Trigonometric and Hyperbolic Polya type inequalities

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Abstract

Here based on trigonometric and hyperbolic type Taylor formulae we derive Polya type inequalities in a number of cases.

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1 Main Results

We present a collection of Polya's type inequalities.

Theorem 1 *Let $f \in C^2([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We set*

$$M_1 := \max \left\{ \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \|f'' + f\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}, \tag{1}$$

$$M_2 := \max \left\{ \|f'' + f\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \|f'' + f\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)} \right\}, \tag{2}$$

and

$$M_3 := \max \left\{ \|f'' + f\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}, \|f'' + f\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)} \right\}. \tag{3}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq \min \left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1, \\ \frac{(b-a)^2}{4} M_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}} (p+1)^{\frac{1}{p}} \left(2+\frac{1}{p}\right)} M_3 \end{array} \right\}. \tag{4}$$

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Proof. Here $f \in C^2([a,b], K)$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1$. By Corollary 3.4 of [1], we have

$$f(x) = \int_a^x (f''(t) + f(t)) \sin(x-t)\, dt, \tag{5}$$

and

$$f(x) = \int_x^b (f''(t) + f(t)) \sin(x-t)\, dt. \tag{6}$$

By using $|\sin x| \leq |x|$, $\forall\, x \in \mathbb{R}$, we obtain

$$|f(x)| \leq \int_a^x |f''(t) + f(t)| \, |\sin(x-t)|\, dt \leq$$

$$\int_a^x |f''(t) + f(t)| (x-t)\, dt \leq \left(\int_a^x (x-t)\, dt \right) \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]} = \tag{7}$$

$$\frac{(x-a)^2}{2} \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \ \ \forall\, x \in \left[a, \frac{a+b}{2}\right].$$

Also it holds

$$|f(x)| \leq (x-a) \left(\int_a^x |f''(t) + f(t)|\, dt \right) \leq \tag{8}$$

$$(x-a) \|f'' + f\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \ \ \forall\, x \in \left[a, \frac{a+b}{2}\right].$$

Furthermore, by Hölder's inequality, we have ($p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$)

$$|f(x)| \leq \left(\int_a^x |f''(t) + f(t)|^q\, dt \right)^{\frac{1}{q}} \left(\int_a^x (x-t)^p\, dt \right)^{\frac{1}{p}} \leq \tag{9}$$

$$\|f'' + f\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)} \frac{(x-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}}, \ \ \forall\, x \in \left[a, \frac{a+b}{2}\right].$$

We have found that

$$|f(x)| \leq \left\{ \begin{array}{l} \frac{(x-a)^2}{2} \|f'' + f\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \\ (x-a) \|f'' + f\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \\ \frac{(x-a)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \|f'' + f\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}, \quad p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \end{array} \right\}, \tag{10}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$.

Similarly acting, we get that

$$|f(x)| = \left| \int_x^b (f''(t) + f(t)) \sin(x-t)\, dt \right| \leq$$

$$\int_x^b |f''(t)+f(t)| |\sin(t-x)| dt \leq \int_x^b |f''(t)+f(t)| (t-x) dt, \tag{11}$$

and

$$|f(x)| \leq \left\{ \begin{array}{l} \frac{(b-x)^2}{2} \|f''+f\|_{\infty,\left[\frac{a+b}{2},b\right]}, \\ (b-x) \|f''+f\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}, \\ \frac{(b-x)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \|f''+f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}, \quad p,q>1: \frac{1}{p}+\frac{1}{q}=1 \end{array} \right\}, \tag{12}$$

$\forall\, x \in \left[\frac{a+b}{2}, b\right]$.

Consequently, we obtain

$$\int_a^{\frac{a+b}{2}} |f(x)| dx \leq \left\{ \begin{array}{l} \frac{(b-a)^3}{16} \|f''+f\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \\ \frac{(b-a)^2}{8} \|f''+f\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} \|f''+f\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \ p,q>1: \frac{1}{p}+\frac{1}{q}=1. \end{array} \right\} \tag{13}$$

Similarly, we derive that

$$\int_{\frac{a+b}{2}}^b |f(x)| dx \leq \left\{ \begin{array}{l} \frac{(b-a)^3}{16} \|f''+f\|_{\infty,\left[\frac{a+b}{2},b\right]}, \\ \frac{(b-a)^2}{8} \|f''+f\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} \|f''+f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}, \quad p,q>1: \frac{1}{p}+\frac{1}{q}=1. \end{array} \right\} \tag{14}$$

