J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

GENERALIZED ZWEIER $\mathcal{I}$-CONVERGENT SEQUENCE SPACES OF FUZZY NUMBERS

KAVITA SAINI AND KULDIP RAJ

Abstract. In the present paper we introduce Zweier ideal convergent sequences spaces of fuzzy numbers by using lacunary sequence, infinite matrix and generalized difference matrix operator A_i^p. We study some topological and algebraic properties of these sequence spaces. Some inclusion relations related to these spaces are also establish.

1. Introduction and Preliminaries

Initially the idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al.[10]. Gurdal [7] studied the ideal convergence sequences in 2-normed spaces. Later on, it was further studied by Savas [21], Savas and Hazarika [8], Tripathy and Dutta [25], Tripathy and Hazarika [26], Raj et al.[17]. Let X be a non-empty set, then a family of sets $\mathcal{I} \subset 2^X$ is called an ideal iff for each $X_1, X_2 \in \mathcal{I}$, we have $X_1 \cup X_2 \in \mathcal{I}$ and for each $X_1 \in \mathcal{I}$ and each $X_2 \subset X_1$, we have $X_2 \in \mathcal{I}$. A non-empty family of sets $U \subset 2^X$ is a filter on X iff $\phi \notin U$, for each $X_1, X_2 \in U$, we have $X_1 \cap X_2 \in U$ and each $X_1 \in U$ and each $X_1 \subset X_2$, we have $X_2 \in U$. An ideal $\mathcal{I}$ is said to be non-trivial ideal if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$. Clearly, $\mathcal{I} \subset 2^X$ is a non-trivial ideal iff $U = U(\mathcal{I}) = \{X - X_1 : X_1 \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset 2^X$ is said to be admissible iff $\{x : x \in X\} \subset \mathcal{I}$. A non-trivial ideal is called maximal if there cannot exists any non-trivial ideal $\mathcal{J} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset.
A sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to a real number x_0 if

$$\{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\} \in \mathcal{I},$$

for every $\epsilon > 0$ (see [10]). We denote it by $\mathcal{I} - \lim x_k = x_0$.
Kızmaz [9] introduced the notion of difference sequence spaces and studied $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. Further this notion generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\Delta^i)$, $c(\Delta^i)$ and $c_0(\Delta^i)$. The new type of generalization of the difference sequence spaces was introduced by Tripathy and Esi [27] who studied the spaces $l_\infty(\Delta_v^i)$, $c(\Delta_v^i)$ and $c_0(\Delta_v^i)$. Let i, v be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_v^i) = \{x = (x_k) \in w : (\Delta_v^i x_k) \in Z\},$$

where $\Delta_v^i x = (\Delta_v^i x_k) = (\Delta_v^{i-1} x_k - \Delta_v^{i-1} x_{k+1})$ and $\Delta_v^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^i x_k = \sum_{n=0}^{i} (-1)^n \begin{pmatrix} i \\ n \end{pmatrix} x_{k+vn}.$$

Başar and Atlay [2] introduced and studied the generalized difference matrix $A(m,n) =$

2010 *Mathematics Subject Classification.* 40A05, 40A30.
Key words and phrases. Musielak-Orlicz function, ideal convergence, generalized difference matrix operator, fuzzy real number.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

$(a_{rs}(m,n))$ which is a generalization of $\Delta^1_{(1)}$-difference operator as follows:

$$a_{rs}(m,n) = \begin{cases} m, & (s=r); \\ n, & (s=r-1); \\ 0, & 0 \leq s \leq r-1 \text{ or } s > r. \end{cases}$$

for all $r, s \in \mathbb{N}$ and $m, n \in \mathbb{R} - \{0\}$.
Başarir and Kayikçi [3] introduced the generalized difference matrix A^p of order p and the binomial representation of this operator is

$$A^p(x_k) = \sum_{v=0}^{p} \binom{p}{v} m^{p-v} n^v x_{k-v},$$

where $m, n \in \mathbb{R} - \{0\}$ and $r \in \mathbb{N}$.
Recently, Başarir et al.[4] studied the following generalized difference sequence spaces

