A novel computational analysis of diabetes model with Caputo-Katugampola memory

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Abstract

In this study, we have investigated a diabetes model and its complication using the Katugampola fractional derivative in Caputo sense. We studied a deterministic mathematical model that uses non integer derivatives to precisely depict the dynamics of diabetes mellitus. We have employed q-homotopy analysis generalized transform method (q-HAGTM) to find analytical approximate solution for presented model. Furthermore, the fixed-point theorem is employed to present the existence as well as uniqueness analysis of obtained solution for the discussed model. The obtained results are further complemented by conducting numerical simulations, providing graphical demonstrations that support and illustrate the findings. This approach enables us to understand and develop efficient ways to cure these diseases.

Keywords: Diabetes model; Caputo-Katugampola fractional derivative; Generalized Laplace transform

1 Introduction

Diabetes is a rapidly growing global concern, with its occurrence and prevalence on the rise worldwide. This chronic condition imposes a substantial burden not only on individuals but also on society as a whole due to the numerous complications associated with the disease. It is a chronic metabolic disorder characterized by high levels of glucose in the blood. It occurs when the body either does not produce enough insulin or cannot effectively use the insulin it produces. There are mainly 2 types of diabetes, Type 1 diabetes, often referred to as insulin-dependent diabetes mellitus (IDDM), typically manifests in individuals below the age of 40, although it can occur at any age. It accounts for approximately 10 to 15 percent of the diabetic population. Type 1 diabetes is characterized by an autoimmune response in which the body's immune system

mistakenly attacks and destroys the insulin-producing cells in the pancreas. As a result, people with Type 1 diabetes require lifelong insulin treatment to regulate their blood sugar levels. Type 2 diabetes, previously known as non-insulin-dependent diabetes mellitus (NIDDM), is the most common form of diabetes, accounting for approximately 85 to 90 percent of all cases. In this the body either does not produce enough insulin or becomes resistant to its effects, leading to elevated blood sugar levels. Prolonged hyperglycemia (having blood glucose concentration higher than 70-180 mg/dl), can result in long-term complications, such as neuropathy, retinopathy, and cardiovascular and heart diseases and pose significant risks to individuals and impact their quality of life and overall health [30, 24, 1]. In 2003, around 194 million people had diabetes, making up over 3% (5.1% for ages 20 to 79) of the global population. The prevalence of diabetes is on the rise, and it is projected to reach approximately 333 million people (6.3%) by 2025 [5]. Researchers have produced a multitude of studies aimed at developing mathematical models for predicting the proliferation of diabetes and issues coherent with it, such as Pandit et al. [20], Makroglou et al. [17], Patil et al. [21] and others. Numerous scientists and mathematicians have empirically validated the usefulness of fractional extensions of integer-order mathematical models in systematically representing natural phenomena, exemplified by methodologies such as the Caputo approach, which adeptly captures and represents the inherent characteristics of real-world processes [28, 10]. Singh et al. [26] presented analysis of fractional diabetes model with exponential law using Caputo–Fabrizio fractional derivative. Miller and Ross [18], Podlubny [22] authored their work namely "An introduction to the fractional calculus and fractional differential equations" and "fractional differential equation" respectively in which they focused on derivatives and integrals of fractional order and highlighted the evolution of fractional calculus and its application in modelling physical problems. In this paper we have used Katugampola fractional derivative in Caputo sense to formulate fractional diabetes model including its inherent complication factors such as its occurrence, spreading, healing and natural mortality rate. The Caputo derivative [6] and the Riemann-Liouville derivative [9] are two often used fractional derivatives. A novel fractional order derivative, introduced by Katugampola [14, 13], provides a generalized fractional derivative encompassing both the Riemann-Liouville and Hadamard fractional integrals and derivatives. Lately, Katugampola fractional derivative in Caputo sense has been used due to its ability to capture both local differentiation and integration properties and provides framework for handling systems with fractional exponents, which makes it well-suited for modeling and analyzing systems with fractional dynamics. Almeida et al. in [4] introduced a fractional operator of new kind, namely Katugampola derivative in Caputo type with special cases being the Caputo and the Caputo–Hadamard fractional derivatives. In this article, the diabetes model is developed with the main goal of examining the diabetes model using a novel non integer order derivative and to studying the specifications regarding the existence and uniqueness of the diabetes model's solution. In this work, we study the fractional diabetes model by applying the q-HAGTM. The q-HAGTM [25], which is related to fractional order diabetes modeling is used to find approximate analytical solution. The generalized Laplace transform (GLT) [12] and the q-homotopy analysis method (q-HAM) [7, 8] are combined in the utilized methodology to produce an effective result. An advancement of HAM is called q-HAM, which is a more gener-

alized approach than HAM [15, 16], has been established the search for more elegant methods to enlarge the convergence region. The q-homotopy analysis method has been exploited by the authors and used to interpret non-linear arbitrary PDEs [3, 2, 23]. By Singh et al [27], the q-HATM methodology was presented. In this, unlike prior approaches no discretization, linearization, or perturbation is required. The q-HAGTM employs two convergence parameters denoted as n and $\hbar$ which allows greater flexibility in modifying and regulating the convergence rate and region of convergence for the series solutions. The studied method is novel in sense that it yields a simple optimal solution, a significant region of convergence, and a non-local effect in the attained solution. Hence, In presented work we have utilized the novel Katumgampola based fractional derivative model to simulate spread of diabetes in human populus while also presenting the effect of external factors on it's occurrence, spreading, healing and natural mortality rate. This article is presented in following order- we introduce the definitions of Caputo derivative, Caputo-Katugampola fractional derivative, generalized Laplace transform,Katugampola integral opreator in Section 2. In Section 3, fundamental procedure of the implemented analytical technique that is q-HAGTM is given. Description of discussed model is presented in mathematical form in Section 4. In Section-5, we have obtained analytical solution to fractional order diabetes model and its complication by using q-HAGTM. In Section 6, the fixed-point theory is used to investigate the existence and uniqueness of the system's solutions. Numerical simulation with graphical representation is shown in Section 7. Finally, the conclusion of this research article is presented in Section 8.

