Invariance, solutions, periodicity and asymptotic behavior of a class of fourth order difference equations

Mensah Folly-Gbetoula *

School of Mathematics, University of the Witwatersrand, 2050, Johannesburg, South Africa.

Abstract

We construct Lie symmetry generators of some fourth order difference equations. We use these generators to derive similarity variables that make it possible to obtain exact solutions. In some cases, we study periodicity and asymptotic behavior of the solutions.

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1 Introduction

Several years back, Sophus Lie studied the invariance property of equations under a group of transformations. The approach used was later known as Lie symmetry method. This method has been used to solve differential equations, and recently it has been applied to difference equations. Although Maeda studied difference equations via Lie symmetry analysis in twentieth century [9, 10], it is Hydon who really rekindled the interest for solving difference equations via symmetry. For Hydon's work, refer to [8].
Most often, difference equations arise as a result of discretizing differential equations, especially in phenomena that depend on time. There are many ways in which a differential equation can be discretized (see [4]). Difference equations have numerous applications. For example, biological systems, population dynamics, economics, physics (see [1, 2]). Although difference equations appear simple, finding their solutions can be incredibly difficult. The symmetry approach to finding solutions of difference equations is recent and the reader can refer to [8] and some recent articles [5–7, 11, 12] for further knowledge on this method.
In this paper, we consider the system of difference equations

$$x_{n+4} = \frac{x_n x_{n+1}}{x_{n+3}(a_n + b_n x_n x_{n+1})} \tag{1}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers. For equation (1), we derive all Lie point symmetries and give formulas for solutions in closed form. We also discuss periodicity and asymptotic behavior of solutions in some cases.

*Mensah.Folly-Gbetoula@wits.ac.za

1.1 Preliminaries

In this section, we give a background on symmetry methods for difference equations. Our definitions and notation come from [3, 8, 13].
Consider the difference equations

$$x_{n+4} = \Omega(x_n, x_{n+1}, x_{n+3}), \tag{2}$$

where n denotes the independent variable; x_n the dependent variable. In this case u_{n+i} denotes the 'i-th shift' of u_n.
Consider the group of transformations

$$(n, x_n) \mapsto (n, \tilde{x}_n = x_n + \varepsilon Q_1(n, x_n) + O(\varepsilon^2)), \tag{3}$$

where Q is the characteristic of the group of point transformations. Let

$$X = Q(n, x_n)\frac{\partial}{\partial x_n} \tag{4}$$

be the corresponding infinitesimal generator. The group of transformations (3) is a symmetry group if and only if

$$Q(n+4, \Omega) - \mathcal{X}(\Omega) = 0, \tag{5}$$

whenever (2) holds. Here,

$$\mathcal{X} = Q(n, x_n)\frac{\partial}{\partial x_n} + Q(n, x_{n+1})\frac{\partial}{\partial x_{n+1}} + Q(n+3, x_{n+3})\frac{\partial}{\partial x_{n+3}}$$

denotes the prolongation of X to all shifts of x_n appearing in the right hand sides of equations in (2). Equation (5), known as the linearized symmetry condition, can be solved for Q by applying the appropriate differential operators. The characteristic, together with the canonical coordinate

$$s = \int \frac{dx_n}{Q(n, x_n)}, \tag{6}$$

are necessary in the reduction of order of (2). The following definition can be used to check if a given function is invariant under a given group of transformations.

Definition 1 *[13] Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \textit{for all} \qquad x \in M.$$

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2 Main results

2.1 Symmetry and difference invariant

To obtain the the criterion which gives the Lie point symmetries of (1), we force (5) on

$$x_{n+4} = \frac{x_n x_{n+1}}{x_{n+3}(a_n + b_n x_n x_{n+1})}. \tag{7}$$

This leads to

$$Q(n+4, x_{n+4}) + \frac{x_n x_{n+1}(a_n + b_n x_n x_{n+1})Q(n+3, x_{n+3})}{{x_{n+3}}^2(a_n + b_n x_n x_{n+1})^2} - \frac{a_n[x_n Q(n+1, x_{n+1}) + x_{n+1}Q(n, x_n)]}{x_{n+3}(a_n + b_n x_n x_{n+1})^2} = 0. \tag{8}$$

