L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces

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Abstract

In this paper, we study the existence of solutions and L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces. We no longer require the Azera-Ascoli theorem to prove the existence of continuous solutions of nonlinear differential systems, but instead we apply the regularity results of general linear differential equations to the case the L^2-primitive process for retarded stochastic neutral functional differential systems with unbounded principal operators, delay terms and local Lipschitz continuity of the nonlinear term. Finally, we give a simple example to which our main result can be applied.

Keywords: stochastic neutral differential equations, retarded system, L^2-primitive process, analytic semigroup, fractional power

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1 Introduction

In this paper, we study the existence of solutions and L^2-primitive process for the following retarded stochastic neutral functional differential equations in Hilbert spaces:

$$\begin{cases} d[x(t)+g(t,x_t)] = [Ax(t)+\int_{-h}^{0} a_1(s)A_1x(t+s)ds + k(t)]dt + f(t,x_t)dW(t), \\ x(0)=\phi^0 \in L^2(\Omega,H), \quad x(s)=\phi^1(s),\ s\in[-h,0). \end{cases} \tag{1.1}$$

where $t>0$, $h>0$, $a_1(\cdot)$ is Hölder continuous, k is a forcing term, $W(t)$ stands for K-valued Brownian motion or Winner process with a finite trace nuclear covariance operator Q, and g, f, are given functions satisfying some assumptions. Moreover, $A: D(A)\subset H \to H$ is unbounded and $A_1 \in B(H)$, where $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$.

This kind of systems arises in many practical mathematical models, such as, population dynamics, physical, biological and engineering problems, etc. (see [6, 11, 23]).

Many authors have studied for the theory of stochastic differential equations in a variety of ways (see [4] [7] and reference therein), impulsive stochastic neutral differential equations [14, 21], approximate controllability of stochastic equations [5, 27, 26].

As for the retarded differential equations, Jeong et al [17, 18], Wang [32], and Sukavanam et al. [28] have discussed the regularity of solutions and controllability of the semilinear retarded systems, and see [8, 15, 16, 24] and references therein for the linear retarded systems.

In [10, 12, 13], the authors have discussed the existence of solutions for mild solutions for the neutral differential systems with state-dependence delay. Most studies about the neutral initial value problems governed by retarded semilinear parabolic equation have been devoted to the control problems.

Recently, second order neutral impulsive integrodifferential systems have been studied in [2, 25], and Stochastic differential systems with impulsive conditions in [1, 3, 29]. Further, as for impulsive neutral stochastic differential inclusions with nonlocal initial conditions have been studied for the existence results by Lin and Hu [22], and controllability results by [19].

Let $(\Omega,\mathcal{F},P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t\in I\}$ satisfying $\mathcal{F}_t\subset\mathcal{F}$. An H valued random variables is an $\mathcal{F}$ -measurable function $x(t):\Omega\to H$. Usually we suppress the dependence on $w\in\Omega$ in the stochastic process $\mathcal{S}=\{x(t,w):\Omega\to H:$

$t \in [0,T]\}$and write $x(t)$ instead of $x(t,w)$ and $x(t) : [0,T] \to H$ in the space of $\mathcal{S}$. Then we have to study on results in connection with solutions of random differential and integral equations in Hilbert spaces. It should be ensured that $x(t,w)$ is a H-valued random variable with finite second moments and L^2-primitive process of (1.1) for all $t \in T$ in order to study stationary random function, Brownian motion, Markov process, and etc. But the papers treating the regularity for second moments of the systems and L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces are not many.

In this paper, we propose a different approach of the earlier works used Azera-Ascoli theorem to prove the existence of the mild solutions of functional differential systems in the Banach space of all continuous functions. Our approach is that regularity results of general differential equations results of the linear cases of Di Blasio et al. [8] and semilinear cases of [17] remain valid under the above formulation of the stochastic neutral differential system (1.1) even though the system (1.1) contains unbounded principal operators, delay term and local Lipschitz continuity of the nonlinear term.

The paper is organized as follows. In Section 2, we construct the strict solution of the semilinear functional differential equations and introduce basic properties. In Section 3, by using properties of the strict solutions in dealt in Section 2, we will obtain the L^2-primitive process of (1.1), and a variation of constant formula of L^2-primitive process of (1.1) on the solution space. Finally, we give a simple example to which our main result can be applied.

2 Preliminaries and Lemmas

The inner product and norm in H are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. V is another Hilbert space densely and continuously embedded in H. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. For brevity we may regard that

$$||u||_* \le |u| \le ||u||, \quad u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0. \tag{2.2}$$

Let A be the operator associated with the sesquilinear form $-a(\cdot,\cdot)$:

$$((c_1 - A)u, v) = -a(u,v), \quad u,\ v \in V.$$

It follows from (2.2) that for every $u \in V$

$$\text{Re}\,(Au,u) \ge c_0||u||^2.$$

Then A is a bounded linear operator from V to V^* according to the Lax-Milgram theorem, and its realization in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

is also denoted by A. Then A generates an analytic semigroup $S(t) = e^{tA}$ in both H and V^* as in Theorem 3.6.1 of [30]. Moreover, there exists a constant C_0 such that

$$||u|| \le C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}, \tag{2.3}$$

for every $u \in D(A)$, where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^**(Section 1.3.3 of [31]).*

If X is a Banach space and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into X the p-th powers of norms are integrable.

For the sake of simplicity we assume that the semigroup $S(t)$ generated by A is uniformly bounded, that is, There exists a constant M_0 such that

$$||S(t)||_{B(H)} \le M_0, \quad ||AS(t)||_{B(H)} \le \frac{M_0}{t}. \tag{2.4}$$

The following lemma is from [30, Lemma 3.6.2].

Lemma 2.2. *There exists a constant* M_0 *such that the following inequalities hold:*

$$||S(t)||_{B(H,V)} \le t^{-1/2} M_0, \tag{2.5}$$
$$||S(t)||_{B(V^*,V)} \le t^{-1} M_0, \tag{2.6}$$
$$||AS(t)||_{B(H,V)} \le t^{-3/2} M_0. \tag{2.7}$$

First, consider the following initial value problem for the abstract linear parabolic equation

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + \int_{-h}^{0} a_1(s)A_1x(t+s)ds + k(t), \quad 0 < t \le T, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \ s \in [-h, 0). \end{cases} \tag{2.8}$$

By virtue of Theorem 2.1 of [15] or [8], we have the following result on the corresponding linear equation of (2.8).

