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APPROXIMATE EULER–LAGRANGE QUADRATIC MAPPINGS IN FUZZY BANACH SPACES *

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ABSTRACT. For any rational numbers k, l with $kl(l-1) \neq 0$, we prove the generalized Hyers–Ulam stability of the Euler-Lagrange quadratic functional equation

$$f(kx+ly)+f(kx-ly)+2(l-1)[k^2f(x)-lf(y)] = l[f(kx+y)+f(kx-y)]$$

using both the direct method and fixed point method in fuzzy Banach spaces.

1. INTRODUCTION.

Some mathematicians have established fuzzy spaces with fuzzy norms on linear spaces from various points of view [2, 12, 18, 34]. Xiao and Zhu [34], Cheng and Mordeson [6], and Bag and Samanta [2, 3] gave the idea of fuzzy norms over linear spaces in such a manner that the corresponding fuzzy metric may be of Kramosil and Michalek type [17] and investigated some properties of fuzzy linear operators on fuzzy normed spaces.

Now, we introduce the definition of fuzzy normed spaces given in [2, 21, 22].

Definition 1.1 [2, 21, 22]**.** Let X be a real linear space. A function $N : X \times \mathbf{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbf{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ for $c \neq 0$;
(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N_5) $N(x, \cdot)$ is a non-decreasing function on $\mathbf{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbf{R}$.

The pair (X, N) is called a fuzzy normed (linear) space. The properties of fuzzy normed linear spaces and examples of fuzzy norms are given in [21, 23].

Definition 1.2 [2, 21, 22]**.** Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent or to converge to x if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$, and we denote it by N-$\lim_{n\to\infty} x_n = x$.

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1

Definition 1.3 [2, 21, 22]**.** Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space. They say that a mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at $x_0 \in X$ if for each sequence $\{x_n\}$ converging to each $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [3, 21]).

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [14] gave the first affirmative partial answer to the question of Ulam for additive mappings on Banach spaces. Hyers's theorem has been generalized by Aoki [1], Th.M. Rassias [28] and Găvruta [13] by considering an unbounded Cauchy difference. The classical functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

associated with the parallelogram equality $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ in inner product spaces, is called a quadratic functional equation, and every solution of the quadratic functional equation is said to be a quadratic mapping. First of all, the Hyers-Ulam stability problem for the quadratic functional equation has been established by Skof [32], Cholewa [7] and Czerwik [9]. In particular, Isac and Th.M Rassias [15] have provided a new application of fixed point theorems to prove the stability theory of functional equations. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 8, 31, 23, 27, 26]).

We recall the fixed point theorem from [19], which is needed in Section 3.

Theorem 1.4 [4, 19]**.** Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with the Lipschitz constant $L < 1$. Then, for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

On the other hand, J.M. Rassias investigated the Hyers–Ulam stability for the relative Euler–Lagrange functional equation

$$f(ax + by) + f(bx - ay) = (a^2 + b^2)[f(x) + f(y)]$$

in [29, 30]. The stability problems of several quadratic functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [5, 24, 10, 11]). In the paper [16], the authors have proved the generalized Hyers–Ulam stability of the Euler–Lagrange quadratic functional equation

$$\begin{aligned} f(kx+ly)+f(kx-ly) \\ = kl[f(x+y)+f(x-y)]+2(k-l)[kf(x)-lf(y)] \end{aligned} \tag{1.1}$$

in fuzzy Banach spaces, where k, l are nonzero rational numbers with $k \neq l$.

Motivated to research stability results of the Euler–Lagrange functional equation, we investigate the generalized Hyers–Ulam stability of the following modified Euler–Lagrange functional equation

$$\begin{aligned} f(kx+ly)+f(kx-ly)+2(l-1)[k^2f(x)-lf(y)] \\ = l[f(kx+y)+f(kx-y)] \end{aligned} \tag{1.2}$$

using both the fixed point method and the direct method in fuzzy Banach spaces in the paper, where k, l are nonzero rational numbers with $kl(l-1) \neq 0$. Throughout the paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed space.

