On Ramanujan's asymptotic formula for $n!$

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Abstract

In this paper, we present the following new asymptotic formula of factorial n

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-U(n)}, \qquad n \to \infty$$

where $U(n)=\left(\frac{240}{11}n+\frac{9480}{847}+\frac{919466}{65219\,n}+\frac{1455925}{5021863\,n^2}-\frac{6391301400029}{92804028240\,n^3}+...\right)^{-1}$ depending on Ramanujan's approximation formula for $n!$ and we deduce the following upper bound for gamma function $\Gamma(x+1)<\sqrt{\pi}\,(x/e)^x\left[8x^3+4x^2+x+\frac{1}{30}+\frac{1}{\frac{240x}{11}+\frac{9480}{847}}\right]^{1/6}$, $x>0$.

2010 Mathematics Subject Classification: 41A60, 41A25, 33B15.

Key Words: Factorial, Ramanujan's formula, asymptotic formula, best possible constant, rate of convergence,bounds.

1 Introduction.

In many science branches, we need estimations of big factorials. Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \to \infty$$

is the most well known and used approximation formula for factorial n, which is satisfactory in many branches such as statistical physics and statistics but we need more precise estimates in many pure mathematics studies. For more details about Stirling's formula refinements and its related inequalities, we refer to [2], [12], [22].

Other known formula for estimating $n!$ for large values of n is Ramanujan formula:

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}}, \tag{1}$$

which is a refinement of Stirling's formula and was recorded in the book "The lost notebook and other unpublished papers" as a conjecture of Srinivasa Ramanujan based on some numerical evidence. For more details please refer to [1], [4], [13], [24], [29].

Starting from Ramanujan formula (1), Karatsuba presented the following asymptotic formula [13]

$$\Gamma(x+1) \sim \sqrt{\pi}\,(x/e)^x \left[8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\frac{79}{3360x^2}+\frac{3539}{201600x^3}+...\right]^{1/6}, \qquad (2)$$

where $\Gamma(x)=\int_0^\infty e^{-r}r^{x-1}dr$, $x>0$ is the ordinary gamma function and $n!=\Gamma(n+1)$ for $n\in N$. Mortici [23] improve the Ramanujan formula by establishing the following asymptotic formula:

$$\Gamma(x+1) \sim \sqrt{\pi}\,(x/e)^x \left[8x^3+4x^2+x+\frac{1}{30}\right]^{1/6} \exp\left[-\frac{11}{11520x^4}+\frac{13}{3440x^5}+\frac{1}{691200x^6}+...\right], \qquad (3)$$

which is faster than formula (2).

Dumitrescu and Mortici [9] introduced the following class of approximations:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\,(x/e)^x \sqrt[6]{1+\frac{1}{2(x-\delta)}+\frac{\alpha}{2(x-\delta)^2}+\frac{\beta}{2(x-\delta)^3}}, \qquad \alpha,\beta,\delta\in R \qquad (4)$$

which is a generalization of the Ramanujan's formula (1) at $\delta=0$, $\alpha=1/8$ and $\beta=1/240$.

More various results involving approximations for the gamma function and the factorial can be found in [7], [8], [15], [16], [25], [26], [30] and the references therein.

In sequel, we need the following important Lemma, which is due to Mortici in 2010 and is a very useful tool for constructing asymptotic expansions and measuring the convergence rate of a family of null sequences [19]:

Lemma 1.1. *If $\{\sigma_m\}_{m\in N}$ is a null sequence and there is $s\in R$ and $n>1$ such that*

$$\lim_{m\to\infty} m^n(\sigma_m-\sigma_{m+1})=s, \qquad (5)$$

then we have

$$\lim_{m\to\infty} m^{n-1}\sigma_m=\frac{s}{n-1}.$$

From Lemma (1.1), we can conclude that the convergence rate of the sequence $\{\sigma_m\}_{m\in N}$ will increase with the increasing of the value of n in relation (5). Several approximations, formulas and inequalities have been produced using the technique developed by this Lemma. For more details please refer to [5], [6], [11], [14], [17], [20], [21], [28] and the references therein.

In the rest of this paper, we will present a new asymptotic formula of $n!$ depending on Ramanujan's asymptotic formula (1) and we deduce a new upper bound for the ordinary gamma function related to our new asymptotic formula.

