(\xi) = \frac{1}{(l - i\xi)}$, then we have $q(t) = \mathcal{F}^{-1} \left\{ \frac{1}{(l - i\xi)} \right\}$. Now, we set $y(t) = e^{-lt}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt} e^{ist} dt = 0. \tag{4.3}$$

Hence

$$\mathcal{F}\{y'(t) + l y(t)\} = -i\xi Y(\xi) + l Y(\xi) = (l - i\xi)Y(\xi)$$

Then by using (4.3), we have $\mathcal{F}\{y'(t) + l y(t)\} = 0$, since \mathcal{F} is one-to-one operator, thus $y'(t) + l y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.1). Then by using (4.2) and (4.3) we can obtain

$$\mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}$$

Since the operator \mathcal{F} is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t) q(t-s) ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s) ds \right| \leq K\phi(t)\epsilon E_{\alpha}(t^{\alpha}).$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s) ds \right|$, the integral exists for each value of t and $\phi(t)$ is an integrable function. Then by virtue of Definition 2.9 the differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. \square

Now, we prove the Mittag-Leffler-Hyers-Ulam-Rassias stability of the non-homogeneous linear differential equation (1.2) with the help of Fourier Transforms.

Theorem 4.2. *The differential equation (1.2) has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

PROOF. Let l be a constant in \mathbb{F} . For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ is a continuously differentiable function and $\phi : (0, \infty) \rightarrow (0, \infty)$ an integrable function satisfying

$$|x'(t) + l x(t) - r(t)| \leq \phi(t)\epsilon E_{\alpha}(t^{\alpha}) \tag{4.4}$$

for all $t > 0$. We will now prove that, there exist a solution $y : (0, \infty) \rightarrow \mathbb{F}$, which satisfies the differential equation $y'(t) + l y(t) = r(t)$ such that

$$|x(t) - y(t)| \leq K\phi(t)\epsilon E_{\alpha}(t^{\alpha}),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x'(t) + l x(t) - r(t)$ for each $t > 0$. In view of (4.4), we have $|p(t)| \leq \phi(t)\epsilon E_{\alpha}(t^{\alpha})$. Now, taking Fourier transform to $p(t)$, we have

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{\{P(\xi) + R(\xi)\} (l + i\xi)}{l^2 - \xi^2}. \tag{4.5}$$

Now, let us take $Q(\xi)$ as $\frac{1}{(l - i\xi)}$; then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l - i\xi)} \Rightarrow q(t) = \mathcal{F}^{-1} \left\{ \frac{1}{(l - i\xi)} \right\}.$$

We set $y(t) = e^{-lt} + (r(t) * q(t))$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt} e^{ist} dt + \frac{R(\xi)}{(l - i\xi)} = \frac{R(\xi)}{(l - i\xi)} \tag{4.6}$$

Now,

$$\mathcal{F}\{y'(t) + l y(t)\} = \mathcal{F}\{y'(t)\} + l \mathcal{F}\{y(t)\} = -i\xi Y(\xi) + l Y(\xi) = R(\xi)$$

Then by using (4.6), we have $\mathcal{F}\{y'(t) + l y(t)\} = \mathcal{F}\{r(t)\}$, since \mathcal{F} is one-to-one operator, thus $y'(t) + l y(t) = r(t)$. Hence $y(t)$ is a solution of the differential equation (1.2). Then by using (4.5) and (4.6) we can obtain

$$\mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}.$$

Since the operator \mathcal{F} is one-to-one and linear, it gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t) q(t - s) ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t - s) ds \right| \leq K \phi(t) \epsilon E_{\alpha}(t^{\alpha}).$$

If $K = \left| \int_{-\infty}^{\infty} q(t - s) ds \right|$ the integral exists for each value of t and $\phi(t)$ is an integrable function.

Hence by the virtue of Definition 2.10 the differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. \square

Now, we are going to establish the Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order homogeneous differential equation (1.3).

Theorem 4.3. *The second order linear differential equation (1.3) has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

PROOF. Let l, m are constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m, \mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ is a twice continuously differentiable function and $\phi : (0, \infty) \rightarrow (0, \infty)$ an integrable function satisfying the inequality

$$|x''(t) + l x'(t) + m x(t)| \leq \phi(t) \epsilon E_{\alpha}(t^{\alpha}) \tag{4.7}$$

for all $t > 0$. We will now prove that there exists a solution $y : (0, \infty) \rightarrow \mathbb{F}$ satisfying the homogeneous differential equation (1.3) such that

$$|x(t) - y(t)| \leq K \phi(t) \epsilon E_{\alpha}(t^{\alpha}),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x''(t) + l x'(t) + m x(t)$ for each $t > 0$. In view of (4.7), we have $|p(t)| \leq \phi(t) \epsilon E_{\alpha}(t^{\alpha})$. Now, taking Fourier transform to $p(t)$, we have

$$P(\xi) = \mathcal{F}\{x''(t)\} + l \mathcal{F}\{x'(t)\} + m \mathcal{F}\{x(t)\} = (\xi^2 - i\xi l + m) X(\xi)$$

$$X(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m}.$$

Since l, m be constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)(i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \mu)(i\xi - \nu)}. \tag{4.8}$$

Choosing $Q(\xi)$ as $\frac{1}{(i\xi - \mu)(i\xi - \nu)}$, then we have $\mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)(i\xi - \nu)}$ and we define a function $y(t) = \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} e^{i\xi t} dt = 0. \tag{4.9}$$

Now, $\mathcal{F}\{y''(t) + l y'(t) + m y(t)\} = (\xi^2 - i\xi l + m) Y(\xi)$. Then by using (4.9), we have $\mathcal{F}\{y''(t) + l y'(t) + m y(t)\} = 0$, since \mathcal{F} is one-to-one operator, thus $y''(t) + l y'(t) + m y(t) = 0$. Hence $y(t)$ is a solution of the differential equation (1.3). Then by using (4.8) and (4.9) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= X(\xi) - Y(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m} \\ &= P(\xi) Q(\xi) = \mathcal{F}\{p(t)\} \mathcal{F}\{q(t)\} \\ \Rightarrow \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator \mathcal{F} is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$\begin{aligned} |x(t) - y(t)| &= |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t) q(t-s) ds \right| \\ &\leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s) ds \right| \leq K \phi(t) \epsilon E_{\alpha}(t^{\alpha}). \end{aligned}$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s) ds \right|$ exists for each value of t and $\phi(t)$ is an integrable function.

Then by the virtue of Definition 2.11 the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. \square

Finally, we are going to investigate the Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order non-homogeneous differential equation (1.4).

Theorem 4.4. *The second order linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam-Rassias stability.*

PROOF. Let l, m be constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \rightarrow \mathbb{F}$ is

a twice continuously differentiable function and $\phi : (0, \infty) \rightarrow (0, \infty)$ an integrable function satisfying the inequality

$$|x''(t) + l x'(t) + m x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.10}$$

for all $t > 0$. We have to prove that there exists a solution $y : (0, \infty) \rightarrow \mathbb{F}$ satisfying the non-homogeneous differential equation (1.4) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$.

Let us define a function $p : (0, \infty) \rightarrow \mathbb{F}$ such that $p(t) =: x''(t) + l x'(t) + m x(t) - r(t)$ for each $t > 0$. In view of (4.10), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking the Fourier transform to $p(t)$, we have

$$\begin{aligned} P(\xi) &= \mathcal{F}\{x''(t)\} + l \mathcal{F}\{x'(t)\} + m \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\} \\ &= (\xi^2 - i\xi l + m) X(\xi) - R(\xi) \\ X(\xi) &= \frac{P(\xi) + R(\xi)}{\xi^2 - i\xi l + m}. \end{aligned}$$

Since l, m be constants in \mathbb{F} such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)(i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \mu)(i\xi - \nu)}. \tag{4.11}$$

Assuming $Q(\xi) = \mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)(i\xi - \nu)}$ and defining a function

$$y(t) = \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} + (r(t) * q(t))$$

and also taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} e^{ist} dt + \frac{R(\xi)}{(i\xi - \mu)(i\xi - \nu)} = \frac{R(\xi)}{(i\xi - \mu)(i\xi - \nu)}. \tag{4.12}$$

Now, $\mathcal{F}\{y''(t) + l y'(t) + m y(t)\} = (\xi^2 - i\xi l + m) Y(\xi) = R(\xi)$. Then by using (4.12), we have $\mathcal{F}\{y''(t) + l y'(t) + m y(t)\} = \mathcal{F}\{r(t)\}$, since \mathcal{F} is one-to-one operator; thus

$$y''(t) + l y'(t) + m y(t) = r(t).$$

Hence $y(t)$ is a solution of the differential equation (1.4). Then by using (4.11) and (4.12) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= \frac{P(\xi) + R(\xi)}{(i\xi - \mu)(i\xi - \nu)} - \frac{R(\xi)}{(i\xi - \mu)(i\xi - \nu)} \\ &= P(\xi) Q(\xi) = \mathcal{F}\{p(t)\} \mathcal{F}\{q(t)\} \\ \Rightarrow \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator \mathcal{F} is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$\begin{aligned} |x(t) - y(t)| &= |p(t) * q(t)| \\ &= \left| \int_{-\infty}^{\infty} p(t) q(t-s) ds \right| \\ &\leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s) ds \right| \leq K\phi(t)\epsilon E_{\alpha}(t^{\alpha}). \end{aligned}$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s) ds \right|$, the integral exists for each value of t . Then by the virtue of Definition 2.12 the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. \square

Conclusion: We have proved the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equations of first order and second order with constant co-efficients using the Fourier Transforms method. That is, we established the sufficient criteria for Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation of first order and second order with constant co-efficients using Fourier Transforms method. Additionally, this paper also provides another method to study the Mittag-Leffler-Hyers-Ulam stability of differential equations. Also, this paper shows that the Fourier Transform method is more convenient to study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation with constant co-efficients.

References

- [1] M.R. Abdollahpoura, R. Aghayaria, M.Th. Rassias, Hyers-Ulam stability of associated Laguerre Differential equations in a subclass of analytic functions, *Journal of Mathematical Analysis and Applications*, 437 (2016), 605-612.
- [2] Abbas Najati, Jung Rye Lee, Choonkil Park, and Themistocles M. Rassias, On the stability of a Cauchy type functional equation, *Demonstr. Math.* 51 (2018) 323-331.
- [3] Alsina, C., Ger, R.: On Some inequalities and stability results related to the exponential function. *Journal of Inequalities Appl.* **2**, 373–380 (1998)
- [4] Aoki, T.: On the stability of the linear transformation in Banach Spaces. *J. Math. Soc. Japan*, **2**, 64–66 (1950)
- [5] Alqifiary, Q.H., Jung, S.M.: Laplace Transform And Generalized Hyers-Ulam stability of Differential equations. *Elec., J. Diff., Equations*, **2014 (80)**, 1–11 (2014)
- [6] Bourgin, D.G.: Classes of transformations and bordering transformations. *Bull. Amer. Math. Soc.* **57**, 223–237 (1951)
- [7] I. Fakunle, P.O. Arawomo, Hyers-Ulam stability of certain class of Nonlinear second order differential equations, *Global Journal of Pure and Applied Mathematics*, 14 (8) (2018) 1029-1039.
- [8] P. Gavruta, S. M. Jung and Y. Li, Hyers - Ulam Stability for Second order linear differential equations with boundary conditions, *Elec. J. of Diff. Equations*, Vol. 2011 (2011), No. 80, pp. 1-5.
- [9] Hyers, D.H.: On the Stability of a Linear functional equation. *Proc. Natl. Acad. Sci. USA*, **27**, 222–224 (1941)

- [10] Iz-iddine EL-Fassi, Choonkil Park, and Gwang Hui Kim, Stability and hyperstability of a quadratic functional equation and a characterization of inner product spaces, *Demonstr. Math.* **51** (2018) 295-303.
- [11] Jung, S.M.: Hyers-Ulam stability of linear differential equation of first order. *Appl. Math. Lett.* **17**, 1135–1140 (2004)
- [12] Jung, S.M.: Hyers-Ulam stability of linear differential equations of first order (III). *J. Math. Anal. Appl.* **311**, 139–146 (2005)
- [13] Jung, S.M.: Hyers-Ulam stability of linear differential equations of first order (II). *Appl. Math. Lett.* **19**, 854–858 (2006)
- [14] Jung, S.M.: Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. *J. Math. Anal. Appl.* **320**, 549–561 (2006)
- [15] S. M. Jung, Approximate solution of a Linear Differential Equation of Third Order, *Bull. of the Malaysian Math. Sciences Soc.* (2) **35** (4), 2012, 1063-1073.
- [16] Y. Li, Y. Shen, Hyers-Ulam stability of linear differential equations of second order, *Appl. Math. Lett.* **23** (2010), 306-309.
- [17] Miura, T.: On the Hyers-Ulam stability of a differentiable map. *Sci. Math. Japan*, **55**, 17–24 (2002)
- [18] Miura, T., Jung, S.M., Takahasi, S.E.: Hyers-Ulam-Rassias stability of the Banach space valued linear differential equation $y' = \lambda y$. *J. Korean Math. Soc.* **41**, 995–1005 (2004)
- [19] Miura, T., Miyajima, S., Takahasi, S.E.: A Characterization of Hyers-Ulam Stability of first order linear differential operators. *J. Math. Anal. Appl.* **286**, 136–146 (2003)
- [20] Miura, T., Takahasi, S.E., Choda, H.: On the Hyers-Ulam stability of real continuous function valued differentiable map. *Tokyo J. Math.* **24**, 467–476 (2001)
- [21] Nazarianpoor, M., Rassias, J.M., Sadeghi, GH.: Solution and stability of Quattuorvigintic Functional equation in Intuitionistic Fuzzy Normed spaces. *Iranian Journal of Fuzzy system*, **15** (4), 13–30 (2018)
- [22] R. Murali and A. Ponmana Selvan, On the Generalized Hyers-Ulam Stability of Linear Ordinary Differential Equations of Higher Order, *International Journal of Pure and Applied Mathematics*, **117** (12) (2017) 317-326.
- [23] R. Murali and A. Ponmana Selvan, Hyers-Ulam-Rassias Stability for the Linear Ordinary Differential Equation of Third order, *Kragujevac Journal of Mathematics*, **42** (4) (2018) 579-590.
- [24] R. Murali and A. Ponmana Selvan, *Hyers-Ulam Stability of Linear Differential Equation*, Computational Intelligence, Cyber Security and Computational Models, Models and Techniques for Intelligent Systems and Automation, Communications in Computer and Information Science, **844**. Springer, Singapore (2018), 183-192.
- [25] Obloza, M.: Hyers stability of the linear differential equation. *Rockznik Nauk-Dydakt. Prace Math.* **13**, 259–270 (1993)
- [26] Obloza, M.: Connection between Hyers and Lyapunov stability of the ordinary differential equations. *Rockznik Nauk-Dydakt. Prace Math.* **14**, 141–146 (1997)
- [27] M. N. Qarawani, Hyers-Ulam stability of Linear and Nonlinear differential equation of second order, *Int. Journal of Applied Mathematical Research*, **1** (4), 2012, 422-432.
- [28] M. N. Qarawani, Hyers-Ulam stability of a Generalized second order Nonlinear Differential equation, *Applied Mathematics*, 2012, 3, pp. 1857-1861.
- [29] Rassias, J.M.: On approximation of approximately linear mappings by linear mappings. *J. Funct., Anal.* **46**, 126–130 (1982)
- [30] Eshaghi Gordji, M., Javadian, A., Rassias, J.M.: Stability of systems of bi-quadratic and additive-cubic functional equations in Frechet's spaces. *Functional Analysis, Approximation and Computation*, **4** (1), 85–93 (2012)
- [31] Rassias, J.M., Murali, R., Rassias, M.J., Antony Raj, A.: General Solution, stability and Non-stability of Quattuorvigintic functional equation in Multi-Banach spaces. *Int. J. Math. And Appl.* **5**, 181–194 (2017)
- [32] Ravi, K., Rassias, J.M., Senthil Kumar, B.V.: Ulam-Hyers stability of undecic functional equation in quasi-beta-normed spaces fixed point method. *Tbilisi Mathematical Science*, **9** (2), 83–103 (2016)
- [33] Ravi, K., Rassias, J.M., Pinelas, S., Suresh, S.: General solution and stability of Quattuordecic functional equation in quasi-beta-normed spaces. *Advances in pure mathematics*, **6**, 921–941 (2016) doi: 10.4236/apm.2016.612070

- [34] Xu, T.Z., Rassias, J.M., Xu, W.X.: A fixed point approach to the stability of a general mixed additive-cubic functional equation in quasi fuzzy normed spaces. *International Journal of the Physical Sciences*, **6(2)**, 313–324 (2011)
- [35] Ravi, K., Rassias, J.M., Senthil Kumar, B.V.: Generalized Ulam-Hyers stability of the Harmonic Mean functional equation in two variables. *International Journal of Analysis and Applications*, **1 (1)**, 1–17 (2013)
- [36] Rassias, Th.M.: On the stability of the linear mappings in Banach Spaces. *Proc. Amer. Math. Soc.* **72**, 297–300 (1978)
- [37] Rus, I.A.: Ulam Stabilities of Ordinary Differential Equations in a Banach Space. *Carpathian J. Math.* **26 (1)**, 103–107 (2010)
- [38] K. Ravi, R. Murali and A. Ponmana Selvan, Hyers - Ulam stability of nth order linear differential equation with initial and boundary condition, *Asian Journal of Mathematics and Computer Research*, 11 (3), 2016, 201-207.
- [39] Takahasi, S.E., Miura, T., Miyajima, S.: On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \alpha y$. *Bulletin Korean Math. Soc.* **39**, 309-315 (2002)
- [40] Ulam, S.M.: *Problem in Modern Mathematics*, Chapter IV, Science Editors, Willey, New York (1960)
- [41] Vida Kalvandi, Nasrin Eghbali and John Micheal Rassias, Mittag-Leffler-Hyers-Ulam stability of fractional differential equations of second order, *J. Math. Extension*, **13 (1)** (2019) 1 - 15.
- [42] Wang, G., Zhou, M., Sun, L.: Hyers-Ulam stability of linear differential equations of first order. *Appl. Math. Lett.* **21**, 1024–1028 (2008)
- [43] J. Xue, Hyers - Ulam stability of linear differential equations of second order with constant coefficient, *Italian Journal of Pure and Applied Mathematics*, No. 32, (2014) 419-424.

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On Some Systems of Three Nonlinear Difference Equations

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Abstract

We consider in this paper, the solution of the following systems of difference equation:

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{\pm 1 + y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{\pm 1 + z_{n-2}x_{n-1}y_n}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0$ are arbitrary non zero real numbers.

Keywords: difference equations, recursive sequences, periodic solutions, system of difference equations, stability.

Mathematics Subject Classification: 39A10.

1 Introduction

Difference equations related to differential equations as discrete mathematics related to continuous mathematics. Most of these models are described by nonlinear delay difference equations; see, for example, [9], [10]. The subject of the qualitative study of the nonlinear delay population models is very extensive, and the current research work tends to center around the relevant global dynamics of the considered systems of difference equations such as oscillation, boundedness of solutions, persistence, global

stability of positive steady states, permanence, and global existence of periodic solutions. See [13], [17], [19]-[22], [26], [28], [29] and the references therein. In particular, Agarwal and Elsayed [1] deal with the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}.$$

Camouzis et al. [5] studied the global character of solutions of the difference equation

$$x_{n+1} = \frac{\delta x_{n-2} + x_{n-3}}{A + x_{n-3}}.$$

Clark and Kulenovic [7] investigated the global asymptotic stability of the system

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}.$$

In [9], Din studied the boundedness character, steady-states, local asymptotic stability of equilibrium points, and global behavior of the unique positive equilibrium point of a discrete predator-prey model given by

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \quad y_{n+1} = \frac{\delta x_n y_n}{x_n + \eta y_n}.$$

Elsayed et al. [23] discussed the global convergence and periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}.$$

Elsayed and El-Metwally [24] discussed the periodic nature and the form of the solutions of the nonlinear difference equations systems

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1} (\pm 1 \pm x_n y_{n-2})}, \quad y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1} (\pm 1 \pm y_n x_{n-2})}.$$

Geliskan and Kara [25] studied some behavior of solutions of some systems of rational difference equations of higher order and they showed that every solution is periodic with a period depends on the order.

In [27] Kurbanli discussed a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n - 1}, \quad y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n - 1}, \quad z_{n+1} = \frac{x_n}{z_{n-1} y_n}.$$

Touafek et al. [33] studied the sufficient conditions for the global asymptotic stability of the following systems of rational difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}.$$

with a real number's initial conditions.

Our goal in this paper is to investigate the form of the solutions of the system of three difference equations

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{\pm 1 + y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{\pm 1 + z_{n-2}x_{n-1}y_n}, \quad (1)$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0$ are arbitrary real numbers. Moreover, we obtain some numerical simulation to the equation are given to illustrate our results.

2 The System

$$x_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n}$$

In this section, we study the solution of the following system of difference equations.

$$x_{n+1} = \frac{x_{n-2}}{1 + x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{1 + y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{1 + z_{n-2}x_{n-1}y_n}, \quad (2)$$

where $n \in N_0$ and the initial conditions are arbitrary real numbers.

The following theorem is devoted to the form of the solutions of system (1).

Theorem 1. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (1). Then for $n = 0, 1, 2, \dots$, we have the following formulas

$$\begin{aligned} x_{3n-2} &= x_{-2} \prod_{i=0}^{n-1} \frac{(1 + (3i)x_{-2}y_{-1}z_0)}{(1 + (3i + 1)x_{-2}y_{-1}z_0)}, & x_{3n-1} &= x_{-1} \prod_{i=0}^{n-1} \frac{(1 + (3i + 1)x_{-1}y_0z_{-2})}{(1 + (3i + 2)x_{-1}y_0z_{-2})}, \\ x_{3n} &= x_0 \prod_{i=0}^{n-1} \frac{(1 + (3i + 2)x_0y_{-2}z_{-1})}{(1 + (3i + 3)x_0y_{-2}z_{-1})}, \\ y_{3n-2} &= y_{-2} \prod_{i=0}^{n-1} \frac{(1 + (3i)x_0y_{-2}z_{-1})}{(1 + (3i + 1)x_0y_{-2}z_{-1})}, & y_{3n-1} &= y_{-1} \prod_{i=0}^{n-1} \frac{(1 + (3i + 1)x_{-2}y_{-1}z_0)}{(1 + (3i + 2)x_{-2}y_{-1}z_0)}, \\ y_{3n} &= y_0 \prod_{i=0}^{n-1} \frac{(1 + (3i + 2)x_{-1}y_0z_{-2})}{(1 + (3i + 3)x_{-1}y_0z_{-2})}, \\ z_{3n-2} &= z_{-2} \prod_{i=0}^{n-1} \frac{(1 + (3i)x_{-1}y_0z_{-2})}{(1 + (3i + 1)x_{-1}y_0z_{-2})}, & z_{3n-1} &= z_{-1} \prod_{i=0}^{n-1} \frac{(1 + (3i + 1)x_0y_{-2}z_{-1})}{(1 + (3i + 2)x_0y_{-2}z_{-1})}, \\ z_{3n} &= z_0 \prod_{i=0}^{n-1} \frac{(1 + (3i + 2)x_{-2}y_{-1}z_0)}{(1 + (3i + 3)x_{-2}y_{-1}z_0)}, \end{aligned}$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n - 1$.

$$\begin{aligned}
 x_{3n-5} &= x_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_{-2}y_{-1}z_0)}{(1 + (3i + 1)x_{-2}y_{-1}z_0)}, & x_{3n-4} &= x_{-1} \prod_{i=0}^{n-2} \frac{(1 + (3i + 1)x_{-1}y_0z_{-2})}{(1 + (3i + 2)x_{-1}y_0z_{-2})}, \\
 x_{3n-3} &= x_0 \prod_{i=0}^{n-2} \frac{(1 + (3i + 2)x_0y_{-2}z_{-1})}{(1 + (3i + 3)x_0y_{-2}z_{-1})}, \\
 y_{3n-5} &= y_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_0y_{-2}z_{-1})}{(1 + (3i + 1)x_0y_{-2}z_{-1})}, & y_{3n-4} &= y_{-1} \prod_{i=0}^{n-2} \frac{(1 + (3i + 1)x_{-2}y_{-1}z_0)}{(1 + (3i + 2)x_{-2}y_{-1}z_0)}, \\
 y_{3n-3} &= y_0 \prod_{i=0}^{n-2} \frac{(1 + (3i + 2)x_{-1}y_0z_{-2})}{(1 + (3i + 3)x_{-1}y_0z_{-2})}, \\
 z_{3n-5} &= z_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_{-1}y_0z_{-2})}{(1 + (3i + 1)x_{-1}y_0z_{-2})}, & z_{3n-4} &= z_{-1} \prod_{i=0}^{n-2} \frac{(1 + (3i + 1)x_0y_{-2}z_{-1})}{(1 + (3i + 2)x_0y_{-2}z_{-1})}, \\
 z_{3n-3} &= z_0 \prod_{i=0}^{n-2} \frac{(1 + (3i + 2)x_{-2}y_{-1}z_0)}{(1 + (3i + 3)x_{-2}y_{-1}z_0)}.
 \end{aligned}$$

It follows from Eq.(1) that

$$\begin{aligned}
 x_{3n-2} &= \frac{x_{3n-5}}{1 + x_{3n-5}y_{3n-4}z_{3n-3}} \\
 &= \frac{x_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1 + (x_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)})(y_{-1} \prod_{i=0}^{n-2} \frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)})(z_0 \prod_{i=0}^{n-2} \frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)})} \\
 &= \frac{x_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1 + x_{-2}y_{-1}z_0 \prod_{i=0}^{n-2} ((\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)})(\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)})(\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)}))} \\
 &= \frac{x_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1 + x_{-2}y_{-1}z_0 \prod_{i=0}^{n-2} (\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)})} \\
 &= x_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_{-2}y_{-1}z_0)}{(1 + (3i + 1)x_{-2}y_{-1}z_0)} \frac{1}{1 + (\frac{x_{-2}y_{-1}z_0}{(1+(3n-3)x_{-2}y_{-1}z_0)})} \\
 &= x_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_{-2}y_{-1}z_0)}{(1 + (3i + 1)x_{-2}y_{-1}z_0)} (\frac{(1 + (3n - 3)x_{-2}y_{-1}z_0)}{(1 + (3n - 3)x_{-2}y_{-1}z_0) + x_{-2}y_{-1}z_0}) \\
 &= x_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_{-2}y_{-1}z_0)}{(1 + (3i + 1)x_{-2}y_{-1}z_0)} (\frac{(1 + (3n - 3)x_{-2}y_{-1}z_0)}{(1 + (3n - 2)x_{-2}y_{-1}z_0)}).
 \end{aligned}$$

Then, we see that

$$x_{3n-2} = x_{-2} \prod_{i=0}^{n-1} \frac{(1 + (3i)x_{-2}y_{-1}z_0)}{(1 + (3i + 1)x_{-2}y_{-1}z_0)}.$$

Also, we see from Eq.(1) that

$$\begin{aligned} y_{3n-2} &= \frac{y_{3n-5}}{1 + y_{3n-5}z_{3n-4}x_{3n-3}} \\ &= \frac{y_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}}{1 + (y_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})})(z_{-1} \prod_{i=0}^{n-2} \frac{(1+(3i+1)x_0y_{-2}z_{-1})}{(1+(3i+2)x_0y_{-2}z_{-1})})(x_0 \prod_{i=0}^{n-2} \frac{(1+(3i+2)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})})} \\ &= \frac{y_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}}{1 + x_0y_{-2}z_{-1} \prod_{i=0}^{n-2} \frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})}} \\ &= y_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_0y_{-2}z_{-1})}{(1 + (3i + 1)x_0y_{-2}z_{-1})} \left(\frac{1}{1 + \frac{x_0y_{-2}z_{-1}}{1+(3n-3)x_0y_{-2}z_{-1}}} \right) \\ &= y_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_0y_{-2}z_{-1})}{(1 + (3i + 1)x_0y_{-2}z_{-1})} \left(\frac{1 + (3n - 3)x_0y_{-2}z_{-1}}{1 + (3n - 3)x_0y_{-2}z_{-1} + x_0y_{-2}z_{-1}} \right) \\ &= y_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_0y_{-2}z_{-1})}{(1 + (3i + 1)x_0y_{-2}z_{-1})} \left(\frac{1 + (3n - 3)x_0y_{-2}z_{-1}}{1 + (3n - 2)x_0y_{-2}z_{-1}} \right). \end{aligned}$$

Then, we see that

$$y_{3n-2} = y_{-2} \prod_{i=0}^{n-1} \frac{(1 + (3i)x_0y_{-2}z_{-1})}{(1 + (3i + 1)x_0y_{-2}z_{-1})}$$

Finally, we see that

$$\begin{aligned}
 z_{3n-2} &= \frac{z_{3n-5}}{1 + z_{3n-5}x_{3n-4}y_{3n-3}} \\
 &= \frac{z_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}}{1 + (z_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})})(x_{-1} \prod_{i=0}^{n-2} \frac{(1+(3i+1)x_{-1}y_0z_{-2})}{(1+(3i+2)x_{-1}y_0z_{-2})})(y_0 \prod_{i=0}^{n-2} \frac{(1+(3i+2)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})})} \\
 &= \frac{z_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}}{1 + x_{-1}y_0z_{-2} \prod_{i=0}^{n-2} \frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})}} \\
 &= z_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_{-1}y_0z_{-2})}{(1 + (3i + 1)x_{-1}y_0z_{-2})} \left(\frac{1}{1 + \frac{x_{-1}y_0z_{-2}}{1+(3n-3)x_{-1}y_0z_{-2}}} \right) \\
 &= z_{-2} \prod_{i=0}^{n-2} \frac{(1 + (3i)x_{-1}y_0z_{-2})}{(1 + (3i + 1)x_{-1}y_0z_{-2})} \left(\frac{1 + (3n - 3)x_{-1}y_0z_{-2}}{1 + (3n - 2)x_{-1}y_0z_{-2}} \right).
 \end{aligned}$$

Then

$$z_{3n-2} = z_{-2} \prod_{i=0}^{n-1} \frac{(1 + (3i)x_{-1}y_0z_{-2})}{(1 + (3i + 1)x_{-1}y_0z_{-2})}.$$

This completes the proof.

3 The System

$$x_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n}, y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$$

In this section, we obtain the form of the solutions of the system of three difference equations

$$x_{n+1} = \frac{x_{n-2}}{1 + x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{-1 + z_{n-2}x_{n-1}y_n}, \quad (3)$$

where $n \in N_0$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 2. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (2). Then for $n = 0, 1, 2, \dots$, we have the following formulas

$$\begin{aligned}
 x_{3n-2} &= \frac{x_{-2}}{1 + nx_{-2}y_{-1}z_0}, \quad x_{3n-1} = \frac{x_{-1}(x_{-1}y_0z_{-2} - 1)}{(n + 1)x_{-1}y_0z_{-2} - 1}, \quad x_{3n} = \frac{x_0}{1 + nx_0y_{-2}z_{-1}}, \\
 y_{3n-2} &= \frac{(-1)^{n+1}y_{-2}(1 + (n - 1)x_0y_{-2}z_{-1})}{x_0y_{-2}z_{-1} - 1}, \quad y_{3n-1} = (-1)^ny_{-1}(1 + nx_{-2}y_{-1}z_0), \\
 y_{3n} &= \frac{(-1)^ny_0((n + 1)x_{-1}y_0z_{-2} - 1)}{x_{-1}y_0z_{-2} - 1},
 \end{aligned}$$

$$z_{3n-2} = \frac{(-1)^{n+1}z_{-2}}{nx_{-1}y_0z_{-2} - 1}, \quad z_{3n-1} = \frac{(-1)^{n+1}z_{-1}(x_0y_{-2}z_{-1} - 1)}{(n-1)x_0y_{-2}z_{-1} + 1}, \quad z_{3n} = \frac{(-1)^nz_0}{1 + nx_{-2}y_{-1}z_0}.$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n - 1$.

$$x_{3n-5} = \frac{x_{-2}}{1 + (n-1)x_{-2}y_{-1}z_0}, \quad x_{3n-4} = \frac{x_{-1}(x_{-1}y_0z_{-2} - 1)}{nx_{-1}y_0z_{-2} - 1}, \quad x_{3n-3} = \frac{x_0}{1 + (n-1)x_0y_{-2}z_{-1}},$$

$$y_{3n-5} = \frac{(-1)^ny_{-2}(1 + (n-2)x_0y_{-2}z_{-1})}{x_0y_{-2}z_{-1} - 1}, \quad y_{3n-4} = (-1)^{n-1}y_{-1}(1 + (n-1)x_{-2}y_{-1}z_0),$$

$$y_{3n-3} = \frac{(-1)^{n-1}y_0(nx_{-1}y_0z_{-2} - 1)}{x_{-1}y_0z_{-2} - 1},$$

$$z_{3n-5} = \frac{(-1)^nz_{-2}}{(n-1)x_{-1}y_0z_{-2} - 1}, \quad z_{3n-4} = \frac{(-1)^nz_{-1}(x_0y_{-2}z_{-1} - 1)}{(n-2)x_0y_{-2}z_{-1} + 1}, \quad z_{3n-3} = \frac{(-1)^{n-1}z_0}{1 + (n-1)x_{-2}y_{-1}z_0},$$

from system (2) we can prove as follow

$$\begin{aligned} x_{3n-2} &= \frac{x_{3n-5}}{1 + x_{3n-5}y_{3n-4}z_{3n-3}} \\ &= \frac{\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0}}{1 + \left(\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0}\right)\left((-1)^{n-1}y_{-1}(1 + (n-1)x_{-2}y_{-1}z_0)\right)\left(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0}\right)} \\ &= \frac{x_{-2}}{1 + (n-1)x_{-2}y_{-1}z_0 + x_{-2}y_{-1}z_0} = \frac{x_{-2}}{1 + nx_{-2}y_{-1}z_0} \end{aligned}$$

Also, we get

$$\begin{aligned} y_{3n-1} &= \frac{y_{3n-4}}{-1 + y_{3n-4}z_{3n-3}x_{3n-2}} \\ &= \frac{(-1)^{n-1}y_{-1}(1 + (n-1)x_{-2}y_{-1}z_0)}{-1 + \left((-1)^{n-1}y_{-1}(1 + (n-1)x_{-2}y_{-1}z_0)\right)\left(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0}\right)\left(\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0}\right)} \\ &= \frac{(-1)^ny_{-1}(1 + (n-1)x_{-2}y_{-1}z_0)(1 + nx_{-2}y_{-1}z_0)}{1 + (n-1)x_{-2}y_{-1}z_0} = (-1)^ny_{-1}(1 + nx_{-2}y_{-1}z_0) \end{aligned}$$

$$\begin{aligned} z_{3n} &= \frac{z_{3n-3}}{-1 + z_{3n-3}x_{3n-2}y_{3n-1}} \\ &= \frac{\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0}}{-1 + \left(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0}\right)\left(\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0}\right)\left((-1)^ny_{-1}(1 + nx_{-2}y_{-1}z_0)\right)} \\ &= \frac{(-1)^nz_0}{1 + (n-1)x_{-2}y_{-1}z_0 + x_{-2}y_{-1}z_0} = \frac{(-1)^nz_0}{1 + nx_{-2}y_{-1}z_0}. \end{aligned}$$

4 The System

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$$

In this section, we study the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}, \quad (4)$$

where $n \in N_0$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 3. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (3). Then for $n = 0, 1, 2, \dots$, we have the following formulas

$$\begin{aligned} x_{3n-2} &= \frac{x_{-2}}{nx_{-2}y_{-1}z_0 - 1}, \quad x_{3n-1} = \frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2} - 1)}{(n-1)x_{-1}y_0z_{-2} + 1}, \quad x_{3n} = \frac{(-1)^n x_0}{1 + nx_0y_{-2}z_{-1}}, \\ y_{3n-2} &= \frac{y_{-2}}{nx_0y_{-2}z_{-1} + 1}, \quad y_{3n-1} = \frac{y_{-1}(x_{-2}y_{-1}z_0 - 1)}{(n+1)x_{-2}y_{-1}z_0 - 1}, \quad y_{3n} = \frac{y_0}{nx_{-1}y_0z_{-2} + 1}, \\ z_{3n-2} &= \frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2} + 1)}{x_{-1}y_0z_{-2} - 1}, \quad z_{3n-1} = (-1)^n z_{-1}(nx_0y_{-2}z_{-1} + 1), \\ z_{3n} &= \frac{(-1)^n z_0((n+1)x_{-2}y_{-1}z_0 - 1)}{x_{-2}y_{-1}z_0 - 1}. \end{aligned}$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n - 1$

$$\begin{aligned} x_{3n-5} &= \frac{x_{-2}}{(n-1)x_{-2}y_{-1}z_0 - 1}, \quad x_{3n-4} = \frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2} - 1)}{(n-2)x_{-1}y_0z_{-2} + 1}, \quad x_{3n-3} = \frac{(-1)^{n-1} x_0}{1 + (n-1)x_0y_{-2}z_{-1}}, \\ y_{3n-5} &= \frac{y_{-2}}{(n-1)x_0y_{-2}z_{-1} + 1}, \quad y_{3n-4} = \frac{y_{-1}(x_{-2}y_{-1}z_0 - 1)}{nx_{-2}y_{-1}z_0 - 1}, \quad y_{3n-3} = \frac{y_0}{(n-1)x_{-1}y_0z_{-2} + 1}, \\ z_{3n-5} &= \frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2} + 1)}{x_{-1}y_0z_{-2} - 1}, \quad z_{3n-4} = (-1)^{n-1} z_{-1}((n-1)x_0y_{-2}z_{-1} + 1), \\ z_{3n-3} &= \frac{(-1)^{n-1} z_0(nx_{-2}y_{-1}z_0 - 1)}{x_{-2}y_{-1}z_0 - 1}, \end{aligned}$$

from system (3) we can prove as follow

$$\begin{aligned} x_{3n-1} &= \frac{x_{3n-4}}{-1 + x_{3n-4}y_{3n-3}z_{3n-2}} \\ &= \frac{\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2} - 1)}{(n-2)x_{-1}y_0z_{-2} + 1}}{-1 + \left(\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2} - 1)}{(n-2)x_{-1}y_0z_{-2} + 1}\right)\left(\frac{y_0}{(n-1)x_{-1}y_0z_{-2} + 1}\right)\left(\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2} + 1)}{x_{-1}y_0z_{-2} - 1}\right)} \\ &= \frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2} - 1)}{-((n-2)x_{-1}y_0z_{-2} + 1) + ((-1)^n x_{-1})((-1)^{n+1}y_0z_{-2})} \\ &= \frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2} - 1)}{(n-1)x_{-1}y_0z_{-2} + 1}. \end{aligned}$$

Also, we get

$$\begin{aligned}
 y_{3n} &= \frac{y_{3n-3}}{1 + y_{3n-3}z_{3n-2}x_{3n-1}} \\
 &= \frac{\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1}}{1 + \left(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1}\right)\left(\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}\right)\left(\frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-1)x_{-1}y_0z_{-2}+1}\right)} \\
 &= \frac{y_0}{(n-1)x_{-1}y_0z_{-2} + 1 + y_0((-1)^{n+1}z_{-2})((-1)^{n+1}x_{-1})} \\
 &= \frac{y_0}{nx_{-1}y_0z_{-2} + 1} \\
 z_{3n-2} &= \frac{z_{3n-5}}{-1 + z_{3n-5}x_{3n-4}y_{3n-3}} \\
 &= \frac{\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}}{-1 + \left(\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}\right)\left(\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1}\right)\left(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1}\right)} \\
 &= \frac{\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}}{\frac{-((n-2)x_{-1}y_0z_{-2}+1)}{(n-1)x_{-1}y_0z_{-2}+1}} \\
 &= \frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2} + 1)}{x_{-1}y_0z_{-2} - 1}.
 \end{aligned}$$

This completes the proof.

5 The System

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n}$$

In this section, we investigate the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{1 + z_{n-2}x_{n-1}y_n}, \quad (5)$$

where the initial conditions $n \in N_0$ are arbitrary non zero real numbers. The following theorem is devoted to the form of the solutions of system (4).