$$Z(A_i^p) = \{x = (x_k) \in w : (A_i^p x_k) \in Z\},$$

for $Z = l_\infty,\ \bar{c},\ \bar{c_0}$, where $\bar{c}, \bar{c_0}$ are the sets of statistically convergent and statistically null convergent respectively and the binomial representation of operator A_i^p is as follows:

$$A_i^p(x_k) = \sum_{v=0}^{p} \binom{p}{v} m^{p-v} n^v x_{k-iv}.$$

Şengönül [22] defined the sequence $y = (y_k)$ which is frequently used as the $Z-$transformation of the sequence $x = (x_k)$ that is,

$$y_k = \beta x_k + (1-\beta) x_{k-1},$$

where $x_{-1} = 0,\ k \neq 0,\ 1 < k < \infty$ and Z denotes the matrix $Z = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} \beta, & (i=k); \\ 1-\beta, & (i-1=k)(i,k \in \mathbb{N}); \\ 0, & \text{otherwise}. \end{cases}$$

Şengönül [22] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows:

$$\mathcal{Z} = \{x = (x_k) \in w : Z(x) \in c\}$$

and

$$\mathcal{Z}_0 = \{x = (x_k) \in w : Z(x) \in c_0\}.$$

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex, continuous and non-decreasing function which also satisfy $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is called as an Orlicz sequence space. An Orlicz function is said to satisfy Δ_2-condition if for a constant R, $M(Qx) \leq RQM(x)$ for all values of $x \geq 0$ and for $Q > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called as Musielak-Orlicz function.To know more about sequence spaces see ([1], [15], [16], [24], [18], [19] and [28]) and references therein.
An increasing non-negative integer sequence $\theta = (k_r)$ with $k_0 = 0$ and $k_r - k_{r-1} \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and q_r denotes the ratio $\frac{k_r}{k_{r-1}}$. The space of

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

lacunary strongly convergent sequence was defined by Freedman et al. [6] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0, \text{ for some } L\Big\}.$$

The space N_θ is a $BK-$ space with the norm

$$\|x\| = \sup\bigg(\frac{1}{h_r}\sum_{k\in I_r}|x_k|\bigg).$$

Let $\lambda = (\lambda_{nk})$ be an infinite matrix of real or complex numbers λ_{nk}, where $n, k \in$ N. Then a matrix transformation of $x = (x_k)$ is denoted as λx and $\lambda x = \lambda_n(x))$ if $\lambda_n(x) = \sum_{k=1}^{\infty} \lambda_{nk}x_k$ converges for each $n \in$ N.

The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [29] in 1965. Subsequently many authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. The theory of sequences of fuzzy numbers was first studied by Matloka [12]. He studied some of their properties and showed that every convergent sequences of fuzzy numbers is bounded. Later on Nanda [13] introduced sequences of fuzzy numbers and studied that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Further, the theory of sequences of fuzzy numbers have been discussed by Savas and Mursaleen [20], Tripathy and Nanda [23], Hazarika and Savas [8] and many more.

Let B denote the set of all closed bounded intervals $U = [u_1, u_2]$ on the real line $\mathbb{R}$. For $U, V \in B$, we define $U \leq V$ iff $u_1 \leq v_1$ and $u_2 \leq v_2$ and we define

$$d(U, V) = \max\{|u_1 - v_1|, |u_2, v_2|\}.$$

It is well known that d defines a metric on B and (B, d) is a complete metric space (see [14]).

A fuzzy number is a function $U : \mathbb{R} \to [0, 1]$, which satisfy the following conditions:

(i) U is normal i.e there exits an x_0 such that $U(x_0) = 1$,

(ii) U is convex i.e for $x, y \in \mathbb{R}$ and $0 \leq \tau \leq 1$,

$$U(\tau x + (1-\tau)y) \geq \min\{U(x), U(y)\},$$

(iii) U is upper semi-continuous,

(iv) the closure of the set supp(U) is compact, where supp$(U) = \{x \in \mathbb{R} : U(x) > 0\}$ and it is denoted by $[U]^0$.

