Some Properties of Invariant Submanifolds of Generalized Sasakian-Space-Forms Using Curvature Tensors

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ABSTRACT
In this paper we study various results of invariant submanifolds of generalized Sasakian-space-forms. Here we search the necessary and sufficient condition for invariant submanifolds of generalized Sasakian-space-forms to be totally geodesic satisfying$Q(\sigma,R)=0, Q(S,\sigma)=0,, Q(\sigma,C)=0, Q(\sigma,\bar{C})=0$ and $Q(\sigma,\tilde{C})=0$ where $R,S,C,\bar{C},\tilde{C}$ and σ are curvature tensor, Riccitensor,concircular curvature tensor,conformal curvature tensor, conharmonic curvature tensor and the second fundamental form respectively.

Keywords: Generalized Sasakian-space-forms; invariant submanifold; totally geodesic.

1. INTRODUCTION

In [2] Alegre, Blair and Carriazointroduced the concept of generalized Sasakian-space-forms, providing examples and establishing foundational properties. They demonstrated that any generalized Sasakian-space-form endowed with a K-contact structure is necessarily a Sasakian manifold and, in dimensions greater than five, it is also a Sasakian-space-form. Additionally, they explored the conditions under which a generalized Sasakian-space-form may be a contact metric manifold, presenting more intricate examples of generalized Sasakian-space-forms characterized by non-constant structure functions.

In differential geometry the curvature tensor plays an important role in and the sectional curvatures of a manifold determine the curvature tensor R completely. A Riemannian manifold consists with sectional curvature tensor c is defined as a real-space-form and its curvature tensor R satisfies the following condition

$$R(X,Y)Z=c\{g(Y,Z)X-g(X,Z)Y\}. \qquad (1.1)$$

These spaces are the Euclidean spaces, the spheres and the hyperbolic spaces according as $c=0, c>0$ and $c<0$.

$$R(X,Y)Z=\frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\}+\frac{c-1}{4}\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}. \qquad (1.2)$$

$$R(X,Y)Z=f_1\{g(Y,Z)X-g(X,Z)Y\}+f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}+f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\} \qquad (1.3)$$

for all vector fields X, Y, Z on M, where R is the curvature tensor of M and such a manifold of dimension $(2n+1), n>1$ (the condition $n>1$ is assumed throughout the paper), is denoted by $M^{2n+1}(f_1, f_2, f_3)$.

The authors in[24] investigated doubly warped product manifolds and derived a general inequality for such manifolds when they are isometrically immersed in arbitrary Riemannian manifolds. Several applications of this inequality were also obtained.In [7] the authorsderived a sharp inequality relating the squared norm of the second fundamental form to the warping function for contact CR-warped products in isometrically immersed cosymplectic space forms. They also examined the equality case and explored various applications of the result.The authorsin [25] derived relationships between the totally real sectional curvature and the scalar curvature for invariant submanifolds of a generalized Sasakian-space-form.In [19] the authors investigated the necessary and sufficient conditions for a submanifold, tangent to the structure vector field ξ of a Sasakian manifold, to qualify as a contact CR submanifold. They also examined the integrability conditions of the distributions that define the contact CR structure of these submanifolds. Additionally, they explored contact CR submanifolds of a Sasakian manifold with flat normal connections and minimal contact CR submanifolds within a Sasakian manifold.The authorsin [4] examined submanifolds of a generalized Sasakian-space-form by characterizing invariant and anti-invariant submanifolds through the action of the curvature tensor. For an almost semi-invariant

