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A Novel Family of Distribution with Application in Engineering Problems: A Simulation Study

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Abstract

We establish a novel family of Kumaraswamy-X probability distributions in the present investigation. We discussed the Kumaraswamy-Exponential univariate probability distribution. The new distribution with three parameters possesses density function with unimodal and reverse J-shape and hazard rate function of bathtub shaped. We study various statistical properties for it and derive the expressions for its density function, distribution function, survival and hazard rate function, Probability weighted Moments, lth moment, moment generating function, quantile function and Shannon entropy. For the derived distribution order statistics is also discussed. The parameters are estimated using the maximum likelihood estimation approach, and the performance of the estimators was evaluated using a Monte Carlo simulation. Through extensive Monte Carlo simulations and comparative analyses, we assess the performance of the Kumaraswamy-X distribution against other common probability distributions used in engineering contexts. When we apply it to real datasets, it offers a more suitable fit than other existing distributions. We explore the characteristics and potential applications of the Kumaraswamy-X distribution in the context of engineering problems through a comprehensive simulation-based investigation.

Keywords: T-X family of distributions, Probability weighted Moments, Shannon entropy, Order Statistics, Monte Carlo simulation, Maximum likelihood estimation.
AMS 2000 Subject Code: 62F10, 62F03.

1 Introduction

In probability distribution theory, the selection of a specific probability distribution for modeling real-world phenomena depends on the flexibility of the distribution. It is practice to apply probability distributions that better match the set of data that is available, instead of transforming the current data collected. Because of this, there have been numerous recent attempts to guarantee that the classical probability distributions are updated and developed, since this could boost their adaptability and improve their ability to predict real-world data sets. The Kumaraswamy-X probability distribution, an extension of the well-known Kumaraswamy distribution, has application in modeling a wide range of lifetime problems. This study explores the characteristics and potential applications of the Kumaraswamy-X distribution in the context of engineering problems through a comprehensive simulation-based investigation.

The concept of creating customised distributions is still a hot topic in the literature today. Several approaches could be used to extend an existing standard distribution. For instance, generalization, which entails leveraging the widely available generalized family of distributions, can boost a distribution's adaptability. To generalize the distribution an additional shape parameter(s) may be added to the family of distributions. These extra shape parameter(s) are responsible for altering the tail weight of the resulting compound distribution and introducing skewness. The extension of classical distributions is a long-standing practice and an important issue in statistics, just like many other real-world issues.

The distributions could be used in different domains, like engineering, economics, industrial and physical fields, among a great number of others. To increase the flexibility of traditional distributions, statisticians developed methods for creating new probability distribution families. In many relevant fields, these improvements give practitioners more flexible model options for results that fit them better and are ultimately more accurate. For instance, some of the well-known families are the beta-G family (B-G) by Eugene et al. [8], Kumaraswamy-G family (Kw-G) by Cordeiro and de Castro [6], McDonald-G family (Mc-G) by Alexander et al. [2], T-X family introduced by Alzaatreh et al. [4], gamma-X family by Alzaatreh et al. [3], Exponentiated T-X family by Alzaghal et al. [5], Logistic-X family by Tahir et al. [15], new Weibull-X family by Ahmad et al. [1] and some new member of T-X family by Jamal and Nasir [10] among others. A new family of Distribution with application on two real datasets on survival problem by Modi et al. [12]. Power Exponentiated Family of Distributions proposed by Modi [11]. In this article, we have proposed a new lifetime family of distributions which can be used to fit data in different fields. The paper is organized as follows. In **Section 2**, we define T-X family of distributions. Kumaraswamy distribution and Exponential distribution in **Section 3** and **Section 4** respectively. In **Section 5**, we provide the probability density function (pdf) and the cumulative distribution function (cdf) of the Kumaraswamy-Exponential distribution. In **Section 6**, we examine the survival function and hazard rate function for the new distribution. Formulas for moments, moment generating function and probability weighted moments of the Kumaraswamy-Exponential distribution (KED) are given in **Section 7**, **Section 8** & **Section 9** respectively. Mean, median and mode are discussed in **Section 10** and quantile function in **Section 11**. The Simulation study and Shannon entropy in **Section 12** & **Section 13** respectively. The distribution of the order statistics for the new distribution are discussed in **Section 14**. In **Section 15**, we estimate its parameters using the method of maximum likelihood estimation. In **Section 16**, we show the application of Kumaraswamy-Exponential distribution on two real datasets and compare it with some well known distributions. We need the following Lemmas to complete the derivations:

Lemma 1.1. *From Gradshteyn and Ryzhik [9], Equation (1.110), Page 25. If q is a positive real non integer and $|z| \leq 1$, then by binomial series expansion we have:*

$$(1-z)^{\Upsilon-1} = \sum_{p=0}^{\infty} (-1)^p \binom{\Upsilon-1}{p} z^p.$$

Lemma 1.2. *From Prudnikov et al. [14], Equation (18), Page 241, the integral expression is defined as follows:*

$$\int z^{\zeta} \ln z \, dz = z^{\zeta+1} \left[\frac{\ln z}{\zeta+1} - \frac{1}{(\zeta+1)^2} \right].$$

Lemma 1.3. *From Gradshteyn and Ryzhik [9], Equation (3.383.1), Page 347. For $Re\Omega > 0$, $\quad Re\varsigma > 0$*

$$\int_0^{\kappa} x^{\Omega-1} (\kappa - x)^{\varsigma-1} e^{\beta x} \, dx = B(\varsigma, \Omega) \, \kappa^{\varsigma+\Omega-1} {}_1F_1(\Omega; \varsigma+\Omega; \beta\kappa).$$