2 Mathematical preliminaries

Definition 1: The fractional order (ρ) Caputo derivative [6] is described as follows,

$$ {}_{a}^{C}D_{\tau}^{\rho}\xi(\tau) = \frac{1}{\Gamma(l-\rho)}\int_{a}^{\tau}\frac{\xi^{(l)}(v)\,dv}{(\tau-v)^{\rho+1-l}}, \quad (l-1<\rho\leq l),\ l\in\mathbb{N}. \tag{1} $$

Definition 2: The Katugampola fractional derivative (KFD) [14, 13] in Caputo kind of order $0<\rho\leq 1$ of the function $\xi(\tau)$ can be given as

$$ {}_{a}^{kc}D_{\tau}^{\rho,\eta}\xi(\tau) = \frac{1}{\Gamma(1-\rho)}\int_{a}^{\tau}\left(\frac{\tau^{\eta}-v^{\eta}}{\eta}\right)^{-\rho}\zeta\frac{\xi(v)}{v^{1-\eta}}dv, \tag{2} $$

where the differential operator ζ is defined by $\zeta = \tau^{1-\eta}\frac{d}{d\tau}$. If we consider $\eta = 1$, the fractional derivative (FD) in Eq. (2) becomes the Caputo fractional derivative with order ρ . If η approaches 0, then the FD of Eq. (2) results into Caputo-Hadamard FD of order ρ.

Definition 3: Let $\xi, m : [a,\infty) \to R$ be real valued function s.t. $m(\tau)$ is continuous and $m'(\tau) > 0$ on $[a,\infty)$. Now, if the GLT [12] of $\xi(\tau)$ exists, then

$$ L_m\{\xi(\tau)\}(s) = \int_{a}^{\infty} e^{-s(m(\tau)-m(a))}\xi(\tau)\,m'(\tau)\,d\tau, \tag{3} $$

$'s'$ being the GLT parameter.

Note that if we set $a = 0$ and $m(\tau)=\tau$ in Eq. (3), then GLT transforms into the classical Laplace transform (LT) but if we set $m(\tau) = \frac{\tau^\eta}{\eta}$ and $a = 0$, then GLT reduces to η−LT [11]. This inclusive study is represented as the GLT with $m(\tau) = \frac{\tau^\eta}{\eta}$ with $a = 0$ by $\frac{\tau^\eta}{\eta}$−LT. Henceforth, the $\frac{\tau^\eta}{\eta}$- LT is described as

$$L_{\frac{\tau^\eta}{\eta}}\{\xi(\tau)\}(s) = \int_a^\infty e^{-s\frac{\tau^\eta}{\eta}}\xi(\tau)\frac{d\tau}{\tau^{1-\eta}}. \tag{4}$$

The $\frac{\tau^\eta}{\eta}$−LT of the KFD in Caputo kind [11, 29] can be stated as follows

$$L_{\frac{\tau^\eta}{\eta}}\left\{\left(\frac{\tau^\eta}{\eta}\right)^\rho\right\}(s) = \frac{\Gamma(1+\rho)}{s^{1+\rho}},$$

$$L_{\frac{\tau^\eta}{\eta}}\left\{\left(kc D_\tau^{\rho,\eta}\xi(\tau)\right)\right\}(s) = s^\rho L_{\frac{\tau^\eta}{\eta}}\xi(\tau)(s) - s^{\rho-1}\xi(0). \tag{5}$$

Definition 4: The Katugampola integral operator [14] of fractional order ρ is defined as

$$({}_a+I^{\rho,\eta}\xi)(\tau) = \frac{\eta^{1-\rho}}{\Gamma(\rho)}\int_a^\tau \frac{v^{\eta-1}}{(\tau^\eta - v^\eta)^{1-\rho}}\xi(v)dv. \tag{6}$$

3 Fundamental plan of *q*-homotopy analysis generalized transform method (*q*-HAGTM)

Principal scheme of proposed method is discussed by studying a nonlinear differential equation associated to the Katugampola derivative. It can be stated as follows

$$kc D_\tau^{\rho,\eta}\xi(\tau) + M\xi(\tau) + Q\xi(\tau) = f(\tau), \qquad l-1 < \rho \le l, \tag{7}$$

here $\xi(\tau)$ is a function in time τ and $kc D_\tau^{\rho,\eta}$ represents the Caputo-Katugamopla derivative of order ρ, R denotes the linear bounded operator, Q denotes common nonlinear differential operator,which is Lipschitz continuous and $f(\tau)$ stands for source term. Apply GLT operator on Eq. (7), we have

$$L_{\frac{\tau^\eta}{\eta}}\left[kc D_\tau^{\rho,\eta}\xi(t)\right] + L_{\frac{\tau^\eta}{\eta}}\left[M\xi(\tau) + Q\xi(\tau)\right] = L_{\frac{\tau^\eta}{\eta}}[f(\tau)]. \tag{8}$$

