Characteristics of Mexican hat wavelet transform in a class of generalized quotient space

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Abstract

In this paper, Mexican hat wavelet transformation is defined on the space of tempered generalized quotients by employing the structure of exchange property. We study the exchange property for the Mexican hat wavelet transform by applying the theory of the Mexican hat wavelet transform of distributions. Further, different properties of Mexican hat wavelet transform are investigated on the space of tempered generalized quotients.

Key words: Wavelet transform; Exchange property; Distribution space; Tempered generalized quotient

Mathematics Subject Classification(2010): 44A15; 44A35; 46F99; 54B15

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1 Introduction

The wavelet transform $(Wf)(b,a)$ of a square integrable function f, is given by

$$(Wf)(b,a) = \int_{-\infty}^{\infty} f(t)\overline{\psi_{b,a}}(t)dt, \tag{1.1}$$

where

$$\psi_{b,a}(t) = (\sqrt{a})^{-1}\psi\left(\frac{t-b}{a}\right), \quad b, t \in \mathbb{R}^n, \text{ and } a > 0. \tag{1.2}$$

The inversion formula for (1.1) is given by

$$\frac{2}{C_\psi}\int_0^\infty \left[\int_{-\infty}^{\infty} (\sqrt{a})^{-1}(Wf)(b,a)\psi\left(\frac{x-b}{a}\right)db\right]\frac{da}{a^2} = f(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

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where the admissibility condition C_ψ is given by

$$\frac{C_\psi}{2} = \int_0^\infty \frac{|\hat{\psi}(u)|^2}{|u|} dv = \int_0^\infty \frac{|\hat{\psi}(-u)|^2}{|u|} du < \infty \qquad [3, \text{p. } 64].$$

The Mexican hat wavelet is constructed by taking the negative second derivative of a Gaussian function and is given by [24]

$$\psi(t) = e^{-(\frac{t^2}{2})}(1-t^2) = -\frac{d^2}{dt^2} e^{-(\frac{t^2}{2})} \tag{1.4}$$

such that

$$\psi_{b,a}(t) = -a^{\frac{3}{2}} D_t^2 e^{-\frac{(b-t)^2}{2a^2}}, \qquad \left(D_t = \frac{d}{dt}\right). \tag{1.5}$$

Thus, (1.1) can be reduced to

$$(Wf)(b,a) = -a^{\frac{3}{2}} \int_{\mathbb{R}} f(t)\ D_t^2 e^{-\frac{(b-t)^2}{2a^2}} dt, \qquad a \in \mathbb{R}_+ \tag{1.6}$$

which then, under certain conditions on f is

$$(Wf)(b,a) = -a^{\frac{3}{2}} \int_{\mathbb{R}} f^{(2)}(t)\ \ e^{-\frac{(b-t)^2}{2a^2}} dt, \qquad a \in \mathbb{R}_+. \tag{1.7}$$

Let a function $k_a(b-t)$ be defined by

$$k_a(b-t) = \frac{1}{\sqrt{2\pi a}} e^{\left(\frac{-(b-t)^2}{2a}\right)}, \tag{1.8}$$

where $t \in \mathbb{R},\ b = \sigma + i\omega$ and $a \in \mathbb{R}_+$. Then

$$D_t^2 k_{a^2}(b-t) = \frac{1}{\sqrt{2\pi a}} D_t^2 \left(e^{\frac{-(b-t)^2}{2a^2}}\right). \tag{1.9}$$

Therefore, by (1.5)

$$\psi_{b,a}(t) = -(2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} D_t^2 k_{a^2}(b-t)$$

and hence the Mexican hat wavelet transform is given by

$$\begin{aligned} (Wf)(b,a) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} f(t) D_t^2 k_{a^2}(b-t) dt \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} f^{(2)}(t) k_{a^2}(b-t) dt \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} (f^{(2)} * k_{a^2})(b), \quad b \in \mathbb{C},\ a \in \mathbb{R}_+, \end{aligned} \tag{1.10}$$

where $k_{a^2}(b-t) = \frac{1}{\sqrt{2\pi a}} e^{\frac{-(b-t)^2}{2a^2}}$.

