An inertial extragradient subgradient method for solving bilevel equilibrium problems

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Abstract

In this paper, we propose an algorithm by combining an inertial term with the extragradient subgradient method for finding some solutions of bilevel equilibrium problems in a real Hilbert space. Then, we establish a strongly convergent theorem of the proposed algorithm under some sufficient assumptions on the bifunctions involving pseudomonotone and Lipschitz-type conditions. Some numerical experiments are tested to illustrate the advantage performance of our algorithm.

Keywords: Bilevel equilibrium problem; Extragradient subgradient method; Inertial method; Strong convergence

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H, and let f and g be bifunctions from $H \times H$ to $\mathbb{R}$ such that $f(x,x) = 0$ and $g(x,x) = 0$ for all $x \in H$. The equilibrium problem associated with g and C is denoted by $EP(C,g)$: Find $x^* \in C$ such that

$$g(x^*, y) \geq 0 \quad \text{for every} \quad y \in C, \tag{1.1}$$

which was considered by Blum and Oettli [4]. The solution set of problem (1.1) is denoted by Ω.

It can be seen that the equilibrium problem is related to science in various fields and is very important because many problems arise in applied areas such as the fixed point problem, the (generalized) Nash equilibrium problem in game theory, the saddle point problem, the variational inequality problem, the optimization problem and others.

The simple basic method for solving some monotone equilibrium problems is the proximal point method (see [20, 22, 27]). In 2008, Tran et al. [37] proposed the extragradient algorithm for solving the equilibrium problem by using the strongly convex minimization problem to solve at each iteration. Furthermore, Hieu [16] introduced subgradient extragradient methods for pseudomonotone equilibrium problem and the other methods (see the details in [1, 12, 21, 23, 31, 39]).

In this paper, we consider the bilevel equilibrium problems, that is, the equilibrium problem whose constraints are the solution sets of equilibrium problems: Find $x^* \in \Omega$ such that

$$f(x^*, y) \geq 0 \quad \text{for every} \quad y \in \Omega. \tag{1.2}$$

The solution set of problem (1.2) is denoted by Ω^*.

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The bilevel equilibrium problems were introduced by Chadli et al. [7] in 2000. This kind of problems is very important and interesting because it is a generalization class of problems such as optimization problems over equilibrium constraints, variational inequality over equilibrium constraints, hierarchical minimization problems, and complementarity problems. Furthermore, the particular case of the bilevel equilibrium can be applied to a real word model such as the variational inequality over the fixed point set of a firmly nonexpansive mapping applied to the power control problem of CDMA networks which were introduced by Iiduka [18]. For more on the relation of bilevel equilibrium with particular cases, see [10, 19, 30].

Methods for solving bilevel equilibrium problems have been studied extensively by many authors. In 2010, Moudafi [28] introduced a simple proximal method and proved the weak convergence to a solution of problem (1.2). In 2014, Quy [33] introduced the algorithm by combining the proximal method with the Halpern method for solving bilevel monotone equilibrium and fixed point problem. For more details and most recent works on the methods for solving bilevel equilibrium problems, we refer the reader to [2, 8, 36]. The authors considered the method for monotone and pseudoparamonotone equilibrium problem. If a bifunction is more generally monotone, we cannot use the above methods for solving bilevel equilibrium problem, for example, the pseudomonotone property.

In 2018, Yuying et. al [40] proposed a method for finding the solution for bilevel equilibrium problems where f is strongly monotone and g is pseudomonotone and Lipschitz-type continuous. They obtained the convergent sequence by combining an extragradient subgradient method with the Halpern method.

On the other hand, an inertial-type algorithm was first proposed by Polyak [32] as an acceleration process in solving a smooth convex minimisation problem. An inertial-type algorithm is a two-step iterative method in which the next iterate is defined by making use of the previous two iterates. It is well known that incorporating an inertial term in an algorithm speeds up or accelerates the rate of convergence of the sequence generated by the algorithm. Consequently, a lot of research interest is now devoted to the inertial-type algorithm(see e.g. [5, 13, 24] and the references contained in them).

Motivated and inspired by the research work in this direction, in this work, we provide an algorithm which is generated by an inertial term and the extragradient subgradient method for solving bilevel equilibrium problems in a real Hilbert space. Then, the strong convergence theorem of the proposed algorithm are established under some sufficient assumptions on the bifunctions involving pseudomonotone and Lipschitz-type conditions. The numerical experiments are investigated to illustrate the advantage performance together with some improvement of our algorithm.

2. Preliminaries

Throughout this paper, H is a real Hilbert space, C is a nonempty closed convex subset of H. Denote that $x_n \rightharpoonup x$ and $x_n \to x$ are the weak convergence and the strong convergence of a sequence $\{x_n\}$ to x, respectively. For every $x \in H$, there exists a unique element $P_C x$ defined by

$$P_C x = \text{argmin}\{\|x - y\| : y \in C\},$$

which can be found, e.g., in [[6], Sect. 1.2.2, Theorem 1.7], [[11], Theorem 3.4(2)], [[14], Theorem 7.43], [[17], Chap. III, Sect. 3.1] or [[29], Theorem 8.25].

Lemma 2.1 ([15]). *The metric projection P_C has the following basic properties:*

(i) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ *for all* $x \in H$ *and* $y \in C$*;*
(ii) $\langle x - P_C x, P_C x - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$*;*
(iii) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ *for all* $x, y \in H$.