Theorem 4. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (4). Then for $n = 0, 1, 2, \dots$, we have the following formulas

$$\begin{aligned}
 x_{3n-2} &= \frac{(-1)^{n+1}x_{-2}((n-1)x_{-2}y_{-1}z_0 + 1)}{x_{-2}y_{-1}z_0 - 1}, \quad x_{3n-1} = (-1)^n x_{-1}(nx_{-1}y_0z_{-2} + 1), \\
 x_{3n} &= \frac{(-1)^n x_0((n+1)x_0y_{-2}z_{-1} - 1)}{x_0y_{-2}z_{-1} - 1},
 \end{aligned}$$

$$y_{3n-2} = \frac{(-1)^{n+1}y_{-2}}{nx_0y_{-2}z_{-1} - 1}, \quad y_{3n-1} = \frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0 - 1)}{(n-1)x_{-2}y_{-1}z_0 + 1}, \quad y_{3n} = \frac{(-1)^ny_0}{nx_{-1}y_0z_{-2} + 1},$$

$$z_{3n-2} = \frac{z_{-2}}{nx_{-1}y_0z_{-2} + 1}, \quad z_{3n-1} = \frac{z_{-1}(x_0y_{-2}z_{-1} - 1)}{(n+1)x_0y_{-2}z_{-1} - 1}, \quad z_{3n} = \frac{z_0}{nx_{-2}y_{-1}z_0 + 1}.$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n - 1$

$$x_{3n-5} = \frac{(-1)^nx_{-2}((n-2)x_{-2}y_{-1}z_0 + 1)}{x_{-2}y_{-1}z_0 - 1}, \quad x_{3n-4} = (-1)^{n-1}x_{-1}((n-1)x_{-1}y_0z_{-2} + 1),$$

$$x_{3n-3} = \frac{(-1)^{n-1}x_0(x_0y_{-2}z_{-1} - 1)}{x_0y_{-2}z_{-1} - 1},$$

$$y_{3n-5} = \frac{(-1)^ny_{-2}}{(n-1)x_0y_{-2}z_{-1} - 1}, \quad y_{3n-4} = \frac{(-1)^ny_{-1}(x_{-2}y_{-1}z_0 - 1)}{(n-2)x_{-2}y_{-1}z_0 + 1}, \quad y_{3n-3} = \frac{(-1)^{n-1}y_0}{(n-1)x_{-1}y_0z_{-2} + 1},$$

$$z_{3n-5} = \frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2} + 1}, \quad z_{3n-4} = \frac{z_{-1}(x_0y_{-2}z_{-1} - 1)}{nx_0y_{-2}z_{-1} - 1}, \quad z_{3n-3} = \frac{z_0}{(n-1)x_{-2}y_{-1}z_0 + 1},$$

from system (4) we can prove as follow

$$x_{3n} = \frac{x_{3n-3}}{-1 + x_{3n-3}y_{3n-2}z_{3n-1}}$$

$$= \frac{\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}}{-1 + \left(\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}\right)\left(\frac{(-1)^{n+1}y_{-2}}{nx_0y_{-2}z_{-1}-1}\right)\left(\frac{z_{-1}(x_0y_{-2}z_{-1}-1)}{(n+1)x_0y_{-2}z_{-1}-1}\right)}$$

$$= \frac{\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}}{\frac{x_0y_{-2}z_{-1}-((n+1)x_0y_{-2}z_{-1}-1)}{(n+1)x_0y_{-2}z_{-1}-1}} = \frac{(-1)^nx_0((n+1)x_0y_{-2}z_{-1} - 1)}{x_0y_{-2}z_{-1} - 1}$$

Also, we get

$$y_{3n-1} = \frac{y_{3n-4}}{-1 + y_{3n-4}z_{3n-3}x_{3n-2}}$$

$$= \frac{\frac{(-1)^ny_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1}}{-1 + \left(\frac{(-1)^ny_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1}\right)\left(\frac{z_0}{(n-1)x_{-2}y_{-1}z_0+1}\right)\left(\frac{(-1)^{n+1}x_{-2}((n-1)x_{-2}y_{-1}z_0+1)}{x_{-2}y_{-1}z_0-1}\right)}$$

$$= \frac{\frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1}}{\frac{(n-2)x_{-2}y_{-1}z_0+1+x_{-2}y_{-1}z_0}{(n-2)x_{-2}y_{-1}z_0+1}} = \frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0 - 1)}{(n-1)x_{-2}y_{-1}z_0 + 1}$$

$$z_{3n-2} = \frac{z_{3n-5}}{1 + z_{3n-5}x_{3n-4}y_{3n-3}}$$

$$= \frac{\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}{1 + \left(\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}\right)\left((-1)^{n-1}x_{-1}((n-1)x_{-1}y_0z_{-2} + 1)\right)\left(\frac{(-1)^{n-1}y_0}{(n-1)x_{-1}y_0z_{-2}+1}\right)}$$

$$= \frac{\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}{\frac{(n-1)x_{-1}y_0z_{-2}+1+x_{-1}y_0z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}} = \frac{z_{-2}}{nx_{-1}y_0z_{-2} + 1}$$

This completes the proof.

The following cases can be proved similarly.

6 On The System

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}, \quad (6)$$

where the initial conditions $n \in N_0$ are arbitrary non zero real numbers.

Theorem 5. Let $\{x_n, y_n, z_n\}_{n=-2}^{+\infty}$ be solutions of system (5). Then

1- $\{x_n\}_{n=-2}^{+\infty}$, $\{y_n\}_{n=-2}^{+\infty}$ and $\{z_n\}_{n=-2}^{+\infty}$ and are periodic with period six i.e.,

$$x_{n+6} = x_n, \quad y_{n+6} = y_n, \quad z_{n+6} = z_n.$$

2- We have the following form

$$\begin{aligned} x_{6n-2} &= x_{-2}, \quad x_{6n-1} = x_{-1}, \quad x_{6n} = x_0, \\ x_{6n+1} &= \frac{x_{-2}}{x_{-2}y_{-1}z_0 - 1}, \quad x_{6n+2} = x_{-1}(x_{-1}y_0z_{-2} - 1), \quad x_{6n+3} = \frac{x_0}{x_0y_{-2}z_{-1} - 1}, \\ y_{6n-2} &= y_{-2}, \quad y_{6n-1} = y_{-1}, \quad y_{6n} = y_0, \\ y_{6n+1} &= \frac{y_{-2}}{x_0y_{-2}z_{-1} - 1}, \quad y_{6n+2} = y_{-1}(x_{-2}y_{-1}z_0 - 1), \quad y_{6n+3} = \frac{y_0}{x_{-1}y_0z_{-2} - 1}, \\ z_{6n-2} &= z_{-2}, \quad z_{6n-1} = z_{-1}, \quad z_{6n} = z_0, \\ z_{6n+1} &= \frac{z_{-2}}{x_{-1}y_0z_{-2} - 1}, \quad z_{6n+2} = z_{-1}(x_0y_{-2}z_{-1} - 1), \quad z_{6n+3} = \frac{z_0}{x_{-2}y_{-1}z_0 - 1}, \end{aligned}$$

Or equivalently

$$\begin{aligned} \{x_n\}_{n=-2}^{+\infty} &= \left\{ x_{-2}, x_{-1}, x_0, \frac{x_{-2}}{x_{-2}y_{-1}z_0 - 1}, x_{-1}(x_{-1}y_0z_{-2} - 1), \frac{x_0}{x_0y_{-2}z_{-1} - 1} \right\}, \\ \{y_n\}_{n=-2}^{+\infty} &= \left\{ y_{-2}, y_{-1}, y_0, \frac{y_{-2}}{x_0y_{-2}z_{-1} - 1}, y_{-1}(x_{-2}y_{-1}z_0 - 1), \frac{y_0}{x_{-1}y_0z_{-2} - 1} \right\}. \\ \{z_n\}_{n=-2}^{+\infty} &= \left\{ z_{-2}, z_{-1}, z_0, \frac{z_{-2}}{x_{-1}y_0z_{-2} - 1}, z_{-1}(x_0y_{-2}z_{-1} - 1), \frac{z_0}{x_{-2}y_{-1}z_0 - 1} \right\}. \end{aligned}$$

7 On The System

$$x_{n+1} = \frac{x_{n-2}}{-1-x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1-y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{-1-z_{n-2}x_{n-1}y_n}$$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1-x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1-y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{-1-z_{n-2}x_{n-1}y_n}, \tag{7}$$

where the initial conditions $n \in N_0$ are arbitrary non zero real numbers.

Theorem 6. Let $\{x_n, y_n, z_n\}_{n=-2}^{+\infty}$ be solutions of system (6). Then

1- $\{x_n\}_{n=-2}^{+\infty}$, $\{y_n\}_{n=-2}^{+\infty}$ and $\{z_n\}_{n=-2}^{+\infty}$ and are periodic with period six i.e.,

$$x_{n+6} = x_n, \quad y_{n+6} = y_n, \quad z_{n+6} = z_n.$$

2- We have the following form

$$\begin{aligned} x_{6n-2} &= x_{-2}, \quad x_{6n-1} = x_{-1}, \quad x_{6n} = x_0, \\ x_{6n+1} &= -\frac{x_{-2}}{x_{-2}y_{-1}z_0 + 1}, \quad x_{6n+2} = -x_{-1}(x_{-1}y_0z_{-2} + 1), \quad x_{6n+3} = -\frac{x_0}{x_0y_{-2}z_{-1} + 1}, \\ y_{6n-2} &= y_{-2}, \quad y_{6n-1} = y_{-1}, \quad y_{6n} = y_0, \\ y_{6n+1} &= -\frac{y_{-2}}{x_0y_{-2}z_{-1} + 1}, \quad y_{6n+2} = -y_{-1}(x_{-2}y_{-1}z_0 + 1), \quad y_{6n+3} = -\frac{y_0}{x_{-1}y_0z_{-2} + 1}, \\ z_{6n-2} &= z_{-2}, \quad z_{6n-1} = z_{-1}, \quad z_{6n} = z_0, \\ z_{6n+1} &= -\frac{z_{-2}}{x_{-1}y_0z_{-2} + 1}, \quad z_{6n+2} = -z_{-1}(x_0y_{-2}z_{-1} + 1), \quad z_{6n+3} = -\frac{z_0}{x_{-2}y_{-1}z_0 + 1}, \end{aligned}$$

Or equivalently

$$\begin{aligned} \{x_n\}_{n=-2}^{+\infty} &= \left\{ x_{-2}, x_{-1}, x_0, -\frac{x_{-2}}{x_{-2}y_{-1}z_0 + 1}, -x_{-1}(x_{-1}y_0z_{-2} + 1), -\frac{x_0}{x_0y_{-2}z_{-1} + 1} \right\}, \\ \{y_n\}_{n=-2}^{+\infty} &= \left\{ y_{-2}, y_{-1}, y_0, -\frac{y_{-2}}{x_0y_{-2}z_{-1} + 1}, -y_{-1}(x_{-2}y_{-1}z_0 + 1), -\frac{y_0}{x_{-1}y_0z_{-2} + 1} \right\}. \\ \{z_n\}_{n=-2}^{+\infty} &= \left\{ z_{-2}, z_{-1}, z_0, -\frac{z_{-2}}{x_{-1}y_0z_{-2} + 1}, -z_{-1}(x_0y_{-2}z_{-1} + 1), -\frac{z_0}{x_{-2}y_{-1}z_0 + 1} \right\}. \end{aligned}$$

8 The System

$$x_{n+1} = \frac{x_{n-2}}{1-x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{1-y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{1-z_{n-2}x_{n-1}y_n}$$

In this section, we study the solution of the following system of difference equations.

$$x_{n+1} = \frac{x_{n-2}}{1-x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{1-y_{n-2}z_{n-1}x_n}, \quad z_{n+1} = \frac{z_{n-2}}{1-z_{n-2}x_{n-1}y_n} \quad (8)$$

where $n \in N_0$ and the initial conditions are arbitrary nonzero real numbers.

The following theorem is devoted to the form of the solutions of system (7).

Theorem 7. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (7). Then for $n = 0, 1, 2, \dots$, we have the following formulas

$$\begin{aligned} x_{3n-2} &= -x_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (3i)x_{-2}y_{-1}z_0)}{(-1 + (3i+1)x_{-2}y_{-1}z_0)}, & x_{3n-1} &= x_{-1} \prod_{i=0}^{n-1} \frac{(-1 + (3i+1)x_{-1}y_0z_{-2})}{(-1 + (3i+2)x_{-1}y_0z_{-2})}, \\ x_{3n} &= x_0 \prod_{i=0}^{n-1} \frac{(-1 + (3i+2)x_0y_{-2}z_{-1})}{(-1 + (3i+3)x_0y_{-2}z_{-1})}, \\ y_{3n-2} &= -y_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (3i)x_0y_{-2}z_{-1})}{(-1 + (3i+1)x_0y_{-2}z_{-1})}, & y_{3n-1} &= y_{-1} \prod_{i=0}^{n-1} \frac{(-1 + (3i+1)x_{-2}y_{-1}z_0)}{(-1 + (3i+2)x_{-2}y_{-1}z_0)}, \\ y_{3n} &= y_0 \prod_{i=0}^{n-1} \frac{(-1 + (3i+2)x_{-1}y_0z_{-2})}{(-1 + (3i+3)x_{-1}y_0z_{-2})}, \\ z_{3n-2} &= -z_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (3i)x_{-1}y_0z_{-2})}{(-1 + (3i+1)x_{-1}y_0z_{-2})}, & z_{3n-1} &= z_{-1} \prod_{i=0}^{n-1} \frac{(-1 + (3i+1)x_0y_{-2}z_{-1})}{(-1 + (3i+2)x_0y_{-2}z_{-1})}, \\ z_{3n} &= z_0 \prod_{i=0}^{n-1} \frac{(-1 + (3i+2)x_{-2}y_{-1}z_0)}{(-1 + (3i+3)x_{-2}y_{-1}z_0)}, \end{aligned}$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n - 1$.

$$\begin{aligned} x_{3n-5} &= -x_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_{-2}y_{-1}z_0)}{(-1 + (3i+1)x_{-2}y_{-1}z_0)}, & x_{3n-4} &= x_{-1} \prod_{i=0}^{n-2} \frac{(-1 + (3i+1)x_{-1}y_0z_{-2})}{(-1 + (3i+2)x_{-1}y_0z_{-2})}, \\ x_{3n-3} &= x_0 \prod_{i=0}^{n-2} \frac{(-1 + (3i+2)x_0y_{-2}z_{-1})}{(-1 + (3i+3)x_0y_{-2}z_{-1})}, \\ y_{3n-5} &= -y_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_0y_{-2}z_{-1})}{(-1 + (3i+1)x_0y_{-2}z_{-1})}, & y_{3n-4} &= y_{-1} \prod_{i=0}^{n-2} \frac{(-1 + (3i+1)x_{-2}y_{-1}z_0)}{(-1 + (3i+2)x_{-2}y_{-1}z_0)}, \\ y_{3n-3} &= y_0 \prod_{i=0}^{n-2} \frac{(-1 + (3i+2)x_{-1}y_0z_{-2})}{(-1 + (3i+3)x_{-1}y_0z_{-2})}, \end{aligned}$$

$$z_{3n-5} = -z_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_{-1}y_0z_{-2})}{(-1 + (3i+1)x_{-1}y_0z_{-2})}, \quad z_{3n-4} = z_{-1} \prod_{i=0}^{n-2} \frac{(-1 + (3i+1)x_0y_{-2}z_{-1})}{(-1 + (3i+2)x_0y_{-2}z_{-1})},$$

$$z_{3n-3} = z_0 \prod_{i=0}^{n-2} \frac{(-1 + (3i+2)x_{-2}y_{-1}z_0)}{(-1 + (3i+3)x_{-2}y_{-1}z_0)},$$

It follows from Eq.(7) that

$$x_{3n-2} = \frac{x_{3n-5}}{1 + x_{3n-5}y_{3n-4}z_{3n-3}}$$

$$= \frac{-x_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1+(-x_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)})(y_{-1} \prod_{i=0}^{n-2} \frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)})(z_0 \prod_{i=0}^{n-2} \frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)})}$$

$$= \frac{-x_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1 - x_{-2}y_{-1}z_0 \prod_{i=0}^{n-2} ((\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)})(\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)})(\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)}))}$$

$$= \frac{-x_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1 - x_{-2}y_{-1}z_0 \prod_{i=0}^{n-2} (\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)})}$$

$$= -x_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_{-2}y_{-1}z_0)}{(-1 + (3i+1)x_{-2}y_{-1}z_0)} \frac{1}{1 + (\frac{x_{-2}y_{-1}z_0}{(-1+(3n-3)x_{-2}y_{-1}z_0)})}$$

$$= -x_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_{-2}y_{-1}z_0)}{(-1 + (3i+1)x_{-2}y_{-1}z_0)} (\frac{(-1 + (3n-3)x_{-2}y_{-1}z_0)}{(-1 + (3n-3)x_{-2}y_{-1}z_0) + x_{-2}y_{-1}z_0})$$

$$= -x_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_{-2}y_{-1}z_0)}{(-1 + (3i+1)x_{-2}y_{-1}z_0)} (\frac{(-1 + (3n-3)x_{-2}y_{-1}z_0)}{(-1 + (3n-2)x_{-2}y_{-1}z_0)})$$

Then, we see that

$$x_{3n-2} = -x_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (3i)x_{-2}y_{-1}z_0)}{(-1 + (3i+1)x_{-2}y_{-1}z_0)}$$

Also, we see from Eq.(1) that

$$\begin{aligned}
 y_{3n-2} &= \frac{y_{3n-5}}{1 + y_{3n-5}z_{3n-4}x_{3n-3}} \\
 &= \frac{-y_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}}{1+(-y_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})})(z_{-1} \prod_{i=0}^{n-2} \frac{(-1+(3i+1)x_0y_{-2}z_{-1})}{(-1+(3i+2)x_0y_{-2}z_{-1})})(x_0 \prod_{i=0}^{n-2} \frac{(-1+(3i+2)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})})} \\
 &= \frac{-y_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}}{1 - x_0y_{-2}z_{-1} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})}} \\
 &= -y_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_0y_{-2}z_{-1})}{(-1 + (3i + 1)x_0y_{-2}z_{-1})} \left(\frac{1}{1 + \frac{x_0y_{-2}z_{-1}}{-1+(3n-3)x_0y_{-2}z_{-1}}} \right) \\
 &= -y_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_0y_{-2}z_{-1})}{(-1 + (3i + 1)x_0y_{-2}z_{-1})} \left(\frac{-1 + (3n - 3)x_0y_{-2}z_{-1}}{-1 + (3n - 3)x_0y_{-2}z_{-1} + x_0y_{-2}z_{-1}} \right) \\
 &= -y_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_0y_{-2}z_{-1})}{(-1 + (3i + 1)x_0y_{-2}z_{-1})} \left(\frac{-1 + (3n - 3)x_0y_{-2}z_{-1}}{-1 + (3n - 2)x_0y_{-2}z_{-1}} \right)
 \end{aligned}$$

Then, we see that

$$y_{3n-2} = -y_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (3i)x_0y_{-2}z_{-1})}{(-1 + (3i + 1)x_0y_{-2}z_{-1})}$$

Finally, we see that

$$\begin{aligned}
 z_{3n-2} &= \frac{z_{3n-5}}{1 + z_{3n-5}x_{3n-4}y_{3n-3}} \\
 &= \frac{-z_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}}{1+(-z_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})})(x_{-1} \prod_{i=0}^{n-2} \frac{(-1+(3i+1)x_{-1}y_0z_{-2})}{(-1+(3i+2)x_{-1}y_0z_{-2})})(y_0 \prod_{i=0}^{n-2} \frac{(-1+(3i+2)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})})} \\
 &= \frac{-z_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}}{1 - x_{-1}y_0z_{-2} \prod_{i=0}^{n-2} \frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})}} \\
 &= -z_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_{-1}y_0z_{-2})}{(-1 + (3i + 1)x_{-1}y_0z_{-2})} \left(\frac{1}{1 + \frac{x_{-1}y_0z_{-2}}{-1+(3n-3)x_{-1}y_0z_{-2}}} \right) \\
 &= -z_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (3i)x_{-1}y_0z_{-2})}{(-1 + (3i + 1)x_{-1}y_0z_{-2})} \left(\frac{-1 + (3n - 3)x_{-1}y_0z_{-2}}{-1 + (3n - 2)x_{-1}y_0z_{-2}} \right)
 \end{aligned}$$

Then,

$$z_{3n-2} = -z_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (3i)x_{-1}y_0z_{-2})}{(-1 + (3i+1)x_{-1}y_0z_{-2})}$$

This completes the proof

8.1 Numerical Examples

For confirming the results of this section, we consider the following numerical example which represent solutions to the previous systems.

Example 1. We consider interesting numerical example for the difference equations system (1) with the initial conditions $x_{-2} = 13$, $x_{-1} = 0.4$, $x_0 = 3$, $y_{-2} = 0.5$, $y_{-1} = 7$, $y_0 = 3.7$, $z_{-2} = 0.9$, $z_{-1} = 17$ and $z_0 = 0.72$. (See Fig. 1).

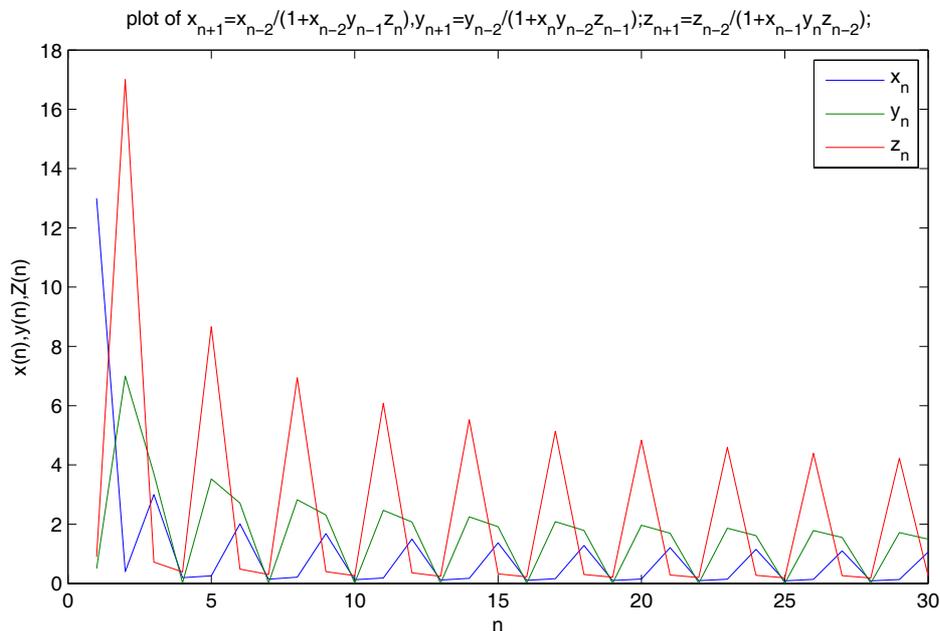


Figure 1.

Example 2. We put the initial conditions for system (2) as follows: $x_{-2} = 1.3$, $x_{-1} = -0.4$, $x_0 = 0.3$, $y_{-2} = 0.5$, $y_{-1} = 0.1$, $y_0 = -0.7$, $z_{-2} = -0.9$, $z_{-1} = 0.7$ and $z_0 = 0.2$. (See Fig. 2).

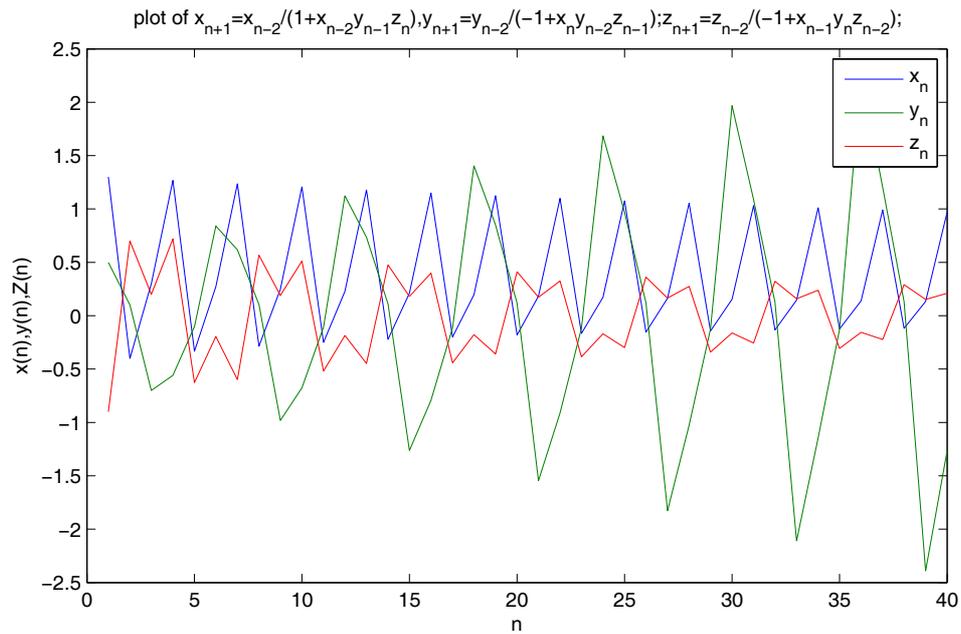


Figure 2.

Example 3. For the difference equations system (3) where the initial conditions $x_{-2} = 1.3$, $x_{-1} = 0.4$, $x_0 = 0.3$, $y_{-2} = 0.25$, $y_{-1} = 0.1$, $y_0 = 0.7$, $z_{-2} = 0.9$, $z_{-1} = 0.7$ and $z_0 = 0.2$. (See Fig. 3).

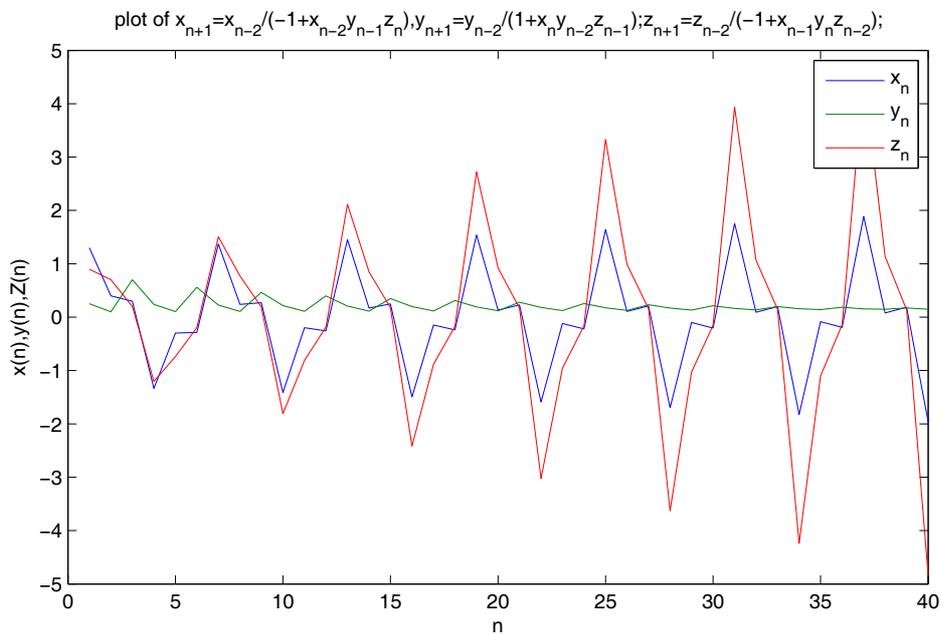


Figure 3.

Example 4. We assume $x_{-2} = 1.3$, $x_{-1} = 0.4$, $x_0 = 0.3$, $y_{-2} = 0.25$, $y_{-1} = 0.1$, $y_0 = 0.7$, $z_{-2} = 0.9$, $z_{-1} = 0.7$ and $z_0 = 0.2$ for system (4) see Fig. 4.

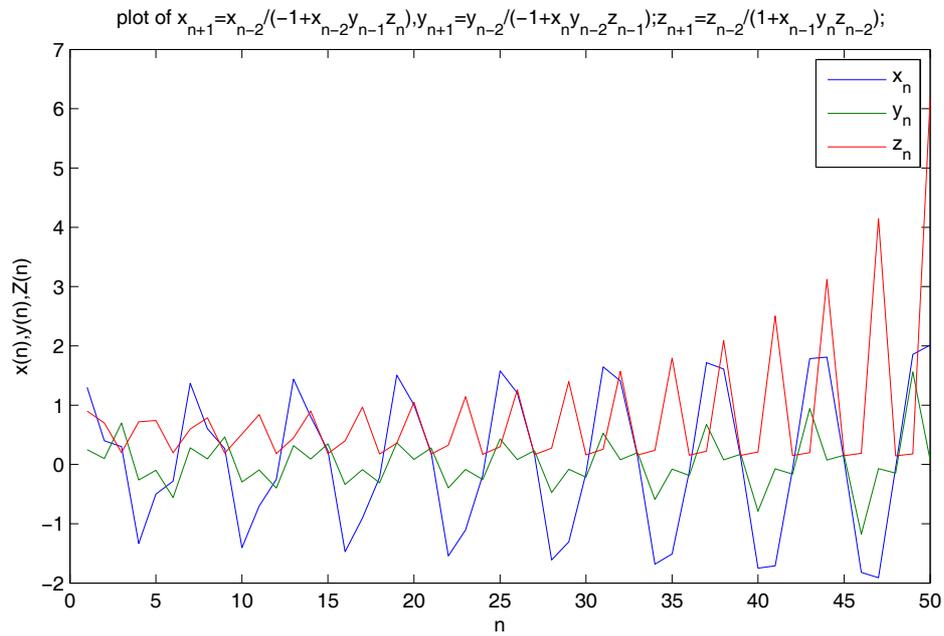


Figure 4.

Example 5. See Fig. 5, if we take system (5) with $x_{-2} = 3$, $x_{-1} = -0.4$, $x_0 = 2$, $y_{-2} = -0.5$, $y_{-1} = 0.9$, $y_0 = 0.7$, $z_{-2} = 0.19$, $z_{-1} = -0.4$ and $z_0 = 0.1$.

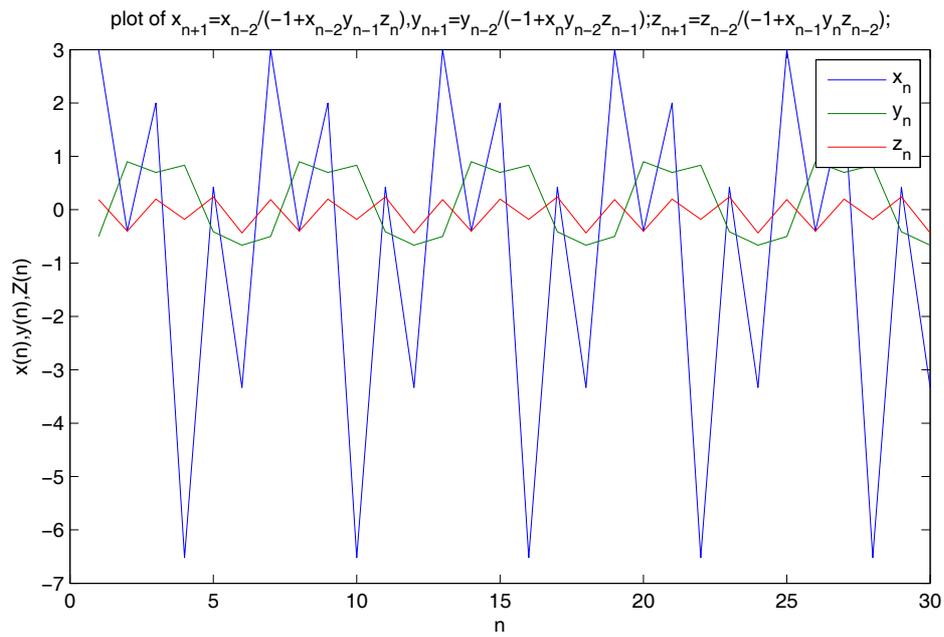


Figure 5.

Example 6. See Fig. 6, if we consider system (6) with $x_{-2} = -9$, $x_{-1} = 0.4$, $x_0 = -2$, $y_{-2} = 0.2$, $y_{-1} = 0.7$, $y_0 = 1.8$, $z_{-2} = 9$, $z_{-1} = -0.4$ and $z_0 = -2$.

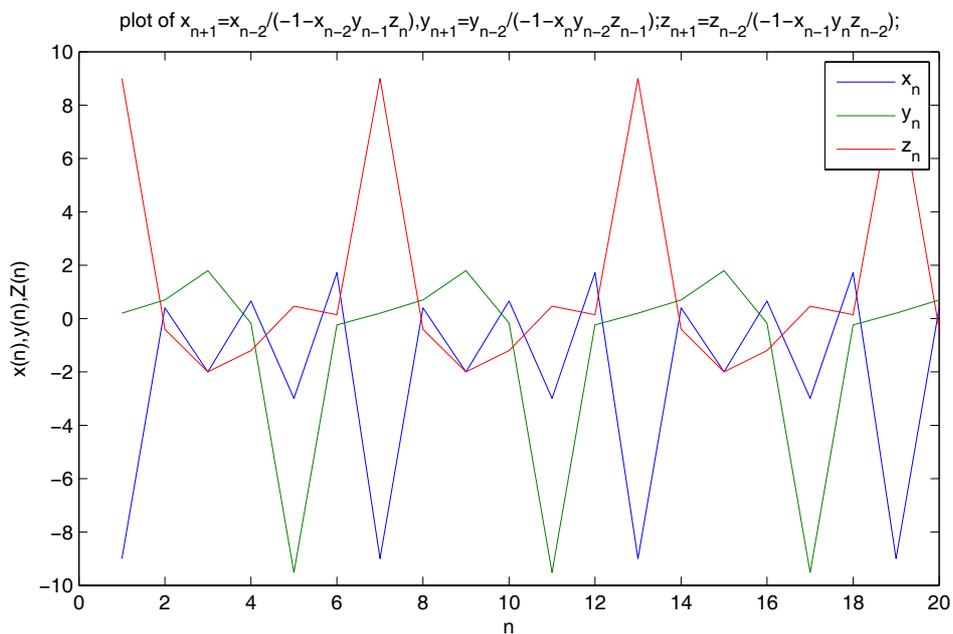


Figure 6.

Example 7. We take the difference equations system (7) with the initial conditions $x_{-2} = 9$, $x_{-1} = 4$, $x_0 = 2$, $y_{-2} = 3$, $y_{-1} = 7$, $y_0 = 18$, $z_{-2} = 11$, $z_{-1} = -4$ and $z_0 = 5$. (See Fig. 7).

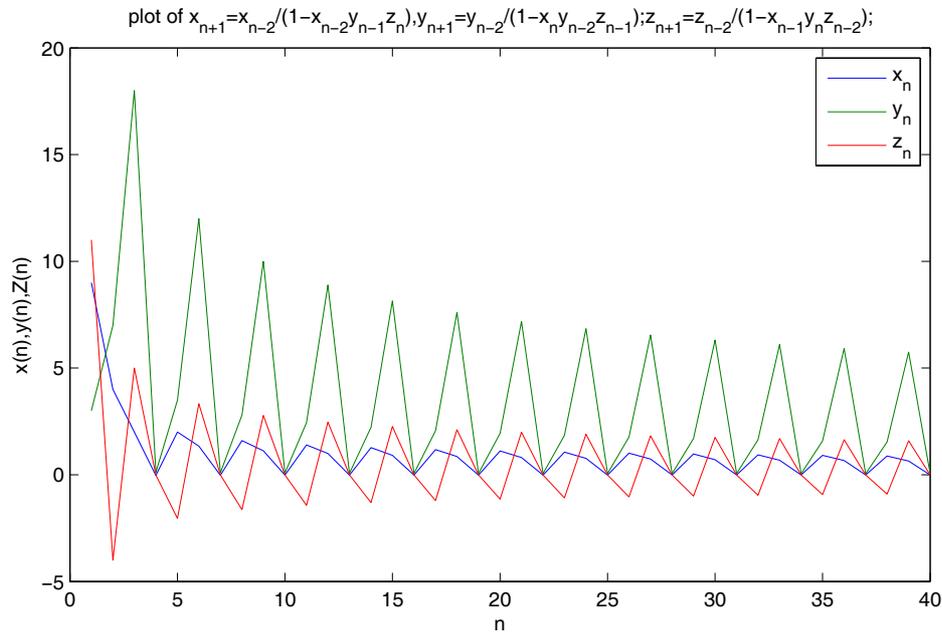


Figure 7.

References

- [1] R. P. Agarwal and E. M. Elsayed, On the solution of fourth-order rational recursive sequence, *Advanced Studies in Contemporary Mathematics*, 20 (4), (2010), 525–545.
- [2] A. M. Ahmed and A.M. Youssef, A solution form of a class of higher-order rational difference equations, *Journal of the Egyptian Mathematical Society*, 21 (3) (2013), 248-253.
- [3] M. Avotina, On three second-order rational difference equations with period-two solutions, *International Journal of Difference Equations*, 9 (1) (2014), 23–35.
- [4] N. Battaloglu, C. Cinar and I. Yalçınkaya, The dynamics of the difference equation, *ARS Combinatoria*, 97 (2010), 281-288.
- [5] E. Camouzis, and E. Chatterjee, On the dynamics of $x_{n+1} = \frac{\delta x_{n-2} + x_{n-3}}{A + x_{n-3}}$, *Journal of Mathematical Analysis and Applications*, 331 (2007), 230-239.

- [6] C. Cinar, I. Yalcinkaya and R. Karatas, On the positive solutions of the difference equation system $x_{n+1} = \frac{m}{y_n}$, $y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}}$, Journal of Institute of Mathematics & Computer Sciences, 18 (2005), 135-136.
- [7] D. Clark, and M. R. S. Kulenovic, A coupled system of rational difference equations, Computers & Mathematics with Applications, 43 (2002) 849-867.
- [8] X. Deng, X. Liu, Y. Zhang, Periodic and subharmonic solutions for a 2nth-order difference equation involving p-Laplacian, Indagationes Mathematicae New Series, 24 (3) (2013), 613-625.
- [9] Q. Din, Qualitative nature of a discrete predator-prey system, Contemporary Methods in Mathematical Physics and Gravitation, 1 (1) (2015), 27-42.
- [10] Q. Din, On a system of rational difference equation, Demonstratio Mathematica, Vol. XLVII (2) (2014), 324-335.
- [11] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Global behavior of the solutions of difference equation, Advances in Difference Equations, 2011, (2011): 28.
- [12] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Some properties and expressions of solutions for a class of nonlinear difference equation, Utilitas Mathematica, 87 (2012), 93-110.
- [13] M. M. El-Dessoky, and E. M. Elsayed, On the solutions and periodic nature of some systems of rational difference equations, Journal of Computational Analysis and Applications, 18 (2) (2015), 206-218.
- [14] E. M. Elsayed, On the global attractivity and the solution of recursive sequence, Studia Scientiarum Mathematicarum Hungarica, 47 (3) (2010), 401-418.
- [15] E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 982309, 17 pages.
- [16] E. M. Elsayed, On the solution of some difference equations, European Journal of Pure and Applied Mathematics, 4 (3) (2011), 287-303.
- [17] E. M. Elsayed, Solutions of rational difference system of order two, Mathematical and Computer Modelling, 55 (2012), 378-384.
- [18] E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, Journal of Computational Analysis and Applications, 15 (1) (2013), 73-81.

- [19] E. M. Elsayed, On the solutions and periodic nature of some systems of difference equations, *International Journal of Biomathematics*, 7 (6) (2014), 1450067, (26 pages).
- [20] E. M. Elsayed, Solution for systems of difference equations of rational form of order two, *Computational and Applied Mathematics*, 33 (3) (2014), 751-765.
- [21] E. M. Elsayed and C. Cinar, On the solutions of some systems of Difference equations, *Utilitas Mathematica*, 93 (2014), 279-289.
- [22] E. M. Elsayed, On a system of two nonlinear difference equations of order two, *Proceedings of the Jangjeon Mathematical Society*, 18 (3) (2015), 353-368.
- [23] E. M. Elsayed, M. M. El-Dessoky and Asim Asiri, Dynamics and behavior of a second order rational difference equation, *Journal of Computational Analysis and Applications*, 16 (4) (2014), 794-807.
- [24] E. M. Elsayed and H. A. El-Metwally, On the solutions of some nonlinear systems of difference equations, *Advances in Difference Equations* 2013, 2013:16, Published: 7 June 2013.
- [25] A. Gelisken and M. Kara, Some General Systems of Rational Difference Equations, *Journal of Difference Equations*, Volume 2015, Article ID 396757, 7 pages.
- [26] A. S. Kurbanli, C. Cinar and I. Yalçınkaya, On the behavior of positive solutions of the system of rational difference equations, *Mathematical and Computer Modelling*, 53 (2011), 1261-1267.
- [27] A. Kurbanli, C. Cinar and M. Erdoğan, On the behavior of solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n - 1}$, $y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n - 1}$, $z_{n+1} = \frac{x_n}{z_{n-1}y_n}$, *Applied Mathematics*, 2 (2011), 1031-1038.
- [28] H. Ma and H. Feng, On Positive Solutions for the Rational Difference Equation Systems, *International Scholarly Research Notices*, Volume 2014, Article ID 857480, 4 pages.
- [29] G. Papaschinopoulos and C. J. Schinas, On the dynamics of two exponential type systems of difference equations, *Computers & Mathematics with Applications*, 64 (7) (2012), 2326-2334.
- [30] H. Sedaghat, *Nonlinear Difference Equations, Theory with Applications to Social Science Models*, Kluwer Academic Publishers, Dordrecht, 2003.
- [31] S. Stević, Domains of undefinable solutions of some equations and systems of difference equations, *Applied Mathematics and Computation*,

219 (2013), 11206-11213.

[32] S. Stevic, M. A. Alghamdi, A. Alotaibi, and E. M. Elsayed, Solvable product-type system of difference equations of second order, *Electronic Journal of Differential Equations*, Vol. 2015 (2015) (169), 1–20.

[33] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, *Mathematical and Computer Modelling*, 55 (2012), 1987—1997.

APPROXIMATION OF SOLUTIONS OF THE INHOMOGENEOUS GAUSS DIFFERENTIAL EQUATIONS BY HYPERGEOMETRIC FUNCTION

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ABSTRACT. In this paper, we solve the inhomogeneous Gauss differential equation and apply this result to estimate the error bound occurring when an analytic function is approximated by an appropriate hypergeometric function.

1. INTRODUCTION

More than a half century ago, Ulam [22] posed the famous Ulam stability problem which was partially solved by Hyers [7] in the framework of Banach spaces. The Hyers' theorem was generalized by Aoki [4] for additive mappings. In 1978, Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy difference inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \quad (\varepsilon \geq 0, p \in [0, 1)).$$

Since then, the stability problems of various functional equations have been studied by many authors (see [1, 6, 8, 9, 13, 15, 17, 18, 19, 20]).

Alsina and Ger [3] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved that if a differentiable function $f : I \rightarrow \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leq \epsilon$, where I is an open subinterval of \mathbb{R} , then there exists a solution $f_0 : I \rightarrow \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leq 3\epsilon$ for any $t \in I$. From then on, many research papers about the Hyers-Ulam stability of differential equations have appeared in the literature, see [2, 5, 10, 11, 12, 21, 23] for instance.

The form of the homogeneous Gauss differential equation has the form

$$x(1 - x)y'' + [r - (1 + s + t)x]y' - sty = 0. \tag{1.1}$$

It is easy to see that

$$y_1 = 1 + \frac{st}{1!r}x + \frac{(st)(s + 1)(t + 1)}{2!r(r + 1)}x^2 + \frac{(st)(s + 1)(s + 2)(t + 1)(t + 2)}{3!r(r + 1)(r + 2)}x^3 + \dots$$

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and

$$y_2 = x^{1-r} \left[1 + \frac{(s-r+1)(t-r+1)}{1!(2-r)}x + \frac{(s-r+1)(s-r+2)(t-r+1)(t-r+2)}{2!(2-r)(3-r)}x^2 + \frac{(s-r+1)(s-r+2)(s-r+3)(t-r+1)(t-r+2)(t-r+3)}{3!(2-r)(3-r)(4-r)}x^3 + \dots \right]$$

are a fundamental set of solutions of equation (1.1) (if $r \neq 1$). The series y_1 known the hypergeometric function is convergent for $|x| < 1$ and is represented by

$$y_1 = F(s, t, r, x).$$

Note that

$$y_2 = x^{1-r} F(s-r+1, t-r+1, 2-r, x)$$

is of the same type. Thus the general solution is

$$y_c = c_1 y_1 + c_2 y_2 = c_1 F(s, t, r, x) + c_2 x^{1-r} F(s-r+1, t-r+1, 2-r, x).$$

2. INHOMOGENEOUS GAUSS DIFFERENTIAL EQUATION

In this section, we consider the solution of inhomogeneous Gauss differential equation of the form

$$x(1-x)y'' + [r - (1+s+t)x]y' - sty = \sum_{m=0}^{+\infty} a_m x^m, \tag{2.1}$$

where the coefficients a_n 's of the power series are given such that the radius of convergence is positive.