On using the GLT of Caputo-Katugampola fractional derivative, we have

$$s^\rho L_{\frac{\tau^\eta}{\eta}}\left[\xi(\tau)\right](s) - s^{\rho-1}\xi(0) + L_{\frac{\tau^\eta}{\eta}}\left[M\xi(\tau) + Q\xi(\tau)\right] - L_{\frac{\tau^\eta}{\eta}}[f(\tau)] = 0. \tag{9}$$

On refining the Eq. (9), we get

$$L_{\frac{\tau^\eta}{\eta}}\left[\xi(\tau)\right](s) - \frac{1}{s}\xi(0) + \frac{1}{s^\rho}\left[L_{\frac{\tau^\eta}{\eta}}\left[M\xi(\tau) + Q\xi(\tau)\right] - L_{\frac{\tau^\eta}{\eta}}[f(\tau)\right] = 0. \tag{10}$$

Now, we present a nonlinear operator which is given as follows

$$\begin{aligned} Q[\phi(\tau;q)] = L_{\frac{\tau^\eta}{\eta}}[\phi(\tau;q)] - \frac{1}{s}\phi(0;q)(0^+) \\ + \frac{1}{s^\rho}\left[L_{\frac{\tau^\eta}{\eta}}\left[M\phi(\tau;q) + Q\phi(\tau;q)\right] - L_{\frac{\tau^\eta}{\eta}}[f(\tau,q)\right]. \end{aligned} \tag{11}$$

Here $q \in \left[0, \frac{1}{n}\right]$ and $\phi(\tau; q)$ denotes a real valued function. Further, in subsequent approach, we set a homotopy

$$(1 - nq) L_{\frac{\tau^\eta}{\eta}} \left[\phi(\tau; q) - \xi_0(\tau)\right] = \hbar q Q[\phi(\tau; q)], \tag{12}$$

where $L_{\frac{\tau^\eta}{\eta}}$ indicates that the GLT operator, the auxiliary parameter $\hbar \neq 0$ and $\phi(\tau; q)$ is an unknown function, $\xi_0(\tau)$ is an initial approximation of $\xi(\tau)$. Furthermore, by substituting the embedding parameter values of $q = 0$ and $q = 1/n$, it gives

$$\phi(\tau; 0) = \xi_0(\tau) \qquad \phi\left(\tau; \frac{1}{n}\right) = \xi(\tau). \tag{13}$$

Therefore, when q progresses from 0 to $\frac{1}{n}$, $\phi(\tau; q)$ transforms from $\xi_0(\tau)$ to the solution $\xi(\tau)$. Expanding $\phi(\tau; q)$ into a series form by employing Taylor's theorem about parameter q, we get

$$\phi(\tau; q) = \xi_0(\tau) + \sum_{k=1}^{\infty} \xi_k(\tau) q^k, \tag{14}$$

where,

$$\xi_k(\tau) = \frac{1}{k!} \frac{\partial^k}{\partial q^k} \{\phi(\tau; q)\}\Big|_{q=0}. \tag{15}$$

If the initial condition $\xi_0(\tau)$, asymptotic parameter n and convergence control parameter $\hbar$ are expressed appropriately, then Eq. (15) converges at $q = \frac{1}{n}$, then, we obtain the subsequent equation

$$\xi(\tau) = \xi_0(\tau) + \sum_{k=1}^{\infty} n^{-k} \xi_k(\tau). \tag{16}$$

The solution given by Eq. (16) is a solution of discussed Eq. (7). Using Eq. (16) and Eq. (12). Solution of governing equation can be attained as

$$\xi_k(\tau) = \{\xi_0(\tau), \xi_1(\tau), \ldots \xi_k(\tau)\}. \tag{17}$$

On differentiating Eq. (12) k-times w.r.t q and then multiplying by $1/k!$ and substituting $q = 0$, we get

$$L_{\frac{\tau^\eta}{\eta}} \left[\xi_k(\tau) - \alpha_k \xi_{k-1}(\tau)\right] = \hbar[\Re_k(\xi_{k-1})]. \tag{18}$$

Employing the inverse GLT operator on Eq. (18), we attain the subsequent result

$$\xi_k(\tau) = \alpha_k \xi_{k-1}(\tau) + \hbar L^{-1}_{\frac{\tau^\eta}{\eta}}[\Re_k(\xi_{k-1})]. \tag{19}$$

Where α_k is defined as

$$\alpha_k = \begin{cases} 0, & \text{if } k \leq 1 \\ n, & k > 1 \end{cases} \tag{20}$$

and we represent the value of $\Re_k(\xi_{k-1})$ as follows

$$\Re_k(\xi_{k-1}) = L_{\frac{\tau^\eta}{\eta}} \left[\xi_{k-1}(\tau)\right] - \left(1 - \frac{\alpha_k}{n}\right)\left[s^{-1}\xi(0) + s^{-\rho} L_{\frac{\tau^\eta}{\eta}} f(\tau)\right] + s^{-\rho} L_{\frac{\tau^\eta}{\eta}} \left[R\xi_{k-1} + A_{k-1}\right]. \tag{21}$$

