Some new odd-even congruence graphs

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Received: 18.04.2024 Revised : 16.05.2024 Accepted: 20.05.2024

ABSTRACT
Introduction: The concept of labeling of graph has been introduced in mid-1960. A graph labeling is an assignment of integers to the nodes or edges or both depends on specific conditions. Many types of graph labeling techniques have been studied by several authors.
Objectives: Labeling of graphs has remarkable applications in diverse fields like the study of the physical cosmology, debug circuit design, X-ray, crystallography, astronomy, radar, broadcasting network, secret sharing, database management, coding theory and much more. Besides these graph labeling techniques analyze and solve various graph related problems.
Methods: This research paper focuses on Odd-even congruence labeling of different types of graphs. Assignment of natural numbers as labels for the edges and vertices of a graph based on modular arithmetic property known as congruence graph labeling.For congruence graph labeling it entails the assignment of odd integers to vertices and even integers to edges with property of congruence graph labeling.
Results: This labeling method has been identified on triangular snake graph, ladder graph, coconut tree, triangular book, friendship graph.

Keywords: Labeling, Congruence labeling, Odd-even congruence labeling, triangular snake graph, ladder graph, coconut tree, triangular book, friendship graph.

INTRODUCTION
Informative labelling techniques serve the purpose of representing graphs in highly distributed manner. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions . An extensive survey of various graph labeling problems is available in [2].In 1967, Rosa [3] published a pioneering paper on graph labeling problems. Thereafter many types of graph labeling methods have been studied by several authors. For standard terminology of graph theory, we used [1].
G.Thamizhendhi and K.Kanakambika [4] have introduced Odd-even congruence labeling and they proved behaviourof several graphs like bipartite graph, comb graph, star graph, graph acquired by connecting two copies of even cycle C_r by a path P_t , shadow graph of the path P_t and the tensor product of $K_{1,t}$&P_2 as Odd-even congruence graph. A graph G is said to be Odd-even congruence graph, if vertex and edge set are assigned by distinct odd and even integers respectively, further $f(u_p) \equiv f(u_q)(\text{mod } g(e))$, u_p and u_q are adjacent vertices in G. In this paper we investigate the Odd-even congruence labeling of triangular snake, ladder graph, triangular book, coconut tree graph and tree related graphs.

Preliminaries
Definition 2.1 A Triangular snake T_n is obtained from a path $u_1, u_2, u_3, \dots\dots\dots, u_n$ by joining u_i and u_{i+1} to a new vertex w_i for $1 \leq i \leq n-1$
Definition 2.2 A ladder graph [3] is defined by $L_n \times P_2$ where P_n is a path with n vertices and $\times$ denotes the cartesian product and K_2 is a complete graph with two vertices.
Definition 2.3 An open ladder OL_n , $n \geq 2$ is obtained from two graphs [8]of length $n-1$ with $V(G) = \{u_i, v_i: 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 2 \leq i \leq n-1\}$
Definition 2.4 The triangular book graph $B_{3,n}$ for $n \geq 1$ is a planar undirected graph [11] having $n+2$ vertices $v_0, v_0', v_1, v_{2,\dots\dots\dots}, v_n$ and $2n+1$ edges constructed by n triangles sharing a common edge v_0v_0'.
Definition 2.5 Coconut tree graph is obtained by identifying the central vertex of $K_{1,m}$ with a pendant vertex of the path P_n.
Definition 2.6F- tree on $n+2$ vertices, denoted by F_n is obtained from a path P_n by attaching exactly two

pendant vertices of the $n-1$ and n^{th} vertex of P_n.

Definition 2.7 Y- tree on $n+1$ vertices, denoted by Y_n is obtained from a path P_n by attaching a pendant vertex of the n^{th} vertex of P_n.

Definition 2.8 A bijection defined as $h: V \to \{1,2, \dots \dots \dots d\}$ and $k: E \to \{1,2, \dots \dots \dots d-1\}$ of G is known as [9] congruence graph, if $h(s_p) \equiv h(s_q) \left(\bmod k(w_p)\right)$, where $d = \min\{2|V|, 2|E|\}$

Definition 2.9 A bijection $h: V \to \{1,2, \dots \dots \dots 2d+1\}$ and $k: E \to \{2,4, \dots \dots \dots 2d\}$ of G is [10] odd-even congruence graph, if $h(s_p) \equiv h(s_q) \left(\bmod k(w_p)\right)$, where $d = \min\{2|V|, 2|E|\}$.

