CF Pebbling Number of Path and Path Related Graphs

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ABSTRACT
Assume G is a graph with some pebbles distributed over its vertices. A CF pebbling move is when x pebbles are removed from one vertex, $\left\lfloor \frac{x}{2} \right\rfloor$ pebbles are thrown away and $\left\lceil \frac{x}{2} \right\rceil$ pebbles are moved to an adjacent vertex. The CF pebbling number λ(G), of a connected graph G is the least positive integer n such that any distribution of n pebbles on G, allows one pebble to be carried to any arbitrary vertex using a sequence of CF pebbling moves. The CF pebbling number of path and path related graphs are determined in this study.

Keywords: CF pebbling move, CF pebbling number, path related graphs.

1. INTRODUCTION

Graph theory, an extensive and vibrant area of mathematics, delves into the properties and applications of graphs, which are structures used to model relationships between objects. Among the many intriguing topics within graph theory, pebbling problems have significant attention due to their combinatorial complexity and practical relevance in areas such as network optimization and resource management. This paper examines a particular variant known as the CF (Ceiling Floor) pebbling.

The CF pebbling number of a graph quantifies the minimum number of pebbles needed to guarantee that, regardless of their initial distribution, a pebble can be moved to any target vertex through a series of CF pebbling moves. A CF pebbling move involves removing x pebbles from a vertex, discarding $\left\lfloor \frac{x}{2} \right\rfloor$ pebbles, and placing $\left\lceil \frac{x}{2} \right\rceil$ pebbles on an adjacent vertex. This variant, incorporating the ceiling and floor functions, introduces additional complexity to the pebbling process, making the determination of the CF pebbling number a challenging and intriguing problem.

In this paper we study the CF pebbling number of path and path-related graphs, analyzing how this number evolves with varying graph configurations. We present precise formulations for the CF pebbling number of path P_n of length n and extend our analysis to path related graphs. Here, $p(v)$ denotes the number of pebbles placed in the vertex v in a graph G.

2. Preliminaries

Definition 2.1: Assume G is a graph with some pebbles distributed over its vertices. A CF pebbling move is when x Pebbles are removed from one vertex, $\left\lfloor \frac{x}{2} \right\rfloor$ pebbles are thrown away and $\left\lceil \frac{x}{2} \right\rceil$ pebbles are moved to an adjacent vertex. The CF pebbling number λ(G), of a connected graph G is the least positive integer n such that any distribution of n pebbles on G allows one pebble to carried to any arbitrary vertex using a sequence of CF pebbling moves.

Definition 2.2: A CF pebbling number $\lambda(G, v)$, of a vertex v of a graph G is the smallest number $\lambda(G, v)$ such that atleast one pebble may be moved to target vertex v using a sequence of CF pebbling moves, for any placement of $\lambda(G, v)$ pebbles on the vertices of G. The maximum $\lambda(G, v)$ over all the vertices of G is the CF pebbling number of a graph denoted as λ(G).

Definition 2.3: A pendant vertex is a vertex that has a degree 1, meaning it is only connected to one edge. Pendant vertices are also known as leaf vertices or end vertices. In trees, pendant vertices are called terminal nodes or simply leaves.

3. The CF pebbling number of path and path related graphs

Theorem 3.1. For the path P_1, $\lambda(P_1)$ is 2.

Proof. Let $V(P_1) = \{v_1, v_2\}$ and $E(P_1) = \{v_1v_2\}$. Without loss of generality, assume v_1 is our target vertex and has zero pebbles. By placing a single pebble on v_2, a pebble cannot be moved to v_1. So $\lambda(P_1) \geq 2$. If vertex v_1 receives a pebble then there is nothing to prove. Assume that v_1 has zero pebbles, then with two pebbles in v_2, a pebble can be moved to v_1. So $\lambda(P_1) \leq 2$.

Theorem 3.2. For the path P_2, $\lambda(P_2)$ is 3.

