SUPERSTABILITY OF THE PEXIDER TYPE SINE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we find solutions and investigate the superstability bounded by function for the sine functional equation (S) from the approximate inequality of the Pexider type functional equation:

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)k(y).$$

Furthermore, the results are extended to Banach algebras. As a consequence, we obtain the superstability for the exponential functional equations, the hyperbolic functional equations, and the jointed Pexider Lobacevski equation.

Keywords: stability, superstability, sine functional equation, cosine functional equation.

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1. Introduction

In 1979, Baker *et al.* [4] announced the *superstability* as the new concept as follows: If f satisfies $|f(x+y) - f(x)f(y)| \le \varepsilon$ for some fixed $\varepsilon > 0$, then either f is bounded or f satisfies the exponential functional equation

$$f(x+y) = f(x)f(y). \tag{E}$$

D'Alembert, in 1769, introduced the cosine (d'Alembert) functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{C}$$

whose superstability was proved on Abelian group by Baker [3] in 1980.

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y) + f(x-y) = 2f(x)g(y), \tag{W}$$

$$f(x+y) + f(x-y) = 2g(x)f(y), \tag{K}$$

in which (W) is called the Wilson equation, and (K) was raised by Kim [9].

The superstability of the cosine (C), Wilson (W) and Kim (K) was founded in Badora [1], Ger [2], Kannappan and Kim [9], Kim [13, 15, 16, 20], and in [5, 7, 22].

In 1983, Cholewa [6] investigated the superstability of the sine functional equation

$$f\big(\frac{x+y}{2}\big)^2 - f\big(\frac{x-y}{2}\big)^2 = f(x)f(y) \tag{S}$$

under the condition bounded by constant (Hyers sense). His result was improved to the condition bounded by a function (Găvruta's sense in [8]) in Badora and Ger [2].

Their results were also improved by Kim [11, 12, 14], which are the superstability of the generalized sine functional equations:

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = f(x)g(y) \tag{S_{fg}}$$

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = g(x)f(y) \tag{S_{gf}}$$

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = g(x)g(y) \tag{S_{gg}}$$

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = g(x)h(y) \tag{S_{gh}}$$

under the condition bounded by a constant or a function.

The aim of this paper is to find solutions and to investigate the superstability bounded by the function (Găvruta sense in [8]) for the sine functional equation (S) from an approximate inequality of the Pexider type functional equation:

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)k(y), \tag{S_{fghk}}$$

which is represented by the exponential equations, hyperbolic cosine(sine) equations, and the jointed Pexider Lobacevski equation (PL).

As corollaries, we obtain the superstability bounded by a constant or the function for the sine functional equation (S) from an approximate inequality of the sine type functional equations (S_{fg}), (S_{gf}), (S_{gg}), (S_{gh}), and the Pexider type functional equations:

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)h(y) \tag{S_{fghh}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)f(y) \tag{S_{fghf}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = h(x)g(y) \tag{S_{fghg}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = f(x)h(y) \tag{S_{fgfh}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = f(x)g(y) \tag{S_{fgfg}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = f(x)f(y) \tag{S_{fgff}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)h(y) \tag{S_{fggh}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)g(y) \tag{S_{fggg}}$$

$$f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)f(y). \qquad (S_{fggf})$$

Furthermore, the obtained results are extended to Banach algebras.

In this paper, let $(G,+)$ be an uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, and G the field of real numbers. f, g, h, k are nonzero functions and ε is a nonnegative real constant, $\varphi : G \to \mathbb{R}$ be a mapping.

2. Creation of the equations and its solution.

The purpose of this chapter is to show the creation and the solution for the frequently risen function equations dued by the trigonometric function.

Let us recall the trigonometric formula, except for (C), (W) (K).

