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MIZOGUCHI- TAKAHASHI'S FIXED POINT THEOREM IN ν-GENERALIZED METRIC SPACES

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ABSTRACT. Our main work is to prove Mizogchi- Takahashi theorem in ν-generalized metric space in the sense of Brancairi. In the same setting we prove two more theorems which are generalizations of the main one.

1. INTRODUCTION

A metric is defined as a mapping $d : X \times X \to [0, \infty)$, for any non-empty set X which satisfying the following axioms, for any $x, y, z \in X$

(i) $d(x, y) = 0$ iff $x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

We said that the pair (X, d) is a metric space. The theory of metric spaces form a basic environment for a lot of concepts in mathematics such as the fixed point theorems which have an important rules in various branches of mathematical analysis. One of the famous result of fixed point theorems is Banach Contraction Principle which state that,

Theorem 1.1. [11]*(Banach Contraction Principle)*
Let (X, d) be a complete metric space. Let $T : X \to X$ be a self map on X such that

$$d(Tx, Ty) \leq rd(x, y),$$

hold for any $x, y \in X$, where $r \in [0, 1)$. Then T has a unique fixed point.

Many authors explored the importance of this theorem and extended it in different directions. For examples, we refer the reader to the following papers [2, 9, 8, 6], and the references therein. In (1969) Nadler extended theorem 1.1 for multi-valued mapping. Recall that the set of all non- empty, closed and bounded subsets of X is denoted by $CB(X)$ and let A, B be any sets in $CB(X)$. A Hausdorff metric is defined as

$$\mathcal{H}(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

Theorem 1.2. [12]*(Nadler's theorem) Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multi-valued map. Assume that*

$$\mathcal{H}(Tx, Ty) \leq rd(x, y),$$

holds for each $x, y \in X$ and $r \in [0, 1)$. Then T has a fixed point.

Key words and phrases. Mizoguchi-Takahashi's theorem, ν- generalized metric space,Generalized of MT- theorem, Fixed point theory.

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Many attempts have been done to generalize Nadler's theorem. One of these generalizations is Mizoguchi- Takahashi's theorem which stats that:

Theorem 1.3. [10] *Let (X,d) be a complete metric space. Let $T : X \to CB(X)$ be a multi-valued mapping. Assume that*

$$\mathcal{H}(Tx,Ty) \leq \beta(d(x,y))d(x,y),$$

hold for each $x,y \in X$, where $\beta : [0,\infty) \to [0,1)$ is a function such that $\limsup_{s\to t^+} \beta(s) < 1$. Then T has a fixed point.

Remark 1.4. *The function β in theorem 1.3, which satisfies $\limsup_{s\to t^+} \beta(s) < 1$ is called Mizoguchi- Takahashi function (MT- function for short).*

Starting with Mizoguchi and Takahashi's paper, many generalizations of their theorem have been established see [3, 13]. Recently, Eldred et al [4], claimed that Nadler's and Mizoguchi- Takahashi's theorems are equivalent. However, in [14], Suzuki proved that their claim is not true and he shown that Mizoguchi- Takahashi's theorem (1.3) is a real extension of Nadler's theorem. This is why we are interesting in such theorem.

In another direction, in (2000) Branciari created a new concept of generalized metric spaces by modifying the triangle inequality to involve more points.

Definition 1.5. [1] *Let X be a non- empty set and $d : X \times X \to [0,\infty)$. For $\nu \in \mathbb{N}$, a pair (X,d) is called a ν- generalized metric space if the following hold:*

(M1) $d(x,y) = 0$ *iff* $x = y$

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,y) \leq d(x,u_1) + d(u_1,u_2) + ... + d(u_\nu,y)$,
for any $x,u_1,u_2,...u_\nu,y \in X$, such that $x,u_1,u_2,...u_\nu,y$ are all different.

It is not difficult to show that the new space is not the same as the original one. Moreover, the new space is hard to deal with because it does not satisfy all topological properties that metric space has, see [15] for more details. Recently, in [16], Suzuki proved Nadler's theorem in ν- generalized metric spaces. The main work of this paper is to prove Mizoguchi -Takahashi's theorem in ν- generalized metric spaces. Firstly, we will list all the necessary definitions and some results that we will need. Then, we will be able to prove our main results.

