Under Quasi Nonexpansive Mapping Generalization of Weak Convergence and Study of Fixed Point in Hilbert Space

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Abstract

This manuscript proposes a generalization of weak convergence and study of fixed point in a real Hilbert space for quasi-nonexpanding maps. In this work, we introduce a class of fixed points theorems for nonexpansive mapping and generalized form of nonexpansive mapping under the Hilbert space. In addition, we obtained under quasi nonexpansive mapping weak convergence with respect to Hilbert space by Mann's Type.

Key Words and Phrases: Hilbert space, quasi-nonexpansive mappings, weak convergence.

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1 Introduction

A significant area of research in pure as well as applied mathematics is fixed point theory. The Fixed-Point Theory has many applications in various fields namely Approximation Theory, Integral Equations, Game Theory, Optimization, Economics, and several others [1]. How to solve nonlinear equations like $Tx = 0$ is one of the fundamental issues in mathematics. We can utilise iterative techniques like Newton methods and its variations to solve these problems. We must therefore employ approximation techniques because the zeros of a nonlinear equation cannot be stated in closed forms. Nowadays, we frequently employ iterative techniques to obtain a system's approximate solution. The

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general Newton's method is frequently used approach. Recent advancements in the solution system have made it possible for us to reach iterative formulae by employing Taylor's polynomial, quadrature formulas, and other methods. One of the powerful and versatile solution technique for solving nonlinear equations is Fixed point iterative method [2]. Recently many fixed-point results have been discussed in different type of non-expansive mappings [3, 4].

Let H be any Hilbert space having convex closed subset of K which is non empty. Now define a continuous mapping S from convex subset K to convex subset K. A point $a \in K$ known as a fixed point of continuous mapping S if $S(a) = a$. Additionally, the $F(S)$ denotes the collection of all fixed points for S. A fixed point's existence theorems of single-valued nonexpansive mappings has been studied by a few authors [5]. A mapping $T : B \to B$ defined on space B if is known as nonexpansive mapping $\|T\mu - T\zeta\| \leq \|\mu - \zeta\|, \forall \mu, \zeta$ in space B. A general map $T : B \to B$ defined on space B is called quasi-nonexpansive mapping provided it has fixed point in space B and if $\vartheta \in B$ is fixed point of T, then $\|T\mu - \vartheta\| \leq \|\mu - \vartheta\|, \forall \mu \in B$. Thus every nonexpansive mapping becomes quasi-nonexpansive if it has a minimum one fixed point. A mapping $T : B \to B$ described as being generalised nonexpansive if $\forall \mu, \vartheta \in B$ and $m, n, o \geq 0$, the mapping T satisfy
$\|T\mu - T\vartheta\| \leq m\|\mu - \vartheta\| + n\{\|\mu - T\mu\| + \|\vartheta - T\vartheta\|\} + o\{\|\mu - T\vartheta\| + \|\vartheta - T\mu\|\}$
with $m + 2n + 2o \leq 1$ [6].

Let B be convex subset of X. For $\mu_j \in B$, define a sequence $\{\mu\}_{n=1}^{\infty}$ such that $\mu_{n+1} = (1 - \beta_n)\mu_n + \beta_n T\mu_n$, where $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of positive number $\beta_n \in [a, b]$ for all $n \in \mathbb{N}$ and $0 < a < b < 1$. A mapping $T : B \to B$ is demiclosed with respect to $\omega \in X$ if for each sequence $\{\mu_n\} \subset B$ and each $\mu \in X$ it follows from $\mu_n \rightharpoonup \mu$ and $\lim T(\mu_n) = \omega$ that $\mu \in B$ and $T(\mu) = \omega$. The set of fixed point of T is denoted by the abbreviation $F(T)$ [7].

2 Preliminaries

Firstly we introduce some lemmas and definitions.

Definition 2.1 [7] Any Banach space X is called a Hilbert space if there exit a scalar product defined on space X such that the norm defined in space X is same as the norm defined by the relation $\tau = <\tau, \tau>^{1/2}$.