In Eq. (21) A_k exhibit the homotopy polynomial [19] and given as

$$A_k = \frac{1}{\Gamma k}\left[\frac{\partial^k}{\partial q^k} Q\phi(\tau;q)\right]_{q=0}, \tag{22}$$

and

$$\phi(\tau;q) = \phi_0 + q\phi_1 + q^2\phi_2 + \ldots. \tag{23}$$

On utilizing Eq. (21) in Eq. (19), we attain the subsequent equation

$$\xi_k(\tau) = (\alpha_k + \hbar)\,\xi_{k-1}(\tau) - \hbar\left(1 - \frac{\alpha_k}{n}\right) L^{-1}_{\frac{\tau^\eta}{\eta}}\left[s^{-1}\xi(0) + s^{-\rho} L_{\frac{\tau^\eta}{\eta}} f(\tau)\right] + \hbar L^{-1}_{\frac{\tau^\eta}{\eta}}\left[s^{-\rho} L_{\frac{\tau^\eta}{\eta}}\left[R\xi_{k-1} + A_{k-1}\right]\right]. \tag{24}$$

Hence, by utilizing Eq. (24), we can determine various components of $\xi_k(\tau)$ for $n \geq 1$ and q-HAGTM solution can be given by the following equation

$$\xi(\tau) = \sum_{k=0}^{\infty}\left(\frac{1}{n}\right)^k \xi_k(\tau). \tag{25}$$

4 Fractional diabetes mathematical model

The diabetes model in classical form, along with its complications [5] can be given as

$$\begin{aligned}\frac{dD}{d\tau} &= P - (\gamma + \varpi) D + C,\\ \frac{dC}{d\tau} &= P + \gamma D - (\eta + \varpi + \alpha + \beta) C.\end{aligned} \tag{26}$$

Here C indicates number of diabetics with complications. D denotes number of diabetics without complications at time τ. $N = N(\tau) = C(\tau) + D(\tau)$ represents size of population having diabetes at the time τ. The occurrence of diabetic mellitus is denoted by P. γ indicates probability of developing a complication. α shows rate at which patients with complications become severely disabled. β denotes the mortality rate due to complications. ϖ indicates the rate of natural mortality. η shows rate at which complications are cured.

$$\begin{aligned}\frac{dC}{d\tau} &= -(\gamma + \sigma) C + \gamma N,\\ \frac{dN}{d\tau} &= P - (\alpha + \beta) C - \varpi N.\end{aligned} \tag{27}$$

Where $\sigma = \eta + \varpi + \alpha + \beta$, with the initial condition

$$C(0) = C_0, N(0) = N_0. \tag{28}$$

As the classical order derivative does not attribute memory to the system, hence, in order to contain the whole memory of the system, we change the model (27) from integer order derivative to the Katugampola fractional derivative in the Caputo sense.

$$\begin{aligned}{}_{k}c D_\tau^{\rho,\eta} C(\tau) &= -(\gamma + \sigma) C + \gamma N,\\ {}_{k}c D_\tau^{\rho,\eta} N(\tau) &= P - (\alpha + \beta) C - \varpi N.\end{aligned} \tag{29}$$

5 q-HAGTM algorithm for fractional diabetes model and its complication

In this section, we solve the fractional order diabetes model and its complication Eq. (29) subjecting to the initial condition Eq. (28) by q-HAGTM, we get

$$L_{\frac{\tau^\eta}{\eta}}\left[{}_{kc}D_\tau^{\rho,\eta}C(\tau)\right](s) = L_{\frac{\tau^\eta}{\eta}}\left[-(\gamma+\sigma)C+\gamma N\right],$$

$$L_{\frac{\tau^\eta}{\eta}}\left[{}_{kc}D_\tau^{\rho,\eta}N(\tau)\right](s) = L_{\frac{\tau^\eta}{\eta}}\left[P-(\alpha+\beta)C-\varpi N\right]. \tag{30}$$

After employing the GLT formula for the Katugampola fractional derivative and further simplification, we get

$$s^\rho L_{\frac{\tau^\eta}{\eta}}\{C(\tau)\}(s) - s^{\rho-1}C(0) = L_{\frac{\tau^\eta}{\eta}}\left[-(\gamma+\sigma)C+\gamma N\right],$$

$$s^\rho L_{\frac{\tau^\eta}{\eta}}\{N(\tau)\}(s) - s^{\rho-1}N(0) = L_{\frac{\tau^\eta}{\eta}}\left[P-(\alpha+\beta)C-\varpi N\right]. \tag{31}$$

On simplification Eq. (31), we have

$$L_{\frac{\tau^\eta}{\eta}}\{C(\tau)\}(s) - \frac{C_0}{s} - \frac{1}{s^\rho}\left\{L_{\frac{\tau^\eta}{\eta}}\left[-(\gamma+\sigma)C+\gamma N\right]\right\},$$

$$L_{\frac{\tau^\eta}{\eta}}\{N(\tau)\}(s) - \frac{N_0}{s} - \frac{1}{s^\rho}\left\{L_{\frac{\tau^\eta}{\eta}}\left[P-(\alpha+\beta)C-\varpi N\right]\right\} = 0. \tag{32}$$

