Boros Integral Involving the Product of Family of Polynomials and the Incomplete I-Function

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Abstract

The current manuscript's goal is to determine the Boros integral with three parameters, which comprises of the multiplication of the incomplete I-function and a family of polynomials. Some interesting corollaries are provided as a specific case of our primary findings.

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1 Introduction and Preliminaries

Due to new hurdles in applied science and technology in the present period, the popularity of special functions is growing by the day. Special functions have been used widely in the varity of fields of fluid problems, biological problems, communication and other probelms of physics (see [1, 6–9, 13, 16, 21–24]). However, it has been noted that there are many issues in the fields of astrophysics and heat conduction for which the answers provided by the most prominent groups of special functions are insufficient. In this instance, the illustration makes use of the definition of incomplete gamma functions and its generalisations. So, The investigation of incomplete hypergeometric functions, incomplete H-functions, and incomplete $\overline{H}$-functions has been made possible by the use of incomplete type of gamma functions. For more details, one can see [2, 3] about incomplete functions and their recent applications.

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Jangid et al. [10] recently introduced a new category of incomplete I- functions ${}^{\gamma}I^{m,\,n}_{p,\,q}(y)$ and ${}^{\Gamma}I^{m,\,n}_{p,\,q}(y)$, which is the generalization of Rathie's I-function [18] and it is described as:

$$\begin{aligned} {}^{\gamma}I^{m,\,n}_{p,\,q}(y) &= {}^{\gamma}I^{m,\,n}_{p,\,q}\left[y \,\middle|\, \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_2,\varsigma_2;\mathcal{F}_2),\cdots,(f_p,\varsigma_p;\mathcal{F}_p) \\ (g_1,\varrho_1;\mathcal{G}_1),\cdots,(g_q,\varrho_q;\mathcal{G}_q) \end{matrix} \right] \\ &= {}^{\gamma}I^{m,\,n}_{p,\,q}\left[y \,\middle|\, \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (g_j,\varrho_j;\mathcal{G}_j)_{1,\,q} \end{matrix} \right] = \frac{1}{2\pi i}\int_{\pounds} \phi(r,t)\; y^r\; dr, \end{aligned} \tag{1}$$

and

$$\begin{aligned} {}^{\Gamma}I^{m,\,n}_{p,\,q}(y) &= {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[y \,\middle|\, \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_2,\varsigma_2;\mathcal{F}_2),\cdots,(f_p,\varsigma_p;\mathcal{F}_p) \\ (g_1,\varrho_1;\mathcal{G}_1),\cdots,(g_q,\varrho_q;\mathcal{G}_q) \end{matrix} \right] \\ &= {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[y \,\middle|\, \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (g_j,\varrho_j;\mathcal{G}_j)_{1,\,q} \end{matrix} \right] = \frac{1}{2\pi i}\int_{\pounds} \Phi(r,t)\; y^r\; dr, \end{aligned} \tag{2}$$

$\forall\ y \neq 0$, where

$$\phi(r,t) = \frac{\{\gamma(1-f_1+\varsigma_1 r,\,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \tag{3}$$

and

$$\Phi(r,t) = \frac{\{\Gamma(1-f_1+\varsigma_1 r,\,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \tag{4}$$

where, $\gamma(.,t)$ and $\Gamma(.,t)$ are the lower and upper incomplete gamma functions described in (6) and (7).
The incomplete I-functions ${}^{\gamma}I^{m,\,n}_{p,\,q}(y)$ and ${}^{\Gamma}I^{m,\,n}_{p,\,q}(y)$ exist $\forall\ t \geq 0$ in accordance with Rathie's parameters and contour mentioned in [18] with,

$$\Delta > 0,\ |arg(y)| < \Delta\frac{\pi}{2},$$

where

$$\Delta = \sum_{j'=1}^{m}\mathcal{G}_{j'}\,\varrho_{j'} - \sum_{j'=m+1}^{q}\mathcal{G}_{j'}\,\varrho_{j'} + \sum_{j'=1}^{n}\mathcal{F}_{j'}\,\varsigma_{j'} - \sum_{j'=n+1}^{p}\mathcal{F}_{j'}\,\varsigma_{j'}\,.$$

For $\mathcal{F}_1 = 1$, the following relation is satisfied by the incomplete I-functions:

$${}^{\gamma}I^{m,\,n}_{p,\,q}(y) + {}^{\Gamma}I^{m,\,n}_{p,\,q}(y) = I^{m,\,n}_{p,\,q}(y), \tag{5}$$

for the well known Rathie's I-function [18]. Some additional properties regarding the incomplete I- function can be found in [4].