• Let l^2 be the set contains the elements of the form $\tau = (\tau_1, \tau_2, ...)$ such that

$$\|\tau\| = \sum_{i=1}^{\infty} |\tau_i^2|^{1/2} < \infty.$$

• The inner product space $(R^n, < .,. >)$ equipped with the induced norm given by

$$\|\tau\| = < \tau, \tau >^{1/2} = \left(\sum_{i=1}^{\infty} |\eta_i|^2 \right)^{1/2} \textit{such that } (\eta_1, \eta_2, ...\eta_n) \in R^n$$

Definition 2.2 [**6**] Let H be any Hilbert space, mapping T defined on Hilbert space H is nonexpansive if

$$d(T\zeta, T\vartheta) \leq d(\zeta, \vartheta), \forall \zeta, \vartheta \in H$$

Definition 2.3 [**6**] Let H be any Hilbert space then the mapping T on Hilbert space H is quasi nonexpansive if

$$d(T\mu, \vartheta) \leq d(\mu, \vartheta), \forall \mu \in H, \forall \vartheta \in F(T)$$

such that mapping T has at least one fixed point.
Definition 2.4 [**7**] The Opial's condition is crucial for understanding the demi-closed Ness the nonlinear mappings principle as well as the geometry of spaces and sequence convergence. Any If Space X meets the requirement of the Opial, a sequence ζ_n defined on space X converges weakly to any $\zeta_0 \in X$ then

$$\lim_{n\to\infty} inf\|\zeta_n - \zeta_0\| < \lim_{n\to\infty} inf\|\zeta_n - \zeta\|, \forall \zeta \in X \ and \ \zeta \neq \zeta_0$$

Here if we replace the strict inequality $<$ by the inequality $\leq$ then, we obtain weak Opial's condition.
Definition 2.5 [**5**] Let $E \subseteq H$ where H is a Hilbert space and $T : E \to H$ is a map defined from E to Hilbert space H. Then the mapping T is demiclosed at any $s \in H$ if for any corresponding sequence $\zeta_n \in E$ the mapping T follow the condition as:

$$\zeta_n \to \beta \in E \ and \ T\zeta_n \to \vartheta \Rightarrow T\beta = \vartheta$$

An Opial's Condition defined on reflexive Banach space X such that E is a not empty space X closed convex subset containing a nonexpansive mapping $T : E \to X$ then $I - T$ is demi closed.
Lemma 2.6 [**8**] Assume a Hilbert space H such that $E \in H$ then $S : E \to CB(E)$ is called mapping of Condition (A) if

$$\|\mu - \vartheta\| = d(\mu, S\vartheta), \ for \ all \ \mu \in H \ and \ \vartheta \in F(s).$$

Lemma 2.7 [**9**] Let H be a real Hilbert space and $K \in H$ such that a quasi nonexpansive map from $S : H \to CB(H)$ with $F(S)$ *nonempty*. Then, $F(S)$ is said to be closed and if S fullfill above Condition (A), then $F(S)$ is said to be convex. The mapping S is said to hybrid if

$$3H(S\mu, S\omega)^2 \leq \|\mu - \omega\|^2 + d(\omega, S\mu)^2 + d(\mu, S\omega)^2, \forall \mu, \omega \in K$$

Lemma 2.8 [**5**] Let H be a Hilbert space such that $E \in H$ and $S : E \to E(E)$ is hybrid mapping. Assume ϑ_n be a sequence in mapping E such that $\vartheta_n \to \vartheta$ and $\lim_{n\to\infty} \|\vartheta_n - y_n\| = 0$ for sequence $y_n \in S\vartheta_n$. Then, $\vartheta \in S\vartheta$.

3 Quasi Nonexpansive Mapping with Respect to Hilbert Space

Lemma 3.1 Let X be a normed space with a convex subset C and mapping $T : C \to C$ defined on space C be a quasi-nonexpansive mapping. Suppose that $\{\zeta_n\}_{n=1}^{\infty}$ is a sequence such that $\zeta_1 \in C$. Then, the limit $\lim_{n\to\infty} \|\zeta_n - \zeta\|$ exists for each $\zeta \in F(T)$.

