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Laplace Variational Iteration Method for Solving fractional Wave like Equations

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This paper introduces the latest procedure for explaining certain types of fractional wave equations using the variation iteration method (VIM) and Laplace transform. The Laplace variation iteration method is a type of semi-analytical technique applied to both linear and non-linear equations without requiring linearization, discretization, or perturbation. It is not a time-consuming method and converses the solution rapidly with the exact and less error solution. This approach is delineated and then explained through several example cases. The outcomes demonstrate that this alternate strategy yields reliable outcomes and the results are displayed graphically.

1 Introduction

Mathematics, engineering, and sciences are full of amazing phenomena that can be precisely described by using mathematical techniques from fractional calculus, such as the perception of fractional order derivatives and integrals [6,14,15,19]. Differential equations of fractional order [25,26,27,28] have been gaining a lot of attention newly owing to the precise understanding of nonlinear phenomena.

The Wave equations are the linear partial differential equations of the second order. This equation describes the waves, which are a common part of classic physics. These include water waves, sound waves, and light waves. Over the last few years, there has been a new application of wave-like models to physical problems. These models can be used in different fields [2,12,16,17]. Due to the importance of wave-like equations, many researchers [2,20,21] have considered solutions to these equations. For the current issue, we take into consideration the following fractional wave equations with variable

$$D_t^{\alpha} v(x,t) = F(x',y',z')\frac{\partial^2 v}{\partial x'^2} + G(x',y',z')\frac{\partial^2 v}{\partial y'^2} + H(x',y',z')\frac{\partial^2 v}{\partial z'^2}; \quad 1 < \alpha \leq 2 \tag{1}$$

with the initial conditions

$$u(x',y',z',0) = h(x',y',z'), u_t(x',y',z',0) = m(x',y',z') \tag{2}$$

An analytical approach that is more powerful than the traditional variational technique is called the "Variational iteration method" (VIM). It was initially recommended by He [8]. The "Laplace variational iteration method" is a combination of the "Laplace transform" and "variational iteration method." Applications of VIM to fractional differential equations are slow to converge, mainly because they directly use the Lagrange multipliers of ordinary differential equations (ODEs) [23]. Wu and Baleanu [24] pointed out that it can be difficult to apply integrals by parts of the Riemann-Liouville (RL) integral resulting from the constructed correction function. To overcome this shortcoming, they proposed to identify generalized Lagrange multipliers via the Laplace transform. This method has been utilized by many authors to solve several difficulties [1,3,7]. The novelty of this work lies in applying the "Laplace variational iteration method" (LVIM) for solving heat equations of fractional order.

2 Preliminaries

Definition 1 The Caputo derivative of arbitrary order [4] of function v(x,t) is presented as

$$D_t^{\alpha} v\,(\,x,t\,) = \frac{1}{\Gamma(m-\mu} \int_0^t (t-\delta)^{m-\mu-1}\, v^{(m)}\,(\,x,\delta\,)\,d\delta = J_t^{m-\mu} D^m u(\,v,t\,);$$
$$m-1 < \mu \leq \; m, m \in \; N, \tag{3}$$

where $\frac{d^{\alpha}}{dt^{\alpha}}$ and J_t^{α} shows the Riemann- Liouville integral operator of fractional order [19], $\alpha > 0$

$$J_t^{\alpha} v(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\delta)^{\alpha-1} v(x,\delta)\,d\delta \;; m-1 < \mu \leq m, m \in N \tag{4}$$

Definition 2 The Laplace Transform [18,19] of f(t), $t > 0$ is defined as

$$L[f(t)] = F(s) = \int_0^t e^{-st} f(t)\,dt \tag{5}$$

Definition 3 The Laplace transform of $D_t^{\alpha} v(x,t)$ is explained as [18,19]

$$L[D_t^{\alpha} v(x,t)] = L[v(x,t)] - \sum_{n=0}^{m-1} v^n(x,0) s^{\alpha-n-1}; m-1 < \alpha \leq m, m \in N \tag{6}$$

Definition 4 The Mittag-Leffler function is explained as [18]

$$E_{\alpha}\,(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma\,(\alpha n+1)} \tag{7}$$

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+\beta)} \tag{8}$$

3 Variational Iteration Method

He [10] established a method VIM for solving problems. This is a common technique used to evaluate solutions for linear and non-linear problems. Illustrate the VIM model, we take into consideration the subsequent non-linear equation with given constraints::

$$Pv(x,t) + Qv(x,t) = f(x,t) \tag{9}$$

where $'v'$ is the unknown function, ‘P’, ‘Q’ are linear and nonlinear operators, and f is the source term. The correction functional for (7) is given as follows:

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda[Pv_n(\xi,t) + Qv_n(\xi,t) - f(\xi,t)]d\xi \tag{10}$$

where λ is a general Lagrange multiplier that can be identified optimally via the variation theory. The subscript n indicates the nth approximation and u_n is considered as a restricted variation $\delta u_n = 0$.