The most general theory of the MHWT is investigated on the generalized function space $(\mathscr{W}_{\alpha,\beta}^{\gamma})'$ developed by Pathak *et al.* [8]. It is proved that the MHWT $(Wf)(b,a)$ of $f \in (\mathscr{W}_{\alpha,\beta}^{\gamma})'$, is given by $\langle f^{(2)}(t), k_{a^2}(b-t)\rangle$ is an analytic function in the strip $\frac{\alpha}{\gamma} < \operatorname{Re} b < \frac{\beta}{\gamma}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Recently, the wavelet transform has been comprehensively studied in many functions, distributions, and tempered distribution spaces. Several interesting properties and applications in generalized function spaces have been developed (See, for example, [6, 9, 10, 11, 12, 13, 17, 18, 19, 20, 21]. On the other hand, Mikusiński's algebraic approach gave a new transformation to the theory of functional analysis. The space of generalized quotients (Boehmians) is the recent generalization of the Schwartz distribution and the motivation for the expansion is in the core of Mikusiński operators. Its application to function spaces with the involvement of convolution provides different generalized function spaces. Hence, many integral transforms have been investigated in such spaces [1, 5, 7, 14, 15, 16, 22, 23].

Let $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}_+)$ be the spaces of functions with continuous derivatives which are rapidly decreasing on $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}_+$. The dual of $\mathscr{S}$ is represented by $\mathscr{S}'$ that is known as the space of tempered distributions. The spaces $\mathscr{S}$ and $\mathscr{S}'$ have been introduced and developed in [2]. The class $\mathscr{S}'$ of tempered distributions is contained in $(\mathscr{W}_{\alpha,\beta}^{\gamma})'$. Therefore the Mexican hat wavelet transform theory can be made applicable to $\mathscr{S}'$. Further, the Mexican hat wavelet transform can be expanded to the space of tempered generalized quotient, as the space is a natural expansion of tempered distributions. Here, we extend the Mexican hat wavelet transformation to a class of generalized quotient space that have quotients of sequences in the form of f_n/φ_n, where the numerator contains terms of the sequence from some set $\mathscr{S}'$ and the denominator is a delta sequence such that it satisfies the following condition

$$f_n * \varphi_m = f_m * \varphi_m, \quad \forall m, n \in \mathbb{N}. \tag{1.11}$$

Further, the delta sequences are defined as sequences of functions $\{\varphi_n\} \in \mathscr{S}$ that satisfies

1. $\int_{\mathbb{R}^n} \varphi_n(x) dx = 1$ for all $n = 1, 2, 3, \cdots$.

2. There exists a constant $C > 0$ such that
$$\int_{\mathbb{R}^n} |\varphi_n(x)|\, dx \leq C \text{ for all } n = 1, 2, 3, \cdots.$$

3. $\lim_{n\to\infty} \int_{\|x\|\geq\epsilon} \|x\|^k \, |(\varphi_j(x))|\, dx = 0$ for every $k \in \mathbb{N}$ and $\epsilon > 0$.

In particular, we extend the transformation to generalized quotient space by defining an exchange property for the Mexican hat wavelet transform. In the next section, we introduce some of the basic results required for the investigation of MHWT on the generalized quotient space. Section 3 describes some algebraic properties of MHWT in the context of tempered generalized quotients.

2 The exchange property

In this section, the space of tempered generalized quotients is constructed by applying the exchange property. This construction for generalized quotients indicates that the role of convergence is not necessary.

Theorem 2.1. *For a function $f \in \mathscr{S}'$ and $t \in \mathbb{R}$,*

$$(Wf)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} (f^{(2)} * k_{a^2})(b) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((f^{(2)} * k_{a^2}) e^{-\frac{t^2}{2n}})(b).$$

Proof. Consider,

$$\begin{aligned} (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((f^{(2)} * k_{a^2}) e^{-\frac{t^2}{2n}})(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} \int_{\mathbb{R}} f^{(2)}(t) k_{a^2}(b) e^{-\frac{t^2}{2n}} dt \\ &= a^{\frac{3}{2}} \lim_{n\to\infty} \int_{\mathbb{R}} f^{(2)}(t) e^{-\frac{(b-t)^2}{2a^2}} \; e^{-\frac{t^2}{2n}} dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} f^{(2)}(t) e^{-\frac{(b-t)^2}{2a^2}} dt. \end{aligned}$$

Therefore,

$$(Wf)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((f^{(2)} * k_{a^2}) e^{-\frac{t^2}{2n}})(b).$$

□

Theorem 2.2. *For $f \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$, we have*

$$(W(f*\varphi))(b,a) = (Wf)(b,a) * \varphi.$$

Proof. By using [4, Lemma 4.3.8], $(f*\varphi) \in \mathscr{S}'$ and hence $(W(f*\varphi))(b,a)$ is defined. Also, by Theorem 2.1

$$(W(f*\varphi))(b,a) \;=\; (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} (((f^{(2)}*\varphi) * k_{a^2}) e^{-\frac{t^2}{2n}})(b).$$

