BCI-implicative ideals of BCI-algebras using neutrosophic quadruple structure

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Abstract. Neutrosophic quadruple structure is used to study BCI-implicative ideal in BCI-algebra. The conceot of neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra is introduced, and their related properties are investigated. Relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal are consulted. Conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal are provided. A characterization of a neutrosophic quadruple BCI-implicative ideal is displayed, and the extension property of neutrosophic quadruple BCI-implicative ideal is established.

1. Introduction

In [14], Smarandche has introduced the neutrosophic quadruple numbers for the first time. Using the notion of Smarandache's neutrosophic quadruple numbers, Akinleye et al. [2] presented the notion of neutrosophic quadruple algebraic structures. In particular, they studied neutrosophic quadruple rings. Agboola et al. [1] studied neutrosophic quadruple algebraic hyperstructures, in particular, they developed neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings. Using BCK/BCI-algebras, Jun et al. [7] have established neutrosophic quadruple BCK/BCI-algebra, and have studied neutrosophic quadruple (positive implicative) ideal in neutrosophic quadruple BCK-algebra and neutrosophic quadruple closed ideal in neutrosophic quadruple BCI-algebra. Muhiuddin et al. [13] have studied neutrosophic quadruple q-ideal and (regular) neutrosophic quadruple ideal in neutrosophic quadruple BCI-algebra. Muhiuddin et al. [12] also have studied implicative neutrosophic quadruple ideal in neutrosophic quadruple BCK-algebra.

In this article, we study BCI-implicative ideal in BCI-algebra using neutrosophic quadruple structure. We define neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra, and investigate their related properties. We consult relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal. We provide conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal. We discuss a characterization of an neutrosophic quadruple BCI-implicative ideal, and establish the extension property of neutrosophic quadruple BCI-implicative ideal.

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2. Preliminaries

A BCK/BCI-algebra, which is an important class of logical algebras, is introduced by K. Iséki (see [4, 5]) and it is being studied by many researchers.

A *BCI-algebra* is a set X with a binary operation "$\cdot$" and a special element "0" that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x \cdot y) \cdot (x \cdot z)) \cdot (z \cdot y) = 0)$,

(II) $(\forall x, y \in X)\ ((x \cdot (x \cdot y)) \cdot y = 0)$,

(III) $(\forall x \in X)\ (x \cdot x = 0)$,

(IV) $(\forall x, y \in X)\ (x \cdot y = 0,\ y \cdot x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 \cdot x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x \cdot 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \ \Rightarrow\ x \cdot z \leq y \cdot z,\ z \cdot y \leq z \cdot x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x \cdot y) \cdot z = (x \cdot z) \cdot y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x \cdot z) \cdot (y \cdot z) \leq x \cdot y) \tag{2.4}$$

where $x \leq y$ if and only if $x \cdot y = 0$.

Any BCI-algebra X satisfies the following conditions (see [3]):

$$(\forall x, y \in X)(x \cdot (x \cdot (x \cdot y)) = x \cdot y), \tag{2.5}$$

$$(\forall x, y \in X)(0 \cdot (x \cdot y) = (0 \cdot x) \cdot (0 \cdot y)), \tag{2.6}$$

$$(\forall x, y \in X)(0 \cdot (0 \cdot (x \cdot y)) = (0 \cdot y) \cdot (0 \cdot x)). \tag{2.7}$$

An element a in a BCI-algebra X is said to be *minimal* (see [3]) if the following assertion is valid.

$$(\forall x \in X)(x \leq a \ \Rightarrow\ x = a). \tag{2.8}$$

Note that the zero element 0 in a BCI-algebra X is minimal (see [3]).

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x \cdot y \in S$ for all $x, y \in S$. A subset G of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in G, \tag{2.9}$$

$$(\forall x \in X)\,(\forall y \in G)\,(x \cdot y \in G \ \Rightarrow\ x \in G)\,. \tag{2.10}$$

A subset G of a BCI-algebra X is called

- a *closed ideal* of X (see [3]) if it is an ideal of X which satisfies:

$$(\forall x \in X)(x \in G \ \Rightarrow\ 0 \cdot x \in G), \tag{2.11}$$

- a *BCI-positive implicative ideal* of X (see [8, 9]) if it satisfies (2.9) and

$$(\forall x, y, z \in X)\,(((x \cdot z) \cdot z) \cdot (y \cdot z) \in G,\ y \in G \ \Rightarrow\ x \cdot z \in G)\,, \tag{2.12}$$

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- a *BCI-commutative ideal* of X (see [10]) if it satisfies (2.9) and

$$\begin{aligned}&(x\cdot y)\cdot z\in G,\ z\in G\\ &\Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in G\end{aligned}\tag{2.13}$$

for all $x, y, z \in X$,

- a *BCI-implicative ideal* of X (see [8]) if it satisfies (2.9) and

$$\begin{aligned}&(((x\cdot y)\cdot y)\cdot(0\cdot y))\cdot z\in G,\ z\in G\\ &\Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in G\end{aligned}\tag{2.14}$$

for all $x, y, z \in X$.

Note that every BCI-implicative ideal is an ideal, but the converse is not true (see [8]).

Lemma 2.1 ([8]). *A subset K of X is a BCI-implicative ideal of a BCI-algebra X if and only if it is an ideal of X that satisfies the following condition.*

$$((x\cdot y)\cdot y)\cdot(0\cdot y)\in K\ \Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K\tag{2.15}$$

for all $x, y \in X$.

