"H-Function-Induced Distributions for Mixed Sums of Independent Random Variables"

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Received: 19.04.2024 Revised : 12.05.2024 Accepted: 15.05.2024

ABSTRACT
"We develop a systematic methodology for computing the distribution of mixed sums of independent random variables, encompassing finite-range and infinite-range probability density functions, with the latter being expressible as a product of polynomials and H-functions. By harnessing the power of Laplace transforms and their inverses, our approach yields a general, parameterizable result that accommodates a broad spectrum of specializations, thereby providing a valuable tool for theorists and practitioners alike. The technical framework established herein paves the way for further research into the distributional properties of complex random systems."

Keywords: PDF, CDF, Generalized Hypergeometric Functions .

INTRODUCTION
A natural generalization of ${}_2F_1$ is the generalized hypergeometric function, the so-called ${}_pF_q$, where p parameters of nature of a, b and q parameters of the nature of c. This ensuring series

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p; \\ \\ b_1, \ldots, b_q; \end{matrix} z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n \quad z^n}{(b_1)_n \ldots\ldots (b_q)_n \quad n!} $$

$$ = \sum_{n=0}^{\infty} \frac{\prod^{p} (a_i)_n \, z^n}{\prod^{q} (b_j)_n \; n!}, \qquad (1) $$

is known as the generalized hypergeometric series and the function ${}_pF_q$ is called generalized hypergeometric function of variable z. ${}_pF_q$ is not defined if any denominator parameter b_q is a negative integer or zero. If any numerator parameter a_p is zero or a negative integer, the series terminates. If ${}_pF_q$ does not terminate, it converges

(i) for all finite z if $p \le q$;
(ii) for $|z| < 1$ if $p = q + 1$;
(iii) for $|z| = 1$ if $p = q + 1$ and $R\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$

and diverges for all $z \neq 0$ if $p > q + 1$.
Recently Mittal and Gupta [2, p. 117] has given the following notation of the H-function of two variables as:

$$ H^{0,\, n_1:m_2,\, n_2;m_3,\, n_3}_{p_1,\, q_1:p_2,\, q_2;p_3,\, q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j;\, \alpha_j,\, A_j)_{1,\, p_1}: (c_j,\, \gamma_j)_{1,\, p_2}: (e_j,\, E_j)_{1,\, p_3} \\ (b_j;\, \beta_j,\, B_j)_{1,\, q_1}: (d_j,\, \delta_j)_{1,\, q_2}: (f_j,\, F_j)_{1,\, q_3} \end{matrix}\right] $$

$$ = \frac{-1}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta)\, \theta_2(\xi)\, \theta_3(\eta) x^{\xi} y^{\eta}\, d\xi\, d\eta, \qquad (2) $$

Where

$$ \phi_1(\xi, \eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j\xi + A_j\eta)}{\prod^{p_1} \Gamma(a_j - \alpha_j\xi - A_j\eta) \prod^{q_1} \Gamma(1 - b_j + \beta_j\xi + B_j\eta)}, $$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j\xi) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \gamma_j\xi)}{\prod^{q_2} \Gamma(1 - d_j + \delta_j\xi) \prod^{p_2} \Gamma(c_j - \gamma_j\xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j\eta) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j\eta)}{\prod^{q_3} \Gamma(1 - f_j + F_j\eta) \prod^{p_3} \Gamma(e_j - E_j\eta)},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, $(i = 1, 2, 3; j = 2, 3)$. Also, all the A's, α's, B's, β's, γ's, δ's, E's, and F's are assumed to the positive quantities for standardization purpose.
The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j\xi)$ $(j = 1, ..., m_2)$ lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ $(j = 1, ..., n_2)$, $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ $(j = 1, ..., n_1)$ to the left of the contour.
The contour L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ $(j = 1, ..., m_3)$ lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ $(j = 1, ..., n_3)$, $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ $(j = 1, ..., n_1)$ to the left of the contour.
The function, defined by (2), is analytic function of x and y if

$$R = \sum^{p_1} \alpha_j + \sum^{p_2} \gamma_j - \sum^{q_1} \beta_j - \sum^{q_2} \delta_j < 0,$$