2. General solution of (1.2).

The following lemma can be found in the paper [16].

Lemma 2.1. [16] A mapping $f : X \to Y$ between linear spaces satisfies the functional equation

$$f(rx+y)+f(rx-y) = r[f(x+y)+f(x-y)]+2(r-1)[rf(x)-f(y)]$$

for any fixed rational numbers r with $r \neq 0, 1$ if and only if f is quadratic.

Now, we present the general solution of the functional equation (1.2).

Theorem 2.2. A mapping $f : X \to Y$ between vector spaces satisfies the functional equation (1.2) if and only if $f - f(0)$ is quadratic, where $f(0) = 0$ whenever $k^2 \neq l+1$.

Proof. First of all, replacing $(x, y) := (0, 0)$ in the functional equation (1.2), we find $f(0) = 0$ whenever $k^2 \neq l + 1$. Substituting $(x, y) := (x, 0)$ in (1.2), we get $f(kx) = k^2 f(x)$ for all $x \in X$. Putting $(x, y) := (0, x)$ in (1.2), one has

$$f(lx)+f(-lx) = (2l^2-l)f(x)+lf(-x) \tag{2.1}$$

for all $x \in X$. Replacing x by $-x$ in (2.1), one gets

$$f(-lx)+f(lx) = (2l^2-l)f(-x)+lf(x) \tag{2.2}$$

for all $x \in X$. Subtracting equation (2.1) from (2.2), we find $f(-x) = f(x)$ and so $f(lx) = l^2 f(x)$ for all $x \in X$. Thus the equation (1.2) can be rewritten as

$$f(x+\frac{ly}{k})+f(x-\frac{ly}{k}) = l[f(x+\frac{y}{k})+f(x-\frac{y}{k})]-2(l-1)[f(x)-lf(\frac{y}{k})],$$

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which yields by switching (x, y) with (y, kx)

$$f(lx+y)+f(lx-y)=l[f(x+y)+f(x-y)]+2(l-1)[lf(x)-f(y)]$$

for all $x, y \in X$. Therefore, it follows from Lemma 2.1 that f is quadratic.

Conversely, if a mapping f is quadratic, then it is obvious that f satisfies the equation (1.2).

3. Stability of equation (1.2) by fixed point method.

For notational convenience, we define the difference operator $D_{kl}f : X^2 \to Y$ of the equation (1.2) for a given mapping $f : X \to Y$ as

$$\begin{aligned} D_{kl}f(x,y) \ &:= \ f(kx+ly)+f(kx-ly)+2(l-1)[k^2f(x)-lf(y)] \\ &\quad -l[f(kx+y)+f(kx-y)] \end{aligned}$$

for all $x, y \in X$. Now, we are going to consider a stability problem concerning the stability of equation (1.2) by using the fixed point theorem for contraction mappings on generalized complete metric spaces.

Theorem 3.1. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality

$$N(D_{kl}f(x,y), t_1+t_2) \geq \min\{N'(\varphi(x), t_1^q), N'(\varphi(y), t_2^q)\} \tag{3.1}$$

for all $x, y \in X$ and all $t_i > 0$ $(i = 1, 2)$, and for some $q > 0$, and assume in addition that there exists a constant $s \in \mathbf{R}$ with $|s| \neq 1$, $0 < |s|^{\frac{1}{q}} < k^2$ such that a constrained function $\varphi : X \to Z$ satisfies the inequality

$$N'(\varphi(kx), t) \geq N'(s\varphi(x), t), \tag{3.2}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x, y) = 0$ and the approximate functional inequality

$$\begin{aligned} N(f(x)-Q(x),t) \ &\geq \ \min\Big\{N'\Big(\frac{\varphi(x)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, t^q\Big), \\ &\qquad N'\Big(\frac{\varphi(0)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, t^q\Big)\Big\} \end{aligned} \tag{3.3}$$

near f for all $x \in X$ and all $t > 0$.