Lemma 2.2 ([10]). *An ideal K of X is a BCI-commutative ideal of X if and only if it satisfies:*

$$x\cdot y\in K\ \Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K\tag{2.16}$$

for all $x, y, z \in X$.

Lemma 2.3 ([9]). *An ideal K of X is a BCI-positive implicative ideal of X if and only if it satisfies:*

$$((x\cdot y)\cdot y)\cdot(0\cdot y)\in K\ \Rightarrow\ x\cdot y\in K\tag{2.17}$$

for all $x, y, z \in X$.

We refer the reader to the books [3, 11] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let X be a set. A *neutrosophic quadruple X-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in X$ and T, I, F have their usual neutrosophic logic meanings (see [7]).

The set of all neutrosophic quadruple X-numbers is denoted by $N_q(X)$, that is,

$$N_q(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $N_q(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let X be a BCK/BCI-algebra. We define a binary operation $\boxdot$ on $N_q(X)$ by

$$(a, xT, yI, zF)\boxdot(b, uT, vI, wF) = (a\cdot b, (x\cdot u)T, (y\cdot v)I, (z\cdot w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in N_q(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

Then $(N_q(X); \boxdot, \tilde{0})$ is a BCK/BCI-algebra (see [7]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $N_q(X)$.

We define an order relation "$\ll$" and the equality "$=$" on $N_q(X)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in N_q(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $N_q(X)$.

Let X be a BCK/BCI-algebra. Given nonempty subsets K and J of X, consider the set

$$N_q(K, J) := \{(a, xT, yI, zF) \in N_q(X) \mid a, x \in K \ \& \ y, z \in J\},$$

which is called the neutrosophic quadruple set based on K and J.

The set $N_q(K, K)$ is denoted by $N_q(K)$, and it is called the neutrosophic quadruple set based on K.

3. Neutrosophic quadruple BCI-implicative ideals

In what follows, let X denote a BCI-algebra unless otherwise specified.

Definition 3.1. Let K and J be nonempty subsets of X. Then the neutrosophic quadruple set based on K and J is called a *neutrosophic quadruple BCI-implicative ideal* (briefly, *NQ-BCI-implicative ideal*) over (X, K, J) if it is a BCI-implicative ideal of $N_q(X)$. If $K = J$, then we say that it is an *NQ-BCI-implicative ideal* over (X, K).

Example 3.2. Consider a BCI-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the binary operation $\cdot$, which is given in Table 1.

Table 1. Cayley table for the binary operation "$\cdot$"

$\cdot$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 6^4 elements. If we take $K = \{0, 1, 2\}$, then the neutrosophic quadruple set based on K has 81-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\zeta}_i \mid i = 1, 2, \cdots, 80\},$$

and it is an NQ-BCI-implicative ideal over (X, K) where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\zeta}_1 = (0, 0T, 0I, 1F)$, $\tilde{\zeta}_2 = (0, 0T, 0I, 2F)$,
$\tilde{\zeta}_3 = (0, 0T, 1I, 0F)$, $\tilde{\zeta}_4 = (0, 0T, 1I, 1F)$, $\tilde{\zeta}_5 = (0, 0T, 1I, 2F)$,
$\tilde{\zeta}_6 = (0, 0T, 2I, 0F)$, $\tilde{\zeta}_7 = (0, 0T, 2I, 1F)$, $\tilde{\zeta}_8 = (0, 0T, 2I, 2F)$,
$\tilde{\zeta}_9 = (0, 1T, 0I, 0F)$, $\tilde{\zeta}_{10} = (0, 1T, 0I, 1F)$, $\tilde{\zeta}_{11} = (0, 1T, 0I, 2F)$,

