Two step Newton's method with multiplicative calculus to solve the non-linear equations

Gurjeet Singh[1,a], **Sonia Bhalla**[1,b]

[1] Department of Mathematics, Chandigarh University, Gharuan, Punjab, India, 140413
[a] Corresponding author: gs42477@gmail.com
[b] sonia.e8843@cumail.in

Abstract

For solving non-linear equations, iterative root-finding methods are important because of the broad range of applications in science and engineering. We have constructed an iterative method based on multiplicative calculus in this paper. Some numerical results are performed to exposed the efficiency of proposed and earlier method.

Keywords: Multiplicative calculus, Non linear equations, Iterative methods, Newton's-Raphson method, Order of convergence

1 Introduction

In the field of engineering and sciences, solving nonlinear equations effectively is one of the interesting task. Sometimes it is difficult to solve these problems. Then, we rely on iterative schemes to execute the root of non-linear function $g(t) = 0$. One of the popular methods for approximating the root of a non-linear function is the Newton's method [14] defined as

$$t_{q+1} = t_q - \frac{g(t_q)}{g'(t_q)}, \quad q = 0, 1, 2, 3... \tag{1}$$

The convergence order of Newton's method is two for the simple root. Several variants of Newton's method are developed to improve the convergence order in the literature such as Halley method [23], super-Halley method [16], Euler's method [21], Weerakoon and Fernando [22] etc. All of the above mentioned methods consist second-order derivatives except Weerakoon and Fernando. From 1964 to 2012, researchers [1],[9],[15],[24] has developed fourth-order methods to find the non-linear equations roots like Traub and Ostrowski [15], Chun and Ham [9], Cordero and Torregrosa [1], Kanwar et al. [24] etc. Out of them, Kanwar et. al. introduced a method which consists second-order derivatives while other listed methods have first-order derivative. Sometime it is difficult to achieve the second-order derivative at each step of the method. So some authors [10-11] developed second-order derivative free methods to solve the non-linear equations.

But still handling of first-order or second-order derivative in iterative techniques is difficult task. Nowdays, non-linear equations $g(t) + 1 = 1$ are solved using multiplicative calculus instead of function $g(t) = 0$. Initially, in 2008 Bashirov et al. [3] discussed the theoretical foundations and various applications of multiplicative calculus. In 2009 and 2011 Misirli and Gurefe [12], Riza et al. [18], and Ozyapici & Misirli [13] used multiplicative calculus to develop multiplicative numerical methods and in 2010 Filip and Piatecki [11] used it to examine economic growth and Uzer [8] extended the multiplicative calculus to include complex valued functions of complex variables, which was previously applicable only to positive real valued functions of real variables. In 2011 Bashirov et al. [4] used it to develop multiplicative differential equations. Bashirov & Riza [5] and in 2012 Florack and van Assen [17] used in biomedical image analysis. Currently, in 2016 Ozyapici, Sensoy and Karanfiller constructed a Multiplicative Newton's method. Keeping the same fact in mind, we consider the joint four-order multiplicative Newton's method.

This paper is structured as follows. Some basic terms of Multiplicative Calculus forms Section 2. As described in Section 3, a convergence analysis is conducted to determine the fourth-order of convergence of the proposed method. In Section 4, we presents comparisons of results obtained by proposed method with some other fourth-order methods. Finally, the conclusions form Section 5.

2 Some basic terms of Multiplicative Calculus

Definition: Let $g(t)$ be a real positive valued function in the open interval (a, b). Assume function be changes in $t \in (a, b)$ s.t. $g(t)$ changes in $g(t + h)$. Then [13] multiplicative forward operator denoted as Δ^* defined as follows

$$\Delta^* g(t) = \frac{g(t+h)}{g(t)} \tag{2}$$

By considring the operator Δ^* in (2), multiplicative derivative can be defined as below

$$g^*(t) = \lim_{h \to 0} (\Delta^* g)^{1/h} \tag{3}$$

The function $g^*(t)$ is said to be multiplicative differentiable at t if the limit on R.H.S exists.