The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathbb{F}}$. Let $[U]^0 = \overline{x \in \mathbb{R} : u(x) > 0}$ and the r-level set is $[U]^r = \{x \in \mathbb{R} : u(x) \geq r\}, (0 \leq r \leq 1)$. The set $[U]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $U, V \in \mathbb{R}_{\mathbb{F}}$ and $\lambda \in \mathbb{R}$, it is positive to define uniquely the sum $U \oplus V$ and the product $U \odot V$ as follows:

$$[U \oplus V]^r = [U]^r + [V]^r \text{ and } [\lambda \odot U]^r = \lambda[U]^r.$$

Now, denote the interval $[U]^r$ by $[u_1^{(r)}, u_2^{(r)}]$, where $u_1^{(r)} \leq u_2^{(r)}$ and $u_1^{(r)}, u_2^{(r)} \in \mathbb{R}$, for $r \in [0, 1]$.

Now, define $\hat{d} : \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}$ by

$$\hat{d}(U, V) = \sup_{r\in[0,1]} d([U]^r, [V]^r).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Definition 1.1. A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to a fuzzy number x_0 if for every $\epsilon > 0$ there exist a positive integer n_0 such that

$$\hat{d}(x_k, x_0) < \epsilon, \text{ for } k > n_0.$$

Definition 1.2. A sequence $x = (x_k)$ of fuzzy numbers is said to be $\mathcal{I}$- convergent to a fuzzy number x_0 if for every $\epsilon > 0$ such that

$$\{k \in \mathbb{N} : \hat{d}(x_k, x_0) \geq \epsilon\} \in \mathcal{I}.$$

Throughout the article, we denote Zweier fuzzy number sequence $Z(x)$ by x' for $x \in \omega^F$.

Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ be a bounded sequence of positive real numbers, $\lambda = (\lambda_{nk})$ be an infinite matrix, θ be a lacunary sequence and ω^F is the set of all sequences of fuzzy real numbers. In the present paper we define lacunary Zweier $\mathcal{I}-$convergent, lacunary Zweier $\mathcal{I}-$null and lacunary Zweier $\mathcal{I}-$bounded sequence spaces of fuzzy numbers as follows:

$$\mathcal{Z}^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Bigg\{x = (x_k) \in \omega^F : \Bigg\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Bigg[M_k\Bigg(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\Bigg)\Bigg]^{q_k} \geq \epsilon\Bigg\} \in \mathcal{I} \text{ for some } \rho > 0 \text{ and } x_0 \in \mathbb{R}_{\mathbb{F}}\Bigg\},$$

$$\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Bigg\{x = (x_k) \in \omega^F : \Bigg\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Bigg[M_k\Bigg(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\Bigg)\Bigg]^{q_k} \geq \epsilon\Bigg\} \in \mathcal{I} \text{ for some } \rho > 0\Bigg\}$$

and

$$\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Bigg\{x = (x_k) \in \omega^F : \exists K > 0 \text{ s.t. } \Bigg\{n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Bigg[M_k\Bigg(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\Bigg)\Bigg]^{q_k} \geq K\Bigg\} \in \mathcal{I} \text{ for some } \rho > 0\Bigg\},$$

where,

$$\bar{0}(t) = \begin{cases} 1, & if \ \ t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

If $0 < q_k \leq \sup q_k = D, C = \max(1, 2^{D-1})$. Then

$$|c_k + d_k|^{q_k} \leq C(|c_k|^{q_k} + |d_k|^{q_k}), \tag{1.1}$$

for all $c_k, d_k \in \mathbb{R}$ and for all $k \in \mathbb{N}$.

The main purpose of this paper is to study some classes of lacunary Zweier sequences of fuzzy numbers defined by means of generalized difference matrix operator, Musielak-Orlicz function and infinite matrix. We shall make an effort to study some interesting algebraic and topological properties of concerning sequence spaces. Also, we examine some interrelations between these sequence spaces.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ be a bounded sequence of positive real numbers and θ be a lacunary sequence. Then the sequence spaces $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$, $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ and $\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ are closed under addition and scalar multiplication.*

Proof. Consider $x = (x_k), y = (y_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ and α, β are scalars. Then there exist positive numbers $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \in \mathcal{I}$$

and

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \in \mathcal{I}.$$

Since A_i^p is linear and by using the continuity of Musielak-Orlicz function $\mathcal{M}$, we have the following inequality:

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p(\alpha(x_k') + \beta(y_k')))}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\right]^{q_k} \\
&\quad\leq D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \\
&\quad+ D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k} \\
&\quad\leq DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \\
&\quad+ DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k},
\end{aligned}$$

where $K = \max\left\{1, \frac{|\alpha|\rho_1}{|\alpha|\rho_1+|\beta|\rho_2}, \frac{|\beta|\rho_2}{|\alpha|\rho_1+|\beta|\rho_2}\right\}$.