2. Preliminary

Definition 2.1. *A point $x \in X$ is said to be a fixed point of multi-valued map T if $x \in Tx$.*

Definition 2.2. [1] *Let (X,d) be a ν- generalized metric space. A sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ is said to be Cauchy sequence if*

$$\lim_n \sup_{n>m} d(x_n,x_m) = 0$$

Definition 2.3. [16] *A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $(\sum,\neq)$- Cauchy sequence if all x_n's are different and*

$$\sum_{j=1}^{\infty} d(x_j,x_{j+1}) < \infty$$

Definition 2.4. [16] *Let (X,d) be a ν- generalized metric space. We said that, X is a $(\sum,\neq)$- complete if every $(\sum,\neq)$- Cauchy sequence converges.*

Lemma 2.5. [16, 5] *Let (X,d) be a ν- generalized metric space.*

- *Every converge $(\sum, \neq)$- Cauchy sequence is Cauchy.*
- *Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence converges to some $y \in X$ and $\{y_n\} \in X$ be a sequence such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then, $\{y_n\}$ also converges to y.*

Lemma 2.6. [14] *Let $\beta : [0,\infty) \to [0,1)$ is a MT-function. Then, for all $s \in [0,\infty)$, there exist $r_s \in [0,1)$ and $\varepsilon_s > 0$ such that $\beta(t) \leq r_s$ for all $t \in [s, s+\varepsilon_s)$*

Lemma 2.7. [12] *Let (X,d) be a metric space. For any $A, B \in CB(X)$ and $\varepsilon > 0$, there exist $a \in A$ and $b \in B$ such that $d(a,b) \leq \mathcal{H}(A,B) + \varepsilon$*

3. Main Result

In this section we prove Mizoguchi -Takahashi's theorem in ν -generalized metric spaces and some of its generalizations in the space.

Theorem 3.1. *Let (X,d) be a $(\sum, \neq)$ complete, ν- generalized metric space. and let T be a multi-valued map defined from X into $CB(X)$ satisfies the following:*

(i) If $\{y_n\} \in Tx$ and $\{y_n\}$ converges to y then $y \in Tx$.
(ii) For any $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq \alpha(d(x,y))d(x,y)$,

where α is MT-function. Then T has a fixed point.

Proof. Let define a function $\gamma : [0,\infty) \to [0,1)$ as $\gamma(t) = \dfrac{1+\alpha(t)}{2}$. It is not difficult to show that $\alpha(t) < \gamma(t)$, for any $t \in [0,\infty)$ and $\lim_{s\to t^+} \sup \gamma(s) < 1$. Moreover, for each $x, y \in X$ and $v \in Tx$, there exist $u \in Ty$ such that

$$d(v,u) \leq \gamma(d(x,y))d(x,y).$$

Putting $v = y$, we will get that

$$d(y,u) \leq \gamma(d(x,y))d(x,y)$$

Define $f(x) = \inf\{d(x,b) : b \in Tx\}$ and suppose that T does not have a fixed point (i.e., for all $x \in X, f(x) > 0$). Let $x_1 \in X$ be arbitrary and choose $x_2 \in Tx_1$ satisfying

$$d(x_1,x_2) < \frac{1}{\gamma(d(x_1,x_2))} f(x_1). \tag{1}$$

Since $Tx_2 \neq \emptyset$, we can choose an arbitrary element $x_3 \in Tx_2$ such that

$$f(x_2) \leq d(x_2,x_3) \leq \gamma(d(x_1,x_2))d(x_1,x_2). \tag{2}$$

Also, as in equation (1), we have

$$d(x_2,x_3) < \frac{1}{\gamma(d(x_2,x_3))} f(x_2). \tag{3}$$

From (2) and (3), we have

$$d(x_2,x_3) \leq \min\{\gamma(d(x_1,x_2))d(x_1,x_2), \frac{1}{\gamma(d(x_2,x_3))} f(x_2)\}.$$

Thus

$$\gamma(d(x_2,x_3))d(x_2,x_3) < f(x_2) \leq \gamma(d(x_1,,x_2))d(x_1,x_2) < f(x_1).$$