We present the non-linear operator given as follows

$$Q_1\{C,\tau;q\} = L_{\frac{\tau^\eta}{\eta}}\{C,\tau;q\}(s) - \frac{C_0}{s} - \frac{1}{s^\rho}\left\{L_{\frac{\tau^\eta}{\eta}}\left[-(\gamma+\sigma)\{C,\tau;q\}+\gamma\{N,\tau;q\}\right]\right\},$$

$$Q_2\{N,\tau;q\} = L_{\frac{\tau^\eta}{\eta}}\{N,\tau;q\}(s) - \frac{N_0}{s} - \frac{1}{s^\rho}\left\{L_{\frac{\tau^\eta}{\eta}}\left[P-(\alpha+\beta)\{C,\tau;q\}-\varpi\{N,\tau;q\}\right]\right\}. \tag{33}$$

The term of the k^{th} order deformation equation are as follows

$$L_{\frac{\tau^\eta}{\eta}}\{C_k(\tau)-\alpha_k C_{k-1}(\tau)\} = \hbar\Re_{1,k}(C_{k-1}(\tau)),$$

$$L_{\frac{\tau^\eta}{\eta}}\{N_k(\tau)-\alpha_k N_{k-1}(\tau)\} = \hbar\Re_{2,k}(N_{k-1}(\tau)). \tag{34}$$

Where

$$\Re_{1,k}(C_{k-1}) = L_{\frac{\tau^\eta}{\eta}}\{C_{k-1}\} - \left(1-\frac{\alpha_k}{n}\right)\left(\frac{C_0}{s}\right) - \frac{1}{s^\rho}\left\{L_{\frac{\tau^\eta}{\eta}}\left[-(\gamma+\sigma)C_{k-1}+\gamma N_{k-1}\right]\right\},$$

$$\Re_{2,k}(N_{k-1}) = L_{\frac{\tau^\eta}{\eta}}\{N_{k-1}\} - \left(1-\frac{\alpha_k}{n}\right)\left(\frac{N_0}{s}\right) - \frac{1}{s^\rho}\left\{L_{\frac{\tau^\eta}{\eta}}\left[P-(\alpha+\beta)C_{k-1}-\varpi N_{k-1}\right]\right\}. \tag{35}$$

By using the inverse GLT on Eq. (34), we get

$$C_k(\tau) = \alpha_k C_{k-1}(\tau) + \hbar L^{-1}_{\frac{\tau^\eta}{\eta}}\Re_{1,k}(C_{k-1}(\tau)),$$

$$N_k(\tau) = \alpha_k N_{k-1}(\tau) + \hbar L^{-1}_{\frac{\tau^\eta}{\eta}} \Re_{2,k}(N_{k-1}(\tau)). \tag{36}$$

Solution to the k^{th} order deformation equation is expressed by

$$C_k(\tau) = \alpha_k C_{k-1}(\tau) + \hbar L^{-1}_{\frac{\tau^\eta}{\eta}} \left\{ L_{\frac{\tau^\eta}{\eta}} \{C_{k-1}\} - \left(1 - \frac{\alpha_k}{n}\right)\left(\frac{C_0}{s}\right) - \frac{1}{s^\rho} \left\{ L_{\frac{\tau^\eta}{\eta}} \left[-(\gamma+\sigma) C_{k-1} + \gamma N_{k-1} \right] \right\} \right\},$$

$$N_k(\tau) = \alpha_k N_{k-1}(\tau) + \hbar L^{-1}_{\frac{\tau^\eta}{\eta}} \left\{ L_{\frac{\tau^\eta}{\eta}} \{N_{k-1}\} - \left(1 - \frac{\alpha_k}{n}\right)\left(\frac{N_0}{s}\right) - \frac{1}{s^\rho} \left\{ L_{\frac{\tau^\eta}{\eta}} \left[P - (\alpha+\beta) C_{k-1} - \varpi N_{k-1} \right] \right\} \right\}. \tag{37}$$

Putting $k = 1, 2\ldots$ in Eq. (37), we obtain

$$C_1(\tau) = \hbar \left[(\gamma+\sigma) C_0 - \gamma N_0 \right] \frac{1}{\Gamma(1+\rho)} \left(\frac{\tau^\eta}{\eta}\right)^\rho,$$

$$N_1(\tau) = \hbar \left[-P + (\alpha+\beta) C_0 + \varpi N_0 \right] \frac{1}{\Gamma(1+\rho)} \left(\frac{\tau^\eta}{\eta}\right)^\rho. \tag{38}$$

Similarly

$$C_2(\tau) = (n+\hbar) \{ \hbar (\gamma+\sigma) C_0 - \gamma N_0 \} \frac{1}{\Gamma(1+\rho)} \left(\frac{\tau^\eta}{\eta}\right)^\rho + \hbar^2 \left\{ \left[(\gamma+\sigma)^2 C_0 - \gamma(\gamma+\sigma) N_0 \right] - \gamma \left[-P + (\alpha+\beta) C_0 + \varpi N_0 \right] \right\} \frac{1}{\Gamma(1+2\rho)} \left(\frac{\tau^\eta}{\eta}\right)^{2\rho},$$

$$N_2(\tau) = (n+\hbar) \{ \hbar \left[-P + (\alpha+\beta) C_0 + \varpi N_0 \right] \} \frac{1}{\Gamma(1+\rho)} \left(\frac{\tau^\eta}{\eta}\right)^\rho - \hbar \{P\} \frac{1}{\Gamma(1+\rho)} \left(\frac{\tau^\eta}{\eta}\right)^\rho + \hbar^2 \{ (\alpha+\beta) \left[(\gamma+\sigma) C_0 - \gamma N_0 \right] + \mu \left[-P + (\alpha+\beta) C_0 + \varpi N_0 \right] \} \frac{1}{\Gamma(1+2\rho)} \left(\frac{\tau^\eta}{\eta}\right)^{2\rho}. \tag{39}$$