RESULTS

Theorem 3.1 The Triangular snake T_n is odd -even congruence graph for all $n \geq 2$.

Proof:

Let the vertex set of T_n be

$$V(T_n) = \{v_i / 1 \leq i \leq n\} \cup \{u_i / 1 \leq i \leq n-1\}$$

and the edge set be

$$E(T_n) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i u_i / 1 \leq i \leq n-1\}$$

Where $p_i = v_i u_i$, $q_i = v_{i+1} u_i$ and $r_i = v_{i+1} v_i$

With $|V| = 2n-1$ and $|E| = 3n-3$

For $G = T_n$, we have $d = \min\{2(2n-1), 2(3n-3)\}$, $n \geq 2$

$$= \min\{(4n-2), (6n-6)\}$$

$$= 4n-2$$

There exists two independent and disjoint vertex such as

$V_1 = \{v_1, v_2, \dots\dots\dots\dots, v_i\}$ where $1 \leq i \leq n$

and $V_2 = \{u_1, u_2, \dots\dots\dots\dots u_i\}$ where $1 \leq i \leq n-1$

Define the bijection function $h_1 : v_i \to \{1,11,31,71, \dots\dots\dots\dots, 5(2^{i+1}) - 9\}$ as $h_1(v_1) = 1$ and $h_1(v_i) = 5(2^{i+1}) - 9$ for $2 \leq i \leq n$ and

$h_2 : u_i \to \{5,19,47,103, \dots\dots\dots\dots 7(2^{n+1}) - 9\}$ as $h_2(u_i) = 7(2^{n+1}) - 9$ for $i \geq 2$ and $h_2(u_1) = 5$

The edge labeling

$s_1 : p_i \to \{4,8,16, \dots\dots\dots\dots 2^{i+1}\}$ as $s_1(p_i) = 2^{i+1}$ for $1 \leq i \leq n$

$s_2 : q_i \to \{6,12,24, \dots\dots\dots\dots, 3(2^i)\}$ as $s_2(q_i) = 3(2^i)$ for $1 \leq i \leq n$

$s_3 : r_i \to \{10,20,40, \dots\dots\dots\dots 5(2^i)\}$ as $s_3(r_i) = 5(2^i)$ for $1 \leq i \leq n$

Clearly

$s_1(p_i)$ divides $\left|\left(h_1(v_i) - h_2(u_i)\right)\right|$ for $1 \leq i \leq n$

$s_2(q_i)$ divides $\left|\left(h_1(v_{i+1}) - h_2(u_i)\right)\right|$ for $1 \leq i \leq n$

$s_3(r_i)$ divides $\left(h_2(v_{i+1}) - h_2(v_i)\right)$ for $1 \leq i \leq n$

Hence the triangular snake (T_n) is odd-even congruence graph for all $n \geq 2$.

Example 3.2

Consider the graph (T_9) with $|V| = 17$ and $|E| = 24$

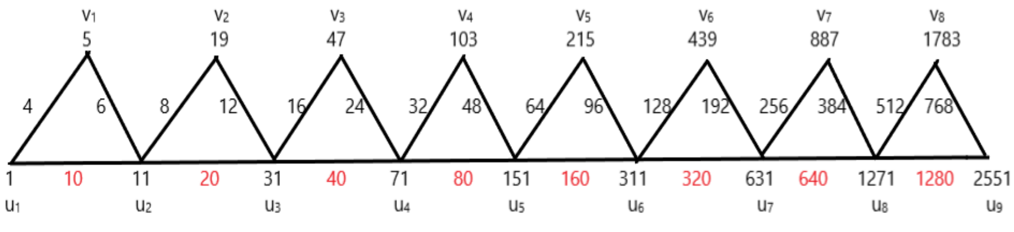

Figure 1. shows that T_9 receives odd-even congruence labeling.

Theorem: 3.3 The Graph Ladder L_n is an odd even congruence graph.

Proof:

Let G be a graph of Ladder L_n

Let $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertices of L_n and

$E(T_n) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i v_i / 1 \leq i \leq n\}$

Where $p_i = u_i u_{i+1}$, $q_i = u_{i+1} v_{i+1}$ and $r_i = v_i v_{i+1}$ be the edges of L_n.

