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# Caputo $\psi$-fractional Ostrowski and Grüss inequalities for several functions

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**Abstract**

Very general univariate mixed Caputo $\psi$-fractional Ostrowski and Grüss type inequalities for several functions are presented. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. We give also applications.

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## 1 Introduction

In 1938, A. Ostrowski [5] proved the following important inequality.

**Theorem 1** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on $(a,b)$, i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty , \tag{1}$$

*for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.*

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

In this article we are greatly motived and inspired by Theorem 1, see also [2]. Here we present various $\psi$-fractional Ostrowski and Grüss type inequalities for several functions and we give interesting applications.

## 2 Background

Here we follow [1].

Let $\alpha > 0$, $[a,b] \subset \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ which is integrable and $\psi \in C^1([a,b])$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a,b]$. Consider $n = \lceil \alpha \rceil$, the ceiling of $\alpha$. The left and right fractional integrals are defined, respectively, as follows:

$$I_{a+}^{\alpha,\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t)\, dt, \tag{2}$$

and

$$I_{b-}^{\alpha,\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t)\, dt, \tag{3}$$

for any $x \in [a,b]$, where $\Gamma$ is the gamma function.

The following semigroup property is valid for fractional integrals: if $\alpha, \beta > 0$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f(x) = I_{a+}^{\alpha+\beta,\psi} f(x), \text{ and } I_{b-}^{\alpha,\psi} I_{b-}^{\beta,\psi} f(x) = I_{b-}^{\alpha+\beta,\psi} f(x).$$

We mention

**Definition 2** *([1]) Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f, \psi \in C^n([a,b])$ with $\psi$ being increasing and $\psi'(x) \neq 0$, for all $x \in [a,b]$. The left $\psi$-Caputo fractional derivative of $f$ of order $\alpha$ is given by*

$$^C D_{a+}^{\alpha,\psi} f(x) := I_{a+}^{n-\alpha,\psi} \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x), \tag{4}$$

*and the right $\psi$-Caputo fractional derivative of $f$ is given by*

$$^C D_{b-}^{\alpha,\psi} f(x) := I_{b-}^{n-\alpha,\psi} \left( -\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x). \tag{5}$$

*To simplify notation, we will use the symbol*

$$f_\psi^{[n]}(x) := \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x), \tag{6}$$

*with $f_\psi^{[0]}(x) = f(x)$.*

By the definition, whrn $\alpha = m \in \mathbb{N}$, we have

$$\begin{array}{l} ^C D_{a+}^{\alpha,\psi} f(x) = f_\psi^{[m]}(x) \\ \text{and} \\ ^C D_{b-}^{\alpha,\psi} f(x) = (-1)^m f_\psi^{[m]}(x). \end{array} \tag{7}$$

If $\alpha \notin \mathbb{N}$, we have

$$^C D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f_\psi^{[n]}(t)\, dt, \tag{8}$$

and

$$ {}^{C}D_{b-}^{\alpha,\psi}f(x)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}\psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1}f_{\psi}^{[n]}(t)\,dt, \tag{9} $$

$\forall\ x\in[a,b]$.

In particular, when $\alpha\in(0,1)$, we have

$$ \begin{array}{l} {}^{C}D_{a+}^{\alpha,\psi}f(x)=\frac{1}{\Gamma(1-\alpha)}\int_{a}^{x}(\psi(x)-\psi(t))^{-\alpha}f'(t)\,dt,\\ \text{and}\\ {}^{C}D_{b-}^{\alpha,\psi}f(x)=\frac{-1}{\Gamma(1-\alpha)}\int_{x}^{b}(\psi(t)-\psi(x))^{-\alpha}f'(t)\,dt \end{array} \tag{10} $$

$\forall\ x\in[a,b]$.

Clearly the above is a generalization of left and right Caputo fractional derivatives.

For more see [1].

