On solutions of semilinear second-order impulsive functional differential equations

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Abstract

This paper deals with the regularity for solutions of second-order semilinear impulsive differential equations contained the nonlinear convolution with cosine families, and obtain a variation of constant formula for solutions of the given equations.

*Keywords:*semilinear second-order equations, regularity for solutions, cosine family, sine family

AMS Classification Primary 35F25; Secondary 35K55

1 Introduction

In this paper we are concerned with the regularity of the following second-order semilinear impulsive differential system

$$\begin{cases} w^{''}(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), \quad 0 < t \leq T, \\ w(0) = x_0, \quad w^{'}(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1,2,...,m \end{cases} \tag{1.1}$$

in a Banach space X. Here k belongs to $L^2(0,T)$ and $g : [0,T] \times D(A) \to X$ is a nonlinear mapping such that $w \mapsto g(t,w)$ satisfies Lipschitz continuous. In (1.1),

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the principal operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$. The impulsive condition

$$\Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m$$

is combination of traditional evolution systems whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc.

In recent years the theory of impulsive differential systems has been emerging as an important area of investigation in applied sciences. The reason is that it is richer than the corresponding theory of classical differential equations and it is more adequate to represent some processes arising in various disciplines. The theory of impulsive systems provides a general framework for mathematical modeling of many real world phenomena(see [1, 2] and references therein). The theory of impulsive differential equations has seen considerable development. Impulsive differential systems have been studied in [3, 4, 5, 6], second-order impulsive integrodifferential systems in [7, 8], and Stochastic differential systems with impulsive conditions in [9, 10, 11].

In this paper, we allow implicit arguments about L^2-regularity results for semilinear hyperbolic equations with impulsive condition. These consequences are obtained by showing that results of the linear cases [12, 13] and semilinear case [14] on the L^2-regularity remain valid under the above formulation of (1.1). Earlier works prove existence of solution by using Azera Ascoli theorem. But we propose a different approach from that of earlier works to study mild, strong and classical solutions of Cauchy problems by using the properties of the linear equation in the hereditary part.

This paper is organized as follows. In Section 2, we give some definition, notation and the regularity for the corresponding linear equations. In Section 3, by using properties of the strict solutions of linear equations in dealt in Section 2, we will obtain the L^2-regularity of solutions of (1.1), and a variation of constant formula of solutions of (1.1). Finally, we also give an example to illustrate the applications of the abstract results..

2 Preliminaries

In this section, we give some definitions, notations, hypotheses and Lemmas. Let X be a Banach space with norm denoted by $||\cdot||$.

Definition 2.1. *[15] A one parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in X is called a strongly continuous cosine family if* *c(1)* $C(s+t)+C(s-t)=2C(s)C(t), \quad \textit{for all } s,\ t \in \mathbb{R},$

c(2) $C(0)=I,$

c(3) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$.

If $C(t),\ t \in \mathbb{R}$ is a strongly continuous cosine family in X , then $S(t),\ t \in \mathbb{R}$ is the one parameter family of operators in X defined by

$$S(t)x=\int_0^t C(s)xds,\ \ x \in X,\ \ t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in \mathbb{R}$ is the operator $A : X \to X$ defined by

$$Ax=\frac{d^2}{dt^2}C(0)x.$$

We endow with the domain $D(A)=\{x \in X : C(t)x$ is a twice continuously differentiable function of $t\}$ with norm

$$||x||_{D(A)}=||x||+\sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\}+||Ax||.$$

We shall also make use of the set

$$E=\{x \in X : C(t)x \ \text{ is a once continuously differentiable function of } t\}$$

with norm

$$||x||_E=||x||+\sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\}.$$

It is not difficult to show that $D(A)$ and E with given norms are Banach spaces.

The following Lemma is from Proposition 2.1 and Proposition 2.2 of [1].