Lemma 1.4. *From Gradshteyn and Ryzhik [9], Equation (2.729.1), Page 239, the integral expression is defined as follows:*

$$\int x^{\xi} \ln(a+bx) \, dx$$
$$= \frac{1}{\xi+1} \left[x^{\xi+1} - \frac{(-a)^{\xi+1}}{b^{\xi+1}} \right] \ln(a+bx) + \frac{1}{\xi+1} \sum_{k=1}^{\xi+1} \frac{(-1)^k x^{\xi-k+2} a^{k-1}}{(\xi-k+2) b^{k-1}}.$$

2 T-X family of distributions

The cumulative distribution function (cdf) of the T-X family introduced by Alzaatreh et al. [4], is given by $U\{W(Q(x))\}$. Let T be a continuous random variable (r.v.) with pdf u(t) defined on [0, 1], can be defined as:

$$G_X(x) = \int_0^{W(Q_X(x))} u(t)\, dt,$$

where Q is the cdf of X, U(t) is the cdf of a r.v. T and W is a non-decreasing function having the support of U as its range defined on [0, 1]. Thus, we have:

$$G_X(x) = U\left[-\log\left(1 - Q_X(x)\right)\right] \qquad x > 0, \tag{2.1}$$

$$g_X(x) = \frac{q_X(x)}{(1 - Q_X(x))} u\left[-\log\left(1 - Q_X(x)\right)\right]. \tag{2.2}$$

Thus substituting the different cdf Q(x)and pdf q(x), we can obtain a number of distributions.

3 Kumaraswamy distribution

A continuous random variable X is said to have Kumaraswamy distribution, if its pdf $f_X(x)$ and cdf $F_X(x)$ are, respectively, given by:

$$f(x) = \kappa b x^{\kappa-1} \left(1 - x^{\kappa}\right)^{b-1}, \qquad 0 \le x \le 1, b > 0, \kappa > 0 \tag{3.1}$$

and

$$F(x) = 1 - \left(1 - x^{\kappa}\right)^{b}. \tag{3.2}$$

4 Exponential distribution

A continuous random variable X is said to have Exponential distribution, if its pdf $f_X(x)$ and cdf $F_X(x)$ are, respectively, given by:

$$f(x) = \eta e^{-\eta x}, \quad x \ge 0,\ \eta > 0, \tag{4.1}$$

and

$$F(x) = 1 - e^{-\eta x}. \tag{4.2}$$

5 Kumaraswamy - Exponential distribution

Using Kumaraswamy distribution in T-X family, we obtain the Kumaraswamy-X family of distributions:

$$G_X(x) = 1 - \left[1 - \left(-\log\left(1 - Q\left(x\right)\right)\right)^{\kappa}\right]^{b}.$$

Using cdf given in equation (4.2), we obtain cdf of Kumaraswamy - Exponential distribution as:

$$G_X(x) = 1 - \left[1 - (\eta x)^{\kappa}\right]^{b}. \tag{5.1}$$

Using Kumaraswamy distribution in T-X family density given in equation (2.2), we obtain the pdf of Kumaraswamy-X family of distributions:

$$g_X(x) = \frac{q(x)}{1 - Q(x)} \kappa b \left(-\log\left(1 - Q(x)\right)\right)^{\kappa - 1} \left[1 - \left(-\log\left(1 - Q(x)\right)\right)^{\kappa}\right]^{b-1}.$$

Using pdf in equation (4.1) and cdf in equation (4.2), we obtain pdf of Kumaraswamy - Exponential distribution

$$g_X(x) = \eta^{\kappa} \kappa b x^{\kappa - 1} \left[1 - (\eta x)^{\kappa}\right]^{b-1}, \tag{5.2}$$

Using Lemma 1, in the above expression then we have

$$g_X(x) = \eta^{\kappa} \kappa b x^{\kappa - 1} \sum_{v=0}^{\infty} (-1)^v \binom{b-1}{v} (\eta x)^{v\kappa} \quad x > 0, b > 0, \eta > 0, \kappa > 0. \tag{5.3}$$

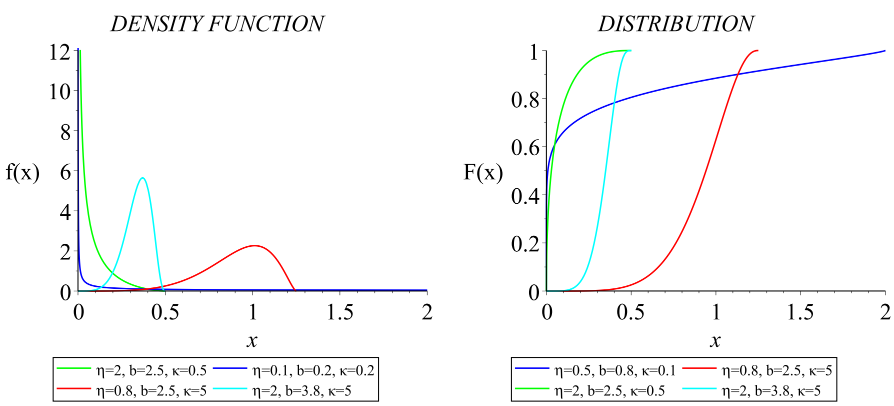


Figure 1: Density function (Left) and distribution function (Right) graphs of Kumaraswamy - Exponential distribution for different values of its parameters η, b, κ.