Proof: We have given that the mapping T is a quasi-nonexpansive mapping. Hence, we have

$$\|\zeta_{n+1} - \zeta\| = \|(\alpha_n T\zeta_n + (1-\alpha_n)\zeta_n) - \zeta\|$$
$$\leq \alpha_n\|T\zeta_n - \zeta\| + (1-\alpha_n)\|\zeta_n - \zeta\| = \|\zeta_n - \zeta\|,$$

For each $\zeta \in F(T)$. Hence, the sequence $\{\|\zeta_n - \zeta\|\}_{n=1}^{\infty}$ is a bounded below and nonincreasing sequence, so from this we conclude that the limit $\lim_{n\to\infty} \|\zeta_n - \zeta\|$ exists for each $\zeta \in F(T)$.

Lemma 3.2 Let us consider a uniformly convex Hilbert space X, $0 < b < d < 1$, $\beta \geq 0$, $t_n \in [b, d]$ and $\{\zeta_n\}_{n=1}^{\infty}$ and $\{\vartheta_n\}_{n=1}^{\infty}$ are sequences defined on Hilbert space X such that

$$\limsup\|\zeta_n\| \leq \beta, \ \limsup\|\vartheta_n\| \leq \beta, \ \textit{and} \ \lim_{n\to\infty} \|t_n\zeta_n + (1-t_n)\vartheta_n\| = \beta$$

then the $\lim_{n\to\infty} \|\zeta_n - \vartheta_n\| = 0$.

Lemma 3.3 Let us consider uniformly convex Hilbert space X with a convex subset C of X and $T : C \to C$ be a quasi-nonexpansive mapping. Assume that $\mu_1 \in C$ and $\{\mu_n\}_{n=1}^{\infty}$ is a sequence then the limit $\lim_{n\to\infty} \|\mu_n - T\mu_n\| = 0$.

Proof: Let μ be fixed point of quasi-nonexpansive mapping T. Now, we know that limit $d = \lim_{n\to\infty} \|\mu_n - \mu\| = 0$. is well-defined by Lemma 3.1 and $\lim_{n\to\infty} Sup\|T\mu_n - \mu\| \leq d$ Since, $\|T\mu_n - \mu\| \leq \|\mu_n - \mu\|$ for all natural numbers. Additionally, we know that

$$\lim_{n\to\infty} \|\alpha_n(T\mu_n - \mu) + (1-\alpha_n)(\mu_n - \mu)\| = \lim_{n\to\infty} \|\mu_{n+1} - \mu\| = d$$

So, from lemma 3.2 we conclude that $\lim_{n\to\infty} \|T\mu_n - \mu_n\| = 0$.

Theorem 3.4 Let us consider a uniformly convex Hilbert space X satisfying Opial's condition and C be a closed subset of Hilbert space X, and mapping $T : C \to C$ be a quasi-non-expansive mapping with $I - T$ demiclosed with respect to zero. Suppose that $\zeta_1 \in C$ Then the sequence $\{\zeta_n\}_{n=1}^{\infty}$ converges weakly to some fixed point of quasi-non-expansive mapping T.

Proof: Let us consider two weakly convergent subsequences $\{\zeta_{\emptyset_n}\}$ and $\{\zeta_{\psi_n}\}$ of sequence $\{\zeta_n\}$ which are weakly convergent to some points ζ and ϑ in C, respectively. Since $\lim_{n\to\infty} \|\zeta_n - T\zeta_n\| = 0$ by Lemma 3.3 and $I - T$ is demiclosed with respect to zero such that $T\zeta = \zeta$ and $T\vartheta = \vartheta$