$\tilde{\zeta}_{12} = (0, 1T, 1I, 0F)$, $\tilde{\zeta}_{13} = (0, 1T, 1I, 1F)$, $\tilde{\zeta}_{14} = (0, 1T, 1I, 2F)$,
$\tilde{\zeta}_{15} = (0, 1T, 2I, 0F)$, $\tilde{\zeta}_{16} = (0, 1T, 2I, 1F)$, $\tilde{\zeta}_{17} = (0, 1T, 2I, 2F)$,
$\tilde{\zeta}_{18} = (0, 2T, 0I, 0F)$, $\tilde{\zeta}_{19} = (0, 2T, 0I, 1F)$, $\tilde{\zeta}_{20} = (0, 2T, 0I, 2F)$,
$\tilde{\zeta}_{21} = (0, 2T, 1I, 0F)$, $\tilde{\zeta}_{22} = (0, 2T, 1I, 1F)$, $\tilde{\zeta}_{23} = (0, 2T, 1I, 2F)$,
$\tilde{\zeta}_{24} = (0, 2T, 2I, 0F)$, $\tilde{\zeta}_{25} = (0, 2T, 2I, 1F)$, $\tilde{\zeta}_{26} = (0, 2T, 2I, 2F)$,
$\tilde{\zeta}_{27} = (1, 0T, 0I, 0F)$, $\tilde{\zeta}_{28} = (1, 0T, 0I, 1F)$, $\tilde{\zeta}_{29} = (1, 0T, 0I, 2F)$,
$\tilde{\zeta}_{30} = (1, 0T, 1I, 0F)$, $\tilde{\zeta}_{31} = (1, 0T, 1I, 1F)$, $\tilde{\zeta}_{32} = (1, 0T, 1I, 2F)$,
$\tilde{\zeta}_{33} = (1, 0T, 2I, 0F)$, $\tilde{\zeta}_{34} = (1, 0T, 2I, 1F)$, $\tilde{\zeta}_{35} = (1, 0T, 2I, 2F)$,
$\tilde{\zeta}_{36} = (1, 1T, 0I, 0F)$, $\tilde{\zeta}_{37} = (1, 1T, 0I, 1F)$, $\tilde{\zeta}_{38} = (1, 1T, 0I, 2F)$,
$\tilde{\zeta}_{39} = (1, 1T, 1I, 0F)$, $\tilde{\zeta}_{40} = (1, 1T, 1I, 1F)$, $\tilde{\zeta}_{41} = (1, 1T, 1I, 2F)$,
$\tilde{\zeta}_{42} = (1, 1T, 2I, 0F)$, $\tilde{\zeta}_{43} = (1, 1T, 2I, 1F)$, $\tilde{\zeta}_{44} = (1, 1T, 2I, 2F)$,
$\tilde{\zeta}_{45} = (1, 2T, 0I, 0F)$, $\tilde{\zeta}_{46} = (1, 2T, 0I, 1F)$, $\tilde{\zeta}_{47} = (1, 2T, 0I, 2F)$,
$\tilde{\zeta}_{48} = (1, 2T, 1I, 0F)$, $\tilde{\zeta}_{49} = (1, 2T, 1I, 1F)$, $\tilde{\zeta}_{50} = (1, 2T, 1I, 2F)$,
$\tilde{\zeta}_{51} = (1, 2T, 2I, 0F)$, $\tilde{\zeta}_{52} = (1, 2T, 2I, 1F)$, $\tilde{\zeta}_{53} = (1, 2T, 2I, 2F)$,
$\tilde{\zeta}_{54} = (2, 0T, 0I, 0F)$, $\tilde{\zeta}_{55} = (2, 0T, 0I, 1F)$, $\tilde{\zeta}_{56} = (2, 0T, 0I, 2F)$,
$\tilde{\zeta}_{57} = (2, 0T, 1I, 0F)$, $\tilde{\zeta}_{58} = (2, 0T, 1I, 1F)$, $\tilde{\zeta}_{59} = (2, 0T, 1I, 2F)$,
$\tilde{\zeta}_{60} = (2, 0T, 2I, 0F)$, $\tilde{\zeta}_{61} = (2, 0T, 2I, 1F)$, $\tilde{\zeta}_{62} = (2, 0T, 2I, 2F)$,
$\tilde{\zeta}_{63} = (2, 1T, 0I, 0F)$, $\tilde{\zeta}_{64} = (2, 1T, 0I, 1F)$, $\tilde{\zeta}_{65} = (2, 1T, 0I, 2F)$,
$\tilde{\zeta}_{66} = (2, 1T, 1I, 0F)$, $\tilde{\zeta}_{67} = (2, 1T, 1I, 1F)$, $\tilde{\zeta}_{68} = (2, 1T, 1I, 2F)$,
$\tilde{\zeta}_{69} = (2, 1T, 2I, 0F)$, $\tilde{\zeta}_{70} = (2, 1T, 2I, 1F)$, $\tilde{\zeta}_{71} = (2, 1T, 2I, 2F)$,
$\tilde{\zeta}_{72} = (2, 2T, 0I, 0F)$, $\tilde{\zeta}_{73} = (2, 2T, 0I, 1F)$, $\tilde{\zeta}_{74} = (2, 2T, 0I, 2F)$,
$\tilde{\zeta}_{75} = (2, 2T, 1I, 0F)$, $\tilde{\zeta}_{76} = (2, 2T, 1I, 1F)$, $\tilde{\zeta}_{77} = (2, 2T, 1I, 2F)$,
$\tilde{\zeta}_{78} = (2, 2T, 2I, 0F)$, $\tilde{\zeta}_{79} = (2, 2T, 2I, 1F)$, $\tilde{\zeta}_{80} = (2, 2T, 2I, 2F)$.

Theorem 3.3. *Every NQ-BCI-implicative ideal is a neutrosophic quadruple ideal.*

Proof. It is straightforward since every BCI-implicative ideal is an ideal in BCI-algebras. □

The converse of Theorem 3.3 is not true in general as seen in the following example.

Example 3.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $\cdot$, which is given in Table 2.

TABLE 2. Cayley table for the binary operation "$\cdot$"

$\cdot$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	0	4
2	2	2	0	0	4
3	3	3	2	0	4
4	4	4	4	4	0

Then X is a BCI-algebra (see [8]), and the neutrosophic quadruple BCI-algebra $N_q(X)$ has 625 elements. If we take $K = \{0, 1\}$, then the neutrosophic quadruple set based on K has 16-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\zeta}_i \mid i = 1, 2, \cdots, 15\},$$

and it is a neutrosophic quadruple ideal over (X, K) where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\zeta}_1 = (0, 0T, 0I, 1F)$,
$\tilde{\zeta}_2 = (0, 0T, 1I, 0F)$, $\tilde{\zeta}_3 = (0, 0T, 1I, 1F)$,
$\tilde{\zeta}_4 = (0, 1T, 0I, 0F)$, $\tilde{\zeta}_5 = (0, 1T, 0I, 1F)$,
$\tilde{\zeta}_6 = (0, 1T, 1I, 0F)$, $\tilde{\zeta}_7 = (0, 1T, 1I, 1F)$,
$\tilde{\zeta}_8 = (1, 0T, 0I, 0F)$, $\tilde{\zeta}_9 = (1, 0T, 0I, 1F)$,
$\tilde{\zeta}_{10} = (1, 0T, 1I, 0F)$, $\tilde{\zeta}_{11} = (1, 0T, 1I, 1F)$,
$\tilde{\zeta}_{12} = (1, 1T, 0I, 0F)$, $\tilde{\zeta}_{13} = (1, 1T, 0I, 1F)$,
$\tilde{\zeta}_{14} = (1, 1T, 1I, 0F)$, $\tilde{\zeta}_{15} = (1, 1T, 1I, 1F)$.
If we take $\tilde{x} = (2, 2T, 2I, 2F)$ and $\tilde{y} = (3, 3T, 3I, 3F)$ in $N_q(X)$, then