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Continuously, $\{x_n\}_{n\in\mathbb{N}} \in X$ is a sequence constructed such that $x_{n+1} \in Tx_n$ and satisfying

(4)
$$\gamma(d(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) < f(x_{n+1}) \leq \gamma(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < f(x_n),$$

and

$$d(x_{n+1}, x_{n+2}) \leq \gamma(d(x_{n+1}, x_n))d(x_{n+1}, x_n). \tag{5}$$

Since $\gamma(t) < 1$, we have $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$. Hence, from (4) and (5), the sequences $\{f(x_n)\}$ and $\{d(x_n, x_{n+1})\}$ are strictly decreasing. Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a $(\sum, \neq)$- Cauchy sequence in two steps:

Step 1 we show that all terms different. Suppose not i.e suppose $x_n = x_m$ for some $n > m$, where $m, n \in \mathbb{N}$. Hence

$$\begin{aligned} f(x_m) &= \inf\{d(x_m, b) : b \in Tx_m\} \\ &= \inf\{d(x_n, b) : b \in Tx_n\} \\ &= f(x_n), \end{aligned}$$

which contradicts $\{f(x_n)\}$ being strictly decreasing.

Step 2 We show that $\sum d(x_n, x_{n+1}) < \infty$. Since $\{d(x_n, x_{n+1})\}$ is an decreasing sequence in R and bounded below, it converges to some positive real number (say δ). Also, we have $\lim_{s\to t^+} \sup \gamma(s) < 1$, thus, there exist $r \in [0, 1)$ and $\varepsilon > 0$ such that $\gamma(s) \leq r$ for all $s \in [\delta, \delta + \varepsilon)$. For any $n \in \mathbb{N}$, we can choose $\mu \in \mathbb{N}$ satisfying $\delta \leq d(x_n, x_{n+1}) \leq \delta + \varepsilon$ with $n \geq \mu$. So,

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=\mu+1}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n d(x_\mu, x_{\mu+1}) \\ &< \infty. \end{aligned}$$

Thus $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence in $(\sum, \neq)$ complete ν- generalize metric space. Then, it is converge to some $z \in X$ and by lemma (2.5), $\{x_n\}$ is a Cauchy sequence. From our assumption we choose $\{u_n\} \in Tz$ satisfy

$$d(x_{n+1}, u_n) \leq \mathcal{H}(Tx_n, Tz) \leq \gamma(d(x_n, z))d(x_n, z),$$

for any $n \in \mathbb{N}$. But $\{x_n\}$ converges to z, so $d(x_{n+1}, u_n) \to 0$ as $n \to \infty$. Thus we have $x_{n+1} \to z$ and $x_{n+1} \to u_n$. Therefore, by lemma(2.5) $d(u_n, z) = 0$ as $n \to \infty$. So $d(Tz, z) = 0$ implies $f(z) = 0$ which is a contradiction. Therefore, there exist $z \in X$ such that $f(z) = 0$ and hence $z \in Tz$ is a fixed point. □

Definition 3.2. [7] *A multi- valued map T from X into $CB(X)$ is called α- admissible if for any $x \in X$ and $y \in Tx$, $\alpha(x, y) \geq 1$ implies $\alpha(y, z) \geq 1$ for any $z \in Ty$, where $\alpha : X \times X \to [0, \infty)$.*

The up coming lemma proved in [18], for single-valued map here, we prove it for multi- valued map.

Lemma 3.3. *Let (X,d) be a ν- generalized metric space. Let T be a multi-valued mapping from X into 2^X and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X defined by $x_{n+1}\in Tx_n$ such that $x_n\neq x_{n+1}$. Assume that*

$$d(x_n,x_{n+1})\leq \delta d(x_{n-1},x_n) \tag{6}$$

hold for any $\delta\in[0,1)$. Then $x_n\neq x_m$ $\forall n\neq m\in\mathbb{N}$.