Still we need from [1] the following left and right fractional Taylor's formulae:

**Theorem 3** *([1]) Let $\alpha>0$, $n\in\mathbb{N}$ such that $n=\lceil\alpha\rceil$, $[a,b]\subset\mathbb{R}$ and $f,\psi\in C^{n}([a,b])$ with $\psi$ being increasing and $\psi'(x)\neq0$, for all $x\in[a,b]$. Then, the left fractional Taylor formula follows,*

$$ f(x)=\sum_{k=0}^{n-1}\frac{f_{\psi}^{[k]}(a)}{k!}(\psi(x)-\psi(a))^{k}+I_{a+}^{\alpha,\psi}\ {}^{C}D_{a+}^{\alpha,\psi}f(x), \tag{11} $$

*and the right fractional Taylor formula follows,*

$$ f(x)=\sum_{k=0}^{n-1}(-1)^{k}\frac{f_{\psi}^{[k]}(b)}{k!}(\psi(b)-\psi(x))^{k}+I_{b-}^{\alpha,\psi}\ {}^{C}D_{b-}^{\alpha,\psi}f(x), \tag{12} $$

$\forall\ x\in[a,b]$.

*In particular, given $\alpha\in(0,1)$, we have*

$$ \begin{array}{l} f(x)=f(a)+I_{a+}^{\alpha,\psi}\ {}^{C}D_{a+}^{\alpha,\psi}f(x),\\ \text{and}\\ f(x)=f(b)+I_{b-}^{\alpha,\psi}\ {}^{C}D_{b-}^{\alpha,\psi}f(x), \end{array} \tag{13} $$

$\forall\ x\in[a,b]$.

**Remark 4** *For convenience we can rewrite (11)-(13) as follows:*

$$ f(x)=\sum_{k=0}^{n-1}\frac{f_{\psi}^{[k]}(a)}{k!}(\psi(x)-\psi(a))^{k}+ \tag{14} $$

$$ \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt, $$

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*and*

$$f(x)=\sum_{k=0}^{n-1}\frac{(-1)^k f_{\psi}^{[k]}(b)}{k!}(\psi(b)-\psi(x))^k+ \tag{15}$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^C D_{b-}^{\alpha,\psi} f(t)\,dt,$$

$\forall\ x\in[a,b]$.

*When* $\alpha\in(0,1)$, *we get:*

$$\begin{array}{l} f(x)=f(a)+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^C D_{a+}^{\alpha,\psi} f(t)\,dt, \\ \text{and} \\ f(x)=f(b)+\frac{1}{\Gamma(\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^C D_{b-}^{\alpha,\psi} f(t)\,dt, \end{array} \tag{16}$$

$\forall\ x\in[a,b]$.

Again from [1] we have the following:

Consider the norms $\|\cdot\|_{\infty}:C([a,b])\to\mathbb{R}$ and $\|\cdot\|_{C_{\psi}^{[n]}}:C^n([a,b])\to\mathbb{R}$, where $\|f\|_{C_{\psi}^{[n]}}:=\sum_{k=0}^{n}\left\|f_{\psi}^{[k]}\right\|_{\infty}$.

We have

**Theorem 5** *([1]) The* $\psi$*-Caputo fractional derivatives are bounded operators. For all* $\alpha>0$ $(n=\lceil\alpha\rceil)$

$$\left\|{}^C D_{a+}^{\alpha,\psi}\right\|_{\infty}\le K\|f\|_{C_{\psi}^{[n]}} \tag{17}$$

*and*

$$\left\|{}^C D_{b-}^{\alpha,\psi}\right\|_{\infty}\le K\|f\|_{C_{\psi}^{[n]}}, \tag{18}$$

*where*

$$K=\frac{(\psi(b)-\psi(a))^{n-\alpha}}{\Gamma(n+1-\alpha)}>0. \tag{19}$$

# 3 Main Results

At first we present the following $\psi$-fractional Ostrowski type inequalities for several functions:

**Theorem 6** *Let* $\alpha>0$, $n\in\mathbb{N}:n=\lceil\alpha\rceil$, $[a,b]\subset\mathbb{R}$ *and* $f_i,\psi\in C^n([a,b])$, $i=1,...,r$; *with* $\psi$ *being increasing and* $\psi'(x)\neq0$, *for all* $x\in[a,b]$. *Let* $x_0\in[a,b]$ *and assume that* $f_{i\psi}^{[k]}(x_0)=0$, *for* $k=1,...,n-1$; $i=1,...,r$. *Set*

$$\Phi(f_1,...,f_r)(x_0):=r\int_a^b\left(\prod_{k=1}^{r}f_k(x)\right)d\psi(x)- \tag{20}$$

$$\sum_{i=1}^{r}\left[f_i(x_0)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)d\psi(x)\right].$$