Now, put $a = \lim_{n\to\infty} \|\zeta_n - \vartheta\|$ by lemma 3.1. Assume that $\zeta \neq \vartheta$ and consider the fact that $\zeta_{\emptyset_n} \rightharpoonup \zeta$ and $\zeta_{\psi_n} \rightharpoonup \vartheta$ then from the Opial's condition we get

$$a = \liminf\|\zeta_{\emptyset_n} - \zeta\| < \liminf\|\zeta_{\emptyset_n} - \vartheta\| = b,$$
$$b = \liminf\|\zeta_{\psi_n} - \vartheta\| < \liminf\|\zeta_{\psi_n} - \zeta\| = a,$$

Which is a contradiction. Hence $\zeta = \vartheta$.
This shows that the above sequence $\{\zeta_n\}_{n=1}^{\infty}$ has exactly one weak cluster point, from which we conclude that the sequence $\{\zeta_n\}_{n=1}^{\infty}$ converges weakly to some $\tau \in C$. On Repeating the above concept we conclude that $T\tau = \tau$. Hence, the sequence $\{\zeta_n\}_{n=1}^{\infty}$ converges weakly to some fixed point of T.

4 Generalized Quasi-Nonexpansive Mapping under Hilbert Space

A map $T : H \to H$ defined on Hilbert space H is said to be generalized quasi nonexpansive mapping if $\forall \mu, \vartheta \in C$ and $m, n, o \geq 0$, mapping T satisfies

$$\|T\mu - T\vartheta\| \leq m\|\mu - \vartheta\| + n\{\|\mu - T\mu\| + \|\vartheta - T\vartheta\|\} + o\{\|\mu - T\vartheta\| + \|\vartheta - T\mu\|\}$$

with $m + 2n + 2o \leq 1$.
Theorem 4.1 Let H be a uniformly convex Hilbert space with a bounded convex subset C of H. Mapping T be a generalized quasi nonexpansive mapping. Then, for any small $\epsilon \geq 0$ there exists a small $\delta(\epsilon) > 0$ will be such that for each pair of points ζ_0, ζ_1 in C with $\|T\zeta_n - \zeta_1\| \leq \delta(\epsilon)$, and for any point ζ lies on the line segment joining point ζ_0 to point ζ_1 with $\|T\zeta - \zeta\| \leq \epsilon$.
Proof: We have given that the point ζ lies on the line segment joining point ζ_1 to point ζ_2. Therefore,

$$\zeta = (1 - \lambda)\zeta_1 + \lambda\zeta_2,\ 0 \leq \lambda \leq 1.$$

Now, define $f = b + c$. Suppose $\|\zeta_1 - \zeta_2\| \leq \epsilon(1 - f)/4$. Then, for each ζ lies on the line segment joining the points ζ_0 to ζ_1,

$$\|\zeta - \zeta_1\| \leq \epsilon(1 - f)/4$$
$$\|T\zeta - \zeta\| \leq \|T\zeta - T\zeta_1\| + \|T\zeta_1 - \zeta_1\| + \|\zeta_1 - \zeta\|$$

therefore, we get

$$\begin{aligned}(1 - c)\|T\zeta - T\zeta_1\| &\leq \alpha\|\zeta - \zeta_1\| + b\{\|\zeta - T\zeta\| + \|\zeta_1 - T\zeta_1\|\}\\ &\quad + c\{\|\zeta - T\zeta_1\| + \|\zeta_1 - T\zeta\|\} - c\|T\zeta - T\zeta_1\|\\ &\leq \alpha\|\zeta - \zeta_1\| + b\{\|\zeta - T\zeta\| + \|\zeta_1 - T\zeta_1\|\}\\ &\quad + c\{\|\zeta - T\zeta_1\| + \|\zeta_1 - T\zeta_1\|\}\\ &\leq (a + c)\|\zeta - \zeta_1\| + b\|\zeta - T\zeta\| + (b + 2c)\|\zeta_1 - T\zeta_1\|\end{aligned}$$