3.1 Laplace Variational Iteration Method (LVIM)

To demonstrate the elementary purpose of (LVIM), deliberate a general fractional non-linear nonhomogeneous partial differential equation through the primary situations of the type

$$D_t^{\alpha} v(x,t) + Pv(x,t) + Qv(x,t) = f(x,t); m-1 < \alpha \leq m, m \in N \tag{11}$$

$$v_n(x,0) = h_k(x); n = 0,1,2,3,...m-1 \tag{12}$$

where D_t^{α} is the Caputo derivative. P and Q are linear and nonlinear operators, respectively, and f is the source term. By applying Laplace transform pertaining to t, on both sides of (9), we get

$$L[v(x,t)] = \frac{1}{s^{\alpha}} \sum_{n=0}^{m-1} v^n(x,0)s^{\alpha-n-1} + \frac{1}{s^{\alpha}} L[f(x,t)] - \frac{1}{s^{\alpha}} L[Pv(x,t) + Qv(x,t)] \tag{13}$$

taking inverse Laplace transform on equation (13)

$$v(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} \sum_{n=0}^{m-1} s^{\alpha-1-n} v^n(x,0) + \frac{1}{s^{\alpha}} L[f(x,t)]\right] - L^{-1}\left[\frac{1}{s^{\alpha}} L[Pv(x,t) + Qv(x,t)]\right] \tag{14}$$

by differentiating (14), concerning t, we get

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial}{\partial t}\left\{L^{-1}\left[\frac{1}{s^{\alpha}} \sum_{n=0}^{m-1} s^{\alpha-1-n} v^n(x,0) + \frac{1}{s^{\alpha}} L[f(x,t)]\right] - L^{-1}\left[\frac{1}{s^{\alpha}} L[Pv(x,t) + Qv(x,t)]\right]\right. \tag{15}$$

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The correction functional for (15)

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda \left[\frac{\partial v_n(x,\varepsilon)}{\partial \varepsilon} - \frac{\partial}{\partial \varepsilon}\left\{L^{-1}\left[\frac{1}{s^{\alpha}}\sum_{n=0}^{m-1} s^{\alpha-1-n} v^n(x,0) + \frac{1}{s^{\alpha}} L[f(x,t)]\right]\right\} - L^{-1}\left[\frac{1}{s^{\alpha}} L[Pv(x,\varepsilon) + Qv(x,\varepsilon)]\right]\right] d\varepsilon \tag{16}$$

The general Lagrange multiplier for (16) can be identified optimally via variation theory to get

$$1 + (\lambda)_{\epsilon=t} = 0 \tag{17}$$

From (17), we get

$$\lambda = -1 \tag{18}$$

Substituting $\lambda = -1$ into (16), then the iterative formula for $n = 0, 1, 2, \ldots$, as follows:

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left[\frac{\partial v_n(x,\varepsilon)}{\partial \varepsilon} - \frac{\partial}{\partial \varepsilon}\left\{L^{-1}\left[\frac{1}{s^{\alpha}}\sum_{n=0}^{m-1} s^{\alpha-1-n} v^n(x,0) + \frac{1}{s^{\alpha}} L[f(x,t)]\right] - L^{-1}\left[\frac{1}{s^{\alpha}} L[Pv(x,\varepsilon) + Qv(x,\varepsilon)]\right]\right] d\varepsilon \tag{19}$$

Begin with the early iteration

$$v_0(x,t) = v(x,0) + t v_t(x,0) \tag{20}$$

As a limit of the subsequent approximations, the exact answer is provided $v_n(x,t)$, $n = 0, 1, 2, \ldots$; alternatively in other words