*Then*

$$|\Phi(f_1,...,f_r)(x_0)|\leq\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right. \tag{21}$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right].$$

*If* $0<\alpha\leq1$, *then (21) is valid without any initial conditions.*

**Proof.** By Theorem 3 we have that

$$f_i(x)-f_i(x_0)=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi}f_i(t)\,dt, \tag{22}$$

$\forall\ x\in[x_0,b]$,
and

$$f_i(x)-f_i(x_0)=\frac{1}{\Gamma(\alpha)}\int_{x}^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi}f_i(t)\,dt, \tag{23}$$

$\forall\ x\in[a,x_0]$;
for all $i=1,...,r$.

Multiplying (22) and (23) by $\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)$ we obtain, respectively,

$$\prod_{k=1}^{r}f_k(x)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x_0)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)}{\Gamma(\alpha)}\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi}f_i(t)\,dt, \tag{24}$$

$\forall\ x\in[x_0,b]$,

$$\prod_{k=1}^{r}f_k(x)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x_0)=$$

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$$\frac{\left(\prod\limits_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{\Gamma(\alpha)} \int_x^{x_0} \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} \, {}^C D_{x_0-}^{\alpha,\psi} f_i(t)\,dt, \tag{25}$$

$\forall\ x \in [a, x_0]$;

for all $i = 1, ..., r$.

Adding (24) and (25), separately, we obtain

$$r\left(\prod_{k=1}^{r} f_k(x)\right) - \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{x_0}^{x} \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} \, {}^C D_{x_0+}^{\alpha,\psi} f_i(t)\,dt\right], \tag{26}$$

$\forall\ x \in [x_0, b]$,

and

$$r\left(\prod_{k=1}^{r} f_k(x)\right) - \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{x}^{x_0} \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} \, {}^C D_{x_0-}^{\alpha,\psi} f_i(t)\,dt\right], \tag{27}$$

$\forall\ x \in [a, x_0]$.

Next we integrate (26) and (27) with respect to $\psi(x)$, $x \in [a, b]$. We have

$$r\int_{x_0}^{b}\left(\prod_{k=1}^{r} f_k(x)\right) d\psi(x) - \sum_{i=1}^{r}\left[f_i(x_0)\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) d\psi(x)\right] =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) [\int_{x_0}^{x} \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} \, {}^C D_{x_0+}^{\alpha,\psi} f_i(t)\,dt] d\psi(x)\right], \tag{28}$$

and

$$r\int_{a}^{x_0}\left(\prod_{k=1}^{r} f_k(x)\right) d\psi(x) - \sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) d\psi(x)\right] =$$

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$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)[\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi}f_i(t)\,dt]d\psi(x)\right]. \tag{29}$$

Adding (28) and (29) we derive the identity:

$$\Phi(f_1,...,f_r)(x_0) := r\int_a^b\left(\prod_{k=1}^{r} f_k(x)\right)d\psi(x) -$$

$$\sum_{i=1}^{r}\left[f_i(x_0)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)d\psi(x)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi}f_i(t)\,dt\right)d\psi(x)\right]\right. \tag{30}$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi}f_i(t)\,dt\right)d\psi(x)\right]\right].$$

Hence it holds

$$|\Phi(f_1,...,f_r)(x_0)| \leq$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} |f_j(x)|\right)\left(\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ \left|{}^{C}D_{x_0-}^{\alpha,\psi}f_i(t)\right|dt\right)d\psi(x)\right]\right.$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} |f_j(x)|\right)\left(\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ \left|{}^{C}D_{x_0+}^{\alpha,\psi}f_i(t)\right|dt\right)d\psi(x)\right]\right] =: (*). \tag{31}$$

We observe that

$$(*) \leq \frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} |f_j(x)|\right)(\psi(x_0)-\psi(x))^{\alpha}\,d\psi(x)\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} |f_j(x)|\right)(\psi(x)-\psi(x_0))^{\alpha}\,d\psi(x)\right]\right] = \tag{32}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$
$$\left.\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right].$$

By Theorem 4.10, p. 98 of [3], we get that $I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j|\right)\in C([a,b])$ and so at any $x_0\in[a,b]$ is finite, $i=1,...,r$. Similarly, by Theorem 4.11, p. 101 of [3], we get that $I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j|\right)\in C([a,b])$ and so at any $x_0\in[a,b]$ is finite, $i=1,...,r$. Arguing similarly, we get that ${}^{C}D_{a+}^{\alpha,\psi}f_i$, ${}^{C}D_{b-}^{\alpha,\psi}f_i\in C([a,b])$, for all $i=1,...,r$.

