On linear fuzzy real numbers

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Abstract. In this paper we introduce the notion of liner fuzzy real numbers, and show that the set of all positive (or negative) symmetric linear fuzzy real numbers forms a semiring. Moreover, we discuss a complex transform of linear fuzzy real numbers.

1. Introduction

A. Kaufmann and M. M. Gupta [2] introduced the notion of a trapezoidal fuzzy number, and D. Dubois and H. Prade [1] generalized the notion of trapezoidal fuzzy numbers. X. F. Zhang and G. W. Meng [12] introduced the notion of an isoceles triangular fuzzy number, and discussed the simplification of addition and subtraction operations of fuzzy numbers. A. Kumar et al. [2] studied an RM approach for ranking of generalized trapezoidal fuzzy numbers, and showed that the ranking function satisfies all the reasonable properties of fuzzy quantities proposed by X. Wang and E. E. Kerre [11]. Neggers and Kim researched fuzzy posets [7] and created Linear Fuzzy Real numbers [8]. Linear Fuzzy Real numbers were used by Monk [4]. In [9], Rogers et al. focused on linear fuzzy programming problems. Rogers [10] focused on solving and manipulating Fuzzy Nonlinear problems in the Linear Fuzzy Real number system using the Gradient Descent. In texts on fuzzy logic, fuzzy subsets of the real numbers may also be referred to as *fuzzy real numbers*, thus obviating the needs to talk about fuzzy subsets of the real numbers. Actually, equating these concepts may be a disadvantage since in some way we expect numbers, whether fuzzy or not, to behave differently from (sub)sets, whether fuzzy or not. It is with this in mind that we seek to introduce among several models of systems of fuzzy real numbers, the system of *linear fuzzy real numbers* discussed below. For general concepts for fuzzy set theory we refer to [5, 6].

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2. Linear fuzzy real numbers

A mapping $\mu : \mathbf{R} \to [0,1]$ is called a *linear fuzzy real number* [8, 4] if there is a triple of real numbers a, b, c $(a \le b \le c)$ such that

(1) $\mu(b) = 1$,
(2) $\mu(x) = 0$ if $x < a$ or $x > c$,
(3) $\mu(x) = \frac{x-a}{b-a}$ if $a \le x < b$,
(4) $\mu(x) = \frac{c-x}{c-b}$ if $b < x \le c$.

We denote such a linear fuzzy real number by a triple $\mu =< a, b, c >$ or $\mu = \mu(a, b, c)$ where $a \le b \le c$. Notice that the integral $\int_{-\infty}^{\infty} \mu(t)dt = \frac{b-a}{2} + \frac{c-b}{2} = \frac{c-a}{2}$, i.e., if the *goodness* of the fuzzy subset $\mu : (-\infty, \infty) \to [0,1]$ is defined by

$$G(\mu) := \frac{e^{\gamma(\mu)} - 1}{e^{\gamma(\mu)} + 1}, \quad \gamma(\mu) = [\int_{-\infty}^{\infty} \mu(t)dt]^{-1},$$

then in the case of a linear fuzzy real number $\mu =< a, b, c >$, it follows that

$$G(\mu) = \frac{e^{\frac{2}{c-a}} - 1}{e^{\frac{2}{c-a}} + 1}$$

In particular, if we let $c - a \to 0^+$, then $G(\mu) \to 1$, so that for any $\mu =< a, a, a >$, we set $G(\mu) = 1$. On the other hand, if $c - a \to \infty$, then $G(\mu) \to 0^+$, and $0 \le G(\mu) \le 1$. If $\mu =< a, a, a >$, then $\mu(t) = 0$ if $t \ne a$, while $\mu(a) = 1$, i.e., $\mu = \delta_a$, the *characteristic function* of the real number a. For any two linear fuzzy real numbers $\mu_i =< a_i, b_i, c_i >$ $(i = 1, 2)$, we define $\mu_1 + \mu_2 :=< a_1 + a_2, b_1 + b_2, c_1 + c_2 >$.