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) \tag{21}$$

3.2 Applications of LVIM for Solving fractional wave-like equations

Problem 1 Deliberate the succeeding 1-D fractional wave-like equation:

$$D_t^{\alpha} v(x,t) = \frac{1}{2} x^2 \frac{\partial^2 v}{\partial x^2}; 1 < \alpha \leq 2 \tag{22}$$

initial condition:

$$v_0(x,y,0) = 0, v_t(x,0) = x^2 \tag{23}$$

taking the Laplace transformation on (22) and result specified by equation (23) we obtain

$$L\left[v(\ x,t\ )\right] = \frac{x}{s} + \frac{x^2}{s^2} + \frac{1}{2s^{\alpha}} x^2 L\left[\frac{\partial^2 u}{\partial x^2}\right] \tag{24}$$

applying inverse Laplace transformation to the Equation (24), we have

$$v(\ x,t\ ) = x + x^2 t + L^{-1}\left[\frac{1}{2s^{\alpha}} x^2 L\left[\frac{\partial^2 v}{\partial x^2}\right]\right] \tag{25}$$

differentiating Equation (25) concerning t, we have

$$\frac{\partial v}{\partial t} = x^2 + \frac{\partial}{\partial t} L^{-1}\left[\frac{1}{2s^{\alpha}} x^2 L\left[\frac{\partial^2 v}{\partial x^2}\right]\right] \tag{26}$$

the correction functional for $\lambda = -1$ is offered by

$$v_{n+1}(\ x,t\ ) = v_n(\ x,t\ ) - \int_0^t \left[\frac{\partial v_n(\ x,\varepsilon\ )}{\partial \varepsilon} - x^2 - \frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{2s^{\alpha}} x^2 L\left(\frac{\partial^2 v_n}{\partial x^2}\right)\right\}\right] d\varepsilon \tag{27}$$

the initial iteration

$$v_0(x,0) = x + x^2 t \tag{28}$$

using the equation in equation (26), we have

$$v_0(x,t) = v_0(x,0) = x + x^2 t \tag{29}$$

$$v_1(x,t) = x + x^2 t + x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{30}$$

$$v_2(x,t) = x + x^2 t + x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + x^2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \tag{31}$$

therefore it is expected that the general term in the successive approximation

$$v_n(x,t) = x + x^2[t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + ...] \tag{32}$$

assumed that the solution was in closed form by

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = x + x^2 t E_{\alpha,2}(t^\alpha) \tag{33}$$

where $E_{\alpha,2}(t^\alpha)$ is the Mittag- Laffler Function defined in equation (6) letting $\alpha = 2$ then

$$v(x,t) == x + x^2 t + \frac{sinht}{t} \tag{34}$$

Numerical and Graphical discussion

In this part we found a record for numerical explanation of equation (31) and plot some graphs for $\alpha = 0.25, 0.5, 0.75, 1$.

Table 1: The values of $v(x,t)$ for $\alpha = 0.25$

$\alpha = 0.25$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	22.907618	50.042141	135.672615	263.118327	432.379275
4	13.010819	111.097371	305.270476	595.530113	981.876342
6	21.343969	186.09572	513.599243	1003.854451	1656.861548
8	30.896476	272.068286	752.411907	1471.927338	2430.614579
10	41.483595	367.352360	1017.089089	1990.696185	3288.171245

Table 2: The values of $v(x,t)$ for $\alpha = 0.50$

$\alpha = 0.50$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	49.149229	58.14922947	133.192304	258.256916	424.343065
4	16.018022	138.162200	380.450555	742.883089	1225.459800
6	31.055812	273.502315	756.395319	1479.734827	2443.520836
8	51.021537	453.193835	1255.538432	2458.055328	4060.744521
10	75.788321	676.094894	1874.708039	3671.627756	6066.854046

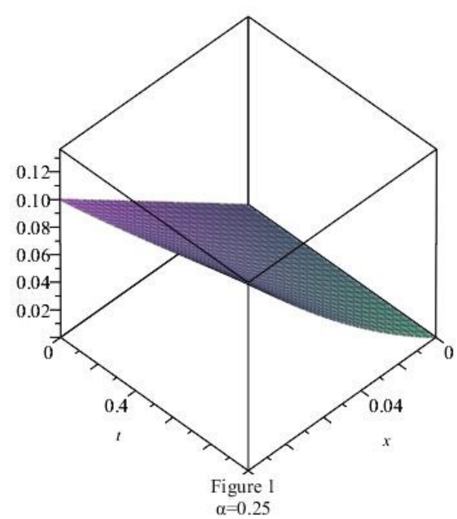
Table 3:The values of $v(x,t)$ for $\alpha = 0.75$