Thus, we have

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p(\alpha(x_k') + \beta(y_k')))}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\right]^{q_k} \geq \epsilon\right\} \\
&\quad\subseteq \left\{n \in \mathbb{N} : DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \\
&\quad\cup \left\{n \in \mathbb{N} : DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\}.
\end{aligned}$$

Since the sets on right hand side of above relation belong to $\mathcal{I}$. Thus, the sequence space $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ is closed under addition and scalar multiplication . Similarly, we can prove others. $\square$

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ and $v = (v_k)$ be two bounded sequences of positive real numbers with $0 < q_k \leq v_k$ for each k and $(\frac{v_k}{q_k})$ be bounded. Then*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

(i) $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$,
(ii) $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$,
(iii)$\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$.

Proof. The proof of the theorem is straightforward, so we omit it. □

Theorem 2.3. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and $q=(q_k)$ be a bounded sequence of positive numbers. Then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q],\subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}$ $[A_i^P,\theta,\lambda,\mathcal{M},q]$.*

Proof. We know that the first inclusion is obvious. Next, we show that $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$. Let $(x_k)\in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]$. Then we have

$$\begin{aligned}
&\tfrac{1}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}\\
&\quad\le \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',x_0)}{\rho}\Big)\Big]^{q_k}\\
&\quad+ \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(x_0,\bar{0})}{\rho}\Big)\Big]^{q_k}\\
&\quad\le \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',x_0)}{\rho}\Big)\Big]^{q_k}\\
&\quad+ C\max\Big\{1,\sup\Big(\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(x_0,\bar{0})}{\rho}\Big)\Big]\Big)^D\Big\},
\end{aligned}$$

where $\sup q_k=D$ and $C=\max(1,2^{D-1})$. Therefore, $(x_k)\in \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]$. This completes the proof of the theorem. □

Theorem 2.4. *Let $\mathcal{M}=(M_k)$ and $\mathcal{M}'=(M_k')$ be two Musielak-Orlicz functions. Then the folowing inclusions holds:*
(i) $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$,
(ii) $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$,
(iii) $\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$.

Proof. Suppose $(x_k)\in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]$. Then, we have

$$\begin{aligned}
&\lambda_{nk}\Big[(M_k+M_k')\Big(\tfrac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}\\
&\quad\le C\Big[\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]\Big]^{q_k}+C\Big[\lambda_{nk}\Big[M_k'\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]\Big]^{q_k},
\end{aligned}$$

which consequently implies that

$$\begin{aligned}
&\tfrac{1}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[(M_k+M_k')\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}\\
&\quad\le \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}\\
&\quad+ \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k'\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}.
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

This implies $(x_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M} + \mathcal{M}', q]$. We can prove the other cases in the same way. □

Theorem 2.5. *Let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M'_k)$ be two Musielak-Orlicz functions. Then the folowing inclusion holds:*

$$\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}', q] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}.\mathcal{M}', q].$$

Proof. For given $\epsilon > 0$ and choose ϵ_0 such that $\sup_n \left(\sum_{k \in I_r} \lambda_{nk}\right) \max\{\epsilon_0^h, \epsilon_0^D\} < \epsilon$. Choose $0 < \varphi < 1$ such that $M_k(t) < \epsilon_0$, for all $k \in \mathbb{N}$. Let $x = (x_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}', q]$. Then for some $\rho > 0$, we have

$$B_1 = \left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)\right]^{q_k} \geq \varphi^D\right\} \in \mathcal{I}.$$

If $n \notin B_1$, then we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)\right]^{q_k} < \varphi^D.$$

This implies

$$\left[M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)\right]^{q_k} < \varphi^D \quad \text{for all } k \in \mathbb{N}.$$