With $|V(G)| = 2n$ and $|E(G)| = 3n-2$ of Ladder L_n.

we have $d = min\{2(2n), 2(3n-2)\}$

$= \min(4n, 6n-4) = 4n$
Label the vertices and edges as follows:
The vertices of L_n are labeled as given below.
The function $f_1 : u_i \to \{1,7,31,\ldots\ldots\ldots\ldots,2^{2i-1}-1\}$ as $f(u_i) = 2^{2i-1}-1$ for $1 \le i \le n$
And $f_2 : v_i \to \{3,15,63,\ldots\ldots\ldots\ldots,2^{2i}-1\}$ as $f(v_i) = 2^{2i}-1$, $1 \le i \le n$
Edges of L_n are labeled as follows:
The function $s_1 : p_i \to \{6,24,96,384\ldots\ldots\ldots\ldots,3(2^{i-1})\}, s_1(p_i) = 3(2^{2i-1})$ for $1 \le i \le n$
$s_2 : q_i \to \{8,32,128,512,\ldots\ldots\ldots\ldots,2^{2i+1}\}, s_2(q_i) = 2^{2i+1}$ for $1 \le i \le n$
$s_3 : r_i \to \{12,48,192,768,\ldots\ldots\ldots\ldots,3(2^{2i})\}, s_3(r_i) = 3(2^{2i})$ for $1 \le i \le n$
Clearly $s_1(p_i)$divides $|(f_1(u_i) - f_1(u_{i+1}))|$ for $1 \le i \le n$
$s_2(q_i)$divides $|(f_1(u_{i+1}) - f_2(v_{i+1}))|$ for $1 \le i \le n-2$
$s_3(r_i)$divides $|(f_2(v_{i+1}) - f_2(v_i))|$ for$1 \le i \le n$
Hence the ladder (L_n) is an odd-even congruence graph for all $n \ge 2$.

Example: 3.4
Considerthe ladder graph (L_n) for $n = 8$

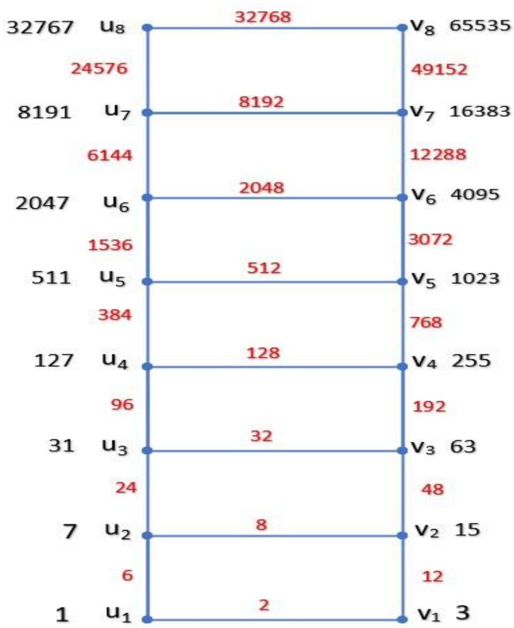

Figure 2. shows that L_n admits odd-even congruence labeling.

Corollary: 3.5 The open ladder graph OL_n admits odd-even congruence graph for $n \ge 3$.
Proof:
Let G be a graph of Open Ladder
Let $V(G) = \{u_i, v_i : 1 \le i \le n\}$ be the vertices of L_n and
$E(T_n) = \{u_i u_{i+1}, v_i v_{i+1} \,/\, 1 \le i \le n-1\} \cup \{u_i v_i \,/\, 2 \le i \le n-1\}$
Where $p_i = u_i u_{i+1}$, $q_i = u_i v_i$ for $2 \le i \le n-1$ and $r_i = v_i v_{i+1}$ be the edges of L_n.
Suppose $|V(G)| = 2n$ and $|E(G)| = 3n-4$ of open Ladder OL_n.
we have $d = min\{2(2n), 2(3n-4)\}$
$= \min(4n, 6n-8) = 4n$
Label the vertices and edges are as follows:
The vertices of L_n are labeled as given below.
The function $f_1 : u_i \to \{1,7,31,\ldots\ldots\ldots\ldots,2^{2i-1}-1\}$ as $f(u_i) = 2^{2i-1}-1$ for $1 \le i \le n$
And $f_2 : v_i \to \{3,15,63,\ldots\ldots\ldots\ldots,2^{2i}-1\}$ as $f(v_i) = 2^{2i}-1$, $1 \le i \le n$
Edges of L_n are labeled as follows:
The function $s_1 : p_i \to \{6,24,96,384\ldots\ldots\ldots\ldots,3(2^{i-1})\}, s_1(p_i) = 3(2^{2i-1})$ for $1 \le i \le n$
$s_2 : q_i \to \{8,32,128,512,\ldots\ldots\ldots\ldots,2^{2i+1}\}, s_2(q_i) = or\ 2^{2i-1}2^{2i+1}$ for $1 \le i \le n$
$s_3 : r_i \to \{12,48,192,768,\ldots\ldots\ldots\ldots,3(2^{2i})\}, s_3(r_i) = 3(2^{2i})$ for $1 \le i \le n$