6 Hazard Rate Function and Survival Function

To study the life phenomena we can use hazard rate function as an important characteristic. Using the pdf defined in equation (5.2), we define h(x) as:

$$h(x) = \frac{\eta^{\kappa} \kappa b x^{\kappa - 1}}{1 - (\eta x)^{\kappa}}, \tag{6.1}$$

also, its survival function obtained as:

$$S(x) = \left[1 - (\eta x)^{\kappa}\right]^{b}. \tag{6.2}$$

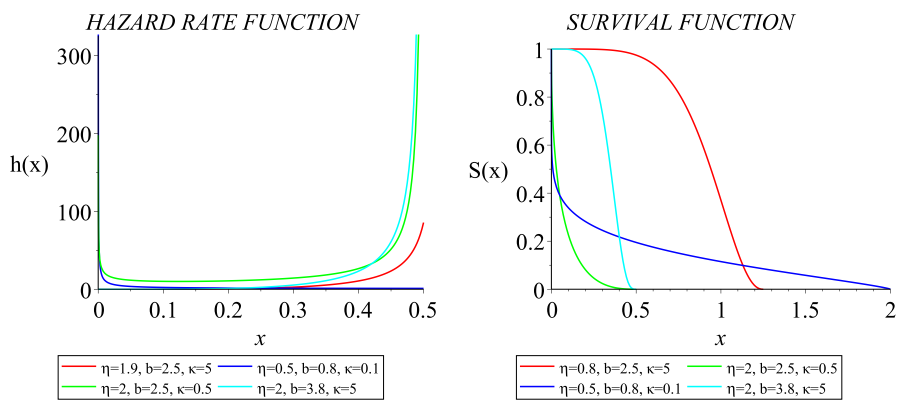


Figure 2: Hazard rate function (Left) and survival function (Right) graphs of Kumaraswamy - Exponential distribution for different values of its parameters η, b, κ.

7 Moments

The l^{th} moment of a random variable X with pdf defined in equation (5.2), can be calculated as:

$$\mu_l^{'} = E(x^l) = \eta^{\kappa}\kappa b \int_0^1 x^{l+\kappa-1} \sum_{v=0}^{\infty} (-1)^v \begin{pmatrix} b-1 \\ v \end{pmatrix} (\eta x)^{v\kappa}\, dx,$$

$$= \kappa b \sum_{v=0}^{\infty} (-1)^v \begin{pmatrix} b-1 \\ v \end{pmatrix} \eta^{v\kappa+\kappa} \int_0^1 x^{l+v\kappa+\kappa-1} dx.$$

On integration, we obtain

$$\mu_l^{'} = \kappa b \sum_{v=0}^{\infty} (-1)^v \begin{pmatrix} b-1 \\ v \end{pmatrix} \eta^{v\kappa+\kappa} \frac{1}{l+v\kappa+\kappa}. \tag{7.1}$$

8 Moment Generating Function

The mgf for the pdf defined in equation (5.2), is given by:

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \eta^\kappa \kappa b \int_0^1 e^{tx}.x^{\kappa-1} \sum_{v=0}^\infty (-1)^v \binom{b-1}{v} (\eta x)^{v\kappa} dx$$

$$= \kappa b \sum_{v=0}^\infty (-1)^v \binom{b-1}{v} \eta^{v\kappa+\kappa} \int_0^1 e^{tx} x^{v\kappa+\kappa-1} dx$$

Using Lemma 3, in the above expression then we have

$$E\left(e^{tx}\right) = \kappa b \sum_{v=0}^\infty (-1)^v \binom{b-1}{v} \times$$
$$\eta^{v\kappa+\kappa} B(1, \kappa+v\kappa) (1)^{\kappa+v\kappa} . {}_1F_1(\kappa+v\kappa; 1+\kappa+v\kappa; t.1)$$

$$E\left(e^{tx}\right) = \kappa b \sum_{v=0}^\infty (-1)^v \binom{b-1}{v} \frac{\eta^{v\kappa+\kappa}}{(\kappa+v\kappa)} . {}_1F_1(\kappa+v\kappa; 1+\kappa+v\kappa; t) \quad (8.1)$$

9 Probability Weighted Moments

For the pdf of the proposed distribution, corresponding p^{th} probability weighted moment is given by:

$$\rho = E\left(x^p (G(x))^\phi\right) = \int_0^1 x^p (G(x))^\phi .g(x) dx,$$

$$= \kappa b \eta^\kappa \int_0^1 x^{p+\kappa-1} \left(1 - [1-(\eta x)^\kappa]^b\right)^\phi . [1-(\eta x)^\kappa]^{b-1} dx,$$

using Lemma 1, in the above expression then we have

$$= \eta^\kappa \kappa b \sum_{t=0}^\infty (-1)^t \binom{\phi}{t} \int_0^1 x^{p+\kappa-1} [1-(\eta x)^\kappa]^{bt+b-1} dx,$$

again we apply Lemma 1, then we obtained

$$= \kappa b \sum_{t=0}^\infty \sum_{u=0}^\infty (-1)^{t+u} \binom{\phi}{t} \binom{bt+b-1}{u} \eta^{\kappa+\kappa u} \int_0^1 x^{p+\kappa+\kappa u-1} dx.$$

$$\rho = \kappa b \sum_{t=0}^{\infty} \sum_{u=0}^{\infty} (-1)^{t+u} \binom{\phi}{t} \binom{bt+b-1}{u} \frac{\eta^{\kappa+\kappa u}}{p+\kappa+\kappa u}. \tag{9.1}$$

10 Mean, Median and Mode

The mean of a probability distribution is defined as:

$$E(x) = \int_0^{\infty} x.g(x)\,dx$$

$$E(x) = \kappa b \sum_{d=0}^{\infty} (-1)^d \binom{b-1}{d} \eta^{d\kappa+\kappa} \frac{1}{1+d\kappa+\kappa}. \tag{10.1}$$