Consider,

$$\begin{aligned} (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} (((f^{(2)}*\varphi) * k_{a^2}) e^{-\frac{t^2}{2n}})(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} (f^{(2)}*\varphi)(t) k(b-t,a^2) e^{-\frac{t^2}{2n}} \; dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} (f^{(2)}*\varphi)(t) e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}} \; dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} \langle f^{(2)}(s), \varphi(t-s)\rangle e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}} \; dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} \langle f^{(2)}(s), \varphi(t-s)\rangle \psi_n(t) dt, \qquad (2.1) \end{aligned}$$

where $\psi_n(t) = e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}}$.

By [8, Lemma 4.3], we have

$$a^{\frac{3}{2}} \int_{-m}^{m} \langle f^{(2)}(s), \varphi(t-s)\rangle \psi_n(t)dt = a^{\frac{3}{2}} \left\langle f^{(2)}(s), \int_{-m}^{m} \varphi(t-s)\psi_n(t)dt \right\rangle, \ \forall m > 0,$$

which converges to

$$a^{\frac{3}{2}} \left\langle f^{(2)}(s), \int_{-m}^{m} \varphi(t-s)\psi_n(t)dt \right\rangle \text{ as } m \to \infty,$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \langle f^{(2)}(s), \varphi(t-s)\rangle e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}} \ dt &= \left\langle f^{(2)}(s), \int_{-\infty}^{\infty} \varphi(t-s)\psi_n(t) \ dt \right\rangle \\ &= \langle f^{(2)}(s), (\varphi * \psi_n)(s)\rangle. \qquad (2.2) \end{aligned}$$

Let us now consider,

$$\begin{aligned} (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} ((f^{(2)} * k_{a^2}) * \varphi)(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} (f^{(2)} * k_{a^2})(b-t)\varphi(t) \ dt \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{-M}^{M} \langle f^{(2)}(s), k_{a^2}(b-t-s)\rangle \varphi(t) \ dt, \end{aligned}$$

where supp $\varphi \subseteq [-P, P]$. Now by [8, Lemma 4.3],

$$\begin{aligned} (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} ((f^{(2)} * k_{a^2}) * \varphi)(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{-M}^{M} \langle f^{(2)}(s), k_{a^2}(b-t-s)\rangle \varphi(t) \ dt \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \left\langle f^{(2)}(s), \int_{-\infty}^{\infty} k_{a^2}(b-t-s)\varphi(t) \ dt \right\rangle \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \left\langle f^{(2)}(s), \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} a} \psi(t-s)\varphi(t) \ dt \right\rangle \\ &= a^{\frac{3}{2}} \left\langle f^{(2)}(s), \int_{-\infty}^{\infty} \psi(t-s)\varphi(t) \ dt \right\rangle \\ &= a^{\frac{3}{2}} \langle f^{(2)}(s), (\varphi * \psi)(s)\rangle. \qquad (2.3) \end{aligned}$$

From (2.2) and (2.3), we obtain

$$(W(f * \varphi))(b, a) = (Wf)(b, a) * \varphi.$$

□

Definition 2.3. For a family $\{\varphi_j\}_{j \in J}$, where $\varphi_j \in \mathcal{S}$, we define

$$M\bigg(\{\varphi_j\}_J \bigg) = \{x \in \mathbb{R}^n : \varphi_j(x) = 0, \ \ \forall j \in J\}. \qquad (2.4)$$

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A family of pairs $\{(f_j, \varphi_j)\}_J$, where $f_j \in \mathcal{S}'$ and $\varphi_j \in \mathcal{S}$, have the exchange property if

$$f_j * \varphi_k = f_k * \varphi_j, \forall j, k \in J. \tag{2.5}$$

Let set $\mathcal{A}$ denotes the collection of $\{(f_j, \varphi_j)\}_J$, where $f_j \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$, $\forall j \in J$, with exchange property such that $M\left(\{\varphi_j\}_J \right) = \emptyset$.

If $M\left(\{\varphi_j\}_J\right) = \emptyset$ and $M\left(\{\lambda_k\}_K\right) = \emptyset$, then $M\left(\{\varphi_j * \lambda_k\}_{J\times K}\right) = \emptyset$.