Lemma 2.1. *Let* $C(t)(t \in \mathbb{R})$ *be a strongly continuous cosine family in* X*. The following are true :*

c(4) $C(t)=C(-t)$ *for all* $t \in \mathbb{R}$,

c(5) $C(s), S(s), C(t)$ *and* $S(t)$ *commute for all* $s,t \in \mathbb{R}$,

c(6) $S(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$, *c(7)* *there exist constants* $K \geq 1$ *and* $\omega \geq 0$ *such that*

$$||C(t)|| \leq Ke^{\omega|t|} \text{ for all } t \in \mathbb{R},$$
$$||S(t_1) - S(t_2)|| \leq K\Big|\int_{t_2}^{t_1} e^{\omega|s|}ds\Big| \text{ for all } t_1, t_2 \in \mathbb{R},$$

c(8) *if* $x \in E$*, then* $S(t)x \in D(A)$ *and*

$$\frac{d}{dt}C(t)x = AS(t)x = S(t)Ax = \frac{d^2}{dt^2}S(t)x,$$

c(9) *if* $x \in D(A)$*, then* $C(t)x \in D(A)$ *and*

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax,$$

c(10) *if* $x \in X$ *and* $r, s \in \mathbb{R}$*, then*

$$\int_r^s S(\tau)xd\tau \in D(A) \quad \text{and} \quad A(\int_r^s S(\tau)xd\tau) = C(s)x - C(r)x,$$

c(11) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$*,*

c(12) $S(s+t) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in \mathbb{R}$*,*

c(13) $C(s+t) = C(t)C(s) - S(t)S(s)$ *for all* $s, t \in \mathbb{R}$*,*

c(14) $C(s+t) - C(t-s) = 2AS(t)S(s)$ *for all* $s, t \in \mathbb{R}$*.*

The following Lemma is from Proposition 2.4 of [15].

Lemma 2.2. *Let* $C(t)(t \in \mathbb{R})$ *be a strongly continuous cosine family in* X *with infinitesimal generator* A*. If* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A)$*,* $y_0 \in E$*, and*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \;\; t \in \mathbb{R},$$

then $w(t) \in D(A)$ *for* $t \in \mathbb{R}$*,* w *is twice continuously differentiable, and* w *satisfies*

$$w''(t) = Aw(t) + f(t), \;\; t \in R, \;\; w(0) = x_0, \;\; w'(0) = y_0. \tag{2.2}$$

Conversely, if $f : \mathbb{R} \to X$ *is continuous,* $w(t) : \mathbb{R} \to X$ *is twice continuously differentiable,* $w(t) \in D(A)$ *for* $t \in \mathbb{R}$*, and* w *satisfies* (2.2)*, then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \;\; t \in \mathbb{R}.$$

Proposition 2.1. *Let $f : \mathbb{R} \to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$. Then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R}$$

is a solution of (2.2) belonging to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$. Moreover, we have that there exists a positive constant C_1 such that for any $T > 0$,

$$||w||_{L^2(0,T;D(A))} \le C_1(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{2.3}$$

3 Nonlinear equations

This section is to investigate the regularity of solutions of a second-order nonlinear impulsive differential system

$$\begin{cases} w''(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w'(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1,2,...,m \end{cases} \tag{3.1}$$

in a Banach space X.

Assumption (G) Let $g : [0,T] \times D(A) \to X$ be a nonlinear mapping such that $t \mapsto g(t,w)$ is measurable and

(g1) $||g(t,w_1) - g(t,w_2)||_{D(A)} \le L||w_1 - w_2||$,

for a positive constant L.

Assumption (I) Let $I_k^1 : D(A) \to X$, $I_k^2 : E \to X$ be continuous and there exist positive constants $L(I_k^1)$, $L(I_k^2)$ such that

(i1) $||I_k^1(w_1) - I_k^1(w_2)|| \le L(I_k^1)||w_1 - w_2||_{D(A)}$, for each $w_1, w_2 \in D(A)$
$||I_k^1(w)|| \le L(I_k^1)$, for $w \in D(A)$

(i2) $||I_k^2(w_1') - I_k^2(w_2')|| \le L(I_k^2)||w_1' - w_2'||_E$, for each $w_1', w_2' \in E$
$||I_k^2(w')|| \le L(I_k^2)||$, for $w' \in E$.