Clearly $s_1(p_i)$divides $\left|\left(f_1(u_i) - f_1(u_{i+1})\right)\right|$ for $1 \leq i \leq n$
$s_2(q_i)$divides $\left|\left(f_1(u_{i+1}) - f_2(u_{i+1})\right)\right|$ for $1 \leq i \leq n-2$
$s_3(r_i)$divides $\left|\left(f_2(v_{i+1}) - f_2(v_i)\right)\right|$ for$1 \leq i \leq n$
Hence the open ladder (OL_n) is an odd-even congruence graph for all $n \geq 2$.

Example: 3.6
Considerthe open ladder graph (OL_n) for $n = 8$

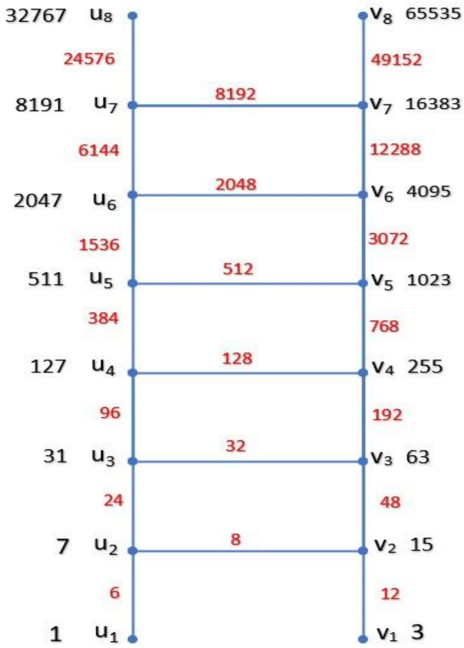

Figure 3. shows that OL_nadmits odd-even congruence labeling.

Theorem: 3.7 The triangular book $B_{3,n}$ admits odd-even congruence graph.
Proof:
Let $B_{3,n}$ be a triangular book.
Then $V(B_{3,n}) = \{x, y, z_i : 1 \leq i \leq n\}$ and $E(B_{3,n}) = \{w, w_i, q_i : 1 \leq i \leq n\}$
Also $|V(B_{3,n})| = n + 2$ and $|E(B_{3,n})| = 2n + 1$
Then d =min $(2(n+2), 2(2n+1)) = 2(n+2)$
Define a bijective function ⍰ : $V(G) \rightarrow \{1,3,7, \ldots\ldots\ldots, 4i+3\}$ is defined as follows
$f(x) = 1, f(y) = 3, f(w) = 2$
⍰$(z_i) = 4i + 3$ for $1 \leq i \leq n$
The edge labeling $k : E(G) \rightarrow \{2,4,6,8, \ldots\ldots\ldots, 4i+2\}$ is defined as
$k(w_i) = 4i + 2$ for $1 \leq i \leq n$
$k(q_i) = 4i$ for $1 \leq i \leq n$
Clearly $k(w)$divides $|f(x) - f(y)|$ for $1 \leq i \leq n$
$k(w_i)$divides $\left|\left(f(x) - ⍰(z_i)\right)\right|$ for $1 \leq i \leq n$ and
$k(q_i)$divides $\left|\left(f(y) - ⍰(z_i)\right)\right|$ for $1 \leq i \leq n$
Hence the graph triangular book $B_{3,n}$ is an odd-even congruence graph.

