J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Isolation numbers of matrices over nonbinary Boolean semiring

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Abstract. Let $\mathbb{B}_k$ be the nonbinary Boolean semiring and A be a $m \times n$ Boolean matrix over $\mathbb{B}_k$. The Boolean rank of a Boolean matrix A is the smallest k such that A can be factored as an $m \times k$ Boolean matrix times a $k \times n$ Boolean matrix. The isolation number of A is the maximum number of nonzero entries in A such that no two are in any row or any column, and no two are in a 2×2 submatrix of all nonzero entries. We have that the isolation number of A is a lower bound on the Boolean rank of A. We also compare the isolation number with the binary Boolean rank of the support of A, and determine the equal cases of them.

1. Introduction

There are many papers on the study of rank of matrices over several semirings containing binary Boolean algebra, fuzzy semiring, semiring of nonegative integers, and so on ([2], [3], [6], and [7]). But there are few papers on isolation numbers of matrices. Gregory et al ([7]) introduced set of isolated entries and compared binary Boolean rank with biclique covering number. Recently Beasley ([2]) introduced isolation number of Boolean matrix and compare it with binary Boolean rank.

In this paper, we investigate the possible isolation number of Boolean matrix and compare it with Boolean rank of Boolean matrix and the binary Boolean rank of the support of the Boolean matrix.

2. Preliminaries

Definition 2.1. A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ with two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (the identity is denoted by 0);
- $\mathcal{S}$ is a monoid under multiplication (the identity is denoted by 1, $1 \neq 0$);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

Definition 2.2. A semiring $\mathcal{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

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[0]**2010 Mathematics Subject Classification**: 15A23; 15A03; 15B15.

[0]**Keywords**: Boolean rank; nonbinary Boolean semiring; binary Boolean algebra; isolation number.

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Definition 2.3. A semiring $\mathcal{S}$ is called a *Boolean semiring* if $\mathcal{S}$ is equivalent to a set of subsets of a given set $\mathbb{X}$, the sum of two subsets is their union, and the product is their intersection. The zero element 0 is the empty set and the identity element 1 is the whole set $\mathbb{X}$.

Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k. Then $\mathcal{P}(S_k)$ is the Boolean semiring of all subsets of S_k with operations in above definition. Let $\mathbb{B}_k$ be a Boolean semiring of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, that is a subset of $\mathcal{P}(S_k)$. It is straightforward to see that a Boolean semiring $\mathbb{B}_k$ is a commutative and antinegative semiring. Moreover, all of its elements, except 0 and 1, are zero-divisors. If $\mathbb{B}_k$ consists of only 0 (the empty subset) and 1 (the whole set S_k) then it is called a *binary Boolean semiring*, which is denoted as $\mathbb{B}_1$. If $\mathbb{B}_k$ is not a binary Boolean semiring then it is called a *nonbinary Boolean semiring*.

Throughout the paper, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes a nonbinary Boolean semiring, which contains at least 3 elements. Let $\mathcal{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from a Boolean semiring $\mathbb{B}_k$.

Let $\mathcal{M}_n(\mathbb{B}_k) = \mathcal{M}_{m,n}(\mathbb{B}_k)$ if $m = n$, let I_m denote the $m \times m$ identity matrix, $O_{m,n}$ denote the zero matrix in $\mathcal{M}_{m,n}(\mathbb{B}_k)$, $J_{m,n}$ denote the matrix of all ones in $\mathcal{M}_{m,n}(\mathbb{B}_k)$. The subscripts are usually omitted if the order is obvious, and we write I, O, J.

Definition 2.4. The matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ is said to be of *Boolean rank r* if there exist matrices $B \in \mathcal{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathcal{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer such that such a factorization exists. We denote $b(A) = r$.

By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

Now let $\mathcal{M}_{m,n}(\mathbb{B}_1)$ denote the set of all $m \times n$ binary Boolean matrices with entries in $\mathbb{B}_1$. The *binary Boolean rank* of $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ is the Boolean rank over $\mathbb{B}_1$ and denoted $b_1(A)$.

Definition 2.5. For two (binary) Boolean matrices A and B, A *dominates* B if $a_{i,j} = 0$ implies $b_{i,j} = 0$.

Given a matrix $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, we let $\mathbf{x}^{(j)}$ denote the j^{th} column of X and $\mathbf{x}_{(i)}$ denote the i^{th} row. Now if $b(A) = r$ and $A = BC$ is a factorization of $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}$. Since each of the terms $\mathbf{b}^{(i)}\mathbf{c}_{(i)}$ is a Boolean rank one matrix, the Boolean rank of A is also the minimum number of Boolean rank one matrices whose sum is A.