The incomplete gamma functions $\gamma(r,t)$ and $\Gamma(r,t)$ are described in the following way:

$$\gamma(r,t) = \int_0^t u^{r-1}\, e^{-u}\, du, \qquad \big(t \geqq 0; \Re(r) > 0\big), \tag{6}$$

and

$$\Gamma(r,t) = \int_t^\infty u^{r-1}\, e^{-u}\, du, \qquad \big(t \geqq 0;\ \Re(r) > 0 \quad \text{when} \quad t = 0\big), \tag{7}$$

recognized as the lower and upper incomplete gamma functions respectively. The following relation is satisfied by the incomplete gamma functions.

$$\gamma(r,t) + \Gamma(r,t) = \Gamma(r), \qquad \big(\Re(r) > 0\big). \tag{8}$$

A general class of polynomials was studied by the Srivastava [19,20], defined in the following way:

$$S_V^U[t] = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, t^R\,, \tag{9}$$

where $U \in \mathbf{Z}^+$ and $A_{V,\,R}$ are real or complex numbers arbitrary constant. The notations $[k]$ indicates the Floor function and $(\kappa)_\mu$ denote the Pochhammer symbol described by:

$$(\kappa)_0 = 1 \qquad \textit{and} \qquad (\kappa)_\mu = \frac{\Gamma(\kappa+\mu)}{\Gamma(\kappa)} \qquad (\mu \in \mathcal{C}),$$

in the form of the Gamma function.

Lemma 1. *Let $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$ and $P > \frac{1}{2}$, we have the integral depending upon the three parameters, see Boros and Molls [5, 14].*

$$\int_0^\infty \left[\frac{h^2}{bh^4 + 2ah^2 + c}\right]^P dh = \frac{B\left(P - \frac{1}{2},\, \frac{1}{2}\right)}{2^{P+1/2}\sqrt{b}\left[a + \sqrt{bc}\right]^{P-1/2}}, \tag{10}$$

where $B(m,\,n)$ denotes the Beta function. Equation (10) can also be expressed in the following way, by using the relation $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

$$\int_0^\infty \left[\frac{h^2}{bh^4 + 2ah^2 + c}\right]^P dh = \frac{\sqrt{\pi}\,\Gamma\left(P - \frac{1}{2}\right)}{\Gamma(P)\sqrt{b}\,2^{P+\frac{1}{2}}(a+\sqrt{bc})^{P-\frac{1}{2}}}. \tag{11}$$

Concerning the proof, see Boros and Moll [5] and Quershi et al. [17].

Hundreds of special functions have been employed in applied mathematics and computing sciences for many centuries due to their outstanding features and wide range of applications. The application of image formulas involving one or more variable special functions under various definite integrals is crucial from the perspective of the usefulness of these consequences in the evaluation of generalised integrals, applied physics, and many engineering areas. A variety of improper integrals involving incomplete I-functions and the family of polynomials have been examined in this study, primarily motivated by various applications of these findings. A significant amount of additional findings can be constructed as special instances from our main results because of the unified character of our results.

2 Main Results

For $X = \frac{h^2}{bh^4+2ah^2+c}$, the following is the outcomes:

Theorem 1. *For $c \geq 0, b > 0\,, a > -\sqrt{bc}$, $P > \frac{1}{2}$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P {}^\Gamma I_{p,\,q}^{m,\,n}(X^e\,y)dh = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}}$$

$$\times\ {}^\Gamma I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P,\,e;\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P,\,e;\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{12}$$

Proof. The LHS of equation (12) is:

$$G' = \int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P {}^\Gamma I_{p,\,q}^{m,\,n}(X^e\,y)dh. \tag{13}$$

Replace the incomplete I- function ${}^\Gamma I_{p,\,q}^{m,\,n}(y)$ by (2), we get:

$$G' = \int_0^\infty X^P \frac{1}{2\pi i}\int_{\pounds} \Phi(r,t)\ (X^e\,y)^r\ dh\,dr, \tag{14}$$

where $\Phi(r,t)$ is given by (4).
Interchange the integration order in the above equation gives:

$$G' = \frac{1}{2\pi i}\int_{\pounds} \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}} y^r$$

$$\times \int_0^\infty X^{P+er}\,dh\,dr. \tag{15}$$

Now with the help of Lemma 1, evaluate the integral, we get:

$$\int_0^\infty X^{P+er}dh = \frac{\sqrt{\pi}\Gamma(P+er-\frac{1}{2})}{\Gamma(P+er)\sqrt{b}\,2^{P+er+\frac{1}{2}}(a+\sqrt{bc})^{P+er-\frac{1}{2}}}. \tag{16}$$