We now recall the concept of proximity operator introduced by Moreau [26]. For a proper, convex and lower semicontinuous function $g : H \to (-\infty, \infty]$ and $\gamma > 0$, the Moreau envelope of g of parameter γ is the convex function

$$^{\gamma}g(x) = \inf_{y \in H} \left\{ g(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\} \quad \forall x \in H.$$

For all $x \in H$, the function $y \mapsto g(y)+\frac{1}{2\gamma}\|y-x\|^2$ is proper, strongly convex and lowe semicontinuous, thus the infimum is attained, i.e. $^{\gamma}g : H \to \mathbb{R}$.

The unique minimum of $y \mapsto g(y) + \frac{1}{2\gamma}\|y - x\|^2$ is called proximal point of g at x and it is denoted by $\text{prox}_g(x)$. The operator

$$\text{prox}_g(x) : H \to H$$

$$x \mapsto \arg\min_{y \in H} \left\{ g(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\}$$

is well-defined and is said to be the proximity operator of g. When $g = i_C$ (the indicator function of the convex set C), one has

$$\text{prox}_{i_C}(x) = P_C(x)$$

for all $x \in H$.

We also recall that the subdifferential of $g : H \to (-\infty, \infty]$ at $x \in \text{dom}g$ is defined as the set of all subgradient of g at x

$$\partial g(x) := \{w \in H : g(y) - g(x) \geq \langle w, y - x\rangle \, \forall y \in H\}.$$

The function g is called subdifferentiable at x if $\partial g(x) \neq \emptyset$, g is said to be subdifferentiable on a subset $C \subset H$ if it is subdifferentiable at each point $x \in C$, and it is said to be subdifferentiable, if it is subdifferentiable at each point $x \in H$, i.e., if $\text{dom}(\partial g) = H$.

The normal cone of C at $x \in C$ is defined by

$$N_C(x) := \{q \in H : \langle q, y - x\rangle \leq 0, \forall y \in C\}.$$

Definition 2.2 ([34, 35])**.** A bifunction $\psi : H \times H \to \mathbb{R}$ is called:

(i) β-strongly monotone on C if there exists $\beta > 0$ such that

$$\psi(x, y) + \psi(y, x) \leq -\beta\|x - y\|^2 \quad \forall x, y \in C;$$

(ii) monotone on C if

$$\psi(x, y) + \psi(y, x) \leq 0 \quad \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$\psi(x, y) \geq 0 \Rightarrow \psi(y, x) \leq 0 \quad \forall x, y \in C.$$

(iv) β-strongly pseudomonotone on C if there exists $\beta > 0$ such that

$$\psi(x, y) \geq 0 \Rightarrow \psi(y, x) \leq -\beta\|x - y\|^2 \quad \forall x, y \in C.$$

It is easy to see from the aforementioned definitions that the following implications hold,

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \quad \text{and} \quad \text{(i)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)}$$

The converses in general are not true.

In this paper, we consider the bifunctions f and g under the following conditions.

Condition A

(A1) $f(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
(A2) $f(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
(A3) f is β-strongly monotone on $H \times H$.
(A4) For each $x, y \in H$, there exists $L > 0$ such that

$$\|w - v\| \leq L\|x - y\|, \quad \forall w \in \partial f(x,\cdot)(x), v \in \partial f(y,\cdot)(y).$$

(A5) The function $x \mapsto \partial f(x,\cdot)(x)$ is bounded on the bounded subsets of H.

Condition B

(B1) $g(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
(B2) $g(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
(B3) g is pseudomonotone on C with respect to Ω, i.e.,

$$g(x, x^*) \leq 0, \quad \forall x \in C,\ x^* \in \Omega.$$

(B4) g is Lipschitz-type continuous, i.e., there exist two positive constants L_1, L_2 such that

$$g(x,y) + g(y,z) \geq g(x,z) - L_1\|x-y\|^2 - L_2\|y-z\|^2,\ \forall x,y,z \in H.$$

(B5) g is jointly weakly continuous on $H \times H$ in the sense that, if $x, y \in H$ and $\{x_n\}, \{y_n\} \in H$ converge weakly to x and y, respectively, then $g(x_n, y_n) \to g(x,y)$ as $n \to +\infty$.

Example 2.3 ([40]). Let $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,y) = 5y^2 - 7x^2 + 2xy$ and $g(x,y) = 2y^2 - 7x^2 + 5xy$. It follows that f and g satisfy Condition A and Condition B, respectively.

Lemma 2.4 ([3], Propositions 3.1, 3.2). *If the bifunction g satisfies Assumptions (B1), (B2), and (B3), then the solution set Ω is closed and convex.*

Remark 2.5. Let the bifunction f satisfy Condition A and the bifunction g satisfy Condition B. If $\Omega \neq \emptyset$, then the bilevel equilibrium problem (1.2) has a unique solution, see the details in [33].

Lemma 2.6 ([9]). *Let $\phi : C \to \mathbb{R}$ be a convex, lower semicontinuous, and subdifferentiable function on C. Then x^* is a solution to the convex optimization problem*

$$\min\{f(x) : x \in C\}$$

if and only if

$$0 \in \partial\phi(x^*) + N_C(x^*).$$

The following lemmas will be used in the proof of the convergence result.

