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Approximate solutions of space and time fractional telegraph equations using Taylor series expansion method

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The study investigates the telegraph equations by considering space and time fractional derivatives. Caputo's concept of fractional derivatives is used here. We are focusing to generalize the solutions of integer order telegraph equations to fractional order telegraph equations. In this case, approximate solutions for fractional order telegraph equations have been obtained using Taylor series expansion. Additionally, it has been shown quantitatively how the solutions converge by using the number of terms in the series solutions.

Keywords

Telegraph equation, Caputo fractional derivative, Taylor series.

Mathematics Subject Classification

65M99

1 Introduction

Recent research suggests that fractional order derivatives are essential for a wide range of physical phenomena viz. rheology, damping law, heat-diffusion, wave dynamics, signal processing, etc. Researchers have used different types of analytical and numerical methods to handle such problems. Techniques including modified extended tanh method [1], novel analytical technique [2], coupled transformation method [3], Galerkin and collocation methods [4], etc. are some

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recent endeavors in this paradigm. Moreover, the semi-analytical homotopy perturbation method (HPM) has been used to tackle fractional models such as the heat-conduction equation [5], convection-diffusion problem [6], and wave equation [7] etc. Additionally, Dubey et al. have used local fractional natural homotopy analysis method [8, 9] and some coupling techniques such as the local fractional variational iteration technique with the local fractional natural transform [10] and local fractional homotopy perturbation method with local fractional natural transform operator [11] to solve various types of physical problems.

The studied of fractional telegraph equations (FTEs) have gained popularity due to various applications they can be applied to, such as modeling reaction diffusion, transmission and propagation of electrical signals, etc.

The general FTE [12] is given as,

$$D_x^p \phi(x,t) = a D_t^q \phi(x,t) + b D_t^r \phi(x,t) + c\phi(x,t) + h(x,t)$$

,

where $1 < p,\, q \leq 2,\, 0 < r \leq 1,\ x, t \geq 0,\, \phi(0,t) = f_1(t),\, \phi_x(0,t) = f_2(t)$ and a, b, c are constants.

Various authors have solved telegraph equations using different numerical and analytical techniques. The analytical solution for FTE with respect to time has been obtained by chen et al. [13] by using method of separating variables. The space-fractional telegraph equation (SFTE) and time-fractional telegraph equation (TFTE) and related telegraph process have been disscussed by Orsingher and Zaho [14] and Orsingher and Beghin [15]. Moreover, some other methods such as variational iteration method [16], HPM [17], differential transform method [18] have also been used to handle FTEs.

The Taylor series expansion method (TSEM) is applied by Demir et al. [19] for different fractional partial differential equation (PDE). In this study, we use Taylor series of an analytical solution of the integer order differential equation. This Taylor series solution can be reach out to the approximate or exact solution of fractional differential equation (FDE). Our approach changes the terms of Taylor series expansion for derivatives in the sense of fractionals and integers so that their relationship remains unaltered. Applications of this method demonstrate that it may be used to solve any differential equation derived from FDEs with ease and effectiveness, provided that the differential equation has an analytical or approximative solution. Here, the Caputo concept of the fractional derivative is utilised.

Here, Taylor series is applied on SFTEs and TFTEs. Firstly, the Taylor series expansion for analytical solution obtained from integer order telegraph equation is determined and then the expansion has been extended for FDE.

The rest of the paper follows this format: Methodology of the Taylor series expansion method is described in Section 2. Section 3 includes solutions of SFTEs and TFTEs. This section also covers convergency tables and graphical solutions. Finally, Section 4 concludes this article with a brief summary.

2 Taylor series expansion method

In this section, TSEM has been discussed to handle space and time fractional PDEs.

2.1 Fundamental approach to solve space-fractional PDEs

Let us consider a genaral form space-fractional PDE as

$$D_x^{\alpha}\phi(x,t) = \eta\left(\phi, \frac{\partial\phi}{\partial t}, \cdots \frac{\partial^n\phi}{\partial t^n}, x, t\right), \quad k-1 < \alpha \leq k,\ x > 0,\ t > 0. \tag{1}$$

To find the solution of Eq. (1), we must first calculate the solution to its integer order version by using the expression $\alpha = k$, which is represented as

$$D_x^{k}\phi(x,t) = \eta\left(\phi, \frac{\partial\phi}{\partial t}, \cdots \frac{\partial^n\phi}{\partial t^n}, x, t\right), \quad t > 0,\ x > 0. \tag{2}$$

From the exact answer of Eq. (1), one can get the approximation or exact solution (2). To do this, we must modify the terms in the Taylor series expansion of integer order differential equation (2). In the infinite Taylor series expansion of the solution of Eq. (2), the first k terms remain the same. Moreover, the fractional derivative with respect to x is used in place of the integer order derivative with respect to x in order to maintain the relationship between the terms of the Taylor series and to satisfy the boundary conditions of the fractional differential equation.