We have that

$$\int_a^b |f(x)| dx = \int_a^{\frac{a+b}{2}} |f(x)| dx + \int_{\frac{a+b}{2}}^b |f(x)| dx \leq$$

$$\left\{ \begin{array}{l} \frac{(b-a)^3}{16} \left[\|f''+f\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \|f''+f\|_{\infty,\left[\frac{a+b}{2},b\right]}\right], \\ \frac{(b-a)^2}{8} \left[\|f''+f\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)} + \|f''+f\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}\right], \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} \left[\|f''+f\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, + \|f''+f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]. \end{array} \right. \leq \tag{15}$$

$$\left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1, \\ \frac{(b-a)^2}{4} M_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} M_3 \end{array} \right\}. \tag{16}$$

The claim is proved. ■

We continue with

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Theorem 2 *All as in Theorem 1. Denote*

$$M_1^* := \max\left\{\|f'' - f\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \|f'' - f\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}, \tag{17}$$

$$M_2^* := \max\left\{\|f'' - f\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \|f'' - f\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}\right\}, \tag{18}$$

and

$$M_3^* := \max\left\{\|f'' - f\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \|f'' - f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right\}. \tag{19}$$

Then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \leq \min \cosh(b-a) \times \left\{\begin{array}{l} \frac{(b-a)^3}{8} M_1^*, \\ \frac{(b-a)^2}{4} M_2^*, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} M_3^* \end{array}\right\}. \tag{20}$$

Proof. As similar to Theorem 1 it is omitted. It based on Corollary 3.5 of [1]. Also we use that $|\sinh x| \leq \cosh(b-a)|x|$, $\forall\, x \in [-(b-a), b-a]$, by the mean value theorem. ■

It follows

Theorem 3 *Let $f \in C^4([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0,1,2,3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We set*

$$A_1 := \max\left\{\left\|f^{(4)} - f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|f^{(4)} - f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}, \tag{21}$$

$$A_2 := \max\left\{\left\|f^{(4)} - f\right\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \left\|f^{(4)} - f\right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}\right\}, \tag{22}$$

and

$$A_3 := \max\left\{\left\|f^{(4)} - f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \left\|f^{(4)} - f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right\}. \tag{23}$$

Then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \leq$$

$$\min \frac{(\cosh(b-a)+1)}{4} \times \left\{\begin{array}{l} \frac{(b-a)^3}{4} A_1, \\ \frac{(b-a)^2}{2} A_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} A_3 \end{array}\right\}. \tag{24}$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.6 of [1]. ■

We continue with

Theorem 4 *Let $f \in C^4([a,b],K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0,1,2,3$, and $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let also $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. We set*

$$B_1 := \max\left\{ \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}, \tag{25}$$

$$B_2 := \max\left\{ \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)} \right\}, \tag{26}$$

and

$$B_3 := \max\left\{ \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \left\| f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)} \right\}. \tag{27}$$

Then

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \leq$$

$$\min \frac{1}{\left|\beta^2 - \alpha^2\right|} \times \begin{cases} \frac{(b-a)^3}{4} B_1, \\ \frac{(b-a)^2}{2} B_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} B_3 \end{cases}. \tag{28}$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.7 of [1]. ■

We finish with

Theorem 5 *All as in Theorem 4, $|\alpha|, |\beta| < 1$.However here, instead of B_1, B_2, B_3, we set*

$$D_1 := \max\left\{ \left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}, \tag{29}$$

$$D_2 := \max\left\{ \left\| f^{(4)} - \left(\alpha^2 + \beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)}, \right.$$

$$\left\| f^{(4)} - \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)} \Bigg\}, \tag{30}$$

and

$$D_3 := \max \Bigg\{ \left\| f^{(4)} - \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)},$$

$$\left\| f^{(4)} - \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)} \Bigg\}. \tag{31}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq$$

$$\min \frac{\cosh(b-a)}{\left|\beta^2-\alpha^2\right|} \times \begin{cases} \frac{(b-a)^3}{4} D_1, \\ \frac{(b-a)^2}{2} D_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}\left(2+\frac{1}{p}\right)} D_3 \end{cases}. \tag{32}$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.9 of [1]. ∎

References

[1] Ali Hasan Ali and Zsolt Páles, *Taylor-type expansions in terms of exponential polynomials*, Mathematical Inequalities & Applications, 25(4) (2022), 1123-1141.