The median of the proposed distribution is given by

$$G_X(M) = \frac{1}{2} \text{ and } M = \frac{1}{\eta}\left[1-\{0.5\}^{1/b}\right]^{1/\kappa}. \tag{10.2}$$

The mode of the proposed distribution given in equation (5.2), can be obtained as

$$g_X(x) = \eta^{\kappa}\kappa b x^{\kappa-1}\left[1-(\eta x)^{\kappa}\right]^{b-1}$$

$$g'(x) = g(x)\,.\left[\frac{\kappa-1}{x} - \frac{\eta^{\kappa}\kappa x^{\kappa-1}(b-1)}{1-(\eta x)^{\kappa}}\right] \tag{10.3}$$

Thus $g(x)$ has mode at $x = \frac{1}{\eta}\left[\frac{\kappa-1}{b\kappa-1}\right]^{1/\kappa}$ with $g(0)=0,\ g(\infty)=\infty$. Clearly $g'(x) > 0,\ \forall b, \kappa, \eta$ this shows that $g(x)$ is a growing function of x.

11 Quantile Function

Using equation (5.1) the quantile function of proposed distribution is given by:

$$Q(x) = \frac{1}{\eta}\left[1-\{1-U\}^{1/b}\right]^{1/\kappa}. \tag{11.1}$$

where for interval [0,1], U follows the Uniform distribution.

12 Simulation Study

To judge the MLEs estimators performance for a finite sample of size n, here we carry out a Monte Carlo simulation analysis. To explore the average biases (ABs), root mean square errors (RMSEs), mean square errors (MSEs) and maximum likelihood estimates (MLEs), a simulation study based on the Kumaraswamy - Exponential distribution is conducted for the distribution parameters η, b and κ. Multiple simulations with different sample sizes and parameter

settings were used to conduct the simulation experiment. We use the quantile function to produce random samples for the KED. The simulation study was performed for sample sizes $n = 50, 100, ..., 1500$ each repeated 1500 times, for the following parameter values $\eta = 2.2,\ b = 3.9,\ \kappa = 4.8$.
The MLEs of the KE model are calculated using the optim ( ) R-function with method = "SANN ". For every set of simulated data, say, (estimates) for $i = 1,\ 2,\ \ldots, 1500$, the AB, MSE, and RMSE of the parameters were computed for $\eta = 2.2,\ b = 3.9,\ \kappa = 4.8$. For different sample sizes, the AB, MLE and RMSE of the parameters, $\eta,\ b$ and κ are shown. These results lead us to the conclusion that the MLEs are best to estimate the model parameters, with more stability and closer to the genuine values. Table 1 and Fig. 3 demonstrate that the RMSE, AB, and MSE drop as sample size grows as would be predicted. The MLEs of the model's parameters are also quite near to their actual values. Thus even small samples can be fitted with derived distribution with better precision.

Table 1: Results obtained for Monte Carlo simulation of MLE, AB and RMSE for the KE Distribution.

Para.	n	MLE	AB	RMSE
	50	0.7829606	-1.4170393	1.518000
	100	0.8087420	-1.3912579	1.555102
	300	1.0488439	-1.1511561	1.544286
η	600	1.5011235	-0.6988764	1.269147
	900	1.8463882	-0.3536118	0.973826
	1200	2.0019255	-0.1980744	0.792184
	1500	2.1223060	-0.0776939	0.615142
	50	1.009512	-2.8904885	3.968965
	100	3.642784	-0.2572159	5.646243
	300	5.213961	1.3139606	5.896080
b	600	5.129874	1.2298743	4.692855
	900	4.670879	0.7708793	3.495491
	1200	4.528949	0.6289491	2.829144
	1500	4.269290	0.3692901	2.041111
	50	13.287630	8.48763013	10.84302
	100	11.627627	6.82762727	9.0838924
	300	7.172703	2.37270337	3.9406621
κ	600	5.465790	0.66579011	2.0529831
	900	4.919511	0.11951053	1.2649680
	1200	4.738959	-0.06104107	0.9195160
	1500	4.633040	-0.16696001	0.7118000

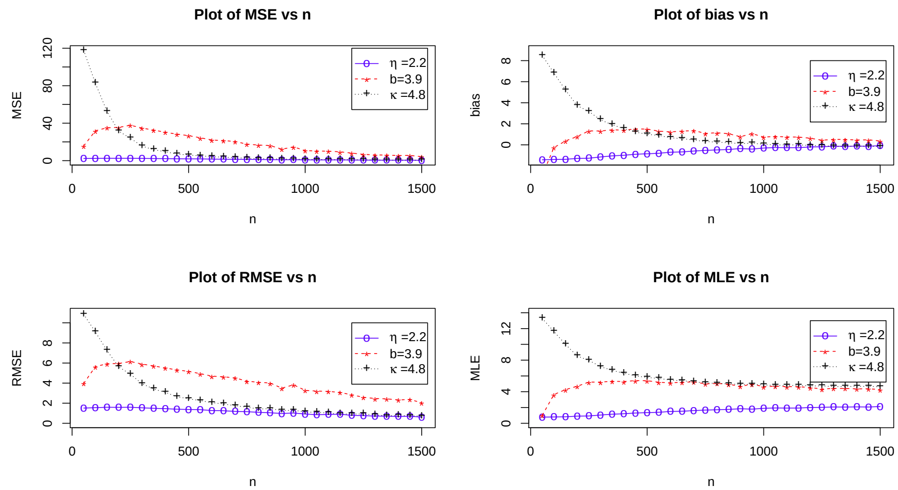


Figure 3: Plots for MLE, bias, MSE, and RMSE for Kumaraswamy - Exponential distribution for parameter values $\eta = 2.2, b = 3.9, \kappa = 4.8$.