Theorem 2.1. *Assume that the radius of convergence of the power series $\sum_{m=0}^{+\infty} a_m x^m$ is $R_0 > 0$ and*

$$R_1 = \lim_{k \rightarrow \infty} \left| \frac{c_k}{c_{k+1}} \right| > 0. \tag{2.2}$$

Let ρ be a positive number defined by $\rho = \min\{1, R_0, R_1\}$. Then every solution $y : (-\rho, \rho) \rightarrow \mathbb{C}$ of differential equation (2.1) can be expressed by

$$y(x) = y_c(x) + \sum_{m=1}^{+\infty} c_m x^m, \tag{2.3}$$

where $c_1 = \frac{1}{r} a_0$ and

$$c_m = \frac{a_{m-1}}{m(m-1+r)} + \frac{1}{m!} \sum_{i=1}^{m-1} a_{m-i-1} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \prod_{j=1}^i (m+s-j)(m+t-j) \tag{2.4}$$

for any $m \in \{2, 3, \dots\}$.

Proof. We will show that each function $y : (-\rho, \rho) \rightarrow \mathbb{C}$ defined by (2.3) is a solution of the inhomogeneous Gauss differential equation (2.1), where y_c is a solution of homogeneous Gauss differential equation (1.1). For this purpose, it is only necessary to show that $y_p(x) = \sum_{m=1}^{\infty} c_m x^m$ satisfies differential equation (2.1). Therefore, letting $y_p(x) = \sum_{m=1}^{\infty} c_m x^m$ in differential equation (2.1), we obtain

$$\begin{aligned} \sum_{m=1}^{+\infty} m(m+1)c_{m+1}x^m + r \sum_{m=0}^{+\infty} (m+1)c_{m+1}x^m - \sum_{m=2}^{+\infty} m(m-1)c_mx^m \\ - (1+s+t) \sum_{m=1}^{+\infty} mc_mx^m - st \sum_{m=1}^{+\infty} c_mx^m = \sum_{m=0}^{+\infty} a_mx^m. \end{aligned}$$

Hence

$$rc_1 + \sum_{m=1}^{+\infty} [(m+1)(m+r)c_{m+1} - (m+s)(m+t)c_m]x^m = \sum_{m=0}^{+\infty} a_mx^m.$$

Therefore, we get $c_1 = \frac{1}{r}a_0$ and

$$c_{m+1} = \frac{1}{(m+1)(m+r)}a_m + \frac{(m+s)(m+t)}{(m+1)(m+r)}c_m, \quad (m = 1, 2, \dots).$$

By some manipulations, we obtain

$$\begin{aligned} c_m &= \frac{a_{m-1}}{m(m-1+r)} \\ &+ \frac{1}{m!} \sum_{i=1}^{m-1} a_{m-i-1} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \prod_{j=1}^i (m+s-j)(m+t-j) \end{aligned} \tag{2.5}$$

for any $m \in \{2, 3, \dots\}$. The condition (2.2) implies that the radius of convergence of $y_p(x) = \sum_{m=1}^{+\infty} c_mx^m$ is R_1 . By using the ratio test, we can easily show that the radius of convergence of y_c is 1. Thus y is certainly defined on $(-\rho, \rho)$. \square

Corollary 2.2. *Assume that the assumptions of Theorem 2.1 hold. Then there exists $C > 0$ such that*

$$\begin{aligned} \sum_{m=1}^{+\infty} c_mx^m &\leq \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)}x^m \\ &+ \sum_{i=1}^{+\infty} \sum_{m=2}^{+\infty} \frac{Ca_{m-2}}{(m+i-1)^2} \prod_{j=0}^i \left(1 - \frac{-st}{(m+i-j-1)(m+i-j+s+t-1)}\right) x^{m+i-1}. \end{aligned}$$

Proof. Since there exists a constant $C > 0$ with

$$\frac{1}{m!} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \leq \frac{C}{m^2} \prod_{j=0}^i \frac{1}{(m-j)(m-j+s+t)}$$

for any $m = 2, 3, \dots$ and for any $i = 1, 2, \dots$, it follows from (2.5) that

$$\begin{aligned} \sum_{m=1}^{+\infty} c_m x^m &= c_1 x + \sum_{m=2}^{+\infty} c_m x^m = \frac{1}{r} a_0 x + \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m \\ &+ \sum_{m=2}^{+\infty} \frac{1}{m!} \sum_{i=1}^{m-1} a_{m-i-1} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \prod_{j=0}^i (m+s-j)(m+t-j) x^m \\ &\leq \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{m=2}^{+\infty} \sum_{i=1}^{m-1} \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^i \frac{(m+s-j)(m+t-j)}{(m-j)(m-j+s+t)} x^m \\ &= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{m=2}^{+\infty} \sum_{i=1}^{m-1} \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^i \left(1 - \frac{-st}{(m-j)(m-j+s+t)}\right) x^m \\ &= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{i=1}^{+\infty} \sum_{m=i+1}^{+\infty} A_{mi} x^m \\ &= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{i=1}^{+\infty} \sum_{m=2}^{+\infty} A_{m+i-1,i} x^{m+i-1}, \end{aligned}$$

where we define

$$A_{mi} := \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^i \left(1 - \frac{-st}{(m-j)(m-j+s+t)}\right)$$

for all $i = 1, 2, \dots$ and $m = 2, 3, \dots$. □

3. APPROXIMATION PROPERTY OF HYPERGEOMETRIC FUNCTION

In this section, we investigate an approximation property of hypergeometric functions. More precisely, we will prove that if an analytic function satisfies the condition (2.2), then it can be approximated by a hypergeometric function. Suppose that y is a given function expressed as a power series of the form

$$y(x) = \sum_{m=0}^{\infty} b_m x^m, \tag{3.1}$$

whose radius of convergence is $R_0 > 0$. Then we obtain

$$\begin{aligned} x(1-x)y'' + [r - (1+s+t)x]y' - sty &= \sum_{m=0}^{\infty} [(m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m] x^m \\ &= \sum_{m=0}^{\infty} a_m x^m, \end{aligned} \tag{3.2}$$

where we define

$$a_m := (m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m \tag{3.3}$$

for all $m \in \{0, 1, 2, 3, \dots\}$.

Lemma 3.1. *If the a_m 's, the b_m 's and the c_m 's are as defined in (3.3), (3.1) and (2.4), then*

$$c_m = b_m - \frac{b_0}{m!} \prod_{j=1}^m \frac{1}{m-j+r} \prod_{j=1}^{m-1} (m+s-j)(m+t-j) \tag{3.4}$$

for all $m \in \{0, 1, 2, 3, \dots\}$.

Proof. The proof is clear by induction on m . For $m = 1$ and by (3.3) we have

$$c_1 = \frac{a_0}{r} = \frac{1}{r}(rb_1 - stb_0) = b_1 - \frac{st}{r}b_0. \tag{3.5}$$

Assume now that formula (3.3) is true for some m . It follows from (2.4), (3.3) and (3.4) that

$$\begin{aligned} c_{m+1} &= \frac{a_m}{(m+1)(m+r)} + \frac{(m+s)(m+t)}{(m+1)(m+r)}c_m \\ &= \frac{1}{(m+1)(m+r)} \left((m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m \right) \\ &\quad + \frac{(m+s)(m+t)}{(m+1)(m+r)} \left(b_m - \frac{b_0}{m!} \prod_{j=1}^m \frac{1}{m-j+r} \prod_{j=1}^{m-1} (m+s-j)(m+t-j) \right) \\ &= b_{m+1} - \frac{b_0}{(m+1)!} \prod_{j=1}^{m+1} \frac{1}{m+1-j+r} \prod_{j=1}^m (m+1+s-j)(m+1+t-j), \end{aligned}$$

as desired. □

Theorem 3.2. *Let R and R_0 be positive constants with $R < R_0$. Assume that $y : (-R, R) \rightarrow \mathbb{C}$ is a function of the form (3.1) whose radius of convergence is R_1 . Also, b_m 's and c_m 's are given by (3.3) and (3.4), respectively. If $R < \min\{1, R_0, R_1\}$, then there exist a hypergeometric function $y_h : (-R, R) \rightarrow \mathbb{C}$ and a constant $d > 0$ such that $|y(x) - y_h(x)| \leq d \frac{x}{1-x}$ for all $x \in (-R, R)$.*

Proof. We assume that y can be represented by a power series (3.1) whose radius of convergence is $R < R_0$. So

$$x(1-x) \sum_{m=2}^{+\infty} m(m-1)b_m x^{m-2} + [r - (1+s+t)x] \sum_{m=1}^{+\infty} mb_m x^{m-1} - st \sum_{m=0}^{+\infty} mb_m x^m$$

is also a power series whose radius of convergence is R_0 , more precisely, in view of (3.2) and (3.3), we have

$$\begin{aligned} x(1-x) \sum_{m=0}^{+\infty} m(m-1)b_m x^{m-2} + [r - (1+s+t)x] \sum_{m=0}^{+\infty} mb_m x^{m-1} \\ - st \sum_{m=0}^{+\infty} mb_m x^m = \sum_{m=0}^{+\infty} a_m x^m \end{aligned}$$

for all $x \in (-R, R)$. Since the power series $\sum_{m=0}^{+\infty} a_m x^m$ is absolutely convergent on its interval of convergence, which includes the interval $[-R, R]$ and the power series $\sum_{m=0}^{+\infty} |a_m x^m|$ is continuous on $[-R, R]$. So there exists a constant $d_1 > 0$ with

$$\sum_{m=0}^n |a_m x^m| \leq d_1$$

for all integers $n \geq 0$ and for any $x \in (-R, R)$.

On the other hand, since

$$\sum_{k=1}^{+\infty} \left| \frac{-st}{(m-k-1)(m-k-1+t+s)} \right| \leq \frac{st\pi^2}{6} =: d_2, \quad (m = 2, 3, \dots),$$

we have

$$\left| \prod_{k=1}^{+\infty} \left(1 - \frac{-st}{(m-k-1)(m-k-1+t+s)} \right) \right| \leq d_2, \quad (m = 2, 3, \dots)$$

(see [16, Theorem 6.6.2]). Hence, substituting $i - j$ for k in the above infinite product, there exists a constant d_3 with

$$\left| \prod_{j=0}^i \left(1 - \frac{-st}{(m-i-j-1)(m-i-j-1+t+s)} \right) \right| \leq d_3$$

for all $i = 1, 2, \dots$ and $m = 2, 3, \dots$. Therefore, it follows Lemma 2.2 that

$$\left| \sum_{m=0}^{\infty} c_m x^m \right| \leq d_1 d_3 \frac{x}{1-x} \tag{3.6}$$

for all $x \in (-R_0, R_0)$. This completes the proof of our theorem. □

Corollary 3.3. *Assume that R and R_0 are positive constants with $R < R_0$. Let $y : (R, R_0) \rightarrow \mathbb{C}$ be a function which can be represented by a power series of the form (3.1) whose radius of convergence is R_0 . Moreover, assume that there exists a positive number R_1 satisfying the condition (2.2) with b_m 's and c_m 's given in (3.1) and Lemma 3.1. If $R < \min\{1, R_0, R_1\}$ then there exists a hypergeometric function $y_h : (-R, R) \rightarrow \mathbb{C}$ such that $|y(x) - y_h(x)| = O(x)$ as $x \rightarrow 0$.*

Example 3.4. *Now, we will introduce an example concerning the hypergeometric function for differential equation (2.1) with $st = \frac{1}{16}$. Given a constant R with $0 < R < 1$ and assume that a function $y : (-R, R) \rightarrow \mathbb{C}$ can be expressed as a power series of the form (3.1), where*

$$b_m = \begin{cases} 0, & m=0 \\ \frac{1}{4^m}, & m \geq 1. \end{cases}$$

It is easy to see that the radius of convergence of the above power series is $R_1 = 4$. Since $b_0 = 0$ it follows from Lemma 3.1 that $c_m = b_m$ for each $m \in \{0, 1, 2, 3, \dots\}$. Moreover, there exists a positive constant R_1 such that the condition (2.2) is satisfied

$$R_1 = \lim_{k \rightarrow +\infty} \left| \frac{c_k}{c_{k+1}} \right| = \lim_{k \rightarrow +\infty} \left| \frac{b_k}{b_{k+1}} \right| = 4.$$

Now we assume $r = s = t = \frac{1}{4}$. Then we get

$$\begin{aligned} \sum_{m=0}^{+\infty} |a_m x^m| &\leq \frac{1}{16} + \frac{15}{64}|x| + \sum_{m=2}^{+\infty} \frac{4(m + \frac{1}{4})^2 - (m + 1)(m + \frac{1}{4})}{4^{m+2}} |x|^m \\ &\leq \frac{1}{16} + \frac{15}{64} + \sum_{m=2}^{+\infty} \frac{3m(m + \frac{1}{4})}{4^{m+2}} \\ &\leq \frac{1}{16} + \frac{15}{64} + \sum_{m=2}^{+\infty} \frac{1}{2^{m+2}} \leq \frac{1}{16} + \frac{15}{64} + \frac{1}{8} = \frac{27}{64} \end{aligned}$$

for all $x \in (-R, R)$. Since $R < \min\{1, R_0, R_1\} = 1$, we can conclude from (3.6) that there exists a solution function $y_h : (-R, R) \rightarrow \mathbb{C}$ of the Gauss differential equation (2.1) with $r = s = t = \frac{1}{4}$ satisfying $|y(x) - y_h(x)| \leq \frac{27}{64} \frac{x}{1-x}$ for all $x \in (-R, R)$.

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REFERENCES

- [1] M. Adam and S. Czerwik, *On the stability of the quadratic functional equation in topological spaces*, Banach J. Math. Anal. **1** (2007), 245–251.
- [2] Z. Ali, P. Kumam, K. Shah and A. Zada, *Investigation of Ulam stability results of a coupled system of nonlinear implicit fractional differential equations*, Math. **7** (2019), Art. No. 341.
- [3] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373–380.
- [4] T. Aoki, *On the stability of linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
- [5] E. Bicer and C. Tunc, *On the Hyers-Ulam stability of certain partial differential equations of second order*, Nonlinear Dyn. Syst. Theory **17** (2017), 150–157.
- [6] M. Eshaghi Gordji, C. Park and M. B. Savadkouhi, *The stability of a quartic type functional equation with the fixed point alterbative*, Fixed Point Theory **11** (2010), 265–272.
- [7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.
- [8] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.
- [9] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
- [10] S. Jung, *Legendre’s differential equation and its Hyers-Ulam stability*, Abst. Appl. Anal. **2007** (2007), Article ID 56419.
- [11] B. Kim and S. Jung, *Bessel’s differential equation and its Hyers-Ulam stability*, J. Inequal. Appl. **2007** (2007), Article ID 21640.
- [12] K. Liu, M. Fečkan, D. O’Regan and J. Wang, *Hyers-Ulam stability and existence of solutions for differential equations with Caputo-Fabrizio fractional derivative*, Math. **7** (2019), Art. No. 333.
- [13] S. Ostadbashi and M. Soleimaninia, *On Pexider difference for a Pexider cubic functional equation*, Math. Reports **18(68)** (2016), 151–162.
- [14] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.
- [15] J. M. Rassias, *On the Ulam stability of mixed type mappings on restricted domains*, J. Math. Anal. Appl. **276** (2002), 747–762.
- [16] M. Reed, *Fundamental Ideas of Analysis*, John Wiley and Sons, New York, 1998.

- [17] S. Shaghali, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Comput. Anal. Appl. **13** (2011), 1106–1114.
- [18] S. Shaghali, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.
- [19] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^* -homomorphisms*, J. Comput. Anal. Appl. **16** (2014), 964–973.
- [20] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^* -homomorphisms and J^* -derivations for a generalized Cauchy-Jensen equation*, J. Comput. Anal. Appl. **17** (2014), 125–134.
- [21] C. Tuğ, *A study of the stability and boundness of the solutions of nonlinear differential equations of the fifth order*, Indian J. Pure Appl. Math. **33** (2002), 519–529.
- [22] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley and Sons, New York, NY, USA, Science edition, 1964.
- [23] E. Zadrzynska and W. M. Zajackowski, *On stability of solutions to equations describing incompressible heat-conducting motions under Navier's boundary conditions*, Acta Appl. Math. **152** (2017), 147–170.

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ON TOPOLOGICAL ROUGH GROUPS

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ABSTRACT. In this paper, we give an introduction for rough groups and rough homomorphisms. Then we present some properties related to topological rough subgroups and rough subsets. Finally we construct the product of topological rough groups and give an illustrated example.

1. INTRODUCTION

In [2], Bagirmaz *et al.* introduced the concept of topological rough groups. They extended the notion of a topological group to include algebraic structures of rough groups. In addition, they presented some examples and properties.

The main purpose of this paper is to introduce some basic definitions and results about topological rough groups and topological rough subgroups. We also introduce the Cartesian product of topological rough groups.

This paper is as follows: Section 2 gives basic results and definitions on rough groups and rough homomorphisms. In Section 3, following results and definitions of [2], we give some more interesting and nice results about topological rough groups. Finally, in Section 4 we prove that the product of topological rough groups is a topological rough group. Further, an example is provided.

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2. ROUGH GROUPS AND ROUGH HOMOMORPHISMS

First, we give the definition of rough groups introduced by Biswas and Nanda in 1994 [3].

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Let (U, R) be an approximation space. For a subset $X \subseteq U$,

$$\overline{X} = \{[x]_R : [x]_R \cap X \neq \emptyset\}$$

and

$$\underline{X} = \{[x]_R : [x]_R \subseteq X\}.$$

Suppose that $*$ is a binary operation defined on U . We will use xy instead of $x * y$ for each composition of elements $x, y \in U$ as well as for composition of subsets XY , where $X, Y \subseteq U$.

Definition 2.1. [2] Let $G = (\underline{G}, \overline{G})$ be a rough set in the approximation space (U, R) . Then $G = (\underline{G}, \overline{G})$ is called a rough group if the following conditions are satisfied:

- (1) for all $x, y \in G$, $xy \in \overline{G}$ (closed);
- (2) for all $x, y, z \in \overline{G}$, $(xy)z = x(yz)$ (associative law);
- (3) for all $x \in G$, there exists $e \in \overline{G}$ such that $xe = ex = x$ (e is the rough identity element);
- (4) for all $x \in G$, there exists $y \in G$ such that $xy = yx = e$ (y is the rough inverse element of x . It is denoted as x^{-1}).

Definition 2.2. [2] A nonempty rough subset $H = (\underline{H}, \overline{H})$ of a rough group $G = (\underline{G}, \overline{G})$ is called a rough subgroup if it is a rough group itself.

A rough set $G = (\underline{G}, \overline{G})$ is a trivial rough subgroup of itself. Also the rough set $e = (e, \overline{e})$ is a trivial rough subgroup of the rough group G if $e \in G$.

Theorem 2.1. [2] A rough subset H is a rough subgroup of the rough group G if the two conditions are satisfied:

- (1) for all $x, y \in H$, $xy \in \overline{H}$;
- (2) for all $y \in H$, $y^{-1} \in H$.

Also, a rough normal subgroup can be defined. Let N be a rough subgroup of the rough group G . Then N is called a rough normal subgroup of G if for all $x \in G$, $xN = Nx$.

Definition 2.3. [4] Let (U_1, R_1) and (U_2, R_2) be two approximation spaces and $*, *'$ be two binary operations on U_1 and U_2 , respectively. Suppose that $G_1 \subseteq U_1$, $G_2 \subseteq U_2$ are rough groups. If the mapping $\varphi : \overline{G_1} \rightarrow \overline{G_2}$ satisfies $\varphi(x * y) = \varphi(x) *' \varphi(y)$ for all $x, y \in \overline{G_1}$, then φ is called a **rough homomorphism**.

Definition 2.4. [4] A rough homomorphism φ from a rough group G_1 to a rough group G_2 is called:

- (1) a rough epimorphism (or surjective) if $\varphi : \overline{G_1} \rightarrow \overline{G_2}$ is onto.
- (2) a rough embedding (or monomorphism) if $\varphi : \overline{G_1} \rightarrow \overline{G_2}$ is one-to-one.
- (3) a rough isomorphism if $\varphi : \overline{G_1} \rightarrow \overline{G_2}$ is both onto and one-to-one.

3. TOPOLOGICAL ROUGH GROUPS

We study a topological rough group, which has an ordinary topology on a rough group, i.e., a topology τ on \overline{G} induced a subspace topology τ_G on G . Suppose that (U, R) is an approximation space with a binary operation $*$ on U . Let G be a rough group in U .

Definition 3.1. [2] *A topological rough group is a rough group G with a topology τ_G on \overline{G} satisfying the following conditions:*

- (1) *the product mapping $f : G \times G \rightarrow \overline{G}$ defined by $f(x, y) = xy$ is continuous with respect to a product topology on $G \times G$ and the topology τ on G induced by τ_G ;*
- (2) *the inverse mapping $\iota : G \rightarrow G$ defined by $\iota(x) = x^{-1}$ is continuous with respect to the topology τ on G induced by τ_G .*

Elements in the topological rough group G are elements in the original rough set G with ignoring elements in approximations.

Example 3.1. *Let $U = \{\overline{0}, \overline{1}, \overline{2}\}$ be any group with 3 elements. Let $U/\mathcal{R} = \{\{\overline{0}, \overline{2}\}, \{\overline{1}\}\}$ be a classification of equivalent relation. Let $G = \{\overline{1}, \overline{2}\}$. Then $\underline{G} = \{\overline{1}\}$ and $\overline{G} = \{\overline{0}, \overline{1}, \overline{2}\} = U$. A topology on \overline{G} is $\tau_G = \{\varphi, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\}$ and the relative topology is $\tau = \{\varphi, G, \{\overline{1}\}, \{\overline{2}\}\}$. The conditions in Definition 3.1 are satisfied and hence G is a topological rough group.*

Example 3.2. *Let $U = \mathbb{R}$ and $U/\mathcal{R} = \{\{x : x \geq 0\}, \{x : x < 0\}\}$ be a partition of \mathbb{R} . Consider $G = \mathbb{R}^* = \mathbb{R} - 0$. Then G is a rough group with addition. It is also a topological rough group with the standard topology on \mathbb{R} .*

Example 3.3. *Consider $U = S_4$ the set of all permutations of four objects. Let $(*)$ be the multiplication operation of permutations. Let*

$$U/\mathcal{R} = \{E_1, E_2, E_3, E_4\}$$

be a classification of U , where

$$\begin{aligned} E_1 &= \{1, (12), (13), (14), (23), (24), (34)\}, \\ E_2 &= \{(123), (132), (142), (124), (134), (143), (234), (243)\}, \\ E_3 &= \{(1234), (1243), (1342), (1324), (1423), (1432)\}, \\ E_4 &= \{(12)(34), (13)(24), (14)(23)\}. \end{aligned}$$

Let $G = \{(12), (123), (132)\}$. Then $\overline{G} = E_1 \cup E_2$. Clearly, G is a rough group. Consider a topology on \overline{G} as $\tau_G = \{\varphi, \overline{G}, \{(12)\}, \{1, (123), (132)\}, \{1, (12), (123), (132)\}\}$. Then the relative topology on G is $\tau = \{\varphi, G, \{(12)\}, \{(123), (132)\}\}$. The conditions in Definition 3.1 are satisfied and hence G is a topological rough group.

Proposition 3.1. [2] *Let G be a topological rough group and fix $a \in G$. Then*

- (1) the mapping $L_a : G \rightarrow \overline{G}$ defined by $L_a(x) = ax$, is one-to-one and continuous for all $x \in G$.
- (2) the mapping $R_a : G \rightarrow \overline{G}$ defined by $R_a(x) = xa$, is one-to-one and continuous for all $x \in G$.
- (3) the inverse mapping $\iota : G \rightarrow G$ is a homeomorphism for all $x \in G$.

Proposition 3.2. [2] *Let G be a topological rough group. Then $G = G^{-1}$.*

Proposition 3.3. [2] *Let G be a topological rough group and $V \subseteq G$. Then V is open (resp. closed) if and only if V^{-1} is open (resp. closed).*

Proposition 3.4. [2] *Let G be a topological rough group and W be an open set in \overline{G} with $e \in W$. Then there exists an open set V with $e \in V$ such that $V = V^{-1}$ and $VV \subseteq W$.*

Proposition 3.5. [2] *Let G be a rough group. If $G = \overline{G}$, then G is a topological group.*

Definition 3.2. *Let G be a topological rough group. Then a subset U of G is called rough symmetric if $U = U^{-1}$.*

From the definition of rough subgroups, we obtain the following result.

Corollary 3.1. *Every rough subgroup of a topological rough group is rough symmetric.*

Theorem 3.1. *Let G be a topological rough group. Then the closure of any rough symmetric subset A of G is again rough symmetric.*

Proof. Since the inverse mapping $\iota : G \rightarrow G$ is a homeomorphism, $cl(A) = (cl(A))^{-1}$. \square

Theorem 3.2. *Let G be a topological rough group and H be a rough subgroup. Then $cl(H)$ is a rough group in \overline{G} .*

Proof. (1) Identity element: $H \subseteq cl(H)$ implies that $\overline{H} \subseteq \overline{cl(H)}$ and so $e \in \overline{cl(H)}$. Since $cl(H) \subseteq G$, we have $ex = xe = x$ for all $x \in cl(H)$.

(2) Inverse element: $cl(H)^{-1} \subseteq cl(H^{-1}) = cl(H)$.

(3) Closed under product: Let $x, y \in cl(H)$. Then $xy \in \overline{G}$, which implies that there exists an open set $U \in \overline{G}$ such that $xy \in U$. We will prove that $U \wedge H \neq \varphi$. Consider the multiplication mapping $\mu : G \times G \rightarrow \overline{G}$. This implies that there exist open sets W, V of G such that $x \in W, y \in V, W \wedge H \neq \varphi, V \wedge H \neq \varphi$. Since the topology on G is a relative topology on \overline{G} , there exist open sets W', V' of \overline{G} such that $W \subseteq W', V \subseteq V'$. Hence $W' \wedge H \neq \varphi, V' \wedge H \neq \varphi$. Then $\mu(W \times V) \wedge H \neq \varphi$, but we have $\mu(W \times V) \subseteq U$, which implies $H \wedge U \neq \varphi$. So $xy \in cl(H) \subseteq \overline{cl(H)}$. This implies that $cl(H)$ is a rough group of \overline{G} .

Thus $cl(H)$ is a rough group in \overline{G} . \square

Definition 3.3. Let (X, τ) be a topological rough space of approximation space (U, \mathcal{R}) , and let $\mathcal{B} \subseteq \tau$ be a base for τ . For $x \in X$, the family

$$\mathcal{B}_x = \{O \in \mathcal{B} : x \in O\} \subseteq \mathcal{B}$$

is called a base at x .

Theorem 3.3. Let G be a topological rough space with \overline{G} group. For $g \in \overline{G}$, the base at g is equal to

$$\mathcal{B}_g = \{gO : O \in \mathcal{B}_e\},$$

where e is the identity element of a rough group G .

4. PRODUCT OF TOPOLOGICAL ROUGH GROUPS

Let (U, \mathcal{R}_1) and (V, \mathcal{R}_2) be approximation spaces with binary operations $*_1$ and $*_2$, respectively. Consider the Cartesian product of U and V : let $x, x' \in U$ and $y, y' \in V$. Then $(x, y), (x', y') \in U \times V$. Define $*$ as $(x, y) * (x', y') = (x *_1 x', y *_2 y')$. Then $*$ is a binary operation on $U \times V$. In [1], Alharbi *et al.* proved that the product of equivalence relations is also an equivalence relation on $U \times V$.

Theorem 4.1. [1] Let $G_1 \subseteq U$ and $G_2 \subseteq V$ be two rough groups. Then the Cartesian product $G_1 \times G_2$ is a rough group.

The following conditions are satisfied:

- (1) For all $(x, y), (x', y') \in G_1 \times G_2$, $(x_1, y'_1) * (x_2, y'_2) = (x_1 *_1 x_2, y'_1 *_2 y'_2) \in \overline{G_1} \times \overline{G_2}$.
- (2) Associative law is satisfied over all elements in $\overline{G_1} \times \overline{G_2}$.
- (3) There exists an identity element $(e, e') \in \overline{G_1} \times \overline{G_2}$ such that $\forall (x, x') \in G_1 \times G_2$, $(x, x') \times (e, e') = (e, e') \times (x, x') = (ex, e'x') = (x.x')$.
- (4) For all $(x, x') \in G_1 \times G_2$, there exists an element $(y, y') \in G_1 \times G_2$ such that $(x, x') * (y, y') = (y, y') * (x, x') = (e, e')$.

Example 4.1. Consider Example 3.1 where $U = \{\overline{0}, \overline{1}, \overline{2}\}$ and $U/R = \{\{\overline{0}, \overline{2}\}, \{\overline{1}\}\}$. Then the Cartesian product $U \times U$ is as follows:

$$U \times U = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{0}, \overline{1}), (\overline{2}, \overline{0}), (\overline{2}, \overline{2}), (\overline{2}, \overline{1}), (\overline{1}, \overline{0}), (\overline{1}, \overline{2}), (\overline{1}, \overline{1})\}.$$

Then the new classification is

$$\{\{\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{2}, \overline{0}), (\overline{2}, \overline{2})\}, \{(\overline{0}, \overline{1}), (\overline{2}, \overline{1})\}, \{(\overline{1}, \overline{0}), (\overline{1}, \overline{2})\}, \{(\overline{1}, \overline{1})\}\}.$$

Consider the rough group $G = \{\overline{1}, \overline{2}\}$. Then the Cartesian product $G \times G$ is

$$G \times G = \{(\overline{2}, \overline{2}), (\overline{2}, \overline{1}), (\overline{1}, \overline{2}), (\overline{1}, \overline{1})\},$$

where $\overline{G \times G} = \overline{G} \times \overline{G} = U \times U$. From the definition of a rough group, we have that

- (1) the multiplication of elements in $G \times G$ is closed under $\overline{G} \times \overline{G}$, i.e. $(\overline{2}, \overline{2})(\overline{2}, \overline{2}) = (\overline{1}, \overline{1})$, $(\overline{2}, \overline{2})(\overline{2}, \overline{1}) = (\overline{1}, \overline{0})$, $(\overline{2}, \overline{2})(\overline{1}, \overline{1}) = (\overline{0}, \overline{0})$, $(\overline{2}, \overline{2})(\overline{1}, \overline{2}) = (\overline{0}, \overline{1})$, $(\overline{2}, \overline{1})(\overline{2}, \overline{1}) = (\overline{1}, \overline{2})$,
 $(\overline{2}, \overline{1})(\overline{1}, \overline{1}) = (\overline{0}, \overline{2})$, $(\overline{2}, \overline{1})(\overline{1}, \overline{2}) = (\overline{0}, \overline{0})$, $(\overline{1}, \overline{1})(\overline{1}, \overline{1}) = (\overline{2}, \overline{2})$, $(\overline{1}, \overline{1})(\overline{1}, \overline{2}) = (\overline{2}, \overline{0})$;
- (2) there exists $(\overline{0}, \overline{0}) \in \overline{G} \times \overline{G}$ such that for every $(g, g') \in G \times G$, we have $(\overline{0}, \overline{0})(g, g') = (g, g')$;
- (3) for every element of $G \times G$, there exists an inverse element in $G \times G$, where $(\overline{1}, \overline{1})^{-1} = (\overline{2}, \overline{2}) \in G \times G$, $(\overline{2}, \overline{1})^{-1} = (\overline{1}, \overline{2}) \in G \times G$;
- (4) the associative law is satisfied.

Hence $G \times G$ is a rough group.

From Example 3.1, we have $\tau_G = \{\varphi, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\}$ as a topology on \overline{G} . Then $\tau_G \times \tau_G$ is the product topology of $\overline{G} \times \overline{G}$. Also we have $\tau = \{\varphi, G, \{\overline{1}\}, \{\overline{2}\}\}$ as a relative topology on G . So $\tau \times \tau$ is a topology on $G \times G$ induced by $\tau_G \times \tau_G$.

Consider the multiplication mapping $\mu : (G \times G) \times (G \times G) \rightarrow \overline{G} \times \overline{G}$. This mapping is continuous with respect to topology $\tau \times \tau$ and the product topology on $(G \times G) \times (G \times G)$. Also, we can show that the inverse mapping $\iota : G \times G \rightarrow G \times G$ is continuous. Hence $G \times G$ is a topological rough group.

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REFERENCES

- [1] N. Alharbi, H. Aydi, C. Özel, *Rough spaces on rough sets* (preprint).
- [2] N. Bagirmaz, I. Icen, A.F. Ozcan, *Topological rough groups*, *Topol. Algebra Appl.* **4** (2016), 31–38.
- [3] R. Biswas, S. Nanda, *Rough groups and rough subgroups*, *Bull. Polish Acad. Sci. Math.* **42** (1994), 251–254.
- [4] C.A. Neelima, P. Isaac, *Rough anti-homomorphism on a rough group*, *Global J. Math. Sci. Theory Pract.* **6**, (2014), no. 2, 79–85.

ON THE FARTHEST POINT PROBLEM IN BANACH SPACES

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ABSTRACT. A long standing conjecture in theory of Banach spaces is: "Every uniquely remotal set in a Banach is a singleton". This is known as the farthest point Conjecture. In an attempt to solve this problem, we give our contribution toward solving it, in the positive direction, by proving that every such subset E in the sequence space ℓ^1 is a singleton.

1. INTRODUCTION

Let X be a normed space, and E be a closed and bounded subset of X . We define the real valued function $D(., E) : X \rightarrow \mathbb{R}$ by

$$D(x, E) = \sup\{\|x - e\| : e \in E\},$$

the farthest distance function. We say that E is remotal if for every $x \in X$, there exists $e \in E$ such that $D(x, E) = \|x - e\|$. In this case, we denote the set $\{e \in E : D(x, E) = \|x - e\|\}$ by $F(x, E)$. It is clear that $F(., E) : X \rightarrow E$ is a multi-valued function. However, if $F(., E) : X \rightarrow E$ is a single-valued function, then E is called uniquely remotal. In such case, we denote $F(x, E)$ by $F(x)$, if no confusion arises.

The study of remotal and uniquely remotal sets has attracted many mathematicians in the last decades, due to its connection with the geometry of Banach spaces. We refer the reader to [1], [3], [5], [6] and [8] for samples of these studies. However, uniquely remotal sets are of special interest. In fact, one of the most interesting and hitherto unsolved problems in the theory of farthest points, known as the the farthest point problem, which is stated as: If every point of a normed space X admits a unique farthest point in a given bounded subset E , then must E be a singleton ?.

This problem gained its importance when Klee [4] proved that: singletonness of uniquely remotal sets is equivalent to convexity of Chebyshev sets in Hilbert spaces (which is an open problem too, in the theory of nearest points).

Since then, a considerable work has been done to answer this question, and many partial results have been obtained toward solving this problem. We refer the reader to [1], [3], [6] and [8] for some related work on uniquely remotal sets.

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Centers of sets have played a major role in the study of uniquely remotal sets, see [1], [2] and [3]. Recall that a center c of a subset E of a normed space X is an element $c \in X$ such that

$$D(c, E) = \inf_{x \in X} D(x, E).$$

Whether a set has a center or not is another question. However, in inner product spaces, any closed bounded set does have a center [1].

In [7] it was proved that if E is a uniquely remotal subset of a normed space, admitting a center c , and if F , restricted to the line segment $[c, F(c)]$ is continuous at c , then E is a singleton. Then recently, a generalization has been obtained in [9], where the authors proved the singletonness of uniquely remotal sets if the farthest point mapping F restricted to $[c, F(c)]$ is partially continuous at c . Furthermore, a generalization of Klee's result in [4], "If a compact subset E , with a center c , is uniquely remotal in a normed space X , then E must be a singleton", was also obtained in [9].

In this article, we prove that every uniquely remotal subset of the sequence space $\ell^1(\mathbb{R})$ is a singleton. Recall that $\ell^1(\mathbb{R}) = \{x = (x_n) : x_n \in \mathbb{R} \text{ and } \sum_{n=1}^{\infty} |x_n| < \infty\}$.

2. PRELIMINARIES

In this section, we prove the following propositions that play a key role in the proof of the main result.

Throughout the rest of the paper, F will denote the farthest distance single-valued function associated with a uniquely remotal set E .

Proposition 2.1. *Let E be a uniquely remotal subset of a Banach space X . Let (x_n) be a sequence in X such that (x_n) converges to $x \in X$. If $F(x_n) = y$ for all n , where $y \in E$, then $F(x) = y$.*

Proof. Suppose that $F(x) \neq y$. Since E is uniquely remotal, then there exists $w \in E$ such that $F(x) = w$. Further, there exists $\epsilon > 0$ such that $\|x - w\| > \|x - y\| + \epsilon$. Also, there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x\| < \frac{\epsilon}{2}$ for all $n \geq n_0$. Therefore, for $m \geq n_0$

$$\begin{aligned} \|x_m - w\| &\geq \|x - w\| - \|x_m - x\| \\ &> \|x - y\| + \epsilon - \frac{\epsilon}{2} \\ &> \|x_m - y\| + \frac{\epsilon}{2} - \|x_m - x\| \\ &> \|x_m - y\|. \end{aligned}$$

This contradicts that $y = F(x_m)$. Hence, we must have $F(x) = y$. □

Proposition 2.2. *Let K be a compact subset of a Banach space X and E be uniquely remotal in X . Then there exist $x \in K$ and $e \in E$ such that*

$$D(E, K) = \sup\{\|y - \theta\| : y \in K, \theta \in E\} = \|e - x\|.$$

Proof. From the definition of $D(E, K)$, there exist two sequences (e_n) and (x_n) in E and K respectively such that

$$D(E, K) = \lim_{n \rightarrow \infty} \|e_n - x_n\|.$$

Since K is compact, then there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges to x in K . So,

$$D(E, K) = \lim_{k \rightarrow \infty} \|e_{n_k} - x_{n_k}\|.$$

The definition of $D(E, K)$ implies that $D(E, K) \geq \|e' - x'\|$ for all $e' \in E$ and $x' \in K$. Therefore,

$$\lim_{k \rightarrow \infty} \|e_{n_k} - x_{n_k}\| \geq \|x - F(x)\|.$$

But

$$\|e_{n_k} - x_{n_k}\| \leq \|e_{n_k} - x\| + \|x - x_{n_k}\| \leq \|x_{n_k} - x\| + \|x - F(x)\|.$$

Thus

$$\lim_{k \rightarrow \infty} \|x_{n_k} - y_{n_k}\| \leq \|x - F(x)\|.$$

Since $x \in K$ and $F(x) \in E$, it follows that $D(E, K) = \|x - F(x)\|$, which ends the proof. □

3. MAIN RESULTS

Let E be a uniquely remotal subset of a Banach space X . Let x_0 be an element in X and $e_0 \in E$ be the unique farthest point from x_0 , i.e $F(x_0) = e_0$. Consider the closed ball

$$B[x_0, \|x_0 - e_0\|] = B[x_0, D(x_0, E)].$$

Then clearly e_0 lies on the boundary of $B[x_0, D(x_0, E)]$.

Let $J = \{B[y, \|y - e_0\|] : F(y) = e_0\}$, and define the relation " \leq " on J as follows:

$$B_1 \leq B_2 \text{ if } B_2 \subseteq B_1.$$

It is easy to see that the relation " \leq " is a partial order.

Now, we claim the following.

Theorem 3.1. *J has a maximal element.*

Proof. Let T be any chain in J . Consider the net $\{\|y_\alpha - e_0\| : \alpha \in I\}$. Notice that if $B_{\alpha_1} \leq B_{\alpha_2}$ then $\|y_{\alpha_2} - e_0\| \leq \|y_{\alpha_1} - e_0\|$. Let $r = \inf_{\alpha \in I} \|y_\alpha - e_0\|$. Then it is easy to see that if the infimum is attained at some α_0 , then $B_{\alpha_0}[y_{\alpha_0}, \|y_{\alpha_0} - e_0\|]$ is an upper bound for T . If the infimum is not attained then there exists a sequence

(B_n) in T such that $\lim_{n \rightarrow \infty} \|y_n - e_0\| = \inf_{\alpha \in I} \|y_\alpha - e_0\| = r$.

We claim that (y_n) has a convergent subsequence. If not, then there exists $\epsilon > 0$ such that $\|y_n - y_m\| > \epsilon$ for all n, m . Clearly we can assume that $\epsilon < r$.

Since $\lim_{n \rightarrow \infty} \|y_n - e_0\| = r$, then there exists $n_0 \in \mathbb{N}$ such that $\|y_n - e_0\| < r + \frac{\epsilon}{2}$ for all $n \geq n_0$. But $\|y_{n_0} - y_{n_0+1}\| > \epsilon$, so $B_{n_0} \subseteq B_{n_0+1}$. Farther, $r \leq \|y_{n_0} - e_0\|$ and $\|y_{n_0+1} - e_0\| < r + \frac{\epsilon}{2}$. Without loss of generality, we can assume, for simplicity, that $y_{n_0} = 0$. Then the element $v = (1 + \frac{r}{\|y_{n_0+1}\|} y_{n_0+1}) \in B_{n_0+1}$.

Now, $\|v - 0\| = \|v\| = \|y_{n_0+1}\| + r > r + \epsilon$. Thus, $v \notin B_{n_0}$ which contradicts the fact that $B_{n_0+1} \subseteq B_{n_0}$. Hence, there is a subsequence (y_{n_k}) that converges to some element, say y . By assumption $F(y_{n_k}) = e_0$ for all n_k , which implies by Proposition 2.1 that $F(y) = e_0$. Thus, $B[y, \|y - e_0\|] \in J$.

It suffices now to show that $B[y, \|y - e_0\|] \subseteq B_\alpha$ for all $\alpha \in I$. If this is not true then there exists $w \in B[y, \|y - e_0\|]$ such that $w \notin B_{m_1}$ for some m_1 . Since (B_n) is a chain, then $w \notin B_{n_k}$ for all $n_k > m_1$. Furthermore, $\|w - y_{n_k}\| > r + \epsilon'$ for some $\epsilon' > 0$ and all $n_k > m_1$.

But $\|w - y_{n_k}\| \leq \|y_{n_k} - y\| + \|y - w\|$, where $\|y_{n_k} - y\| \rightarrow 0$ and $\|y - w\| < \|y - F(y)\| = \|y - r\|$. It follows that $\liminf_{n_k} \|w - y_{n_k}\| \leq r$, which contradicts the fact that $\|w - y_{n_k}\| > r + \epsilon'$. This means that $B[y, \|y - e_0\|]$ is an upper bound for the chain T . Hence, By Zorn's lemma J has a maximal element. □

Now we are ready to prove the main result of this paper.