Hence,

$$M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)^{q_k} < \varphi \quad \text{for all } k \in \mathbb{N}.$$

Therefore,

$$M_k\left(M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)\right)^{q_k} < \epsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Thus, we get

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)\right)\right]^{q_k} < \sup_n \left(\sum_{k \in I_r} \lambda_{nk}\right) \max\{\epsilon_0^h, \epsilon_0^D\} < \epsilon.$$

Now, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)\right)\right]^{q_k} < \epsilon.$$

This implies

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(M'_k\left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho}\right)\right)\right]^{q_k} \geq \epsilon\right\} \subset B_1 \in \mathcal{I}.$$

This completes the proof. □

Theorem 2.6. *If $\lim q_k > 0$ and $x = (x_k) \to x_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$, then x_0 is unique.*

Proof. Let $\lim q_k = u_0$. Consider that $(x_k) \to x_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$ and $(x_k) \to y_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$. So, there exist $\rho_1, \rho_2 > 0$, such that

$$X_1 = \left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho_1}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \in \mathcal{I} \tag{2.1}$$

and

$$X_2 = \left\{ n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x_k', y_0)}{\rho_2} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \in \mathcal{I}. \tag{2.2}$$

Define $\rho = \max\{2\rho_1, 2\rho_2\}$. Then we have

$$\sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k}$$

$$\leq \quad D \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho} \right) \right]^{q_k} + D \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x_k', y_0)}{\rho} \right) \right]^{q_k}.$$

Then from (2.1) and (2.2), we have

$$\left\{ n \in \mathbb{N} : \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k} \geq \epsilon \right\}$$

$$\begin{aligned} &\subseteq \quad \left\{ n \in \mathbb{N} : D \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \\ &\cup \quad \left\{ n \in \mathbb{N} : D \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x_k', y_0)}{\rho_2} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \\ &\subseteq \quad X_1 \cup X_2 \in \mathcal{I}. \end{aligned}$$

Also,

$$\left[M_k \left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k} \to \left[M_k \left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{u_0} \quad \text{as } \ k \to \infty.$$

Then, we have

$$\lim_{k\to\infty} \left[M_k \left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{q_k} = \left[M_k \left(\frac{\hat{d}(x_0, y_0)}{\rho} \right) \right]^{u_0} = 0.$$

Thus, $x_0 = y_0$. □

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $q = (q_k)$ be a bounded sequence of positive real numbers,*
(a) If $0 < \inf q_k \leq q_k \leq 1$ for all k, then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$ and $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q] \subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$.
(b) If $1 \leq q_k \leq \sup q_k = D < \infty$ for all k, then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$ and $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}] \subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$.

Proof. (a) Suppose $(x_k) \in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$. Since $0 < \inf q_k \leq q_k \leq 1$, then we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho} \right) \right]$$

$$\leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho} \right) \right]^{q_k}.$$

Thus,

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\right)\right] \geq \epsilon\right\}$$

$$\subseteq \quad \left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\right)\right]^{q_k} \geq \epsilon\right\} \in \mathcal{I}.$$

The other part can be proved in the same way.

(ii) Suppose $(x_k) \in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$. Since $1 \leq q_k \leq \sup q_k = D < \infty$. Then for each $0 < \epsilon < 1$, there exists a positive integer m_0 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\right)\right] \leq \epsilon < 1,$$

for all $n \geq m_0$. This implies

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\right)\right]^{q_k}$$

$$\leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\right)\right].$$

Thus,

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\right)\right]^{q_k} \geq \epsilon\right\}$$

$$\subseteq \quad \left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho}\right)\right] \geq \epsilon\right\} \in \mathcal{I}.$$

The other part can be proved in the same way. □

References

[1] R. Anand, C. Sharma and K. Raj, *Seminormed double sequence spaces of four dimensional matrix and Musielak-Orlicz function*, J. Inequal. Appl., (2018), **2018**:285.

[2] E. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J., **55** (2003), 136-147.

[3] M. Başarir and M. Kayikçi, *On the generalized B^m-Riesz difference sequence spaces and β-property*, J. Inequal. Appl., **2009** (2009), Article ID 385029, 18 pages.