Proof. Let $V(P_2) = \{v_1, v_2, v_3\}$ and $E(P_2) = \{v_1v_2, v_2v_3\}$. By placing two pebbles on v_3, a pebble cannot be moved to a target vertex v_1. So $\lambda(P_2) \geq 3$. Without loss of generality, assume that v_1 is our target vertex and distribute three pebbles on vertices of P_2. If vertex v_1 receives a pebble, then there is nothing to prove, so assume v_1 has zero pebbles. If v_2 receives atleast two pebbles then a pebble can be moved from v_2 to v_1. so assume v_2 receives atmost one pebble.
If v_2 has a single pebble then from v_3, a pebble can be moved to v_2 and from v_2 a pebble can be moved to v_1.
If v_2 has zero pebbles, then from v_3 using CF pebbling move, two pebbles can be moved to v_2 and from v_2 a pebble can be moved to v_1.
If v_2 is our target vertex, then at least one of v_1 or v_3 receives at least two pebbles, then a pebble can be moved to v_2. So $\lambda(P_2) \leq 3$.

Theorem 3.3. For a path P_n of length n, $\lambda(P_n) = 2^{n-1} + 1, \forall n \geq 2$.

Proof: Let $V(P_n) = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ and
$E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_{n+1}\}$. Assume v_1 is our target vertex and it has zero pebbles.
By placing 2^{n-1} pebbles on v_{n+1}, a pebble cannot be moved to v_1, hence $\lambda(P_n) \geq 2^{n-1} + 1$. Distributing $2^{n-1} + 1$ pebbles on all vertices of path P_n. The result is true when n = 2. Assume the result is true for a path P_{k-1} (i.e., $\lambda(P_{k-1}) = 2^{k-2} + 1$).
To prove that the result is true for path P_k. Any path P_k can be divided into two paths say P_{k_1} and P_{k_2}.
Let $V(P_{k_1}) = \{v_1\}$ and $V(P_{k_2}) = \{v_2, \ldots, v_n, v_{n+1}\}$, $E(P_{k_2}) = \{v_2v_3, v_3v_4, \ldots, v_nv_{n+1}\}$.

Case (i): If P_{k_1} has no pebble and all pebbles are placed on P_{k_2}. Let v_1 be our target vertex. Using $2.\,2^{n-2} + 1$ pebbles, two pebbles can be moved to v_2 and a pebble can be moved to v_1.

Case (ii): If P_{k_1} has all pebbles and P_{k_2} receives no pebbles. Let any vertex $v_i, i \neq 1$, be our target vertex. Then from v_1, $2^{n-2} + 1$ pebbles can be moved to v_2, using $2^{n-2} + 1$ pebbles, a pebble can be moved to any target vertex of P_{k_2} using induction. So $\lambda(P_n) \leq 2^{n-1} + 1$.

Theorem 3.4. For star graph $K_{1,n}$, $\lambda(K_{1,n}) = n + 1$ for $n \geq 1$.

Proof: Let $V(K_{1,n}) = \{v_0, v_1, \ldots v_n\}$ such that $\deg(v_0) = n$ and $\deg(v_i) = 1; 1 \leq i \leq n$.
Now, put 2 pebbles on v_n and one pebble on each of the vertices $v_i, 1 < i < n$. Then no pebble could be moved to v_1. Thus $\lambda(K_{1,n}) \geq n + 1$.
Now, Consider a distribution L of n+1 pebbles on the vertices of $K_{1,n}$.

Case (i): Assume that v_1 be our target vertexand $p(v_1) = 0$. If n+1 pebbles are distributed to n vertices of $K_{1,n} - \{v_1\}$ such that there exists a vertex with at least 2 pebbles.
Suppose, $p(v_0) = 2$. Since $d(v_0, v_i) = 1; 1 \leq i \leq n$, one pebble could easily moved to v_1 by CF pebbling move.
On the other hand if $p(v_0) \leq 1$ then at least n pebbles are distributed in n-1 pendant vertices other than v_1.
Then either any one pendant vertex has at least 3 pebbles or two pendant vertices has at least 2 pebbles. However, by CF pebbling move two pebbles could be moved to v_0 and hence could place a pebble in v_1.

