An $M/G/1$ Feedback retrial queue with working vacation and a waiting server

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Abstract

An $M/G/1$ feedback retrial queue with working vacation and a waiting server is taken into consideration in this study. Both retrial times and service times are assumed to follow general distribution and the waiting server follows an exponential distribution. During the working vacation period customers are served at a lesser rate of service. Before going for a vacation the server waits for some arbitrary amount of time and so is called a waiting server. We obtain the probability generating function (PGF) for the number of customers and the mean number of customers in the invisible waiting area by utilizing the supplementary variable technique. We compute the mean waiting time. Out of interest a few special cases are conferred. Numerical outcomes are exhibited.

Keywords: Retrial queue, Working vacation, Supplementary variable technique, Waiting server, Feedback.

Mathematics Subject Classification 2010: 60K25, 90B22

1 Introduction

Retrial queues are expressed by the fact that if a customer observed that the server is occupied then they are entered into the invisible waiting area called an orbit. In recent years numerous researchers have examined the retrial queue. For a more in-depth analysis of the retrial queues, we can refer[1,2,3].
In Queueing theory queueing models with server vacation has a most impactful application. In addition to the vacation strategy Servi and Finn [4] developed a newest vacation strategy, called as Working Vacation (WV). In the WV period the server provides a lesser rate of service to the customers than during the regular service period. Wu and Takagi [5] examined M/G/1/Multiple Working

Vacation (MWV). Pazhani Bala Murugan and Santhi[7] studied the $M/G/1$ retrial queue with MWV. For a comprehensive study on WV we can refer [8]. Whenever the system becomes empty the server leaves from the regular service period (RS) and goes on a WV, but in a waiting server model the server wait for a arbitrary amount of time before going to WV.

The server wait option is representative of many queueing mechanisms used in real life, especially when interacting with people. For a detailed study on waiting server model we can refer [9,10,11]. In queueing models, some customers rejoin the orbit after receiving the service to receive it again out of dissatisfaction. This is referred to as feedback. For a broad analysis of feedback retrial queues we can refer[12,13,14,15].

In this article, we consider an $M/G/1$ feedback retrial queue with multiple WV and a waiting server. This article has the following structure. We explain the model in segment 2. In segment 3 performance measures are established. Segment 4 discusses some special cases. In segment 5 numerical outcomes are exhibited . The conclusion is given in segment 6.

2 Model Description

We examined an $M/G/1$ retrial queue with switch over time to working vacation where the primary customer's arrival follows a Poisson process with a rate of λ and service discipline is first-in-first-out(FIFO). If an approaching customer discovers that the server is occupied, then they exit the service area because we assume that there is no waiting area and they join the orbit. At a service completion instant, only the customer at the head of the orbit is permitted to approach the server. The retrial time follows a general distribution with a distribution function G(x) for the regular service period, let $g(x)$ and $G^*(\theta)$ signifies the pdf and LST respectively, and for WV period, let $L(x), l(x), L^*(\theta)$ signifies the distribution function, pdf and LST respectively. On the service completion epoch of each customer, if there is a contest between the primary customer and an orbit customer. The service time in the RS period follows a general distribution with the distribution function $R_s(x), r_s(x)$ and $R_s^*(\theta)$ as its pdf and LST respectively. The service delivered in the WV period follows a general distribution with $W_v(x), w_v(x), W_v^*(\theta)$ as its distribution function, pdf, LST. The server waits for an arbitrary period of time once the orbit turns empty, which follows an exponential distribution with a rate of α. After completion of the waiting time, the server goes for WV, which follows an exponential distribution with a rate of β. After getting the service, customers either rejoin the orbit with probability m or depart the system with probability $n(= 1-m)$. Inter-arrival times, retrial times, service time in RS periods and WV periods are all presumed to be independent of one another.

Let's use the subsequent random variables.

$O(t)$- Size of the orbit at time "t".

$G^0(t), R_s^0(t)$- the remaing retrial time and remaining service time in RS period.

$L^0(t), W_v^0(t)$- the remaing retrial time and remaining service time in WV period.

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At time "t" the four distinct states of the server are

$$E(t) = \begin{cases} 0 \text{ - if the server is not being occupied in WV} \\ 1 \text{ - if the server is not being occupied in RS period} \\ 2 \text{ - if the server is being occupied in WV} \\ 3 \text{ - if the server is being occupied in RS period} \end{cases}$$

so that the supplementary variables $L^0(t), G^0(t), W_v^0(t)$ and $R_s^0(t)$ are introduced in order to obtain the bivariate Markov Processes $\{(O(t), B(t)); t \geq 0\}$, where

$$B(t) = \begin{cases} L^0(t), & \text{if } E(t) = 0; \\ G^0(t), & \text{if } E(t) = 1; \\ W_v^0(t), & \text{if E(t)=2;} \\ R_s^0(t), & \text{if E(t)=3.} \end{cases}$$

$$\begin{aligned}
W_{0,0} &= \lim_{t\to\infty} P[O(t) = 0,\ E(t) \ = 0] \\
R_{0,0} &= \lim_{t\to\infty} P[O(t) = 0,\ E(t) \ = 1] \\
W_{0,h} &= \lim_{t\to\infty} P[O(t) = h,\ E(t) \ = 0,\ x < L^0(t) \ \leq x + dx];\ h \geq 1 \\
R_{0,h} &= \lim_{t\to\infty} P[O(t) = h,\ E(t) \ = 1,\ x < G^0(t) \ \leq x + dx];\ h \geq 1 \\
W_{1,h} &= \lim_{t\to\infty} P[O(t) = h,\ E(t) \ = 2,\ x < W_v^0(t) \ \leq x + dx];\ h \geq 0 \\
R_{1,h} &= \lim_{t\to\infty} P[O(t) = h,\ E(t) \ = 3,\ x < R_s^0(t) \ \leq x + dx];\ h \geq 0
\end{aligned}$$

The above mentioned are the limiting probabilities which we have defined.

$$R_s^*(\theta) = \int_0^\infty e^{-\theta x} r_s(x)dx \qquad W_v^*(\theta) = \int_0^\infty e^{-\theta x} w_v(x)dx$$

$$L^*(\theta) = \int_0^\infty e^{-\theta x} l(x)dx \qquad G^*(\theta) = \int_0^\infty e^{-\theta x} g(x)dx$$

$$W_{0,h}^*(\theta) = \int_0^\infty e^{-\theta x} W_{0,h}(x)dx \qquad W_{0,h}^*(0) = \int_0^\infty W_{0,h}(x)dx$$

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$$W_0^*(z,\theta) = \sum_{h=1}^{\infty} W_{0,h}^*(\theta) z^h \qquad W_0^*(z,0) = \sum_{h=1}^{\infty} W_{0,h}^*(0) z^h$$
$$W_0(z,0) = \sum_{h=1}^{\infty} W_{0,h}(0) z^h \qquad W_1^*(z,\theta) = \sum_{h=0}^{\infty} W_{1,h}^*(\theta) z^h$$
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The above mentioned are the Laplace Steiltjes Transform and PGF which we have defined.
In steady state the system was illustrated by the subsequent differential difference equations:

$$\lambda W_{0,0} = nW_{1,0}(0) + \alpha R_{0,0} \tag{1}$$
$$-\frac{d}{dx}W_{0,h}(x) = -(\beta+\lambda)W_{0,h}(x) + nW_{1,h}(0)l(x) + mW_{1,h-1}(0)l(x); \quad h \geq 1 \tag{2}$$
$$-\frac{d}{dx}W_{1,0}(x) = -(\lambda+\beta)W_{1,0}(x) + W_{0,1}(0)w_v(x) + \lambda W_{0,0}w_v(x) \tag{3}$$
$$-\frac{d}{dx}W_{1,h}(x) = -(\lambda+\beta)W_{1,h}(x) + \lambda W_{1,h-1}(x) + W_{0,h+1}(0)w_v(x) + \lambda\int_0^{\infty} W_{0,h}(x)dx\; w_v(x); h \geq 1 \tag{4}$$
$$(\lambda+\alpha)R_{0,0} = nR_{1,0}(0) \tag{5}$$
$$-\frac{d}{dx}R_{0,h}(x) = -\lambda R_{0,h}(x) + nR_{1,h}(0)g(x) + mR_{1,h-1}(0)g(x) + \beta\int_0^{\infty} W_{0,h}(x)dx\; g(x); \quad h \geq 1 \tag{6}$$
$$-\frac{d}{dx}R_{1,0}(x) = -\lambda R_{1,0}(x) + R_{0,1}(0)r_s(x) + \beta r_s(x)\int_0^{\infty} W_{1,0}(x)dx \tag{7}$$