hence,

$$\begin{aligned}\|T\zeta-\zeta\|&\leq\|T\zeta-T\zeta_1\|+\|T\zeta_1-\zeta_1\|+\|\zeta_1-\zeta\|\\&\leq\left(1+\frac{a+c}{1-c}\right)\|\zeta-\zeta_1\|+\frac{b}{1-c}\|\zeta-T\zeta\|\\&\quad+\left(1+\frac{b+2c}{1-c}\right)\|\zeta_1-T\zeta_1\|\end{aligned}$$

and

$$\|T\zeta-\zeta\|\leq\frac{1+a}{1-b-c}\|\zeta-\zeta_1\|+\frac{1+b+c}{1-b-c}\|\zeta_2-T\zeta_1\|\leq\frac{\epsilon}{2}+\frac{2}{1-f}\delta(\epsilon)\leq\epsilon$$

if $\delta(\epsilon)<(1-f)\epsilon/4$.
Therefore, we consider only that couple of points ζ_1,ζ_2 which satisfying the condition $\|\zeta_1-\zeta_2\|\geq(1-f)\epsilon/4$.
Let $d_0=diam(C)$. Then, for $\lambda<\epsilon\frac{1-f}{4d_0}$, $\|\zeta-\zeta_1\|=\lambda\|\zeta_2-\zeta_1\|<\epsilon(1-f)/4$ and by the argument, $\|T\zeta-\zeta\|<\epsilon$. Hence, we must consider only $\lambda\geq\frac{\epsilon(1-f)}{4d_0}$. If $1-\lambda<\epsilon(1-f)/4d_0$, then $\|\zeta-\zeta_2\|=(1-\lambda)\|\zeta_2-\zeta_1\|<(1-f)/4d_0$. And, applying the same argument with ζ_2 replacing ζ_1, again we get $\|T\zeta-\zeta\|<\epsilon$. Therefor we get

$$\lambda\in\left[\frac{\epsilon(1-f)}{4d_0}\right].$$

set $y=T\zeta$. Then

$$\|y-\zeta_1\|\leq\|T\zeta-T\zeta_2\|+\|T\zeta_1-\zeta_1\|$$

and

$$\begin{aligned}\|T\zeta-T\zeta_1\|\leq\alpha\|\zeta-\zeta_1\|&+b[\|\zeta-\zeta_1\|+\|\zeta_1-T\zeta\|+\|\zeta_1-T\zeta_1\|]\\&+c[\|\zeta-\zeta_1\|+\|\zeta_1-T\zeta_1\|+\|\zeta_1-T\zeta\|].\end{aligned}$$

thus,

$$(1-b-c)\|T\zeta-\zeta_1\|\leq(a+b+c)\|\zeta-\zeta_1\|+(1+b+c)\|\zeta_1-T\zeta_1\|,$$

and

$$\|T\zeta-\zeta_1\|\leq\|\zeta-\zeta_1\|+2(1-f)^{-1}\|\zeta_1-T\zeta_2\|\leq\lambda\|\zeta_1-\zeta_2\|+2\delta(\epsilon)/(1-f).$$

similarly,

$$\|T\zeta-\zeta_2\|\leq(1-\lambda)\|\zeta_1-\zeta_2\|+2\delta(\epsilon)/(1-f)$$

set

$$\begin{aligned}z_0&=\lambda^{-1}\|\zeta_1-\zeta_2\|^{-1}(y-\zeta_1),\\z_1&=(1-\lambda)^{-1}\|\zeta_1-\zeta_2\|^{-1}(\zeta_2-y).\end{aligned}$$

then

$$\|z_0\| \leq 1 + \frac{64 d_0 \delta(\epsilon)}{\epsilon^2 (1-f)^3}.$$

similarly

$$\|z_1\| \leq 1 + \frac{64 d_0 \delta(\epsilon)}{\epsilon^2 (1-f)^3}.$$

but we know that

$$\|\lambda z_0 + (1-\lambda) z_1\| = 1.$$

and H is a Uniformly convex Hilbert space. Hence, if we choose positive $\delta(\epsilon)$ as small as possible then, we have $\|z_0 - z_1\| < \epsilon/d_0$. Thus,

$$\|y - \zeta\| = \|((1-\lambda)(y - \zeta_1) - \lambda(\zeta - y) = \lambda(1-\lambda)\|\zeta_1 - \zeta_2\| \|z_0 - z_1\| < \epsilon)\|.$$