Theorem 2.1. *If $\mu_i =< a_i, b_i, c_i >$ $(i = 1, 2)$ be linear fuzzy real numbers, then $G(\mu_1 + \mu_2) \le \min\{G(\mu_1), G(\mu_2)\}$.*

Proof. If we let $F(x) := \frac{e^{\frac{2}{x}}-1}{e^{\frac{2}{x}}+1}$ then

$$\begin{aligned} F'(x) &= \frac{(e^{\frac{2}{x}} + 1)e^{\frac{2}{x}}(-\frac{2}{x^2}) - (e^{\frac{2}{x}} - 1)e^{\frac{2}{x}}(-\frac{2}{x^2})}{(e^{\frac{2}{x}} + 1)^2} \\ &= \frac{-4e^{\frac{2}{x}}}{x^2(e^{\frac{2}{x}} + 1)^2} \\ &< 0. \end{aligned}$$

Hence, if $x_1 \geq x_2$, then $F(x_1) \leq F(x_2)$. Since $(c_1 + c_2) - (a_1 + a_2) \geq c_i - a_i$, $(i = 1, 2)$,

$$\begin{aligned} G(\mu_1 + \mu_2) &= F((c_1 + c_2) - (a_1 + a_2)) \\ &\leq F(c_i - a_i) \\ &= G(\mu_i), \end{aligned}$$

proving the theorem. □

Corollary 2.2. *If $G(\mu_1 + \mu_2) = G(\mu_1)$, then μ_2 is the characteristic function of a real number.*

Proof. If $G(\mu_1 + \mu_2) = G(\mu_1)$, then $(c_1 + c_2) - (a_1 + a_2) = c_1 - a_1$ and hence $c_2 - a_2 = 0$, which means that $\mu_2 =< b_2, b_2, b_2 >= \delta_{b_2}$ for some $b_2 \in \mathbf{R}$. □

Proposition 2.3. *Let $\mu_i =< a_i, b_i, c_i >$, $(i = 1, 2, 3)$, be linear fuzzy real numbers and $\delta_0 =< 0, 0, 0 >$. Then*
(i) $\mu_i + \delta_0 = \mu_i$,
(ii) $\mu_1 + \mu_2 = \mu_2 + \mu_1$,
(iii) $(\mu_1 + \mu_2) + \mu_3 = \mu_1 + (\mu_2 + \mu_3)$,
(iv) *If $\mu_1 + \mu_2 = \delta_0$, then $\mu_1 = \delta_{b_1}, \mu_2 = \delta_{b_2}$ and $b_2 = -b_1$.*

Proof. (iv). If we let $\mu_1 + \mu_2 = \delta_0$, then $G(\mu_1 + \mu_2) = G(\delta_0) = 1 \leq \min\{G(\mu_1), G(\mu_2)\}$ by Theorem 2.1. Hence $G(\mu_1) = G(\mu_2) = 1$, i.e., $\mu_1 = \delta_{b_1}$ and $\mu_2 = \delta_{b_2}$ for some $b_1, b_2 \in \mathbf{R}$ and obviously $b_2 = -b_1$. □

By Proposition 2.3, we obtain the following theorem.

Theorem 2.4. *If $\mathcal{L}$ is the collection of all linear fuzzy real numbers with the operation "+", then $(\mathcal{L}, +)$ is a commutative semigroup with a neutral element δ_0.*

Remark. In view of Theorem 2.1, there exists a function $G : \mathcal{L} \to [0, 1]$ such that $G(\mu_1 + \mu_2) \leq \min\{G(\mu_1), G(\mu_2)\}$, denoting "the goodness" of the linear fuzzy real number.

Let $\mu_i :=< a_i, b_i, c_i >$, $(i = 1, 2)$, be linear fuzzy real numbers. We define a new linear fuzzy real number $\mu_1 \ominus \mu_2 :=< a_1 - a_2, b_1 - b_2, c_1 - c_2 >$ when $a_1 - a_2 \leq b_1 - b_2 \leq c_1 - c_2$, which is called the *subtraction* of μ_1 by μ_2.