The theorem is proved. ■

We continue with

**Theorem 7** *All as in Theorem 6 with $\alpha\geq 1$. Then*

$$|\Phi(f_1,...,f_r)(x_0)|\leq\sum_{i=1}^{r}\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_1([a,x_0],\psi)} I_{a+}^{\alpha,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right.$$
$$\left.+\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_1([x_0,b],\psi)} I_{b-}^{\alpha,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]. \tag{33}$$

**Proof.** From (31) we get

$$(*)\leq\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_1([a,x_0],\psi)}\right.\right.$$
$$\left.\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)(\psi(x_0)-\psi(x))^{\alpha-1}d\psi(x)\right)\right]+$$
$$\left.\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_1([x_0,b],\psi)}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)(\psi(x)-\psi(x_0))^{\alpha-1}d\psi(x)\right)\right]\right] \tag{34}$$

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$$= \sum_{i=1}^{r} \left[ \left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)} I_{a+}^{\alpha,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) + \right.$$

$$\left. \left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)} I_{b-}^{\alpha,\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right], \tag{35}$$

proving the theorem. ■

We continue with

**Theorem 8** *All as in Theorem 6 with* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq 1$. *Then*

$$|\Phi(f_1, ..., f_r)(x_0)| \leq \frac{\Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r} \left[ \left[ \left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_q([a,x_0],\psi)} I_{a+}^{\alpha+\frac{1}{p},\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right. \tag{36}$$

$$\left. + \left[ \left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_q([x_0,b],\psi)} I_{b-}^{\alpha+\frac{1}{p},\psi} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right].$$

**Proof.** From (31) we obtain

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \left[ \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) \right. \right.$$

$$\left. \left( \int_x^{x_0} (\psi(t) - \psi(x))^{p(\alpha-1)} d\psi(t) \right)^{\frac{1}{p}} \left( \int_x^{x_0} \left| {}^{C}D_{x_0-}^{\alpha,\psi} f_i(t) \right|^q d\psi(t) \right)^{\frac{1}{q}} d\psi(x) \right] +$$

$$\left[ \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) \left( \int_{x_0}^{x} (\psi(x) - \psi(t))^{p(\alpha-1)} d\psi(t) \right)^{\frac{1}{p}} \right. \tag{37}$$

$$\left. \left. \left( \int_{x_0}^{x} \left| {}^{C}D_{x_0+}^{\alpha,\psi} f_i(t) \right|^q d\psi(t) \right)^{\frac{1}{q}} d\psi(x) \right] \right] \leq \frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_q([a,x_0],\psi)}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(\psi(x_0)-\psi(x))^{\alpha-1+\frac{1}{p}}d\psi(x)\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_q([x_0,b],\psi)}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(\psi(x)-\psi(x_0))^{\alpha-1+\frac{1}{p}}d\psi(x)\right]\right]= \tag{38}$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_q([a,x_0],\psi)}I_{a+}^{\alpha+\frac{1}{p},\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_q([x_0,b],\psi)}I_{b-}^{\alpha+\frac{1}{p},\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right], \tag{39}$$

proving the theorem. ■

We mention as motivation for Grüss type inequalities the following:

**Theorem 9** *(1882, Čebyšev [4]) Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions with $f',g'\in L_\infty([a,b])$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right|$$

$$\leq\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty. \tag{40}$$

*The above integrals are assumed to exist.*

Next follow $\psi$-Caputo fractional Grüss type inequalities for several functions.