Hence, for any small $\epsilon > 0$ there exists a small $\delta(\epsilon) > 0$ such that for each pair of points ζ_0, ζ_1 in C with $\|T\zeta_n - \zeta_1\| \leq \delta(\epsilon)$, and for any point ζ lies on the line segment joining point ζ_1 to point ζ_2 with $\|T\zeta - \zeta\| \leq \epsilon$.

5 Weak Convergence of Quasi-Nonexpansive Mapping with Respect to Hilbert Space by Mann's Type

In 1953, Mann[11] created the standard Mann's iteration technique. Since Mann's iterative approach for creating fixed points for nonexpansive mapping has been thoroughly studied by other authors. The typical Mann's iterative procedure produces the sequence $\{\vartheta_n\}$ as follows:

$$\vartheta_1 = \vartheta \in K$$

$$\vartheta_{n+1} = (1 - \zeta_n)\vartheta_n + \zeta_n T \vartheta_n, \ \forall\ n \geq 1$$

Where $< \zeta_n >$ is a sequence lies in 0 to 1. Mann's Type weak convergence theorem for quasi-nonexpansive mapping.

For quasi-nonexpansive mapping in Hilbert space, we provide a weak convergence theorem of mann's kind [11] in this section. Before demonstrating this, we address several common findings, such as:

Lemma 5.1 Let T be quasi-nonexpansive map defined from closed convex subset C of Hilbert space H to C. Then, $I - T$ is demiclosed.

Proof: We have given that $T : C \to C$ be a quasi nonexpansive mapping defined on Hilbert space H. Then, for any real $\gamma, \delta \in R$ we have

$$\gamma\|T\zeta - T\vartheta\|^2 + (1-\gamma)\|\zeta - T\vartheta\|^2 \leq \delta\|T\zeta - \vartheta\|^2 + (1-\delta)\|\zeta - \vartheta\|^2 \tag{5.1}$$

for all $\zeta, \vartheta \in C$.
Now, suppose $\zeta_n \rightharpoonup r$ and $\vartheta_n \rightharpoonup T\zeta_n \to 0$. Let us consider

$$\gamma\|T\zeta_n - Tr\|^2 + (1-\gamma)\|\zeta_n - Tr\|^2 \leq \delta\|T\zeta_n - r\|^2 + (1-\delta)\|\zeta_n - r\|^2 \tag{5.2}$$

from these inequalities, we have

$$\gamma\|T\zeta_n - \zeta_n + \zeta_n - Tr\|^2 + (1-\gamma)\|\zeta_n - Tr\|^2 \leq \delta\|T\zeta_n - \zeta_n + \zeta_n - r\|^2 + (1-\delta)\|\zeta_n - r\|^2$$

and hence

$$\begin{aligned}&\gamma(\|T\zeta_n - \zeta_n\|^2 + \|\zeta_n - Tr\|^2 + 2\langle T\zeta_n - \zeta_n,\ \zeta_n - Tr\rangle) + (1-\gamma)\|\zeta_n - Tr\|^2 \\ &\leq \delta(\|T\zeta_n - \zeta_n\|^2 + \|\zeta_n - r\|^2 + 2\langle T\zeta_n - \zeta_n,\ \zeta_n - Tr\rangle) + (1-\delta)\|\zeta_n - r\|^2\end{aligned}$$

now, we apply a Hilbert limit μ on both the sides of the above inequality, then we have

$$\begin{aligned}&\gamma\mu_n(\|T\zeta_n - \zeta_n\|^2 + \|\zeta_n - Tr\|^2 + 2\langle T\zeta_n - \zeta_n,\ \zeta_n - Tr\rangle) + (1-\gamma)\mu_n\|\zeta_n - Tr\|^2 \\ &\leq \delta\mu_n(\|T\zeta_n - \zeta_n\|^2 + \|\zeta_n - r\|^2 + 2\langle T\zeta_n - \zeta_n,\ \zeta_n - Tr\rangle) + (1-\delta)\mu_n\|\zeta_n - r\|^2\end{aligned}$$