$$S = \sum^{p_1} A_j + \sum^{p_3} F_j - \sum^{q_1} B_j - \sum^{q_3} F_j < 0,$$

The H-function of two variables given by (2) is convergent if

$$U = -\sum^{p_1} \alpha_j - \sum^{q_1} \beta_j - \sum^{m_2} \delta_j - \sum^{q_2} \delta_j + \sum^{n_2} \gamma_j - \sum^{p_2} \gamma_j > 0, \quad (3)$$

$$V = -\sum^{p_1} A_j - \sum^{q_1} B_j - \sum^{m_3} F_j - \sum^{q_3} F_j + \sum^{n_3} E_j - \sum^{p_3} E_j > 0, \quad (4)$$

and $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$.

The distribution of sum of random variables

The distribution of sum of random variables is of great importance in many areas of physics and engineering. For example, sums of independent gamma random variables have application in problems of queuing theory such as determination of total waiting time, in civil engineering such as determination of the total excess water flow in a dam. They also appear in obtaining the inter arrival time of drought events which is the sum of the drought duration and the successive non drought duration. Various authors notably Linhart [3], Jackson [4] and Grice and Bain [5] have studied the applications of distribution of sum of random variables. The distribution of the sum of two independent random variables has been obtained by several authors, particularly when both the variates come from the same family of distribution. In this context the works of Albert [6] for uniform variates, Holm and Alouini [7], and Moschopoulos [8] for gamma variates, Loaiciga and Leipnik [9] for Gumbel variates are worth mentioning.
Moreover, Nason [10] has obtained the distribution of the sum of t and Gaussian random variables and pointed out its application in Bayesian wavelet shrinkage. In recent papers Chaurasia and Kumar [11] and Chaurasia and Singh [12] have investigated about the distributions of random variables associated with special functions. We know that the distribution of sum of several independent random variables

when each random variable is of simply infinite or doubly infinite range can easily be obtained by means of characteristic function or moment generating function. But, when the random variables are distributed over finite range, these methods are not much useful and the power of integral transform method comes sharply into focus. Here, we shall obtain the distribution of sum of two independent random variables, X_1 and X_2, where X_1 possess finite uniform probability density function and X_2 follows infinite probability density function involving the product of general class of polynomials and H-function, given by the equation (5) and (6) respectively.
Thus

$$f_1(x_1) = \begin{cases} \frac{1}{a}, & 0 \le x \le a \\ 0, & otherwise\ a > 0 \end{cases} \tag{5}$$

$$f_2(x_2) = \begin{cases} C x_2^{\lambda-1} e^{-\mu x_2} H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix} \zeta x_2^{-\gamma} \\ \eta \end{matrix} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3} \end{matrix} \right], & x_2 \ge 0 \\ 0, & otherwise \end{cases} \tag{6}$$

where

$$C^{-1} = \mu^{-\lambda} H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}\left[\begin{matrix} \zeta y^{\gamma} \\ \eta \end{matrix} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(\lambda,\gamma):(f_j,F_j)_{1,q_3} \end{matrix} \right] \tag{7}$$

and the following conditions are satisfied:

(i) $\gamma > 0, \mu > 0, \lambda - \gamma \min_{1 \le j \le m_2}(\frac{d_j}{\delta_j}) > 0$,

(ii) $|\arg\zeta| < \frac{1}{2}U\pi$, $|\arg\eta| < \frac{1}{2}V\pi$, where U and V are given in (3) and (4) respectively.

(iii) The parameter of H-function are real and so restricted that $f_2(x_2)$ remains non-negative.