Theorem 2.4. *If $\{(f_j, \varphi_j)\}_J \in \mathcal{A}$, then there exists a unique $F \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}_+)$ such that F is the Mexican hat wavelet transform of the family of functions $\{(f_j, \varphi_j)\}_J$, i.e., $F = (W\{(f_j, \varphi_j)\}_J)$.*

Proof. Let us consider family of sequences $\{(f_j, \varphi_j)\}_J \in \mathcal{A}$, where $f_j \in \mathscr{S}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}$, $\forall j \in J$, with exchange property such that $|\varphi(x)| > \epsilon$, for some $\epsilon > 0$, and $x \in M\left(\{\varphi_j\}_J\right)^c$. Then, in some open neighborhood of x, we define

$$F = \frac{(W f_j)}{\varphi_j}. \tag{2.6}$$

Case 1: We show that for some open neighborhood of x we have a quotient F that is unique in that neighborhood, i.e., F does not depend on $j \in J$. Let U and V be some open neighborhood of x such that $|\varphi_j(x)| > \epsilon$, $\forall x \in U$ and $|\varphi_k(x)| > \epsilon$, $\forall x \in V$. Then since $\{(f_j, \varphi_j)\} \in \mathcal{A}$, hence it satisfy the exchange property and therefore,

$$f_j * \varphi_k = f_k * \varphi_j, \ \forall j, k \in J. \tag{2.7}$$

Applying Mexican hat wavelet transform to (2.7), we get

$$\begin{aligned} (W(f_j * \varphi_k)) &= (W(f_k * \varphi_j)) \\ (W f_j) * \varphi_k &= (W f_k) * \varphi_j \quad \text{(by Theorem 2.2)} \\ \frac{(W f_j)}{\varphi_j} &= \frac{(W f_k)}{\varphi_k}. \end{aligned} \tag{2.8}$$

Hence, we get a quotient $F = \dfrac{(W f_j)}{\varphi_j}$ on $U \cap V$.

Case 2: We need to show that $F \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}_+)$ is unique. From (2.6) and (2.8), for any $j, k \in J$, we have

$$(W f_k) = F \varphi_k, \ \forall k \in J \tag{2.9}$$

such that there exists a unique $F \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}_+)$ which implies exchange property.

Clearly, for a total sequence, say $\{\varphi_j\}_{\mathbb{N}}$, where $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ for all $j \in \mathbb{N}$, there is an $f_j \in \mathcal{S}'(\mathbb{R}^n)$ such that $(W f_j) = \varphi_j F$. Hence, $\{(f_j, \varphi_j)\}_{\mathbb{N}} \in \mathcal{A}$ and $F = (W(\{(f_j, \varphi_j)\}_{\mathbb{N}}))$. □

For the family of pairs of sequences $\{(f_j,\varphi_j)\}_J$, $\{(g_k,\lambda_k)\}_K \in \mathcal{A}$ has an Equivalence Relation, i.e., $\{(f_j,\varphi_j)\}_J$ $\{(g_k,\phi_k)\}_K$ if

$$f_j * \lambda_k = g_k * \varphi_j, \quad \forall j \in J,\ k \in K. \tag{2.10}$$

Theorem 2.5. *Let* $\{(f_j,\varphi_j)\}_J, \{(g_k,\lambda_k)\}_K \in \mathcal{A}$. *Then* $\{(f_j,\varphi_j)\}_J \sim \{(g_k,\lambda_k)\}_K$ *iff* $(W(\{(f_j,\varphi_j)\}_J)) = (W(\{(g_k,\lambda_k)\}_K))$.

Proof. Let $\{(f_j,\varphi_j)\}_J \sim \{(g_k,\lambda_k)\}_K$, hence, they satisfy the exchange property, defined as

$$f_j * \lambda_k = g_k * \varphi_k,\ \forall j \in J, k \in K.$$

Let F and G denotes the Mexican hat wavelet transform of some family of sequences such that $F = (W(\{(f_j,\varphi_j)\}_J))$ and $G = (W(\{(g_k,\lambda_k)\}_K))$. Now, consider,

$$\begin{aligned} \varphi_j F * \lambda_k &= (Wf_j) * \lambda_k \\ &= (W(f_j * \lambda_k)) \\ &= (W(g_k * \varphi_j)) \\ &= (Wg_k) * \varphi_j \\ &= \lambda_k G * \varphi_j. \end{aligned}$$