Put (16) in (15), we get:

$$G^{'} = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \frac{1}{2\pi i}$$

$$\int_{\pounds} \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\{\Gamma\left(P+er-\frac{1}{2}\right)\}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}\{\Gamma\left(P+er\right)\}}$$

$$\times\, y^r[2(a+\sqrt{bc})]^{-er}\,dr. \tag{17}$$

Now convert equation (17) in incomplete I- function to obtain the desired result. □

Theorem 2. *For $b>0$, $c\geq 0$, $a>-\sqrt{bc}$, $P>\frac{1}{2}$then we have the following result:*

$$\int_0^\infty\left(\frac{h^2}{h^4+2ah^2+c}\right)^P {}^{\gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}}$$

$$\times\, {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\;\middle|\;\begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P;\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\(1-P,\,e;\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{18}$$

Theorem 2 is proved in the same way as theorem 1 with the same conditions.

Theorem 3. *For $b>0$, $c\geq 0$, $a>-\sqrt{bc}$, $P>\frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in Z^+$ then we have the following result:*

$$\int_0^\infty\left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\,{}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}}\times\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{V,R}\,w^R.\,\frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\, {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\;\middle|\;\begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\(1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{19}$$

Proof. The LHS of (19) is:

$$G = \int_0^\infty \left(\frac{h^2}{bh^4 + 2ah^2 + c} \right)^P S_V^U[wX] \ {}^\Gamma I_{p,\,q}^{m,\,n}(X^e\, y) dh. \tag{20}$$

Replace the Srivastava Polynomial $S_V^U[t]$ and incomplete I- function ${}^\Gamma I_{p,\,q}^{m,\,n}(y)$ by (9) and (2) respectively, we get:

$$G = \int_0^\infty X^P \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, (w\, X)^R \frac{1}{2\pi i} \int_\pounds \Phi(r,t)\; (X^e\, y)^r\; dh\, dr, \tag{21}$$

where $\Phi(r,t)$ is given by (4).
Interchange the integration order in the above equation gives:

$$\sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R \frac{1}{2\pi i} \int_\pounds \frac{\{\Gamma(1 - f_1 + \varsigma_1 r, t)\}^{\mathcal{F}_1} \prod_{j=1}^{m} \{\Gamma(g_j - \varrho_j r)\}^{\mathcal{G}_j} \prod_{j=2}^{n} \{\Gamma(1 - f_j + \varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p} \{\Gamma(f_j - \varsigma_j r)\}^{\mathcal{F}_j} \prod_{j=m+1}^{q} \{\Gamma(1 - g_j + \varrho_j r)\}^{\mathcal{G}_j}}$$
$$\times\, y^r \times \int_0^\infty X^{P+R+er}\, dh\, dr. \tag{22}$$

Now with the help of Lemma 1, evaluate the integral, we get:

$$\int_0^\infty X^{P+R+er} dh = \frac{\sqrt{\pi}\Gamma(P + R + er - \frac{1}{2})}{\Gamma(P + R + er)\sqrt{b}\, 2^{P+R+er+\frac{1}{2}}(a + \sqrt{bc})^{P+R+er-\frac{1}{2}}}. \tag{23}$$

Put (23) in (22), we get:

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a + \sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R \cdot \frac{1}{[2(a + \sqrt{bc})]^R} \times \frac{1}{2\pi i}$$
$$\int_\pounds \frac{\{\Gamma(1 - f_1 + \varsigma_1 r, t)\}^{\mathcal{F}_1} \{\Gamma\left(P + R + er - \frac{1}{2}\right)\} \prod_{j=1}^{m} \{\Gamma(g_j - \varrho_j r)\}^{\mathcal{G}_j} \prod_{j=2}^{n} \{\Gamma(1 - f_j + \varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p} \{\Gamma(f_j - \varsigma_j r)\}^{\mathcal{F}_j} \prod_{j=m+1}^{q} \{\Gamma(1 - g_j + \varrho_j r)\}^{\mathcal{G}_j} \{\Gamma\left(P + R + er\right)\}}$$
$$\times\, y^r [2(a + \sqrt{bc})]^{-er}\, dr. \tag{24}$$

Now convert equation (24) in incomplete I- function to obtain the desired result. □

Theorem 4. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$, $P > \frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in Z^+$ then we have the following*

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result:

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ {}^\gamma I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^\gamma I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e} \left| \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2}-P-R,\, e,\, 1\right), (f_j, \varsigma_j; \mathcal{F}_j)_{2,\,p} \\ (1-P-R,\, e,\, 1), (g_j, \varrho_j; \mathcal{G}_j)_{1,\,q} \end{matrix} \right. \right]. \tag{25}$$

Theorem (4) is proved in the same way as theorem (3) with the same conditions.
Remark: If we set $U=1, A_{V,0}=1$ and $A_{V,R}=0\ \forall\ R\neq 0$ in theorem 3 and 4 then the result is same as that of theorem 1 and 2.