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$$-\frac{d}{dx}R_{1,h}(x) = -\lambda R_{1,h}(x)+\lambda R_{1,h-1}(x)+\beta r_s(x)\int_0^\infty W_{1,h}(x)dx + R_{0,h+1}(0)r_s(x)+\lambda r_s(x)\int_0^\infty R_{0,h}(x)dx; \quad h\geq 1 \tag{8}$$

Taking the LST from (2) to (8) on both sides results

$$\theta W^*_{0,h}(\theta)-W_{0,h}(0) = (\lambda+\beta)W^*_{0,h}(\theta)-nW_{1,h}(0)L^*(\theta) -mW_{1,h-1}(0)L^*(\theta); \quad h\geq 1 \tag{9}$$

$$\theta W^*_{1,0}(\theta)-W_{1,0}(0) = (\lambda+\beta)W^*_{1,0}(\theta)-W_{0,1}(0)W^*_v(\theta) -\lambda W_{0,0}W^*_v(\theta) \tag{10}$$

$$\theta W^*_{1,h}(\theta)-W_{1,h}(0) = (\lambda+\beta)W^*_{1,h}(\theta)-W_{0,h+1}(0)W^*_v(\theta) -\lambda W^*_{1,h-1}(\theta)-\lambda W^*_{0,h}(0)W^*_v(\theta); \quad h\geq 1 \tag{11}$$

$$\theta R^*_{0,h}(\theta)-R_{0,h}(0) = \lambda R^*_{0,h}(\theta)-nR_{1,h}(0)G^*(\theta)-\beta G^*(\theta)W^*_{0,h}(0) -mR_{1,h-1}(0)G^*(\theta); \quad h\geq 1 \tag{12}$$

$$\theta R^*_{1,0}(\theta)-R_{1,0}(0) = \lambda R^*_{1,0}(\theta)-R_{0,1}(0)R^*_s(\theta)-\beta R^*_s(\theta)W^*_{1,0}(0) -\lambda R_{0,0}R^*_s(\theta) \tag{13}$$

$$\theta R^*_{1,h}(\theta)-R_{1,h}(0) = \lambda R^*_{1,h}(\theta)-\lambda R^*_{1,h-1}(\theta)-R^*_s(\theta)R_{0,h+1}(0) -\beta R^*_s(\theta)W^*_{1,h}(0)-\lambda R^*_s(\theta)R^*_{0,h}(0); \quad h\geq 1 \tag{14}$$

Summing over h from 1 to infinity $\times$ (9) with z^h and results,

$$W^*_0(z,\theta)[\theta-(\lambda+\beta)] = W_0(z,0)-L^*(\theta)[(n+mz)W_1(z,0) -nW_{1,0}(0)] \tag{15}$$

Summing over h from 1 to infinity $\times$ (11) with z^h and comprise with (10) results,

$$W^*_1(z,\theta)[\theta-(\lambda+\beta-\lambda z)] = W_1(z,0)-\frac{W^*_v(\theta)}{z}W_0(z,0) -\lambda W_{0,0}W^*_v(\theta)-\lambda W^*_v(\theta)W^*_0(z,0) \tag{16}$$

Placing $\theta=\lambda+\beta$ in (15), results

$$W_0(z,0) = L^*(\lambda+\beta)[(n+mz)W_1(z,0)-nW_{1,0}(0)] \tag{17}$$

Placing $\theta=0$ and (Sub.) (17) in (15), results

$$W^*_0(z,0) = \frac{(1-L^*(\lambda+\beta))((n+mz)W_1(z,0)-nW_{1,0}(0))}{\lambda+\beta} \tag{18}$$

Placing $\theta = \lambda + \beta - \lambda z$ and (Sub.) (17) and (18) in (16), results

$$W_1(z,0) = \frac{\begin{bmatrix} [\lambda z(\lambda+\beta)W_{0,0} - n[L^*(\lambda+\beta)(\lambda+\beta-\lambda z) \\ +\lambda z]W_{1,0}(0)]W_v^*(\lambda+\beta-\lambda z) \end{bmatrix}}{Dr_1(z)} \tag{19}$$

(Sub.)(19) in (17), results

$$W_0(z,0) = \frac{\begin{bmatrix} (\lambda+\beta)z[(n+mz)\lambda W_v^*(\lambda+\beta-\lambda z)W_{0,0} \\ -nW_{1,0}(0)]L^*(\lambda+\beta) \end{bmatrix}}{Dr_1(z)} \tag{20}$$

Let $f(z) = (\lambda+\beta)z - W_v^*(\lambda+\beta-\lambda z)(L^*(\lambda+\beta)(\lambda+\beta-\lambda z)+\lambda z)$, for $f(z) = 0$ we obtain $f(0) < 0$ and $f(1) > 0$ which $\Rightarrow$ that $\exists$ a real root $z_1 \in (0,1)$.
At $z = z_1$(20) is converted in to

$$W_{1,0}(0) = \lambda W_v^*(\lambda - \lambda z_1 + \beta)(n+mz)W_{0,0} \tag{21}$$

(Sub.) (21) in (19), results

$$W_1(z,0) = \frac{\lambda W_v^*(\lambda+\beta-\lambda z)UP(z)}{Dr_1(z)}W_{0,0} \tag{22}$$

where,

$$UP(z) = z(\lambda+\beta) - (n+mz)W_v^*(\lambda+\beta-\lambda z_1)[\lambda z + L^*(\lambda+\beta)(\lambda+\beta-\lambda z)]$$

(Sub.) (21) in (20), results

$$W_0(z,0) = \frac{\begin{bmatrix} \begin{bmatrix} (n+mz)\lambda z(\lambda+\beta)[W_v^*(\lambda+\beta-\lambda z) \\ -W_v^*(\lambda-\lambda z_1+\beta)] \end{bmatrix} L^*(\lambda+\beta) \end{bmatrix}}{Dr_1(z)} \tag{23}$$

(Sub.) (21) and (22) in (18), results

$$W_0^*(z,0) = \frac{\begin{bmatrix} \begin{bmatrix} (1-L^*(\lambda+\beta))\lambda z(n+mz)[W_v^*(\lambda+\beta-\lambda z) \\ -W_v^*(\lambda+\beta-\lambda z_1)] \end{bmatrix} \end{bmatrix}}{Dr_1(z)} \tag{24}$$

Placing $\theta = 0$ and (Sub.) (22), (23) and (24) in (16), results

$$W_1^*(z,0) \ = \ \frac{W_{0,0}\lambda(1-W_v^*(\lambda+\beta-\lambda z))UP(z)}{(\lambda+\beta-\lambda z)Dr_1(z)} \tag{25}$$

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Summing over h from 1 to infinity $\times$ (12) with z^h and results

$$\begin{aligned} R_0^*(z,\theta)(\theta-\lambda) &= R_0(z,0) - G^*(\theta)[(n+mz)R_1(z,0) - nR_{1,0}(0)] \\ &\quad -W_0^*(z,0)\beta G^*(\theta) \end{aligned} \tag{26}$$