Case(ii): Now, assume v_0 as the target vertex such that $p(v_0) = 0$. Then the distribution of n+1 pebbles in the n pendant vertices of $K_{1,n}$, one vertex has at least 2 pebbles. Clearly, $d(v_0, v_i) = 1; 1 \leq i \leq n$, a pebble could be moved to v_0. Hence $\lambda(K_{1,n}) \leq n + 1$.

Definition 3.1:
The addition of two graphs G_1 and G_2 is a graph with a vertex set that is the union of G_1 and G_2 and an edge set that is the union of G_1 and G_2.

Theorem 3.5. Let P_n be a path of length n. Then $\lambda(P_n + K_1) = n + 2$ for $n \geq 1$.

Proof: Let $V(P_n + K_1) = \{v_0, v_1, v_2, \ldots, v_{n+1}\}$. Let $\deg(v_0) = n + 1$, $\deg(v_i) = 2$,

$i = 1, n+1$ and $\deg(v_j) = 3$ for all j = 2, 3,..,n.
By putting one pebble each on the vertices $v_1, v_2, \ldots, v_{n+1}$,we cannot move a pebble to v_0. Thus, $\lambda(P_n + K_1) \geq n+2$
Case (i): Suppose there are n+2 pebbles, which has been distributed on the vertices of $P_n + K_1$.
Let v_0 be the target vertex. If $p(v_0) = 0$, then there exists some $i \in \{1,2,\ldots,n+1\}$ such that $p(v_i) \geq 2$. So, we can move one pebble to v_0 by CF move from such v_i.
Case (ii): Let v_k be the target vertex such that $p(v_k) = 0$ and $1 \leq k \leq n+1$.
Sub case a: If $p(v_0) \geq 2$ or $p(v_i) \geq 3$ for some $i \neq k$, then $\{v_0, v_k\}$ or $\{v_i, v_0, v_k\}$ forms a transmitting sub graph. Hence we can move a pebble to v_k.
Sub case b: If $p(v_0) = 1$,then there exist some $v_j, j \neq 0, k$ such that $p(v_j) \geq 2$. Then $\{v_j, v_0, v_k\}$ forms a transmitting sub graph and we are done.
Sub case c: If $p(v_0) = 0$ then there should be at least one vertex v_s such that $p(v_s) \geq 3$ or there exists atleast two vertices v_j and v_t such that $p(v_j) \geq 2$ and $p(v_t) \geq 2$, then two pebbles can be moved to v_o each one from v_j and v_t and hence apebble could be moved from v_0 to v_k. Thus $\lambda(P_n + K_1) \leq n+2$.

Definition 3.2:[2] A graph which joins the empty graph K_m on m nodes and the path graph P_n on n nodes is called fan graph. If m = 1 then it is called fan graph and if m = 2 it is called double fan.