Lemma 2.3. *1) For $(\phi^0, \phi^1) \in V \times L^2(-h, 0; D(A))$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (2.8) belonging to*

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||\phi^0|| + ||\phi^1||_{L^2(-h,0;D(A))} + ||k||_{L^2(0,T;H)}), \tag{2.9}$$

where C_1 is a constant depending on T and

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} = \max\{||x||_{L^2(0,T;D(A))}, ||x||_{W^{1,2}(0,T;H)}\}$$

(2) Let $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (2.8) belonging to

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}), \tag{2.10}$$

where C_1 is a constant depending on T.

Let the solution spaces $\mathcal{W}_0(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}_0(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}_0(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

Thus, there exists a constant $c_1 > 0$ such that

$$||x||_{C([0,T];V)} \le c_1||x||_{\mathcal{W}_0(T)}, \quad ||x||_{C([0,T];H)} \le c_1||x||_{\mathcal{W}_1(T)}. \tag{2.11}$$

Lemma 2.4. *Suppose that* $k \in L^2(0,T;H)$ *and* $x(t) = \int_0^t S(t-s)k(s)ds$ *for* $0 \le t \le T$. *Then there exists a constant* C_2 *such that*

$$\begin{aligned} ||x||_{L^2(0,T;D(A))} &\le C_1||k||_{L^2(0,T;H)}, \\ ||x||_{L^2(0,T;H)} &\le C_2T||k||_{L^2(0,T;H)}, \end{aligned} \tag{2.12}$$

and

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.13}$$

Proof. The first assertion is immediately obtained by (2.9). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2dt \\ &\le M_0\int_0^T (\int_0^t |k(s)|ds)^2dt \\ &\le M_0\int_0^T t\int_0^t |k(s)|^2dsdt \\ &\le M_0\frac{T^2}{2}\int_0^T |k(s)|^2ds, \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M_0/2}||k||_{L^2(0,T;H)}. \tag{2.14}$$

From (2.3), (2.12), and (2.14) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1T}(M_0/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M_0/2}, C_0\sqrt{C_1}(M_0/2)^{1/4}\},$$

the proof is complete. □

In what follows in this section, we assume $c_1 = 0$ in (2.2) without any loss of generality. So we have that $0 \in \rho(A)$ and the closed half plane $\{\lambda : \operatorname{Re}\lambda \ge 0\}$ is contained in the resolvent set of A. In this case, it is possible to define the fractional power A^α for $\alpha > 0$. The subspace $D(A^\alpha)$ is dense in H and the expression

$$||x||_\alpha = ||A^\alpha x||, \quad x \in D(A^\alpha)$$

defines a norm on $D(A^\alpha)$. It is also well known that A^α is a closed operator with its domain dense and $D(A^\alpha) \supset D(A^\beta)$ for $0 < \alpha < \beta$. Due to the well known fact that $A^{-\alpha}$ is a bounded operator, we can assume that there is a constant $C_{-\alpha} > 0$ such that

$$||A^{-\alpha}||_{\mathcal{L}(H)} \le C_{-\alpha}, \quad ||A^{-\alpha}||_{\mathcal{L}(V^*,V)} \le C_{-\alpha}. \tag{2.15}$$

Lemma 2.5. *For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$:*

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \leq \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(H,V)} \leq \frac{C_\alpha}{t^{3\alpha/2}}. \tag{2.16}$$

Proof. The relation is from the inequalities (2.6) and (2.7) by properties of fractional power of A and the definition of $S(t)$. □

3 Existence of solutions

In this paper $(H, |\cdot|)$ and $(K, |\cdot|_K)$ denote real separable Hilbert spaces. Consider the following retarded semilinear impulsive neutral differential system in Hilbert space H:

$$\begin{cases} d[x(t) + g(t, x_t)] = [Ax(t) + \int_{-h}^0 a_1(s)A_1 x(t+s)ds + k(t)]dt + f(t, x_t)dW(t), \\ x(0) = \phi^0 \in L^2(\Omega, H), \quad x(s) = \phi^1(s), \ s \in [-h, 0). \end{cases} \tag{3.1}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t \in I\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

An H valued random variables is an $\mathcal{F}$ -measurable function $x(t) : \Omega \to H$ and the collection of random variables $\mathcal{S} = \{x(t, w) : \Omega \to H \ : \ t \in [0, T], \ w \in \Omega\}$ is a stochastic process. Generally, we just write $x(t)$ instead of $x(t, w)$ and $x(t) : [0, T] \to H$ in the space of $\mathcal{S}$

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal basis of K, and let $Q \in B(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $\mathrm{Tr}(Q) = \sum_{n=1}^\infty \sqrt{\lambda_n} = \lambda < \infty$ (Tr denotes the trace of the operator), where $\lambda_n \geq 0 (n = 1, 2, \cdots)$, and $B(K, K)$ denotes the space of all bounded linea operators from K into K.

$\{W(t) : t \geq 0\}$ be a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator Q over $(\Omega, \mathcal{F}, P)$, which satisfies that

$$W(t) = \sum_{n=1}^\infty \sqrt{\lambda_n} w_i(t) e_n, \quad t \geq 0,$$

where $\{w_i(t)\}_{i=1}^\infty$ be mutually independent one dimensional standard Wiener processes over $(\Omega, \mathcal{F}, P)$. Then the above K-valued stochastic process $W(t)$ is called a Q-Wiener process.

We assume that $\mathcal{F}_t = \sigma\{W(s) : 0 \le s \le t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\psi \in B(K, H)$ and define

$$|\psi|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} |\sqrt{\lambda_n}\psi e_n|^2.$$

If $|\psi|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. $B_Q(K, H)$ stands for the space of all Q-Hilbert-Schmidt operators. The completion $B_Q(K, H)$ of $B(K, H)$ with respect to the topology induced by the norm $|\psi|_Q$, where $|\psi|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology.

