Effective and Difference Mean Edges in Fuzzy Graphs

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ABSTRACT
A difference mean edge and an effective difference mean edge relate the membership value of an edge with the membership values of its end vertices.This study discusses the propertiesof difference mean edges and effective difference mean edges. Based on them, few features of the difference mean fuzzy graph have been derived. We provide a necessary criterion for an edge to be a difference mean edge. The requirements that must be fulfilled for a difference mean edge that acts as an effective edge are determined. We analyze specific aspects of an effective difference mean fuzzy graph on path, star and explain the membership values of vertices in an effective difference mean fuzzy graph.

Keywords: Difference mean edge, Difference Mean fuzzy graph, Effective difference mean edge, Effective difference mean fuzzy graph, Order, Size.

1. INTRODUCTION

The vertices, edges, weights, and other elements of a graph are taken to be specified in graph theory.That is, there is no doubt as to these objects' existence. However, there are a lot of unknowns in the actual world. This indicates that the weights, edges, and vertices are not always known. For instance, it might not be precisely known how long a car would take to drive or how much capacity it has on a network of roads. In 1975, A. Rosenfeld created the concept of a fuzzy graph to depict these types of graphs. Similar to set theory, fuzzy graph theory draws its foundation from the fuzzy set theory that L. A. Zadeh created in 1965 [8]. Rosenfeld devised the fuzzy graph's structure while taking into consideration fuzzy relations on fuzzy sets. In the fields of economics, communication systems, mathematics, and other fields, the development of fuzzy graph theory has created new avenues for study.Fuzzy graphs are best illustrated by social networks like Facebook, Twitter, LinkedIn, and so on.Difference mean fuzzy graphs are a special type of fuzzy graph where the edges are all difference mean edges. They will be helpful in analyzing situations that can be modeled into fuzzy graphs and also its ability to model imprecise and uncertain information in a more nuanced way. Difference-mean fuzzy graphs have implications in network analysis and optimization. This incorporation of the difference mean concept provides a different perspective on edge memberships, which can lead to more accurate modeling of network structures and dynamics. This in turn allows for improved routing, resource allocation and fault tolerance in complex systems such as communication networks and transportation networks.Nagoorgani and BasheerAhamed explored the order and size of fuzzy graphs [1] whereas Nagoorgani and Rajalaxmi examined a fuzzy labeling and the characteristics of a fuzzy labeling graph [3, 4]. Radha and Renganathan investigated the idea of effective fuzzy semigraphs[6].A difference-mean fuzzy graph was constructed by Radha and Sri Harini by placing a constraint on the membership value of the edges along with a few characteristics of the difference-mean edge involved in the direct sum of two fuzzy graphs[7]. The properties of difference mean edge in fuzzy graphs and effective difference mean edge in fuzzy graphs are discussed in this paper.Also conditions for a fuzzy graph on path and star graphs to effective difference mean fuzzy graphs are discussed.

Let $G:(\sigma,\mu)$ be a fuzzy graph on a graph $G^*:(V,E)$ is a pair of functions $\sigma: V\to[0,1]$ and $\mu: V\times V\to[0,1]$, in such a way that, $\mu(a,b)\le\sigma(a)\wedge\sigma(b), \forall a,b\in V$. If $\mu(ab)=\sigma(a)\wedge\sigma(b)$, $\forall ab\in E$, then the fuzzy graph is effective

Let $G:(\sigma,\mu)$be fuzzy graph. Then the order of$G$is$O(G)=\sum_{v\in V}\mu(v)$and the size of$G$is$S(G)=\sum_{u\neq v}\mu(uv)$

A path P in a fuzzy graph is a sequence of distinct vertices$v_0v_1v_2\ldots\ldots v_n$such that$\mu(v_{i-1}v_i)>0, 1\le i\le n$,here$n\ge1$ is called the length of the path P.

A star in a fuzzy graph consists of two node sets V = {v} and U = $\{u_1, u_2, \ldots, u_n\}$such that $\mu(vu_i) > 0$ for everyiand$\mu(u_i u_j)=0$ for everyi, j.It is denoted by$K_{1,n}$. vis called the apex vertex.