$$\begin{aligned}
&(((\tilde{y} \boxdot \tilde{x}) \boxdot \tilde{x}) \boxdot (\tilde{0} \boxdot \tilde{x})) \boxdot \tilde{0} \\
&= ((((3, 3T, 3I, 3F) \boxdot (2, 2T, 2I, 2F)) \boxdot (2, 2T, 2I, 2F))\boxdot \\
&((0, 0T, 0I, 0F) \boxdot (2, 2T, 2I, 2F))) \boxdot (0, 0T, 0I, 0F) \\
&= (0, 0T, 0I, 0F) \in N_q(K).
\end{aligned}$$

But

$$\begin{aligned}
&\tilde{y} \boxdot ((\tilde{x} \boxdot (\tilde{x} \boxdot \tilde{y})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{y} \boxdot \tilde{x})))) \\
&= (3, 3T, 3I, 3F) \boxdot (((2, 2T, 2I, 2F) \boxdot ((2, 2T, 2I, 2F) \boxdot (3, 3T, 3I, 3F)))\boxdot \\
&((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot ((3, 3T, 3I, 3F) \boxdot (2, 2T, 2I, 2F))))) \\
&= (2, 2T, 2I, 2F) \notin N_q(K).
\end{aligned}$$

Hence $N_q(K)$ is not a BCI-implicative ideal of $N_q(X)$, and so it is not an NQ-BCI-implicative ideal over (X, K).

Lemma 3.5 ([7]). *If K and J are ideals of X, then the neutrosophic quadruple set based on K and J is a neutrosophic quadruple ideal over (X, K, J).*

Theorem 3.6. *The neutrosophic quadruple set based on BCI-implicative ideals K and J of X is an NQ-BCI-implicative ideal over (X, K, J).*

Proof. Let K and J be BCI-implicative ideals of X. Since $0 \in K \cap J$, we get $\tilde{0} \in N_q(K, J)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $N_q(X)$ such that

$$(((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot \tilde{z} \in N_q(K, J)$$

and $\tilde{z} \in N_q(K, J)$. Then $\tilde{z} = (z_1,\ z_2T,\ z_3I,\ z_4F) \in N_q(K, J)$ and

$$\begin{aligned}
&(((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot \tilde{z} \\
&= ((((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F))) \boxdot (z_1, z_2T, z_3I, z_4F) \\
&= (((((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1)) \cdot z_1), ((((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2)) \cdot z_2)T, \\
&((((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3)) \cdot z_3)I, ((((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4)) \cdot z_4)F) \\
&\in N_q(K, J).
\end{aligned}$$

Hence $z_i \in K$ and $(((x_i \cdot y_i) \cdot y_i) \cdot (0 \cdot y_i)) \cdot z_i \in K$ for $i = 1, 2$; and $z_j \in J$ and $(((x_j \cdot y_j) \cdot y_j) \cdot (0 \cdot y_j)) \cdot z_j \in K$ for $j = 3, 4$. Since K and J are BCI-implicative ideals of X, it follows that $x_i \cdot ((y_i \cdot (y_i \cdot x_i)) \cdot (0 \cdot (0 \cdot (x_i \cdot y_i)))) \in K$ and $x_j \cdot ((y_j \cdot (y_j \cdot x_j)) \cdot (0 \cdot (0 \cdot (x_j \cdot y_j)))) \in J$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\begin{aligned}
&\tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \cdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y})))) \\
&= (x_1, x_2T, x_3I, x_4F) \cdot (((y_1, y_2T, y_3I, y_4F) \cdot ((y_1, y_2T, y_3I, y_4F) \cdot \\
&(x_1, x_2T, x_3I, x_4F))) \cdot ((0, 0T, 0I, 0F) \cdot ((0, 0T, 0I, 0F) \cdot \\
&((x_1, x_2T, x_3I, x_4F) \cdot (y_1, y_2T, y_3I, y_4F))))) \\
&= (x_1 \cdot ((y_1 \cdot (y_1 \cdot x_1)) \cdot (0 \cdot (0 \cdot (x_1 \cdot y_1)))), \\
&(x_2 \cdot ((y_2 \cdot (y_2 \cdot x_2)) \cdot (0 \cdot (0 \cdot (x_2 \cdot y_2)))))T, \\
&(x_3 \cdot ((y_3 \cdot (y_3 \cdot x_3)) \cdot (0 \cdot (0 \cdot (x_3 \cdot y_3)))))I, \\
&(x_4 \cdot ((y_4 \cdot (y_4 \cdot x_4)) \cdot (0 \cdot (0 \cdot (x_4 \cdot y_4)))))F) \\
&\in N_q(K, J).
\end{aligned}$$

Hence $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J). □

Corollary 3.7. *The neutrosophic quadruple set based on a BCI-implicative ideal K of X is an NQ-BCI-implicative ideal over (X, K).*

Proposition 3.8. *Every neutrosophic quadruple set based on BCI-implicative ideals K and J of X satisfies the following condition.*

$$\begin{aligned}
&((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J) \\
&\Rightarrow \tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y})))) \in N_q(K, J).
\end{aligned} \tag{3.1}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$.