**Theorem 10** *Let $0<\alpha\leq 1$, $[a,b]\subset\mathbb{R}$ and $f_i,\psi\in C^1([a,b])$, $i=1,...,r\in\mathbb{N}-\{1\}$; with $\psi$ being increasing and $\psi'(x)\neq 0$, for all $x\in[a,b]$. Assume that $\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}<\infty$, and $\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}<\infty$, $i=1,...,r$. Set*

$$\Delta^\psi(f_1,...,f_r):=r(\psi(b)-\psi(a))\left(\int_a^b\left(\prod_{k=1}^{r}f_k(x)\right)d\psi(x)\right)-$$

$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)d\psi(x)\right)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)d\psi(x)\right)\right]. \tag{41}$$

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*Then*

$$\left|\Delta^{\psi}\left(f_1,...,f_r\right)\right| \leq \left(\psi\left(b\right)-\psi\left(a\right)\right)$$
$$\left\{\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right.\right.$$
$$\left.\left.+\left[\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right]\right\}. \quad (42)$$

**Proof.** Here $0<\alpha\leq 1$, i.e. $n=1$. Then (30) is valid without any initial conditions. Clearly $\Phi\left(f_1,...,f_r\right)\in C\left([a,b]\right)$. Thus, by integrating (30) against $\psi$ we obtain

$$\Delta^{\psi}\left(f_1,...,f_r\right)=\int_a^b\Phi\left(f_1,...,f_r\right)\left(x_0\right)d\psi\left(x_0\right)=$$
$$r\left(\psi\left(b\right)-\psi\left(a\right)\right)\left(\int_a^b\left(\prod_{k=1}^{r}f_k\left(x\right)\right)d\psi\left(x\right)\right)-$$
$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i\left(x\right)d\psi\left(x\right)\right)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)d\psi\left(x\right)\right)\right]=$$
$$\frac{1}{\Gamma\left(\alpha\right)}\int_a^b\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\right.\right.\right.$$
$$\left.\left(\int_x^{x_0}\psi'\left(t\right)\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha-1}\,{}^{C}D_{x_0-}^{\alpha,\psi}f_i\left(t\right)dt\right)d\psi\left(x\right)\right]+ \quad (43)$$
$$\left.\left.\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{x_0}^{x}\psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1}\,{}^{C}D_{x_0+}^{\alpha,\psi}f_i\left(t\right)dt\right)d\psi\left(x\right)\right]\right]\right\}d\psi\left(x_0\right).$$

Hence it holds

$$\left|\Delta^{\psi}\left(f_1,...,f_r\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\int_a^b\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x\right)\right|\right)\right.\right.\right.$$

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$$\left(\int_x^{x_0} \psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\left|{}^C D_{x_0-}^{\alpha,\psi} f_i(t)\right| dt\right) d\psi(x)\Bigg] + \Bigg[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)$$

$$\left(\int_{x_0}^{x} \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|{}^C D_{x_0+}^{\alpha,\psi} f_i(t)\right| dt\right) d\psi(x)\Bigg]\Bigg]\Bigg\} d\psi(x_0) =: (**). \tag{44}$$

Using (21) we derive

$$(**) \leq (\psi(b)-\psi(a))$$

$$\left\{\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0-}^{\alpha,\psi} f_i\right\|_{\infty,[a,x_0]} \sup_{x_0\in[a,b]} I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right.\right.$$

$$\left.\left.+\left[\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0+}^{\alpha,\psi} f_i\right\|_{\infty,[x_0,b]} \sup_{x_0\in[a,b]} I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]\right\}, \tag{45}$$

proving the theorem. ■

We make

**Remark 11** *Let $\alpha > 0$, $[a,b] \subset \mathbb{R}$, $f \in C([a,b])$ and $\psi \in C^1([a,b])$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a,b]$. Let $x_0 \in [a,b]$. We observe the following*

$$\left|I_{a+}^{\alpha,\psi} f(x_0)\right| \overset{(2)}{\leq} \frac{1}{\Gamma(\alpha)}\int_a^{x_0} \psi'(t)(\psi(x_0)-\psi(t))^{\alpha-1}|f(t)|\,dt \leq$$

$$\frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha)}\int_a^{x_0}(\psi(x_0)-\psi(t))^{\alpha-1} d\psi(t) = \frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(\psi(x_0)-\psi(a))^{\alpha}. \tag{46}$$

*That is*

$$\left|I_{a+}^{\alpha,\psi} f(x_0)\right| \leq \frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(\psi(x_0)-\psi(a))^{\alpha}. \tag{47}$$