submanifold of a generalized Sasakian-space-form, they introduced the concepts of D_λ-sectional curvature and (D_λ,D_μ)-sectional curvature. Additionally, they derived results concerning the Ricci tensor, the scalar curvature, and totally umbilical submanifolds. In [5] the authors investigated Trans-Sasakian manifolds through D-conformal deformations. They analyzed the curvature tensor of a generalized Sasakian-space-form following a D-conformal deformation and derived conditions for the resulting manifold to qualify as a new generalized Sasakian-space-form. They also provided examples illustrating these conditions.The authors in [8] explored the differential geometric theory of submanifolds immersed in a Kenmotsu manifold. They derived new integrability conditions for the distributions defining contact CR-submanifolds and established characterizations for when the induced structure is parallel.
In [18] the authorsinvestigated W_2-flat generalized Sasakian-space-forms and derived a necessary and sufficient condition for a generalized Sasakian-space-form to be W_2-flat. They also examined generalized Sasakian-space-forms satisfying the condition W_2.S=0 and studied W_2-semisymmetric generalized Sasakian-space-forms. They showed that if a generalized Sasakian-space-form satisfies W_2.R=0, then either the manifold is W_2-flat, or the curvature tensor R of the manifold satisfies a specific condition.The authors in[10] explored the pseudo-symmetry properties of Sasakian-space-forms.In [6] the authors derived the necessary and sufficient conditions for a contact CR-submanifold to be classified as a CR-product, D-geodesic, and $D^\perp$-geodesic. They further analyzed the properties of contact CR-products and totally umbilical contact CR-submanifolds (CR-products) within the context of a Kenmotsu space form.
The authors in [14] explored the geometry of distributions for semi-slant submanifolds in $(\alpha,\ \beta)$ trans-Sasakian manifolds, examining cohomology groups and Bott connection-related forms. They investigated variational problems for slant submanifolds in generalized Sasakian-space-forms and derived conditions under which the first normal Chern class of integral submanifolds is trivial.
In dimensions ≥5, the authors in [3] proved that the space must be a Sasakian manifold with constant functions f_1, f_2, f_3. For 3D manifolds, they derived the curvature tensor for non-Sasakian contact metric generalized Sasakian-space-forms and studied trans-Sasakian generalized Sasakian-space-forms, showing that any 3D $(\alpha,\ \beta)$trans-Sasakian manifold is a generalized Sasakian-space-form. M. M. Tripathi and et al. [1] initiated the new type of curvature tensor called τ-curvature tensor. Nagaraja and Somashekhara [30] studied τ–curvature tensor in (k, μ)-contact manifold.
This paper is organized as follows: Section 2 covers preliminaries on Generalized Sasakian-Space-Forms and curvature tensors. Section 3 discusses invariant submanifolds. The main results are presented in Sections 4 through 8.

2.Preliminaries

A$(2n+1)$and $n > 1$-dimensional Riemannian manifold M is called almost contact metric manifold [11], if there exists on $M^{2n+1}(f_1, f_2, f_3)$ a (1, 1) tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-formη such that

$$\phi^2(X) = -X + \eta(X)\xi, \phi\xi = 0, \tag{2.1}$$
$$\eta(\xi) = 1, g(X, \xi) = \eta(X), \eta(\phi X) = 0, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$
$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$
$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \tag{2.5}$$

From (1.3) we have in a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ and $(n > 1)$[2]:

$$(\nabla_X \phi)(Y) = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \tag{2.6}$$
$$\nabla_X \xi = -(f_1 - f_3)\phi X \tag{2.7}$$
$$QX = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y), \tag{2.8}$$
$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y), \tag{2.9}$$
$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.10}$$
$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.11}$$
$$R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.12}$$
$$\eta(R(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{2.13}$$
$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.14}$$
$$S(\xi, \xi) = 2n(f_1 - f_3) \tag{2.15}$$
$$Q\xi = 2n(f_1 - f_3)\xi \tag{2.16}$$
$$C(X, Y)\xi = \left[(f_1 - f_3) - \frac{r}{2n(2n+1)}\right][\eta(Y)X - \eta(X)Y] \tag{2.17}$$
$$\bar{C}(X, Y)\xi = \left[(f_1 - f_3) - \frac{r}{2n(2n+1)}\right][\eta(Y)X - \eta(X)Y] - \frac{1}{(2n-1)}[\eta(Y)QX - \eta(X)QY] \tag{2.18}$$

$$\tilde{C}(X,Y)\xi = \frac{r-2n(f_1-f_3)}{2n(2n-1)}[\eta(Y)X-\eta(X)Y] - \frac{1}{(2n-1)}[\eta(Y)QX-\eta(X)QY] \quad (2.19)$$

3.Some important results of invariant submanifold of generalized sasakian-space-forms

Let N be a submanifoldof a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$.

Also let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TN and the normal bundle $T^{\perp}N$ of N respectively. Then the Gauss-Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \quad (3.1)$$

$$\overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \quad (3.2)$$

for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$, where$\nabla^{\perp}$ is the connection in the normal bundle,σ and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of N into $M^{2n+1}(f_{1,} f_{2,} f_3)$. The second fundamental form σ and the shape operator A_V are related by$g(\sigma(X,Y),V) = g(A_V X, Y)$. N is called a totally geodesic submanifold if σ vanishes identically.

$$\sigma(X,\xi) = 0 \quad (3.3)$$

4. Invariant submanifold of generalized sasakian-space-forms satisfying $Q(\sigma, R) = 0$

This section presents the necessary and sufficient condition for invariant submanifolds of generalized Sasakian space forms to be totally geodesic under the condition that$Q(\sigma, R) = 0$.