and hence

$$\gamma\mu_n\|\zeta_n - Tr\|^2 + (1-\gamma)\mu_n\|\zeta_n - Tr\|^2 \leq \delta\mu_n\|\zeta_n - r\|^2 + (1-\delta)\mu_n\|\zeta_n - r\|^2$$

so, we have

$$\mu_n\|\zeta_n - Tr\|^2 \leq \mu_n\|\zeta_n - r\|^2$$

since,

$$\mu_n\|\zeta_n - r\|^2 + \mu_n\|\zeta_n - r + r - Tr\|^2 \leq \mu_n\|\zeta_n - r\|^2$$

therefore, finally we get

$$\mu_n\|\zeta_n - r\|^2 + \mu_n\|r - Tr\|^2 + 2\mu_n\langle \zeta_n - r,\ r - Tr\rangle \leq \mu_n\|\zeta_n - r\|^2$$

so, we from all these we get $\mu_n\|r - Tr\|^2 \leq 0$ and $\|r - Tr\|^2 \leq 0$. Which implies, $Tr = r$. Therefore, $I - T$ is demiclosed.

Theorem 5.2 Let T be quasi-nonexpansive map defined from closed convex subset C of Hilbert space H to C with at least one fixed point i.e. $F(T) = \{\rho \in C : T\rho = \rho\}$ and $F(T) \neq \emptyset$. Let G be a metric protection of Hilbert space H onto $F(T)$ and $\{x_n\}$ be a real number sequence lies between 0 and 1 such that $\lim_{n\to\infty} \inf x_n(1 - x_n) > 0$. Let $\langle x_n\rangle$ generates a sequence $\langle \rho_n\rangle$ such that

$$\rho_{n+1} = x_n\rho_n + (1 - x_n)T\rho_n,\ n = 1,\ 2,\ 3.\ldots.,\ \rho_1 = \rho \in C$$

then the sequence $\langle \rho_n\rangle$ converges weakly to a member ϑ of $F(T)$ such that $\vartheta = \lim_{n\to\infty} G\rho_n$.

Proof: : Let $\zeta \in F(T)$ and T be a quasi-nonexpansive mapping defined on Hilbert space H. Then, we have

$$\|\rho_{n+1} - \zeta\|^2 = \|x\rho_n + (1 - x_n)T\rho_n - \zeta\|^2 \leq x_n\|\rho_n - \zeta\|^2 + (1 - x_n)\|T\rho_n - \zeta\|^2$$

$$\leq x_n\|\rho_n - \zeta\|^2 + (1 - x_n)\|\rho_n - \zeta\|^2 = \|\rho_n - \zeta\|^2$$

For all natural numbers. Hence, the limit $\lim_{n\to\infty} \|\rho_n - \zeta\|^2$ exists. So, we can say that the sequence $\{\rho_n\}$ is bounded.
we also have

$$\begin{aligned}\|\rho_{n+1} - \zeta\|^2 &= \|x_n\rho_n + (1 - x_n)T\rho_n - \zeta\|^2 \\ &= x_n\|\rho_n - \zeta\|^2 + (1 - x_n)\|T\rho_n - \zeta\|^2 - x_n(1 - x_n)\|T\rho_n - \rho_n\|^2 \\ &\leq x_n\|\rho_n - \zeta\|^2 + (1 - x_n)\|\rho_n - \zeta\|^2 - x_n(1 - x_n)\|T\rho_n - \rho_n\|^2 \\ &= \|\rho_n - \zeta\|^2 - x_n(1 - x_n)\|T\rho_n - \rho_n\|^2\end{aligned}$$

so, we have

$$x_n(1 - x_n)\|T\rho_n - \rho_n\|^2 \leq \|\rho_n - \zeta\|^2 - \|\rho_{n+1} - \zeta\|^2$$