Lemma 2.7 ([38]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$, and $\{\xi_n\}$ be a sequence in $\mathbb{R}$ satisfying the condition*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \xi_n, \quad \forall n \geq 0,$$

where $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \xi_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 ([25]). *Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that*

$$a_{n_j} < a_{n_j+1} \quad \textit{for all} \quad j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ *defined, for all* $n \geq n_0$*, by*

$$\tau(n) = \max\{k \leq n \mid a_k < a_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying*

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$*, the following two estimates hold:*

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \text{and} \quad a_n \leq a_{\tau(n)+1}.$$

Lemma 2.9 ([40])**.** *Suppose that* f *is* β*-strongly monotone on* H *and satisfies (A4). Let* $0 < \alpha < 1$*,* $0 \leq \eta \leq 1-\alpha$*, and* $0 < \mu < \frac{2\beta}{L^2}$*. For each* $x, y \in H$*,* $w \in \partial f(x,\cdot)(x)$*, and* $v \in \partial f(y,\cdot)(y)$*, we have*

$$\|(1-\eta)x - \alpha\mu w - [(1-\eta)y - \alpha\mu v]\| \leq (1-\eta-\alpha\sigma)\|x-y\|,$$

where $\sigma = 1 - \sqrt{1-\mu(2\beta-\mu L^2)} \in (0,1]$.

3. Main Result

In this section, we propose the algorithm for finding the solution of a bilevel equilibrium problem under the strong monotonicity of f and the pseudomonotonicity and Lipschitztype continuous conditions on g.

Algorithm 3.1. Initialization: Choose $x_0, x_1 \in H$, $0 < \mu < \frac{2\beta}{L^2}$, $\theta \in [0,1)$, the sequences $\{\alpha_n\} \subset (0,1)$, $\{\epsilon_n\} \subset [0,+\infty)$ and $\{\eta_n\}$ are such that

$$\begin{cases} \lim_{n\to\infty}\alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty, \\ 0 \leq \eta_n \leq 1-\alpha_n \ \forall n \geq 0, \ \lim_{n\to\infty}\eta_n = \eta < 1, \\ \sum_{n=0}^{\infty} \epsilon_n < \infty. \end{cases}$$

Select initial $x_0, x_1 \in C$ and set $n \geq 1$.
Step 1.: Given x_{n-1} and x_n $(n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where

$$\theta_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{if otherwise.} \end{cases} \tag{3.1}$$

Choose $\{\lambda_n\}$ such that

$$0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \min\left(\frac{1+\theta_n}{2L_1}, \frac{1+\theta_n}{2L_2}\right).$$

Compute

$$s_n = x_n + \theta_n(x_n - x_{n-1}),$$

$$y_n = \underset{y\in C}{\arg\min}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - s_n\|^2\right\},$$

$$z_n = \underset{y\in C}{\arg\min}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}.$$

Step 2. Compute $w_n \in \partial f(z_n,\cdot)(z_n)$ and

$$x_{n+1} = \eta_n x_n + (1-\eta_n)z_n - \alpha_n\mu w_n.$$

Set $n := n+1$ and return to **Step 1**.

Remark 3.2. Some remarks on the algorithm are in order now.

(1) Evidently, we have from (3.1) that

$$\sum_{n=0}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty, \tag{3.2}$$

due to $\theta_n \|x_n - x_{n-1}\| \le \bar{\theta_n} \|x_n - x_{n-1}\| \le \epsilon_n$.

(2) When $\theta_n = 0$, Algorithm 3.1 reduces to Algorithm 1 of [40].

Theorem 3.3. *Let bifunctions f and g satisfy Condition A and Condition B, respectively. Assume that $\Omega \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution of the bilevel equilibrium problem* (1.2).

Proof. Under assumptions of two bifunctions f and g, we get the unique solution of the bilevel equilibrium problem (1.2), denoted by x^*.

Step 1: Show that

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - (1+\theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1+\theta_n - 2\lambda_n L_2)\|y_n - z_n\|^2 - \theta_n \|x_n - x_{n-1}\|^2. \tag{3.3}$$

The definition of y_n and Lemma 2.6 imply that

$$0 \in \partial \left\{ \lambda_n g(x_n, y) + \frac{1}{2}\|y - s_n\|^2 \right\} (y_n) + N_C(y_n).$$

There are $w \in \partial g(x_n, \cdot)(y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$\lambda_n w + y_n - s_n + \bar{w} = 0. \tag{3.4}$$

Since $\bar{w} \in N_C(y_n)$, we have

$$\langle \bar{w}, y - y_n \rangle \le 0 \quad \text{for all}\, y \in C. \tag{3.5}$$

By using (3.4) and (3.5), we obtain $\lambda_n \langle w, y - y_n \rangle \ge \langle s_n - y_n, y - y_n \rangle$ for all $y \in C$. Since $z_n \in C$, we have

$$\lambda_n \langle w, z_n - y_n \rangle \ge \langle s_n - y_n, z_n - y_n \rangle. \tag{3.6}$$

It follows from $w \in \partial g(x_n, \cdot)(y_n)$ that

$$g(x_n, y) - g(x_n, y_n) \ge \langle w, y - y_n \rangle \quad \text{for all}\, y \in H. \tag{3.7}$$

By using (3.6) and (3.7), we get

$$\lambda_n \{g(x_n, z_n) - g(x_n, y_n)\} \ge \langle s_n - y_n, z_n - y_n \rangle. \tag{3.8}$$

Similarly, the definition of z_n implies that

$$0 \in \partial \left\{ \lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\} (z_n) + N_C(z_n).$$