Proposition 2.5. *If $\mu_1 \ominus \mu_2 = \delta_b$ for some real number b, then $G(\mu_1) = G(\mu_2)$.*

Proof. If $\mu_1 \ominus \mu_2 = \delta_b$, then $a_1 - a_2 = b_1 - b_2 = c_1 - c_2 = b$, i.e., μ_1 and μ_2 have the same "shape", and $G(\mu_1) = G(\mu_2)$. □

Proposition 2.6. *If $\mu_1 \ominus \mu_2$ is defined, then $G(\mu_1) \leq \min\{G(\mu_1 \ominus \mu_2), G(\mu_2)\}$.*

Proof. If $\mu_1 \ominus \mu_2$ is defined, then $\mu_1 = (\mu_1 \ominus \mu_2) + \mu_2$. By applying Theorem 2.1, we obtain the result. □

Remark. The fact $G(\mu_1) \leq G(\mu_2)$ is not sufficient to determine that $\mu_1 \ominus \mu_2$ is defined. For example, consider $\mu_1 =< 2, 3, 7 >$ and $\mu_2 =< 1, 3, 4 >$. Then $G(\mu_1) = \frac{e^{2/5}-1}{e^{2/5}+1} \leq \frac{e^{2/3}-1}{e^{2/3}+1} = G(\mu_2)$, but $\mu_1 \ominus \mu_2$ is not defined, since $2 - 1 > 3 - 3, 3 - 3 < 7 - 4$.

3. Multiplcations of linear fuzzy real numbers

Let $\mathcal{A}$ be the set of all linear fuzzy real numbers $\mu =< a, b, c >$ with $a \neq c$. Given $\mu_i =< a_i, b_i, c_i > \in \mathcal{A}$ $(i = 1, 2)$, we construct $\mu_1 \odot \mu_2$ as follows:

$$\begin{aligned} a &= \inf\{t_1 t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ c &= \sup\{t_1 t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ b &= \frac{a + c}{2}\{\frac{b_1 - a_1}{c_1 - a_1} + \frac{b_2 - a_2}{c_2 - a_2}\}. \end{aligned}$$

For example, $< -3, -2, -1 > \odot < -5, -1, 4 >=< -12, b, 15 >$ where $b = \frac{3}{2}\{\frac{1}{2} + \frac{4}{9}\} = \frac{17}{12}$, and $< -3, -2, 2 > \odot < -5, -1, 4 >=< -12, \frac{29}{30}, 15 >$.

Remark. The associative law for the product $\odot$ fails for linear fuzzy real numbers. Consider $\mu_1 =< -10, -9, -1 >, \mu_2 =< -8, 1, 2 >$ and $\mu_3 =< 1, 4, 5 >$. Then $(\mu_1 \odot \mu_2) \odot \mu_3 =< -20, \frac{91}{3}, 80 > \odot < 1, 4, 5 >=< -100, 94, 400 >$, but $\mu_1 \odot (\mu_2 \odot \mu_3) =< -10, -9, -1 > \odot < -40, -\frac{94}{5}, 10 >=< -100, \frac{1204}{15}, 400 >$. Hence $(\mu_1 \odot \mu_2) \odot \mu_3 \neq \mu_1 \odot (\mu_2 \odot \mu_3)$.