13 Shannon Entropy

The entropy of a random variable is a measure of deviation of the uncertainty. The Shannon entropy defined as:

$$E\left[-\ln g_X(x)\right] = -\int_0^\infty \ln g_X(x).g_X(x)dx$$

Using pdf defined in equation (5.2), we get

$$\begin{aligned}
E\left[-\ln g_X(x)\right] = \\
& -\eta^\kappa \kappa b \int_0^1 \ln\left[\eta^\kappa \kappa b x^{\kappa-1}\left[1-(\eta x)^\kappa\right]^{b-1}\right] x^{\kappa-1}\left[1-(\eta x)^\kappa\right]^{b-1} dx \\
& = -\eta^\kappa \kappa b \ln\left(\eta^\kappa \kappa b\right) \int_0^1 x^{\kappa-1}\left[1-(\eta x)^\kappa\right]^{b-1} dx - \eta^\kappa \kappa b\left(\kappa-1\right) \times \\
& \int_0^1 \ln\left(x\right).x^{\kappa-1}\left[1-(\eta x)^\kappa\right]^{b-1} dx - \eta^\kappa \kappa b\left(b-1\right) \times \\
& \int_0^1 \ln\left(1-(\eta x)^\kappa\right).x^{\kappa-1}\left[1-(\eta x)^\kappa\right]^{b-1} dx
\end{aligned}$$

$$= I_1 + I_2 + I_3. \tag{13.1}$$

where,

$$I_1 = -\eta^{\kappa}\kappa b \ln\left(\eta^{\kappa}\kappa b\right) \int_0^1 x^{\kappa-1}\left[1-(\eta x)^{\kappa}\right]^{b-1} dx,$$

$$= -\kappa b \ln\left(\eta^{\kappa}\kappa b\right) \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} b-1 \\ i \end{pmatrix} \eta^{\kappa+i\kappa} \int_0^1 x^{\kappa+i\kappa-1} dx$$

$$I_1 = -\kappa b \ln\left(\eta^{\kappa}\kappa b\right) \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} b-1 \\ i \end{pmatrix} \frac{\eta^{\kappa+i\kappa}}{\kappa+i\kappa}.$$

$$I_2 = -\eta^{\kappa}\kappa b (\kappa-1) \int_0^1 \ln(x).x^{\kappa-1}\left[1-(\eta x)^{\kappa}\right]^{b-1} dx,$$

putting $1-(\eta x)^{\kappa} = t \Rightarrow -\kappa\eta^{\kappa}x^{\kappa-1}dx = dt$ and $x = \frac{1}{\eta}(1-t)^{1/\kappa}$, we get

$$I_2 = -\eta^{\kappa}\kappa b (\kappa-1) \int_{1-\eta^{\kappa}}^{1} \ln\left(\frac{(1-t)^{1/\kappa}}{\eta}\right) t^{b-1} \frac{dt}{\eta^{\kappa}\kappa},$$

$$= -b(\kappa-1) \int_{1-\eta^{\kappa}}^{1} \left[\ln\left(\frac{1}{\eta}\right) + \frac{1}{\kappa}\ln(1-t)\right] t^{b-1} dt,$$

$$= -b(\kappa-1)\ln\left(\frac{1}{\eta}\right) \int_{1-\eta^{\kappa}}^{1} t^{b-1}dt - b\frac{(\kappa-1)}{\kappa} \int_{1-\eta^{\kappa}}^{1} \ln(1-t).t^{b-1}dt,$$

by using Lemma 4, we found that

$$I_2 = -(\kappa-1)\ln\left(\frac{1}{\eta}\right)\left[1-(1-\eta^{\kappa})^b\right] - b\frac{(\kappa-1)}{\kappa b} \left[\left(t^b-1\right)\ln(1-t) - \sum_{g=1}^{b}\frac{t^{b-g+1}}{(b-g+1)}\right]_{1-\eta^{\kappa}}^{1} \tag{13.2}$$

$$= -(\kappa-1)\ln\left(\frac{1}{\eta}\right)\left[1-(1-\eta^{\kappa})^b\right] - \frac{(\kappa-1)}{\kappa} \left[-\sum_{g=1}^{b}\frac{1}{(b-g+1)} - \left((1-\eta^{\kappa})^b-1\right)\ln(\eta^{\kappa}) + \sum_{g=1}^{b}\frac{(1-\eta^{\kappa})^{b-g+1}}{(b-g+1)}\right].$$

$$I_3 = -\eta^\kappa \kappa b\,(b-1)\int_0^1 \ln\left(1-(\eta x)^\kappa\right).x^{\kappa-1}\left[1-(\eta x)^\kappa\right]^{b-1}dx,$$

putting $1-(\eta x)^\kappa = t \Rightarrow -\kappa\eta^\kappa x^{\kappa-1}dx = dt$ and $x = \frac{1}{\eta}(1-t)^{1/\kappa}$, we get

$$I_3 = -b\,(b-1)\int_{1-\eta^\kappa}^{1} \ln(t)\,.t^{b-1}dt,$$

by using Lemma 2, we found that

$$\begin{aligned} I_3 &= -\,(b-1)\left[t^b\ln(t)-\frac{t^b}{b}\right]_{1-\eta^\kappa}^{1} \\ &= (b-1)\left[\frac{1}{b}+(1-\eta^\kappa)^b\ln(1-\eta^\kappa)-\frac{(1-\eta^\kappa)^b}{b}\right]. \end{aligned}$$

putting values of I_1, I_2 and I_3 in equation (13.1) we can obtain required result.