Proof. We consider the set of functions

$$\Omega := \{g : X \to Y | g(0) = 0\}$$

and define a generalized metric on Ω as follows:

$$\begin{aligned} d_\Omega(g,h) := \inf\Big\{K \in [0,\infty] : N(g(x)-h(x), Kt) \geq \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\}, \\ \forall x \in X, \forall t > 0\Big\}. \end{aligned}$$

Then one can easily see that (Ω, d_Ω) is a complete generalized metric space [20].

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Now, we define an operator $J : \Omega \to \Omega$ as

$$Jg(x) = \frac{g(kx)}{k^2}$$

for all $g \in \Omega$, $x \in X$.

We first prove that J is strictly contractive on Ω. For any $g, h \in \Omega$, let $\varepsilon \in [0, \infty)$ be any constant with $d_\Omega(g, h) \le \varepsilon$. Then it follows from the use of (3.2) and the definition of $d_\Omega(g, h) \le \varepsilon$ that

$$\begin{aligned}
&N(g(x) - h(x), \varepsilon t) \ge \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\},\\
&\quad \Rightarrow N\Big(\frac{g(kx)}{k^2} - \frac{h(kx)}{k^2}, \frac{|s|^{\frac{1}{q}}}{k^2}\varepsilon t\Big) \ge \min\{N'(\varphi(kx), |s|t^q), N'(\varphi(0), |s|t^q)\},\\
&\quad \Rightarrow N\Big(Jg(x) - Jh(x), \frac{|s|^{\frac{1}{q}}}{k^2}\varepsilon t\Big) \ge \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\},\\
&\quad \Rightarrow d_\Omega(Jg, Jh) \le \frac{|s|^{\frac{1}{q}}}{k^2}\varepsilon, \qquad \forall x \in X, t > 0.
\end{aligned}$$

Since ε is an arbitrary constant with $d_\Omega(g, h) \le \varepsilon$, we see that for any $g, h \in \Omega$,

$$d_\Omega(Jg, Jh) \le \frac{|s|^{\frac{1}{q}}}{k^2} d_\Omega(g, h),$$

which implies J is strictly contractive with the constant $\frac{|s|^{\frac{1}{q}}}{k^2} < 1$ on Ω.

We now want to show that $d_\Omega(f, Jf) < \infty$. If we put $y := 0$, $t_i := t$ $(i = 1, 2)$ in (3.1), then we arrive at

$$N\Big(f(x) - \frac{f(kx)}{k^2}, \frac{t}{|l-1|k^2}\Big) \ge \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\},$$

which yields $d_\Omega(f, Jf) \le \frac{1}{|l-1|k^2} < \infty$, and so

$$d_\Omega(J^n f, J^{n+1} f) \le d_\Omega(f, Jf) \le \frac{1}{|l-1|k^2}$$

for all $n \in \mathbf{N}$. Now, applying the fixed point theorem of the alternative for contractions on generalized complete metric spaces due to Margolis and Diaz [19], we obtain the following approximate functional inequalities (a), (b) and (c):

(a) There is a mapping $Q : X \to Y$ with $Q(0) = 0$ such that

$$d_\Omega(f, Q) \le \frac{1}{1 - \frac{|s|^{\frac{1}{q}}}{k^2}} d_\Omega(f, Jf) \le \frac{1}{|l-1|(k^2 - |s|^{\frac{1}{q}})},$$

and thus Q is a fixed point of the operator J, that is, $\frac{1}{k^2}Q(kx) = JQ(x) = Q(x)$ for all $x \in X$. Thus we arrive at

$$\begin{aligned} &N\Big(f(x)-Q(x), \frac{t}{|l-1|(k^2-|s|^{\frac{1}{q}})}\Big) \geq \min\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\},\\ &N(f(x)-Q(x), t) \geq \min\Big\{N'\Big(\varphi(x), |l-1|^q(k^2-|s|^{\frac{1}{q}})^q\, t^q\Big),\\ &\qquad\qquad N'\Big(\varphi(0), |l-1|^q(k^2-|s|^{\frac{1}{q}})^q\, t^q\Big)\Big\} \end{aligned}$$

for all $t>0$ and all $x \in X$, which implies the approximation (3.3).