There are $u \in \partial g(y_n, \cdot)(z_n)$ and $\bar{u} \in N_C(x)$ such that

$$\lambda_n u + z_n - x_n + \bar{u} = 0. \tag{3.9}$$

Since $\bar{u} \in N_C(z_n)$, we have

$$\langle \bar{u}, y - z_n \rangle \le 0 \quad \text{for all}\, y \in C. \tag{3.10}$$

By using (3.9) and (3.10), we obtain $\lambda_n\langle u, y - z_n\rangle \geq \langle x_n - z_n, y - z_n\rangle$ for all $y \in C$. Since $x^* \in C$, we have

$$\lambda_n\langle u, x^* - z_n\rangle \geq \langle x_n - z_n, x^* - z_n\rangle \tag{3.11}$$

It follows from $u \in \partial g(y_n, \cdot)(z_n)$ that

$$g(y_n, y) - g(y_n, z_n) \geq \langle u, y - z_n\rangle \quad \text{for all}\, y \in H. \tag{3.12}$$

By using (3.11) and (3.12), we get

$$\lambda_n\{g(y_n, x^*) - g(y_n, z_n)\} \geq \langle x_n - z_n, x^* - z_n\rangle.$$

Since $x^* \in \Omega$, we have $g(x^*, y_n) \geq 0$. If follows from the pseudomonotonicity of g on C with respect to Ω that $g(y_n, x^*) \leq 0$. This implies that

$$\langle x_n - z_n, z_n - x^*\rangle \geq \lambda_n g(y_n, z_n). \tag{3.13}$$

Since g is Lipschitz-type continuous, there exist two positive constants L_1, L_2 such that

$$g(y_n, z_n) \geq g(x_n, z_n) - g(x_n, y_n) - L_1\|x_n - y_n\|^2 - L_2\|y_n - z_n\|^2. \tag{3.14}$$

By using (3.13) and (3.14), we get

$$\langle x_n - z_n, z_n - x^*\rangle \geq \lambda_n\{g(x_n, z_n) - g(x_n, y_n)\} - \lambda_n L_1\|x_n - y_n\|^2 - \lambda_n L_2\|y_n - z_n\|^2.$$

From (3.8) and the above inequality, we obtain

$$2\langle x_n - z_n, z_n - x^*\rangle \geq 2\langle s_n - y_n, z_n - y_n\rangle - 2\lambda_n L_1\|x_n - y_n\|^2 - 2\lambda_n L_2\|y_n - z_n\|^2. \tag{3.15}$$

By the definition of s_n, we have that

$$\begin{aligned} 2\langle s_n - y_n, z_n - y_n\rangle &= 2\langle x_n + \theta_n(x_n - x_{n-1}) - y_n, z_n - y_n\rangle \\ &= -2\langle x_n - y_n, y_n - z_n\rangle + 2\theta_n\langle x_n - x_{n-1}, z_n - y_n\rangle. \end{aligned}$$

We know that

$$\begin{aligned} 2\langle x_n - z_n, z_n - x^*\rangle &= \|x_n - x^*\|^2 - \|z_n - x_n\|^2 - \|z_n - x^*\|^2 \\ -2\langle x_n - y_n, y_n - z_n\rangle &= -\|x_n - z_n\|^2 + \|x_n - y_n\|^2 + \|y_n - z_n\|^2 \\ 2\theta_n\langle x_n - x_{n-1}, z_n - y_n\rangle &= \theta_n(\|x_n - y_n\|^2 - \|x_n - x_{n+1}\|^2 - \|y_n - z_n\|^2). \end{aligned}$$

From (3.15), we can conclude that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1+\theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1+\theta_n - 2\lambda_n L_2)\|y_n - z_n\|^2 - \theta_n\|x_n - x_{n-1}\|^2. \tag{3.16}$$

Step 2: The sequences $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.
Since $0 < \lambda_n < a$, where $a = \min\left\{\frac{1+\theta_n}{2L_1}, \frac{1+\theta_n}{2L_2}\right\}$, we have

$$(1 + \theta_n - 2\lambda_n L_1) > 0 \quad \text{and} \quad (1 + \theta_n - 2\lambda_n L_2) > 0.$$

It follows from (3.3) and the above inequalities that

$$\|z_n - x^*\| \leq \|x_n - x^*\| \quad \text{for all}\, n \in \mathbb{N}. \tag{3.17}$$

By Lemma 2.9 and (3.17), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\eta_n x_n + (1-\eta_n)z_n - \alpha\mu w_n - x^* + \eta_n x^* - \eta_n x^* + \alpha_n\mu v - \alpha_n\mu v\| \\ &= \|(1-\eta_n)z_n - \alpha_n\mu w_n - (1-\eta_n)x^* + \alpha_n\mu v + \eta_n(x_n - x^*) - \alpha_n\mu v\| \\ &\le \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v]\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\ &\le (1-\eta_n-\alpha_n\sigma)\|z_n - x^*\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\ &\le (1-\eta_n-\alpha_n\sigma)\|x_n - x^*\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\ &= (1-\alpha_n\sigma)\|x_n - x^*\| + \alpha_n\tau\left(\frac{\mu\|v\|}{\sigma}\right), \end{aligned} \tag{3.18}$$

where $w_n \in \partial f(z_n,\cdot)(z_n)$ and $v \in \partial f(x^*,\cdot)(x^*)$. This implies that

$$\|x_{n+1}-x^*\| \le \max\left\{\|x_n - x^*\|, \frac{\mu\|v\|}{\sigma}\right\}.$$

By induction, we obtain

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\mu\|v\|}{\sigma}\right\}.$$

Thus the sequence $\{x_n\}$ is bounded. By using (3.17), we have $\{z_n\}$, and using Condition (A5), we can conclude that $\{w_n\}$ is also bounded.