Let V be a dense subspace of H as mentioned in Section 2. For $T > 0$ we define

$$M^2(-h, T; V) = \{x : [-h, T] \to V : E(\int_{-h}^{T} ||x(s)||^2 ds) < \infty\}$$

with norm defined by

$$||x||_{M^2(0,T;V)} = \big[E(\int_{-h}^{T} ||x(s)||^2 ds)\big]^{1/2}.$$

The spaces $M^2(-h, 0; V)$, $M^2(0, T; V)$, and $M^2(0, T; V^*)$ are also defined as the same way and the basic theory of the class of all nonanticipative functions can be founded in [9]. For $h > 0$, we assume that $\phi^1 : [-h, 0) \to V$ is a given initial value satisfying

$$E(\int_{-h}^{0} ||\phi^1(s)||^2 ds) < \infty,$$

that is, $\phi^1 \in M^2(-h, 0; V)$. In this note, a random variable $x(t) : \Omega \to H$ will be called an L^2-primitive process if $x \in M^2(-h, T; V)$.

For each $s \in [0, T]$, we define $x_s : [-h, 0] \to H$ as

$$x_s(r) = x(s + r), \quad -h \le r \le 0.$$

We will set

$$\Pi = M^2(-h, 0; V).$$

Definition 3.1. *A stochastic process $x : [-h, T] \times \Omega \to H$ is called a solution of (3.1) if*

(i) *$x(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \ge 0$.*

(ii) *$x(t) \in H$ has cádlág paths on $t \in (0,T)$ such that*

$$\begin{aligned} x(t) =& S(t)[\phi^0 + g(0,x_0)] - g(t,x_t) + \int_0^t AS(t-s)g(s,x_s)ds \\ &+ \int_0^t S(t-s)\{\int_{-h}^0 a_1(\tau)A_1 x(s+\tau)d\tau ds + f(s,x_s)dW(s)\} \\ &+ \int_0^t S(t-s)k(s)ds, \\ x(0) =& \phi^0, \quad x(s) = \phi^1(s), \quad s \in [-h,0). \end{aligned} \tag{3.2}$$

(iii) *$x \in M^2(0,T;V)$ i.e., $E(\int_0^T ||x(s)||^2 ds) < \infty$ and $C([0,T];H)$.*

To establish our results, we introduce the following assumptions on system (3.1).
Assumption (A). We assume that $a_1(\cdot)$ is Hölder continuous of order ρ:

$$|a_1(0)| \le H_1, \quad |a_1(s) - a_1(\tau)| \le H_1(s-\tau)^\rho.$$

Assumption (G). Let $g : [0,T] \times \Pi \to H$ be a nonlinear mapping satisfying the following conditions hold:

(i) For any $x \in \Pi$, the mapping $g(\cdot, x)$ is strongly measurable.

(ii) There exist positive constants L_g and $\beta > 2/3$ such that

$$\begin{aligned} &E|A^\beta g(t,x)|^2 \le L_g(||x||_\Pi + 1)^2, \\ &E|A^\beta g(t,x) - A^\beta g(t,\hat{x})|^2 \le L_g||x - \hat{x}||_\Pi^2, \end{aligned}$$

for all $t \in [0,T]$, and $x, \hat{x} \in \Pi$.

Assumption (F). Let $f : \mathbb{R} \times \Pi \to B(K,H)$ be a nonlinear mapping satisfying the following:

(i) For any $x \in \Pi$, the mapping $f(\cdot, x)$ is strongly measurable.

(ii) There exists a function $L_f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$E|f(t,x) - f(t,y)|^2 \le L_f(r)||x - y||_\Pi^2, \quad t \in [0,T]$$

hold for $||x||_\Pi \le r$ and $||y||_\Pi \le r$.

(iii) The inequality

$$E|f(t,x)|^2 \le L_f(r)(||x||_\Pi + 1)^2$$

holds for every $t \in [0,T]$ and $||x||_\Pi \le r$.

Lemma 3.1. *Let $x \in M^2(-h,T;V)$. Then the mapping $s \mapsto x_s$ belongs to $C([0,T];\Pi)$, and for each $0 < t \le T$*

$$\begin{aligned} &||x_t||_\Pi \le ||x||_{M^2(-h,t;V)} = ||\phi^1||_\Pi + ||x||_{M^2(0,t;V)}, \\ &E(||x||^2_{L^2(0,t;V)}) = ||x||^2_{M^2(0,t;V)}, \\ &||x.||_{L^2(0,t;\Pi)} \le \sqrt{t}||x||_{M^2(-h,t;V)}. \end{aligned} \tag{3.3}$$

Proof. The first paragraph is easy to verify. In fact, it is from the following inequality;

$$||x_t||^2_\Pi = E\big(\int_{-h}^0 ||x(t+\tau)||^2 d\tau\big) \le E\big[\int_{-h}^t ||x(\tau)||^2 d\tau\big] \le ||x||^2_{M^2(-h,t;V)}, \ t > 0.$$

The second paragraph is immediately obtained by definition. From the inequality (3.3), we have

$$\begin{aligned} \int_0^t ||x_s||^2_\Pi ds &= \int_0^t \big[E\big(\int_{-h}^0 ||x(s+\tau)||^2 d\tau\big)\big]^2 ds \\ &= \int_0^t \big[E\big(\int_{s-h}^s ||x(\tau)||^2 d\tau\big)\big]^2 ds \le t||x||^2_{M^2(-h,t;V)}, \end{aligned}$$

which completes the last paragraph. □

One of the main useful tools in the proof of existence theorems for nonlinear functional equations is the following Sadvoskii's fixed point theorem.

Lemma 3.2. *(Krasnoselski [20]) Suppose that Σ is a closed convex subset of a Banach space X. Assume that K_1 and K_2 are mappings from Σ into X such that the following conditions are satisfied:*

(i) $(K_1 + K_2)(\Sigma) \subset \Sigma$,

(ii) K_1 is a completely continuous mapping,

(iii) K_2 is a contraction mapping.

Then the operator $K_1 + K_2$ has a fixed point in Σ.

From now on, we establish the following results on the solvability of the equation (3.1).