For $w \in L^2(0,T : D(A))$, we set

$$F(t,w) = \int_0^t k(t-s)g(s,w(s))ds$$

where k belongs to $L^2(0,T)$. Then we will seek a mild solution of (3.1), that is, a solution of the integral equation

$$\begin{aligned} w(t) =& C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s,w)+f(s)\}ds \\ &+ \sum_{0<t_k<t} C(t-t_k)I_k^1(w(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w'(t_k^+)), \ \ t\in\mathbb{R}. \end{aligned} \tag{3.2}$$

Remark 3.1. *If $g:[0,T]\times X\to X$ is a nonlinear mapping satisfying*

$$||g(t,w_1)-g(t,w_2)|| \le L||w_1-w_2||$$

for a positive constant L, then our results can be obtained immediately.

Lemma 3.1. *Let $w\in L^2(0,T;D(A))$, $T>0$. Then $F(\cdot,w)\in L^2(0,T;X)$ and*

$$||F(\cdot,w)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w||_{L^2(0,T;D(A))}.$$

Moreover if $w_1,\ w_2\in L^2(0,T;D(A))$, then

$$||F(\cdot,w_1)-F(\cdot,w_2)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w_1-w_2||_{L^2(0,T;D(A))}.$$

Lemma 3.2. *If $k\in W^{1,2}(0,T)$, $T>0$, then*

$$\begin{aligned} A\int_0^t S(t-s)F(s,w)ds &= -F(t,w) \\ &+ \int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\, ds \\ &+ \int_0^t (C(t-s)-I)k(0)g(s,w(s))ds. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Suppose that the Assumptions (G) and Assumption (I) are satisfied. If $f:\mathbb{R}\to X$ is continuously differentiable, $x_0\in D(A)$, $y_0\in E$, and $k\in W^{1,2}(0,T)$, $T>0$, then there exists a time $T\ge T_0>0$ such that the functional differential equation (3.1) admits a unique solution w in $L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)$.*

Proof. Let us fix $T_0 > 0$ so that

$$\begin{aligned} C_2 \equiv &\omega^{-1} K L T_0^{3/2} (e^{\omega T_0} - 1) ||k||_{L^2(0,T_0)} \\ &+ \{\omega^{-1} K (e^{\omega T_0} - 1) + 1\} T_0^{3/2} / \sqrt{3} L ||K e^{\omega T_0} + 1|| ||k||_{W^{1,2}(0,T_0)} \\ &+ \{\omega^{-1} K (e^{\omega T_0} - 1) + 1\} T_0 / \sqrt{2} L ||K e^{\omega T_0} + 1|| ||k(0)|| \\ &+ \{w^{-1} K (e^{w T_0} - 1) + 2\} \sum_{0<t_k<t} L(I_k^1) K e^{w T_0} \\ &+ \{2 w^{-1} K (e^{w T_0} - 1) + 1\} \sum_{0<t_k<t} L(I_k^2) < 1 \end{aligned} \tag{3.4}$$

where $K,\ L,\ L(I_k^1)$ and $L(I_k^2)$ are constants in c(7), (g1) and Assumption (I) respectively. Invoking Proposition 2.1, for any $v \in L^2(0, T_0; D(A))$ we obtain the equation

$$\begin{cases} w''(t) = Aw(t) + F(t, v) + f(t), \quad 0 < t \le T_0, \\ w(0) = x_0, \quad w'(0) = y_0 \\ \Delta w(t_k) = I_k^1(v(t_k)), \quad \Delta w'(t_k) = I_k^2(v'(t_k^+)), \quad k = 1, 2, ..., m \end{cases} \tag{3.5}$$

has a unique solution $w \in L^2(0, T_0; D(A)) \cap W^{1,2}(0, T_0; E)$. Let $w_1,\ w_2$ be the solutions of (3.5) with v replaced by $v_1,\ v_2 \in L^2(0, T_0; D(A))$, respectively. Put

$$\begin{aligned} J(w)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s, v) + f(s)\} ds \\ + \sum_{0<t_k<t} C(t - t_k) I_k^1(v(t_k)) + \sum_{0<t_k<t} S(t - t_k) I_k^2(v'(t_k^+)). \end{aligned}$$