2 Main results.

In our first step, we will try to find the best possible constants k_1 and k_2 in the approximation formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{k_1 n+k_2}}, \qquad n\to\infty \tag{6}$$

by defining a sequence A_n satisfies

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{k_1 n+k_2}}\, e^{A_n}, \qquad n\geq 1.$$

Then

$$A_n - A_{n+1} = \left(\frac{1}{12k_1}-\frac{11}{2880}\right)\frac{1}{n^5}+\left(-\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}\right)\frac{1}{n^6}$$
$$+\left(\frac{-9031k_1^3+158200k_1^2+100800k_1k_2+33600k_2^2}{268800k_1^3}\right)\frac{1}{n^7}+O(n^{-8}).$$

If $\left(\frac{1}{12k_1}-\frac{11}{2880}\right)\neq 0$ and $\left(-\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}\right)\neq 0$, then the sequence $A_n - A_{n+1}$ has a rate of worse than n^{-6}. So, we will consider

$$\begin{cases} \frac{1}{12k_1}-\frac{11}{2880}=0 \\ -\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}=0 \end{cases}$$

that is, $k_1=\frac{240}{11}$ and $k_2=\frac{9480}{847}$. Now by Lemma (1.1), we obtain the following result:

Lemma 2.1. *The sequence*

$$A_n = \ln n! - \ln\sqrt{\pi} - n\ln n - n - \frac{1}{6}\ln n\left(8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}}\right) \tag{7}$$

has a rate of convergence equal to n^{-6}, where

$$\lim_{n\to\infty} n^7(A_n - A_{n+1}) = \frac{459733}{124185600}.$$

In our second step, we will try to find the best possible constants T_1, T_2 and T_3 in the approximation formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}+\frac{T_1}{n}+\frac{T_2}{n^2}+\frac{T_3}{n^3}}}, \qquad n\to\infty \tag{8}$$

by defining a sequence B_n satisfies

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}+\frac{T_1}{n}+\frac{T_2}{n^2}+\frac{T_3}{n^3}}}\, e^{B_n}, \qquad n\geq 1.$$

Hence

$$\begin{aligned} B_n - B_{n+1} &= \frac{(919466 - 65219T_1)}{248371200\ n^7} + \frac{(454576437T_1 - 100437726T_2 - 637955952)}{32784998400\ n^8} \\ &+ \frac{1}{265066712064000\ n^9}(42535179617T_1^2 - 1277759560770T_1 + 4664306354407T_2 \\ &- 92804028240T_3 + 16394247383595) \\ &+ \frac{1}{54427031543808000\ n^{10}}(-59336575555595T_1^2 + 1965125297982T_1T_2 \\ &+ 750735798062481T_1 - 361540539736530T_2 + 118464342048360T_3 \\ &- 8420494064916176) \\ &+ \frac{1}{301743462878871552000\ n^{11}}(-2774101878984597T_1^3 + 143136026144382810T_1^2 \\ &- 79155247002714960T_1T_2 + 12105171835569120T_1T_3 \\ &- 1055004771223149285T_1 + 60525859177845607T_2^2 + 6180552136457196960T_2 \\ &- 2679997511635567200T_3 + 10139336461783525554 0) \\ &+ O(n^{-12}). \end{aligned}$$

To obtain the best possible values of the constants T_1, T_2 and T_3, we put

$$\begin{cases} 65219T_1 = 919466 \\ 454576437T_1 - 100437726T_2 = 637955952 \\ 42535179617T_1^2 - 1277759560770T_1 + 4664306354407T_2 - 92804028240T_3 = -16394247383595 \end{cases},$$

that is, $T_1 = \frac{919466}{65219}$, $T_2 = \frac{1455925}{5021863}$ and $T_3 = -\frac{639130140029}{92804028240}$. Hence by Lemma (1.1), we get the following result:

Lemma 2.2. *The sequence*

$$B_n = \ln n! - \ln\sqrt{\pi} - n\ln n - n - \frac{1}{6}\ln n\left(8n^3 + 4n^2 + n + \frac{1}{30} - \frac{1}{\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\ n} + \frac{1455925}{5021863\ n^2} - \frac{639130140029}{92804028240\ n^3}}\right) \tag{9}$$

has a rate of convergence equal to n^{-9}, where

$$\lim_{n\to\infty} n^{10}(B_n - B_{n+1}) = \frac{142970656174139}{1088540630876160000}.$$

In our third step, we can follow the same technique to get the following result:

Lemma 2.3. *The sequence C_n defined by*

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3 + 4n^2 + n + \frac{1}{30} - V(n)}\ e^{C_n},$$

where

$$V(n) = \frac{1}{\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\ n} + \frac{1455925}{5021863\ n^2} - \frac{639130140029}{92804028240\ n^3} + \frac{T_4}{n^4} + \frac{T_5}{n^5} + \frac{T_6}{n^6}},$$

converges to zero as n^{-12} *with the best possible constants* $T_4 = \frac{142970656174139}{42875461046880}$, $T_5 = \frac{288878734012247231}{22009403337398400}$ *and* $T_6 = -\frac{5422052608484409095873}{3965654293332443712000}$ *since*

$$\lim_{n\to\infty} n^{13}(C_n - C_{n+1}) = -\frac{38437701554879448131979}{19141959578859903385600000}.$$