Theorem 3.6. $\lambda(P_n + 2K_1) = n+3$ for $n \geq 1$.
Proof: Let $V(P_n + K_1) = \{x_0, y_0, v_1, v_2, \ldots, v_{n+1}\}$. Let $\deg(x_0)$ *and* $\deg(y_0) = n+1$, $\deg(v_i) = 3$, $i = 1, n+1$ and $\deg(v_j) = 4$ for all j = 2, 3,..,n.
Placing n+2 pebbles, one on each vertices $x_0, y_0, v_1, v_2, \ldots, v_n$leaves the vertex v_{n+1} unpebbled. Thus $\lambda(P_n + 2K_1) \geq n+3$.
Case (i): Suppose there are n+3 pebbles, which has been distributed on the vertices of $P_n + 2K_1$.
Let x_0 be the target vertex such that$p(x_0) = 0$, then there exists some $i \in \{1,2,\ldots,n+1\}$ such that $p(v_i) \geq 2$. So, we can move one pebble to x_0 by CF move from such v_i's.
Suppose all $p(v_i) < 2$, then
i) If $p(y_0) \geq 3$ thenx_0 can be pebbled as $d(x_0, y_0) = 2$.
ii) If $p(y_0) \leq 2$, then there exists atleast one v_i such that $p(v_i) \geq 1$, thus $\{y_0, v_i, x_0\}$forms a transmitting subgraph and hence $x_0$can be pebbled.
A similar case holds if y_0 is the target vertex.
Case (ii): Let v_k be the target vertex such that $p(v_k) = 0$ and $1 \leq k \leq n+1$.
Sub case (a): If $p(x_0) \geq 2$ or $p(v_i) \geq 3$ for some $i \neq k$, then $\{x_0, v_k\}$ or $\{v_i, x_0, v_k\}$ forms a transmitting sub graph. Hence we can move a pebble to v_k. A similar case holds if x_0 is replaced by y_0.
Sub case (b): If $p(x_0) = 1$,then there exist some v_i such that $p(v_i) \geq 2$. Thus$\{v_i, x_0, v_k\}$ forms a transmitting sub graph and we are done . If all v_i such that $p(v_i) < 2$, then $p(y_0) \geq 2$. Thus, $\{y_0, v_j, x_0, v_k\}$ forms a transmitting subgraph such that $p(v_j) = 1$, $j \neq k$ and hence a pebble can be moved to v_k.
Sub case (c): If $p(x_0) = 0$then there exist some v_i such that $p(v_i) \geq 2$. Thus $\{v_i, x_0, v_k\}$ forms a transmitting sub graph and we are done . If all v_i such that $p(v_i) < 2$, then $p(y_0) \geq 3$. Thus, $\{y_0, v_j, x_0, v_k\}$ forms a transmitting subgraph such that $p(v_j) = 1$, $j \neq k$ and hence a pebble can be moved to v_k.Thus $\lambda(P_n + 2K_1) \leq n+3$.

Theorem 3.7:
For cycle C_n with n vertices, $n < 8, \lambda(C_n) = n$.
Proof:
Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ where $n < 8$.
Without loss of generality, let us assume that v_1 be our target vertex and $p(v_1) = 0$.
Let n-1 pebbles be placed on the vertices of C_n in such a way that $p(v_i) = 1$ for all $i \neq 1$. Then v_1 cannot be pebbled. Hence $\lambda(C_n) \geq n$.
Case (i) : n is even
Let n pebbles be placed on the vertices of C_n other than v_1. That is, n pebbles are placed on n-1 vertices.
Consider the following distribution.
Subcase 1: If all pebbles are placed on $v_{\frac{n}{2}+1}$, then by CF pebbling move v_1 can be pebbled.
Subcase 2: If $p\left(v_{\frac{n}{2}+1}\right) = 0$. Consider the paths

$$P_1: v_1, v_2, \dots, v_{\frac{n}{2}}.$$
$$P_2: v_{\frac{n}{2}+2}, \dots, v_n, v_1 .$$

Here P_1 and P_2 are paths of lengths $\frac{n}{2}-1$. Since the pebbles are distributed in these paths, either P_1 or P_2 has atleast $\frac{n}{2}$ pebbles. Since $\frac{n}{2} > 1 + 2^{\frac{n}{2}-2}$ for n= 4, 6, v_1 can be pebbled.

Subcase 3: If $0 < p\left(v_{\frac{n}{2}+1}\right) < n$. Consider the paths

$$P_1: v_1, v_2, \dots, v_{\frac{n}{2}+1}.$$
$$P_2: v_{\frac{n}{2}+1}, \dots, v_n, v_1 .$$

Both P_1 and P_2 are paths of length $\frac{n}{2}$. If either the path has atleast $1 + 2^{\frac{n}{2}-1}$ pebbles then v_1 can be pebbled. If both path has less than $1 + 2^{\frac{n}{2}-1}$ pebbles, then

1) For n=4, v_1 can be pebbled by CF pebbling move.
2) For n = 6, either P_1 or P_2 has atleast $\frac{n}{2}$ pebbles, thus v_1 can be pebbled.

Case (ii): n is odd

Let n pebbles be placed on the vertices of C_n other than v_1. Consider the paths

$$P_1: v_1, v_2, \dots, v_{\left\lceil\frac{n}{2}\right\rceil}.$$
$$P_2: v_{\left\lceil\frac{n}{2}\right\rceil+1}, \dots, v_n, v_1 .$$

The paths P_1 and P_2 are of length $\left\lceil\frac{n}{2}\right\rceil$.Clearly either P_1 or P_2 has atleast $\left\lceil\frac{n}{2}\right\rceil$ pebbles. Thus when n=3,5 v_1 can be pebbled.