Example3.8
Consider a graph $G = B_{3,n}$ with $n = 6$

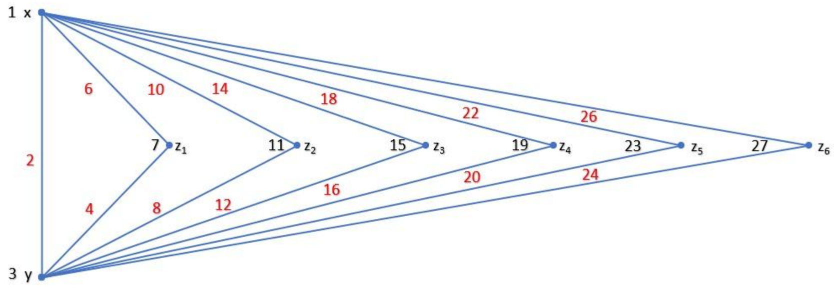

Figure 4. exhibits the odd-even congruence labeling of the graph triangular book.

Theorem: 3.9 Coconut tree $CT(m,n)$ is an odd-even congruence graph.

Proof:

Let $V(G) = \{u_i,\ v_j /1 \le i \le m, 1 \le i \le n\}$ and the edge set

$E(G) = \{u_i v_1\ /1 \le i \le m\} \cup \{v_i v_{i+1}\ /1 \le i \le n-1\}$ denoted as $e_i = \{u_i v_1 /1 \le i \le m\}$ and $w_p = \{v_i v_{i+1} /1 \le i \le n-1\}$

Now $|V(CT(m,n))| = m+n$ and $|E(CT(m,n))| = m+n-1$

Then $d = \min\left(2(m+n), 2(m+n-1)\right)$

$$= 2(m+n-1)$$

Define the bijection $f : V(G) \rightarrow \{1,3,7, \dots\dots\dots, 2^{m+n}-1\}$ is defined as follows

$f(u_i) = 2^{n+i} - 1$ for $1 \le i \le m$

$f(v_i) = 2^i - 1$ for $1 \le i \le n$

The edge labeling $k : E(G) \rightarrow \{2,4,8,30,62, \dots\dots\dots, 2^{n+i+1} - 2\}$ is defined as follows

$k(e_i) = 2^{n+i+1} - 2$ for $1 \le i \le m$

$k(w_i) = 2^i$for $1 \le i \le n$

Clearly $k(w_p)$divides $\left|\left(f(v_{i+1}) - f(v_i)\right)\right|$ for $1 \le i \le n$ and

$k(e_i)$divides $\left|\left(f(u_i) - f(v_1)\right)\right|$ for $1 \le i \le n$

Hence the graph coconut tree $CT(m,n)$ is an odd-even congruence graph.

Example: 3.10

Consider a graph $G = CT(m,n)$ with $m = 7$ and $n = 4$

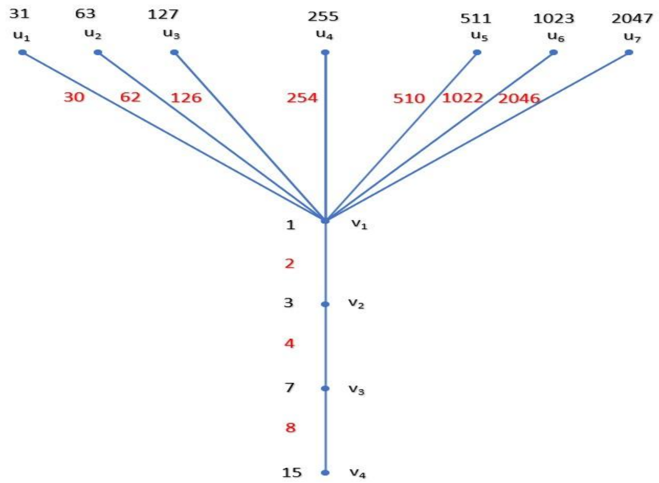

Fig 5. shows that Coconut tree graph admits odd-even congruence labeling.

Theorem: 3.11F-tree FP_n $n \ge 3$ is an odd even congruence graph.