$\sin(x+y) + \cos(x-y) = [\sin(x) + \cos(x)][\sin(y) + \cos(y)]$ *implies*

$$f(x+y) + g(x-y) = [f(x) + g(x)][f(y) + g(y)] = h(x)h(y). \qquad (fghh)$$

$\cos(x+y) + \sin(x-y) = [\cos(x) + \sin(x)][\cos(y) - \sin(y)]$ *implies*

$$f(x+y) + g(x-y) = [f(x) + g(x)][f(y) - g(y)] = h(x)k(y). \qquad (fghk)$$

$\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$ *implies*

$$f(x+y) - f(x-y) = 2g(x)f(y). \qquad (T_{gf})$$

$\cos(x+y) - \cos(x-y) = -2\sin(x)\sin(y)$ *implies*

$$f(x+y) - f(x-y) = -2g(x)g(y) = 2g(x)h(y). \qquad (T_{gh})$$

$\cos(x+y) - \sin(x-y) = [\cos(x) - \sin(x)][\cos(y) + \sin(y)]$ *implies*

$$f(x+y) - g(x-y) = [f(x) - g(x)][f(y) + g(y)] = h(x)k(y). \qquad (T_{fghk})$$

$\sin(x+y) - \cos(x-y) = [\sin(x) - \cos(x)][\cos(y) - \sin(y)]$ *implies*

$$f(x+y) - g(x-y) = [f(x) - g(x)][g(y) - f(y)] = h(x)k(y). \qquad (T_{fghk})$$

$$f(x+y) - f(x-y) = 2f(x)f(y). \qquad (T)$$

Like the cosine and the sine, the above functional equations are also derived simultaneously by the hyperbolic cosine (sine), exponential equation, and Jensen equation, as can be seen in the following relations:

$$\cosh(x+y) \pm \cosh(x-y) = 2\cosh(x)\cosh(y)\big(= -2\sinh(x)\sinh(y)\big)$$
$$\sinh(x+y) \pm \sinh(x-y) = 2\sinh(x)\cosh(y)\big(= 2\cosh(x)\sinh(y)\big)$$
$$a^{x+y} \pm a^{x-y} = 2a^x \tfrac{a^y \pm a^{-y}}{2} \approx 2e^x \tfrac{e^y \pm e^{-y}}{2} = 2e^x\cosh(y)\big(= 2e^x\sinh(y)\big)$$
$$(n(x+y)+c) \pm (n(x-y)+c) = 2(nx+c)\big(= 2n(y)\big)$$
$$: \text{Jensen equation, for } f(x) = nx + c, g(y) = 1,$$

where the subtraction corresponds into parentheses ( ).

Since the trigonometric and hyperbolic functions are expressed by an exponential function as following: $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, respectively, all of the above functional equations naturally have exponential and hyperbolic functions as solution.

Now, let's bring the quadratic functional equation generated by a product or a square of the above equations, which is the target of this paper.

It is well known that the sine functional equation (S) is derived as follows:

$$f\left(\tfrac{x+y}{2}\right)^2 - f\left(\tfrac{x-y}{2}\right)^2 = \sin\left(\tfrac{x+y}{2}\right)^2 - \sin\left(\tfrac{x-y}{2}\right)^2 \\ = \sin(x)\sin(y) = f(x)f(y).$$

Eq. (S) has simultaneously an exponential solution as follows :

$$\left(\frac{1}{2i}\left(e^{i\frac{x+y}{2}} - e^{-i\frac{x+y}{2}}\right)\right)^2 - \left(\frac{1}{2i}\left(e^{i\frac{x-y}{2}} - e^{-i\frac{x-y}{2}}\right)\right)^2 = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right).$$

Also, simultaneously, (S) is satisfied for the hyperbolic sine function as follows:

$$f\left(\tfrac{x+y}{2}\right)^2 - f\left(\tfrac{x-y}{2}\right)^2 = \sinh\left(\tfrac{x+y}{2}\right)^2 - \sinh\left(\tfrac{x-y}{2}\right)^2 \\ = \left(\tfrac{1}{2}\left(e^{\frac{x+y}{2}} - e^{-\frac{x+y}{2}}\right)\right)^2 - \left(\tfrac{1}{2}\left(e^{\frac{x-y}{2}} - e^{-\frac{x-y}{2}}\right)\right)^2 = \left(\tfrac{1}{2}\left(e^{x}-e^{-x}\right)\right)\left(\tfrac{1}{2}\left(e^{y}-e^{-y}\right)\right) \\ = \sinh(x)\sinh(y) = f(x)f(y),$$

which is added solutions as the hyperbolic sine, exponential function.

