FRACTIONAL CALCULUS OPERATORS OF THE GENERALIZED HURWITZ-LERCH ZETA FUNCTION

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Abstract. In this paper, our aim is to establish certain generalized Marichev-Saigo-Maeda fractional integral and derivative formulas involving generalized p–extended Hurwitz-Lerch zeta function by using the Hadamard product (or the convolution) of two analytic functions. We then obtain their composition formulas by using fractional integral and derivative formulas and certain Integral transforms associated with Beta, Laplace and Whittaker transforms involving generalized p–extended Hurwitz-Lerch Zeta function.

1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The study of fractional integrals and fractional derivatives has a long history, and they have many real-world applications due to their properties of interpolation between operators of integer order and its applications in various fields of science and engineering, such as fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, and probability. This field has covered classical fractional operators such as Riemann-Liouville, Weyl, Caputo, Grnwald-Letnikov, etc. Also, especially in the last two decades, many new operators have appeared, often defined using integrals with special functions in the kernel, such as Atangana-Baleanu, Prabhakar, Marichev-Saigo-Maeda, and tempered, as well as their extended or multivariable forms. These have been intensively studied because they can also be useful in modelling and analysing real-world processes because of their different properties and behaviours, which are comparable to those of the classical operators[8, 22, 23, 24]. Special functions, such as the Hurwitz-Lerch Zeta function, Mittag-Leffler functions, hypergeometric functions, Foxs H-functions, Wright functions, Bessel and hyper-Bessel functions, etc., also have some more classical and fundamental connections with fractional calculus [13]. Some of them, such as the Mittag-Leffler function and its generalisations, appear naturally as solutions of fractional differential equations or fractional difference equations. Furthermore, many interesting relationships between different special functions may be discovered using the operators of fractional calculus. Because of their significance and potential for applications, fractional calculus operators (such as the Riemann-Liouville, Weyl, Liouville-Caputo, and other operators of

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fractional integrationandfractional derivative) have undergone extensive development and study (for more information, seein [12], [18] and [26]).

We begin by recalling a general pair of fractional integral operators known as Marichev-Saigo-Maeda that have the third-order Appell'stwo-variable hypergeometric function $F_3(.)$ as their kernel (see for more information, [15, 20, 21]), which is defined by:

Definition 1. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi \in \mathrm{C}$ and $x > 0$, then for $\Re(\xi) > 0$,*

$$\left(I_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} f\right)(x) = \frac{x^{-\varpi_1}}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1} t^{-\varpi_1'} \times F_3\left(\varpi_1, \varpi_1', \nu_1, \nu_1'; \xi; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, \mathrm{d}t. \tag{1.1}$$

and

$$\left(I_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} f\right)(x) = \frac{x^{-\varpi_1'}}{\Gamma(\xi)} \int_x^\infty (t-x)^{\xi-1} t^{-\varpi_1} \times F_3\left(\varpi_1, \varpi_1', \nu_1, \nu_1'; \xi; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, \mathrm{d}t. \tag{1.2}$$

Here, the Appell's hypergeometric function of two variables, [25], is denoted by $F_3(.)$.

Definition 2. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi \in \mathrm{C}$ and $x > 0$, then for $\Re(\xi) > 0$,*

$$\begin{aligned}\left(D_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} f\right)(x) &= \left(I_{0+}^{-\varpi_1',-\varpi_1,-\nu_1',-\nu_1,-\xi} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\varpi_1',-\varpi_1,-\nu_1'+n,-\nu_1,-\xi+n} f\right)(x) \quad (n = [\Re(\xi)]+1) \\ &= \frac{1}{\Gamma(n-\xi)} \left(\frac{d}{dx}\right)^n x^{\varpi_1'} \int_0^x (x-t)^{n-\xi-1} t^{\sigma} \\ &\quad \times F_3\left(-\varpi_1', -\varpi_1, n-\nu_1', -\nu_1; n-\xi; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, \mathrm{d}t.\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\left(D_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} f\right)(x) &= \left(I_{-}^{-\varpi_1',-\varpi_1,-\nu_1',-\nu_1,-\xi} f\right)(x) \\ &= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-\varpi_1',-\varpi_1,-\nu_1',-\nu_1'+n,-\xi+n} f\right)(x) \quad (n = [\Re(\xi)]+1) \\ &= \frac{1}{\Gamma(n-\xi)} \left(-\frac{d}{dx}\right)^n x^{\varpi_1'} \int_x^\infty (t-x)^{n-\xi-1} t^{\sigma'} \\ &\quad \times F_3\left(-\varpi_1', -\varpi_1, \nu_1', n-\nu_1; n-\xi; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, \mathrm{d}t.\end{aligned} \tag{1.4}$$

These operators include Riemann-Liouville, Erdélyi-Kober, and Saigo hypergeometric fractional calculus operators as special examples for various parameter choices (see for more information, [12], [18] and [26]). Early on, the p–extended Bessel function, p–modified Bessel function, p–extended Sturve function,and p–extended Mathieu series were used by a number of authors to create some intriguing generalized fractional formulas, (see, for details, [5, 10]).