Step 3: Show that the sequence $\{x_n\}$ converges strongly to x^*.

Since $x^* \in \Omega^*$, we have $f(x^*, y) \ge 0$ for all $y \in \Omega$. Note that $f(x^*, x^*) = 0$. Thus x^* is a minimum of the convex function $f(x^*,\cdot)$ over Ω. By Lemma 2.6, we obtain $0 \in \partial f(x^*,\cdot)(x^*) + N_\Omega(x^*)$. Then there exists $v \in \partial f(x^*,\cdot)(x^*)$ such that

$$\langle v, z - x^*\rangle \ge 0 \quad \text{for all}\, z \in \Omega. \tag{3.19}$$

Note that

$$\|x-y\|^2 \le \|x\|^2 - 2\langle y, x-y\rangle \quad \text{for all}\, x, y \in H. \tag{3.20}$$

From Lemma 2.9 and (3.20), we obtain

$$\begin{aligned} &\|x_{n+1}-x^*\|^2 \\ &= \|\eta_n x_n + (1-\eta_n)z_n - \alpha\mu w_n - x^*\|^2 \\ &= \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*) - \alpha_n\mu v\|^2 \\ &\le \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*)\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le \left\{\|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*)\|^2\right\} - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le [(1-\eta_n-\alpha_n\sigma)\|z_n - x^*\| + \eta_n\|x_n - x^*\|]^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le (1-\eta_n-\alpha_n\sigma)\|z_n - x^*\|^2 + \eta_n\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le (1-\eta_n-\alpha_n\sigma)\|x_n - x^*\|^2 + \eta_n\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &= (1-\alpha_n\sigma)\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \end{aligned} \tag{3.21}$$

It follows that

$$\|x_{n+1}-x^*\|^2 \le (1-\alpha_n\sigma)\|x_n - x^*\|^2 + 2\alpha_n\mu\langle v, x^* - x_{n+1}\rangle. \tag{3.22}$$

Let us consider two cases.

Case 1: There exists n_0 such that $\|x_n - x^*\|$ is decreasing for $n \geq n_0$. Therefore the limit of sequence $\|x_n - x^*\|$ exists. By using (3.17) and (3.22), we obtain

$$\begin{aligned}
0 &\leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2 \\
&\leq -\frac{\alpha_n \sigma}{1-\eta_n}\|z_n - x^*\|^2 - \frac{2\alpha_n \mu}{1-\eta_n}\langle v, x_{n+1} - x^*\rangle \\
&\quad + \frac{1}{1-\eta_n}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2).
\end{aligned}$$

Since $\lim_{n\to\infty} = \eta_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and the limit of $\|x_n - x^*\|$ exists, we have

$$\lim_{n\to\infty}(\|x_n - x^*\|^2 - \|z_n - x^*\|^2) = 0. \tag{3.23}$$

From $0 < \lambda_n < a$ and inequality (3.3), we get

$$(1 + \theta_n - a)\|x_n - y_n\|^2 \leq (1 + \theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2.$$

By using (3.23), we obtain $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Next, we show that

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle \leq 0. \tag{3.24}$$

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle = \limsup_{k\to\infty}\langle v, x^* - x_{n_k}\rangle.$$

Since $\{x_{n_k}\}$ is bounded, we may assume that $\{x_{n_k}\}$ converges weakly to some $\bar{x} \in H$. Therefore

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle = \limsup_{k\to\infty}\langle v, x^* - x_{n_k}\rangle = \langle v, x^* - \bar{x}\rangle. \tag{3.25}$$

Since $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we have $y_{n_k} \rightharpoonup \bar{x}$. Let us consider that

$$\lim_{n\to\infty}\|s_n - y_n\| \leq \lim_{n\to\infty}\|s_n - x_n\| + \lim_{n\to\infty}\|x_n - y_n\|.$$

By the definition of s_n, we have that

$$\begin{aligned}
\lim_{n\to\infty}\|s_n - x_n\| &= \lim_{n\to\infty}\|x_n - \theta_n(x_n - x_{n-1}) - x_n\| \\
&= \lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\|.
\end{aligned}$$

Using the assumption $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < \infty$, it implies that $\lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| = 0$. Thus $\lim_{n\to\infty}\|s_n - x_n\| = 0$. Since $\lim_{n\to\infty}\|s_n - x_n\| = 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we have $s_{n_k} \rightharpoonup \bar{x}$. Since C is closed and convex, it is also weakly closed and thus $\bar{x} \in C$. Next, we show that $\bar{x} \in \Omega$. From the definition of $\{y_n\}$ and Lemma 2.6, we obtain

$$0 \in \partial\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|s_n - y\|^2\right\}(y_n) + N_C(y_n).$$

There exist $\bar{w} \in N_C(y_n)$ and $w \in \partial g(x_n, \cdot)(y_n)$ such that

$$\lambda_n w + y_n - s_n + \bar{w} = 0. \tag{3.26}$$

Since $\bar{w} \in N_C(y_n)$, we have $\langle \bar{w}, y - y_n\rangle \leq 0$ for all $y \in C$. From (3.26), we obtain

$$\lambda_n\langle w, y - y_n\rangle \geq \langle s_n - y_n, y - y_n\rangle \quad \text{for all}\, y \in C. \tag{3.27}$$