Theorem 3.1. *Let Assumptions (A), (G) and (F) be satisfied. Assume that* $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ *and* $k \in M^2(0, T; V^*)$ *for* $T > 0$. *Then, there exists a solution* x *of the system (3.1) such that*

$$x \in M^2(0, T; V) \cap C([0, T]; H).$$

Moreover, there is a constant C_3 *independent of the initial data* (ϕ^0, ϕ^1) *and the forcing term* k *such that*

$$||x||_{M^2(-h,T;V)} \leq C_3(1 + E(|\phi^0|^2) + ||\phi^1||_\Pi + ||k||_{M^2(0,T;V^*)}). \tag{3.4}$$

Proof. Let

$$r := 2\big[C_1 C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) + \sqrt{3}C_1\big(E(|\phi^0|^2) + ||\phi^1||_\Pi^2 + ||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\big],$$

and

$$\begin{aligned} N := &\sqrt{3}C_{-\alpha}\sqrt{L_g}\big(||\phi^1||_\Pi + r + 1\big) \\ &+ (3\beta - 2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}(||\phi^1||_\Pi + r + 1) \\ &+ C_2\mathrm{Tr}(Q)\sqrt{L_f(r)}(||\phi^1||_\Pi + r + 1), \end{aligned}$$

where $\beta > 2/3$, C_1 and C_2 is constants in Lemma 2.3 and Lemma 2.4, respectively. Let

$$T_1^\gamma := \max\{T_1^{1/2}, T_1^{3\beta/2}\}$$

and choose $0 < T_1 < T$ such that

$$T_1^\gamma N \leq \frac{r}{2} = C_1 C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) + \sqrt{3}C_1\big(\sqrt{E(|\phi^0|^2)} + ||\phi^1||_\Pi + ||k||_{M^2(0,T_1;V^*)}, \tag{3.5}$$

and

$$\hat{N} := T_1^\gamma\big\{\sqrt{3}C_{-\alpha}\sqrt{L_g} + (3\beta - 1)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g} + C_2\mathrm{Tr}(Q)\sqrt{L_f(r)}\big\} < 1. \tag{3.6}$$

Let J be the operator on $M^2(0,T_1;V)$ defined by

$$\begin{aligned}(Jx)(t) =&S(t)[\phi^0+g(0,\phi^1)]-g(t,x_t)+\int_0^t AS(t-s)g(s,x_s)ds\\&+\int_0^t S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau ds+f(s,x_s)dW(s)\big\}\\&+\int_0^t S(t-s)k(s)ds.\end{aligned}$$

It is easily seen that J is continuous from $C([0,T_1];H)$ into itself. Let

$$\Sigma=\{x\in M^2(-h,T;V):x(0)=\phi^0,\ \text{and } x(s)=\phi^1(s)(s\in[-h,0))\}.$$

and

$$\Sigma_r=\{x\in\Sigma:||x||_{M^2(0,T_1;V)}\le r\},$$

which is a bounded closed subset of $M^2(0,T_1;V)$. Now, we give the proof of Theorem 3.1 in the following several steps:

Now, in order to show that the operator J has a fixed point in $\Sigma_r\subset M^2(0,T_1;V)$, we take the following steps.

Step 1. J maps Σ_r into Σ_r.

By (2.10), (2.15) and Assumption (G), and noting $x_0=\phi^1$, we know

$$\begin{aligned}E\big[\int_0^{T_1}\|S(t)g(0,x_0)\|^2dt\big]&=E\big[C_1^2|g(0,\phi^1)|^2\big]\\&=E\big[C_1^2\big(||A^{-\beta}||_{B(H)}|A^\beta g(0,\phi^1)|\big)^2\big]\\&\le(C_1C_{-\alpha})^2L_g(||\phi^1||_\Pi+1)^2.\end{aligned}\tag{3.7}$$

From (2.10) of Lemma 2.3 it follows

$$\begin{aligned}&E\big[\int_0^{T_1}\big\|S(t)\phi^0+\int_0^t S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau+k(s)\big\}ds\big\|^2dt\big]\\&\le E\big[C_1^2\{|\phi^0|+||\phi^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T_1;V^*)}\}^2\big]\\&\le 3C_1^2\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big).\end{aligned}\tag{3.8}$$

By using Assumption (G) and Lemma 3.1, we have

$$\begin{aligned}||g(\cdot,x.)||^2_{M^2(0,T_1;V)}&=E\big(\int_0^{T_1}\big\|A^{-\beta}A^\beta g(t,x_t)\big\|^2dt\big)\\&\le C_{-\alpha}^2E\big(\int_0^{T_1}\big|A^\beta g(t,x_t)\big|^2dt\big)\le C_{-\alpha}^2L_gT_1\big(||x_t||_\Pi+1\big)^2\\&\le 3C_{-\alpha}^2L_gT_1\big(||\phi^1||_\Pi^2+||x||_{M^2(0,T_1;V)}+1\big)\end{aligned}\tag{3.9}$$

Define $H_1 : M^2(0,T_1;V) \to M^2(0,T_1;V)$ by

$$(H_1x)(t) = \int_0^t AS(t-s)g(s,x_s)ds.$$

Then from Lemma 2.5 and Assumption (G) we have

$$\begin{aligned}||AS(t-s)g(s,x_s)|| &= ||A^{1-\beta}S(t-s)||_{B(H,V)}|A^\beta g(s,x_s)| \\ &\le \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^\beta g(s,x_s)|,\end{aligned}$$

and hence, by using Hólder inequality and Assumption (G),

$$\begin{aligned}&||H_1x||^2_{M^2(0,T_1;V)} = E\big[\int_0^{T_1} \Big\|\int_0^t AS(t-s)g(s,x_s)ds\Big\|^2 dt\big] \qquad (3.10)\\ &\le E\big[\int_0^{T_1}\big(\int_0^t \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^\beta g(s,x_s)|ds\big)^2 dt\big]\\ &\le E\big[\int_0^{T_1} C^2_{1-\beta}(3\beta-2)^{-1}t^{3\beta-2}\int_0^t |A^\beta g(s,x_s)|^2 dsdt\big]\\ &\le (3\beta-2)^{-1}C^2_{1-\beta}L_g(||x_s||_\Pi+1)^2\int_0^{T_1} t^{3\beta-1}dt\\ &\le (3\beta-2)^{-1}(3\beta)^{-1}C^2_{1-\beta}L_gT_1^{3\beta}(||x||_{M^2(-h,T_1;V)}+1)^2\\ &= (3\beta-2)^{-1}(3\beta)^{-1}C^2_{1-\beta}L_gT_1^{3\beta}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)^2\end{aligned}$$