Theorem 3.2. *Every uniquely remotal set in $\ell^1(\mathbb{R})$ is a singleton.*

Proof. Let E be a uniquely remotal set in ℓ^1 , and let \hat{e} be the unique farthest point in E from 0, i.e. $F(0) = \hat{e}$. By Theorem 3.1, $J = \{B[y, \|y - \hat{e}\|] : F(y) = \hat{e}\}$ has a maximal element say $B[\hat{v}, \|\hat{v} - \hat{e}\|]$.

Without loss of generality, we may assume that $\hat{v} = 0$ and $\|\hat{e}\| = 1$ so that the maximal element is the unit ball of ℓ^1 . Let $\hat{e} = (b_1, b_2, b_3, \dots)$. Since $\|\hat{e}\| = 1$ then with no loss of generality we can assume that $b_1 \neq 0$. Further, assume $b_1 > 0$. So, $b_1 > \frac{1}{m_0}$ for some $m_0 \in \mathbb{N}$.

Let $\delta_1 = (1, 0, 0, \dots)$ and consider the sequence $(\frac{\delta_1}{n})$ in ℓ^1 , where $n > m_0$. Then $F(\frac{\delta_1}{n}) \neq \hat{e}$ for all $n > m_0$, since if $F(\frac{\delta_1}{n}) = \hat{e}$ for some $n > m_0$, then for $w \in B[\frac{\delta_1}{n}, \|\frac{\delta_1}{n} - \hat{e}\|]$, we have $\|w\| - \|\frac{\delta_1}{n}\| \leq \|w - \frac{\delta_1}{n}\| \leq \|\frac{\delta_1}{n} - \hat{e}\|$. But $b_1 > \frac{1}{n}$, so $\|\frac{\delta_1}{n} - \hat{e}\| = \|\hat{e}\| - \frac{1}{n} = \|\hat{e}\| - \|\frac{\delta_1}{n}\|$. Thus, $\|w\| \leq \|\hat{e}\| = 1$ and accordingly $w \in B[0, 1]$, which contradicts the maximality of $B[0, 1]$. Hence, $F(\frac{\delta_1}{n}) \neq \hat{e}$ for all $n > m_0$.

Let $F(\frac{\delta_1}{n}) = z_n = (c_1^n, c_2^n, c_3^n, \dots)$. Then we must have $c_1^n < \frac{1}{n}$ for all $n > m_0$. Otherwise, we obtain that $\|z_n - \frac{\delta_1}{n}\| = \|z_n\| - \|\frac{\delta_1}{n}\| \leq 1 - \frac{1}{n} = \|\hat{e} - \frac{\delta_1}{n}\|$, which contradicts the fact that $F(\frac{\delta_1}{n}) = z_n$.

Now, since $\frac{\delta_1}{n} \rightarrow 0$, then $\|z_n\| \rightarrow 1$. Further, the sequence c_1^n converges to λ , where $\lambda \leq 0$.

Consider the set $P = \{b_1\delta_1\}$. Then, clearly $D(\hat{e}, P) = \sum_{j=2}^{\infty} |b_j| < 1$. Also, $D(z_n, P) = \|z_n - b_1\delta_1\| = |c_1^n - b_1| + \sum_{j=2}^{\infty} |c_j^n|$. Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} D(z_n, P) &= (b_1 + |\lambda|) + \lim_{n \rightarrow \infty} \sum_{j=2}^{\infty} |c_j^n| \\ &= b_1 + |\lambda| + (1 - |\lambda|) \\ &= 1 + b_1 \end{aligned}$$

Since $D(P, E) \geq D(P, z_n)$ for all n , we get that $D(P, E) \geq 1 + b_1$. On the other hand, $D(P, E) = \sup_{e \in E} \|b_1\delta_1 - e\| \leq b_1 + 1$, since $\|e\| \leq 1$ for every $e \in E$. Thus,

$$D(P, E) = 1 + b_1.$$

By Proposition 2.2, $D(P, E) = \|b_1\delta_1 - e'\|$ for some $e' \in E$. So,

$$1 + b_1 \leq b_1 + \|e'\| \leq 1 + b_1,$$

which implies that $\|e'\| = 1$. Therefore, e' is another farthest point in E from 0 , i.e. $F(0) = \{e', \hat{e}\}$, which contradicts the unique remotality of E . Hence, E must be a singleton. \square

REFERENCES

- [1] A. Astaneh, On Uniquely Remotal Subsets of Hilbert Spaces, Indian Journal Of Pure and Applied Mathematics. 14(10) (1983) 1311–1317.
- [2] A. Astaneh, On Singletonness of Uniquely Remotal Sets, Indian Journal Of Pure and Applied Mathematics 17(9)(1986) 1137–1139.
- [3] M. Baronti, A note on remotal sets in Banach spaces, Publications de L'institute mathematique 53(67) (1993) 95–98.
- [4] Klee, V., Convexity of Chebychev sets, Math. Ann. 142 (1961) 292–304.
- [5] M. Martin, T.S.S.R.K Rao, On Remotality for Convex Sets in Banach Spaces, Journal of Approximation Theory. 162(2)(2010) 392–396.
- [6] T. D. Narang, On singletonness of uniquely remotal sets , Periodica Mathematica Hungarica 21(1990) 17–19.
- [7] A. Niknam, On Uniquely Remotal Sets, Indian Journal Of Pure and Applied Mathematics 15(10) (1984) 1079–1083.
- [8] M. Sababheh, R. Khalil, A study of Uniquely Remotal Sets, Journal of Computational Analysis and Applications 13(7)(2010) 1233–1239.
- [9] M. Sababheh, A. Yousef and R. Khalil, Uniquely Remotal Sets in Banach Spaces, Filomat 31:9 (2017), 2773–2777.

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On the stability of 3-Lie homomorphisms and 3-Lie derivations

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Abstract. In this paper, we prove the Hyers-Ulam stability of 3-Lie homomorphisms in 3-Lie algebras for Cauchy-Jensen functional equation. We also prove the Hyers-Ulam stability of 3-Lie derivations on 3-Lie algebras for Cauchy-Jensen functional equation.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations had been first raised by Ulam [21]. In 1941, Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. The generalizations of this result have been published by Aoki [2] for $(0 < p < 1)$, Rassias [19] for $(p < 0)$ and Gajda [8] for $(p > 1)$ for additive mappings and linear mappings by a general control function $\theta(\|x\|^p + \|y\|^p)$, respectively. In 1994, Găvruta [9] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions, i.e., who replaced $\theta(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Several stability problems for various functional equations have been investigated in [1, 4, 6, 7, 12, 14, 15, 16, 17, 18, 20].

A Lie algebra is a Banach algebra endowed with the Lie product

$$[x, y] := \frac{(xy - yx)}{2}.$$

Similarly, a 3-Lie algebra is a Banach algebra endowed with the product

$$[[x, y], z] := \frac{[x, y]z - z[x, y]}{2}.$$

Let A and B be two 3-Lie algebras. A \mathbb{C} -linear mapping $H : A \rightarrow B$ is called a 3-Lie homomorphism if

$$H([[x, y], z]) = [[H(x), H(y)], H(z)]$$

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for all $x, y, z \in A$. A \mathbb{C} -linear mapping $D : A \rightarrow A$ is called a 3-Lie derivation if

$$D\left([x, y, z]\right) = [[D(x), y], z] + [[x, D(y)], z] + [[x, y,], D(z)]$$

for all $x, y, z \in A$ (see [22]).

Throughout this paper, we suppose that A and B are two 3-Lie algebras. For convenience, we use the following abbreviation for a given mapping $f : A \rightarrow B$

$$D_\mu f(x, y, z) := f\left(\frac{\mu x + \mu y}{2} + \mu z\right) + f\left(\frac{\mu x + \mu z}{2} + \mu y\right) + f\left(\frac{\mu y + \mu z}{2} + \mu x\right) - 2\mu f(x) - 2\mu f(y) - 2\mu f(z)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in A$.

Throughout this paper, assume that A is a 3-Lie algebra with norm $\|\cdot\|$ and that B is a 3-Lie algebra with norm $\|\cdot\|$.

2. STABILITY OF 3-LIE HOMOMORPHISMS IN 3-LIE ALGEBRAS

We need the following lemmas which have been given in for proving the main results.

Lemma 2.1. ([11]) *Let X be a uniquely 2-divisible abelian group and Y be linear space. A mapping $f : X \rightarrow Y$ satisfies*

$$f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+z}{2} + y\right) + f\left(\frac{y+z}{2} + x\right) = 2[f(x) + f(y) + f(z)] \tag{2.1}$$

for all $x, y, z \in X$ if and only if $f : X \rightarrow Y$ is additive.

Lemma 2.2. *Let X and Y be linear spaces and let $f : X \rightarrow Y$ be a mapping such that*

$$D_\mu f(x, y, z) = 0 \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then the mapping $f : X \rightarrow Y$ is \mathbb{C} -linear.

Proof. By Lemma 2.2, f is additive. Letting $y = z = 0$ in (2.1), we get $2f\left(\mu\frac{x}{2}\right) = \mu f(x)$ and so $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $f : X \rightarrow Y$ is \mathbb{C} -linear. \square

In the following, we investigate the Hyers-Ulam stability of (2.1).

Theorem 2.3. *Let $\varphi : A^3 \rightarrow [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y, z) := \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) < \infty \tag{2.3}$$

for all $x, y, z \in A$. Suppose that $f : A \rightarrow B$ is a mapping satisfying

$$\|D_\mu f(x, y, z)\| \leq \varphi(x, y, z), \tag{2.4}$$

$$\|f([x, y, z]) - [[f(x), f(y)], f(z)]\| \leq \varphi(x, y, z) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \rightarrow B$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{6} \tilde{\varphi}(x, x, x) \tag{2.6}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.4), we get

$$\|3f(2x) - 6f(x)\| \leq \varphi(x, x, x) \tag{2.7}$$

for all $x \in A$. If we replace x by $2^n x$ in (2.7) and divide both sides by $3 \cdot 2^{n+1}$. then we get

$$\left\| \frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^n} f(2^n x) \right\| \leq \frac{1}{3 \cdot 2^{n+1}} \varphi(2^n x, 2^n x, 2^n x)$$

for all $x \in A$ and all nonnegative integers n . Hence

$$\begin{aligned} \left\| \frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^m} f(2^m x) \right\| &= \left\| \sum_{k=m}^n \left[\frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^k} f(2^k x) \right] \right\| \\ &\leq \sum_{k=m}^n \left\| \frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^k} f(2^k x) \right\| \\ &\leq \frac{1}{6} \sum_{k=m}^n \frac{1}{2^k} \varphi(2^k x, 2^k x, 2^k x) \end{aligned} \tag{2.8}$$

for all $x \in A$ and all nonnegative integers $n \geq m \geq 0$. It follows from (2.3) and (2.8) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence in B for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in A$. Thus one can define the mapping $H : A \rightarrow B$ by

$$H(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (2.8), we get (2.6). It follows from (2.3) that

$$\begin{aligned} \|D_\mu H(x, y, z)\| &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|D_\mu f(2^n x, 2^n y, 2^n z)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0 \end{aligned}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. So $D_\mu H(x, y, z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.2, the mapping $H : A \rightarrow B$ is \mathbb{C} -linear.

It follows from (2.5) that

$$\begin{aligned} &\|H([x, y], z) - [[H(x), H(y)], H(z)]\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{8^n} \|f([2^n x, 2^n y], 2^n z) - [[f(2^n x), f(2^n y)], f(2^n z)]\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{8^n} \varphi(2^n x, 2^n y, 2^n z) \leq \lim_{n \rightarrow \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0 \end{aligned}$$

for all $x, y, z \in A$. Thus

$$H([x, y], z) = [[H(x), H(y)], H(z)]$$

for all $x, y, z \in A$.

Therefore, the mapping $H : A \rightarrow B$ is a 3-Lie homomorphism. □

Corollary 2.4. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 < 3$. Suppose that $f : A \rightarrow B$ is a mapping such that*

$$\|D_\mu f(x, y, z)\| \leq \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}), \tag{2.9}$$

$$\|f([[x, y], z]) - [[f(y), f(z)], f(x)]\| \leq \varepsilon(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3}) \tag{2.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \rightarrow B$ such that

$$\|f(x) - H(x)\| \leq \frac{\theta}{3} \left\{ \frac{1}{2 - 2^{p_1}} \|x\|^{p_1} + \frac{1}{2 - 2^{p_2}} \|x\|^{p_2} + \frac{1}{2 - 2^{p_3}} \|x\|^{p_3} \right\}$$

for all $x \in A$.

Theorem 2.5. *Let $\Phi : A^3 \rightarrow [0, \infty)$ be a function such that*

$$\sum_{n=1}^{\infty} 8^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) < \infty \tag{2.11}$$

for all $x, y, z \in A$. Suppose that $f : A \rightarrow B$ is a mapping such that

$$\|D_\mu f(x, y, z)\|_B \leq \psi(x, y, z),$$

$$\|f([[x, y], z]) - [[f(x), f(y)], f(z)]\| \leq \psi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \rightarrow B$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{6} \tilde{\psi}(x, x, x) \tag{2.12}$$

for all $x \in A$, where $\tilde{\psi}(x, y, z) := \sum_{n=1}^{\infty} 2^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)$ for all $x, y, z \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique 3-Lie homomorphism $H : A \rightarrow B$ satisfying (2.12). The mapping $H : A \times A \rightarrow B$ is given by

$$H(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

The rest of the proof is similar to the proof of Theorem 2.3 □

Corollary 2.6. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and q_3 be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \rightarrow B$ is a mapping satisfying (2.9) and (2.10). Then there exists a unique 3-Lie homomorphism $H : A \rightarrow B$ such that*

$$\|f(x) - H(x)\| \leq \frac{\theta}{3} \left\{ \frac{1}{2^{p_1} - 2} \|x\|^{p_1} + \frac{1}{2^{p_2} - 2} \|x\|^{p_2} + \frac{1}{2^{p_3} - 2} \|x\|^{p_3} \right\}$$

for all $x \in A$.

3. STABILITY OF 3-LIE DERIVATIONS ON 3-LIE ALGEBRAS

In this section, we prove the Hyers-Ulam stability of 3-Lie derivations on 3-Lie algebras for the functional equation $D_\mu f(x, y, z) = 0$.

Theorem 3.1. *Let $\varphi : A^3 \rightarrow [0, \infty)$ be a function satisfying (2.3). Suppose that $f : A \rightarrow A$ is a mapping satisfying*

$$\|D_\mu f(x, y, z)\| \leq \varphi(x, y, z),$$

$$\|f([[x, y], z]) - [[f(x), y], z] - [[x, f(y)], z] - [[x, y], f(z)]\| \leq \varphi(x, y, z) \tag{3.1}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie derivation $D : A \rightarrow A$ such that

$$\|f(x) - D(x)\| \leq \frac{1}{6} \tilde{\varphi}(x, x, x) \tag{3.2}$$

for all $x \in A$, where $\tilde{\varphi}$ is given in Theorem 2.3.

Proof. By the proof of Theorem 2.3, there exists a unique \mathbb{C} -linear mapping $D : A \rightarrow A$ satisfying (3.2) and

$$D(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. It follows from (3.1) that

$$\begin{aligned} & \|D([[x, y], z]) - [[D(x), y], z] - [[x, D(y)], z] - [[x, y], D(z)]\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{8^n} \|f([[2^n x, 2^n y], 2^n z]) - [[f(2^n x), 2^n y], 2^n z] - [[2^n x, f(2^n y)], 2^n z] - [[2^n x, 2^n y], f(2^n z)]\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{8^n} \varphi(2^n x, 2^n y, 2^n z) = 0 \end{aligned}$$

for all $x, y, z \in A$. So

$$D([[x, y], z]) = [[D(x), y], z] + [[x, D(y)], z] + [[x, y], D(z)]$$

for all $x, y, z \in A$. Therefore, the mapping $D : A \rightarrow A$ is a 3-Lie derivation. □

Corollary 3.2. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 < 3$. Suppose that $f : A \rightarrow A$ is a mapping such that*

$$\|D_\mu f(x, y, z)\| \leq \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}), \tag{3.3}$$

$$\|f([[x, y], z]) - [[f(x), y], z] - [[x, f(y)], z] - [[x, y], f(z)]\| \leq \varepsilon(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3}) \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie derivation $D : A \rightarrow A$ such that

$$\|f(x) - D(x)\| \leq \frac{\theta}{3} \left\{ \frac{1}{2 - 2^{p_1}} \|x\|^{p_1} + \frac{1}{2 - 2^{p_2}} \|x\|^{p_2} + \frac{1}{2 - 2^{p_3}} \|x\|^{p_3} \right\}$$

for all $x \in A$.

Theorem 3.3. Let $\psi : A^3 \rightarrow [0, \infty)$ be a function satisfying (2.11). Suppose that $f : A \rightarrow A$ is a mapping satisfying

$$\|D_\mu f(x, y, z)\| \leq \psi(x, y, z),$$

$$\|f([[x, y], z]) - [[f(x), y], z] - [[x, f(y)], z] - [[x, y], f(z)]\| \leq \psi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie derivation $D : A \rightarrow A$ such that

$$\|f(x) - D(x)\| \leq \frac{1}{6} \tilde{\psi}(x, x, x) \tag{3.5}$$

for all $x \in A$, where $\tilde{\psi}$ is given in Theorem 2.5.

Proof. By the proof of Theorem 2.3, there exists a unique \mathbb{C} -linear mapping $D : A \rightarrow A$ satisfying (3.5) and

$$D(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

The rest of proof is similar to the proof Theorem 3.1. □

Corollary 3.4. Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and q_3 be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \rightarrow B$ is a mapping satisfying (3.3) and (3.4). Then there exists a unique 3-Lie derivation $D : A \rightarrow A$ such that

$$\|f(x) - H(x)\| \leq \frac{\theta}{3} \left\{ \frac{1}{2^{p_1} - 2} \|x\|^{p_1} + \frac{1}{2^{p_2} - 2} \|x\|^{p_2} + \frac{1}{2^{p_3} - 2} \|x\|^{p_3} \right\}$$

for all $x \in A$.

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REFERENCES

- [1] M. Almahalebi, A. Charifi, C. Park and S. Kabbaj, *Hyerstability results for a generalized radical cubic functional equation related to additive mapping in non-Archimedean Banach spaces*, J. Fixed Point Theory Appl. **20** (2018), 2018:40.
- [2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
- [3] M. Eshaghi Gordji, S. Bazeghi, C. Park and S. Jang, *Ternary Jordan ring derivations on Banach ternary algebras: A fixed point approach*, J. Comput. Anal. Appl. **21** (2016), 829–834.
- [4] M. Eshaghi Gordji, V. Keshavarz, J. Lee and D. Shin, *Stability of ternary m-derivations on ternary Banach algebras*, J. Comput. Anal. Appl. **21** (2016), 640–644.
- [5] M. Eshaghi Gordji, V. Keshavarz, J. Lee, D. Shin and C. Park, *Approximate ternary Jordan ring homomorphisms in ternary Banach algebras*, J. Comput. Anal. Appl. **22** (2017), 402–408.
- [6] M. Eshaghi Gordji, V. Keshavarz, C. Park and S. Jang, *Ulam-Hyers stability of 3-Jordan homomorphisms in C^* -ternary algebras*, J. Comput. Anal. Appl. **22** (2017), 573–578.
- [7] M. Eshaghi Gordji, V. Keshavarz, C. Park and J. Lee, *Approximate ternary Jordan bi-derivations on Banach Lie triple systems*, J. Comput. Anal. Appl. **21** (2017), 45–51.

- [8] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci. **14** (1991), 431–434.
- [9] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [10] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.
- [11] A. Najati and A. Ranjbari, *Stability of homomorphisms for a 3D CauchyJensen type functional equation on C^* -ternary algebras*, J. Math. Anal. Appl. **341** (2008), 62–79.
- [12] L. Naranjani, M. Hassani and M. Mirzavaziri, *Local higher derivations on C^* -algebras are higher derivations*, Int. J. Nonlinear Anal. Appl. **9** (2018), 111–115.
- [13] C. Park, *Homomorphisms between Poisson JC^* -algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.
- [14] C. Park, *C^* -ternary biderivations and C^* -ternary bihomomorphisms*, Math. **6** (2018), Art. 30.
- [15] C. Park, *Bi-additive s -functional inequalities and quasi- $*$ -multipliers on Banach algebras*, Math. **6** (2018), Art. 31.
- [16] C. Park, J. Lee and D. Shin, *Stability of J^* -derivations*, Int. J. Geom. Methods Mod. Phys. **9** (2012), Art. ID 1220009, 10 pages.
- [17] H. Piri, S. Aslani, V. Keshavarz, Th. M. Rassias, C. Park and Y. Park, *Approximate ternary quadratic 3-derivations on ternary Banach algebras and C^* -ternary rings*, J. Comput. Anal. Appl. **24** (2018), 1280–1291.
- [18] M. Raghebi Moghadam, Th. M. Rassias, V. Keshavarz, C. Park and Y. Park, *Jordan homomorphisms in C^* -ternary algebras and JB^* -triples*, J. Comput. Anal. Appl. **24** (2018), 416–424.
- [19] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.
- [20] R. F. Rostami, *Lie ternary (σ, τ, xi) -derivations on Banach ternary algebras*, Int. J. Nonlinear Anal. Appl. **9** (2018), 41–53.
- [21] S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science ed., Wiley, New York, 1940.
- [22] H. Yuan and L. Chen, *Lie n superderivations and generalized Lie n superderivations of superalgebras*, Open Math. **16** (2018), 196–209.

NEUTROSOPHIC EXTENDED TRIPLET GROUPS AND HOMOMORPHISMS IN C^* -ALGEBRAS

JUNG RYE LEE, CHOONKIL PARK*, AND XIAOHONG ZHANG

ABSTRACT. Çelik, Shalla and Olgun [2] defined neutro-homomorphisms in neutrosophic extended triplet groups and Zhang et al. [8] investigated neutro-homomorphisms in neutrosophic extended triplet groups.

In this note, we apply the results on neutro-homomorphisms in neutrosophic extended triplet groups to investigate C^* -algebra homomorphisms in unital C^* -algebras.

1. INTRODUCTION AND PRELIMINARIES

As an extension of fuzzy sets and intuitionistic fuzzy sets, Smarandache [4] proposed the new concept of neutrosophic sets.

Definition 1.1. ([5, 6]) Let N be a nonempty set together with a binary operation $*$. Then N is called a neutrosophic extended triplet set if, for any $a \in N$, there exist a neutral of a (denoted by $\text{neut}(a)$) and an opposite of a (denoted by $\text{anti}(a)$) such that $\text{neut}(a) \in N$, $\text{anti}(a) \in N$ and

$$\begin{aligned} a * \text{neut}(a) &= \text{neut}(a) * a = a, \\ a * \text{anti}(a) &= \text{anti}(a) * a = \text{neut}(a). \end{aligned}$$

The triplet $(a, \text{neut}(a), \text{anti}(a))$ is called a neutrosophic extended triplet.

Note that, for a neutrosophic triplet set $(N, *)$ and $a \in N$, $\text{neut}(a)$ and $\text{anti}(a)$ may not be unique.

Definition 1.2. ([5, 6]) Let $(N, *)$ be a neutrosophic extended triplet set.

Then N is called a neutrosophic extended triplet group (NETG) if the following conditions hold:

- (1) $(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$;
- (2) $(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

N is called a commutative neutrosophic extended triplet group if, for all $a, b \in N$, $a * b = b * a$.

Let A be a unital C^* -algebra with multiplication operation \bullet , unit e and unitary group $U(A) := \{u \in A \mid u^* \bullet u = u \bullet u^* = e\}$. Then $u \bullet v \in U(A)$ and $(u \bullet v) \bullet w = u \bullet (v \bullet w)$ for all $u, v, w \in U(A)$ (see [3]). So $(U(A), \bullet)$ is an NETG.

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Proposition 1.3. ([7]) *Let $(N, *)$ be an NETG. Then*

- (1) *neut(a) is unique for each $a \in N$;*
- (2) *neut(a) * neut(a) = neut(a) for each $a \in N$.*

Note that $u \bullet e = e \bullet u = u$ for any $u \in (U(A), \bullet)$. By Proposition 1.3, $neut(u) = e$ for each $u \in U(A)$.

Definition 1.4. ([7]) Let $(N, *)$ be an NETG. Then N is called a weak commutative neutrosophic extended triplet group (briefly, WCNETG) if $a * neut(b) = neut(b) * a$ for all $a, b \in N$.

Since $neut(v) = e$ for all $v \in U(A)$, $u \bullet neut(v) = neut(v) \bullet u$ for all $u, v \in U(A)$. So $(U(A), \bullet)$ is a WCNETG.

2. NEUTROSOPHIC EXTENDED TRIPLET GROUPS AND C^* -ALGEBRA HOMOMORPHISMS IN UNITAL C^* -ALGEBRAS

Definition 2.1. ([8]) Let $(N, *)$ be a WCNETG. Then N is called a perfect NETG if $anti(neut(a)) = neut(a)$ for all $a \in N$.

Since $anti(e) = e$ and $neut(u) = e$ for all $u \in U(A)$, $anti(neut(u)) = neut(u) = e$ for all $u \in U(A)$. Thus $(U(A), \bullet)$ is a perfect NETG.

Definition 2.2. ([1, 2]) Let $(N_1, *)$ and $(N_2, *)$ be neutrosophic extended triplet groups. A mapping $f : N_1 \rightarrow N_2$ is called a neutro-homomorphism if

$$f(x * y) = f(x) * f(y)$$

for all $x, y \in N_1$.

From now on, assume that A is a unital C^* -algebra with multiplication operation \bullet , unit e and unitary group $U(A)$ and that B is a unital C^* -algebra with multiplication operation \bullet and unitary group $U(B)$.

Definition 2.3. Let $(U(A), \bullet)$ and $(U(B), \bullet)$ be unitary groups of unital C^* -algebras A and B , respectively. A mapping $h : U(A) \rightarrow U(B)$ is called a neutro- $*$ -homomorphism if

$$\begin{aligned} h(u \bullet v) &= h(u) \bullet h(v), \\ h(u^*) &= h(u)^* \end{aligned}$$

for all $u, v \in U(A)$.

Theorem 2.4. *Let A and B be unital C^* -algebras. Let $H : A \rightarrow B$ be a \mathbb{C} -linear mapping and let $h : (U(A), \bullet) \rightarrow (U(B), \bullet)$ be a neutro- $*$ -homomorphism. If $H|_{U(A)} = h$, then $H : A \rightarrow B$ is a C^* -algebra homomorphism.*

Proof. Since $H : A \rightarrow B$ is \mathbb{C} -linear and $x, y \in A$ are finite linear combinations of unitary elements (see [3]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$, $y = \sum_{i=1}^n \mu_i v_i$ ($\lambda_j, \mu_i \in \mathbb{C}$, $u_j, v_i \in U(A)$),

$$\begin{aligned} H(x \bullet y) &= H\left(\left(\sum_{j=1}^m \lambda_j u_j\right) \bullet \left(\sum_{i=1}^n \mu_i v_i\right)\right) = H\left(\sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i (u_j \bullet v_i)\right) \\ &= \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i H(u_j \bullet v_i) = \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i h(u_j \bullet v_i) \\ &= \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i h(u_j) \bullet h(v_i) = \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i H(u_j) \bullet H(v_i) \\ &= H\left(\sum_{j=1}^m \lambda_j u_j\right) \bullet H\left(\sum_{i=1}^n \mu_i v_i\right) = H(x) \bullet H(y) \end{aligned}$$

for all $x, y \in A$.

Since $H : A \rightarrow B$ is \mathbb{C} -linear and each $x \in A$ is a finite linear combination of unitary elements (see [3]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$),

$$\begin{aligned} H(x^*) &= H\left(\left(\sum_{j=1}^m \lambda_j u_j\right)^*\right) = H\left(\sum_{j=1}^m \bar{\lambda}_j u_j^*\right) = \sum_{j=1}^m \bar{\lambda}_j H(u_j^*) = \sum_{j=1}^m \bar{\lambda}_j h(u_j^*) = \sum_{j=1}^m \bar{\lambda}_j h(u_j)^* \\ &= \sum_{j=1}^m \bar{\lambda}_j H(u_j)^* = \left(\sum_{j=1}^m \lambda_j H(u_j)\right)^* = H\left(\sum_{j=1}^m \lambda_j u_j\right)^* = H(x)^* \end{aligned}$$

for all $x \in A$. Thus the \mathbb{C} -linear mapping $H : A \rightarrow B$ is a C^* -algebra homomorphism. \square

3. CONCLUSIONS

In this note, we have studied unitary groups of unital C^* -algebras as neutrosophic extended triplet groups and have extended neutro-homomorphisms in neutrosophic extended triplet groups to neutro- $*$ -homomorphisms in unitary groups of unital C^* -algebras. We have obtained C^* -algebra homomorphisms in unital C^* -algebras by using neutro- $*$ -homomorphisms in unitary groups of unital C^* -algebras.

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COMPETING INTERESTS

The authors declare that they have no competing interests.

REFERENCES

- [1] M. Bal, M.M. Shalla, N. Olgun, *Neutrosophic triplet cosets and quotient groups*, Symmetry **10** (2018), 10:126. doi:10.3390/sym10040126.
- [2] M. Çelik, M.M. Shalla, N. Olgun, *Fundamental homomorphism theorems for neutrosophic extended triplet groups*, Symmetry **10** (2018), 10:321. doi:10.3390/sym10080321.

- [3] R.V. Kadison, J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras: Elementary Theory*, Academic Press, New York, 1983.
- [4] F. Smarandache, *Neutrosophic set—A generalization of the intuitionistic fuzzy sets*, Int. J. Pure Appl. Math. **3** (2005), 287–297.
- [5] F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras and Applications*, Pons Publishing House, Brussels, Belgium, 2017.
- [6] F. Smarandache, M. Ali, *Neutrosophic triplet group*, Neural Comput. Appl. **29** (2018), 595–601.
- [7] X. Zhang, Q.Q. Hu, F. Smarandache, X.G. An, *On neutrosophic triplet groups: Basic properties, NT-subgroups and some notes*, Symmetry **10** (2018), 10:289. doi:10.3390/sym10070289.
- [8] X. Zhang, X. Mao, F. Smarandache, C. Park, *On homomorphism theorem for perfect neutrosophic extended triplet groups*, Information **9** (2018), 9:237. doi:10.3390/info9090237.

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Orthogonal stability of a quadratic functional inequality: a fixed point approach

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Abstract. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a mapping from an orthogonality space (\mathcal{X}, \perp) into a real Banach space $(\mathcal{Y}, \|\cdot\|)$. Using fixed point method, we prove the Hyers-Ulam stability of the orthogonally quadratic functional inequality

$$\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) \right\| \leq \|f(z)\| \quad (0.1)$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y$, $x \perp z$ and $y \perp z$.

Keywords: Hyers-Ulam stability; quadratic functional equation; fixed point method; quadratic functional inequality; orthogonality space.

1. INTRODUCTION AND PRELIMINARIES

Studying functional equations by focusing on their approximate and exact solutions conduces to one of the most substantial significant study brunches in functional equations, what we call “*the theory of stability of functional equations*”. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable* if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable*.

In 1940, the most preliminary form of stability problems was proposed by Ulam [58]. He gave a talk and asked the following: “when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?”

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [26] for the Cauchy’s functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [47] for linear mappings by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, ($\varepsilon >$

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$0, p \in [0, 1)$). In 1994, Găvruta [23] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by the general control function $\varphi(x, y)$ for the existence of a unique linear mapping. The first author treating the stability of the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ was Skof [55] by proving that if f is a mapping from a normed space \mathcal{X} into a Banach space \mathcal{Y} satisfying $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon$, for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : \mathcal{X} \rightarrow \mathcal{Y}$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. Cholewa [13] extended the Skof's theorem by replacing \mathcal{X} by an abelian group \mathcal{G} . The Skof's result was later generalized by Czerwik [14] in the spirit of Ulam-Hyers-Rassias. For more epochal information and various aspects about the stability of functional equations theory, we refer the reader to the monographs ([6, 11, 12, 15, 16, 20, 27, 30, 41, 42, 43, 46], [48]–[51], [54]), which also include many interesting results concerning the stability of different functional equations in many various spaces.

Assume that $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is a real inner product space with the usual Hilbert norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. Moreover, consider the orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x \perp y$$

in which \perp is an abstract orthogonality relation. By the Pythagorean theorem, $f : \mathcal{X} \rightarrow \mathbb{R}$ defined by $f(x) = \|x\|^2 = \langle x, x \rangle$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonally Cauchy functional equation is not equivalent to the classic Cauchy equation on the whole inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$.

Pinsker [44] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [56] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation was first investigated by Gudder and Strawther [25]. They defined \perp by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [52] introduced his new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Furthermore, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [53] investigated the problem in a rather more general framework.

We now recall the concept of orthogonality space in the sense of Rätz [52], and then proceed it to prove our results for the orthogonally functional inequality (0.1).

Definition 1.1. Suppose \mathcal{X} is a real vector space with $\dim \mathcal{X} \geq 2$ and \perp is a binary relation on \mathcal{X} with the following properties:

- (\mathcal{O}_1) totality of \perp for zero: $x \perp 0, 0 \perp x$ for all $x \in \mathcal{X}$;
- (\mathcal{O}_2) independence: if $x, y \in \mathcal{X} - 0, x \perp y$, then x, y are linearly independent;
- (\mathcal{O}_3) homogeneity: if $x, y \in \mathcal{X}, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- (\mathcal{O}_4) the Thalesian property: if \mathcal{P} is a 2-dimensional subspace of $\mathcal{X}, x \in \mathcal{P}$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in \mathcal{P}$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair (\mathcal{X}, \perp) is called an orthogonality space and it becomes an orthogonality normed space when the orthogonality space equipped with a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space \mathcal{X} defined by (\mathcal{O}_1) , and for non-zero elements $x, y \in \mathcal{X}$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(\mathcal{X}, \|\cdot\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation \perp is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in \mathcal{X}$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [3]–[5], [10, 18, 29]).

Ger and Sikorska [24] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space \mathcal{X} into a real Banach space \mathcal{Y} and $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$, for all $x, y \in \mathcal{X}$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : \mathcal{X} \rightarrow \mathcal{Y}$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$, for all $x \in \mathcal{X}$.

Consider the classic quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ on the real inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$. Then the important parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

which holds entirely in a square norm on an inner product space, shows that $f : \mathcal{X} \rightarrow \mathbb{R}$ defined by $f(x) = \|x\|^2 = \langle x, x \rangle$, is a solution for the quadratic functional equation on the whole inner product space \mathcal{X} , (particularly in where $x \perp y$).

The orthogonally quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x \perp y$$

was first investigated by Vajzović [59] when \mathcal{X} is a Hilbert space, \mathcal{Y} is the scalar field, f is continuous and \perp means the Hilbert space orthogonality. Later, Drljević [19], Fochi [22], Moslehian [34, 35] and Szabó [57] generalized the Vajzović's results. See also [36, 37, 40].

The following quadratic 3-variables functional equation

$$\begin{aligned} f\left(\frac{x + y + z}{2}\right) + f\left(\frac{x - y - z}{2}\right) + f\left(\frac{y - x - z}{2}\right) + f\left(\frac{z - x - y}{2}\right) \\ = f(x) + f(y) + f(z) \end{aligned} \tag{1.1}$$

has been introduced and solved by S. Farhadabadi, J. Lee and C. Park on vector spaces in [21]. It has been also shown that the functional equation (1.1) is equivalent to the classic quadratic functional

equation in vector spaces. In any inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$, it is easy to verify that

$$\begin{aligned} & \left\langle \frac{x+y+z}{2}, \frac{x+y+z}{2} \right\rangle + \left\langle \frac{x-y-z}{2}, \frac{x-y-z}{2} \right\rangle + \left\langle \frac{y-x-z}{2}, \frac{y-x-z}{2} \right\rangle \\ & + \left\langle \frac{z-x-y}{2}, \frac{z-x-y}{2} \right\rangle = \langle x, x \rangle + \langle y, y \rangle + \langle z, z \rangle \end{aligned}$$

for all $x, y, z \in \mathcal{X}$. For this obvious reason, similar to the classic quadratic functional equation, the mapping $f(x) = \langle x, x \rangle$ can also be a solution for the 3-variables equation (1.1) on the whole inner product space \mathcal{X} (particularly, for the case $x \perp y, y \perp z$ and $x \perp z$).

Fixed point theory has a basic role in applications of many considerable branches in mathematics specially in stability problems. In 1996, Isac and Rassias [28] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In view of the fact that, we will use methods related to fixed point theory, we give briefly some useful information, a definition and a fundamental result in fixed point theory.

Definition 1.2. Let \mathcal{X} be a set. A function $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ is called a *generalized metric* on \mathcal{X} if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Theorem 1.3. ([7, 17]) Let (\mathcal{X}, d) be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \rightarrow \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in \mathcal{X}$, either

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty, \quad \forall n \geq n_0$;
- (2) the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of \mathcal{J} ;
- (3) y^* is the unique fixed point of \mathcal{J} in the set $\mathcal{Y} = \{y \in \mathcal{X} \mid d(\mathcal{J}^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, \mathcal{J}y)$ for all $y \in \mathcal{Y}$.

In 2003, Cădariu and Radu [7, 8, 45] exerted the above definition and fixed point theorem to prove some stability problems for the Jensen and Cauchy functional equations. During the last decade, by applying fixed point methods, stability problems of several functional equations have been extensively investigated by a number of authors (see [2, 8, 9, 31, 33, 38, 39, 45]).

Throughout this paper, (\mathcal{X}, \perp) is an orthogonality space and $(\mathcal{Y}, \|\cdot\|)$ is a real Banach space.

2. SOLUTION AND HYERS-ULAM STABILITY OF THE FUNCTIONAL INEQUALITY (0.1)

In this section, we first solve the orthogonally quadratic functional inequality (0.1) by proving an orthogonal superstability proposition, and then we prove its Hyers-Ulam stability in orthogonality spaces.

Definition 2.1. A mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called an (exact) *orthogonally quadratic mapping* if

$$f(x + y) + f(x - y) = f(x) + f(y) \tag{2.1}$$

for all $x, y \in \mathcal{X}$, with $x \perp y$. And it is called an *approximate orthogonally quadratic mapping* if

$$\begin{aligned} & \left\| f\left(\frac{x + y + z}{2}\right) + f\left(\frac{x - y - z}{2}\right) + f\left(\frac{y - x - z}{2}\right) \right. \\ & \left. + f\left(\frac{z - x - y}{2}\right) - f(x) - f(y) \right\| \leq \|f(z)\| \end{aligned} \tag{2.2}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y, y \perp z$ and $x \perp z$.

Proposition 2.2. *Each approximate orthogonally quadratic mapping in the form of (2.2) is also an (exact) orthogonally quadratic mapping satisfying (2.1).*

Proof. Assume that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an approximate orthogonally quadratic mapping satisfying (2.2).

Since $0 \perp 0$, letting $x = y = z = 0$ in (2.2), we have

$$\|2f(0)\| \leq \|f(0)\| = 0$$

and so $f(0) = 0$.

Since $(x + y) \perp 0$ for all $x, y \in \mathcal{X}$, replacing x, y and z by $x + y, 0$ and 0 in (2.2), respectively, we conclude that

$$\left\| 2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{-x - y}{2}\right) - f(x + y) \right\| \leq \|f(0)\| = 0,$$

which implies

$$f\left(\frac{x + y}{2}\right) + f\left(\frac{-x - y}{2}\right) = \frac{1}{2}f(x + y) \tag{2.3}$$

for all $x, y \in \mathcal{X}$ (particularly, with $x \perp y$).

Replacing y by $-y$ in the above equality, we get

$$f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) = \frac{1}{2}f(x - y) \tag{2.4}$$

for all $x, y \in \mathcal{X}$ (particularly, with $x \perp y$).

Since $x \perp 0$ for all $x \in \mathcal{X}$, letting $z = 0$ in (2.2), we obtain

$$\begin{aligned} & \left\| f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) + f\left(\frac{-x - y}{2}\right) \right. \\ & \left. - f(x) - f(y) \right\| \leq \|f(0)\| = 0 \end{aligned}$$

and so

$$f\left(\frac{x + y}{2}\right) + f\left(\frac{-x - y}{2}\right) + f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) = f(x) + f(y) \tag{2.5}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$.

It follows from (2.3), (2.4) and (2.5) that

$$\frac{1}{2}f(x + y) + \frac{1}{2}f(x - y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$ with $x \perp y$, which is the equation (2.1). Hence $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an (exact) orthogonally quadratic mapping. □

Theorem 2.3. *Let $\varphi : \mathcal{X}^3 \rightarrow [0, \infty)$ be a function such that $\varphi(0, 0, 0) = 0$ and there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \leq 4\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{2.6}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be an even mapping satisfying

$$\begin{aligned} & \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right. \\ & \left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) \right\| \leq \|f(z)\| + \varphi(x, y, z) \end{aligned} \tag{2.7}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \frac{\alpha}{1-\alpha}\varphi(x, 0, 0) \tag{2.8}$$

for all $x \in \mathcal{X}$.

Proof. Consider the set $\mathcal{S} := \{h : \mathcal{X} \rightarrow \mathcal{Y}\}$ and introduce the generalized metric on \mathcal{S} :

$$d(g, h) = \inf \left\{ \mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x, 0, 0), \quad \forall x \in \mathcal{X} \right\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (\mathcal{S}, d) is complete (see [32]).

Now we consider the linear mapping $\mathcal{J} : \mathcal{S} \rightarrow \mathcal{S}$ such that

$$\mathcal{J}g(x) := \frac{1}{4}g(2x)$$

for all $g \in \mathcal{S}$ and all $x \in \mathcal{X}$.

Since $0 \perp 0$, letting $x = y = z = 0$ in (2.7), we have

$$2\|f(0)\| \leq \|f(0)\| + \varphi(0, 0, 0).$$

So $f(0) = 0$.