The methodology of solving these functional equations is given as follows:

- Firstly, apply the differential operator $\frac{\partial}{\partial x_n} + \frac{x_{n+1}}{x_n}\frac{\partial}{\partial x_{n+1}}$ on equation (8). This leads (after simplification) to

  $$x_{n+1}Q'(n+1, x_{n+1}) - x_{n+1}Q'(n, x_n) - Q(n+1, x_{n+1}) + \frac{a_n}{x_n}Q(n, x_n) = 0.$$

- Secondly, differentiate with respect to x_n, separate by powers of x_{n+1} and solve the resulting system of over determining equations for Q. This gives

  $$Q(n, x_n) = \alpha(n)x_n + \beta(n)$$

  for some functions α and β of n.

- Lastly, substitute the latter in (8) to eliminate any dependency among the arbitrary functions that appear in Q. This leads to the constraints

  $$\alpha(n) + \alpha(n+1) = 0 \text{ and } \quad \beta(n) = 0. \tag{9}$$

We have omitted the details in the computation. The constraints in (9) are readily solved ($\alpha(n) = (-1)^n$) and we have

$$Q = (-1)^n x_n. \tag{10}$$

Consequently, Equation (1) admits a one dimensional Lie algebra:

$$X = (-1)^n x_n \frac{\partial}{\partial x_n}. \tag{11}$$

The canonical coordinate is given by

$$s_n = \int \frac{dx_n}{(-1)^n x_n} = (-1)^n \ln|x_n| \tag{12}$$

and the difference invariant which is inspired by the form of the final constraints (9) is given by

$$\mathbf{u}_n = (-1)^n s_n + (-1)^{n+1} s_{n+1}. \tag{13}$$

It is not difficult to verify, using Definition 1 together with (11), that (13) is indeed invariant under the group of transformations of (1). For simplicity, we prefer using the compatible variable

$$|u_n| = \exp(-\mathbf{u}_n) \tag{14}$$

which is also invariant. This gives a convenient choice of the change variables which does not require lucky guesses. With this variable u_n, it follows that

$$u_{n+3} = a_n u_n + b_n \tag{15}$$

whose solution is given by

$$u_{3n+j} = u_j \left(\prod_{k_1=0}^{n-1} a_{3k_1+j} \right) + \sum_{l=0}^{n-1} \left(b_{3l+j} \prod_{k_2=l+1}^{n-1} a_{3k_2+j} \right), \quad j = 0, 1, 2. \tag{16}$$

To obtain the solutions of (1), we go up the hierarchy created by the changes of variables. By evaluating (13) as a telescoping series, we have

$$(-1)^n s_n = (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{k_1} \mathbf{u}_{k_1} + (-1)^n s_0 \quad (= \ln |x_n| \quad \text{from} \quad (12)), \tag{17}$$

i.e.

$$x_n = \exp\left\{ (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{k_1} \mathbf{u}_{k_1} + (-1)^n s_0 \right\}, \tag{18}$$

$$= \exp\{ \sum_{k_1=0}^{n-1} (-1)^{n+k_1} \ln u_{k_1} + \ln x_0 \}, \tag{19}$$

where all the u_{k_1}'s are obtained using (16).
Note. Equation (19) gives the closed form solution of (1) in a unified manner. Looking at the form of u_l in (16), we rephrase (19) as follows:

$$x_{6n+j} = \exp\left\{ \sum_{k_1=0}^{6n+j-1} (-1)^{6n+j+k_1} \ln u_{k_1} + \ln x_0 \right\}, \tag{20}$$

$$= x_j \prod_{i=0}^{n-1} \left(\prod_{r=0}^{2} \frac{u_{6i+j+2r}}{u_{6i+j+2r+1}} \right), \tag{21}$$

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$j = 0, 1, \ldots, 5$. More clearly,

