Exact solutions of conformable fractional Harry Dym equation

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Abstract: The aim of this paper is to find exact solutions for the conformable fractional Harry Dym Equation. In this work we deal with three different forms of conformable fractional Harry Dym Equation and for each form a suitable wave variable substitution is found. Each substitution transform its corresponding problem to an ordinary differential equation, What is more, the resulted ordinary differential equations in the three cases are the same. General solutions are obtained by applying the direct integration method on the resulted ordinary differential equation. These obtained solutions are found for some particular choices for the constants values. The behavior of every solution is discussed and illustrated in graphs. The tedious integrals and difficult computations associated with calculations in this paper are performed and simplified by using Mathematica 9.0.

Keywords: Conformable fractional derivative, Harry Dym Equation, Conformable Harry Dym Equation, Exact solutions.

1. Introduction

Recently, differential equations with fractional derivatives attracted the interest of many researchers; since such equations describe effectively many phenomena in applied sciences such as physics, biology, technology, and engineering [3, 7, 14].

Harry Dym equation (HD) was so named related to the name of its discoverer Harry Dym in his unpublished paper 1973-1974, although it appeared to first time in Kruskal and Moser [9]. HD equation represents a system which gathers non-linearity and dispersion, also it is a completely integrable nonlinear evolution equation which obeys an infinite number of conservation laws, but it does not have the Painleve property. More properties for HD equation discussed in details can be found in the reference [4]. Moreover HD equation can be connected to the Korteweg-ge Vries equation which has many applications in hydrodynamics [4, 15].

Many efforts have been done to find exact and approximate solutions for both HD equation and fractional HD equation like algebraic geometric solution of the HD equation[13], solitions solutions of the (2+1) dimensional HD equation via Darboux transformation [2], explicit solutions for HD equation [1], exact solution of the HD equation [12], an efficient approach for fractional HD equation by using sumudu transform [10], symmetries and exact solutions of the time fractional HD equation with Rieman-Liouville derivative [5], and a fractional model of HD equation and its approximate solution [11].

Fractional derivatives have many definitions [14] but the most used of these definitions are Riemann-Liouville derivative and Caputo derivative. They were defined as follows:

(i) Riemann - Liouville Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

(ii) Caputo Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Recently, a new definition called conformable fractional derivative was introduced by authors in [6], Since then the interest of it keeps growing and many equations were solved using such definition [8]. In this paper we intend to find exact solutions for fractional HD equation in the sense of this definition rather than Rieman-Liouville definition or Caputo definition. The rest of the paper is organized as follows: Basics of conformable fractional derivative are stated in section 2, in section 3 solutions for conformable fractional HD equation are found, in section 4 some examples are discussed.

2. Basic results on conformable fractional derivatives.

Now, Let us summarize the basic properties of the conformable fractional derivative definition.

Definition [6]: Given a function$f: [0, \infty) \to \mathbb{R}$. And $t > 0, \alpha \in (0, 1]$, then the conformable fractional derivative of order α is defined as

$T_\alpha\,(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}$,

T_α is called the conformable fractional derivative of f of order α .

Let $f^{\alpha}(t)$ stands for $T_\alpha\,(f)(t) = \frac{d^\alpha f}{dt^\alpha}$.

If f is α-differentiable in some$(0, b), b > 0$, and $\lim_{t\to 0^+} f^{\alpha}(t)$ exists, then by definition:

$f^{\alpha}(0) = \lim_{t\to 0^+} f^{\alpha}(t)$

Theorem 1 [6]: Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then

1. $T_\alpha\,(af + bg) = a\,T_\alpha\,(f) + b\,T_\alpha\,(g),\ for\ all\ a, b \in\ \mathbb{R}$.
2. $T_\alpha\,(t^p) = pt^{p-\alpha}\ for\ all\ p \in\ \mathbb{R}$.
3. $T_\alpha\,(\lambda) = 0\ for\ all\ constants\ functions\ f(t) = \lambda$.
4. $T_\alpha\,(fg) = f\,T_\alpha\,(g) + g\,T_\alpha\,(f)$.
5. $T_\alpha\,\left(\frac{f}{g}\right) = \frac{g\,T_\alpha\,(f)-f\,T_\alpha\,(g)}{g^2}$.
6. If, in addition, f is differentiable, then $T_\alpha\,(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Theorem 2 [8]: let f be an α-differentiable function in conformable sense and differentiable and suppose that g is also differentiable and defined in the range of f. Then