Since $x \perp 0$ for all $x \in \mathcal{X}$, letting $y = z = 0$ in (2.7), we get $\|4f\left(\frac{x}{2}\right) - f(x)\| \leq \varphi(x, 0, 0)$ for all $x \in \mathcal{X}$. Dividing both sides by 4, putting $2x$ instead of x and then using (2.6), we obtain

$$\left\| \frac{1}{4}f(2x) - f(x) \right\| \leq \frac{1}{4}\varphi(2x, 0, 0) \leq \alpha\varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$, which clearly yields

$$d(\mathcal{J}f, f) \leq \alpha. \tag{2.9}$$

Let $g, h \in \mathcal{S}$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\| \leq \varepsilon\varphi(x, 0, 0)$ for all $x \in \mathcal{X}$. Hence the definition of $\mathcal{J}g$ and (2.6), result that

$$\|\mathcal{J}g(x) - \mathcal{J}h(x)\| = \left\| \frac{1}{4}g(2x) - \frac{1}{4}h(2x) \right\| \leq \frac{1}{4}\varepsilon\varphi(2x, 0, 0) \leq \alpha\varepsilon\varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$, which implies that $d(\mathcal{J}g, \mathcal{J}h) \leq \alpha\varepsilon = \alpha d(g, h)$ for all $g, h \in \mathcal{S}$.

Thus \mathcal{J} is a strictly contractive mapping with Lipschitz constant $\alpha < 1$.

According to Theorem 1.3, there exists a mapping $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following:

(1) \mathcal{Q} is a fixed point of \mathcal{J} , i.e., $\mathcal{J}\mathcal{Q} = \mathcal{Q}$, and so

$$\frac{1}{4}\mathcal{Q}(2x) = \mathcal{Q}(x) \tag{2.10}$$

for all $x \in \mathcal{X}$. The mapping \mathcal{Q} is a unique fixed point of \mathcal{J} in the set

$$\mathcal{M} = \{g \in \mathcal{S} : d(g, f) < \infty\}.$$

This signifies that \mathcal{Q} is a unique mapping satisfying (2.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - \mathcal{Q}(x)\| \leq \mu\varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$;

(2) $d(\mathcal{J}^n f, \mathcal{Q}) \rightarrow 0$ as $n \rightarrow \infty$. So, we conclude that

$$\lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x) = \mathcal{Q}(x) \tag{2.11}$$

for all $x \in \mathcal{X}$;

(3) $d(f, \mathcal{Q}) \leq \frac{1}{1-\alpha}d(f, \mathcal{J}f)$, which gives by (2.9) the inequality

$$d(f, \mathcal{Q}) \leq \frac{\alpha}{1-\alpha}.$$

This proves that the inequality (2.8) holds.

To end the proof we show that \mathcal{Q} is an orthogonally quadratic mapping.

By (2.11), (2.7), (2.6) and the fact that $\alpha < 1$,

$$\begin{aligned} & \left\| \mathcal{Q}\left(\frac{x+y+z}{2}\right) + \mathcal{Q}\left(\frac{x-y-z}{2}\right) + \mathcal{Q}\left(\frac{y-x-z}{2}\right) + \mathcal{Q}\left(\frac{z-x-y}{2}\right) \right. \\ & \qquad \qquad \qquad \left. - \mathcal{Q}(x) - \mathcal{Q}(y) \right\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \left\| f(2^{n-1}(x+y+z)) + f(2^{n-1}(x-y-z)) \right. \\ & \qquad \qquad \qquad \left. + f(2^{n-1}(y-z-x)) + f(2^{n-1}(z-x-y)) - f(2^n x) - f(2^n y) \right\| \\ & \leq \left\| \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n z) \right\| + \lim_{n \rightarrow \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z) \\ & \leq \|\mathcal{Q}(z)\| + \lim_{n \rightarrow \infty} \alpha^n \varphi(x, y, z) \\ & = \|\mathcal{Q}(z)\| \end{aligned}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $x \perp z$ and $y \perp z$. And, now applying Proposition 2.2, we obtain that \mathcal{Q} is an orthogonally quadratic mapping and the proof is complete. \square

Theorem 2.4. *Let $\varphi : \mathcal{X}^3 \rightarrow [0, \infty)$ be a function such that $\varphi(0, 0, 0) = 0$ and there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \leq \frac{\alpha}{4}\varphi(2x, 2y, 2z)$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be an even mapping satisfying (2.7). Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \frac{1}{1-\alpha}\varphi(x, 0, 0) \tag{2.12}$$

for all $x \in \mathcal{X}$.

Proof. Let (\mathcal{S}, d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $\mathcal{J} : \mathcal{S} \rightarrow \mathcal{S}$ such that

$$\mathcal{J}g(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in \mathcal{X}$.

Similar to the proof of Theorem 2.3, from (2.7) one can get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$, which means $d(f, \mathcal{J}f) \leq 1$.

We can also show that \mathcal{J} is a strictly contractive mapping with Lipschitz constant $\alpha < 1$. So by applying Theorem 1.3 again, we have

$$d(f, \mathcal{Q}) \leq \frac{1}{1-\alpha}d(f, \mathcal{J}f) \leq \frac{1}{1-\alpha}$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of the previous theorem. □

Corollary 2.5. *Let \mathcal{X} be a normed orthogonality space. Let δ be a nonnegative real number and $p \neq 2$ be a positive real number. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned} & \left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y)\right\| \\ & \leq \left\|f(z)\right\| + \delta(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \frac{2^p}{|2^p - 4|} \delta \|x\|^p$$

for all $x \in \mathcal{X}$.

Proof. Define $\varphi(x, y, z) := \delta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in \mathcal{X}$.

First assume that $0 < p < 2$.

Take $\alpha := 2^{p-2}$. Since $p < 2$, obviously $\alpha < 1$. Hence there exists an $\alpha < 1$ such that

$$\begin{aligned} \varphi(x, y, z) &= \delta(\|x\|^p + \|y\|^p + \|z\|^p) \\ &= 4\alpha 2^{-p} \delta(\|x\|^p + \|y\|^p + \|z\|^p) \\ &= 4\alpha \delta\left(\left\|\frac{x}{2}\right\|^p + \left\|\frac{y}{2}\right\|^p + \left\|\frac{z}{2}\right\|^p\right) \\ &= 4\alpha \varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$ (particularly, with $x \perp y$, $y \perp z$ and $x \perp z$). The recent term allows to use Theorem 2.3. So by applying Theorem 2.3, it follows from (2.8) that

$$\|f(x) - \mathcal{Q}(x)\| \leq \frac{2^p}{4 - 2^p} \delta \|x\|^p$$

for all $x \in \mathcal{X}$.

For the case $p > 2$, taking $\alpha := 2^{2-p}$, and then applying Theorem 2.4, we similarly obtain the desired result. □

REFERENCES

- [1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
- [2] S. Alizadeh and F. Moradlou, *Approximate a quadratic mapping in multi-Banach spaces, a fixed point approach*, Int. J. Nonlinear Anal. Appl. **7** (2016), no. 1, 63–75.
- [3] J. Alonso and C. Benítez, *Orthogonality in normed linear spaces: a survey I. Main properties*, Extracta Math. **3** (1988), 1–15.
- [4] J. Alonso and C. Benítez, *Orthogonality in normed linear spaces: a survey II. Relations between main orthogonalities*, Extracta Math. **4** (1989), 121–131.
- [5] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), 169–172.
- [6] L. Cădariu, L. Găvruta and P. Găvruta, *On the stability of an affine functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 60–67.
- [7] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen’s functional equation*, J. Inequal. Pure & Appl. Math. **4** (1) Art. ID 4 (2003).
- [8] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
- [9] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).
- [10] S.O. Carlsson, *Orthogonality in normed linear spaces*, Ark. Mat. **4** (1962), 297–318.
- [11] A. Chahbi and N. Bounader, *On the generalized stability of d’Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
- [12] L. S. Chadli, S. Melliani, A. Moujahid and M. Elomari, *Generalized solution of sine-Gordon equation*, Int. J. Nonlinear Anal. Appl. **7** (2016), no. 1, 87–92.
- [13] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
- [14] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
- [15] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, London, Singapore and Hong Kong, 2002.
- [16] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, Florida, 2003.
- [17] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
- [18] C.R. Diminnie, *A new orthogonality relation for normed linear spaces*, Math. Nachr. **114**, (1983), 197–203.
- [19] F. Drljević, *On a functional which is quadratic on A-orthogonal vectors*, Publ. Inst. Math. (Beograd) **54**, (1986), 63–71.
- [20] M. Eshaghi Gordji, Y. Cho, H. Khodaei and M. Ghanifard, *Solutions and stability of generalized mixed type QCA-functional equations in random normed spaces*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **59** (2013), 299–320.
- [21] S. Farhadabadi J. Lee and C. Park, *A new quadratic functional equation version and its stability and superstability*, J. Computat. Anal. Appl. **23** (2017), 544–552.
- [22] M. Fochi, *Functional equations in A-orthogonal vectors*, Aequationes Math. **38** (1989), 28–40.
- [23] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [24] R. Ger and J. Sikorska, *Stability of the orthogonal additivity*, Bull. Pol. Acad. Sci. Math. **43** (1995), 143–151.
- [25] S. Gudder and D. Strawther, *Orthogonally additive and orthogonally increasing functions on vector spaces*, Pacific J. Math. **58** (1975), 427–436.
- [26] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl Acad. Sci. USA, **27** (1941), 222–224.

- [27] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
- [28] G. Isac and Th.M. Rassias, *Stability of ψ -additive mappings: Applications to nonlinear analysis*, Internat. J. Math. & Math. Sci. **19** (1996), 219–228.
- [29] R.C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292.
- [30] S. Jung, *Hyers-Ulam-Rassias stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Florida, 2001.
- [31] Y. Jung and I. Chang, *The stability of a cubic type functional equation with the fixed point alternative*, J. Math. Anal. Appl. **306** (2005), 752–760.
- [32] D. Mihet and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
- [33] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
- [34] M.S. Moslehian, *On the orthogonal stability of the Pexiderized quadratic equation*, J. Difference Equ. Appl. **11** (2005), 999–1004.
- [35] M.S. Moslehian, *On the stability of the orthogonal Pexiderized Cauchy equation*, J. Math. Anal. Appl. **318** (2006), 211–223.
- [36] M.S. Moslehian and Th.M. Rassias, *Orthogonal stability of additive type equations*, Aequationes Math. **73** (2007), 249–259.
- [37] L. Paganoni and J. Rätz, *Conditional functional equations and orthogonal additivity*, Aequationes Math. **50** (1995), 135–142.
- [38] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2007**, Art. ID 50175 (2007).
- [39] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl. **2008**, Art. ID 493751 (2008).
- [40] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
- [41] C. Park, S. Kim, J. Lee and D. Shin, *Quadratic ρ -functional inequalities in β -homogeneous normed spaces*, Int. J. Nonlinear Anal. Appl. **6** (2015), no. 2, 21–26.
- [42] C. Park and A. Najati, *Generalized additive functional inequalities in Banach algebras*, Int. J. Nonlinear Anal. Appl. **1** (2010), no. 2, 54–62.
- [43] C. Park and J. Park, *Generalized Hyers-Ulam stability of an Euler-Lagrange type additive mapping*, J. Difference Equ. Appl. **12** (2006), 1277–1288.
- [44] A.G. Pinsker, *Sur une fonctionnelle dans l'espace de Hilbert*, C. R. (Dokl.) Acad. Sci. URSS, n. Ser. **20** (1983), 411–414.
- [45] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
- [46] J.M. Rassias, *The Ulam stability problem in approximation of approximately quadratic mappings by quadratic mappings*, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), Art. 52.
- [47] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [48] Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Stud. Univ. Babeş-Bolyai Math. **43** (1998), 89–124.
- [49] Th.M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
- [50] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
- [51] Th.M. Rassias (ed.), *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.
- [52] J. Rätz, *On orthogonally additive mappings*, Aequationes Math. **28** (1985), 35–49.

- [53] J. Rätz and Gy. Szabó, *On orthogonally additive mappings IV*, Aequationes Math. **38** (1989), 73–85.
- [54] K. Ravi, E. Thandapani and B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables*, J. Nonlinear Sci. Appl. **7** (2014), 18–27.
- [55] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano. **53** (1983), 113–129.
- [56] K. Sundaresan, *Orthogonality and nonlinear functionals on Banach spaces*, Proc. Amer. Math. Soc. **34** (1972), 187–190.
- [57] Gy. Szabó, *Sesquilinear-orthogonally quadratic mappings*, Aequationes Math. **40** (1990), 190–200.
- [58] S.M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.
- [59] F. Vajzović, *Über das Funktional H mit der Eigenschaft: $(x, y) = 0 \Rightarrow H(x + y) + H(x - y) = 2H(x) + 2H(y)$* , Glasnik Mat. Ser. III **2** (22) (1967), 73–81.

**INTEGRAL INEQUALITIES FOR ASYMMETRIZED
SYNCHRONOUS FUNCTIONS**

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ABSTRACT. In this paper we establish some integral inequalities for the product of asymmetrized synchronous/asynchronous functions. Some examples for integrals of monotonic functions, including power, logarithmic and sin functions are also provided.

1. INTRODUCTION

For a function $f : [a, b] \rightarrow \mathbb{C}$ we consider the *symmetrical transform of f* on the interval $[a, b]$, denoted by $\check{f}_{[a,b]}$ or simply \check{f} , when the interval $[a, b]$ is implicit, as defined by

$$(1.1) \quad \check{f}(t) := \frac{1}{2} [f(t) + f(a + b - t)], \quad t \in [a, b].$$

The *anti-symmetrical transform of f* on the interval $[a, b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply \tilde{f} and is defined by

$$\tilde{f}(t) := \frac{1}{2} [f(t) - f(a + b - t)], \quad t \in [a, b].$$

It is obvious that for any function f we have $\check{f} + \tilde{f} = f$.

If f is convex on $[a, b]$, then for any $t_1, t_2 \in [a, b]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have

$$\begin{aligned} \check{f}(\alpha t_1 + \beta t_2) &= \frac{1}{2} [f(\alpha t_1 + \beta t_2) + f(a + b - \alpha t_1 - \beta t_2)] \\ &= \frac{1}{2} [f(\alpha t_1 + \beta t_2) + f(\alpha(a + b - t_1) + \beta(a + b - t_2))] \\ &\leq \frac{1}{2} [\alpha f(t_1) + \beta f(t_2) + \alpha f(a + b - t_1) + \beta f(a + b - t_2)] \\ &= \frac{1}{2} \alpha [f(t_1) + f(a + b - t_1)] + \frac{1}{2} \beta [f(t_2) + f(a + b - t_2)] \\ &= \alpha \check{f}(t_1) + \beta \check{f}(t_2), \end{aligned}$$

which shows that \check{f} is convex on $[a, b]$.

Consider the real numbers $a < b$ and define the function $f_0 : [a, b] \rightarrow \mathbb{R}$, $f_0(t) = t^3$. We have [6]

$$\check{f}_0(t) := \frac{1}{2} [t^3 + (a + b - t)^3] = \frac{3}{2} (a + b) t^2 - \frac{3}{2} (a + b)^2 t + \frac{1}{2} (a + b)^3$$

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for any $t \in \mathbb{R}$.

Since the second derivative $(\check{f}_0)''(t) = 3(a + b)$, $t \in \mathbb{R}$, then \check{f}_0 is strictly convex on $[a, b]$ if $\frac{a+b}{2} > 0$ and strictly concave on $[a, b]$ if $\frac{a+b}{2} < 0$. Therefore if $a < 0 < b$ with $\frac{a+b}{2} > 0$, then we can conclude that f_0 is not convex on $[a, b]$ while \check{f}_0 is convex on $[a, b]$.

We can introduce the following concept of convexity [6], see also [9] for an equivalent definition.

Definition 1. We say that the function $f : [a, b] \rightarrow \mathbb{R}$ is symmetrized convex (concave) on the interval $[a, b]$ if the symmetrical transform \check{f} is convex (concave) on $[a, b]$.

Now, if we denote by $Con[a, b]$ the closed convex cone of convex functions defined on $[a, b]$ and by $SCon[a, b]$ the closed convex cone of symmetrized convex functions, then from the above remarks we can conclude that

$$(1.2) \quad Con[a, b] \subsetneq SCon[a, b].$$

Also, if $[c, d] \subset [a, b]$ and $f \in SCon[a, b]$, then this does not imply in general that $f \in SCon[c, d]$.

We have the following result [6], [9] :

Theorem 1. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is symmetrized convex and integrable on the interval $[a, b]$. Then we have the Hermite-Hadamard inequalities

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}.$$

We also have [6]:

Theorem 2. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is symmetrized convex on the interval $[a, b]$. Then for any $x \in [a, b]$ we have the bounds

$$(1.4) \quad f\left(\frac{a+b}{2}\right) \leq \check{f}(x) \leq \frac{f(a) + f(b)}{2}.$$

For a monograph on Hermite-Hadamard type inequalities see [8].

In a similar way, we can introduce the following concept as well:

Definition 2. We say that the function $f : [a, b] \rightarrow \mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a, b]$ if the anti-symmetrical transform \tilde{f} is monotonic nondecreasing (nonincreasing) on the interval $[a, b]$.

If f is monotonic nondecreasing on $[a, b]$, then for any $t_1, t_2 \in [a, b]$ we have

$$\begin{aligned} \tilde{f}(t_2) - \tilde{f}(t_1) &= \frac{1}{2} [f(t_2) - f(a + b - t_2)] - \frac{1}{2} [f(t_1) - f(a + b - t_1)] \\ &= \frac{1}{2} [f(t_2) - f(t_1)] + \frac{1}{2} [f(a + b - t_1) - f(a + b - t_2)] \\ &\geq 0, \end{aligned}$$

which shows that $f : [a, b] \rightarrow \mathbb{R}$ is asymmetrized monotonic nondecreasing on the interval $[a, b]$.

Consider the real numbers $a < b$ and define the function $f_0 : [a, b] \rightarrow \mathbb{R}$, $f_0(t) = t^2$. We have

$$\tilde{f}_0(t) := \frac{1}{2} [t^2 - (a + b - t)^2] = (a + b)t - \frac{1}{2}(a + b)^2$$

and $(\tilde{f}_0)'(t) = a + b$, therefore $f : [a, b] \rightarrow \mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a, b]$ provided $\frac{a+b}{2} > 0 (< 0)$. So, if we take $a < 0 < b$ with $\frac{a+b}{2} > 0$, then f is asymmetrized monotonic nondecreasing on $[a, b]$ but not monotonic nondecreasing on $[a, b]$.

If we denote by $\mathcal{M}^\nearrow[a, b]$ the closed convex cone of monotonic nondecreasing functions defined on $[a, b]$ and by $\mathcal{AM}^\nearrow[a, b]$ the closed convex cone of asymmetrized monotonic nondecreasing functions, then from the above remarks we can conclude that

$$(1.5) \quad \mathcal{M}^\nearrow[a, b] \subsetneq \mathcal{AM}^\nearrow[a, b].$$

Also, if $[c, d] \subset [a, b]$ and $f \in \mathcal{AM}^\nearrow[a, b]$, then this does not imply in general that $f \in \mathcal{AM}^\nearrow[c, d]$.

We recall that the pair of functions (f, g) defined on $[a, b]$ are called *synchronous* (*asynchronous*) on $[a, b]$ if

$$(1.6) \quad (f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0$$

for any $t, s \in [a, b]$. It is clear that if both functions (f, g) are monotonic nondecreasing (nonincreasing) on $[a, b]$ then they are synchronous on $[a, b]$. There are also functions that change monotonicity on $[a, b]$, but as a pair they are still synchronous. For instance if $a < 0 < b$ and $f, g : [a, b] \rightarrow \mathbb{R}$, $f(t) = t^2$ and $g(t) = t^4$, then

$$(f(t) - f(s))(g(t) - g(s)) = (t^2 - s^2)(t^4 - s^4) = (t^2 - s^2)^2(t^2 + s^2) \geq 0$$

for any $t, s \in [a, b]$, which show that (f, g) is synchronous.

Definition 3. We say that the pair of functions (f, g) defined on $[a, b]$ is called *asymmetrized synchronous* (*asynchronous*) on $[a, b]$ if the pair of transforms (\tilde{f}, \tilde{g}) is *synchronous* (*asynchronous*) on $[a, b]$, namely

$$(1.7) \quad (\tilde{f}(t) - \tilde{f}(s))(\tilde{g}(t) - \tilde{g}(s)) \geq (\leq) 0$$

for any $t, s \in [a, b]$.

It is clear that if f, g are asymmetrized monotonic nondecreasing (nonincreasing) on $[a, b]$ then they are asymmetrized synchronous on $[a, b]$.

One of the most important results for synchronous (asynchronous) and integrable functions f, g on $[a, b]$ is the well-known Čebyšev's inequality:

$$(1.8) \quad \frac{1}{b-a} \int_a^b f(t)g(t) dt \geq (\leq) \frac{1}{b-a} \int_a^b f(t) dt \frac{1}{b-a} \int_a^b g(t) dt.$$

For integral inequalities of Čebyšev's type, see [1]-[5], [7], [10]-[18] and the references therein.

Motivated by the above results, we establish in this paper some inequalities for asymmetrized synchronous (asynchronous) functions on $[a, b]$. Some examples for power, logarithm and sin functions are provided as well.

2. MAIN RESULTS

We have the following result:

Theorem 3. *Assume that f, g are asymmetrized synchronous (asynchronous) and integrable functions on $[a, b]$. Then*

$$(2.1) \quad \int_a^b \tilde{f}(t) g(t) dt \geq (\leq) 0.$$

Proof. We consider only the case of symmetrized synchronous and integrable functions.

1. By the Čebyšev's inequality (1.8) for (\tilde{f}, \tilde{g}) we get

$$(2.2) \quad \frac{1}{b-a} \int_a^b \tilde{f}(t) \tilde{g}(t) dt \geq \frac{1}{b-a} \int_a^b \tilde{f}(t) dt \frac{1}{b-a} \int_a^b \tilde{g}(t) dt.$$

We have

$$\int_a^b \tilde{f}(t) dt = \frac{1}{2} \left[\int_a^b f(t) dt - \int_a^b f(a+b-t) dt \right] = 0$$

since, by the change of variable $s = a + b - t, t \in [a, b]$,

$$\int_a^b f(a+b-t) dt = \int_a^b f(s) ds.$$

Also,

$$(2.3) \quad \begin{aligned} \int_a^b \tilde{f}(t) \tilde{g}(t) dt &= \frac{1}{4} \int_a^b [f(t) - f(a+b-t)] [g(t) - g(a+b-t)] dt \\ &= \frac{1}{4} \int_a^b [f(t)g(t) + f(a+b-t)g(a+b-t)] dt \\ &\quad - \frac{1}{4} \int_a^b [f(t)g(a+b-t) + f(a+b-t)g(t)] dt \\ &= \frac{1}{4} \left[\int_a^b f(t)g(t) dt + \int_a^b f(a+b-t)g(a+b-t) dt \right] \\ &\quad - \frac{1}{4} \left[\int_a^b f(t)g(a+b-t) dt + \int_a^b f(a+b-t)g(t) dt \right] \\ &= \frac{1}{2} \left(\int_a^b f(t)g(t) dt - \int_a^b f(a+b-t)g(t) dt \right) \\ &= \int_a^b \tilde{f}(t)g(t) dt \end{aligned}$$

since, by the change of variable $s = a + b - t, t \in [a, b]$, we have

$$\int_a^b f(a+b-t)g(a+b-t) dt = \int_a^b f(t)g(t) dt$$

and

$$\int_a^b f(t)g(a+b-t) dt = \int_a^b f(a+b-t)g(t) dt.$$

By (2.2) we then get the desired result (2.1).

2. An alternative proof is as follows. Since (\tilde{f}, \tilde{g}) are synchronous, then

$$\left[\tilde{f}(t) - \tilde{f}\left(\frac{a+b}{2}\right) \right] \left[\tilde{g}(t) - \tilde{g}\left(\frac{a+b}{2}\right) \right] \geq 0$$

for any $t \in [a, b]$, which is equivalent to

$$(2.4) \quad \tilde{f}(t)\tilde{g}(t) \geq 0 \text{ for any } t \in [a, b],$$

or to

$$[f(t) - f(a+b-t)][g(t) - g(a+b-t)] \geq 0 \text{ for any } t \in [a, b].$$

This is a property of interest for asymmetrized synchronous functions.

If we integrate the inequality (2.4) and use the identity (2.3) we get the desired result (2.1). \square

Remark 1. The inequality (2.1) can be written in an equivalent form as

$$\int_a^b f(t)g(t) dt \geq \int_a^b f(a+b-t)g(t) dt,$$

or as

$$\int_a^b f(t)g(t) dt \geq \int_a^b \tilde{f}(t)g(t) dt.$$

Theorem 4. If both f, g are asymmetrized monotonic nondecreasing (nonincreasing) and integrable functions on $[a, b]$, then

$$(2.5) \quad \frac{1}{4} |f(b) - f(a)| |g(b) - g(a)| \geq \frac{1}{b-a} \int_a^b \tilde{f}(t)g(t) dt \geq 0,$$

and

$$(2.6) \quad \frac{1}{2} \min \left\{ |f(b) - f(a)| \frac{1}{b-a} \int_a^b |g(t)| dt, |g(b) - g(a)| \frac{1}{b-a} \int_a^b |f(t)| dt \right\} \\ \geq \frac{1}{b-a} \int_a^b \tilde{f}(t)g(t) dt \geq 0.$$

Proof. Assume that both f, g are asymmetrized monotonic nondecreasing and integrable functions on $[a, b]$, then they are asymmetrized synchronous and by (2.1) we get the second inequality in (2.5).

We also have

$$\tilde{f}(a) \leq \tilde{f}(t) \leq \tilde{f}(b)$$

for any $t \in [a, b]$, namely

$$-\frac{1}{2} [f(b) - f(a)] \leq \frac{1}{2} [f(t) - f(a+b-t)] \leq \frac{1}{2} [f(b) - f(a)],$$

for any $t \in [a, b]$, which implies that $\frac{1}{2} [f(b) - f(a)] \geq 0$ and

$$(2.7) \quad \frac{1}{2} |f(t) - f(a+b-t)| \leq \frac{1}{2} [f(b) - f(a)]$$

for any $t \in [a, b]$.

Similarly, we have $\frac{1}{2} [g(b) - g(a)] \geq 0$ and

$$(2.8) \quad \frac{1}{2} |g(t) - g(a+b-t)| \leq \frac{1}{2} [g(b) - g(a)]$$

for any $t \in [a, b]$.

If we multiply (2.7) and (2.8), then we get

$$\begin{aligned}
 (2.9) \quad & \frac{1}{4} [f(t) - f(a+b-t)] [g(t) - g(a+b-t)] \\
 &= \frac{1}{4} |[f(t) - f(a+b-t)] [g(t) - g(a+b-t)]| \\
 &\leq \frac{1}{4} [f(b) - f(a)] [g(b) - g(a)]
 \end{aligned}$$

for any $t \in [a, b]$.

Since

$$\begin{aligned}
 0 &\leq \int_a^b \tilde{f}(t) g(t) dt = \frac{1}{4} \int_a^b [f(t) - f(a+b-t)] [g(t) - g(a+b-t)] dt \\
 &\leq \frac{1}{4} [f(b) - f(a)] [g(b) - g(a)] (b-a),
 \end{aligned}$$

where for the last inequality we used (2.9), hence we get the first inequality in (2.5).

Also, we observe that

$$0 \leq \int_a^b \tilde{f}(t) g(t) dt = \int_a^b \left| \tilde{f}(t) g(t) \right| dt \leq \frac{1}{2} [f(b) - f(a)] \int_a^b |g(t)| dt$$

and since

$$\int_a^b \tilde{f}(t) g(t) dt = \int_a^b f(t) \tilde{g}(t) dt,$$

then also

$$\int_a^b f(t) \tilde{g}(t) dt \leq \frac{1}{2} [g(b) - g(a)] \int_a^b |f(t)| dt$$

and the inequality (2.6) is also proved. □

Remark 2. *If the functions $f, g : [a, b] \rightarrow \mathbb{R}$ are either both of them nonincreasing or nondecreasing on $[a, b]$, then they are integrable and we have the inequalities (2.5) and (2.6).*

We have the following refinement of the inequality in (2.1).

Theorem 5. *Assume that f, g are asymmetrized synchronous and integrable functions on $[a, b]$. Then*

$$\begin{aligned}
 (2.10) \quad & \frac{1}{b-a} \int_a^b \tilde{f}(t) g(t) dt \\
 &\geq \left| \frac{1}{b-a} \int_a^b |\tilde{f}(t)| |\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b |\tilde{f}(t)| dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt \right| \geq 0.
 \end{aligned}$$

Proof. By the continuity property of modulus, we have

$$\begin{aligned}
 \left| \tilde{f}(t) - \tilde{f}(s) \right| \left| \tilde{g}(t) - \tilde{g}(s) \right| &= \left| \left[\tilde{f}(t) - \tilde{f}(s) \right] \left[\tilde{g}(t) - \tilde{g}(s) \right] \right| \\
 &= \left| \tilde{f}(t) - \tilde{f}(s) \right| \left| \tilde{g}(t) - \tilde{g}(s) \right| \\
 &\geq \left| \left| \tilde{f}(t) \right| - \left| \tilde{f}(s) \right| \right| \left| \tilde{g}(t) - \tilde{g}(s) \right| \\
 &= \left| \left(\left| \tilde{f}(t) \right| - \left| \tilde{f}(s) \right| \right) \left(\tilde{g}(t) - \tilde{g}(s) \right) \right|
 \end{aligned}$$

for any $t, s \in [a, b]$.

Taking the double integral mean on $[a, b]^2$ and using the properties of the integral versus the modulus, we have

$$(2.11) \quad \begin{aligned} & \frac{1}{(b-a)^2} \int_a^b \int_a^b [\tilde{f}(t) - \tilde{f}(s)] [\tilde{g}(t) - \tilde{g}(s)] dt ds \\ & \geq \left| \frac{1}{(b-a)^2} \int_a^b \int_a^b (|\tilde{f}(t)| - |\tilde{f}(s)|) (|\tilde{g}(t)| - |\tilde{g}(s)|) dt ds \right|. \end{aligned}$$

Since, by Korkine's identity we have

$$\begin{aligned} & \frac{1}{(b-a)^2} \int_a^b \int_a^b [\tilde{f}(t) - \tilde{f}(s)] [\tilde{g}(t) - \tilde{g}(s)] dt ds \\ & = 2 \left[\frac{1}{b-a} \int_a^b \tilde{f}(t) \tilde{g}(t) dt - \frac{1}{b-a} \int_a^b \tilde{f}(t) dt \frac{1}{b-a} \int_a^b \tilde{g}(t) dt \right] \\ & = \frac{2}{b-a} \int_a^b \tilde{f}(t) \tilde{g}(t) dt \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(b-a)^2} \int_a^b \int_a^b (|\tilde{f}(t)| - |\tilde{f}(s)|) (|\tilde{g}(t)| - |\tilde{g}(s)|) dt ds \\ & = 2 \left[\frac{1}{b-a} \int_a^b |\tilde{f}(t)| |\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b |\tilde{f}(t)| dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt \right], \end{aligned}$$

hence by (2.11) we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \tilde{f}(t) \tilde{g}(t) dt \\ & \geq \left| \frac{1}{b-a} \int_a^b |\tilde{f}(t)| |\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b |\tilde{f}(t)| dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt \right|. \end{aligned}$$

By using the identity (2.3) we get the desired result (2.10). □

Remark 3. We remark that, if (\tilde{f}, g) are synchronous, then by a similar argument to the one above for $g \leftrightarrow \tilde{g}$ we have

$$(2.12) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b \tilde{f}(t) g(t) dt \\ & \geq \left| \frac{1}{b-a} \int_a^b |\tilde{f}(t)| |g(t)| dt - \frac{1}{b-a} \int_a^b |\tilde{f}(t)| dt \frac{1}{b-a} \int_a^b |g(t)| dt \right| \geq 0. \end{aligned}$$

Also, since

$$\frac{1}{b-a} \int_a^b \tilde{f}(t) g(t) dt = \frac{1}{b-a} \int_a^b f(t) \tilde{g}(t) dt,$$

then if we assume that (f, \tilde{g}) are synchronous we also have

$$(2.13) \quad \frac{1}{b-a} \int_a^b \tilde{f}(t) g(t) dt \geq \left| \frac{1}{b-a} \int_a^b |f(t)| |\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b |f(t)| dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt \right| \geq 0.$$

Now, if f and g have the same monotonicity, then $(\tilde{f}, \tilde{g}), (\tilde{f}, g), (f, \tilde{g})$ are synchronous and we have

$$(2.14) \quad \frac{1}{b-a} \int_a^b \tilde{f}(t) g(t) dt \geq \max \left\{ \left| C(\tilde{f}, \tilde{g}) \right|, \left| C(\tilde{f}, g) \right|, \left| C(f, \tilde{g}) \right| \right\} \geq 0,$$

where

$$C(h, \ell) := \frac{1}{b-a} \int_a^b |h(t) \ell(t)| dt - \frac{1}{b-a} \int_a^b |h(t)| dt \frac{1}{b-a} \int_a^b |\ell(t)| dt$$

provided h and ℓ are integrable on $[a, b]$.

We say that the function $h : [a, b] \rightarrow \mathbb{R}$ is H - r -Hölder continuous with the constant $H > 0$ and power $r \in (0, 1]$ if

$$(2.15) \quad |h(t) - h(s)| \leq H |t - s|^r$$

for any $t, s \in [a, b]$. If $r = 1$ we call that h is L -Lipschitzian when $H = L > 0$.

Theorem 6. Assume that f, g are asymmetrized synchronous with f is H_1 - r_1 -Hölder continuous and g is H_2 - r_2 -Hölder continuous on $[a, b]$. Then

$$(2.16) \quad \frac{1}{4(r_1 + r_2 + 1)} H_1 H_2 (b-a)^{r_1+r_2} \geq \frac{1}{b-a} \int_a^b \tilde{f}(t) g(t) dt \geq 0.$$

If particular, if f is L_1 -Lipschitzian and g is L_2 -Lipschitzian, then

$$(2.17) \quad \frac{1}{12} L_1 L_2 (b-a)^2 \geq \frac{1}{b-a} \int_a^b \tilde{f}(t) g(t) dt \geq 0.$$

Proof. From (2.3) we have

$$\begin{aligned} 0 &\leq \int_a^b \tilde{f}(t) g(t) dt = \frac{1}{4} \int_a^b [f(t) - f(a+b-t)][g(t) - g(a+b-t)] dt \\ &= \frac{1}{4} \int_a^b |[f(t) - f(a+b-t)][g(t) - g(a+b-t)]| dt \\ &\leq \frac{1}{4} H_1 H_2 \int_a^b |2t - a - b|^{r_1+r_2} dt = \frac{2^{r_1+r_2}}{4} H_1 H_2 \int_a^b \left| t - \frac{a+b}{2} \right|^{r_1+r_2} dt \\ &= \frac{2}{2^{2-r_1-r_2}} H_1 H_2 \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^{r_1+r_2} dt = \frac{2}{2^{2-r_1-r_2}} H_1 H_2 \frac{\left(\frac{b-a}{2}\right)^{r_1+r_2+1}}{r_1+r_2+1} \\ &= \frac{1}{4(r_1+r_2+1)} H_1 H_2 (b-a)^{r_1+r_2+1}, \end{aligned}$$

which is equivalent to the desired result (2.16). □

3. SOME EXAMPLES

Consider the identity function $\ell : [a, b] \rightarrow \mathbb{R}$ defined by $\ell(t) = t$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we have

$$(3.1) \quad \frac{1}{4}(b-a)[g(b)-g(a)] \geq \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) g(t) dt \geq \max\{|C_{1,\ell}(g)|, |C_{2,\ell}(g)|, |C_{3,\ell}(g)|\} \geq 0,$$

where

$$C_{1,\ell}(g) := \frac{1}{b-a} \int_a^b \left| \left(t - \frac{a+b}{2}\right) \tilde{g}(t) \right| dt - \frac{1}{4} \int_a^b |\tilde{g}(t)| dt,$$

$$C_{2,\ell}(g) := \frac{1}{b-a} \int_a^b \left| \left(t - \frac{a+b}{2}\right) g(t) \right| dt - \frac{1}{4} \int_a^b |g(t)| dt$$

and

$$C_{3,\ell}(g) := \frac{1}{b-a} \int_a^b |t\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b |t| dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt.$$

If g is monotonic nondecreasing and L -Lipschitzian on $[a, b]$, then by (2.17) we get

$$(3.2) \quad \frac{1}{12}L(b-a)^2 \geq \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) g(t) dt (\geq 0).$$

Consider the power function $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t^p$ with $p > 0$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we get

$$(3.3) \quad \frac{1}{4}(b^p - a^p)[g(b) - g(a)] \geq \frac{1}{b-a} \int_a^b \left[\frac{t^p - (a+b-t)^p}{2} \right] g(t) dt \geq \max\{|C_{1,p}(g)|, |C_{2,p}(g)|, |C_{3,p}(g)|\} \geq 0,$$

where

$$C_{1,p}(g) := \frac{1}{b-a} \int_a^b \left| \frac{t^p - (a+b-t)^p}{2} \right| |\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b \left| \frac{t^p - (a+b-t)^p}{2} \right| dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt,$$

$$C_{2,p}(g) := \frac{1}{b-a} \int_a^b \left| \frac{t^p - (a+b-t)^p}{2} \right| |g(t)| dt - \frac{1}{b-a} \int_a^b \left| \frac{t^p - (a+b-t)^p}{2} \right| dt \frac{1}{b-a} \int_a^b |g(t)| dt$$

and

$$C_{3,p}(g) := \int_a^b t^p |\tilde{g}(t)| dt - \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt.$$

If g is monotonic nondecreasing and L -Lipschitzian on $[a, b]$, then by (2.17) we get

$$(3.4) \quad \frac{p}{12}L(b-a)^2 \begin{cases} b^{p-1} & \text{if } p \geq 1 \\ a^{p-1} & \text{if } p \in (0, 1) \end{cases} \geq \frac{1}{b-a} \int_a^b \left[\frac{t^p - (a+b-t)^p}{2} \right] g(t) dt (\geq 0).$$

Consider the function $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$, $f = \ln$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we have

$$(3.5) \quad \frac{1}{4} \ln \left(\frac{b}{a} \right) [g(b) - g(a)] \geq \frac{1}{2(b-a)} \int_a^b \ln \left(\frac{t}{a+b-t} \right) g(t) dt \geq \max \{ |C_{1,\ln}(g)|, |C_{2,\ln}(g)|, |C_{3,\ln}(g)| \} \geq 0,$$

where

$$C_{1,\ln}(g) := \frac{1}{b-a} \int_a^b \left| \ln \left(\frac{t}{a+b-t} \right) \right|^{1/2} |\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b \left| \ln \left(\frac{t}{a+b-t} \right) \right|^{1/2} dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt,$$

$$C_{2,\ln}(g) := \frac{1}{b-a} \int_a^b |\ln t| |\tilde{g}(t)| dt - \frac{1}{b-a} \int_a^b |\ln t| dt \frac{1}{b-a} \int_a^b |\tilde{g}(t)| dt$$

and

$$C_{1,\ln}(g) := \frac{1}{b-a} \int_a^b \left| \ln \left(\frac{t}{a+b-t} \right) \right|^{1/2} |g(t)| dt - \frac{1}{b-a} \int_a^b \left| \ln \left(\frac{t}{a+b-t} \right) \right|^{1/2} dt \frac{1}{b-a} \int_a^b |g(t)| dt.$$

If g is monotonic nondecreasing and L -Lipschitzian on $[a, b]$, then by (2.17) we get

$$(3.6) \quad \frac{1}{6a} L(b-a)^2 \geq \frac{1}{b-a} \int_a^b \ln \left(\frac{t}{a+b-t} \right) g(t) dt \ (\geq 0).$$

Consider the function $f : [a, b] \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}$, $f = \sin$. If g is monotonic nondecreasing, then by (2.5) we have

$$(3.7) \quad \frac{1}{2} \sin \left(\frac{b-a}{2} \right) [g(b) - g(a)] \geq \frac{1}{b-a} \int_a^b \sin \left(t - \frac{a+b}{2} \right) g(t) dt \geq 0.$$

If g is monotonic nondecreasing and L -Lipschitzian on $[a, b]$, then by (2.17) we get

$$(3.8) \quad \frac{1}{12} L(b-a)^2 \times \begin{cases} \cos b & \text{if } -\frac{\pi}{2} \leq a < b \leq 0, \\ \max \{ \cos a, \cos b \} & \text{if } -\frac{\pi}{2} \leq a < 0 < b \leq \frac{\pi}{2}, \\ \cos a & \text{if } 0 \leq a < b \leq \frac{\pi}{2} \end{cases} \geq \frac{1}{b-a} \cos \left(\frac{a+b}{2} \right) \int_a^b \sin \left(t - \frac{a+b}{2} \right) g(t) dt \ (\geq 0).$$

REFERENCES

[1] A. Aglić Aljinović, R. Hoxha and J. Pečarić, On some Čebyšev-Grüss type integral inequalities. *Mat. Bilten* No. **30** (2006), 49–62
 [2] P. R. Beesack and J. E. Pečarić, Integral inequalities of Čebyšev type. *J. Math. Anal. Appl.* **111** (1985), no. 2, 643–659.
 [3] K. Boukerrioua and A. Guezane-Lakoud, On generalization of Čebyšev type inequalities. *J. Inequal. Pure Appl. Math.* **8** (2007), No. 2, Article 55, 4 pp.

- [4] S. S. Dragomir, On some improvements of Čebyšev's inequality for sequences and integrals. *Studia Univ. Babeş-Bolyai Math.* **35** (1990), no. 4, 35–40.
- [5] S. S. Dragomir, A refinement of Ostrowski's inequality for the Čebyšev functional and applications. *Analysis (Munich)* **23** (2003), No. 4, 287–297.
- [6] S. S. Dragomir, Symmetrized convexity and Hermite-Hadamard type inequalities, *J. Math. Ineq.* **10** (2016), No. 4, 901–918. Preprint *RGMA Res. Rep. Coll.* **17** (2014), Art. 2. [Online <http://rgmia.org/papers/v17/v17a02.pdf>].
- [7] S. S. Dragomir and B. Mond, Some mappings associated with Cebysev's inequality for sequences of real numbers. *Bull. Allahabad Math. Soc.* **8/9** (1993/94), 37–55 (1997).
- [8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].
- [9] A. El Farissi, M. Benbachir and M. Dahmane, An extension of the Hermite-Hadamard inequality for convex symmetrized functions. *Real Anal. Exchange*, **38** (2012/13), No. 2, 467–474.
- [10] A. Guezane-Lakoud and F. Aissaoui, New Čebyšev type inequalities for double integrals. *J. Math. Inequal.* **5** (2011), No. 4, 453–462.
- [11] S.-R. Hwang, Ostrowski-Grüss-Čebyšev type inequalities involving several functions. *Tamsui Oxf. J. Math. Sci.* **23** (2007), No. 1, 105–125.
- [12] Z. Liu, Generalizations of some new Čebyšev type inequalities. *J. Inequal. Pure Appl. Math.* **8** (2007), No. 1, Article 13, 6 pp.
- [13] B. G. Pachpatte, On Ostrowski-Grüss-Čebyšev type inequalities for functions whose modulus of derivatives are convex. *J. Inequal. Pure Appl. Math.* **6** (2005), No. 4, Article 128, 15 pp.
- [14] B. G. Pachpatte, New inequalities of Čebyšev type for double integrals, *Demonstratio Math.* **40** (2007), no. 1, 43–50.
- [15] B. G. Pachpatte, A note on Čebyšev type inequalities. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **53** (2007), no. 1, 97–102.
- [16] R. T. Rahmail, Inversion of Čebyšev's inequality for bounded functions. (Russian) *Izv. Vysš. Učebn. Zaved. Matematika* 1977, No. **12** (187), 77–84.
- [17] E. Set, M. Z. Sarikaya and F. Ahmad, A generalization of Čebyšev type inequalities for first differentiable mappings. *Miskolc Math. Notes* **12** (2011), No. 2, 245–253.
- [18] F. Zafar and N. A. Mir, A note on the generalization of some new Čebyšev type inequalities. *Tamsui Oxf. J. Inf. Math. Sci.* **27** (2011), No. 2, 149–157.