Since $w \in \partial g(x_n, \cdot)(y_n)$, we have

$$g(x_n, y) - g(x_n, y_n) \geq \langle w, y - y_n \rangle \quad \text{for all}\, y \in H. \tag{3.28}$$

Combining (3.27) and (3.28), we get

$$\lambda_n \{g(x_n, y) - g(x_n, y_n)\} \geq \langle s_n - y_n, y - y_n \rangle \quad \text{for all}\, y \in C. \tag{3.29}$$

Taking $n = n_k$ and $k \to \infty$ in (3.29), the assumption of λ_n and (B5), we obtain $g(\bar{x}, y) = 0$ for all $y \in C$. This implies that $\bar{x} \in \Omega$. By inequality (3.19), we obtain $\langle v, \bar{x} - x^* \rangle \geq 0$. It follows from (3.25) that

$$\limsup_{n\to\infty} \langle v, x^* - x_{n+1} \rangle \leq 0. \tag{3.30}$$

We can write inequality (3.22) in the following form:

$$\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n \sigma)\|x_n - x^*\|^2 + \alpha_n \sigma \xi_n,$$

where $\xi_n = \frac{2\mu}{\sigma} \langle v, x^* - x_{n+1} \rangle$. It follows from (3.30) that $\limsup_{n\to\infty} \xi_n \leq 0$. By Lemma 2.7, we can conclude that $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$. Hence $x_n \to x^*$ as $n \to \infty$.

Case 2: There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - x^*\| \leq \|x_{n_{j+1}} - x^*\|$ for all $j \in \mathbb{N}$. By Lemma 2.8, there exists a nondecreasing sequence $\{\tau(n)\}$ of N such that $\lim_{n\to\infty} \tau(n) = \infty$, and for each sufficiently large $n \in \mathbb{N}$, we have

$$\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \quad \text{and} \quad \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\|. \tag{3.31}$$

Combining (3.18) and (3.31), we have

$$\begin{aligned} \|x_{\tau(n)} - x^*\| &\leq \|x_{\tau(n)+1} - x^*\| \\ &\leq (1 - \eta_{\tau(n)} - \alpha_{\tau(n)}\sigma)\|z_{\tau(n)} - x^*\| + \eta_{\tau(n)}\|x_{\tau(n)} - x^*\| + \alpha_{\tau(n)}\mu\|v\|. \end{aligned} \tag{3.32}$$

From (3.17) and (3.32), we get

$$0 \leq \|x_{\tau(n)} - x^*\| - \|z_{\tau(n)} - x^*\| \leq -\frac{\alpha_{\tau(n)}\sigma}{1 - \eta_{\tau(n)}}\|z_{\tau(n) - x^*}\| + \frac{\alpha_{\tau(n)}\sigma}{1 - \eta_{\tau(n)}}\|v\|. \tag{3.33}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \eta_n = \eta < 1$, $\{z_n\}$ is bounded, and (3.33), we have $\lim_{n\to\infty}(\|x_{\tau(n)} - x^*\| - \|z_{\tau(n)} - x^*\|) = 0$. It follows from the boundedness of $\{x_n\}$ and $\{z_n\}$ that

$$\lim_{n\to\infty} (\|x_{\tau(n)} - x^*\|^2 - \|z_{\tau(n)} - x^*\|^2) = 0. \tag{3.34}$$

By using the assumption of $\{\lambda_n\}$, we get the following two inequalities:

$$1 + \theta_n - 2\lambda_{\tau(n)}L_1 > 1 + \theta_n - 2aL_1 > 0 \quad \text{and} \quad 1 + \theta_n - 2\lambda_{\tau(n)}L_2 > 1 + \theta_n - 2aL_2 > 0.$$

From (3.3), we obtain

$$\begin{aligned} \|z_{\tau(n)} - x^*\|^2 &\leq \|x_{\tau(n)} - x^*\|^2 - (1 + \theta_n - 2\lambda_{\tau(n)}L_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad - (1 + \theta_n - 2\lambda_{\tau(n)}L_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - x^*\|^2 - (1 + \theta_n - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad - (1 + \theta_n - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} 0 &< (1 + \theta_n - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 + (1 + \theta_n - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - x^*\|^2 - \|z_{\tau(n)} - x^*\|^2. \end{aligned}$$

It follows from (3.34) and the above inequality that

$$\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{3.35}$$

Note that $\|x_{\tau(n)} - z_{\tau(n)}\| \leq \|x_{\tau(n)} - y_{\tau(n)}\| + \|y_{\tau(n)} - z_{\tau(n)}\|$. From (3.35), we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{3.36}$$