$$T_\alpha\,(fog)\,(t) = t^{1-\alpha}\; g^{'}(t) f^{'}\big(g(t)\big).$$

More properties, definitions and theorems as Roll's Theorem and Mean Value Theorem for conformable fractional derivative are expressed in the work [6],

3. Fractional Harry Dym Equation.

The classical HD equation is:

$$u_t \;= u^3 u_{xxx} \qquad\qquad (*)$$

Where $u(x,t)$ is a function of two real variables $x\; and\; t$.

Let us write:

$$u_t^\alpha \;= T_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha}\;, \qquad u_x^\alpha \;= T_x^\alpha u = \frac{\partial^\alpha u}{\partial x^\alpha}\;, \qquad u_x^{(3\alpha)} = T_x^{(3\alpha)} u = T_x^\alpha\, T_x^\alpha\, T_x^\alpha u\,.$$

Now we will solve three fractional forms of$(*)$:

(i) $u_t^\alpha \;= u^3 u_{xxx}\;.$ (1)

(ii) $u_t \;= u^3 u_x^{(3\alpha)}.$ (2)

(iii) $u_t^\alpha \;= u^3 u_x^{(3\alpha)}.$ (3)

Where $\alpha \in (0, 1]$.

Using suitable wave variable substitution in each form will transform the equation to an ordinary differential equation as follows:

1. For form (i) let the wave variable substitution $\eta = x + \frac{c}{\alpha} t^\alpha$ and $u(x,t) = v(\,\eta)$. So one can write $u = v \circ \eta$, now apply Theorem 2 to find u_t^α . You will get that $u_t^\alpha \;= t^{1-\alpha}\, \eta^{'}(t) v^{'}\big(\eta(t)\big) = cv^{'}, also\; u^3 \;= v^3 and\; u_{xxx} \;= v^{'''}$. Hence equation (1) is transformed to:

$$cv^{'} \;= v^3 v^{'''} \qquad\qquad (4)$$

2. For form (ii) let the wave variable substitution $\eta = \frac{1}{\alpha} x^\alpha + ct$ and $u(x,t) = v(\,\eta) = v \circ \eta.$ so

$u_t^\alpha \;= cv^{'}, u^3 \;= v^3 and\; u_x^{(3\alpha)} \;= v^{'''}$. Then equation (2) is transformed to:

$$cv^{'} \;= v^3 v^{'''} \qquad\qquad (4)$$

3. For form (iii) let the wave variable substitution $\eta = \frac{1}{\alpha}x^{\alpha} + \frac{c}{\alpha}t^{\alpha}$ and $u(x,t) = v(\eta)$. so $u_t^{\alpha} = cv', u^3 = v^3 and\ u_x^{(3\alpha)} = v'''$. Then equation (3) is transformed to:

$$cv' = v^3 v''' \qquad (4)$$

Now to solve the resulted ordinary differential equation (4), rewrite it as:

$$v''' + \left(\frac{c}{2v^2}\right)' = 0 \qquad (5)$$

Integrate (5) with respect to η, gets

$$v'' + \frac{c}{2v^2} = \frac{c_1}{2} \qquad (6)$$

Multiply (6) by v' then integrate with respect to η yields

$$(v')^2 = \frac{c}{v} + c_1 v + c_2 \qquad (7)$$

Using the separation of variables changes (7) to

$$d\eta = \pm\sqrt{\frac{v}{c_1 v^2 + c_2 v + c}}\ dv \qquad (8)$$

Integrate both sides of (8) using Mathematica 9.0 you will obtain

$$\eta = \pm\int\sqrt{\frac{v}{c_1 v^2 + c_2 v + c}}\, dv + c_3 \qquad (9)$$

$$\eta = \pm i\ \frac{ABCD}{c_1 G}[Elliptic\ E(i\ \sinh^{-1}(G), K) - Elliptic\ F(i\ \sinh^{-1}(G), K)] + c_3 \qquad (10)$$