By following the same procedure remaining terms for $k \geq 2$ find the series solution of model. So, the solution of fractional order diabetes and its complication model Eq. (29) is given by

$$C_k(\tau) = C_0(\tau) + \frac{1}{n} C_1(\tau) + \left(\frac{1}{n}\right)^2 C_2(\tau) + \ldots,$$

$$N_k(\tau) = N_0(\tau) + \frac{1}{n} N_1(\tau) + \left(\frac{1}{n}\right)^2 N_2(\tau) + \ldots. \tag{40}$$

6 Analysis of existence and uniqueness of the obtained solution

Here, we investigate the existence of a solution for the fractional diabetic model through the fixed point assumption.

Now, using the Katugamola integral oprator given by Eq. (4) to the system (29), we obtain the subsequent integral equations

$$C(\tau) - C(0) = \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau [-(\gamma+\sigma) C + \gamma N] \cdot v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1} dv,$$

$$N(\tau) - N(0) = \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau [P - (\alpha+\beta) C - \varpi N] \cdot v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1} dv. \tag{41}$$

For ingenuity, we find out

$$K_1(\tau, C) = [-(\gamma+\sigma) C + \gamma N],$$

$$K_2(\tau, N) = [P - (\alpha+\beta) C - \varpi N]. \tag{42}$$

Theorem 1: The kernels K_i, $i = 1, 2$ satisfy the Lipschitz condition, when $0 \leq \Psi_1 < 1$, $i = 1, 2$.

Proof. Suppose $K_1(\tau, C) = [-(\gamma+\sigma) C + \gamma N]$, is the kernel and $C(\tau)$ and $C_1(\tau)$ be two functions, consequently we obtain the following

$$\|K_1(\tau, C) - K_1(\tau, C_1)\| = \left\|[-(\gamma+\sigma) C + \gamma N] - [-(\gamma+\sigma) C_1 + \gamma N]\right\|,$$

$$= \|-(\gamma+\sigma) \cdot (C(\tau) - C_1(\tau))\|$$

$$\leq -(\gamma+\sigma) \cdot \|(C(\tau) - C_1(\tau))\|$$

$$\leq \Psi_1 \|(C(\tau) - C_1(\tau))\|. \tag{43}$$

Now consider $\Psi_1 = -(\gamma+\sigma) < 1$, let $P_1 = max_{t\in R}\|C(\tau)\|$ and $P_2 = max_{t\in R}|N(\tau)\|$ are bounded function then, we find

$$\|K_1(\tau, C) - K_1(\tau, C_1)\| \leq \Psi_1 \|(C(\tau) - C_1(\tau))\|. \tag{44}$$

Obviously, which is Lipschitz condition for K_1. In addition if $0 \leq \Psi_1 < 1$. Then $C(\tau)$ has an upper bound.

In the same way, we can show that

$$\|K_2(\tau, N) - K_2(\tau, N_1)\| \leq \Psi_2 \|(N(\tau) - N_1(\tau))\|. \tag{45}$$

From Eq. (42) K_1 and K_2 are the kernels. Then the associate integrals are found

$$C(\tau) = C(0) + \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau K_1(v, C) \cdot v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1} dv,$$

$$N(\tau) = N(0) + \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau K_2(v, N) \cdot v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1} dv. \tag{46}$$

Further, we get

$$C_n(\tau) = C(0) + \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau K_1(v, C_{n-1}) \cdot v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1} dv,$$

$$N_n(\tau) = N(0) + \frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau K_2(v, N_{n-1})\cdot v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}dv. \tag{47}$$

Where the initial conditions are

$$C(0) = C_0 \quad and\ N(0) = N_0. \tag{48}$$

After subtracting consecutive terms, we have

$$\Xi_n = C_n(\tau) - C_{n-1}(\tau) = \frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau (K_1(v, C_{n-1}) - K_1(v, C_{n-2}))\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv,$$

$$\Delta_n = N_n(\tau) - N_{n-1}(\tau) = \frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau (K_2(v, N_{n-1}) - K_2(v, N_{n-2}))\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv. \tag{49}$$

Taking the below

$$C_n(\tau) = \sum_{j=1}^{n}\Xi_j(\tau),$$

$$N_n(\tau) = \sum_{j=1}^{n}\Delta_j(\tau). \tag{50}$$

Hence forth by applying the tri-angular and norm properties on equation of (49), we arrive at the following equation

$$\|\Xi_n\| = \|\,C_n(\tau) - C_{n-1}(\tau)\| \le \frac{\eta^{1-\rho}}{\Gamma\rho}\left\|\int_0^\tau (K_1(v, C_{n-1}) - K_1(v, C_{n-2}))\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv\right\|,$$

$$\|\Delta_n\| = \|\,N_n(\tau) - N_{n-1}(\tau)\| \le \frac{\eta^{1-\rho}}{\Gamma\rho}\left\|\int_0^\tau (K_2(v, N_{n-1}) - K_2(v, N_{n-2}))\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv\right\|. \tag{51}$$