Let

$$(H_2x)(t) = \int_0^t S(t-s)f(s,x_s)dW(s).$$

For (2.13) of Lemma 2.4 it follows

$$\begin{aligned}||H_2x||^2_{M^2(0,T_1;V)} &= E\big[\int_0^{T_1}\Big|\int_0^t S(t-s)f(s,x_s)dW(s)\Big|^2 dt\big] \qquad (3.11)\\ &\le E[C_2^2\mathrm{Tr}(Q)^2T_1||f(s,x_s)||^2_{L^2(0,T;V^*)}]\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1||f(s,x_s)||^2_{M^2(0,T;V^*)}\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1L_f(r)(||x_s||_\Pi+1)^2\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1L_f(r)(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)^2\end{aligned}$$

Therefore, from (3.7)-(3.11) it follows that

$$\begin{aligned}||Jx||_{M^2(0,T_1;V)} \leq & C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) \\ & + \sqrt{3}C_1\big(E[|\phi^0|^2] + ||\phi^1||_\Pi^2 + ||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2} \\ & + \sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1\big) \\ & + (3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1) \\ & + C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1) \\ \leq & C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) \\ & + \sqrt{3}C_1\big(E[|\phi^0|^2] + ||\phi^1||_\Pi^2 + ||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2} + T_1^\gamma N \leq r,\end{aligned}$$

and so, J maps Σ_r into Σ_r.

Define mapping $K_1 + K_2$ on $L^2(0, T_1; V)$ by the formula

$$\begin{aligned}&(Jx)(t) = (K_1x)(t) + (K_2x)(t), \\ &(K_1x)(t) = \int_0^t S(t-s)\int_0^s a_1(\tau - s)A_1x(\tau)d\tau ds,\end{aligned}$$

and

$$\begin{aligned}(K_2x)(t) =& S(t)[\phi^0 + g(0,x_0)] - g(t,x_t) + \int_0^t AS(t-s)g(s,x_s)ds \\ & + \int_0^t S(t-s)\big\{\int_{s-h}^0 a_1(\tau-s)A_1\phi^1(\tau)d\tau ds + f(s,x_s)dW(s)\big\} \\ & + \int_0^t S(t-s)k(s)ds.\end{aligned}$$

Step 2. K_1 is a completely continuous mapping.

We can now employ Lemma 3.2 with Σ_r. Assume that a sequence $\{x_n\}$ of $M^2(0, T_1; V)$ converges weakly to an element $x_\infty \in M^2(0, T_1; V)$, i.e., $w-\lim_{n\to\infty} x_n = x_\infty$. Then we will show that

$$\lim_{n\to\infty} ||K_1x_n - K_1x_\infty||_{M^2(0,T_1;V)} = 0, \tag{3.12}$$

which is equivalent to the completely continuity of K_1 since $M^2(0, T_1; V)$ is reflexive. For a fixed $t \in [0, T_1]$, let $x_t^*(x) = (K_1x)(t)$ for every $x \in M^2(0, T_1; V)$. Then $x_t^* \in M^2(0, T_1; V^*)$ and we have $\lim_{n\to\infty} x_t^*(x_n) = x_t^*(x_\infty)$ since $w-\lim_{n\to\infty} x_n = x_\infty$. Hence,

$$\lim_{n\to\infty}(K_1x_n)(t) = (K_1x_\infty)(t), \quad t \in [0, T_1].$$

Set

$$h(s)=\int_0^s a_1(\tau-s)A_1x(\tau)d\tau.$$

Then by using Hólder inequality we obtain the following inequality

$$\begin{aligned}|h(s)|&\le \Big|\int_0^s (a_1(\tau-s)-a_1(0))A_1x(\tau)d\tau\Big| \\ &\quad+\Big|\int_0^s a_1(0)A_1x(\tau)d\tau\Big| \\ &\le \big\{\big((2\rho+1)^{-1}s^{2\rho+1}\big)^{1/2}+\sqrt{s}\big\}H_1\|A_1\|_{B(H)}\big(\int_0^s \|x(\tau)\|^2d\tau\big)^{1/2}.\end{aligned}\tag{3.13}$$

Thus, by (2.5) and (3.13) it holds

$$\begin{aligned}&\|(K_1x)(t)\|^2=\Big\|\int_0^t S(t-s)h(s)ds\Big\|^2\\ &\le (H_1\|A_1\|_{B(H)})^2\big(\int_0^t \|x(\tau)\|^2d\tau\big)\Big\|\int_0^t\frac{1}{(t-s)^{1/2}}\{((2\rho+1)^{-1}s^{(2\rho+1)/2}+\sqrt{s}\}ds\Big\|^2\\ &\le (H_1\|A_1\|_{B(H)})^2\|x\|^2_{L^2(0,t;V)}\big\{(2\rho+1)^{-1}B(1/2,(2\rho+3)/2)t^{\rho+1}+B(1/2,3/2)t\big\}^2.\\ &:=c_2\|x\|^2_{L^2(0,t;V)},\end{aligned}$$

where c_2 is a constant and $B(\cdot,\cdot)$ is the Beta function. Here we used

$$B(1/2,(2\rho+3)/2)t^{\rho+1}=\int_0^t(t-s)^{-1/2}s^{(2\rho+1)/2}ds.$$

And we know

$$\sup_{0\le t\le T_1}\|E[(K_1x)(t)]^2\|\le c_2\|x\|^2_{M^2(0,T_1;V)}\le\infty.$$

Therefore, by Lebesgue's dominated convergence theorem it holds

$$\lim_{n\to\infty}E\big(\int_0^{T_1}\|(K_1x_n)(t)\|^2dt\big)=E\big(\int_0^{T_1}\|(K_1x_\infty)(t)\|^2dt\big),$$

i.e., $\lim_{n\to\infty}\|K_1x_n\|_{M^2(0,T_1;V)}=\|K_1x_\infty\|_{M^2(0,T_1;V)}$. Since $M^2(0,T_1;V)$ is a reflexive space, it holds (3.12).

Step 3. K_2 is a contraction mapping.