The binary Boolean rank has many applications in combinatorics, especially graph theory, for example, if $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ is the adjacency matrix of the bipartite graph G with bipartition (X, Y), the binary Boolean rank of A is the minimum number of bicliques that cover the edges of G, called the *biclique covering number*.

Definition 2.6. Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, a set of *isolated entries* ([7]) is a set of entries, usually written as $E = \{a_{i,j}\}$ such that $a_{i,j} \neq 0$, no two entries in E are in the same row, no two entries in E are in the same column, and, if $a_{i,j}, a_{k,l} \in E$ then, $a_{i,l} = 0$ or $a_{k,j} = 0$. That is, isolated entries are independent entries and any two isolated entries $a_{i,j}$ and $a_{k,l}$ do not lie in a submatrix of A of the form $\begin{bmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{bmatrix}$ with all entries nonzero. The *isolation number of* A, $\iota(A)$, is the maximum size of a set of isolated entries.

Note that $\iota(A) = 0$ if and only if $A = O$.

Example 2.7. Let $\sigma \in \mathbb{B}_k$ be neither 0 nor 1 and

$$A = \begin{bmatrix} 1 & 1 & \sigma & 0 & 0 \\ \sigma & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \sigma \\ 0 & \sigma & 0 & 1 & 1 \\ 0 & 0 & 1 & \sigma & 1 \end{bmatrix}$$

be a Boolean matrix over $\mathbb{B}_k$ and E_1 is the set of $\sigma's$ which are located at the positions $\{a_{1,3}, a_{2,1}, a_{3,5}, a_{4,2}, a_{5,4}\}$ of A. The entries $\sigma's$ of A are isolated entries and hence $\iota(A) = 5$. But the entries of A in the position in $E_2 = \{a_{1,1}, a_{2,2}, a_{3,5}, a_{4,4}, a_{5,3}\}$ are not isolated.

Suppose that $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ and $b(A) = r$. Then there are r Boolean rank one matrices A_i such that

$$A = A_1 + A_2 + \cdots + A_r. \tag{2.1}$$

Because each Boolean rank one matrix can be permuted to a matrix of the form $\begin{bmatrix} N & O \\ O & O \end{bmatrix}$ with all nonzero entries in N, it is easily seen that the matrix consisting of two isolated entries of A cannot be dominated by any one A_i among the Boolean rank one summand of A in (2.1). Thus

$$i(A) \leq b(A). \tag{2.2}$$

Many functions, sets and relations concerning matrices do not depend upon the magnitude or nature of the individual entries of a matrix, but rather only on whether the entry is zero or nonzero. These combinatorially significant matrices have become increasingly important in recent years. Of primary interest is the binary Boolean rank. Finding the binary Boolean rank of a $(0, 1)$-matrix is an NP-Complete problem ([8]), and consequently finding bounds on the binary Boolean rank of a matrix is of interest to those researchers that would use binary Boolean rank in their work. If the $(0, 1)$-matrix is the reduced adjacency matrix of a bipartite graph, the isolation number of the matrix is the maximum size of a non-competitive matching in the bipartite graph. This is related to the study of such combinatorial problems as the patient hospital problem, the stable marriage problem, etc. An additional reason for studying the isolation number is that it is a lower bound on the Boolean rank of a Boolean matrix over $\mathbb{B}_k$. While finding the isolation number as well as finding the Boolean rank of a Boolean matrix is an NP-Complete problem ([1]), for some matrices finding the isolation number can be easier than finding the Boolean rank especially if the matrix is sparse:

Example 2.8. Let $\sigma \in \mathbb{B}_k$ and

$$F = \begin{bmatrix} 1 & 1 & 1 & \sigma & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \sigma & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & \sigma & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a Boolean matrix in $\mathcal{M}_9(\mathbb{B}_k)$.

Then we can easily see $b(F) \leqq 6$ from first 3 rows and columns, however to find that Boolean rank is not 5, requires much calculation if the isolation number is not considered. However, the isolation number is easily seen to be 6, both computationally and visually, the σ's in this matrix represent a set of isolated entries. Thus we have $b(F) = 6$ by (2.2).

Note that if any of the 1's in F are replaced by zeros, the resulting matrix still has Boolean rank 6 as well as isolation number 6.

Terms not specifically defined here can be found in Brualdi and Ryser [5] for matrix terms, or Bondy and Murty [4] for graph theoretic terms.

For our use in the next section, we define the support matrix of a Boolean matrix. If $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then the *support* of A is the binary Boolean matrix $\overline{A} = (\overline{a_{i,j}}) \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ such that $\overline{a_{i,j}} = 1$ if $a_{i,j} \neq 0$ and $\overline{a_{i,j}} = 0$ if $a_{i,j} = 0$.