$\alpha = 0.75$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	5.239601	41.156416	110.990046	214.740490	352.407748
4	19.958726	173.628535	478.968154	935.977582	1544.656820
6	57.104733	507.942604	1407.618347	2756.131960	4553.483443
8	129.591614	1160.324531	3219.790365	6307.989114	10424.92078
10	251.599886	2258.398981	6269.997168	12286.39445	20307.59083

Table 4:The values of $v(x,t)$ for $\alpha = 1$

$\alpha = 1.0$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	5.33	42	113.33	219.33	360
4	20.66	180	496.66	970.66	1602
6	56	498	1380	2702	4464
8	119.33	1068	2963.33	5805.33	9594
10	218.66	1962	5446.66	10672.66	17640

The solution is graphically presented in Figures 1,2,3, and 4 for various fractional orders of α



Figur 1: The behaviour of $v(x,t)$ w.r.t. x and t for $\alpha = 0.25$

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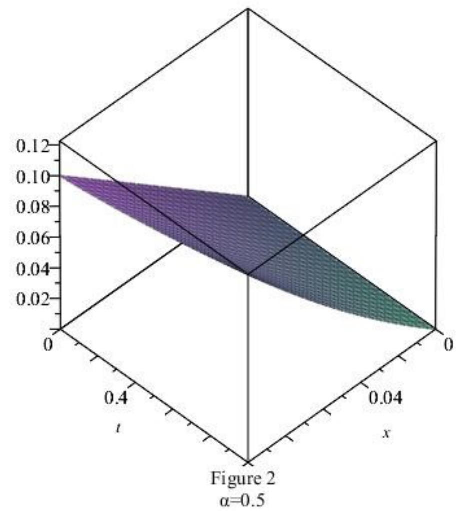
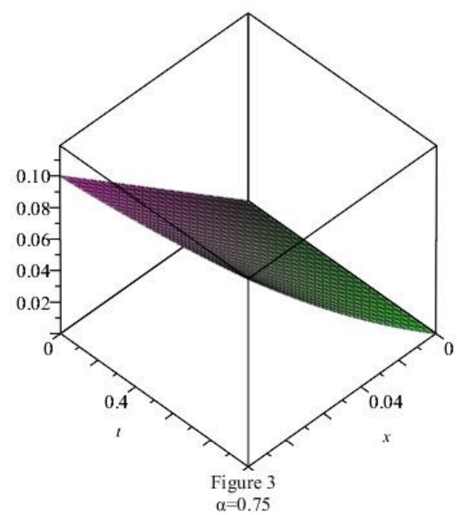
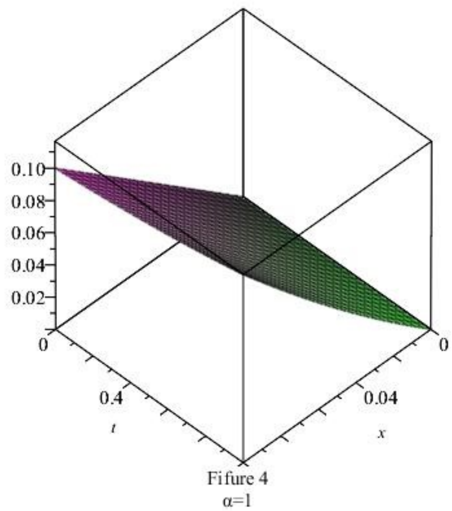


Figur 2: The behaviour of $v(x,t)$ w.r.t. x and t for $\alpha = 0.50$



Figur 3: The behaviour of $v(x,t)$ w.r.t. x and t for $\alpha = 0.75$



Figur 4: The behaviour of $v(x,t)$ w.r.t x and t for $\alpha = 1$

Problem 2 Deliberate the succeeding 2-D fractional wave-like equation:

$$D_t^\alpha v(x,y,t) = \frac{1}{2}[x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2}], 1 < \alpha \leq 2 \tag{35}$$

initial condition:

$$v_0(x,y,0) = x^4, v_t(x,y,0) = y^4 \tag{36}$$

taking the Laplace transformation on (34) and using the result specified by (35), we achieve,

$$v(\ x,y,t\ ) = x^4 + y^4 t + L^{-1}\left[\frac{1}{12s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2}\right]\right] \tag{37}$$

apply inverse Laplace transform we have

$$\frac{\partial v}{\partial t} = y^4 + \frac{\partial}{\partial t}L^{-1}\left[\frac{1}{12s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2}\right]\right] \tag{38}$$