14 Order statistics

In this section, we develop the distribution of the q^{th} order statistic of the Kumaraswamy-Exponential distribution (KED). Let $X_{(1:n)} \leq \ldots \leq X_{(r:n)} \leq \ldots \leq X_{(n:n)}$ represents the ordered sample of n random variables for KED. The distribution of the q^{th} order statistics $X_{q:p}, q = 1, 2, \ldots, p$ can be defined as:

$$g_{q:p}(x) = C_{q:p}\left[G(x;\eta,b,\kappa)\right]^{q-1} g(x;\eta,b,\kappa)\left[1-G(x;\eta,b,\kappa)\right]^{p-q} \quad x>0 \quad (14.1)$$

where $G(.)$ and $g(.)$ are given by equation (5.1) and equation (5.2) respectively, thus

$$C_{q:p}(x) = \frac{p!}{(q)!\,(p-q)!}.$$

Thus, Using binomial expansion given in Lemma 1, we get

$$g_{q:p}(x) = C_{q:p}\sum_{k=0}^{\infty}(-1)^k\binom{p-q}{k}\left[G(x;\eta,b,\kappa)\right]^{q+k-1} g(x;\eta,b,\kappa),$$

$$\begin{aligned} g_{q:p}(x) = C_{q:p}\sum_{k=0}^{\infty}(-1)^k\binom{p-q}{k}\left[1-\left[1-(\eta x)^\kappa\right]^b\right]^{q+k-1} \times \\ \eta^\kappa\kappa b x^{\kappa-1}\left[1-(\eta x)^\kappa\right]^{b-1}, \end{aligned}$$

now using Lemma 1, we obtained

$$g_{q:p}(x) = C_{q:p} \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} (-1)^{k+u} \binom{p-q}{k} \binom{q+k-1}{u} \times \eta^{\kappa} \kappa b x^{\kappa-1} \left[1-(\eta x)^{\kappa}\right]^{b+bu-1}. \quad (14.2)$$

It's s^{th} moment can be calculated as:

$$E(x^s) = C_{q:p} \sum_{w=0}^{\infty} \sum_{u=0}^{\infty} (-1)^{w+u} \binom{p-q}{w} \binom{q+w-1}{u} \times \eta^{\kappa} \kappa b. \int_0^1 x^{\kappa+s-1} \left[1-(\eta x)^{\kappa}\right]^{b+bu-1} dx,$$

Again applying Lemma 1, we have

$$E(x^s) = C_{q:p} \sum_{w=0}^{\infty} \sum_{u=0}^{\infty} \sum_{r=0}^{\infty} (-1)^{w+u+r} \binom{p-q}{w} \binom{q+w-1}{u} \times \binom{b+bu-1}{r} \eta^{\kappa+\kappa r} \kappa b. \int_0^1 x^{\kappa+\kappa r+s-1} dx,$$

$$= C_{q:p} \sum_{w=0}^{\infty} \sum_{u=0}^{\infty} \sum_{r=0}^{\infty} (-1)^{w+u+r} \binom{p-q}{w} \binom{q+w-1}{u} \times \binom{b+bu-1}{r} \frac{\eta^{\kappa+\kappa r} \kappa b}{\kappa+\kappa r+s} \quad (14.3)$$

15 Maximum Likelihood Estimators

Let X is a random variable having the pdf of Kumaraswamy-Exponential distribution defined as:

$$g_X(x) = \eta^{\kappa} \kappa b x^{\kappa-1} \left[1-(\eta x)^{\kappa}\right]^{b-1}.$$

Then its log-likelihood function can be written as:

$$L(x; \eta, b, \kappa) = n \ln \kappa + n\kappa \ln \eta + n \ln b + (\kappa-1) \sum_{i=1}^{n} \ln(x_i) + (b-1) \sum_{i=1}^{n} \ln\left(1-(\eta x_i)^{\kappa}\right). \quad (15.1)$$

Thus the non-linear normal equations are given as follows:

$$\frac{\partial L(x;\eta,b,\kappa)}{\partial \eta}=\frac{n\kappa}{\eta}-(b-1)\sum_{i=1}^{n}\frac{\kappa\eta^{\kappa-1}x_i^{\kappa}}{\left(1-(\eta x_i)^{\kappa}\right)}. \tag{15.2}$$

$$\frac{\partial L(x;\eta,b,\kappa)}{\partial \kappa}=n\ln\eta+\frac{n}{\kappa}+\sum_{i=1}^{n}\ln(x_i)-(b-1)\sum_{i=1}^{n}\frac{\eta^{\kappa}x_i^{\kappa}\ln(\eta x_i)}{\left(1-(\eta x_i)^{\kappa}\right)}. \tag{15.3}$$

$$\frac{\partial L(x;\eta,b,\kappa)}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln\left(1-(\eta x_i)^{\kappa}\right). \tag{15.4}$$

To find the estimate of the unknown parameters by using the maximum likelihood method equate the equation (15.2) - equation (15.4) to zero and we can obtain solution.