(b) Since $d_\Omega(J^n f, Q) \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned} &N\Big(\frac{f(k^n x)}{k^{2n}} - Q(x), t\Big)\\ &= N(f(k^n x) - Q(k^n x), k^{2n} t)\\ &\geq \min\Big\{N'\Big(\frac{\varphi(k^n x)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, k^{2nq}t^q\Big), N'\Big(\frac{\varphi(0)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, k^{2nq}t^q\Big)\Big\}\\ &\geq \min\Big\{N'\Big(\frac{\varphi(x)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, \Big(\frac{k^{2q}}{|s|}\Big)^n t^q\Big), N'\Big(\frac{\varphi(0)}{|l-1|^q(k^2-|s|^{\frac{1}{q}})^q}, \Big(\frac{k^{2q}}{|s|}\Big)^n t^q\Big)\Big\}\\ &\to 1 \quad \text{as} \quad n \to \infty \quad \Big(\frac{k^{2q}}{|s|} > 1\Big) \end{aligned}$$

for all $t > 0$ and all $x \in X$, that is, the mapping $Q : X \to Y$ given by

$$N\text{-}\lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}} = Q(x) \tag{3.4}$$

is well defined for all $x \in X$. In addition, it follows from the conditions (3.1), (3.2) and (N_4) that

$$\begin{aligned} N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, t\Big) \quad &\geq \quad \min\Big\{N'\Big(\varphi(k^n x), \frac{k^{2nq}t^q}{2^q}\Big), N'\Big(\varphi(k^n y), \frac{k^{2nq}t^q}{2^q}\Big)\Big\}\\ &\geq \quad \min\Big\{N'\Big(|s|^n\varphi(x), \frac{k^{2nq}t^q}{2^q}\Big), N'\Big(|s|^n\varphi(y), \frac{k^{2nq}t^q}{2^q}\Big)\Big\}\\ &\geq \quad \min\Big\{N'\Big(\varphi(x), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{2^q}\Big), N'\Big(\varphi(y), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{2^q}\Big)\Big\}\\ &\to \quad 1 \quad \text{as} \quad n \to \infty, \quad t > 0, \end{aligned} \tag{3.5}$$

for all $x \in X$. Therefore we obtain, by use of (N_4), (3.4) and (3.5),

$$\begin{aligned} N(D_{kl}Q(x,y),t) &\geq \min\Big\{N\Big(D_{kl}Q(x,y) - \frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big), \\ &\qquad N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big)\Big\} \\ &= N\Big(\frac{D_{kl}f(k^n x, k^n y)}{k^{2n}}, \frac{t}{2}\Big), \qquad \text{(for sufficiently large } n) \\ &\geq \min\Big\{N'\Big(\varphi(x), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{4^q}\Big), N'\Big(\varphi(y), \Big(\frac{k^{2q}}{|s|}\Big)^n \frac{t^q}{4^q}\Big)\Big\} \\ &\to 1 \quad \text{as} \quad n \to \infty, \quad t > 0, \end{aligned}$$

which implies $D_{kl}Q(x,y) = 0$ by (N_2), and so the mapping Q is quadratic satisfying equation (1.2).

(c) The mapping Q is a unique fixed point of the operator J in the set $\Delta = \{g \in \Omega | d_\Omega(f,g) < \infty\}$. Thus, if we assume that there exists another Euler-Lagrange type quadratic mapping $Q' : X \to Y$ satisfying inequality (3.3), then

$$Q'(x) = \frac{Q'(kx)}{k^2} = JQ'(x), \quad d_\Omega(f, Q') \leq \frac{1}{|l-1|(k^2 - |s|^{\frac{1}{q}})} < \infty,$$

and so Q' is a fixed point of the operator J and $Q' \in \Delta = \{g \in \Omega | d_\Omega(f,g) < \infty\}$. By the uniqueness of the fixed point of J in Δ, we find that $Q = Q'$, which proves the uniqueness of Q satisfying inequality (3.3). This ends the proof of the theorem.