(Sub.) $W_{1,0}(0) = (n+mz)\lambda W_v^*(\lambda+\beta-\lambda z_1)W_{0,0}$ in (1), we get

$$\alpha R_{0,0} = \lambda(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))W_{0,0}$$

Placing $\theta = \lambda$ and (Sub.) $R_{1,0}(0) = \lambda(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))W_{0,0} - \lambda R_{0,0}$ in (26), results

$$\begin{aligned} R_0(z,0) &= [(n+mz)R_1(z,0) - \lambda(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))W_{0,0} \\ &\quad -\lambda R_{0,0} + \beta W_0^*(z,0)]G^*(\lambda) \end{aligned} \tag{27}$$

Summing over h from 1 to infinity $\times$ (14) with z^h and comprise with (13) results

$$R_1^*(z,\theta)[\theta-\lambda+\lambda z)] = R_1(z,0) - \left[\frac{R_0(z,0)}{z} + \beta W_1^*(z,0) + \lambda R_0^*(z,0) + \lambda R_{0,0}\right] R_s^*(\theta) \tag{28}$$

Placing $\theta = 0$ and (Sub.) (27) and
$nR_{1,0}(0) = (1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))\lambda W_{0,0} - \lambda R_{0,0}$ in (26), results

$$\begin{aligned} R_0^*(z,0) &= \Big[(n+mz)R_1(z,0) - (1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))\lambda W_{0,0} \\ &\quad -\lambda R_{0,0} + \beta W_0^*(z,0)\Big]\Big[\frac{(1-G^*(\lambda))}{\lambda}\Big] \end{aligned} \tag{29}$$

Placing $\theta = \lambda - \lambda z$ and (Sub.) (27) and (29) in (28), results

$$R_1(z,0) = \frac{\left[\begin{aligned} R_s^*(\lambda-\lambda z)\Big[&\beta z W_1^*(z,0) + \beta[(1-z)G^*(\lambda)+z]W_0^*(z,0) \\ &-[(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))\lambda W_{0,0} + \lambda R_{0,0}] \\ &[(1-z)G^*(\lambda)+z] + \lambda z R_{0,0}\Big] \end{aligned}\right]}{z-(n+mz)R_s^*(\lambda-\lambda z)[z+G^*(\lambda)(1-z)]} \tag{30}$$

(Sub.) (30) in (27), results

$$R_0(z,0) = \frac{\left[\begin{aligned} zG^*(\lambda)\ \Big[&\beta(n+mz)R_s^*(\lambda-\lambda z)W_1^*(z,0) + \beta W_0^*(z,0) \\ &-\lambda(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))W_{0,0} \\ &-\lambda(1-(n+mz)R_s^*(\lambda-\lambda z))R_{0,0}\Big] \end{aligned}\right]}{z-(n+mz)R_s^*(\lambda-\lambda z)\big[(1-z)G^*(\lambda)+z\big]} \tag{31}$$

(Sub.) (30) in (29), results

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$$R_0^*(z,0) = \frac{\begin{bmatrix}(1-G^*(\lambda))z\ [\beta W_1^*(z,0)(n+mz)R_s^*(\lambda-\lambda z)+\beta W_0^*(z,0)\\ -\lambda(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))W_{0,0}\\ -\lambda(1-(n+mz)R_s^*(\lambda-\lambda z))R_{0,0}]\end{bmatrix}}{\lambda\big\{z-(n+mz)R_S^*(\lambda-\lambda z)\big[(1-z)G^*(\lambda)+z\big]\big\}} \tag{32}$$

Placing $\theta = 0$ and (Sub.) (30), (31) and (32) in (28), results

$$R_1^*(z,0) = \frac{\begin{bmatrix}\Big\{W_0^*(z,0)[G^*(\lambda)(1-z)+z]\beta+\lambda z R_{0,0}-[G^*(\lambda)(1-z)+z]\\ \times[\lambda(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))W_{0,0}+\lambda R_{0,0}]\\ +\beta W_1^*(z,0)z\Big\}(1-R_s^*(\lambda-\lambda z))\end{bmatrix}}{(\lambda-\lambda z)\ \Big[\ z-(n+mz)R_s^*(\lambda-\lambda z)[z+G^*(\lambda)(1-z)]\Big]} \tag{33}$$

We define $W_v(z) = W_0^*(z,0) + W_1^*(z,0) + W_{0,0}$

$$\begin{aligned}W_v(z) =& \frac{W_{0,0}}{(\lambda+\beta-\lambda z)D_1(z)}\Big\{\lambda z(W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1))\\ &\times(\lambda+\beta-\lambda z)(1-L^*(\lambda+\beta))+\lambda(1-W_v^*(\lambda+\beta-\lambda z))\\ &\times[z(\lambda+\beta)-[\lambda z+L^*(\lambda+\beta)(\lambda+\beta-\lambda z)]W_v^*(\lambda+\beta-\lambda z_1)\\ &\times(n+mz)]+(\lambda+\beta-\lambda z)[z(\lambda+\beta)-(n+mz)W_v^*(\lambda+\beta-\lambda z)\\ &\times(\lambda z+L^*(\lambda+\beta)(\lambda+\beta-\lambda z))]\Big\}\end{aligned} \tag{34}$$

when the server is on WV period, as the PGF for the number of customers in orbit.
(Sub.) (24) and (25) in (32), results

$$\begin{aligned}R_0^*(z,0) =& \frac{z(1-G^*(\lambda))W_{0,0}}{(\lambda+\beta-\lambda z)Dr_1(z)Dr_2(z)}\Big\{\beta(n+mz)(1-W_v^*(\lambda+\beta-\lambda z))\\ &\times R_s^*(\lambda-\lambda z)\big\{(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z_1)[\lambda z\\ &+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)]\big\}+\beta z(\lambda+\beta-\lambda z)(1-L^*(\lambda+\beta))\\ &\times(n+mz)(W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1))-(\lambda+\beta-\lambda z)\\ &\times(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))\big\{(\lambda+\beta)z-W_v^*(\lambda+\beta-\lambda z)\\ &\times(n+mz)[\lambda z+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)]\big\}-\frac{\lambda}{\alpha}(\lambda+\beta-\lambda z)\\ &\times(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))(1-(n+mz)R_s^*(\lambda-\lambda z))\\ &\times\big\{(\lambda+\beta)z-W_v^*(\lambda+\beta-\lambda z)[\lambda z+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)]\big\}\Big\}\end{aligned} \tag{35}$$

(Sub.) (24), (25) in (33), results

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$$
\begin{aligned}
R_1^*(z,0) \quad = \quad & \frac{(1-R_s^*(\lambda-\lambda z))W_{0,0}}{Dr_2(z)(\lambda+\beta-\lambda z)Dr_1(z)}\Big\{[\lambda z+G^*(\lambda)(\lambda+\beta-\lambda z)](n+mz) \\
& \times[1-L^*(\lambda+\beta)][W_v^*(\lambda+\beta-\lambda z)-W_v^*(\beta+\lambda-\lambda z_1)]\beta z \\
& -(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))[\lambda z+G^*(\lambda)(\lambda+\beta-\lambda z)] \\
& \times\big\{(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z)[\lambda z+(\lambda+\beta-\lambda z) \\
& \times L^*(\lambda+\beta)]\big\}+(W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1))\beta z \\
& \times(\lambda+\beta)L^*(\lambda+\beta)-\frac{\lambda}{\alpha}(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))G^*(\lambda) \\
& \times(\lambda+\beta-\lambda z)\big\{(\lambda+\beta)z-[\lambda z+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)] \\
& \times W_v^*(\lambda+\beta-\lambda z)(n+mz)\big\}+m\Big[[\lambda\beta z^2W_v^*(\lambda+\beta-\lambda z_1) \\
& +\beta^2 zW_v^*(\lambda+\beta-\lambda z_1)W_v^*(\lambda+\beta-\lambda z)L^*(\lambda+\beta) \\
& \times+W_v^*(\lambda+\beta-\lambda z_1)\lambda\beta zL^*(\lambda+\beta)W_v^*(\lambda+\beta-\lambda z)(1-z)] \\
& \times(n+mz)-[\beta^2z^2+\lambda\beta z^2]W_v^*(\lambda+\beta-\lambda z)\Big]\Big\} \qquad (36)
\end{aligned}
$$