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{u_{3(2i)}}{u_{3(2i)+1}} \frac{u_{3(2i)+2}}{u_{3(2i+1)}} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}}, \tag{22a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{u_{3(2i)+1}}{u_{3(2i)+2}} \frac{u_{3(2i+1)}}{u_{3(2i+1)+1}} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}}, \tag{22b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{u_{3(2i)+2}}{u_{3(2i+1)}} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}} \frac{u_{3(2i+2)}}{u_{3(2i+2)+1}}, \tag{22c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)}}{u_{3(2i+1)+1}} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}} \frac{u_{3(2i+2)+1}}{u_{3(2i+2)+2}}, \tag{22d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}} \frac{u_{3(2i+2)}}{u_{3(2i+2)+1}} \frac{u_{3(2i+2)+2}}{u_{3(2i+3)}}, \tag{22e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}} \frac{u_{3(2i+2)+1}}{u_{3(2i+2)+2}} \frac{u_{3(2i+3)}}{u_{3(2i+3)+1}}. \tag{22f}$$

We then substitute the expressions given in (16) in (22) to get

$$\begin{aligned} x_{6n} = & x_0 \prod_{i=0}^{n-1} \frac{u_0 \prod_{l_1=0}^{2i-1} a_{3l_1} + \sum_{j=0}^{2i-1} b_{3j} \prod_{l_2=j+1}^{2i-1} a_{3l_2}}{u_1 \prod_{l=0}^{2i-1} a_{3l+1} + \sum_{j=0}^{2i-1} b_{3j+1} \prod_{l_2=j+1}^{2i-1} a_{3l_2+1}} \frac{u_2 \prod_{l=0}^{2i-1} a_{3l+2} + \sum_{j=0}^{2i-1} b_{3j+2} \prod_{l_2=j+1}^{2i-1} a_{kl_2+2}}{u_0 \prod_{l=0}^{2i} a_{3l} + \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{kl_2}} \\ & \frac{u_1 \prod_{l=0}^{2i} a_{3l+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{kl_2+1}}{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{kl_2+2}}, \end{aligned} \tag{23a}$$

$$\begin{aligned} x_{6n+1} = & x_1 \prod_{i=0}^{n-1} \frac{u_1 \prod_{l_1=0}^{2i-1} a_{3l_1+1} + \sum_{j=0}^{2i-1} b_{3j+1} \prod_{l_2=j+1}^{2i-1} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i-1} a_{3l+2} + \sum_{j=0}^{2i-1} b_{3j+2} \prod_{l_2=j+1}^{2i-1} a_{3l_2+2}} \frac{u_0 \prod_{l=0}^{2i} a_{3l} + \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}}{u_1 \prod_{l=0}^{2i} a_{3l+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}} \\ & \frac{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}, \end{aligned} \tag{23b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{u_2 \prod_{l_1=0}^{2i-1} a_{3l_1+2} + \sum_{j=0}^{2i-1} b_{3j+2} \prod_{l_2=j+1}^{2i-1} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i} a_{3l} + \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}} \frac{u_1 \prod_{l=0}^{2i} a_{3l+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}$$
$$\frac{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}, \tag{23c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{u_0 \prod_{l_1=0}^{2i} a_{3l_1} + \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}}{u_1 \prod_{l=0}^{2i} a_{3l+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}} \frac{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}$$
$$\frac{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i+1} a_{3l+2} + \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}}, \tag{23d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{u_1 \prod_{l_1=0}^{2i} a_{3l_1+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}} \frac{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{kl_2+1}}$$
$$\frac{u_2 \prod_{l=0}^{2i+1} a_{3l+2} + \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i+2} a_{3l} + \sum_{j=0}^{2i+2} b_{3j} \prod_{l_2=j+1}^{2i+2} a_{3l_2}}, \tag{23e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{u_2 \prod_{l_1=0}^{2i} a_{3l_1+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}} \frac{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i+1} a_{3l+2} + \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}}$$
$$\frac{u_0 \prod_{l=0}^{2i+2} a_{3l} + \sum_{j=0}^{2i+2} b_{3j} \prod_{l_2=j+1}^{2i+2} a_{3l_2}}{u_1 \prod_{l=0}^{2i+2} a_{3l+1} + \sum_{j=0}^{2i+2} b_{3j+1} \prod_{l_2=j+1}^{2i+2} a_{3l_2+1}}. \tag{23f}$$