If g is positive function and the derivative of g at t exist, then q^{th} multiplicative derivatives of g exist and

$$g^{*(q)}(t) = exp\left\{(ln \circ g)^{(q)}(t)\right\} \tag{4}$$

2

Theorem 1: (Multiplicative Taylor Theorem in one variable) [5] Let $g(t)$ be a function in open interval (a, b) s.t the functions is $q+1$ times $*$ differentiable on (a, b). Then for any $t, t+h \in A(a, b)$, there is a number $\theta \in (a, b)$ such that

$$g(t+h) = \prod_{p=0}^{n} \left(g^{*(p)}(t)\right)^{\frac{h^p}{p!}} . \left(g^{*(q+1)}(t+\theta h)\right)^{\frac{h^{q+1}}{(q+1)!}} \tag{5}$$

Theorem 2: (Multiplicative Newton's-Raphson method) [7] Assume that $g \in \mathcal{C}^2[a, b]$ and there exist a number $p \in [a, b]$ such that $g(p) = 1$. If $g^*(p) \neq 1$ and $h(t) = t - \dfrac{ln\ g(t)}{ln\ g^*(t)}$ then there exist a $\delta > 0$ such that the sequence $pk_{k=1}^{\infty}$ defined by iteration will converge to m for any initial value $p_0 \in [p-\delta, p+\delta]$

$$p_k = p_{k-1} - \frac{ln\ g(p_{k-1})}{ln\ g^*(p_{k-1})} \tag{6}$$

with error $e_{q+1} = b_2 e_q^2 + 2(b_3 - b_2^2)e_q^3 + \mathcal{O}(e_q^4)$

3 The Proposed Method and Analysis of Convergence

Here we constructed two step iterative method by considering first step as multiplicative Newton's-Raphson method and second step as considering ordinary Newton's-Raphson Scheme.

$$y_q = t_q - \frac{ln\ g(t_q)}{ln\ g^*(t_q)},$$

$$t_{q+1} = y_q - \frac{g(y_q)}{g'(y_q)}. \tag{7}$$

Where $q = 1, 2, 3, ...$ is the iteration level .
For convergence analysis, we have proved the following theorem.

Theorem 3: Suppose that for an open interval I, the function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ has only one root, $s \in I$. Let $g(t)$ be a sufficiently ordinary differentiable and then multiplicative differentiable in the neighborhood of s. Then the proposed method (7) has fourth-order of convergence.
Proof: Let s be the simple root of $g(t)$ and $e_q = t_q - s$. Consider the function $H(t) = t_{q+1}$ defined by

$$H(t) = y_q - \frac{g(y_q)}{g'(y_q)},$$

3

where

$$y_q = t_q - \frac{ln\ g(t_q)}{ln\ g^*(t_q)} \tag{8}$$

By using Mathematica version 11.1.1 with the fact that $y'(s) = 0$ from Theorem 2, the function $H(t)$ satisfies

$$H(s) = r \quad and \quad H^{(q)}(s) = 0, q = 1, 2, 3. \tag{9}$$

Thus, $H^{(4)}(s)$ can be given as

$$H^{(4)}(s) = \frac{3(g'(t)^2 - g''(t))^2 g''(t)}{g'(t)^3}$$

By Taylor expansion of $H(t_n)$ around s with condition (9), one obtain

$$t_{q+1} = H(t_q) = H(s) + \frac{H^{(4)}}{4!} e_q^4 + \mathcal{O}(e_q^5).$$

Hence,

$$e_{q+1} = \frac{H^{(4)}(s)}{4!} e_q^4 + \mathcal{O}(e_q^5)$$

Hence, the method (8) has fourth-order of convergence.

4 Numerical Examples

Several examples are given in this section to illustrate the applicability of the proposed method. The results of proposed method denoted as (PM) is also compared with earlier methods such as two-step Newton's Method [19] denoted as (NM), Chun method [10] denoted as (CM) and Maheshwari method [6] denoted as (MM) reprsented in Table 1 - Table 4. All computations can be done in Mathematica version 11.1.1 software and the stopping criteria $|t_{q+1} - t_q| < \epsilon$ and $\epsilon = 10^{-14}$ is used. The obtained results are compared for first three iterations. Moreover, the Approximated computational order of convergence(ACOC) is computed by using the following.

$$\rho \cong \frac{ln\left|\dfrac{t_{q+1} - s}{t_q - s}\right|}{ln\left|\dfrac{t_q - s}{t_{q-1} - s}\right|}.$$

4

Example 1: A fraction conversion problem is considered firstly, in which nitrogen-hydrogen feed is converted to ammonia fractionally. A temperature of 500°C and a pressure of 250 atm have been used in this problem. The nonlinear form of this problem is as follows::

$$g_1(t) = -0.186 - \frac{8t^2(t-4)^2}{9(t-2)^3}, \tag{10}$$

The simplified form of equation (10) is reduces to non-linear function as

$$g_1(t) = t^4 - 7.79075t^3 + 14.7445t^2 + 2.511t - 1.674 \tag{11}$$

Since the polynomial above has a degree of four, there must be exactly four roots. Due to its definition, fraction conversion lies in the interval (0,1), so there can be only one root in this interval, and that is 0.2777595428. Using the initial guess $t_0 = 0.4$ in Table 1, it is clear that our suggested method takes fewer iterations than others.