Also, the other examples of the Pexider type quadratic functional equations

$$f(x)f(y) = \begin{cases} (i)\ f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \\ (ii)\ g\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 \\ (iii)\ g\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \\ (iv)\ -\left(f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2\right) \end{cases}$$

have solutions to the hyperbolic sine(cosine) as follows:

$$\sinh(x)\sinh(y) = \begin{cases} (i)\ \sinh^2\left(\frac{x+y}{2}\right) - \sinh^2\left(\frac{x-y}{2}\right) \\ (ii)\ \cosh^2\left(\frac{x+y}{2}\right) - \cosh^2\left(\frac{x-y}{2}\right) \\ (iii)\ \cosh^2\left(\frac{x+y}{2}\right) - \sinh^2\left(\frac{x-y}{2}\right) \\ (iv)\ -\left(\sinh^2\left(\frac{x+y}{2}\right) - \cosh^2\left(\frac{x-y}{2}\right)\right). \end{cases}$$

Next, the Lobacevski equation

$$f\left(\frac{x+y}{2}\right)^2 = f(x)f(y) \tag{L}$$

is considered to the exponential equation (E) by $f\left(\frac{x+y}{2}\right)^2 = \left(e^{\frac{x+y}{2}}\right)^2 = e^{x+y} = e^x e^y = f(x)f(y)$.

The Lobacevski equation (L) was generalized by Kim [17, 18] Kim and Park [21] to the Pexider type Lobacevski equations

$$f\left(\frac{x+y}{2}\right)^2 = g(x)h(y), \qquad f\left(\frac{x+y}{n}\right)^m = g(x)h(y). \tag{PL}$$

Hence (S_{fghk}) and all (S) type equations are also represented as joint of (L) and (PL) as follows:

$$\begin{aligned} f\left(\frac{x+y}{m}\right)^m - g\left(\frac{x-y}{n}\right)^n &= \left(\sqrt[m]{p}\left(a^{\frac{x+y}{m}}\right)\right)^m - \left(\sqrt[n]{q}\left(a^{-\frac{x-y}{n}}\right)\right)^n \\ &= p\left(a^{x+y}\right) - q\left(a^{-x+y}\right) = \left(pa^x - qa^{-x}\right)a^y \\ &= h(x)k(y), \end{aligned}$$

where $f(x) = \sqrt[m]{p}(a^x)$, $g(x) = \sqrt[n]{q}(a^{-x})$, $h(x) = pa^x - qa^{-x}$, $k(x) = a^x$.

As a result, the target function equation (S_{fghk}) has solutions as the trigonometric, exponential, hyperbolic function, jointed Pexider Lobacevski equation.

In the following, we show examples of solution applied to the trigonometric function for (S_{gg}), (S_{gh}), (S_{fghk}), (S_{fghh}). Of course, it is also natural to have the its solutions as exponential function, hyperbolic sine(cosine) function, Pexider Lobacevski equation. Their description will be skip.

Solution 1. *The functions $f, g, h : G \longrightarrow \mathbb{C}$ satisfy (S_{gh}) if and only if f, g, h are solutions with $f(x) = \cos x$, $g(x) = \sin x$, and $h(x) = -\sin x$.*

In particular, if the functions $f, g : G \longrightarrow \mathbb{C}$ satisfy the functional equation (S_{gg}) if and only if f, g are solutions with $f(x) = \cos x$ and $g(x) = i \sin x$.