Proof:

Let $V(G) = \{u, v, v_i /1 \le i \le n-1\}$

$E(G) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{uv_2, vv_1\}$

where $e_i = v_i v_{i+1}$, $w_1 = vv_1$ and $w_2 = uv_2$

be the vertex set and edge set of FP_n
Also $|V(FP_n)| = n+2$ and $|E(FP_n)| = n+1$
Then d =min $\left(2(n+2), 2(n+1)\right)$
=min $\left((2n+4), (2n+2)\right) = 2n+2$
Define a bijective function $f : V(G) \rightarrow \{1,3,7, \ldots\ldots\ldots, 2n+3\}$ is defined as follows
$f(u) = 3$
$f(v) = 1$
$f(v_i) = 2^{i+2} - 1$ for $1 \leq i \leq n$
The edge labeling $k : E(G) \rightarrow \{6,12, \ldots\ldots\ldots, 4i+2\}$ is defined as

$$k(w_1) = 6$$
$$k(w_2) = 12$$

$k(e_i) = 2^{i+2}$ for $1 \leq i \leq n-1$
Clearly $k(w_1)$divides $|f(v_1) - f(v)|$
$k(w_2)$divides $\left|\left(f(v_2) - f(u)\right)\right|$ for $1 \leq i \leq n$ and
$k(e_i)$divides $\left|\left(f(v_i) - f(v_{i+1})\right)\right|$ for $1 \leq i \leq n-1$
Hence F-tree FP_n , $n \geq 3$ is an odd even congruence graph.

Example3.12
Consider FP_6 with $|V| = 8$ and $|E| = 7$

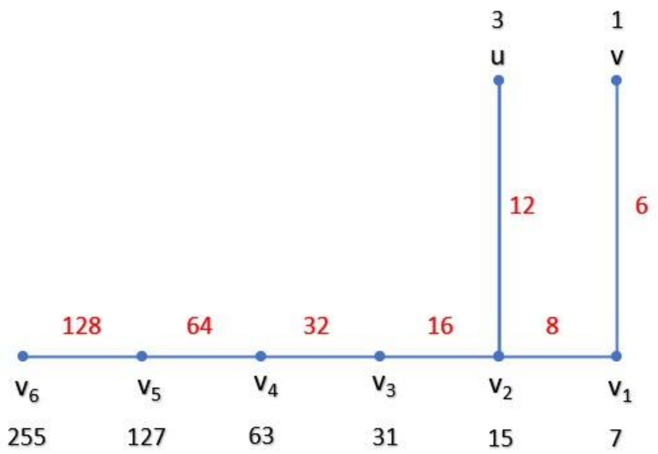

Fig 6. shows that FP_6graph admits odd-even congruence labeling.

Theorem: 3.13 Let G be the Graph obtained by identifying a pendant vertex of P_m with a leaf of $K_{1,n}$ then $G$is an odd even congruence graph.
Proof:
Let $V(G) = \{u_i,\ v_j / 1 \leq i \leq n, 1 \leq i \leq m\}$ and the edge set
$E(G) = \{u_1 v_i\ \ /1 \leq i \leq n\} \cup \{u_i u_{i+1}\ \ /1 \leq i \leq n\}$
Where $w_i = u_i u_{i+1}$ and $e_i = u_1 v_i$
Now $|V(G)| = m + n$ and $|E(G)| = m + n - 1$
Then d =min $\left(2(m+n), 2(m+n-1)\right)$
=min $\left((2m+2n), (2m+2n-2)\right) = 2m + 2n - 2$
Define a labeling $f : V(G) \rightarrow \{1,3,7,15,31,63, \ldots\ldots\ldots, 2^{n+i} - 1\}$ is defined as follows
$f(v_i) = 2^{n+i} - 1$ for $1 \leq i \leq m$
$f(u_i) = 2^i - 1$ for $1 \leq i \leq n$
The edge labeling $k : E(G) \rightarrow \{2,4,8,30,62, \ldots\ldots\ldots, 2^{n+i+1} - 2\}$ is defined as follows
$k(e_i) = 2^{n+i+1} - 2$ for $1 \leq i \leq m$
$k(w_i) = 2^i$for $1 \leq i \leq n$
Clearly $k(w_i)$divides $\left|\left(f(u_i) - f(u_{i+1})\right)\right|$ for $1 \leq i \leq n$ and
$k(e_i)$divides $\left|\left(f(v_i) - f(u_1)\right)\right|$ for $1 \leq i \leq n$
Thus f admits odd even congruence labeling on Gand hence G is an odd even congruence graph.