the correction functional for $\lambda = -1$ is given as follows

$$v_{n+1}(\ x, y, t\ ) = v_n(\ x, y, t\ ) - \int_0^t \left[\frac{\partial v_n(\ x, y, \varepsilon\ )}{\partial \varepsilon} - y^4 - \right. \\ \left. \frac{\partial}{\partial \varepsilon} L^{-1} \left\{ \frac{1}{12 s^{\alpha}} L \left(x^2 \frac{\partial^2 v_n}{\partial x^2} + y^2 \frac{\partial^2 v_n}{\partial y^2} \right) \right\} d\varepsilon \right. \tag{39}$$

the initial iteration

$$v_0(x, y, 0) = x^4 + y^4 t \tag{40}$$

using the equation in equation (38), we have

$$v_0(x, y, t) = x^4 + y^4 t \tag{41}$$

$$v_1(x, y, t) = x^4 + y^4 t + x^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{42}$$

$$v_2(x, y, t) = x^4 + y^4 t + x^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + x^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + y^4 \frac{t^{2\alpha+1}}{\Gamma(\alpha+2)} \tag{43}$$

assumed that the solution was in closed form by

$$v(x, y, t) = \lim_{n \to \infty} v_n(x, y, t) = x^4 E_{\alpha}(t^{\alpha}) + y^4 E_{\alpha,2}(t^{\alpha}) \tag{44}$$

where $E_{\alpha,2}(t^{\alpha})$ is the Mittag- Laffler Function defined in equation (6) letting $\alpha = 2$ then

$$v(x, y, t) == x^4 cosht + y^4 sinht \tag{45}$$

Numerical and Graphical discussion

In this part we found a record for numerical explanation of equation (42) and plot some graphs for $\alpha = 0.25, 0.5, 0.75, 1$.

Table 5: The values of $v(x, y, t)$ for $\alpha = 0.25$

$\alpha = 0.25$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	10.723461	322.384991	2448.587477	9388.799277	25645.15124
4	13.010819	111.097371	305.270476	595.530113	981.876342
6	30.834622	470.086827	3457.001819	13208.40076	36049.51540
8	42.943473	524.703270	3814.269888	14553.73738	39709.24682
10	56.013753	578.426352	4130.832026	15728.39172	42893.84687

Table 6:The values of $v(x,y,t)$ for $\alpha = 0.50$

$\alpha = 0.50$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	10.723461	378.384991	2878.483393	11040.56935	30158.96889
4	25.274780	605.815447	4553.491981	17441.49478	47629.60945
6	44.819765	825.936021	6137.526560	23478.30743	64096.35272
8	69.213075	1044.536135	7676.732939	29328.90486	80045.70396
10	98.356569	1263.816429	9188.943467	35062.15232	95666.06497

Table 7:The values of $v(x,y,t)$ for $\alpha = 0.75$

$\alpha = 0.75$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	9.535236	406.142637	3103.072959	11907.75724	32531.34205
4	23.880947	831.523787	6323.495098	24253.16615	66250.59382
6	44.109711	1342.276394	10169.80983	38989.11017	106493.7576
8	70.420350	1926.202514	14545.52123	55743.88528	152244.5578
10	103.057275	2575.614244	19389.00163	74279.76634	202852.7288

Table 8:The values of $v(x,y,t)$ for $\alpha = 1$

$\alpha = 1.0$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	10.33	410.33	3130.33	12010.33	32810.33
4	35.66	1075.66	8147.66	31235.66	85315.66
6	85	2085	15685	60085	164085
8	166.33	3446.33	25750.33	98566.33	269126.33
10	287.66	5167.66	38351.66	146687.66	400447.66

The solution is graphically presented in Figures 5,6,7, and 8 for various fractional orders of α