Consider a fuzzy graph $G:(\sigma,\mu)$on $G^*:(V,E)$. Let z_1z_2 be an edge in G. If$\mu(z_1z_2) = \frac{|\sigma(z_1)-\sigma(z_2)|}{2}$ or if $2\mu(z_1z_2) = |\sigma(z_1) - \sigma(z_2)|$, then z_1z_2 is a difference mean edge. If every edge in G is a difference mean edge, then G is a difference mean fuzzy graph.

A difference mean fuzzy graph G is an effective difference mean fuzzy graph if each edge in G is an effective difference mean edge.

Example 1.1

The following fuzzy graph on path P_4 is a difference mean fuzzygraph.

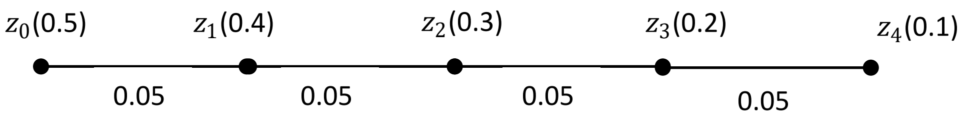

Fig. 1.1: Difference Mean Fuzzy Graph

Example 1.2

The following are fuzzy graph on star $K_{1,3}$ is an effective difference mean fuzzy graph.

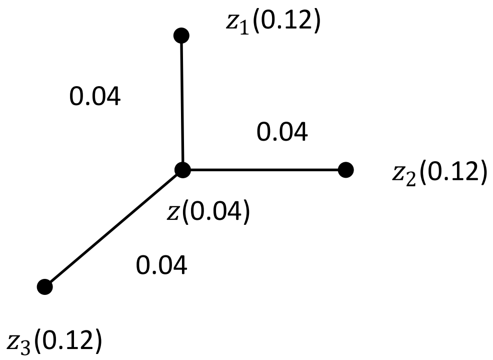

Fig. 1.2: Effective Difference Mean Fuzzy Graph

2.Properties of Difference Mean Edges in Fuzzy Graphs

Theorem 2.1

If wz is a difference mean edge with $\sigma(w) > \sigma(z)$ in afuzzy graph$G:(\sigma,\mu)$on $G^*:(V,E)$, then $\sigma(z) < \sigma(w) \leq 3\sigma(z)$.

Proof

Since wz is a difference mean edge and $\sigma(w) > \sigma(z)$,$\mu(wz) = \frac{|\sigma(w)-\sigma(z)|}{2} = \frac{\sigma(w)-\sigma(z)}{2}$

Therefore,

$\mu(wz) \leq \sigma(w) \wedge \sigma(z) \Rightarrow \frac{\sigma(w)-\sigma(z)}{2} \leq \sigma(z) \Rightarrow \sigma(w) - \sigma(z) \leq 2\sigma(z) \Rightarrow \sigma(w) \leq 3\sigma(z)$.Hence $\sigma(z) < \sigma(w) \leq 3\sigma(z)$.

Theorem 2.2

If wz is a difference mean edge in a fuzzy graph$G:(\sigma,\mu)$on$G^*:(V,E)$, then $\sigma(w) \neq \sigma(z)$.

Proof

Let wz be a difference mean edge.

If $\sigma(w) = \sigma(z)$, then $\mu(wz) = \frac{|\sigma(w)-\sigma(w)|}{2} = 0$. This is not possible. Hence$\sigma(w) \neq \sigma(z)$.

Theorem 2.3

If $G:(\sigma,\mu)$ has a difference mean edge, then σ cannot be a constant function.

Proof

If wz is a difference mean edge, then $\sigma(w) \neq \sigma(z)$. Therefore σ is not a constant function.

Theorem 2.4

If $G: (\sigma, \mu)$ is a difference mean fuzzy graph, then σ is not a constant function.

Proof

Since each edge of G is a difference mean edge, σ cannot be a constant function.

Remark 2.5

If $G: (\sigma, \mu)$ is a difference mean fuzzy graph, then σ assumes atleast two values. But the fuzzy graphs in Fig 1.1 and Fig 1.2 show that μ can be a constant function in difference mean fuzzy graphs.