The more generalized form of Hurwitz-Lerch zeta function has been considered very recently by Luo *et al.* [17] in the following form

$$\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}(z,s,a;p) := \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B^{(\theta,\theta')}(\vartheta+n,\, \nu-\vartheta\,;\, p)}{B(\vartheta,\, \nu-\vartheta)} \frac{z^n}{(n+a)^s} \tag{1.5}$$

$$\big(\Re(\theta)>0, \Re(\theta')>0, \Re(p)\geq 0;\ p,\lambda,\vartheta,s\in\mathbb{C};\ \nu,\, a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\, |z|<1\big).$$

where $\mathrm{B}^{(\theta,\theta')}(x,y;p)$ denotes the generalized Beta function, that is introduced by Chaudhry *et al.* [2]

$$\mathrm{B}^{(\theta,\theta')}(x,y;p) = \mathrm{B}^{(\theta,\theta')}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, {}_1F_1(\theta;\theta';-\frac{p}{t})\mathrm{d}t\,, \tag{1.6}$$

when $\min\{\Re(\theta)>0, \Re(\theta')>0, \Re(x), \Re(y)\}>0; \Re(p)\geq 0$. They also introduced p-extended of hypergeometric function as [3]:

$$F^{(\theta,\theta')}_p(a,b;c;z) = \sum_{n\geq 0} (a)_n \frac{\mathrm{B}^{(\theta,\theta')}(b+n,\, c-b\,;\, p)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!} \qquad p \geqq 0; |z|<1;\ \Re(c)>\Re(b)>0\,, \tag{1.7}$$

Additionally provided in [4]are related properties, multiple integral representations, differentiation formulæ, Mellin transforms, recurrence relations, and summations.

The definition of the Hadamard product (or convolution) of two analytical functions, such as in[5], is necessary for the present study. If the R_f and R_g be the radii of convergence of the two power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n\ (|z|<R_f)\ \text{ and }\ g(z) := \sum_{n=0}^{\infty} b_n z^n\ (|z|<R_g),$$

respectively. Then the Hadamard product is the new emerged series defined by

$$(f\, *\, g)(z) := \sum_{n=0}^{\infty} a_n\, b_n z^n = (g\, *\, f)(z)\ (|z|<R) \tag{1.8}$$

where

$$R = \lim_{n\to\infty}\left|\frac{a_n\, b_n}{a_{n+1}\, b_{n+1}}\right| = \left(\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\right)\cdot\left(\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|\right) = R_f\, .R_g,$$

so that, in general, we have $R \geqq R_f \cdot R_g$.

In the following study, we seek to broaden the compositions of the generalized fractional integral and differential operators (1.1), (1.2), (1.3) and (1.4) for the p–extended Hurwitz-Lerch zeta function (1.5) by using the Hadamard product (1.8) in terms of p–extended Hurwitz-Lerch zeta function and Wright hypergeometric function.

2. Fractional formulas of the p–extended Hurwitz-Lerch zeta function

The Wright hypergeometric function ${}_r\Psi_s(z)$ $(r,\, s\in\mathbb{N}_0)$ having numerator and denominator parameters r and s, respectively, defined for $\zeta_1,\ldots,\,\zeta_r\in\mathbb{C}$ and $\kappa_1,\ldots,\,\kappa_s\in\mathbb{C}\setminus\mathbb{Z}_0^-$ by (see, for example, [11, 14, 18, 25]):

$${}_r\Psi_s\left[\begin{array}{l}(\zeta_1,A_1),\cdots,(\zeta_r,A_r);\\ (\kappa_1,B_1),\cdots,(\kappa_s,B_s);\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\Gamma(\zeta_1+A_1n)\cdots\Gamma(\zeta_r+A_rn)}{\Gamma(\kappa_1+B_1n)\cdots\Gamma(\kappa_s+B_sn)}\frac{z^n}{n!} \tag{2.1}$$

$$\left(A_j \in \mathbb{R}^+ \ (j=1,\ldots,r);\ B_j \in \mathbb{R}^+ \ (j=1,\ldots,s);\ 1+\sum_{j=1}^{s} B_j - \sum_{j=1}^{r} A_j \geqq 0\right),$$

with

$$|z| < \nabla := \left(\prod_{j=1}^{r} A_j^{-A_j}\right) \ . \ \left(\prod_{j=1}^{s} B_j^{B_j}\right).$$

Also, if we take $A_j = B_k = 1$ $(j = 1,\ldots,r; \quad k = 1,\ldots,s)$ in (2.1), reduces to the generalized hypergeometric function ${}_rF_s$ $(r,\, s \in \mathbb{N}_0)$ (see, *e.g.*, [25]):

$$ {}_rF_s\left[\begin{array}{c} \zeta_1,\ldots,\zeta_r; \\ \kappa_1,\ldots,\kappa_s; \end{array} z\right] = \frac{\Gamma(\kappa_1)\cdots\Gamma(\kappa_s)}{\Gamma(\zeta_1)\cdots\Gamma(\zeta_r)}\, {}_r\Psi_s\left[\begin{array}{c} (\zeta_1,1),\cdots,(\zeta_r,1); \\ (\kappa_1,1),\cdots,(\kappa_s,1); \end{array} z\right]. \tag{2.2}$$

In the context of our investigation, the image formulas or power functions below, referencing [1],are significant.