Consider a linear fuzzy real number $\mu =< a, \frac{a+c}{2}, c >$ with $a \neq c$. We call such a fuzzy subset μ a *symmetric linear fuzzy real number.* Let $\mathcal{B}$ be the set of all symmetric linear fuzzy real numbers $\mu =< a, \frac{a+c}{2}, c >$ with $a \neq c$. It is easy to show that if $\sigma_1, \sigma_2 \in \mathcal{B}$, then $\sigma_1 + \sigma_2 \in \mathcal{B}$, and $\sigma_1 \odot \sigma_2 \in \mathcal{B}$. Furthermore, it is easy to show that $(\sigma_1 \odot \sigma_2) \odot \sigma_3 = \sigma_1 \odot (\sigma_2 \odot \sigma_3)$ and $\sigma_1 \odot \sigma_2 = \sigma_2 \odot \sigma_1$. We summarize:

Theorem 3.1. *Let $\mathcal{B}$ be the set of all symmetric linear fuzzy real numbers $\mu =< a, \frac{a+c}{2}, c >$ with $a \neq c$. Then $(\mathcal{B}, \odot)$ is a commutative semigroup. Moreover, $\mu \odot \delta_0 = \delta_0$ for all $\mu \in \mathcal{B}$.*

Remark. Given the symmetric linear fuzzy real numbers $\mu_1 =< -2, -1.5, -1 >, \mu_2 =< -3, -0.5, 2 >, \mu_3 =< -4, -1, 2 > \in \mathcal{B}$, we have $(\mu_1 + \mu_2) \odot \mu_3 =< -10, 5, 20 >$, but $\mu_1 \odot \mu_2 + \mu_2 \odot \mu_3 =< -12, 4, 20 >$. Hence the distributive law fails.

A linear fuzzy real number $\mu =< a, b, c >$ is said to be *positive*(*negative*, resp.) if $a > 0$ $(c < 0$, resp.).

Proposition 3.2. *Let $\mu_i \in \mathcal{B}$ $(i = 1, 2, 3)$. If μ_3 is positive (or negative), then $(\mu_1 + \mu_2) \odot \mu_3 = \mu_1 \odot \mu_3 + \mu_2 \odot \mu_3$. If μ_1 is positive (or negative), then $\mu_1 \odot (\mu_2 + \mu_3) = \mu_1 \odot \mu_2 + \mu_1 \odot \mu_3$.*

Proof. Straightforward. ☐

Given $\mu_i =< a_i, b_i, c_i >\in \mathcal{A}$ $(i = 1, 2)$, we consider a "*weighted product*" $\mu_1 \otimes \mu_2 :=< a, b, c >$ where

$$\begin{aligned} a &= \inf\{t_1 t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ c &= \sup\{t_1 t_2 \,|\, t_i \in [a_i, c_i], i = 1, 2\}, \\ b &= (a + c)\Big[\frac{\frac{b_1 - a_1}{(c_1 - a_1)^2} + \frac{b_2 - a_2}{(c_2 - a_2)^2}}{\frac{1}{c_1 - a_1} + \frac{1}{c_2 - a_2}}\Big]. \end{aligned}$$

Thus, if $c_1 - a_1 = c_2 - a_2$, then we obtain for b the formula:

$$b = (a + c)\frac{(b_1 - a_1) + (b_2 - a_2)}{(c_1 - a_1) + (c_2 - a_2)}$$

If $\frac{b_1 - a_1}{c_1 - a_1} = \frac{b_2 - a_2}{c_2 - a_2} = \frac{1}{2}$, then

$$\begin{aligned} b &= (a + c)\,\frac{\frac{1}{2}(\frac{1}{c_1 - a_1}) + \frac{1}{2}(\frac{1}{c_2 - a_2})}{\frac{1}{c_1 - a_1} + \frac{1}{c_2 - a_2}} \\ &= \frac{a + c}{2} \end{aligned}$$

so that for $\mu_1, \mu_2 \in \mathcal{B}$, we obtain $\mu_1 \otimes \mu_2 = \mu_1 \odot \mu_2$. We summarize:

Proposition 3.3. *If $\mu_1, \mu_2 \in \mathcal{B}$, then $\mu_1 \otimes \mu_2 = \mu_1 \odot \mu_2$.*

Remark. The weighted product $\sigma_1 \otimes \sigma_2$ is not associative in general. For example, let $\sigma_1 :=< -10, -9, -1 >, \sigma_2 :=< -8, 1, 2 >$ and $\sigma_3 :=< 1, 4, 5 >$. Then $\sigma_1 \otimes \sigma_2 =< -20, \frac{1{,}658}{57}, 80 >$ and $\sigma_2 \otimes \sigma_3 =< -40, -\frac{333}{14}, 10 >$ and so $(\sigma_1 \otimes \sigma_2) \otimes \sigma_3 =< -100, \frac{109{,}673}{494}, 400 >$ and $\sigma_1 \otimes (\sigma_2 \otimes \sigma_3) =< -100, \frac{106{,}774}{2{,}478}, 400 >$.