Proof. $f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = \cos\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = -\sin x \sin y = g(x)h(y)$. In particular case, it is established that $f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = \cos\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = -\sin x \sin y = i^2 \sin x \sin y = g(x)g(y)$. □

Solution 2. *The functions $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy (S_{fghk}) if and only if f, g, h, k are solutions with $f(x) = \sin(x)$, $g(x) = \cos(x)$, $h(x) = (\sin^2 - \cos^2)(x)$, $k(x) = (\cos^2 - \sin^2)(x)$.*

Proof. $f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = \sin\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = (\sin^2 x - \cos^2 x)(\cos^2 y - \sin^2 y) = h(x)k(y)$. □

Solution 3. *(i) The functions $f, g, h : G \longrightarrow \mathbb{C}$ satisfy (S_{fghh}) if and only if f, g, h are solutions with $f(x) = \cos(2x)$, $g(x) = \sin(2x)$, $h(x) = (\cos^2 - \sin^2)(x)$.*

(ii) The functions $f, g, h : G \longrightarrow \mathbb{C}$ satisfy (S_{fghh}) if and only if f, g, h are solutions with $f(x) = \sin(x)$, $g(x) = \cos(x)$, $h(x) = i(\sin^2 - \cos^2)(x)$.

Proof. (i) $f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = \cos\left(2\frac{x+y}{2}\right)^2 - \sin\left(2\frac{x-y}{2}\right)^2 = \cos\left(x+y\right)^2 - \sin\left(x-y\right)^2 = (\cos^2 x - \sin^2 x)(\cos^2 y - \sin^2 y) = h(x)h(y)$.

(ii) $f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = \sin\left(\frac{x+y}{2}\right)^2 - \cos\left(\frac{x-y}{2}\right)^2 = (\sin^2 x - \cos^2 x)(\cos^2 y - \sin^2 y) = i^2(\sin^2 x - \cos^2 x)(\sin^2 y - \cos^2 y) = h(x)h(y)$. □

Remark 1. *(i) It is trivial that all 1∼3 have solutions as an exponential functions, hyperbolic sine (cosine) functions, and Lovachevsky equations.*

(ii) The investigation of solutions associated with the generative method for (S_{fghk}) can be further extended to that for the Pexider type function equation:

$$f\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x-y}{2}\right)^2 = h(x)k(y). \tag{P_{fghk}}$$

3. Superstability of (S) from the approximate inequality of (S_{fghk})

We investigate the superstability of the sine functional equation (S) from the approximate inequality of the Pexider type functional equation (S_{fghk}) related to (S). As a corollary, we obtain the superstability of the sine functional equation (S).

Theorem 1. *Assume that $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right| \leq \varphi(y) \quad \forall x, y \in G, \tag{3.1}$$

which satisfies one of the cases $k(0) = 0$, $f(x)^2 = g(x)^2$.

Then either h is bounded or k satisfies (S). In addition, if h satisfies (C), *k and h satisfy $(T_{gf}) := k(x+y) - k(x-y) = 2h(x)k(y)$.*

Proof. The inequality (3.1) may equivalently be written as

$$|f(x+y)^2 - g(x-y)^2 - h(2x)k(2y)| \leq \varphi(2y), \quad \forall\, x, y \in G. \tag{3.2}$$

Let h be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \neq |h(2x_n)| \to \infty, \quad \text{as} \quad n \to \infty. \tag{3.3}$$

Taking $x = x_n$ in (3.2), we obtain

$$\left| \frac{f(x_n+y)^2 - g(x_n-y)^2}{h(2x_n)} - k(2y) \right| \leq \frac{\varphi(2y)}{|h(2x_n)|},$$

and so by (3.3), we have

$$k(2y) = \lim_{n\to\infty} \frac{f(x_n+y)^2 - g(x_n-y)^2}{h(2x_n)}. \tag{3.4}$$

Using (3.1) we have

$$\begin{aligned} 2\varphi(y) \geq & \left| h(2x_n+x)k(y) - f\left(\frac{2x_n+x+y}{2}\right)^2 + g\left(\frac{2x_n+x-y}{2}\right)^2 \right| \\ & + \left| h(2x_n-x)k(y) - f\left(\frac{2x_n-x+y}{2}\right)^2 + g\left(\frac{2x_n-x-y}{2}\right)^2 \right| \\ \geq & \,|\, (h(2x_n+x) + h(2x_n-x))\, k(y) \\ & - \left(f\left(x_n + \frac{x+y}{2}\right)^2 - g\left(x_n - \frac{x+y}{2}\right)^2 \right) \\ & - \left(f\left(x_n + \frac{-x+y}{2}\right)^2 - g\left(x_n - \frac{-x+y}{2}\right)^2 \right) \Bigg| \end{aligned} \tag{3.5}$$