We observe that if $0 < |s| < 1$ in Theorem 3.1, then

$$\min\big\{N'(\varphi(x), t^q), N'(\varphi(0), t^q)\big\} = N'(\varphi(x), t^q)$$

for all $x \in X$ and all $t > 0$ since $N'(\varphi(0), t^q) \geq N'\Big(\varphi(0), \frac{t^q}{|s|^n}\Big) \to 1$ as $n \to \infty$ by the condition (3.2).

Theorem 3.2 Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl}f(x,y), t_1 + t_2) \geq \min\{N'(\varphi(x), t_1^q), N'(\varphi(y), t_2^q)\}$$

for all $x, y \in X$ and all $t_i > 0$ $(i = 1, 2)$ and for some $q > 0$, and furthermore assume that there exists a constant $s \in \mathbf{R}$ with $|s| \neq 1$, $|s|^{\frac{1}{q}} > k^2$ such that a constrained function $\varphi : X \to Z$ satisfies

$$N'\Big(\varphi\Big(\frac{x}{k}\Big), t\Big) \geq N'\Big(\frac{1}{s}\varphi(x), t\Big)$$

for all $x \in X$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x,y) = 0$ and the approximate functional inequality

$$N(f(x)-Q(x),t) \geq \min\Big\{N'\Big(\frac{\varphi(x)}{|l-1|^q(|s|^{\frac{1}{q}}-k^2)^q}, t^q\Big), N'\Big(\frac{\varphi(0)}{|l-1|^q(|s|^{\frac{1}{q}}-k^2)^q}, t^q\Big)\Big\}, \tag{3.6}$$

for all $t > 0$ and all $x \in X$.

Proof. Finally, applying the same argument as in the proof of Theorem 3.1, we can find a mapping $Q : X \to Y$ defined by

$$N\text{-}\lim_{n\to\infty} k^{2n} f(\frac{x}{k^n}) = Q(x)$$

satisfying the equation $D_{kl}Q(x,y) = 0$ and the approximate functional inequality (3.6) near f.

4. Stability of equation (1.2) by direct method.

In the following, we are going to investigate alternatively generalized Hyers–Ulam stability of the Euler–Lagrange functional equation (1.2) via the direct method in fuzzy Banach spaces.

Theorem 4.1. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl}f(x,y),t) \geq N'(\varphi(x,y),t) \tag{4.1}$$

and assume in addition that there exists a constant $s \in \mathbf{R}$ subject to $0 < |s| < k^2$ such that a constrained function $\varphi : X^2 \to Z$ satisfies the functional inequality

$$N'(\varphi(kx,ky),t) \geq N'(s\varphi(x,y),t) \tag{4.2}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x,y) = 0$ and the approximate inequality

$$N(f(x)-Q(x),t) \geq N'\Big(\frac{\varphi(x,0)}{2|l-1|(k^2-|s|)}, t\Big), \quad t > 0, \tag{4.3}$$

for all $x \in X$.

Proof. It follows from the assumption (4.2) that

$$\begin{aligned} N'(\varphi(k^nx,k^ny),t) &\geq N'(s^n\varphi(x,y),t) \\ &= N'\Big(\varphi(x,y), \frac{t}{|s|^n}\Big), \quad t > 0, \end{aligned}$$

which yields

$$N'(\varphi(k^nx,k^ny),|s|^nt) \geq N'(\varphi(x,y),t), \quad t > 0, \tag{4.4}$$

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for all $x, y \in X$. Putting $(x, y) := (x, 0)$ in (4.1), we have

$$N(2(l-1)f(kx) - 2(l-1)k^2 f(x), t) \geq N'(\varphi(x,0), t),$$

$$\text{or,} \quad N\Big(f(x) - \frac{f(kx)}{k^2}, \frac{t}{2|l-1|k^2}\Big) \geq N'(\varphi(x,0), t) \tag{4.5}$$