3. Comparisons between isolation numbers and Boolean ranks over $\mathcal{M}_{m,n}(\mathbb{B}_k)$

In this section, we compare the isolation number with Boolean rank of a Boolean matrix, and also we compare the isolation number with binary Boolean rank of the support of a Boolean matrix.

Lemma 3.1. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $b(A + B) \leq b(A) + b(B)$. And for $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_1)$, $b_1(A + B) \leq b_1(A) + b_1(B)$.*

Proof. It follows from the definition of Boolean rank and equation (2.1). ■

Lemma 3.2. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $\overline{A + B} = \overline{A} + \overline{B}$ in $\mathcal{M}_{m,n}(\mathbb{B}_1)$.*

Proof. It follows from the facts that $\mathbb{B}_k$ is an antinegative semiring and $1 + 1 = 1$ in $\mathbb{B}_1$. ■

Lemma 3.3. *For $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $b_1(\overline{A}) \leq b(A)$.*

Proof. If $b(A) = r$, then A has a Boolean rank one factorization such that $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}$ with $B = [\mathbf{b}^{(1)}\mathbf{b}^{(2)} \cdots \mathbf{b}^{(r)}] \in \mathcal{M}_{m,k}(\mathbb{B}_k)$ and $C = [\mathbf{c}_{(1)}\mathbf{c}_{(2)} \cdots \mathbf{c}_{(k)}]^t \in \mathcal{M}_{k,n}(\mathbb{B}_k)$ from (2.1). Therefore
$b_1(\overline{A}) = b_1(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}}) = b_1(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)}} + \overline{\mathbf{b}^{(2)}\mathbf{c}_{(2)}} + \cdots + \overline{\mathbf{b}^{(r)}\mathbf{c}_{(r)}}) \leq r$, from Lemma 3.2. Hence $b_1(\overline{A}) \leq b(A)$. ■

We may have strict inequality in Lemma 3.3 as we see in the following example.

Example 3.4. Let $S_3 = \{x, y, z\}$ and $\mathbb{B}_3 = \{0, \{x\}, \{x, y\}, 1\}$ with $1 = \{x, y, z\}$. Consider $X = \begin{bmatrix} 1 & \{x\} \\ \{x, y\} & \{x, y\} \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & \{x\} \\ \{x, y\} & \{x\} \end{bmatrix}$ in $\mathcal{M}_2(\mathbb{B}_3)$. Then $b(X) = 2$ but $b_1(\overline{X}) = b_1(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = 1$. Hence $b_1(\overline{X}) < b(X)$. But $b(Y) = b_1(\overline{Y}) = 1$ since $Y = \begin{bmatrix} 1 \\ \{x, y\} \end{bmatrix} \begin{bmatrix} 1 & \{x\} \end{bmatrix}$ over $\mathbb{B}_3$.

Lemma 3.5. *For $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $\iota(A) = \iota(\overline{A})$.*

Proof. If $a_{i,j}$ and $a_{k,l}$ are any isolated entries in A, then $i \neq k$ and $j \neq l$, and that $a_{i,l} = 0$ or $a_{k,j} = 0$. Hence $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are isolated entries in $\overline{A}$, so we have $\iota(A) \leq \iota(\overline{A})$.

Conversely, if $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are any isolated entries in $\overline{A}$, then $a_{i,j} \neq 0$ and $a_{k,l} \neq 0$ and that $a_{i,l} = \overline{a_{i,l}} = 0$ or $a_{k,j} = \overline{a_{k,j}} = 0$. Hence $a_{i,j}$ and $a_{k,l}$ are isolated entries in A, so we have $\iota(\overline{A}) \leq \iota(A)$. ■

Theorem 3.6. *If $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then $\iota(A) = 1$ if and only if $b_1(\overline{A}) = 1$.*

Proof. Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. If $b_1(\overline{A}) = 1$ then $A \neq O$ so that $\iota(A) \neq 0$ and since $\iota(A) = \iota(\overline{A}) \leq b_1(\overline{A})$ by (2.2), we have $\iota(A) = 1$.

Conversely, suppose on the contrary that there exists a matrix $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ such that $\iota(A) = 1$, $b_1(\overline{A}) > 1$. Then, there exists two non-equal and nonzero rows of $\overline{A}$, say ith and jth. Hence, without loss of generality, there exists a k such that $\overline{a_{i,k}} = 1$ and $\overline{a_{j,k}} = 0$. Then, $\overline{a_{i,k}}$ and any unit entry in jth row of $\overline{B}$ constitute a set of two isolated entries. Thus, $\iota(A) = \iota(\overline{A}) > 1$, a contradiction. ■

It follows that the subset of $\mathcal{M}_{m,n}(\mathbb{B}_k)$ of matrices with isolation number 1 is the same as the set of matrices whose support has Boolean rank 1.