3. Properties of Effective Difference Mean Edge in Fuzzy Graphs

Theorem 3.1

A fuzzy graph $G: (\sigma, \mu)$on $G^*: (V, E)$. If wz is an effective difference mean edge with$\sigma(w) > \sigma(z)$, then $\sigma(w) = 3\sigma(z)$

Proof

Since wz is a difference mean edge and $\sigma(w) > \sigma(z)$,

$$\mu(wz) = \frac{|\sigma(w) - \sigma(z)|}{2} = \frac{\sigma(w) - \sigma(z)}{2}$$

Since wz is effective,$\mu(wz) = \sigma(w) \wedge \sigma(z) \Rightarrow \frac{\sigma(w)-\sigma(z)}{2} = \sigma(z) \Rightarrow \sigma(w) = 3\,\sigma(z)$.

Theorem 3.2

If $G: (\sigma, \mu)$ is an effective difference mean fuzzy graph and if σ assumes k values, then the k values of σ are of the form $a, 3a, 3^2a, \ldots\ldots, 3^{k-1}a$, where 0 < a < 1.

Proof

Since σ assumes k different values, G has at least k vertices.Let w_1 be a vertex with minimum membership value.Let $\sigma(w_1) = a$. Then a> 0.

Since G is an effective difference mean fuzzy graph and $w_1$has minimum membership value, a vertex,say w_2, adjacent to w_1 has membership value3a using theorem 3.1.

Since $\sigma(w_2) = 3a$, by theorem 3.1, a vertex adjacent to w_2 may have membership value a or 3^2a. Since σ has k different values, there is a vertex, say w_3, different from w_1 with membership value 3^2a.

A vertex adjacent to w_3 may have membership value 3aor 3^3a. Since σ has k different values, there is a vertex w_4 different from w_2 with membership value 3^3a.

Proceeding like this, since σ assumes k different values, there is a vertex w_k with membership value $3^{k-1}a$.

Since $3^i a < 1 \Rightarrow a < 1/3^i$ for every ivarying from 0 to k – 1, a< 1. Hence the theorem.

4. Effective difference-mean fuzzy graph on path and stargraphs

In this section, the properties of effective difference mean fuzzy graph and difference mean fuzzy graphs on path and star graphs are presented.

Theorem 4.1

Let $G: (\sigma, \mu)$ be an effective difference fuzzy graph on a path $P_n{}^*: w_0 w_1 w_2 \ldots\ldots w_n$of length n with $n + 1$ vertices. Thenσassumes exactly $n + 1$ values if and only if either $\sigma(w_0) < \sigma(w_1) < \sigma(w_2) <\ldots< \sigma(w_n)$or $\sigma(w_0) > \sigma(w_1) > \sigma(w_2) >\ldots> \sigma(w_n)$.

Proof

Let G be an effective difference mean fuzzy graph on the path $P_n{}^*: w_0 w_1 w_2 \ldots\ldots w_n$.

Assume that σassumes exactly $n + 1$ values. Then since G has n+1 vertices, no two vertices of G have the same membership value.

Suppose $\sigma(w_{j-1}) < \sigma(w_j)$ and $\sigma(w_j) > \sigma(w_{j+1})$ for some j.

Then $\sigma(w_j) = 3\sigma(w_{j-1})$ and $\sigma(w_j) = 3\sigma(w_{j+1})$ using theorem 3.1

Therefore $3\sigma(w_{j-1}) = 3\sigma(w_{j+1})$

$\Rightarrow \sigma(w_{j-1}) = \sigma(w_{j+1})$

$\Rightarrow w_{j-1}$andw_{j+1} have same membership value.

This is a contradiction to our assumption.

The proof is similar if $\sigma(w_{j-1}) > \sigma(w_j)$ and $\sigma(w_j) < \sigma(w_{j+1})$ for some j.

Hence either$\sigma(w_0) < \sigma(w_1) < \sigma(w_2) < \cdots < \sigma(w_n)or\sigma(w_0) > \sigma(w_1) > \cdots > \sigma(w_{n+1})$.