By using the definition of x_{n+1} and Lemma 2.9, we obtain

$$\begin{aligned}
\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 &= \|\eta_{\tau(n)} x_{\tau(n)} + (1-\eta_{\tau(n)}) z_{\tau(n)} - \alpha_{\tau(n)} \mu t_{\tau(n)} - x_{\tau(n)}\| \\
&= \|(1-\eta_{\tau(n)}) z_{\tau(n)} - \alpha_{\tau(n)} \mu t_{\tau(n)} \\
&\quad - [(1-\eta_{\tau(n)}) x_{\tau(n)} - \alpha_{\tau(n)} w_{\tau(n)}] - \alpha_{\tau(n)} w_{\tau(n)}\| \\
&\leq \|(1-\eta_{\tau(n)}) z_{\tau(n)} - \alpha_{\tau(n)} t_{\tau(n)} \\
&\quad - [(1-\eta_{\tau(n)}) x_{\tau(n)} - \alpha_{\tau(n)} w_{\tau(n)}]\| + \alpha_{\tau(n)} \|w_{\tau(n)}\| \\
&\leq (1-\eta_{\tau(n)} - \alpha_{\tau(n)}\sigma) \|z_{\tau(n)} - x_{\tau(n)}\| + \alpha_{\tau(n)} \|w_{\tau(n)}\| \\
&\leq \|z_{\tau(n)} - x_{\tau(n)}\| + \alpha_{\tau(n)} \|w_{\tau(n)}\|,
\end{aligned}$$

where $t_{\tau(n)} \in \partial f(z_{\tau(n)}, \cdot)(z_{\tau(n)})$ and $w_{\tau(n)} \in \partial f(x_{\tau(n)}, \cdot)(x_{\tau(n)})$. Since $\lim_{n\to\infty} \alpha_n = 0$, the boundedness of $\{w_{\tau(n)}\}$ and (3.36), we have $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$. As proved in the first case, we can conclude that

$$\limsup_{n\to\infty} \langle v, x^* - x_{\tau(n)+1} \rangle = \limsup_{n\to\infty} \langle v, x^* - x_{\tau(n)} \rangle \leq 0. \tag{3.37}$$

Combining (3.22) and (3.31), we obtain

$$\begin{aligned}
\|x_{\tau(n)+1} - x^*\|^2 &\leq (1-\alpha_{n(\tau)}\sigma) \|x_{\tau(n)} - x^*\|^2 + 2\alpha_{n(\tau)} \mu \langle v, x^* - x_{\tau(n)+1} \rangle \\
&\leq (1-\alpha_{n(\tau)}\sigma) \|x_{\tau(n)+1} - x^*\|^2 + 2\alpha_{n(\tau)} \mu \langle v, x^* - x_{\tau(n)+1} \rangle.
\end{aligned}$$

By using (3.31) again, we have

$$\|x_n - x^*\|^2 \leq \|x_{\tau(n)+1} - x^*\|^2 \leq \frac{2\mu}{\sigma} \langle v, x^* - x_{\tau(n)+1} \rangle.$$

From (3.37), we can conclude that $\limsup_{n\to\infty} \|x_n - x^*\|^2 \leq 0$. Hence $x_n \to x^*$ as $n \to \infty$. This completes the proof. ■

4. Numerical example

In this section, we provide a numerical example to test our algorithm. All Matlab colds were performed on a computer with CPU Intel Core i7-7500U, up to 3.5GHz, 4GB of RAM under version MATLAB R2015b. In the following example, we use the standard Euclidean norm and inner product.

Example 4.1. We compare our algorithm with Algorithm 1 proposed in Yuying et al. [40]. Let us consider a problem when $H = \mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n : -5 \leq x_i \leq 5, \forall i \in \{1, 2, ..., n\}\}$. Let the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$g(x, y) = \langle Px + Qy, y - x \rangle \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$

where P and Q are randomly symmetric positive semidefinite matrices such that $P - Q$ is positive semidefinite. Then g is pseudomonotone on $\mathbb{R}^n$. Next, we obtain that g is Lipschitz-type continuous with $L_1 = L_2 = \frac{1}{2}\|P - Q\|$. Furthermore, we define the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$f(x, y) = \langle Ax + By, y - x \rangle \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$

Table 1: Comparison: proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall i = 1, 2, ..., n\}$.

n	Algorithm 3.1				Yuying et al. Algorithm	
	$\theta = 0.6$		$\theta = 0.9$			
	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)
5	29	1.0015	28	1.0178	34	1.3618
10	43	1.7310	38	1.3645	54	1.9099
50	90	4.3222	88	4.6822	98	5.5028

Table 2: Comparison: proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall i = 1, 2, ..., n\}$.

n	Algorithm 3.1				Yuying et al. Algorithm	
	$\theta = 0.6$		$\theta = 0.9$			
	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)
5	32	1.1074	30	1.0388	37	1.3528
10	50	1.8239	45	1.8472	61	2.3260
50	108	6.6858	105	6.5254	116	6.7247

with A and B being positive definite matices defined by

$$B = N^T N + nI_n \quad \text{and} \quad A = B + M^T M + nI_n, \tag{4.1}$$

where M, N are randomly $n \times n$ matrices and I_n is the identity matrix.
Moreover, $\partial f(x, \cdot)(x) = \{(A+B)x\}$ and $\|(A+B)x - (A+B)y\| \leq \|A+B\|\|x-y\|$ for all $x, y \in \mathbb{R}^n$. Thus the mapping $x \to \partial f(x, \cdot)(x)$ is bounded and $\|A+B\|$-Lipschitz continuous on every bounded subset of H.