We rewrite (23) in terms of initial conditions only as follows:

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{\prod_{l_1=0}^{2i-1} a_{3l_1} + x_0x_1 \sum_{j=0}^{2i-1} b_{3j} \prod_{l_2=j+1}^{2i-1} a_{3l_2}}{\prod_{l=0}^{2i-1} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i-1} b_{3j+1} \prod_{l_2=j+1}^{2i-1} a_{3l_2+1}} \frac{\prod_{l=0}^{2i-1} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i-1} b_{3j+2} \prod_{l_2=j+1}^{2i-1} a_{kl_2+2}}{\prod_{l=0}^{2i} a_{3l} + x_0x_1 \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{kl_2}} \frac{\prod_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{kl_2+1}}{\prod_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{kl_2+2}}, \tag{24a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{\prod_{l_1=0}^{2i-1} a_{3l_1+1} + x_1x_2 \sum_{j=0}^{2i-1} b_{3j+1} \prod_{l_2=j+1}^{2i-1} a_{3l_2+1}}{\prod_{l=0}^{2i-1} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i-1} b_{3j+2} \prod_{l_2=j+1}^{2i-1} a_{3l_2+2}} \frac{\prod_{l=0}^{2i} a_{3l} + x_0x_1 \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}}{\prod_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}} \frac{\prod_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}, \tag{24b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{\prod_{l_1=0}^{2i-1} a_{3l_1+2} + x_2x_3 \sum_{j=0}^{2i-1} b_{3j+2} \prod_{l_2=j+1}^{2i-1} a_{3l_2+2}}{\prod_{l=0}^{2i} a_{3l} + x_0x_1 \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}} \frac{\prod_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}}{\prod_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}} \frac{\prod_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}{\prod_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}, \tag{24c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{\prod_{l_1=0}^{2i} a_{3l_1} + x_0x_1 \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}}{\prod_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}} \frac{\prod_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}} \frac{\prod_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}{\prod_{l=0}^{2i+1} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}}, \tag{24d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{\prod_{l_1=0}^{2i} a_{3l_1+1} + x_1x_2 \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}}{\prod_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}} \frac{\prod_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}{\prod_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{kl_2+1}} \frac{\prod_{l=0}^{2i+1} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}}{\prod_{l=0}^{2i+2} a_{3l} + x_0x_1 \sum_{j=0}^{2i+2} b_{3j} \prod_{l_2=j+1}^{2i+2} a_{3l_2}}, \tag{24e}$$

$$
\begin{aligned}
x_{6n+5} =& x_5 \prod_{i=0}^{n-1} \frac{\prod_{l_1=0}^{2i} a_{3l_1+2} + x_2x_3 \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}} \frac{\prod_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}{\prod_{l=0}^{2i+1} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}} \\
& \frac{\prod_{l=0}^{2i+2} a_{3l} + x_0x_1 \sum_{j=0}^{2i+2} b_{3j} \prod_{l_2=j+1}^{2i+2} a_{3l_2}}{\prod_{l=0}^{2i+2} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i+2} b_{3j+1} \prod_{l_2=j+1}^{2i+2} a_{3l_2+1}},
\end{aligned}
\tag{24f}
$$

where $x_4 = x_0x_1/(x_3(a_0 + b_0x_0x_1))$ and $x_5 = x_2x_3(a_0 + b_0x_0x_1)/(x_0(a_1 + b_1x_1x_2))$. In the following subsections, we study some special cases.