For $A = A_1 + A_2 + \cdots + A_r$ with $b(A) = r$, let $\mathcal{R}_i$ denote the indices of the nonzero rows of A_i and $\mathcal{C}_j$ denote the indices of the nonzero columns of A_j, $i, j = 1. \cdots, k$. Let $r_i = |\mathcal{R}_i|$, the number of nonzero rows of A_i and $c_j = |\mathcal{C}_j|$, the number of nonzero columns of A_j.

Lemma 3.7. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then if $b(A) \geq b_1(\overline{A}) = 2$ then $\iota(A) = 2$, and if $\iota(A) = 2$ then $b_1(\overline{A}) \neq 3$.*

Proof. If $b_1(\overline{A}) = 2$, then $\iota(A) > 1$ by Theorem 3.6. Since $\iota(A) = \iota(\overline{A}) \leq b_1(\overline{A})$ from Lemma 3.5 and (2.2), we have that $\iota(A) = \iota(\overline{A}) = 2$.

Now, suppose that $\iota(A) = 2$ and that $b_1(\overline{A}) = 3$. Then, we have a factorization of $\overline{A}$ as $\overline{A} = C \times D$ with $C \in \mathcal{M}_{m,3}(\mathbb{B}_1)$ and $D \in \mathcal{M}_{3,n}(\mathbb{B}_1)$. Then, the three rows of D generate all the rows of $\overline{A}$. Since $b_1(\overline{A}) = 3$, D cannot have binary Boolean rank 2 or less. Thus, we have $b_1(D) = 3$. Therefore, we have a factorization of D as $D = E \times F$ with $E \in \mathcal{M}_{3,3}(\mathbb{B}_1)$ and $F \in \mathcal{M}_{3,n}(\mathbb{B}_1)$. Then, the three column of E generate all the columns of D and $b_1(E) = 3$. Therefore, it is sufficient to consider 3×3 matrices of binary Boolean rank 3. However, there are only 10 following 3×3 matrices of binary Boolean rank 3 up to permutations:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ B_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_6 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ B_7 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ B_8 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B_9 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_{10} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since B_5 can be permuted to B_2 and B_7 can be permuted to B_4, and B_9 can be permuted to B_6 with transposing. Therefore, there are only seven non-equivalent 3×3 matrices of binary Boolean rank 3. However, these matrices have three isolation entries on the main diagonal. Thus, we have a contradiction to the conditions that $\iota(B) = 2$ and $r_{\mathbb{B}_1}(\overline{B}) = 3$. Thus, if $\iota(A) = 2$ then $b_1(\overline{A}) \neq 3$. ■

Theorem 3.8. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then, $\iota(A) = 2$ if and only if $b_1(\overline{A}) = 2$.*

Proof. From Lemma 3.7, we have the sufficiency. So we only need show the necessity.

Suppose there exists $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ with $\iota(A) = \iota(\overline{A}) = 2$ and $b_1(\overline{A}) > 2$. By Lemma 3.7, $b_1(\overline{A}) \neq 3$, and hence $b_1(\overline{A}) \geq 4$. Thus we choose A such that if $b_1(\overline{A}) > b_1(\overline{C}) > 2$ then $\iota(C) > 2$. Suppose that $\overline{A} = \overline{A_1} + \overline{A_2} + \cdots + \overline{A_r}$ for $r = b_1(\overline{A})$ where each $\overline{A_i}$ is binary Boolean rank 1, i.e., r is the minimum r such that $b_1(\overline{A}) = r$ and $\iota(A) = 2$. Suppose that $\overline{A_1}$ has the fewest number of nonzero rows of the $\overline{A_i}$'s. As in the proof of the above lemma 3.7, permute the rows of $\overline{A}$ so that $\overline{A_1}$ has nonzero rows $1, 2, \cdots, r_1$. For $j = 1, \cdots, r_1$, let $\overline{B_j}$ be the matrix whose first j rows are the first j rows of $\overline{A}$ and whose last $m-j$ rows are all zero. Let $\overline{C_j}$ be the matrix whose first j rows are all zero and whose last $m-j$ rows are the last $m-j$ rows of $\overline{A}$. Then $\overline{A} = \overline{B_j} + \overline{C_j}$. Further any set of isolated entries of $\overline{C_j}$ is a set of isolated entries for $\overline{A}$. Now, from $b_1(\overline{A}) \leq b_1(\overline{B_j}) + b_1(\overline{C_j})$, and the fact that $b_1(\overline{C_j}) = b_1(\overline{C_{j-1}})$ or $b_1(\overline{C_j}) = b_1(\overline{C_{j-1}}) - 1$, there is some t such that $b_1(\overline{C_t}) = b_1(\overline{A}) - 1$. Since $b_1(\overline{C_t}) < r$ by the choice of $\overline{A}$, for this t, we have that $\iota(\overline{C_t}) > 2$ since $b_1(\overline{C_t}) \geq 3$. That is, $\iota(A) = \iota(\overline{A}) > 2$, which is impossible since $\iota(A) = 2$. Therefore $b_1(\overline{A}) = 2$. ■