3 Special Case

In this section, as a particular instance of Theorem 3 and Theorem 4, we establish the Boros integral for the multiplication of Srivastava polynomial with the incomplete $\bar{I}$- function and the incomplete $\bar{H}$- function. Further, some special value will be given to Srivastava polynomial in order to get the outcomes in the form of Hermite and Laguerre polynomials. If we provide the parameter of particular features, we get the following special cases to delineate the use of fundamental outcomes.
(i) Incomplete $\bar{I}$- function: If we set $\mathcal{G}_j=1$ for $1\leq j\leq m$ in (2) and making use of the connection, that is (see [11])

$$\begin{aligned} {}^\Gamma\overline{I}_{p,\,q}^{m,\,n}(y) &= {}^\Gamma\overline{I}_{p,\,q}^{m,\,n}\left[y \left| \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), (f_j, \varsigma_j; \mathcal{F}_j)_{2,\,p} \\ (g_j, \varrho_j; 1)_{1,\,m}, (g_j, \varrho_j; \mathcal{G}_j)_{m+1,\,q} \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i}\int_\pounds \overline{\phi}(r,t) y^r\, dr, \end{aligned} \tag{26}$$

where,

$$\overline{\phi}(r,t) = \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1} \prod_{j=1}^m \Gamma(g_j-\varrho_j r) \prod_{j=2}^n \{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^p \{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j} \prod_{j=m+1}^q \{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \tag{27}$$

in (19) and (25), then we obtain the corollaries as follows:

Corollary 1. *For $b>0$, $c\geq 0$, $a>-\sqrt{bc}$, $P>\frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in Z^+$ then we have the following*

result:

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ {}^{\Gamma}\bar{I}_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R\, \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\Gamma}\bar{I}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e;\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (1-P-R,\,e;\,1),(g_j,\varrho_j;\,1)_{1,\,m},\,(g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q} \end{array}\right]. \tag{28}$$

Corollary 2. *For $b>0$, $c\geq 0$, $a>-\sqrt{bc}$, $P>\frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in Z^+$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ {}^{\gamma}\bar{I}_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R.\, \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\gamma}\bar{I}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e;\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (1-P-R,\,e;\,1),(g_j,\varrho_j;\,1)_{1,\,m},\,(g_j,\varrho_j;G_j)_{m+1,\,q} \end{array}\right]. \tag{29}$$

(ii) Incomplete $\bar{H}$- function: If we set $\mathcal{F}_j=1$ for $n+1\leq j\leq p$ and $\mathcal{G}_j=1$ for $1\leq j\leq m$ in (2) and making use of the connection, that is (see [12], [15])

$$\begin{aligned} \overline{\Gamma}_{p,\,q}^{m,\,n}(y) &= \overline{\Gamma}_{p,\,q}^{m,\,n}\left[y\ \middle|\ \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,n},(f_j,\varsigma_j;1)_{n+1,\,p} \\ (g_j,\varrho_j;1)_{1,\,m},(g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q} \end{array}\right] \\ &= \overline{\Gamma}_{p,\,q}^{m,\,n}\left[y\ \middle|\ \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,n},(f_j,\varsigma_j)_{n+1,\,p} \\ (g_j,\varrho_j)_{1,\,m},(g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q} \end{array}\right] \\ &= \frac{1}{2\pi i}\int_{\pounds} \overline{\psi}(r,t)y^r\,dr, \end{aligned} \tag{30}$$

where,

$$\overline{\psi}(r,t) = \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\Gamma(g_j-\varrho_j r)\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\Gamma(f_j-\varsigma_j r)\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \tag{31}$$

in (19) and (25), then we obtain the corollaries as follows.