For every x_1 and $x_2 \in \Sigma_r$, we have

$$\begin{aligned}(K_2x_1)(t) - (K_2x_2)(t) =& g(t, x_{2_t}) - g(t, x_{1_t})\\ &+ \int_0^t AS(t-s)\big(g(t, x_{1_s}) - g(t, x_{2_s})\big)ds\\ &+ \int_0^t S(t-s)\{f(s, x_1(s)) - f(s, x_2(s))\}dW(s).\end{aligned}$$

In a similar way to (3.9)-(3.11) and Proposition 2.3, we have

$$\begin{aligned}||K_2x_1 - K_2x_2||_{M^2(0,T_1;V)} \le& \big\{C_{-\alpha}\sqrt{T_1L_g} + (3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}T_1^{3\beta/2}\sqrt{L_g}\\ &+ C_2\sqrt{T_1L_f(r)}\mathrm{Tr}(Q)\big\}||x_1 - x_2||_{M^2(0,T_1;V)}\\ \le& \hat{N}||x_1 - x_2||_{M^2(0,T_1;V)}.\end{aligned}$$

So by virtue of the condition (3.6) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $M^2(0, T_1; V)$. This has proved the local existence and uniqueness of the solution of (3.1).

Step 4. We drive a priori estimate of the solution.

To prove the global existence, we establish a variation of constant formula (3.4) of solution of (3.1). Let x be a solution of (3.1) and $\phi^0 \in H$. Then we have that from (3.7)-(3.11) it follows that

$$\begin{aligned}||x||_{M^2(0,T_1;V)} \le& C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1)\\ &+ \sqrt{3}C_1\big(E[|\phi^0|^2] + ||\phi^1||_\Pi^2 + ||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\ &+ \sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1\big)\\ &+ (3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1)\\ &+ C_2\mathrm{Tr}(Q)T_1\sqrt{T_1L_f(r)}(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1)\\ \le& \hat{N}||x||_{L^2(0,T_1;V)} + \hat{N}_1,\end{aligned}$$

where

$$\begin{aligned}\hat{N}_1 =& C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1)\\ &+ \sqrt{3}C_1\big(E[|\phi^0|^2] + ||\phi^1||_\Pi^2 + ||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\ &+ \sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi + 1\big)\\ &+ (3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi + 1)\\ &+ C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi + 1).\end{aligned}$$

Taking into account (3.6) there exists a constant C_3 such that

$$\begin{aligned} ||x||_{L^2(0,T_1;V)} \leq& (1-\hat{N})^{-1}\hat{N}_1 \\ \leq& C_3(1+E(|\phi^0|^2)+||\phi^1||_\Pi+||k||_{M^2(0,T_1;V^*)}), \end{aligned}$$

which obtain the inequality (3.4).

Now we will prove that $E[x(T_1)^2]<\infty$ in order that the solution can be extended to the interval $[T_1,2T_1]$.

Define a mapping $H_3 : L^2(0,T_1;V)\to L^2(0,T_1;V)$ as

$$(H_3x)(t)=S(t)[\phi^0+g(0,x_0)]+\int_0^t S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau+k(s)\big\}ds$$

The from (2.11) and Lemma 2.3 it follows that

$$\begin{aligned} E|(H_3x)(T_1)|^2 &\leq c_1E||H_3x||^2_{\mathcal{W}_1} \\ &\leq 3c_1C_1E\big\{|\phi^0+g(0,\phi^1)|+||\phi^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T_1;V^*)}\big\}^2 \\ &\leq c_1C_1\big\{E|\phi^0+g(0,\phi^1)|^2+||\phi^1||^2_{M^2(-h,0;V)}+||k||^2_{M^2(0,T_1;V^*)}\big\} := I, \end{aligned} \tag{3.14}$$

and from (2.4) and Assumption (F),

$$\begin{aligned} E|(H_2x)(T_1)|^2 =& E\big|\int_0^{T_1} S(T_1-s)f(s,x_s)dW(s)\big|^2 \\ \leq& M_0^2\mathrm{Tr}(Q)^2T_1L_f(r)(||x_s||_\Pi+1)^2 \\ \leq& M_0^2\mathrm{Tr}(Q)^2T_1L_f(r)(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)^2 := II. \end{aligned} \tag{3.15}$$

Moreover, by using Assumption (G) we have

$$\begin{aligned} E|g(T_1,x_{T_1})|^2 &\leq E\big\|A^{-\beta}A^\beta g(t,x_{T_1})\big\|^2, \\ &\leq C\_ALPHA^2L_g\big(||x_{T_1}||_\Pi+1\big)^2 \\ &\leq C\_ALPHA^2L_g\big(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1\big)^2 := III, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} E|(H_1x)(T_1)|^2 &= E|\int_0^{T_1} AS(T_1-s)g(s,x_s)ds|^2 \\ &= E|\int_0^{T_1} A^{1-\beta}W(T_1-s)A^\beta g(s,x_s)ds|^2 \\ &\leq E\big[\int_0^{T_1}\frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^\beta(g(s,x_s)|ds\big]^2 \\ &\leq E\big[C_{1-\beta}^2(3\beta-2)^{-1}T_1^{3\beta-2}\int_0^t|A^\beta(g(s,x_s)|^2ds\big]^2 \\ &= C_{1-\beta}^2(3\beta-2)^{-1}T_1^{3\beta-1}L_g(||x||_{M^2(0,T_1;V)}+||\phi^1||_{M^2(-h,0;V)}+1)^2 := IV. \end{aligned} \tag{3.17}$$

Thus,by (3.14)-(3.17) we have

$$\begin{aligned} E|x(T_1)|^2 &= E\Big|(H_3x)(T_1) - g(T_1, x_{T_1}) + \int_0^{T_1} AS(T_1 - s)g(s, x_s)ds \\ &+ \int_0^{T_1} S(T_1 - s)f(s, x_s)dW(s)\Big| \\ &\leq I + II + III + IV < \infty. \end{aligned}$$

Hence we can solve the equation in $[T_1, 2T_1]$ with the initial $(x(T_1), x_{T_1})$ and an analogous estimate to (3.4). Since the condition (3.6) is independent of initial values, the solution can be extended to the interval $[0, nT_1]$ for any natural number n, and so the proof is complete. □

Remark 3.1. *Thanks for Lemma 2.3, we note that the solution of (3.1) with the conditions of Theorem 3.1 satisfies also that*

$$E(\int_{-h}^{T} ||x'(s)||_*^2 ds) < \infty.$$

Here we note that by a simple calculation using the properties of analytic semigroup, it is immediately seen that $x \in M^2(-h, T; H)$.