for all $x \in X$. Therefore it follows from (4.4), (4.5) that

$$N\Big(\frac{f(k^n x)}{k^{2n}} - \frac{f(k^{n+1}x)}{k^{2(n+1)}}, \frac{|s|^n t}{2|l-1|k^{2(n+1)}}\Big) \geq N'(\varphi(k^n x, 0), |s|^n t) \geq N'(\varphi(x,0), t)$$

for all $x \in X$ and any integer $n \geq 0$. Thus, we deduce the functional inequality

$$\begin{aligned} N\Big(f(x) - \frac{f(k^n x)}{k^{2n}}, \sum_{i=0}^{n-1} \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \qquad (4.6)\\ = N\Big(\sum_{i=0}^{n-1}\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(i+1)}}\Big), \sum_{i=0}^{n-1} \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \\ \geq \min_{0\leq i\leq n-1}\Big\{N\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(i+1)}}, \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big)\Big\} \\ \geq N'(\varphi(x,0), t), \quad t > 0, \end{aligned}$$

which implies

$$\begin{aligned} & N\Big(\frac{f(k^m x)}{k^{2m}} - \frac{f(k^{m+p}x)}{k^{2(m+p)}}, \sum_{i=m}^{m+p-1} \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \\ & = N\Big(\sum_{i=m}^{m+p-1}\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(i+1)}}\Big), \sum_{i=m}^{m+p-1} \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big) \\ & \geq \min_{m\leq i\leq m+p-1}\Big\{N\Big(\frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2(n+1)}}, \frac{|s|^i t}{2|l-1|k^{2(i+1)}}\Big)\Big\} \\ \geq \quad & N'(\varphi(x,0), t), \quad t > 0, \end{aligned}$$

for all $x \in X$ and any integers $p > 0, m \geq 0$. Therefore, one concludes

$$N\Big(\frac{f(k^m x)}{k^{2m}} - \frac{f(k^{m+p}x)}{k^{2(m+p)}}, t\Big) \geq N'\Big(\varphi(x,0), \frac{t}{\sum_{i=m}^{m+p-1} \frac{|s|^i}{2|l-1|k^{2(i+1)}}}\Big) \tag{4.7}$$

for all $x \in X$ and any integers $p > 0, m \geq 0, t > 0$. Since $\sum_{i=m}^{m+p-1} \frac{|s|^i}{k^{2i}}$ is a convergent series, we know that the sequence $\{\frac{f(k^n x)}{k^{2n}}\}$ is Cauchy in the fuzzy Banach space (Y, N), and so it converges in Y. Therefore a mapping $Q : X \to Y$ defined by

$$Q(x) := N\text{-}\lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}} \Leftrightarrow \lim_{n\to\infty} N(\frac{f(k^n x)}{k^{2n}} - Q(x), t) = 1, \ \forall t > 0,$$

is well defined for all $x \in X$. In addition, we see from (4.6) that

$$N\Big(f(x) - \frac{f(k^n x)}{k^{2n}}, t\Big) \geq N'\Big(\varphi(x,0), \frac{t}{\sum_{i=0}^{n-1} \frac{|s|^i}{2|l-1|k^{2(i+1)}}}\Big), \tag{4.8}$$

and thus for any ε with $0 < \varepsilon < 1$ the following inequality

$$
\begin{aligned}
N(f(x)-Q(x),t) &\geq \min\Big\{N\Big(f(x)-\frac{f(k^n x)}{k^{2n}},(1-\varepsilon)t\Big),\\
&\qquad\qquad N\Big(\frac{f(k^n x)}{k^{2n}}-Q(x),\varepsilon t\Big)\Big\}\\
&\geq N'\Big(\varphi(x,0),\frac{(1-\varepsilon)t}{\sum_{i=0}^{n-1}\frac{|s|^i}{2|l-1|k^{2(i+1)}}}\Big)\\
&\geq N'(\varphi(x,0),2|l-1|(1-\varepsilon)(k^2-|s|)t),
\end{aligned}
\tag{4.9}
$$

holds good for sufficiently large n, and for all $x \in X$ and all $t > 0$. Since ε is arbitrary and N' is a left continuous function, we obtain

$$N(f(x)-Q(x),t) \geq N'(\varphi(x,0),2|l-1|(k^2-|s|)t), \quad t>0,$$

for all $x \in X$, which yields the approximation (4.3).