Conversely assume that $\sigma(w_0) < \sigma(w_1) < \sigma(w_2) < \cdots < \sigma(w_n)$.
Let $\sigma(w_0) = a$. Since $\sigma(w_0) < \sigma(w_1)$,usingtheorem 3.1, $\sigma(w_1) = 3a$.
Since $\sigma(w_1) < \sigma(w_2)$, using theorem 3.1, $\sigma(w_2) = 3\sigma(w_1) = 3^2\sigma(w_0)$.
Since $\sigma(w_2) < \sigma(w_3)$, using theorem 3.1,$\sigma(w_3) = 3\sigma(w_2) = 3^3\sigma(w_0)$.
Proceeding like this,$\sigma(w_i) = 3^i\sigma(w_1), i = 0,1,\cdots,n$.
The proof is similar if $\sigma(w_0) > \sigma(w_1) > \sigma(w_2) > \cdots > \sigma(w_n)$.

Theorem 4.2
Let $G:(\sigma,\mu)$ be an effective difference mean fuzzy graph on a path $P_n{}^*: w_0w_1w_2\cdots w_n$. Then(i)$\sigma(w_0) < \sigma(w_1) < \sigma(w_2) < \cdots < \sigma(w_n)$if and only ifthere exists $a \in (0,1)$ such that $\sigma(w_i) = 3^i a, i = 0,1,2,\cdots,n$and $\mu(w_{i-1}w_i) = 3^{i-1}a, i = 1,2,\dots,n$. (ii)$\sigma(w_0) > \sigma(w_1) > \sigma(w_2) > \cdots > \sigma(w_n)$if and only ifthere exists $a \in (0,1)$ such that $\sigma(w_i) = 3^{n-i}a, i = 0,1,2,\cdots,n$ and $\mu(w_{i-1}w_i) = 3^{n-i}a, i = 1,2,\dots,n$.
Proof
(i) Let $\sigma(w_0) = a$. Then $0 < a < 1$.
Proceeding as in the proof of the theorem 4.1,$\sigma(w_i) = 3^i\sigma(w_0) = 3^i a, i = 0,1,2,\dots,n$
Since G is effective,$\mu(w_{i-1}w_i) = \sigma(w_{i-1}) \wedge \sigma(w_i) = 3^{i-1}a \wedge 3^i a = 3^{i-1}a$for $i = 1,2,\dots,n$.
Conversely assume thatthere existsa $\in (0,1)$such that$\sigma(w_i) = 3^i a, i = 0,1,2,\dots,n$and $\mu(w_{i-1}w_i) = 3^{i-1}a, i = 1,2,\dots,n$.
Since $a< 3a<3^2a<\dots<3^n a$, $\sigma(w_0) < \sigma(w_1) < \sigma(w_2) <\dots< \sigma(w_n)$.
The proof of (ii) is similar.

Theorem 4.3
If $G:(\sigma,\mu)$ is an effective difference mean fuzzy graph on a path $P_n{}^*: w_0w_1w_2\dots w_n$ such that $\sigma(w_0) < \sigma(w_1) < \sigma(w_2) < \cdots < \sigma(w_{n+1})$or $\sigma(w_0) > \sigma(w_1) > \sigma(w_2) > \cdots > \sigma(w_n)$,then there exists $a \in (0,1)$ such that $O(G) = \frac{a}{2}[3^{n+1} - 1]$ and $S(G) = \frac{a}{2}[3^n - 1]$.
Proof
By theorem 4.2, there exists $a \in (0,1)$ such that
$\sigma(w_i) = 3^i a,\ i = 0,1,2,\dots,n$and
$\mu(w_{i-1}w_i) = 3^{i-1}a,\ i = 1,2,\dots,n$
Therefore $O(G) = \sigma(w_0) + \sigma(w_1) + \sigma(w_2) + \cdots + \sigma(w_n)$

$$= a + 3a + \cdots + 3^n a = a[1 + 3 + \cdots + 3^n] = a\left[\frac{3^{n+1} - 1}{3 - 1}\right] = \frac{a}{2}[3^{n+1} - 1]$$

$$S(G) = \mu(w_0w_1) + \mu(w_1w_2) + \cdots + \mu(w_{n-1}w_n)$$
$$= a + 3a + \cdots + 3^{n-1}a$$
$$= \frac{a}{2}[3^n - 1]$$

Example 4.4
Consider the following effective difference mean fuzzy graph on path P_3.