Proof. Let $((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J)$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$. Then

$$\begin{aligned}
&((((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1)) \cdot 0, ((((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2)) \cdot 0)T,\\
&((((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3)) \cdot 0)I, ((((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4)) \cdot 0)F)\\
&= (((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1), (((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2))T,\\
&(((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3))I, (((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4))F)\\
&= (((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot\\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F))\\
&= ((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J),
\end{aligned}$$

and so $(((x_i \cdot y_i) \cdot y_i) \cdot (0 \cdot y_i)) \cdot 0 \in K$ for $i = 1, 2$ and $(((x_j \cdot y_j) \cdot y_j) \cdot (0 \cdot y_j)) \cdot 0 \in J$ for $j = 3, 4$. Since $0 \in K \cap J$, and since K and J are BCI-implicative ideals of X, it follows that $x_i \cdot ((y_i \cdot (y_i \cdot x_i)) \cdot (0 \cdot (0 \cdot (x_i \cdot y_i)))) \in K$ for $i = 1, 2$, and $x_j \cdot ((y_j \cdot (y_j \cdot x_j)) \cdot (0 \cdot (0 \cdot (x_j \cdot y_j)))) \in J$ for $j = 3, 4$. Hence we have

$$\begin{aligned}
&\tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y}))))\\
&= (x_1 \cdot ((y_1 \cdot (y_1 \cdot x_1)) \cdot (0 \cdot (0 \cdot (x_1 \cdot y_1)))),\\
&(x_2 \cdot ((y_2 \cdot (y_2 \cdot x_2)) \cdot (0 \cdot (0 \cdot (x_2 \cdot y_2)))))T,\\
&(x_3 \cdot ((y_3 \cdot (y_3 \cdot x_3)) \cdot (0 \cdot (0 \cdot (x_3 \cdot y_3)))))I,\\
&(x_4 \cdot ((y_4 \cdot (y_4 \cdot x_4)) \cdot (0 \cdot (0 \cdot (x_4 \cdot y_4)))))F)\\
&\in N_q(K, J).
\end{aligned}$$

This completes the proof. □

We provide conditions for a neutrosophic quadruple set to be an NQ-BCI-implicative ideal.

Theorem 3.9. *Let K and J be ideals of X such that*

$$\begin{aligned}
&((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J)\\
&\Rightarrow\ x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J)
\end{aligned} \tag{3.2}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Assume that $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z \in K$ (resp., J) for all $x, y \in X$ and $z \in K$ (resp., J). Then $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) since K and J are ideals of X. It follows from the condition (3.2) that $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J). Hence K and J are BCI-implicative ideals of X, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 3.10. *Let K be an ideal of X such that*

$$\begin{aligned}
&((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K\\
&\Rightarrow\ x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K
\end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X, K).

Theorem 3.11. *Let K and J be ideals of X such that*

$$0 \cdot x \in K \text{ (resp., } J), \tag{3.4}$$

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J) \;\Rightarrow\; x \cdot (y \cdot (y \cdot x)) \in K \text{ (resp., } J) \tag{3.5}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Assume that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) for all $x, y \in X$. Then $x \cdot (y \cdot (y \cdot x)) \in K$ (resp., J) by (3.5). Using (I), (II), (2.3), (2.5), (2.6) and (3.4), we have

$$\begin{aligned}
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y))))) \cdot (x \cdot (y \cdot (y \cdot x)))\\
&\leq (y \cdot (y \cdot x)) \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y))))\\
&\leq 0 \cdot (0 \cdot (x \cdot y))\\
&= 0 \cdot ((0 \cdot x) \cdot (0 \cdot y))\\
&= 0 \cdot ((((0 \cdot y) \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y))\\
&= 0 \cdot ((((0 \cdot (0 \cdot (0 \cdot y))) \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y))\\
&= 0 \cdot ((((0 \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y)) \cdot (0 \cdot (0 \cdot y)))\\
&= 0 \cdot (((0 \cdot (x \cdot y)) \cdot (0 \cdot y)) \cdot (0 \cdot (0 \cdot y)))\\
&= 0 \cdot (0 \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y)))\\
&\in K \text{ (resp., } J).
\end{aligned}$$

It follows that $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J). Hence K and J are BCI-implicative ideals of X by Lemma 2.1. Therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 3.12. *Let K be an ideal of X such that*

$$0 \cdot x \in K, \tag{3.6}$$

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \;\Rightarrow\; x \cdot (y \cdot (y \cdot x)) \in K \tag{3.7}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X, K).

Theorem 3.13. *Let X be a BCI-algebra satisfying the conditions:*

$$(\forall x, y \in X)(x \cdot y = ((x \cdot y) \cdot y) \cdot (0 \cdot y)), \tag{3.8}$$

$$(\forall x, y \in X)(x \cdot (x \cdot y) = y \cdot (y \cdot (x \cdot (x \cdot y)))). \tag{3.9}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Let K and J be closed ideals of X. Assume that $((x\cdot y)\cdot y)\cdot(0\cdot y)\in K$ (resp., J). Then $0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\in K$ (resp., J). Using the conditions (3.8), (3.9), (2.3), (2.5), (I) and (III), we have

$$\begin{aligned}
&(x\cdot(y\cdot(y\cdot x)))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&=(x\cdot(y\cdot(y\cdot x)))\cdot(x\cdot y)\\
&=(x\cdot(x\cdot y))\cdot(y\cdot(y\cdot x))\\
&=(y\cdot(y\cdot(x\cdot(x\cdot y))))\cdot(y\cdot(y\cdot x))\\
&=(y\cdot(y\cdot(y\cdot x)))\cdot(y\cdot(x\cdot(x\cdot y)))\\
&=(y\cdot x)\cdot(y\cdot(x\cdot(x\cdot y)))\\
&\leq(x\cdot(x\cdot y))\cdot x\\
&=0\cdot(x\cdot y)\\
&=0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&\in K \text{ (resp., } J).
\end{aligned}\tag{3.10}$$