[4] M. Başarir, S. Kayikci and E. E. Kara, *Some generalized difference statistically convergent sequence spaces in 2-normed space*, J. Inequal. Appl., **37** (2013), 2013: 177.

[5] M. Et and R. Çolak, *On some generalized difference sequence spaces and related matrix transformations*, Hokkaido Math. J., **26** (1997), 483-492.

[6] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesaro-type summability spaces*, Proc. London Math. Soc., **37** (1978), 508-520.

[7] M. Gurdal, *On ideal convergent sequences in 2-normed spaces*, Thai J. Math., **4** (2006), 85-91.

[8] B. Hazarika and E. Savas, *Some I-convergent lambda-summable difference sequences spaces of fuzzy real numbers defined by a sequence of Orlicz functions*, Math. Comput. Modelling, **54** (2011), 2986-2998.

[9] H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull., **24** (1981), 169-176.

[10] P. Kostyrko, T. Salat and W. Wilczynski, *I-convergence*, Real Anal. Exchange, **26** (2000-2001), 669-686.

[11] J. Lindenstrauss and L. Tzafriri, *An Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379-390.

[12] M. Matloka, *Sequences of fuzzy numbers*, J. Math. Anal. Appl., **28** (1986), 28-37.

[13] S. Nanda., *On sequences of fuzzy number*, Fuzzy Sets Syst., **33** (1989), 123-126.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

[14] M. L. Puri and D. A. Ralescu, *Differentials of fuzzy functions*, J. Math. Anal. Appl., **91** (1983), 552-558.
[15] K. Raj, A. Choudhary and C. Sharma, *Almost strongly Orlicz double sequence spaces of regular matrices and their applications to statistical convergence*, Asian-Eur. J. Math., **11** 1850073 (2018), doi.org/10.1142/S1793557118500730.
[16] K. Raj, C. Sharma and A. Choudhary, *Applications of Tauberian theorem in Orlicz spaces of double difference sequences of fuzzy numbers*, J. Intell. Fuzzy Systems, **35** (2018), 2513-2524.
[17] K. Raj, A. Abzhapborav and A. Khassymkan, *Some generalized difference sequences of ideal convergence and Orlicz functions*, J. Comput. Anal. Appl., **22** (2017), 52-63.
[18] K. Raj, and R. Anand, *Double difference spaces of almost null and almost convergent sequences for Orlicz function*, J. Comput. Anal. Appl., **24** (2018), 773-783.
[19] A. Choudhary and K. Raj, *Applications of double difference fractional order operators to originate some spaces of sequences*, J. Comput. Anal. Appl., **28** (2020), 94-103.
[20] E. Savas and M. Mursaleen, *On statistically convergent double sequence of fuzzy numbers*, Inform. Sci., **162** (2004), 183-192.
[21] E. Savas, *A sequence spaces in 2-normed space defined by ideal convergence and an Orlicz function*, Abst. Appl. Anal., (2011), Article ID 741382, 1-8.
[22] M. Şengönül, *On The Zweier Sequence Space*, Demonstratio Math., **40** (2007), 181-196.
[23] B. C. Tripathy and S. Nanda, *Absolute value of fuzzy real number and fuzzy sequence spaces*, Jour. Fuzzy Math., **8** (2000), 883-892
[24] B. C. Tripathy, S. Debnath and S. Saha, *On some difference sequence spaces of interval numbers*, Proyecciones, **37** (2018), 603-612.
[25] B. C. Tripathy and A. J. Dutta, *On I-acceleration convergence of sequences of fuzzy numbers*, Math. Modell. Analysis, **17** (2012), 549-557.
[26] B. C. Tripathy and B. Hazarika, *Some I-convergent sequence spaces defined by Orlicz functions*, Acta. Math. Appl. Sinica, **27** (2011), 149-154.
[27] B. C. Tripathy and A. Esi, *A new type of difference sequences spaces*, Int. J. Sci. and Tech., **1** (2006), 11-14.
[28] B. C. Tripathy and R. Goswami, *Statistically convergent multiple sequences in probalistic normed spaces*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **78** (2016), 83-94.
[29] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338-353.

School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
E-mail address: kavitasainitg3@gmail.com
E-mail address: kuldipraj68@gmail.com