We define $R_S(z)=R_0^*(z,0)+R_1^*(z,0)+R_{0,0}$

$$
\begin{aligned}
R_S(z) \quad = \quad & \frac{W_{0,0}}{(\lambda+\beta-\lambda z)(Dr_1(z)Dr_2(z))}\Big\{z(1-G^*(\lambda))\Big\{(1-W_v^*(\lambda+\beta-\lambda z)) \\
& \times R_s^*(\lambda-\lambda z)\beta(n+mz)\Big[(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z_1) \\
& \times[\lambda z+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)]\Big]+\beta z(1-L^*(\lambda+\beta))(\lambda+\beta-\lambda z) \\
& \times(n+mz)(W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1))-(\lambda+\beta-\lambda z) \\
& \times(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))\Big[(\lambda+\beta)z-W_v^*(\lambda+\beta-\lambda z) \\
& \times[\lambda z+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)](n+mz)\Big]\Big\}+(1-R_s^*(\lambda-\lambda z)) \\
& \times\Big\{[\lambda z+(\lambda+\beta-\lambda z)G^*(\lambda)][W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1)] \\
& \times\beta z(n+mz)[1-L^*(\lambda+\beta)]-(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1)) \\
& \times[\lambda z+G^*(\lambda)(\lambda+\beta-\lambda z)][(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z) \\
& \times[\lambda z+L^*(\lambda+\beta)(\lambda+\beta-\lambda z)]]+\beta z(\lambda+\beta)(W_v^*(\lambda+\beta-\lambda z) \\
& -W_v^*(\lambda+\beta-\lambda z_1))L^*(\lambda+\beta)\Big\}+m\Big[[\beta^2zW_v^*(\lambda+\beta-\lambda z_1) \\
& \times W_v^*(\lambda+\beta-\lambda z)L^*(\lambda+\beta)+\lambda\beta z^2W_v^*(\lambda+\beta-\lambda z_1)+(1-z) \\
& \times\lambda\beta zW_v^*(\lambda+\beta-\lambda z_1)L^*(\lambda+\beta)](n+mz)-[\beta^2z^2+\lambda\beta z^2] \\
& \times W_v^*(\lambda+\beta-\lambda z)\Big]+\frac{\lambda}{\alpha}(\lambda+\beta-\lambda z)(1-W_v^*(\lambda+\beta-\lambda z_1)) \\
& \times(n+mz)[(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z)[\lambda z+L^*(\lambda+\beta) \\
& \times(\lambda+\beta-\lambda z)][mG^*(\lambda)R_s^*(\lambda-\lambda z)-G^*(\lambda)](1-z)\Big\} \qquad (37)
\end{aligned}
$$

where,

$$Dr_1(z) = z(\lambda+\beta) - (n+mz)W_v^*(\lambda+\beta-\lambda z)[\lambda z + L^*(\lambda+\beta)(\lambda+\beta-\lambda z)] \quad (38)$$

$$Dr_2(z) = z - (n+mz)R_s^*(\lambda-\lambda z)[G^*(\lambda)(1-z)+z] \quad (39)$$

when the server is on RS period, as the PGF for the number of customers in the orbit.
Again, we define $R(z) = R_S(z) + W_v(z)$

$$\begin{aligned} R(z) = {} & \frac{W_{0,0}}{(\lambda+\beta-\lambda z)D_1(z)D_2(z)}\Bigg\{ z(1-G^*(\lambda))\Big[(1-W_v^*(\lambda+\beta-\lambda z))\beta \\ & \times(n+mz)R_s^*(\lambda-\lambda z)\big[(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z_1)[\lambda z \\ & +(\lambda+\beta-\lambda z)L^*(\lambda+\beta)]\big] + \beta z(\lambda-\lambda z)(n+mz)(1-L^*(\lambda+\beta)) \\ & \times(W_v^*(\lambda+\beta-\lambda z) - W_v^*(\lambda+\beta-\lambda z_1)) - (\lambda+\beta-\lambda z)(1-(n+mz) \\ & \times W_v^*(\lambda+\beta-\lambda z_1))\big[(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z)[L^*(\lambda+\beta) \\ & \times(\lambda+\beta-\lambda z)+\lambda z]\big]\Big] + (1-R_s^*(\lambda-\lambda z))\Big[[\lambda z+(\lambda+\beta-\lambda z)G^*(\lambda)] \\ & \times\beta z(n+mz)[1-L^*(\lambda+\beta)][W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1)] \\ & -(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))[\lambda z+G^*(\lambda)(\lambda+\beta-\lambda z)]\big[(\lambda+\beta)z \\ & -(n+mz)W_v^*(\lambda+\beta-\lambda z)[\lambda z+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)]\big] + (\lambda+\beta) \\ & \times\beta z(W_v^*(\lambda+\beta-\lambda z)) - W_v^*(\lambda+\beta-\lambda z_1))L^*(\lambda+\beta) + m\Big\{[\lambda\beta z^2 \\ & \times W_v^*(\lambda+\beta-\lambda z_1) + \beta^2 z W_v^*(\lambda+\beta-\lambda z_1)W_v^*(\lambda+\beta-\lambda z)L^*(\lambda+\beta) \\ & +W_v^*(\lambda+\beta-\lambda z_1)\lambda\beta z W_v^*(\lambda+\beta-\lambda z)L^*(\lambda+\beta)(1-z)](n+mz) \\ & -[\beta^2 z^2+\lambda\beta z^2]W_v^*(\lambda+\beta-\lambda z)\Big] + \frac{\lambda}{\alpha}(\lambda+\beta-\lambda z)(1-(n+mz) \\ & \times W_v^*(\lambda+\beta-\lambda z_1))[(\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z)[\lambda z \\ & +(\lambda+\beta-\lambda z)L^*(\lambda+\beta)][mG^*(\lambda)R_s^*(\lambda-\lambda z)-G^*(\lambda)](1-z)\Big] \\ & +\Big[(\lambda+\beta-\lambda z)(W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1))(1-L^*(\lambda+\beta)) \\ & \times\lambda z+\lambda(1-W_v^*(\lambda+\beta-\lambda z))[z(\lambda+\beta)-(n+mz)W_v^*(\lambda+\beta-\lambda z_1) \\ & \times(\lambda z+L^*(\lambda+\beta)(\lambda+\beta-\lambda z))] + (\lambda+\beta-\lambda z)[z(\lambda+\beta)-(n+mz) \\ & \times W_v^*(\lambda+\beta-\lambda z)(\lambda z+L^*(\lambda+\beta)(\lambda+\beta-\lambda z))]\Big]\Big[z-R_s^*(\lambda-\lambda z) \\ & \times[z+(1-z)G^*(\lambda)]\Big]\Bigg\} \end{aligned} \quad (40)$$

where $Dr_1(z)$ and $Dr_2(z)$ are given in (38) and (39) as the PGF for the number of customers in the orbit. Make use of the normalizing condition $R(1) = 1$ to findout that $W_{0,0}$ araise in (41). Using L'Hospitals rule and (Sub.) $z = 1$ in