Then

$$\begin{aligned} J(w_1)(t) - J(w_2)(t) = &\int_0^t S(t-s)\{F(s, v_1) - F(s, v_2)\} ds \\ &+ \sum_{0<t_k<t} C(t - t_k)\{I_k^1(v_1(t_k)) - I_k^1(v_2(t_k))\} \\ &+ \sum_{0<t_k<t} S(t - t_k)\{I_k^2(v_1'(t_k^+)) - I_k^1(v_2'(t_k^+))\}, \\ = &I_1 + I_2 + I_3. \end{aligned}$$

So, from Lemmas 3.1, 3.2, it follows that for $0 \le t \le T_0$,

$$\begin{aligned} &|| \int_0^t S(t-s)\{F(s, v_1) - F(s, v_2)\} ds || \\ &\le \omega^{-1} K L T_0 (e^{\omega T_0} - 1) ||k||_{L^2(0,T_0)} ||v_1 - v_2||_{L^2(0,T_0;D(A))}, \end{aligned}$$

$$
\begin{aligned}
&||\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||\\
&\le ||AS(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||\\
&= ||S(t)A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||,
\end{aligned}
$$

and

$$
\begin{aligned}
&||A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||\\
&\le ||\int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)(g(\tau,v_1(\tau))-g(\tau,v_2(\tau)))d\tau\, ds||\\
&\quad + ||\int_0^t (C(t-s)-I)k(0)(g(s,v_1(s))-g(s,v_2(s)))ds||\\
&\le tL||Ke^{\omega t}+1||||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&\quad + \sqrt{t}L||Ke^{\omega t}+1||||k(0)||||v_1-v_2||_{L^2(0,T_0;D(A))}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&||I_1||_{L^2(0,T_0;D(A))} \le \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \qquad (3.6)\\
&+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||_{L^2(0,T_0;D(A))}.
\end{aligned}
$$

By Assumption (i1), we obtain

$$
||\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1(t_k^-))-I_k^1(v_2(t_k^-))\}|| \le \sum_{0<t_k<T_0} Ke^{wT_0}L(I_k^1)||v_1-v_2||_{D(A)},
$$

$$
\begin{aligned}
&||\frac{d}{dt}C(t)\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||\\
&\le ||AS(t)\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||\\
&= ||S(t)A\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||,
\end{aligned}
$$

and

$$\begin{aligned}||A\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1(t_k^-))-I_k^1(v_2(t_k^-))\}|| &= ||\sum_{0<t_k<t}C(t-t_k)A\{I_k^1(v_1)-I_k^1(v_2)\}||\\ &\le \sum_{0<t_k<t}Ke^{wt}||I_k^1(v_1)-I_k^1(v_2)||_{D(A)}\\ &\le \sum_{0<t_k<t}Ke^{wt}L(I_k^1)||v_1-v_2||_{D(A)}.\end{aligned}$$

Therefore, we have

$$||I_2||_{L^2(0,T_0;D(A))}\le\{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t}L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))}. \tag{3.7}$$

We also obtain from Assumption (i2),

$$||\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1'(t_k^+))-I_k^1(v_2'(t_k^+))\}||\le\sum_{0<t_k<T_0}Kw^{-1}(e^{wT_0}-1)L(I_k^2)||v_1-v_2||_{D(A)},$$

$$\begin{aligned}&||\frac{d}{dt}C(t)\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||\\ &\le ||AS(t)\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||\\ &= ||S(t)A\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||,\end{aligned}$$

and

$$\begin{aligned}||A\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1'(t_k^+))-I_k^1(v_2'(t_k^+))\}|| &= ||\sum_{0<t_k<t}\frac{d}{dt}C(t)\{I_k^2(v_1')-I_k^1(v_2')\}||\\ &\le \sum_{0<t_k<t}||I_k^2(v_1')-I_k^1(v_2')||_E\\ &\le \sum_{0<t_k<t}L(I_k^2)||v_1'-v_2'||_E.\end{aligned}$$