Lemma 1. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho \in \mathrm{C}$ and $x > 0$. The relation that follows is then:*

(a) *If $\Re(\xi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\varpi_1 + \varpi_1' + \nu_1 - \xi), \Re(\varpi_1' - \nu_1')\}$, then*

$$\left(I_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} t^{\varrho-1}\right)(x) = \frac{\Gamma(\varrho)\Gamma(\varrho+\xi-\varpi_1-\varpi_1'-\nu_1)\Gamma(\varrho+\nu_1'-\varpi_1')}{\Gamma(\varrho+\nu_1')\Gamma(\varrho+\xi-\varpi_1-\varpi_1')\Gamma(\varrho+\xi-\varpi_1'-\nu_1)} x^{\varrho+\xi-\varpi_1-\varpi_1'-1} \tag{2.3}$$

(b) *If $\Re(\xi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(-\nu_1), \Re(\varpi_1 + \varpi_1' - \xi), \Re(\varpi_1 + \nu_1' - \xi)\}$, then*

$$\left(I_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} t^{\varrho-1}\right)(x) = \frac{\Gamma(1-\varrho-\nu_1)\Gamma(1-\varrho-\xi+\varpi_1+\varpi_1')\Gamma(1-\varrho-\xi+\varpi_1+\nu_1')}{\Gamma(1-\varrho)\Gamma(1-\varrho-\xi+\varpi_1+\varpi_1'+\nu_1')\Gamma(1-\varrho+\varpi_1-\nu_1)} x^{\varrho+\xi-\varpi_1-\varpi_1'-1}. \tag{2.4}$$

Lemma 2. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho \in \mathrm{C}$ and $x > 0$. The relation that follows is then:*

(a) *If $\Re(\xi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\xi - \varpi_1 - \varpi_1' + \nu_1'), \Re(\nu_1 - \varpi_1)\}$, then*

$$\left(D_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} t^{\varrho-1}\right)(x) = \frac{\Gamma(\varrho)\Gamma(\varrho-\xi+\varpi_1+\varpi_1'+\nu_1')\Gamma(\varrho-\nu_1+\varpi_1)}{\Gamma(\varrho-\nu_1)\Gamma(\varrho-\xi+\varpi_1+\varpi_1')\Gamma(\varrho-\xi+\varpi_1+\nu_1')} x^{\varrho-\xi+\varpi_1+\varpi_1'-1} \tag{2.5}$$

(b) *If $\Re(\xi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(\nu_1'), \Re(\xi - \varpi_1 - \varpi_1'), \Re(\xi - \varpi_1' - \nu_1)\}$, then*

$$\left(D_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} t^{\varrho-1}\right)(x) = \frac{\Gamma(1-\varrho-\nu_1')\Gamma(1-\varrho+\xi-\varpi_1-\varpi_1')\Gamma(1-\varrho+\xi-\varpi_1'-\nu_1)}{\Gamma(1-\varrho)\Gamma(1-\varrho+\xi-\varpi_1-\varpi_1'-\nu)\Gamma(1-\varrho-\varpi_1'-\nu_1')} x^{\varrho-\xi+\varpi_1+\varpi_1'-1}. \tag{2.6}$$

We begin the key outcomes exposition with showing the composition formulae for generalized fractional operators (1.1), (1.2), (1.3) and (1.4) involving the p–extended Hurwitz-Lerch zeta function by making use of the Hadamard product (1.8) in terms of p–extended Hurwitz-Lerch zeta function (1.5) and Fox-Wright function (2.1).

Theorem 1. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho, p, \lambda, \vartheta, s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu,\, a \in \mathbb{C}\setminus\mathbb{Z}_0^-$ such that $\Re(\xi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\varpi_1 + \varpi_1' + \nu_1 - \xi), \Re(\varpi_1' - \nu_1')\}$ with $|t| < 1$. Then for $\Re(p) \geq 0$, the fractional integration formula shown below is valid:*

$$\left(I_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}(t^{\gamma}, s, a; p)\right\}\right)(x)$$

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$$= x^{\varrho+\xi-\varpi_1-\varpi_1'-1}\,\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}(x^\gamma,s,a;p)$$
$$*\ {}_4\Psi_3\left[\begin{array}{c}(1,1),(\varrho,\gamma),(\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma),(\varrho+\nu_1'-\varpi_1',\gamma);\\(\varrho+\nu_1',\gamma),(\varrho+\xi-\varpi_1-\varpi_1',\gamma),(\varrho+\xi-\varpi_1'-\nu_1,\gamma);\end{array}\ x^\gamma\right].$$