Actually, in general case, products $\mu_1 \odot \delta_{b_2}$ and $\mu_1 \otimes \delta_{b_2}$ are troublesome to define in a simple way since $a_2 = b_2 = c_2$ produce singularities.

Theorem 3.4. *Let $\mathcal{C}$ be the set of all positive (or negative) symmetric linear fuzzy real numbers. Then $(\mathcal{C}, +, \odot)$ is a semiring.*

Proof. It follows from Theorems 2.4 and 3.1, and Proposition 3.2. ☐

4. Complex transforms of linear fuzzy real numbers

Given the linear fuzzy real number $\mu = \mu(a, b, c)$ we may associate with it the complex transform $T(\mu)$, where

$$T(\mu)(s) = \int_{-\infty}^{\infty} s^2 e^{st} \mu(a, b, c)(t) dt \tag{5}$$

By integration by parts we obtain

$$\begin{aligned} T(\mu)(s) &= \int_a^c s^2 e^{st} \mu(a, b, c)(t) dt \\ &= \int_a^b s^2 e^{st} \frac{t-a}{b-a} dt + \int_b^c s^2 e^{st} \frac{c-t}{c-b} dt \\ &= \frac{1}{c-b}(e^{sc} - e^{sb}) - \frac{1}{b-a}(e^{sb} - e^{sa}) \end{aligned}$$

We summarize:

Proposition 4.1. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number, then its associated complex transformation $T(\mu)$ is*

$$T(\mu)(s) = \frac{1}{c-b}(e^{sc} - e^{sb}) - \frac{1}{b-a}(e^{sb} - e^{sa}) \tag{6}$$

Example 4.2. If $\mu_0 = \mu(-1, 0, 1)$, then $T(\mu_0)(s) = (e^s - 1) - (1 - e^{-s}) = (e^s + e^{-s}) - 2 = 2\cosh s - 2$ or "inversely" $\cosh s = \frac{T(\mu_0)(s)+2}{2}$ indicating that we may consider the functions $T(\mu)(s)$ to be "pseudo-hyperbolic" in nature.

Proposition 4.3. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda \neq 0$, then*

$$T(\mu)(\lambda s) = \frac{1}{c-b}(e^{\lambda sc} - e^{\lambda sb}) - \frac{1}{b-a}(e^{\lambda sb} - e^{\lambda sa}) \tag{7}$$

Proof. If $\lambda \neq 0$, then

$$\begin{aligned} T(\mu)(\lambda s) &= \int_{-\infty}^{\infty} (\lambda s)^2 e^{\lambda st} \mu(a, b, c)(t) dt \\ &= \lambda^2 \int_a^b s^2 e^{\lambda st} \frac{t-a}{b-a} dt + \lambda^2 \int_b^c s^2 e^{\lambda st} \frac{c-t}{c-b} dt \end{aligned}$$

By integration by parts, we obtain $\int_a^b s^2 e^{\lambda st} \frac{t-a}{b-a} dt = \frac{s}{\lambda} e^{\lambda sb} - \frac{1}{b-a} \frac{1}{\lambda^2}(e^{\lambda sb} - e^{\lambda sa})$ and $\int_b^c s^2 e^{\lambda st} \frac{c-t}{c-b} dt = -\frac{s}{\lambda} e^{\lambda sb} + \frac{1}{c-b} \frac{1}{\lambda^2}(e^{\lambda sc} - e^{\lambda sb})$, which proves the proposition. □