Where: $A = \sqrt{\frac{v}{c_1 v^2 + c_2 v + c}}$, $B = -c_2 + \sqrt{-4cc_1 + c_2{}^2}$, $C = \sqrt{1 + \frac{2c_1 v}{c_2 - \sqrt{-4cc_1 + c_2{}^2}}}$

$D = \sqrt{1 + \frac{2c_1 v}{c_2 + \sqrt{-4cc_1 + c_2{}^2}}}$, $G = \sqrt{\frac{2c_1 v}{c_2 + \sqrt{-4cc_1 + c_2{}^2}}}$ and $K = \frac{c_2 + \sqrt{-4cc_1 + c_2{}^2}}{c_2 - \sqrt{-4cc_1 + c_2{}^2}}$.

$Elliptic\ F$ and $Elliptic\ E$ are elliptic integrals of the first and second kind respectively.

For some particular choices to the constants c, c_1 and c_2 in equation (9) one can get simpler solutions as follows:

- Let $c_1 = c_2 = 0$, then $\eta = \pm \frac{2}{3}\, v \sqrt{\frac{v}{c}} + c_3$, hence

$$v = (c_3 \pm \frac{3}{2}\sqrt{c}\ \eta)^{\frac{2}{3}} \qquad (11)$$

- Let $c_1 = 0, c_2 \neq 0$, then $\eta = \pm\left(\frac{\sqrt{cv + c_2 v^2}}{c_2} - \frac{c}{c_2^{\frac{3}{2}}} \log(2c_2\sqrt{v} + 2\sqrt{vc_2{}^2 + c_2 c}\ )\right) + c_3$

Other suggested constants are:

1. Let $c_2 = 2\sqrt{cc_1}$.
2. Let $c_2 = -2\sqrt{cc_1}$.

You can easily using Mathematica 9.0 to perform the integration of equation (9) to get formula of η after you determine the suggested constants, however the difficulty that faces is how to get v with respect to η explicitly , except the formula in (11), this what was discussed in [12], Hence it seems that formula (11) is the only explicit solution for equations (1), (2) and (3). So results can be summarized as follows:

· The solution of equation (1) is $u(x,t) = (c_3 \pm \frac{3}{2}\sqrt{c}\ \left(x + \frac{c}{\alpha} t^{\alpha}\right))^{\frac{2}{3}}$.

· The solution of equation (2) is $u(x,t) = (c_3 \pm \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha} x^{\alpha} + ct)\ )^{\frac{2}{3}}$.

· The solution of equation (3) is $u(x,t) = (c_3 \pm \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha} x^{\alpha} + \frac{c}{\alpha} t^{\alpha})\ )^{\frac{2}{3}}$.

Remarks:

1. The same ordinary differential equation is obtained from the three different forms of conformable fractional Harry Dym- Equation after using special wave variable for each form.

2. A function could be α-differentiable at a point but not differentiable, illustrating example was discussed in [6].

4. Examples.

Example 1: Let $\alpha = 0.7$, for the graph of equation (1) solution $u(x,t) = (c_3 + \frac{3}{2}\sqrt{c}\left(x + \frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4$ and $c = 1$ see Figure 1.