*Similarly, we obtain*

$$\left|I_{b-}^{\alpha,\psi} f(x_0)\right| \overset{(3)}{\leq} \frac{1}{\Gamma(\alpha)}\int_{x_0}^{b} \psi'(t)(\psi(t)-\psi(x_0))^{\alpha-1}|f(t)|\,dt \leq$$

$$\frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha)}\int_{x_0}^{b}(\psi(t)-\psi(x_0))^{\alpha-1} d\psi(t) = \frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(\psi(b)-\psi(x_0))^{\alpha}. \tag{48}$$

*That is*

$$\left|I_{b-}^{\alpha,\psi} f(x_0)\right| \leq \frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(\psi(b)-\psi(x_0))^{\alpha}. \tag{49}$$

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We make

**Remark 12** *Let $\alpha \geq 1$, the rest as in Remark 11. We observe that*

$$\left| I_{a+}^{\alpha,\psi} f(x_0) \right| \overset{(2)}{\leq} \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \psi'(t) \left(\psi(x_0) - \psi(t)\right)^{\alpha-1} |f(t)| \, dt \leq$$

$$\frac{\left(\psi(x_0) - \psi(a)\right)^{\alpha-1}}{\Gamma(\alpha)} \int_a^{x_0} |f(t)| \, d\psi(t) = \frac{\left(\psi(x_0) - \psi(a)\right)^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([a,x_0],\psi)}. \tag{50}$$

*That is*

$$\left| I_{a+}^{\alpha,\psi} f(x_0) \right| \leq \frac{\left(\psi(x_0) - \psi(a)\right)^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([a,x_0],\psi)}. \tag{51}$$

*Similarly, we get*

$$\left| I_{b-}^{\alpha,\psi} f(x_0) \right| \overset{(3)}{\leq} \frac{1}{\Gamma(\alpha)} \int_{x_0}^{b} \psi'(t) \left(\psi(t) - \psi(x_0)\right)^{\alpha-1} |f(t)| \, dt \leq$$

$$\frac{\left(\psi(b) - \psi(x_0)\right)^{\alpha-1}}{\Gamma(\alpha)} \int_{x_0}^{b} |f(t)| \, d\psi(t) = \frac{\left(\psi(b) - \psi(x_0)\right)^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([x_0,b],\psi)}. \tag{52}$$

*That is*

$$\left| I_{b-}^{\alpha,\psi} f(x_0) \right| \leq \frac{\left(\psi(b) - \psi(x_0)\right)^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([x_0,b],\psi)}. \tag{53}$$

Next, we simplify our main theorems:

**Proposition 13** *All as in Theorem 6. Then*

$$|\Phi(f_1, ..., f_r)(x_0)| \leq \frac{1}{\Gamma(\alpha+2)}$$

$$\sum_{i=1}^{r} \left[ \left[ \left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{r} f_j \right\|_{\infty,[a,x_0]} \left(\psi(x_0) - \psi(a)\right)^{\alpha+1} \right] \right.$$

$$\left. + \left[ \left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{r} f_j \right\|_{\infty,[x_0,b]} \left(\psi(b) - \psi(x_0)\right)^{\alpha+1} \right] \right]. \tag{54}$$

*If $0 < \alpha \leq 1$, then (54) is valid without any initial conditions.*

**Proof.** By (21), (47) and (49). ■

Next comes

**Proposition 14** *All as in Theorem 6 with $\alpha \geq 1$. Then*

$$|\Phi(f_1,...,f_r)(x_0)| \leq \frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^C D_{x_0-}^{\alpha,\psi} f_i\right\|_{L_1([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{L_1([a,x_0],\psi)}(\psi(x_0)-\psi(a))^{\alpha-1}\right]\right.$$

$$\left.+\left[\left\|{}^C D_{x_0+}^{\alpha,\psi} f_i\right\|_{L_1([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{L_1([x_0,b],\psi)}(\psi(b)-\psi(x_0))^{\alpha-1}\right]\right]. \quad (55)$$