Therefore, we have

$$||I_3||_{L^2(0,T_0;D(A))} \le \{w^{-1}K(e^{wT_0}-1)+2\} \sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))}. \tag{3.8}$$

Thus, from (3.6),(3.7), and (3.8), we conclude that

$$\begin{aligned} &||J(w_1)-J(w_2)||_{L^2(0,T_0;D(A))} \\ &\le \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}L||k||_{L^2(0,T_0)}\sqrt{T_0}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{2w^{-1}K(e^{wT_0}-1)+1\}\sum_{0<t_k<t} L(I_k^2)||v_1-v_2||_{W^{1,2}(0,T_0;D(A))}. \end{aligned} \tag{3.9}$$

Moreover, it is easily seen that

$$||J(w_1)-J(w_2)||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \le C_2||v_1-v_2||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}.$$

So by virtue of the condition (3.4) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $[0,T_0]$. □

Theorem 3.2. *Suppose that the Assumptions (G) and (I) are satisfied. If $f:\mathbb{R}\to X$ is continuously differentiable, $x_0\in D(A), y_0\in E$, and $k\in W^{1,2}(0,T)$, $T>0$, then the solution w of (3.1) exists and is unique in $L^2(0,T;D(A))\cap W^{1,2}(0,T;E)$, and there exists a constant C_3 depending on T such that*

$$||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \le C_3(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T;X)}). \tag{3.10}$$

Proof. Let $w(\cdot)$ be the solution of (3.1) in the interval $[0,T_0]$ where T_0 is a constant in (3.4) and $v(\cdot)$ be the solution of the following equation

$$\begin{aligned} v''(t) &= Av(t)+f(t),\ \ 0<t, \\ v(0) &= x_0,\ \ v'(0)=y_0. \end{aligned}$$

Then

$$(w-v)(t) = \int_0^t S(t-s)F(s,w)ds + \sum_{0<t_k<t} C(t-t_k)I_k^1(w(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w'(t_k^+)),$$

and in view of (3.9)

$$||w-v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \le C_2||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}, \tag{3.11}$$

that is, combining (3.11) with Proposition 2.1 we have

$$\begin{aligned} ||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} &\le \frac{1}{1-C_2}||v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \\ &\le \frac{C_1}{1-C_2}(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T_0;X)}). \end{aligned} \tag{3.12}$$

Now from

$$\begin{aligned} &A\int_0^{T_0} S(T_0-s)\{F(s,w)+f(s)\}ds \\ &= C(T_0)f(0)-f(T_0)+\int_0^{T_0}(C(T_0-s)-I)f^{'}(s)ds \\ &- F(T_0,w)+\int_0^{T_0}(C(T_0-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\, ds \\ &+\int_0^{T_0}(C(T_0-s)-I)k(0)g(s,w(s))ds, \end{aligned}$$

$$||A\sum_{0<t_k<t} C(t-t_k)I_k^1(w_1)|| \le Kw^{-1}(e^{wT_0-1})Ke^{wT_0}\sum_{0<t_k<t} L(I_k^1)||w(t_k)||_{D(A)},$$

$$||\sum_{0<t_k<t} S(t-t_k)I_k^2(v_1')|| \le \sum_{0<t_k<t} L(I_k^2)||w'(t_k^+)||_E,$$

and since

$$\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,w)+f(s)\}ds = S(t)A\int_0^t S(t-s)\{F(s,w)+f(s)\}ds,$$

$$\frac{d}{dt}C(t)\sum_{0<t_k<t} C(t-t_k)I_k^1(w) \le S(t)A\sum_{0<t_k<t} C(t-t_k)I_k^1(w).$$

$$\frac{d}{dt}C(t)\sum_{0<t_k<t} S(t-t_k)I_k^2(w') \le S(t)A\sum_{0<t_k<t} S(t-t_k)I_k^2(w').$$