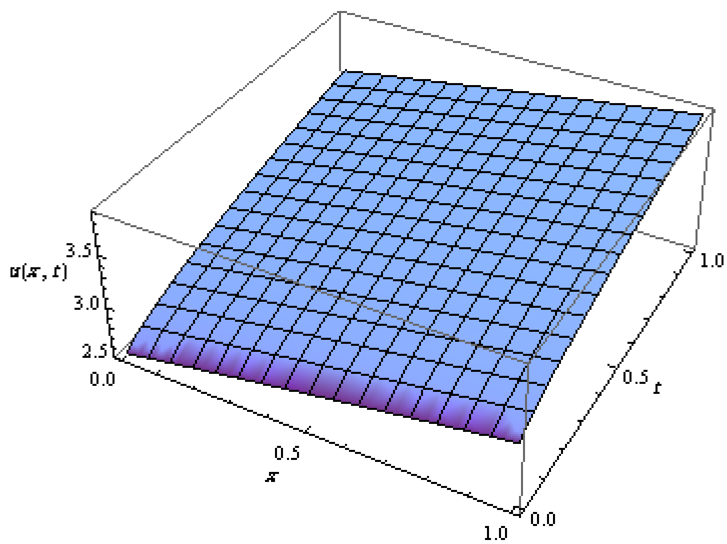

Fig. 1 The graph of $u(x,t) = (4 + \frac{3}{2}\left(x + \frac{1}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ at $\alpha = 0.7$ for example 1

Example 2: The graph of equation (1) solution $u(x,t) = (c_3 + \frac{3}{2}\sqrt{c}\left(x + \frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ versus x at $t = 1$, $c_3 = 4$ *and* $c = 1$ for different values of α is in Figure 2.

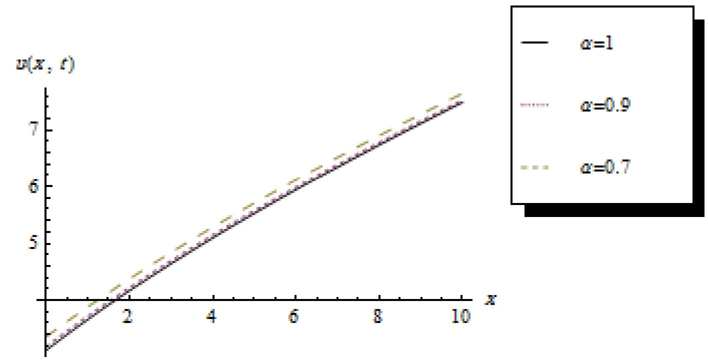

Fig. 2 The graph of $u(x,t) = (4 + \frac{3}{2}\left(x + \frac{1}{\alpha}\right))^{\frac{2}{3}}$ versus x at $t = 1$ at $\alpha = 1, 0.9$ and 0.7 for example 2

Example 3: Let $\alpha = 0.9$, for the graph of equation (2) solution $u(x,t) = (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\,(\frac{1}{\alpha}x^{\alpha} + ct)\,)^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4$ and $c = 1$ see Figure 3.

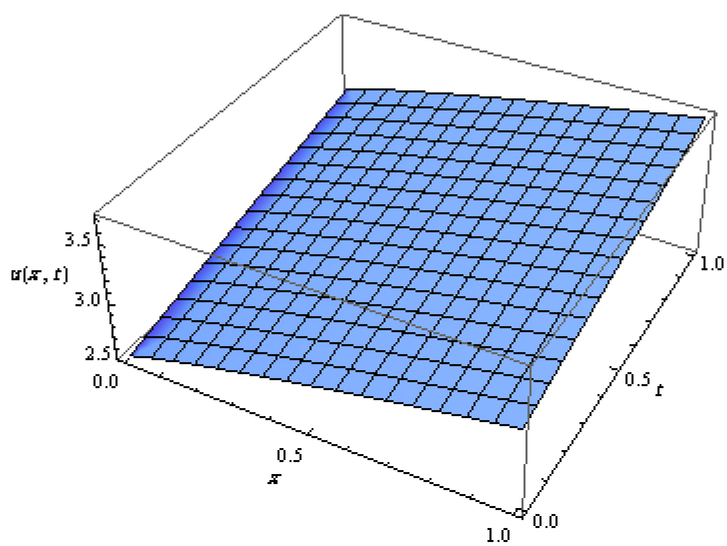

Fig. 3 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + t\right))^{\frac{2}{3}}$ at $\alpha = 0.9$ for example 3

Example 4: The graph of equation (2) solution $u(x,t) = \left(4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + t\right)\right)^{\frac{2}{3}}$ versus x at $t = 0, c_3 = 4$ and $c = 1$ for different values of α is in Figure 4.