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FURTHER INEQUALITIES FOR HEINZ OPERATOR MEAN

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ABSTRACT. In this paper we obtain some new inequalities for Heinz operator mean.

1. INTRODUCTION

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0, 1]$

$$A\nabla_\nu B := (1 - \nu)A + \nu B,$$

the *weighted operator arithmetic mean*, and

$$A\sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean* [13]. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

Define the *Heinz operator mean* by

$$H_\nu(A, B) := \frac{1}{2} (A\sharp_\nu B + A\sharp_{1-\nu} B).$$

The following interpolatory inequality is obvious

$$(1.1) \quad A\sharp B \leq H_\nu(A, B) \leq A\nabla B$$

for any $\nu \in [0, 1]$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$(1.2) \quad a^{1-\nu} b^\nu \leq (1 - \nu)a + \nu b$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν -weighted arithmetic-geometric mean inequality*.

We consider the *Kantorovich's constant* defined by

$$(1.3) \quad K(h) := \frac{(h+1)^2}{4h}, \quad h > 0.$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K(\frac{1}{h})$ for any $h > 0$.

In the recent paper [1] we have obtained the following additive and multiplicative reverse of Young's inequality

$$(1.4) \quad 0 \leq (1 - \nu)a + \nu b - a^{1-\nu} b^\nu \leq \nu(1 - \nu)(a - b)(\ln a - \ln b)$$

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and

$$(1.5) \quad 1 \leq \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^\nu} \leq \exp \left[4\nu(1-\nu) \left(K \left(\frac{a}{b} \right) - 1 \right) \right],$$

for any $a, b > 0$ and $\nu \in [0, 1]$, where K is Kantorovich's constant.

The operator version of (1.4) is as follows [1]:

Theorem 1. *Let A, B be two positive operators. For positive real numbers m, m', M, M' , put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0, 1]$.*

(i) *If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then*

$$(1.6) \quad 0 \leq A\nabla_\nu B - A\sharp_\nu B \leq \nu(1-\nu)(h-1) \ln hA$$

and, in particular

$$(1.7) \quad 0 \leq A\nabla B - A\sharp B \leq \frac{1}{4}(h-1) \ln hA.$$

(ii) *If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then*

$$(1.8) \quad 0 \leq A\nabla_\nu B - A\sharp_\nu B \leq \nu(1-\nu) \frac{h-1}{h} \ln hA$$

and, in particular

$$(1.9) \quad 0 \leq A\nabla B - A\sharp B \leq \frac{1}{4} \frac{h-1}{h} \ln hA.$$

The operator version of (1.5) is [1]:

Theorem 2. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions*

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$;

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;

we have

$$(1.10) \quad A\nabla_\nu B \leq \exp [4\nu(1-\nu)(K(h)-1)] A\sharp_\nu B$$

and, in particular

$$(1.11) \quad A\nabla B \leq \exp [K(h)-1] A\sharp B.$$

For other recent results on geometric operator mean inequalities, see [2]-[12], [14] and [16]-[17].

We recall that *Specht's ratio* is defined by [15]

$$(1.12) \quad S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln \left(h^{\frac{1}{h-1}} \right)} & \text{if } h \in (0, 1) \cup (1, \infty), \\ 1 & \text{if } h = 1. \end{cases}$$

It is well known that $\lim_{h \rightarrow 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0, h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

In the recent paper [6] we obtained amongst other the following result for the Heinz operator mean of A, B that are positive invertible operators that satisfy the condition $mA \leq B \leq MA$ for some constants $M > m > 0$,

$$(1.13) \quad \omega(m, M) A\sharp B \leq H_\nu(A, B) \leq \Omega(m, M) A\sharp B,$$

where

$$\Omega(m, M) := \begin{cases} S(m^{|2\nu-1|}) & \text{if } M < 1, \\ \max\{S(m^{|2\nu-1|}), S(M^{|2\nu-1|})\} & \text{if } m \leq 1 \leq M, \\ S(M^{|2\nu-1|}) & \text{if } 1 < m, \end{cases}$$

and

$$\omega(m, M) := \begin{cases} S(M^{|\nu-\frac{1}{2}|}) & \text{if } M < 1, \\ 1 & \text{if } m \leq 1 \leq M, \\ S(m^{|\nu-\frac{1}{2}|}) & \text{if } 1 < m. \end{cases}$$

Motivated by the above results we establish in this paper some new additive and multiplicative reverse inequalities for the Heinz operator mean.

2. ADDITIVE REVERSE INEQUALITIES FOR HEINZ MEAN

We have the following generalization of Theorem 1:

Theorem 3. *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that*

$$(2.1) \quad mA \leq B \leq MA.$$

Then for any $\nu \in [0, 1]$ we have

$$(2.2) \quad (0 \leq) A\nabla_{\nu}B - A\sharp_{\nu}B \leq \nu(1-\nu)\Omega(m, M)A$$

where

$$(2.3) \quad \Omega(m, M) := \begin{cases} (m-1)\ln m & \text{if } M < 1, \\ \max\{(m-1)\ln m, (M-1)\ln M\} & \text{if } m \leq 1 \leq M, \\ (M-1)\ln M & \text{if } 1 < m. \end{cases}$$

In particular, we have

$$(2.4) \quad (0 \leq) A\nabla B - A\sharp B \leq \frac{1}{4}\Omega(m, M)A.$$

Proof. We consider the function $D : (0, \infty) \rightarrow [0, \infty)$ defined by $D(x) = (x-1)\ln x$. We have that $D'(x) = \ln x + 1 - \frac{1}{x}$ and $D''(x) = \frac{x+1}{x^2}$ for $x \in (0, \infty)$. This shows that the function is convex on $(0, \infty)$, monotonic decreasing on $(0, 1)$ and monotonic increasing on $[1, \infty)$ with the minimum 0 realized in $x = 1$.

From the inequality (1.4) we have

$$(0 \leq) (1-\nu) + \nu x - x^{\nu} \leq \nu(1-\nu)D(x)$$

for any $x > 0$, $\nu \in [0, 1]$ and hence

$$(2.5) \quad (0 \leq) (1-\nu)I + \nu X - X^{\nu} \leq \nu(1-\nu) \max_{m \leq x \leq M} D(x)$$

for the positive operator X that satisfies the condition $0 < mI \leq X \leq MI$ for $0 < m < M$ and $\nu \in [0, 1]$.

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (2.5) we get

$$(2.6) \quad (1 - \nu)I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^\nu \leq \nu(1 - \nu) \max_{m \leq x \leq M} D(x)$$

Now, if we multiply (2.6) in both sides with $A^{1/2}$ we get

$$(2.7) \quad \begin{aligned} (0 \leq) (1 - \nu)A + \nu B - A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2} \\ \leq \nu(1 - \nu) \max_{m \leq x \leq M} D(x)A \end{aligned}$$

for any $\nu \in [0, 1]$.

Finally, since

$$\max_{m \leq x \leq M} D(x) = \begin{cases} (m - 1) \ln m & \text{if } M < 1, \\ \max \{(m - 1) \ln m, (M - 1) \ln M\} & \text{if } m \leq 1 \leq M, \\ (M - 1) \ln M & \text{if } 1 < m, \end{cases}$$

then by (2.7) we get the desired result (2.2). □

Corollary 1. *With the assumptions of Theorem 3 we have*

$$(2.8) \quad (0 \leq) A \nabla B - H_\nu(A, B) \leq \nu(1 - \nu) \Omega(m, M)A.$$

Proof. From (2.2) we have by replacing ν with $1 - \nu$ that

$$(2.9) \quad (0 \leq) A \nabla_{1-\nu} B - A \sharp_{1-\nu} B \leq \nu(1 - \nu) \Omega(m, M)A.$$

Adding (2.2) with (2.9) and dividing by 2 we get (2.8). □

Corollary 2. *Let A, B be two positive operators. For positive real numbers m, m', M, M' , put $h := \frac{M}{m}, h' := \frac{M'}{m'}$ and let $\nu \in [0, 1]$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$(2.10) \quad (0 \leq) A \nabla B - H_\nu(A, B) \leq \nu(1 - \nu)(h - 1) \ln hA.$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$(2.11) \quad (0 \leq) A \nabla B - H_\nu(A, B) \leq \nu(1 - \nu) \left(\frac{h - 1}{h}\right) \ln hA.$$

Proof. If the condition (i) is valid, then we have

$$I < \frac{M'}{m'}I = h'I \leq X \leq hI = \frac{M}{m}I,$$

which, by (2.8) gives the desired result (2.10).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \leq X \leq \frac{1}{h'}I < I,$$

which, by (2.8) gives

$$(0 \leq) A \nabla B - H_\nu(A, B) \leq \nu(1 - \nu) \left(\frac{1}{h} - 1\right) \ln \frac{1}{h}$$

that is equivalent to (2.11). □

Theorem 4. *With the assumptions of Theorem 3 we have*

$$(2.12) \quad (0 \leq) H_\nu(A, B) - A\sharp B \leq \frac{1}{4m^{1-\nu}} \max_{x \in [m, M]} D(x^{2\nu-1}) A,$$

where the function $D : (0, \infty) \rightarrow [0, \infty)$ is defined by $D(x) = (x - 1) \ln x$ (see the proof of Theorem 3).

Proof. From the inequality (1.4) we have for $\nu = \frac{1}{2}$

$$(2.13) \quad (0 \leq) \frac{c+d}{2} - \sqrt{cd} \leq \frac{1}{4} (c-d) (\ln c - \ln d)$$

for any $c, d > 0$.

If we take in (2.13) $c = a^{1-\nu}b^\nu$ and $d = a^\nu b^{1-\nu}$ then we get

$$(2.14) \quad \frac{a^{1-\nu}b^\nu + a^\nu b^{1-\nu}}{2} - \sqrt{ab} \leq \frac{1}{4} (a^{1-\nu}b^\nu - a^\nu b^{1-\nu}) (\ln a^{1-\nu}b^\nu - \ln a^\nu b^{1-\nu})$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

This inequality is of interest in itself.

Now, if we take in (2.14) $a = 1$ and $b = x$, then we get

$$(2.15) \quad \begin{aligned} 0 &\leq \frac{x^\nu + x^{1-\nu}}{2} - \sqrt{x} \leq \frac{1}{4} (x^\nu - x^{1-\nu}) (\ln x^\nu - \ln x^{1-\nu}) \\ &= \frac{2\nu - 1}{4} (x^\nu - x^{1-\nu}) \ln x = \frac{1}{4x^{1-\nu}} (x^{2\nu-1} - 1) \ln x^{2\nu-1} \\ &= \frac{1}{4x^{1-\nu}} D(x^{2\nu-1}) \end{aligned}$$

for any $x > 0$ and $\nu \in [0, 1]$.

Now, if $x \in [m, M] \subset (0, \infty)$, then by (2.15) we get the upper bound

$$(0 \leq) \frac{x^\nu + x^{1-\nu}}{2} - \sqrt{x} \leq \frac{1}{4m^{1-\nu}} \max_{x \in [m, M]} D(x^{2\nu-1}).$$

Using the continuous functional calculus, we then have

$$(2.16) \quad (0 \leq) \frac{X^\nu + X^{1-\nu}}{2} - X^{1/2} \leq \frac{1}{4m^{1-\nu}} \max_{x \in [m, M]} D(x^{2\nu-1})$$

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (2.16) we get

$$(2.17) \quad \begin{aligned} 0 &\leq \frac{(A^{-1/2}BA^{-1/2})^\nu + (A^{-1/2}BA^{-1/2})^{1-\nu}}{2} - (A^{-1/2}BA^{-1/2})^{1/2} \\ &\leq \frac{1}{4m^{1-\nu}} \max_{x \in [m, M]} D(x^{2\nu-1}) \end{aligned}$$

for any $\nu \in [0, 1]$.

Now, if we multiply (2.17) in both sides with $A^{1/2}$ we get the desired result (2.12). \square

Corollary 3. *Let A, B be as in Corollary 2.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$(2.18) \quad (0 \leq) H_\nu(A, B) - A\sharp B \leq \frac{1}{4(h')^{1-\nu}} \begin{cases} (h^{2\nu-1} - 1) \ln h^{2\nu-1} & \text{if } \nu \in [\frac{1}{2}, 1], \\ ((h')^{2\nu-1} - 1) \ln (h')^{2\nu-1} & \text{if } \nu \in [0, \frac{1}{2}]. \end{cases}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$(2.19) \quad (0 \leq) H_\nu(A, B) - A\sharp B \leq \frac{1}{4} h^{1-\nu} \begin{cases} (h^{-2\nu+1} - 1) \ln h^{-2\nu+1} & \text{if } \nu \in [\frac{1}{2}, 1], \\ ((h')^{-2\nu+1} - 1) \ln (h')^{-2\nu+1} & \text{if } \nu \in [0, \frac{1}{2}]. \end{cases}$$

Proof. If the condition (i) is valid, then we have

$$I < \frac{M'}{m'} I = h' I \leq X \leq h I = \frac{M}{m} I,$$

which, by (2.12) gives

$$(2.20) \quad 0 \leq H_\nu(A, B) - A\sharp B \leq \frac{1}{4(h')^{1-\nu}} \max_{x \in [h', h]} D(x^{2\nu-1}) A.$$

Observe that, if $\nu \in [\frac{1}{2}, 1]$, then

$$\max_{x \in [h', h]} D(x^{2\nu-1}) = D(h^{2\nu-1}) = (h^{2\nu-1} - 1) \ln h^{2\nu-1}.$$

If $\nu \in [0, \frac{1}{2}]$, then

$$\max_{x \in [h', h]} D(x^{2\nu-1}) = D((h')^{2\nu-1}) = ((h')^{2\nu-1} - 1) \ln (h')^{2\nu-1}.$$

By (2.20) we get the desired result (2.18).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h} I \leq X \leq \frac{1}{h'} I < I,$$

which, by (2.12) gives

$$(2.21) \quad 0 \leq H_\nu(A, B) - A\sharp B \leq \frac{1}{4(\frac{1}{h})^{1-\nu}} \max_{x \in [\frac{1}{h}, \frac{1}{h'}]} D(x^{2\nu-1}) A.$$

If $\nu \in [\frac{1}{2}, 1]$, then

$$\max_{x \in [\frac{1}{h}, \frac{1}{h'}]} D(x^{2\nu-1}) = D\left(\left(\frac{1}{h}\right)^{2\nu-1}\right) = D((h)^{-2\nu+1}).$$

If $\nu \in [0, \frac{1}{2}]$, then

$$\max_{x \in [\frac{1}{h}, \frac{1}{h'}]} D(x^{2\nu-1}) = D\left(\left(\frac{1}{h'}\right)^{2\nu-1}\right) = D((h')^{-2\nu+1}).$$

By (2.21) we get the desired result (2.19). □

3. MULTIPLICATIVE REVERSE INEQUALITIES FOR HEINZ MEAN

We have the following generalization of Theorem 2:

Theorem 5. *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that the condition (2.1) is valid. Then for any $\nu \in [0, 1]$ we have*

$$(3.1) \quad A\nabla_{\nu}B \leq A\sharp_{\nu}B \exp [4\nu (1 - \nu) (F (m, M) - 1)]$$

where

$$F (m, M) := \begin{cases} K (m) & \text{if } M < 1, \\ \max \{K (m), K (M)\} & \text{if } m \leq 1 \leq M, \\ K (M) & \text{if } 1 < m, \end{cases}$$

In particular, we have

$$(3.2) \quad A\nabla B \leq A\sharp B \exp [F (m, M) - 1].$$

Proof. From the inequality (1.5) we have for $a = 1$ and $b = x$ that

$$(3.3) \quad (1 - \nu) + \nu x \leq x^{\nu} \exp \left[4\nu (1 - \nu) \left(K \left(\frac{1}{x} \right) - 1 \right) \right] \\ = x^{\nu} \exp [4\nu (1 - \nu) (K (x) - 1)]$$

for any $x > 0$ and hence

$$(3.4) \quad (1 - \nu) I + \nu X \leq X^{\nu} \max_{m \leq x \leq M} \exp [4\nu (1 - \nu) (K (x) - 1)] \\ = X^{\nu} \exp \left[4\nu (1 - \nu) \left(\max_{m \leq x \leq M} K (x) - 1 \right) \right]$$

for any operator X with the property that $0 < mI \leq X \leq MI$ and for any $\nu \in [0, 1]$.

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (3.4) we get

$$(3.5) \quad (1 - \nu) I + \nu A^{-1/2}BA^{-1/2} \\ \leq \left(A^{-1/2}BA^{-1/2} \right)^{\nu} \max_{m \leq x \leq M} \exp [4\nu (1 - \nu) (K (x) - 1)] \\ = \left(A^{-1/2}BA^{-1/2} \right)^{\nu} \exp \left[4\nu (1 - \nu) \left(\max_{m \leq x \leq M} K (x) - 1 \right) \right]$$

for any $\nu \in [0, 1]$.

Now, if we multiply (3.5) in both sides with $A^{1/2}$ we get

$$(3.6) \quad (1 - \nu) A + \nu BA \\ \leq A^{1/2} \left(A^{-1/2}BA^{-1/2} \right)^{\nu} A^{1/2} \max_{m \leq x \leq M} \exp [4\nu (1 - \nu) (K (x) - 1)] \\ = A^{1/2} \left(A^{-1/2}BA^{-1/2} \right)^{\nu} A^{1/2} \exp \left[4\nu (1 - \nu) \left(\max_{m \leq x \leq M} K (x) - 1 \right) \right]$$

for any $\nu \in [0, 1]$.

Since

$$\max_{m \leq x \leq M} K(x) = \begin{cases} K(m) & \text{if } M < 1, \\ \max\{K(m), K(M)\} & \text{if } m \leq 1 \leq M, \\ K(M) & \text{if } 1 < m, \end{cases}$$

then by (3.6) we get the desired result (3.1). □

Corollary 4. *With the assumptions of Theorem 5 we have*

$$(3.7) \quad A \nabla B \leq \exp [4\nu(1-\nu)(F(m, M) - 1)] H_\nu(A, B).$$

Corollary 5. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:*

- (i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$;
- (ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;

we have

$$(3.8) \quad A \nabla B \leq \exp [4\nu(1-\nu)(K(h) - 1)] H_\nu(A, B).$$

We also have:

Theorem 6. *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that the condition (2.1) is valid. Then for any $\nu \in [0, 1]$ we have*

$$(3.9) \quad H_\nu(A, B) \leq \exp [\Theta_\nu(m, M) - 1] A \sharp B$$

where

$$(3.10) \quad \Theta_\nu(m, M) := \begin{cases} K(m^{2\nu-1}) & \text{if } M < 1, \\ \max\{K(m^{2\nu-1}), K(M^{2\nu-1})\} & \text{if } m \leq 1 \leq M, \\ K(M^{2\nu-1}) & \text{if } 1 < m. \end{cases}$$

Proof. From the inequality (1.5) we have for $\nu = \frac{1}{2}$

$$(3.11) \quad \frac{\frac{c+d}{2}}{\sqrt{cd}} \leq \exp \left(K \left(\frac{c}{d} \right) - 1 \right)$$

for any $c, d > 0$.

If we take in (3.11) $c = a^{1-\nu}b^\nu$ and $d = a^\nu b^{1-\nu}$ then we get

$$(3.12) \quad \frac{a^{1-\nu}b^\nu + a^\nu b^{1-\nu}}{2} \leq \exp \left(K \left(\left(\frac{a}{b} \right)^{1-2\nu} \right) - 1 \right) \sqrt{ab}$$

for any $a, b > 0$ for any $\nu \in [0, 1]$.

This is an inequality of interest in itself.

If we take in (2.19) $a = x$ and $b = 1$, then we get

$$(3.13) \quad \frac{x^{1-\nu} + x^\nu}{2} \leq \exp (K(x^{1-2\nu}) - 1) \sqrt{x},$$

for any $x > 0$.

Now, if $x \in [m, M] \subset (0, \infty)$ then by (2.20) we have

$$(3.14) \quad \frac{x^{1-\nu} + x^\nu}{2} \leq \sqrt{x} \exp \left(\max_{x \in [m, M]} K(x^{1-2\nu}) - 1 \right)$$

for any $x \in [m, M]$.

If $\nu \in (0, \frac{1}{2})$, then

$$\max_{x \in [m, M]} K(x^{1-2\nu}) = \begin{cases} K(m^{1-2\nu}) & \text{if } M < 1, \\ \max \{K(m^{1-2\nu}), K(M^{1-2\nu})\} & \text{if } m \leq 1 \leq M, \\ K(M^{1-2\nu}) & \text{if } 1 < m. \end{cases}$$

If $\nu \in (\frac{1}{2}, 1)$, then

$$\begin{aligned} \max_{x \in [m, M]} K(x^{1-2\nu}) &= \max_{x \in [m, M]} K(x^{2\nu-1}) \\ &= \begin{cases} K(m^{2\nu-1}) & \text{if } M < 1, \\ \max \{K(m^{2\nu-1}), K(M^{2\nu-1})\} & \text{if } m \leq 1 \leq M, \\ K(M^{2\nu-1}) & \text{if } 1 < m. \end{cases} \end{aligned}$$

Therefore, by (3.14) we have

$$(3.15) \quad \frac{x^{1-\nu} + x^\nu}{2} \leq \exp[\Theta(m, M) - 1] \sqrt{x}$$

for any $x \in [m, M] \subset (0, \infty)$ and for any $\nu \in [0, 1]$.

If X is an operator with $mI \leq X \leq MI$, then by (3.15) we have

$$\frac{X^{1-\nu} + X^\nu}{2} \leq \exp[\Theta(m, M) - 1] X^{1/2}.$$

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (3.15) we get

$$(3.16) \quad \begin{aligned} &\frac{1}{2} \left[\left(A^{-1/2}BA^{-1/2} \right)^{1-\nu} + \left(A^{-1/2}BA^{-1/2} \right)^\nu \right] \\ &\leq \exp[\Theta(m, M) - 1] \left(A^{-1/2}BA^{-1/2} \right)^{1/2}. \end{aligned}$$

Now, if we multiply (3.16) in both sides with $A^{1/2}$ we get the desired result (3.9). \square

Finally, we have

Corollary 6. For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$,

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$,

we have for $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$ that

$$(3.17) \quad H_\nu(A, B) \leq \exp \left[K \left(h^{|2\nu-1|} \right) - 1 \right] A \sharp B,$$

where $\nu \in [0, 1]$.

REFERENCES

- [1] S. S. Dragomir, Some new reverses of Young's operator inequality, Preprint *RGMA Res. Rep. Coll.* **18** (2015), Art. 130. [<http://rgmia.org/papers/v18/v18a130.pdf>].
- [2] S. S. Dragomir, On new refinements and reverses of Young's operator inequality, Preprint *RGMA Res. Rep. Coll.* **18** (2015), Art. 135. [<http://rgmia.org/papers/v18/v18a135.pdf>].
- [3] S. S. Dragomir, Some inequalities for operator weighted geometric mean, Preprint *RGMA Res. Rep. Coll.* **18** (2015), Art. 139. [<http://rgmia.org/papers/v18/v18a139.pdf>].
- [4] S. S. Dragomir, Refinements and reverses of Hölder-McCarthy operator inequality, Preprint *RGMA Res. Rep. Coll.* **18** (2015), Art. 143. [<http://rgmia.org/papers/v18/v18a143.pdf>].
- [5] S. S. Dragomir, Some reverses and a refinement of Hölder operator inequality, Preprint *RGMA Res. Rep. Coll.* **18** (2015), Art. 147. [<http://rgmia.org/papers/v18/v18a147.pdf>].
- [6] S. S. Dragomir, Some inequalities for Heinz operator mean, Preprint *RGMA Res. Rep. Coll.* **18** (2015), Art. 163. [<http://rgmia.org/papers/v18/v18a163.pdf>].
- [7] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.
- [8] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46-49.
- [9] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.
- [10] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Lin. Multilin. Alg.*, **59** (2011), 1031-1037.
- [11] F. Kittaneh, M. Krnić, N. Lovrićević and J. Pečarić, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators, *Publ. Math. Debrecen*, 2012, **80**(3-4), 465-478.
- [12] M. Krnić and J. Pečarić, Improved Heinz inequalities via the Jensen functional, *Cent. Eur. J. Math.* **11** (9) 2013,1698-1710.
- [13] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264** (1980), 205-224.
- [14] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
- [15] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.
- [16] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.
- [17] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.*, **5** (2011), 551-556.

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Global Dynamics of Monotone Second Order Difference Equation

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Abstract. We investigate the global character of the difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots$$

with several period-two solutions, where f is decreasing in the first variable and is increasing in the second variable. We show that the boundaries of the basins of attractions of different locally asymptotically stable equilibrium solutions or period-two solutions are in fact the global stable manifolds of neighboring saddle or non-hyperbolic equilibrium solutions or period-two solutions. We illustrate our results with the complete study of global dynamics of a certain rational difference equation with quadratic terms.

Keywords. asymptotic stability, attractivity, bifurcation, difference equation, global, local stability, period two;

AMS 2000 Mathematics Subject Classification: 37B25, 37D10, 39A20, 39A30.

1 Introduction and Preliminaries

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$ be such that $f(I \times I) \subseteq \mathcal{K}$ where $\mathcal{K} \subseteq I$ is a compact set. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{1}$$

where f is a continuous and decreasing in the first variable and increasing in the second variable. The following result gives a general information about global behavior of solutions of Equation (1).

Theorem 1 ([4])

Let $I \subseteq \mathbb{R}$ and let $f \in C[I \times I, I]$ be a function which is non-decreasing in first and non-increasing in second argument. Then for every solution of Equation (1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of (1) converges to either an equilibrium or period-two solution or to the singular point on the boundary. Consequently, most important question becomes determining the basins of attraction of these solutions as well as the unbounded solutions. The answer to this question follows from an application of the theory of monotone maps in the plane which will be presented in Preliminaries.

In [1, 2, 3] authors consider difference equation (1) with several equilibrium solutions as well as the period-two solutions and determine the basins of attraction of different equilibrium solutions and the period-two solutions. In this paper we consider Equation (1) which has up to two equilibrium solutions and up to two minimal period-two solutions which are in South-East ordering. More precisely, we will give sufficient conditions for the precise description of the basins of attraction of different equilibrium solutions and period-two solutions. The results can be immediately extended to the case of any number of the equilibrium solutions and the period-two solutions by replicating our main results.

This paper is organized as follows. In the rest of this section we will recall several basic results on competitive systems in the plane from [7, 15, 16, 17] which are included for completeness of presentation. Our main results about some global dynamics scenarios for monotone systems in the plane and their application to global dynamics of

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Equation (1) are given in section 2. As an application of the results from section 2 in section 3 the global dynamics of difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \dots \tag{2}$$

with all non-negative parameters and initial conditions is presented. All global dynamic scenarios for Equation (1) will be illustrated in the case of Equation (2), which global dynamics can be shortly described as the sequence of exchange of stability bifurcations between an equilibrium and one or two period-two solutions.

We now give some basic notions about monotone maps in the plane.

Definition 2 Let R be a subset of \mathbb{R}^2 with nonempty interior, and let $T : R \rightarrow R$ be a map (i.e., a continuous function). Set $T(x, y) = (f(x, y), g(x, y))$. The map T is competitive if $f(x, y)$ is non-decreasing in x and non-increasing in y , and $g(x, y)$ is non-increasing in x and non-decreasing in y . If both f and g are nondecreasing in x and y , we say that T is cooperative. If T is competitive (cooperative), the associated system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}, \quad n = 0, 1, \dots, \quad (x_{-1}, x_0) \in R \tag{3}$$

is said to be competitive (cooperative). The map T and associated difference equations system are said to be strongly competitive (strongly cooperative) if the adjectives non-decreasing and non-increasing are replaced by increasing and decreasing.

Consider a partial ordering \preceq on \mathbb{R}^2 . Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on \mathcal{R} if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on \mathcal{R} if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the North-East ordering (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$. Now we can show that a map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is cooperative and a map monotone with respect to the South-East ordering is competitive.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \dots, 4$ to be the usual four quadrants based at $x = (x_1, x_2)$ and numbered in a counterclockwise direction, for example, $Q_1(x) = \{y = (y_1, y_2) \in \mathbb{R}^2 : x_1 \leq y_1, x_2 \leq y_2\}$. Basin of attraction of a fixed point (\bar{x}, \bar{y}) of a map T , denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to (\bar{x}, \bar{y}) . Similarly, we define a basin of attraction of a periodic point of period p . The fixed point $A(x, y)$ of the map T is said to be non-hyperbolic point of stable type if one of the roots of characteristic equation evaluated in A is 1 or -1 and the second root is in $(-1, 1)$.

The next four results, from [16, 17], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [7, 19] and in [18].

Theorem 3 Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\bar{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \text{int}(Q_1(\bar{x}) \cup Q_3(\bar{x}))$ is nonempty (i.e., \bar{x} is not the NW or SE vertex of \mathcal{R}), and T is strongly competitive on Δ . Suppose that the following statements are true.

- a. The map T has a C^1 extension to a neighborhood of \bar{x} .
- b. The Jacobian $J_T(\bar{x})$ of T at \bar{x} has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through \bar{x} that is invariant and a subset of the basin of attraction of \bar{x} , such that \mathcal{C} is tangential to the eigenspace E^λ at \bar{x} , and \mathcal{C} is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of \mathcal{C} in the interior of \mathcal{R} are either fixed points or minimal period-two points. In the latter case, the set of endpoints of \mathcal{C} is a minimal period-two orbit of T .

Theorem 4 For the curve \mathcal{C} of Theorem 3 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.

- i. The map T has no fixed points nor periodic points of minimal period two in Δ .
- ii. The map T has no fixed points in Δ , $\det J_T(\bar{x}) > 0$, and $T(x) = \bar{x}$ has no solutions $x \in \Delta$.
- iii. The map T has no points of minimal period-two in Δ , $\det J_T(\bar{x}) < 0$, and $T(x) = \bar{x}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 3 reduces just to $|\lambda| < 1$. This follows from a change of variables [19] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 5 *Assume the hypotheses of Theorem 3, and let \mathcal{C} be the curve whose existence is guaranteed by Theorem 3. If the endpoints of \mathcal{C} belong to $\partial\mathcal{R}$, then \mathcal{C} separates \mathcal{R} into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \quad (4)$$

such that the following statements are true.

(i) \mathcal{W}_- is invariant, and $\text{dist}(T^n(x), Q_2(\bar{x})) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_-$.

(ii) \mathcal{W}_+ is invariant, and $\text{dist}(T^n(x), Q_4(\bar{x})) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), \bar{x} is an interior point of \mathcal{R} and T is C^2 and strongly competitive in a neighborhood of \bar{x} , then T has no periodic points in the boundary of $Q_1(\bar{x}) \cup Q_3(\bar{x})$ except for \bar{x} , and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int } Q_2(\bar{x})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int } Q_4(\bar{x})$ for $n \geq n_0$.

If T is a map on a set \mathcal{R} and if \bar{x} is a fixed point of T , the stable set $\mathcal{W}^s(\bar{x})$ of \bar{x} is the set $\{x \in \mathcal{R} : T^n(x) \rightarrow \bar{x}\}$ and unstable set $\mathcal{W}^u(\bar{x})$ of \bar{x} is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^0 \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1}, x_0 = x, \text{ and } \lim_{n \rightarrow -\infty} x_n = \bar{x} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\bar{x})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\bar{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on \mathcal{R} , the sets $\mathcal{W}^s(\bar{x})$ and $\mathcal{W}^u(\bar{x})$ are the stable and unstable manifolds of \bar{x} .

Theorem 6 *In addition to the hypotheses of part (B) of Theorem 5, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve \mathcal{C} of Theorem 3 has endpoints in $\partial\mathcal{R}$, then \mathcal{C} is the stable set $\mathcal{W}^s(\bar{x})$ of \bar{x} , and the unstable set $\mathcal{W}^u(\bar{x})$ of \bar{x} is a curve in \mathcal{R} that is tangential to E^μ at \bar{x} and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\bar{x})$ in \mathcal{R} are fixed points of T .*

Remark 7 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative D_1f negative and first partial derivative D_2f positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (1) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (1) is a strictly competitive map on $I \times I$, see [16].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (1) to obtain the equivalent system

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= f(v_n, u_n) \end{aligned}, \quad n = 0, 1, \dots$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u), f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [16].

Remark 8 The characteristic equation of Equation (1) at an equilibrium solution (\bar{x}, \bar{x}) :

$$\lambda^2 - D_1f(\bar{x}, \bar{x})\lambda - D_2f(\bar{x}, \bar{x}) = 0, \quad (5)$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 3-6 depends on the nonexistence of minimal period-two solution.

2 Main Results

In this section we present some global dynamics scenarios which feasibility will be illustrated in Section 3.

Theorem 9 Consider the competitive map T generated by the system (3) on a rectangular region \mathcal{R} with nonempty interior. Suppose T has no minimal period-two solutions in \mathcal{R} , is strongly competitive on $\text{int } \mathcal{R}$, is C^2 in a neighborhood of any fixed point and b . of Theorem 3 holds.

- (a) Assume that T has a saddle fixed points E_1, E_3 and locally asymptotically stable fixed point E_2 , such that $E_1 \preceq_{se} E_2 \preceq_{se} E_3$, and E_0 , which is South-west corner of the region \mathcal{R} is either repeller or singular point. Furthermore assume that $E_1 \preceq_{se} E_0 \preceq_{se} E_3$ and that the ray through E_0 and E_1 (resp. E_0 and E_2) is stable manifold of E_1 (resp. E_2). If T has no period-two solutions then every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2 .
- (b) Assume that T has locally asymptotically stable fixed points E_1, E_3 and a saddle fixed point E_2 , such that $E_1 \preceq_{se} E_2 \preceq_{se} E_3$, and E_0 , which is South-west corner of the region \mathcal{R} is either repeller or singular point. Furthermore assume that $E_1 \preceq_{se} E_0 \preceq_{se} E_3$ and that the ray through E_0 and E_1 (resp. E_0 and E_3) is attracted to E_1 (resp. E_3). If T has no period-two solutions then every solution which starts below (resp. above) the stable manifold $\mathcal{W}^s(E_2)$ converges to E_1 (resp. E_3).
- (c) Assume that T has exactly five fixed points E_1, \dots, E_5 , $E_1 \preceq_{se} E_2 \preceq_{se} E_3 \preceq_{se} E_4 \preceq_{se} E_5$ where E_1, E_3, E_5 are saddle points, and E_2, E_4 are locally asymptotically stable points. Assume that E_0 , which is South-west corner of the region \mathcal{R} , is either repeller or singular point such that $E_1 \preceq_{se} E_0 \preceq_{se} E_5$ and that the ray through E_0 and E_1 (resp. E_0 and E_5) is part of the basin of attraction of E_1 (resp. E_5). If T has no period-two solutions then every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2 while every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_3)$ and $\mathcal{W}^s(E_5)$ converges to E_4 .
- (d) Assume that T has exactly five fixed points E_1, \dots, E_5 , $E_1 \preceq_{se} E_2 \preceq_{se} E_3 \preceq_{se} E_4 \preceq_{se} E_5$ where E_1, E_3, E_5 are locally asymptotically stable points, and E_2, E_4 are saddle points. Assume that E_0 , which is South-west corner of the region \mathcal{R} , is either repeller or singular point such that $E_1 \preceq_{se} E_0 \preceq_{se} E_5$ and that the ray through E_0 and E_1 (resp. E_0 and E_5) is part of the basin of attraction of E_1 (resp. E_5). If T has no period-two solutions then every solution which starts below (resp. above) the stable manifold $\mathcal{W}^s(E_4)$ (resp. $\mathcal{W}^s(E_2)$) converges to E_5 (resp. E_1). Every solution which starts between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ converges to E_3 .

Proof.

- (a) The existence of the global stable and unstable manifolds of the saddle point equilibria E_1 and E_3 is guaranteed by Theorems 3 - 6. In view of uniqueness of these manifolds we have that $\mathcal{W}^s(E_1)$ has end points in E_0 and $(0, \infty)$ while $\mathcal{W}^s(E_3)$ has end points in E_0 and $(\infty, 0)$. Furthermore $\mathcal{W}^u(E_1)$ and $\mathcal{W}^u(E_3)$ have end points in E_2 . Now, by Corollary 2 in [16] every solution which starts in the interior of the ordered interval $[[E_1, E_2]]$ is attracted to E_2 and similarly every solution which starts in the interior of the ordered interval $[[E_2, E_3]]$ is attracted to E_2 . Furthermore, for every $(x_0, y_0) \in [[E_1, E_3]] \setminus ([[E_1, E_2]] \cup [[E_2, E_3]] \cup \{E_0\})$ one can find the points $(x_l, y_l) \in [[E_1, E_2]]$ and $(x_u, y_u) \in [[E_1, E_2]]$ such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$ and so $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u)), n \geq 1$, which implies that $T^n((x_0, y_0)) \rightarrow E_2$. Finally, for every $(x_0, y_0) \in \mathcal{R} \setminus ([[E_1, E_3]] \cup \{E_0\})$ one can find the points $(x_L, y_L) \in \mathcal{W}^u(E_1), (x_U, y_U) \in \mathcal{W}^u(E_3)$ such that $(x_L, y_L) \preceq_{se} (x_0, y_0) \preceq_{se} (x_U, y_U)$ which implies that $T^n((x_0, y_0))$ will eventually enter $[[E_1, E_3]]$ and so it will converge to E_2 .
- (b) The existence of the stable and unstable manifolds of the saddle point equilibrium E_2 is guaranteed by Theorems 3-6. The endpoints of the unstable manifold are E_1 and E_3 . First one can assume that the initial point $(x_0, y_0) \in [[E_1, E_2]] \setminus \{E_0\}$. In view of Corollary 2 in [16] the interior of $[[E_1, E_2]]$ is subset of the basin of attraction of E_1 . If the initial point $(x_0, y_0) \notin [[E_1, E_2]]$ but it is between $\mathcal{W}^s(E_1)$ and the ray through E_0 and E_1 then one can find te points (x_l, y_l) the ray through E_0 and E_1 and $(x_u, y_u) \in \mathcal{W}^s(E_1)$ such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$ and so $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u)), n \geq 1$, which means $T^n((x_0, y_0))$ will eventually enter $[[E_1, E_2]]$ and so $T^n((x_0, y_0)) \rightarrow E_2$.
The proof when the initial point (x_0, y_0) is below $\mathcal{W}^s(E_2)$ is similar.
- (c) The proof is similar to the one in case (a) and will be ommitted. This dynamic scenario is a replication of dynamic scenario in (a).

- (d) The proof is similar to the one in case (b) and will be omitted. This dynamic scenario is exactly replication of dynamic scenario in (b).

■

In the case of Equation (1) we have the following results which are direct application of Theorem 9.

Theorem 10 Consider Equation (1) and assume that f is decreasing in first and increasing in the second variable on the set $(a, b)^2$, where a is either the repeller or a singular point of f , such that f is C^2 in a neighborhood of any fixed point.

- (a) Assume that Equation (1) has locally asymptotically stable equilibrium solutions $\bar{x} > a$ and the unique saddle point minimal period-two solution $\{P_1, Q_1\}$, $P_1 \preceq_{se} (a, a) \preceq_{se} Q_1$. Assume that the stable manifold of P_1 (resp. Q_1) is the line through (a, a) and P_1 (resp. the line through (a, a) and Q_1). Then the equilibrium \bar{x} is globally asymptotically stable for all $x_{-1}, x_0 > a$.
- (b) Assume that Equation (1) has the saddle equilibrium solution $\bar{x} > a$ and the unique locally asymptotically stable minimal period-two solution $\{P_1, Q_1\}$, $P_1 \preceq_{se} (a, a) \preceq_{se} Q_1$. Assume that the stable manifold of P_1 (resp. Q_1) is the line through (a, a) and P_1 (resp. the line through (a, a) and Q_1). Then the period-two solution $\{P_1, Q_1\}$ attracts all initial points off the global stable manifold $\mathcal{W}^s(E(\bar{x}, \bar{x}))$.
- (c) Assume that Equation (1) has a saddle equilibrium solution $\bar{x} > a$. Assume that Equation (1) has two minimal period-two solutions $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ such that $P_1 \preceq_{se} P_2 \preceq_{se} E(\bar{x}, \bar{x}) \preceq_{se} Q_2 \preceq_{se} Q_1$, where $\{P_2, Q_2\}$ is locally asymptotically stable and $\{P_1, Q_1\}$ is a saddle point and assume that the global stable manifold of P_1 (resp. Q_1) is the line through (a, a) and P_1 (resp. the line through (a, a) and Q_1). Then every solution which starts off the union of global stable manifolds $\mathcal{W}^s(E(\bar{x}, \bar{x})) \cup \mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1)$ converges to the period-two solution $\{P_2, Q_2\}$.
- (d) Assume that Equation (1) has locally asymptotically stable equilibrium solution $\bar{x} > a$. Assume that Equation (1) has two minimal period-two solutions $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ such that $P_1 \preceq_{se} P_2 \preceq_{se} E(\bar{x}, \bar{x}) \preceq_{se} Q_2 \preceq_{se} Q_1$, where $\{P_1, Q_1\}$ is locally asymptotically stable and $\{P_2, Q_2\}$ is a saddle point. If the line through (a, a) and P_1 (resp. the line through (a, a) and Q_1) is a part of the basin of attraction of $\{P_1, Q_1\}$ then every solution which starts between the stable manifolds $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ converges to \bar{x} while every solution which starts below $\mathcal{W}^s(Q_2)$ (resp. above $\mathcal{W}^s(P_2)$) converges to the period-two solution $\{P_1, Q_1\}$.

Proof.