for all $x, y \in G$ and all $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}\frac{2\varphi(y)}{|h(2x_n)|} \geq \Bigg| \frac{h(2x_n+x)+h(2x_n-x)}{h(2x_n)}k(y) \\ - \frac{f\left(x_n+\frac{x+y}{2}\right)^2 - g\left(x_n-\frac{x+y}{2}\right)^2}{h(2x_n)} \\ - \frac{f\left(x_n+\frac{-x+y}{2}\right)^2 - g\left(x_n-\frac{-x+y}{2}\right)^2}{h(2x_n)}\Bigg| \quad (3.6)\end{aligned}$$

for all $x, y \in G$ and all $n \in \mathbb{N}$.

Taking the limit as $n \longrightarrow \infty$ with the use of (3.4) and (3.6), we conclude that, for every $x \in G$, there exists the limit function

$$L_1(x) := \lim_{n\to\infty} \frac{h(2x_n+x)+h(2x_n-x)}{h(2x_n)},$$

where the obtained function $L_1 : G \to \mathbb{C}$ satisfies the equation as even

$$k(x+y)+k(-x+y) = L_1(x)k(y) \quad \forall x, y \in G. \tag{3.7}$$

First, let us consider the case $k(0) = 0$. Then it forces by (3.7) that k is odd. So (3.7) is

$$k(x+y)-k(x-y) = L_1(x)k(y) \quad \forall x, y \in G. \tag{3.8}$$

By means of (3.8) and the oddness of k, we have the following

$$\begin{aligned}k(x+y)^2 - k(x-y)^2 &= [k(x+y)+k(x-y)]L_1(x)k(y) \qquad (3.9)\\ &= \big[k(2x+y)+k(2x-y)\big]k(y)\\ &= \big[k(y+2x)-k(y-2x)\big]k(y)\\ &= L_1(y)k(2x)k(y).\end{aligned}$$

Putting $x = y$ in (3.8), we conclude that

$$k(2y) = L_1(y)k(y) \text{ for all } x, y \in G. \tag{3.10}$$

The equation (3.9), in return, leads with (3.10) to the equation

$$k(x+y)^2 - k(x-y)^2 = k(2x)k(2y), \tag{3.11}$$

which, by 2-divisibility of G, states nothing else but (S).

In addition, if h satisfies (C), L_1 forces $2h$, so (3.8) forces that k and h satisfy (T_{gf}).

For the other case $f(x)^2 = g(x)^2$, it is enough to show that $k(0) = 0$. Suppose that this is not the case. Then, we may assume that $k(0) = c$: constant.

Putting $y = 0$ in (3.1), from the above assumption, we obtain the inequality

$$|h(x)| \leq \frac{\varphi(0)}{c} \qquad \forall\, x \in G.$$

This inequality means that h is globally bounded, which is a contradiction by unboundedness assumption. Thus the claimed $k(0) = 0$ holds, so the proof is completed. □

Theorem 2. *Suppose that $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right| \le \varphi(x) \quad \forall x, y \in G, \tag{3.12}$$

which satisfies one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$.

Then either k is bounded or h satisfies (S). In addition, if k satisfies (C)*, h and k satisfy the Wilson equation (W):= $h(x+y) + h(x-y) = 2h(x)k(y)$.*

Proof. Let k be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that $k(2y_n)| \to \infty$ as $n \to \infty$. An obvious slight change in the proof steps applied in the start of Theorem 1 gives us

$$h(2x) = \lim_{n\to\infty} \frac{f(x+y_n)^2 - g(x-y_n)^2}{k(2y_n)}. \tag{3.13}$$

Replacing y by $y + 2y_n$ and $-y + 2y_n$ in (3.12), the same procedure of (3.5) and (3.6) allows, with an applying of (3.13), one to state the existence of a limit function

$$L_2(y) := \lim_{n\to\infty} \frac{k(y+2y_n) + k(-y+2y_n)}{k(2y_n)},$$

where $L_2 : G \longrightarrow \mathbb{C}$ satisfies the equation

$$h(x+y) + h(x-y) = h(x)L_2(y) \quad \forall x, y \in G. \tag{3.14}$$

For the case $h(0) = 0$, it forces by (3.14) that h is odd.