Now, by applying Lemma 2, we get $F = G$.
Conversely, we need to show that the family of sequences $\{(f_j,\varphi_j)\}_J$ and $\{(g_k,\lambda_k)\}_K$ are equivalent. Let us consider

$$\begin{aligned} & F = G \\ \Longrightarrow\quad & (Wf_j) * \lambda_k = (Wg_k) * \varphi_j \\ \Longrightarrow\quad & (W(f_j * \lambda_k)) = (W(g_k * \varphi_j)) \\ \Longrightarrow\quad & f_j * \lambda_k = g_k * \varphi_j. \end{aligned} \tag{2.11}$$

Hence, $\{(f_j,\varphi_j)\}_J \sim \{(g_k,\lambda_k)\}_K$. □

From the above theorem it is shown that there is an equivalence relation on $\mathcal{A}$ and hence splits $\mathcal{A}$ into equivalence classes. The equivalence class contains the generalized quotient $\frac{f_n}{\varphi_n}$ and is denoted by $\left[\frac{f_n}{\varphi_n}\right]$. These equivalence classes are called generalized quotients or Boehmians and the space of all such generalized quotients is denoted by $\mathcal{B}$.

Definition 2.6. Let $X = \left[\frac{f_n}{\varphi_n}\right] \in \mathscr{B}$, then the MHWT of X as a generalized quotient is defined by,

$$Y = (WX)(b,a) = \left[\frac{(Wf_n)(b,a)}{\varphi_n}\right].$$

It is well defined since, if $X = \left[\frac{f_n}{\varphi_n}\right] = Y = \left[\frac{g_n}{\psi_n}\right]$ in $\mathscr{B}$, then

$$\begin{aligned} f_m * \psi_n &= g_n * \varphi_m \ \ \forall m, n \in \mathbb{N} \\ (W(f_m * \psi_n))(b,a) &= (W(g_n * \varphi_m))(b,a) \\ (Wf_m)(b,a) * \psi_n &= (Wg_n)(b,a) * \varphi_m \quad \text{(by Theorem 2.2)} \\ \left[\frac{(Wf_n)(b,a)}{\varphi_n}\right] &= \left[\frac{(Wg_n)(b,a)}{\psi_n}\right]. \end{aligned}$$

Further, by considering the map $f \ \to \left[\frac{f*\delta_n}{\delta_n}\right]$, any $f \in \mathcal{W}'(-\infty,\infty)$ can be considered as an element of $\mathscr{B}$ by [4, Theorem 4.3.9], i.e., if $X = \left[\frac{f*\delta_n}{\delta_n}\right]$, then

$$(WX)(b,a) = \left[\frac{W(f*\delta_n)(b,a)}{\delta_n}\right] = \left[\frac{(Wf)(b,a)*\delta_n}{\delta_n}\right] = (Wf)(b,a).$$

This definition extends the theory of MHWT to more general spaces than $(\mathscr{W}^{\gamma}_{\alpha,\beta})'$.

From Theorem 2.4 and Theorem 2.5, it is clear that the Mexican hat wavelet transform is a bijection from the space of generalized quotients to the space of distributions.

Theorem 2.7. *For every $\mathcal{X} \in \mathcal{B}_{\mathscr{S}'(\mathbb{R}^n)}$ there exists a delta sequence (φ_n) such that $\mathcal{X} = [\{(f_n, \varphi_n)\}_{\mathbb{N}}]$ for some $f_n \in \mathscr{S}'(\mathbb{R}^n)$.*

Proof. Let $(\phi_n) \in \mathscr{S}(\mathbb{R}^n)$, be a delta sequence and $X \in \mathcal{B}_{\mathscr{S}'(\mathbb{R}^n)}$. Then, $(WX) * \phi_n \in \mathscr{S}'$, since $(WX) \in \mathscr{S}'$. Consequently, $(WX) * \phi_n = (Wg_n)$ for some $g_n \in \mathscr{S}'$. Therefore, we have

$$X = \left[\frac{g_n * \phi_n}{\phi_n * \phi_n}\right]. \tag{2.12}$$

Hence, $f_n = (g_n * \phi_n) \in \mathscr{S}'$ and by using the property of delta sequences $\phi_n * \phi_n \in \mathscr{S}$ is a delta sequence. This completes the proof. ☐

Conclusions

The space of generalized quotients includes regular operators, distributions, ultra-distributions and also objects which are neither regular operators nor distributions. It may be concluded here that the space of tempered generalized quotient is constructed in a simple way by using the exchange property. This new construction is further used to represent the Mexican hat wavelet transform of tempered generalized quotients with its algebraic properties. This space of generalized quotient can be applied to examine Mexican hat wavelet transformation on various manifolds.

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