Example 2: Consider a Kepler's Equation

$$g_2(t) = t - \alpha_1 Sin(t) - K, \tag{12}$$

where $0 \leq \alpha_1 < 1$ and $0 \leq K \leq \pi$. We solve the equation by taking $K = 0.1$ and $\alpha_1 = 0.25$. For this set of values the root is 0.13320215082857313... which is approximated by proposed and earlier methods at the initial root $t_0 = 2$ and results are shown in Table 2.

Example 3: Problems of transcendental and algebraic nature. The following equations are used to numerically analyze the proposed technique:

$$(a) \qquad g_3(t) = e^{-t} + Cost, with\ exact\ root\ s = 1.7461. \tag{13}$$

$$(b)\ g_4(t) = te^{t^2} - Sin^2t + 3Cost - 4, with\ exact\ root\ s = 1.0651. \tag{14}$$

Table 3 and Table 4 shows the numerical outcomes starting with the initial guess 2.0 and 1.0 respectively. According to the numerical results, the proposed method requires fewer steps and reduces computation time.

5

Method	q	$\lvert t_q - t_{q-1}\rvert$	$\lvert g(t_q)\rvert$	ρ
NM	1	2.7402×10^{-1}	22.441	4.000
	2	2.7338×10^{-2}	9.6722×10^{-1}	
	3	3.4387×10^{-4}	4.3475×10^{-15}	
CM	1	2.5637×10^{-1}	22.4486	3.9987
	2	4.4424×10^{-2}	1.7056	
	3	5.9249×10^{-4}	1.8635×10^{-2}	
MM	1	2.5941×10^{-1}	22.4486	4.000
	2	4.1567×10^{-2}	1.5720	
	3	4.0948×10^{-4}	1.2868×10^{-2}	
PM	1	5.46149×10^{-1}	23.4486	3.999
	2	3.07797×10^{-2}	2.13315	
	3	3.44644×10^{-4}	1.0109	

Table 1: Fraction Conversion of Nitrogen-Hydrogen to Ammonia

Method	q	$\lvert t_q - t_{q-1}\rvert$	$\lvert g(t_q)\rvert$	ρ
NM	1	1.6621	1.5227	4.000
	2	6.5678×10^{-3}	5.0169×10^{-3}	
	3	2.8775×10^{-13}	2.1973×10^{-13}	
CM	1	1.6495	1.5226	4.000
	2	1.9131×10^{-2}	1.4623×10^{-2}	
	3	3.3555×10^{-10}	-2.5622×10^{-10}	
MM	1	1.6490	1.5226	4.000
	2	1.9652×10^{-2}	1.5022×10^{-2}	
	3	3.2303×10^{-10}	-2.4667×10^{-10}	
PM	1	1.66756	2.5226	4.000
	2	1.12051×10^{-3}	1.0008	
	3	9.0489×10^{-15}	1.0000	

Table 2: Kepler's Equation

6

Method	q	$\lvert t_q - t_{q-1}\rvert$	$\lvert g(t_q)\rvert$	ρ
NM	1	2.5389×10^{-1}	-2.8081×10^{-1}	4.000
	2	3.2728×10^{-5}	3.7935×10^{-5}	
	3	3.9104×10^{-21}	4.5325×10^{-21}	
CM	1	2.5424×10^{-1}	-2.8081×10^{-1}	4.000
	2	3.7917×10^{-4}	4.3955×10^{-4}	
	3	7.1375×10^{-16}	8.2731×10^{-16}	
MM	1	2.5418×10^{-1}	-2.8081×10^{-1}	4.000
	2	3.2177×10^{-4}	3.7299×10^{-4}	
	3	3.3377×10^{-16}	3.8688×10^{-16}	
PM	1	2.5398×10^{-1}	7.1919×10^{-1}	4.000
	2	1.1458×10^{-4}	1.0001	
	3	4.7761×10^{-18}	1.0000	

Table 3: $e^{-t} + Cost$

Method	q	$\lvert t_q - t_{q-1}\rvert$	$\lvert g(t_q)\rvert$	ρ
NM	1	4.2454×10^{-1}	103.12	3.8484
	2	3.6870×10^{-1}	13.838	
	3	1.3811×10^{-1}	1.3718	
CM	1	3.4554×10^{-1}	103.121	3.9221
	2	3.3105×10^{-1}	20.3079	
	3	2.1605×10^{-1}	3.4207	
MM	1	3.6468×10^{-1}	103.121	3.9615
	2	3.4233×10^{-1}	18.5257	
	3	2.0077×10^{-1}	2.7788	
PM	1	9.2639×10^{-1}	104.12	4.000
	2	8.4709×10^{-3}	1.0580	
	3	7.6367×10^{-9}	1.0000	

Table 4: $te^{t^2} - Sin^2 t + 3Cost - 4$

5 Conclusion

Here, we developed the Joint Multiplicative Newton's method which is mixture of multiplicative Newton's method and Ordinary Newton's method. We tested the proposed method for approximating the roots of nonlinear equations and compared it with ordinary methods. The obtained results are efiicient as compared with earlier ones in terms of residual error, consecutive error and order of convergence.