If n = 7, placing a pebble in an intermediate vertex of either P_1 or P_2 means $\left\lceil\frac{n}{2}\right\rceil$ pebbles are enough to pebble v_1 by CF pebbling moves through the corresponding path.

Without loss of generality, let us assume that $p\left(v_{\left\lceil\frac{n}{2}\right\rceil}\right) = 4$ and the path P_2 has 3 pebbles. If $p\left(v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) \geq 2$, then by CF pebbling move 2 pebbles can be moved to $v_{\left\lceil\frac{n}{2}\right\rceil}$ and thus v_1 can be pebbled.

If $p\left(v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) \leq 1$, then atleast one of the intermediate vertices of P_2 has pebbles. Thus from $v_{\left\lceil\frac{n}{2}\right\rceil}$, by CF pebbling move 2 pebbles can be moved to $v_{\left\lceil\frac{n}{2}\right\rceil+1}$ and hence v_1 can be pebbled .

Thus, $\lambda(C_n) \leq n$.

Lemma 3.1: For n = 1,2, $\lambda(P_1 \times K_2) = 4$ and $\lambda(P_2 \times K_2) = 6$.

Proof:

Since $P_1 \times K_2$ is a cycle C_4 and $\lambda(C_4) = 4$.

Now consider $P_2 \times K_2$. Let $V(P_2 \times K_2) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$ and $E(P_2 \times K_2) = \{\{v_i v_{i+1}\} \cup \{u_i u_{i+1}\} \cup \{v_j u_j\}: 1 \leq i \leq 2, 1 \leq j \leq 3\}$. Let $\deg(v_1) = \deg(v_3) = \deg(u_1) = \deg(u_3) = 2$ and $\deg(v_2) = \deg(u_2) = 3$.

Let u_3 be the target vertex. If five pebbles are placed in such a way that p(v)=1 for all $v \neq u_3$. Then no pebble can reach u_3 . Thus $\lambda(P_2 \times K_2) \geq 6$.

Case (1): Let v be the target vertex such that deg(v)=2 and p(v)=0. Let 6 pebbles be placed on the graph as follows.

i) If any vertex w such $w \neq v$ and w is adjacent to v has atleast 2 pebbles then one pebble can be moved to the target vertex.

ii) If for all vertex w, $p(w) = 1$, such that $w \neq v$ and w is adjacent to v and if there exists any x such that p(x)≥ 2 and x is adjacent to w, then one pebble can be moved to w and v can be pebbled.

iii) If for all vertex w, $p(w) = 1$, such that $w \neq v$ and w is adjacent to v and if x is adjacent to w and

a) for all x, p(x) =0 then the remaining vertex y has four pebbles and thus one pebble can be moved to w and from w, v can be pebbled.

b)for all x, p(x)=1 then the remaining vertex y has two pebbles and thus v can be pebbled.

c) either of the vertex x adjacent to w has one pebble then the remaining vertex y has 3 pebbles, then two pebbles can be moved to x and from there one pebble can be moved to w and hence v can be pebbled.

iv) if for all vertex w, $p(w) = 0$, such that $w \neq v$ and w is adjacent to v , then consider the pathP_1: {v, w, x, y} and its symmetric path $P_2: \{v, w, x\}$. If P_1 has atleast 5 pebbles then v can be pebbled. If P_1 has less than 5 pebbles then the remaining pebbles must be in P_2. If P_2 has atleast 3 pebbles then one pebble can be moved to v through the path P_2. Otherwise P_2has 2 pebbles then a pebble can be moved to P_1 and hence the target can be reached.