While satisfying the Lipschitz conditions the kernels yield the following outcomes

$$\begin{aligned}\|\,C_n(\tau) - C_{n-1}(\tau)\| &\le \frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau \|(K_1(v, C_{n-1}) - K_1(v, C_{n-2}))\|\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv\\ &\le \Psi_1\frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau \|C_{n-1} - C_{n-2}\|\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv,\\ \|\,N_n(\tau) - N_{n-1}(\tau)\| &\le \frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau \|(K_2(v, N_{n-1}) - K_2(v, N_{n-2}))\|\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv\\ &\le \Psi_2\frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau \|N_{n-1} - N_{n-2}\|\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv.\end{aligned} \tag{52}$$

Therefore, we obtain the following

$$\|\Xi_n\| \le \Psi_1\frac{\eta^{1-\rho}}{\Gamma\rho}\int_0^\tau \|\Xi_{n-1}(\tau)\|\times v^{\eta-1}(\tau^\eta - v^\eta)^{\rho-1}\,dv,$$

$$\|\Delta_n\| \leq \Psi_2 \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau \|\Delta_{n-1}(\tau)\| \times v^{\eta-1} (\tau^\eta - v^\eta)^{\rho-1} \, dv. \tag{53}$$

Theorem 2. The Katugampola fractional derivative non-integer order fractional diabetic model has unique solution provided that the following conditions are satisfied for τ_{max}.

$$\frac{\Psi_i}{\Gamma\rho + 1} \left(\frac{\tau^\eta{}_{max}}{\eta} \right)^\rho < 1, i = 1, 2. \tag{54}$$

Proof. Here, we assume that $C(\tau)$ and $N(\tau)$ are bounded functions and fulfills the Lipschitz condition, then using Eq. (53), and using recursive techniques, we have

$$\|\Xi_n\| \leq \|C_0\| \left[\frac{\Psi_1}{\Gamma\rho + 1} \left(\frac{\tau^\eta{}_{max}}{\eta} \right)^\rho \right]^n,$$

$$\|\Delta_n\| \leq \|N_0\| \left[\frac{\Psi_2}{\Gamma\rho + 1} \left(\frac{\tau^\eta{}_{max}}{\eta} \right)^\rho \right]^n. \tag{55}$$

All above functions exist and result in Eq. (55), therefore, we will show that these functions are the solutions to the diabetic model. Then, we have

$$C(\tau) - C(0) = C_n(\tau) - A_n(\tau),$$

$$N(\tau) - N(0) = N_n(\tau) - B_n(\tau). \tag{56}$$

Further, we calculate the following norms of $A_n(\tau)$

$$\begin{aligned} \|A_n(\tau)\| &\leq \frac{\eta^{1-\rho}}{\Gamma\rho} \left\| \int_0^\tau (K_1(v, C) - K_1(v, C_{n-1})) \times v^{\eta-1} (\tau^\eta - v^\eta)^{\rho-1} \, dv \right\| \\ &\leq \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau \|(K_1(v, C) - K_1(v, C_{n-1}))\| \times v^{\eta-1} (\tau^\eta - v^\eta)^{\rho-1} \, dv \\ &\leq \Psi_1 \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau \|C - C_{n-1}\| \times v^{\eta-1} (\tau^\eta - v^\eta)^{\rho-1} \, dv \\ &\leq \frac{\Psi_1}{\Gamma\rho + 1} \left(\frac{\tau^\eta}{\eta} \right)^\rho \|C - C_{n-1}\|. \end{aligned} \tag{57}$$

The following equation follows a recursive process

$$\|A_n(\tau)\| \leq \|C_0\| \left[\frac{1}{\Gamma\rho + 1} \left(\frac{\tau^\eta}{\eta} \right)^\rho \right]^{n+1} {\Psi_1}^n F.$$

At τ_{max} we obtain

$$\|A_n(\tau)\| \leq \|C_0\| \left[\frac{1}{\Gamma\rho + 1} \left(\frac{\tau^\eta_{max}}{\eta} \right)^\rho \right]^{n+1} {\Psi_1}^n F. \tag{58}$$

On the above equation, when we take the limits of both sides, we get $\|A_n(\tau)\| \to 0$ at $n \to \infty$. It is possible to attain $\|B_n(\tau)\| \to 0$. Hence, the proof is concluded.

Uniqueness of Solution: The uniqueness of solutions that is attained in this segment of the diabetic mathematical model is considered. We assume $C_1(\tau)$ and $N_1(\tau)$ are the other solutions of the proposed system, then we have

$$C(\tau) - C_1(\tau) = \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau (K_1(v, C) - K_1(v, C_1)) \times v^{\eta-1} (\tau^\eta - v^\eta)^{\rho-1} \, dv. \tag{59}$$

The following result is obtained by applying the norm to each side of Eq. (59)

$$\|C(\tau) - C_1(\tau)\| \leq \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau \|(K_1(v, C) - K_1(v, C_1))\| \times v^{\eta-1} (\tau^\eta - v^\eta)^{\rho-1} \, dv. \tag{60}$$