Proof. Using the relation (2.3) and the definitions (1.5) and (1.1),we can shift the order of integration. Thus, we obtain for $x > 0$

$$\left(I^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}_{0,x}\left\{t^{\varrho-1}\,\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}(t^\gamma,s,a;p)\right\}\right)(x)$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k\,\mathrm{B}^{(\theta,\theta')}(\vartheta+k,\,\nu-\vartheta\,;\,p)}{(k+a)^s\,\mathrm{B}(\vartheta,\,\nu-\vartheta)\,k!}\ \left(I^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}_{0,x}\{t^{\varrho+\gamma k-1}\}\right)(x)$$
$$=x^{\varrho+\xi-\varpi_1-\varpi_1'-1}\ \sum_{k=0}^{\infty}\frac{(\lambda)_k\ \mathrm{B}(\vartheta+k,\,\nu-\vartheta\,;\,p)}{(k+a)^s\,\mathrm{B}(\vartheta,\,\nu-\vartheta)\,k!}$$
$$\times\,\frac{\Gamma(\varrho+\gamma k)\Gamma(\varrho+\xi-\varpi_1-\varpi_1'-\nu_1+\gamma k)\Gamma(\varrho+\nu_1'-\varpi_1'+\gamma k)}{\Gamma(\varrho+\nu_1'+\gamma k)\Gamma(\varrho+\xi-\varpi_1-\varpi_1'+\gamma k)\Gamma(\varrho+\xi-\varpi_1'-\nu_1+\gamma k)}x^{\gamma k}.\quad(2.7)$$

Subsequently,the necessary formula is obtained by utilizing the Hadamard product (1.8)in (2.7), which, in light of (1.5) and (2.1). □

Theorem 2. *Let* $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,\,a\in\mathbb{C}\backslash\mathbb{Z}_0^-$ *such that* $\Re(\xi)>0$ *and* $\Re(\varrho)<1+\min\{\Re(-\nu_1),\Re(\varpi_1+\varpi_1'-\xi),\Re(\varpi_1+\nu_1'-\xi)\}$ *with* $|1/t|<1$. *Then for* $\Re(p)\geq 0$, *the fractional integration formula shown below is valid:*

$$\left(I^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}_{x,\infty}\left\{t^{\varrho-1}\,\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}\left(\frac{1}{t^\gamma},s,a;p\right)\right\}\right)(x)$$
$$=x^{\varrho+\xi-\varpi_1-\varpi_1'-1}\,\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}\left(\frac{1}{x^\gamma},s,a;p\right)$$
$$*\ {}_4\Psi_3\left[\begin{array}{c}(1,1),(1-\varrho-\nu_1,\gamma),(1-\varrho-\xi+\varpi_1+\varpi_1',\gamma),(1-\varrho-\xi+\varpi_1+\nu_1',\gamma);\\(1-\varrho,\gamma),(1-\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),(1-\varrho+\varpi_1-\nu_1,\gamma);\end{array}\ \frac{1}{x^\gamma}\right].$$

Theorem 3. *Let* $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,\,a\in\mathbb{C}\backslash\mathbb{Z}_0^-$ *such that* $\Re(\xi)>0$ *and* $\Re(\varrho)>\max\{0,\Re(\xi-\varpi_1-\varpi_1'-\nu_1'),\Re(\nu_1-\varpi_1)\}$ *with* $|t|<1$. *Then for* $\Re(p)\geq 0$, *the fractional integration formula shown below is valid:*

$$\left(D^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}_{0,x}\left\{t^{\varrho-1}\,\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}(t^\gamma,s,a;p)\right\}\right)(x)$$
$$=x^{\varrho-\xi+\varpi_1+\varpi_1'-1}\,\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}(x^\gamma,s,a;p)$$
$$*\ {}_4\Psi_3\left[\begin{array}{c}(1,1),(\varrho,\gamma),(\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),(\varrho-\nu_1+\varpi_1,\gamma);\\(\varrho-\nu_1,\gamma),(\varrho-\xi+\varpi_1+\varpi_1',\gamma),(\varrho-\xi+\varpi_1+\nu_1',\gamma);\end{array}\ x^\gamma\right].$$

Proof. Using the relation (2.5), and the definitions (1.5), (1.3), we can shift the order of integration. Thus, we obtain for $x > 0$

$$\left(D^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}_{0,x}\left\{t^{\varrho-1}\,\Phi^{(\theta,\theta')}_{\lambda,\vartheta;\nu}(x^\gamma,s,a;p)\right\}\right)(x)$$

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$$= \sum_{k=0}^{\infty} \frac{(\lambda)_k \, \mathrm{B}^{(\theta,\theta')}(\vartheta+k,\, \nu-\vartheta\,;\, p)}{(k+a)^s \, \mathrm{B}(\vartheta,\, \nu-\vartheta)\, k!} \left(D_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} \{ t^{\varrho+\gamma k-1} \} \right)(x)$$

$$= x^{\varrho-\xi+\varpi_1+\varpi_1'-1} \sum_{k=0}^{\infty} \frac{(\lambda)_k \, \mathrm{B}(\vartheta+k,\, \nu-\vartheta\,;\, p)}{(k+a)^s \, \mathrm{B}(\vartheta,\, \nu-\vartheta)\, k!}$$

$$\times \frac{\Gamma(\varrho+\gamma k)\Gamma(\varrho-\xi+\varpi_1+\varpi_1'+\nu_1'+\gamma k)\Gamma(\varrho-\nu_1+\varpi_1+\gamma k)}{\Gamma(\varrho-\nu_1+\gamma k)\Gamma(\varrho-\xi+\varpi_1+\varpi_1'+\gamma k)\Gamma(\varrho-\xi+\varpi_1+\nu_1'+\gamma k)} x^{\gamma k}. \tag{2.8}$$

Subsequently,the necessary formula (1.3) is obtained by utilizing the Hadamard product (1.8) in (2.8), which in light of (1.5) and (2.1). □