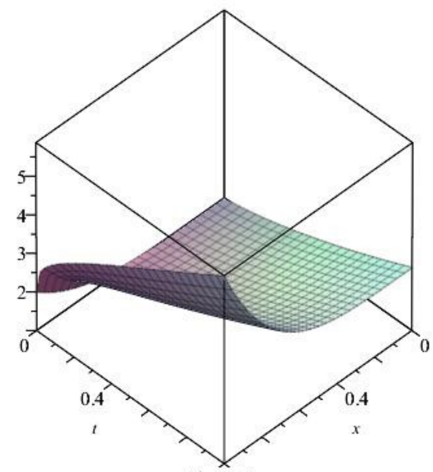


Figure 5
α=0.25

Figur 5: The behaviour of $v(x, y, t)$ w.r.t. x and t for $\alpha = 0.25$

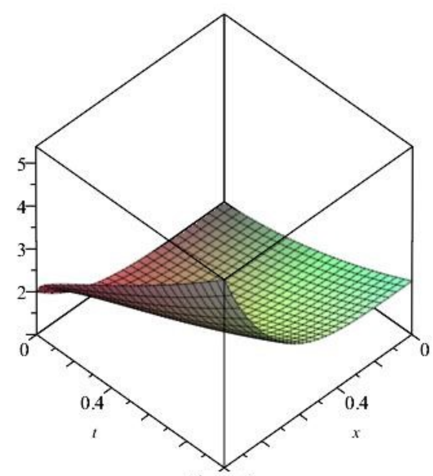
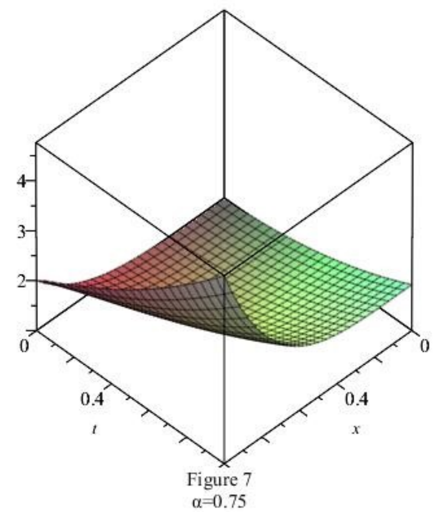
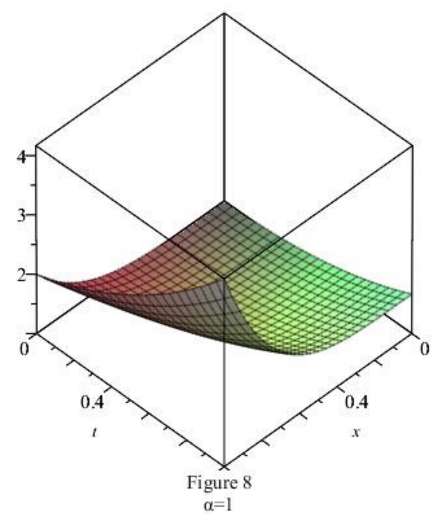


Figure 6
α=0.5

Figur 6: The behaviour of $v(x, y, t)$ w.r.t. x and t for $\alpha = 0.50$



Figur 7: The behaviour of $v(x, y, t)$ w.r.t. x and t for $\alpha = 0.75$



Figur 8: The behaviour of $v(x, y, t)$ w.r.t.x and t for $\alpha = 1$

Problem 3 Deliberate the succeeding 3-D fractional wave-like equation:

$$D_t^{\alpha} v(\, x, y, z, t\,) = x^2 + y^2 + z^2 + \frac{1}{2}\left[x^2 \frac{\partial^2 v}{\partial x^2} + y^2 \frac{\partial^2 v}{\partial y^2} + z^2 \frac{\partial^2 v}{\partial z^2}\right]; 1 < \alpha \leq 2 \quad (46)$$

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initial condition:

$$v_0(x,y,z,0)=0, v_t(x,y,z,0)=x^2+y^2-z^2 \tag{47}$$

taking the laplace transform of equation (45) and result obtained by equation (46) we obtain

$$\begin{aligned} L\left[v(\ x,y,z,t\ )\right]=\frac{x^2+y^2-z^2}{s^2}+\frac{1}{s^\alpha}L\left(x^2+y^2+z^2\right)+\\ \frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right] \end{aligned} \tag{48}$$

apply inverse Laplace transform we have

$$\begin{aligned} v(\ x,y,z,t\ )=t\left(x^2+y^2-z^2\right)+\left(x^2+y^2+z^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)}+\\ L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right]\right] \end{aligned} \tag{49}$$

differentiating Equation (48) regarding t, we have

$$\begin{aligned} \frac{\partial v}{\partial t}=\left(x^2+y^2-z^2\right)+\left(x^2+y^2+z^2\right)\alpha\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}+\\ \frac{\partial}{\partial t}\left\{L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right]\right]\right\} \end{aligned} \tag{50}$$