On the other hand, it is clear from (4.1) and (N_5) that the relation

$$
\begin{aligned}
N\Big(\frac{D_{kl}f(k^n x,k^n y)}{k^{2n}},t\Big) &\geq N'(\varphi(k^n x,k^n y),k^{2n}t)\\
&\geq N'\Big(\varphi(x,y),\frac{k^{2n}}{|s|^n}t\Big)\\
&\to 1 \quad \text{as} \quad n\to\infty
\end{aligned}
$$

holds for all $x, y \in X$ and all $t > 0$. Therefore, we figure out by definition of $\lim_{n\to\infty} N(\frac{f(k^n x)}{k^{2n}} - Q(x), t) = 1$ for all $(t > 0)$ that

$$
\begin{aligned}
N(D_{kl}Q(x,y),t) &\geq \min\Big\{N\Big(D_{kl}Q(x,y)-\frac{D_{kl}f(k^n x,k^n y)}{k^{2n}},\frac{t}{2}\Big),\\
&\qquad\qquad N\Big(\frac{D_{kl}f(k^n x,k^n y)}{k^{2n}},\frac{t}{2}\Big)\Big\}\\
&= N\Big(\frac{D_{kl}f(k^n x,k^n y)}{k^{2n}},\frac{t}{2}\Big) \qquad \text{(for sufficiently large } n)\\
&\geq N'\Big(\varphi(x,y),\frac{k^{2n}}{2|s|^n}t\Big), \quad t>0\\
&\to 1 \quad \text{as} \quad n\to\infty,
\end{aligned}
$$

which implies $D_{kl}Q(x,y) = 0$ by (N_2). Thus we find that Q is a quadratic mapping satisfying equation (1.2) and inequality (4.3) near the approximate quadratic mapping $f : X \to Y$.

To prove the uniqueness, we now assume that there is another quadratic mapping $Q' : X \to Y$ which satisfies the approximate inequality (4.3). Then it follows from the equality $Q'(k^n x) = k^{2n}Q'(x)$, $Q(k^n x) = k^{2n}Q(x)$ and (4.3) that

$$\begin{aligned} N(Q(x)-Q'(x),t) &= N\Big(\frac{Q(k^n x)}{k^{2n}} - \frac{Q'(k^n x)}{k^{2n}}, t\Big) \\ &\geq \min\Big\{N\Big(\frac{Q(k^n x)}{k^{2n}} - \frac{f(k^n x)}{k^{2n}}, \frac{t}{2}\Big), N\Big(\frac{f(k^n x)}{k^{2n}} - \frac{Q'(k^n x)}{k^{2n}}, \frac{t}{2}\Big)\Big\} \\ &\geq N'(\varphi(k^n x,0),(k^2-|s|)k^{2n}t) \\ &\geq N'\Big(\varphi(x,0), \frac{(k^2-|s|)k^{2n}t}{|s|^n}\Big), \quad t>0, \end{aligned}$$

for all $n \in \mathbf{N}$, which tends to 1 as $n \to \infty$ by (N_5). Therefore one obtains $Q(x) = Q'(x)$ for all $x \in X$, completing the proof of uniqueness. This completes the proof of the theorem.

We remark that if $k = 1$ in Theorem 4.1, then

$$N'(\varphi(x,y),t) \geq N'\big(\varphi(x,y), \frac{t}{|s|^n}\big) \to 1$$

as $n \to \infty$, and so $\varphi(x,y) = 0$ for all $x, y \in X$. Hence, $D_{kl}f(x,y) = 0$ for all $x, y \in X$ and thus f is itself an Euler–Lagrange quadratic mapping.