Case (2): Let v be the target vertex such that deg(v)=3 and p(v)=0. Let 6 pebbles be placed on the graph as follows.

i) If any u such that p(u)≥ 2 such that u is adjacent to v then one pebble can be moved to v.

ii) Let p(u) < 2 for all u such that u is adjacent to v.Let the remaining vertices be x and y and deg(x)=deg(y)=2 and d(x,v)=d(y,v)=2. Then if p(x)=3 or p(y)=3 then one pebble can be moved to v. If p(x)=0, then there exists a y-v path with atleast 4 pebbles thus v can be pebbled. If p(x)=1, then there exists a {y,u,v} path with 3 pebbles such that p(y)=2 and p(u)=1 or p(y)≥3. Thus, {y,u,v}forms a transmitting subgraph and hence v can be pebbled. A similar case holds if x is replaced by y.

Thus $\lambda(P_2 \times K_2) \leq 6$.

Theorem 3.8. Let P_n be a path of length n. Then$\lambda(P_n \times K_2) = 1 + 2^n$ for n ≥ 3.

Proof: Let $V(P_n \times K_2) = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n+2}\}$ and $E(P_n \times K_2) = \left\{\{v_i v_{i+1}\} \cup \{v_j v_{j+1}\right\} \cup \{v_k v_{k+n+1}\}: 1 \leq i \leq n, n+1 \leq j \leq 2n+1, 1 \leq k \leq n+1 \}$ Let $\deg(v_1) = \deg(v_n) = \deg(v_{n+1}) = \deg(v_{2n+2}) = 2$ and $\deg(v_i)=3$ for all $i \neq 1, n, n+1, 2n+2$.

If the vertex v_{2n+2} contains 2^npebbles, then by CF pebbling move, no pebbles could be shifted to v_1. Thus $\lambda(P_n \times K_2) \geq 1 + 2^n$.

Now prove that $\lambda(P_n \times K_2) \leq 2^n + 1$.

Consider a distribution L of $2^n + 1$ pebbles on $V(P_n \times K_2)$.

Case (i): Let y be a target vertex such that deg(y) = 2. Without loss of generality, let $y = v_1$. Then $d(v_1, v_t) \leq n + 1$, $2 \leq t \leq 2n + 2$. And there is a path from v_{n+2} to v_1, a pebble could be shifted to the target $y = v_1$ by the distribution of $2^n + 1$ pebbles on either path $P_1 = \{v_{2n+2}, v_{n+1}, \dots, v_2, v_1\}$ or path$P_2 = \{v_{2n+2}, v_{2n+1}, \dots v_{n+2}, v_1\}$.

If pebbles are distributed on both the paths, then one of the paths has atleast $1 + 2^{n-1}$ pebbles. If all $1 + 2^{n-1}$ are placed on v_{2n+2} then through the path that contains all $1 + 2^n$ pebbles, one pebble can be moved to the target vertex. If $1 + 2^{n-1}$ pebbles are distributed on some vertices of any one of the path say P_1 then clearly atleast $2^{n-1} - n \geq 1$ pebbles can be moved to P_1 from the path P_2 and as these pebbles are distributed in the intermediate vertices of the path, also one pebble in the intermediate vertex at a distance i from the initial vertex of the path is equivalent to $1 + 2^{i-1}$pebbles placed at the initial vertex of the path, the target vertex can be pebbled.

Case (ii): Let y be any target vertex, deg(y) = 3and p(y) = 0. Consider two paths $P_1 = \{v_1, v_2, \dots, v_i, \dots, v_{n+1}\}$ and $P_2 = \{v_{n+2}, v_{n+3}, \dots, v_{n+i+1}, \dots, v_{2n+2}\}$.

Let y =v_i. Then if $p(P_1) \geq 1 + 2^{n-1}$ then $v_i$can be pebbled as d(v,v_i)< n for all $v \in V(P_1)$. If $p(P_2) \geq 1 + 2^{n-1}$ then two pebbles can be moved to v_{n+i+1} as d(v,v_{n+1+i})< n for all $v \in V(P_2)$ and hence one pebble can be moved to v_i.

A similar proof holds if the target vertex is on the path P_2.

Thus $\lambda(P_n \times K_2) \leq 1 + 2^n$ for n ≥ 3.

Theorem 3.9.$\lambda(P_n \odot K_1) = 2^{n+1} + n$ for n ≥ 1.