Putting $y = x$ in (3.14), we get

$$h(2x) = h(x)L_2(x) \quad \forall x, \in G. \tag{3.15}$$

From (3.14), the oddness of h and (3.15), we obtain the equation

$$\begin{aligned} h(x+y)^2 - h(x-y)^2 &= h(x)L_2(y)[h(x+y) - h(x-y)] \\ &= h(x)[h(x+2y) - h(x-2y)] \\ &= h(x)[h(2y+x) + h(2y-x)] \\ &= h(x)h(2y)L_2(x) \\ &= h(2x)h(2y), \end{aligned}$$

which, by 2-divisibility of G, states (S).

In addition, if k satisfies (C), L_2 forces $2k$, so (3.14) forces that h and k satisfy (W).

The other case $f(x)^2 = g(-x)^2$ also is established $h(0) = 0$ for the same reason as that of Theorem 1, so the proof is completed. □

From Theorems 1 and 2, we obtain the following result as a corollary.

Theorem 3. *Suppose that $f, g, h, k : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right| \le \min\{\varphi(x), \varphi(y)\} \tag{3.16}$$

for all $x, y \in G$. Then

(i) either h under the cases $k(0) = 0$ or $f(x)^2 = g(x)^2$ is bounded or k satisfies (S). In addition, if h satisfies (C), *k and h satisfy (T_{gf}):= $k(x+y) - k(x-y) = 2h(x)k(y)$;*

(ii) either k under the cases $h(0) = 0$ or $f(x)^2 = g(-x)^2$ is bounded, or h satisfies (S). In addition, if k satisfies (C), *h and k satisfy the Wilson equation (W):=$h(x+y) + h(x-y) = 2h(x)k(y)$.*

As a corollary, we obtain the stability of the sine functional equation (S) from Theorems 1, 2, 3.

Corollary 1. *Assume that $f : G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)f(y) \right| \le \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

Then, either f is bounded or f satisfies (S).

Proof. Assumption $f(0) = 0$ in Theorems is simply eliminated (see [2, Theorem 5]). □

4. Application of the equations (S_{fghh}), (S_{fghf}), (S_{fgfh}), (S_{fgfg})

Replacing according to the location by f, g, or h for the functions k, h in Theorems 1, 2, and 3, as corollaries, we obtain the stability of the sine functional equation (S) from the approximate inequalities of (S_{fghh}), (S_{fghf}), (S_{fgfh}), (S_{fgfg}). Other cases are skipped. All proofs follow from that of Theorems 1, 2, 3.

4.1. Stability of the equation (S_{fghh}).

Corollary 2. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)h(y) \right| \le \begin{cases} (i)\ \varphi(y) \\ (ii)\ \varphi(x) \\ (iii)\ \min\{\varphi(x), \varphi(y)\} \end{cases} \qquad \forall x, y \in G.$$

Then, either h is bounded or h satisfies (S) under one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$, $f(x)^2 = g(-x)^2$, respectively.

4.2. Stability of the equation (S_{fghf}).

Corollary 3. *Assume that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)f(y) \right| \le \varphi(y), \quad \forall x, y \in G$$

which satisfies one of the cases $f(0) = 0$, $f(x)^2 = g(x)^2$.

Then, either h is bounded or f satisfy (S). In addition, if h satisfies (C), *then f and h satisfy (T_{gf}):= $f(x+y) - f(x-y) = 2h(x)f(y)$.*

Corollary 4. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)f(y) \right| \leq \varphi(x), \quad \forall x, y \in G$$

which satisfies one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$.

Then, either f is bounded or h satisfies (S). In addition, if f satisfies (C), *h and f satisfy the Wilson equation (W):=$h(x+y) + h(x-y) = 2h(x)f(y)$.*

The following result follows from Corollaries 3 and 4.