We have

$$\begin{aligned}||w(T_0)||_{D(A)} = ||C(T_0)x_0 + S(T_0)y_0 + \int_0^{T_0} S(T_0-s)\{F(s,w)+f(s)\}ds \\ + \sum_{0<t_k<t} C(t-t_k)I_k^1(w) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w')||_{D(A)}\end{aligned}$$

$$\begin{aligned}&\le (\omega^{-1}K(e^{\omega T_0}-1)+1)\{Ke^{\omega T_0}||x_0||_{D(A)} + ||y_0||_E + T_0L||k||_{L^2(0,T_0)}||w||_{L^2(0,T_0;D(A))} \\ &+ ||Ke^{\omega T_0}f(0)|| + ||f(0)|| + ||K(e^{\omega T_0}+1)\sqrt{T_0}||f||_{W^{1,2}(0,T;X)} \\ &+ tL||Ke^{\omega t}+1||\,||k||_{W^{1,2}(0,T_0)}||w||_{L^2(0,T_0;D(A))} \\ &+ \sqrt{t}L||Ke^{\omega t}+1||\,||k(0)||||w||_{L^2(0,T_0;D(A))}\} \\ &+ \{2+Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t} Ke^{wT_0}L(I_k^1) \\ &+ \{1+2Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t} L(I_k^2).\end{aligned}$$

Hence, from (3.12), there exists a positive constant $C>0$ such that

$$||w(T_0)||_{D(A)} \le C(1+||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T_0;X)}).$$

Since the condition (3.4) is independent of initial values, the solution of (3.1) can be extended to the interval $[0, nT_0]$ for every natural number n. An analogous estimate to (3.12) holds for the solution in $[0, nT_0]$, and hence for the initial value $(w(nT_0), w^{'}(nT_0)) \in D(A)\times E$ in the interval $[nT_0, (n+1)T_0]$. $\square$

Example. We consider the following partial differential equation

$$\begin{cases} w^{''}(t,x) = Aw(t,x) + F(t,w) + f(t), \quad 0<t, \quad 0<x<\pi, \\ w(t,0) = w(t,\pi) = 0, \quad t\in\mathbb{R} \\ w(0,x) = x_0(x), \quad w^{'}(0,x) = y_0(x), \quad 0<x<\pi \\ \Delta w(t_k,x) = I_k^1(w(t_k)) = (\gamma_k||w''(t_k,x)|| + t_k), \quad 1\le k\le m, \\ \Delta w'(t_k,x) = I_k^2(w'(t_k)) = \delta_k||w'(t_k,x)||, \end{cases} \tag{E}$$

where constants γ_k and $\delta_k (k=1,\cdots,m)$ are small.

Let $X = L^2([0,\pi];\mathbb{R})$, and let $e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$. Then $\{e_n : n = 1, \cdots\}$ is an orthonormal base for X. Let $A : X \to X$ be defined by

$$Aw(x) = w''(x),$$

where $D(A) = \{w \in X : w,\ w^{'}$ are absolutely continuous, $w^{''} \in X,\ w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2(w, e_n)e_n, \quad w \in D(A),$$

and A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt (w, e_n)e_n, \quad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(w, e_n)e_n, \quad w \in X.$$

Let $g_1(t, x, w, p), \quad p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t, \delta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \le \delta$ such that

(f1) $g_1(t, x, 0, 0) = 0$,

(f2) $|g_1(t, x, w, p) - g_1(t, x, w, q)| \le \rho(t, |w|)|p - q|$,

(f3) $|g_1(t, x, w_1, p) - g_1(t, x, w_2, p)| \le \rho(t, |w_1| + |w_2|)|w_1 - w_2|$.