- (a) In view of Remark 7 the second iterate T^2 of the map T associated with Equation (1) is strictly competitive. Applying Theorem 9 part (a) to T^2 , where we set $E_1 = P_1, E_2 = (\bar{x}, \bar{x}), E_3 = Q_1$ we complete the proof.
- (b) The proof follows from Theorem 9 part (b) applied to T^2 , where we set $E_1 = P_1, E_2 = (\bar{x}, \bar{x}), E_3 = Q_1$ and observation that locally asymptotically stable fixed point (resp. saddle point) for T has the same character for T^2 .
- (c) The proof is similar to the proof in case (a) and will be omitted.
- (d) The proof follows from Theorem 9 part (d) applied to T^2 , where we set $E_1 = P_1, E_2 = P_2, E_3 = (\bar{x}, \bar{x}), E_4 = Q_2, E_5 = Q_1$ and the observation that locally asymptotically stable fixed point (resp. saddle point) for T has the same character for T^2 .

■

Remark 11 The term "saddle point" in formulation of statements of Theorems 9 and 10 can be replaced by the term "non-hyperbolic point of stable type". Results related to Theorem 9 were obtained in [1, 2] and the results related to Theorem 10 were obtained in [6, 9, 10]. Furthermore Cases (b) and (c) of Theorem 9 can be extended to the case when we have any odd number of the equilibrium points which alternate its stability between two types: locally asymptotically stable and saddle points or non-hyperbolic equilibrium points of the stable type. The transition from Case (a) to Case (b) and from Case (c) to Case (d) in Theorem 9 is an exchange of stability bifurcation, while in the case of Theorem 10 these two bifurcations are two global period doubling bifurcations.

3 Case study: Equation $x_{n+1} = \frac{\gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}$

We investigate global behavior of Equation (2), where the parameters γ, A, B, C are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. Equation (2) is a special case of equations

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \dots \tag{6}$$

and

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \dots \tag{7}$$

The comprehensive linearized stability analysis of Equation (6) was given in [9] and some special cases were considered in [10]. Some special cases of Equation (7) have been considered in the series of papers [5, 6, 11, 12, 19]. Describing the global dynamics of Equation (7) is a formidable task as this equation contains as a special cases many equations with complicated dynamics, such as the linear fractional difference equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, \dots \tag{8}$$

Equation (2) has 0 as a singular point and the first quadrant as the region \mathcal{R} .

3.1 Local stability analysis

By using the substitution $y_n = \frac{C}{\gamma} x_n$ Equation (2) is reduced to the equation

$$x_{n+1} = \frac{x_{n-1}}{A'x_n^2 + B'x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \tag{9}$$

where $A' = \frac{\gamma^2}{C^2} A$ and $B' = \frac{\gamma^2}{C^2} B$. In the sequel we consider Equation (9) where A' and B' will be replaced with A and B respectively.

First, we notice that under the conditions on parameters all solutions of Equation (9) are in interval $(0, 1]$ and that 0 is a singular point.

Equation (9) has the unique positive equilibrium \bar{x} given by

$$\bar{x} = \frac{-1 + \sqrt{1 + 4(A+B)}}{2(A+B)}. \tag{10}$$

The partial derivatives associated to Equation (9) at the equilibrium \bar{x} are

$$f'_x = \frac{-y(2Ax + By)}{(Ax^2 + Bxy + y)^2} \Big|_{\bar{x}} = -\frac{4(2A+B)}{(1 + \sqrt{1 + 4A + 4B})^2}, \quad f'_y = \frac{Ax^2}{(Ax^2 + Bxy + y)^2} \Big|_{\bar{x}} = \frac{4A}{(1 + \sqrt{1 + 4A + 4B})^2}.$$

Characteristic equation associated to Equation (9) at the equilibrium is

$$\lambda^2 + \frac{4(2A+B)}{(1 + \sqrt{1 + 4A + 4B})^2} \lambda - \frac{4A}{(1 + \sqrt{1 + 4A + 4B})^2} = 0.$$

By applying the linearized stability Theorem, see [13], we obtain the following result.

Theorem 12 *The unique positive equilibrium solution \bar{x} of Equation (9) is:*

- i) *locally asymptotically stable when $B + 3A > 4A^2$;*
- ii) *a saddle point when $B + 3A < 4A^2$;*
- iii) *a non-hyperbolic point of stable type (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{4A} < 1$) when $B + 3A = 4A^2$.*

In the next lemma we prove the existence of period two solutions of Equation (9).

Lemma 13 *Equation (9) has the minimal period-two solution $\{(0, 1), (1, 0)\}$ and the minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$, where*

$$\phi = \frac{A - \sqrt{(A-B)(A(-3+4A)-B)-B}}{2A(A-B)} \quad \text{and} \quad \psi = \frac{A + \sqrt{(A-B)(A(-3+4A)-B)-B}}{2A(A-B)} \tag{11}$$

if and only if

$$\frac{3}{4} < A < 1 \quad \text{and} \quad B + 3A < 4A^2 \quad \text{or} \quad A > 1 \quad \text{and} \quad B + 3A > 4A^2.$$

Proof. A minimal period-two solution is a positive solution of the following system

$$\begin{cases} x + (B - A)y - 1 = 0 \\ -Axy + y = 0. \end{cases} \tag{12}$$

where $\phi + \psi = x$ and $\phi\psi = y$. The system (12) has two solutions $x = 1, y = 0$ and

$$x = \frac{1}{A}, \quad y = \frac{A - 1}{A(B - A)}.$$

For second solution we have that $x, y, x^2 - 4y > 0$ if and only if

$$\frac{3}{4} < A < 1 \text{ and } B + 3A < 4A^2 \quad \text{or} \quad A > 1 \text{ and } B + 3A > 4A^2.$$

Now, ϕ and ψ are solution of equation

$$t^2 - \frac{1}{A}t - \frac{A - 1}{A(B - A)} = 0,$$

and the proof is complete. ■

The following theorem describes the local stability nature of the period-two solutions.

Theorem 14 Consider Equation (9).

i) The minimal period two solution $\{(0, 1), (1, 0)\}$ is:

- a) locally asymptotically stable when $A > 1$;
- b) a saddle point when $A < 1$;
- c) a non-hyperbolic point of the stable type when $A = 1$.

ii) The minimal period two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$, given by (11) is:

- a) locally asymptotically stable when $\frac{3}{4} < A < 1$ and $B + 3A < 4A^2$;
- b) a saddle point when $A > 1$ and $B + 3A > 4A^2$.

iii) If $A = B = 1$ the minimal period two solution $\{\phi, 1 - \phi\}$ ($0 < \phi < 1$) is non-hyperbolic.

Proof. In order to prove this theorem, we associate the second iterate map to Equation (9). We have

$$T^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u, v) \\ h(u, v) \end{pmatrix}$$

where

$$g(u, v) = \frac{u}{Av^2 + Buv + u}, \quad h(u, v) = \frac{v}{v + \frac{Au^2}{(Av^2 + Buv + u)^2} + \frac{Buv}{Av^2 + Buv + u}}.$$

The Jacobian of the map T^2 has the following form

$$J_{T^2} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} g'_u(\phi, \psi) & g'_v(\phi, \psi) \\ h'_u(\phi, \psi) & h'_v(\phi, \psi) \end{pmatrix}$$

where

$$\begin{aligned} g'_u &= \frac{Av^2}{(Av^2 + Buv + u)^2}, & g'_v &= -\frac{u(Bu + 2Av)}{(Av^2 + Buv + u)^2}, \\ h'_u &= -\frac{Av^3(u + Buv + Av^2)(Buv(1 + Bv) + A(2u + Bv^3))}{(A^2v^5 + u^2v(1 + Bv)(1 + B + Bv) + Au(u + v^3(2 + B + 2Bv)))^2}, \\ h'_v &= \frac{u(u + Buv + Av^2)(B^2u^2v^2(1 + Bv) + A^2v^2(5u + 2Bv^3) + Au(u + 3Buv + Bv^3(2 + 3Bv^3)))}{(A^2v^5 + u^2v(1 + Bv)(1 + B + Bv) + Au(u + v^3(2 + B + 2Bv)))^2}. \end{aligned}$$

Set

$$S = g'_u(\phi, \psi) + h'_v(\phi, \psi)$$

and

$$D = g'_u(\phi, \psi)h'_v(\phi, \psi) - g'_v(\phi, \psi)h'_u(\phi, \psi).$$

After some lengthy calculation one can see that:

i) for the minimal period-two solution $\{(0, 1), (1, 0)\}$ we have

$$S = \frac{1}{A} \text{ and } \mathcal{D} = 0$$

and applying the linearized stability Theorem [13] we obtain that the minimal period-two solution $\{(0, 1), (1, 0)\}$ of Equation (9) is:

- a) locally asymptotically stable when $A > 1$;
- b) a saddle point when $A < 1$;
- c) a non-hyperbolic point of the stable type when $A = 1$.

ii) For the positive minimal period two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ we have

$$S = \frac{6A^4 + A(B-2)B - B^2 - 3A^3(3+2B) + A^2(4+B(6+B))}{A^2(A-B)^2}, \quad \mathcal{D} = \frac{(A-1)^2}{(A-B)^2}.$$

Applying the linearized stability Theorem [13] we obtain that the minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ of Equation (9) is:

- a) locally asymptotically stable when $\frac{3}{4} < A < 1$ and $B + 3A < 4A^2$;
- b) a saddle point when $A > 1$ and $B + 3A > 4A^2$.

iii) If $A = B = 1$ then

$$S = 1 + \phi^2(1 - \phi)^2, \quad \mathcal{D} = \phi^2(1 - \phi)^2$$

from which the proof follows.

■

3.2 Global results and basins of attraction

In this section we present global dynamics results for Equation (9).

Theorem 15 *If $B + 3A > 4A^2$ and $0 < A < 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$ given by (10) which is locally asymptotically stable and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is a saddle point. Furthermore, the global stable manifold of the period-two solution $\{P, Q\}$ is given by $\mathcal{W}^s(\{P, Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are the coordinate axes. The basin of attraction $\mathcal{B}(E) = \{(x, y) : x \geq 0, y \geq 0\}$. More precisely*

- i) *If $(u_0, v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P , and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q .*
- ii) *If $(u_0, v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q , and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P .*
- iii) *If $(u_0, v_0) \in \mathcal{R} \setminus (\mathcal{W}^s(P) \cup \mathcal{W}^s(Q))$ (the region between $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$) then the sequence $\{(u_n, v_n)\}$ is attracted to $E(\bar{x}, \bar{x})$.*

See Figure 1 for visual illustration.

Proof. The proof is direct application of Theorem 10 part (a). ■

Theorem 16 *If $B + 3A > 4A^2$ and $A = 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$ which is locally asymptotically stable and the minimal period-two solution, $\{P(0, 1), Q(1, 0)\}$ which is a non-hyperbolic point of stable type. Furthermore, the global stable manifold of the period-two solution $\{P, Q\}$ is given by $\mathcal{W}^s(\{P, Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are the coordinate axes. The global dynamics is given in Theorem 15.*

Proof. In view of Remark 11 the proof is direct application of Theorem 10 part (a).

■

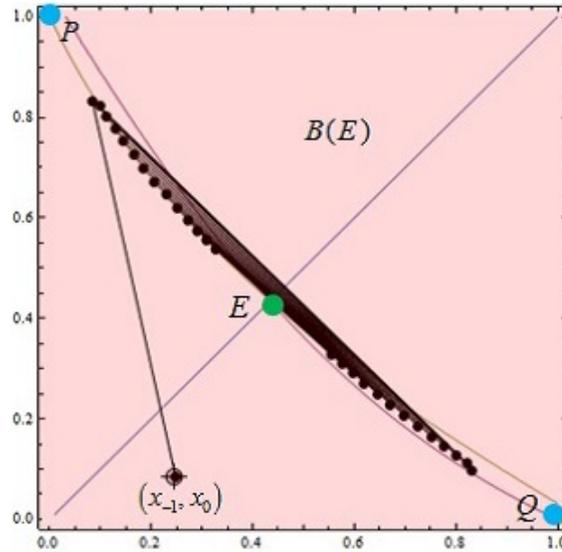


Figure 1: Visual illustration of Theorem 15.

Theorem 17 *If $B + 3A > 4A^2$ and $A > 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$ which is locally asymptotically stable and two minimal period-two solutions $\{P_1(0, 1), Q_1(1, 0)\}$ which is locally asymptotically stable and $\{P_2(\phi, \psi), Q_2(\psi, \phi)\}$ given by (11), which is a saddle point. Furthermore, the global stable manifold of the period-two solution $\{P_2, Q_2\}$ is given by $\mathcal{W}^s(\{P_2, Q_2\}) = \mathcal{W}^s(P_2) \cup \mathcal{W}^s(Q_2)$ where $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ are continuous increasing curves, that divide the first quadrant into two connected components, namely*

$$\mathcal{W}_1^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_2) : \exists y \in \mathcal{W}^s(P_2) \text{ with } y \preceq_{se} x\} \text{ and } \mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_2) : \exists y \in \mathcal{W}^s(P_2) \text{ with } x \preceq_{se} y\}$$

$$\mathcal{W}_2^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q_2) : \exists y \in \mathcal{W}^s(Q_2) \text{ with } y \preceq_{se} x\} \text{ and } \mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q_2) : \exists y \in \mathcal{W}^s(Q_2) \text{ with } x \preceq_{se} y\}$$

respectively such that the following statements are true.

- i) *If $(u_0, v_0) \in \mathcal{W}^s(P_2)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_2 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q_2 .*
- ii) *If $(u_0, v_0) \in \mathcal{W}^s(Q_2)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q_2 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2 .*
- iii) *If $(u_0, v_0) \in \mathcal{W}_1^-$ (the region above $\mathcal{W}^s(P_2)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to Q_1 .*
- iv) *If $(u_0, v_0) \in \mathcal{W}_2^+$ (the region below $\mathcal{W}^s(Q_2)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to Q_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to P_1 .*
- v) *If $(u_0, v_0) \in \mathcal{W}_1^+ \cap \mathcal{W}_2^-$ (the region between $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$) then the sequence $\{(u_n, v_n)\}$ is attracted to $E(\bar{x}, \bar{x})$.*

Shortly the basin of attraction of E is the region between $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ while the rest of the first quadrant without $\mathcal{W}^s(P_2) \cup \mathcal{W}^s(Q_2) \cup (0, 0)$ is the basin of attraction of $\{P_1, Q_1\}$.

See Figure 2 for visual illustration.

Proof. The proof is direct application of Theorem 10 part (d).

■

Theorem 18 *If $B + 3A < 4A^2$ and $\frac{3}{4} < A < 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$ which is a saddle point and minimal period-two solution $\{P_1(0, 1), Q_1(1, 0)\}$ which is a saddle point and $\{P_2(\phi, \psi), Q_2(\psi, \phi)\}$, given by (11) which is locally asymptotically stable. Furthermore, there exists a set \mathcal{C}_E which is an invariant subset of the basin of attraction of E . The set \mathcal{C}_E is a graph of a strictly increasing continuous function of the first variable*

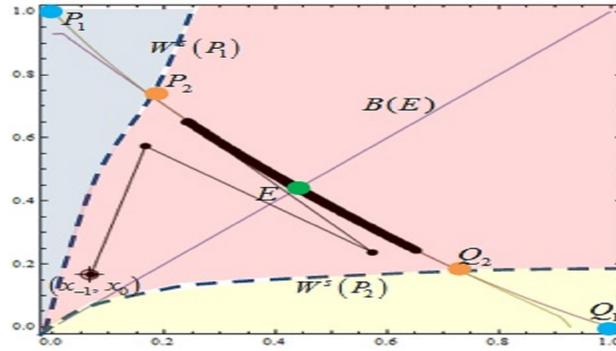


Figure 2: Visual illustration of Theorem 17.

on $(0, \infty)$ interval and separates $\mathcal{R} \setminus (0, 0)$, where $\mathcal{R} = (0, \infty) \times (0, \infty)$ into two connected and invariant components $\mathcal{W}^-(\bar{x}, \bar{x})$ and $\mathcal{W}^+(\bar{x}, \bar{x})$. The global stable manifold of the period-two solution $\{P_1, Q_1\}$ is given by $\mathcal{W}^s(\{P_1, Q_1\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1)$ where $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$ are continuous nondecreasing curves which represent the coordinate axes. The basin of attraction of $\{P_2, Q_2\}$ is the first quadrant without $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1) \cup (0, 0) \cup \mathcal{C}_E$. More precisely

- i) Every initial point (u_0, v_0) in \mathcal{C}_E is attracted to E .
- ii) If $(u_0, v_0) \in \mathcal{W}^s(P_1)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q_1 .
- iii) If $(u_0, v_0) \in \mathcal{W}^s(Q_1)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1 .
- iv) If $(u_0, v_0) \in \mathcal{W}^-(\bar{x}, \bar{x})$ (the region between \mathcal{C}_E and $\mathcal{W}^s(P_1)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_2 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to Q_2 .
- v) If $(u_0, v_0) \in \mathcal{W}^+(\bar{x}, \bar{x})$ (the region between \mathcal{C}_E and $\mathcal{W}^s(Q_1)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to Q_2 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to P_2 .

See Figure 3 for visual illustration.

Proof. Theorem 12 implies that there exists a unique equilibrium solution $E(\bar{x}, \bar{x})$ which is a saddle point and Theorem 14 implies that minimal period-two solution $\{P_1(0, 1), Q_1(1, 0)\}$ is a saddle point and $\{P_2(\phi, \psi), Q_2(\psi, \phi)\}$ is locally asymptotically stable. Now the proof is direct application of Theorem 10 part (c). ■

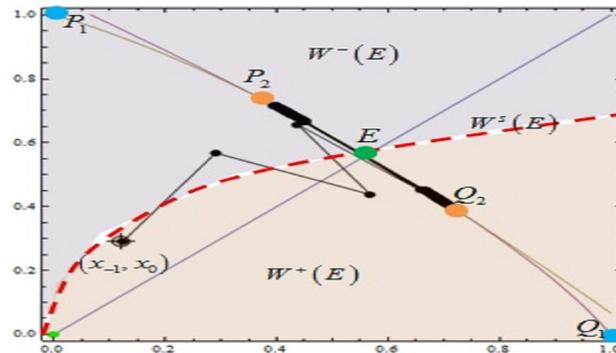


Figure 3: Visual illustration of Theorem 18.

Theorem 19 If $B + 3A < 4A^2$ and $A = 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$, which is a saddle point and the minimal period-two solution $\{P_1(0, 1), Q_1(1, 0)\}$ which is a non-hyperbolic point of stable type.

Furthermore, the global stable manifold $\mathcal{W}^s(E)$ is continuous increasing curve which divides first quadrant and the global stable manifold of the period-two solution $\{P_1, Q_1\}$ is given by $\mathcal{W}^s(\{P_1, Q_1\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1)$ where $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$ are the coordinate axes. The basin of attraction $\mathcal{B}(\{P_1, Q_1\}) = \{(x, y) : x \geq 0, y \geq 0\} \setminus (\mathcal{W}^s(E) \cup (0, 0))$. More precisely

- i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E .
- ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1 .
- iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q_1 .

See Figure 4 for visual illustration.

Proof. From Theorem 12 Equation (9) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point. Theorem 14 implies that the period-two solution $\{P, Q\}$ is a non-hyperbolic point. In view of Remark 11 the proof is direct application of Theorem 10 part (b). ■

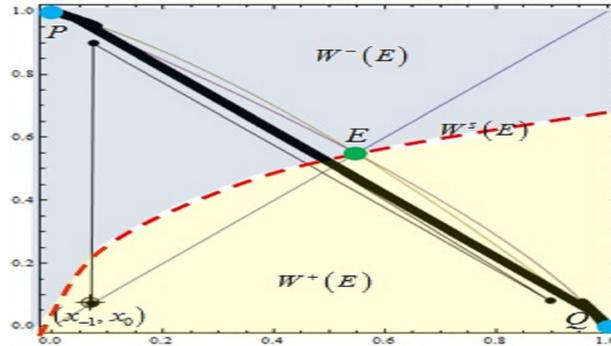


Figure 4: Visual illustration of Theorem 19.

Theorem 20 If $B + 3A < 4A^2$ and $A > 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$ which is a saddle point and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is locally asymptotically stable. The global behavior is the same as in Theorem 19.

Proof. The proof is direct application of Theorem 10 part (b). ■

Theorem 21 Assume that $B + 3A = 4A^2$.

- a) If $\frac{3}{4} < A < 1$ then Equation (9) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a non-hyperbolic point of stable type and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is a saddle point. Then every initial point (u_0, v_0) in \mathcal{R} is attracted to E .
- b) If $A > 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$ which is a non-hyperbolic point of the stable type and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is locally asymptotically stable. The global behavior is the same as in Theorem 19.
- c) If $A = 1$ then Equation (9) has a unique equilibrium solution $E(\bar{x}, \bar{x})$ and infinitely many minimal period-two solution $\{P(\phi, 1 - \phi), Q(1 - \phi, \phi)\}$ ($0 < \phi < 1$) which are a non-hyperbolic points of stable type.
 - i) There exists a continuous increasing curve \mathcal{C}_E which is a subset of the basin of attraction of E
 - ii) For every minimal period-two solution $\{P(\phi, 1 - \phi), Q(1 - \phi, \phi)\}$ ($0 < \phi < 1$) there exists the global stable manifold given by $\mathcal{W}^s(\{P, Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are continuous increasing curves. If $(u_0, v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to P and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to Q . If $(u_0, v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to Q and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to P . The union of these stable manifolds and \mathcal{C}_E foliates the first quadrant without the singular point $(0, 0)$.

See Figure 5 for visual illustration.

Proof.

- a) From Theorem 12 Equation (9) has a unique equilibrium point $E(\bar{x}, \bar{x}) = (\frac{1}{2A}, \frac{1}{2A})$ which is non-hyperbolic of stable type. From Theorem 14 Equation (9) has a unique minimal period-two solution $\{P_1(0, 1), Q_1(1, 0)\}$ which is a saddle point. In view of Remark 11 the proof is direct application of Theorem 10 part (a).
- b) From Theorem 12 Equation (9) has a unique equilibrium point $E(\bar{x}, \bar{x}) = (\frac{1}{2A}, \frac{1}{2A})$, which is non-hyperbolic of stable type. From Theorem 14 Equation (9) has a unique minimal period-two solution $\{P_1(0, 1), Q_1(1, 0)\}$ which is locally asymptotically stable point. In view of Remark 11 the proof is direct application of Theorem 10 part (b).
- c) From Theorem 12 Equation (9) has a unique equilibrium point $E(\bar{x}, \bar{x}) = (\frac{1}{2A}, \frac{1}{2A})$ which is non-hyperbolic. All conditions of Theorem 5 are satisfied, which yields the existence a continuous increasing curve C_E which is a subset of the basin of attraction of E . The proof of the statement ii) follows from Theorems 3, 5, 14 and Theorem 5 in [8].

■

Remark 22 The global dynamics of Equation (9) can be described in the language of bifurcation theory as follows: when $B + 3A \neq 4A^2$, then the period-doubling bifurcation happens when A is passing through the value 1 in such a way that for $A > 1$ new interior period-two solution emerges and exchange stability with already existing period-two solution on the boundary. Another bifurcation happens when $B + 3A < 4A^2$ in which case the stability of the unique equilibrium changes from local attractor to the saddle point. Finally, there is a bifurcation at another critical value $B + 3A = 4A^2$ when A is passing through the critical value 1, which is one of exchange stability between the unique equilibrium and unique period-two solution, with specific dynamics at $A = 1$, when there is an infinite number of period-two solutions which basins of attraction filled up the first quadrant without the origin. See [16] for similar results.

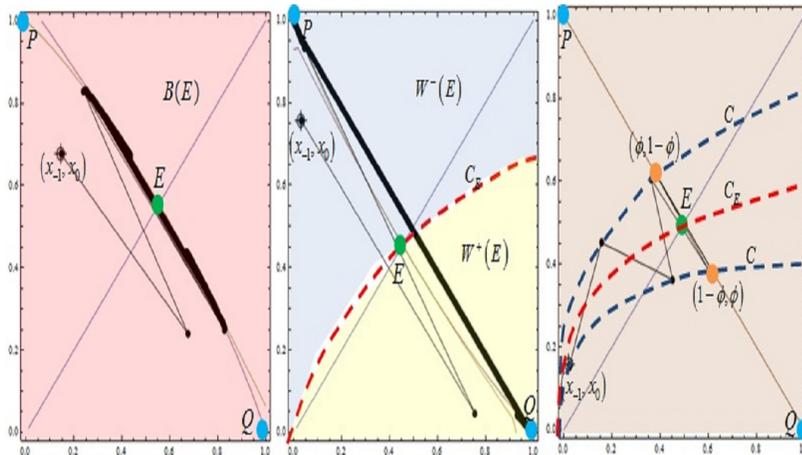


Figure 5: Visual illustration of Theorem 21.

References

- [1] E. Bertrand and M. R. S. Kulenović, Global Dynamic Scenarios for Competitive Maps in the Plane, *Adv. Difference Equ.*, Volume 2018 (2018):307, 28 p.
- [2] A. Bilgin, M. R. S. Kulenović and E. Pilav, Basins of Attraction of Period-Two Solutions of Monotone Difference Equations, *Adv. Difference Equa.*, Volume 2016 (2016), 25p.

- [3] A. Brett and M. R. S. Kulenović, Basins of Attraction of Equilibrium Points of Monotone Difference Equations, *Sarajevo J. Math.*, 5(2009), 211-233 .
- [4] E. Camouzis and G. Ladas. When does local asymptotic stability imply global attractivity in rational equations? *J. Difference Equ. Appl.*, 12 (2006), 863–885.
- [5] M. Dehghan, C. M. Kent, R. Mazrooei-Sebdani, N. L. Ortiz and H. Sedaghat, Monotone and oscillatory solutions of a rational difference equation containing quadratic terms, *J. Difference Equ. Appl.*, 14 (2008), 1045–1058.
- [6] M. Garić Demirović, M. R. S. Kulenović and M. Nurkanović, Basins of Attraction of Equilibrium Points of Second Order Difference Equations, *Appl. Math. Letters*, 25(2012), 2110–2115.
- [7] M. Hirsch and H. L. Smith, Monotone Maps: A Review, *J. Difference Equ. Appl.* 11(2005), 379–398.
- [8] S. Kalabušić, M. R. S. Kulenović and E. Pilav, Global Dynamics of Anti-Competitive Systems in the Plane, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 20(2013), 477-505.
- [9] S. Kalabušić, M. R. S. Kulenović and M. Mehuljić, Global Period-doubling Bifurcation of Quadratic Fractional Second Order Difference Equation, *Discrete Dyn. Nat. Soc.*, (2014), Art. ID 920410, 13p.
- [10] S. Kalabušić, M. R. S. Kulenović and M. Mehuljić, Global Dynamics and Bifurcations of Two Quadratic Fractional Second Order Difference Equations, *J. Comp. Anal. Appl.*, 20(2016), 132–143.
- [11] C. M. Kent and H. Sedaghat, Global attractivity in a quadratic-linear rational difference equation with delay. *J. Difference Equ. Appl.* 15 (2009), 913–925.
- [12] C. M. Kent and H. Sedaghat, Global attractivity in a rational delay difference equation with quadratic terms, *J. Difference Equ. Appl.*, 17 (2011), 457–466.
- [13] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman and Hall/CRC, Boca Raton, London, 2001.
- [14] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.
- [15] M. R. S. Kulenović and O. Merino, Competitive-Exclusion versus Competitive-Coexistence for Systems in the Plane, *Discrete Cont. Dyn. Syst. Ser. B* 6(2006), 1141–1156.
- [16] M. R. S. Kulenović and O. Merino, Global Bifurcation for Competitive Systems in the Plane, *Discrete Contin. Dyn. Syst. B*, 12 (2009), 133–149.
- [17] M. R. S. Kulenović and O. Merino, Invariant manifolds for competitive discrete systems in the plane. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 20 (2010), 2471-2486.
- [18] M. R. S. Kulenović and O. Merino, Invariant Curves for Planar Competitive and Cooperative Maps, *J. Difference Equ. Appl.*, 24(2018), 898–915.
- [19] H. L. Smith, Planar Competitive and Cooperative Difference Equations, *J. Difference Equ. Appl.* 3(1998), 335-357.

Global Dynamics of Generalized Second-Order Beverton–Holt Equations of Linear and Quadratic Type

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Abstract. We investigate second-order generalized Beverton–Holt difference equations of the form

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \dots,$$

where f is a function nondecreasing in both arguments, the parameter a is a positive constant, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers in the domain of f . We will discuss several interesting examples of such equations and present some general theory. In particular, we will investigate the local and global dynamics in the event f is a certain type of linear or quadratic polynomial, and we explore the existence problem of period-two solutions.

Keywords. attractivity, difference equation, invariant sets, periodic solutions, stable set .

AMS 2010 Mathematics Subject Classification: 39A20, 39A28, 39A30, 92D25

1 Introduction and Preliminaries

Consider the following second-order difference equation:

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \dots \tag{1}$$

Here f is a continuous function nondecreasing in both arguments, the parameter a is a positive real number, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers in the domain of f . Equation (1) is a generalization of the first-order Beverton–Holt equation

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \quad n = 0, 1, \dots, \tag{2}$$

where $a > 0$ and $x_0 \geq 0$. The global dynamics of Equation (2) may be summarized as follows, see [9, 15]:

$$\lim_{n \rightarrow \infty} x_n = \begin{cases} 0 & \text{if } a \leq 1 \\ a - 1 & \text{if } a > 1 \text{ and } x_0 > 0. \end{cases} \tag{3}$$

Many variations of Equation (2) have been studied. German biochemist Leonor Michaelis and Canadian physician Maud Menten used the model in their study of enzyme kinetics in 1913; see [20]. Additionally, Jacques Monod, a French biochemist, happened upon the model empirically in his study of microorganism growth around 1942; see [20]. It was not until 1957 that fisheries scientists Ray Beverton and Sidney Holt used the model in their study of population dynamics, see [1, 9]. The so-called Monod differential equation [20] is given by

$$\frac{1}{N} \cdot \frac{dN}{dt} = \frac{rS}{a + S}, \tag{4}$$

where $N(t)$ is the concentration of bacteria at time t , $\frac{dN}{dt}$ is the growth rate of the bacteria, $S(t)$ is the concentration of the nutrient, r is the maximum growth rate of the bacteria, and a is a half-saturation

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constant (when $S = a$, the right-hand side of Equation (4) equals $r/2$). Based on experimental data, the following system of two differential equations for the nutrient S and bacteria N , as presented in [20], is given by

$$\frac{dS}{dt} = -\frac{1}{\gamma}N \frac{rS}{a+S}, \quad \frac{dN}{dt} = N \frac{rS}{a+S}, \tag{5}$$

where the constant γ is called the growth yield. Both Equation (4) and System (5) contain the function $f(x) = rx/(a+x)$ known as the Monod function, Michaelis-Menten function, Beverton–Holt function, or Holling function of the first kind; see [1, 5, 9, 11].

One possible two-generation population model based on Equation (2),

$$x_{n+1} = \frac{a_1x_n}{1+x_n} + \frac{a_2x_{n-1}}{1+x_{n-1}}, \quad n = 0, 1, \dots, \tag{6}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [18]. The global dynamics of Equation (6) may be summarized as follows:

$$\lim_{n \rightarrow \infty} x_n = \begin{cases} 0 & \text{if } a_1 + a_2 \leq 1 \\ a_1 + a_2 - 1 & \text{if } a_1 + a_2 > 1 \text{ and } x_0 + x_{-1} > 0. \end{cases}$$

This result was extended in [5] to the case of a k -generation population model based on Equation (2) of the form

$$x_{n+1} = \sum_{i=0}^{k-1} \frac{a_i x_{n-i}}{1+x_{n-i}}, \quad n = 0, 1, \dots, \tag{7}$$

where $a_i \geq 0$ for $i = 0, 1, \dots, k-1$, $\sum_{i=0}^{k-1} a_i > 0$, and $x_{1-k}, \dots, x_0 \geq 0$. It was shown that the global dynamics of Equation (7) may be given precisely by (3), where $a = \sum_{i=0}^{k-1} a_i$ and we consider all initial conditions positive.

The simplest model of Beverton–Holt type which exhibits two coexisting attractors and the Allee effect is the sigmoid Beverton–Holt (or second-type Holling) difference equation

$$x_{n+1} = \frac{ax_n^2}{1+x_n^2}, \quad n = 0, 1, \dots, \tag{8}$$

where $a > 0$ and $x_0 \geq 0$. The dynamics of Equation (8) may be concisely summarized as follows:

$$\lim_{n \rightarrow \infty} x_n = \begin{cases} 0 & \text{if } a < 2 \text{ or } (a \geq 2 \text{ and } x_0 < \bar{x}_-) \\ \bar{x}_- & \text{if } a \geq 2 \text{ and } x_0 = \bar{x}_- \\ \bar{x}_+ & \text{if } a \geq 2 \text{ and } x_0 > \bar{x}_-, \end{cases} \tag{9}$$

where \bar{x}_- and \bar{x}_+ are the two positive equilibria when $a \geq 2$; see [1, 5]. One possible two-generation population model based on Equation (8),

$$x_{n+1} = \frac{a_1x_n^2}{1+x_n^2} + \frac{a_2x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \dots, \tag{10}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [4]. However, the summary of the global dynamics of Equation (10) is not an immediate extension of the global dynamics of Equation (8) as given in

(9); see [4]. Equation (10) can have up to three equilibrium solutions and up to three period-two solutions. In the case when Equation (10) has three equilibrium solutions and three period-two solutions, the zero equilibrium, the larger positive equilibrium, and one period-two solution are attractors with substantial basins of attraction, which together with the remaining equilibrium and the global stable manifolds of the saddle-point period-two solutions exhaust the first quadrant of initial conditions. This behavior happens when the coefficient a_2 is in some sense dominant to a_1 ; see [4]. Such behavior is typical for other models in population dynamics such as

$$x_{n+1} = \frac{a_1 x_n}{1 + x_n} + \frac{a_2 x_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \dots$$

and

$$x_{n+1} = a_1 x_n + \frac{a_2 x_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \dots,$$

which were also investigated in [4]. In the case of a k -generation population model based on the sigmoid Beverton–Holt difference equation with $k > 2$, one can expect to have attractive period- k solutions as well as chaos.

The first model of the form given in Equation (1), where f is a linear function in both variables (that is, $f(u, v) = cu + dv$ for $c, d, u, v \geq 0$) was considered in [19] to describe the global dynamics in part of the parametric space. Here we will extend the results from [19] to the whole parametric space. In this paper we will then restrict ourselves to the case when $f(u, v)$ is a quadratic polynomial, which will give similar global dynamics to that presented for Equation (10). The corresponding dynamic scenarios will be essentially the same for any polynomial function of the type $f(u, v) = cu^k + dv^m$ where $c, d \geq 0$ and m, k are positive integers. Higher values of m and k may only create additional equilibria and period-two solutions but should replicate the global dynamics seen in the quadratic case presented in this paper. The global dynamics of some higher-order transcendental-type generalized Beverton-Holt equation was considered in [3].

Let the function $F : [0, \infty)^2 \rightarrow [0, a)$ be defined as follows:

$$F(u, v) = \frac{af(u, v)}{1 + f(u, v)}. \tag{11}$$

Then Equation (1) becomes $x_{n+1} = F(x_n, x_{n-1})$ for all $n = 0, 1, \dots$, where $F(u, v)$ is nondecreasing in both of its arguments.

The following theorem from [2] immediately applies to Equation (1).

Theorem 1 *Let I be a set of real numbers and $F : I \times I \rightarrow I$ be a function which is nondecreasing in the first variable and nondecreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^\infty$ of the equation*

$$x_{n+1} = F(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, \dots, \tag{12}$$

the subsequences $\{x_{2n}\}_{n=0}^\infty$ and $\{x_{2n-1}\}_{n=0}^\infty$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of Equation (12) converges to either an equilibrium, a period-two solution, or to a singular point on the boundary. It should be noticed that Theorem 1 is specific for second-order difference equations and does not extend to difference equations of order higher than two. Furthermore, the powerful theory of monotone maps in the plane [16, 17] can be applied to Equation (1) to determine the boundaries of the basins of attraction of the equilibrium

solutions and period-two solutions. Finally, when $f(u, v)$ is a polynomial function, all computation needed to determine the local stability of all equilibrium solutions and period-two solutions is reduced to the theory of counting the number of zeros of polynomials in a given interval, as given in [12]. This theory will give more precise results than the global attractivity and global asymptotic stability results in [7, 8]. However, in the case of difference equations of the form

$$x_{n+1} = \frac{ag(x_n, x_{n-1}, \dots, x_{n+1-k})}{1 + g(x_n, x_{n-1}, \dots, x_{n+1-k})}, \quad n = 0, 1, \dots, \quad k \geq 1,$$

where $a > 0$ and g is nondecreasing in all its arguments, Theorem 1 does not apply for $k > 2$, but the results from [7, 8, 13] can give global dynamics in some regions of the parametric space.

The following theorem from [10] is often useful in determining the global attractivity of a unique positive equilibrium.

Theorem 2 *Let $I \subseteq [0, \infty)$ be some open interval and assume that $F \in C[I \times I, (0, \infty)]$ satisfies the following conditions:*

- (i) $F(x, y)$ is nondecreasing in each of its arguments;
- (ii) Equation (12) has a unique positive equilibrium point $\bar{x} \in I$ and the function $F(x, x)$ satisfies the **negative feedback condition**:

$$(x - \bar{x})(F(x, x) - x) < 0 \text{ for every } x \in I \setminus \{\bar{x}\}.$$

Then every positive solution of Equation (12) with initial conditions in I converges to \bar{x} .

The following result from [4] will be used to describe the global dynamics of Equation (1).

Theorem 3 *Assume that difference equation (12) has three equilibrium points $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$ where the equilibrium points \bar{x}_0 and \bar{x}_{NE} are locally asymptotically stable. Further, assume that there exists a minimal period-two solution $\{\Phi_1, \Psi_1\}$ which is a saddle point such that $(\Phi_1, \Psi_1) \in \text{int}(Q_2(E_{SW}))$. In this case there exist four continuous curves $\mathcal{W}^s(\Phi_1, \Psi_1)$, $\mathcal{W}^s(\Psi_1, \Phi_1)$, $\mathcal{W}^u(\Phi_1, \Psi_1)$, $\mathcal{W}^u(\Psi_1, \Phi_1)$, where $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1)$ are passing through the point E_{SW} , and are graphs of decreasing functions. The curves $\mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$ are the graphs of increasing functions and are starting at E_0 . Every solution which starts below $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$ in the North-east ordering converges to E_0 and every solution which starts above $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$ in the North-east ordering converges to E_{NE} , i.e. $\mathcal{W}^s(\Phi_1, \Psi_1) = \mathcal{C}_1^+ = \mathcal{C}_2^+$ and $\mathcal{W}^s(\Psi_1, \Phi_1) = \mathcal{C}_1^- = \mathcal{C}_2^-$.*

This paper is organized as follows. The next section deals with the local stability of equilibrium solutions and period-two solutions of the general second-order difference equation (12), where $F(u, v)$ is nondecreasing in both of its arguments. In view of the results for monotone maps in [16, 17] and their applications to second-order difference equations in [4, 5], the local dynamics of the equilibrium solutions and period-two solutions will determine the global dynamics in hyperbolic cases and some nonhyperbolic cases as well. The third section will provide some examples of global dynamic scenarios of Equation (1) when the function $f(u, v)$ is either linear in both variables or linear in one variable and quadratic in the other variable. The obtained results will be interesting from a modeling point of view as they show that the appearance of period-two solutions with substantial basins of attraction (sets which contain open subsets) is controlled by the coefficient of the x_{n-1} term that is affected by the size of the grandparents' population. The same phenomenon was observed in the case of Equation (10).

2 Local Stability

In this section we provide general conditions to determine the local stability of equilibrium solutions and period-two solutions.

It is clear that $x_n \leq a$ for all $n \geq 1$. In light of Theorem 1, since all solutions are bounded, if there are no singular points on the boundary of the domain of F , it immediately follows that all solutions to Equation (1) converge to an equilibrium or a period-two solution.

An equilibrium \bar{x} of Equation (1) satisfies

$$\bar{x}(1 + f(\bar{x}, \bar{x})) = af(\bar{x}, \bar{x}). \tag{13}$$

Clearly $\bar{x}_0 = 0$ is an equilibrium point if and only if $(0, 0)$ is in the domain of f and $f(0, 0) = 0$.

The linearized equation of Equation (1) about an equilibrium \bar{x} is

$$z_{n+1} = F_u(\bar{x}, \bar{x})z_n + F_v(\bar{x}, \bar{x})z_{n-1}, \quad n = 0, 1, \dots$$

Since f is a nondecreasing function, it follows that $F_u(\bar{x}, \bar{x}) \geq 0, F_v(\bar{x}, \bar{x}) \geq 0$. Therefore, if

$$\lambda(\bar{x}) = F_u(\bar{x}, \bar{x}) + F_v(\bar{x}, \bar{x}) = \frac{a(f_u(\bar{x}, \bar{x}) + f_v(\bar{x}, \bar{x}))}{(1 + f(\bar{x}, \bar{x}))^2}, \tag{14}$$

then in view of Corollary 2 of [13] we may conclude that

$$\bar{x} \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } \lambda(\bar{x}) < 1 \\ \text{nonhyperbolic} & \text{if } \lambda(\bar{x}) = 1 \\ \text{unstable} & \text{if } \lambda(\bar{x}) > 1. \end{cases}$$

Further, Theorem 2.13 of [15] implies that if \bar{x} is unstable, then

$$\bar{x} \text{ is } \begin{cases} \text{a repeller} & \text{if } \delta(\bar{x}) > 1 \\ \text{nonhyperbolic} & \text{if } \delta(\bar{x}) = 1 \\ \text{a saddle point} & \text{if } \delta(\bar{x}) < 1, \end{cases}$$

where

$$\delta(\bar{x}) = F_v(\bar{x}, \bar{x}) - F_u(\bar{x}, \bar{x}) = \frac{a(f_v(\bar{x}, \bar{x}) - f_u(\bar{x}, \bar{x}))}{(1 + f(\bar{x}, \bar{x}))^2}. \tag{15}$$

Let (ϕ, ψ) be a period-two solution of Equation (1). The Jacobian matrix of the corresponding map $T = G^2$, where $G(u, v) = (v, F(v, u))$ and F is given by Equation (11), is given in Theorem 12 of [6]. The linearized equation evaluated at (ϕ, ψ) is

$$\lambda^2 - Tr J_T(\phi, \psi)\lambda + Det J_T(\phi, \psi) = 0,$$

where

$$Tr J_T(\phi, \psi) = D_2F(\psi, \phi) + D_1F(F(\psi, \phi), \psi) \cdot D_1F(\psi, \phi) + D_2F(F(\psi, \phi), \psi)$$

and

$$Det J_T(\phi, \psi) = D_2F(F(\psi, \phi), \psi) \cdot D_2F(\psi, \phi).$$

3 Examples

In this section we present four examples of different forms of Equation (1) where the transition function $f(u, v)$ is linear or quadratic polynomial in its variables which effects the global dynamics.