the correction functional for $\lambda=-1$ is given as follows

$$\begin{aligned} v_{n+1}\left(x,y,z,t\right)=v_n\left(x,y,z,t\right)-\int_0^t\left[\frac{\partial v_n\left(x,y,z,\varepsilon\right)}{\partial\varepsilon}-\left(x^2+y^2-z^2\right)-\right.\\ \left.\left(x^2+y^2+z^2\right)\frac{\alpha\varepsilon^{\alpha-1}}{\Gamma\left(\alpha+1\right)}-\frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v_n}{\partial x^2}+y^2\frac{\partial^2 v_n}{\partial y^2}+z^2\frac{\partial^2 v_n}{\partial z^2}\right]\right]\right\}\right]d\varepsilon \end{aligned} \tag{51}$$

the initial iteration

$$v_0(x,y,z,0)=(x^2+y^2-z^2)t+(x^2+y^2+z^2\ )\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{52}$$

then, we have

$$v_1(\ x,y,z,t\ ) = v_0(\ x,y,z,t\ ) - \int_0^t \left[\frac{\partial v_0(\ x,y,z,\ \varepsilon\ )}{\partial \varepsilon} - \left(x^2+y^2-z^2\right) - \left(x^2+y^2+z^2\right)\frac{\alpha\varepsilon^{\alpha-1}}{\Gamma\left(\alpha+1\right)} - \frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v_0}{\partial x^2}+y^2\frac{\partial^2 v_0}{\partial y^2}+z^2\frac{\partial^2 v_0}{\partial z^2}\right]\right]\right\} d\varepsilon \tag{53}$$

$$v_1(\ x,y,z,t\ ) = t\left(x^2+y^2-z^2\right) + \left(x^2+y^2+z^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \left(x^2+y^2-z^2\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \left(x^2+y^2+z^2\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{54}$$

$$v_2(\ x,y,z,t\ ) = v_1(\ x,y,z,t\ ) - \int_0^t \left[\frac{\partial v_1(x,y,z,\varepsilon)}{\partial \varepsilon} - \left(x^2+y^2-z^2\right) - \left(x^2+y^2+z^2\right)\frac{\alpha\varepsilon^{\alpha-1}}{\Gamma\left(\alpha+1\right)} - \frac{\partial}{\partial\varepsilon}\left\{L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v_1}{\partial x^2}+y^2\frac{\partial^2 v_1}{\partial y^2}+z^2\frac{\partial^2 v_1}{\partial z^2}\right]\right]\right\} d\varepsilon \tag{55}$$

$$v_2(\ x,y,z,t\ ) = t\left(x^2+y^2-z^2\right) + \left(x^2+y^2+z^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \left(x^2+y^2-z^2\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \left(x^2+y^2+z^2\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(x^2+y^2-z^2\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \left(x^2+y^2+z^2\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \tag{56}$$

assumed that the solution was in closed form by

$$v(\ x,y,z,t\ ) = limn \to \infty v_n(\ x,y,z,t\ ) = \ t\left(x^2+y^2-z^2\right)E_{\alpha,2}\left(t^\alpha\right) + \left(x^2+y^2+z^2\right)\left[E_\alpha\left(t^\alpha\right)-1\right] \tag{57}$$

where $E_\alpha\left(t^\alpha\right)$ and $E_{\alpha,2}(t^\alpha)$ are the Mittag-Laffer Function defined in equations (7) and (8) letting $\alpha = 2$ then

$$v(x,y,z,t) = (x^2+y^2)e^t + z^2e^{-t} - (x^2+y^2+z^2) \tag{58}$$

Numerical and Graphical discussion

In this part we found a record for numerical explanation of equation (51) and plot some graphs for different values of $\alpha = 0.25, 0.5, 0.75, 1$.

Table 9:The values of $v(x, y, z, t)$ **for** $\alpha = 0.25$

$\alpha = 0.25$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	8.513426	60.620834	164.835650	321.157875	529.587508
4	13.992796	109.935171	301.819920	589.647043	973.416542
6	20.636626	169.729635	467.915654	915.194682	36049.51540
8	28.188647	237.697827	656.716187	1285.243727	2123.280446
10	36.506662	312.559961	864.666558	1692.826455	27107.039651

Table 10:The values of $v(x, y, z, t)$ **for** $\alpha = 0.50$

$\alpha = 0.50$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	9.127692	66.149229	180.192304	351.256916	579.343065
4	21.018022	173.162200	477.450555	933.883089	1542.459800
6	38.05581275	326.502315	903.395319	23478.30743	64096.35272
8	60.021537	524.193835	1452.538432	2845.055328	4701.744521
10	86.788321	765.094894	2121.708039	4156.62775	6869.854046