Theorem 4.1. Let N be an invariant submanifold of a generalized Sasakian-space-forms M.Then N is totally geodesic if and only if N satisfies$Q(\sigma, R) = 0$, provided that$(1-2n)(f_1 - f_3) \neq 0$.

Proof. Let N be an invariant submanifold of a generalized Sasakian-space-formsM satisfying $Q(\sigma, R) = 0$. Therefore

$$0 = Q(\sigma,R) = Q(\sigma,R)(X,Y,Z;U,V) = ((U \wedge_\sigma V).R)(X,Y)Z = -R\big((U \wedge_\sigma V)X, Y\big)Z - R(X,(U \wedge_\sigma V)Y)Z - R(X,Y)(U \wedge_\sigma V)Z \quad (4.1)$$

Where $U \wedge_\sigma V$ is defined by

$$(U \wedge_\sigma V)P = \sigma(V,P)U - \sigma(U,P)V \quad (4.2)$$

Using (4.2) in (4.1) we get

$$-\sigma(V,X)R(U,Y)Z + \sigma(U,X)R(V,Y)Z - \sigma(V,Y)R(X,U)Z + \sigma(U,Y)R(X,V)Z - \sigma(V,Z)R(X,Y)U + \sigma(U,Z)R(X,Y)V = 0. \quad (4.3)$$

Putting $Z = V = \xi$ in (4.3) and using (3.3), we get

$$\sigma(U,X)R(\xi,Y)\xi + \sigma(U,Y)R(X,\xi)\xi = 0 \quad (4.4)$$

Using (2.12) in (4.4) we obtain

$$\sigma(U,X)(f_1-f_3)\{\eta(Y)\xi - Y\} + \sigma(U,Y)(f_1-f_3)\{X - \eta(X)\xi\} = 0 \quad (4.5)$$

Taking inner product with W, we get

$$\sigma(U,X)(f_1-f_3)\{\eta(Y)\eta(W) - g(Y,W)\} + \sigma(U,Y)(f_1-f_3)\{g(X,W) - \eta(X)\eta(W)\} = 0 \quad (4.6)$$

Contracting Y and W we get

$$\sigma(U,X)(1-2n)(f_1-f_3) = 0 \quad (4.7)$$

Hence $\sigma(U,X) = 0$, provided

$(1-2n)(f_1-f_3) \neq 0$.

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

5. Invariant submanifold of generalized sasakian-space-forms satisfying $Q(S, \sigma) = 0$

This section presents the necessary and sufficient condition for invariant submanifolds of generalized Sasakian space forms to be totally geodesic under the condition that$Q(S, \sigma) = 0$.

Theorem 5.1. Let N be an invariant submanifold of a generalized Sasakian-space-forms M. Then N is totally geodesic if and only if N satisfies$Q(S,\sigma) = 0$, provided that$2n(f_1 - f_3) \neq 0$.

Proof. Let N be an invariant submanifold of a generalized Sasakian-space-forms M satisfying $Q(S,\sigma) = 0$. Therefore

$$0 = Q(S,\sigma) = Q(S,\sigma)(X,Y;U,V) = ((U \wedge_S V).\sigma)(X,Y) = -\sigma\big((U \wedge_S V)X, Y\big) - \sigma(X,(U \wedge_S V)Y) \quad (5.1)$$

Using (4.2) in (5.1) we get

$$-S(V,X)\,\sigma(U,Y) + S(U,X)\,\sigma(V,Y) - S(V,Y)\,\sigma(X,U) + S(U,Y)\,\sigma(X,V) = 0 \quad (5.2)$$

Putting $Y = U = \xi$ in (5.2) and using (3.3) and (2.15) we get

$$2n(f_1-f_3)\sigma(X,V) = 0 \quad (5.3)$$

Hence $\sigma(U,X)=0$, provided
$2n(f_1-f_3)\neq 0$.
Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

6. Invariant submanifold of generalized sasakian-space-forms satisfying $Q(\sigma,C)=0$

This section presents the necessary and sufficient condition for invariant submanifolds of generalized Sasakian space forms to be totally geodesic under the condition that$Q(\sigma,C)=0$.

Theorem 6.1. Let N be an invariant submanifold of a generalized Sasakian-space-forms M. Then N is totally geodesic if and only if N satisfies$Q(\sigma,C)=0$, provided that$(1-2n)[(f_1-f_3)-\frac{r}{2n(2n+1)}]\neq 0$.