To solve Eq. (2) with respect to x, a primitive Taylor series form is shown below.

$$\phi(x,t) = \sum_{n=0}^{\infty} \frac{\partial^n\phi(0,t)}{\partial x^n}\frac{x^n}{n!}. \tag{3}$$

The solution of Eq. (1) is thus expressed in the following way:

$$\phi(x,t) = \sum_{n=0}^{k-1} \frac{\partial^n\phi(0,t)}{\partial x^n}\frac{x^n}{n!} + \sum_{n=1}^{\infty}\sum_{j=0}^{k-1} \frac{\partial^{kn+j}\phi(0,t)}{\partial x^{kn+j}}\frac{x^{n\alpha+j}}{\Gamma(n\alpha+j+1)}. \tag{4}$$

2.2 Fundamental approach to solve time-fractional PDEs

Let us consider a genaral form time-fractional PDE as

$$D_t^{\alpha}\phi(x,t) = \eta\left(\phi, \frac{\partial \phi}{\partial x}, \cdots \frac{\partial^n \phi}{\partial x^n}, x, t\right), \quad k-1 < \alpha \leq k,\, t > 0. \tag{5}$$

For obtaing the solution of Eq. (5), we have to determine the solution for integer order form of Eq. (5) by taking $\alpha = k$, which is written as

$$D_t^{k}\phi(x,t) = \eta\left(\phi, \frac{\partial \phi}{\partial x}, \cdots \frac{\partial^n \phi}{\partial x^n}, x, t\right), \quad t > 0. \tag{6}$$

If we alter the terms of the Taylor series expansion in the solution of the integer order differential equation, we can obtain the approximate or precise solution of Equation (5) from the exact solution of Eq. (6). The initial k terms of the infinite Taylor series expansion of solution of Eq. (6) are unaltered. Additionally, in order to maintain the relationship between the terms of the Taylor series and to satisfy the boundary conditions of the fractional differential equation, the integer order derivative with respect to t is substituted by the fractional derivative with respect to t.

For the solution of Eq. (6) with regard to t, the sketched Taylor series form is shown below.

$$\phi(x,t) = \sum_{n=0}^{\infty} \frac{\partial^n \phi(x,0)}{\partial t^n}\frac{t^n}{n!}. \tag{7}$$

Then the solution of Eq. (5) is given in the following format.

$$\phi(x,t) = \sum_{n=0}^{k-1} \frac{\partial^n \phi(x,0)}{\partial t^n}\frac{t^n}{n!} + \sum_{n=1}^{\infty}\sum_{j=0}^{k-1} \frac{\partial^{kn+j} \phi(x,0)}{\partial t^{kn+j}}\frac{t^{n\alpha+j}}{\Gamma(n\alpha+j+1)}. \tag{8}$$

3 Numerical examples

3.1 Solution of SFTE

Take a look at a general SFTE example [20].

$$\frac{\partial^{2\alpha}\phi}{\partial x^{2\alpha}} = \frac{\partial^2 \phi}{\partial t^2} + 4\frac{\partial \phi}{\partial t} + 4\phi \quad t \geq 0,\, 0 < \alpha \leq 1, \tag{9}$$

with initial conditions

$$\phi(0,t) = 1 + e^{-2t},\ \phi_x(0,t) = 2. \tag{10}$$

The exact solution of Eq. (9) with initial conditions (10) is given as

$$\phi(x,t) = e^{2x} + e^{-2t}. \tag{11}$$

The Taylor series expansion of the exact solution (11) is below.

$$\phi(x,t) = \left(e^{-2t} + 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots\right). \tag{12}$$

As discussed in the procedure, the solution to Eq. (9) is given by

$$\phi(x,t) = \left(1 + e^{-2t} + \frac{2x^\alpha}{\Gamma(\alpha+1)} + \frac{4x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{8x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right). \tag{13}$$

To demonstrate how well the approach works, we will check its convergence by increasing the number of terms in the series solution. In perspective of this, Convergency Table 1 contains values for the fractional value α, with $\alpha = 0.8$ and a fixed value of $t = 0.1$, while x ranges from 0 to 1.