Corollary 5. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)f(y) \right| \leq \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in G$. Then

(i) either h is bounded under one of the cases $f(0) = 0$, $f(x)^2 = g(x)^2$ or f satisfy (S). In addition, if h satisfies (C), *f and h satisfy (T_{gf}):= $f(x+y) - f(x-y) = 2h(x)f(y)$;*

(ii) either f is bounded under one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$ or h satisfies (S). In addition, if f satisfies (C), *h and f satisfy the Wilson equation (W):=$h(x+y) + h(x-y) = 2h(x)f(y)$.*

4.3. **Stability of the equation** (S_{fgfh})**.**

Corollary 6. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right| \leq \varphi(y),$$

which satisfies one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$.

Then, either f is bounded or h satisfies (S). In addition, if f satisfies (C), *h and f satisfy (T_{gf}):= $h(x+y) - h(x-y) = 2f(x)h(y)$.*

Corollary 7. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right| \leq \varphi(x),$$

which satisfies one of the cases $f(0) = 0$, $f(x)^2 = g(-x)^2$.

Then, either h is bounded or f satisfies (S). In addition, if h satisfies (C), *f and h satisfy the Wilson equation (W):= $f(x+y) + f(x-y) = 2f(x)h(y)$.*

Corollary 8. *Suppose that $f, g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right| \leq \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in G$. Then

(i) either f is bounded under one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$ or h satisfies (S). In addition, if f satisfies (C), *h and f satisfy (T_{gf}):= $h(x+y) - h(x-y) = 2f(x)h(y)$;*

(ii) either h is bounded under one of the cases $f(0)=0$, $f(x)^2=g(-x)^2$ or f satisfies (S). In addition, if h satisfies (C)*, f and h satisfy the Wilson equation (W):=$f(x+y)+f(x-y)=2f(x)h(y)$.*

4.4. **Stability of the equation** (S_{fgfg})**.**

Corollary 9. *Suppose that $f,g:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-f(x)g(y)\right|\leq\varphi(y),$$

which satisfies one of the cases $g(0)=0$, $f(x)^2=g(x)^2$.

Then, either f is bounded or g satisfies (S). In addition, if f satisfies (C)*, g and f satisfy (T_{gf}):= $g(x+y)-g(x-y)=2f(x)g(y)$.*

Corollary 10. *Suppose that $f,g:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-f(x)g(y)\right|\leq\varphi(x),$$

which satisfies one of the cases $f(0)=0$, $f(x)^2=g(-x)^2$.

Then, either g is bounded or f satisfies (S). In addition, if g satisfies (C)*, then f and g satisfy the Wilson equation (W).*

Corollary 11. *Suppose that $f,g:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2-f(x)g(y)\right|\leq\min\{\varphi(x),\varphi(y)\}$$

for all $x,y\in G$. Then

(i) either f is bounded under one of the cases $g(0)=0$, $f(x)^2=g(x)^2$ or g satisfies (S). In addition, if f satisfies (C)*, g and f satisfy (T_{gf}):= $g(x+y)-g(x-y)=2f(x)g(y)$;*

(ii) either g is bounded under one of the cases $f(0)=0$, $f(x)^2=g(-x)^2$ or f satisfies (S). In addition, if g satisfies (C)*, then f and g satisfy the Wilson equation (W).*

Remark 2. *As corollaries, we obtain more stability results for the following reduced equations of (S_{fghk}).*

(i) The stability for the functional equations (S_{fghg}), (S_{fggh}), (S_{fggf}), (S_{fgff}), (S_{fggg}), and (S_{gh}), (S_{gf}), (S_{fg}), (S_{gg}) is skipped by same reason as the cases (S_{fghh}), (S_{fghf}), (S_{fgfh}), (S_{fgfg}). In particular, the stability for the equations (S_{gh}), (S_{gf}), (S_{fg}), (S_{gg}) is found in papers (see [11, 14, 19]*).*

(ii) Applying $\varphi(x)=\varphi(y)=\varepsilon$ in all results containing (i), then it imply the stability results.

5. Extension of the stability results to Banach algebras

All the results in Sections 3 and 4 can be also extended to Banach algebras. The following theorem is an extension dued by Theorem 1, Theorem 2, and Theorem 3.