Proof: Let $V(P_n) = \{v_1, \dots, v_{n+1}\}$ and $V(P_n \odot K_1) = \{v_1, \dots, v_{n+1}, v_1', \dots, v_{n+1}'\}$ and $E(P_n \odot K_1) = E(P_n) \cup \{v_i v_i' : 1 \leq i \leq n+1\}$.

Let v_{n+1}' be the target vertex.

Without loss of generality place 2^{n+1} pebbles on v_1' and remaining n-1 pebbles on the pendant vertices except v_{n+1}'.

By the successive CF pebbling moves, v_1 will receive $\left\lceil \frac{2^{n+1}}{2} \right\rceil$ pebbles, then v_2 will receive $\left\lceil \frac{2^{n+1}}{4} \right\rceil$pebbles in the next move.

proceeding like this v_{n+1} will receive exactly one pebble, this will make v_{n+1}' as unreachable.

Hence $\lambda(P_n \odot K_1) \geq 2^{n+1} + n$.

Consider a distribution of $2^{n+1} + n$ pebbles.

Case (i): Choose any vertex $v_i, 1 \leq i \leq n+1$ such that $p(v_i) = 0$ as a target vertex.Consider the path $P' = \{v_1', v_1, \dots, v_{n+1}, v_{n+1}'\}$ of length n+2.

If all $2^{n+1} + n$ pebbles are distributed on the path P' then our target vertex can be easily pebbled. If all $2^{n+1} + n$ pebbles are distributed in a way that $p(P') = 0$ then atleast $2^n + 1$ pebbles will reach the intermediate vertices v_j, $2 \leq j \leq n, i \neq j$, of P'. Thus, our target vertex can be pebbled as the distance between the target vertex and the intermediate vertices v_j, $2 \leq j \leq n, i \neq j$, of P' is atmost n-1.

If $2^{n+1} + n$ pebbles are distributed on both $P_n$and pendant vertices. Let s > 0 be the number of pebbles distributed on the pendant vertices such that $p(P_n) = 2^{n+1} + n - s$. If s = 1, then $2^{n+1} + n - 1$ pebbles on P_n are enough to reach the target vertex with one pebble.

Suppose n > s > 1. Then the number of pebbles on the path P_n will be atleast $2^{n+1} + 1$. Hence one pebble can reach the target vertex. Suppose $s \geq n$. Then the number of pebbles the pathP_n can receive from both the path P_n and the pendant vertices will be atleast $1+2^n$. Since P_n is a path of length n, one pebble can be moved to the target vertex.

Case (ii): Let v_i' be the target vertex such that $p(v_i')$ = 0 and $1 \leq i \leq n + 1$.

If all the pebbles are placed either on path P_n or on the pendant vertices except the target vertex then the target vertex can be easily pebbled as the distance between the target vertex and any other vertex in the graph is atmost n+2.

If $2^{n+1} + n$ pebbles are distributed on both P_n and pendant vertices. Let s > 0 be the number of pebbles distributed on the pendant vertices such that $p(P_n) = 2^{n+1} + n - s$.

Now, proceeding as above in case (i), for any values of s, the target vertex can be easily pebbled as each pendant vertex is non-adjacent,an intermediate vertex of path P_n at a distance of i from the initial vertex of path P_n has one pebble is equivalent to placing $1+2^{i-1}$ pebbles on the initial vertex of path P_n and the distance between the target vertex and anyother vertex in the graph is atmost n+2.

Thus $\lambda(P_n \odot K_1) \leq 2^{n+1} + n$.

Hence $\lambda(P_n \odot K_1) = 2^{n+1} + n$.

4. CONCLUSION

In this paper we find the CF pebbling number of path and path related graphs. The CF pebbling number of other standard graphs is an open problem.

REFERENCES

[1] F.R.K. Chung, Pebbling in Hypercubes, SIAM J. Discrete Math.2 (1989), 467-472.

[2] J. Jannet Raji and S. Meenakshi, " Roman Domination Number Of Double Fan Graphs",Advances and Applications in Mathematical Sciences Volume 21, Issue 1, November 2021, Pages 485-491 © 2021 Mili Publications, India.