Theorem 4.2. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl}f(x,y),t) \geq N'(\varphi(x,y),t) \tag{4.10}$$

and assume in addition that there exists a constant $s \in \mathbf{R}$ subject to $|s| > k^2$ such that a constrained function $\varphi : X^2 \to Z$ satisfies the inequality

$$N'\Big(\varphi\Big(\frac{x}{k}, \frac{y}{k}\Big), t\Big) \geq N'\Big(\frac{1}{s}\varphi(x,y), t\Big), \quad t > 0, \tag{4.11}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x,y) = 0$ and the approximate inequality

$$N(f(x)-Q(x),t) \geq N'\Big(\frac{\varphi(x,0)}{2|l-1|(|s|-k^2)}, t\Big), \quad t>0, \tag{4.12}$$

for all $x \in X$.

Proof. It follows from (4.5) and (4.11) that

$$N\Big(f(x) - k^2 f\Big(\frac{x}{k}\Big), \frac{t}{2|l-1||s|}\Big) \geq N'(\varphi(x,0),t), \quad t>0,$$

for all $x \in X$. Therefore it follows that

$$N\Big(f(x) - k^{2n} f\Big(\frac{x}{k^n}\Big), \sum_{i=0}^{n-1} \frac{k^{2i}}{2|l-1||s|^{i+1}} t\Big) \geq N'(\varphi(x,0),t), \quad t>0,$$

for all $x \in X$ and any integer $n > 0$. Thus we see from the last inequality that

$$\begin{aligned} N\Big(f(x) - k^{2n} f\Big(\frac{x}{k^n}\Big), t\Big) &\geq N'\Big(\varphi(x,0), \frac{t}{\sum_{i=0}^{n-1} \frac{k^{2i}}{2|l-1||s|^{i+1}}}\Big) \\ &\geq N'(\varphi(x,0), 2|l-1|(|s|-k^2)t), \quad t > 0. \end{aligned}$$

The remaining assertions go through the corresponding part of Theorem 4.1 by the similar way.

We also observe that if $k = 1$ in Theorem 4.2, then

$$N'(\varphi(x,y),t) \geq N'(\varphi(x,y), |s|^n t) \to 1$$

as $n \to \infty$, and so $\varphi(x,y) = 0$ for all $x, y \in X$. Hence, $D_{kl}f = 0$ and thus f is itself an Euler–Lagrange quadratic mapping.

Corollary 4.3. Let X be a normed space and $(\mathbf{R}, N')$ be a fuzzy normed space. Assume that there exist real numbers $\theta_1 \geq 0$, $\theta_2 \geq 0$ and that p is a real number such that either $0 < p < 2$ or $p > 2$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl}f(x,y),t) \geq N'(\theta_1\|x\|^p + \theta_2\|y\|^p, t)$$

for all $x, y \in X$ and all $t > 0$, then we can find a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x,y) = 0$ and the inequality

$$N(f(x)-Q(x),t) \geq \begin{cases} N'\big(\frac{\theta_1\|x\|^p}{2|l-1|(k^2-|k|^p)}, t\big), & \text{if } 0 < p < 2,\ |k| > 1,\ (p > 2,\ |k| < 1) \\ N'\big(\frac{\theta_1\|x\|^p}{2|l-1|(|k|^p-k^2)}, t\big), & \text{if } p > 2,\ |k| > 1,\ (0 < p < 2,\ |k| < 1) \end{cases}$$

for all $x \in X$ and all $t > 0$.

Proof. Taking $\varphi(x,y) = \theta_1\|x\|^p + \theta_2\|y\|^p$ and applying Theorem 4.1 and Theorem 4.2, we obtain the desired approximations, respectively.

Corollary 4.4. Assume that for $k \neq 1$, there exists a real number $\theta \geq 0$ such that the mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_{kl}f(x,y),t) \geq N'(\theta, t)$$

for all $x, y \in X$ and all $t > 0$. Then we can find a unique Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{kl}Q(x,y) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq N'\Big(\frac{\theta}{2|l-1||k^2-1|}, t\Big)$$

for all $x \in X$ and all $t > 0$.

We remark that if $\theta = 0$, then $N(D_{kl}f(x,y),t) \geq N'(0,t) = 1$, and so $D_{kl}f(x,y) = 0$. Thus we get $f = Q$ is itself an Euler–Lagrange quadratic mapping.

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