It follows that $x\cdot(y\cdot(y\cdot x))\in K$ (resp., J), and so that

$$x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K \text{ (resp., } J)$$

in the proof of Theorem 3.18. Thus K and J are BCI-implicative ideals of X by Lemma 2.1, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J) by Theorem 3.6. □

Corollary 3.14. *Let X be a BCI-algebra satisfying the conditions* (3.8) *and* (3.9)*. If K is a closed ideal of X, then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X,K).*

Corollary 3.15. *Let X be a BCI-algebra satisfying the condition:*

$$(\forall x,y\in X)((x\cdot(x\cdot y))\cdot(y\cdot x)=y\cdot(y\cdot x)).\tag{3.11}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J).

Proof. If X satisfies the condition (3.11), then it satisfies two conditions (3.8) and (3.9) (see [**?**, **?**]). Hence the result is induced from Theorem 3.13. □

Corollary 3.16. *Let X be a BCI-algebra satisfying the condition* (3.11)*. If K is a closed ideal of X, then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X,K).*

Theorem 3.17. *Let X be a BCI-algebra satisfying the condition* (3.9) *and*

$$(\forall x,y\in X)((x\cdot(y\cdot x))\cdot(0\cdot(y\cdot x))=x).\tag{3.12}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J).

Proof. Let K and J be closed ideals of X. The conditions (3.12) and (III) lead to the following fact.

$$(z\cdot y)\cdot(((z\cdot y)\cdot(z\cdot(z\cdot y)))\cdot(0\cdot(z\cdot(z\cdot y))))=0.\tag{3.13}$$

It follows from (2.1), (I), (2.2), (2.3) and (III) that

$$\begin{aligned}
&(z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y)) = ((z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y)))\cdot 0\\
&= ((z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y)))\cdot((z\cdot y)\cdot(((z\cdot y)\cdot(z\cdot(z\cdot y)))\cdot\\
&(0\cdot(z\cdot(z\cdot y)))))\\
&\leq (((z\cdot y)\cdot(z\cdot(z\cdot y)))\cdot(0\cdot(z\cdot(z\cdot y))))\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y))\\
&\leq (((z\cdot y)\cdot y)\cdot(0\cdot(z\cdot(z\cdot y))))\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y))\\
&\leq (0\cdot y)\cdot(0\cdot(z\cdot(z\cdot y)))\\
&\leq (z\cdot(z\cdot y))\cdot y\\
&= (z\cdot y)\cdot(z\cdot y) = 0.
\end{aligned} \tag{3.14}$$

Hence $(z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y)) = 0$ since 0 is a minimal element of X, that is,

$$z\cdot y \leq ((z\cdot y)\cdot y)\cdot(0\cdot y). \tag{3.15}$$

On the other hand, we get

$$\begin{aligned}
&(((z\cdot y)\cdot y)\cdot(0\cdot y))\cdot(z\cdot y) = (((z\cdot y)\cdot y)\cdot(z\cdot y))\cdot(0\cdot y)\\
&= (((z\cdot y)\cdot(z\cdot y))\cdot y)\cdot(0\cdot y) = (0\cdot y)\cdot(0\cdot y) = 0
\end{aligned}$$

by (2.3) and (III), that is,

$$((z\cdot y)\cdot y)\cdot(0\cdot y) \leq z\cdot y. \tag{3.16}$$

Conditions (3.15) and (3.16) induce

$$z\cdot y = ((z\cdot y)\cdot y)\cdot(0\cdot y).$$

Therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.13. □

We now consider extension property of NQ-BCI-implicative ideal.

Theorem 3.18. *For any nonempty subsets K and J of X, let A and B be closed ideals of X such that $K \subseteq A$ and $J \subseteq B$. If K and J are BCI-implicative ideals of X, then the neutrosophic quadruple set based on A and B is an NQ-BCI-implicative ideal over (X, A, B), which is larger than the NQ-BCI-implicative ideal over (X, K, J).*

Proof. Assume that K and J are BCI-implicative ideals of X. It is clear that $N_q(K, J) \subseteq N_q(A, B)$. Let $((x\cdot y)\cdot y)\cdot(0\cdot y) \in A$ (resp., B) for all $x, y \in X$. Then $0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)) \in A$ (resp., B) since A and B are closed ideals of X. Using (2.3) and (III) induce

$$\begin{aligned}
&(((x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot y)\cdot y)\cdot(0\cdot y)\\
&= (((x\cdot y)\cdot y)\cdot(0\cdot y))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&= 0 \in K \text{ (resp., } J),
\end{aligned} \tag{3.17}$$

which implies from Lemma 2.1 that

$$\begin{aligned}
&(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot((y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\cdot\\
&(0\cdot(0\cdot((x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot y))))\\
&\in K \subseteq A \text{ (resp., } J \subseteq B).
\end{aligned} \tag{3.18}$$