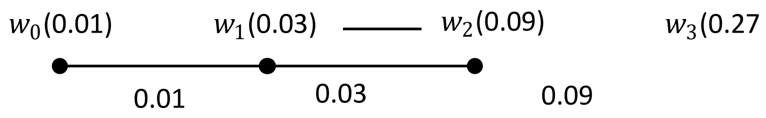

Fig.4.4

Here$(w_0) < \sigma(w_1) < \sigma(w_2) < \sigma(w_3)$. Alsoa $= 0.01, n = 3$.
Therefore by theorem 4.3,
$O(G) = \frac{a}{2}[3^{n+1} - 1] = \frac{0.01}{2}[3^4 - 1] = 0.4$.
and $S(G) = \frac{a}{2}[3^n - 1] = \frac{0.01}{2}[3^3 - 1] = 0.13$.
They can be verified by direct calculation,
$O(G) = 0.01 + 0.03 + 0.09 + 0.27 = 0.4$and $S(G) = 0.01 + 0.03 + 0.09 = 0.13$.

Theorem 4.5
If $G:(\sigma,\mu)$ is an effective difference mean fuzzy graph on a star, then there exists $a \in (0,1)$ such that σ takes exactly two values a and $3a$ or exactly three values $a, 3a$,and $9a$.

Proof
Let G be an effective difference mean fuzzy graph on a star $K_{1,n}{}^{*}$ with $n+1$ vertices $s, s_1, \dots, s_n$ having s as the apex vertex.
If $n = 1$, then $\sigma(s)$ and $\sigma(s_1)$ have membership values of the form a and $3a$.
Let $n \geq 2$. Then $ss_i, i = 1, \dots, n$ are the only edges of G.
Case (1): $\sigma(s) < \sigma(s_i)$ for all i or, $\sigma(s) > \sigma(s_i)$ for all i.
Suppose $\sigma(s) < \sigma(s_i)$ for all i. Then $\sigma(s_i) = 3\sigma(s)$ for all i.
Therefore if $\sigma(s) = a$, then σ takes the values a and $3a$.
Suppose $\sigma(s) > \sigma(s_i)$. Then $\sigma(s) = 3\sigma(s_i)$ for all i
Therefore if $\sigma(s_i) = a$, then σ takes the values a and $3a$.
Case (2): $\sigma(s) < \sigma(s_j)$ for some indices j and $\sigma(s) > \sigma(s_k)$ for all other indices k.
If $\sigma(s) < \sigma(s_j)$, then $\sigma(s_j) = 3\sigma(s)$

$$\sigma(s) > \sigma(s_k) \Rightarrow \sigma(s) = 3\sigma(s_k) \Rightarrow \sigma(s_k) = \frac{\sigma(s)}{3}$$

Let $\sigma(s_k) = a$ then $\sigma(s) = 3a$ and $\sigma(s_j) = 9a$.

Therefore $\sigma(s) = 3a$ and $\sigma(s_i) = \begin{cases} a, if\ \sigma(s) < \sigma(s_i) \\ 9a, if\ \sigma(s) > \sigma(s_i) \end{cases}$

Hence σ takes the three values a , 3a and 9a.

Theorem 4.6
If $G: (\sigma, \mu)$ is an effective difference mean fuzzy graph on a star with vertices $s, s_1, \dots, s_n$ having s as the apex vertex such that either $\sigma(s) < \sigma(s_i)$, $i = 1, \dots, n$ or $\sigma(s) > \sigma(s_i)$, $i = 1, \dots, n$, then μ is a constant function.
Proof
Since G is an effective difference mean fuzzy graph, each edge ss_i is an effective difference mean edge.
Therefore, $\sigma(s) < \sigma(s_i)$, $i = 1, \dots, n \Rightarrow \sigma(s_i) = 3\sigma(s), i = 1, \dots, n$.
Let $\sigma(s) = a$. Then $\sigma(s_i) = 3a, i = 1, \dots, n$
Since G is effective, $\mu(ss_i) = \sigma(s) \wedge \sigma(s_i) = a \wedge 3a = a$, for $i = 1, \dots, n$.
If $\sigma(s) > \sigma(s_i)$, $i = 1, \dots, n$, then $(ss_i) = \sigma(s) \wedge \sigma(s_i) = 3a \wedge a = a$, for $i = 1, \dots, n$.
Hence μ is a constant function.