2.2 The case where (a_n) and (b_n) are 3 periodic sequences

Let $a_n = \{a_0, a_1, a_2, a_0, a_1, a_2, \dots\}$ and $b_n = \{b_0, b_1, b_2, b_0, b_1, b_2, \dots\}$. Equations in (23) reduce to

$$
\begin{aligned}
x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{a_0^{2i} + b_0x_0x_1 \sum_{j=0}^{2i-1} a_0^j}{a_1^{2i} + b_1x_1x_2 \sum_{j=0}^{2i-1} a_1^j} \frac{a_2^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a_2^j}{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j} \frac{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j}, \\
x_{6n+1} =& x_1 \prod_{i=0}^{n-1} \frac{a_1^{2i} + b_1x_1x_2 \sum_{j=0}^{2i-1} a_1^j}{a_2^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a_2^j} \frac{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j}{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j} \frac{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j}, \\
x_{6n+2} =& x_2 \prod_{i=0}^{n-1} \frac{a_2^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a_2^j}{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j} \frac{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j} \frac{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j}{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j}, \\
x_{6n+3} =& x_3 \prod_{i=0}^{n-1} \frac{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j}{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j} \frac{u_2a_2^{2i+1} + b_2 \sum_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j} \frac{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j}{a_2^{2i+2} + b_2x_2x_3 \sum_{j=0}^{2i+1} a_2^j}, \\
x_{6n+4} =& x_4 \prod_{i=0}^{n-1} \frac{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j} \frac{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j}{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j} \frac{a_2^{2i+2} + b_2x_2x_3 \sum_{j=0}^{2i+1} a_2^j}{a_0^{2i+3} + b_0x_0x_1 \sum_{j=0}^{2i+2} a_0^j},
\end{aligned}
$$

$$
x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j} \frac{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j}{a_2^{2i+2} + b_2x_2x_3 \sum_{j=0}^{2i+1} a_2^j} \frac{a_0^{2i+3} + b_0x_0x_1 \sum_{j=0}^{2i+2} a_0^j}{a_1^{2i+3} + b_1x_1x_2 \sum_{j=0}^{2i+2} a_1^j}.
$$

2.3 The case where (a_n) and (b_n) are real constants

Let $a_n = a$ and $b_n = b$. Equations in (23) give rise to

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_0x_1 \sum_{j=0}^{2i-1} a^j}{a^{2i} + bx_1x_2 \sum_{j=0}^{2i-1} a^j} \frac{a^{2i} + bx_2x_3 \sum_{j=0}^{2i-1} a^j}{a^{2i+1} + bx_0x_1 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + bx_1x_2 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j}, \tag{25a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_1x_2 \sum_{j=0}^{2i-1} a^j}{a^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a^j} \frac{a^{2i+1} + bx_0x_1 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_1x_2 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j}, \tag{25b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_2x_3 \sum_{j=0}^{2i-1} a^j}{a^{2i+1} + bx_0x_1 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j} \frac{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j}, \tag{25c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{a^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a^j}{a^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_2x_3 \sum_{j=0}^{2i+1} a^j}, \tag{25d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{a^{2i+1} + bx_1x_2 \sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j} \frac{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_2x_3 \sum_{j=0}^{2i+1} a^j}{a^{2i+3} + bx_0x_1 \sum_{j=0}^{2i+2} a^j}, \tag{25e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{a^{2i+1} + bx_2x_3 \sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_1x_2 \sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_2x_3 \sum_{j=0}^{2i+1} a^j} \frac{a^{2i+3} + bx_0x_1 \sum_{j=0}^{2i+2} a^j}{a^{2i+3} + bx_1x_2 \sum_{j=0}^{2i+2} a^j}. \tag{25f}$$

