Multivariate Gudermannian function based neural network approximation

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Abstract

Here we present multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We examine also the case of approximation by iterated operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the Gudermannian sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

2020 AMS Mathematics Subject Classification: 41A17, 41A25, 41A30, 41A36.

Keywords and Phrases: Gudermannian sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated approximation.

1 Introduction

G.A. Anastassiou in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities.

He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Motivations for this work are the article [17] of Z. Chen and F. Cao, and [4], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15], [18], [19].

Here we perform multivariate Gudermannian sigmoid function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$, and also iterated approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by Gudermannian sigmoid function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the Gudermannian sigmoid function. About neural networks see [20], [21], [22].

2 Background

See also [13], [24].

Here we consider $gd(x)$ the Gudermannian function [24], which is a sigmoid function, as a generator function:

$$\sigma(x) = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right) = \int_0^x \frac{dt}{\cosh t} =: gd(x), \; x \in \mathbb{R}. \tag{1}$$

Let the normalized generator sigmoid function

$$f(x) := \frac{2}{\pi}\sigma(x) = \frac{2}{\pi}\int_0^x \frac{dt}{\cosh t} = \frac{4}{\pi}\int_0^x \frac{1}{e^t + e^{-t}} dt, \quad x \in \mathbb{R}. \tag{2}$$

Here

$$f'(x) = \frac{2}{\pi \cosh x} > 0, \quad \forall\, x \in \mathbb{R},$$

hence $f$is strictly increasing on $\mathbb{R}$.

Notice that $\tanh(-x) = -\tanh x$ and $\arctan(-x) = -\arctan x$, $x \in \mathbb{R}$.

So, here the neural network activation function will be:

$$W(x) = \frac{1}{4}\left[f(x+1) - f(x-1)\right], \; x \in \mathbb{R}. \tag{3}$$

By [3], we get that

$$W(x) = W(-x), \quad \forall\, x \in \mathbb{R}, \tag{4}$$

i.e. it is even and symmetric with respect to the y-axis. Here we have $f(+\infty) = 1$, $f(-\infty) = -1$ and $f(0) = 0$. Clearly it is

$$f(-x) = -f(x), \;\; \forall\, x \in \mathbb{R}, \tag{5}$$

an odd function, symmetric with respect to the origin. Since $x+1 > x-1$, and $f(x+1) > f(x-1)$, we obtain $W(x) > 0$, $\forall\, x \in \mathbb{R}$.

By [13], we have that

$$W(0) = \frac{1}{\pi} gd(1) \cong 0.2757. \tag{6}$$

By [13] W is strictly decreasing on $(0, +\infty)$, and strictly increasing on $(-\infty, 0)$, and $W'(0) = 0$.

Also we have that

$$\lim_{x \to +\infty} W(x) = \lim_{x \to -\infty} W(x) = 0, \tag{7}$$

that is the x-axis is the horizontal asymptote for W.

Conclusion, W is a bell shaped symmetric function with maximum $W(0) \cong 0.2757$.

We need

Theorem 1 *([13]) It holds that*

$$\sum_{i=-\infty}^{\infty} W(x-i) = 1, \;\; \forall\, x \in \mathbb{R}. \tag{8}$$

Theorem 2 *([13]) We have that*

$$\int_{-\infty}^{\infty} W(x)\, dx = 1. \tag{9}$$

So $W(x)$ is a density function.

Theorem 3 *([13]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} W(nx-k) < \frac{2}{\pi e^{(n^{1-\alpha}-2)}} = \frac{2e^2}{\pi e^{n^{1-\alpha}}}. \tag{10}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 4 *([13]) Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k)} < \frac{2\pi}{gd(2)} \cong 4.824, \tag{11}$$

$\forall\ x \in [a,b]$.

We make

Remark 5 *([13])*

(i) We have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \neq 1, \tag{12}$$

for at least some $x \in [a,b]$.