16 Application To Real Life Data

Now we apply the proposed Kumaraswamy-Exponential distribution on two engineering data sets. We compare its flexibility with some pre-defined distributions. To analyse the present study, we obtain the results using R software. Following distributions are considered for discussion:
Exponentiated Exponential Distribution

$$f(j)=r\varpi\left(1-e^{-\varpi.j}\right)^{r-1}.e^{-\varpi.j}$$

Exponentiated Weibull distribution

$$f(w)=a\beta\sigma^{\beta}w^{\beta-1}.\exp\left(-(\sigma w)^{\beta}\right)\left(1-\exp\left(-(\sigma w)^{\beta}\right)\right)^{a-1}$$

Beta distribution

$$f(x)=\frac{\Gamma(\alpha+b)}{\Gamma(\alpha).\Gamma(b)}x^{\alpha-1}(1-x)^{b-1}$$

Burr-XII exponential distribution

$$f(l)=cp\varpi.\left(e^{\varpi l}-1\right)^{c-1}e^{\varpi l}\left(1+\left(e^{\varpi l}-1\right)^{c}\right)^{-p-1}$$

Gompertz distribution

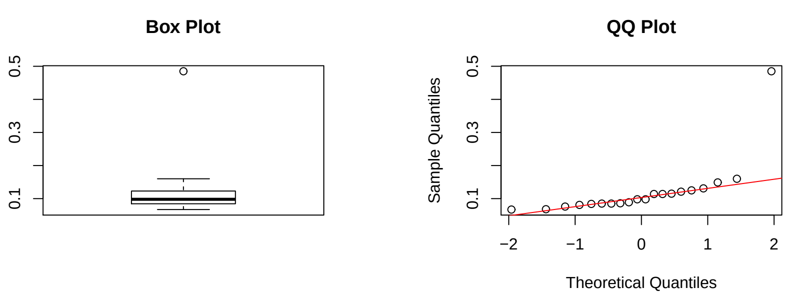
$$f(x)=\lambda\exp\left[\alpha x-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)\right]$$

At $\alpha=1\%$ LOS assume the hypothesis as,
H_0: The data fit the Kumaraswamy Exponential distribution
H_1: The data do not fit the Kumaraswamy Exponential distribution

Table 2: Table containing estimates and AIC values.

Distributions	Estimates	p-value	D	LL	AIC
Kumaraswamy exponential distribution	$\eta = 0.583923$ $b = 54.427161$ $\kappa = 1.596120$	0.1178	0.26605	-26.13379	58.26758
Beta distribution	$\alpha = 3.11202$ $b = 21.81905$	0.1521	0.25378	-27.8813	59.7626
Exponentiated Weibull distribution	$a = 9.388397$ $\beta = 0.975218$ $\sigma = 24.898336$	0.5196	0.18229	-32.83807	71.67614
Exponentiated exponential distribution	$\alpha = 13.82227$ $\theta = 27.75196$	0.6835	0.16024	-3297643	69.95286
Burr-XII exponential distribution	$c = 12.2957340$ $p = 0.1133163$ $\tau = 9.6519132$	0.924	0.12272	-37.99018	81.98036

Data Set1: The data from Murthy et al. [13] representing of the breakdown time of 20 mechanical parts. The records are:
0.085, 0.114, 0.068, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.121, 0.115, 0.125, 0.131, 0.081, 0.149, 0.076, 0.160, 0.084, 0.485, 0.067.



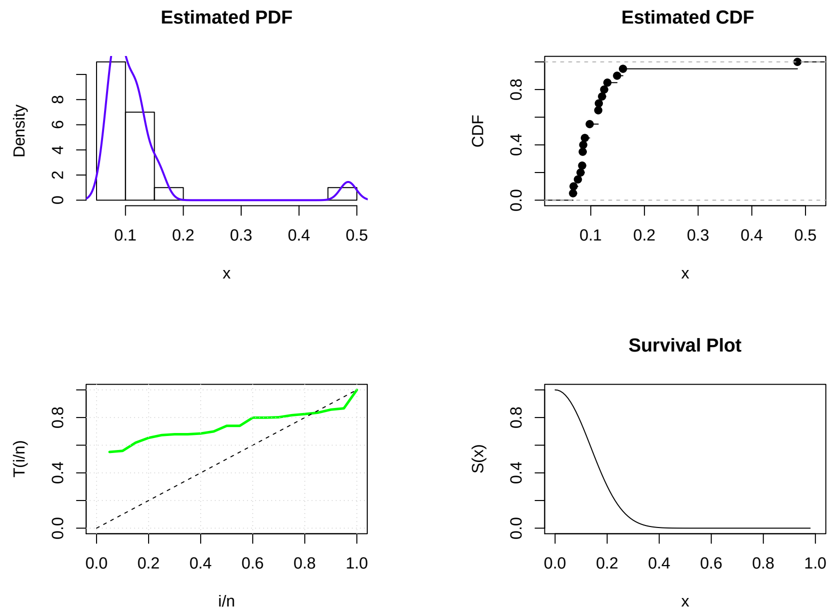


Figure 4: Plots of fitted Kumaraswamy Exponential distribution for breakdown time data.

Plots of fitted KE distribution for breakdown time data are displayed in Figure 4. Box plot revels that data is positively skewed. The TTT plot in Figure 4 for data 1 has concave than convex shape which suggests that hazard shape is upside-down bathtub (unimodal). The empirical visualization suggests that the KE distribution provides an improved fit for the breakdown time data.