2.3.1 The case where $a = 1$

Equations in (25) simplify to

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{1 + 2ibx_0x_1}{1 + 2ibx_1x_2} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_0x_1} \frac{1 + (2i+1)bx_1x_2}{1 + (2i+1)bx_2x_3}, \tag{26a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{1 + 2ibx_1x_2}{1 + 2ibx_2x_3} \frac{1 + (2i+1)bx_0x_1}{1 + (2i+1)bx_1x_2} \frac{1 + (2i+1)bx_2x_3}{1 + (2i+2)bx_0x_1}, \tag{26b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_0x_1} \frac{1 + (2i+1)bx_1x_2}{1 + (2i+1)bx_2x_3} \frac{1 + (2i+2)bx_0x_1}{1 + (2i+2)bx_1x_2}, \tag{26c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_0x_1}{1+(2i+1)bx_1x_2} \frac{1+(2i+1)bx_2x_3}{1+(2i+2)bx_0x_1} \frac{1+(2i+2)bx_1x_2}{1+(2i+2)bx_2x_3}, \tag{26d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_1x_2}{1+(2i+1)bx_2x_3} \frac{1+(2i+2)bx_0x_1}{1+(2i+2)bx_1x_2} \frac{1+(2i+2)bx_2x_3}{1+(2i+3)bx_0x_1}, \tag{26e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_2x_3}{1+(2i+2)bx_0x_1} \frac{1+(2i+2)bx_1x_2}{1+(2i+2)bx_2x_3} \frac{1+(2i+3)bx_0x_1}{1+(2i+3)bx_1x_2}. \tag{26f}$$

2.3.2 The case where $a = -1$

Let $a_n = -1$ and $b_n = b$. Equations in (23) result in

$$x_{6n} = \frac{x_0(x_1x_2b-1)^n}{(x_0x_1b-1)^n(x_2x_3b-1)^n}, x_{6n+1} = \frac{x_1(x_0x_1b-1)^n(x_2x_3b-1)^n}{(x_1x_2b-1)^n},$$

$$x_{6n+2} = \frac{x_2(x_1x_2b-1)^n}{(x_0x_1b-1)^n(x_2x_3b-1)^n}, x_{6n+3} = \frac{x_3(x_0x_1b-1)^n(x_2x_3b-1)^n}{(x_1x_2b-1)^n},$$

$$x_{6n+4} = \frac{x_0x_1(x_1x_2b-1)^n}{x_3(x_0x_1b-1)^{n+1}(x_2x_3b-1)^n}, x_{6n+5} = \frac{x_2x_3(x_0x_1b-1)^{n+1}(x_2x_3b-1)^n}{x_0(x_1x_2b-1)^{n+1}}.$$

2.4 Existence of six periodic solutions

From (26), if $a = 1$ and $b = 0$, then the solution of (1) is periodic with period six as long as $u_0 \neq x_2$ or $x_1 \neq x_3$. It should also be noted that the solutions are periodic with period two when $x_0 = x_2$ and $x_1 = x_3$.
The graphs below are cases where the solutions are six periodic.

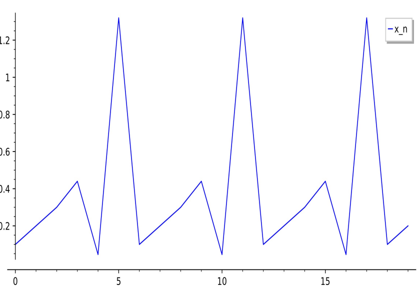

Figure 1: $a = 1, b = 0, x_0 = 0.1, x_1 = 0.2, x_2 = 0.3, x_3 = 0.44$.

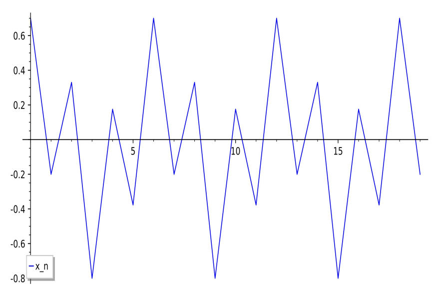

Figure 2: $a = 1, b = 0, x_0 = 0.7, x_1 = -0.2, x_2 = 0.33, x_3 = -0.8$.

2.5 Existence of 12-periodic solutions

Using (27), we have that if $a = -1$ and $b = 0$, then the solution of (1) is periodic with period twelve.
The graphs below are cases where the solutions are twelve periodic.