(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \leq 1. \tag{13}$$

We introduce

$$Z(x_1,...,x_N) := Z(x) := \prod_{i=1}^{N} W(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{14}$$

It has the properties:

(i) $Z(x) > 0,\ \forall\ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1,...,x_N-k_N) = 1, \tag{15}$$

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where $k := (k_1, ..., k_n) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \tag{16}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1, \tag{17}$$

that is Z is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\begin{gathered} \lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil), \\ \lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor), \end{gathered} \tag{18}$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} W(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} W(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} W(nx_i - k_i) \right). \tag{19}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k). \tag{20}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

(v) As in [10], pp. 379-380, we derive that

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} Z(nx-k) \overset{(10)}{<} \frac{2e^2}{\pi e^{n^{1-\beta}}},\ \ 0<\beta<1,\ m\in\mathbb{N}, \tag{21}$$

with $n\in\mathbb{N}: n^{1-\beta}>2$, $x\in\prod_{i=1}^{N}[a_i,b_i]$.

(vi) By Theorem 4 we get that

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}<\left(\frac{2\pi}{gd(2)}\right)^N \cong (4.824)^N, \tag{22}$$

$\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n\in\mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\infty} Z(nx-k) < \frac{2e^2}{\pi e^{n^{1-\beta}}}, \tag{23}$$

$0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}>2$, $x\in\mathbb{R}^N$, $m\in\mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\neq 1, \tag{24}$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Here $\left(X,\|\cdot\|_{\gamma}\right)$ is a Banach space.

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$, $x=(x_1,...,x_N)\in\prod_{i=1}^{N}[a_i,b_i]$, $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We introduce and define the following multivariate linear normalized neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$):

$$A_n(f,x_1,...,x_N):=A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}=$$

$$\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}W(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} W(nx_i-k_i)\right)}. \tag{25}$$

For large enough $n\in\mathbb{N}$ we always obtain $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$. Also $a_i\leq\frac{k_i}{n}\leq b_i$, iff $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

When $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} g\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}. \tag{26}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1,x) = 1, \ \ \forall \ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$\|A_n(f,x)\|_\gamma \leq \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} = \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{27}$$

$\forall \ x \in \prod_{i=1}^{N}[a_i,b_i]$.

Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma \leq \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{28}$$

$\forall \ x \in \prod_{i=1}^{N}[a_i,b_i]$, $\forall \ n \in \mathbb{N}$, $\forall \ f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then $cg \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Furthermore it holds

$$A_n(cg,x) = c\widetilde{A}_n(g,x), \ \ \forall \ x \in \prod_{i=1}^{N}[a_i,b_i]. \tag{29}$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$A_n(c) = c, \ \ \forall \ c \in X. \tag{30}$$

We call $\widetilde{A}_n$ the companion operator of A_n.

For convinience we call

$$A_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) =$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} \dots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} W(nx_i - k_i)\right), \tag{31}$$

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$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

That is

$$A_n(f,x) := \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}, \tag{32}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}. \tag{33}$$

Consequently we derive

$$\left\| A_n(f,x) - f(x) \right\|_\gamma \overset{(22)}{\leq} (4.824)^N \left\| A_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right\|_\gamma, \tag{34}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

We will estimate the right hand side of (34).

For the last and others we need

Definition 6 *([11], p. 274) Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M,X)$. We define the first modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \leq diam(M). \tag{35}$$

If $\delta > diam(M)$, then

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{36}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 7 *([11], p. 274) We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M,X)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (35). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} W(nx_i - k_i)\right), \tag{37}$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x) := C_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z(nx-k) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,...,t_N)\,dt_1...dt_N\right)$$

$$\cdot \left(\prod_{i=1}^{N} W(nx_i - k_i)\right), \tag{38}$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1,r_2,...r_N} \geq 0$, such that $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) =$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{39}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x) := D_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f)\, Z(nx-k) = \tag{40}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,k_2,...,k_N}(f) \left(\prod_{i=1}^{N} W(nx_i - k_i)\right),$$

$\forall\ x \in \mathbb{R}^N$.

In this article we study the approximation properties of $A_n, B_n, C_n,\ D_n$ neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 8 *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m, N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

1)

$$\left\|A_n(f,x) - f(x)\right\|_\gamma \leq (4.824)^N \left[\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4e^2 \left\|\left\|f\right\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}}\right] =: \lambda_1(n), \tag{41}$$

and

2)

$$\left\|\left\|A_n(f) - f\right\|_\gamma\right\|_\infty \leq \lambda_1(n). \tag{42}$$

We notice that $\lim_{n\to\infty} A_n(f) \stackrel{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

Above ω_1 *is with respect to* $p=\infty$.