References

[1] A. Cordero, J. L. Hueso, E. M. Juan, R. Torregrosa, Steffensen type methods for solving nonlinear equations, J. Comput. Appl. Math. 236(12) 3058-3064 (2012).

[2] A. Cordero, N. Garrido, J. R. Torregrosa, P. T. Nvarro, Iterative schemes for finding all roots simultaneously of nonlinear equations, Appl. Math. Lett., Volume 134,(2022)

[3] A.E. Bashirov, E.M.Kurpınar, A.Ozyapıcı, Multiplicative calculus and its applications, J. Math. Anal. Appl. 337(1), 36–48 (2008)

[4] A.E. Bashirov, E. Mısırlı, Y. Tandogdu, A. Ozyapıcı, On modeling with multiplicative differential equations. Appl. Math. J Chin. Univ. 26(4), 425–438 (2011)

[5] A.E. Bashirov, M. Riza, On complex multiplicative differentiation. TWMS J. Appl. Eng. Math. 1(1), 51–61 (2011)

[6] A. K. Maheshwari, A fourth order iterative method for solving nonlinear equations, Appl. Math. Comput. 211(2) 383 –391 (2009)

[7] A Ozyapıcı, Effective Root-Finding Methods for Nonlinear Equations Based on Multiplicative Calculi, J. Math. (2016)

[8] A. Uzer, Multiplicative type complex calculus as an alternative to the classical calculus. Comput. Math. Appl. 60(10), 2725–2737 (2010)

[9] C. Chun and Y. Ham, Some fourth-order modifications of Newton's method, Appl. Math. Comput. 197(2) 654-658 (2008).

[10] C. Chun, M. Y. Lee, B. Neta, and J. Dzunic, On optimal fourth-order iterative methods free from second derivative and their dynamics, Appl. Math. Comput. 218(11) 6427–6438 (2012).

[11] D.A. Filip, C. Piatecki, A non-Newtonian examination of the theory of exogenous economic growth, CNCSIS-UEFISCSU (project number PNII IDEI 2366/2008) and Laboratoire d'Economie d'Orleans (LEO) (2010)

[12] E. Mısırlı, Y. Gurefe, Multiplicative Adams–Bashforth–Moulton methods. Numer. Algo. 57(4), 425–439 (2011)

[13] E. Mısırlı, A. Ozyapıcı, Exponential approximations on multiplicative calculus. Proc. Jangjeon Math. Soc. 12(2), 227–236 (2009)

[14] E. Suli and D. Mayers, An Introduction to Numerical Analysis. Cambridge University Press. ISBN 0-521-00794-1. (2003).

[15] J. F. Traub, Prentice-Hall, Iterative methods for the solution of equations, Englewood Cliffs, New Jersey, 1964.

[16] J. Gutierrez and M. Hernandez, An accerlation of Newton's method, Appl. Math. Comput. 117(2-3) 223-239 (2001).

[17] L. Florack, H. van Assen, Multiplicative calculus in biomedical image analysis, J. Math. Imaging Vis. 42(1), 64–75 (2012)

[18] M. Riza, A. Ozyapıcı, E. Mısırlı, Multiplicative finite difference methods. Q. Appl. Math. 67(4),745 (2009)

[19] M. Ruiz, A. Alberto, I. K. Argyros, Two-step Newton methods. J. Complex, 30(4), 533–553 (2014)

[20] M. Shams, N. Rafiq, N. Kausar, On iterative techniques for estimating all roots of nonlinear equation and its system with application in differential equation. Adv Differ Equ 2021, 480 (2021)

[21] S. Amat, S. Busquier, and J. M. Gutierrez,Geometric Construction of Iterative Functions to Solve Nonlinear Equations, J. Comput. Appl. Math. 157(1), 197-205 (2003).

[22] S. Weekaroon and TGI. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. 13 87-93 (2002).

[23] V. Kanwar and S.K. Tomar, Modified families of Newton, Halley and Chebyshev methods, Appl. Math. Comput. 192 20-26 (2007).

[24] V. Kanwar, R. Behl and K. K. Sharma, Simply constructed family of a Ostrowski's method with optimal order of convergence, J. Comput. Appl. Math. 62(11) 4021-4027 (2011).