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Corollary 3. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$, $P > \frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in Z^+$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ \bar{\Gamma}_{p,\,q}^{m,\,n}(X^e\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R\, \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\ \bar{\Gamma}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e} \left| \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right), \\ (1-P-R,\,e,\,1),(g_j,\varrho_j;1)_{1,\,m}, \end{matrix} \right.\right.$$
$$\left.\begin{matrix} (f_j,\varsigma_j;\mathcal{F}_j)_{2,\,n},\,(f_j,\varsigma_j;1)_{n+1,\,p} \\ (g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q} \end{matrix}\right]. \tag{32}$$

Corollary 4. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$, $P > \frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in Z^+$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ \bar{\gamma}_{p,\,q}^{m,\,n}(X^e\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R.\, \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\ \bar{\gamma}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e} \left| \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right), \\ (1-P-R,\,e,\,1),(g_j,\varrho_j;1)_{1,\,m}, \end{matrix} \right.\right.$$
$$\left.\begin{matrix} (f_j,\varsigma_j;\mathcal{F}_j)_{2,\,n},\,(f_j,\varsigma_j;1)_{n+1,\,p} \\ (g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q} \end{matrix}\right]. \tag{33}$$

(iii) Hermite Polynomial: If we set $A_{V,\,R} = (-1)^R$ and $U = 2$ in (9) then $S_V^2[t] \to t^{V/2} H_V\left(\frac{1}{2\sqrt{t}}\right)$ and making use of the connection, that is (see [20]):

$$H_V(t) = \sum_{R=0}^{[V/2]} (-1)^R \frac{V!}{R!(V-2R)!} (2t)^{V-2R}, \tag{34}$$

in (19) and (25), then we obtain the corollaries as follows.

Corollary 5. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$ and $P > \frac{1}{2}$ then we have the following*

result:

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P (Xw)^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{Xw}}\right) {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\, y)dh =$$
$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/2]} \frac{(-1)^R\, V!}{R!\,(V-2R)!}\, w^R \cdot \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\, {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e} \,\middle|\, \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q} \end{matrix}\right]. \tag{35}$$

Corollary 6. *For* $b>0,\ c\geq 0,\ a>-\sqrt{bc}$ *and* $P>\frac{1}{2}$ *then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P (Xw)^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{Xw}}\right) {}^{\gamma}I_{p,\,q}^{m,\,n}(X^e\, y)dh =$$
$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/2]} \frac{(-1)^R\, V!}{R!\,(V-2R)!}\, w^R \cdot \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\, {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e} \,\middle|\, \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q} \end{matrix}\right]. \tag{36}$$

(iv) Laguerre Polynomial: If we set $A_{V,\,R}=\binom{V+\alpha}{V}\frac{1}{(\alpha+1)_R}$ and $U=1$ in (9) then $S_V^1[t]\to L_V^{(\alpha)}(t)$ and making use of the connection, that is (see [20]).

$$L_V^{\alpha}(t)=\sum_{R=0}^{V}\binom{V+\alpha}{V-R}\frac{(-t)^R}{R!}, \tag{37}$$

in (19) and (25), then we obtain the corollaries as follows.

Corollary 7. *For* $b>0,\ c\geq 0,\ a>-\sqrt{bc}$ *and* $P>\frac{1}{2}$ *then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P L_V^{(\alpha)}[wX]\ {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\, y)dh =$$
$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{V} \binom{V+\alpha}{V-R}\frac{(-w)^R}{R!}\, \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\, {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e} \,\middle|\, \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q} \end{matrix}\right]. \tag{38}$$

Corollary 8. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$ and $P > \frac{1}{2}$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4 + 2ah^2 + c}\right)^P L_V^{(\alpha)}[wX]\ {}^\gamma I_{p,\,q}^{m,\,n}(X^e\, y)dh =$$
$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{V} \binom{V+\alpha}{V-R} \frac{(-w)^R}{R!} \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\ {}^\gamma I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e} \,\middle|\, \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P - R,\, e,\, 1\right) (f_j, \varsigma_j; \mathcal{F}_j)_{2,\,p} \\ (1-P-R,\, e,\, 1), (g_j, \varrho_j; \mathcal{G}_j)_{1,\,q} \end{matrix}\right]. \tag{39}$$

Remark: If we set $U = 1, A_{V,0} = 1$ and $A_{V,R} = 0\ \forall\ R \neq 0$ for the the first four corollaries (Corollary 1- Corollary 4) then it becomes the special case for Theorem 1 and 2.

4 Concluding Remarks

In this article, we obtain the Boros integral with three parameter for the incomplete I- function which is the extension of the I- function investigated by Jangid et al. [10] and we also study Boros integral for the product of incomplete I- function and Srivastava Polynomial. As the incomplete I- function generalize varity of incomplete functions like: I- function, Meijer G- function, hypergeometric function, H-function, $\bar{I}$- function and many other functions.

Also, Srivastava Polynomial generalize various other polynomial like: Hermite polynomial, Jacobi polynomial, Laguerre polynomial, Gegenbauer polynomial, Legendre polynomial, Tchebycheff polynomial, Gould-Hopper Polynomial and several other polynomials. Our main findings are therefore important and can be used to count the many Boros integral forms associated with various special functions and polynomials.

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