It is easy to see that all the conditions of Theorem 3.3 and of Theorem 3.1 in [40] are satisfied. New, we compare the performance of our algorithm and algorithm of Yuying et al. [40], we take $\lambda_k = \frac{1}{k+5}$, $\alpha_k = \frac{1}{k+4}$, $\eta_k = \frac{k+1}{3(k+4)}$, $\mu = \frac{2}{\|A+B\|^2}$, the same starting point $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall\, i = 1, 2, ..., n\}$ and $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall\, i = 1, 2, ..., n\}$ for all the algorithms. For Algorithm 3.1, we choose $\epsilon_k = \frac{1}{k^{1.1}}$, $\theta \in [0, 1)$ and θ_k such that $0 \leq \theta_k \leq \bar{\theta_k}$, where

$$\theta_k = \begin{cases} \min\left\{\theta, \dfrac{1}{k^{1.1}\,\|x_k - x_{k-1}\|}\right\} & \text{if } x_k \neq x_{k-1}, \\ \theta & \text{if otherwise.} \end{cases}$$

To terminate the algorithm, we used the stopping criteria $\|x_{k+1} - x_k\| < \varepsilon$ with $\varepsilon = 10^{-6}$ is a tolerance. The results are reported in the Table 1 and Table 2, we can see that the number of iterations (No. of Iter.) by Algorithm 3.1 with different inertial parameters ($\theta = 0.6$ and $\theta = 0.9$) is less than that of Yuying et al. Algorithm [40], for two different starting points, we can see that in this example the starting points $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall\, i = 1, 2, ..., n\}$ give better performance than $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall\, i = 1, 2, ..., n\}$. Moreover, Figure 1 and Figure 2 illustrate the numerical behavior of both algorithms. In these figures, the value of errors $\|x_{k+1} - x_k\|$ is represented by the y-axis, number of iterations is represented by the x-axis.

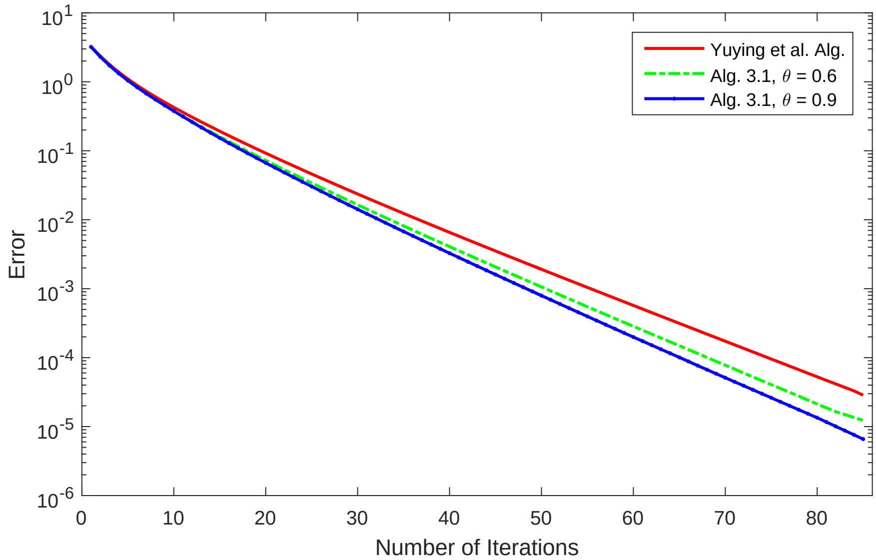

Figure 1: Comparison of proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = (1, 1, ..., 1)^T$ and n=50.

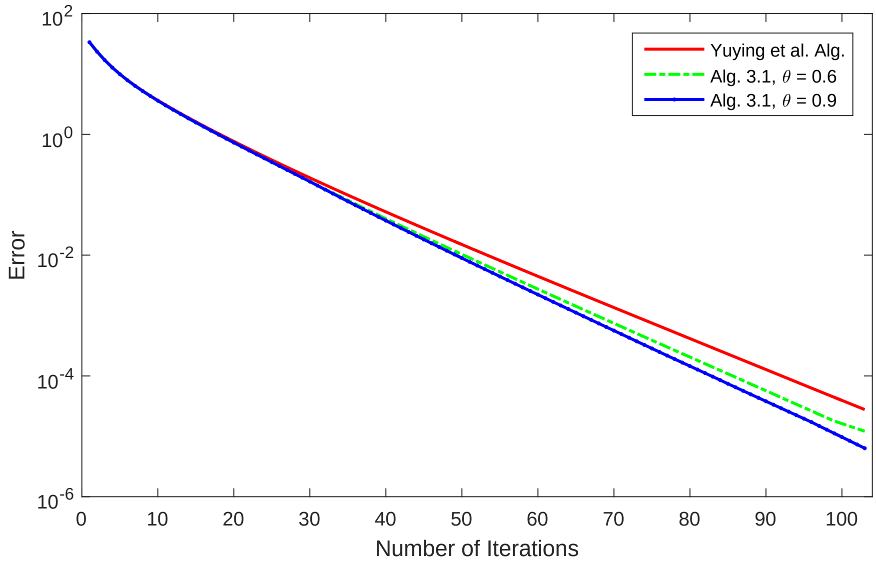

Figure 2: Comparison of proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = (1, 2, ..., 50)^T$ and n=50.

5. Conclusions

In this article, we introduced an iterative algorithm for finding the solution of a bilevel equilibrium problem in real Hilbert space. Under some suitable conditions imposed on parameters, we proved the strong convergence of the algorithm. We showed the efficiency of the proposed algorithm is verified by a numerical experiment and preliminary comparison. These numerical results have also confirmed that the algorithm with inertial effects seems to work better than without inertial effects.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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