3.1 Linear-Linear: $f(u, v) = cu + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1})}{1 + cx_n + dx_{n-1}}, \quad n = 0, 1, \dots, \tag{16}$$

where $c \geq 0$ and $d > 0$. If $d = 0$, then Equation (16) becomes Equation (2) after a reduction of parameters. By Equation (13) we know that $\bar{x}_0 = 0$ is always a fixed point and $\bar{x}_+ = \frac{a(c+d)-1}{c+d}$ is a unique positive fixed point for $a(c + d) > 1$.

Since $\lambda(\bar{x}_0) = a(c + d)$, we have that

$$\bar{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } a(c + d) < 1 \\ \text{nonhyperbolic} & \text{if } a(c + d) = 1 \\ \text{unstable} & \text{if } a(c + d) > 1. \end{cases}$$

Further, notice that

$$\lambda(\bar{x}_+) = \frac{a(c + d)}{\left(1 + \left(\frac{a(c+d)-1}{c+d}\right) \cdot (c + d)\right)^2} = \frac{1}{a(c + d)} < 1$$

for all values of parameters for which \bar{x}_+ exists. Therefore

$$\bar{x}_+ = \frac{a(c + d) - 1}{c + d} \text{ is always locally asymptotically stable.}$$

Note that there is an exchange in stability from \bar{x}_0 to \bar{x}_+ as the parametric value $a(c + d)$ passes through 1.

We next search for period-two solutions. Suppose there exists such a solution $\{\psi, \phi, \psi, \phi, \dots\}$ with $\phi \neq \psi$. Then $\{\psi, \phi\}$ satisfies the following system:

$$\begin{cases} \psi = \frac{af(\phi, \psi)}{1 + f(\phi, \psi)} = \frac{a(c\phi + d\psi)}{1 + c\phi + d\psi} \\ \phi = \frac{af(\psi, \phi)}{1 + f(\psi, \phi)} = \frac{a(c\psi + d\phi)}{1 + c\psi + d\phi} \end{cases} \tag{17}$$

Notice that

$$\psi - \phi = \frac{a(d - c)(\psi - \phi)}{(1 + c\phi + d\psi)(1 + c\psi + d\phi)},$$

whence we deduce that $d > c$ and $(1 + c\phi + d\psi)(1 + d\psi + d\phi) = a(d - c)$. Now

$$\psi + \phi = \frac{a((c + d)(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi))}{a(d - c)},$$

or equivalently,

$$2c(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi) = 0.$$

Since $\psi + \phi > 0$, it must be the case that $c = 0$, and then $2d^2\psi\phi = 0$ so that one of either ϕ or ψ equals zero. Without loss of generality assume $\phi = 0$. But then $\psi = \frac{ad\psi}{1+d\psi}$, and hence $\psi = \frac{ad-1}{d} = \bar{x}_+$. Thus the only non-equilibrium solution of System (17) is the period-two solution $\{\bar{x}_+, 0, \bar{x}_+, 0, \dots\}$, which exists for $ad > 1$ and $c = 0$. Now we formulate our main result about the global dynamics of Equation (16).

Theorem 4 Consider Equation (16).

- (a) If $a(c + d) \leq 1$, then $\bar{x}_0 = 0$ is a global attractor of all solutions.
- (b) If $c = 0$ and $ad > 1$, then there exists a period-two solution $\{\frac{ad-1}{d}, 0, \frac{ad-1}{d}, 0, \dots\}$. \bar{x}_+ is a global attractor of all solutions with positive initial conditions. Any solution with exactly one initial condition equal to zero will converge to the period-two solution.
- (c) If $c > 0$ and $a(c + d) > 1$, \bar{x}_+ is a global attractor of all nonzero solutions.

Proof.

- (a) If $a(c + d) \leq 1$, then $\bar{x}_0 = 0$ is the only equilibrium, and no period-two solutions exist. By Theorem 1 all solutions must converge to zero.
- (b) Suppose $c = 0$ and $ad > 1$, and consider $I = (0, \infty)$. Notice that

$$F(x, x) = \frac{adx}{1 + dx} \geq x \iff \bar{x}_+ \geq x,$$

and therefore by Theorem 2 we have that all solutions with initial conditions in I converge to \bar{x}_+ . Now suppose one initial condition is zero, so without loss of generality assume $x_{-1} = 0$ and $x_0 > 0$. Then $x_1 = 0$ and

$$x_2 = \frac{adx_0}{1 + dx_0} \geq x_0 \iff \frac{ad - 1}{d} = \bar{x}_+ \geq x_0.$$

Further, one can show $x_2 \leq \bar{x}_+ \iff x_0 \leq \bar{x}_+$. By induction, $\lim_{k \rightarrow \infty} x_{2k} = \bar{x}_+$ and $x_{2k-1} = 0$ for all $k = 0, 1, \dots$. Thus all solutions with exactly one initial condition equal to zero will converge to the period-two solution $\{\bar{x}_+, 0, \bar{x}_+, 0, \dots\}$.

- (c) When $c > 0$ and $a(c + d) > 1$, \bar{x}_+ is locally asymptotically stable while \bar{x}_0 is unstable. As in the proof of (b) we can employ Theorem 2 to show that all solutions with positive initial conditions must converge to \bar{x}_+ . Since $c > 0$ and $d > 0$, if $x_0 + x_{-1} > 0$, then $x_1 = F(x_0, x_{-1}) > 0$ (and also $x_2 > 0$), so the solution eventually has consecutive positive terms and must converge to \bar{x}_+ .

□

3.2 Translated Linear-Linear: $f(u, v) = cu + dv + k$

We briefly consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1} + k)}{1 + cx_n + dx_{n-1} + k}, \quad n = 0, 1, \dots, \tag{18}$$

where $c \geq 0$, $d \geq 0$, $c + d > 0$, and $k > 0$. We notice in this example $f(0, 0) = k > 0$, so the origin cannot be an equilibrium. More specifically, an equilibrium of Equation (18) must satisfy

$$(c + d)\bar{x}^2 + (k + 1 - a(c + d))\bar{x} - ak = 0$$

Since $c + d > 0$ and $ak > 0$ by Descartes' Rule of Signs it must be the case that there exists a unique positive equilibrium \bar{x}_+ .

Theorem 5 Consider Equation (18) such that $c + d > 0$ and $k > 0$. The unique positive equilibrium \bar{x}_+ is a global attractor.

Proof. The result follows from a straightforward application of Theorem 1.4.8 of [14]. □

3.3 Quadratic-Linear: $f(u, v) = cu^2 + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n^2 + dx_{n-1})}{1 + cx_n^2 + dx_{n-1}}, \quad n = 0, 1, \dots \tag{19}$$

Remark 1 For the analysis that follows, we will consider Equation (19) with $c > 0$ and $d > 0$. Notice that when $c = 0$ Equation (19) is a special case of Equation (16), and the global dynamics for this case is discussed in Theorem 4. When $d = 0$ Equation (19) is essentially Equation (8), the dynamics of which may be seen in (9).

An equilibrium solution of Equation (19) satisfies

$$c\bar{x}^3 + d\bar{x}^2 + \bar{x} = ac\bar{x}^2 + ad\bar{x}$$

so that all nonzero equilibria satisfy

$$c\bar{x}^2 + (d - ac)\bar{x} + (1 - ad) = 0, \tag{20}$$

whence we easily deduce the possible solutions

$$\bar{x}_{\pm} = \frac{ac - d \pm \sqrt{(d - ac)^2 + 4c(ad - 1)}}{2c},$$

which are real if and only if $R = (d - ac)^2 + 4c(ad - 1) \geq 0$.

Notice that

$$R \geq 0 \iff d^2 - 2acd + a^2c^2 + 4acd - 4c \geq 0 \iff (ac + d)^2 \geq 4c. \tag{21}$$

Here we have that

$$\lambda(\bar{x}) = \frac{a(2c\bar{x} + d)}{(1 + c\bar{x}^2 + d\bar{x})^2}.$$

Theorem 6 Equation (19) always has the zero equilibrium $\bar{x}_0 = 0$, and

$$\bar{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } ad < 1 \\ \text{nonhyperbolic} & \text{if } ad = 1 \\ \text{a repeller} & \text{if } ad > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\bar{x}_0) = \delta(\bar{x}_0) = ad$. □

The next result gives the local stability of positive equilibrium solutions.

Theorem 7 Assume $c > 0$ and $d > 0$.

(1) Suppose either

(a) $d \geq ac$ and $1 \geq ad$, or

(b) $d < ac$, $1 > ad$, and $R < 0$.

Then Equation (19) has no positive equilibria.

(2) Suppose either

(a) $1 < ad$, or

(b) $d < ac$ and $1 = ad$.

Then Equation (19) has the positive equilibrium solution \bar{x}_+ , and it is locally asymptotically stable.

(3) Suppose $d < ac$, $1 > ad$, and $R = 0$. Then Equation (19) has the positive equilibrium solution \bar{x}_\pm , and it is nonhyperbolic of stable type (that is one characteristic value is $\lambda_1 = \pm 1$ and the other $|\lambda_1| < 1$).

(4) Suppose $d < ac$, $1 > ad$, and $R > 0$. Then Equation (19) has two positive equilibria, \bar{x}_+ and \bar{x}_- ; \bar{x}_+ is locally asymptotically stable, and \bar{x}_- is a saddle point.

Proof. The existence of positive equilibria follows from Descartes' Rule of Signs. Using Equation (14), notice that

$$\lambda(\bar{x}) = \frac{a(2c\bar{x} + d)}{(1 + c\bar{x}^2 + d\bar{x})^2} = \frac{a(2c\bar{x} + d)}{(a(c\bar{x} + d))^2} = \frac{2c\bar{x} + d}{a(c\bar{x} + d)^2} = \frac{1}{a(c\bar{x} + d)} + \frac{c\bar{x}}{a(c\bar{x} + d)^2}.$$

Further, for the parametric values for which \bar{x}_+ exists,

$$\begin{aligned} \lambda(\bar{x}_+) \leq 1 &\iff \frac{c\bar{x}_+}{a(c\bar{x}_+ + d)^2} \leq \frac{a(c\bar{x}_+ + d) - 1}{a(c\bar{x}_+ + d)} \\ &\iff c\bar{x}_+ \leq (c\bar{x}_+ + d)(a(c\bar{x}_+ + d) - 1) = (c\bar{x}_+ + d)(c\bar{x}_+^2 + d\bar{x}_+) \\ &\iff c \leq (c\bar{x}_+ + d)^2 \\ &\iff 4c \leq (2c\bar{x}_+ + 2d)^2 = (ac + d + \sqrt{R})^2, \end{aligned}$$

which is true by (21). Thus if $R > 0$, \bar{x}_+ is locally asymptotically stable, and if $R = 0$, \bar{x}_\pm is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (19) about \bar{x}_\pm , $y^2 = F_u(\bar{x}_\pm, \bar{x}_\pm)y + F_v(\bar{x}_\pm, \bar{x}_\pm)$, reduces to $acy^2 - (ac - d)y - d = 0$, which has characteristic values $y_1 = 1$ and $y_2 = -\frac{d}{ac}$, where $-1 < y_2 < 0$ since $ac > d$. Thus in this case \bar{x}_\pm is nonhyperbolic of stable type. When \bar{x}_- exists, then

$$\begin{aligned} \lambda(\bar{x}_-) > 1 &\iff 4c > (ac + d - \sqrt{R})^2 \\ &\iff 4c + (ac + d)\sqrt{R} > (ac + d)^2 \\ &\iff (ac + d)\sqrt{R} > (ac + d)^2 - 4c = R \\ &\iff (ac + d)^2 > R = (ac + d)^2 - 4c, \end{aligned}$$

which is true since $c > 0$. To show more specifically that \bar{x}_- is a saddle point when $R > 0$, we must show that $\delta(\bar{x}_-) < 1$, where δ is defined by Equation (15). Notice

$$\delta(\bar{x}_-) = \frac{a(d - 2c\bar{x}_-)}{(1 + c\bar{x}_-^2 + d\bar{x}_-)^2} = \frac{a(d - 2c\bar{x}_-)}{(a(c\bar{x}_- + d))^2} = \frac{4(d - 2c\bar{x}_-)}{a(2c\bar{x}_- + 2d)^2} = \frac{4(2d - ac + \sqrt{R})}{a(ac + d - \sqrt{R})^2},$$

and so we have that

$$\begin{aligned} \delta(\bar{x}_-) < 1 &\iff 4\left(2d - ac + \sqrt{R}\right) < a\left(ac + d - \sqrt{R}\right)^2 \\ &\iff (2 + a(ac + d))\sqrt{R} < a(ac + d)^2 - 4d. \end{aligned}$$

The right-hand side of the latter inequality is positive since $a(ac + d)^2 - 4d > 4ac - 4d = 4(ac - d) > 0$ by assumption. But then

$$\begin{aligned} \delta(\bar{x}_-) < 1 &\iff (2 + a(ac + d))^2((ac + d)^2 - 4c) < (a(ac + d)^2 - 4d)^2 \\ &\iff 3a^3c^2d + 6a^2cd^2 + 3ad^3 - 3a^2c^2 - 2acd - 3d^2 - 4c < 0 \\ &\iff (ad - 1)(3d^2 + 3a^2c^2 + 2c(3ad + 2)) < 0, \end{aligned}$$

which is automatically true since the latter factor is strictly positive and $ad < 1$. Thus indeed \bar{x}_- is a saddle point when it exists for $R > 0$. \square

Theorem 8 *There exist no minimal period-two solutions to Equation (19) if $c, d > 0$.*

Proof. Suppose there exist $\phi, \psi > 0$ with $\phi \neq \psi$ such that

$$\begin{cases} \psi = \frac{af(\phi, \psi)}{1 + f(\phi, \psi)} = \frac{a(c\phi^2 + d\psi)}{1 + c\phi^2 + d\psi} \\ \phi = \frac{af(\psi, \phi)}{1 + f(\psi, \phi)} = \frac{a(c\psi^2 + d\phi)}{1 + c\psi^2 + d\phi} \end{cases}. \tag{22}$$

From System (22) we notice that

$$\psi - \phi = \frac{a(\psi - \phi)(d - c(\psi + \phi))}{(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi)},$$

whence it immediately follows that $(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi) = a(d - c(\psi + \phi))$. But then

$$\psi + \phi = \frac{2(c\phi^2 + d\psi)(c\psi^2 + d\phi) + c(\psi^2 + \phi^2) + d(\psi + \phi)}{d - c(\psi + \phi)}.$$

Thus we have that necessarily

$$2\phi\psi = \frac{2a^2(c\phi^2 + d\psi)(c\psi^2 + d\phi)}{a(d - c(\psi + \phi))} = a\left((\psi + \phi) - \frac{c(\psi^2 + \phi^2) + d(\psi + \phi)}{d - c(\psi + \phi)}\right) > 0$$

since both $\psi, \phi > 0$. But this implies that

$$\begin{aligned} (\psi + \phi)(d - c(\psi + \phi)) &> c(\psi^2 + \phi^2) + d(\psi + \phi) \\ \iff d(\psi + \phi) - c(\psi + \phi)^2 &> c(\psi^2 + \phi^2) + d(\psi + \phi) \\ \iff 0 > c(\psi^2 + \phi^2) + c(\psi + \phi)^2, \end{aligned}$$

a clear contradiction since $c > 0$.

Now suppose there exists a period-two solution $\{\phi, \psi, \phi, \psi, \dots\}$ with $\phi \neq \psi$ but $\phi\psi = 0$. Suppose without loss of generality that $\phi = 0$. Now

$$\begin{cases} \psi &= \frac{af(0, \psi)}{1 + f(0, \psi)} = \frac{ad\psi}{1 + d\psi} \\ 0 &= \frac{af(\psi, 0)}{1 + f(\psi, 0)} = \frac{ac\psi^2}{1 + c\psi^2} \end{cases},$$

which immediately leads to the contradiction $\psi = \phi = 0$ for $c > 0$. Thus Equation (19) has no minimal period-two solutions. \square

The next result describes the global dynamics of Equation (19).

Theorem 9 Consider Equation (19) under the condition $c > 0$ and $d > 0$.

(1) Suppose either

- (a) $d \geq ac$ and $1 \geq ad$, or
- (b) $d < ac$, $1 > ad$, and $R < 0$.

Then \bar{x}_0 is a global attractor of all solutions.

(2) Suppose either

- (a) $1 < ad$, or
- (b) $d < ac$ and $1 = ad$.

Then \bar{x}_+ is a global attractor of all nonzero solutions.

(3) Suppose $d < ac$, $1 > ad$, and $R = 0$. Then Equation (19) has the equilibria $\bar{x}_0 = 0$, which is locally asymptotically stable, and \bar{x}_\pm , which is nonhyperbolic of stable type. There exists a continuous curve \mathcal{C} passing through $E = (\bar{x}_\pm, \bar{x}_\pm)$ such that \mathcal{C} is the graph of a decreasing function. The set of initial conditions $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of two disjoint basins of attraction, namely $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E)$, where $E_0 = (\bar{x}_0, \bar{x}_0)$,

$$\begin{aligned} \mathcal{B}(E_0) &= \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{C}\}, \text{ and} \\ \mathcal{B}(E) &= \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{C}\} \cup \mathcal{C}. \end{aligned}$$

(4) Suppose $d < ac$, $1 > ad$, and $R > 0$. Then Equation (19) has the equilibria $\bar{x}_0 = 0$, which is locally asymptotically stable, \bar{x}_- , which is a saddle point, and \bar{x}_+ , which is locally asymptotically stable. There exist two continuous curves $\mathcal{W}^s(E_-)$ and $\mathcal{W}^u(E_-)$, both passing through $E_- = (\bar{x}_-, \bar{x}_-)$, such that $\mathcal{W}^s(E_-)$ is the graph of a decreasing function and $\mathcal{W}^u(E_-)$ is the graph of an increasing function. The set of initial conditions $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of three disjoint basins of attraction, namely $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_-) \cup \mathcal{B}(E_+)$, where $E_0 = (\bar{x}_0, \bar{x}_0)$, $E_+ = (\bar{x}_+, \bar{x}_+)$, $\mathcal{B}(E_-) = \mathcal{W}^s(E_-)$,

$$\begin{aligned} \mathcal{B}(E_0) &= \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{W}^s(E_-)\}, \text{ and} \\ \mathcal{B}(E_+) &= \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{W}^s(E_-)\} \end{aligned}$$

Proof. (1) The proof in this case follows from Theorems 1, 7, and 8 along with the fact that $\bar{x}_0 = 0$ is the sole equilibrium of Equation (19).

(2) The proof used to show that all solutions with positive initial conditions converge to \bar{x}_+ follows from an application of Theorem 2 (as used above in the proof of Theorem 4). Notice that $x_1 = F(x_0, x_{-1}) > 0$ if either $x_0 > 0$ or $x_{-1} > 0$ (and similar for x_2), so $I = (0, \infty)$ is an attracting and invariant interval. Thus all nonzero solutions must converge to \bar{x}_+ .

(3) The proof follows from an application of Theorems 1-4 of [17] applied to the *cooperative* second iterate of the map corresponding to Equation (19). The proof is completely analogous to the proof of Theorem 5 in [4], so we omit the details.

(4) The proof follows from an immediate application of Theorem 5 in [4]. □

3.4 Linear-Quadratic: $f(u, v) = cu + dv^2$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1}^2)}{1 + cx_n + dx_{n-1}^2}, \quad n = 0, 1, \dots \tag{23}$$

Remark 2 For the analysis that follows, we will consider Equation (23) with $c > 0$ and $d > 0$. Notice that when $d = 0$ Equation (23) reduces to Equation (2), a special case of Equation (16). When $c = 0$ Equation (23) is essentially Equation (8) with delay.

An equilibrium of (23) satisfies

$$d\bar{x}^3 + c\bar{x}^2 + \bar{x} = ac\bar{x} + ad\bar{x}^2$$

so that all nonzero equilibria satisfy

$$d\bar{x}^2 + (c - ad)\bar{x} + (1 - ac) = 0, \tag{24}$$

whence we easily deduce the possible solutions

$$\bar{x}_{\pm} = \frac{ad - c \pm \sqrt{(c - ad)^2 + 4d(ac - 1)}}{2d},$$

which are real if and only if $R = (c - ad)^2 + 4d(ac - 1) \geq 0$.

Notice that

$$R \geq 0 \iff c^2 - 2acd + a^2d^2 + 4acd - 4d \geq 0 \iff (ad + c)^2 \geq 4d. \tag{25}$$

Here we have that

$$\lambda(\bar{x}) = \frac{a(c + 2d\bar{x})}{(1 + c\bar{x} + d\bar{x}^2)^2}.$$

Theorem 10 Equation (23) always has the zero equilibrium $\bar{x}_0 = 0$, and

$$\bar{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } ac < 1 \\ \text{nonhyperbolic} & \text{if } ac = 1 \\ \text{unstable} & \text{if } ac > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\bar{x}_0) = ac$. □

Theorem 11 Consider Equation (23) and assume $c > 0$ and $d > 0$.

- (1) *Suppose either*
 (a) $c \geq ad$ and $1 \geq ac$, or
 (b) $c < ad$, $1 > ac$, and $R < 0$.
 Then Equation (23) has no positive equilibria.
- (2) *Suppose either*
 (a) $1 < ac$, or
 (b) $c < ad$ and $1 = ac$.
 Then Equation (23) has the positive equilibrium solution \bar{x}_+ , and it is locally asymptotically stable.
- (3) *Suppose $c < ad$, $1 > ac$, and $R = 0$. Then Equation (23) has the positive equilibrium solution \bar{x}_\pm , and it is nonhyperbolic of stable type.*
- (4) *Suppose $c < ad$, $1 > ac$, and $R > 0$. Then Equation (23) has two positive equilibria, \bar{x}_+ and \bar{x}_- ; \bar{x}_+ is locally asymptotically stable, and \bar{x}_- is unstable.
 Let $K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d$.
 (i) If $K < 0$, then \bar{x}_- is a saddle point.
 (ii) If $K > 0$, then \bar{x}_- is a repeller.
 (iii) If $K = 0$, then \bar{x}_- is nonhyperbolic of unstable type (that is one characteristic value is $\lambda_1 = \pm 1$ and the other $|\lambda_1| > 1$).*

Proof. Much of the analysis is similar to the considerations in the proof of Theorem 7. Notice that

$$\lambda(\bar{x}) = \frac{a(c + 2d\bar{x})}{(1 + c\bar{x} + d\bar{x}^2)^2} = \frac{a(c + 2d\bar{x})}{(a(c + d\bar{x}))^2} = \frac{c + 2d\bar{x}}{a(c + d\bar{x})^2} = \frac{1}{a(c + d\bar{x})} + \frac{d\bar{x}}{a(c + d\bar{x})^2}.$$

For the parametric values for which \bar{x}_+ exists,

$$\begin{aligned} \lambda(\bar{x}_+) \leq 1 &\iff \frac{d\bar{x}_+}{a(c + d\bar{x}_+)^2} \leq \frac{a(c + d\bar{x}_+) - 1}{a(c + d\bar{x}_+)} \\ &\iff d\bar{x}_+ \leq (c + d\bar{x}_+)(a(c + d\bar{x}_+) - 1) = (c + d\bar{x}_+)(c\bar{x}_+ + d\bar{x}_+^2) \\ &\iff d \leq (c + d\bar{x}_+)^2 \\ &\iff 4d \leq (2c + 2d\bar{x}_+)^2 = (ad + c + \sqrt{R})^2, \end{aligned}$$

which is true by (25). Thus if $R > 0$, \bar{x}_+ is locally asymptotically stable, and if $R = 0$, \bar{x}_\pm is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (23) about \bar{x}_\pm , $y^2 = F_u(\bar{x}_\pm, \bar{x}_\pm)y + F_v(\bar{x}_\pm, \bar{x}_\pm)$, reduces to $ady^2 - cy + c - ad = 0$, which has characteristic values $y_1 = 1$ and $y_2 = \frac{c-ad}{ad}$, where $-1 < y_2 < 0$ since $ad > c$. Thus in this case \bar{x}_\pm is nonhyperbolic of stable type. When \bar{x}_- exists,

$$\begin{aligned} \lambda(\bar{x}_-) > 1 &\iff 4d > (ad + c - \sqrt{R})^2 \\ &\iff 4d + (ad + c)\sqrt{R} > (ad + c)^2 \\ &\iff (ad + c)\sqrt{R} > (ad + c)^2 - 4d = R \\ &\iff (ad + c)^2 > R = (ad + c)^2 - 4d \end{aligned}$$

which is true since $d > 0$. To more specifically classify \bar{x}_- , we must calculate $\delta(\bar{x}_-)$. Notice

$$\delta(\bar{x}_-) = \frac{a(2d\bar{x}_- - c)}{(1 + c\bar{x}_- + d\bar{x}_-^2)^2} = \frac{a(2d\bar{x}_- - c)}{(a(c + d\bar{x}_-))^2} = \frac{4(2d\bar{x}_- - c)}{a(2c + 2d\bar{x}_-)^2} = \frac{4(ad - 2c - \sqrt{R})}{a(ad + c - \sqrt{R})^2},$$

and so we have that

$$\begin{aligned} \delta(\bar{x}_-) \geq 1 &\iff 4(ad - 2c - \sqrt{R}) \geq a(ad + c - \sqrt{R})^2 \\ &\iff (a(ad + c) - 2)\sqrt{R} \geq a(ad + c)^2 - 4ad + 4c = aR + 4c. \end{aligned}$$

Notice that $R > 0$ automatically implies $a(ad + c) > 2$, as

$$0 < (ad + c)^2 - 4d < a^2d^2 + 2acd + a^2d^2 - 4d = 2d(a(ad + c) - 2)$$

since $c < ad$. Therefore we may square both sides to obtain

$$\begin{aligned} \delta(\bar{x}_-) \geq 1 &\iff (a(ad + c) - 2)^2 R \geq (aR + 4c)^2 \\ &\iff R(a^2(ad + c)^2 - 4a(ad + c) + 4) \geq a^2R^2 + 8acR + 16c^2 \\ &\iff R(a^2R - 4ac + 4) \geq a^2R^2 + 8acR + 16c^2 \\ &\iff R(1 - 3ac) - 4c^2 \geq 0 \\ &\iff a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d \geq 0. \end{aligned}$$

Thus if

$$K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d, \tag{26}$$

$K < 0$ implies \bar{x}_- is a saddle point and $K > 0$ implies it is a repeller. If $K = 0$, \bar{x}_- is nonhyperbolic, and we expect in such case to be nonhyperbolic of unstable type. Indeed one can show that in the event $K = 0$, the characteristic equation of the linearization of Equation (23) about \bar{x}_- , $y^2 = F_u(\bar{x}_-, \bar{x}_-)y + F_v(\bar{x}_-, \bar{x}_-)$, has roots $y_1 = -1$ and $y_2 = F_u(\bar{x}_-, \bar{x}_-) + 1 > 1$, which immediately shows the desired result. \square

The investigation of the existence of periodic solutions of Equation (23) is an interesting one that involves a thorough analysis of potential parametric cases. This analysis will reveal the potential for the existence of several nonzero periodic solutions. The juxtaposition of Equation (19) with Equation (23) illustrates an interesting phenomenon in which, loosely speaking, the dominance of the delay term x_{n-1} contributes to the possibility of periodic solutions arising.

A minimal period-two solution $\{\phi, \psi, \phi, \psi, \dots\}$ with $\phi, \psi > 0$ and $\phi \neq \psi$ must satisfy

$$\begin{cases} \psi = \frac{af(\phi, \psi)}{1 + f(\phi, \psi)} = \frac{a(c\phi + d\psi^2)}{1 + c\phi + d\psi^2} \\ \phi = \frac{af(\psi, \phi)}{1 + f(\psi, \phi)} = \frac{a(c\psi + d\phi^2)}{1 + c\psi + d\phi^2} \end{cases}. \tag{27}$$

Eliminating either ψ or ϕ from System (27) we obtain

$$(d\phi^2 + (c - ad)\phi + (1 - ac))h(\phi) = 0, \quad \text{or} \quad (d\psi^2 + (c - ad)\psi + (1 - ac))h(\psi) = 0,$$

where

$$\begin{aligned} h(x) &= -d^3x^6 + d^2(c + 2ad)x^5 - d(c^2 + 2d + 3acd + a^2d^2)x^4 + d(c + 3ac^2 + 2ad + 3a^2cd)x^3 \\ &\quad - (c^2 + ac^3 + d + 2acd + 3a^2c^2d + a^3cd^2)x^2 + ac(1 + ac)(2c + ad)x - a^2c^2(1 + ac). \end{aligned} \tag{28}$$

Since $dx^2 + (c - ad)x + (1 - ac) \neq 0$ for any x that is not a solution of the equilibrium equation (24), minimal period-two solutions must be the solutions of the equation

$$h(x) = 0. \tag{29}$$

Theorem 12 *Any real solutions of Equation (29) are positive numbers for $c, d > 0$, and there exist up to three minimal period-two solutions of Equation (23). Furthermore, let K be as defined in Equation (26), and define the following expressions:*

$$J = 4a^5cd^4 - 8a^4c^2d^3 + 12a^3c^3d^2 - 24a^3cd^3 - 8a^2c^4d + 28a^2c^2d^2 - a^2d^3 + 4ac^5 + 4ac^3d + 32acd^2 + 4c^4 + 8c^2d + 4d^2$$

$$\Delta_1 = 6d^6$$

$$\Delta_2 = d^{10} (8a^2d^2 - 16acd - 7c^2 - 24d)$$

$$\Delta_3 = -2d^{12} (8a^5cd^5 + 13a^4c^2d^4 + 10a^3c^3d^3 - 44a^3cd^4 + 4a^2c^4d^2 - 34a^2c^2d^3 - 4a^2d^4 - 19ac^5d + 14ac^3d^2 + 44acd^3 + 6c^6 + 7c^4d + 5c^2d^2 + 16d^3)$$

$$\Delta_4 = c^2d^{13} (-16a^9cd^8 - 12a^8c^2d^7 + 24a^7c^3d^6 + 152a^7cd^7 - 68a^6c^4d^5 + 80a^6c^2d^6 + 8a^6d^7 + 48a^5c^5d^4 - 164a^5c^3d^5 - 464a^5cd^6 - 60a^4c^6d^3 + 20a^4c^4d^4 - 180a^4c^2d^5 - 64a^4d^6 + 56a^3c^7d^2 - 332a^3c^5d^3 + 388a^3c^3d^4 + 488a^3cd^5 - 48a^2c^8d + 272a^2c^6d^2 + 255a^2c^4d^3 + 152a^2c^2d^4 + 136a^2d^5 + 24ac^9 + 8ac^7d + 124ac^5d^2 + 180ac^3d^3 - 152acd^4 + 24c^8 + 68c^6d + 32c^4d^2 - 44c^2d^3 - 32d^4)$$

$$\Delta_5 = 2c^4d^{13}J (3a^8c^2d^6 + 2a^7cd^6 - 18a^6c^2d^5 - a^6d^6 + 6a^5c^5d^3 + 10a^5c^3d^4 - 8a^5cd^5 - 10a^4c^4d^3 + 44a^4c^2d^4 + 6a^4d^5 + 54a^3c^5d^2 - 25a^3c^3d^3 - 6a^3cd^4 + 3a^2c^8 - 8a^2c^6d + 35a^2c^4d^2 - 39a^2c^2d^3 - 9a^2d^4 + 6ac^7 + 2ac^5d + 4ac^3d^2 + 14acd^3 + 3c^6 + 10c^4d + 11c^2d^2 + 4d^3)$$

$$\Delta_6 = a^2c^6d^{14}(ac + 1)KJ^2.$$

- (1) *If $\Delta_i > 0$ for all $2 \leq i \leq 6$ then Equation (29) has six real roots. Consequently, Equation (23) has three minimal period-two solutions.*
- (2) *If $\Delta_j \leq 0$ for some $2 \leq j \leq 5$ and $\Delta_i > 0$ for $i \neq j$, then Equation (29) has two distinct real roots and two pairs of conjugate imaginary roots. Consequently, Equation (23) has one minimal period-two solution.*
- (3) *If $\Delta_i \leq 0$, $\Delta_{i+1} \geq 0$ (such that at least one of these is strict) for some $2 \leq i \leq 4$, and if $\Delta_6 < 0$, then Equation (29) has three pairs of conjugate imaginary roots. Consequently, Equation (23) has no minimal period-two solutions.*

Proof. The proof of the first statement follows from Descartes' Rule of Signs.

Let $\text{disc}(h)$ denote the 12×12 discrimination matrix as defined in [12]:

$$\text{disc}(h) = \begin{bmatrix} a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 \end{bmatrix}.$$

Here a_k equals the coefficient of the degree- k term of h as defined in Equation (28); that is, $a_6 = -d^3$, $a_5 = d^2(c + 2ad)$, $a_4 = -d(c^2 + 2d + 3acd + a^2d^2)$, $a_3 = d(c + 3ac^2 + 2ad + 3a^2cd)$, $a_2 = -(c^2 + ac^3 + d + 2acd + 3a^2c^2d + a^3cd^2)$, $a_1 = ac(1 + ac)(2c + ad)$, and $a_0 = -a^2c^2(1 + ac)$. Let Δ_k denote the determinant of the submatrix of $\text{disc}(h)$ formed by its first $2k$ rows and $2k$ columns for $k = 1, 2, \dots, 6$. Then the values of Δ_k are listed above, and the veracity of the statements above may now be verified by employing Theorem 1 of [12]. Notice that $\Delta_1 > 0$ for all $d > 0$. \square

Remark 3 The parametric conditions discussed above do not exhaust all of the parametric space but cover a substantial region of parameters for which Equation (23) possesses hyperbolic dynamics.

We will use the sufficient conditions provided in Theorems 10, 11, and 12 to obtain some global dynamic scenarios discussed in [4]. We will not investigate the dynamics of Equation (23) when it has one or no positive fixed point since in such cases the dynamics should be similar to the dynamics of Equation (19) discussed in Theorem 9. The following theorem relies on results from [4] and summarizes potential hyperbolic dynamic scenarios for Equation (23) in the event it possesses three fixed points and zero, one, or three pairs of hyperbolic period-two points. In particular, Theorem 3 is applicable to case (ii) of the following result. See also the statement and proof of Theorem 11 in [4].

Theorem 13 Consider Equation (23) and assume $0 < c < ad, ac < 1$ such that $R > 0$.

- (i) If $\Delta_i > 0$ for all $2 \leq i \leq 6$ then Equation (23) has three equilibria $\bar{x}_0 < \bar{x}_- < \bar{x}_+$, where \bar{x}_0 and \bar{x}_+ are locally asymptotically stable and \bar{x}_- is a repeller, and three minimal period-two solutions $\{\phi_1, \psi_1\}$, $\{\phi_2, \psi_2\}$, and $\{\phi_3, \psi_3\}$. Here $(\phi_1, \psi_1) \prec_{ne} (\phi_2, \psi_2) \prec_{ne} (\phi_3, \psi_3)$, $\{\phi_1, \psi_1\}$ and $\{\phi_3, \psi_3\}$ are saddle points, and $\{\phi_2, \psi_2\}$ is locally asymptotically stable. The global behavior of Equation (23) is described by Theorem 8 of [4]. In this case there exist four continuous curves $\mathcal{W}^s(\phi_1, \psi_1)$, $\mathcal{W}^s(\psi_1, \phi_1)$, $\mathcal{W}^s(\phi_3, \psi_3)$, $\mathcal{W}^s(\psi_3, \phi_3)$ that have endpoints at $E_- = (\bar{x}_-, \bar{x}_-)$ and are graphs of decreasing functions. Every solution which starts below $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ in the northeast ordering converges to $E_0 = (\bar{x}_0, \bar{x}_0)$ and every solution which starts above $\mathcal{W}^s(\phi_3, \psi_3) \cup \mathcal{W}^s(\psi_3, \phi_3)$ in the northeast ordering converges to $E_+ = (\bar{x}_+, \bar{x}_+)$. Every solution which starts above $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ and below $\mathcal{W}^s(\phi_3, \psi_3) \cup \mathcal{W}^s(\psi_3, \phi_3)$ in the northeast ordering converges to $\{\phi_2, \psi_2\}$. For example, this happens for $a = 1, c = \frac{389}{2176}$, and $d = \frac{249}{64}$.

- (ii) If $\Delta_j \leq 0$ for some $2 \leq j \leq 5$ and $\Delta_i > 0$ for $i \neq j$, then Equation (23) has three equilibria $\bar{x}_0 < \bar{x}_- < \bar{x}_+$, where \bar{x}_0 and \bar{x}_+ are locally asymptotically stable and \bar{x}_- is a repeller, and one period-two solution $\{\phi_1, \psi_1\}$, which is a saddle point. The global behavior of Eq. (23) is described by Theorem 7 of [4]. In this case there exist four continuous curves $\mathcal{W}^s(\phi_1, \psi_1), \mathcal{W}^s(\psi_1, \phi_1), \mathcal{W}^u(\phi_1, \psi_1), \mathcal{W}^u(\psi_1, \phi_1)$, where $\mathcal{W}^s(\phi_1, \psi_1), \mathcal{W}^s(\psi_1, \phi_1)$ have endpoints at $E_- = (\bar{x}_-, \bar{x}_-)$ and are graphs of decreasing functions. The curves $\mathcal{W}^u(\phi_1, \psi_1), \mathcal{W}^u(\psi_1, \phi_1)$ are graphs of increasing functions and start at $E_0 = (\bar{x}_0, \bar{x}_0)$. Every solution which starts below $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ in the northeast ordering converges to E_0 and every solution which starts above $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ in the northeast ordering converges to $E_+ = (\bar{x}_+, \bar{x}_+)$. For example, this happens for $a = 1, c = \frac{1}{5}$, and $d = \frac{237}{64}$.
- (iii) If $\Delta_i \leq 0$ and $\Delta_{i+1} \geq 0$ (such that at least one of these is strict) for some $2 \leq i \leq 4$, and if $\Delta_6 < 0$, then Eq. (23) has three equilibria $\bar{x}_0 < \bar{x}_- < \bar{x}_+$, where \bar{x}_0 and \bar{x}_+ are locally asymptotically stable and \bar{x}_- is a saddle point, and no period-two solution. The global behavior of Equation (23) is described by Theorem 5 of [4] or Theorem 9 case (4). For example, this happens for $a = 1, c = \frac{493}{1024}$, and $d = \frac{157}{48}$.

Equation (23) exhibits global dynamics similar to that of Equation (10), which was investigated in [4]. Therefore, we pose the following conjecture.

Conjecture 1 *There exists a topological conjugation between the maps in Equations (10) and (23).*

References

- [1] L. J. S. Allen. *An Introduction to Mathematical Biology*, Prentice Hall, 2006.
- [2] A. M. Amleh, E. Camouzis, and G. Ladas, On the Dynamics of a Rational Difference Equation, Part 1. *Inter. J. Difference Equ.*, 3(1): 1–35, 2008.
- [3] E. Bertrand and M. R. S. Kulenović, Global Dynamics of Higher-Order Transcendental-Type Generalized Beverton–Holt Equations, *Int. J. of Difference Equations*, 13(2018), 71–84.
- [4] A. Bilgin, M. R. S. Kulenović, and E. Pilav, Basins of Attraction of Period-Two Solutions of Monotone Difference Equations. *Adv. Difference Equ.*, (2016), 2016: 74.
- [5] A. Bilgin and M. R. S. Kulenović, Global Asymptotic Stability for Discrete Single Species Population Models, *Discrete Dyn. Nat. Soc.*, Volume 2017, 2017, 15 p.
- [6] A. Brett and M. R. S. Kulenović, Basins of Attraction of Equilibrium Points of Monotone Difference Equations. *Sarajevo J. Math.*, 5(18): 211–233, 2009.
- [7] A. Brett, E. J. Janowski, and M. R. S. Kulenović, Global Asymptotic Stability for Linear Fractional Difference Equation, *J. Difference Equ. Appl.*, 1(2014), 12 p.
- [8] M. DiPippo, E. J. Janowski, and M. R. S. Kulenović, Global Asymptotic Stability for Quadratic Fractional Difference Equation, *Adv. Difference Equ.*, (2015), 2015:179, 13 p.
- [9] S. Elaydi. *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 3rd edition, 2005.
- [10] E. A. Grove, C. M. Kent, R. Levins, G. Ladas, and S. Valicenti, Global Stability in Some Population Models, *Proceedings of the Fourth International Conference on Difference Equations and Applications*, August 27-31, 1998, Poznan, Poland, Gordon and Breach Science Publishers, 2000, 149–176.

- [11] C. S. Holling, The functional response of predators to prey density and its role in mimicry and population regulation, *Memoirs of the Entomological Society of Canada*, vol. 97, supplement 45, pp. 5–60, 1965.
- [12] X. Hou, L. Yang, and Z. Zeng, A complete discrimination system for polynomials. *Sci. China Ser. E*, 39(6), 628–646 (1996).
- [13] E. J. Janowski and M. R. S. Kulenović, Attractivity and global stability for linearizable difference equations, *Comp. Math. Appl.* 57(2009), 1592–1607.
- [14] M. R. S. Kulenović and G. Ladas. *Dynamics of Second Order Rational Difference Equations*. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [15] M. R. S. Kulenović and O. Merino. *Discrete Dynamical Systems and Difference Equations with Mathematica*. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [16] M. R. S. Kulenović and O. Merino, Global bifurcations for competitive systems in the plane. *Discrete Contin. Dyn. Syst. Ser. B*, 12 (2009), 133–149.
- [17] M. R. S. Kulenović and O. Merino, Invariant manifolds for competitive discrete systems in the plane, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 20 (2010), 2471–2486.
- [18] M. R. S. Kulenović and A. A. Yakubu, Compensatory versus Overcompensatory Dynamics in Density-dependent Leslie Models, *J. Difference Equ. Appl.* 10 (2004), 1251–1266.
- [19] H. L. Smith and H. Thieme, Persistence and global stability for a class of discrete time structured population models, *Discrete Contin. Dyn. Syst. Ser. A*, (33), 4627–4646.
- [20] H. L. Smith, Bacterial Growth. Arizona State University. 14 June 2006. <https://math.la.asu.edu/hal-smith/bacteriagrow.pdf>

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