Theorem 4. *Let* $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho, p, \lambda, \vartheta, s \in \mathbb{C}$ *with* $\gamma \in \mathbb{R}^+$ *and* $\nu,\, a \in \mathbb{C} \backslash \mathbb{Z}_0^-$ *such that* $Re(\xi) > 0$ *and* $Re(\varrho) < 1 + \min\{Re(\nu_1'), Re(\xi-\varpi_1-\varpi_1', Re(\xi-\varpi_1'-\nu_1)\}$ *with* $|1/t| < 1$. *Then for* $\Re(p) \geq 0$, *the fractional integration formula shown below is valid:*

$$\left(D_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} \left\{ t^{\varrho-1} \, \Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')} \left(\frac{1}{t^\gamma}, s, a; p \right) \right\} \right)(x)$$

$$= x^{\varrho-\xi+\varpi_1+\varpi_1'-1} \, \Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')} \left(\frac{1}{x^\gamma}, s, a; p \right)$$

$$* \; {}_4\Psi_3 \left[\begin{matrix} (1,1), (1-\varrho-\nu_1',\gamma), (1-\varrho+\xi-\varpi_1-\varpi_1',\gamma), (1-\varrho+\xi-\varpi_1'-\nu_1,\gamma); \\ (1-\varrho,\gamma), (1-\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma), (1-\varrho-\varpi_1'-\nu_1',\gamma); \end{matrix} \; \frac{1}{x^\gamma} \right].$$

3. Certain integral transforms

With the aid of the findings from the previous section, we will give severalextremely intriguing theorems relating to the Beta, Laplace, and Whitaker transformations in this section. First, we would like to define these transformations for this.

Definition 3. *As is customary, the Euler-Beta transform* [19]*of the function* $f(z)$ *is set forth by*

$$\mathcal{B}\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\, dz. \tag{3.1}$$

Definition 4. *As is customary, the Laplace transform (see,* e.g., [19]*) of the function* $f(z)$ *is set forth by*

$$L\{f(z); t\} = \int_0^\infty e^{-tz} f(z)\, dz. \quad (Re(t) > 0) \tag{3.2}$$

The following integral involving Whittaker function (see Mathai *et al.* [14, p. 79]):

$$\int_0^\infty t^{\rho-1} e^{-\frac{1}{2}at} \, W_{\kappa,\nu}(at)\, dt = a^{-\rho} \frac{\Gamma(\frac{1}{2} \pm \nu + \rho)}{\Gamma(1-\kappa+\rho)} \quad (Re(a) > 0,\ Re(\rho \pm \nu) > -\frac{1}{2}), \tag{3.3}$$

is significant to the subject at hand, where $W_{\kappa,\nu}$ is the Whittaker function [16, p. 334].

In this portion, the following captivating results in the form of theorems shall be demonstrated. These findings are put forward here without further justification because they follow directly from the definitions (3.1), (3.2), (3.3)and Theorems 1 to 4.

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Theorem 5. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho, p, \lambda, \vartheta, s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu,\, a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) > \max\{0, Re(\varpi_1 + \varpi_1' + \nu_1 - \xi), Re(\varpi_1' - \nu_1')\}$ with $|t| < 1$. Then for $\Re(p) \geq 0$, the Beta-transform formula shown below is valid:*

$$B\left\{\left(I_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}((tz)^\gamma, s, a; p)\right\}\right)(x) : l, m\right\}$$
$$= x^{\varrho+\xi-\varpi_1-\varpi_1'-1}\,\Gamma(m)\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}(x^\gamma, s, a; p)$$
$$*\ {}_5\Psi_4\left[\begin{matrix}(1,1),(l,\gamma),(\varrho,\gamma),(\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma),(\varrho+\nu_1'-\varpi_1',\gamma);\\(l+m,\gamma),(\varrho+\nu_1',\gamma),(\varrho+\xi-\varpi_1-\varpi_1',\gamma),(\varrho+\xi-\varpi_1'-\nu_1+\gamma,\gamma);\end{matrix}\ x^\gamma\right].$$

Theorem 6. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho, p, \lambda, \vartheta, s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu,\, a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) < 1 + \min\{Re(-\nu_1), Re(\varpi_1 + \varpi_1' - \xi), Re(\varpi_1 + \nu_1' - \xi)\}$ with $|1/t| < 1$. Then for $\Re(p) \geq 0$, the Beta-transform formula shown below is valid:*

$$B\left\{\left(I_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{z}{t}\right)^\gamma, s, a; p\right)\right\}\right)(x) : l, m\right\}$$
$$= x^{\varrho+\xi-\varpi_1-\varpi_1'-1}\,\Gamma(m)\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\frac{1}{x^\gamma}, s, a; p\right)$$
$$*\ {}_5\Psi_4\Big[\begin{matrix}(1,1),(l,\gamma),(1-\varrho-\nu_1,\gamma),\\(l+m,\gamma),(1-\varrho,\gamma),\end{matrix}$$
$$\begin{matrix}(1-\varrho-\xi+\varpi_1+\varpi_1',\gamma),(1-\varrho-\xi+\varpi_1+\nu_1',\gamma);\\(1-\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),(1-\varrho+\varpi_1-\nu_1,\gamma);\end{matrix}\ \frac{1}{x^\gamma}\Big].$$