Theorem 4.7
If $G: (\sigma, \mu)$ is an effective difference mean fuzzy graph on a star with vertices $s, s_1, \dots, s_n$ having s as the apex vertex such that $\sigma(s) < \sigma(s_j)$ for some indices j and $\sigma(s) > \sigma(s_k)$ for all other indices k, then μ takes exactly two values.
Proof: If $\sigma(s) < \sigma(s_j)$, then $\sigma(s_j) = 3\sigma(s)$.

$$\sigma(s) > \sigma(s_k) \Rightarrow \sigma(s) = 3\sigma(s_k) \Rightarrow \sigma(s_k) = \frac{\sigma(s)}{3}$$

Let $\sigma(s_k) = a$. Then $\sigma(s) = 3a$ and $\sigma(s_j) = 9a$.

Therefore $\sigma(s) = 3a$ and $\sigma(s_i) = \begin{cases} a, \text{if } \sigma(s) < \sigma(s_i) \\ 9a, \text{if } \sigma(s) > \sigma(s_i) \end{cases}$

Hence $\mu(s_i) = \begin{cases} a, \text{if } \sigma(s) < \sigma(s_i) \\ 3a, \text{if } \sigma(s) > \sigma(s_i) \end{cases}$

Theorem 4.8
If $G: (\sigma, \mu)$ is an effective difference mean fuzzy graph on a star with vertices $s, s_1, \dots, s_n$ having s as the apex vertex such that either $\sigma(s) < \sigma(s_i)$, $i = 1, \dots, n$ or $\sigma(s) > \sigma(s_i)$, $i = 1, \dots, n$, then there exist $a \in (0,1)$ such that $O(G) = (3n+1)a$ and $S(G) = na$.
Proof
Let $\sigma(s) = a$. Then $a \in (0, 1)$.
Since G is an effective difference mean fuzzy graph, $\sigma(s_i) = 3a$, $i = 1, \dots, n$.
Therefore $O(G) = \sum_{i=1}^{n} \sigma(s_i) + \sigma(s) = \sum_{i=1}^{n} 3a + a = 3na + a = a(3n+1)$.
Now $\mu(ss_i) = \sigma(s) \wedge \sigma(s_i) = a \wedge 3a = a$, $i = 1, \dots, n$. Therefore

$$S(G) = \sum_{i=1}^{n} \mu(ss_i) = \sum_{i=1}^{n} a = na.$$

Example 4.9
Consider the following effective difference mean fuzzy graph on star graph $K_{1,3}$

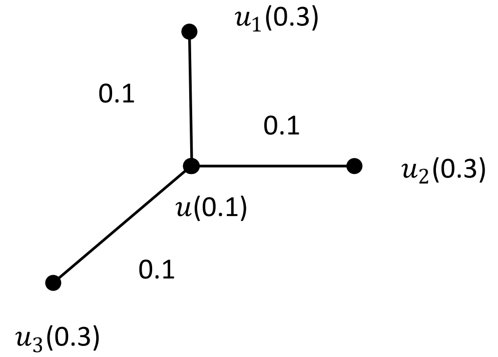

Fig 4.9

Here $a = 0.1, n = 3$.
Therefore using theorem 4.8, $O(G) = (3n + 1)a = 1$ and $S(G) = na\ 0.3$
They can be verified from direct calculation $O(G) = 0.1 + 0.3 + 0.3 + 0.3 = 1$
and$S(G) = 0.1 + 0.1 + 0.1 = 0.3$

5. CONCLUSION

A contribution towards the properties of the difference mean edges in fuzzy graphs and difference mean fuzzy graphs. In this paper, some properties of difference mean fuzzy graph have been discussed. Further we investigated about some properties of difference mean fuzzy graphs on path and star graphs.Also the formulas for the order and size in effective difference mean fuzzy graph on path and star have been derived. They will be helpful in developing further properties of fuzzy graphs.

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