Proof. We observe that

$$\Delta(x) := A_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x) Z(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx-k). \tag{43}$$

Thus

$$\left\|\Delta(x)\right\|_\gamma \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_{\gamma} Z(nx-k) +$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_{\gamma} Z(nx-k) \overset{(16)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^{\beta}}\right) + 2\left\| \|f\|_{\gamma} \right\|_{\infty} \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) \overset{(21)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^{\beta}}\right) + \frac{4e^2 \left\| \|f\|_{\gamma} \right\|_{\infty}}{\pi e^{n^{1-\beta}}}. \tag{44}$$

So that

$$\|\Delta(x)\|_{\gamma} \leq \omega_1\left(f, \frac{1}{n^{\beta}}\right) + \frac{4e^2 \left\| \|f\|_{\gamma} \right\|_{\infty}}{\pi e^{n^{1-\beta}}}. \tag{45}$$

Now using (34) we finish the proof. ■

We make

Remark 9 *([11], pp. 263-266) Let $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $N \in \mathbb{N}$; where $\|\cdot\|_p$ is the L_p-norm, $1 \leq p \leq \infty$. $\mathbb{R}^N$ is a Banach space, and $\left(\mathbb{R}^N\right)^j$ denotes the j-fold product space $\mathbb{R}^N \times ... \times \mathbb{R}^N$ endowed with the max-norm $\|x\|_{(\mathbb{R}^N)^j} := \max_{1 \leq \lambda \leq j} \|x_{\lambda}\|_p$, where $x := (x_1, ..., x_j) \in \left(\mathbb{R}^N\right)^j$.*

Let $\left(X, \|\cdot\|_{\gamma}\right)$ be a general Banach space. Then the space $L_j := L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ of all j-multilinear continuous maps $g : \left(\mathbb{R}^N\right)^j \to X$, $j = 1, ..., m$, is a Banach space with norm

$$\|g\| := \|g\|_{L_j} := \sup_{\left(\|x\|_{(\mathbb{R}^N)^j} = 1\right)} \|g(x)\|_{\gamma} = \sup \frac{\|g(x)\|_{\gamma}}{\|x_1\|_p ... \|x_j\|_p}. \tag{46}$$

Let M be a non-empty convex and compact subset of $\mathbb{R}^N$ and $x_0 \in M$ is fixed.

Let O be an open subset of $\mathbb{R}^N : M \subset O$. Let $f : O \to X$ be a continuous function, whose Fréchet derivatives (see [23]) $f^{(j)} : O \to L_j = L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ exist and are continuous for $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in \left(\mathbb{R}^N\right)^j$, $x \in M$.

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We will work with $f|_M$.

Then, by Taylor's formula ([16]), ([23], p. 124), we get

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} + R_m(x, x_0), \quad \text{all } x \in M, \tag{47}$$

where the remainder is the Riemann integral

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0) \right) (x-x_0)^m du, \tag{48}$$

here we set $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.

We consider

$$w := \omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \le h}} \left\| f^{(m)}(x) - f^{(m)}(y) \right\|, \tag{49}$$

$h > 0$.

We obtain

$$\left\| \left(f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0) \right) (x-x_0)^m \right\|_\gamma \le$$

$$\left\| f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0) \right\| \cdot \|x-x_0\|_p^m \le$$

$$w \|x-x_0\|_p^m \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil, \tag{50}$$

by Lemma 7.1.1, [1], p. 208, where $\lceil \cdot \rceil$ *is the ceiling.*

Therefore for all $x \in M$ *(see [1], pp. 121-122):*

$$\|R_m(x, x_0)\|_\gamma \le w \|x-x_0\|_p^m \int_0^1 \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil \frac{(1-u)^{m-1}}{(m-1)!} du$$

$$= w\Phi_m\left(\|x-x_0\|_p\right) \tag{51}$$

by a change of variable, where

$$\Phi_m(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t|-s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t|-jh)_+^m \right), \quad \forall\, t \in \mathbb{R}, \tag{52}$$

is a (polynomial) spline function, see [1], p. 210-211.

Also from there we get

$$\Phi_m(t) \le \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h|t|^{m-1}}{8(m-1)!} \right), \quad \forall\, t \in \mathbb{R}, \tag{53}$$

with equality true only at $t = 0$.