Theorem 7. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho, p, \lambda, \vartheta, s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu,\, a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) > \max\{0, Re(\xi - \varpi_1 - \varpi_1' - \nu_1'), Re(\nu_1 - \varpi_1)\}$ with $|t| < 1$. Then for $\Re(p) \geq 0$, the Beta-transform formula shown below is valid:*

$$B\left\{\left(D_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}((tz)^\gamma), s, a; p\right\}\right)(x) : l, m\right\}$$
$$= x^{\varrho-\xi+\varpi_1+\varpi_1'-1}\,\Gamma(m)\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}(x^\gamma, s, a; p)$$
$$*\ {}_5\Psi_4\left[\begin{matrix}(1,1),(l,\gamma),(\varrho,\gamma),(\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),(\varrho-\nu_1+\varpi_1,\gamma);\\(l+m,\gamma),(\varrho-\nu_1,\gamma),(\varrho-\xi+\varpi_1+\varpi_1',\gamma),(\varrho-\xi+\varpi_1+\nu_1'+\gamma,\gamma);\end{matrix}\ x^\gamma\right].$$

Theorem 8. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho, p, \lambda, \vartheta, s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu,\, a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) < 1 + \min\{Re(\nu_1'), Re(\xi - \varpi_1 - \varpi_1', Re(\xi - \varpi_1' - \nu_1)\}$ with $|1/t| < 1$. Then for $\Re(p) \geq 0$, the Beta-transform formula shown below is valid:*

$$B\left\{\left(D_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{z}{t}\right)^\gamma, s, a; p\right)\right\}\right)(x) : l, m\right\}$$
$$= x^{\varrho-\xi+\varpi_1+\varpi_1'-1}\,\Gamma(m)\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\frac{1}{x^\gamma}, s, a; p\right)$$
$$*\ {}_5\Psi_4\Big[\begin{matrix}(1,1),(l,\gamma),(1-\varrho-\nu_1',\gamma),\\(1-\varrho,\gamma),(1-\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma),\end{matrix}$$

$$\left.\begin{array}{c}(1-\varrho+\xi-\varpi_1-\varpi_1',\gamma),(1-\varrho+\xi-\varpi_1'-\nu_1,\gamma);\\(l+m,\gamma),(1-\varrho-\varpi_1'-\nu_1',\gamma);\end{array}\frac{1}{x^\gamma}\right].$$

Theorem 9. *Let $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s\in\mathbb{C}$ with $\gamma\in\mathbb{R}^+$ and $\nu,\, a\in\mathbb{C}\backslash\mathbb{Z}_0^-$ such that $Re(\xi)>0$ and $Re(\varrho)>\max\{0,Re(\varpi_1+\varpi_1'+\nu_1-\xi),Re(\varpi_1'-\nu_1')\}$ with $|t|<1$. Then for $\Re(p)\geq 0$, the Laplace-transform formula shown below is valid:*

$$L\left\{z^{l-1}\left(I_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}((tz)^\gamma),s,a;p\right\}\right)(x)\right\}$$
$$=\frac{x^{\varrho+\xi-\varpi_1-\varpi_1'-1}}{s^l}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{x}{s}\right)^\gamma,s,a;p\right)$$
$$*\ {}_5\Psi_3\left[\begin{array}{c}(1,1),(l,\gamma),(\varrho,\gamma),(\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma),(\varrho+\nu_1'-\varpi_1',\gamma);\\(\varrho+\nu_1',\gamma),(\varrho+\xi-\varpi_1-\varpi_1',\gamma),(\varrho+\xi-\varpi_1'-\nu_1,\gamma);\end{array}\left(\frac{x}{s}\right)^\gamma\right].$$

Theorem 10. *Let $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s\in\mathbb{C}$ with $\gamma\in\mathbb{R}^+$ and $\nu,\, a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ such that $Re(\xi)>0$ and $Re(\varrho)<1+\min\{Re(-\nu_1),Re(\varpi_1+\varpi_1'-\xi),Re(\varpi_1+\nu_1'-\xi)\}$ with $|1/t|<1$. Then for $\Re(p)\geq 0$, the Laplace-transform formula shown below is valid:*

$$L\left\{z^{l-1}\left(I_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{z}{t}\right)^\gamma,s,a;p\right)\right\}\right)(x)\right\}$$
$$=\frac{x^{\varrho+\xi-\varpi_1-\varpi_1'-1}}{s^l}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{1}{xs}\right)^\gamma,s,a;p\right)$$
$$*\ {}_5\Psi_3\Big[\begin{array}{c}(1,1),(l,\gamma),(1-\varrho-\nu_1,\gamma),(1-\varrho-\xi+\varpi_1+\varpi_1',\gamma),\\(1-\varrho,\gamma),(1-\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),\end{array}$$
$$\left.\begin{array}{c}(1-\varrho-\xi+\varpi_1+\nu_1',\gamma);\\(1-\varrho+\varpi_1-\nu_1,\gamma);\end{array}\left(\frac{1}{xs}\right)^\gamma\right].$$