Since

$$\begin{aligned}0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)) &= ((0\cdot(x\cdot y))\cdot(0\cdot y))\cdot(0\cdot(0\cdot y))\\ &= (((0\cdot x)\cdot(0\cdot y))\cdot(0\cdot y))\cdot(0\cdot(0\cdot y))\\ &= (((0\cdot(0\cdot(0\cdot y)))\cdot x)\cdot(0\cdot y))\cdot(0\cdot y)\\ &= (((0\cdot y)\cdot x)\cdot(0\cdot y))\cdot(0\cdot y)\\ &= (0\cdot x)\cdot(0\cdot y)\\ &= 0\cdot(x\cdot y)\end{aligned} \tag{3.19}$$

by (2.6), (2.3), (2.5) and (III), we have

$$\begin{aligned}&0\cdot(0\cdot((x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot y))\\ &= 0\cdot(0\cdot((x\cdot y)\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))\\ &= 0\cdot((0\cdot(x\cdot y))\cdot(0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))\\ &= 0\cdot((0\cdot(x\cdot y))\cdot(0\cdot(x\cdot y)))\\ &= 0.\end{aligned} \tag{3.20}$$

Combining (3.18) and (3.20) implies that

$$\begin{aligned}&(x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\ &= (x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\\ &\in A \text{ (resp., } B).\end{aligned} \tag{3.21}$$

Since A and B are ideals of X, it follows that

$$x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))) \in A \text{ (resp., } B). \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}&(x\cdot(y\cdot(y\cdot x)))\cdot(x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))))\\ &\le (y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\cdot(y\cdot(y\cdot x))\\ &\le (y\cdot x)\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))\\ &\le (x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot x\\ &= 0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\ &\in A \text{ (resp., } B).\end{aligned} \tag{3.23}$$

By (3.22) and (3.23), we get $x\cdot(y\cdot(y\cdot x)) \in A$ (resp., B). Using (3.19), (I), (II) we get

$$\begin{aligned}&(x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y)))))\cdot(x\cdot(y\cdot(y\cdot x)))\\ &\le (y\cdot(y\cdot x))\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\\ &\le 0\cdot(0\cdot(x\cdot y))\\ &= 0\cdot(0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))) \in A \text{ (resp., } B).\end{aligned} \tag{3.24}$$

It follows that $x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y)))) \in A$ (resp., B). Hence A and B are BCI-implicative ideals of X by Lemma 2.1. Therefore the neutrosophic quadruple set based on A and B is an NQ-BCI-implicative ideal over (X, A, B) by Theorem 3.6. □

Corollary 3.19. *For any nonempty subset K of X, let A be a closed ideal of X such that $K \subseteq A$. If K is a BCI-implicative ideals of X, then the neutrosophic quadruple set based on A is an NQ-BCI-implicative ideal over (X, A), which is larger than the NQ-BCI-implicative ideal over (X, K).*

4. Relations between NQ-BCI-commutative ideal, NQ-BCI-positive implicative ideal and NQ-BCI-implicative ideal

Theorem 4.1. *For any nonempty subsets K and J of X, every NQ-BCI-implicative ideal over (X, K, J) is an NQ-BCI-commutative ideal over (X, K, J).*

Proof. Let K and J be nonempty subsets of X such that the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J). Let $x, y, z \in X$ be such that $z \in K$ (resp., J) and $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z \in K$ (resp., J). Then $(z, zT, zI, zF) \in N_q(K, J)$ and

$$\begin{aligned}
&((((x, xT, xI, xF) \boxdot (y, yT, yI, yF)) \boxdot (y, yT, yI, yF)) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y, yT, yI, yF))) \boxdot (z, zT, zI, zF) \\
&= ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z, ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)T, \\
&((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)I, ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)F) \\
&\in N_q(K, J)
\end{aligned}$$

Since $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$, it follows that

$$\begin{aligned}
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))), (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))T, \\
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))I, (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))F) \\
&= (x, xT, xI, xF) \boxdot (((y, yT, yI, yF) \boxdot ((y, yT, yI, yF) \boxdot (x, xT, xI, xF))) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot ((x, xT, xI, xF) \boxdot (y, yT, yI, yF))))) \\
&\in N_q(K, J).
\end{aligned}$$

Hence $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J), and so K and J are BCI-implicative ideals of X. Thus K and J are ideals of X. Assume that $x \cdot y \in K$ (resp., J) for all $x, y \in X$. Then

$$(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot (x \cdot y) = (0 \cdot y) \cdot (0 \cdot y) = 0 \in K \text{ (resp., } J)$$

by using (2.3) and (III), which implies that

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J).$$

Hence $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot 0 \in K$ (resp., J) and $0 \in K$ (resp., J). Since K (resp., J) is a BCI-implicative ideal of X, it follows that

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J).$$

Therefore K (resp., J) is a BCI-commutative ideal of X by Lemma 2.2, and consequently the neutrosophic quadruple set based on K and J is an NQ-BCI-commutative ideal over (X, K, J). □

The converse of Theorem 4.1 is not true in general. In fact, $N_q(K)$ in Example 3.4 is not a BCI-implicative ideal of $N_q(X)$. But it is routine to verify that $N_q(K)$ is a BCI-commutative ideal of $N_q(X)$.