Let

$$g(t, w)x = g_1(t, x, w, Dw, D^2 w).$$

Then noting that

$$\begin{aligned} ||g(t, w_1) - g(t, w_2)||_{0,2}^2 \le\ & 2\int_\Omega |g_1(t, x, w_1, p) - g_1(t, x, w_2, q)|^2 dx \\ & + 2\int_\Omega |g_1(t, x, w_1, q) - g_1(t, x, w_2, q)|^2 dx \end{aligned}$$

where $p = (Dw_1, D^2 w_1)$ and $q = (Dw_2, D^2 w_2)$, it follows from (f1), (f2) and (f3) that

$$||g(t, w_1) - g(t, w_2)||_{0,2}^2 \le L(||w_1||_{D(A)}, ||w_2||_{D(A)})||w_1 - w_2||_{D(A)}$$

where $L(||w_1||_{D(A)}, ||w_2||_{D(A)})$ is a constant depending on $||w_1||_{D(A)}$ and $||w_2||_{D(A)}$. We set

$$F(t, w) = \int_0^t k(t-s)g(s, w(s))ds$$

where k belongs to $L^2(0, T)$. Then, from the results in section 3, the solution w of (E) exists and is unique in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$, and there exists a constant C_3 depending on T such that

$$||w||_{L^2(0,T;D(A))} \leq C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}).$$

References

[1] V.Lakshmikantham, D.D.Bainov and P.S.Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[2] A.M.Samoilenko and N.A.Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[3] A.Anguraj and M.Mallika Arjunan, *Existence results for an impulsive neutral integro-differential equations in Banach spaces*, Nonlinear Studies, 16(1) (2009), 33-48.

[4] A.Anguraj and K.Karthikeyan, *Existence of solutions for impulsive neutral functional differential equations with nonlocal conditions*, , Nonlinear Anal. 70(7) (2009), 2717-2721.

[5] B.Radhakrishnan and K.Balachandran, *Controllability of impulsive neutral functional evolution integrodifferential systems with infinite delay*, Nonlinear Anal. Hybrid Syst. 5(2011), 655-670.

[6] L.Zhu, Q.Dong and G.Li, *Impulsive differential equations with nonlocal condition in general Banach spaces*, Adances in Difference Eqautions, (10)(2012), 1-21.

[7] G.Arthi and K.K.Balachandran, *Controllability of damped second-order neutral functional differential systems with impulses*, Taiwanese J.Mathe. 16 (2012), 89-106.

[8] B.Radhakrishnan and K.Balachandran, *Controllability results for second order neutral impulsive integrodifferential systems*, J.Optim.Theory Appl. 151 (2011), 589-612.

[9] A.Anguraj and A.Vinodkumar, *Existence, uniqueness and stability of impulsive Stochastic partial neutral functional differential equations with infinite delays*, J.Appl.Math.andInformatics 28 (2010), 3-4.

[10] K.Balachandran and R.Sathaya, *Controllability of nonlocal Stochastic quasilinear integrodifferential systems*, Electron J.Qual.Theory differ. 50 (2011), 1-16.

[11] L.Hu and Y.Ren, *Existence for impulsive Stochastic functional integrodifferential equations*, Acta Appl.Math. 111 (2010), 303-317.

[12] M.Ikawa, *Mixed problems for hyperbolic equations of second order*, J.Math.Soc.Japan, 20(4) (1968), 580-608.

[13] V.Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press Limited, 1993.

[14] J.M.Jeong and H.J.Hwang, *Regularity for solutions of second order semilinear equations with cosine families*, Sylwan Journal, 158(9) (2014), 22-35.

[15] C.C.Travis and G.F.Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta.Math. 32 (1978), 75-96.

[16] R.C.Maccamy, *A model for one-dimensional, nonlinear viscoelasticity*, Quart.Appl.Math. 35 (1997), 21-33.

[17] R.C.Maccamy, *An integro-differntial equation with applications in heat flow*, Ibid. 35 (1977), 1-19.

[18] J.M.Jeong, Y.C.Kwun and J.Y.Park, *Regularity for solutions of nonlinear evolution equations with nonlinear perturbations,* , Comput.Math.Appl. 43 (2002), 1231-1237.

[19] J.M.Jeong, Y.C.Kwun and J.Y.Park, *Approximate controllability for semilinear retarded differential equations*, J.Dynam.Control Systems, 15(3) (1998), 329-346.

[20] C.C.Travis and G.F.Webb, *An abstract second order semilinear Volterra integrodiferential equation*, SIAM J.Math.Anal. 10(2) (1979), 412-423.