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(40) results,

$$W_{0,0} = \frac{1-\rho_s}{\left[\begin{array}{l}\left\{\dfrac{O}{\beta G^*(\lambda)[\lambda+\beta-W_v^*(\beta)(\lambda+\beta L^*(\lambda+\beta))]}\right\} \\ -\left\{\dfrac{P+mT}{G^*(\lambda)[\lambda+\beta-W_v^*(\beta)(\lambda+\beta L^*(\lambda+\beta))]}\right\} \\ +\left\{\dfrac{\beta W_v^*(\lambda+\beta-\lambda z_1)L^*(\lambda+\beta)(1-G^*(\lambda))}{G^*(\lambda)[\lambda+\beta-W_v^*(\beta)(\lambda+\beta L^*(\lambda+\beta))]}+Q\right\}\end{array}\right]} \tag{41}$$

$$R_{0,0} = \frac{\lambda}{\alpha}(1-W_v^*(\lambda+\beta-\lambda z_1))W_{0,0} \tag{42}$$

where,

$$\begin{aligned} O &= (\lambda-\lambda W_v^*(\lambda+\beta-\lambda z_1)+\beta)[\lambda+\beta G^*(\lambda) \\ &\quad -W_v^*(\beta)(\lambda+\beta L^*(\lambda+\beta))] \\ P &= \lambda E(R_s)W_v^*(\beta)[\lambda+\beta-W_v^*(\lambda+\beta-\lambda z_1)(\lambda+\beta L^*(\lambda+\beta))] \\ T &= \beta[L^*(\lambda+\beta)[W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1)] \\ &\quad -W_v^*(\lambda+\beta-\lambda z)[1-L^*(\lambda+\beta)W_v^*(\lambda+\beta-\lambda z_1)]] \\ &\quad -\frac{\lambda}{\beta^2}(1-W_v^*(\lambda+\beta-\lambda z_1))\left[1+\frac{\beta}{\alpha}\right] \\ Q &= \frac{\lambda}{\alpha}(1-W_v^*(\lambda+\beta-\lambda z_1))G^*(\lambda) \\ \rho_s &= \frac{\lambda E(R_s)}{G^*(\lambda)}-\frac{m}{G^*(\lambda)} \end{aligned}$$

$E(R_s)$ is the mean service time and the system's stability condition $\rho_s < 1$ is obtained from (41).

3 The Model's Performance Measures

Mean Orbit Length in WV period:
We assume that W_v - mean orbit size and L_v - mean waiting time of the customer in the orbit during WV period.
Then

$$\begin{aligned} W_v &= \frac{d}{dz}W_v(z)\Big|_{z=1} \\ &= \frac{d}{dz}[W_1^*(z,0)+W_0^*(z,0)]\Big|_{z=1} \\ &= \frac{d}{dz}\left[\frac{S(z)}{(\lambda+\beta-\lambda z)Dr_1(z)}+\frac{K(z)}{Dr_1(z)}\right]W_{0,0}\Big|_{z=1} \end{aligned}$$

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$$= \left[\frac{\left[\begin{array}{c} Dr_1(z)(\lambda+\beta-\lambda z)S'(z) \\ -S(z)[(\lambda+\beta-\lambda z)Dr_1'(z) - Dr_1(z)\lambda] \end{array}\right]}{Dr_1(z))^2(\lambda+\beta-\lambda z)} + \left[\frac{K'(z)Dr_1(z) - Dr_1'(z)K(z)}{(Dr_1(z))^2}\right] \right] W_{0,0}\Big|_{z=1}$$

where,

$$\begin{aligned} K(z) &= \lambda z(1-L^*(\lambda+\beta))(n+mz)[W_v^*(\lambda+\beta-\lambda z) - W_v^*(\lambda+\beta-\lambda z_1)] \\ Dr_1(z) &= z(\lambda+\beta) - (n+mz)W_v^*(\lambda+\beta-\lambda z)(L^*(\lambda+\beta) \\ &\quad (\lambda+\beta-\lambda z)+\lambda z) \\ S(z) &= \lambda(1-W_v^*(\lambda+\beta-\lambda z))\big[z(\lambda+\beta) - (n+mz)W_v^*(\lambda+\beta-\lambda z_1) \\ &\quad \times(L^*(\lambda+\beta)(\lambda+\beta-\lambda z)+\lambda z)\big] \end{aligned}$$

Differentiating $S(z), K(z)$ and $Dr_1(z)$ with respect to z, we get

$$\begin{aligned} S'(z) &= \lambda^2 W_v^{*'}(\lambda+\beta-\lambda z)[z(\lambda+\beta) - (n+mz)W_v^*(\lambda+\beta-\lambda z_1) \\ &\quad \times(\lambda+\beta-\lambda z)L^*(\lambda+\beta)+\lambda z] + (1-W_v^*(\lambda+\beta-\lambda z))\lambda \\ &\quad \times[\lambda+\beta-(n+mz)(\lambda-\lambda L^*(\lambda+\beta))W_v^*(\lambda+\beta-\lambda z_1)] \\ &\quad -m[(\lambda+\beta-\lambda z)L^*(\lambda+\beta)+\lambda z]W_v^*(\lambda+\beta-\lambda z_1) \\ K'(z) &= (1-L^*(\lambda+\beta))(n+mz)\lambda[W_v^*(\lambda+\beta-\lambda z) - W_v^*(\lambda+\beta-\lambda z_1)] \\ &\quad +(n+mz)\lambda z(1-L^*(\lambda+\beta))(-\lambda W_v^{*'}(\lambda+\beta-\lambda z)) \\ &\quad +m\lambda z(1-L^*(\lambda+\beta))[W_v^*(\lambda+\beta-\lambda z) - W_v^*(\lambda+\beta-\lambda z_1)] \\ Dr_1'(z) &= (\lambda+\beta) + (n+mz)\lambda W_v^{*'}(\lambda+\beta-\lambda z)(\lambda z + (n+mz)L^*(\lambda+\beta) \\ &\quad (\lambda+\beta-\lambda z)) - W_v^*(\lambda+\beta-\lambda z)(\lambda-\lambda L^*(\lambda+\beta)) \\ &\quad -m(\lambda z+(n+mz)L^*(\lambda+\beta)(\lambda+\beta-\lambda z))W_v^*(\lambda+\beta-\lambda z) \end{aligned}$$

At $z=1$ L_v turns

$$= \left[\left[\frac{\beta Dr_1(1)S'(1) - S(1)[\beta Dr_1'(1) - \lambda Dr_1(1)]}{(\beta Dr_1(1))^2}\right] + \left[\frac{Dr_1(1)K'(1) - K(1)Dr_1'(1)}{(Dr_1(1))^2}\right] \right] W_{0,0}$$

we know that $L_v = \dfrac{W_v}{\lambda}$
where,

$$\begin{aligned} S(1) &= \lambda(1-W_v^*(\beta))[\beta+\lambda-W_v^*(\lambda+\beta-\lambda z_1)(\lambda+\beta L^*(\lambda+\beta))] \\ S'(1) &= \lambda^2 W_v^{*'}(\beta)[\lambda+\beta-W_v^*(\lambda+\beta-\lambda z_1)(\lambda+\beta L^*(\lambda+\beta))] \\ &\quad +\lambda(1-W_v^*(\beta))[\lambda+\beta-W_v^*(\lambda+\beta-\lambda z_1)(\lambda-\lambda L^*(\lambda+\beta))] \\ &\quad -m(\lambda+\beta L^*(\lambda+\beta))W_v^*(\lambda+\beta-\lambda z_1) \end{aligned}$$

$$\begin{aligned}
K(1) &= \lambda(1-L^*(\lambda+\beta))(W_v^*(\beta)-W_v^*(\beta-\lambda z_1+\lambda)) \\
K'(1) &= \lambda(1-L^*(\lambda+\beta))\big[W_v^*(\beta)-W_v^*(\beta+\lambda-\lambda z_1)-W_v^{*'}(\beta)\lambda \\
&\quad +m[W_v^*(\beta)-W_v^*(\beta+\lambda-\lambda z_1)]\big] \\
Dr_1(1) &= \beta-(\lambda+\beta L^*(\lambda+\beta))W_v^*(\beta)+\lambda \\
Dr_1'(1) &= \beta+[\lambda W_v^{*'}(\beta)+mW_v^*(\beta)][\lambda+L^*(\lambda+\beta)\beta] \\
&\quad +\lambda-[\lambda-\lambda L^*(\lambda+\beta)]W_v^*(\beta)
\end{aligned}$$