Now, we obtain that the solution mapping is continuous in the following result, which is useful for the control problem and physical applications of the given equation.

Theorem 3.2. *Let Assumptions (A), (G) and (F) be satisfied. Assuming that the initial data* $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ *and the forcing term* $k \in M^2(0, T; V^*)$. *Then the solution* x *of the equation (3.1) belongs to* $x \in M^2(0, T; V)$ *and the mapping*

$$L^2(\Omega, H) \times \Pi \times M^2(0, T; V^*) \ni (\phi^0, \phi^1, k) \mapsto x \in M^2(0, T; V) \tag{3.18}$$

is continuous.

Proof. From Theorem 3.1, it follows that if $(\phi^0, \phi^1, k) \in L^2(\Omega, H) \times \Pi \times M^2(0, T; V^*)$ then x belongs to $M^2(0, T; V)$. Let $(\phi_i^0, \phi_i^1, k_i)$ and x^i be the solution of (3.1) with

$(\phi_i^0, \phi_i^1, k_i)$ in place of (ϕ^0, ϕ^1, k) for $i = 1,\ 2$. Let $x_i (i = 1, 2) \in \Sigma_r$. Then it holds

$$\begin{aligned}
&x^1(t) - x^2(t) = S(t)[(\phi_1^0 - \phi_2^0) + (g(0, x_0^1) - g(0, x_0^2))] \\
&- (g(t, x_t^1) - g(t, x_t^2)) + \int_0^t AS(t-s)(g(s, x_s^1) - g(t, x_s^2))ds \\
&+ \int_0^t S(t-s)\{ \int_{-h}^0 a_1(\tau)A_1(x^1(s+\tau) - x^2(s+\tau))d\tau ds \\
&+ \int_0^t S(t-s)\{((Fx^1)(s) - (Fx^2)(s)) + (k_1(s) - k_2(s))\}ds. \\
&+ \int_0^t S(t-s)(k_1(s) - k_2(s))ds
\end{aligned}$$

Hence, by applying the same argument as in the proof of Theorem 3.1, we have

$$||x_1 - x_2||_{M^2(0,T_1;V)} \leq \hat{N}||x_1 - x_2||_{L^2(0,T_1;V)} + \hat{N}_2,$$

where

$$\begin{aligned}
\hat{N}_2 =& C_1 C_ALPHA\sqrt{L_g}(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ \sqrt{3}C_1\big(E[|\phi_1^0 - \phi_1^0|^2] + ||\phi_1^1 - \phi_2^1||_\Pi^2 + ||k_1 - k_2||^2_{M^2(0,T_1;V^*)}\big)^{1/2} \\
&+ \sqrt{3}C_ALPHA\sqrt{T_1 L_g}\big(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ (3\beta - 2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{(3\beta+1)/2}(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ C_2\mathrm{Tr}(Q)\sqrt{T_1 L_f(r)}(||\phi_1^1 - \phi_2^1||_\Pi),
\end{aligned}$$

which implies

$$||x||_{M^2(0,T_1;V)} \leq \hat{N}_2(1 - \hat{N})^{-1}.$$

Therefore, it implies the inequality (3.18). □

4 Example

Let

$$H = L^2(0, \pi),\ V = H_0^1(0, \pi),\ V^* = H^{-1}(0, \pi).$$

Consider the following retarded neutral stochastic differential system in Hilbert space H:

$$\begin{cases} d[x(t,y)+g(t,x_t(t,y))] = [Ax(t,y)+\int_{-h}^{0} a_1(s)A_1x(t+s,y)ds+k(t,y)]dt \\ \qquad\qquad\qquad\qquad +F(t,x(t,y))dW(t), \quad (t,y)\in[0,T]\times[0,\pi], \\ x(0,y)=\phi^0(y)\in L^2(\Omega,H), \quad x(s,y)=\phi^1(s,y), \quad (s,y)\in[-h,0)\times[0,\pi], \end{cases} \tag{3.19}$$

where $h>0$, $a_1(\cdot)$ is Hölder continuous, $A_1\in B(H)$, and $W(t)$ stands for a standard cylindrical Winner process in H defined on a stochastic basis $(\Omega,\mathcal{F},P)$. Let

$$a(u,v)=\int_0^{\pi}\frac{du(y)}{dy}\overline{\frac{dv(y)}{dy}}dy.$$

Then

$$A=\partial^2/\partial y^2 \quad \text{with} \quad D(A)=\{x\in H^2(0,\pi): x(0)=x(\pi)=0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n=-n^2$ and $z_n(y)=(2/\pi)^{1/2}\sin ny$, respectively. Moreover,

(a1) $\{z_n : n\in N\}$ is an orthogonal basis of H and

$$S(t)x=\sum_{n=1}^{\infty} e^{n^2t}(x,z_n)z_n, \quad \forall x\in H,\ t>0.$$

Moreover, there exists a constant M_0 such that $||S(t)||_{B(H)}\le M_0$.

(a2) Let $0<\alpha<1$. Then the fractional power $A^{\alpha}: D(A^{\alpha})\subset H\to H$ of A is given by

$$A^{\alpha}x=\sum_{n=1}^{\infty} n^{2\alpha}(x,z_n)z_n,\ D(A^{\alpha}):=\{x: A^{\alpha}x\in H\}.$$

In particular,

$$A^{-1/2}x=\sum_{n=1}^{\infty}\frac{1}{n}(x,z_n)z_n, \text{ and } ||A^{-1/2}||=1.$$

The nonlinear mapping f is a real valued function belong to $C^2([0,\infty))$ which satisfies the conditions

(f1) $f(0)=0$, $f(r)\ge 0$ for $r>0$,

(f2) $|f^{'}(r) \leq c(r+1)$ and $|f''(r)| \leq c$ for $r \geq 0$ and $c > 0$.

If we present

$$F(t, x(t,y)) = f^{'}(|x(t,y)|^2)x(t,y),$$

Then it is well known that F is a locally Lipschitz continuous mapping from the whole V into H by Sobolev's imbedding theorem (see [30, Theorem 6.1.6]). As an example of q in the above, we can choose $q(r) = \mu^2 r + \eta^2 r^2/2$ (μ and η is constants).