since the limit $\lim_{n\to\infty} \|\rho_n - \zeta\|^2$ exists and $\lim \inf_{n\to\infty} x_n(1 - x_n) > 0$, we have $\|T\rho_n - \rho_n\|^2 \to 0$. The above defined sequence $\{\rho_n\}$ is bounded. Hence, there exists a subsequence $\{\rho_{n_i}\}$ of sequence $\{\rho_n\}$ such that $\rho_{n_i} \to v$. By lemma 5.1, we obtained a fixed point $\vartheta \in F(T)$. Similarly, let us assume that $\{\rho_{n_i}\}$ and $\{\rho_{n_j}\}$ are the two sub sequences of sequence $\{\rho_n\}$ such that $\rho_{n_i} \to \vartheta_1 and \rho_{n_j} \to \vartheta_2$. Now, for proving the given theorem we must show that $\vartheta_1 = \vartheta_2$. we know $\vartheta_1, \vartheta_2 \in F(T)$ and hence, the limits $\lim_{n\to\infty} \|\rho_n - \vartheta_1\|^2$ and $\lim_{n\to\infty} \|\rho_n - \vartheta_2\|^2$ are exist.
now, Put

$$\alpha = \lim_{n\to\infty} (\|\rho_n - \vartheta_1\|^2 - \|\rho_n - \vartheta_2\|^2), \ for\ all\ positive\ integers.$$

$$\|\rho_n - \vartheta_1\|^2 - \|\rho_n - \vartheta_2\|^2 = 2\langle\rho_n, \ \vartheta_2 - \vartheta_1\rangle + \|\vartheta_1\|^2 - \|\vartheta_2\|^2$$

such that $\rho_{n_i} \rightharpoonup \vartheta_1$ *and* $\rho_{n_j} \rightharpoonup \vartheta_2$ then, we get

$$\alpha = 2\langle\vartheta_2, \ \vartheta_2 - \vartheta_1\rangle + \|\vartheta_1\|^2 - \|\vartheta_2\|^2 \tag{5.3}$$

and

$$\alpha = 2\langle\vartheta_2, \ \vartheta_2 - \vartheta_1\rangle + \|\vartheta_1\|^2 - \|\vartheta_2\|^2 \tag{5.4}$$

Now, by combining these two equations, we obtain $0 = 2\langle\vartheta_2 - \vartheta_1, \ \vartheta_2 - \vartheta_1\rangle$ and hence $\|\vartheta_2 - \vartheta_1\|^2 = 0$. So, we get $\vartheta_2 = \vartheta_1$. This implies that the sequence $\{\rho_n\}$ converges weakly to an fixed point ϑ of $F(T)$.

Since $\|\rho_{n+1} - \zeta\|^2 \leq \|\rho_n - \zeta\|$ for all $\zeta \in F(T)$ and $n \in N$ and we already see that the sequence $\{Gx_n\}$ firmly converges to a fixed point g of $F(T)$. From the property of g we have

$$\langle x_n - Gx_n,\ Gx_n - y\rangle \geq 0$$

for all fixed points $y \in F(T)$ and $n \in N$. Since the sequence ρ_n converges to ϑ and the sequence Gx_n converges to g. So, we can say that

$$\langle \vartheta - g,\ g - y\rangle \geq 0$$

for all $y \in F(T)$. Putting $y = \vartheta$, we get $g = \vartheta$. This means $\vartheta = \lim_{n\to\infty} G\rho_n$. Therefore, the sequence $\langle \rho_n \rangle$ converges weakly to a member ϑ of $F(T)$ such that $\vartheta = \lim_{n\to\infty} G\rho_n$.

Conclusion

This article's objective is to give a common approach for talking about the fixed point of generalized quasi-nonexpansive mapping and quasi-nonexpansive mapping with respect to Hilbert space. The research piece concludes by presenting a novel approach to examining the fixed-point theorems for quasi-nonexpansive mapping and its generalized form about Hilbert space.

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