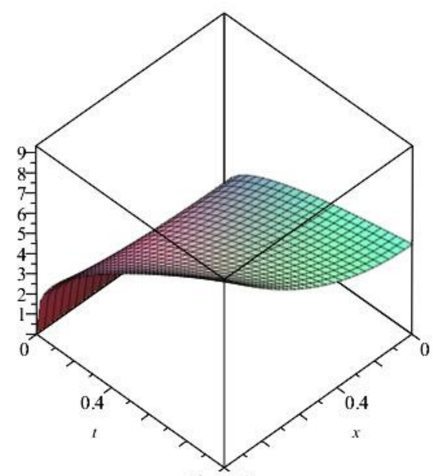
Table 11 The values of $v(x, y, z, t)$ **for** $\alpha = 0.75$

$\alpha = 0.75$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	8.793468	63.1412156	171.836710	334.879951	552.270940
4	23.663151	196.968364	543.578790	1063.494429	1756.71528
6	49.835450	432.519053	1197.886259	2345.937068	3876.671480
8	89.129443	786.164995	2180.236099	4271.342754	7059.484963
10	143.116946	1272.052519	3529.923663	6916.730379	903.395319

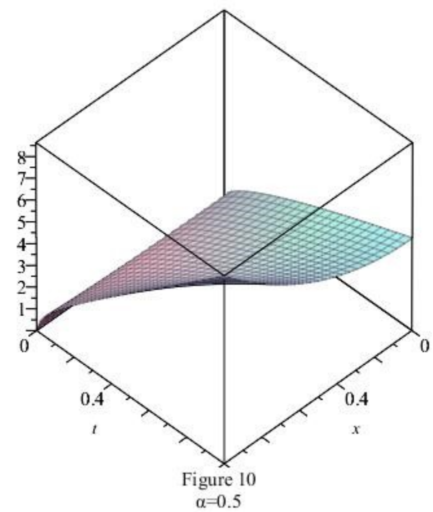
Table 12:The values of $v(x, y, z, t)$ for $\alpha = 1$

$\alpha = 1.0$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	8.3333	59	160.33	312.33	515
4	25.33	215	593.66	1161.66	1919
6	63	551	1527	2991	4943
8	128.33	1139	3160.33	6192.33	10235
10	229.66	2051	5693.66	11157.66	18443

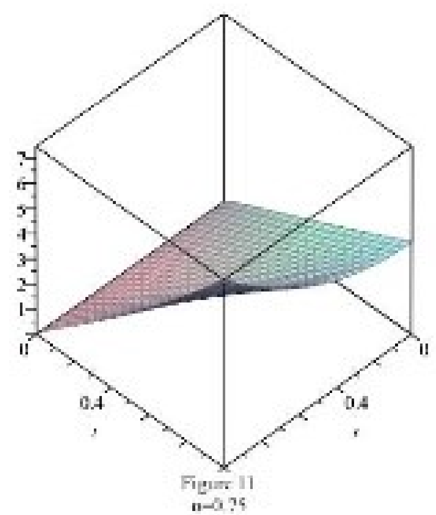
The solution is graphically presented in Figures 9,10,11, and 12 for various fractional orders of α



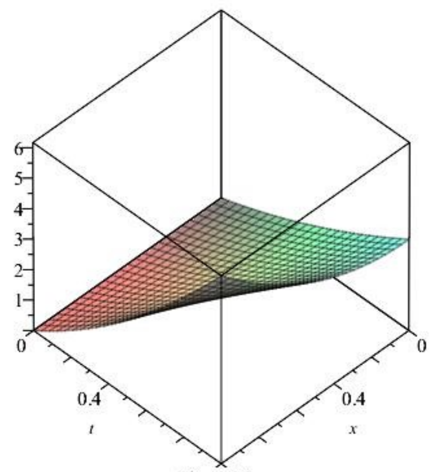
Figur 9: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 0.25$



Figur 10: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 0.50$



Figur 11: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 0.75$



Figur 12: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 1$

4 Conclusion:

We review the Laplace variational iteration method to show why it works well for obtaining approximate analytical solutions of nonlinear equations governing nonlinear phenomena. In the conferred document, the "Laplace Variational Iteration Method" is productively executed for the fractional wave equation, wherever we put in the fractional derivative in form of Caputo sense. The analytical, consequent, and comprehensive outcomes have been specified in expressions of a power series that come together to the exact solutions. The graphical consequences of the analysis are also manifested. In the future authors and scholars may use this paper for reference purposes and different values for parameters may be used for the graphical presentation so that the related phenomena may well be understood.

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