Theorem 11. *Let $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s\in\mathbb{C}$ with $\gamma\in\mathbb{R}^+$ and $\nu,\, a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ such that $Re(\xi)>0$ and $Re(\varrho)>\max\{0,Re(\xi-\varpi_1-\varpi_1'-\nu_1'),Re(\nu_1-\varpi_1)\}$ with $|t|<1$. Then for $\Re(p)\geq 0$, the Laplace-transform formula shown below is valid:*

$$L\left\{z^{l-1}\left(D_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}((tz)^\gamma,s,a;p)\right\}\right)(x):\right\}$$
$$=\frac{x^{\varrho-\xi+\varpi_1+\varpi_1'-1}}{s^l}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{x}{s}\right)^\gamma,s,a;p\right)$$
$$*\ {}_5\Psi_3\left[\begin{array}{c}(1,1),(l,\gamma),(\varrho,\gamma),(\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),(\varrho-\nu_1+\varpi_1,\gamma);\\(\varrho-\nu_1,\gamma),(\varrho-\xi+\varpi_1+\varpi_1',\gamma),(\varrho-\xi+\varpi_1+\nu_1',\gamma);\end{array}\left(\frac{x}{s}\right)^\gamma\right].$$

Theorem 12. *Let $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s\in\mathbb{C}$ with $\gamma\in\mathbb{R}^+$ and $\nu,\, a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ such that $Re(\xi)>0$ and $Re(\varrho)<1+\min\{Re(\nu_1'),Re(\xi-\varpi_1-\varpi_1',Re(\xi-\varpi_1'-\nu_1)\}$ with $|1/t|<1$. Then for $\Re(p)\geq 0$, the Laplace-transform formula shown below is valid:*

$$L\left\{z^{l-1}\left(D_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{z}{t}\right)^\gamma,s,a;p\right)\right\}\right)(x)\right\}$$
$$=\frac{x^{\varrho-\xi+\varpi_1+\varpi_1'-1}}{s^l}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{1}{xs}\right)^\gamma,s,a;p\right)$$

$$* \ {}_5\Psi_3\Big[\begin{matrix} (1,1),(l,\gamma),(1-\varrho-\nu_1',\gamma), \\ (1-\varrho,\gamma),(1-\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma), \end{matrix}$$

$$\begin{matrix} (1-\varrho+\xi-\varpi_1-\varpi_1',\gamma),(1-\varrho+\xi-\varpi_1'-\nu_1,\gamma); \\ (1-\varrho-\varpi_1'-\nu_1',\gamma); \end{matrix} \ \left(\frac{1}{xs}\right)^{\gamma}\Big].$$

Theorem 13. *Let $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu, a \in \mathbb{C}\setminus\mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) > \max\{0, Re(\varpi_1+\varpi_1'+\nu_1-\xi), Re(\varpi_1'-\nu_1')\}$ with $|t|<1$. Then for $\Re(p) \geq 0$, the Laplace-transform formula shown below is valid:*

$$\int_0^\infty z^{l-1}e^{-\frac{1}{2}\delta z}\, W_{\tau,\varsigma}(\delta z)\left\{\left(I_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}((wtz)^{\gamma}),s,a;p\right\}\right)(x)\right\}\mathrm{d}z$$

$$= \frac{x^{\varrho+\xi-\varpi_1-\varpi_1'-1}}{\delta^l}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{wx}{\delta}\right)^{\gamma},s,a;p\right)$$

$$* \ {}_6\Psi_4\Big[\begin{matrix} (1,1),(\tfrac{1}{2}+\zeta+l,\gamma),(\tfrac{1}{2}-\zeta+l,\gamma), \\ (\tfrac{1}{2}-\tau+l,\gamma),(\varrho+\nu_1',\gamma), \end{matrix}$$

$$\begin{matrix} (\varrho,\gamma),(\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma),(\varrho+\nu_1'-\varpi_1',\gamma); \\ (\varrho+\xi-\varpi_1-\varpi_1',\gamma),(\varrho+\xi-\varpi_1'-\nu_1,\gamma); \end{matrix} \ \left(\frac{wx}{\delta}\right)^{\gamma}\Big].$$

Theorem 14. *Let $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu, a \in \mathbb{C}\setminus\mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) < 1+\min\{Re(-\nu_1), Re(\varpi_1+\varpi_1'-\xi), Re(\varpi_1+\nu_1'-\xi)\}$ with $|1/t|<1$. Then for $\Re(p) \geq 0$, the integral formula shown below is valid:*

$$\int_0^\infty z^{l-1}e^{-\frac{1}{2}\delta z}\, W_{\tau,\varsigma}(\delta z)\left\{\left(I_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{wz}{t}\right)^{\gamma},s,a;p\right)\right\}\right)(x)\right\}\mathrm{d}z$$

$$= \frac{x^{\varrho+\xi-\varpi_1-\varpi_1'-1}}{\delta^l}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}\left(\left(\frac{w}{x\delta}\right)^{\gamma},s,a;p\right)$$

$$* \ {}_6\Psi_4\Big[\begin{matrix} (1,1),(\tfrac{1}{2}+\zeta+l,\gamma),(\tfrac{1}{2}-\zeta+l,\gamma),(1-\varrho-\nu_1,\gamma), \\ (\tfrac{1}{2}-\tau+l,\gamma),(1-\varrho,\gamma), \end{matrix}$$

$$\begin{matrix} (1-\varrho-\xi+\varpi_1+\varpi_1',\gamma),(1-\varrho-\xi+\varpi_1+\nu_1',\gamma); \\ (1-\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),(1-\varrho+\varpi_1-\nu_1,\gamma); \end{matrix} \ \left(\frac{w}{x\delta}\right)^{\gamma}\Big].$$