Lemma 4.2 ([6]). *If K and J are BCI-positive implicative ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-positive implicative ideal over (X, K, J).*

Theorem 4.3. *For any nonempty subsets K and J of X, every NQ-BCI-implicative ideal over (X, K, J) is an NQ-BCI-positive implicative ideal over (X, K, J).*

Proof. Let K and J be nonempty subsets of X such that $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$. Then K and J are ideals of X (see the proof of Theorem 4.1). Let $x, y \in X$ be such that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J). Then

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J)$$

by Lemma 2.1. Note that

$$\begin{aligned}
&(x \cdot y) \cdot (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))\\
&\leq ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \cdot y\\
&= (0 \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))\\
&= (0 \cdot (x \cdot y)) \cdot (y \cdot x)\\
&= ((0 \cdot x) \cdot (0 \cdot y)) \cdot (y \cdot x)\\
&= (0 \cdot (0 \cdot x)) \cdot x\\
&= 0 \in K \text{ (resp., } J).
\end{aligned}$$

It follows that $x \cdot y \in K$ (resp., J). Hence K and J are BCI-positive implicative ideals of X by Lemma 2.3, and therefore $N_q(K, J)$ is a BCI-positive implicative ideal of $N_q(X)$ by Lemma 4.2. □

In the following example, we can see that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation "·", which is given in Table 3.

TABLE 3. Cayley table for the binary operation "·"

·	0	1	2	3	4
0	0	0	0	0	4
1	1	0	1	0	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then X is a BCI-algebra (see [8]), and the neutrosophic quadruple BCI-algebra $N_q(X)$ has 625 elements. If we take $K = \{0, 2\}$, then the neutrosophic quadruple set based on K has 16-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\rho}_i \mid i = 1, 2, \cdots, 15\},$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\rho}_1 = (0, 0T, 0I, 2F)$, $\tilde{\rho}_2 = (0, 0T, 2I, 0F)$,
$\tilde{\rho}_3 = (0, 0T, 2I, 1F)$, $\tilde{\rho}_4 = (0, 2T, 0I, 0F)$, $\tilde{\rho}_5 = (0, 2T, 0I, 2F)$,

$\tilde{\rho}_6 = (0, 2T, 2I, 0F)$, $\tilde{\rho}_7 = (0, 2T, 2I, 2F)$, $\tilde{\rho}_8 = (2, 0T, 0I, 0F)$,
$\tilde{\rho}_9 = (2, 0T, 0I, 2F)$, $\tilde{\rho}_{10} = (2, 0T, 2I, 0F)$, $\tilde{\rho}_{11} = (2, 0T, 2I, 2F)$,
$\tilde{\rho}_{12} = (2, 2T, 0I, 0F)$, $\tilde{\rho}_{13} = (2, 2T, 0I, 2F)$, $\tilde{\rho}_{14} = (2, 2T, 2I, 0F)$,
$\tilde{\rho}_{15} = (2, 2T, 2I, 2F)$.

It is routine to verify that $N_q(K)$ is an NQ-BCI-positive implicative ideal over (X, K). If we take $\tilde{\alpha}_1 = (1, 1T, 1I, 1F)$ and $\tilde{\alpha}_3 = (3, 3T, 3I, 3F)$ in $N_q(X)$, then $\tilde{0} \in N_q(K)$ and

$$(((\tilde{\alpha}_1 \boxdot \tilde{\alpha}_3) \boxdot \tilde{\alpha}_3) \boxdot (\tilde{0} \boxdot \tilde{\alpha}_3)) \boxdot \tilde{0} = \tilde{0} \in N_q(K).$$

But,

$$\tilde{\alpha}_1 \boxdot ((\tilde{\alpha}_3 \boxdot (\tilde{\alpha}_3 \boxdot \tilde{\alpha}_1)) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{\alpha}_1 \boxdot \tilde{\alpha}_3)))) = \tilde{\alpha}_1 \boxdot (\tilde{0} \boxdot \tilde{0}) = \tilde{\alpha}_1 \notin N_q(K).$$

Hence $N_q(K)$ is not an NQ-BCI-implicative ideal over (X, K).

We display a characterization of an NQ-BCI-implicative ideal.

Theorem 4.5. *For any nonempty subsets K and J of X, the neutrosophic quadruple set based on K and J is both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K, J) if and only if it is an NQ-BCI-implicative ideal over (X, K, J).*

Proof. For the sufficiency, see Theorems 4.1 and 4.3. Let $N_q(K, J)$ be both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K, J). Then K and J are both a BCI-commutative ideal and a BCI-positive implicative ideal of X. Assume that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) for all $x, y \in X$. Then $x \cdot y \in K$ (resp., J) by Lemma 2.3, and so

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K(\text{resp., } J)$$

by Lemma 2.2. It follows from Lemma 2.1 that K and J are BCI-implicative ideals of X. Therefore the neutrosophic quadruple set based on K and J is an NQ-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 4.6. *For any nonempty subset K of X, the neutrosophic quadruple set based on K is both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K) if and only if it is an NQ-BCI-implicative ideal over (X, K).*

5. Conclusions

Smarandache introduced the notion of neutrosophic quadruple numbers by considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where a, b, c and d are real or complex numbers and T, I, F have their usual neutrosophic logic meanings. Jun et al. made up neutrosophic quadruple BCK/BCI-algebras and (positive) implicative neutrosophic quadruple BCK-algebras using neutrosophic quadruple numbers based on BCK/BCI-algebras (instead of real or complex numbers). In this article, we have studied BCI-implicative ideal in BCI-algebra using neutrosophic quadruple structure. We have introduced neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra, and have investigated their related properties. We have consulted relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal. We have provided conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal. We have discussed a characterization of an NQ-BCI-implicative ideal, and have established the extension property of neutrosophic quadruple BCI-implicative ideal. Based on the contents and ideas of this manuscript, we will study neutrosophic quadruple structure for various algebraic sub-structures in the future.

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