Distribution Of The Mixed Sum Of Two Independent Random Variables
Theorem

If X_1 and X_2 are two independent random variables having the probability density function defined by (5) and (6) respectively. Then the probability density function of

$$Y = X_1 + X_2 \tag{8}$$

is given by

$$g(y) = g_1(y),\ 0 \le y \le a$$
$$= g_1(y) - g_2(y),\ a < y < \infty \tag{9}$$

where

$$g_1(y) = \frac{C}{a}\sum_{n=0}^{\infty} y^{\lambda}\frac{(-\mu y)^n}{n!} H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}\left[\begin{matrix} \zeta y^{-\gamma} \\ \eta \end{matrix} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\lambda+n,\gamma):(e_j,E_j)_{1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(\lambda+n,\gamma):(f_j,F_j)_{1,q_3} \end{matrix} \right],\ y \ge 0 \tag{10}$$

and

$$g_2(y) = \frac{C}{a}\sum_{n=0}^{\infty} (y-a)^{\lambda}\frac{\{-\mu(y-a)\}^n}{n!} H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}\left[\begin{matrix} \zeta (y-a)^{-\gamma} \\ \eta \end{matrix} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\lambda+n,\gamma):(e_j,E_j)_{1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(\lambda+n,\gamma):(f_j,F_j)_{1,q_3} \end{matrix} \right],\ y \ge 0 \tag{11}$$

C is given by (7) and the following conditions are satisfied:

(i) $\gamma > 0, \mu > 0, \lambda - \gamma \min_{1 \le j \le m_2}(\frac{d_j}{\delta_j}) > 0$,

(ii) $|\arg\zeta| < \frac{1}{2}U\pi$, $|\arg\eta| < \frac{1}{2}V\pi$, where U and V are given in (3) and (4) respectively.

(iii) The parameter of H-function are real and so restricted that $g_1(y)$ and $g_2(y)$ remains non-negative.

Proof

To obtain the probability density function of $Y = X_1 + X_2$, we use the method of Laplace transform and its inverse. Let the Laplace transform of Y be denoted by $\phi_y(t)$, then

$$\phi_y(t) = L\{f_1(x_1); t\}\ L\{f_2(x_2); t\} \tag{12}$$

The Laplace transform of $f_1(x_1)$ is a simple integral so it can easily be evaluated and for the Laplace transform of $f_2(x_2)$, we express the H-function in terms of Mellin-Barnes type contour integral (2). Now, we interchange the order of x_2- and s-integrals and evaluate x_2-integral as gamma integral to get

$$\phi_y(s) = \frac{C}{a}\frac{(1-e^{-at})}{t}(t+\mu)^{-\lambda} H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}$$

$$\left[\frac{\zeta(t+\mu)^{\gamma}}{\eta}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(\lambda,\gamma):(f_j,F_j)_{1,q_3}\end{matrix}\right] \qquad (13)$$

Now, we break the above expression in two parts, as follows:

$$\begin{aligned}\phi_y(s) = {} & \frac{C}{a}\frac{(t+\mu)^{-\lambda}}{t}\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}\left[\frac{\zeta(t+\mu)^{\gamma}}{\eta}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(\lambda,\gamma):(f_j,F_j)_{1,q_3}\end{matrix}\right]\\ & -\frac{C}{a}\frac{e^{-at}(t+\mu)^{-\lambda}}{t}\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}\left[\frac{\zeta(t+\mu)^{\gamma}}{\eta}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(\lambda,\gamma):(f_j,F_j)_{1,q_3}\end{matrix}\right] \qquad (14)\end{aligned}$$

To obtain the inverse Laplace transform of first term of equation (14), we express the H-function in contour integral, collect the terms involving ‘t’ and take its inverse Laplace transform and using the known result (Erdélyi [13], p.238, eq.9)). Further, writing the confluent hypergeometric function thus obtained in series form and interpreting the result by equation (2), we get the value of $g_1(y)$. The inverse Laplace transform of second term easily follows by the value of $g_1(y)$ and second shifting property for Laplace transform, we get the value of $g_2(y)$.

CONCLUSION

The importance of our result lies in its manifold generality. In view of the generality of the H-function, on specializing the various parameters in the H-function, we obtain, from our results, several pdfs such as the gamma pdf, beta pdf, Rayleigh pdf, Weibull pdf, Nakagami-m pdf, Chi-Squared pdf, half-Gaussian pdf, one-sided exponential pdf, half-Cauchy pdf, lognormal pdf, Rician pdf, K_v pdf etc. Thus, the results presented in this paper would at once yield a very large number of pdfs occurring in the problems of statistics, applied mathematics, mathematical physics and engineering.

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