Example: 3.14
Consider the Graph obtained by identifying a pendant vertex of P_m with a leaf of $K_{1,n}$ where m = 7 and n = 4

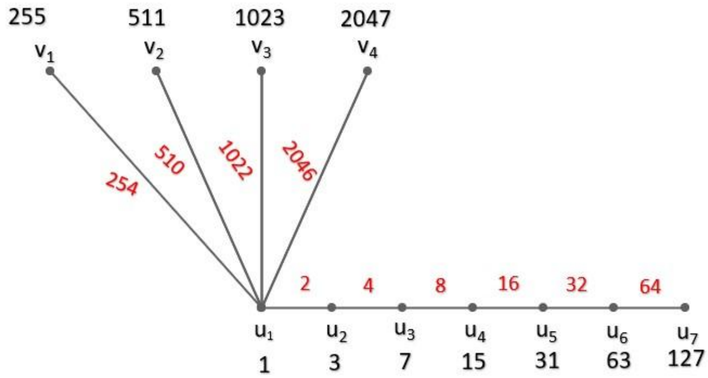

Figure 7.admits odd even congruence labeling.

Theorem: 3.15 Y- tree is an Odd-even congruence graph.
Proof:
Suppose $V(G) = \{u, v_i /1 \leq i \leq n\}$
$E(G) = \{v_i v_{i+1}, v_{n-1}u \ : 1 \leq i \leq n-1\}$
where $e_i = v_i v_{i+1}$, $w = v_{n-1}u$
be the vertex set and edge set of Y-tree.
Also $|V(G)| = n + 1$ and $|E(G)| = n$
Then d =min $\{2|V| , 2|E|\} = \min\{2(n + 1), 2n\}$
$= 2n$
Define a bijective function $f : V(G) \rightarrow \{1,3,7, \ldots\ldots\ldots, 2^{n+1} - 1\}$ is defined as follows
$f(v_i) = 2^i - 1$ for $1 \leq i \leq n$
$f(u) = 2^{n+1} - 1$
The edge labeling $k : E(G) \rightarrow \{2,4,8, \ldots\ldots\ldots, 2^i\}$ is defined as
$k(e_i) = 2^i$ for $1 \leq i \leq n$
$k(w) = 3 \times 2^{n-1}$
Clearly $k(e_i)$divides $|f(v_{i+1}) - f(v_i)|$ for $1 \leq i \leq n$ and
$k(w)$divides $|(f(u) - f(v_{n-1}))|$
Hence Y-tree is an odd even congruence graph.

Example: 3.16
Consider the Y-tree graph (Y_9) with $|V| = 9$ and $|E| = 8$

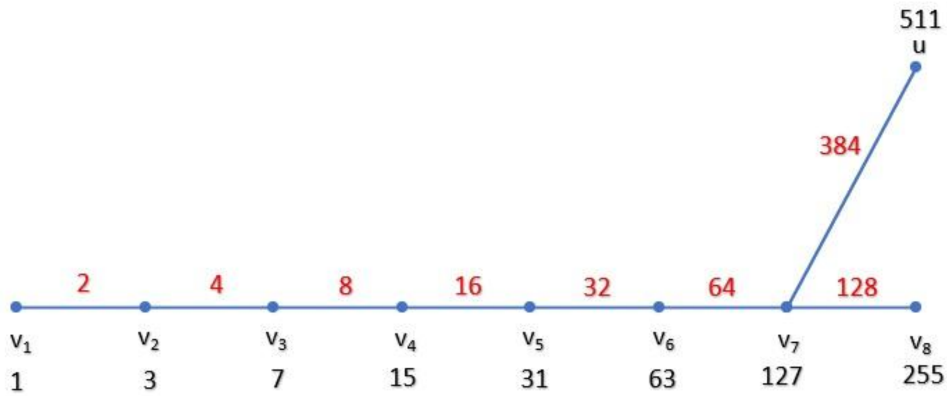

Figure 8. shows that Y_8 receives odd-even congruence labeling.

CONCLUSION

The labelling of graphs is an interesting and vast research area which is very useful and it is extended in various topics by several people. In this paper we have studied Odd-even congruence labeling behaviour of triangular snake, ladder graph, triangular book, coconut tree graph and tree related graphs. To derive similar results for other graph families is an open problem.

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