Now, as we can see in the following example, there is a Boolean matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ such that $\iota(\overline{A}) = 3$ and $b_1(\overline{A})$ is relative large, depending on m and n.

Example 3.9. For $n \geq 3$, let $\overline{D_n} = J \setminus I \in \mathcal{M}_n(\mathbb{B}_1)$. Then, it is easily shown that $\iota(\overline{D_n}) = 3$ while $b_1(\overline{D_n}) = r$ where $r = min\left\{h : n \leq \begin{pmatrix} h \\ \frac{h}{2} \end{pmatrix}\right\}$, see [6](Corollary 2). So, $\iota(\overline{D_{20}}) = 3$ while $b_1(\overline{D_{20}}) = 6$.

Definition 3.10. A *tournament matrix* $[T] \in \mathcal{M}_n(\mathbb{B}_k)$ is the adjacency matrix of a directed graph called a tournament, T. It is characterized by $[T] \circ [T]^t = O$ and $[T] + [T]^t = J - I$, where $\circ$ denotes entrywise multiplication of two matrices.

Now, for each $r = 1, 2, \cdots, \min\{m, n\}$, can we characterize the matrices in $\mathcal{M}_{m,n}(\mathbb{B}_k)$ for which $\iota(A) = b_1(\overline{A})$? Of course it is done if $r = 1$ or $r = 2$ in the above theorems, but only in those cases. For $r = m$ we can also find a characterization:

Theorem 3.11. *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then, $\iota(A) = b_1(\overline{A}) = m$ if and only if there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B}_1)$ and $Q \in \mathcal{M}_n(\mathbb{B}_1)$ such that $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T} \in \mathcal{M}_m(\mathbb{B}_1)$ where $\overline{T} \in \mathcal{M}_m(\mathbb{B}_1)$ is dominated by a tournament matrix. (There are no restrictions on C.)*

Proof. Suppose that $\iota(A) = m$. Then we permute A by permutation matrices P and Q so that the set of isolated entries are in the (d, d) positions, $d = 1, \cdots, m$. That is, if $X = PAQ$ then $I = \{x_{1,1}, x_{2,2}, \cdots, x_{m,m}\}$ is the set of isolated entries in X. Therefore $X = [B|C]$, with $\overline{b_{i,i}} = \overline{x_{i,i}} = 1$ and $\overline{b_{i,j}} \cdot \overline{b_{j,i}} = 0$ for every i and $j \neq i$ from the definition of the isolated entries. Thus, $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix. Thus, $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T}$ and clearly there are no conditions on C.

Conversely, if $PAQ = [B|C]$ and $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix, then the diagonal entries of B form a set of isolated entries for PAQ and hence A has a set of m isolated entries. Thus $\iota(A) = b_1(\overline{A}) = m$. ∎

Corollary 3.12. *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. If there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B}_1)$ and $Q \in \mathcal{M}_n(\mathbb{B}_1)$ such that $PAQ = [B|C]$ where $B \in \mathcal{M}_m(\mathbb{B}_k)$ is a diagonal matrix or a triangular matrix with nonzero diagonal entries, then $\iota(A) = b_1(\overline{A}) = m$.*

4. Conclusions

In this paper, we investigated the nonbinary Boolean rank of a matrix A and the rank of its support for the given isolation number k over nonbinary Boolean semirings. Thus, we proved that the isolation number of A is the same as the Boolean rank of the support of it if the isolation numbers are 1 and 2. If the isolation number were greater than 2, then we showed by example that binary Boolean rank of the support of the given nonbinary Boolean matrix may be strictly greater than the isolation number of the matrix. In addition, in some special cases involving tournament matrices, we obtained that the isolation number of the given matrix and the Boolean rank of its support of the nonbinary Boolean matrix are the same.

Acknowledgement The third author, Seok-Zun Song, was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

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