Proof. Let N be an invariant submanifold of a generalized Sasakian-space-formsM satisfying $Q(\sigma,C)=0$. Therefore

$$0=Q(\sigma,C)=Q(\sigma,C)(X,Y,Z;U,V)=((U\wedge_\sigma V).C)(X,Y)Z=-C\big((U\wedge_\sigma V)X,Y\big)Z-C(X,(U\wedge_\sigma V)Y)Z-C(X,Y)(U\wedge_\sigma V)Z \quad (6.1)$$

Using (4.2) in (6.1) we get

$$-\sigma(V,X)C(U,Y)Z+\sigma(U,X)C(V,Y)Z-\sigma(V,Y)C(X,U)Z+\sigma(U,Y)C(X,V)Z-\sigma(V,Z)C(X,Y)U+\sigma(U,Z)C(X,Y)V=0 \quad (6.2)$$

Putting $Z=V=\xi$ in (6.2) and using (3.3), we get

$$\sigma(U,X)C(\xi,Y)\xi+\sigma(U,Y)C(X,\xi)\xi=0 \quad (6.3)$$

Using (2.17) in (6.3) we obtain

$$\sigma(U,X)\left[(f_1-f_3)-\frac{r}{2n(2n+1)}\right]\{\eta(Y)\xi-Y\}$$
$$+\sigma(U,Y)\left[(f_1-f_3)-\frac{r}{2n(2n+1)}\right]\{X-\eta(X)\xi\}=0 \quad (6.4)$$

Taking inner product with W, we get

$$\sigma(U,X)\left[(f_1-f_3)-\frac{r}{2n(2n+1)}\right]\{\eta(Y)\eta(W)-g(Y,W)\}$$
$$+\sigma(U,Y)\left[(f_1-f_3)-\frac{r}{2n(2n+1)}\right]\{g(X,W)-\eta(X)\eta(W)\}=0 \quad (6.5)$$

Contracting Y and W we get

$$\sigma(U,X)(1-2n)\left[(f_1-f_3)-\frac{r}{2n(2n+1)}\right]=0. \quad (6.6)$$

Hence $\sigma(U,X)=0$, provided

$$(1-2n)\left[(f_1-f_3)-\frac{r}{2n(2n+1)}\right]\neq 0.$$

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

7. Invariant submanifold of generalized sasakian-space-forms satisfying $Q(\sigma,\bar{C})=0$

This section presents the necessary and sufficient condition for invariant submanifolds of generalized Sasakian space forms to be totally geodesic under the condition that$Q(\sigma,\ \bar{C})=0$.

Theorem 7.1. Let N be an invariant submanifold of a generalized Sasakian-space-forms M. Then N is totally geodesic if and only if N satisfies$Q(\sigma,\ \overline{C})=0$, provided that$(2n-1)[(2n+1)f_1+3f_2-2f_3]\neq 0$.

Proof. Let N be an invariant submanifold of a generalized Sasakian-space-forms N $Q(\sigma,\ \bar{C})=0$.Therefore

$$0=Q(\sigma,\ \bar{C})=Q(\sigma,\ \bar{C})(X,Y,Z;U,V)=((U\wedge_\sigma V).\bar{C})(X,Y)Z=-\bar{C}\big((U\wedge_\sigma V)X,Y\big)Z-\bar{C}(X,(U\wedge_\sigma V)Y)Z-\bar{C}(X,Y)(U\wedge_\sigma V)Z \quad (7.1)$$

Using (4.2) in (7.1) we get

$$-\sigma(V,X)\bar{C}(U,Y)Z+\sigma(U,X)\bar{C}(V,Y)Z-\sigma(V,Y)\bar{C}(X,U)Z+\sigma(U,Y)\bar{C}(X,V)Z-\sigma(V,Z)\bar{C}(X,Y)U+\sigma(U,Z)\bar{C}(X,Y)V=0 \quad (7.2)$$

Putting $Z=V=\xi$ in (7.2) and using (3.3), we get

$$\sigma(U,X)\bar{C}(\xi,Y)\xi+\sigma(U,Y)\bar{C}(X,\xi)\xi=0 \quad (7.3)$$

Using (2.18) in (7.3) we obtain

$$\sigma(U,X)\left[\frac{(f_3-f_1)}{(2n-1)}\{\eta(Y)\xi-Y\}-\frac{1}{(2n-1)}\{2n(f_1-f_3)\eta(Y)\xi-QY\}\right]$$
$$+\sigma(U,Y)\left[\frac{(f_3-f_1)}{(2n-1)}\{X-\eta(X)\xi\}-\frac{1}{(2n-1)}\{QX-2n(f_1-f_3)\eta(X)\xi\}\right]=0 \quad (7.4)$$