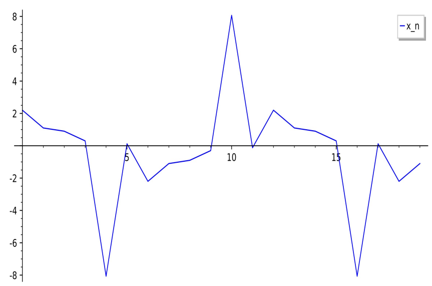

Figure 3: $a = -1, b = 0, x_0 = 2.2, x_1 = 1.1, x_2 = 0.9, x_3 = 0.3$.

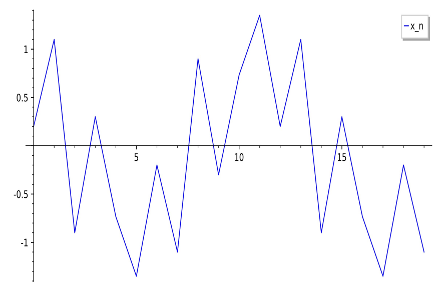

Figure 4: $a = -1, b = 0, x_0 = 0.2, x_1 = 1.1, x_2 = -0.9, x_3 = 0.3$.

3 Asymptotic behavior of the solutions for constant coefficients

Theorem 1 *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be the solution to the sequence in* (1) *where* $a_n = 1$ *for all* $n \geq 0$ *and* $b_n = b \neq 0$. *Then*

$$\lim_{n\to\infty} x_n = 0.$$

Proof 1 *Using* (26), *we have that*

$$\begin{aligned} x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{1 + 2ibx_0x_1}{1 + 2ibx_1x_2} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_0x_1} \frac{1 + (2i+1)bx_1x_2}{1 + (2i+1)bx_2x_3} \\ =& x_0 \prod_{i=0}^{n-1} \frac{1 + 2ibx_0x_1}{1 + (2i+1)bx_0x_1} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_2x_3} \frac{1 + (2i+1)bx_1x_2}{1 + (2i)bx_1x_2} \\ =& x_0 \prod_{i=0}^{n-1} \left(1 + \frac{bx_0x_1}{1 + 2ibx_0x_1}\right)^{-1} \left(1 + \frac{bx_2x_3}{1 + 2ibx_2x_3}\right)^{-1} \left(1 + \frac{bx_1x_2}{1 + 2ibx_1x_2}\right) \end{aligned}$$

We know that $1 + 2ix_kx_{k+1} \to \infty$ *as* $i \to \infty$. *Hence, there is a sufficiently large integer* t *such that for* $i \geq t$, *we have*

$$1 + 2ix_kx_{k+1} \sim 2ix_kx_{k+1}.$$

Thus

$$\begin{aligned} x_{6n} =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\left(1+\frac{1}{2i}\right)^{-1}\left(1+\frac{1}{2i}\right)^{-1}\left(1+\frac{1}{2i}\right) \\ =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\exp\left[\ln\left(1+\frac{1}{2i}\right)^{-1}+\ln\left(1+\frac{1}{2i}\right)^{-1}+\ln\left(1+\frac{1}{2i}\right)\right], \end{aligned}$$

where

$$\Gamma(t)=\prod_{i=0}^{t}\left(1+\frac{bx_0x_1}{1+2ibx_0x_1}\right)^{-1}\left(1+\frac{bx_2x_3}{1+2ibx_2x_3}\right)^{-1}\left(1+\frac{bx_1x_2}{1+2ibx_1x_2}\right).$$

Utilizing the expansion $\ln(1+x)=x+O(x^2),(1+x)^{-1}=1-x+O(x^2)$, *for* $x\to 0$, *we obtain*

$$\begin{aligned} x_{6n} =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\exp\left[-\frac{1}{2i}+O\left(\frac{1}{i^2}\right)\right] \\ =& x_0\Gamma(t)\exp\left[-\sum_{i=t+1}^{n-1}\left(\frac{1}{2i}\right)\right]\prod_{i=t+1}^{n-1}\exp\left[O\left(\frac{1}{i^2}\right)\right]. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} x_{6n}=0 \quad as \quad n\to\infty.$$

Similarly,

$$\lim_{n\to\infty} x_{6n+j}=0 \quad as \quad n\to\infty,$$

for $j=1,2,3,4,5$.

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