Proof. We prove that $x_{n+\ell}\neq x_n$ for all $n\in\mathbb{N}$ and $\ell\geq 1$. Suppose the contrary that is $x_{n+\ell}=x_n$ for some $n\in\mathbb{N}$ and $\ell\geq 1$. By assumption, we have that $x_{n+\ell+1}=x_{n+1}$. Then from (6) we get

(7)

$$d(x_n,x_{n+1})=d(x_{n+\ell},x_{n+\ell+1})\leq \delta d(x_{n+\ell-1},x_{n+\ell})\leq ...\leq \delta^{\ell}d(x_n,x_{n+1})<d(x_n,x_{n+1})$$

which is contradiction. Thus, we get $x_m\neq x_n$ for all $m\neq n$ in $\mathbb{N}$. □

Let Φ be the family of all functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfying the following conditions:

(a) $\varphi(s)=0$ iff $s=0$.
(b) φ is non-decreasing and lower semi-continuous
(c) $\lim_{s\to0^+}\sup\dfrac{s}{\varphi(s)}<\infty$.

Theorem 3.4. *Let (X,d) be a $(\sum,\neq)$ complete ν- generalized metric space. Let $T:X\to CB(X)$ be an α- admissible multi-valued mapping satisfying:*

(i) There exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$
(ii) If $(y_n)\in Tx$ and (y_n) converge to y then $y\in Tx$
(iii) $\alpha(x,y)\mathcal{H}(Tx,Ty)\leq\phi(d(x,y))d(x,y)$ for any $x,y\in X$, and ϕ is MT-function.

Then T has a fixed point.

Proof. Let $\beta:[0,\infty)\to[0,1)$ as $\beta(t)=\dfrac{1+\phi(t)}{2}$ such that $\lim_{s\to t^+}\sup\beta(s)<1$. Clearly $\phi(t)<\beta(t)$ for each $t\in[0,\infty)$. Let $x_0\in X$ and choose $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$. Assume $x_0\neq x_1$ so, $\dfrac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)>0$. Since $Tx_1\neq\emptyset$, choose $x_2\in Tx_1$ such that

$$\begin{aligned}
d(x_1,x_2)&\leq\mathcal{H}(Tx_0,Tx_1)+\frac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)\\
&\leq\alpha(x_0,x_1)\mathcal{H}(Tx_0,Tx_1)+\frac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)\\
&\leq\phi(d(x_0,x_1))d(x_0,x_1)+\frac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)\\
&\leq\beta(d(x_0,x_1))d(x_0,x_1).
\end{aligned}$$

Since T is α- admissible, $x_1\in Tx_0$ and $\alpha(x_0,x_1)\geq 1$ then, $\alpha(Tx_0,Tx_1)\geq 1$ which implies $\alpha(x_1,x_2)\geq 1$. Similarly assume $x_1\neq x_2$ we have $\dfrac{1-\phi(d(x_1,x_2))}{2}d(x_1,x_2)>$

0 and choose $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\leq \mathcal{H}(Tx_1,Tx_2) + \frac{1-\phi(d(x_1,x_2))}{2}d(x_1,x_2) \\ &\leq \alpha(x_1,x_2)\mathcal{H}(Tx_1,Tx_2) + \frac{1-\phi(d(x_1,x_2))}{2}d(x_1,x_2) \\ &\leq \phi(d(x_1,x_2))d(x_1,x_2) + \frac{1-\phi(d(x_1,x_2))}{2}d(x_1,x_2) \\ &\leq \beta(d(x_1,x_2))d(x_1,x_2). \end{aligned}$$

Similarly, using the same method of proving theorem (3.1), we have our result. □

Theorem 3.5. *Let (X,d) be a $(\sum,\neq)$ complete ν- generalized metric space. Let $T : X \to CB(X)$ be a multi-valued map satisfying:*

$$\varphi(\mathcal{H}(Tx,Ty)) \leq \alpha(\varphi(d(x,y)))\varphi(d(x,y)),$$

for each $x,y \in X$, where α is a MT- function and $\varphi \in \Phi$. Then T has a fixed point.