Therefore it holds

$$\left\|R_m(x, x_0)\right\|_\gamma \leq w\left(\frac{\left\|x - x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\|x - x_0\right\|_p^m}{2m!} + \frac{h\left\|x - x_0\right\|_p^{m-1}}{8(m-1)!}\right), \quad \forall\ x \in M. \tag{54}$$

We have found that

$$\left\|f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!}\right\|_\gamma \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left(\frac{\left\|x - x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\|x - x_0\right\|_p^m}{2m!} + \frac{h\left\|x - x_0\right\|_p^{m-1}}{8(m-1)!}\right) < \infty, \tag{55}$$

$\forall\ x, x_0 \in M$.

Here $0 < \omega_1\left(f^{(m)}, h\right) < \infty$, by M being compact and $f^{(m)}$ being continuous on M.

One can rewrite (55) as follows:

$$\left\|f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}\right\|_\gamma \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left(\frac{\left\|\cdot - x_0\right\|_p^{m+1}}{(m+1)!h} + \frac{\left\|\cdot - x_0\right\|_p^m}{2m!} + \frac{h\left\|\cdot - x_0\right\|_p^{m-1}}{8(m-1)!}\right), \ \forall\ x_0 \in M, \tag{56}$$

a pointwise functional inequality on M.

Here $(\cdot - x_0)^j$ maps M into $\left(\mathbb{R}^N\right)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps $\left(\mathbb{R}^N\right)^j$ into X and it is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into X.

Clearly $f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, hence $\left\|f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}\right\|_\gamma \in C(M)$.

Let $\left\{\widetilde{L}_N\right\}_{N \in \mathbb{N}}$ be a sequence of positive linear operators mapping $C(M)$ into $C(M)$.

Therefore we obtain

$$\left(\widetilde{L}_N\left(\left\|f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}\right\|_\gamma\right)\right)(x_0) \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left[\frac{\left(\widetilde{L}_N\left(\left\|\cdot - x_0\right\|_p^{m+1}\right)\right)(x_0)}{(m+1)!h} + \frac{\left(\widetilde{L}_N\left(\left\|\cdot - x_0\right\|_p^m\right)\right)(x_0)}{2m!}\right.$$

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$$+\frac{h\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{m-1}\right)\right)(x_0)}{8(m-1)!}\Bigg], \tag{57}$$

$\forall\ N\in\mathbb{N}$, $\forall\ x_0\in M$.

Clearly (57) is valid when $M=\prod_{i=1}^{N}[a_i,b_i]$ and $\widetilde{L}_n=\widetilde{A}_n$, see (26).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators A_n, $\widetilde{A}_n$ fulfill its assumptions, see (25), (26), (28), (29) and (30).

We present the following high order approximation results.

Theorem 10 *Let O open subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$, such that $\prod_{i=1}^{N}[a_i,b_i]\subset O\subseteq\mathbb{R}^N$, and let $\left(X,\|\cdot\|_\gamma\right)$ be a general Banach space. Let $m\in\mathbb{N}$ and $f\in C^m(O,X)$, the space of $\overline{m}$-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $r>0$. Then*

1)

$$\left\|\left(A_n(f)\right)(x_0)-\sum_{j=0}^{m}\frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right], \tag{58}$$

2) additionally if $f^{(j)}(x_0)=0$, $j=1,\ldots,\overline{m}$, we have

$$\left\|\left(A_n(f)\right)(x_0)-f(x_0)\right\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{59}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right],$$

3)

$$\left\|\left(A_n(f)\right)(x_0)-f(x_0)\right\|_\gamma\leq\sum_{j=1}^{m}\frac{1}{j!}\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma+$$

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$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{60}$$

and

4)

$$\left\|\|A_n(f) - f\|_\gamma\right\|_{\infty, \prod_{i=1}^N [a_i,b_i]} \leq$$
$$\sum_{j=1}^m \frac{1}{j!}\left\|\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma\right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]} +$$
$$\frac{\omega_1\left(f^{(m)}, r\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]}^{\frac{1}{m+1}}\right)}{rm!}$$
$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]}^{\left(\frac{m}{m+1}\right)} \tag{61}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right].$$

We need

Lemma 11 *The function* $\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^m\right)\right)(x_0)$ *is continuous in* $x_0 \in \left(\prod_{i=1}^N [a_i,b_i]\right)$, $m \in \mathbb{N}$.