Lipschitz condition applied to the kernel gives us

$$\|C(\tau) - C_1(\tau)\| \leq \Psi_1 \frac{\eta^{1-\rho}}{\Gamma\rho} \int_0^\tau \|C - C_1\| \times v^{\eta-1} (\tau^\eta - v^\eta)^{\rho-1} \, dv, \tag{61}$$

$$\leq \frac{\Psi_1}{\Gamma\rho + 1} \left(\frac{\tau^\eta}{\eta}\right)^\rho \|C - C_1\|. \tag{62}$$

The following result are obtained

$$\|C(\tau) - C_1(\tau)\| \cdot \left[1 - \frac{\Psi_1}{\Gamma\rho + 1} \left(\frac{\tau^\eta}{\eta}\right)^\rho\right] \leq 0,$$

$$\|C(\tau) - C_1(\tau)\| = 0,$$

$$\implies C(\tau) = C_1(\tau) \tag{63}$$

Considering the above, we can conclude that the first differential equation of the diabetic model has a unique solution. Similarly, we also prove that $N(\tau)$ have unique solutions.

7 Numerical simulations

In this section, we have examined the results of numerical simulations for fractional order diabetes model by using effective and powerful method q-HAGTM. Numerical values have been computed for $C(\tau)$ and $N(\tau)$ at $\rho = 0.85, 0.90$ and 1. Fig.1 explains the nature of trend followed by diabetic population inhibiting complications C corresponding to time τ for discrete values of fractional derivative order ρ. It shows that with increase in time, diabetic population having complications increases. Fig. 2 indicates that the size of diabetic N at time τ also shows incremental behavior with w.r.t time for distinct values of ρ.

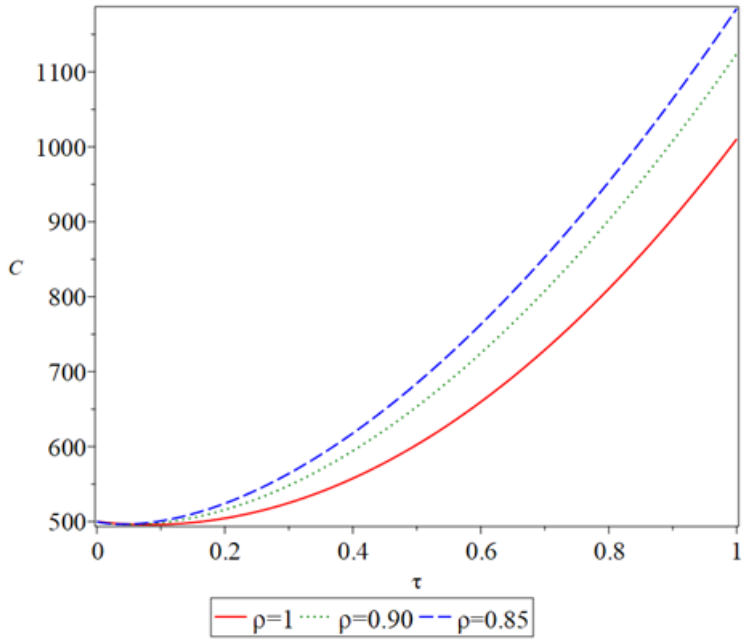

Figure 1: Nature of $C(\tau)$ w.r.t to time τ for distinct values of ρ

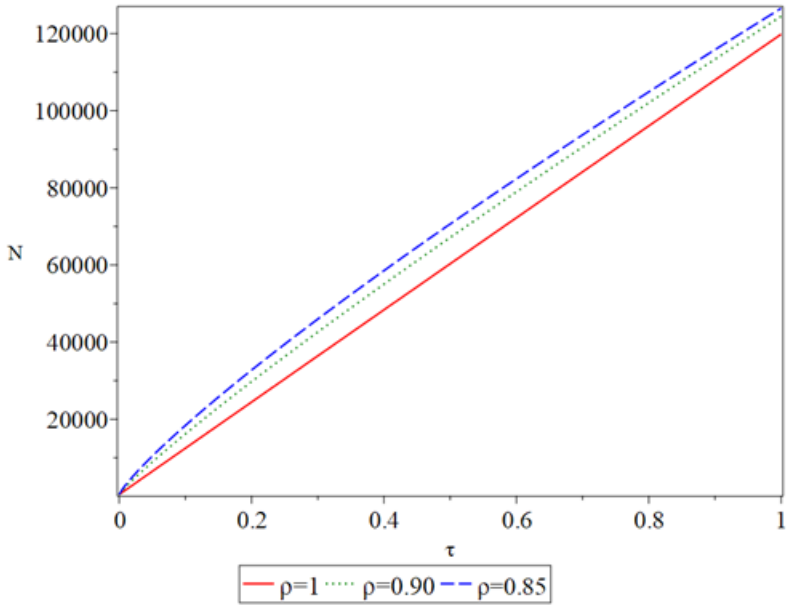

Figure 2: Responses of $N(\tau)$ w.r.t to time τ for distinct values of ρ

8 Conclusions

In this study, the nonlinear fractional diabetic model is investigated with the help of q-HAGTM. The existence and uniqueness of the obtained results are presented using fixed point theory and the Katugampola fractional integral operator. Some numerical results are analyzed to describe the effect of the arbitrary order. The effect of various parameters on the number of diabetic patients with complications and the size of diabetic patients over time is shown graphically. The results of this study are very helpful for medical practitioners dealing with diabetes and related problems. Thus, we have concluded that the implemented technique is efficient for analyzing the behavior of these types of problems arising in various fields.

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