Mean Orbit Length in RS period:
We assume that L_s - mean orbit size and W_s - mean waiting time of the customer in the orbit during WV period.

$$\begin{aligned}
L_s &= \frac{d}{dz}R_S(z)\Big|_{z=1} \\
&= \frac{d}{dz}[R_1^*(z,0)+R_0^*(z,0)]\Big|_{z=1} \\
&= \frac{d}{dz}\left[\frac{Nr_1(z)(1-G^*(\lambda))+Nr_2(z)Nr_3(z)}{Dr_1(z)(\lambda+\beta-\lambda z)Dr_2(z)}\right]W_{0,0}\Big|_{z=1}
\end{aligned}$$

$$L_s = \frac{\left[\begin{array}{r}\big[Dr_2'(z)2Nr_1'(z)(\lambda Dr_1(z)-(\lambda+\beta-\lambda z)Dr_1'(z))\\ +(\lambda+\beta-\lambda z)Dr_1(z)Nr_1''(z)(Dr_2'(z)-Dr_2''(z)Nr_1'(z))\big]\\ \times(1-G^*(\lambda))+2(\lambda+\beta-\lambda z)Nr_2'(z)Dr_2'(z)(Nr_3'(z)Dr_1(z)\\ -Nr_3(z)Dr_1'(z))+Nr_3(z)[2\lambda Nr_2'(z)Dr_2'(z)+(\lambda+\beta-\lambda z)\\ Dr_2'(z)Nr_2''(z)-(\lambda+\beta-\lambda z)Dr_2''(z)Nr_2'(z)]Dr_1(z)\end{array}\right]}{2(Dr_1(z)(\lambda+\beta-\lambda z)Dr_2'(z))^2}W_{0,0}\Big|_{z=1}$$

where,

$$\begin{aligned}
Nr_1(z) &= \beta z(n+mz)R_s^*(\lambda-\lambda z)(1-W_v^*(\lambda+\beta-\lambda z))\big\{(\lambda+\beta)z \\
&\quad -(n+mz)W_v^*(\lambda+\beta-\lambda z_1)[(\lambda+\beta-\lambda z)L^*(\lambda+\beta)+\lambda z]\big\} \\
&\quad +(\lambda+\beta-\lambda z)\beta z^2(n+mz)(1-L^*(\lambda+\beta))[W_v^*(\lambda+\beta-\lambda z) \\
&\quad -W_v^*(\lambda+\beta-\lambda z_1)]-(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))\big\{(\lambda+\beta) \\
&\quad \times z-(n+mz)W_v^*(\lambda+\beta-\lambda z)[(\lambda+\beta-\lambda z)L^*(\lambda+\beta)+\lambda z]\big\} \\
&\quad \times z(\lambda+\beta-\lambda z)-\frac{z\lambda}{\alpha}(1-(n+mz)R_s^*(\lambda-\lambda z))(\lambda+\beta-\lambda z) \\
&\quad \times(1-(n+mz)W_v^*(\lambda+\beta-\lambda z_1))\big\{(\lambda+\beta)z-(n+mz) \\
&\quad \times W_v^*(\lambda+\beta-\lambda z)[(\lambda+\beta-\lambda z)L^*(\lambda+\beta)+\lambda z]\big\} \\
Nr_2(z) &= (1-R_s^*(\lambda-\lambda z)) \\
Dr_1(z) &= (\lambda+\beta)z-(n+mz)W_v^*(\lambda+\beta-\lambda z)[L^*(\lambda+\beta) \\
&\quad \times(\lambda+\beta-\lambda z)+\lambda z] \\
Dr_2(z) &= z-(n+mz)R_s^*(\lambda-\lambda z)[(1-z)G^*(\lambda)+z]
\end{aligned}$$

$$\begin{aligned}
Nr_3(z) &= \beta z[(\lambda+\beta-\lambda z)G^*(\lambda)+\lambda z](1-L^*(\lambda+\beta))(W_v^*(\lambda+\beta-\lambda z)\\
&\quad -W_v^*(\lambda+\beta-\lambda z_1))-(1-W_v^*(\lambda+\beta-\lambda z_1))[(\lambda+\beta-\lambda z)\\
&\quad \times G^*(\lambda)+\lambda z]\{(\lambda+\beta)z-W_v^*(\lambda+\beta-\lambda z)(\lambda+\beta-\lambda z)L^*(\lambda+\beta)\\
&\quad +\lambda z)\}+\beta z(\lambda+\beta)(W_v^*(\lambda+\beta-\lambda z)-W_v^*(\lambda+\beta-\lambda z_1))\\
&\quad \times L^*(\lambda+\beta)-\frac{\lambda}{\alpha}(1-W_v^*(\lambda+\beta-\lambda z_1))G^*(\lambda)(\lambda+\beta-\lambda z)\\
&\quad \times\big\{(\lambda+\beta)z-W_v^*(\lambda+\beta-\lambda z)[\lambda z+(\lambda+\beta-\lambda z)L^*(\lambda+\beta)\big\}\\
&\quad +m\Big[\big[\lambda\beta z^2W_v^*(\lambda+\beta-\lambda z_1)+\beta^2 zW_v^*(\lambda+\beta-\lambda z_1)L^*(\lambda+\beta)\\
&\quad \times W_v^*(\lambda+\beta-\lambda z)+\lambda\beta zW_v^*(\lambda+\beta-\lambda z_1)W_v^*(\lambda+\beta-\lambda z)\\
&\quad \times L^*(\lambda+\beta)(1-z)\big](n+mz)-[\beta^2z^2+\lambda\beta z^2]W_v^*(\lambda+\beta-\lambda z)\Big]
\end{aligned}$$

At $z=1$ L_s turns

$$L_s=\frac{\begin{bmatrix}(1-G^*(\lambda))\big[2Nr_1'(1)Dr_2'(1)(\lambda Dr_1(1)-\beta Dr_1'(1))+\beta Dr_1(1)\\ (Dr_2'(1)Nr_1''(1)-Nr_1'(1)Dr_2''(1))\big]+2\beta Nr_2'(1)Dr_2'(1)\\ (Dr_1(1)Nr_3'(1)-Nr_3(1)Dr_1'(1))+Nr_3(1)Dr_1(1)[2\lambda\\ Nr_2'(1)Dr_2'(1)+\beta Dr_2'(1)Nr_2''(1)-\beta Nr_2'(1)Dr_2''(1)]\end{bmatrix}}{2(\beta Dr_1(1)Dr_2'(1))^2}W_{0,0}$$