Proof. By Lemma 10.3, [11], p. 272. ■

We give

Corollary 12 *(to Theorem 10, case of* $m = 1$*) Then*

1)

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \left\|\left(A_n\left(f^{(1)}(x_0)(\cdot - x_0)\right)\right)(x_0)\right\|_\gamma +$$
$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right)\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}} \tag{62}$$
$$\left[1 + r + \frac{r^2}{4}\right],$$

and

2)

$$\left\| \left\| \left(A_n \left(f \right) \right) - f \right\|_{\gamma} \right\|_{\infty, \prod_{i=1}^{N} [a_i, b_i]} \leq$$

$$\left\| \left\| \left(A_n \left(f^{(1)} \left(x_0 \right) \left(\cdot - x_0 \right) \right) \right) \left(x_0 \right) \right\|_{\gamma} \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} +$$

$$\frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) \left(x_0 \right) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \right)$$

$$\left\| \left(\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_p^2 \right) \right) \left(x_0 \right) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right], \tag{63}$$

$r > 0$.

We make

Remark 13 *We estimate* $0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$,

$$\widetilde{A}_n \left(\left\| \cdot - x_0 \right\|_{\infty}^{m+1} \right) \left(x_0 \right) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(n x_0 - k \right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z \left(n x_0 - k \right)} \overset{(22)}{<}$$

$$(4.824)^N \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(n x_0 - k \right) = \tag{64}$$

$$(4.824)^N \left\{ \sum_{\substack{k = \lceil na \rceil \\ \left\{ : \left\| \frac{k}{n} - x_0 \right\|_{\infty} \leq \frac{1}{n^{\alpha}} \right.}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(n x_0 - k \right) + \right.$$

$$\left. \sum_{\substack{k = \lceil na \rceil \\ \left\{ : \left\| \frac{k}{n} - x_0 \right\|_{\infty} > \frac{1}{n^{\alpha}} \right.}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^{m+1} Z \left(n x_0 - k \right) \right\} \overset{(23)}{\leq}$$

$$(4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \left\| b - a \right\|_{\infty}^{m+1}}{\pi e^{n^{1-\alpha}}} \right\}, \tag{65}$$

(where $b - a = \left(b_1 - a_1, ..., b_N - a_N \right)$*).*

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We have proved that ($\forall$ $x_0 \in \prod_{i=1}^{N} [a_i, b_i]$)

$$\widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) < (4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \|b-a\|_\infty^{m+1}}{\pi e^{n^{1-\alpha}}} \right\} =: \varphi_1(n) \tag{66}$$

($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$).

And, consequently it holds

$$\left\| \widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i,b_i]} <$$

$$(4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \|b-a\|_\infty^{m+1}}{\pi e^{n^{1-\alpha}}} \right\} = \varphi_1(n) \to 0, \quad \text{as } n \to +\infty. \tag{67}$$

So, we have that $\varphi_1(n) \to 0$, *as* $n \to +\infty$. *Thus, when* $p \in [1, \infty]$, *from Theorem 10 we have the convergence to zero in the right hand sides of parts (1), (2).*

Next we estimate $\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma$.

We have that

$$\left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}. \tag{68}$$

When $p = \infty$, $j = 1, ..., m$, *we obtain*

$$\left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma \leq \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j. \tag{69}$$

We further have that

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \overset{(22)}{<}$$

$$(4.824)^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma Z(nx_0 - k) \right) \leq$$

$$(4.824)^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) = \tag{70}$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) =$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left\{ \sum_{\substack{k = \lceil na \rceil \\ \left\{ : \left\| \frac{k}{n} - x_0 \right\|_\infty \leq \frac{1}{n^\alpha} \right.}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right.$$

$$\left. + \sum_{\substack{k = \lceil na \rceil \\ \left\{ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha} \right.}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right\} \overset{(23)}{\leq} \tag{71}$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \|b-a\|_\infty^j}{\pi e^{n^{1-\alpha}}} \right\} \to 0, \ as \ n \to \infty.$$

That is

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \to 0, \ as \ n \to \infty.$$