**Proof.** By (33), (51) and (53). ■
Next follows

**Proposition 15** *All as in Theorem 6 with $p,q>1: \frac{1}{p}+\frac{1}{q}=1$, $\alpha \geq 1$. Then*

$$|\Phi(f_1,...,f_r)(x_0)| \leq \frac{1}{\Gamma(\alpha)\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^C D_{x_0-}^{\alpha,\psi} f_i\right\|_{L_q([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[a,x_0]}(\psi(x_0)-\psi(a))^{\alpha+\frac{1}{p}}\right]\right.$$

$$\left.+\left[\left\|{}^C D_{x_0+}^{\alpha,\psi} f_i\right\|_{L_q([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[x_0,b]}(\psi(b)-\psi(x_0))^{\alpha+\frac{1}{p}}\right]\right]. \quad (56)$$

**Proof.** By (36), (47) and (49). ■
We continue with

**Proposition 16** *All as in Theorem 10. Then*

$$\left|\Delta^{\psi}(f_1,...,f_r)\right| \leq \frac{(\psi(b)-\psi(a))^{\alpha+2}}{\Gamma(\alpha+2)}$$

$$\left\{\sum_{i=1}^{r}\left[\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0-}^{\alpha,\psi} f_i\right\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0+}^{\alpha,\psi} f_i\right\|_{\infty,[x_0,b]}\right]\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[a,b]}\right\}. \quad (57)$$

**Proof.** By (42), (47) and (49). ■

Next we make some applications of our main results.

We need

**Remark 17** *We have that* ($r = 2$)

$$\Phi\left(f_1, f_2\right)\left(x_0\right) = 2\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right) - \tag{58}$$

$$f_1\left(x_0\right)\int_a^b f_2\left(x\right) d\psi\left(x\right) - f_2\left(x_0\right)\int_a^b f_1\left(x\right) d\psi\left(x\right),$$

*and* ($r = 3$)

$$\Phi\left(f_1, f_2, f_3\right)\left(x_0\right) = 3\int_a^b f_1\left(x\right) f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right) - f_1\left(x_0\right)\int_a^b f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right) \tag{59}$$

$$-f_2\left(x_0\right)\int_a^b f_1\left(x\right) f_3\left(x\right) d\psi\left(x\right) - f_3\left(x_0\right)\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right),$$

*etc.*

*Furthermore we derive* ($r = 2$)

$$\Delta^{\psi}\left(f_1, f_2\right) = 2\left[\left(\psi\left(b\right) - \psi\left(a\right)\right)\left(\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right)\right) - \right.$$

$$\left.\left(\int_a^b f_1\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_2\left(x\right) d\psi\left(x\right)\right)\right], \tag{60}$$

*and* ($r = 3$)

$$\Delta^{\psi}\left(f_1, f_2, f_3\right) = 3\left(\psi\left(b\right) - \psi\left(a\right)\right)\left(\int_a^b f_1\left(x\right) f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_1\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_2\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_1\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_3\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right)\right), \tag{61}$$

*etc.*

We give the special cases of fractional Ostrowski type inequalities.

**Proposition 18** *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f_1, f_2 \in C^n([a,b])$. Let $x_0 \in [a,b]$ and assume that $f_{1e^x}^{[k]}(x_0) = f_{2e^x}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$. Then*

$$\left| 2\int_a^b f_1(x) f_2(x) e^x dx - f_1(x_0) \int_a^b f_2(x) e^x dx - f_2(x_0) \int_a^b f_1(x) e^x dx \right| \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \sum_{i=1}^{2} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,e^x} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{2} f_j \right\|_{\infty,[a,x_0]} \left(e^{x_0} - e^a\right)^{\alpha+1} \right] \right.$$

$$\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,e^x} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{2} f_j \right\|_{\infty,[x_0,b]} \left(e^b - e^{x_0}\right)^{\alpha+1} \right] \right]. \quad (62)$$