as it is known that $W_s=\dfrac{L_s}{\lambda}$,
where,

$$\begin{aligned}
Nr_1'(1) &= -\beta\lambda W_v^*(\beta)+\beta\lambda E(R_s)(1-W_v^*(\beta))[\lambda+\beta-\beta W_v^*(\lambda+\beta-\lambda z_1)\\
&\quad \times L^*(\lambda+\beta)-\lambda W_v^*(\lambda+\beta-\lambda z_1)]-\beta^2L^*(\lambda+\beta)W_v^*(\beta)+\beta^2\\
&\quad \times L^*(\lambda+\beta)W_v^*(\lambda+\beta-\lambda z_1)+\lambda[\beta+\lambda-\beta W_v^*(\beta)L^*(\lambda+\beta)\\
&\quad -\lambda W_v^*(\beta)]-\lambda^2W_v^*(\lambda+\beta-\lambda z_1)+\lambda^2W_v^*(\lambda+\beta-\lambda z_1)W_v^*(\beta)\\
&\quad +\lambda\beta W_v^*(\lambda+\beta-\lambda z_1)W_v^*(\beta)L^*(\lambda+\beta)+\frac{\lambda}{\alpha}[-m+\lambda E(R_s)]\\
&\quad \times\beta(1-W_v^*(\lambda+\beta-\lambda z_1))\{(\lambda+\beta)-W_v^*(\beta)[\lambda+\beta L^*(\lambda+\beta)]\}\\
&\quad +\beta m\{(\lambda+\beta)-W_v^*(\beta)[\lambda+\beta L^*(\lambda+\beta)]\}W_v^*(\lambda+\beta-\lambda z_1)\\
Nr_3(1) &= (1-W_v^*(\beta))\{\beta G^*(\lambda)W_v^*(\lambda+\beta-\lambda z_1)(\lambda+\beta L^*(\lambda+\beta))-\beta\lambda\\
&\quad -\beta\lambda G^*(\lambda)-\beta^2G^*(\lambda)\}+(1-W_v^*(\lambda+\beta-\lambda z_1))\{\beta\lambda W_v^*(\beta)\\
&\quad \times L^*(\lambda+\beta)-\lambda^2(1-W_v^*(\beta))\}+\beta^2L^*(\lambda+\beta)(W_v^*(\beta)\\
&\quad -W_v^*(\lambda+\beta-\lambda z_1))-\frac{\lambda}{\alpha}\beta G^*(\lambda)(1-W_v^*(\lambda+\beta-\lambda z_1))\\
&\quad \times\{\lambda+\beta-W_v^*(\beta)(L^*(\lambda+\beta)\beta+\lambda)\}+\big\{\lambda\beta W_v^*(\lambda+\beta-\lambda z_1)\\
&\quad -\beta W_v^*(\beta)[\lambda+\beta-\beta W_v^*(\lambda+\beta-\lambda z_1)L^*(\lambda+\beta)]\big\}m\\
Nr_2'(1) &= -\lambda E(R_s)
\end{aligned}$$

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$$
\begin{aligned}
Nr_1''(1) &= (W_v^*(\beta) - W_v^*(\lambda+\beta-\lambda z_1))[(1-L^*(\lambda+\beta))(\beta\lambda(1-G^*(\lambda)) \\
&\quad +\beta^2 G^*(\lambda)) + \beta^2 L^*(\lambda+\beta)] + G^*(\lambda)(\lambda+\beta+\lambda W_v^*(\beta)) \\
&\quad \times(\beta W_v^*(\lambda+\beta-\lambda z_1)-\lambda) - \beta\lambda L^*(\lambda+\beta)(1-W_v^*(\lambda+\beta-\lambda z_1)) \\
&\quad \times(W_v^*(\beta)+\beta W_v^{*'}(\beta)) + G^*(\lambda)W_v^*(\lambda+\beta-\lambda z_1)[\lambda+\beta-\beta W_v^*(\beta) \\
&\quad \times L^*(\lambda+\beta)+\lambda W_v^*(\beta)]\lambda - \beta G^*(\lambda)[\lambda+\beta-\lambda^2 W_v^{*'}(\beta)+\lambda W_v^*(\beta)] \\
&\quad +[\lambda W_v^{*'}(\beta)(\beta L^*(\lambda+\beta)-\lambda)+\lambda W_v^*(\beta)W^*(\lambda+\beta)][\beta G^*(\lambda) \\
&\quad \times W_v^*(\lambda+\beta-\lambda z_1) - \lambda(1-W_v^*(\lambda+\beta-\lambda z_1))] + \frac{\lambda}{\alpha}G^*(\lambda) \\
&\quad \times(1-W_v^*(\lambda+\beta-\lambda z_1))\big\{\lambda(\lambda+\beta)-\lambda^2 W_v^*(\beta)-2\lambda\beta W_v^*(\beta) \\
&\quad \times L^*(\lambda+\beta)-\lambda^2\beta W_v^{*'}(\beta)-\lambda\beta^2 W_v^{*'}(\beta)L^*(\lambda+\beta)+\lambda\beta W_v^*(\beta) \\
&\quad -\beta(\lambda+\beta)\big\} + m\Big[\beta W_v^*(\lambda+\beta-\lambda z_1)\Big[[L^*(\lambda+\beta)\beta+\lambda] \\
&\quad \times[3(W_v^*(\beta)-W_v^{*'}(\beta)-\lambda E(R_s))+m(1-W_v^*(\beta))-4] \\
&\quad -3(1-W_v^*(\beta))[\lambda-\lambda L^*(\lambda+\beta)]\Big] + \beta(\lambda+\beta)\Big[2(1-W_v^*(\beta) \\
&\quad +\lambda W_v^{*'}(\beta))+\lambda E(R_s)[2+W_v^*(\beta)]\Big] + \beta[1-L^*(\lambda+\beta)][2(2\beta-\lambda) \\
&\quad \times[W_v^*(\beta)-W_v^*(\lambda+\beta-\lambda z_1)]-\lambda\beta W_v^{*'}(\beta)] - \lambda\beta^2[1-L^*(\lambda+\beta)] \\
&\quad \times W_v^*(\beta)+(\lambda+\beta)W_v^*(\lambda+\beta-\lambda z_1)(3\beta-2\lambda)+[L^*(\lambda+\beta)\beta+\lambda] \\
&\quad \times\Big[\beta\lambda W_v^{*'}(\beta)[3W_v^*(\lambda+\beta-\lambda z_1)-2]+W_v^*(\beta)[W_v^*(\lambda+\beta-\lambda z_1) \\
&\quad \times(2\lambda-m\beta-\beta)+\beta]\Big]\frac{\lambda}{\alpha}\Big[\{(\lambda+\beta)(1-W_v^*(\lambda+\beta-\lambda z_1))[\beta+2\beta \\
&\quad -\lambda+\lambda\beta E(R_s)]-2\beta m W_v^*(\lambda+\beta-\lambda z_1)\}+\Big\{[\lambda\beta(W_v^{*'}(\beta) \\
&\quad -E(R_s))-m\beta W_v^*(\beta)-2\beta+\lambda](1-W_v^*(\lambda+\beta-\lambda z_1))+2\beta m \\
&\quad \times W_v^*(\lambda+\beta-\lambda z_1)\Big\}[L^*(\lambda+\beta)\beta+\lambda]-\beta(1-W_v^*(\lambda+\beta-\lambda z_1)) \\
&\quad \times[\lambda-\lambda L^*(\lambda+\beta)]W_v^*(\beta)\Big]\Big] \\
Dr_1(1) &= \beta+\lambda-(\lambda+\beta L^*(\lambda+\beta))W_v^*(\beta) \\
Dr_1'(1) &= \beta+[\lambda W_v^{*'}(\beta)+mW_v^*(\beta)][\lambda+L^*(\lambda+\beta)\beta] \\
&\quad +\lambda-[\lambda-\lambda L^*(\lambda+\beta)]W_v^*(\beta) \\
Dr_2'(1) &= G^*(\lambda)-E(R_s)\lambda-m \\
Dr_2''(1) &= -2(1-G^*(\lambda))[\lambda E(R_s)+m]-E(R_s^2)\lambda^2-2\lambda m E(R_s) \\
Nr_2''(1) &= -\lambda^2 E(R_s^2)
\end{aligned}
$$