Table 1: Convergency chart for the solution of the SFTE.

x	$\phi(x, t = 0.1)$						
	3 terms	5 terms	7 terms	9 terms	13 terms	16 terms	17 terms
0	1.818730	1.818730	1.818730	1.818730	1.818730	1.818730	1.818730
0.1	2.159061	2.240025	2.241460	2.241473	2.241473	2.241473	2.241473
0.2	2.411281	2.680719	2.694812	2.695188	2.695194	2.695194	2.695194
0.3	2.638323	3.195128	3.249704	3.252453	3.252541	3.252541	3.252541
0.4	2.850420	3.793880	3.937923	3.949311	3.949894	3.949895	3.949895
0.5	3.052055	4.483454	4.791251	4.825739	4.828286	4.828291	4.828291
0.6	3.245725	5.268850	5.843929	5.929557	5.938111	5.938144	5.938144
0.7	3.433014	6.154317	7.133256	7.318560	7.342541	7.342702	7.342703
0.8	3.615007	7.143644	8.699801	9.062343	9.121244	9.121873	9.121876
0.9	3.792495	8.240293	10.587472	11.244084	11.374845	11.376950	11.376964
1.0	3.966073	9.447488	12.843557	13.962366	14.230392	14.236641	14.236689

The surface of the graph in Fig. 1 depicts the exact solution of the telegraph equation given in Eq. (11) for $\alpha = 1$. Further, the surface of graphs in Figs. 2 and 3 show the approximate solutions of the SFTE for $\alpha = 0.5$ and 0.3 respectively.