Theorem 4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h, k : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)k(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition $x^* \circ h$ under the cases $k(0) = 0$ or $f(x)^2 = g(x)^2$ is bounded or k satisfies (S), In addition, if h satisfies* (C)*, k and h satisfy (T_{gf}):= $k(x+y) - k(x-y) = 2h(x)k(y)$;*

(ii) *either the superposition $x^* \circ k$ under the cases $h(0) = 0$ or $f(x)^2 = g(-x)^2$ is bounded or h satisfies (S). In addition, if k satisfies* (C)*, h and k satisfy the Wilson equation (W):= $h(x+y) + h(x-y) = 2h(x)k(y)$;*

(iii) (i) *and* (ii) *hold.*

Proof. Assume that (i) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned} \varphi(y) &\geq \left\| h(x)k(y) - f\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x-y}{2}\right)^2 \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(h(x)k(y) - f\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x-y}{2}\right)^2 \right) \right| \\ &\geq \left| x^*(h(x)) \cdot x^*(k(y)) - x^*\left(f\left(\frac{x+y}{2}\right)\right) + x^*\left(g\left(\frac{x-y}{2}\right)^2\right) \right|, \end{aligned}$$

which states that the superpositions $x^* \circ h$ and $x^* \circ k$ yield a solution of stability inequality (3.1) of Theorem 1. Since, by assumption, the superposition $x^* \circ h$ is unbounded, an appeal to Theorem 1 forces that the function $x^* \circ k$ solves the sine equation (S). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference $\mathcal{D}S : G \longrightarrow E$ defined by

$$\mathcal{D}S(x, y) := k\left(\frac{x+y}{2}\right)^2 - k\left(\frac{x-y}{2}\right)^2 - k(x)k(y)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}S(x, y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathcal{D}S(x, y) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed. The cases(ii), (iii) also are the same. □

Corollary 12. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)h(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ h$ is bounded or h satisfies (S) under one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$, $f(x)^2 = g(-x)^2$, respectively.

Corollary 13. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - h(x)f(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition $x^* \circ h$ under one of the cases $f(0) = 0$, $f(x)^2 = g(x)^2$ is bounded or f satisfies (S), In addition, if h satisfies* (C), *f and h satisfy (T_{gf});*

(ii) *either the superposition $x^* \circ f$ under one of the cases $h(0) = 0$, $f(x)^2 = g(-x)^2$ is bounded or h satisfies (S). In addition, if f satisfies* (C), *h and f satisfy the Wilson equation (W);*

(iii) (i) *and* (ii) *hold.*

Corollary 14. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 - f(x)h(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}. \end{cases}$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i)*) either the superposition $x^* \circ f$ under one of the cases $h(0) = 0$, $f(x)^2 = g(x)^2$ is bounded or h satisfy (S);*

In addition, if f satisfies (C), *h and f satisfy (T_{gf}):= $h(x + y) - h(x - y) = 2f(x)h(y)$.*

(ii) *either the superposition $x^* \circ h$ under the cases $f(0) = 0$ or $f(x)^2 = f(-x)^2$ is bounded or f satisfies (S). In addition, if h satisfies* (C), *f and h satisfy the Wilson equation (W):= $f(x + y) + f(x - y) = f(x)h(y)$;*

(iii) (i) *and* (ii) *hold.*

Corollary 15. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f : G \longrightarrow E$ satisfies the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)f(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ f$ is bounded or f satisfies (S).

Remark 3. *All items of Remark 2 also hold to same results for all functional equations on Banach algebras.*

6. Conclusion

We investigated the superstability bounded by function for the sine functional equation (S) from the approximate inequality of the Pexider type functional equation (S_{fghk}), and we studied a creative process for the sine, cosine(d'Alembert), Wilson, Kim's, (S) type functional equations, which are a frequently arisen function equations related for the sine functional equation (S) and the Pexider type functional equation(S_{fghk}).

As a result, all (S) types functional equations related with (S) and (S_{fghk}) can be represented by the trigonometric, exponential, hyperbolic function, jointed Pexider Lobacevski equation. Furthermore, we showed the application of our results to a myriad of equations and the results were extended to Banach algebra.

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