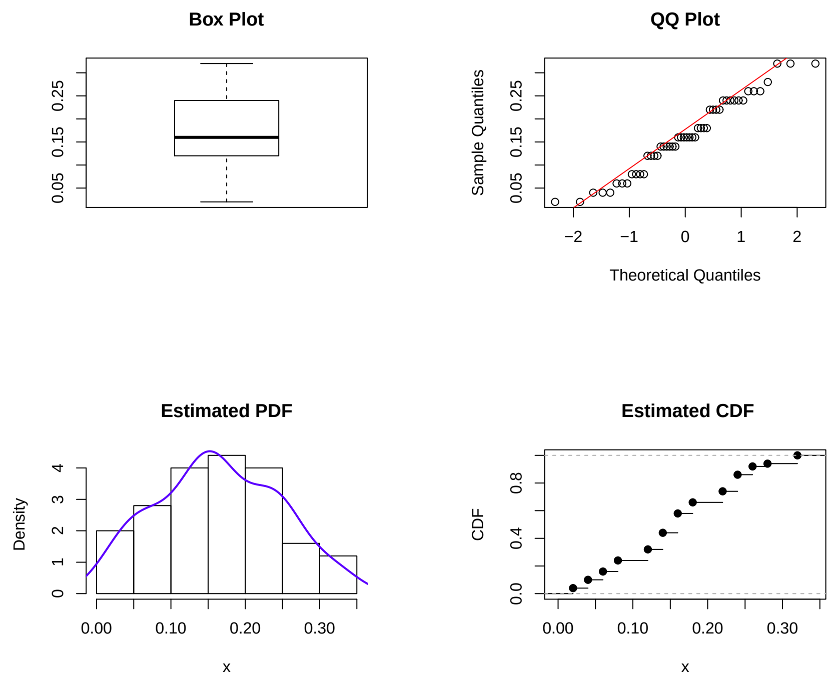
Data Set2: This data set referred from Dasgupta [7] for the 50 observations with opening of 12 mm and sheet thickness of 3.15 mm by the drilling machine. The records are:
0.32, 0.04, 0.02, 0.24, 0.08, 0.22, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.08, 0.32, 0.28, 0.14, 0.24, 0.26, 0.24, 0.22, 0.12, 0.18, 0.16, 0.06, 0.24, 0.14, 0.26, 0.16, 0.14, 0.16, 0.24, 0.16, 0.32, 0.18, 0.16, 0.12, 0.06, 0.02, 0.18, 0.22, 0.16, 0.06, 0.04, 0.14, 0.18, 0.16.

From Table 2 and Table 3, the Kumaraswamy-Exponential distribution has the AIC with lowest value and greater log-likelihood value for three parameter distribution, thus providing better fit than the Burr-XII exponential distribution, Exponentiated exponential distribution, Beta distribution, Exponentiated Weibull distribution and Gompertz distribution. So, since $p - value > \alpha$, we suppose that data follows the Kumaraswamy-Exponential distribution and can-

Table 3: Table containing estimates and AIC values.

Distributions	Estimates	p-value	D	LL	AIC
Kumaraswamy exponential distribution	$\eta = 1.011943$ $b = 33.421670$ $\kappa = 2.099506$	0.613	0.10726	-56.06933	118.13866
Burr-XII exponential distribution	$c = 1.991607$ $p = 17.947926$ $\tau = 1.161502$	0.5666	0.1112	-56.12203	118.24406
Gompertz distribution	$\lambda = 1.590379$ $\alpha = 10.274716$	0.6522	0.10397	-57.07532	118.15064
Exponentiated Weibull distribution	$a = 0.2970342$ $\beta = 4.9819583$ $\sigma = 3.8439353$	0.7083	0.09924	-57.53448	121.06896

not reject the null hypothesis.



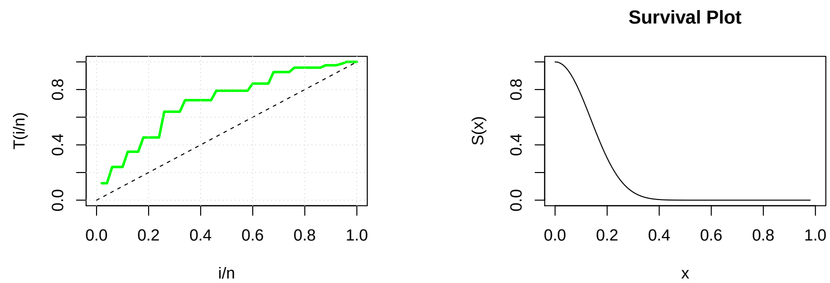


Figure 5: Plots of fitted Kumaraswamy Exponential distribution for drilling machine data.

Plots of fitted KE distribution for drilling machine data are displayed in Fig. 5. Box plot revels that data is normal. The TTT plot in Fig. 5 for data 2 has a concave shape which suggests hrf is increasing. The empirical visualization suggests that the KE distribution provides an improved fit for the drilling machine data.

Conclusion

In this manuscript, we establish a new family of Kumaraswamy-X probability distributions. Particularly, we developed the Kumaraswamy exponential distribution's cdf and pdf expressions. We have studied characteristic properties for the proposed distribution. From density graph, we conclude the proposed distribution has reverse-J shape or unimodal. The graphs for survival and hazard rate function for new distribution are also given. Further the mean, median and mode are discussed. The formulae for the l^{th} moment, probability weighted moments and moment generating function are also derived. We derived the Shannon entropy formula and the distribution of its q^{th} order statistics for proposed distribution. The MLE technique is used to estimate its parameters. We measure the accuracy of the estimators for a finite sample of size n using a Monte Carlo simulation analysis. The distribution is applied on two real datasets and its efficiency measured with some existing distributions. It is clearly visible from findings that the Kumaraswamy Exponential distribution exhibits a better fit for the considered data sets. This study contributes to the expanding body of knowledge on the Kumaraswamy-X probability distribution by offering insights into its theoretical foundations and practical applications in engineering problems. The simulation-based evaluation highlights its potential to enhance the accuracy and reliability of probabilistic modelling in various engineering disciplines, promoting its adoption as a valuable tool in engineering research and practice.

Furthermore, the simulation study has demonstrated that the Kumaraswamy-X distribution can provide a suitable alternative to other well-established distri-

butions, offering a fresh perspective and potentially improving the accuracy of predictive models. Its robust performance in various scenarios, as evidenced by our study, suggests that it should be considered an essential and important tool for the scientist and engineers.
Our future work will also focus on the determination of Bayesian estimators of the proposed distribution. One aspect that will also be the focus of our attention will be the determination of the performance of estimators using various estimation methods.

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