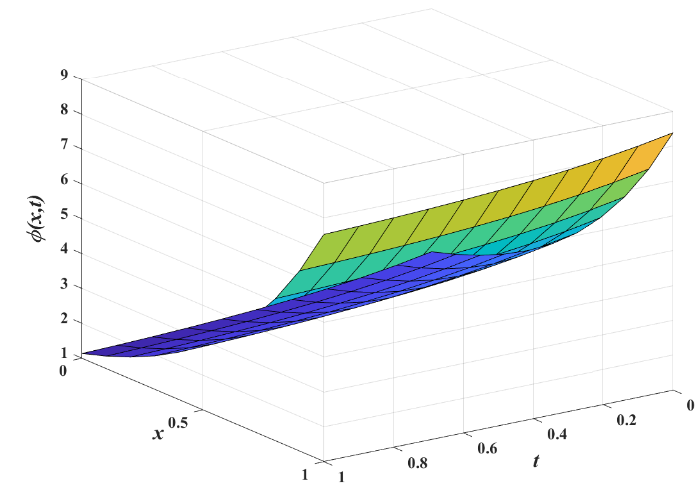

Figure 1: Exact solution of SFTE (11) for $\alpha = 1$

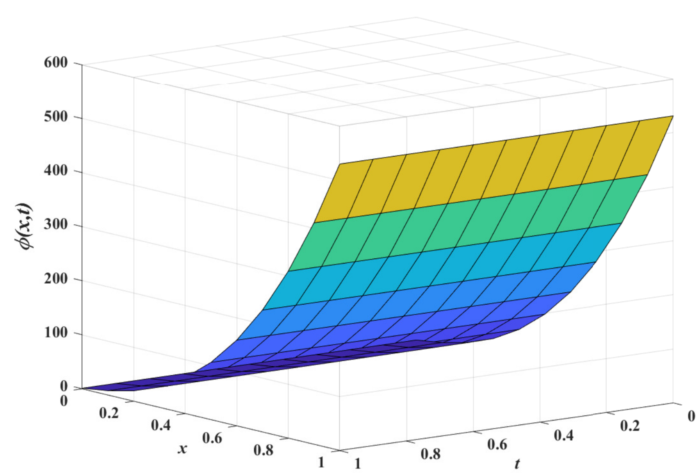

Figure 2: Approximate solution of SFTE (13) for $\alpha = 0.3$

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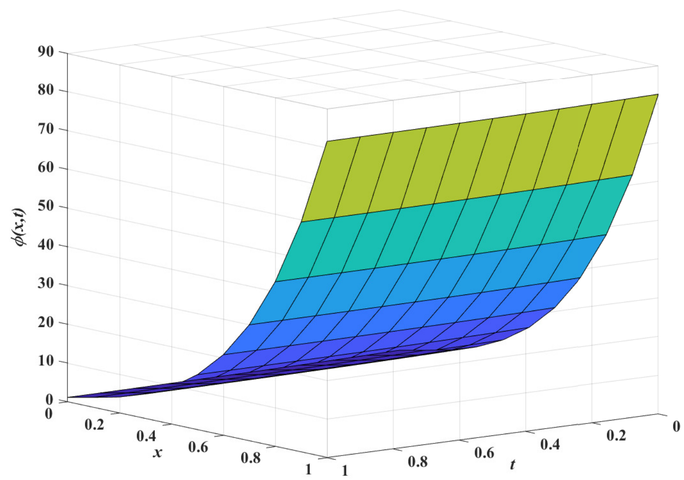

Figure 3: Approximate solution of SFTE (13) for $\alpha = 0.5$

3.2 Solution of TFTE

Take a look at the following TFTE [20],

$$\frac{\partial^\alpha \phi(x,t)}{\partial t^\alpha} = \frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{\partial \phi(x,t)}{\partial t} - \phi(x,t), \quad x,t \geq 0,\ 0 < \alpha \leq 2 \tag{14}$$

with initial conditions

$$\phi(x,0) = e^{-x}, \quad \phi_t(x,0) = -e^{-x}. \tag{15}$$

This equation has the following exact solution for $\alpha = 2$,

$$\phi(x,t) = e^{-(x+t)}. \tag{16}$$

The Taylor series expansion for exact solution (16) as below

$$\phi(x,t) = e^{-2t}\left(1 - t + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots\right). \tag{17}$$

The method allows us to arrive at the following solution to Eq. (14),

$$\phi(x,t) = e^{-x}\left(1 - t + \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right). \tag{18}$$

Here, we examine whether the approximation solution for the fractional value of α is convergent. The calculated values of $\phi(x,t)$ for $\alpha = 1.7, x = 0.1$, and t ranging from 0 to 1 are shown in Table 2. This table indicates that the computed result $\phi(x,t)$ converges up to four places of decimal in the approximation of the tenth term.

The graphical illustration for exact and approximate solutions has been shown. The graph in Fig. 4 has been plotted for the exact solution (16). In Figs. 5 and 6, the surfaces of the graphs show the approximate solutions (18) for $\alpha = 1.2$ and 1.7.

Table 2: Convergency chart for the solution of the TFTE.

t	$\phi(x=0.1,t)$						
	2 terms	3 terms	4 terms	5 terms	6 terms	7 terms	8 terms
0	0.9048374	0.9048374	0.9048374	0.9048374	0.9048374	0.9048374	0.9048374
0.1	0.8426171	0.8499774	0.8482249	0.8485619	0.8485076	0.8485151	0.8485142
0.2	0.7888023	0.8057117	0.8016857	0.8024599	0.8023350	0.8023524	0.8023503
0.3	0.7390122	0.7665189	0.7599697	0.7612292	0.7610260	0.7610542	0.7610508
0.4	0.6920779	0.7309256	0.7216762	0.7234549	0.7231680	0.7232079	0.7232030
0.5	0.6473983	0.6981743	0.6860848	0.6884097	0.6880347	0.6880868	0.6880804
0.6	0.6045998	0.6677937	0.6527476	0.6556410	0.6551744	0.6552392	0.6552313
0.7	0.5634244	0.6394591	0.6213556	0.6248370	0.6242755	0.6243535	0.6243440
0.8	0.5236826	0.6129314	0.5916817	0.5957681	0.5951090	0.5952006	0.5951894
0.9	0.4852285	0.5880266	0.5635508	0.5682577	0.5674985	0.5676040	0.5675911
1.0	0.4479456	0.5645982	0.5368238	0.5421650	0.5413035	0.5414232	0.5414086