Proof. Let $\gamma : [0,\infty) \to [0,1)$ defined by $\gamma(t) = \dfrac{1+\alpha(t)}{2}$. Since φ is non- decreasing function, then

$$\begin{aligned} &\max\left\{\sup_{v\in Tx}\varphi(d(v,Ty)), \sup_{u\in Ty}\varphi(d(u,Tx))\right\} \\ &= \max\left\{\varphi(\sup_{v\in Tx}d(v,Ty)), \varphi(\sup_{u\in Ty}d(u,Tx))\right\} \\ &= \varphi\left(\mathcal{H}(Tx,Ty)\right) \leq \gamma(\varphi(d(x,y)))\varphi(d(x,y)). \end{aligned} \tag{8}$$

There exist an element $z \in Ty$ such that

$$\varphi(d(y,z)) \leq \gamma(\varphi(d(x,y)))\varphi(d(x,y)),$$

for each $x \in X$ and $y \in Tx$. Thus, in the same way a sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ defined as $x_{n+1} \in Tx_n$ is constructed such that

$$\varphi(d(x_n,x_{n+1}) \leq \gamma(\varphi(d(x_{n-1},x_n))\varphi(d(x_{n-1},x_n)) \tag{9}$$

for all $n \in \mathbb{N}$. Since $\gamma(t) < 1$ for any $t \in [0,\infty)$, hence from (9) we get

$$\varphi(d(x_n,x_{n+1}) < \varphi(d(x_{n-1},x_n)). \tag{10}$$

Clearly $\{\varphi(d(x_{n-1},x_n))\}$ is decreasing sequence of positive real numbers. Hence it is converge to some non- negative real number, say ϵ. By contradiction, it is easy to show that $\epsilon = 0$. Note that, φ is a non- decreasing function which implies to $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$. Thus the sequence $\{d(x_n,x_{n+1})\}$ is also decreasing. Hence by lemma (3.3), the terms of the sequence all are different. Now, show that $\sum_{n=0}^{\infty} d(x_n,x_{n+1}) < \infty$. Note that the sequence $\{d(x_n,x_{n+1})\}$ is decreasing and bounded. Thus, it is converges to a positive real number (say δ) which implies that $\varphi(\delta) \leq \varphi(d(x_n,x_{n+1}))$. Thus,

$$\varphi(\delta) \leq \lim_{n\to\infty}\varphi(d(x_n,x_{n+1})) = \epsilon = 0.$$

Since $\varphi(s) = 0$ if and only if $s = 0$ then, $\delta = 0$. By lemma (2.6), there exist $r \in [0, 1)$ such that, $\varphi(d(x_n, x_{n+1})) \le r\varphi(d(x_{n-1}, x_n))$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \varphi(d(x_n, x_{n+1})) &\le \sum_{n=1}^{\mu} \varphi(d(x_n, x_{n+1})) + \sum_{n=\mu+1}^{\infty} \varphi(d(x_n, x_{n+1})) \\ &\le \sum_{n=1}^{\mu} \varphi(d(x_n, x_{n+1})) + \sum_{n=1}^{\infty} r^n \varphi(d(x_\mu, x_{\mu+1})) \\ &< \infty.\end{aligned}$$

By defintion of φ, we have

$$\lim_{n\to\infty} \sup \frac{d(x_n, x_{n+1})}{\varphi(d(x_n, x_{n+1}))} \le \lim_{s\to 0^+} \frac{s}{\varphi(s)} < \infty.$$

Thus, the sequence $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence. Since X is a $(\sum, \neq)$ complete ν- generalized metric space and by lemma (2.5), it is Cauchy and then it is converge to some $z \in X$. From the definition of φ and its increasing we conclude that,

$$\begin{aligned}\varphi d(z, Tz)) &\le \lim_{n\to\infty} \inf \varphi(d(x_{n+1}, Tz) \le \lim_{n\to\infty} \inf \varphi(\mathcal{H}(Tx_n, Tz)) \\ &\le \lim_{n\to\infty} \inf \gamma(\varphi(d(x_n, z)))\varphi(d(x_n, z)) \le \lim_{n\to\infty} \inf \varphi(d(x_n, z)) \\ &= \lim_{s\to 0^+} \varphi(s) = \lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0.\end{aligned}$$

Therefore, $\varphi(d(z, Tz)) = 0$. Thus by the definition of φ and since Tz closed we have $z \in Tz$ is a fixed point. □

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