*If $0 < \alpha \leq 1$, then (62) is valid without any initial conditions.*

**Proof.** Case of $\psi(x) = e^x$, apply Proposition 13 for $r = 2$. ■

We continue with

**Proposition 19** *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a,b] \subset (0,+\infty)$ and $f_1, f_2, f_3 \in C^n([a,b])$. Let $x_0 \in [a,b]$ and assume that $f_{i \ln x}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$; $i = 1,2,3$. Then*

$$\left| 3\int_a^b \frac{f_1(x) f_2(x) f_3(x)}{x} dx - f_1(x_0) \int_a^b \frac{f_2(x) f_3(x)}{x} dx - \right.$$

$$\left. f_2(x_0) \int_a^b \frac{f_1(x) f_3(x)}{x} dx - f_3(x_0) \int_a^b \frac{f_1(x) f_2(x)}{x} dx \right| \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \sum_{i=1}^{3} \left[ \left[ \left\| {}^C D_{x_0-}^{\alpha,\ln x} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{3} f_j \right\|_{\infty,[a,x_0]} \left(\ln\left(\frac{x_0}{a}\right)\right)^{\alpha+1} \right] \right.$$

$$\left. + \left[ \left\| {}^C D_{x_0+}^{\alpha,\ln x} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{3} f_j \right\|_{\infty,[x_0,b]} \left(\ln\left(\frac{b}{x_0}\right)\right)^{\alpha+1} \right] \right]. \quad (63)$$

*If $0 < \alpha \leq 1$, then (63) is valid without any initial conditions.*

**Proof.** Case of $\psi(x) = \ln x$, apply Proposition 13 for $r = 3$. ■

Next we present the special cases of fractional Grüss type inequality:

**Proposition 20** *Let $0<\alpha\leq 1$, $[a,b]\subset\mathbb{R}$ and $f_1,f_2\in C^1([a,b])$. Assume that* $\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0-}^{\alpha,e^x} f_i\right\|_{\infty,[a,x_0]}<\infty$, *and* $\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0+}^{\alpha,e^x} f_i\right\|_{\infty,[x_0,b]}<\infty$, $i=1,2$.

*Then*

$$2\left|\left(e^b-e^a\right)\left(\int_a^b f_1(x) f_2(x) e^x dx\right)-\left(\int_a^b f_1(x) e^x dx\right)\left(\int_a^b f_2(x) e^x dx\right)\right|\leq$$
$$\frac{\left(e^b-e^a\right)^{\alpha+2}}{\Gamma(\alpha+2)}$$
$$\left\{\sum_{i=1}^2\left[\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0-}^{\alpha,e^x} f_i\right\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0+}^{\alpha,e^x} f_i\right\|_{\infty,[x_0,b]}\right]\left\|\prod_{\substack{j=1\\j\neq i}}^2 f_j\right\|_{\infty,[a,b]}\right\}. \tag{64}$$

**Proof.** Apply Proposition 16, for $\psi(x)=e^x$, $r=2$. ■

We finish with another $\psi$-fractional Grüss type inequality:

**Proposition 21** *Let $0<\alpha\leq 1$, $[a,b]\subset(0,\infty)$ and $f_i\in C^1([a,b])$, $i=1,2,3$. Assume that* $\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0-}^{\alpha,\ln x} f_i\right\|_{\infty,[a,x_0]}<\infty$, $\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0+}^{\alpha,\ln x} f_i\right\|_{\infty,[x_0,b]}<\infty$, $i=1,2,3$. *Then*

$$\left|\ln\left(\frac{b}{a}\right)^3\left(\int_a^b \frac{f_1(x) f_2(x) f_3(x)}{x} dx\right)-\left(\int_a^b \frac{f_1(x)}{x} dx\right)\left(\int_a^b \frac{f_2(x) f_3(x)}{x} dx\right)-\right.$$
$$\left.\left(\int_a^b \frac{f_2(x)}{x} dx\right)\left(\int_a^b \frac{f_1(x) f_3(x)}{x} dx\right)-\left(\int_a^b \frac{f_3(x)}{x} dx\right)\left(\int_a^b \frac{f_1(x) f_2(x)}{x} dx\right)\right| \tag{65}$$
$$\leq\frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha+2}}{\Gamma(\alpha+2)}\left\{\sum_{i=1}^3\left[\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0-}^{\alpha,\ln x} f_i\right\|_{\infty,[a,x_0]}+\right.\right.$$
$$\left.\left.\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0+}^{\alpha,\ln x} f_i\right\|_{\infty,[x_0,b]}\right]\left\|\prod_{\substack{j=1\\j\neq 1}}^3 f_j\right\|_{\infty,[a,b]}\right\}.$$

**Proof.** By Proposition 16, for $\psi(x)=\ln x$, $r=3$. ■

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