Given a linear fuzzy real number $\mu = \mu(a, b, c)$ and $\lambda \in \mathbf{R}$, we define a new linear fuzzy real number $\lambda\mu$ as follows:

$$\lambda\mu(a, b, c) := \begin{cases} \mu(\lambda a, \lambda b, \lambda c) & \text{if } \lambda \geq 0 \\ \mu(\lambda c, \lambda b, \lambda a) & \text{otherwise} \end{cases}$$

Proposition 4.4. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda > 0$, then*

$$T(\mu)(\lambda s) = \lambda T(\lambda\mu)(s)$$

Proof. If $\mu = \mu(a, b, c)$ and $\lambda > 0$, then $\lambda\mu = \mu(\lambda a, \lambda b, \lambda c)$ and hence

$$\begin{aligned} T(\lambda\mu)(s) &= \int_{\lambda a}^{\lambda c} s^2 e^{st} \mu(\lambda a, \lambda b, \lambda c)(t) dt \\ &= \int_{\lambda a}^{\lambda b} s^2 e^{st} \frac{t - \lambda a}{\lambda b - \lambda a} dt + \int_{\lambda b}^{\lambda c} s^2 e^{st} \frac{\lambda c - t}{\lambda c - \lambda b} dt \\ &= \frac{1}{\lambda(c - b)}(e^{s\lambda c} - e^{s\lambda b}) - \frac{1}{\lambda(b - a)}(e^{s\lambda b} - e^{s\lambda a}) \end{aligned}$$

By multiplying λ to both sides and by applying Proposition 4.3, we proves the proposition. □

Proposition 4.5. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda < 0$, then*

$$T(\mu)(\lambda s) = (-\lambda)T(\lambda\mu)(s)$$

Proof. If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda < 0$, then $\lambda\mu = \mu(-|\lambda|c, -|\lambda|b, -|\lambda|a)$. By applying Proposition 4.1, we obtain

$$T(\lambda\mu)(s) = \frac{1}{(-|\lambda|a) - (-|\lambda|b)}(e^{s(-|\lambda|a)} - e^{s(-|\lambda|b)}) - \frac{1}{(-|\lambda|b) - (-|\lambda|c)}(e^{s(-|\lambda|b)} - e^{s(-|\lambda|c)}).$$

Using Proposition 4.3 we obtain

$$\begin{aligned} \lambda T(\lambda\mu)(s) &= (-|\lambda|)T(\lambda\mu)(s) \\ &= \frac{1}{a - b}(e^{\lambda sa} - e^{\lambda sb}) - \frac{1}{b - c}(e^{\lambda sb} - e^{\lambda sc}) \\ &= -T(\mu)(\lambda s) \end{aligned}$$

□

Combining Propositions 4.4 and 4.5 we obtain:

Theorem 4.6. *If $\mu = \mu(a, b, c)$ is a linear fuzzy real number and $\lambda \in \mathbf{R}$, then*

$$T(\mu)(\lambda s) = |\lambda| T(\lambda\mu)(s) \tag{8}$$

Proof. For non-zero real number λ, it was proved by Propositions 4.4 and 4.5. If $\lambda = 0$, then $T(\mu)(0s) = 0$, and so (8) holds trivially. □

Given a fuzzy real number $\mu = \mu(a, b, c)$ and $\lambda = -1$, we have $T(\mu)(-s) = T(\mu)((-1)s) = |-1|T((-1)\mu)(s) = T(\mu(-c, -b, -a))(s)$, i.e.,

$$T(\mu(a, b, c))(-s) = T(\mu(-c, -b, -a))(s) \tag{9}$$

If we let $s := -s$ in (9), then we have

$$T(\mu(a, b, c))(s) = T(\mu(-c, -b, -a))(-s) \tag{10}$$

For $\lambda = 2$, we obtain from (8) a "doubling formula".

$$T(\mu)(2s) = 2T(2\mu)(s) \tag{11}$$

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