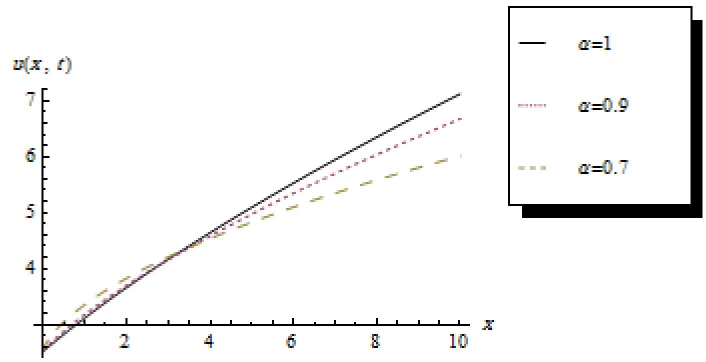

Fig. 4 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}\ x^{\alpha} + t\right))^{\frac{2}{3}}$ versus x at $\mathrm{t} = 0$ at $\alpha = 1, 0.9$ and 0.7 for example 4

Example 5: Let $\alpha = 0.9$, for the graph of equation (3) solution $u(x,t) = (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha}x^{\alpha} + \frac{c}{\alpha}t^{\alpha})\ )^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4$ and $c = 1$ see Figure 5.

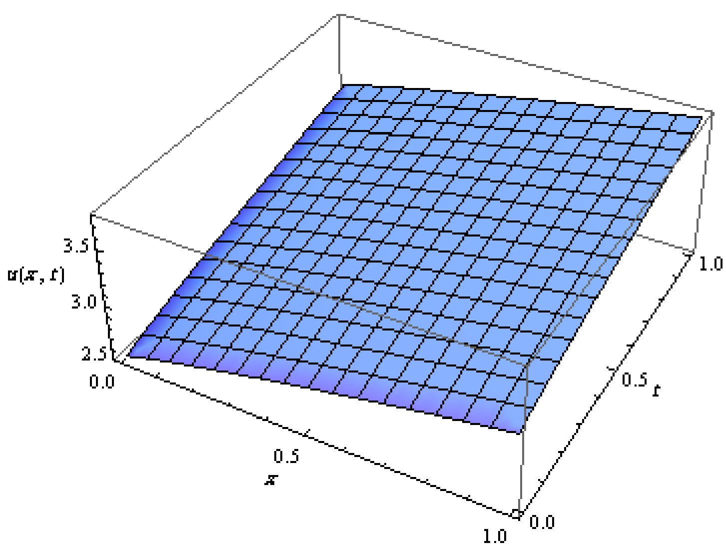

Fig. 5 The graph of $u(x,t) = \ (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + \frac{1}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ at $\alpha = 0.9$ for example 5

Example 6: The graph of equation (3) solution $u(x,t) = (c_3 + \frac{3}{2}\sqrt{c}\,(\frac{1}{\alpha}x^\alpha + \frac{c}{\alpha}t^\alpha)\,)^{\frac{2}{3}}$ versus x at $t = 1$, $c_3 = 4$ *and* $c = 1$ for different values of α is in Figure 6.

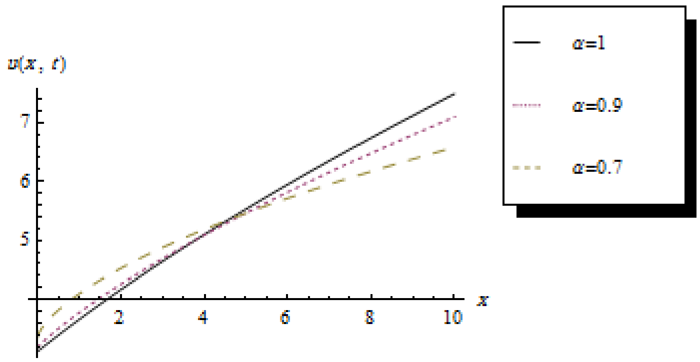

Fig. 6 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^\alpha + \frac{1}{\alpha}t^\alpha\right))^{\frac{2}{3}}$ versus x at $t = 1$ *at* $\alpha = 1, 0.9$ *and* 0.7 for example 6

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