Therefore when $p = \infty$, *for* $j = 1, ..., m$, *we have proved:*

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma <$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \|b-a\|_\infty^j}{\pi e^{n^{1-\alpha}}} \right\} \leq \tag{72}$$

$$(4.824)^N \left\| f^{(j)} \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \|b-a\|_\infty^j}{\pi e^{n^{1-\alpha}}} \right\} =: \varphi_{2j}(n) < \infty,$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 10, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 12, the right hand sides of (62) and (63) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Conclusion 14 *We have proved that the left hand sides of (58), (59), (60), (61) and (62), (63) converge to zero as* $n \to \infty$, *for* $p \in [1, \infty]$. *Consequently* $A_n \to I$ *(unit operator) pointwise and uniformly, as* $n \to \infty$, *where* $p \in [1, \infty]$. *In the presence of initial conditions we achieve a higher speed of convergence, see (59). Higher speed of convergence happens also to the left hand side of (58).*

We further give

Corollary 15 *(to Theorem 10) Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of $\overline{m}$-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$ and $r > 0$. Here $\varphi_1(n)$ as in (67) and $\varphi_{2j}(n)$ as in (72), where $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$, $j = 1, ..., m$. Then*

1)

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{73}$$

2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., \overline{m}$, we have

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{74}$$

3)

$$\left\| \|A_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq \sum_{j=1}^{m} \frac{\varphi_{2j}(n)}{j!} +$$

$$\frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \tag{75}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right] =: \varphi_3(n) \to 0, \text{ as } n \to \infty.$$

We continue with

Theorem 16 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\|B_n(f, x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4e^2 \left\| \|f\|_\gamma \right\|_\infty}{\pi e^{n^{1-\beta}}} =: \lambda_2(n), \tag{76}$$

2)

$$\left\| \|B_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{77}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} B_n(f) = f$, *uniformly.*

Proof. We have that

$$B_n(f,x) - f(x) \overset{(16)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx-k) = \tag{78}$$

$$\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx-k).$$

Hence

$$\|B_n(f,x) - f(x)\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) =$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) +$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) \overset{(16)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\left\|\|f\|_\gamma\right\|_\infty \sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z(nx-k) \overset{(23)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}}, \tag{79}$$

proving the claim. ∎

We give

Theorem 17 *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, ω_1 *is for* $p = \infty$. *Then*

1)

$$\|C_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}} =: \lambda_3(n), \tag{80}$$

2)

$$\left\|\|C_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_3(n). \tag{81}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} C_n(f) = f$, *uniformly.*

Proof. We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, ..., t_N)\, dt_1 dt_2 ... dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} ... \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) dt_1 ... dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \tag{82}$$

Thus it holds (by (38))

$$C_n(f, x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) Z(nx - k). \tag{83}$$

We observe that

$$\|C_n(f, x) - f(x)\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) - f(x)\right) Z(nx - k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n}\right) - f(x)\right) dt\right) Z(nx - k) \right\|_\gamma \leq \tag{84}$$

$$\sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx - k) =$$

$$\sum_{\begin{cases} k = -\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx - k) +$$

$$\sum_{\begin{cases} k = -\infty \\ \left\|\frac{k}{n} - x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx - k) \leq$$

$$\sum_{\begin{cases} k = -\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1\left(f, \|t\|_\infty + \left\|\frac{k}{n} - x\right\|_\infty\right) dt\right) Z(nx - k) +$$

$$2\left\|\|f\|_\gamma\right\|_\infty \left(\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(|nx-k|) \right) \leq$$

$$\omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right) + \frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}}, \tag{85}$$

proving the claim. ■

We also present

Theorem 18 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\|D_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right) + \frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}} = \lambda_4(n), \tag{86}$$

2)

$$\left\|\|D_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_4(n). \tag{87}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} D_n(f) = f$, uniformly.

Proof. Similar to the proof of Theorem 17, as such is omitted. ■

We make

Definition 19 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, where $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space. We define the general neural network operator*

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f)\, Z(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{88}$$

Clearly $l_{nk}(f)$ is an X-valued bounded linear functional such that $\|l_{nk}(f)\|_\gamma \leq \left\|\|f\|_\gamma\right\|_\infty$.

Hence $F_n(f)$ is a bounded linear operator with $\left\|\|F_n(f)\|_\gamma\right\|_\infty \leq \left\|\|f\|_\gamma\right\|_\infty$.

We need

Theorem 20 *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \geq 1$. *Then* $F_n(f) \in C_B\left(\mathbb{R}^N, X\right)$.

Proof. Lengthy and similar to the proof of Theorem 21 of [14], as such is omitted. ■

Remark 21 *By (25) it is obvious that* $\left\| \|A_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty < \infty$, *and* $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, *given that* $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Call L_n *any of the operators* A_n, B_n, C_n, D_n.