Define $g : [0,T] \times \Pi \to H$ as

$$g(t, x_t) = \sum_{n=1}^{\infty} \int_0^t e^{n^2 t}(\int_{-h}^0 a_2(s)x(t+s)ds, z_n)z_n, \quad , t > 0.$$

Then it can be checked that Assumption (G) is satisfied. Indeed, for $x \in \Pi$, we know

$$Ag(t, x_t) = (S(t) - I) \int_{-h}^0 a_2(s)x(t+s)ds,$$

where I is the identity operator form H to itself and

$$|a_2(0)| \leq H_2, \quad |a_2(s) - a_2(\tau)| \leq H_2(s-\tau)^{\kappa}, \quad s, \ \tau \in [-h, 0]$$

for a constant $\kappa > 0$. Hence we have

$$\begin{aligned} E|Ag(t,x_t)|^2 \leq &(M_0+1)^2 \big\{ \big| \int_{-h}^0 (a_2(s) - a_2(0))x(t+s)d\tau \big|^2 \\ &+ \big| \int_{-h}^0 a_2(0)x(t+s)d\tau \big|^2 \big\} \\ \leq &(M_0+1)^2 H_2^2 \big\{ (2\kappa+1)^{-1} h^{2\rho+1} + h \big\} ||x_t||_{\Pi}^2. \end{aligned}$$

It is immediately seen that Assumption (G) has been satisfied. Thus, all the conditions stated in Theorem 3.1 have been satisfied for the equation (3.19), and so there exists a solution x of the equation (3.19) such that

$$E(\int_{-h}^T ||x(s)||^2 ds) < \infty, \text{ and } E(\int_{-h}^T ||x'(s)||_*^2 ds) < \infty.$$

References

[1] A. Anguraj and A. Vinodkumar, *Existence, uniqueness ans stability of impulsive Stochastic partial neutral functional differential equations with infinite delays*, J. Appl. Math. and Informatics 28 (2010), 3–4.

[2] G. Arthi and K. Balachandran, *Controllability of damped second-order neutral functional differential systems with impulses*, Taiwanese J. Math. 16 (2012), 89-106.

[3] K. Balachandran and R. Sathaya, Controllability of nonlocal Stochastic quasilinear integrodifferential systems, Electron J. Qual. Theory Differ. 50 (2011), 1–16.

[4] P. Balasubramaniam, *Existence of solutions of functional stochastic differential inclusions*, Tamkang J. Math. 33 (2002), 35–43.

[5] P. Balasubramaniam, J. Y. Park and P. Muthukumar, *Approximate controllability of neutral stochastic functional differential systems with infinite delay*, Stoch. Anal. Appl. 28 (2010), 389–400.

[6] A. T. Bharucha-Ried, random Integral Equation, Academic Press, Newyork, 1982.

[7] R. F. Curtain, *Stochastic evolution equations with general white noise disturbance*, J. Math. Anal. Appl. 60(1977), 570–595.

[8] G. Di Blasio, K. Kunisch and E. Sinestrari, L^2-*regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl. 102(1984), 38–57.

[9] A. Friedman, Stochastic Differential Equations & Applications, Academic Press, INC. 1975.

[10] X. Fu,*Controllability of neutral functional differential systems in abstract space*, Appl. Math. Comput. 141 (2003), 281-296.

[11] W. Grecksch and C. Tudor, Stochastic Evolution Equations: A Hilbert space Apprauch, Akademic Verlag, Berlin, 1995.

[12] E. Hernández and M. Mckibben, *On state-dependent delay partial neutral functional differential equations*, Appl. Math. Comput. 186(1) (2007). 294–301.

[13] E. Hernández, M. Mckibben, and H. Henrŕquez, *Existence results for partial neutral functional differential equations with state-dependent delay*, Math. Comput. Modell. 49 (2009), 1260–1267.

[14] L. Hu and Y. Ren, *Existence results for impulsive neutral stochastic functional integro-differential equations with infinite dalays*, Acta Appl. Math. 111(2010), 303-317.

[15] J. M. Jeong, *Stabilizability of retarded functional differential equation in Hilbert space*, Osaka J. Math. 28(1991), 347–365.

[16] J. M. Jeong, *Retarded functional differential equations with L^1-valued controller*, Funkcialaj Ekvacioj 36(1993), 71–93.

[17] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynamics and Control Systems, 5 (1999), no. 3, 329–346.

[18] J. M. Jeong and H. G. Kim, Controllability for semilinear functional integrodifferential equations, Bull. Korean Math. Soc. 46 (2009), no. 3, 463–475.

[19] S. Kathikeyan and K. Balachandran, *Controllability of nonlinear stochastic neutral impulsive systems*, Nonlinear Anal. 3(2009), 1117–1135.

[20] M. A. Krannoselski, Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, New York, 1964.

[21] A. Lin and L.hu, *Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal conditions*, Comput. Math. Appl. 59(2010), 64-73.

[22] A. Lin and L. Hu, *Existence results for impulsive neutral stochastic differential inclusions with nonlocal initial conditions*, Comput. Math. Appl. 59(2010), 64–73.

[23] M. Metivier and J. Pellaumail, Stochastic Integration, Academic Press, Newyork, 1980

[24] S. Nakagiri, *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math. 25(1988), 353–398.

[25] B. Radhakrishnan and K. Balachandran, *Controllability results for second order neutral impulsive integrodifferential systems*, J. Optim. Theory Appl. 151 (2011), 589-612.

[26] Y. Ren, L. Hu and R. Sakthivel, *Controllability of impulsive neutral stochastic functional differential systems with infinite delay*, J. Comput. Anal. Appl. 235(2011), 2603–2614.

[27] R. Sakthivel, N. I. Mahmudov and S. G. Lee, *Controllability of non-linear impulsive stochastic systems*, Int. J. Control, 82(2009), 801–807.

[28] N. Sukavanam and Nutan Kumar Tomar, *Approximate controllability of semilinear delay control system*, Nonlinear Func.Anal.Appl. 12(2007), 53–59.

[29] L. Hu and Y. Ren, *Existence for impulsive Stochastic functional integro-differential equations*,Acta Appl. Math. 111(2010), 303–317.

[30] H. Tanabe, *Equations of Evolution*, Pitman-London, 1979.

[31] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, 1978.

[32] L. Wang, *Approximate controllability for integrodifferential equations and multiple delays*, J. Optim. Theory Appl. 143(2009), 185–206.