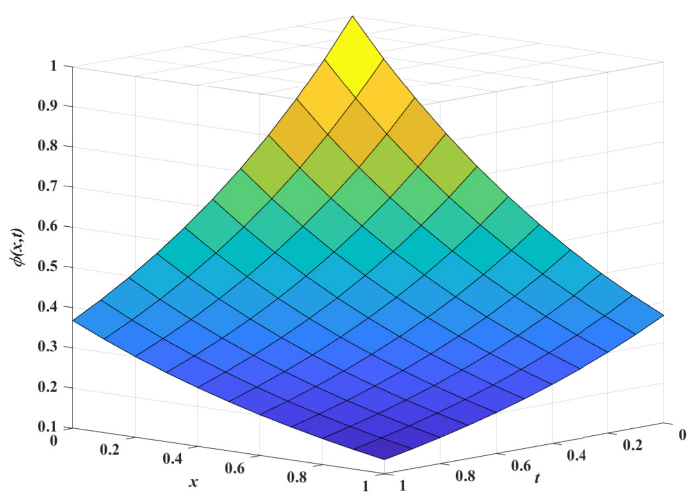

Figure 4: Exact solution of TFTE (16) for $\alpha = 2$

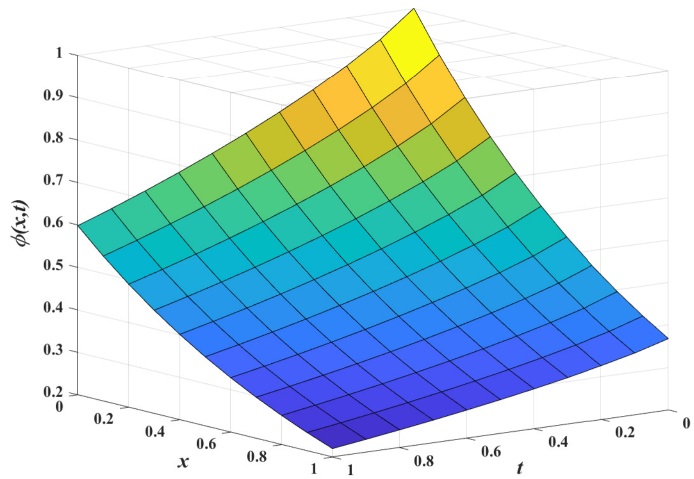

Figure 5: Approximate solution of TFTE (18) for $\alpha = 1.2$

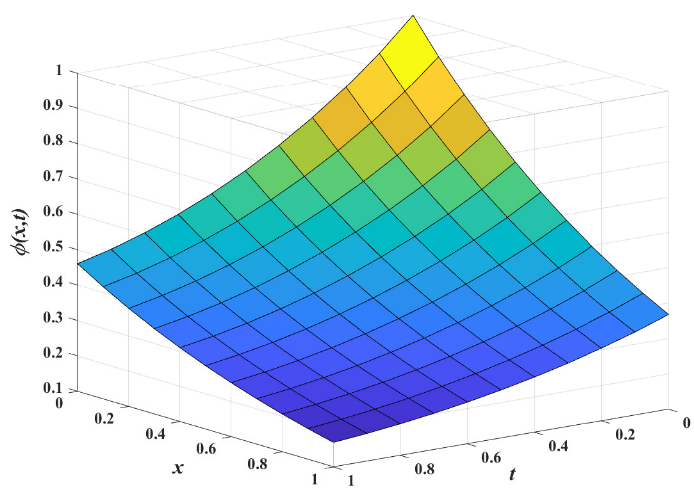

Figure 6: Approximate solution of TFTE (18) for $\alpha = 1.7$

10

4 Conclusion

If the integer order PDEs have analytical solutions, the Taylor series expansion method works well enough for fractional order PDEs. Numerical examples of this method demonstrate that it may be used to solve any differential equation derived from FDEs with ease and effectiveness, provided that the differential equation has an analytical or approximative solution. By extending the Taylor series expansion of the analytical solution, this study has determined the approximate solutions of the SFTE and TFTE. Furthermore, the infinitive series solution obtained in numerical examples 3.1 and 3.2 is identical to that given in [20], confirming the validity of the considered method. Additionally, the convergency Tables 1 and 2 demonstrate the efficacy of the method. As we can observe that the recorded values in Tables 1 and 2 are being closure enough if the number of terms increases. Furthermore, a graphic illustration is used to show both the exact and approximative solutions for integer and fractional values of α. Using the MATLAB programme, the 3D graphs for the solution of SFTE and TFTE are shown. The discussed method can also be used to analyse ODEs and PDEs with fractional derivatives with exponential and Mittag-Leffler kernels in future iterations of the research.

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