Clearly then

$$\left\| \left\| L_n^2(f) \right\|_\gamma \right\|_\infty = \left\| \|L_n(L_n(f))\|_\gamma \right\|_\infty \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \tag{89}$$

etc.

Therefore we get

$$\left\| \left\| L_n^k(f) \right\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \quad \forall\ k \in \mathbb{N}, \tag{90}$$

the contraction property.

Also we see that

$$\left\| \left\| L_n^k(f) \right\|_\gamma \right\|_\infty \leq \left\| \left\| L_n^{k-1}(f) \right\|_\gamma \right\|_\infty \leq \ldots \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty. \tag{91}$$

Here L_n^k *are bounded linear operators.*

Notation 22 *Here* $N \in \mathbb{N}$, $0 < \beta < 1$. *Denote by*

$$c_N := \begin{cases} (4.824)^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{92}$$

$$\varphi(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n,\ B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{93}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^N [a_i, b_i], X\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{94}$$

and

$$Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{95}$$

We give the condensed

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Theorem 23 *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $n, m, N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*
(i)

$$\left\| L_n (f, x) - f (x) \right\|_\gamma \leq c_N \left[\omega_1 (f, \varphi (n)) + \frac{4e^2 \left\| \| f \|_\gamma \right\|_\infty}{\pi e^{n^{1-\beta}}} \right] =: \tau (n), \quad (96)$$

where ω_1 is for $p = \infty$,
and
(ii)

$$\left\| \| L_n (f) - f \|_\gamma \right\|_\infty \leq \tau (n) \to 0, \text{ as } n \to \infty. \quad (97)$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n \to \infty} L_n (f) = f,$$

pointwise and uniformly.

Proof. By Theorems 8, 16, 17, 18. ■

Next we talk about iterated neural network approximation (see also [9]).
We give

Theorem 24 *All here as in Theorem 23 and $r \in \mathbb{N}$, $\tau (n)$ as in (96). Then*

$$\left\| \| L_n^r f - f \|_\gamma \right\|_\infty \leq r \tau (n). \quad (98)$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. As similar to [14] is omitted. ■

We also present

Theorem 25 *Let $f \in \Omega$; $m, N, m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1$; $m_i^{1-\beta} > 2$, $i = 1, ..., r$, $x \in Y$, and let $(L_{m_1}, ..., L_{m_r})$ as $(A_{m_1}, ..., A_{m_r})$ or $(B_{m_1}, ..., B_{m_r})$ or $(C_{m_1}, ..., C_{m_r})$ or $(D_{m_1}, ..., D_{m_r})$, $p = \infty$. Then*

$$\left\| L_{m_r} \left(L_{m_{r-1}} \left(... L_{m_2} \left(L_{m_1} f \right) \right) \right) (x) - f (x) \right\|_\gamma \leq$$

$$\left\| \left\| L_{m_r} \left(L_{m_{r-1}} \left(... L_{m_2} \left(L_{m_1} f \right) \right) \right) - f \right\|_\gamma \right\|_\infty \leq$$

$$\sum_{i=1}^{r} \left\| \| L_{m_i} f - f \|_\gamma \right\|_\infty \leq$$

$$c_N \sum_{i=1}^{r} \left[\omega_1 (f, \varphi (m_i)) + \frac{4e^2 \left\| \| f \|_\gamma \right\|_\infty}{\pi e^{m_i^{1-\beta}}} \right] \leq$$

$$rc_N\left[\omega_1\left(f,\varphi\left(m_1\right)\right)+\frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{m_1^{1-\beta}}}\right]. \tag{99}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. As similar to [14] is omitted. ■

We also give

Theorem 26 *Let all as in Corollary 15, and $r\in\mathbb{N}$. Here $\varphi_3(n)$ is as in (75). Then*

$$\left\|\|A_n^r f-f\|_\gamma\right\|_\infty\leq r\left\|\|A_n f-f\|_\gamma\right\|_\infty\leq r\varphi_3(n). \tag{100}$$

Proof. As similar to [14] is omitted. ■

Application 27 *A typical application of all of our results is when $\left(X,\|\cdot\|_\gamma\right)=(\mathbb{C},|\cdot|)$, where $\mathbb{C}$ are the complex numbers.*

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