Hence, we get the asymptotic formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-U(n)}, \qquad n\to\infty \tag{10}$$

where

$$U(n) = \left(\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\,n} + \frac{1455925}{5021863\,n^2} - \frac{639130140029}{92804028240\,n^3} + \frac{142970656174139}{42875461046880\,n^4}\right.$$
$$\left. + \frac{288878734012247231}{22009403337398400\,n^5} - \frac{5422052608484409095873}{3965654293332443712000\,n^6} + \ldots\right)^{-1}.$$

3 An inequality of Gamma function.

In this section, we will follow a method presented by Elbert and Laforgia in their paper [10] (see also, [3], [27], [32] and its simple modification in [18]):

Corollary 3.1. *Let* $T(t)$ *be a real-valued function defined on* $t > t_0 \in \mathbb{R}$ *with* $\lim_{t\to\infty} T(t) = 0$. *Then* $T(t) > 0$, *if* $T(t) > T(t+1)$ *for all* $t > t_0$ *and* $T(t) < 0$, *if* $T(t) < T(t+1)$ *for all* $t > t_0$.

Now, Consider the following function

$$F(x) = -\frac{1}{6}\ln\left(8x^3+4x^2+x+\frac{1}{\frac{240x}{11}+\frac{9480}{847}}+\frac{1}{30}\right)+x-x\ln(x)+\ln\Gamma(x+1)-\ln(\sqrt{\pi}), \qquad x>0$$

which satisfies

$$\lim_{x\to\infty} F(x) = 0.$$

$$\begin{aligned} F(x)-F(x+1) &= \frac{-1}{6}\ln\left(8x^3+4x^2+x+\frac{847}{18480x+9480}+\frac{1}{30}\right) - x\ln(x) + x\ln(x+1) \\ &+ \frac{1}{6}\ln\left(8(x+1)^3+4(x+1)^2+x+\frac{847}{18480x+27960}+\frac{31}{30}\right) - 1 \\ &\doteqdot H(x) \end{aligned}$$

The function $H(x)$ satisfies

$$H''(x) = \frac{H_1(x)}{H_2(x)} < 0, \qquad x > 0$$

where

$$\begin{aligned} H_1(x) = & - 1.84724\times10^{29}x^{16} - 2.37023\times10^{30}x^{15} - 1.39723\times10^{31}x^{14} - 5.01631\times10^{31}x^{13} \\ & - 1.22596\times10^{32}x^{12} - 2.15964\times10^{32}x^{11} - 2.83269\times10^{32}x^{10} - 2.81806\times10^{32}x^{9} \\ & - 2.14586\times10^{32}x^{8} - 1.25283\times10^{32}x^{7} - 5.57791\times10^{31}x^{6} - 1.86841\times10^{31}x^{5} \\ & - 4.59618\times10^{30}x^{4} - 7.9786\times10^{29}x^{3} - 9.149\times10^{28}x^{2} - 6.15185\times10^{27}x \\ & - 1.83421\times10^{26} < 0 \end{aligned}$$

and

$$H_2(x) = 3x(x+1)^2(154x+79)^2(154x+233)^2\left(147840x^4+149760x^3+56400x^2+10096x+1163\right)^2$$
$$\left(147840x^4+741120x^3+1392720x^2+1163536x+365259\right)^2.$$

Then $H(x)$ is strictly concave function satisfies

$$\lim_{x\to0} H(x) = \frac{1}{6}\left(\log\left(\frac{28855461}{270979}\right)-6\right) < 0$$

and

$$\lim_{x\to\infty} H(x) = 0.$$

So, $F(x) < 0$ for $x > 0$ and hence we get the following inequality

Lemma 3.2.

$$\Gamma(x+1) < \sqrt{\pi}\,(x/e)^x\left[8x^3+4x^2+x+\frac{1}{30}+\frac{1}{\frac{240x}{11}+\frac{9480}{847}}\right]^{1/6}, \qquad x>0. \tag{11}$$

Remark 1. In 2018, Yang and Tian [31] presented the inequality

$$\Gamma(x+1) < \left(\frac{x^2+\frac{6\gamma}{\pi^2-12\gamma}}{x+\frac{6\gamma}{\pi^2-12\gamma}}\right)^{\frac{6\gamma^2}{\pi^2-12\gamma}}, \qquad 0<x<1 \tag{12}$$

which is not included in inequality (11).

Remark 2. From the spirit of the previous inequality (11), we can suggest the following inequality:

$$\Gamma(x+1) > \sqrt[6]{\frac{9480}{1163}}\,(x/e)^x\left[8x^3+4x^2+x+\frac{1}{30}+\frac{1}{\frac{240x}{11}+\frac{9480}{847}}\right]^{1/6}, \qquad x>0$$

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