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$$\begin{aligned} Nr_3'(1) &= \{(1-W_v^*(\beta))(2\beta\lambda E(R_s)+\beta\lambda^2E(R_s^2))+2\beta\lambda^2E(R_s)W_v^{*'}(\beta)\} \\ &\times[\lambda+\beta-W_v^*(\lambda+\beta-\lambda z_1)(\lambda+\beta L^*(\lambda+\beta))]+2\beta\lambda E(R_s) \\ &\times(1-W_v^*(\beta))[\lambda+\beta+\lambda W_v^*(\lambda+\beta-\lambda z_1)(L^*(\lambda+\beta)-1)] \\ &+2\beta\lambda[W_v^{*'}(\beta)(\lambda+\beta L^*(\lambda+\beta))+(1-W_v^*(\beta))(2-L^*(\lambda+\beta) \\ &\times W_v^*(\lambda+\beta-\lambda z_1))]+2\beta^2L^*(\lambda+\beta)(W_v^*(\lambda+\beta-\lambda z_1) \\ &-W_v^*(\beta))+2\lambda^2(1-W_v^*(\beta))(1-W_v^*(\lambda+\beta-\lambda z_1))+2\lambda[\lambda \\ &+\lambda W_v^{*'}(\beta)(\lambda+\beta L^*(\lambda+\beta))+\lambda W_v^*(\beta)(L^*(\lambda+\beta)-1)] \\ &\times(1-W_v^*(\lambda+\beta-\lambda z_1))+\frac{\lambda^2}{\alpha}(1-W_v^*(\lambda+\beta-\lambda z_1))\big\{[\lambda+\beta \\ &-W_v^*(\beta)[\lambda+\beta L^*(\lambda+\beta)]][4\beta+\lambda\beta-1]E(R_s)+2\beta\lambda W_v^{*'}(\beta) \\ &\times[\lambda+\beta L^*(\lambda+\beta)]E(R_s)+\lambda\beta E(R_s^2)[\lambda+\beta-W_v^*(\beta) \\ &(\lambda+\beta L^*(\lambda+\beta))]\big\} \end{aligned}$$

4 Special cases

(a) If the service time distribution follows an exponential distribution, no retrial, no service among the vacation period and there is no feedback then, the present model will be remodeled as analysis of M/M/1 queue with server vacations and a waiting server.
(b) If the server does not wait after the completion of the RS period and there is no feedback then, the present model will be remodeled as an M/G/1 retrial queue with multiple working vacation.

5 Numerical results

The curved graph constructed in Figure 1 and the values tabulated in the Table 1 are obtained by setting the fixed values $\mu_v = 3.6$, $\mu_s = 9.8$, $\mu_{vr} = 3.5$, $\mu_{sr} = 5.3$, $\alpha = 1$, $m = 0.4$ and altering the values of λ from 1 to 2 incremented with 0.2 and increasing the values of β from 0.3 to 1.1 in steps of 0.4, we observed that as λ rises L_v falls and hence the stability of the model is verified.

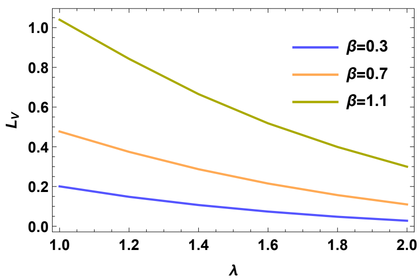

Figure 1: λ versus L_v

λ	$\beta = 0.3$	$\beta = 0.7$	$\beta = 1.1$
1.0	0.2006	0.4771	1.0399
1.2	0.1479	0.3744	0.8428
1.4	0.1066	0.2870	0.6651
1.6	0.0737	0.2149	0.5181
1.8	0.0475	0.1566	0.3990
2.0	0.0275	0.1097	0.2997

Table 1: λ versus L_v

The curved graph constructed in Figure 2 and the values tabulated in the Table 2 are obtained by setting the fixed values $\mu_v = 6.6$, $\mu_s = 10.8$, $\mu_{vr} = 3.5$, $\mu_{sr} = 5.3$, $\alpha = 1.7$, $m = 0.3$ and altering the values of λ from 1 to 2 incremented with 0.2 and increasing the values of β from 0.2 to 0.6 in steps of 0.2. We observed that as λ rises W_v falls which is expected.

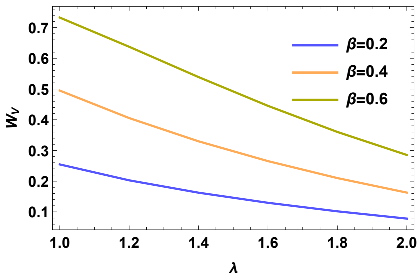

Figure 2: λ versus W_v

λ	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$
1.0	0.2545	0.4949	0.7325
1.2	0.2027	0.4054	0.6375
1.4	0.1624	0.3297	0.5388
1.6	0.1295	0.2652	0.4451
1.8	0.1018	0.2100	0.3602
2.0	0.0781	0.1628	0.2853

Table 2: λ versus W_v

The curved graph constructed in Figure 3 and the values tabulated in the Table 3 are obtained by setting the fixed values $\mu_v = 0.1$, $\mu_s = 10$, $\mu_{vr} = 1.5$, $\mu_{sr} = 4.5$, $\alpha = 0.6$, $m = 0.3$ and altering the values of λ from 1 to 2 incremented with 0.2 and increasing the values of β from 1.3 to 1.7 in steps of 0.2. We observed that as λ rises L_s also rises which shows the stability of the model.

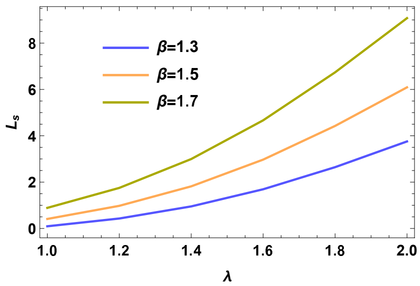

Figure 3: λ versus L_s

λ	$\beta = 1.3$	$\beta = 1.5$	$\beta = 1.7$
1.0	0.0939	0.4101	0.8883
1.2	0.4295	0.9762	1.7499
1.4	0.9517	1.8173	2.9956
1.6	1.6909	2.9705	4.6665
1.8	2.6447	4.4248	6.7373
2.0	3.7557	6.0909	9.0768

Table 3: λ versus L_s

The curved graph constructed in Figure 4 and the values tabulated in the Table 4 are obtained by setting the fixed values $\mu_v = 9.3$, $\mu_s = 11$, $\mu_{vr} = 4.5$, $\mu_{sr} = 5.5$, $\alpha = 1.5$, $m = 0.5$ and altering the values of λ from 1 to 2 incremented with 0.2 and increasing the values of β from 0.3 to 0.5 in steps of 0.1. From the graph, we studied that as λ rises W_s falls which shows the stability of the model.

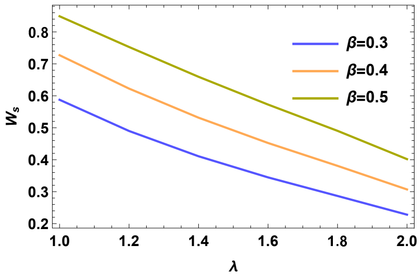

Figure 4: λ versus W_s

λ	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$
1.0	0.5875	0.7271	0.8489
1.2	0.4899	0.6225	0.7527
1.4	0.4107	0.5315	0.6595
1.6	0.3447	0.4523	0.5726
1.8	0.2865	0.3808	0.4904
2.0	0.2280	0.3069	0.4018

Table 4: λ versus W_s

6 Conclusion

In this paper, an $M/G/1$ feedback retrial queue with working vacation and a waiting server is evaluated. We obtained the PGF for the number of customers and the mean number of customers in the orbit. We worked out the mean waiting time. We also derived the performance measures. We performed some particular cases. We illustrated some numerical results.

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