Taking inner product with W, we get

$\sigma(U,X)\left[\frac{(f_3-f_1)}{(2n-1)}\{\eta(Y)\eta(W)-g(Y,W)\}-\frac{1}{(2n-1)}\{2n(f_1-f_3)\eta(Y)\eta(W)-g(QY,W)\}\right]+$
$\sigma(U,Y)\left[\frac{(f_3-f_1)}{(2n-1)}\{g(X,W)-\eta(X)\eta(W)\}-\frac{1}{(2n-1)}\{g(QX,W)-2n(f_1-f_3)\eta(X)\eta(W)\}\right]=0$ (7.5)

Contracting Y and W and using (2.8) we get
$\sigma(U,X)(2n-1)[(2n+1)f_1+3f_2-2f_3]=0.$ (7.6)
Hence $\sigma(U,X)=0$, provided

$$(2n-1)[(2n+1)f_1+3f_2-2f_3]\neq 0.$$

Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

8. Invariant submanifold of generalized sasakian-space-forms satisfying $Q(\sigma,\tilde{C})=0$

This section presents the necessary and sufficient condition for invariant submanifolds of generalized Sasakian space forms to be totally geodesic under the condition that$Q(\sigma,\tilde{C})=0$.

Theorem 8.1. Let N be an invariant submanifold of a generalized Sasakian-space-forms M. Then N is totally geodesic if and only if N satisfies$Q(\sigma,\tilde{C})=0$, provided that$r\neq 2n[(2n+1)f_1+3f_2-2f_3]$.

Proof. Let N be an invariant submanifold of generalized Sasakian-space-formsM satisfying $Q(\sigma,\tilde{C})=0$. Therefore

$0=Q(\sigma,\tilde{C})=Q(\sigma,\tilde{C})(X,Y,Z;U,V)=((U\wedge_\sigma V).\tilde{C})(X,Y)Z=-\tilde{C}((U\wedge_\sigma V)X,Y)Z-\tilde{C}(X,(U\wedge_\sigma V)Y)Z-\tilde{C}(X,Y)(U\wedge_\sigma V)Z$ (8.1)

Using (4.2) in (8.1) we get
$-\sigma(V,X)\tilde{C}(U,Y)Z+\sigma(U,X)\tilde{C}(V,Y)Z-\sigma(V,Y)\tilde{C}(X,U)Z+\sigma(U,Y)\tilde{C}(X,V)Z-\sigma(V,Z)\tilde{C}(X,Y)U+\sigma(U,Z)\tilde{C}(X,Y)V=0$ (8.2)

Putting $Z=V=\xi$ in (8.2) and using (3.3), we get
$\sigma(U,X)\tilde{C}(\xi,Y)\xi+\sigma(U,Y)\tilde{C}(X,\xi)\xi=0$ (8.3)

Using (2.19) in (8.3) we obtain
$\sigma(U,X)\left[\frac{r-2n(f_1-f_3)}{2n(2n-1)}\{\eta(Y)\xi-Y\}-\frac{1}{(2n-1)}\{2n(f_1-f_3)\eta(Y)\xi-QY\}\right]+\sigma(U,Y)\left[\frac{r-2n(f_1-f_3)}{2n(2n-1)}\{X-\eta(X)\xi\}-\frac{1}{(2n-1)}\{QX-2n(f_1-f_3)\eta(X)\xi\}\right]=0$ (8.4)

Taking inner product with W, we get
$\sigma(U,X)\left[\frac{r-2n(f_1-f_3)}{2n(2n-1)}\{\eta(Y)\eta(W)-g(Y,W)\}-\frac{1}{(2n-1)}\{2n(f_1-f_3)\eta(Y)\eta(W)-g(QY,W)\}\right]+$
$\sigma(U,Y)\left[\frac{r-2n(f_1-f_3)}{2n(2n-1)}\{g(X,W)-\eta(X)\eta(W)\}-\frac{1}{(2n-1)}\{g(QX,W)-2n(f_1-f_3)\eta(X)\eta(W)\}\right]=0$ (8.5)

Contracting Y and W and using (2.8) we get
$\sigma(U,X)[r-2n\{(2n+1)f_1+3f_2-2f_3\}]=0.$ (8.6)
Hence $\sigma(U,X)=0$, provided
$r\neq 2n[(2n+1)f_1+3f_2-2f_3]$.
Therefore, the manifold is totally geodesic. The converse part of the theorem is trivial. Therefore, the theorem is proved.

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