Theorem 15. *Let $\varpi_1,\varpi_1',\nu_1,\nu_1',\xi,\varrho,p,\lambda,\vartheta,s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu, a \in \mathbb{C}\setminus\mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) > \max\{0, Re(\xi-\varpi_1-\varpi_1'-\nu_1'), Re(\nu_1-\varpi_1)\}$ with $|t|<1$. Then for $\Re(p) \geq 0$, the integral formula shown below is valid:*

$$\int_0^\infty z^{l-1}e^{-\frac{1}{2}\delta z}\, W_{\tau,\varsigma}(\delta z)\left\{\left(D_{0,x}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi}\left\{t^{\varrho-1}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}((wtz)^{\gamma}),s,a;p\right\}\right)(x)\right\}$$

$$= \frac{x^{\varrho-\xi+\varpi_1+\varpi_1'-1}}{\delta^l}\,\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}(\left(\frac{wx}{\delta}\right)^{\gamma},s,a;p)$$

$$* \ {}_6\Psi_4\Big[\begin{matrix} (1,1),(\tfrac{1}{2}+\zeta+l,\gamma),(\tfrac{1}{2}-\zeta+l,\gamma) \\ (\tfrac{1}{2}-\tau+l,\gamma),(\varrho-\nu_1,\gamma), \end{matrix}$$

$$\begin{matrix} (\varrho+\gamma,\gamma),(\varrho-\xi+\varpi_1+\varpi_1'+\nu_1',\gamma),(\varrho-\nu_1+\varpi_1,\gamma); \\ (\varrho-\xi+\varpi_1+\varpi_1',\gamma),(\varrho-\xi+\varpi_1+\nu_1',\gamma) \end{matrix} \ \left(\frac{wx}{\delta}\right)^{\gamma}\Big].$$

Theorem 16. *Let $\varpi_1, \varpi_1', \nu_1, \nu_1', \xi, \varrho, p, \lambda, \vartheta, s \in \mathbb{C}$ with $\gamma \in \mathbb{R}^+$ and $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ such that $Re(\xi) > 0$ and $Re(\varrho) < 1 + \min\{Re(\nu_1'), Re(\xi - \varpi_1 - \varpi_1', Re(\xi - \varpi_1' - \nu_1)\}$ with $|1/t| < 1$. Then for $\Re(p) \geq 0$, the integral formula shown below is valid:*

$$\int_0^\infty z^{l-1} e^{-\frac{1}{2}\delta z}\, W_{\tau,\varsigma}(\delta z) \left\{ \left(D_{x,\infty}^{\varpi_1,\varpi_1',\nu_1,\nu_1',\xi} \left\{ t^{\varrho-1}\, \Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')} \left(\left(\frac{wz}{t} \right)^\gamma, s, a; p \right) \right\} \right)(x) \right\}$$

$$= \frac{x^{\varrho-\xi+\varpi_1+\varpi_1'-1}}{\delta^l}\, \Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')} \left(\left(\frac{w}{x\delta} \right)^\gamma, s, a; p \right)$$

$$* \; {}_6\Psi_4 \Bigg[\begin{array}{c} (1,1), (\frac{1}{2}+\zeta+l,\gamma), (\frac{1}{2}-\zeta+l,\gamma), (1-\varrho-\nu_1',\gamma), \\ (\frac{1}{2}-\tau+l,\gamma), (1-\varrho,\gamma), \end{array}$$

$$\begin{array}{c} (1-\varrho+\xi-\varpi_1-\varpi_1',\gamma), (1-\varrho+\xi-\varpi_1'-\nu_1,\gamma)p; \\ (1-\varrho+\xi-\varpi_1-\varpi_1'-\nu_1,\gamma), (1-\varrho-\varpi_1'-\nu_1',\gamma); \end{array} \left(\frac{w}{x\delta} \right)^\gamma \Bigg].$$

4. Concluding Remarks and Observations

In the current study, we have found the composition formulas for the generalized Marichev-Saigo-Maeda fractional integrals and differential operators (1.1), (1.2), (1.3) and (1.4) involving the p–extended Hurwitz-Lerch zeta function $\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}(z,s,a;p)$ in terms of the Hadamard product (1.8) of the p–extended Hurwitz-Lerch zeta function $\Phi_{\lambda,\vartheta;\nu}^{(\theta,\theta')}(z,s,a;p)(z,s,a)$ and the Fox–Wright function ${}_r\Psi_s(z)$ employing the Hadamard product (or convolution) of two analytic functions. Additionally, we have derived some image formulas in connection with integral transformations such asthe Euler-Beta, Laplace, and Whittaker transforms. Then, as special cases, we can construct as corollaries the specific image formulas for the Erdélyi-Kober(E-K), Riemann-Liouville(R-L), and Saigo's fractional integral and differential operators. We have left this as an exercise for the readers. The results obtained in this paper are assumed to be have applications in various field of Physical and Engineering Sciences. Another application in real world problems can be developed in recents papers[22, 23]

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